Induced representations of crossed products by coactions

Author: Mansfield, Kevin P.
Publication Date: 1988
DOI: https://doi.org/10.26190/unswworks/9102
License: https://creativecommons.org/licenses/by-nc-nd/3.0/au/
Link to license to see what you are allowed to do with this resource.

Downloaded from http://hdl.handle.net/1959.4/64148 in https://unswworks.unsw.edu.au on 2023-10-02
INDUCED
REPRESENTATIONS OF
CROSSED PRODUCTS
BY COACTIONS

A thesis submitted for the degree of
Doctor of Philosophy
at the University of New South Wales, 1988.

KEVIN MANSFIELD
To Mum and Dad.

Acknowledgements

I wish to record my gratitude to my supervisor, Professor Iain Raeburn, for his encouragement, his helpful advice and his good humour. I am also indebted to the students and staff of the School of Mathematics who have made the University of New South Wales a friendly and congenial place to work.

I gratefully acknowledge the financial support of an Australian Government Postgraduate Research Award.
Abstract

Let \( \delta : A \rightarrow \tilde{M}(A \otimes C_r^*(G)) \) be a coaction of a locally compact group \( G \) on a C\(^*\)-algebra \( A \). Then for any closed normal amenable subgroup \( H \) of \( G \) we define a coaction \( \delta| : A \rightarrow \tilde{M}(A \otimes C_r^*(G/H)) \) of \( G/H \) on \( A \). We present dense \(*\)-subalgebras of the crossed products \( A \times_\delta G \) and \( A \times_{\delta|}(G/H) \) and use these to obtain a process whereby representations of \( A \times_\delta G \) may be constructed from those of \( A \times_{\delta|}(G/H) \). We then classify those representations of \( A \times_\delta G \) which can be obtained in this way. In other words we exhibit an induction process and formulate an imprimitivity theorem for it. We also present an elegant reformulation of Green’s imprimitivity theorem which helped motivate the above results.
## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>List of Notation</td>
<td>1</td>
</tr>
<tr>
<td>Introduction</td>
<td>6</td>
</tr>
<tr>
<td>Chapter 1. Background</td>
<td>10</td>
</tr>
<tr>
<td>§1 The Fourier Algebra</td>
<td>10</td>
</tr>
<tr>
<td>§2 Slice Maps</td>
<td>13</td>
</tr>
<tr>
<td>§3 Integration</td>
<td>14</td>
</tr>
<tr>
<td>§4 Induced Representations of $C^*$-algebras</td>
<td>23</td>
</tr>
<tr>
<td>§5 Actions and their Crossed Products</td>
<td>32</td>
</tr>
<tr>
<td>§6 Coactions and their Crossed Products</td>
<td>35</td>
</tr>
<tr>
<td>Chapter 2. Coactions of Quotients</td>
<td>52</td>
</tr>
<tr>
<td>Chapter 3.</td>
<td>61</td>
</tr>
<tr>
<td>§1 Motivation</td>
<td>61</td>
</tr>
<tr>
<td>§2 A Reformulation of Green's Imprimitivity Theorem</td>
<td>63</td>
</tr>
<tr>
<td>Chapter 4. The Subalgebras</td>
<td>72</td>
</tr>
<tr>
<td>Chapter 5. Induced Representations of Crossed Products by Coactions</td>
<td>91</td>
</tr>
<tr>
<td>Chapter 6. The Imprimitivity Theorem</td>
<td>117</td>
</tr>
<tr>
<td>References</td>
<td>140</td>
</tr>
</tbody>
</table>
Notation

$A$ a $C^*$-algebra

$A(G)$ the Fourier algebra, see page 10

$A_c(G)$ the elements of $A(G)$ which have compact support

$(A, G, \alpha)$ a $C^*$-dynamical system, i.e. an action $\alpha$ of $G$ on $A$

$A \times_\alpha G$ the crossed product of a $C^*$-algebra by the action $\alpha$ of a locally compact group $G$, see page 33

$(A, G, \delta)$ a system comprising a coaction $\delta$ of $G$ on $A$

$A \times_\delta G$ the crossed product of a $C^*$-algebra by the coaction $\delta$ of a locally compact group $G$, see page 38

$A \times_{\delta|} (G/H)$ the crossed product of a $C^*$-algebra by the coaction $\delta$

$A \times_\delta (G/H)$ a subalgebra of $B(\mathcal{H} \otimes L^2(G))$ isomorphic to $A \times_{\delta|} (G/H)$

$A''$ the enveloping von Neumann algebra of a $C^*$-algebra $A$

$Ad W$ the map $T \to WTW^*$

$\alpha$ an action of a locally compact group

$\tilde{\alpha}$ see page 37

$\hat{\alpha}$ the dual coaction, see page 38

$\alpha_G$ the multiplication map, see page 37

$B(G)$ see page 10

$B_r(G)$ see page 10

$B^*$ the dual space of a Banach space $B$

$B(\mathcal{H})$ the bounded linear operators on a Hilbert space $\mathcal{H}$

$B(\mathcal{H})_w$ the weakly continuous functionals on $B(\mathcal{H})$

$\mathbb{C}$ the complex numbers

$C_b(G, A)$ the bounded continuous functions from $G$ to $A$

$C_0(G, A)$ the continuous functions from $G$ to $A$ which vanish at infinity

$C_c(G, A)$ the continuous functions from $G$ to $A$ which have compact support
$C^*_b(G,A)$ the bounded strictly continuous functions from $G$ to $A$

$C_b(G)$ $C_b(G,C)$

$C_o(G)$ $C_o(G,C)$

$C_c(G)$ $C_c(G,C)$

$C_E(G)$ the elements of $C_c(G)$ with support in the compact set $E$

$C^*(G)$ the group $C^*$-algebra

$C^*_c(G)$ the closure of $\lambda_G(L^1(G))$ in $B(L^2(G))$

$\chi_E$ the characteristic function of the set $E$

$\mathcal{D}_H$ see page 72

$\mathcal{D}$ $\mathcal{D}_H$ when $H$ is the trivial subgroup

$\delta$ a coaction of a locally compact group, see page 36

$\delta\mid$ the coaction $\delta$ "restricted" to a quotient, see page 55

$\delta_G$ the comultiplication map, see page 36

$\hat{\delta}$ the dual action, see page 39

$\delta(a)(1 \otimes f)$ an abbreviation for $(\pi \otimes i)(\delta(a))(1 \otimes M_G(f))$, see page 40

$\delta_u(a)$ an abbreviation for $S_u(\delta(a))$

$\Delta$ the modular function

$\bar{f}$ $\bar{f}(t) = \overline{f(t)}$, where $\overline{z}$ is the complex conjugate of $z$

$f^\ast$ $f^\ast(t) = f(t^{-1})$

$f^\dagger$ $f^\dagger(t) = f(t^{-1})$

$f_s$ $f_s(t) = f(s^{-1}t)$

$f^s$ $f^s(t) = f(ts)$

$|f|$ $|f|(t) = |f(t)|$

$f \cdot g$ $f \cdot g(t) = f(t)g(t)$

$f \ast g$ $f \ast g(t) = \int f(s)\alpha_s(g(s^{-1}t))\,ds$ for $f, g \in C_c(G,A)$

$f^\ast$ $f^\ast(t) = (\Delta t)^{-1}\alpha_t(f(t^{-1})^\ast)$ for $f \in C_c(G,A)$

$G, H$ locally compact groups

$\hat{G}$ the dual group of a locally compact abelian group
\(G/H\) the quotient group (for \(H\) normal)
\(\gamma_{x,y}\) see page 122
\(\Gamma\) see page 58
\(\mathcal{H}\) a Hilbert space
\(H^\perp\) the subgroup of \(\hat{G}\) of elements which are trivial on the subgroup \(H\) of \(G\)
\(i_A\) the natural inclusion of \(A\) in \(M(A \times_\alpha G)\), see page 34
\(i_G\) the natural inclusion of \(G\) in \(M(A \times_\alpha G)\), see page 34
\(\text{ind}_{G}^{A} \nu\) the representation of \(A\) induced from \(B\), see page 26
\(I_H\) see page 87
Ideals \(A\) the (closed two-sided) ideals of \(A\)
\(J\) the elements of \(\mathcal{L}(\mathcal{D})\) of length zero, see page 29
\(j_{A \times \alpha G}\) see page 110
\(K(\mathcal{H})\) the closed two sided ideal of compact operators in \(B(\mathcal{H})\)
\(\ker f\) the kernel of the map \(f\)
\(\mathcal{K}(\mathcal{D})\) the “compact operators” of a rigged space \(\mathcal{D}\), see page 29
\(\mathcal{K}(\mathcal{D})\) the completion of \(\mathcal{K}(\mathcal{D})/J\)
\(\mathcal{L}(\mathcal{D})\) the bounded operators of a rigged space \(\mathcal{D}\), see page 29
\(\mathcal{L}(\mathcal{D})\) the completion of \(\mathcal{L}(\mathcal{D})/J\)
\(\lambda_{G}\) the left regular representation of \(L^1(G)\), and also \(G\), on \(L^2(G)\)
\(M(A)\) the multiplier algebra of a \(C^*\)-algebra \(A\)
\(\hat{M}(A \otimes B)\) see page 35
\(M_G\) the representation of \(C_0(G)\) on \(L^2(G)\) by multiplication operators, i.e. \(\{M_{G}(f)(\xi)\}(s) = f(s)\xi(s)\)
\(\mathcal{M}\) the predual of a von Neumann algebra \(\mathcal{M}\)
\(\mu_G\) left Haar measure on \(G\)
\(\varphi\) see page 52
\(\Psi\) see page 100
$\Phi$ see page 54

$\pi$ a representation (usually of $A$)

$\hat{\pi}$ see page 33

$\hat{\pi}$ see page 33

$(\pi, U)$ a covariant representation of a $C^*$-dynamical system $(A, G, \alpha)$, see page 32

$\pi \times U$ the representation of $A \times_{\alpha} G$ corresponding to the covariant representation $(\pi, U)$

$(\pi, \mu)$ a covariant representation of a system $(A, G, \delta)$, see page 39

$\pi \times \mu$ the representation of $A \times_{\delta} G$ corresponding to the covariant representation $(\pi, \mu)$, see page 41

Prim $A$ the primitive ideals of $A$

Rep $A$ the unitary equivalence classes of representations of $A$

$\mathcal{R}$ the real numbers

$\mathcal{R}^+$ the positive real numbers

$\rho_G$ the right regular representation of $L^1(G)$, and also $G$, on $L^2(G)$

$S_n$ a slice map, see page 13

supp $f$ the support of the function $f$

$\Theta$ see page 122

$\sigma$ the right translation action of $G$ on $C_0(G)$ defined by

$\{\sigma_s(f)\}(t) = f(ts)$

$\tau$ the left translation action of $G$ on $C_0(G)$ defined by

$\{\tau_s(f)\}(t) = f(s^{-1}t)$

$T_{x,y}$ see page 29

$Y \otimes_B Z$ the algebraic tensor product of a left $B$-module $Y$ and a right $B$-module $Z$

$Y \otimes Z$ the tensor product $Y \otimes_B Z$ when $B$ is the complex numbers

$A \odot B$ the algebraic tensor product of two $C^*$-algebras
\(A \otimes B\) the algebraic tensor product of two \(C^*\)-algebras completed in the minimum \(C^*\)-norm

\(\mathcal{M} \boxtimes \mathcal{N}\) the tensor product of the von Neumann algebras \(\mathcal{M}\) and \(\mathcal{N}\)

\((u, E, H)\) see page 72

\((u, E)\) \((u, E, H)\) when \(H\) is the trivial subgroup

\(UB(\mathcal{H})\) the unitary group of \(B(\mathcal{H})\)

\(vN(G)\) the group von Neumann algebra

\(\omega_G\) see page 39

\(w^*\) the adjoint of the element \(w\)

\(X\) the bimodule \(\mathcal{D}\) factored (by elements of length zero) and completed

\(\cdot\) a module action

\(f \circ g\) composition of functions

\(A \cong B\) \(A\) is strongly Morita equivalent to \(B\)

\(A \cong B\) \(A\) is isomorphic to \(B\)

\(\S 5\) reference to equation 5 of the current chapter

\(\S 1.5\) reference to equation 5 of chapter 1
Introduction

In quantum mechanics the observables of a physical system are described as non-commuting operators on a Hilbert space, and in some models it is assumed that they form (the self-adjoint part of) a $C^\ast$-algebra. The time evolution, spatial translation and symmetries of the system are then expressed by actions of locally compact groups on this algebra. More generally, we shall be interested in systems comprising a $C^\ast$-algebra $A$, a locally compact group $G$ and an action $\alpha$ of $G$ on $A$.

The usual tool for studying such a system $(A, G, \alpha)$ is the crossed product algebra, $A \times_\alpha G$. This approach is successful because one can recover the action $\alpha$ from $A \times_\alpha G$ by Imai and Takai's theorem [10], which states that there is a coaction $\delta$ of $G$ on $A \times_\alpha G$, that the crossed product $(A \times_\alpha G) \times_\delta G$ is essentially $A$ and that there is an action on $(A \times_\alpha G) \times_\delta G$ which is, in essence, $\alpha$. Thus the study of actions via their crossed products requires a knowledge of coactions and their crossed products. We shall develop the theory of crossed products by coactions by presenting an induction process, and corresponding imprimitivity theorem, for their representations.

Induced representations first appeared in Frobenius' analysis of the representation theory of finite groups [4]. Later, Nakayama [18] and Weil [29] gave a similar construction for compact groups and Wigner [30], by extending these results still further, was able to describe the representations of the (non compact) inhomogeneous Lorentz group, an important result in physics. Building on this Mackey [15, 16, 17] developed a theory of induced representations for arbitrary locally compact groups and proved an imprimitivity theorem characterising those representations that can be obtained from his construction.

Much of the theory of representations of groups is subsumed in the theory of $\ast$-representations of involutive Banach algebras, and, especially, $C^\ast$-algebras. In particular Rieffel [22] has shown that the theory of induced representations of
groups can be reformulated in the framework of $C^*$-algebras. Rieffel's work states that if we can find dense subalgebras $\mathcal{E}$ and $\mathcal{F}$ of the $C^*$-algebras $E$ and $F$, respectively, and a bimodule $X$ which carries a left $\mathcal{E}$ action, a right $\mathcal{F}$ action and an $\mathcal{F}$ valued inner product with the actions and inner product satisfying various conditions, then we can construct representations of $E$ from those of $F$ (see page 25 for details). His work also includes a very general imprimitivity theorem involving the algebra $K(X)$ of the "compact operators" of the $\mathcal{F}$-rigged module $X$.

Green [6] has shown how these ideas can be used to construct representations of crossed products by actions. More precisely, if $\alpha$ is an action of a locally compact group $G$ on a $C^*$-algebra $A$ and $H$ is a closed subgroup of $G$, then $C_c(G, A) (= X)$ is a left $C_c(G, A) (= \mathcal{E})$, right $C_c(H, A) (= \mathcal{F})$ bimodule with a $C_c(H, A)$ valued inner product which satisfies Rieffel's conditions. Hence one can construct representations of $A \times_\alpha G (= E)$ from representations of $A \times_\alpha H (= F)$, that is, one has an induction process. Green also identifies the algebra $K(X)$ of compact operators as the crossed product $(A \otimes C_\alpha(G/H)) \times_{\alpha \otimes \tau} G$, where $\tau$ is the left translation action of $G$ on $C_\alpha(G/H)$, and is thus able to characterise the representations obtained in this way.

It is our intention to present an analogous induction process and imprimitivity theorem for representations of crossed products by coactions. Suppose $\delta$ is a coaction of $G$ on $A$ and $H$ is a closed normal amenable subgroup of $G$. Then we shall show that $\delta$ "restricts" to a coaction $\delta|_A$ of $G/H$ on $A$. Like Green we will use Rieffel's framework with $E$ and $F$ in our case being $A \times_\delta G$ and $A \times_{\delta|_A} (G/H)$, respectively. To establish an induction process we must find candidates for $\mathcal{E}$, $\mathcal{F}$ and $X$, and to prove an imprimitivity theorem for the process we must identify the algebra $K(X)$ in terms of $A \times_\delta G$ and other quantities.

In chapter one we present some background material including a discussion of the Fourier algebra, integration of operator valued functions, induced representations of $C^*$-algebras, actions, coactions and crossed products. In chapter two we
introduce the restriction $\delta|$ of a coaction $\delta$ and show that, given a faithful representation, $\pi$, of $A$ on some Hilbert space $\mathcal{H}$, the crossed product $A \times_{\delta|} (G/H)$ can be faithfully represented on $\mathcal{H} \otimes L^2(G)$. We will generally choose to work with this copy of $A \times_{\delta|} (G/H)$ in $B(\mathcal{H} \otimes L^2(G))$, which we will denote $A \times_{\delta} (G/H)$, rather than $A \times_{\delta|} (G/H)$ itself. Chapter three presents some of the fact and speculation that helped us to formulate our imprimitivity theorem. Similar considerations also led us to an elegant reformulation of Green’s imprimitivity theorem (theorem 3.2) for the case when the subgroup is closed normal and amenable.

We begin the presentation of the main results in chapter four where we show that if $M_G$ is the representation of $C_b(G)$ on $L^2(G)$ by multiplication operators and $\varphi$ averages elements of $C_c(G)$ over $H$-cosets, then the set $\mathcal{D}_H$ of norm limits of sequences $(x_j)_{j=1}^{\infty}$ in $B(\mathcal{H} \otimes L^2(G))$ of the form

$$x_j = \sum_{i=1}^{n_j} (\pi \otimes i)(\delta_\varphi(a_{ij}))(1 \otimes M_G(\varphi(f_{ij})))$$

where the $a_{ij}$ are in $A$, the $f_{ij}$ are elements of $C_c(G)$ all of which have support in some fixed compact subset of $G$ and where $u$ is a fixed compactly supported element of the Fourier algebra $A(G)$, is a dense $*$-subalgebra of $A \times_{\delta} (G/H)$. If $1$ denotes the trivial subgroup of $G$, then $\mathcal{D}_1$ is a dense $*$-subalgebra of $A \times_{\delta} (G/1) = A \times_{\delta} G$ and is our choice for $E$. Our choice for $F$ is $\mathcal{D}_H$. In chapter five we show that $\mathcal{D}_1$ (our candidate for $X$) can be equipped with a $\mathcal{D}_H$ valued inner product. Much of the chapter is devoted to showing that this inner product is well-defined. We also show that $\mathcal{D}_1$ is a left $\mathcal{D}_1$, right $\mathcal{D}_H$, bimodule which fits into Rieffel’s framework. Hence we can induce representations of $A \times_{\delta} G$ from those of $A \times_{\delta} (G/H)$, that is we have our induction process. Now $A \times_{\delta} G$ carries a natural action $\hat{\delta}$ of $G$ (and hence of $H$), called the dual action, and in chapter six we identify the algebra $K(X)$ as the crossed product $(A \times_{\delta} G) \times_{\hat{\delta}} H$. This enables us to present the following imprimitivity theorem for our induction process:

-8-
A representation $\nu$ of $A \times_\delta G$ on $\mathcal{H}$ is induced from a representation of $A \times_\delta (G/H)$ if and only if there exists a unitary representation $U$ of $H$ on $\mathcal{H}$ such that $(\nu, U)$ is a covariant representation of $(A \times_\delta G, H, \hat{\delta})$. 

- 9 -
Chapter 1. Background.

Firstly we establish some notation. Throughout $A$ will denote a $C^*$-algebra and $M(A)$ its multiplier algebra. $B(H)$ will denote the algebra of linear operators on the Hilbert space $H$ with $K(H)$ the ideal of compact operators. $G$ will be a locally compact group with $\lambda_G$ and $\rho_G$, respectively, the left and right regular representations of $G$, and $L^1(G)$, on $B(L^2(G))$. $C^*(G)$ will denote the group $C^*$-algebra and $C^*_r(G)$ will be the subalgebra $\lambda_G(C^*(G))$ of $B(L^2(G))$. The natural inclusion of $G$ in $M(C^*(G))$ will be denoted by $i_G$. $C_b(G, A), C_0(G, A)$ and $C_c(G, A)$ will denote the continuous functions from $G$ to $A$ which (i) are bounded, (ii) vanish at infinity and (iii) have compact support. If $A$ is the complex numbers, then the above algebras will be denoted $C_b(G), C_0(G)$ and $C_c(G)$ respectively. $M_G$ will denote the representation of $C_b(G)$ on $L^2(G)$ by multiplication operators. If $B$ is a Banach space $B^*$ will denote the dual space of $B$. Finally, if $f$ is a function on $G$, we define $\hat{f}$, $\check{f}$, $\bar{f}$, $f_*$ and $f^*$ by $\hat{f}(t) = \overline{f(t)}, \check{f}(t) = f(t^{-1}), \bar{f}(t) = \overline{f(t^{-1})}, f_*(t) = f(s^{-1}t), f^*(t) = f(ts)$.

§1 The Fourier Algebra

Define a map

\[
\Xi : C^*(G)^* \to C_b(G) \quad \text{by} \quad (\Xi(\psi))(s) = \psi(i_G(s)).
\]

(1)

This is possible since any linear functional on a $C^*$-algebra $A$ can be extended to a weakly continuous functional on the enveloping von Neumann algebra $A''$ of $A$. The weak continuity implies the continuity of $\Xi(\psi)$. Now let

\[
B(G) = \Xi(C^*(G)^*) \quad B_r(G) = \Xi(C^*_r(G)^*) \quad A(G) = \Xi(vN(G)_*)
\]

where $vN(G)_*$ is the predual of the group von Neumann algebra $vN(G)$. $A(G)$ is called the Fourier algebra of $G$. Clearly

\[
A(G) \subset B_r(G) \subset B(G) \subset C_b(G).
\]

(2)
Henceforth we shall not differentiate between elements of $C^*(G)^*$ and their images in $C_b(G)$. Note that if $u \in A(G)$, $s \in G$ and $g \in C_c(G)$, then (1) implies

$$u(\lambda_G(s)) = u(s) \quad \text{and} \quad u(\lambda_G(g)) = \int_G u(s)g(s) \, ds . \tag{3}$$

$B(G)$ equipped with the algebraic operations of $C_b(G)$ and the norm inherited from $C^*(G)^*$ is an involutive Banach algebra which has $A(G)$ as a closed subalgebra [3], [20 §7.1-7.2]. So in particular

$$\|u \cdot v\|_{B(G)} \leq \|u\|_{B(G)} \cdot \|v\|_{B(G)} . \tag{4}$$

Now $vN(G)_*$ can be characterised [27 chapter 1 thm. 2.6 (ii)] as those linear functionals on $C^*(G)$ of the form

$$: z \mapsto \sum_{i=1}^{\infty} \langle \lambda_G(z)\xi_i , \eta_i \rangle_{L^2(G)} \quad z \in L^1(G) ,$$

where $\xi_i, \eta_i \in L^2(G)$ with $\sum_{i=1}^{\infty} \|\xi_i\|_2^2 < \infty$ and $\sum_{i=1}^{\infty} \|\eta_i\|_2^2 < \infty$. Since

$$\{ \Xi( : z \mapsto \sum_{i=1}^{\infty} \langle \lambda_G(z)\xi_i , \eta_i \rangle_{L^2(G)} ) \}(s) = \sum_{i=1}^{\infty} \langle \lambda_G(s)\xi_i , \eta_i \rangle_{L^2(G)}$$

$$= \sum_{i=1}^{\infty} \bar{\eta}_i \ast \bar{\xi}_i (s) ,$$

$A(G)$ can be characterised as those those elements of $C_b(G)$ of the form

$$u = \sum_{i=1}^{\infty} \bar{\eta}_i \ast \bar{\xi}_i (s) ,$$

where $\xi_i, \eta_i \in L^2(G)$ with $\sum_{i=1}^{\infty} \|\xi_i\|_2^2 < \infty$ and $\sum_{i=1}^{\infty} \|\eta_i\|_2^2 < \infty$.

We now present a number of related facts that will be used repeatedly later in the discourse, often without reference.
(i) Suppose $E$ is compact and $U$ is open in $G$. Then there exists an element $u$ of $A^+(G)$ such that $0 \leq u \leq 1$ and $u(s) = 1$ on $E$ and is zero off $U$ [3 lemma 3.2].

(ii) Suppose $u \in B(G)$, (respectively $A(G)$). Then $\tilde{u}$, $\tilde{u}$, $u^*$, $u_s \in B(G)$, (respectively $A(G)$) and

$$
\|\tilde{u}\|_{B(G)} = \|\tilde{u}\|_{B(G)} = \|\tilde{u}\|_{B(G)} = \|u^*\|_{B(G)} = \|u_s\|_{B(G)},
$$

[3 rmk. 2.5 (iii) and cor. 2.9].

(iii) $A_c(G) = A(G) \cap C_c(G)$ is $\| \cdot \|_{B(G)}$ dense in $A(G)$ [3 prop. 3.4].

(iv) Suppose $u \in A(G)$ and $\epsilon > 0$. By [3 prop. 3.4] we can choose

$$
v = \sum_{i=1}^{n} \bar{\eta}_i \ast \bar{\xi}_i \quad \text{in } A(G) \text{ such that } \|u - v\|_{B(G)} < \epsilon/3.
$$

By the above $v$ corresponds to the linear functional

$$
: z \rightarrow \sum_{i=1}^{n} \langle \lambda_G(z)\xi_i, \eta_i \rangle_{L^2(G)}.
$$

Since

$$
\left| \sum_{i=1}^{n} \langle \lambda_G(z)\xi_i, \eta_i \rangle_{L^2(G)} \right| \leq \sum_{i=1}^{n} \|\xi_i\|_2 \cdot \|\eta_i\|_2 \cdot \|z\|,
$$

we have that $\|v\|_{B(G)} \leq \sum_{i=1}^{n} \|\xi_i\|_2 \cdot \|\eta_i\|_2$.

Now choose a neighbourhood $V$ of 1 such that

$$
\sum_{i=1}^{n} \|\eta_i - (\eta_i)_s\|_2 < \epsilon/(3 \cdot n \cdot r) \quad \forall s \in V,
$$

where $r = \max \|\xi_i\|_2$. Then for $s \in V$

$$
\|u - u_s\|_{B(G)} \leq \|u - v\|_{B(G)} + \|v - u_s\|_{B(G)} + \|u_s - u_s\|_{B(G)}
$$

$$
\leq \epsilon/3 + \sum_{i=1}^{n} \|\eta_i - (\eta_i)_s\|_2 \cdot \|\xi_i\|_2 + \epsilon/3 \quad \text{(by } t 5\text{)}
$$

$$
< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \quad \text{(by } t 7\text{)}
$$

\[ -12 - \]
and hence the map : \( G \to A(G) : s \to u_s \) is continuous. Similarly the map :
\( G \to A(G) : s \to u^* \) is continuous. (8)

§2 Slice Maps

Let \( A \) and \( B \) be \( C^* \)-algebras. We denote by \( A \otimes B \) the algebraic tensor product of \( A \) and \( B \). \( A \otimes B \) will denote \( A \otimes B \) completed in the minimum \( C^* \)-norm.

For \( u \in B^* \), where \( B^* \) is the dual space of \( B \), define
\[
S_u : A \otimes B \to A \quad \text{by} \quad \sum_{i=1}^{n} a_i \otimes b_i \to \sum_{i=1}^{n} u(b_i)a_i .
\]

Now \( S_u \) is bounded for the minimum \( C^* \)-norm [28] and hence extends to \( A \otimes B \).

Define left and right actions of \( B \) on \( B^* \) by
\[
(b \cdot u)(c) = u(cb) \quad \text{and} \quad (u \cdot b)(c) = u(bc) \quad u \in B^*, \quad b, c \in B .
\]

It is easily checked that
\[
S_{u \otimes b}(z) = S_u((1 \otimes b)z) \quad S_{b \otimes u}(z) = S_u(z(1 \otimes b)) \quad aS_u(z) = S_u((a \otimes 1)z) \quad S_u(z)a = S_u(z(a \otimes 1)) ,
\]
(10)

where \( a \in A, \ b \in B \) and \( z \in A \otimes B \). Now \( B \) is a two sided \( B \)-module with a bounded approximate identity such that \( \|b \cdot u\|_{B^*} \leq \|b\|_B \cdot \|u\|_{B^*} \) for all \( b \in B \) and \( u \in B^* \). So by Cohen's factorisation theorem [8 thm. 32.22] any element \( u \) of \( B^* \) can be written in the following ways
\[
u = b \cdot v = c \cdot \psi \cdot d = w \cdot e ,
\]
(11)

for some \( v, w, \psi \in B^* \) and \( b, c, d, e \in B \). We can use this to extend \( S_u \) to a map
\[
S_u : M(A \otimes B) \to M(A)
\]
determined by
\[
S_u(z)a = S_{b \otimes \psi}(z)a = S_{v \otimes c}(z(a \otimes b))
\]
\[ a S_u(z) = a S_{h \ast v}(z) = S_{h \ast v}((a \otimes c)z) . \]

It is easily checked that the definition is independent of the way we factor \( u \), that \( S_u \) is strictly continuous, that

\[
\| S_u \| = \| u \|_{B^*} \tag{12}
\]

and that the formulas § 10 still hold \[ \text{[13 lemma 1.5]} \]. We shall mainly be interested in the case when \( B = C^*_r(G) \) and \( B^* = B_r(G) \).

§3 Integration

Since we will be dealing extensively with operator valued integrals we take time to establish some of their properties. Throughout, \( \mu_G \) will denote the left Haar measure on \( G \). If there is no danger of confusion \( d\mu_G(s) \) will be abbreviated to \( ds \).

Let \( \mathcal{H} \) be a Hilbert space. Let \( \omega_{\xi, \eta} \) be the linear functional on \( B(\mathcal{H}) \) defined by

\[
\omega_{\xi, \eta}(S) = \langle S(\xi), \eta \rangle_{\mathcal{H}} \quad \forall \ S \in B(\mathcal{H}) . \tag{13}
\]

Let

\[
B(\mathcal{H})_w = \left\{ \sum_{j=1}^{n} \omega_{\xi_j, \eta_j} : \xi_j, \eta_j \in \mathcal{H} \right\}
\]

be the weakly continuous functionals on \( B(\mathcal{H}) \).

