Probability of strength reduction of aged ship structural profiles due to corrosion wear

Author/Contributor:
Chowdhury, Mahiuddin

Publication details:
Proceedings of the Pacific 2006 International Maritime Conference

Event details:
Pacific 2006 International Maritime Conference
Sydney, Australia

Publication Date:
2006

DOI:
https://doi.org/10.26190/unsworks/635

Probability of strength reduction of aged ship structural profiles due to corrosion wear

Mac Chowdhury
The University of New South Wales, Australia

ABSTRACT

A relatively simple, straight-forward statistical method is presented here to assess the probability of reduction of geometrical properties of aged ship structural profiles. These rolled profiles such as bulb plate, tee bar, flat bar etc. begin to shrink mainly due to corrosion wear following the breakdown of protective coatings. Geometrical properties may include the stiffener cross-sectional area with or without the attached plating, moment of inertia, section modulus etc. among many others.

The first set of inputs includes the statistical data about the dimensions or scantlings of the structural profiles usually collected by the manufacturers. For example, the mean and coefficient of variation (c.o.v.) of the web height, web plate thickness etc. The second set of inputs for this analysis is the evenly distributed corrosion wear; the mean wear and the c.o.v. or standard deviation of wear.

Based on certain assumptions the mean and variance of the geometrical properties may be derived statistically from these basic inputs. The outputs may be used to address the following issue.

The Classification Societies set up Renewal Criteria for the stiffeners/profiles in ship hull structures when they do not meet the requirements which may be, for example, cross-sectional area falling below 90% of the original nominal value. However, the means and variances of the geometrical properties as functions of age have many other useful applications.

1. THEORETICAL BACKGROUND

1.1 Mean and Variance of a Function of Random Variables

Let Y be a general function of a number of random variables $x_i;\ i = 1,2,\cdots,n$. To any order of accuracy Y may be expanded as a Taylor series:

$$Y(x_1, x_2, \cdots x_n) = Y(\bar{x}_1, \bar{x}_2, \cdots \bar{x}_n) + \sum_{i=1}^{n}(x_i - \bar{x}_i)\left[\frac{\partial Y}{\partial x_i}\right]_m + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - \bar{x}_i)(x_j - \bar{x}_j)\left[\frac{\partial^2 Y}{\partial x_i \partial x_j}\right]_m + \cdots \quad (1.1)$$

In Eqn (1.1) the function is expanded around the mean values of $x_i = \bar{x}_i$ and the subscript '*m*' also implies that the derivatives are to be evaluated at the mean values of x_i. Here Y may be a geometrical property of the profile and x_i are the dimension variables, all of which are random quantities.

The first assumption is to approximate this function by its linear portion:

$$Y(x_1, x_2, \cdots, x_n) \cong Y(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) + \sum_{i=1}^{n} (x_i - \bar{x}_i)\left[\frac{\partial Y}{\partial x_i}\right]_m \tag{1.2}$$

It immediately follows that the mean value

$$\bar{Y} = Y(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \tag{1.3}$$

To derive the variance of *Y* it is necessary to make a second assumption: the dimension variables x_i are statistically uncorrelated. Therefore the variance of *Y* is given as

$$D_Y = \sum_{i=1}^{n}\left[\frac{\partial Y}{\partial x}\right]_m^2 D_i = \sum_{i=1}^{n}\left\{\left[\frac{\partial Y}{\partial x_i}\right]_m \sigma_i\right\}^2 \tag{1.4}$$

where D_i = the variance of x_i

and σ_i = standard deviation of x_i.

Eqns (1.3) and (1.4) are the basis of estimating the mean and variance/standard deviation of any random quantity which is a general function of *n* other random variables for which means and variances are known.

It will be more convenient to use a non-dimensional form of the random function *Y*:

$$y = \frac{Y}{Y_{\text{nom}}} \tag{1.5a}$$

also

$$\sigma_y = \frac{\sigma_Y}{Y_{\text{nom}}} = \frac{\sqrt{D_Y}}{Y_{\text{nom}}} \tag{1.5b}$$

where Y_{nom} is the nominal value of *Y*. Note the symbol *y*, a lower case for the non-dimensional values. The capital '*Y*' indicates absolute values.