Definition A map \( f : G \to B(\mathcal{H}) \) is said to be integrable if

(i) the maps \( s \to \omega(f(s)) \) are Lebesgue integrable for all \( \omega \in B(\mathcal{H})_w \),

(ii) there exists an element \( T \in B(\mathcal{H}) \) such that

\[
\omega(T) = \int_G \omega(f(s)) \, ds \quad \forall \ \omega \in B(\mathcal{H})_w .
\]

The element \( T \) above is clearly unique and will be denoted

\[
\int_G f(s) \, ds .
\]
Lemma 1.1 Suppose $\alpha, \beta \in \mathbb{C}$ and $f, g : G \to B(\mathcal{H})$ are integrable. Then the map $s \to \alpha f(s) + \beta g(s)$ is integrable and

$$\int_G \alpha f(s) + \beta g(s) \, ds = \alpha \int_G f(s) \, ds + \beta \int_G g(s) \, ds .$$

Also if $\xi : G \to C$ is Lebesgue integrable and $T \in B(\mathcal{H})$. Then the map $s \to \xi(s)T$ is integrable and

$$\int_G \xi(s)T \, ds = \int_G \xi(s) \, ds \, T .$$

Proof Let $\omega \in B(\mathcal{H})_w$. Then $s \to \omega(\alpha f(s) + \beta g(s))$ is Lebesgue integrable and

$$\int_G \omega(\alpha f(s) + \beta g(s)) \, ds = \alpha \int_G \omega(f(s)) \, ds + \beta \int_G \omega(g(s)) \, ds$$

$$= \alpha \omega \left( \int_G f(s) \, ds \right) + \beta \omega \left( \int_G g(s) \, ds \right)$$

(since $f$ and $g$ are integrable)

$$= \omega \left( \alpha \int_G f(s) \, ds + \beta \int_G g(s) \, ds \right) ,$$

so $s \to \alpha f(s) + \beta g(s)$ is integrable and

$$\int_G \alpha f(s) + \beta g(s) \, ds = \alpha \int_G f(s) \, ds + \beta \int_G g(s) \, ds .$$

Now $s \to \omega(\xi(s)T)$ is clearly Lebesgue integrable and

$$\int_G \omega(\xi(s)T) \, ds = \int_G \xi(s) \omega(T) \, ds$$

$$= \int_G \xi(s) \, ds \, \omega(T)$$

$$= \omega \left( \int_G \xi(s) \, ds \, T \right) ,$$

so $s \to \xi(s)T$ is integrable and

$$\int_G \xi(s)T \, ds = \int_G \xi(s) \, ds \, T .$$
Lemma 1.2 Suppose \( f : G \to B(\mathcal{H}) \) is integrable and \( T \in B(\mathcal{H}) \). Then the maps 
\( s \to f(s)T \) and \( s \to Tf(s) \) are integrable and

\[
T \int_G f(s) \, ds = \int_G Tf(s) \, ds,
\]
\[
\int_G f(s) \, ds \cdot T = \int_G f(s)T \, ds.
\]

**Proof** Let \( \omega \in B(\mathcal{H})_w \). Since multiplication with one variable fixed is weakly continuous the maps \( T \cdot \omega \) and \( \omega \cdot T \) defined by

\[
(T \cdot \omega)(S) = \omega(ST) \quad (\omega \cdot T)(S) = \omega(TS) \quad S, T \in B(\mathcal{H}),
\]

are weakly continuous. Hence \( s \to \omega(Tf(s)) \) is Lebesgue integrable by assumption. Similarly \( s \to \omega(f(s)T) \) is Lebesgue integrable. Now

\[
\int_G \omega(Tf(s)) \, ds = \int_G (\omega \cdot T)(f(s)) \, ds
\]
\[
= (\omega \cdot T)\left( \int_G f(s) \, ds \right)
\]
(since \( f \) is integrable)
\[
= \omega \left( T \int_G f(s) \, ds \right).
\]

So \( s \to Tf(s) \) is integrable and

\[
T \int_G f(s) \, ds = \int_G Tf(s) \, ds.
\]

Similarly \( s \to f(s)T \) is integrable and

\[
\int_G f(s) \, ds \cdot T = \int_G f(s)T \, ds.
\]

Lemma 1.3 Suppose \( f : G \to B(\mathcal{H}) \) is integrable. Then the map \( s \to f(s)^* \) is integrable and

\[
\left( \int_G f(s) \, ds \right)^* = \int_G f(s)^* \, ds.
\]
Proof Let $\omega \in B(\mathcal{H})_w$. Since the $*$-operation is weakly continuous the map $\omega^*$ defined by

$$\omega^*(S) = \overline{\omega(S^*)} \quad S \in B(\mathcal{H}),$$

is weakly continuous. Hence: $s \rightarrow \omega(f(s)^*)$ is Lebesgue integrable by assumption. Now

$$\int_G \omega(f(s)^*) \, ds = \int_G \omega^*(f(s)) \, ds$$

$$= \omega \left( \int_G f(s) \, ds \right)$$

(since $f$ is integrable)

$$= \omega \left( \left( \int_G f(s) \, ds \right)^* \right).$$

So: $s \rightarrow f(s)^*$ is integrable and

$$\left( \int_G f(s) \, ds \right)^* = \int_G f(s)^* \, ds.$$

\[ \square \]

Lemma 1.4 Suppose $f: G \rightarrow B(\mathcal{H})$ is integrable. Then

$$\left\| \int_G f(s) \, ds \right\| \leq \int_G \|f(s)\| \, ds.$$

Proof

$$\left\| \int_G f(s) \, ds \right\| = \sup \left\{ \left| \omega \left( \int_G f(s) \, ds \right) \right| : \omega \in B(\mathcal{H})_w, \|\omega\| \leq 1 \right\}$$

$$= \sup \left\{ \int_G |\omega(f(s))| \, ds : \omega \in B(\mathcal{H})_w, \|\omega\| \leq 1 \right\}$$

$$\leq \int_G \sup \{ |\omega(f(s))| : \omega \in B(\mathcal{H})_w, \|\omega\| \leq 1 \} \, ds$$

$$= \int_G \|f(s)\| \, ds.$$

\[ \square \]
Lemma 1.5 Suppose $\mathcal{H}$ and $Q$ are Hilbert spaces, $\gamma : B(\mathcal{H}) \to B(Q)$ is linear and weakly continuous, and $f : G \to B(\mathcal{H})$ is integrable. Then the map $s \to \gamma(f(s))$ is integrable and

$$\gamma\left(\int_{G} f(s) \, ds\right) = \int_{G} \gamma(f(s)) \, ds.$$ 

Proof Let $\omega \in B(Q)_{w}$. Now $\omega \circ \gamma$ is weakly continuous, that is, an element of $B(\mathcal{H})_{w}$, so $s \to \omega(\gamma(f(s)))$ is Lebesgue integrable by assumption. Now

$$\int_{G} \omega(\gamma(f(s))) \, ds = \int_{G} (\omega \circ \gamma)(f(s)) \, ds$$

$$= (\omega \circ \gamma)\left(\int_{G} f(s) \, ds\right)$$

(since $f$ is integrable)

$$= \omega\left(\gamma\left(\int_{G} f(s) \, ds\right)\right),$$

so $s \to \gamma(f(s))$ is integrable and

$$\int_{G} \gamma(f(s)) \, ds = \gamma\left(\int_{G} f(s) \, ds\right).$$

Lemma 1.6 Suppose $f : G \to B(\mathcal{H})$ is weakly continuous and compactly supported. Then $f$ is integrable.

Proof Let $\omega$ be an element of the pre-dual $B(\mathcal{H})_{*}$ of $B(\mathcal{H})$. Then $\omega \circ f$ is continuous and compactly supported, hence bounded. Now consider the maps

$$\zeta_{s} : B(\mathcal{H})_{*} \to C : \omega \to \omega(f(s)) \quad s \in G.$$ 

Then for each $s \in G$

$$|\zeta_{s}(\omega)| = |\omega(f(s))| \leq \|\omega \circ f\|_{C_{c}(G)}.$$
So by the uniform boundedness principle there exists a constant $M$ such that

$$\|\zeta_s\| \leq M \quad \forall \ s \in G,$$

i.e.

$$|\omega(f(s))| < M \cdot \|\omega\|_{B(\mathcal{H})} \quad \forall \ \omega \in B(\mathcal{H}), s \in G.$$

Hence

$$\left| \int_G \omega(f(s)) \, ds \right| < \mu_G(\text{supp} f) \cdot M \cdot \|\omega\|_{B(\mathcal{H})}.$$

So $s \to \omega(f(s))$ is Lebesgue integrable. Now the map $\omega \to \int_G \omega(f(s)) \, ds$ is bounded and defines an element $T$ of $B(\mathcal{H}) = (B(\mathcal{H}^*))^*$ such that

$$\omega(T) = T(\omega) = \int_G \omega(f(s)) \, ds \quad \forall \ \omega \in B(\mathcal{H}) \supset B(\mathcal{H})_w.$$

So $f$ is integrable as claimed. \hfill \square

**Lemma 1.7** Let $\mathcal{H}$ and $\mathcal{Q}$ be Hilbert spaces. Let $A$ and $B$ be $C^*$-subalgebras of $B(\mathcal{H})$ and $B(\mathcal{Q})$ respectively.

(i) Suppose $\gamma : A \to B$ is linear and norm continuous, and $f : G \to A$ is norm continuous with compact support. Then

$$\int_G f(s) \, ds \in A$$

and

$$\gamma\left(\int_G f(s) \, ds\right) = \int_G \gamma(f(s)) \, ds.$$

(ii) Suppose $\gamma : A \to B$ is linear and strictly continuous, and $f : G \to A$ is strictly continuous with compact support. Then

$$\int_G f(s) \, ds \in M(A)$$
and
\[ \gamma \left( \int_G f(s) \, ds \right) = \int_G \gamma(f(s)) \, ds. \]

**Proof**  
(i) First note that \( f \) and \( \gamma \circ f \) are integrable by lemma 1.6. Let \( \epsilon > 0 \). Since \( f \) is norm continuous and compactly supported a standard argument [27 chapter 1 prop. 7.3] shows that we can find \( \xi_j \in C_c(G) \) and \( s_j \in \text{supp} f \), for \( j = 1, \ldots, n \), with \( \text{supp} \xi_j \subset \text{supp} f \), such that

\[ \| f(s) - \sum_{j=1}^{n} \xi_j(s)f(s_j) \| \leq \min\left( \frac{\epsilon}{\mu_G(\text{supp} f)}, \frac{\epsilon}{2 \cdot \| \gamma \| \cdot \mu_G(\text{supp} f)} \right) \]

for all \( s \in G \). Let \( \nu_j = \int_G \xi_j(s) \, ds \). Then by lemmas 1.1 and 1.4

\[ \| \int_G f(s) \, ds - \sum_{j=1}^{n} \nu_j f(s_j) \| = \| \int_G (f(s) - \sum_{j=1}^{n} \xi_j(s)f(s_j)) \, ds \| \]

\[ \leq \int_G \| f(s) - \sum_{j=1}^{n} \xi_j(s)f(s_j) \| \, ds \]

\[ < \mu_G(\text{supp} f) \cdot (\epsilon/\mu_G(\text{supp} f)) \]

\[ < \epsilon, \]

and since \( \sum \nu_j f(s_j) \in A \) for all \( j \) we have that \( \int_G f(s) \, ds \in A \). Now

\[ \| \gamma \left( \int_G f(s) \, ds \right) - \int_G \gamma(f(s)) \, ds \| \]

\[ \leq \| \gamma \left( \int_G f(s) \, ds - \sum_{j=1}^{n} \nu_j f(s_j) \right) \| + \sum_{j=1}^{n} \nu_j \gamma(f(s_j)) - \int_G \gamma(f(s)) \, ds \| \]

\[ \leq \| \gamma \| \cdot \int_G \| f(s) - \sum_{j=1}^{n} \xi_j(s)f(s_j) \| \, ds + \int_G \| \sum_{j=1}^{n} \xi_j(s)\gamma(f(s_j)) - \gamma(f(s)) \| \, ds \]

\[ \leq 2 \cdot \| \gamma \| \cdot \mu_G(\text{supp} f) \cdot (\epsilon/(2 \cdot \| \gamma \| \cdot \mu_G(\text{supp} f))) \]

- 20 -
So
\[ \gamma \left( \int_G f(s) \, ds \right) = \int_G \gamma(f(s)) \, ds. \]

(ii) Once again \( f \) and \( \gamma \circ f \) are integrable by lemma 1.6. Let \( a \in A \) and \( b \in B \).

Then the maps
\[ : s \to af(s) \quad \text{and} \quad : s \to b\gamma(f(s)) \]
are norm continuous with compact support. So by lemma 1.2 and the above
\[ a \int_G f(s) \, ds = \int_G af(s) \, ds \in A \]
and
\[ b \int_G \gamma(f(s)) \, ds = \int_G b\gamma(f(s)) \, ds \in B \]
Hence
\[ \int_G f(s) \, ds \in M(A) \quad \text{and} \quad \int_G \gamma(f(s)) \, ds \in M(B). \]

Also by lemma 1.2 and the above we have that
\[ b \int_G \gamma(f(s)) \, ds = \int_G b\gamma(f(s)) \, ds = b\gamma \left( \int_G f(s) \, ds \right) \]
for all \( b \in B \). Hence
\[ \int_G \gamma(f(s)) \, ds = \gamma \left( \int_G f(s) \, ds \right). \]

I am not sure that the following lemmas are sharp, but they are enough for our purposes.

Lemma 1.8 Let \( H \) and \( K \) be locally compact groups. Suppose the maps \( f : H \times K \to \mathcal{B}(\mathcal{H}) \), \( : k \to f(h, k) : K \to \mathcal{B}(\mathcal{H}) \) and \( : h \to f(h, k) : H \to \mathcal{B}(\mathcal{H}) \) are integrable. Then the maps
\[ : h \to \int_K f(h, k) \, d\mu_K \quad \text{and} \quad : k \to \int_H f(h, k) \, d\mu_H \]
are integrable and

\[ \int_H \left( \int_K f(h, k) \, d\mu_K \right) \, d\mu_H = \int_{H \times K} f(h, k) \, d\mu_{H \times K} = \int_K \left( \int_H f(h, k) \, d\mu_H \right) \, d\mu_K . \]

In particular the lemma holds for \( f : H \times K \to B(H) \) weakly continuous and compactly supported.

**Proof** Let \( \omega \in B(H)_w \) then \( \omega \circ f \) is Lebesgue integrable by assumption. Hence

\[ : h \to \int_K \omega \circ f(h, k) \, d\mu_K \quad \text{and} \quad : k \to \int_H \omega \circ f(h, k) \, d\mu_H \]

are Lebesgue integrable by Fubini's theorem. Now

\[ \int_H \omega \left( \int_K f(h, k) \, d\mu_K \right) \, d\mu_H = \int_H \left( \int_K \omega(f(h, k)) \, d\mu_K \right) \, d\mu_H \]

(since \( : k \to f(h, k) \) is integrable)

\[ = \int_{H \times K} \omega(f(h, k)) \, d\mu_{H \times K} \]

(by Fubini's theorem)

\[ = \omega \left( \int_{H \times K} f(h, k) \, d\mu_{H \times K} \right) . \]

Hence \( : h \to \int_K f(h, k) \, d\mu_K \) is integrable as claimed and

\[ \int_H \left( \int_K f(h, k) \, d\mu_K \right) \, d\mu_H = \int_{H \times K} f(h, k) \, d\mu_{H \times K} . \]

A similar argument shows \( : k \to \int_H f(h, k) \, d\mu_H \) is integrable and

\[ \int_{H \times K} f(h, k) \, d\mu_{H \times K} = \int_K \left( \int_H f(h, k) \, d\mu_H \right) \, d\mu_K . \]

\[ \square \]

**Lemma 1.9** Suppose \( f, f_n : G \to B(H) \) are integrable, \( f_n(s) \to f(s) \) weakly and for all \( \omega \in B(H)_w \) there exists a function \( g_\omega \in L^1(G)^+ \) such that

\[ |\omega(f_n(s))| \leq g_\omega(s) \quad \text{for } n \text{ sufficiently large.} \]
Then \[ \int_G f(s)\, ds = \lim \text{weak limit} \int_G f_n(s)\, ds. \]

Proof Let \( \omega \in B(\mathcal{H})_w \) then \( \omega \circ f(s) \to \omega \circ f_n(s) \) for all \( s \in G \) and

\[ |\omega(f_n(s))| \leq g_\omega(s) \quad \forall \, s \in G \quad \text{and} \quad n \text{ sufficiently large}. \]

So by the dominated convergence theorem \( \omega \circ f \) is Lebesgue integrable. Now

\[ \omega\left( \int_G f(s)\, ds \right) = \int_G \omega(f(s))\, ds \]

(since \( f \) is integrable)

\[ = \int_G \lim_{n \to \infty} \omega(f_n(s))\, ds \]

\[ = \lim_{n \to \infty} \int_G \omega(f_n(s))\, ds \]

(by the dominated convergence theorem)

\[ = \lim_{n \to \infty} \omega\left( \int_G f_n(s)\, ds \right) \]

(since \( f_n \) is integrable). Hence

\[ \int_G f(s)\, ds = \lim \text{weak limit} \int_G f_n(s)\, ds. \]

\[ \square \]

§4 Induced Representations of C*-algebras

Here and throughout all representations will be assumed non-degenerate. Let \( A \) and \( B \) be pre-C*-algebras with completions \( A \) and \( B \) respectively.

Definitions A pre-\( B \)-valued inner product on a complex vector space \( D \), is a sesquilinear map \( \langle \cdot, \cdot \rangle_D : D \times D \to B \) (conjugate linear in one variable and linear in the other) such that

(i) \( \langle x, x \rangle_D \geq 0 \) in the completion of \( B \),

- 23 -
(ii) \[ \langle x, y \rangle_D^* = \langle y, x \rangle_D. \] \hspace{1cm} (14)

Suppose \( \langle \cdot, \cdot \rangle_D \) also has the property that

(iii) \( \langle x, x \rangle_D = 0 \) implies \( x = 0 \).

Then we will say \( \langle \cdot, \cdot \rangle_D \) is \textit{definite} and will call it a \textit{B-valued inner product}.

A vector space \( D \) equipped with a pre-\( B \)-valued inner product will be called a \textit{pre-Hilbert} \( B \) \textit{module}.

A pre-Hilbert \( B \) module \( D \) comes equipped with a semi-norm \( \| \cdot \|_D \) defined by

\[ \|x\|_D = \|\langle x, x \rangle_D\|_B^{1/2}. \] \hspace{1cm} (15)

A vector space \( D \) which is equipped with a \( B \)-valued inner product and which is complete with respect to the norm \( \| \cdot \|_D \) is called a \textit{Hilbert} \( B \) \textit{module}.

A right \( B \)-rigged space \( D \) is a right \( B \)-module with a pre-\( B \)-valued inner product (conjugate linear in the first variable and linear in the second) such that

(i) \( (\mu x) \bullet b = \mu(x \bullet b) = x \bullet (\mu b) \),

(ii) \( \langle x, y \bullet b \rangle_D = \langle x, y \rangle_D b \), \hspace{1cm} (16)

(iii) \( \) the closed linear span of \( \{\langle x, y \rangle_D : x, y \in D\} \) is dense in \( B \),

for all \( x, y \in D, b \in B \) and complex numbers \( \mu \), where \( \bullet \) denotes the module action.

Left \( B \)-rigged spaces are defined similarly but with the pre-\( B \)-valued inner product conjugate linear in the second variable and linear in the first.

A \textit{left} \( \textit{pre-Hermitian} \) \textit{B-rigged} \( A \) \textit{module} is a \( B \)-rigged space \( D \) equipped with a left \( A \) action which satisfies

(i) \( (a \bullet x) \bullet b = a \bullet (x \bullet b) \),

(ii) \( \langle a \bullet x, y \rangle_D = \langle x, a^* \bullet y \rangle_D \), \hspace{1cm} (17)

(iii) \( \langle a \bullet x, a \bullet x \rangle_D \leq \|a\|^2_A \langle x, x \rangle_D \).
(iv) \( \mathcal{AD} \) is dense in \( \mathcal{D} \) with respect to the semi-norm of \( \|15 \),

where \( a \in A, \ b \in B \) and \( x \in D \). Right pre-Hermitian (left) \( B \)-rigged (right) \( A \) modules are defined similarly. If condition (iv) does not necessarily hold we will say that \( \mathcal{D} \) is a possibly degenerate pre-Hermitian \( B \)-rigged \( A \) module. If the \( B \)-valued inner product on \( \mathcal{D} \) is definite and if \( \mathcal{D} \) is complete with respect to the norm of \( \|15 \), then we will call \( \mathcal{D} \) a (left) Hermitian \( B \)-rigged \( A \) module.

Now we can use pre-Hermitian \( B \)-rigged \( A \) modules to construct representations of \( A \) from representations of \( B \), that is, to induce representations of \( B \) to representations of \( A \) [22 thm. 5.1]. This is achieved as follows:

Suppose \( \nu : B \to B(\mathcal{H}) \) is a representation of \( B \) on \( \mathcal{H} \). Then we can define a pre-inner product on \( \mathcal{D} \otimes \mathcal{H} \) by

\[
\langle x \otimes \xi, x \otimes \eta \rangle_{\mathcal{D} \otimes \mathcal{H}} = \langle \{ \nu((y, x)_{\mathcal{D}}) \}(\xi), \eta \rangle_{\mathcal{H}},
\]

Now we obtain a Hilbert space \( \mathcal{D}-\text{ind}^A_B \mathcal{H} \) from \( \mathcal{D} \otimes \mathcal{H} \) by factoring out by the vectors of length zero and completing. It is easily shown that the representation \( \mathcal{D}-\text{ind}^A_B \nu : A \to B(\mathcal{D}-\text{ind}^A_B \mathcal{H}) \) of \( A \) on \( \mathcal{D}-\text{ind}^A_B \mathcal{H} \) given by

\[
\{ \{ \mathcal{D}-\text{ind}^A_B \nu \}(a) \}(x \otimes \xi) = (a \cdot x) \otimes \xi \quad a \in A, \ x \in \mathcal{D}, \ \xi \in \mathcal{D}-\text{ind}^A_B \mathcal{H},
\]

is well-defined. \( \mathcal{D}-\text{ind}^A_B \nu \) is called the representation induced from the representation \( \nu \) of \( B \) via \( \mathcal{D} \). If the module \( \mathcal{D} \) is understood, then we will write \( \text{ind}^A_B \nu \) and drop the words via \( \mathcal{D} \).

Rieffel’s construction of induced representations [22 pg. 222] uses \( \mathcal{D} \otimes_B \mathcal{H} \) instead of \( \mathcal{D} \otimes \mathcal{H} \). To see that our construction is the equivalent, we note that the map

\[
\gamma : \mathcal{D} \otimes \mathcal{H} \to \mathcal{D} \otimes_B \mathcal{H} : x \otimes \xi \to x \otimes \xi,
\]

is well-defined and that its kernel is the submodule generated by the set

\[
\{ (x \cdot b) \otimes \xi - x \otimes (b \cdot \xi) : x \in \mathcal{D}, \ b \in B, \ \xi \in \mathcal{H} \}.
\]
If $P$ is the submodule of $\mathcal{D} \otimes \mathcal{H}$ consisting of elements which are zero for the above inner product and if $R$ be the submodule of $\mathcal{D} \otimes_B \mathcal{H}$ consisting of elements which are zero for the inner product of [22 thm. 5.9], then $\gamma(P) = R$ and $\ker \gamma \subset P$. Hence, by for example, [9 IV cor. 5.9] we have that (as $\mathcal{A}$ modules)

$$(\mathcal{D} \otimes \mathcal{H})/P \cong (\mathcal{D} \otimes_B \mathcal{H})/R.$$ 

Thus the completions are isomorphic as $\mathcal{A}$ modules, that is, the (induced) representations of $\mathcal{A}$ are unitarily equivalent. The reason for adopting the above definition over Rieffel's is that to show a map is well-defined we need only check it is bounded and bilinear, whereas, otherwise, we would also have to check the map is $\mathcal{B}$-balanced.

The above induction process is really a map

$$\mathcal{D}-\text{ind}^A_B : \text{Rep}(\mathcal{B}) \to \text{Rep}(\mathcal{A}) : \nu \to \mathcal{D}-\text{ind}^A_B \nu.$$ 

Now if $\mathcal{D}$ is a pre-Hermitian $\mathcal{B}$-rigged $\mathcal{A}$ module as above, then we can factor (by the vectors of length zero) and complete with respect the semi-norm of $\mathcal{B}15$ to obtain a Banach space $X$. It is easily shown that the action of $\mathcal{A}$ is well-defined on the quotient and extends to an action of $\mathcal{A}$ on $X$. Hence $X$ is a pre-Hermitian $\mathcal{B}$-rigged $\mathcal{A}$ module and thus can be used to induce representations of $\mathcal{B}$ to representations of $\mathcal{A}$. Now it is not difficult to check that the map

$$\zeta : \mathcal{D} \otimes \mathcal{H} \to X \otimes \mathcal{H} : x \otimes \xi \to [x] \otimes \xi,$$

where $x \in X$, $[x]$ is its equivalence class in $X$ and $\xi \in \mathcal{H}$, extends to a unitary

$$\zeta : \mathcal{D}-\text{ind}^A_B \mathcal{H} \to X-\text{ind}^A_B \mathcal{H},$$

which intertwines the $\mathcal{A}$ actions. That is $\mathcal{D}-\text{ind}^A_B \nu$ and $X-\text{ind}^A_B \nu$ are unitarily equivalent. So whether we use $\mathcal{D}$ or $X$ we obtain the same induction process, that is, the same map

$$\text{ind}^A_B : \text{Rep}(\mathcal{B}) \to \text{Rep}(\mathcal{A}) : \nu \to \text{ind}^A_B \nu.$$
Although it is clearly easier to construct induced representations using the module \( D \), rather than its completion \( X \), we shall use completion \( X \) since we shall often require a module which carries actions of the (complete) \( C^* \)-algebras \( A \) and \( B \).

Suppose \( H \) is a subgroup of a locally compact group \( G \) and \( \nu \) is a unitary representation of \( H \). Then by the Mackey's theory of induction [17] one can construct a representation, which we will denote \( \text{ind}^G_H \nu \), of \( G \) from \( \nu \). Rieffel has shown [22] that Mackey's construction is a special case of the above. In this case

\[
A = C^*(G), \quad B = C^*(H), \quad A = D = C_c(G) \quad \text{and} \quad B = C_c(H),
\]

and the actions and inner products are given by

\[
(f \star x)(s) = \int_G f(t)x(t^{-1}s) \, dt
\]

\[
(x \cdot g)(s) = \int_H \sqrt{\frac{\Delta_H(r)}{\Delta_G(r)}} x(sr)g(r^{-1}) \, dr
\]

\[
\langle x, y \rangle_{C_c(G)}(h) = \int_G \sqrt{\frac{\Delta_G(h)}{\Delta_H(h)}} x(s)y(sh) \, ds.
\]

If \( \mu \) is the integrated form of \( \nu \), then \( \text{ind}^G_H \nu \) is unitarily equivalent to \( \text{ind}^{C^*(G)}_{C^*(H)} \mu \).

Let \( A \) and \( B \) be \( C^* \)-algebras with dense \(*\)-subalgebras (pre-\( C^* \)-algebras) \( A \) and \( B \). Now suppose we have a pre-Hermitian \( B \)-rigged \( A \) module which is also a pre-Hermitian \( A \)-rigged \( B \) module, then we can induce a representation of \( B \) to a representation of \( A \), which can then be induced back to a representation of \( B \). It is evident that such a module, with (perhaps) a few extra conditions, could establish an equivalence between the representation theories of \( A \) and \( B \). This is an important idea and is formalised as follows.

**Definition** Let \( A \) and \( B \) be \( C^* \)-algebras. Then \( A \) is said to be **strongly Morita equivalent** to \( B \), denoted \( A \approx B \), if and only if there exists dense \(*\)-subalgebras \( A \)
and $\mathcal{B}$ of $A$ and $B$ respectively, and an $\mathcal{A}$-$\mathcal{B}$ bimodule $\mathcal{D}$ which is a left $\mathcal{A}$-rigged and right $\mathcal{B}$-rigged (with pre-inner products $\langle \cdot , \cdot \rangle^A_D$ and $\langle \cdot , \cdot \rangle^B_D$) such that

(i) $x \cdot \langle y, z \rangle^B_D = \langle x, y \rangle^A_D \cdot z$,

(ii) $\langle a \cdot x, a \cdot x \rangle^B_D \leq \|a\|^2_A \cdot \langle x, x \rangle^B_D$, \hspace{1cm} (19)

(iii) $\langle x \cdot b, x \cdot b \rangle^A_D \leq \|b\|^2_B \cdot \langle x, x \rangle^A_D$,

where $x, y, z \in \mathcal{D}$, $a \in \mathcal{A}$ and $b \in \mathcal{B}$.

We call the bimodule implementing a strong Morita equivalence an **equivalence bimodule**. We note that if $\mathcal{D}$ is an $\mathcal{A}$-$\mathcal{B}$ equivalence bimodule, then $\mathcal{D}$ is a pre-Hermitian $\mathcal{B}$-rigged $\mathcal{A}$ module and also a pre-Hermitian $\mathcal{A}$-rigged $\mathcal{B}$ module [22 prop. 6.14]. That strongly Morita equivalent algebras do indeed have the same representation theory was shown by Rieffel [22 thm. 6.23].

Now we wish to relate ideals of the (complete) $C^*$-algebra $A$ to ideals of the (complete) $C^*$-algebra $B$. In order to do this we need to have actions of $A$ and $B$ on the $\mathcal{A}$-$\mathcal{B}$ equivalence bimodule $\mathcal{D}$, hence we factor and complete $\mathcal{D}$ with respect to the semi-norm $\| \cdot \|_D$ of $\| \cdot \|_D$ is unambiguously defined since

$$\|x\|_D = \|\langle x, x \rangle^A_D\|^{1/2} = \|\langle x, x \rangle^B_D\|^{1/2},$$

for all $x \in \mathcal{D})$ to obtain a Banach space $X$. The conditions on the actions of $\mathcal{A}$ and $\mathcal{B}$ ensure they lift to the quotient and extend to actions of $\mathcal{A}$ and $\mathcal{B}$ on $X$. The inner products also lift to the quotient and extend to the completion so that $X$ becomes an $\mathcal{A}$-$\mathcal{B}$ equivalence bimodule.

Let $\text{Ideals}(A)$ denote the set of (closed two-sided) ideals of a $C^*$-algebra $A$. We define the (inner) **hull-kernel topology** on $\text{Ideals}(A)$ to be the topology determined by the sub-base $(O_J)_{J \in \text{Ideals}(A)}$, where

$$O_J = \{I \in \text{Ideals}(A) : I \nsubseteq J\}.$$
This topology, when restricted to the primitive ideals, is the usual hull-kernel topology.

Rieffel has shown that if $D$ and $X$ are as above, then the $A$-$B$ equivalence bimodule $X$ establishes a lattice isomorphism, and hence a homeomorphism

$$ h : \text{Ideals} (B) \to \text{Ideals} (A), $$

and that if $\nu$ is a representation of $B$, then

$$ h(\ker \nu) = \ker (\text{ind}^A_B \nu), $$

[24 prop. 3.3 and cor. 3.3]. Also since $\nu$ is irreducible if and only if $\text{ind}^A_B \nu$ is [22 cor. 6.25], we see that $h$ restricts to a homeomorphism

$$ h : \text{Prim} (B) \to \text{Prim} (A). $$

For a fuller account see [24 §3].

Let $D$ be a $B$-rigged space. We call an operator $T : D \to D$ bounded if

(i) $\exists \kappa \geq 0$ such that $\langle Tx, Tx \rangle_D \leq \kappa^2 \langle x, x \rangle_D$,

(ii) $\exists T^* : D \to D$ such that $\langle Tx, y \rangle_D = \langle x, T^* y \rangle_D$,

(iii) $T(x \cdot b) = T(x) \cdot b$ for all $x, y \in D$ and $b \in B$.

We define a semi-norm on the set of bounded operators by

$$ \|T\| = \inf \{ \kappa \geq 0 : \langle Tx, Tx \rangle_D \leq \kappa^2 \langle x, x \rangle_D \}. $$

If we let $J$ be the set of operators in $L(D)$ of length zero, then $L(D)/J$ is a pre-$C^*$-algebra [22 prop. 2.5]. We will denote the completion of $L(D)/J$ by $L(D)$.

The operators

$$ T_{w,x} : D \to D : y \mapsto w \cdot \langle x, y \rangle_D, $$

where $w, x \in D$, form a subalgebra of $L(D)$, which we will denote by $\mathcal{K}(D)$. We will denote the completion of $\mathcal{K}(D)/J$ in $L(D)$ by $K(D)$. We will call $K(D)$ the
imprimitivity algebra of the $B$-rigged space $\mathcal{D}$. As examples, the imprimitivity algebra $K(C_c(G))$ for Mackey's induction is $C_o(G/H) \times_r G$ and for Green's induction 
[6] $K(C_c(G, A))$ is $(A \otimes C_o(G/H)) \times_{\alpha \otimes r} G$.

Theorem 1.10 (Rieffel [22 prop's 6.5 and 6.6]) Suppose $B$ is a $C^*$-algebra with a dense $*$-subalgebra $B$ and that $\mathcal{D}$ is a $B$-rigged space. Then

$$K(\mathcal{D}) \approx B.$$ 

Suppose that the semi-norm of § 15 on $\mathcal{D}$ is in fact a norm and that $\mathcal{D}$ is complete with respect to it. Then $L(\mathcal{D})$ can be alternatively described as those operators of $B(\mathcal{D})$ which have an adjoint which is also in $B(\mathcal{D})$ [19].

Lemma 1.11 The map

$$\alpha : L(\mathcal{D}) \rightarrow L(X),$$

$X$ and $\mathcal{D}$ as above, determined by $\{\alpha(T)\}(\{x]\) = [T(x)],$ for all $x \in \mathcal{D},$ where $[x]$ is its equivalence class in $X$, is an isometric $*$-homomorphism which maps $K(\mathcal{D})$ onto $K(X)$.

Proof It is easily seen that $\alpha$ is a well-defined isometric $*$-homomorphism. To see that $\alpha$ maps $K(\mathcal{D})$ onto $K(X)$, note that

$$\{\alpha(T_{x,y})\}(\{z]\) = [z \cdot \langle y, z \rangle_{\mathcal{D}}]$$

$$= [z] \cdot \langle y, z \rangle_{\mathcal{D}}$$

$$= [z] \cdot \langle [y], [z] \rangle_{X}$$

$$= T_{\{z\}, [y]}(\{x]\).$$

and that if $x = \lim [x_i]$ and $y = \lim [y_j]$, then $T_{x,y} = \lim T_{\{x_i\}, [y_j]}$.

This shows that $\alpha$ maps $K(\mathcal{D})/J$ onto a dense subspace of $K(X)$ and hence $K(\mathcal{D})$ onto $K(X)$.
Lemma 1.12 Suppose $\mathcal{D}$ is a possibly degenerate pre-Hermitian $B$-rigged $A$ module. Then the left $A$ action determines a norm decreasing $*$-homomorphism

$$\beta : a \rightarrow [\theta_a] : A \rightarrow \mathcal{L}(\mathcal{D})/J ,$$

where $[\theta_a]$ is the equivalence class of the operator

$$\theta_a(y) = a \cdot y \quad \forall \ y \in \mathcal{D} .$$

Proof First we show that $\theta_a \in \mathcal{L}(\mathcal{D})$.

(i) $\theta_a$ is clearly linear.

(ii) \[
\langle \theta_a(x) , \theta_a(x) \rangle_{\mathcal{D}} = \langle a \cdot x , a \cdot x \rangle_{\mathcal{D}} \\
\leq \|a\|_A^2 \cdot \langle x , x \rangle_{\mathcal{D}} . \quad \text{(by (17))}
\]
So

$$\|\theta_a\|_{\mathcal{L}(\mathcal{D})} \leq \|a\|_A ,$$

and in particular $\theta_a$ is bounded.

(iii) \[
\langle \theta_a(x) , y \rangle_{\mathcal{D}} = \langle a \cdot x , y \rangle_{\mathcal{D}} \\
= \langle x , a^* \cdot y \rangle_{\mathcal{D}} \quad \text{(by (17))} \\
= \langle x , \theta_a^*(y) \rangle_{\mathcal{D}} .
\]
So $\theta_a^* = \theta_a^*$ and in particular $\theta_a$ has an adjoint.

So $\theta_a \in \mathcal{L}(\mathcal{D})$.

Now we show that $\beta$ is a $*$-homomorphism. Let $a, b \in A$. Then

(iv) \[
\{\beta(ab)\}(x) = (ab) \cdot x \\
= a \cdot (b \cdot x) \\
= \{\beta(a) \circ \beta(b)\}(x) ,
\]
so $\beta$ is multiplicative.
(v) by (iii) \( \beta(b)^* = \beta(b^*) \), so \( \beta \) is \(*\)-preserving.

(vi) by (ii) it is clear that \( \beta \) is norm decreasing.

§5 Actions and their Crossed Products

In this section we present some of the results about actions and their crossed products that we will use later in the discourse.

Definitions An action \( \alpha \) of a locally compact group on a \( C^* \)-algebra \( A \) is a \(*\)-homomorphism

\[
\alpha : G \to \text{Aut} A,
\]

where \( \text{Aut} A \) is the set of \(*\)-automorphisms of \( A \), such that for each \( a \in A \) the function \( G \to A : s \to \alpha_s(a) \) is continuous.

We will call a triple \( (A, G, \alpha) \), where \( \alpha \) is an action of \( G \) on \( A \), a \( C^* \)-dynamical system.

A covariant representation of a \( C^* \)-dynamical system \( (A, G, \alpha) \) on a Hilbert space \( \mathcal{H} \) is a pair \( (\pi, U) \), where \( \pi \) is a representation of \( A \) on \( \mathcal{H} \) and \( U \) is a unitary representation \( U \) of \( G \) on \( \mathcal{H} \) such that

\[
\pi(\alpha_s(a)) = U_s \pi(a) U_s^* \quad \forall a \in A, s \in G.
\]

Let \( \alpha \) be an action of \( G \) on \( A \). We define an involution and product on \( C_c(G, A) \) by

\[
y^*(t) = \frac{1}{\Delta t} \alpha_t(y(t^{-1})^*)
\]

\[
yz(t) = \int_G y(s) \alpha_s(z(s^{-1}t)) \, ds,
\]

for all \( y, z \in C_c(G, A) \). We define a norm

\[
\|y\|_1 = \int_G \|y(s)\| \, ds.
\]
Then $C_c(G,A)$ with the above operations and norm is a normed $\ast$-algebra, which we will denote $L^1(G,A)$. We call the $C^\ast$-completion of $L^1(G,A)$ the crossed product of $A$ by (the action $\alpha$ of) $G$ and denote it by $A \rtimes_\alpha G$.

**Theorem 1.13** (Doplicher-Kastler-Robinson [2]) Let $(\pi,U)$ be a covariant representation of the $C^\ast$-dynamical system $(A, G, \alpha)$ on the Hilbert space $\mathcal{H}$. Then the map

$$ : y \rightarrow \int_G \pi(y(s))U_s \, ds : C_c(G,A) \rightarrow B(\mathcal{H}) $$

determines a representation of $L^1(G,A)$, and hence of $A \rtimes_\alpha G$, on $\mathcal{H}$, which we will denote by $\pi \times U$. Moreover the correspondence

$$ \pi \times U \longleftrightarrow (\pi,U) $$

is bijective.

**Definitions** Let $(A, G, \alpha)$ be a $C^\ast$-dynamical system. Let $\pi$ be a representation of $A$ on $\mathcal{H}$. Define $\tilde{\pi} : A \rtimes_\alpha G \rightarrow B(L^2(\mathcal{H}))$ and $1 \otimes \lambda_G : G \rightarrow B(L^2(G, \mathcal{H}))$ by

$$ \{\{\tilde{\pi}(a))((\xi))(s) = \{\alpha_s^{-1}(a))((\xi))(s) \text{ and } \{\{(1 \otimes \lambda_G)(r))((\xi))(s) = \xi((r^{-1}s), (23) $$

for all $s \in G$ and $\xi \in C_c(G, \mathcal{H})$, respectively. Then $(\tilde{\pi}, 1 \otimes \lambda_G)$ is a covariant representation of $(A, G, \alpha)$ on $L^2(G, \mathcal{H})$, called the **left regular representation**.

If we define $\tilde{\pi} : A \rtimes_\alpha G \rightarrow B(L^2(\mathcal{H}))$ and $1 \otimes \rho_G : G \rightarrow B(L^2(G, \mathcal{H}))$ by

$$ \{\{\tilde{\pi}(a))((\xi))(s) = \{\alpha_s(a))((\xi))(s) \text{ and } \{\{(1 \otimes \rho_G)(r))((\xi))(s) = \Delta r \xi((sr), (24) $$

for all $s \in G$ and $\xi \in C_c(G, \mathcal{H})$, respectively, then $(\tilde{\pi}, 1 \otimes \rho_G)$ is also a covariant representation of $(A, G, \alpha)$ on $L^2(G, \mathcal{H})$, called the **right regular representation**.

Let $\pi_U$ denote the universal representation of $A$ on $\mathcal{H}_U$. Then the reduced crossed product of $A$ by $G$, which we will denote by $A \rtimes_{\alpha,r} G$, is the $C^\ast$-algebra

$$ \{\tilde{\pi}_U \times (1 \otimes \lambda_G))(A \rtimes_\alpha G) \subset B(L^2(G, \mathcal{H}_U)) \} . $$
Theorem 1.14 (Takai [26], Zeller-Meier [31]) If $G$ is amenable, then the left and right regular representations of the $C^*$-dynamical system $(A, G, \alpha)$ are faithful, and hence

$$A \times_\alpha G \cong A \times_{\alpha, r} G.$$ 

Proposition 1.15 Let $i_A : A \to M(A \times_\alpha G)$ and $i_G : G \to M(A \times_\alpha G)$ denote the canonical embeddings of $A$ and $G$ into $M(A \times_\alpha G)$ described in [20 lemma 7.6.2], and let $(\pi, U)$ be a covariant representation of the $C^*$-dynamical system $(A, G, \alpha)$. Then

$$(\pi \times U) \circ i_A = \pi \quad \text{and} \quad (\pi \times U) \circ i_G = U.$$ 

Proposition 1.16 Let $(\pi, U)$ and $(\pi', U')$ be covariant representations of the $C^*$-dynamical system $(A, G, \alpha)$ on $\mathcal{H}$ and $\mathcal{H}'$ respectively. Then an operator $V : \mathcal{H} \to \mathcal{H}'$ intertwines $\pi \times U$ and $\pi' \times U'$ if and only if $V$ intertwines $\pi$ and $\pi'$, and also intertwines $U$ and $U'$.

Proof Suppose $V$ intertwines $\pi$ and $\pi'$, and also intertwines $U$ and $U'$. Then

$$\pi \times U(\gamma) V = \int_G \pi(\gamma(s)) U_s ds V$$

$$= \int_G \pi(\gamma(s)) U_s V ds$$

$$= \int_G V \pi'(\gamma(s)) U'_s ds$$

$$= V \int_G \pi'(\gamma(s)) U'_s ds$$

$$= V \pi' \times U'(\gamma).$$

On the other hand if $V$ intertwines $\pi \times U$ and $\pi' \times U'$ it also intertwines their extenstions to $M(A \times_\alpha G)$ and the result follows from the previous proposition. \[\square\]


§6 Coactions and their Crossed Products

In what follows $A$ and $B$ will be $C^*$-algebras and $M(A)$ will denote the multiplier algebra of $A$. We will often consider the strict topology on $M(A)$, that is, the locally convex vector space topology induced by the semi-norms $m \mapsto \|am\|$ and $m \mapsto \|ma\|$ for $a \in A$ and $m \in M(A)$. Let

$$\tilde{M}(A \otimes B) = \{ x \in M(A \otimes B) : x(1 \otimes z), (1 \otimes z)x \in A \otimes B \quad \forall \ z \in B \}.$$ 

The significance of the algebra $\tilde{M}(A \otimes B)$ lies in the fact that it is the non-abelian analogue of the continuous functions from a locally compact space $X$ to $A$, as can be seen by noting that

$$\tilde{M}(A \otimes C_0(X)) \cong C_b(X, A).$$

An important property of $\tilde{M}(A \otimes B)$ is that the slice maps $S_u$ map $\tilde{M}(A \otimes B)$ into $A$. This follows from §10 and §11 since we can write $u = b \cdot v$ for some $b \in B$ and $v \in B^*$, so that if $m \in \tilde{M}(A \otimes B)$, then

$$S_u(m) = S_{b \cdot v}(m) = S_v(m(1 \otimes b)) \in A,$$

since $m(1 \otimes b) \in A \otimes B$. This fact will be used repeatedly without explicit mention.

We will call a homomorphism $\gamma$ from a $C^*$-algebra $A$ to a multiplier algebra $M(B)$ non-degenerate if $A$ has an approximate identity $(e_i)_{i \in I}$ such that $\gamma(e_i) \to 1$ strictly in $M(B)$. If $\gamma$ is non-degenerate, then it has a unique strictly continuous extension, which we will continue to denote $\gamma$, to $M(A)$ [13 lemma 1.1]. Note that if $\gamma$ is non-degenerate, then $\gamma(a_j) \to 1$ strictly in $M(A)$ for every approximate identity $(a_j)_{j \in J}$ of $A$.

Let $g \in C_c(G)$. Then the map

$$: s \to g(s)\lambda_G(s) \otimes \lambda_G(s) : G \to M(C^*_r(G) \otimes C^*_r(G)) \subset B(L^2(G) \otimes L^2(G))$$

- 35 -
is strictly continuous and compactly supported. So by lemma 1.7 it is integrable and
\[ \int_G g(s) \lambda_G(s) \otimes \lambda_G(s) \, ds \in M(C^*_r(G) \otimes C^*_r(G)) . \]

Let \( W_G \in UB(L^2(G \times G)) \) be defined by
\[ \{W_G \xi\}(s,t) = \xi(s, s^{-1}t) \quad \xi \in C_c(G \times G) . \]

Then identifying \( B(L^2(G) \otimes L^2(G)) \) with \( B(L^2(G \times G)) \) it is easy to show that
\[ \int_G g(s) \lambda_G(s) \otimes \lambda_G(s) \, ds = W_G(\lambda_G(g) \otimes 1)W_G^* . \quad (25) \]

Hence
\[ : \lambda_G(g) \to \int_G g(s) \lambda_G(s) \otimes \lambda_G(s) \, ds , \]
determines a well-defined \(*\)-homomorphism
\[ \delta_G : C^*_r(G) \to M(C^*_r(G) \otimes C^*_r(G)) , \]
called the comultiplication map.

**Definition A** coaction of a locally compact group \( G \) on a \( C^* \)-algebra \( A \) is an injective non-degenerate \(*\)-homomorphism \( \delta : A \to \tilde{M}(A \otimes C^*_r(G)) \) such that
\[ (\delta \otimes i) \circ \delta = (i \otimes \delta_G) \circ \delta . \]

We will call a coaction non-degenerate if for each \( \zeta \in A^* \) there exists \( \psi \in C^*_r(G)^* \) such that \( (\zeta \otimes \psi) \circ \delta \neq 0 \). We hope that this terminology will not cause confusion and note that a coaction of an amenable group is automatically non-degenerate [12], [11 prop. 6].

The non-degeneracy of \( \delta \) as a \(*\)-homomorphism ensures
\[ \delta \otimes i : A \otimes C^*_r(G) \to \tilde{M}(A \otimes C^*_r(G)) \otimes C^*_r(G) \to M(A \otimes C^*_r(G) \otimes C^*_r(G)) , \]
extends to (a unique strictly continuous \$\ast\$-homomorphism of) \(\tilde{M}(A \otimes C^*_\varepsilon(G))\).

Similarly \(i \otimes \delta_G\) extends and hence the condition \((\delta \otimes i) \circ \delta = (i \otimes \delta_G) \circ \delta\) makes sense.

To motivate the definition, suppose \(G\) is abelian, \(\alpha : \hat{G} \to \text{Aut}A\) is an action of the dual group \(\hat{G}\) on \(A\) and that the \textit{multiplication map}

\[
\alpha_G : C_b(\hat{G}) \cong M(C_o(\hat{G})) \to C_b(\hat{G} \times \hat{G}) \cong M(C_o(\hat{G}) \otimes C_o(\hat{G}))
\]

is defined by \(\{\alpha_G(f)\}(\sigma, \tau) = f(\sigma \tau)\). Spatially \(\alpha_G\) is given by

\[
\alpha_G(z) = V_G(z \otimes 1)V\textsuperscript{*}_G,
\]

where \(V_G \in UB(L^2(\hat{G} \times \hat{G}))\) is defined by

\[
\{V_G\xi\}(\sigma, \tau) = \xi(\sigma^{-1} \tau), \quad \xi \in C_o(\hat{G} \times \hat{G})
\]

Then if we define \(\tilde{\alpha} : A \to C_b(\hat{G}, A) \cong \tilde{M}(A \otimes C_o(\hat{G}))\) by \(\{\tilde{\alpha}(a)\}(\sigma) = \alpha_\sigma(a)\) we have that

\[
\{(\tilde{\alpha} \otimes i) \circ \tilde{\alpha}\}(a) = \alpha_{\sigma}(a)
\]

\[
= \alpha_\sigma(\alpha_T(a))
\]

\[
= \{\{(i \otimes \alpha_G) \circ \tilde{\alpha}\}(a)\}(\sigma, \tau) \quad \forall \sigma, \tau \in \hat{G}, \quad a \in A,
\]

i.e.,

\[
(\tilde{\alpha} \otimes i) \circ \tilde{\alpha} = (i \otimes \alpha_G) \circ \tilde{\alpha},
\]

and a moment's reflection shows that this equation captures the multiplicative nature of the action.

Now for any locally compact abelian group \(G\), let

\[
F_G : C_o(G) \to C^*(\hat{G})
\]

be the inverse Gelfand transform and let

\[
\delta = (i \otimes F_G) \circ \tilde{\alpha} : A \to \tilde{M}(A \otimes C^*(G)). \quad (28)
\]
Then it can be checked from §25 and §26 that

\[ \delta_G = (F_G \otimes F_G) \circ \alpha_G \circ F_G^{-1}. \]  

(29)

So

\[ (\delta \otimes i) \circ \delta = \left( ((i \otimes F_G) \circ \tilde{\alpha}) \otimes i \right) \circ \left( (i \otimes F_G) \circ \tilde{\alpha} \right) \]

\[ = (i \otimes F_G \otimes F_G) \circ (i \otimes \alpha_G) \circ \tilde{\alpha} \]

\[ = (i \otimes ((F_G \otimes F_G) \circ \alpha_G \circ F_G^{-1})) \circ ((i \otimes F_G) \circ \tilde{\alpha}) \]

\[ = (i \otimes \delta_G) \circ \delta. \]  

(by §27)

(29)

It can also be shown that \( \delta \) is a non-degenerate \( * \)-homomorphism and is thus a coaction of \( G \) on \( A \). Hence every action of the dual group \( \hat{G} \) on \( A \) gives a coaction of \( G \) on \( A \). Further it can be shown this correspondence is bijective.

**Definition**  If \( \alpha : G \to \text{Aut}A \) is an action of \( G \) on \( A \), then there is a natural coaction

\[ \hat{\alpha} : A \times_\alpha G \to M((A \times_\alpha G) \otimes C^*_r(G)), \]

called the dual coaction, of \( G \) on \( A \times_\alpha G \) determined by \( \hat{\alpha}(i_G(s)) = i_G(s) \otimes \lambda_G(s) \) and \( \hat{\alpha}(i_A(a)) = i_A(a) \otimes 1 \), where \( i_G \) and \( i_A \) are the natural inclusions of \( G \) and \( A \) into \( M(A \times_\alpha G) \).

Another example of a coaction, in this case of \( G \) on \( C^*_r(G) \), is the comultiplication map \( \delta_G \).

**Definition**  Suppose \( \pi : A \to B(\mathcal{H}) \) is a faithful representation of \( A \) on the Hilbert space \( \mathcal{H} \) and \( \delta : A \to \tilde{M}(A \otimes C^*_r(G)) \) is a coaction of \( G \) on \( A \). Then the crossed product \( A \times_\delta G \) of \( A \) by \( \delta \) is the \( C^* \)-subalgebra of \( B(\mathcal{H} \otimes L^2(G)) \) generated by the elements \( (\pi \otimes i)(\delta(a))(1 \otimes M_G(f)) \) for \( a \in A \) and \( f \in C_o(G) \).

Coactions and their crossed products were introduced to \( C^* \)-algebras (from von-Neumann algebras) by Landstad [12]. It can be shown that the crossed product
$A \times_\delta G$ is independent of the choice of $\pi$. In fact it can be given a non-spatial definition [13 def. 2.4].

**Notation** When there is no danger of confusion $(\pi \otimes i)(\delta(a))(1 \otimes M_G(f))$ will be shortened to $\delta(a)(1 \otimes f)$.

To motivate the definition, suppose $G$ is abelian, $\alpha : \hat{G} \to \text{Aut}A$ is an action of the dual group $\hat{G}$ on $A$, $\delta$ is the corresponding coaction of $G$ on $A$ and $\mathcal{F} : L^2(G) \to L^2(\hat{G})$ is the Fourier transform. Then the isomorphism

$$Ad(1 \otimes \mathcal{F}) : B(\mathcal{H} \otimes L^2(G)) \to B(\mathcal{H} \otimes L^2(\hat{G}))$$

maps the generators $(\pi \otimes i)(\delta(a))(1 \otimes M_G(f))$ of the crossed product $A \times_\delta G$ to the elements $(\pi \otimes M_G)(\hat{\alpha}(a))(1 \otimes \lambda_{\hat{G}}(\hat{f}))$. Now it can be shown that elements of the form $(\pi \otimes M)(\hat{\alpha}(a))(1 \otimes \lambda_G(g))$, $a \in A$, $g \in C^*(G)$ generate $\pi \times \lambda_{\hat{G}}(A \times_\alpha \hat{G}) \subset B(\mathcal{H} \otimes L^2(\hat{G}))$ and hence that

$$A \times_\delta G \cong A \times_\alpha \hat{G} \ .$$

(30)

So in the abelian case crossed products by coactions correspond to crossed products by actions of the dual group.

**Definition** If $\delta$ is a coaction of $G$ on $A$, then there is a natural action, the dual action, of $G$ on $A \times_\delta G$ defined by

$$\hat{\delta}_s = Ad(1 \otimes \rho_G(s)) \quad s \in G \ .$$

(31)

Let $G$ be any locally compact group. Then we define a unitary element $\omega_G$ of $C_r^*(G, M(C_r^*(G)))$, the bounded strictly continuous maps from $G$ to $M(C_r^*(G))$, by $\omega_G(s) = \lambda_G(s)$ for all $s \in G$.

**Definition** Let $\delta : A \to \hat{M}(A \otimes C_r^*(G))$ be a coaction of $G$ on $A$. Then a covariant representation for the system $(A, G, \delta)$ on $\mathcal{H}$ is a pair $(\pi, \mu)$ of representations of $A$, respectively, $C_0(G)$ on the Hilbert space $\mathcal{H}$, such that

$$(\pi \otimes i)(\delta(a)) = ((\mu \otimes i)(\omega_G))(\pi(a) \otimes 1)((\mu \otimes i)(\omega_G^*)) \quad \forall a \in A .$$
To see that the above covariance condition [21] is the same as that presented in [13 def. 3.5] we need to prove the following lemma.

**Lemma 1.17** Suppose \( u \in B_r(G) \) and \( \varpi_G \) is as above. Then \( S_u(\varpi_G) = u \) and hence if \( \mu : C_o(G) \to B(\mathcal{H}) \) is a representation of \( C_o(G) \), we have that

\[
S_u((\mu \otimes i)(\varpi_G)) = \mu(u) .
\] (32)

**Proof** Let \( g \in C_c(G) \), \( z \in C^*_r(G) \) and recall that \( \{g \otimes z\}(s) = g(s)z \). Then

\[
\{S_u(g \otimes z)\}(s) = g(s)u(z)
\]

\[
= u(g(s)z)
\]

\[
= u(\{g \otimes z\}(s)) .
\] (33)

Now suppose \( \gamma \in C_o(G, C^*_r(G)) \cong C_o(G) \otimes C^*_r(G) \). Then

\[
\gamma = \lim_{i \to \infty} \sum_{j=1}^{n_i} g_{ij} \otimes z_{ij}
\]

for some \( z_{ij} \in C^*_r(G) \), \( g_{ij} \in C_c(G) \) and in particular

\[
\gamma(s) = \lim_{i \to \infty} \sum_{j=1}^{n_i} g_{ij}(s)z_{ij} \quad \forall \ s \in G .
\]

This, \( \|33 \) and the norm continuity of \( S_u \) and \( u \) give

\[
\{S_u(\gamma)\}(s) = u(\gamma(s)) .
\] (34)

Now suppose \( \gamma \in C_b^*(G, C^*_r(G)) \cong M(C_o(G) \otimes C^*_r(G)) \) [1 cor. 3.4]. By \( \|11 \) we can write \( u = b \bullet v \) for \( b \in C^*_r(G) \) and \( v \in B_r(G) \). Let \( f \in C_o(G) \), \( s \in G \) and let \( \cdot \) denote pointwise multiplication. Then

\[
\{S_u(\gamma) \cdot f\}(s) = \{S_{b \cdot v}(\gamma) \cdot f\}(s)
\]

\[
= \{S_v(\gamma \cdot (f \otimes b))\}(s)
\]

\[
= v(\{\gamma \cdot (f \otimes b)\}(s))
\]

\[
= v(\gamma(s)b)f(s)
\]

\[
= \{b \bullet v\}(\gamma(s))f(s)
\]

\[
= \{(u \circ \gamma) \cdot f\}(s) .
\]
So $S_u(\gamma) = u \circ \gamma$ and in particular

$$(S_u(\varpi_G))(s) = u(\varpi_G(s)) = u(\lambda_G(s)) = u(s). \quad (\text{by } \S \text{I})$$

Hence $S_u((\mu \otimes i)(\varpi_G)) = \mu(S_u(\varpi_G)) = \mu(u)$ as required.

By [13 thm. 3.1] the corepresentation $W \in B(\mathcal{H})\overline{\otimes}vN(G)$ corresponding to the representation $\mu$ on $\mathcal{H}$ is determined by the relation

$$S_u(W) = \mu(u) \quad \forall u \in A(G).$$

By lemma 1.17 $W = (\mu \otimes i)(\varpi_G)$, so it is clear that the definition given here is equivalent to that of [13].

**Theorem 1.18** (Landstad-Phillips-Raeburn-Sutherland [13]) The representations of $A \times_\delta G$ on $\mathcal{H}$ correspond bijectively to the covariant representations of $(A, G, \delta)$ on $\mathcal{H}$.

If $(\pi, \mu)$ is a covariant representation of $(A, G, \delta)$, then we will denote the corresponding representation of $A \times_\delta G$ by $\pi \times \mu$. By [13 thm 3.7] we have the following proposition.

**Proposition 1.19** Let $(\pi, \mu)$ be a covariant representation of $A \times_\delta G$. Then

$$(\pi \times \mu) \circ \delta = \pi \quad \text{and} \quad (\pi \times \mu) \circ (1 \otimes M_G) = \mu. \quad \Box$$

Note that this proposition implies that

$$(\pi \times \mu)(\delta(a)(1 \otimes f)) = \pi(a)\mu(f). \quad (35)$$

We shall show that in the abelian case this notion of covariance for coactions corresponds to the usual notion of covariant representations of crossed products by actions. Let $\delta$ be a coaction of an abelian group $G$ on $A$. Let

$$\tilde{\alpha} = (1 \otimes F^{-1}_G) \circ \delta : A \to M(A \otimes C^*(\widehat{G})) \cong C^*_\delta(\widehat{G}, A).$$
Then the map \( \alpha : A \to \text{Aut}(A) \) given by \( \alpha(a) = \{\tilde{\alpha}(a)\}(s) \) is an action of \( \hat{G} \) on \( A \) and \( A \times_\delta G \cong A \times_\alpha \hat{G} \). Thus the representation theory of \( A \times_\delta G \) is already understood in terms of covariant pairs \((\pi, U)\) of the covariant system \((A, \hat{G}, \alpha)\), that is, in terms of a representation \( \pi \) of \( A \) on \( \mathcal{H} \) and a unitary representation \( U \) of \( \hat{G} \) on \( \mathcal{H} \) such that

\[
\pi(\alpha(a)) = U_\sigma \pi(a) U_\sigma^* .
\] (36)

Now \( U \) determines a representation \( \mu \) of \( C_0(G) \) on \( \mathcal{H} \) by \( \mu = U \circ F_G \), where \( U \) is the integrated form of \( U \). Now if \( \tilde{\alpha} \) is as above and \( \sigma \in \hat{G} \), then

\[
\{(\pi \otimes i)(\tilde{\alpha}(a))\}(\sigma) = \pi(\alpha(a))
\]

\[
= U_\sigma \pi(a) U_\sigma^*
\]

(by \( \text{32} \))

\[
= U(\lambda_{\hat{G}}(\sigma)) \pi(a) U(\lambda_{\hat{G}}^*(\sigma))
\]

\[
= U(\omega_{\hat{G}}(\sigma)) \pi(a) U(\omega_{\hat{G}}^*(\sigma))
\]

\[
= \{(U \otimes i)(\omega_{\hat{G}})\}(\sigma) \pi(a) \{(U \otimes i)(\omega_{\hat{G}})\}(\sigma)
\]

\[
= \{((U \otimes i)(\omega_{\hat{G}})) \cdot (\pi(a) \otimes 1) \cdot ((U \otimes i)(\omega_{\hat{G}}^*))\}(\sigma),
\]

where \( \pi(a) \otimes 1 \) is the constant function with value \( \pi(a) \) and \( \cdot \) denotes pointwise multiplication. So

\[
(\pi \otimes i)(\tilde{\alpha}(a)) = ((U \otimes i)(\omega_{\hat{G}})) \cdot (\pi(a) \otimes 1) \cdot ((U \otimes i)(\omega_{\hat{G}}^*)) .
\]

Now if \( F_G \) is as above and

\[
F_G \otimes F_G^{-1} : M(C_0(G) \otimes C^*(G)) \cong C_0^*(G, M(C^*(G))) \to
\]

\[
M(C^*(\hat{G}) \otimes C_0(\hat{G})) \cong C_0^*(\hat{G}, M(C^*(\hat{G}))) .
\]

Then

\[
F_G \otimes F_G^{-1}(\omega_{\hat{G}}) = \omega_{\hat{G}} ,
\] (37)
where \( \varpi_G \) is being considered an element of \( M(C_0(G) \otimes C^*(G)) \), and \( \varpi_{\widehat{G}} \) as an element of \( M(C^*(\widehat{G}) \otimes C_0(\widehat{G})) \). Hence we have that

\[
(\pi \otimes i)(\delta(a)) = (\pi \otimes F_{\widehat{G}})(\tilde{a}(a)) \quad \text{(by \#28)}
\]

\[
= (i \otimes F_{\widehat{G}})((U \otimes i)(\varpi_{\widehat{G}})) \cdot (\pi(a) \otimes 1) \cdot ((U \otimes i)(\varpi_{\widehat{G}}^*))
\]

\[
= (((U \circ F_G) \otimes i)(\varpi_G))(\pi(a) \otimes 1)((U \circ F_G) \otimes i)(\varpi_G^*)) \quad \text{(by \#37)}
\]

\[
= ((\mu \otimes i)(\varpi_G))(\pi(a) \otimes 1)((\mu \otimes i)(\varpi_G^*)).
\]

So each covariant representation \((\pi, U)\) of the system \((A, \widehat{G}, \alpha)\) corresponds to a covariant representation \((\pi, \mu)\) of \((A, G, \delta)\). Further this correspondence is bijective.

**Example 1.20** If \( i \) is the identity map on \( C^*_r(G) \), then \((i, M_G)\) is a covariant representation of \((C^*_r(G), G, \delta_G)\) on \( L^2(G) \).

**Proof** Let \( u \in A(G) \). Then

\[
(i) \quad S_u(\delta_G(\lambda_G(s))((M_G \otimes i)(\varpi_G))) = S_u((\lambda_G(s) \otimes \lambda_G(s))((M_G \otimes i)(\varpi_G)))
\]

\[
= \lambda_G(s)S_u(\lambda_G(s))((M_G \otimes i)(\varpi_G)) \quad \text{(by \#10)}
\]

\[
= \lambda_G(s)\lambda_G(u \cdot \lambda_G(s)) \quad \text{(by \#32)}
\]

\[
= \lambda_G(s)M_G(\tau_s^{-1}(u)),
\]

where \( \tau \) is the left translation action of \( G \) on \( C_0(G) \) defined by, \( (\tau_s(f))(t) = f(s^{-1}t) \). To see how one obtains the last equality note that

\[
\{u \cdot \lambda_G(s)\}(\lambda_G(g)) = u(\lambda_G(s)\lambda_G(g))
\]

\[
= \int_G g(t)u(st) \, dt
\]

\[
= \int_G g(t)\{\tau_s^{-1}(u)}(t) \, dt
\]

\[
= \{\tau_s^{-1}(u)}(\lambda_G(g)) \quad \forall \ g \in C_c(G).
\]

- 43 -
(ii) \( S_u(((M_G \otimes i)(w_G))(\lambda_G(s) \otimes 1)) = S_u((M_G \otimes i)(w_G))\lambda_G(s) \) (by \( \S 10 \))
\[ = M_G(u)\lambda_G(s) \] (by \( \S 32 \))

Now by (i), (ii) and since \((M_G, \lambda_G)\) is a covariant representation of the system \((C_G(G), G, \tau)\) we have that
\[ S_u(\delta_G(\lambda_G(s))((M_G \otimes i)(w_G))) = \lambda_G(s)M_G(\tau^{-1}(u)) \]
\[ = M_G(u)\lambda_G(s) \]
\[ = S_u((M_G \otimes i)(w_G))(\lambda_G(s) \otimes 1) \]

and since \(u\) was arbitrary in \(A(G)\) we have that
\[ \delta_G(\lambda_G(s))((M_G \otimes i)(w_G)) = ((M_G \otimes i)(w_G))(\lambda_G(s) \otimes 1) . \]

Integrating we obtain the covariance condition, as required.

\[ \square \]

**Example 1.21** Suppose \(\delta : A \rightarrow \hat{M}(A \otimes C^*_r(G))\) is a coaction of \(G\) on \(A\) and \(\pi : A \rightarrow B(\mathcal{H})\) is representation of \(A\) on \(\mathcal{H}\). Then \(((\pi \otimes i) \circ \delta, 1 \otimes M_G)\) is a covariant representation of \((A, G, \delta)\).