1.2 Probability Density Function (pdf)

With the first two moments, that is mean and variance, the obvious choice is a Gaussian distribution. However, it needs to be mentioned that other types such as log-normal and Type I asymptotic extreme value distributions are also defined by only these two parameters [Ochi, 1990]. However, the third assumption made here is that the shipbuilding structural profiles obey the Gaussian distribution. This assumption is supported by Ivanov [1982, 1984, 1987, 1991] among others. Therefore,

$$p(y) = \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left[-\left(\frac{y-\bar{y}}{\sigma_y\sqrt{2}}\right)^2\right] \qquad (-\infty \le y \le \infty) \tag{1.6}$$

1.3 Truncated Normal Distribution for *y*

Clearly, as indicated in Eqn (1.6), the geometrical properties cannot assume a negative value. Hence it is necessary to modify Eqn (1.6) to restrict the negative values of *y* in the pdf of *y*. The truncated normal distribution is:

$$p_C(y) = \frac{p(y)}{\Phi(\bar{y}/\sigma_y)} \qquad (0 \le y \le \infty) \qquad (1.7)$$

However, in the present context the value of the cumulative standard normal distribution function, $\Phi(\bar{y}/\sigma_y)$, is extremely close to unity because $\bar{y}/\sigma_y >> 3$. Finally,

$$p_C \cong \frac{1}{\sigma_y\sqrt{2\pi}} \exp\left[-\left(\frac{y-\bar{y}}{\sigma_y\sqrt{2}}\right)^2\right] \qquad (0 \le y \le \infty) \qquad (1.8)$$

Eqn (1.8) has a number of applications including the topic raised in this paper. The right-hand side of Eqn (1.8) is an ordinary Gaussian probability density function and its cumulative distribution, $\Phi(\ )$, is tabulated in most textbooks on statistics.

2. CORROSION WEAR

Broadly speaking there are two distinct approaches to quantify the corrosion wear in the context of statistical analysis. One is based on actual measurements of ships in operation and the other is based on a more accurate description of the electrochemical mechanisms involved. The first method is termed "empirical" model and the latter as "phenomenological" model. In both cases only overall corrosion wear is considered; pitting and grooving is not treated here.

2.1 Empirical Models

2.1.1 Model proposed by Paik *et al.*

Paik *et al.* [2003, 2004] proposed a model based on extensive measurements of sea-water ballast tanks of large oil tankers and bulk carriers. After analysing data collected from nearly 2000 locations their final recommendations are as follows.

The average or most probable values are:

Coating Life, t_C (yrs)	**Mean Thickness Loss (δ mm/yr)**	**Standard Deviation (σ mm/yr)**
5	0.0466	0.0378
7.5	0.0579	0.0479
10	0.0823	0.0758

The possible severe corrosion losses are:

Coating Life, t_C (yrs)	**Mean Thickness Loss (δ mm/yr)**	**Standard Deviation (σ mm/yr)**
5	0.1469	0.0314
7.5	0.1938	0.0426
10	0.2894	0.0644

It should be noted that these numbers represent the uniform (average) annual rate, not the total loss.

2.1.2 Ivanov's (ABS) corrosion model

In his report Ivanov [2001b] gave some arbitrary figures regarding mean values of average annual corrosion wear together with standard deviation. However, Ivanov distinguished between wear of bulb head in the vertical direction, δ_V and the wear of web plate in the horizontal direction, δ_W. The values are as follows:

	Mean Thickness Loss (mm/yr)	**Standard Deviation (mm/yr)**
Bulb head	δ_V = 0.1000	σ_V = 0.0400
Web plate	δ_W = 0.1000	σ_W = 0.0400

2.2 Phenomenological Corrosion Wastage Model

Melchers [2003a,b] proposed the following phenomenological model for general corrosion of mild and low alloy steels under fully aerated conditions. Due to very limited published data in this area Melchers' model is not quite appropriate for ship structural profiles. However, it will be used here merely for the sake of completeness.