**Proof** Firstly note that
\[ (1 \otimes S_f)((1 \otimes M_G \otimes i)(w_G)) = (1 \otimes M_G)(f) \] (by \( \S 32 \))
\[ = 1 \otimes (M_G(f)) \]
\[ = (1 \otimes S_f)(1 \otimes ((M_G \otimes i)(w_G)) . \) (by \( \S 32 \))

So \((1 \otimes M_G \otimes i)(w_G) = 1 \otimes ((M_G \otimes i)(w_G))\) and we have
\[ (((\pi \otimes i) \circ \delta) \otimes i)(\delta(a)) \]
\[ = (\pi \otimes i \otimes i)((\delta \otimes i)(\delta(a))) \]
\[(\pi \otimes i \otimes i)((i \otimes \delta_G)(\delta(a)))\]
\[= (1 \otimes ((M_G \otimes i)(\varpi_G))((\pi \otimes i)(\delta(a)) \otimes 1)(1 \otimes ((M_G \otimes i)(\varpi_G^*)))\]
\[= ((1 \otimes M_G \otimes i)(\varpi_G))((\pi \otimes i)(\delta(a)) \otimes 1)((1 \otimes M_G \otimes i)(\varpi_G^*))\]

as required. \[\square\]

Suppose \(\delta\) is a coaction of an abelian group \(G\) on \(A\), \(\alpha\) is the corresponding action of \(\widehat{G}\), \(\varrho\) be a representation of \(A\) on \(P\) and that \(1\) is the trivial subgroup of \(G\). Then \((\varrho, 1)\) is a covariant representation of \((A, \alpha, 1)\) and \(\text{ind}_{A \times \alpha 1}^{\text{A} \times \alpha 1} \varrho \times 1\) is unitarily equivalent to the representation \(\tilde{\varrho} \times (1 \otimes \lambda_G)\) of \(A \times \delta G\) on \(B(P \otimes L^2(\widehat{G}))\).

By Fourier transforming we can remove all reference to the dual group as follows.

\[
(\text{Ad}(1 \otimes \mathcal{F}^*)) \circ \text{ind}_{A \times \alpha 1}^{\text{A} \times \alpha 1} \varrho \times 1 = (\text{Ad}(1 \otimes \mathcal{F}^*)) \circ (\mathcal{F} \times (1 \otimes \lambda_G))
\]
\[
= (\text{Ad}(1 \otimes \mathcal{F}^*)) \circ ((\pi \otimes M_G) \circ \tilde{\alpha}) \times (1 \otimes \lambda_G)
\]
\[
= ((\pi \otimes i) \circ \delta) \times (1 \otimes M_G), \quad \text{(by \ref{28})}
\]

where \(\mathcal{F}\) denotes the Fourier transform. Now lemma 1.21 shows that \(((\pi \otimes i) \circ \delta) \times (1 \otimes M_G)\) makes sense for arbitrary locally compact groups. Hence for \(\delta\) a coaction of an arbitrary locally compact group we can define the representation of \(A \times \delta G\) induced from the representation \(\varrho \times 1\) of \(A \times \delta_1(G/G)\) to be \(((\pi \otimes i) \circ \delta) \times (1 \otimes M_G)\).

Later we will introduce a more general induction process and show (proposition 5.12) that it includes the above as a special case. This more traditional approach to induction was investigated but eventually abandoned in favour of Rieffel's when it proved difficult to extend it to non-trivial subgroups. However, Gootman and Lazar have used this idea to good effect in [5].

At this point we wish to prove a number of results that will be needed later in the discourse.

**Notation** Suppose \(u \in A(G) \subset C^*_r(G)^*\) and \(S_u : \tilde{M}(A \otimes C^*_r(G)) \to A\) is a slice map. Then we will abbreviate \(S_u(\delta(a))\) to \(\delta_u(a)\).
Lemma 1.22 Suppose \( v \in A(G) \) and that

\[
\delta(a) = \lim_{i \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \quad \text{in} \quad M(A \otimes C^*_r(G)),
\]

for some \( a_{ij} \in A \) and \( g_{ij} \in C_c(G) \). This is always possible since \( A \odot \lambda_G(C_c(G)) \) is strictly dense in \( M(A \otimes C^*_r(G)) \). Then if \( \cdot \) denotes pointwise multiplication

\[
\delta(\delta_v(a)) = \lim_{i \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(v \cdot g_{ij}) \quad \text{in} \quad M(A \otimes C^*_r(G)).
\]

Proof

\[
\delta(\delta_v(a)) = \delta\left( S_v\left( \lim_{i \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \right) \right)
\]

\[
= \delta\left( \lim_{i \to \infty} \sum_{j=1}^{n_i} a_{ij} v(\lambda_G(g_{ij})) \right)
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(a_{ij}) v(\lambda_G(g_{ij}))
\]

\[
= \lim_{i \to \infty} (i \otimes S_v)\left( \sum_{j=1}^{n_i} \delta(a_{ij}) \otimes \lambda_G(g_{ij}) \right)
\]

\[
= (i \otimes S_v)\left( (\delta \otimes i)\left( \lim_{i \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \right) \right)
\]

\[
= (i \otimes S_v)((\delta \otimes i)(\delta(a)))
\]

\[
= (i \otimes S_v)((i \otimes \delta_G)(\delta(a)))
\]

\[
= (i \otimes S_v)((i \otimes \delta_G)(\delta(a)))
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes S_v(\delta_G(\lambda_G(g_{ij})))
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes S_v\left( \int_G g_{ij}(s)\lambda_G(s) \otimes \lambda_G(s) \, ds \right)
\]
Lemma 1.23 Suppose $u, v \in A_c(G)$ and $a \in A$. Then

$$\delta_u(\delta_v(a)) = \delta_{u \cdot v}(a).$$

**Proof** Since $A \otimes \lambda_G(C_c(G))$ is strictly dense in $M(A \otimes C^*_r(G))$,

$$\delta(a) = \text{strict limit } \lim_{n \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \quad \text{in } M(A \otimes C^*_r(G)),
$$

for some $a_{ij} \in A$ $g_{ij} \in C_c(G)$. By lemma 1.22

$$\delta_u(\delta_v(a)) = S_u\left(\text{strict limit } \lim_{n \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(v \cdot g_{ij})\right)$$

$$= \text{strict limit } \lim_{n \to \infty} \sum_{j=1}^{n_i} a_{ij} u(\lambda_G(v \cdot g_{ij}))$$

$$= \text{strict limit } \lim_{n \to \infty} \sum_{j=1}^{n_i} a_{ij} \left(\int_G u(s)v(s)g_{ij}(s) \, ds\right)$$

$$= \text{strict limit } \lim_{n \to \infty} \sum_{j=1}^{n_i} a_{ij}((u \cdot v)(\lambda_G(g_{ij})))$$

$$= \delta_{u \cdot v}(a),$$
using lemma 1.22 again.

Lemma 1.24 Suppose $a, b \in A$, $u, v, w \in A_c(G)$ and that $w$ is one on $(\text{suppu}) \cdot (\text{suppv})$. Then

$$\delta_w(\delta_u(a)\delta_v(a)) = \delta_u(a)\delta_v(a).$$

Proof Firstly we show that if $f, g \in C_c(G)$, then

$$w(\lambda_G((u \cdot f) \ast (v \cdot g))) = u(\lambda_G(f))v(\lambda_G(g))$$

(38)

$$w(\lambda_G((u \cdot f) \ast (v \cdot g))) = \int_G w(s)((u \cdot f) \ast (v \cdot g))(s) \, ds \quad \text{(by \textsection 3)}$$

$$= \int_G w(s) \int_G u(r)f(r)v(r^{-1}s)g(r^{-1}s) \, dr \, ds$$

$$= \int_G \int_G u(r)f(r)v(r^{-1}s)g(r^{-1}s) \, dr \, ds$$

(since $w$ is identically one on $(\text{suppu}) \cdot (\text{suppv})$)

$$= \int_G u(r)f(r) \, dr \int_G v(s)g(s) \, ds$$

$$= u(\lambda_G(f))v(\lambda_G(g)). \quad \text{(by \textsection 3)}$$

Since $A \otimes \lambda_G(C_c(G))$ is strictly dense in $M(A \otimes C^*_r(G))$, we can write

$$\delta(a) = \lim_{i \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(f_{ij})$$

$$\delta(b) = \lim_{k \to \infty} \sum_{l=1}^{m_k} b_{kl} \otimes \lambda_G(g_{kl}),$$

for some $a_{ij}, b_{kl} \in A, f_{ij}, g_{kl} \in C_c(G)$. By \textsection 11 we can write $w = c \cdot \psi \cdot d, u = \tau \cdot x$.
and \( v = y \circ \sigma \) for some \( c, d, x, y \in C_r(G) \) and \( \psi, \tau, \sigma \in B_r(G) \). Let \( e \in A \). Then

\[
e \delta_w(\delta_u(a) \delta_v(b)) e =
\]

\[
S_\psi((e \otimes c)\delta(\delta_u(a) \delta_v(b))(e \otimes d))
\]

\[
S_\psi\left((e \otimes c)\left(\lim_{j=1}^{n_i} a_{ij} \otimes \lambda_G(f_{ij})\right)\left(\lim_{l=1}^{m_k} b_{kl} \otimes \lambda_G(g_{kl})\right)(e \otimes d)\right)
\]

\[
S_\psi\left(\left(\lim_{i,k}^{n_i}(e a_{ij} b_{kl} e) \otimes (c \lambda_G(f_{ij}))\right)\left(\lim_{l=1}^{m_k} (b_{kl} e) \otimes (\lambda_G(g_{kl})d)\right)\right)
\]

\[
S_\psi\left(\lim_{i,k}^{n_i}(e a_{ij} b_{kl} e (c \lambda_G((u \cdot f_{ij}) \bullet (v \cdot g_{kl}))))\right)
\]

\[
\text{by } \ref{38}
\]

\[
\left(\lim_{j=1}^{n_i} e a_{ij} \lambda_G(f_{ij})\right)\left(\lim_{l=1}^{m_k} b_{kl} e \lambda_G(g_{kl})\right)
\]

\[
\left(\lim_{j=1}^{n_i} e a_{ij} \tau(x \lambda_G(f_{ij}))\right)\left(\lim_{l=1}^{m_k} b_{kl} e \lambda_G(g_{kl} y)\right)
\]

\[
S_r\left(\lim_{j=1}^{n_i} (e a_{ij} \otimes (x \lambda_G(f_{ij})))\right)S_\sigma\left(\lim_{l=1}^{m_k} (b_{kl} e \otimes (\lambda_G(g_{kl} y)))\right)
\]

\[
S_r\left((e \otimes x)\lim_{j=1}^{n_i} a_{ij} \otimes \lambda_G(f_{ij})\right)S_\sigma\left(\lim_{l=1}^{m_k} b_{kl} \otimes \lambda_G(g_{kl}(e \otimes y)))\right)
\]

\[
S_r((e \otimes x)\delta(a))S_\sigma(\delta(b)(e \otimes y))
\]
\begin{align*}
&= e \delta_{r \ast z}(a) \delta_y (b) e \\
&= e \delta_u(a) \delta_v(b) e.
\end{align*}

Now letting \( e \) run over an approximate identity of \( A \) gives the lemma. \( \square \)

**Lemma 1.25** Let \( u \in A(G) \subset C^*_r(G)^* \). Then

\[ (S_u(z))^* = S_{u^*}(z^*) \quad \forall \ z \in \hat{M}(A \otimes C^*_r(G)) . \]

**Proof** Firstly note that if \( g \in C_c(G) \). Then

\[
\overline{u(\lambda_G(g))} = \int_G \overline{u(s)} g(s) \, ds
\]

\[
= \int_G \frac{1}{\Delta s} u(s^{-1}) g(s^{-1}) \, ds
\]

\[
= \int_G \overline{u}(s) g^*(s) \, ds
\]

\[
= \overline{\tilde{u}(\lambda_G(g^*))}
\]

Now suppose \( z = \sum_{i=1}^n a_i \otimes g_i \in A \otimes \lambda_G(C_c(G)) \). Then

\[
(S_u(z))^* = \left( \sum_{i=1}^n a_i \otimes \lambda_G(g_i) \right)^* 
\]

\[
= \left( \sum_{i=1}^n a_i u(\lambda_G(g_i)) \right)^* 
\]

\[
= \sum_{i=1}^n a_i^* u(\lambda_G(g_i)) 
\]

\[
= \sum_{i=1}^n a_i^* \overline{u}(\lambda_G(g_i)) 
\]

\[
= \sum_{i=1}^n a_i^* \lambda_G(g_i) 
\]

\[
= S_{u^*} \left( \sum_{i=1}^n a_i^* \otimes (\lambda_G(g_i))^* \right)
\]
\[ = S_{\tilde{a}}\left(\left(\sum_{i=1}^{n} a_i \otimes \lambda_{G}(g_i)\right)^*\right) \]

\[ = S_{\tilde{a}}(z^*) \]

and the lemma follows since both \( S_{\tilde{a}} \), \( S_{\hat{a}} \) and the \(*\)-operation are strictly continuous. \( \square \)
Chapter 2. Coactions of Quotients.

Let $\delta : A \to \tilde{M}(A \otimes C^*_r(G))$ be a coaction of $G$ on $A$. Then for any closed normal amenable subgroup $H$ of $G$ we define a coaction $\delta| : A \to \tilde{M}(A \otimes C^*_r(G/H))$ of $G/H$ on $A$. We then form the crossed product $A \times_{\delta|} (G/H)$ and show it has a faithful representation on $B(\mathcal{H} \otimes L^2(G))$, where $A$ has been faithfully represented on the Hilbert space $\mathcal{H}$.

Definition Define $\varphi : C_c(G) \to C_c(G/H)$ by

$$\{\varphi(f)\}(sH) = \int_H f(sh) \, dh \quad s \in G, \, f \in C_c(G).$$

It should be noted that $\varphi$ is surjective [7 thm. 15.21].

For any locally compact group $G$, $\mu_G$ will denote Haar measure on $G$. If $H$ is a normal subgroup of $G$ we will choose $\mu_G$ and $\mu_{G/H}$ [14 §33.A] such that

$$\int_G f(s) \, d\mu_G(s) = \int_{G/H} \int_H f(sh) \, d\mu_H(h) \, d\mu_{G/H}(sH) \quad \forall f \in C_c(G).$$

Before defining $\delta|$ we need to prove the following two lemmas.

Lemma 2.1 Suppose $H$ is a closed normal amenable subgroup of $G$, $1_H$ is the integrated form of the trivial representation of $H$ on $C$ and $\Lambda : C^*(G) \to M(C^*_r(G/H))$ is the integrated form of the unitary representation $\varphi : G \to M(C^*_r(G/H)) : s \to \lambda_{G/H}(sH)$. Then $\Lambda$ is unitarily equivalent to $\text{ind}_{C^*_r(H)}^{C^*_r(G)} 1_H$.

Proof The proof uses material presented in chapter 1 §4. Note that since $H$ is normal, $\Delta_G(h) = \Delta_H(h)$ for all $h \in H$. Now $\text{ind}_{C^*_r(H)}^{C^*_r(G)} 1_H : C^*(G) \to B(Z)$ is a representation of $C^*(G)$ on $Z$, where $Z$ is $C_c(G) \otimes C$ factored and completed with

- 52 -
respect to the following pre-inner product:

\[
\langle x \otimes \alpha, y \otimes \beta \rangle_{C_c(G) \otimes C} = \langle \{1_H(\langle y, x \rangle_{C_c(G)})\}(\alpha), \beta \rangle_C \\
= \{1_H(\langle y, x \rangle_{C_c(G)})\}(\alpha)\beta \\
= \int_H \langle y, x \rangle_{C_c(G)}(h) \, dh \, \alpha\beta.
\]

Define \( U : C_c(G) \otimes C \to L^2(G/H) \) by \( x \otimes 1 \to \varphi(x) \).

(i) Since \( \varphi(C_c(G)) = C_c(G/H) \), \( U \) maps onto a dense subspace of \( L^2(G/H) \).

(ii) Let \( x \) and \( y \in C_c(G) \). Then

\[
\langle x \otimes 1, y \otimes 1 \rangle_{C_c(G) \otimes C} = \int_H \langle y, x \rangle_{C_c(G)}(h) \, dh \\
= \int_H \int_G \overline{y(s)x(sh)} \, ds \, dh \\
= \int_{G/H} \left( \int_H \overline{y(sr)} \, dr \right) \left( \int_H x(sh) \, dh \right) \, ds \, H \\
= \langle \varphi(x), \varphi(y) \rangle_{L^2(G/H)} \\
= \langle U(x \otimes 1), U(y \otimes 1) \rangle_{L^2(G/H)}. \\
\]

So \( U \) preserves the pre-inner products.

(iii) Let \( x \) and \( f \in C_c(G) \). Then

\[
\{U\left(\{\text{ind}_{C^*(G)}\{H\} 1_H(f)\}(x \otimes 1)\}\right)(tH) = \{U((f \ast x) \otimes 1)\}(tH) \\
= \int_H \int_G f(s)x(s^{-1}th) \, ds \, dh \\
= \int_G f(s) \int_H x(s^{-1}th) \, dh \, ds \\
= \left\{ \left\{ \int_G f(s)\lambda_{G/H}(sH) \, ds \right\}(\varphi(x)) \right\}(tH) \\
= \{A(f)(\varphi(x))\}(tH) \\
= \{A(f)(U(x \otimes 1))\}(tH). \\
\]
So $U$ intertwines the $C_c(G)$ actions.

This implies $U$ extends to a unitary operator from $X$ onto $L^2(G/H)$ which intertwines the $C^*(G)$ actions. So the two representations are unitarily equivalent as claimed.

Lemma 2.2 Suppose $H$ is a closed normal amenable subgroup of $G$ and $\Lambda : C^*(G) \to C^*_r(G/H)$ is as in lemma 2.1. Then $\ker \Lambda \supset \ker \lambda_G$. Hence the map $\Phi : C^*_r(G) \to C^*_r(G/H)$ determined by : $\lambda_G(s) \to \lambda_{G/H}(sH)$ is well-defined and $\Lambda = \Phi \circ \lambda_G$. Also $\Phi$ is a non-degenerate $*$-homomorphism such that if $f \in C_c(G)$, then $\Phi(\lambda_G(f)) = \lambda_{G/H}(\varphi(f))$.

Proof First we show $\ker \lambda_G \subset \ker \Lambda$. Since $H$ is amenable $\ker \lambda_H$ is trivial [20 thm. 7.7.5] and hence $\ker \lambda_H \subset \ker 1_H$. Now induction preserves weak containment [6 prop 9]. So

$$\ker (\text{ind}_{C^*_r(H)^1} \lambda_H) \subset \ker (\text{ind}_{C^*_r(H)^1} 1_H).$$

Since $\text{ind}_{C^*_r(H)^1} \lambda_H$ is unitarily equivalent to $\lambda_G$ [6 prop. 8] and $\text{ind}_{C^*_r(H)^1} 1_H$ is unitarily equivalent to $\Lambda$ by lemma 2.1, we have that $\ker \lambda_G \subset \ker \Lambda$ and hence that $\Phi$ is well-defined.

To see the last statement of the lemma, note that if $f \in C_c(G)$, then

$$\Phi(\lambda_G(f)) = \int_G f(s) \lambda_{G/H}(sH) \, ds$$

$$= \int_{G/H} \int_H f(sh) \, dh \lambda_{G/H}(sH) \, dsH$$

$$= \lambda_{G/H}(\varphi(f)).$$

To see that $\Phi$ maps into $C^*_r(G/H)$, note that elements of the form $\lambda_G(f)$ are dense in $C^*_r(G)$ and $\Phi$ is continuous, so by the last statement of the lemma

$$\Phi(C^*_r(G)) \subset \text{closure} \left( \lambda_G(C_c(G/H)) \right) = C^*_r(G/H).$$
Now we show that $\Phi$ is a non-degenerate $\ast$-homomorphism. Let $(e_j)_{j \in J}$ be an approximate identity for $C^*_r(G)$ contained in $C_c(G)$, and let $g \in C_c(G/H)$. Since $\varphi$ maps onto $C_c(G/H)$, $g = \varphi(f)$ for some $f \in C_c(G)$, and

$$
\Phi(\lambda_G(e_j))\lambda_{G/H}(g) = \Phi(\lambda_G(e_j))\lambda_{G/H}(\varphi(f))
$$

$$
= \Phi(\lambda_G(e_j)\Phi(\lambda_G(f))
$$

$$
= \Phi(\lambda_G(e_j * f))
$$

$$
\to \Phi(\lambda_G(f))
$$

$$
= \lambda_{G/H}(g).
$$

Similarly one shows $\lambda_{G/H}(g)\Phi(\lambda_G(e_j)) \to \lambda_{G/H}(g)$. So $\Phi(\lambda_G(e_j)) \to 1$ strictly in $M(C^*_r(G/H))$ and $\Phi$ is non-degenerate as claimed.

\[\square\]

**Lemma 2.3** Suppose $\delta : A \rightarrow \tilde{M}(A \otimes C^*_r(G))$ is a coaction of $G$ on $A$ and $H$ is a closed normal amenable subgroup of $G$. Define

$$
\delta : A \rightarrow \tilde{M}(A \otimes C^*_r(G/H)) \text{ by } \delta(a) = (i \otimes \Phi)(\delta(a)).
$$

Then $\delta$ is a coaction of $G/H$ on $A$.

**Proof** (i) Since $\Phi$ is non-degenerate, so is $i \otimes \Phi$, and hence $i \otimes \Phi$ extends to $\tilde{M}(A \otimes C^*_r(G))$. Since both $\delta$ and $\Phi$ are non-degenerate $\ast$-homomorphisms, so is $\delta$.

(ii) To see that $\delta$ maps into $\tilde{M}(A \otimes C^*_r(G/H))$, note that by lemma 2.2, if $f \in C_c(G)$, then

$$
\delta(a)(1 \otimes \lambda_{G/H}(\varphi(f))) = ((i \otimes \Phi)(\delta(a)))(i \otimes \Phi)(1 \otimes \lambda_G(f))
$$

$$
= (i \otimes \Phi)(\delta(a)(1 \otimes \lambda_G(f))) \in A \otimes C^*_r(G/H),
$$

since $\delta(a)(1 \otimes \lambda_G(f)) \in A \otimes C^*_r(G)$. Now since elements of the form $\lambda_{G/H}(\varphi(f))$ are dense in $C^*_r(G/H)$, $\delta(a)(1 \otimes z) \in A \otimes C^*_r(G/H)$ for all $z \in C^*_r(G/H)$. 

- 55 -
(iii) Let $f \in C_c(G)$. Then
\[
\delta_G(\lambda_G(f)) = (i \otimes \Phi)(\delta_G(\lambda_G(f)))
\]
\[
= (i \otimes \Phi) \left( \int_G f(s)\lambda_G(s) \otimes \lambda_G(s) \, ds \right)
\]
\[
= \int_G f(s)\lambda_G(s) \otimes \lambda_{G/H}(sH) \, ds \quad \text{(by lemma 1.7)}
\]
\[
= W(\lambda_G(f) \otimes 1)W^*,
\]
where $W \in UB(L^2(G \times G/H))$ is defined by
\[
\{W(\xi)\}(s, tH) = \xi(s, s^{-1}tH).
\]
The right hand side of the above is clearly injective, hence so is $\delta_G|$. Now since
\[
(i \otimes \delta_G)| \circ \delta = (i \otimes i \otimes \Phi) \circ (i \otimes \delta_G) \circ \delta
\]
\[
= (i \otimes i \otimes \Phi) \circ (\delta \otimes i) \circ \delta
\]
\[
= (\delta \otimes i) \circ \delta|,
\]
the injectivity of $(i \otimes \delta_G)| \circ \delta$ implies that of $(\delta \otimes i) \circ \delta|$, and hence of $\delta|$.

(iv) It remains to show the coaction identity. Firstly we note that
\[
(\Phi \otimes \Phi) \circ \delta_G(\lambda_G(g)) = (\Phi \otimes \Phi) \left( \int_G g(s)\lambda_G(s) \otimes \lambda_G(s) \, ds \right)
\]
\[
= \int_G g(s)\lambda_{G/H}(sH) \otimes \lambda_{G/H}(sH) \, ds
\]
(by lemma 1.7)
\[
= \int_{G/H} \int_H g(sh)\lambda_{G/H}(sH) \otimes \lambda_{G/H}(sH) \, dhdsH
\]
\[
= \delta_{G/H} \circ \Phi(\lambda_G(g)).
\]
Hence
\[
(\delta \otimes i)(\delta(a)) = (i \otimes \Phi \otimes \Phi)((\delta \otimes i)(\delta(a))
\]
\[
= (i \otimes \Phi \otimes \Phi)((i \otimes \delta_G)(\delta(a)))
\]
\[
= (i \otimes \delta_{G/H})(((i \otimes \Phi)(\delta(a)))
\]
(by the above)
\[
= (i \otimes \delta_{G/H})(\delta(a)) .
\]

It should be noted that if \( \delta : A \to \tilde{M}(A \otimes C_r^*(G)) \) is a coaction of an abelian group \( G \) and \( \alpha \) is the corresponding action of \( \hat{G} \), then \( \delta \mid \) corresponds to the restriction of \( \alpha \) to the subgroup \( H^\perp \) (explaining the notation). Further if \( \mathcal{F}_H : L^2(G/H) \to L^2(H^\perp) \) is the partial Fourier transform and \( \pi \) is a faithful representation of \( A \), then

\[
r_H = (\tilde{\pi} \times (1 \otimes \lambda_{H^\perp}))^{-1} \circ \text{Ad}(1 \otimes \mathcal{F}_H) : A \times_{\delta_1} (G/H) \to A \times_\alpha H^\perp
\]
is an isomorphism.

In what follows we shall embed \( C_0(G/H) \) in \( C_b(G) \) via the mapping

\[
q : C_0(G/H) \to C_b(G), \text{ where } \{q(F)\}(s) = F(sH) \quad s \in G, \ F \in C_0(G).
\]

That is, we shall view elements of \( C_0(G/H) \) as functions in \( C_b(G) \) which are constant on \( H \)-cosets.

**Proposition 2.4** Suppose \((\pi, \mu)\) is a covariant representation of \((A, G, \delta)\) on a Hilbert space \( Q \). Then \((\pi, \mu \circ q)\) is a covariant representation of \((A, G/H, \delta \mid)\). Hence we have a map, which we will call the restriction map,

\[
\text{Res}_{A \times_{\delta} G}^{A \times_{\delta} (G/H)} : \text{Rep}(A \times_{\delta} G) \to \text{Rep}(A \times_{\delta_1} (G/H)) : (\pi, \mu) \mapsto (\pi, \mu \circ q).
\]

**Proof** Firstly we note that since \( \mu \) is non-degenerate it has a (unique strictly continuous) extension to \( M(C_0(G)) \cong C_b(G) \). So \( \mu \circ q \) is well-defined. Now

\[
(\pi \otimes i)(\delta(a)) = (i \otimes \Phi)((\pi \otimes i)(\delta(a)))
\]

\[
= (i \otimes \Phi) \{((\mu \otimes i)(\varpi_G))((\pi(a) \otimes i)((\mu \otimes i)(\varpi_G^*)\\}\}
\]

(since \( (\pi, \mu) \) is a covariant representation)

\[
= (((\mu \circ q) \otimes i)(\varpi_{G/H}))((\pi(a) \otimes i)((\mu \circ q) \otimes i)(\varpi_{G/H}^*))
\]

\[-57-\]
(since \((i \otimes \Phi)(\varpi_G) = (q \otimes i)(\varpi_{G/H})\)).

So \((\pi, \mu \circ q)\) is a covariant representation of \((A, G/H, \delta)\).

\[\text{Lemma 2.5} \text{ Let } H \text{ be an amenable subgroup of a locally compact group } G. \text{ Then the representation}\]

\[\left( M_G \circ q \right) \times \lambda_G : C_\circ(G/H) \times_r G \to B(L^2(G))\]

is faithful.

\[\text{Proof}\] In [22 §7] Rieffel shows that \(C_\circ(G)\) is an equivalence bimodule implementing a strong Morita equivalence between \(C_\circ(G/H) \times_r G\) and \(C^*\). It is readily seen that

\[\left( M_G \circ q \right) \times \lambda_G = \text{ind}_{C^*}^{C_\circ(G/H) \times_r G} \lambda_H.\]

Hence

\[\ker\left(\left( M_G \circ q \right) \times \lambda_G\right) = \ker\left(\text{ind}_{C^*}^{C_\circ(G/H) \times_r G} \lambda_H\right)\]

\[= \text{ind}_{C^*}^{C_\circ(G/H) \times_r G} (\ker \lambda_H)\]

by [6 prop. 9] and since \(H\) is amenable this is

\[= 0.\]

Hence \(\left( M_G \circ q \right) \times \lambda_G\) is faithful as required.

\[\text{Proposition 2.6} \text{ Suppose } \pi \text{ is a faithful representation of } A \text{ on } \mathcal{H} \text{ and that } H \text{ is a closed normal amenable subgroup of a locally compact group } G. \text{ Then the map}\]

\[\Gamma = \text{Res}_{A \times \delta_1(G/H)}^{A \times \delta_1(G/H)} \left( ((\pi \otimes i) \circ \delta) \times (1 \otimes M_G) \right) : A \times \delta_1(G/H) \to B(\mathcal{H} \otimes L^2(G))\]

is a faithful representation of \(A \times \delta_1(G/H)\) on \(B(\mathcal{H} \otimes L^2(G))\). If \(a \in A\) and \(F \in C_\circ(G/H)\), then

\[\Gamma((\pi \otimes i)(\delta(a))(1 \otimes M_G(F))) = (\pi \otimes i)(\delta(a))(1 \otimes M_G(q(F))).\]
Proof Example 1.21 shows that \( (\pi \otimes i) \circ \delta, 1 \otimes M_G \) is a covariant representation of \((A, G, \delta)\) on \(B(\mathcal{H} \otimes L^2(G))\). So by lemma 2.4 \( \Gamma \) is a covariant representation of \(A \times_{\delta_l} (G/H)\). That \( \Gamma((\pi \otimes i)(\delta(a))(1 \otimes M_G/H(F)) = (\pi \otimes i)(\delta(a))(1 \otimes M_G(q(F))) \) is immediate from \( \dagger \) 1.35. By lemma 2.5, since \( H \) is amenable, the representation

\[
\pi \otimes ((M_G \circ q) \times \lambda_G) : M(A \otimes (C(G/H) \times r, G)) \to B(\mathcal{H} \otimes L^2(G))
\]

is injective and thus has an inverse. We will show that

\[
(\pi \otimes (M_G/H \times \varphi)) \circ (\pi \otimes ((M_G \circ q) \times \lambda_G))^{-1} \circ \Gamma = \text{id}_{A \times_{\delta_l}(G/H)} ,
\]

where \( \varphi \) is as in lemma 2.1, which will imply that \( \Gamma \) is injective as required.

Suppose \((\pi \otimes i)(\delta(a))(1 \otimes M_G/H(F))\) is a generator of \(A \times_{\delta_l} (G/H)\) and that

\[
\delta(a) = \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) ,
\]

for some \( a_{ij} \in A \) and \( g_{ij} \in C_c(G) \). Then

\[
(\pi \otimes (M_G/H \times \varphi)) \circ (\pi \otimes ((M_G \circ q) \times \lambda_G))^{-1} \circ \Gamma(((\pi \otimes i)(\delta(a))(1 \otimes M_G/H(F)))
\]

\[
= (\pi \otimes (M_G/H \times \varphi)) \circ (\pi \otimes ((M_G \circ q) \times \lambda_G))^{-1}(((\pi \otimes i)(\delta(a))(1 \otimes M_G(q(F)))
\]

\[
= (\pi \otimes (M_G/H \times \varphi)) \circ (\pi \otimes ((M_G \circ q) \times \lambda_G))^{-1}\left( \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes (\lambda_G(g_{ij})M_G(q(F))) \right)
\]

\[
= (\pi \otimes (M_G/H \times \varphi))\left( \sum_{j=1}^{n_i} a_{ij} \otimes (i_G(g_{ij})i_{C_\ast (G/H)}(F)) \right)
\]

\[
= (\pi \otimes \Phi)\left( \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \right)(1 \otimes M_G/H(F))
\]

(since the integrated form of \( \varphi \) equals \( \Phi \circ \lambda_G \))

\[
= (\pi \otimes i)(\delta(a))(1 \otimes M_G/H(F)) .
\]

Hence \( \dagger \) 1 holds and the proof is complete. \( \square \)
Notation  Henceforth we will suppress the map \( q \), in particular, \( M_G(q(F)) \) will be denoted \( M_G(F) \). The image \( \Gamma(A \times_{\delta} (G/H)) \subset B(\mathcal{H} \otimes L^2(G)) \) of \( A \times_{\delta} (G/H) \) under \( \Gamma \) will be denoted by \( A \times_{\delta} (G/H) \).

From proposition 2.7 it is clear that \( A \times_{\delta} (G/H) \) is the \( C^* \)-subalgebra of \( B(\mathcal{H} \otimes L^2(G)) \) generated by the elements \( ((\pi \otimes i)(\delta(a))(1 \otimes M_G(F))) \), where \( a \in A \) and \( F \in C_0(G/H) \). Now this makes sense whether or not the subgroup \( H \) is normal or amenable. Using this subalgebra it should be possible to extend the results of this thesis to include at least the non-normal case. I am presently working toward this end.
Chapter 3. Motivation and a Reformulation of Green's Imprimitivity Theorem.

In this chapter we present some of the fact and speculation that motivated the search for an induction process, and imprimitivity theorem, for representations of crossed products by coactions. We begin with an investigation of the case when the coactions, and actions concerned are by abelian groups. This not only motivates our results on induction, but also provides an elegant reformulation of Green's imprimitivity theorem.

§1 Motivation

We wish to establish a method, analogous to the usual induction procedure, for constructing representations of $A \times_\delta G$ from those of $A \times_\delta (G/H)$ and then to characterise those representations of $A \times_\delta G$ which can be obtained in this way. Suppose $G$ is an abelian group with dual group $\hat{G}$ and $H$ is a closed subgroup of $G$. Mackey's induction (for representations of the dual group) is then given by the pre-Hermitian $C_c(H^\perp)$-rigged $C_c(\hat{G})$ bimodule $C_c(\hat{G})$ (which when completed gives a Hermitian $C^*(H^\perp)$-rigged $C^*(\hat{G})$ bimodule) whose imprimitivity algebra is $C_0(\hat{G}/H^\perp) \times_\tau \hat{G}$, where $\tau$ is the left translation action of $\hat{G}$ on $C_0(\hat{G}/H^\perp)$. So by theorem 1.10

$$C_0(\hat{G}/H^\perp) \times_\tau \hat{G} \cong C^*(H^\perp).$$

If we now "Fourier transform" the algebras, we obtain

$$C^*(H) \times_\delta G \cong C_0(G/H),$$

where $\delta : C^*(H) \to \tilde{M}(C^*(H) \otimes C^*(G))$ is the coaction determined by

$$i_H(h) \to i_H(h) \otimes i_G(h) \quad h \in H.$$
The strong Morita equivalence §1 was to prove important in enabling us to reformulate Green's imprimitivity theorem (see below).

Our first result on the induction of representations of crossed products by non-abelian groups was the extension of the strong Morita equivalence §1 to arbitrary locally compact groups $G$ and closed subgroups $H$ of $G$. In the proof it was shown that $C_c(G)$ is a pre-Hermitian $C_c(G/H)$-rigged $C_c(G)$ module, (which when completed gives a Hermitian $C_o(G/H)$-rigged $C_o(G)$ bimodule). As outlined in chapter 1 §4, pre-Hermitian rigged modules establish induction processes. In this case we obtain a method of constructing representations of $C_o(G)$ from those of $C_o(G/H)$. Note that this is the desired induction process (for representations of crossed products by coactions) when $A = \mathcal{C}$ and $G$ coacts trivially. It can also be shown from the proof of §1 that the imprimitivity algebra $K(C_c(G))$ is $C^*(H) \times_\delta G$.

An application of [22 thm. 6.29] then gives the following imprimitivity theorem:

A representation $\nu$ of $C_o(G)$ on the Hilbert space $\mathcal{H}$ is induced from a representation of $C_o(G/H)$ if, and only if, there exists a representation $\mu$ of $C^*(H)$ such that $(\mu, \nu)$ is a covariant representation of $(C^*(H), G, \delta)$.

Unknown to the author at this time was that Rieffel [23] had shown that

$$C_o(G) \rtimes_\tau H \cong C_o(G/H),$$

where $\tau$ is the left translation action of $G$ on $C_o(G/H)$ restricted to $H$. In the light of the above we see that this should be viewed as a theorem about induced representations of crossed products by coactions. From its proof it can be seen that $C_c(G)$ is the same pre-Hermitian $C_c(G/H)$-rigged $C_c(G)$ module as above and thus establishes the same induction process. However, since $K(C_c(G))$ here is $C_o(G) \rtimes_\tau H$, we obtain a slightly different imprimitivity theorem, namely:

A representation $\nu$ of $C_o(G)$ on the Hilbert space $\mathcal{H}$ is induced from a representation of $C_o(G/H)$ if, and only if, there exists a unitary representation $U$ of $H$ such that $(\mu, U)$ is a covariant representation of $(C_o(G), H, \tau)$. 

- 62 -
The strong Morita equivalence \( b_2 \) proved extremely important in the development of the theory, since from it, it is easy to speculate that

\[
(A \times_\delta G) \times_\delta H \approx A \times_{\delta l} (G/H) .
\] (3)

If this speculation is correct we should be able to find dense *-algebras \( A \) and \( B \) of \( A \times_\delta G \) and \( A \times_{\delta l} (G/H) \), respectively, and a pre-Hermitian \( B \)-rigged \( A \) module \( D \) such that \( K(D) \) is isomorphic to \( (A \times_\delta G) \times_\delta H \). The pre-Hermitian rigged module \( D \) will then establish the desired induction process and the fact that \( K(D) \) is isomorphic to \( (A \times_\delta G) \times_\delta H \) will enable us to conclude the following imprimitivity theorem:

A representation \( \nu \) of \( A \times_\delta G \) on the Hilbert space \( \mathcal{H} \) is induced from a representation of \( A \times_{\delta l} (G/H) \) if, and only if, there exists a unitary representation \( U \) of \( H \) such that \((\mu, U)\) is a covariant representation of \( (A \times_\delta G, H, \delta) \).

§2 A reformulation of Green’s Imprimitivity Theorem

Combining \( b_1 \) and \( b_2 \) and assuming the groups are abelian, we have that

\[
C^*(H^\perp) \times_\delta \tilde{G} \approx C_o(\tilde{G}/H^\perp) \approx C_o(\tilde{G}) \times_\tau H^\perp ,
\]

Fourier transforming the algebras we obtain

\[
C_o(G/H)) \times_\tau G \approx C^*(H) \approx C^* (G) \times_{\tilde{H}} (G/H) ,
\]

Now the first strong Morita equivalence is a special case (when \( A = C \) and \( G \) acts trivially) of the strong Morita equivalence

\[
(A \otimes C_o(G/H)) \times_{\alpha \otimes \tau} G \approx A \times_\alpha H ,
\]

associated with Green’s induction process [6]. This leads one to speculate that

\[
(A \otimes C_o(G/H)) \times_{\alpha \otimes \tau} G \approx A \times_\alpha H \approx (A \times_\alpha G) \times_{\delta l} (G/H) \] (4)
(when $G$ is abelian, Fourier transforming the last two algebras gives the strong
Morita equivalence of § 3. In fact this is how it was first hypothesised). If § 4
is correct, then $(A \otimes C_0(G/H)) \times_{\alpha \otimes \tau} G$ and $(A \times_\alpha G) \times_{\alpha_l} (G/H)$ should have
the same representation theory. The following shows, that for an arbitrary locally
compact group $G$ and a closed normal amenable $H$ of $G$, this is indeed the case.

**Proposition 3.1** Let $\pi$, $\zeta$ and $U$ be representations of $A$, $C_0(G/H)$ and $G$,
respectively, on $\mathcal{H}$. Then $\pi$, $\zeta$ and $U$ are such that they form a representation
$(\pi \otimes \zeta) \times U$ of $(A \otimes C_0(G/H)) \times_{\alpha \otimes \tau} G$ on $\mathcal{H}$, if, and only if, they are such that
they form a representation $(\pi \times U) \times \zeta$ of $(A \times_\alpha G) \times_{\alpha_l} (G/H)$ on $\mathcal{H}$.

**Proof** Suppose $(\pi \otimes \zeta) \times U$ is a representation of $(A \otimes C_0(G/H)) \times_{\alpha \otimes \tau} G$ on $\mathcal{H}$.
Then

(i) $\pi(a)\zeta(F) = \zeta(F)\pi(a)$ \quad $\forall a \in A$, $F \in C_0(G/H)$,

(ii) $(\pi \otimes \zeta)((\alpha \otimes \tau)_s(w)) = U_s\pi \otimes \zeta(w)U_s^*$ \quad $\forall s \in G$, $w \in A \otimes C_0(G/H)$.

Let $\mu = \pi \times U$. Then for all $F \in A(G/H)$ and $a \in A$, we have that

$$S_F\left(\left[\left(\mu \otimes i\right)(i_A(a))\right]\left[\left(\zeta \otimes i\right)(\varpi_{G/H})\right]\right)$$

$$= S_F\left(\left[\left(\mu \otimes i\right)(i_A(a) \otimes 1)\right]\left[\left(\zeta \otimes i\right)(\varpi_{G/H})\right]\right)$$

$$= S_F\left(\left[\pi(a) \otimes 1\right]\left[\left(\zeta \otimes i\right)(\varpi_{G/H})\right]\right)$$

$$= \pi(a)S_F((\zeta \otimes i)(\varpi_{G/H}))$$

$$= \pi(a)\zeta(F)$$ \hspace{1cm} (by lemma 1.17)

$$= \zeta(F)\pi(a)$$

$$= S_F((\zeta \otimes i)(\varpi_{G/H}))\mu(i_A(a))$$ \hspace{1cm} (by lemma 1.17)

$$= S_F\left(\left[\left(\zeta \otimes i\right)(\varpi_{G/H})\right]\left[\mu(i_A(a)) \otimes 1\right]\right).$$

Hence

$$(\mu \otimes i)(i_A(a)) = \left[\left(\zeta \otimes i\right)(\varpi_{G/H})\right][\mu(i_A(a)) \otimes 1]\left[\left(\zeta \otimes i\right)(\varpi_{G/H}^*)\right]. \hspace{1cm} (5)$$
Suppose \( \bullet \) is the action defined in §1.9 of \( \nu N(G/H) = A(G/H)^* \) on (the right of) \( A(G/H) \) and that \( E, F \in C_c(G/H) \). Then

\[
\{F \bullet (\lambda_{G/H}(sH))\}(\lambda_{G/H}(E)) = F(\lambda_{G/H}(sH)\lambda_{G/H}(E))
\]

\[
= \int_{G/H} F(tH)E(s^{-1}tH) \, dtH \quad \text{(by §1.3)}
\]

\[
= \int_{G/H} F(stH)E(tH) \, dtH
\]

\[
= \int_{G/H} \{\tau_s^{-1}(F)\}(tH)E(tH) \, dtH
\]

\[
= \{\tau_s^{-1}(F)\}(\lambda_{G/H}(E)),
\]

so \( F \bullet (\lambda_{G/H}(sH)) = \tau_s^{-1}(F) \). We need to know this in order to show that

\[
S_F\left([\mu \otimes i](i_G(s)))[[\zeta \otimes i](\varpi_{G/H})]\right)
\]

\[
= S_F\left([\mu \otimes i](i_G(s) \otimes \lambda_{G/H}(sH)))[[\zeta \otimes i](\varpi_{G/H})]\right)
\]

\[
= S_F\left([U_s \otimes \lambda_{G/H}(sH))[[\zeta \otimes i](\varpi_{G/H})]\right)
\]

\[
= U_sS_F\left([1 \otimes \lambda_{G/H}(sH))[[\zeta \otimes i](\varpi_{G/H})]\right)
\]

\[
= U_sS_F\lambda_{G/H}(sH)([[\zeta \otimes i](\varpi_{G/H})]) \quad \text{(by §10)}
\]

\[
= U_s\zeta(F \bullet \lambda_{G/H}(sH)) \quad \text{(by lemma 1.17)}
\]

\[
= U_s\zeta(\tau_s^{-1}(F)) \quad \text{(by the above)}
\]

\[
= \zeta(F)U_s
\]

\[
= S_F((\zeta \otimes i)(\varpi_{G/H}))U_s \quad \text{(by lemma 1.17)}
\]

\[
= S_F\left(([\zeta \otimes i](\varpi_{G/H}))[U_s \otimes 1]\right)
\]

\[
= S_F\left(([\zeta \otimes i](\varpi_{G/H}))[\mu(i_G(s)) \otimes 1]\right).
\]

Hence

\[
(\mu \otimes i)(\hat{\varpi}(i_G(s))) = ([\zeta \otimes i](\varpi_{G/H}))[\mu(i_G(s)) \otimes 1]([\zeta \otimes i](\varpi_{G/H}^*)].
\]
By integrating 5 and 6 we see that

\[(\mu \otimes i)(\hat{a}(y)) = [(\zeta \otimes i)(\omega_{G/H})][\mu(y) \otimes 1][((\zeta \otimes i)(\omega_{G/H}^*)]\]

for all \(y \in A \times_\alpha G\). Hence we have that \((\mu, \zeta)\) is a covariant representation of \((A \times_\alpha G, G/H, \delta)\), on \(\mathcal{H}\) and that \((\pi \times U) \times \zeta\) is a representation of \((A \times_\alpha G) \times_{\delta_l} (G/H)\) on \(\mathcal{H}\).

Now suppose \((\pi \times U) \times \zeta\) is a representation of \((A \times_\alpha G) \times_{\delta_l} (G/H)\) on \(\mathcal{H}\). Then

(i) \(\pi(\alpha_s(a)) = U_s \pi(a) U_s^*\quad \forall s \in G, \ a \in A\),

(ii) \((\mu \otimes i)(\hat{a}(y)) = [(\zeta \otimes i)(\omega_{G/H})][\mu(y) \otimes 1][((\zeta \otimes i)(\omega_{G/H}^*)]\) \(\forall y \in A \times_\alpha G\).

For all \(F \in A(G/H)\) and \(a \in A\), we have, by reshuffling the first of the above sets of equations, that

\[\zeta(F)\pi(a) = \pi(a)\zeta(F)\quad \forall F \in C_0(G/H), \ a \in A.\]

A similar reshuffling of the second set of equations gives

\[\zeta(F)U_s = U_s \zeta(\tau_s^{-1}(F))\quad \forall F \in C_0(G/H), \ s \in G.\]

These two facts combined with (i) imply that \((\pi \otimes \zeta) \times U\) is a representation of \((A \otimes C_0(G/H)) \times_{\alpha \otimes r} G\), as required.

By [27 thm. 4.14] and theorem 1.18 all representations of \((A \otimes C_0(G/H)) \times_{\alpha \otimes r} G\) are of the form \((\pi \otimes \zeta) \times U\). Also by theorems 1.13 and 1.18 all representations of \((A \times_\alpha G) \times_{\delta_l} (G/H)\) are of the form \((\pi \times U) \times \zeta\). Hence the two algebras have the same representation theory as claimed. Now proposition 3.1 enables us to reformulate Green's imprimitivity theorem.

Suppose \(\mu\) is a representation of \(A \times_\alpha G\) on the Hilbert space \(\mathcal{H}\) and \((\pi, U)\) is the covariant representation corresponding to \(\mu\) (that is \(\mu = \pi \times U\)). Green's
imprimitivity theorem ([6 thm. 6]) states that \( \mu \) can be constructed (via the bimodule \( C_c(G,A) \) described in [6]) if, and only if, there exists a representation \( \zeta \) of \( C_0(G/H) \) on \( \mathcal{H} \) such that

(i) \( \pi(a)\zeta(F) = \zeta(F)\pi(a) \quad \forall a \in A, \ F \in C_0(G/H) \),

(ii) \( (\pi \otimes \zeta)((\alpha \otimes \tau)_s(w)) = U_s \pi \otimes \zeta(w)U_s^* \quad \forall s \in G, \ w \in A \otimes C_0(G/H) \).

But this is if, and only if, there exists a representation \( \zeta \) of \( C_0(G/H) \) on \( \mathcal{H} \) such that \( (\pi \otimes \zeta) \times U \) is a representation of \( (A \otimes C_0(G/H)) \times_{\alpha \otimes \tau} G \) on \( \mathcal{H} \), but by proposition 3.1 this is if, and only if, there exists a representation \( \zeta \) of \( C_0(G/H) \) on \( \mathcal{H} \) such that \( (\pi \times U) \times \zeta \) is a representation of \( (A \times_{\alpha} G) \times_{\tilde{\alpha}l}(G/H) \) on \( \mathcal{H} \).

So we have the following elegant reformulation of Green's imprimitivity theorem (at least for closed normal amenable subgroups \( H \)).

**Theorem 3.2** Suppose \( \alpha : G \to \text{Aut} A \) is an action of a locally compact group \( G \) on a \( C^* \)-algebra \( A \) and \( H \) is a closed normal amenable subgroup of \( G \). Then a representation \( \mu \) of \( A \times_{\alpha} G \) on \( \mathcal{H} \) is induced from a representation of \( A \times_{\alpha} H \) if and only if there exists a representation \( \zeta \) of \( C_0(G/H) \) on \( \mathcal{H} \) such that \( (\mu, \zeta) \) is a covariant representation \( (A \times_{\alpha} G, G/H, \tilde{\alpha}l) \).

This reformulation is a distinct improvement on the original, in that to check whether or not a representation \( \mu \) is induced, one no longer needs to find the covariant representation corresponding to \( \mu \), but can deal with \( \mu \) directly.

Now if \( G \) is amenable more can be said regarding the relationship between the algebras \( (A \otimes C_0(G/H)) \times_{\alpha \otimes \tau} G \) and \( (A \times_{\alpha} G) \times_{\tilde{\alpha}l}(G/H) \), namely, we have the following proposition, whose proof parallels that of (a section of) the proof of the Imai-Takai duality theorem.

**Proposition 3.3** Suppose \( \alpha : G \to \text{Aut} A \) is an action of an amenable group \( G \) on \( A \), \( H \) is a closed normal subgroup of \( G \), \( \tau \) is the left translation action of \( G \) on
$C_o(G/H)$ and $\alpha \otimes \tau$ is the action of $G$ on $A \otimes C_o(G/H)$ defined on elementary tensors by $(\alpha \otimes \tau)_*(a \otimes f) = \alpha_*(a) \otimes \tau_*(f)$. Then

$$(A \otimes C_o(G/H)) \times_{\alpha \otimes \tau} G \cong (A \times_{\alpha} G) \times_{\alpha_0} (G/H).$$

Proof Let $\tilde{\pi}$ and $1 \otimes \lambda_G$ etc. be as in §1.23. Since $G$ is amenable the representations

$$(\pi \otimes M_{G/H})^\sim \times (1 \otimes \lambda_G) : (A \otimes C_o(G/H)) \times_{\alpha \otimes \tau} G \to B(L^2(G/H \times G, \mathcal{H}))$$

and

$$(\tilde{\pi} \times (1 \otimes \lambda_G)) \otimes I : (A \times_{\alpha} G) \times_{\alpha_0} (G/H) \to B(L^2(G \times G/H, \mathcal{H}))$$

are faithful [13 §2.6 rmk. 3]. Note the implicit use of the isomorphisms

$L^2(G, \mathcal{H}) \otimes L^2(G/H) \cong L^2(G/H \times G, \mathcal{H})$

$L^2(G, \mathcal{H}) \cong L^2(G/H, \mathcal{H})$.

Define unitaries $W \in UB(L^2(G \times G/H, \mathcal{H}))$ by

$$\{W\xi\}(s, tH) = \xi(s, s^{-1}tH)$$

and $\Sigma : L^2(G \times G/H, \mathcal{H}) \to L^2(G/H \times G, \mathcal{H})$ by

$$\{\Sigma\xi\}(tH, s) = \xi(s, tH) \quad \text{for} \quad \xi \in L^2(G \times G/H, \mathcal{H}).$$

Let $\Upsilon = \text{Ad}(W \circ \Sigma^*) : B(L^2(G/H \times G, \mathcal{H})) \to B(L^2(G \times G/H, \mathcal{H}))$.

Then $\Upsilon$ is an isomorphism. We will show that $\Upsilon$ maps a set of generators of

$$(\pi \otimes M_{G/H})^\sim \times (1 \otimes \lambda_G)((A \otimes C_o(G/H)) \times_{\alpha \otimes \tau} G)$$

to a set of generators of

$$(\tilde{\pi} \times (1 \otimes \lambda_G)) \otimes I((A \times_{\alpha} G) \times_{\alpha_0} (G/H)).$$
which implies that these algebras are isomorphic, and since the representations are faithful the proposition follows.

Firstly we determine what $\mathcal{T}$ does to the elements $(\pi \otimes M_{G/H}) (a \otimes F)$ and $(1 \otimes \lambda_G)(r)$. Let $\xi \in L^2(G \times G/H, \mathcal{H})$. Then

(i) $\{\mathcal{T}((1 \otimes \lambda_G)(r)))\}(\xi)\}(s, tH)$

\[
\begin{align*}
&= \left\{ \left( (W)(\Sigma^*)(1 \otimes \lambda_G)(r))(\Sigma)(W^*) \right)(\xi) \right\}(s, tH) \\
&= \left\{ \left( (\Sigma^*)(1 \otimes \lambda_G)(r))(\Sigma)(W^*) \right)(\xi) \right\}(s, tH) \\
&= \left\{ \left( (1 \otimes \lambda_G)(r))(\Sigma)(W^*) \right)(\xi) \right\}(s^{-1}tH, s) \\
&= \left\{ \left( \Sigma)(W^*) \right)(\xi) \right\}(s^{-1}tH, r^{-1}s) \\
&= W^*(\xi))(r^{-1}s, s^{-1}tH) \\
&= \lambda_G(r) \otimes \lambda_{G/H}(rH) \\
&= \{\lambda_G(r) \otimes \lambda_{G/H}(rH)\}(\xi))(s, tH)
\end{align*}
\]

(ii) $\{\mathcal{T}((\pi \otimes M_{G/H}) (a \otimes F)))\}(\xi)\}(s, tH)$

\[
\begin{align*}
&= \left\{ \left( (\pi \otimes M_{G/H}) (a \otimes F))(\Sigma)(W^*) \right)(\xi) \right\}(s^{-1}tH, s) \\
&= \left\{ \left( (\pi \otimes M_{G/H})(a \otimes F))(\Sigma)(W^*) \right)(\xi) \right\}(s^{-1}tH) \\
&= \left\{ \left( \pi(a^\Delta^{-1}(a)) \otimes M_{G/H}(\tau^\Delta^{-1}(F)) \right)(\Sigma)(W^*) \right\}(s^{-1}tH) \\
&= \left\{ \tau^\Delta^{-1}(F) \right\}(s^{-1}tH) \{\pi(a^\Delta^{-1}(a)) \otimes M_{G/H}(F) \right\}(s^{-1}tH, s) \\
&= F(tH)\{\tau^\Delta^{-1}(a)\}\{\pi(a^\Delta^{-1}(a)) \otimes M_{G/H}(F) \right\}(s, tH) \\
&= \{\pi(a) \otimes M_{G/H}(F)\}(\xi))(s, tH)
\end{align*}
\]

So $\mathcal{T}((1 \otimes \lambda_G)(r)) = \lambda_G(r) \otimes \lambda_{G/H}(rH)$

and $\mathcal{T}((\pi \otimes M_{G/H}) (a \otimes F)) = \pi(a) \otimes M_{G/H}(F)$. 

- 69 -
We digress momentarily to show that elements of the form

$$\sum_{j=1}^{n} i_G(g_j)i_A(a_j) \quad g_j \in C_c(G), \ a_j \in A,$$

are dense in $A \times \alpha G$. Suppose

$$f = \sum_{j=1}^{n} g_j \otimes a_j \in C_c(G) \otimes A \subset C_c(G, A).$$

Then: $s \to i_A(f(s))i_G(s) : G \to M(A \times \alpha G)$ is strictly continuous with compact support, and hence is integrable, and

$$\int_G i_A(f(s))i_G(s) \, ds = \sum_{j=1}^{n} i_A(a_j) \int_G g_j(s)i_G(s) \, ds$$

$$= \sum_{j=1}^{n} i_A(a_j)i_G(g_j).$$

Now by [20 §7.6.4] the set of such elements $\int_G i_A(f(s))i_G(s) \, ds$ and hence, the set of elements $\sum_{j=1}^{n} i_A(a_j)i_G(g_j)$ for $a_j \in A$ and $g_j \in C_c(G)$, is dense in $A \times \alpha G$. Since

$$\sum_{j=1}^{n} i_G(g_j)i_A(a_j) = \left(\sum_{j=1}^{n} i_A(a_j)i_G(g_j)\right)^*,$$

the set of elements $\sum_{j=1}^{n} i_G(g_j)i_A(a_j)$ for $g_j \in C_c(G)$ and $a_j \in A$, is dense in $A \times \alpha G$ as claimed.

Now $(A \otimes C_o(G/H)) \times_{\alpha \otimes r} G$ is generated by the elements

$$i_G(g)i_{A\otimes C_o(G/H)}(a \otimes F) \quad g \in C_c(G), \ a \in A, \ F \in C_o(G/H).$$

Hence $((\pi \otimes M_{G/H})^\sim \times (1 \otimes \lambda_G))((A \otimes C_o(G/H)) \times_{\alpha \otimes r} G)$ is generated by the elements

$$(1 \otimes \lambda_G)(g)((\pi \otimes M_{G/H})^\sim(a \otimes F)) \quad g \in C_c(G), \ a \in A, \ F \in C_o(G/H),$$
(where \((1 \otimes \lambda_G)(g)\) and \((\pi \otimes M_{G/H})(a \otimes F) \in B(L^2(G, \mathcal{H} \otimes L^2(G/H)))\) are being considered as elements of \(B(L^2(G/H \times G, \mathcal{H}))\)).

Also since \(A \times_\alpha G\) is generated by the elements \(i_G(g)i_A(a)\) for \(g \in C_c(G)\) and \(a \in A\), \(\tilde{\alpha}(A \times_\alpha G)\) is generated by the elements

\[
\int_G g(s)i_G(s) \otimes \lambda_{G/H}(sH) \ ds (i_A(a) \otimes 1).
\]

Hence \(((\tilde{\pi} \times (1 \otimes \lambda_G)) \otimes i)((A \times_\alpha G) \times_\tilde{\alpha} (G/H))\) is generated by the elements

\[
\int_G g(s)\lambda_G(s) \otimes \lambda_{G/H}(sH) \ ds (\tilde{\pi}(a) \otimes M_{G/H}(F)) \quad F \in C_0(G/H),
\]

where \(\lambda_G(s) \otimes \lambda_{G/H}(sH)\) and \(\tilde{\pi}(a) \otimes M_{G/H}(F) \in B(L^2(G, \mathcal{H} \otimes L^2(G/H)))\) are being considered as elements of \(B(L^2(G \times G/H, \mathcal{H}))\).

Now

\[
\mathcal{T}((1 \otimes \lambda_G)(s)((\pi \otimes M_{G/H})\tilde{\lambda}(a \otimes F)))
\]

\[
= \int_G g(s)\mathcal{T}((1 \otimes \lambda_G(s)) \ ds \mathcal{T}((\pi \otimes M_{G/H})\tilde{\lambda}(a \otimes F))
\]

\[
= \int_G g(s)\lambda_G(s) \otimes \lambda_{G/H}(sH) \ ds (\tilde{\pi}(a) \otimes M_{G/H}(F))
\]

(by the above). So \(\mathcal{T}\) maps a set of generators of \(((\pi \otimes M_{G/H})\tilde{\lambda} \times (1 \otimes \lambda_G))((A \otimes C_0(G/H))\times_{\alpha \otimes r} G)\) to a set of generators of \(((\tilde{\pi} \times (1 \otimes \lambda_G))\otimes i)((A \times_\alpha G) \times_\tilde{\alpha} (G/H))\) as claimed. \(\square\)
Chapter 4. The Subalgebras.

Let \( \pi : A \to B(\mathcal{H}) \) be a fixed faithful representation of \( A \) on the Hilbert space \( \mathcal{H} \). Let \( \delta : A \to \tilde{M}(A \otimes C^*_r(G)) \) be a non-degenerate coaction of \( G \) on \( A \). Then for any closed (not necessarily normal or amenable) subgroup \( H \) of \( G \) we present a dense \(*\)-subalgebra \( \mathcal{D}_H \) of \( A \times_\delta (G/H) \subset B(\mathcal{H} \otimes L^2(G)) \). The proof that \( \mathcal{D}_H \) is a \(*\)-subalgebra is rather technical and hence has been broken into a number of easily digestible lemmas.

**Definitions** Suppose \( \delta : A \to \tilde{M}(A \otimes C^*_r(G)) \) is a coaction of \( G \) on \( A \), \( H \) is a closed subgroup of \( G \), \( E \) is a compact subset of \( G \) and \( u \in A_c(G) \). Then an element \( x \) of \( B(\mathcal{H} \otimes L^2(G)) \) is said to be \((u, E, H)\) if it can be written in the form

\[
x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta_u(a_{ij})(1 \otimes \varphi(f_{ij})) ,
\]

where the \( f_{ij} \in C_c(G) \), with \( \text{supp} f_{ij} \subset E \) for all \( i \) and \( j \). We will denote by \( \mathcal{D}_H \) the set of all elements of \( B(\mathcal{H} \otimes L^2(G)) \) which are \((u, E, H)\) for some \( u \in H \) and \( E \) compact in \( G \). If \( H \) is the trivial subgroup we will abbreviate \( \mathcal{D}_H \) to \( \mathcal{D} \) and \((u, E, H)\) to \((u, E)\).