Based on the information provided in the papers by Melchers, the present author has developed the following equations to calculate the total corrosion wear at any time *t* (years) after the coating breakdown.

$$\left.\begin{array}{ll}
\delta_1(t) = r_0 t\,; & 0 \le t \le t_k \\
\delta_2(t) = c_a - \left(\dfrac{t_a}{t_a - t_k}\right)\left[1 - 2\dfrac{t}{t_a} + \dfrac{t^2}{t_a^2}\right]\dfrac{r_0 t_a}{2}\,; & t_k \le t \le t_a \\
\delta_3(t) = c_a + \left(\dfrac{t_a}{t_\ell - t_a}\right)\left[2\bar{r}\left(\dfrac{t}{t_a} - 1\right) - (r_a - r_s)\left(\dfrac{t^2}{t_a^2} - 1\right)\right]\dfrac{t_a}{2}\,; & t_a \le t \le t_\ell \\
\delta_4(t) = c_s + r_s t\,; & t \ge t_\ell
\end{array}\right\} \tag{2.1}$$

Here the subscripts on $\delta(t)$ indicate the corrosion phases, namely, kinetic, oxygen diffusion and two anaerobic phases. In the above equations the expressions for the intermediate quantities in terms of six parameters are:

$$\left.\begin{array}{l}
t_k = 2\dfrac{c_a}{r_0} - t_a \\
t_\ell = \left(\dfrac{t_a}{r_a - r_s}\right)\left[(c_s - c_a)\dfrac{2}{t_a} + (r_a + r_s)\right] \\
\bar{r} = \dfrac{t_\ell}{t_a} r_a - r_s
\end{array}\right\} \tag{2.2}$$

The six parameters used in the above equations were given by Melchers as follows:

$$\begin{array}{lcl}
t_a = 6.61\exp(-0.088T) & ; & c_a = 0.32\exp(-0.038T) \\
r_0 = 0.076\exp(0.054T) & ; & r_a = 0.066\exp(0.061T) \\
c_s = 0.075 + 5{,}678T^{-4} & ; & r_s = 0.045\exp(0.017T)
\end{array} \tag{2.3}$$

where *T* = seawater temperature, °C.

In the absence of more reliable data on standard deviations the following is proposed.

The coefficient of variation (c.o.v.) on corrosion wear is taken as 0.12, a constant. Then

$$\sigma_C(t) = 0.12\, \delta_i(t) \;; \qquad i = 1, 2, 3, 4 \tag{2.4}$$

where $\sigma_C(t)$ is the standard deviation over an unprotected period of t years. It is further noted that in a phenomenological model the mean value and the standard deviation of corrosion wear are the total values over a period of t years, unlike those of the empirical models where the quantities imply annual averages.

3. CALCULATION OF GEOMETRICAL PROPERTIES AND THEIR DERIVATIVES

As indicated earlier, as an example, this paper will discuss the cross-sectional area, A_s of a bulb plate as it ages. The mean and variances will be calculated at five-year intervals beginning from the moment coating breaks down. In other words, the time t indicates the unprotected life of the profile.

Figure 1 shows the bulb plate with its corroded shape dotted [Ivanov, 2001a].