**Proposition 4.1** Suppose \( \delta : A \to \tilde{M}(A \otimes C^*_r(G)) \) is a coaction of \( G \) on \( A \), \( H \) is a closed subgroup of \( G \), \( a \in A \), \( u, v \in A_c(G) \) and \( f \in C_c(G) \). Then the maps

\[
\begin{align*}
: s & \to \delta(u \star_v(a))(1 \otimes \varphi(f_s)) \quad \text{and} \quad : s & \to (1 \otimes \varphi(f_s))\delta(u \star_v(a)) \\
\end{align*}
\]

are continuous with compact support, and

\[
(1 \otimes \varphi(u \star f))\delta(v(a)) = \int_G \delta(u \star_v(a))(1 \otimes \varphi(f_s)) \, ds
\]

\[
\delta(v(a))(1 \otimes \varphi(u \star f)) = \int_G (1 \otimes \varphi(f_s))\delta(u \star_v(a)) \, ds .
\]
Proof  Firstly we show that if \( g \in C_c(G) \), then

\[
M_G(\phi(u * f)) \lambda_G(g) = \int_G S_u^* (\delta_G(\lambda_G(g))) (M_G(\phi(f_s))) ds .
\]  

(1)

Let \( \xi, \eta \in L^2(G) \). Then

\[
\langle \{M_G(\phi(u * f)) \lambda_G(g)\}(\xi) , \eta \rangle_{L^2(G)}
\]

\[
= \int_G \left( \{M_G(\phi(u * f)) \lambda_G(g)\}(\xi)(p) \overline{\eta(p)} \right) dp
\]

\[
= \int_G (\phi(u * f))(p)((\lambda_G(g))(\xi)(p) \overline{\eta(p)}) dp
\]

\[
= \int_G \left( \int_H \left( \int_G u(r) f(r^{-1} ph) dr \right) dh \right) \left( \int_G g(t) \xi(t^{-1} p) dt \right) \overline{\eta(p)} dp
\]

\[
= \int_G \int_G \int_H \int_G u(r) f(r^{-1} ph) g(t) \xi(t^{-1} p) \overline{\eta(p)} dr dh dt dp
\]

\[
= \int_G \int_G \int_H \int_G u(ts) f(s^{-1} t^{-1} ph) g(t) \xi(t^{-1} p) \overline{\eta(p)} ds dh dt dp .
\]  

(2)

Suppose \( E = (\text{suppg})^{-1} \cdot (\text{suppu}) \). Then

\[
\int_G \int_G \int_H |u(ts) f(s^{-1} t^{-1} ph) g(t) \xi(t^{-1} p) \overline{\eta(p)}| dh ds dt dp
\]

\[
= \int_G \int_G \int_H \left( \int_H |f(s^{-1} t^{-1} ph)| dh \right) |u(ts) g(t) \xi(t^{-1} p) \overline{\eta(p)}| ds dp dt
\]

\[
\leq \mu_G(E) \cdot \|u\|_{C_c(G)} \cdot \|\phi(f)\|_{C_c(G/H)} \cdot \int_G |g(t) \xi(t^{-1} p) \overline{\eta(p)}| dp dt
\]

\[
\leq \mu_G(E) \cdot \|u\|_{C_c(G)} \cdot \|\phi(f)\|_{C_c(G/H)} \cdot \int_G (|g| \cdot (|\eta| * |\xi|)) (t) dt < \infty ,
\]

since by chapter 1 §1 \( |g| \cdot (|\eta| * |\xi|) \in A_c(G) \). So the integrand is integrable over \( G \times G \times G \times H \) and hence over \( G \times G \times H \times G \) and we can change the order of integration obtaining

\[
(2) = \int_G \int_G \int_G g(t) u(ts) \xi(t^{-1} p) \left( \int_H f(s^{-1} t^{-1} ph) dh \right) \overline{\eta(p)} dt ds dp.
\]
\[
\int G \int G \int G g(t)u(ts)\left((M_G(\varphi(f_s)))(\xi)\right)(t^{-1}p)\eta(p) \ dtdsdp
\]

\[
= \int G \left(\int G g(t)u(ts)\lambda_G(t)M_G(\varphi(f_s)) \ dtds\right)(\xi)(p)\eta(p) \ dp
\]

\[
= \left(\int G \int G g(t)u(ts)\lambda_G(t)M_G(\varphi(f_s)) \ dtds\right)(\xi, \eta)_{L^2(G)}.
\]

Hence

\[
M_G(\varphi(u \ast f))\lambda_G(g) = \int G \int G g(t)u(ts)\lambda_G(t)M_G(\varphi(f_s)) \ dtds
\]

\[
= \int G \int G g(t)S_u^*(\lambda_G(t) \otimes \lambda_G(t))M_G(\varphi(f_s)) \ dtds
\]

\[
= \int G S_u^* \left(\int G g(t)\lambda_G(t) \otimes \lambda_G(t)((M_G(\varphi(f_s)) \otimes 1)) \ dt\right)ds
\]

\[
= \int G S_u^* (\delta_G(\lambda_G(g))((M_G(\varphi(f_s)) \otimes 1)) \ ds
\]

\[
= \int G S_u^* (\delta_G(\lambda_G(g)))M_G(\varphi(f_s)) \ ds.
\]

By lemma 1.22 we can find \( a_{ij} \in A \) and \( g_{ij} \in C_c(G) \) with \( \text{supp} g_{ij} \subset \text{supp} u \) such that

\[
\delta(\delta_u(a)) = \lim_{i \to \infty} \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \quad \text{in} \quad M(A \otimes C_r^*(G))
\]

and hence

\[
(\pi \otimes \iota)(\delta(\delta_u(a))) = \lim_{i \to \infty} \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \lambda_G(g_{ij}) \quad \text{in} \quad B(\mathcal{H} \otimes L^2(G)).
\]

So

\[
(1 \otimes M_G(\varphi(u \ast f)))(\pi \otimes \iota)(\delta(\delta_u(a)))
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \left(M_G(\varphi(u \ast f))\lambda_G(g_{ij})\right)
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \left(\int G S_u^* (\delta_G(\lambda_G(g_{ij})))M_G(\varphi(f_s))ds\right) \quad \text{(by 1)}
\]
\[
\begin{align*}
\text{defined by} & \\
& \gamma_i(s) = (\pi \otimes S_{u^*})\left( (i \otimes \delta_G)\left( \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \right) \right) (1 \otimes M_G(\varphi(f_s))) \\
& \text{and} & \\
& \gamma(s) = (\pi \otimes i)(\delta(\delta_{u^*\cdot u}(a)))(1 \otimes M_G(\varphi(f_s))) \\
& \text{Now} & \\
& \gamma_i(s) = (\pi \otimes S_{u^*})\left( (i \otimes \delta_G)\left( \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \right) \right) (1 \otimes M_G(\varphi(f_s))) \\
& \rightarrow (\pi \otimes S_{u^*})((i \otimes \delta_G)(\delta(\delta_v(a))))(1 \otimes M_G(\varphi(f_s))) \\
& = (\pi \otimes S_{u^*})((\delta \otimes i)(\delta(\delta_v(a))))(1 \otimes M_G(\varphi(f_s))) \\
& = (\pi \otimes i)(\delta(S_{u^*}(\delta(\delta_v(a)))))(1 \otimes M_G(\varphi(f_s))) \\
& = (\pi \otimes i)(\delta(\delta_{u^*\cdot v}(a)))(1 \otimes M_G(\varphi(f_s))) & \text{(by lemma 1.23)} \\
& = \gamma(s), \\
\end{align*}
\]

that is, \( \gamma_i(s) \to \gamma(s) \) strongly, hence weakly, for all \( s \in G \). We will now show that \( \gamma \) and the \( \gamma_i \) are continuous and compactly supported hence integrable by lemma 1.6.
\[ \| \gamma_i(s) - \gamma_i(s') \|_{B(\mathcal{H} \otimes L^2(G))} \]

\[ \leq \| (\pi \otimes S(u_{u-su})) \left( (i \otimes \delta_G) \left( \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \right) \right) (1 \otimes M_G(\varphi(f_s))) \| \]

\[ + \| (\pi \otimes S(u_{u'})) \left( (i \otimes \delta_G) \left( \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \right) \right) (1 \otimes M_G(\varphi(f_s) - \varphi(f_{s'}))) \| \]

\[ \leq \| (i \otimes S(u_{u-su})) \left( (i \otimes \delta_G) \left( \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \lambda_G(g_{ij}) \right) \right) \| \cdot \| 1 \otimes M_G(\varphi(f_s)) \| \]

\[ + \| (i \otimes S(u_{u'})) \left( (i \otimes \delta_G) \left( \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \lambda_G(g_{ij}) \right) \right) \| \cdot \| 1 \otimes M_G(\varphi(f_s) - \varphi(f_{s'}))) \| \]

\[ \leq \| u^s - u^{s'} \|_{B(G)} \cdot \| \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \lambda_G(g_{ij}) \| \cdot \| \varphi(f_s) \|_{C_s(G/H)} \]

\[ + \| u^s \|_{B(G)} \cdot \| \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \lambda_G(g_{ij}) \| \cdot \| \varphi(f_s) - \varphi(f_{s'}) \|_{C_s(G/H)} \]

\[ \leq \| \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \lambda_G(g_{ij}) \| \cdot \left( \| u^s - u^{s'} \|_{B(G)} \cdot \| \varphi(f) \|_{C_s(G/H)} \right) \]

\[ + \| u^s \|_{B(G)} \cdot \| (\varphi(f))_{s} - (\varphi(f))_{s'} \|_{C_s(G/H)} \right). \]

So \( \| 1.8 \) and the continuity of \( \varphi(f) \) imply the \( \gamma_i \) are continuous. Now

\[ \gamma_i(s) = (\pi \otimes S) \left( (i \otimes \delta_G) \left( \sum_{j=1}^{n_i} a_{ij} \otimes \lambda_G(g_{ij}) \right) \right) (1 \otimes M_G(\varphi(f_s))) \]

\[ = \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \left( S(\int_G g_{ij}(t) \lambda_G(t) \otimes \lambda_G(t) \, dt) (1 \otimes M_G(\varphi(f_s))) \right) \]

\[ = \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \left( \left( \int_G g_{ij}(t) u(t\varphi) \lambda_G(t) \, dt \right) (1 \otimes M_G(\varphi(f_s))) \right). \]

So the \( \gamma_i \) are compactly supported in \((\text{suppg}_{ij})^{-1} \cdot (\text{suppu}) \subset (\text{suppv})^{-1} \cdot (\text{suppu})\).
Now

\[ \|\gamma(s) - \gamma(s')\|_{B(\mathcal{H} \otimes L^2(G))} \]

\[ \leq \left\| (\pi \otimes i) \left( \delta_{(u^* - u')}(a) \right) \right\| \left\| (1 \otimes M_G(\varphi(f_s))) \right\| \]

\[ + \left\| (\pi \otimes i) \left( \delta_{u'}(a) \right) \right\| \left\| (1 \otimes M_G(\varphi(f_s) - \varphi(f_{s'}))) \right\| \]

\[ \leq \left\| \delta_{(u^* - u')}(a) \right\| \left\| (1 \otimes M_G(\varphi(f_s))) \right\| \]

\[ + \left\| \delta_{u'}(a) \right\| \left\| (1 \otimes M_G(\varphi(f_s) - \varphi(f_{s'}))) \right\| \]

\[ \leq \|(u^* - u') \cdot v\|_{B(G)} \cdot \|(a)\|_{\tilde{M}(A \otimes C_\ast(G))} \cdot \|\varphi(f_s)\|_{C_\ast(G/H)} \]

\[ + \|(u') \cdot v\|_{B(G)} \cdot \|\delta(a)\|_{\tilde{M}(A \otimes C_\ast(G))} \cdot \|\varphi(f_s) - \varphi(f_{s'})\|_{C_\ast(G/H)} \]

\[ \leq \|(u^* - u')\|_{B(G)} \cdot \|v\|_{B(G)} \cdot \|a\|_{A} \cdot \|\varphi(f)\|_{C_\ast(G/H)} \]

\[ + \|(u') \cdot v\|_{B(G)} \cdot \|v\|_{B(G)} \cdot \|a\|_{A} \cdot \|\varphi(f)\|_{C_\ast(G/H)} - \|\varphi(f)\|_{C_\ast(G/H)} \]

\[ \leq \|v\|_{B(G)} \cdot \|a\|_{A} \cdot \left( \|(u^* - u')\|_{B(G)} \cdot \|\varphi(f)\|_{C_\ast(G/H)} \right) \]

\[ + \|u\|_{B(G)} \cdot \|\varphi(f)\|_{C_\ast(G/H)} \]

So \textsection 1.8 and the fact that \( \varphi(f) \) is continuous imply \( \gamma \) is continuous. Clearly \( \gamma \) is compactly supported in \((\text{suppv})^{-1} \cdot \text{suppu}\).

Suppose \( \xi, \eta \in L^2(G) \) and \( F \) is the compact set \((\text{suppv})^{-1} \cdot \text{suppu}\). If \( \omega_{\xi, \eta} \) is as in \textsection 1.13, then

\[ |\omega_{\xi, \eta}(\gamma_i(s))| = |\{(\gamma_i(s))(\xi), \eta)\mathcal{H} \otimes L^2(G)| \]

\[ \leq \|\{(\gamma_i(s))(\xi)\| \cdot \|\eta\| \]

\[ \leq \|u\|_{B(G)} \cdot \left\{ \sum_{j=1}^{n_i} \pi(a_{ij}) \otimes \lambda(g_{ij}) \right\} (\xi) \cdot \|\varphi(f)\|_{C_\ast(G/H)} \cdot \|\eta\| \cdot \chi_F(s) \]

\[ \leq \|u\|_{B(G)} \cdot \left( \|\{(\pi \otimes i)\delta(v(a))\}(\xi)\| + 1 \right) \cdot \|\varphi(f)\|_{C_\ast(G/H)} \cdot \|\eta\| \cdot \chi_F(s) \]

- 77 -
for $i$ sufficiently large since $\sum_{j=1}^{n_i} \varphi(a_{ij}) \otimes \lambda(g_{ij}) \to (\varphi \otimes i)(\delta_\nu(a))$ strongly. Since $F$ is compact the right hand side is clearly a positive $L^1(G)$ function as required, justifying the use of lemma 1.9. The second equation of the proposition follows similarly.

\[ \square \]

**Notation** Let $E$ be a compact subset of $G$. Then $C_E(G)$ will denote those elements of $C_c(G)$ whose support is contained in $E$.

**Lemma 4.2** Suppose $\delta: A \to \bar{M}(A \otimes C_c^*(G))$ is a coaction of $G$ on $A$, $H$ is a closed subgroup of $G$, $a \in A$, $E$ is a compact subset of $G$ and $v \in A_c(G)$. Then

(i) there exists a compact subset $F$ of $G$ such that if $f \in C_E(G)$ and $\epsilon > 0$, then there exist $a_j \in A$ and $f_j \in C_F(G)$ for $j = 1, ..., n$ such that

$$\left\| (1 \otimes \varphi(f)) \delta_\nu(a) - \sum_{j=1}^{n} \delta_\nu(a_j)(1 \otimes \varphi(f_j)) \right\| < \epsilon ,$$

(ii) there exists a compact subset $F'$ of $G$ such that if $f \in C_E(G)$ and $\epsilon > 0$, then there exist $a_j' \in A$ and $f_j' \in C_{F'}(G)$ for $j = 1, ..., n'$ such that

$$\left\| \delta_\nu(a)(1 \otimes \varphi(f)) - \sum_{j=1}^{n'} (1 \otimes \varphi(f_j')) \delta_\nu(a_j') \right\| < \epsilon .$$

**Proof** Suppose $E$ is a compact subset of $G$, $f \in C_E(G)$ and $V$ is a compact neighbourhood of the identity. Let $F$ be the compact subset $(\text{supp}v)^{-1} \cdot V \cdot E$ and let $\epsilon > 0$. Choose $u \in A_c^+(G)$ such that $\text{supp}u \subset V$ and

$$\|\varphi(u \ast f) - \varphi(f)\|_{C_*(G/H)} < \epsilon/(2 \cdot \|\delta_\nu(a)\|_A) .$$

Then

$$\| (1 \otimes \varphi(f)) \delta_\nu(a) - (1 \otimes \varphi(u \ast f)) \delta_\nu(a) \| < \epsilon/2 . \quad (4)$$
Now by proposition 4.1

$$(1 \otimes \varphi(u \ast f))\delta(\delta_v(a)) = \int_G \delta(\delta_{u \ast v}(a))(1 \otimes \varphi(f_s)) \, ds,$$

where the integrand $\gamma : s \to \delta(\delta_{u \ast v}(a))(1 \otimes \varphi(f_s))$ is continuous and compactly supported in $(\text{supp} v)^{-1} \cdot (\text{supp} u) \subset (\text{supp} v)^{-1} \cdot V$. So we can find $\zeta_j \in C_c(G)$ and $s_j \in G$ for $j = 1, \ldots, n$ such that

$$\| \gamma(s) - \sum_{j=1}^n \zeta_j(s) \gamma(s_j) \| < \epsilon/(2\mu_G((\text{supp} v)^{-1} \cdot V)) \quad \forall \, s \in G.$$

So if we let $\nu_j = \int_G \zeta_j(s) \, ds$, then

$$\left\| \int_G \gamma(s) \, ds - \sum_{j=1}^n \nu_j \gamma(s_j) \right\| = \left\| \int_G (\gamma(s) - \sum_{j=1}^n \zeta_j(s) \gamma(s_j)) \, ds \right\|
\leq \int_G \| \gamma(s) - \sum_{j=1}^n \zeta_j(s) \gamma(s_j) \| \, ds
< \epsilon/2.$$

Now this and \textsection 4 imply

$$\left\| (1 \otimes \varphi(f))\delta(\delta_v(a)) - \sum_{j=1}^n \delta(\delta_v(\delta_{u \ast j}(a)))(1 \otimes \varphi(\nu_j f_{s_j})) \right\|
\leq \left\| (1 \otimes \varphi(f))\delta(\delta_v(a)) - (1 \otimes \varphi(u \ast f))\delta(\delta_v(a)) \right\|
+ \left\| (1 \otimes \varphi(u \ast f))\delta(\delta_v(a)) - \sum_{j=1}^n \nu_j \delta(\delta_{u \ast j \cdot v}(a))(1 \otimes \varphi(f_{s_j})) \right\|$$

(since $\delta_v(\delta_{u \ast j}(a)) = \delta_{v \cdot u \ast j}(a) = \delta_{u \ast j \cdot v}(a)$ by lemma 1.23)

$$\leq \epsilon/2 + \left\| \int_G \gamma(s) \, ds - \sum_{j=1}^n \nu_j \gamma(s_j) \right\|
\leq \epsilon/2 + \epsilon/2.$$
\[ \| (1 \otimes \varphi(f)) \delta(\delta_u(a)) - \sum_{j=1}^{n} \delta(\delta_u(a_j))(1 \otimes \varphi(f_j)) \| < \epsilon, \]

where \( a_j = \delta_u^*(a) \) and \( f_j = \nu_j f_{s_j} \in C_F(G) \), as required. (ii) follows similarly. \( \square \)

**Lemma 4.3** Let \( E \) and \( F \) be compact subsets of \( G \). Let \( u \) and \( v \in A_c(G) \). Then there exists a compact subset \( D \) of \( G \) and \( w \in A_c(G) \) such that if \( f \in C_E(G) \) and \( g \in C_F(G) \), then \( \delta(\delta_u(a))(1 \otimes \varphi(f)) + \delta(\delta_v(b))(1 \otimes \varphi(g)) \) is \((w, D, H)\).

**Proof** Let \( D = E \cup F \). Then \( D \) is compact. Let \( w \in A_c(G) \) be such that \( w \) restricted to \((\text{supp} u) \cup (\text{supp} v)\) is identically one. Then \( \delta_w(\delta_u(a)) = \delta_u(a) \) and \( \delta_w(\delta_v(b)) = \delta_v(b) \) (lemma 1.24) so if \( f \in C_E(G) \) and \( g \in C_F(G) \), then

\[ \delta(\delta_u(a))(1 \otimes \varphi(f)) + \delta(\delta_v(b))(1 \otimes \varphi(g)) = \sum_{i=1}^{2} \delta(\delta_w(e_i))(1 \otimes \varphi(c_i)) , \]

where \( e_1 = a, e_2 = b \) with the \( e_i \in A \) and \( c_1 = f, c_2 = g \) with the \( c_i \in C_D(G) \).

So \( \delta(\delta_u(a))(1 \otimes \varphi(f)) + \delta(\delta_v(b))(1 \otimes \varphi(g)) \) is \((w, D, H)\). \( \square \)

**Lemma 4.4** Let \( E \) be a compact subset of \( G \). Let \( u \in A_c(G) \). Then there exists a compact subset \( F \) of \( G \) such that if \( f \in C_E(G) \), then \( (\delta(\delta_u(a))(1 \otimes \varphi(f)))^* \) is \((\hat{u}, F, H)\).

**Proof** By lemma 4.2 there exists a compact subset \( F \) of \( G \) such that if \( f \in C_E(G) \) and \( \epsilon > 0 \), then there exist \( f_j \in C_F(G) \) and \( a_j \in A \) such that

\[ \| \delta(\delta_u(a))(1 \otimes \varphi(f)) - \sum_{j=1}^{n} (1 \otimes \varphi(f_j)) \delta(\delta_u(a_j)) \| < \epsilon . \]

So
\[ \| (\delta(\delta_u(a))(1 \otimes \varphi(f)))^* \| - \sum_{j=1}^{n} \delta(\delta_u(a_j^*)) \| (1 \otimes \varphi(f_j)) \| < \epsilon. \]

(since \((\delta_u(a_j))^* = \delta_u(a_j^*)\) by lemma 1.25). So \((\delta(\delta_u(a))(1 \otimes \varphi(f)))^*\) is \((\bar{u}, F, H)\).

Lemma 4.5 Let \(E\) be compact in \(G\). Let \(u, v \in A_c(G)\). Then there exists \(w \in A_c(G)\) such that if \(g \in C_E(G)\) and \(f \in C_c(G)\), then

\[ \delta(\delta_u(a))(1 \otimes \varphi(f)) \delta(\delta_v(b))(1 \otimes \varphi(g)) \text{ is } (w, E, H). \]

Proof Let \(\epsilon > 0\). By lemma 4.2 there exist \(b_j \in A\) and \(f_j \in C_c(G)\) such that

\[ \| (1 \otimes \varphi(f))\delta(\delta_v(b)) - \sum_{j=1}^{n} \delta(\delta_v(b_j))(1 \otimes \varphi(f_j)) \| < \epsilon/(\|\delta_u(a)\|_A \cdot \|\varphi(g)\|_{C_c(G/H)}). \]

Let \(w \in A_c(G)\) be one on \((\text{supp}u) \cdot (\text{supp}v)\). Then \(\delta_w(\delta_u(a)\delta_v(b_j)) = \delta_u(a)\delta_v(b_j)\) (lemma 1.24). Also \((\varphi(f_j)) \cdot (\varphi(g)) = \varphi(\varphi(f_j) \cdot g)\), so

\[ \| \delta(\delta_u(a))(1 \otimes \varphi(f)) \delta(\delta_v(b))(1 \otimes \varphi(g)) - \sum_{j=1}^{n} \delta(\delta_w(\delta_u(a)\delta_v(b_j)))(1 \otimes \varphi(\varphi(f_j) \cdot g)) \| \]

\[ = \| \delta(\delta_u(a))(1 \otimes \varphi(f)) \delta(\delta_v(b))(1 \otimes \varphi(g)) \]

\[ \quad - \sum_{j=1}^{n} \delta(\delta_u(a)\delta_v(b_j))(1 \otimes ((\varphi f_j) \cdot (\varphi g))) \|
\]

\[ = \| \delta(\delta_u(a))((1 \otimes \varphi(f))\delta(\delta_v(b)) - \sum_{j=1}^{n} \delta(\delta_v(b_j))(1 \otimes \varphi(f_j)))(1 \otimes \varphi(g)) \|
\]

\[ < \epsilon, \]

i.e. \[ \| \delta(\delta_u(a))(1 \otimes \varphi(f)) \delta(\delta_v(b))(1 \otimes \varphi(g)) - \sum_{j=1}^{n} \delta(\delta_w(\delta_v(b_j)))(1 \otimes \varphi(g_j)) \| < \epsilon, \]

- 81 -
where \( d_j = \delta_u(a)\delta_v(b_j) \) and \( \gamma_j = \varphi(f_j) \cdot g \) is an element of \( C_E(G) \).

So \( \delta(\delta_u(a))(1 \otimes \varphi(f))\delta(\delta_v(b))(1 \otimes \varphi(g)) \) is \((w, E, H)\).

\[ \]

**Lemma 4.6** Let \( E \) be a compact subset of \( G \). Let \( u, v \in A_c(G) \). Then

(i) there exists \( w \in A_c(G) \) such that if \( g \in C_E(G) \) and \( f \in C_c(G) \), then \( \delta(\delta_u(a))(1 \otimes \varphi(f))\delta(\delta_v(b))(1 \otimes g) \) is \((w, E)\).

(ii) there exists \( \omega \in A_c(G) \) and a compact subset \( F \) of \( G \) such that if \( f \in C_E(G) \) and \( g \in C_c(G) \), then \( \delta(\delta_u(a))(1 \otimes f)\delta(\delta_v(b))(1 \otimes \varphi(g)) \) is \((\omega, F)\).

**Proof** By lemma 4.2 there exists a compact subset \( F \) of \( G \) such that if \( f \in C_E(G) \) and \( \epsilon > 0 \), then there exist \( b_j \in A \) and \( f_j \in C_F(G) \) such that

\[
\|\{1 \otimes \varphi(f)\} - \sum_{j=1}^{n} \delta(\delta_v(b_j))(1 \otimes \varphi(f_j))\| < \epsilon/(\|\delta_u(a)\|_A \cdot \|g\|_{C_c(G)}) .
\]

Let \( w \in A_c(G) \) be one on \((\text{supp } u) \cdot (\text{supp } v)\). Then \( \delta_w(\delta_u(a)\delta_v(b_j)) = \delta_u(a)\delta_v(b_j) \) (lemma 1.24) and

\[
\|\delta(\delta_u(a))(1 \otimes \varphi(f))\delta(\delta_v(b))(1 \otimes g) - \sum_{j=1}^{n} \delta(\delta_u(a)\delta_v(b_j))(1 \otimes ((\varphi(f_j)) \cdot g))\|
\]

\[
= \|\delta(\delta_u(a))(1 \otimes \varphi(f))\delta(\delta_v(b))(1 \otimes g) - \sum_{j=1}^{n} \delta(\delta_u(a)\delta_v(b_j))(1 \otimes ((\varphi(f_j)) \cdot g))\|
\]

\[
= \|\delta(\delta_u(a))\left(1 \otimes \varphi(f)\right)\delta(\delta_v(b)) - \sum_{j=1}^{n} \delta(\delta_v(b_j))(1 \otimes \varphi(f_j))\right)(1 \otimes g)\| < \epsilon ,
\]

i.e.

\[
\|\delta(\delta_u(a))(1 \otimes \varphi(f))\delta(\delta_v(b))(1 \otimes g) - \sum_{j=1}^{n} \delta(\delta_u(c_j))(1 \otimes \gamma_j)\| < \epsilon ,
\]

where \( c_j = \delta_u(a)\delta_v(b_j) \) and \( \gamma_j = \varphi(f_j) \cdot g \) is an element of \( C_E(G) \).

So \( \delta(\delta_u(a))(1 \otimes \varphi(f))\delta(\delta_v(b))(1 \otimes g) \) is \((w, E)\) establishing (i).
By lemma 4.2, with $H$ the trivial subgroup, there exists a compact subset $F$ of $G$ such that if $f \in C_E(G)$ and $\epsilon > 0$, then there exists $b_j \in A$ and $f_j \in C_F(G)$ such that

$$\| (1 \otimes f) \delta(\nu(b)) - \sum_{j=1}^{n} \delta(\delta_{\nu}(b_j))(1 \otimes f_j) \| < \epsilon / (\|\delta_{\nu}(a)\|_A \cdot \|\varphi(g)\|_{C^*(G/H)}) .$$

If we let $\omega = w$ ($w$ as above), then we have that

$$\| \delta(\delta_{\nu}(a))(1 \otimes f) \delta(\nu(b))(1 \otimes \varphi(g)) - \sum_{j=1}^{n} \delta(\delta_{\nu}(a)\delta_{\nu}(b_j))(1 \otimes (f_j \cdot \varphi(g))) \|$$

$$= \| \delta(\delta_{\nu}(a))(1 \otimes f) \delta(\nu(b))(1 \otimes \varphi(g)) - \sum_{j=1}^{n} \delta(\delta_{\nu}(a)\delta_{\nu}(b_j))(1 \otimes (f_j \cdot \varphi(g))) \|$$

$$= \| \delta(\delta_{\nu}(a))(1 \otimes f \delta(\nu(b)) - \sum_{j=1}^{n} \delta(\nu(b_j))(1 \otimes \varphi(f_j))(1 \otimes \varphi(g)) \|$$

$$< \epsilon ,$$

i.e.,

$$\| \delta(\delta_{\nu}(a))(1 \otimes f) \delta(\nu(b))(1 \otimes \varphi(g)) - \sum_{j=1}^{n} \delta(\delta_{\nu}(d_j))(1 \otimes \xi_j) \| < \epsilon ,$$

where $d_j = \delta_{\nu}(a)\delta_{\nu}(b_j)$ and $\xi_j = f_j \cdot \varphi(g)$ is an element of $C_F(G)$.

So $\delta(\delta_{\nu}(a))(1 \otimes f) \delta(\nu(b))(1 \otimes \varphi(g))$ is $(\omega, F)$ establishing (ii).

**Lemma 4.7** Suppose $E$ is a compact subset of $G$, $u \in A_c(G)$ and $(x_k)_{k=1}^{\infty}$ is a sequence in $D_H$ converging to $x \in B(\mathcal{H} \otimes L^2(G))$ such that each $x_k$ is $(u, E, H)$. Then $x$ is $(u, E, H)$. In particular $x \in D_H$.

**Proof** Let $\epsilon > 0$. Choose $x_k$ such that $\|x_k - x\| < \epsilon / 2$. By assumption there exists $a_j \in A$ and $f_j \in C_D(G)$ such that $\|x_k - \sum_{j=1}^{n} \delta(\delta_{\nu}(a_j))(1 \otimes \varphi(f_j))\| < \epsilon / 2$ so $\|x - \sum_{j=1}^{n} \delta(\delta_{\nu}(a_j))(1 \otimes \varphi(f_j))\| < \epsilon / 2 + \epsilon / 2 = \epsilon$ and $x$ is $(u, E, H)$. 

- 83 -
Lemma 4.8 Suppose $E$ and $F$ are compact subsets of $G$, $u, v \in A_c(G)$, $x$ is $(u, E, H)$ and $y$ is $(v, F, H)$. Then there exists a compact subset $D$ of $G$ and $w \in A_c(G)$, such that $x + y$ is $(w, D, H)$. Hence $D_H$ is closed under addition.

Proof By assumption $x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij}))$

$$y = \lim_{k \to \infty} \sum_{l=1}^{m_k} \delta(\delta_u(b_{kl}))(1 \otimes \varphi(g_{kl}))$$

for some $a_{ij}, b_{kl} \in A, f_{ij} \in C_E(G)$ and $g_{kl} \in C_F(G)$. By the continuity of addition

$$x + y = \lim_{i \to \infty} \left( \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})) + \sum_{l=1}^{n_i} \delta(\delta_u(b_{il}))(1 \otimes \varphi(g_{il})) \right).$$

By lemma 4.3 there exists a compact subset $D$ of $G$ and $w \in A_c(G)$ such that

$$\delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})) + \delta(\delta_u(b_{il}))(1 \otimes \varphi(g_{il}))$$

is $(w, D, H)$ for all $i, j$ and $l$. Since a finite sum of $(w, D, H)$ elements is $(w, D, H)$, each term in the limit is $(w, D, H)$.

So by lemma 4.7 the limit, that is, $x + y$, is $(w, D, H)$.

Lemma 4.9 Suppose $E$ is a compact subset of $G$, $u \in A_c(G)$ and $x$ is $(u, E, H)$. Then there exists a compact subset $F$ of $G$ such that $x^*$ is $(\bar{u}, F, H)$ and hence $D_H$ is closed under adjoints.

Proof By assumption $x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij}))$

for some $a_{ij} \in A, f_{ij} \in C_E(G)$. By the continuity of the adjoint operation

$$x^* = \lim_{i \to \infty} \sum_{j=1}^{n_i} (\delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})))^*.$$

By lemma 4.4 there exists a compact subset $F$ such that $(\delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})))^*$ is $(\bar{u}, F, H)$ for all $i, j, k$ and $l$. Since a finite sum of $(\bar{u}, F, H)$ elements is $(\bar{u}, F, H)$,
Lemma 4.10 Suppose $E$ is a compact subset of $G$, $v \in A_c(G)$, $y$ is $(v, E, H)$ and $x \in D_H$. Then there exists $w \in A_c(G)$, such that $xy$ is $(w, E, H)$ and hence $D_H$ is closed under multiplication.