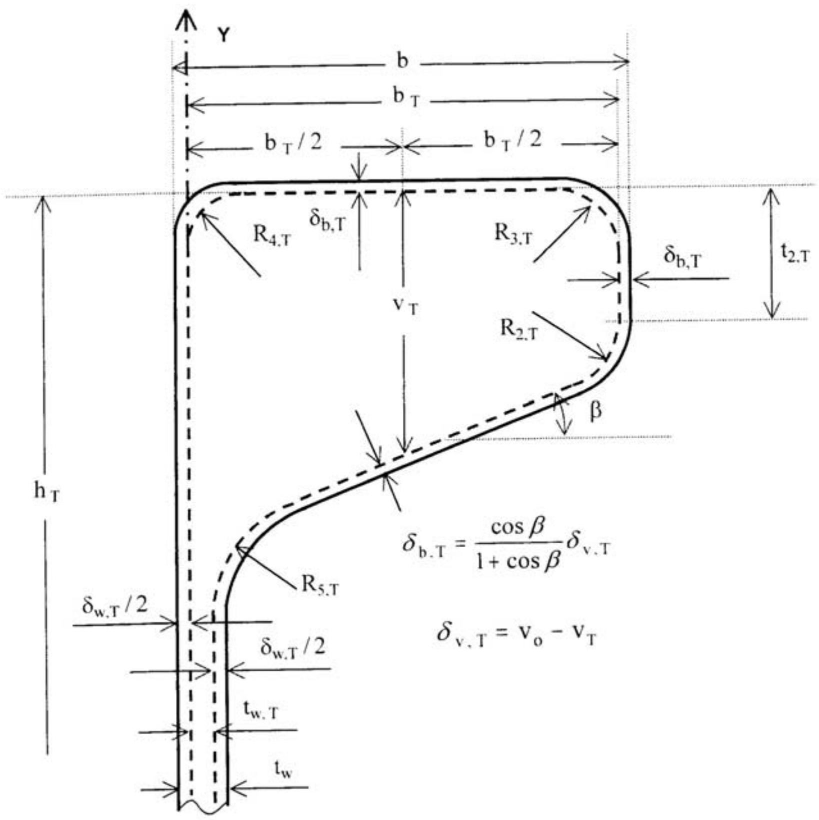

Figure 1: Profile of a bulb bar showing corrosion shrinkage (Ivanov, 2001a).

A bulb plate may be defined by nine independent scantling variables and these are written down as functions of time, *t*:

$$\left.\begin{aligned}
h(t) &= h(0) - f(\beta)\cdot\delta_{V,T} \\
b(t) &= b(0) - f(\beta)\cdot\delta_{V,T} - \frac{1}{2}\delta_{W,T} \\
t_W(t) &= t_W(0) - \delta_{W,T} \\
R_2(t) &= R_2(0) - f(\beta)\delta_{V,T} \\
R_3(t) &= R_3(0) - f(\beta)\delta_{V,T} \\
t_2(t) &= \begin{cases} t_2 - f(\beta)\delta_{V,T}\ ; & \text{if } t_2(0) = R_3(0) \\ t_2 - [1 - f(\beta)(\tan\beta + \tan\eta)]\delta_{V,T}\ ; & \text{if } t_2(0) > R_3(0) \\ R_3 \text{ if } t_2(0) - [1 - f(\beta)(\tan\beta + \tan\eta)]\delta_{V,T} < R_3(t) & \end{cases} \\
R_4(t) &= R_4(0)\,, \quad R_5(t) = R_5(0)\,, \quad \beta(t) = \beta(0)
\end{aligned}\right\} \qquad (3.1)$$

where $\quad f(\beta) = \dfrac{\cos\beta}{1+\cos\beta} \qquad \eta = \dfrac{\pi - 2\beta}{4}$

and $\quad \delta_{V,T}$ and $\delta_{W,T}$ are the total corrosion wear over a period of *t* years.

In other words

and
$$\left.\begin{aligned}
\delta_{V,T} &= t\cdot\delta_V \\
\delta_{W,T} &= t\cdot\delta_W
\end{aligned}\right\} \qquad (3.2)$$

The arguments zero indicate initial (design) values prior to corrosion wear.

Clearly the mean values of these scantlings are readily obtained from Eqn (3.1) simply by substituting the mean values of the right-hand side quantities: $\bar{h}(0)$, $f(\bar{\beta})$, $\bar{\delta}_{V,T}$, etc. These time-dependent dimensions are linear functions of other random variables only if β is treated as a deterministic quantity. For the present, β will be treated as deterministic. This will greatly simplify the calculation of the variance of the time-dependent dimensions, $h(t)$ etc. These are given below.

$$\left.\begin{aligned}
D_h(t) &= D_h(0) + \left[f(\beta)\cdot\sigma_{V,T}\right]^2 \\
D_b(t) &= D_b(0) + \left[\left\{f(\beta)\cdot\sigma_{V,T}\right\}^2 + \frac{1}{4}\left(\sigma_{W,T}\right)^2\right] \\
D_{t_W}(t) &= D_{t_W}(0) + \left(\sigma_{V,T}\right)^2 \\
D_{R_2}(t) &= D_{R_2}(0) + \left[f(\beta)\cdot\sigma_{V,T}\right]^2 \\
D_{R_3}(t) &= D_{R_3}(0) + \left[f(\beta)\cdot\sigma_{V,T}\right]^2 \\
D_{R_4}(t) &= D_{R_4}(0)\,; \quad D_{R_5}(t) = D_{R_5}(0)\,; \quad D_\beta(t) = D_\beta(0) \\
D_{t_2}(t) &= D_{t_2}(0) + \left[f(\beta)\cdot\sigma_{V,T}\right]^2 \quad \text{if } t_2(0) = R_2(0)
\end{aligned}\right\} \qquad (3.3)$$

The corrosion data appearing in the right-hand side of Eqns (3.3) are readily obtained from Section 2 as follows.