Proof By assumption $x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta_{u}(a_{ij})(1 \otimes \varphi(f_{ij}))$

$y = \lim_{k \to \infty} \sum_{l=1}^{m_k} \delta_{v}(b_{kl})(1 \otimes \varphi(g_{kl}))$

for some $a_{ij}, b_{kl} \in A, f_{ij} \in C_c(G)$ and $g_{kl} \in C_E(G)$. By the continuity of multiplication

$$xy = \lim_{i \to \infty} \sum_{j,l} \delta_{u}(a_{ij})(1 \otimes \varphi(f_{ij}))\delta_{v}(b_{il})(1 \otimes \varphi(g_{il})) .$$

By lemma 4.5 there exists $w \in A_c(G)$ such that

$\delta_{u}(a_{ij})(1 \otimes \varphi(f_{ij}))\delta_{v}(b_{il})(1 \otimes \varphi(g_{il}))$ is $(w, E, H)$ for all $i, j$ and $l$. Since a finite sum of $(w, E, H)$ elements is $(w, E, H)$, each term in the limit is $(w, E, H)$. So by lemma 4.7 the limit, that is, $xy$, is $(w, E, H)$. 

Lemma 4.11 Suppose $E$ is a compact subset of $G$, $v \in A_c(G)$, $z$ is $(v, E)$ and $x \in D_H$. Then

$(i)$ there exists $w \in A_c(G)$, such that $xz$ is $(w, E)$ and hence $xz \in D$,

$(ii)$ there exists a compact subset $F$ of $G$ and $\omega \in A_c(G)$ such that $xz$ is $(\omega, F)$ and hence $xz \in D$.

Proof By assumption $x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta_{u}(a_{ij})(1 \otimes \varphi(f_{ij}))$
for some \(a_{ij}, b_{kl} \in A, u \in A_c(G), f_{ij} \in C_c(G)\) and \(g_{kl} \in C_E(G)\).

By the continuity of multiplication

\[
xz = \lim_{i \to \infty} \sum_{j, l} \delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij}))\delta(\delta_v(b_{il}))(1 \otimes g_{il}) .
\]

By lemma 4.6 there exists \(w \in A_c(G)\) such that

\[
\delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij}))\delta(\delta_v(b_{il}))(1 \otimes g_{il}) \text{ is } (w, E)
\]

for all \(i, j\) and \(l\). Since a finite sum of \((w, E)\) elements is \((w, E)\), each term in the limit is \((w, E)\). So by lemma 4.7 (with \(H\) the trivial subgroup) the limit, that is, \(xz\), is \((w, E)\).

By the continuity of multiplication

\[
xx = \lim_{i \to \infty} \sum_{j, l} \delta(\delta_u(b_{il}))(1 \otimes g_{il})\delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij})) .
\]

By lemma 4.6 there exists a compact subset \(F\) of \(G\) and \(w \in A_c(G)\) such that \(\delta(\delta_v(b_{il}))(1 \otimes g_{il})\delta(\delta_u(a_{ij}))(1 \otimes \varphi(f_{ij}))\) is \((w, F)\) for all \(i, j\) and \(l\). Since a finite sum of \((w, F)\) elements is \((w, F)\), each term in the limit is \((w, F)\). So by lemma 4.7 (with \(H\) the trivial subgroup) the limit, that is, \(xx\), is \((w, F)\). □

**Theorem 4.12** Suppose \(\delta : A \to \tilde{M}(A \otimes C_r^*(G))\) is a nondegenerate coaction of \(G\) on \(A\), and \(H\) is a closed subgroup of \(G\). Then \(D_H\) is a dense \(*\)-subalgebra of \(A \times_\delta (G/H)\).

**Proof** Lemmas 4.8 through 4.10 show \(D_H\) is closed under the algebraic operations and involution so it remains to show \(D_H\) is dense in \(A \times_\delta (G/H)\). Firstly we note that

\[
A = \{\delta_u(a) : a \in A, u \in A_c(G)\}
\]
is dense in A. To see this note that $A_c(G)$ is dense in $A(G)$ by §1.6. This and the fact that

$$\|\delta_u(a) - \delta_v(a)\| = \|\delta_{(u-v)}(a)\| \leq \|u - v\|_{B(G)} \cdot \|a\|_A$$

show that $\mathcal{A}$ is dense in $\{\delta_u(a) : a \in A, u \in A(G)\}$, which, since $\delta$ is a non-degenerate coaction, is dense in $A$ by [11 thm. 5]. So $\mathcal{A}$ is dense in $A$ as claimed. Now

$$\{\delta(a)(1 \otimes F) : a \in A, F \in C_0(G/H)\}$$

⊂ closure $\{\delta(a)(1 \otimes F) : a \in A, F \in C_c(G/H)\}$

⊂ closure $\{\delta(\delta_u(a))(1 \otimes \varphi(f)) : u \in A_c(G), a \in A, f \in C_c(G)\}$

(since $\varphi(C_c(G)) = C_c(G/H)$ and $A$ is dense in $A$)

⊂ closure $(\mathcal{D}_H)$

⊂ $A \times_\delta (G/H)$.

But the elements $\delta(a)(1 \otimes F)$ with $a \in A$ and $F \in C_0(G/H)$ generate $A \times_\delta (G/H)$. So the closure of $\mathcal{D}_H$ is $A \times_\delta (G/H)$ i.e. $\mathcal{D}_H$ is dense in $A \times_\delta (G/H)$.

**Definitions** We will denote by $\mathcal{I}_H$ the functions $\xi : H \to \mathcal{D}$ such that

(i) $\xi$ is continuous in the norm topology on $\mathcal{D} \subset B(\mathcal{H} \otimes L^2(G))$,

(ii) $\xi$ is compactly supported,

(iii) $\xi(h)$ is $(u, E)$ for all $h \in H$ for some fixed compact subset $E$ of $G$ and fixed $u \in A_c(G)$.

We will say an element $\xi \in \mathcal{I}_H$ is $(u, E)$ if $\xi(h)$ is $(u, E)$ for all $h \in H$.

**Lemma 4.13** Let $\xi \in \mathcal{I}_H$. Suppose $\xi$ is $(u, E)$. Then $\int_H \xi(h) \, dh$ is $(u, E)$ and hence is in $\mathcal{D}$.
Proof Since $\xi$ is continuous and compactly supported we can find $h_{ij} \in \text{supp}\xi$, $\gamma_{ij} \in C_c(H)$ such that

$$\xi = \text{uniform limit} \sum_{i=1}^{n} \gamma_{ij} \xi(h_{ij}) .$$

Hence

$$\int_H \xi(h) \, dh = \lim_{i \to \infty} \sum_{j=1}^{n} \left( \int_H \gamma_{ij}(h) \, dh \right) \xi(h_{ij}) .$$

Now since $\xi$ is $(u, E)$ each term in the limit is $(u, E)$ so $\int_H \xi(h) \, dh$ is $(u, E)$ by lemma 4.7 and hence $\int_H \xi(h) \, dh$ is an element of $D$.

Lemma 4.14 Suppose $E$ is a compact subset of $G$, $F$ is a compact subset of $H$, $u \in A_c(G)$ and $x$ is $(u, E)$. Then there exists a compact subset $D$ of $G$ such that $\hat{\delta}_h(x)$ is $(u, D)$ for all $h \in F$, where $\hat{\delta}$ is the dual action (page 40) of $G$ on $A \times_{\delta} G$ (restricted to $H$).

Proof Firstly we show that $\hat{\delta}_h(\delta(b)) = \delta(b)$ and $\hat{\delta}_h(1 \otimes f) = 1 \otimes f^h$ for all $b \in A$, $h \in H$ and $f \in C_c(G)$. To see the first equation write

$$\delta(b) = \text{strict limit} \sum_{j=1}^{n_i} b_{ij} \otimes \lambda_G(g_{ij}) ,$$

then

$$\hat{\delta}_h(\delta(b)) = \text{strict limit} \sum_{j=1}^{n_i} b_{ij} \otimes (\rho_G(h)\lambda_G(g_{ij})\rho_G^*(h))$$

$$= \text{strict limit} \sum_{j=1}^{n_i} b_{ij} \otimes \lambda_G(g_{ij})$$

$$= \delta(b) .$$

To see the second equation note that

$$\hat{\delta}_h(1 \otimes M_G(f)) = 1 \otimes (\rho_G(h)M_G(f)\rho_G^*(h)) = 1 \otimes M_G(f^h) .$$
Now by assumption \( x = \lim_{i \to \infty} \sum_{j=1}^{n} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) \)

for some \( a_{ij} \in A \) and \( f_{ij} \in C_E(G) \). By the continuity of \( \hat{\delta}_h \) and the above we have that

\[
\hat{\delta}_h(x) = \lim_{i \to \infty} \sum_{j=1}^{n} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij})
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{n} \delta(\delta_u(a_{ij}))(1 \otimes f^h_{ij}) ,
\]

where \( \text{supp} f^h_{ij} \subset E \cdot F^{-1} \) for all \( i \) and \( j \), hence \( \hat{\delta}_h(x) \) is \((u, E \cdot F^{-1})\) for all \( h \in F \).
So choose \( D \) to be \( E \cdot F^{-1} \).

**Proposition 4.15** Suppose \( \delta : A \to \tilde{M}(A \otimes C^*_c(G)) \) is a non-degenerate coaction of \( G \) on \( A \) and \( H \) is a closed subgroup of \( G \). Then \( \mathcal{I}_H \) is a dense \(*\)-subalgebra of \((A \times_\delta G) \times_\delta H \).

**Proof** Let \( \xi, \gamma \in \mathcal{I}_H \) with \( \xi \) being \((u, E)\) and \( \gamma \) being \((v, F)\) where \( E, F \) are compact subsets of \( G \) and \( u, v \in A_c(G) \).

(i) Now \( \xi(h) \) is \((u, E)\) and \( \gamma(h) \) is \((v, F)\) for all \( h \in H \), so by lemma 4.8 there exists a compact subset \( D \) of \( G \) and \( w \in A_c(G) \) such that \((\xi + \gamma)(h) \) is \((w, D)\) for all \( h \in H \), that is, \( \xi + \gamma = (w, D) \). Clearly \( \xi + \gamma \) is continuous and compactly supported so \( \xi + \gamma \in \mathcal{I}_H \) and \( \mathcal{I}_H \) is closed under addition.

(ii) Now \( \xi(h) \) is \((u, E)\) for all \( h \in H \), so by lemma 4.9 there exists a compact subset \( D \) of \( G \) such that \( \xi(h^{-1})^* \) is \((\bar{u}, D)\) for all \( h \in H \). By lemma 4.14 there exists a compact subset \( L \) of \( G \) such that \( \xi^*(h) = (1/\Delta h)\hat{\delta}_h(\xi(h^{-1})^*) \) is \((\bar{u}, L)\) for all \( h \in (\text{supp} \xi)^{-1} \), hence for all \( h \in H \). Now \( \xi^* \) is continuous and compactly supported so \( \xi^* \in \mathcal{I}_H \) and \( \mathcal{I}_H \) is self adjoint.
(iii) Now $\xi(h)$ is $(u, E)$ and $\gamma(h)$ is $(v, F)$ for all $h \in H$, so by lemma 4.14 there exists a compact subset $D$ of $G$ and $w \in A_c(G)$ such that $\delta_r(\gamma(r^{-1}h))$ is $(v, D)$ for all $r \in \text{supp} \gamma$, $h \in H$, and hence for all $r, h \in H$. By lemma 4.10 there exists $w \in A_c(G)$ such that $\xi(r) \delta_r(\gamma(r^{-1}h))$ is $(w, D)$ for all $r, h \in H$. So by lemma 4.13 $(\xi \ast \gamma)(h) = \int_H \xi(r) \delta_r(\gamma(r^{-1}h)) \, dr$ is $(w, D)$ for all $h \in H$. Now $\xi \ast \gamma$ is continuous and compactly supported so $\xi \ast \gamma \in I_H$ and $I_H$ is closed under multiplication.

(iv) Let $x \in D$. Then the maps $h \mapsto \eta(h)x$, where $\eta \in C_c(H)$, are in $I_H$. So $I_H$ is dense in $C_c(H, A \times_\delta G)$ and hence in $(A \times_\delta G) \times_\delta H$.
Chapter 5. Induced Representations of Crossed Products by Coactions.

In this chapter we show how representations of $A \times \delta G$ can be constructed from those of $A \times \delta (G/H)$. We will achieve this by showing that $D$ is a pre-Hermitian $D_H$-rigged $D$-module, to which we can apply Rieffel’s theory on the induction of representations of $C^*$-algebras [22] (see also chapter 1 §4).

Throughout this chapter $H$ will be a closed normal amenable subgroup of $G$, $\delta$ will be a non-degenerate coaction of $G$ on $A$ and $\pi$ will be a fixed faithful representation of $A$ on the Hilbert space $\mathcal{H}$.

Our immediate goal is to show that $D$ is a $D_H$-rigged space.

**Definition** Define a right action of $D_H$ on $D$ by

$$x \cdot z = xz \quad x \in D, \quad z \in D_H,$$

Note that the action is well defined by lemma 4.11.

In order to define a $D_H$-valued inner product on $D$ we need to establish the following results.

**Lemma 5.1** Let $E$ be a compact subset of $G$. Let $\omega_E \in C^+_c(G)$ be such that $\omega_E$ is identically one on $E$. Then

(i) there exists a positive constant $\alpha_E$ such that if $\xi \in C_E(G)$, then

$$\|\varphi(\xi)\|_{L^2(G/H)} \leq \alpha_E \cdot \|\xi\|_{L^2(G)},$$

(ii) there exists a positive constant $\beta_E$ such that if $\eta \in C_c(G/H)$, then

$$\|\omega_E \cdot \eta\|_{L^2(G)} \leq \beta_E \cdot \|\eta\|_{L^2(G/H)}.$$
Proof (i) Note that by Hölders inequality

\[ \int_H |\xi(sh)| \, dh = \sqrt{\int_H \omega_E(sh)|\xi(sh)| \, dh} \]

\[ \leq \left( \int_H \omega_E(sh)^2 \, dh \right)^{\frac{1}{2}} \cdot \left( \int_H |\xi(sh)|^2 \, dh \right)^{\frac{1}{2}}. \]

Now \( \omega_E^2 \in C_c^+(G) \) so, \( \varphi(\omega_E^2) \in C_c^+(G/H) \) and is thus bounded by say \( m \). Hence

\[ \int_H |\xi(sh)| \, dh \leq m^{\frac{1}{2}} \cdot \left( \int_H |\xi(sh)|^2 \, dh \right)^{\frac{1}{2}}. \]

Now

\[ \|\varphi(\xi)\|^2_{L^2(G/H)} = \int_{G/H} \left( \int_H |\xi(sh)| \, dh \right)^2 \, dsH \]

\[ \leq \int_{G/H} \left( \int_H |\xi(sh)| \, dh \right)^2 \, dsH \]

\[ \leq m \cdot \int_{G/H} \int_H |\xi(sh)|^2 \, dh \, dsH \]

(by the above)

\[ \quad = m \cdot \int_G |\xi(s)|^2 \, ds \]

\[ \quad = m \cdot \|\xi\|_{L^2(G)}. \]

So choose \( \alpha_E = m^{\frac{1}{2}} \).

(ii) \( \|\omega_E \cdot \eta\|^2_{L^2(G)} = \int_G \omega_E(s)^2 |\eta(s)|^2 \, ds \)

\[ = \int_{G/H} \int_H \omega_E^2(sh)|\eta(sh)|^2 \, dh \, dsH \]

\[ = \int_{G/H} \left( \int_H \omega_E^2(sh) \, dh \right)|\eta(s)|^2 \, dsH \]

\[ \leq m \cdot \int_{G/H} |\eta(sH)|^2 \, dsH \]

(m as above)
So choose $\beta_E = m^{\frac{1}{4}}$. 

Lemma 5.2 Suppose $H$ is a Hilbert space, $\gamma_i \in H$ for $i = 1, \ldots, n$, $E$ is a compact subset of $G$ and $\omega_E \in C_c^+(G)$ is identically one on $E$. Then

(i) there exists a positive constant $\alpha_E$ such that if $\xi_i \in C_E(G)$ for $i = 1, \ldots, n$, then

$$\left\| \sum_{i=1}^{n} \gamma_i \otimes \varphi(\xi_i) \right\|_{H \otimes L^2(G/H)} \leq \alpha_E \cdot \left\| \sum_{i=1}^{n} \gamma_i \otimes \xi_i \right\|_{H \otimes L^2(G)}.$$ 

(ii) there exists a positive constant $\beta_E$ such that if $\eta_i \in C_c(G/H)$ for $i = 1, \ldots, n$, then

$$\left\| \sum_{i=1}^{n} \gamma_i \otimes (\omega_E \cdot \eta_i) \right\|_{H \otimes L^2(G/H)} \leq \beta_E \cdot \left\| \sum_{i=1}^{n} \gamma_i \otimes \eta_i \right\|_{H \otimes L^2(G/H)}.$$ 

Proof Let $(\varepsilon_j)_{j \in J}$ be an orthonormal basis of $H$. Let

$$\mathcal{F} = \left\{ \sum_{j \in J} \nu_j \varepsilon_j : \nu_j \in \mathcal{C} \text{ with all but finitely many } \nu_j = 0 \right\}.$$ 

If $\gamma_i = \sum \nu_j \nu_{i,j} \varepsilon_j \in \mathcal{F}$ and $\xi_i \in C_E(G)$ for $i = 1, \ldots, n$, then

$$\left\| \sum_{i=1}^{n} \gamma_i \otimes \varphi(\xi_i) \right\|_{H \otimes L^2(G/H)}^2 = \left\| \sum_{i=1}^{n} \left( \sum_{j \in J} \nu_j \nu_{i,j} \right) \otimes \varphi(\xi_i) \right\|_{H \otimes L^2(G/H)}^2$$

$$= \left\| \sum_{j \in J} \varepsilon_j \otimes \varphi(\sum_{i=1}^{n} \nu_{i,j} \xi_i) \right\|_{H \otimes L^2(G/H)}^2$$

$$= \sum_{j \in J} \left\| \varphi(\sum_{i=1}^{n} \nu_{i,j} \xi_i) \right\|_{H \otimes L^2(G/H)}^2$$

(by the orthonormality of the $\varepsilon_j$)
\[
\leq \sum_{j \in J} \alpha_E^2 \cdot \left\| \sum_{i=1}^{n} \nu_{ij} \xi_i \right\|_{\mathcal{H} \otimes L^2(G)}^2
\]
(by lemma 5.1 since \( \sum \nu_{ij} \in C_E(G) \))

\[
= \alpha_E^2 \cdot \sum_{j \in J} \left\| \left( \sum_{i=1}^{n} \nu_{ij} \xi_i \right) \right\|_{\mathcal{H} \otimes L^2(G)}^2
\]

\[
= \alpha_E^2 \cdot \left\| \sum_{j \in J} \epsilon_j \otimes \left( \sum_{i=1}^{n} \nu_{ij} \xi_i \right) \right\|_{\mathcal{H} \otimes L^2(G)}^2
\]
(by the orthonormality of the \( \epsilon_j \)). Let \( \gamma_i \in \mathcal{H} \). Now \( \mathcal{F} \) is dense in \( \mathcal{H} \) so there exists \( \gamma_{ik} \in \mathcal{F} \) such that \( \gamma_{ik} \to \gamma_i \). Since

\[
\left\| \sum_{i=1}^{n} \gamma_{ik} \otimes \xi_i - \sum_{i=1}^{n} \gamma_i \otimes \xi_i \right\| \leq \sum_{i=1}^{n} \left\| \gamma_{ik} - \gamma_i \right\| \cdot \left\| \xi_i \right\|
\]
we have that \( \sum \gamma_{ik} \otimes \xi_i \to \sum \gamma_i \otimes \xi_i \). Similarly \( \sum \gamma_{ik} \otimes \varphi(\xi_i) \to \sum \gamma_i \otimes \varphi(\xi_i) \). So

\[
\left\| \sum_{i=1}^{n} \gamma_i \otimes \varphi(\xi_i) \right\| = \lim_{k \to \infty} \left\| \sum_{i=1}^{n} \gamma_{ik} \otimes \varphi(\xi_i) \right\|
\]

\[
\leq \alpha_E \cdot \lim_{k \to \infty} \left\| \sum_{i=1}^{n} \gamma_{ik} \otimes \xi_i \right\|
\]

\[
= \alpha_E \cdot \left\| \sum_{i=1}^{n} \gamma_i \otimes \xi_i \right\|
\]
as required. A similar argument gives (ii).

Lemma 5.3 Suppose \( \xi \in C_c(G/H) \), \( E \) is a compact subset of \( G \) and \( f \in C_E(G) \). Let \( \omega_E \in C_c^+(G) \) be identically one on \( E \). Then

\[
\{ \lambda_{G/H}(\varphi g)M_{G/H}(\varphi f) \}(\xi) = \varphi(\{ \lambda_{G}(g)M_{G}(f) \}(\omega_E \cdot \xi)) .
\]

Proof First we note that if \( d, e \in C_c(G) \) and \( \Phi \) is as in lemma 2.2, then
\[
\lambda_{G/H}(\varphi(d * e)) = \Phi(\lambda_G(d * e))
\]
(by lemma 2.2)
\[
= \Phi(\lambda_G(d)) \ast \Phi(\lambda_G(e))
\]
\[
= \lambda_{G/H}((\varphi d) \ast (\varphi e))
\]
which implies \(\varphi(d * e) = (\varphi d) \ast (\varphi e)\). Now
\[
\{\lambda_{G/H}(\varphi g)M_{G/H}(\varphi f)\}(\xi) = (\varphi g) \ast (\varphi f \cdot \xi)
\]
\[
= (\varphi g) \ast (\varphi(f \cdot \xi))
\]
(since \(\xi \in C_c(G/H)\))
\[
= \varphi(g \ast (f \cdot \xi))
\]
(by the above)
\[
= \varphi(g \ast (f \cdot \omega_E \cdot \xi))
\]
(since \(\text{supp} f \subset E\))
\[
= \varphi(\{\lambda_G(g)M_G(f)\}(\omega_E \cdot \xi))
\]
\[
\]
\[
\]
Lemma 5.4 Suppose \(E\) is a compact subset of \(G\) and \(v \in A_c(G)\). Then there exists a positive constant \(\zeta_{v,E}\) such that if \(a_i \in A\) and \(f_i \in C_E(G)\) for \(i = 1, \ldots, n\), then
\[
\left\| \sum_{i=1}^{n} \delta(v(a_i))(1 \otimes \varphi(f_i)) \right\|_{A_{\lambda_4}(G/H)} \leq \zeta_{v,E} \cdot \left\| \sum_{i=1}^{n} \delta(v(a_i))(1 \otimes f_i) \right\|_{A_{\lambda_4}G}
\]
\[
\]
Proof By lemma 1.22 we can find \(a_{ij}^k \in A\) and \(g_{ij}^k \in C_c(G)\) with \(\text{supp} g_{ij}^k \subset \text{supp} v\) such that
\[
\delta(v(a_i)) = \text{strict limit}_{k \to \infty} \sum_{j=1}^{m_{ij}} a_{ij}^k \otimes \lambda_G(g_{ij}^k) \text{ in } M(A \otimes C^*_r(G))
\]
and hence

\[(\pi \otimes i)(\delta(\delta_v(a_i))) = \text{strong limit} \sum_{j=1}^{m_{ij}} \pi(a_{ij}) \otimes \lambda_G(g_{ij}^k) \text{ in } B(\mathcal{H} \otimes L^2(G)) \quad (1)\]

\[(\pi \otimes i)(\delta(\delta_v(a_i))) = \text{strong limit} \sum_{j=1}^{m_{ij}} \pi(a_{ij}) \otimes \lambda_{G/H}(\varphi(g_{ij}^k)) \text{ in } B(\mathcal{H} \otimes L^2(G/H)) \quad (2)\]

Let \(\sum_{i=1}^p \gamma_i \otimes \xi_i \in \mathcal{H} \otimes C_c(G/H)\). Then

\[\left\| \left\{ \sum_{i=1}^n (\pi \otimes i)(\delta(\delta_v(a_i)))(1 \otimes M_{G/H}(\varphi(f_i))) \right\} \left( \sum_{l=1}^p \gamma_l \otimes \xi_l \right) \right\|_{\mathcal{H} \otimes L^2(G/H)}
\]

(by \(\mathcal{H}

\[= \lim_{k \to \infty} \left\| \left\{ \sum_{i,j} \pi(a_{ij}) \otimes (\lambda_{G/H}(\varphi(g_{ij}^k))M_{G/H}(\varphi(f_i))) \right\} \left( \sum_{l=1}^p \gamma_l \otimes \xi_l \right) \right\|_{\mathcal{H} \otimes L^2(G/H)}\]

(by lemma 5.3 where \(\omega_E \in C_c^+(G)\) is identically one on \(E\))

\[\leq \alpha_F \cdot \lim_{k \to \infty} \left\| \left\{ \sum_{i,j} \pi(a_{ij}) \otimes (\lambda_{G/H}(\varphi(g_{ij}^k))M_{G/H}(f_i)) \right\} \left( \sum_{l=1}^p \gamma_l \otimes (\omega_E \cdot \xi_l) \right) \right\|_{\mathcal{H} \otimes L^2(G)}\]

(where \(F\) is the compact set \((\text{supp} \nu) \cdot E\) and \(\alpha_F\) is the positive constant given by lemma 5.2 (i). Note that lemma 5.2 applies since for all \(i, j, k\) and \(l\) the support of \(\{\lambda_{G/H}(\varphi(g_{ij}^k))M_{G/H}(f_i)\}(\omega_E \cdot \xi_l)\) is contained in \(F\))

\[= \alpha_F \cdot \left\| \left\{ \sum_{i=1}^n (\pi \otimes i)(\delta(\delta_v(a_i)))(1 \otimes M_{G/H}(f_i)) \right\} \left( \sum_{l=1}^p \gamma_l \otimes (\omega_E \cdot \xi_l) \right) \right\|_{\mathcal{H} \otimes L^2(G)}\]

(by \(\mathcal{H}

\[\left\| \sum_{i=1}^n \delta(\delta_v(a_i))(1 \otimes \varphi(f_i)) \right\|_{A \times \delta(G/H)}
\]

\[= \left\| \sum_{i=1}^n (\pi \otimes i)(\delta(\delta_v(a_i)))(1 \otimes M_{G/H}(\varphi(f_i))) \right\|_{B(\mathcal{H} \otimes L^2(G/H))}\]

- 96 -
\[
\begin{align*}
&= \sup \left\{ \left\| \sum_{i=1}^{n} (\pi \otimes i)(\delta(\delta_v(a_i))) \left( M_G(H) \varphi(f_i) \right) \right\|_{H \otimes L^2(G/H)} \right. \\
&\quad \left. \left\| \varphi \right\|_{H \otimes L^2(G/H)} \right. \\
&\quad \left. : \varphi \in H \otimes L^2(G/H) \right. \\
&= \sup \left\{ \left\| \sum_{i=1}^{n} (\pi \otimes i)(\delta(\delta_v(a_i))) \left( M_G(H) \varphi(f_i) \right) \right\|_{H \otimes L^2(G/H)} \right. \\
&\quad \left. \left\| \sum_{i=1}^{p} \gamma_i \otimes \xi_i \right\|_{H \otimes L^2(G/H)} \right. \\
&\quad \left. : \sum_{i=1}^{p} \gamma_i \otimes \xi_i \in H \otimes L^2(G/H), \quad \xi_i \in C_c(G/H) \right. \\
&\quad \left. : \sum_{i=1}^{p} \gamma_i \otimes \xi_i \in H \otimes L^2(G/H), \quad \xi_i \in C_c(G/H) \right. \\
&\quad \left. \text{(since the elements } \sum \gamma_i \otimes \xi_i \text{ with } \gamma_i \in C_c(G/H) \text{ are dense in } H \otimes L^2(G/H) \right. \\
&\quad \left. \right. \right. \\
&\leq \alpha_F \sup \left\{ \left\| \sum_{i=1}^{n} (\pi \otimes i)(\delta(\delta_v(a_i))) \left( M_G(H) \varphi(f_i) \right) \right\|_{H \otimes L^2(G)} \right. \\
&\quad \left. \left\| \sum_{i=1}^{p} \gamma_i \otimes \omega \right\|_{H \otimes L^2(G/H)} \right. \\
&\quad \left. : \sum_{i=1}^{p} \gamma_i \otimes \omega \in H \otimes L^2(G), \quad \omega \in C_c(G/H) \right. \\
&\quad \left. : \sum_{i=1}^{p} \gamma_i \otimes \omega \in H \otimes L^2(G), \quad \omega \in C_c(G/H) \right. \\
&\quad \left. \text{(by the above)} \right. \\
&\leq \alpha_F \beta \sup \left\{ \left\| \sum_{i=1}^{n} (\pi \otimes i)(\delta(\delta_v(a_i))) \left( M_G(H) \varphi(f_i) \right) \right\|_{H \otimes L^2(G)} \right. \\
&\quad \left. \left\| \sum_{i=1}^{p} \gamma_i \otimes \omega \right\|_{H \otimes L^2(G/H)} \right. \\
&\quad \left. : \sum_{i=1}^{p} \gamma_i \otimes \omega \in H \otimes L^2(G), \quad \omega \in C_c(G/H) \right. \\
&\quad \left. : \sum_{i=1}^{p} \gamma_i \otimes \omega \in H \otimes L^2(G), \quad \omega \in C_c(G/H) \right. \\
\end{align*}
\]
(where $\beta_E$ is the constant of lemma 5.2 (ii))

\[
\leq \alpha F \beta_E \sup \left\{ \frac{\left\| \left\{ \sum_{i=1}^{n} (\pi \otimes i)(\delta_v(a_i))(1 \otimes M_G(f_i)) \right\}(\varpi) \right\|_{\mathcal{H} \otimes L^2(G)}}{\| \varpi \|_{\mathcal{H} \otimes L^2(G)}} \right\}
\]

: $\varpi \in \mathcal{H} \otimes L^2(G)$

\[
= \alpha F \beta_E \cdot \left\| \sum_{i=1}^{n} (\pi \otimes i)(\delta_v(a_i))(1 \otimes M_G(f_i)) \right\|_{B(\mathcal{H} \otimes L^2(G))}
\]

\[
= \alpha F \beta_E \cdot \left\| \sum_{i=1}^{n} \delta_v(a_i)(1 \otimes f_i) \right\|_{A_c(G)}.
\]

So choose $\zeta_{v,E} = \alpha F \beta_E$. 

\[
: \gamma \in \mathcal{H}, \xi_i \in C_c(G/H)
\]

Proposition 5.5 The map $\Psi : \mathcal{D} \rightarrow \mathcal{D}_H$ given by

\[
\lim_{i \rightarrow \infty} \sum_{j=1}^{n_i} \delta_u(a_{ij})(1 \otimes f_{ij}) \rightarrow \lim_{i \rightarrow \infty} \sum_{j=1}^{n_i} \delta_u(a_{ij})(1 \otimes \varphi(f_{ij}))
\]

is well defined.

Proof Let $x \in \mathcal{D}$. By definition there exists a compact subset $E$ of $G$ and $v \in A_c(G)$ such that

\[
x = \lim_{i \rightarrow \infty} \sum_{j=1}^{n_i} \delta_u(a_{ij})(1 \otimes f_{ij}),
\]
where \( a_{ij} \in A \) and \( f_{ij} \in C_E(G) \). Now

\[
\left\| \sum_{j=1}^{n_i} \delta(u(a_{ij}))(1 \otimes \varphi(f_{ij})) - \sum_{j=1}^{n_k} \delta(u(a_{kj}))(1 \otimes \varphi(f_{kj})) \right\|_{A \times_{\delta}(G/H)}
\]

\[
= \left\| \sum_{j=1}^{n_i} \delta(u(a_{ij}))(1 \otimes \varphi(f_{ij})) - \sum_{j=1}^{n_k} \delta(u(a_{kj}))(1 \otimes \varphi(f_{kj})) \right\|_{A \times_{\delta}(G/H)}
\]

(by proposition 2.6)

\[
\leq \zeta_{u,E} \cdot \left\| \sum_{j=1}^{n_i} \delta(u(a_{ij}))(1 \otimes f_{ij}) - \sum_{j=1}^{n_k} \delta(u(a_{kj}))(1 \otimes f_{kj}) \right\|_{A \times_{\delta} G}
\]

(by lemma 5.4). So the sequence \( \left( \sum_{j=1}^{n_i} \delta(u(a_{ij}))(1 \otimes f_{ij}) \right)_{i=1}^{\infty} \) converges.