Ignoring the directional variation of corrosion wear:

For phenomenological model

$$\sigma_{V,T} = \sigma_{W,T} = \sigma_C(t) \tag{3.4}$$

For empirical models

$$\sigma_{V,T} = \sigma_{W,T} = t\sigma \qquad \text{(Paik \textit{et al.})} \tag{3.5}$$

$$\left.\begin{aligned} \sigma_{V,T} &= t\sigma_V \\ \sigma_{W,T} &= t\sigma_W \end{aligned}\right\} \qquad \text{(Ivanov)} \tag{3.6}$$

Before proceeding further it should be clearly mentioned that treating β as a random variable is not a big problem. In that case some of the time-dependent scantlings will be a non-linear function of random variables, just like cross-sectional area. In that case the Taylor series expansion method outlined in Section 1 is to be followed.

The cross-sectional area as a function of time, $A_S(t)$ is calculated using the current dimensions obtained from Eqns (3.1). Eqns (3.3) together with Eqn (1.4) are used for the variance of $A_S(t)$.

The equations for calculating A_S and its first partial derivatives needed in Eqn (1.4) are given in Appendix A.

4. NUMERICAL EXAMPLE

4.1 Summary of the Steps in Calculations

(a) For any value of *t*, use Eqns (3.1) to calculate the current dimensions. The applicable corrosion wear data are given in Section 2, and the mean values at the beginning $(t = 0)$ is to be specified as inputs.

(b) Calculate the mean cross-sectional area, $A_S(t)$ using the equations given in Appendix A. Also calculate the derivatives of $A_S(t)$.

(c) Use Eqn (3.3) to calculate the variances of the dimension variables at the same value of *t*.

(d) Finally use Eqn (1.4) to calculate the variance of $A_S(t)$.

4.2 A Test Case

As a test case the following inputs are used for a bulb-plate profile [Ivanov, 2001b].

Table I: Means and Standard Deviations of Inputs

Variables x_i	Nominal value (cm)	Initial Mean Value (cm)*	Standard Deviation (cm)	c.o.v. (%)**
h	25.00	25.00	0.1200	0.48
b	4.50	4.50	0.0600	1.33
t_2	1.00	1.00	0.00	-
t_W	1.20	1.23	0.028	2.33
R_2	1.00	1.00	0.00	-
R_3	1.00	1.00	0.00	-
R_4	0.20	0.20	0.00	-
R_5	1.00	1.00	0.00	-
β	30°	30°	0.00	-

* The dimensions prior to coating breakdown; the middle of the specified tolerance range.
** Extremely small c.o.v. indicates the sophisticated manufacturing methods employed today.

Table II: Non-dimensional Means and Standard Deviations of the Bulb Plate Cross-sectional Area
$Y_{\text{nom}} = A_{S,\text{nom}} = 38.1257\text{ cm}^2$; $Y(0) = 38.7716\text{ cm}^2$; $\sigma_Y(0) = 0.6547$ cm

Unprotected Life, t (yr)		2.5	5.0	10	15	20	25
Paik *et al.* Average wear $t_C = 5$ yr	y	1.0085	1.0001	0.9832	0.9664	0.9542	0.9328
	σ_y	.01799	.02026	.02754	.03660	.04641	.05658
Paik *et al.* Severe wear $t_C = 5$ yr	y	0.9904	0.9638	0.9108	0.8580	0.8052	0.7525
	σ_y	.017715	.01931	.02474	.03182	.039695	.0480
Ivanov $\delta_V = \delta_W$	y	0.9988	0.9808	0.9447	0.9086	0.8741	0.8366
	σ_y	.01798	.02058	.02851	.03826	.04873	.05961
Melchers' Phenomenological model	y	0.9961	0.9824	0.9614	0.9403	0.9195	0.8986
	σ_y	.01727	.01750	.01811	.01887	.0202	.02156

Note: The initial values at $t = 0$ yr are: $y = 1.0169$ $\sigma_y = .01717$

5. STATISTICAL MANIPULATION OF THE RESULTS IN TABLE II

5.1 Calculation of Probabilities

The truncated normal probability density function given in Eqn (1.8) can be used to estimate the probabilities of exceedence of A_S as a fraction of its nominal value at any age of the profile after coating breakdown. For example, the probability of the bulb plate cross-sectional area having a value of at least 90% of the nominal value is:

$$\Pr\{y > 0.90\} = 1 - \int_0^{0.90} p_C(y)dy \tag{5.1}$$

However, this integration is equivalent to:

$$1 - \int_{0}^{0.90} p_C(y)dy = 1 - \Phi(z) \tag{5.2}$$

where the normalized random variable

$$z = \frac{0.90 - \bar{y}}{\sigma_y} \tag{5.3}$$

and $\Phi(\ )$ is the cumulative probabilities of the standard normal distribution, extensively tabulated in most texts on statistics [Netter *et al.*, 1988]. The final results are presented as graphs (Figures 2-5).

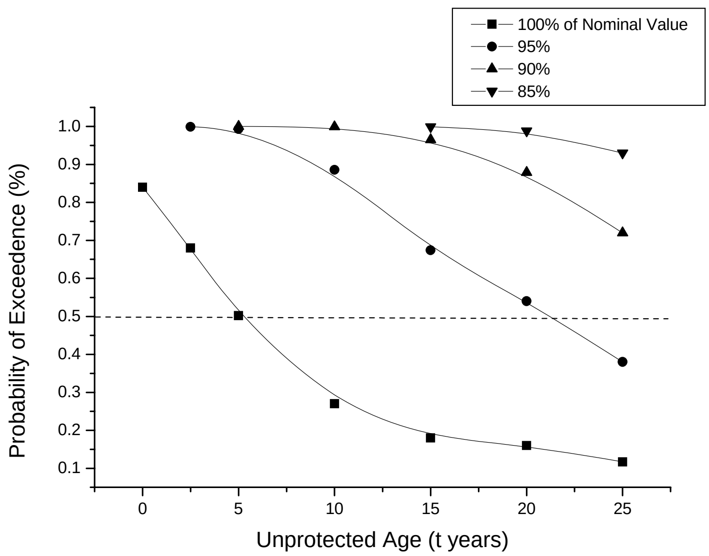

Figure 2: Probabilty based on Paik et al. average.

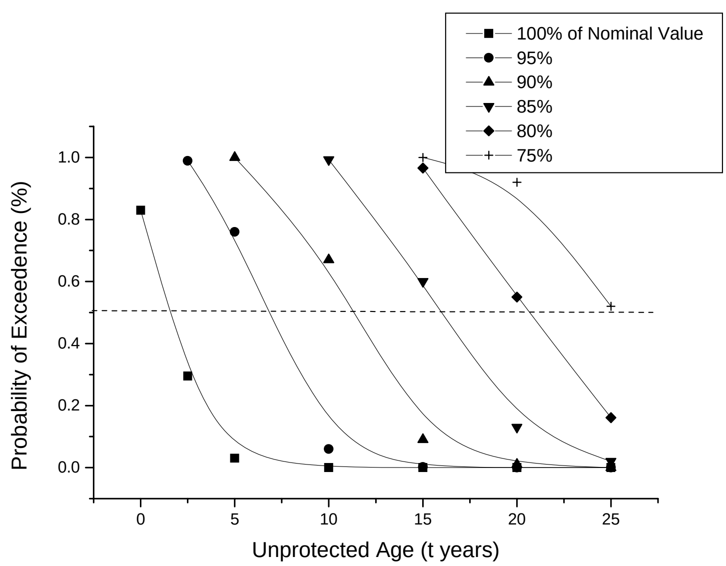

Figure 3: Probability based on Paik et al. severe.

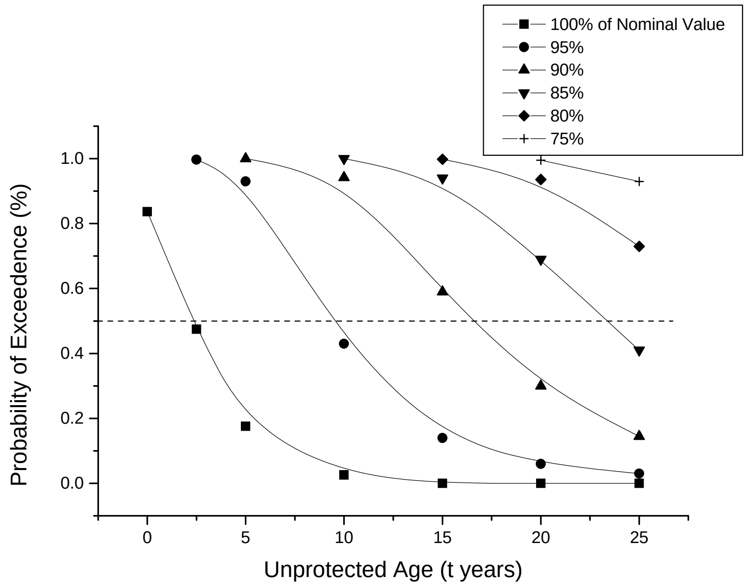

Figure 4: Probabilty based on ABS model.

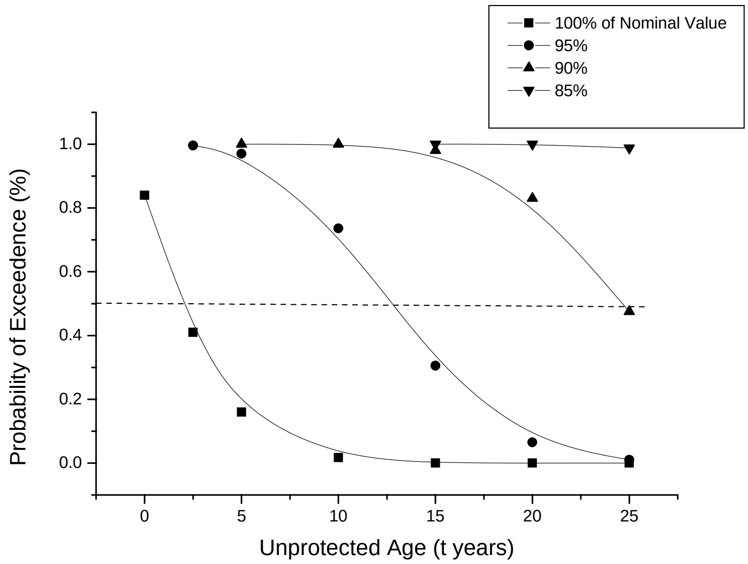

Figure 5: Probabity based on Melchers model.

5.2 Interpretation of the Graphs

Referred to Figure 2: Empirical model, moderate corrosion wear.

The probability of maintaining the cross-sectional area at 95% of its initial (nominal) value falls below 50% at about 21 years of exposure time, whereas at the end of 25 years there is a 72% probability that the area will be 90% of its initial value.

Similarly, referred to Figure 3: Empirical model: severe corrosion wear.

The probability of maintaining the cross-sectional area at 95% of its initial value falls below 50% roughly at about only seven years of exposure, and at the end of about 12 years there is less than a 50% probability of maintaining even 90% of its initial value. Maybe this is the time to replace the stiffeners.

It is probably rational to re-word "below 50% probability" as literally meaning "unlikely". Then for severe corrosion it is unlikely to maintain 90% of the nominal cross-sectional area after about 12 years of exposed service.

6. DISCUSSIONS AND CONCLUSIONS

To highlight the method only one particular geometric property has been discussed. By using an Excel spreadsheet it is easy to program for other geometrical properties such as first moment of area, moment of inertia, section modulus etc. Also it is equally easy to cover all practical structural profiles, such as flat bars, tee bars, fundia T-bars, etc, all with or without attached plating.

The method can be extended to another higher level: to assess the mean and variance of the hull girder mid-ship section modulus as a function of time. In this case the inputs will be means and variances of the stiffeners, side and bottom plating, deck plating, etc. As the profiles age, the effects of corrosion wear are two-fold: the mean value falls while the standard deviation or variance rises. The latter brings in increased uncertainty in maintaining a certain minimum acceptable value of the geometric property concerned.

What is the minimum acceptable value of a certain geometric property is probably the prerogative of the Classification societies. Most of the assumptions introduced here are normally accepted by the community. However, some or all of those may be relaxed at the cost of additional computations. The basic formulation still remains valid.

REFERENCES

[1] Ivanov, L.D. (2001a,b) *Probabilistic Presentation of the Geometric Properties of Shipbuilding Structural Profiles when Assessing Elastic Bending Strength*, Technical Report RD-2001-19, Vols 1 and 2, American Bureau of Shipping, Houston, Texas, USA.

[2] Ivanov, L.D. (1982, 1984) "Statistical estimation of the geometric characteristics of the cross section of welded T-bars during the ship's lifetime", *International Shipbuilding Progress*, No. 339, Nov. 1982 and No. 359, July 1984.

[3] Ivanov, L.D. (1987) "Statistical treatment of the geometric characteristics of the cross section of bulb plates as a function of the ship's lifetime", *Second International Symposium: Ship's Reliability '87*, Varna, Bulgaria, May.

[4] Ivanov, L.D. and Stoyanov, P.S. (1991) "Probabilistic estimation of the cross section geometric characteristics of inverted angles and flat bars", *Fourth International Symposium: Ship's Reliability and Productivity '91*, Varna, Bulgaria, May.

[5] Paik, J.K., Thyamballi, A.K., Park, Y.I. and Hwang, J.S. (2004) "A time-dependent corrosion wastage model for seawater ballast tank structures of ships", *Corrosion Science*, Vol. 46, pp471-486.

[6] Paik, J.K., Thyamballi, A.K., Park, Y.I. and Hwang, J.S. (2003) "A time-dependent corrosion wastage model for bulk carrier structures", *Trans. RINA*, Vol. 145, Part A2.

[7] Ochi, M.K. (1990) *Applied Probability and Stochastic Processes in Engineering and Physical Sciences*, John Wiley & Sons.

[8] Melchers, R.E. (2003a,b) “Modelling of marine immersion corrosion for mild and low-alloy steels – Part 1: Phenomenological model, Part 2: Uncertainty estimation”, *Corrosion*, Vol. 59, No. 4, April.
[9] Neter, J., Wasserman, W. and Whitmore, G.A. (1988) *Applied Statistics*, 3rd Edition, Allyn and Bacon, 1988.

APPENDIX A

The Cross-sectional Area, A_S of a Bulb Plate

Referring to Figure 1, the net area may be split as follows:

$$A_S = A_{B,L} + A_{R_5} - \left(A_{R_2} + A_{R_3} + A_{R_4}\right) \tag{A1}$$

where

$$A_{B,L} = bt + (h-t)t_W - \frac{1}{2}(b-t_w)^2 \tan\beta \tag{A2}$$

and

$$\left.\begin{aligned} t &= t_2 + (b-t_W)\tan\beta + R_2\eta \\ \eta &= \frac{\pi}{4} - \frac{1}{2}\beta \end{aligned}\right\} \tag{A3}$$

also

Here $A_{B,L}$ is the area enclosed by straight lines.

$$\left.\begin{aligned} A_{R_2} &= R_2^2\left(\tan\eta - \eta\right) \\ A_{R_3} &= R_3^2\left(1-\frac{\pi}{4}\right) \\ A_{R_4} &= R_4^2\left(1-\frac{\pi}{4}\right) \\ A_{R_5} &= R_5^2\left(\tan\eta - \eta\right) \end{aligned}\right\} \tag{A4}$$

A_{Ri} are the fillet areas.

Partial Derivatives of the Bulb Plate Cross-sectional Area, A_S

$$\frac{\partial A_S}{\partial h} = t_W \;;\quad \frac{\partial A_S}{\partial b} = t \;;\quad \frac{\partial A_S}{\partial t_2} = b - t_W \;;\quad \frac{\partial A_S}{\partial t_W} = h - t$$

$$\frac{\partial A_S}{\partial R_2} = \left(b - t_W\right)\tan\eta - \frac{2A_{R_2}}{R_2} \;;\quad \frac{\partial A_S}{\partial R_3} = -\frac{2A_{R_3}}{R_3}$$

$$\frac{\partial A_S}{\partial R_4} = -\frac{2A_{R_4}}{R_4} \;;\quad \frac{\partial A_S}{\partial R_5} = \frac{2A_{R_5}}{R_5}$$

$$\frac{\partial A_S}{\partial \beta} = \left(b - t_W\right)\left[\left(b - t_W\right)\sec^2\beta - \frac{1}{2}\left\{\left(b - t_W\right)\sec^2\beta + R_2\sec^2\eta\right\}\right] + \frac{1}{2}\left(R_2^2 - R_5^2\right)\tan^2\eta$$

where t and η are defined in Appendix A and other symbols are for nine scantling variables shown in Figure 1.