To see that \( \Psi(x) \) does not depend on the way we express \( x \) as a limit suppose that

\[
x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(u(a_{ij}))(1 \otimes f_{ij}) = \lim_{l \to \infty} \sum_{k=1}^{m_l} \delta(v(b_{kl}))(1 \otimes g_{kl})
\]

where \( a_{ij}, b_{kl} \in A, u, v \in A_c(G), f_{ij} \in C_D(G) \) and \( g_{kl} \in C_E(G) \) for fixed compact sets \( D \) and \( E \) of \( G \). Then by lemma 4.8 we can find \( \omega \in A_c(G) \) and a compact subset \( F \) of \( G \) such that

\[
\sum_{j=1}^{n_i} \delta(u(a_{ij}))(1 \otimes f_{ij}) - \sum_{k=1}^{m_l} \delta(v(b_{kl}))(1 \otimes g_{kl})
\]

is \( (\omega, F) \) for all \( i \) and \( l \). So by proposition 2.6 and lemma 5.4 we have

\[
\left\| \sum_{j=1}^{n_i} \delta(u(a_{ij}))(1 \otimes \varphi(f_{ij})) - \sum_{k=1}^{m_l} \delta(v(b_{kl}))(1 \otimes \varphi(g_{kl})) \right\|_{A \times_{\delta}(G/H)}
\]

\[
= \left\| \sum_{j=1}^{n_i} \delta(u(a_{ij}))(1 \otimes \varphi(f_{ij})) - \sum_{k=1}^{m_l} \delta(v(b_{kl}))(1 \otimes \varphi(g_{kl})) \right\|_{A \times_{\delta}(G/H)}
\]

\[
\leq \zeta_{\omega,F} \cdot \left\| \sum_{j=1}^{n_i} \delta(u(a_{ij}))(1 \otimes f_{ij}) - \sum_{k=1}^{m_l} \delta(v(b_{kl}))(1 \otimes g_{kl}) \right\|_{A \times_{\delta} G}
\]
for all $i$ and $l$. So $\Psi(x)$ is independent of how we express $x$. 

With $\Psi$ well defined we can define a $\mathcal{D}_H$-valued sesquilinear map $\langle \cdot, \cdot \rangle_D$ on $\mathcal{D}$ by 

$$\langle w, y \rangle_D = \Psi(w^* y).$$

To see that $\langle \cdot, \cdot \rangle_D$ is a pre-$\mathcal{D}_H$-valued inner product we need to establish a few more lemmas.

**Lemma 5.6** Let $x, y \in \mathcal{D}$. Then the map 

$$: h \mapsto x^\delta_h(y) : H \rightarrow B(\mathcal{H} \otimes L^2(G))$$

is norm continuous of compact support, and

$$x \Psi(y) = \int_H x^\delta_h(y) \, dh$$

$$\Psi(y) x = \int_H \delta_h(y)x \, dh .$$

**Proof** Firstly we show that if $f \in C_c(G)$, then the map 

$$: h \mapsto \rho_H(h)M_G(f)\rho_H^*(h) : H \rightarrow B(L^2(G)) ,$$

is integrable with integral $M_G(\varphi(f))$. Note that since

$$\int_G \int_H |f(sh)\xi(s)\overline{\eta(s)}| \, dh \, ds \leq \|\varphi(|f|)\|_{C_c(G/H)} \cdot \|\xi\|_2 \cdot \|\eta\|_2 \leq \infty ,$$

the map $: (h, s) \rightarrow f(sh)\xi(s)\overline{\eta(s)}$ is Lebesgue integrable. Let $\xi, \eta \in L^2(G)$, then

$$\int_H |\omega_{\xi, \eta}(\rho_H(h)M_G(f)\rho_H^*(h))| \, dh$$

$$= \int_H |\langle \{\rho_H(h)M_G(f)\rho_H^*(h)\}\xi, \eta \rangle_{L^2(G)}| \, dh$$

- 100 -
by the above. So the map : \( h \rightarrow \omega_{\xi, \eta}(\rho_H(h)M_G(f)\rho_H^*(h)) \) is Lebesgue integrable. Now

\[
\int_H \langle \{\rho_H(h)M_G(f)\rho_H^*(h)\}(\xi), \eta \rangle_{L^2(G)} \; dh = \int_G \left( \int_H f(s h) \; dh \right) \xi(s)\eta(s) \; ds
\]

\[= \langle \{M_G(\varphi(f))\}(\xi), \eta \rangle_{L^2(G)} .
\]

So : \( h \rightarrow \rho_H(h)M_G(f)\rho_H^*(h) \) is integrable with integral \( M_G(\varphi(f)) \). Now

\[
\int_H \hat{\delta}_h(1 \otimes M_G(f)) \; dh = \int_H 1 \otimes (\rho_H(h)M_G(f)\rho_H^*(h)) \; dh
\]

\[= 1 \otimes \int_H \rho_H(h)M_G(f)\rho_H^*(h) \; dh\]

(by lemma 1.5 with \( \gamma : B(L^2(G)) \rightarrow B(\mathcal{H} \otimes L^2(G)) \) defined by \( \gamma(x) = 1 \otimes x \))

\[= 1 \otimes M_G(\varphi(f)) ,
\]

or in abbreviated notation

\[
\int_H \hat{\delta}_h(1 \otimes f) \; dh = 1 \otimes \varphi(f) . \tag{3}
\]

Now by assumption there exist compact subsets \( E, F \) of \( G \) and \( u, v \in A_c(G) \) such that

\[
x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij})
\]

\[
y = \lim_{k \to \infty} \sum_{l=1}^{m_k} (1 \otimes g_{kl})\delta(\delta_v(b_{kl}))
\]

- 101 -
for some \( a_{ij}, b_{kl} \in A, u, v \in A_c(G), f_{ij} \in C_E(G) \) and \( g_{kl} \in C_F(G) \). So

\[
x \Psi(y) = \lim_{i \to \infty} \sum_{j,l} \delta(\delta_u(a_{ij}))(1 \otimes (f_{ij} \cdot \varphi(g_{il})))\delta(\delta_v(b_{il})))
\]

\[
= \lim_{i \to \infty} \sum_{j,l} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) \int_H \delta_h(1 \otimes g_{il}) \, dh \, \delta(\delta_v(b_{il})))
\text{ (by \S 3)}
\]

\[
= \lim_{i \to \infty} \int_H \sum_{j,l} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) \delta_h(1 \otimes g_{il}) \delta(\delta_v(b_{il}))) \, dh
\]

\[
= \lim_{i \to \infty} \int_H \sum_{j=1}^{n_i} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) \delta_h \left( \sum_{l=1}^{m_i} (1 \otimes g_{il}) \delta(\delta_v(b_{il})) \right) \, dh
\]

(since \( \delta_h(\delta(b_{il})) = \delta(b_{il}) \) by \S 4.5). Now provided we can apply lemma 1.9, this is

\[
= \int_H x \delta_h(y) \, dh,
\]

which will establish the lemma. To see that we may use lemma 1.9, define \( \gamma \) and \( \gamma_i : H \to B(H \otimes L^2(G)) \) by

\[
\gamma_i(h) = \sum_{j,l} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) \delta_h(1 \otimes g_{il}) \delta(\delta_v(b_{il})))
\]

and

\[
\gamma(h) = x \delta_h(y).
\]

Since \( \delta \) is an action of \( H \) on \( B(H \otimes L^2(G)) \) \( \gamma \) and the \( \gamma_i \) are norm continuous. Also since \( (1 \otimes f_{ij}) \delta_h(1 \otimes g_{il}) = 1 \otimes (f_{ij} \cdot g_{il}^h) \) the \( \gamma_i \) are compactly supported in \( (\text{supp} f_{ij})^{-1} \cdot (\text{supp} g_{il}) \subset E \cdot F^{-1} \). Now \( \gamma \), being the pointwise limit of the \( \gamma_i \), also has compact support in \( E \cdot F^{-1} \). Hence \( \gamma \) and the \( \gamma_i \) are integrable. Let \( \xi, \eta \in H \otimes L^2(G) \). If \( \omega_{\xi,\eta} \) is as in \S 1.13, then

\[
|\omega_{\xi,\eta}(\gamma_i(h))| = |\langle \gamma_i(h) \rangle(\xi), \eta \rangle_{H \otimes L^2(G)}| \\
\leq \|\gamma_i(h)\| \cdot \|\xi\| \cdot \|\eta\|
\]

- 102 -
\[
\leq \left\| \sum_{j=1}^{n} \delta(\delta_u(a_{ij}))(1 \otimes f_{ij}) \right\| \cdot \left\| \sum_{i=1}^{m} (1 \otimes f_{ii}) \delta(\delta_u(a_{ii})) \right\| \cdot \|\xi\| \cdot \|\eta\| \cdot \chi_{E \cdot F^{-1}}(h) \\
\leq (\|x\| + 1) \cdot (\|y\| + 1) \cdot \|\xi\| \cdot \|\eta\| \cdot \chi_{E \cdot F^{-1}}(h)
\]

for \(i\) sufficiently large. Since \(E \cdot F^{-1}\) is compact the right hand side is clearly a positive \(L^1(H)\) function as required. This justifies the use of lemma 1.9. The second equation of the lemma follows similarly. \(\square\)

**Lemma 5.7** Let \( (e_i)_{i \in I} \) be an increasing approximate identity of \( A \). Then

(i) \( \delta(e_i) \to 1 \) strictly in \( M(A \times_\delta (G/H)) \),

(ii) there exists a fixed \( \omega \in A_c(G) \) such that \( \delta(\omega(e_i))_{i \in I} \) is an increasing approximate identity of \( A \).

**Proof** Since \( (e_i)_{i \in I} \) is an approximate identity we have

\[ 0 \leq e_i \leq e_j \quad \text{for} \quad i \leq j, \quad \|e_i\| \leq 1 \quad \text{and} \quad e_i \to 1 \text{ strictly in } M(A). \]

(i) Let \( \epsilon > 0 \) and \( z \in A \times_\delta (G/H) \). Then by theorem 4.12 we can find \( a_j \in A \) and \( f_j \in C_c(G) \) such that

\[ \|z - \sum_{j=1}^{n} \delta(a_j)(1 \otimes \varphi(f_j))\| < \epsilon/3. \]

Now choose \( i_0 \) such that \( i \geq i_0 \) implies

\[ \|e_i a_j - a_j\| < \epsilon/(3rn) \quad \text{where} \quad r = \max \|\varphi(f_j)\|_{C_c(G/H)}. \]

Then for \( i \geq i_0 \)

\[
\|\delta(e_i)z - z\| \leq \|\delta(e_i)z - \delta(e_i) \sum_{j=1}^{n} \delta(a_j)(1 \otimes \varphi(f_j))\| \\
+ \|\delta(e_i) \sum_{j=1}^{n} \delta(a_j)(1 \otimes \varphi(f_j)) - \sum_{j=1}^{n} \delta(a_j)(1 \otimes \varphi(f_j))\|
\]

- 103 -
\[ + \left\| \sum_{j=1}^{n} \delta(a_j)(1 \otimes \varphi(f_j)) - z \right\| \]
\[ \leq \|\delta(e_i)\| \cdot \varepsilon/3 + \sum_{j=1}^{n} \|e_i a_j - a_j\| \cdot \|\varphi(f_j)\|_{C_\ast(G/H)} + \varepsilon/3 \]
\[ \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \]
\[ = \varepsilon , \]

i.e. \(\|\delta(e_i)z - z\| \to 0\) as \(i \to \infty\). A similar argument shows \(\|z\delta(e_i) - z\| \to 0\) as \(i \to \infty\). So \(\delta(e_i) \to 1\) strictly in \(M(A \otimes C_\ast(G/H))\).

(ii) Since \(\delta\) is a non-degenerate \(*\)-homomorphism we have that \(\delta(e_i) \to 1\) strictly in \(M(A \otimes C_\ast(G))\). Now choose \(\omega \in A_c(G)\) to be positive definite (i.e. \(\omega\) considered as an element of \(C_\ast(G)^*\) is positive) with \(\omega(1) = \|\omega\|_{B(G)} = 1\). Then

(a) If \(p\) is a positive functional on \(A\) and \(i \leq j\), then
\[ p(\delta(\omega(e_j)) - \delta(\omega(e_i))) = p \otimes \omega(\delta(e_j - e_i)) \geq 0 \]
and since \(p\) was arbitrary \(\delta(\omega(e_i)) \leq \delta(\omega(e_j))\).

(b) By \(\|1.12\|\|\omega(e_i)\|_A \leq \|\omega\|_{B(G)} \cdot \|e_i\|_A \leq 1\).

(c) Let \(a \in A\). By \(\|1.11\|\) we can write \(\omega = b \ast v\) for some \(b \in C_\ast(G), v \in B_r(G)\) and
\[ \delta(\omega(e_i)a = S_\omega(\delta(e_i))a \]
\[ = S_\omega(\delta(e_i)(a \otimes 1)) \]
\[ = S_b \ast v(\delta(e_i)(a \otimes 1)) \]
\[ = S_v(\delta(e_i)(a \otimes b)) \]
\[ \to S_v(a \otimes b) \]
(since \(\delta(e_i) \to 1\) strictly in \(M(A \otimes C_\ast(G))\) and the slice maps are norm continuous)
\[ = S_\omega(a \otimes 1)) = a \omega(1) = a . \]
Similarly $a \delta_{\omega(e_i)} \to a$. Hence $\delta_{\omega(e_i)} \to 1$ strictly in $M(A)$ as required. 

**Lemma 5.8** Let $(\delta_{\omega(e_i)})_{i \in I}$ be an approximate identity of $A$ of the type introduced in lemma 5.7. Let $\mathcal{E}$ (resp. $\mathcal{E}_H$) be the compact subsets of $G$ (resp. $G/H$) ordered by inclusion.

(i) For each $E \in \mathcal{E}_H$ let $\omega_E$ be an element of $C_c^+(G/H)$ with $0 \leq \omega_E \leq 1$ and such that $\omega_E$ is identically one on $E$. If we let

$$z_{(i,E)} = \delta(\omega(e_i))(1 \otimes \omega_E) \quad (i, E) \in I \times \mathcal{E}_H ,$$

then $z_{(i,E)} \to 1$ strictly in $M(A \times \delta(G/H))$.

(ii) For each $E \in \mathcal{E}$ let $\omega_E$ be an element of $C_c^+(G)$ with $0 \leq \omega_E \leq 1$ and such that $\omega_E$ is identically one on $E$. If we let

$$x_{(i,E)} = \delta(\omega(e_i))(1 \otimes \omega_E) \quad (i, E) \in I \times \mathcal{E} ,$$

then $x_{(i,E)} \to 1$ $\ast$-strongly in $B(\mathcal{H} \otimes L^2(G))$.

**Proof** (i) Let $\epsilon > 0$ and $z \in A \times \delta(G/H)$. Then by theorem 4.12 we can find $b_j \in A$, $f_j \in C_c(G)$ such that

$$\left\|z - \sum_{j=1}^{n}(1 \otimes \varphi(f_j))\delta(a_j)\right\| < \epsilon/3 .$$

By lemma 5.7 we can find $i_0$ such that $i \geq i_0$ implies

$$\left\|\delta(\omega(e_i))\sum_{j=1}^{n}(1 \otimes \varphi(f_j))\delta(a_j) - \sum_{j=1}^{n}(1 \otimes \varphi(f_j))\delta(a_j)\right\| < \epsilon/3 .$$

Now note that for $E \geq E_0 = \bigcup_{j=1}^{n}\text{supp}\varphi(f_j)$

$$(1 \otimes \omega_E)\sum_{j=1}^{n}(1 \otimes \varphi(f_j))\delta(a_j) = \sum_{j=1}^{n}(1 \otimes \varphi(f_j))\delta(a_j) .$$
So for \((i, E) \geq (i_0, E_0)\)

\[
\|z_{(i,E)}z - z\| \leq \|\delta(\omega(e_i))(1 \otimes \varphi_E)(z - \delta(\omega(e_i))\sum_{j=1}^{n}(1 \otimes \varphi(f_j))\delta(a_j))\|
\]

\[
+ \|\delta(\omega(e_i))\left(\sum_{j=1}^{n}(1 \otimes \varphi(f_j))\delta(a_j)\right) - \sum_{j=1}^{n}(1 \otimes \varphi(f_j))\delta(a_j)\|
\]

\[
+ \|\sum_{j=1}^{n}(1 \otimes \varphi(f_j))\delta(a_j) - z\|
\]

\[
\leq \epsilon/3 + \epsilon/3 + \epsilon/3
\]

\[
= \epsilon,
\]
i.e. \(\|z_{(i,E)}z - z\| \to 0\) as \(i \to \infty\). A similar argument shows \(\|zz_{(i,E)} - z\| \to 0\).

(ii) With \(H\) the trivial subgroup lemma 5.7 shows that \(\delta(\omega(e_i)) \to 1\) strictly in \(M(A \times \delta G)\) and hence \((\pi \otimes i)(\delta(\omega(e_i))) \to 1\) \(*\)-strongly in \(B(\mathcal{H} \otimes L^2(G))\).

Also it is easy to see that \(1 \otimes M_G(w_E) \to 1\) \(*\)-strongly in \(B(\mathcal{H} \otimes L^2(G))\). Now since the norms \(\|(\pi \otimes i)((\delta(\omega(e_i)))\|\) and \(\|1 \otimes M_G(w_E)\|\) are bounded by one, \((\pi \otimes i)(\delta(\omega(e_i)))(1 \otimes (M_G(w_E)) \to 1\) \(*\)-strongly in \(B(\mathcal{H} \otimes L^2(G))\) as required.

\[\square\]

**Lemma 5.9** Let \(w, y \in \mathcal{D}\). Then

\[
\langle w, y \rangle_{\mathcal{D}} = \Psi(w^*y)
\]

defines a \(\mathcal{D}_H\)-valued inner product on \(\mathcal{D}\). \(\mathcal{D}\) equipped with this inner product is a \(\mathcal{D}_H\)-rigged space.

**Proof** Suppose \(w, y \in \mathcal{D}\) and \((x_j)_{j \in I \times \varepsilon}\) is the net of lemma 5.8 (ii). Then

(i) \(\langle \cdot, \cdot \rangle_{\mathcal{D}}\) is clearly conjugate linear in the first variable and linear in the second.

(ii) \(\langle y, y \rangle_{\mathcal{D}} = \Psi(y^*y)\)
\[ = \text{strong limit} \lim_{h \to 0} x_{j} \Psi(y^{*}y) x_{j}^{*} \]
\[ = \text{strong limit} \int_{H} x_{j} \delta_{h}(y^{*}y) x_{j}^{*} dh. \] (by lemma 5.6)

Now by lemma 5.6 the integrand \( h \to x_{j} \delta_{h}(y^{*}y) x_{j}^{*} \) is norm continuous and compactly supported so it can be uniformly approximated in norm by sums of the form
\[ \sum_{k} \nu_{ijk} x_{j} \delta_{h_{ijk}}(y^{*})(x_{j} \delta_{h_{ijk}}(y^{*}))^{*} \quad \mathcal{H}_{ijk} \in H, \nu_{ijk} \geq 0. \]

Hence \( \langle y, y \rangle_{D} = \text{strong limit} \left( \lim_{h \to 0} \sum_{k} \nu_{ijk} x_{j} \delta_{h_{ijk}}(y^{*})(x_{j} \delta_{h_{ijk}}(y^{*}))^{*} \right) \)
\[ \geq 0, \]
since the positive elements are \(*\)-strongly (and norm) closed.

(iii) \( \langle w, y \rangle_{D}^{*} = (\Psi(w^{*}y))^{*} \)
\[ = \text{strong limit} x_{j} (\Psi(w^{*}y))^{*} \]
\[ = \text{strong limit} (\Psi(w^{*}y) x_{j}^{*})^{*} \]
\[ = \text{strong limit} \left( \int_{H} \delta_{h}(w^{*}y) x_{j}^{*} dh \right)^{*} \] (by lemma 5.6)
\[ = \text{strong limit} \int_{H} x_{j} \delta_{h}(y^{*}w) dh \]
\[ = \text{strong limit} x_{j} \Psi(y^{*}w) \]
\[ = \langle y, w \rangle_{D}, \]

so \( \langle \cdot, \cdot \rangle_{D} \) is a \( D_{H} \)-valued inner product on \( D \) as claimed.

To see that \( D \) equipped with this inner product is a \( D_{H} \)-rigged space we need to show \( \langle w, y \cdot z \rangle_{D} = \langle w, y \rangle_{D} z \) and that the linear span of the range of the inner product is dense in \( A \times_{\delta} (G/H) \).
Suppose \( w, y \in D, z \in D_H \) and \( (x_j)_{j \in I \times \varepsilon} \) is the net of lemma 5.8 (ii). Then
\[
\langle w, y \ast z \rangle_D = \Psi(w^\ast y z)
\]
\[
= \overset{\text{strong limit}}{\star} x_j \Psi(w^\ast y z)
\]
\[
= \overset{\text{strong limit}}{\star} \int_H x_j \hat{\delta}_h(w^\ast y z) \, dh \quad \text{(by lemma 5.6)}
\]
\[
= \overset{\text{strong limit}}{\star} \int_H x_j \hat{\delta}_h(w^\ast y) \, dh \, z
\]
(since \( \hat{\delta}_h(z) = z \) for all \( z \in A \times_\delta (G/H) \), as in \( \| 4.5 \))
\[
= \overset{\text{strong limit}}{\star} x_j \Psi(w^\ast y)z
\]
\[
= \Psi(w^\ast y)z .
\]

Let \( (z(i,E)) \) be the net of lemma 5.8 (i) (i.e. \( z(i,E) = \delta(\omega(e_i))(1 \otimes \omega_E) \)). Choose \( f_E \in C_c^+(G) \) such that \( \varphi(f_E) = \omega_E^2 \) and let \( z \in A \times_\delta (G/H) \). Then
\[
\langle \delta(\delta_\omega(e_i))(1 \otimes f_E^\frac{1}{2}) , \delta(\delta_\omega(e_i))(1 \otimes f_E^\frac{1}{2})z \rangle_D
\]
\[
= \langle \delta(\delta_\omega(e_i))(1 \otimes f_E^\frac{1}{2}) , \delta(\delta_\omega(e_i))(1 \otimes f_E^\frac{1}{2})z \rangle_D z
\]
\[
= \Phi(\delta(\delta_\omega(e_i))(1 \otimes f_E) \delta(\delta_\omega(e_i)))z
\]
\[
= \delta(\delta_\omega(e_i))(1 \otimes \varphi(f_E)) \delta(\delta_\omega(e_i))z
\]
\[
= \delta(\delta_\omega(e_i))(1 \otimes \omega_E^2) \delta(\delta_\omega(e_i))z
\]
\[
= z(i,E)z(i,E)^* z
\]
\[
\rightarrow z \quad \text{in norm.}
\]
So the inner product has dense range.

**Theorem 5.10** \( D \) is a pre-Hermitian \( D_H \)-rigged \( D \)-module.

**Proof** Firstly recall that \( D \), with the inner-product and right action as in lemma 5.9, is \( D_H \)-rigged. Now define an left action of \( D \) on \( D \) by
\[
y \ast w = yw \quad w, y \in D .
\]
This is well defined by lemma 4.10. Now we must show that this action satisfies the conditions of § 1.17. Let \( x, y \in \mathcal{D} \) and \( z \in \mathcal{D}_H \). Then

(i) \( (y \cdot x) \cdot z = yxz = y \cdot (x \cdot z) \),

(ii) \( \langle y \cdot w , x \rangle_{\mathcal{D}} = \Psi(w^*y^*x) = \langle w , y^* \cdot x \rangle_{\mathcal{D}} \),

(iii) by lemma 5.6 we have that

\[
\hat{x} \langle y \cdot w , y \cdot w \rangle_{\mathcal{D}} x^* = \int_H x \hat{\delta_f}(w^*y^*yw) x^* \, dh
\]

\[
\leq \|y\|_{A \times_s G}^2 \int_H x \hat{\delta_f}(w^*w) x^* \, dh
\]

\[
= \|y\|_{A \times_s G}^2 \cdot x \langle w , w \rangle_{\mathcal{D}} x^* .
\]

Let \((x_j)_{j \in I \times \varepsilon}\) be the net of lemma 5.8 (ii). Then

\[
\|y\|_{A \times_s G}^2 \cdot \langle w , w \rangle_{\mathcal{D}} - \langle y \cdot w , y \cdot w \rangle_{\mathcal{D}}
\]

\[
= \text{strong limit } \lim_j (\|y\|_{A \times_s G}^2 \cdot \langle w , w \rangle_{\mathcal{D}} - \langle y \cdot w , y \cdot w \rangle_{\mathcal{D}}) x_j^*
\]

\[
\geq 0 ,
\]

(since the positive elements of \( B(\mathcal{H} \otimes L^2(G)) \) are \(*\)-strongly closed). But an element is positive in \( B(\mathcal{H} \otimes L^2(G)) \) if and only if it is positive in any \( C^* \)-subalgebra of \( B(\mathcal{H} \otimes L^2(G)) \) containing it [27 prop. 4.8]. Hence

\[
\|y\|_{A \times_s G}^2 \cdot \langle w , w \rangle_{\mathcal{D}} - \langle y \cdot w , y \cdot w \rangle_{\mathcal{D}} \geq 0 \tag{4}
\]

in the completion, \( A \times_s (G/H) \), of \( \mathcal{D}_H \), as required.

(iv) Letting \( H \) be the trivial subgroup in lemma 5.8 (i), we see that there is a net \((x_{(i,E)})\) in \( \mathcal{D} \) such that \( z_{(i,E)} x \rightarrow x \) for all \( x \in \mathcal{D} \). This shows that \( \mathcal{D} \mathcal{D} \) is dense in \( \mathcal{D} \) for the norm of \( B(\mathcal{H} \otimes L^2(G)) \). To see that \( \mathcal{D} \mathcal{D} \) is dense in \( \mathcal{D} \) for the semi-norm \( \| \cdot \|_{\mathcal{D}} \) of § 1.15, we note that if \( x \in \mathcal{D} \), then \( x \) is \((u,F)\) for some \( u \in A_c(G) \) and
compact subset \( F \) of \( G \), and that
\[
\|x\|^2_D = \|\langle x, x \rangle_D\| \\
= \|\Phi (x^* x)\| \\
\leq \zeta_{u,F}^2 \cdot \|x\|^2 ,
\]
by lemma 5.4, where \( \zeta_{u,F} \) is the constant given by that lemma.

This shows the left action satisfies the necessary conditions and therefore establishes the theorem.

\[ \square \]

Corollary 5.11  The map
\[
: x \rightarrow [\theta_x] : D \rightarrow \mathcal{L}(D)/J ,
\]
where \([\theta_x]\) is the equivalence class of the operator defined by
\[
\theta_x(y) = x \cdot y \quad \forall y \in D ,
\]
is a \( * \)-homomorphism which is norm-decreasing for the \( A \times \delta G \)-norm and hence extends to a \( * \)-homomorphism
\[
j_{A \times \delta G} : A \times \delta G \rightarrow L(D) .
\]

Proof  Clear from the theorem and lemma 1.12.  \[ \square \]

Now the module \( D \) of theorem 5.10 can be factored and completed with respect to the semi-norm \( \| \cdot \|_D \) of \( \| \cdot \|_{1.15} \) to give a Hermitian \( A \times \delta (G/H) \)-rigged \( A \times \delta G \) module \( X \) which can be used to construct representations of \( A \times \delta G \) (see chapter 1 §4) from those of \( A \times \delta (G/H) \) and hence from those of \( A \times \delta (G/H) \) (since if \( \mu \) is a representation of \( A \times \delta (G/H) \), \( \mu \circ \Gamma^{-1} \), where \( \Gamma \) is the isomorphism of proposition 2.6), is a representation of \( A \times \delta (G/H) \).
To recap: suppose $\nu : A \times G (G/H) \to B(\mathcal{Q})$ is a representation of $A \times G (G/H)$ on $\mathcal{Q}$. Then we can define a pre-inner product on the tensor product $X \otimes \mathcal{Q}$ by

$$\langle [x] \otimes \xi , [y] \otimes \eta \rangle_{X \otimes \mathcal{Q}} = \langle \nu \circ \Gamma^{-1}(\langle y , x \rangle_D) \rangle(\xi) , \eta \rangle_{\mathcal{Q}}.$$

We obtain a Hilbert space $\text{ind}^{A \times G}_{A \times G (G/H)} \mathcal{Q}$ from $X \otimes \mathcal{Q}$ by factoring out by the vectors of length zero and completing. The representation of $A \times G$ induced from $\nu$ is then the representation $\text{ind}^{A \times G}_{A \times G (G/H)} \nu : A \times G \to B(\text{ind}^{A \times G}_{A \times G (G/H)} \mathcal{Q})$, determined by

$$\{ \{ \text{ind}^{A \times G}_{A \times G (G/H)} \nu \}(y) \}(x \otimes \xi) = [yx] \otimes \xi \quad y \in A \times G , \ x \in X , \ \xi \in \mathcal{Q}.$$

Thus we obtain the map

$$\text{ind}^{A \times G}_{A \times G (G/H)} : \text{Rep}(A \times G (G/H)) \to \text{Rep}(A \times G)$$

$$: \mu \to \text{ind}^{A \times G}_{A \times G (G/H)} \mu,$$

which is the desired induction process.

As promised, we shall show that the induced representations presented on page 46 are a special case of the above (when $H = G$).

**Proposition 5.12** Suppose $\varrho$ is a representation of $A$ on the Hilbert space $\mathcal{P}$ and that $1$ is the trivial representation of $G/G$. Then $(\varrho, 1)$ is a covariant representation of $(A, \delta, G/G)$ on $\mathcal{P}$ and $\text{ind}^{A \times G}_{A \times G (G/G)} (\varrho \times 1)$ is unitarily equivalent to $(\varrho \otimes i) \times (1 \otimes M_G)$.

**Proof** $X$-$\text{ind}^{A \times G}_{A \times G (G/G)} (\varrho \times 1)$ is unitarily equivalent to $D$-$\text{ind}^{A \times G}_{A \times G (G/G)} (\varrho \times 1)$ (see page 26) so it will be enough to show that $D$-$\text{ind}^{A \times G}_{A \times G (G/G)} (\varrho \times 1)$ is unitarily equivalent to $(\varrho \otimes i) \times (1 \otimes M_G)$. Let $E$ be the subspace of $D$ consisting of the elements $\sum_{j=1}^{n} (1 \otimes (u * f_j)) \delta(\delta_v(a_j))$ (our abbreviation for $\sum_{j=1}^{n} (1 \otimes M_G(u * f_j))(\pi \otimes i)(\delta(\delta_v(a_j)))$) for $u$ and $v$ fixed elements of $A_c(G)$,
f_j \in C_c(G) \text{ and } a_j \in A. \text{ By theorem 4.12 and the continuity of the involution the set of adjoins of elements of } \mathcal{D} \text{ is dense in } A \times_\delta G. \text{ This and the fact that } A_c(G) \text{ contains an approximate identity for } C^*(G) \text{ (see chapter 1 §1) show that } E \text{ is dense in } \mathcal{D} \text{ (for the } A \times_\delta G \text{ norm). Let } x \in E. \text{ Then } x^*x \in \mathcal{D} \text{ so by assumption there exists a compact set } F \text{ such that }

\begin{equation}
x^*x = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_w(b_{ij}))(1 \otimes g_{ij})
\end{equation}

for some } w \in A_c(G), b_{ij} \in A \text{ and } g_{ij} \in C_F(G). \text{ Let } \xi \in \mathcal{P}. \text{ Then }

\begin{equation}
\|x \otimes \xi\|^2_{\mathcal{D} \otimes \mathcal{P}} = |\langle x \otimes \xi, x \otimes \xi \rangle_{\mathcal{D} \otimes \mathcal{P}}|
\end{equation}

\begin{equation}
= |\langle (\psi(x^*x))(\xi), \xi \rangle_{\mathcal{P}}|
\end{equation}

(\text{where } \Gamma \text{ is the map of proposition 2.6})

\begin{equation}
= |\langle (\psi(x^*x))(\xi), \xi \rangle_{\mathcal{P}}| \leq \|\Gamma^{-1} \circ \psi(x^*x)\|_{A \times_\delta(G/G)} \cdot \|\xi\|^2
\end{equation}

\begin{equation}
= \| \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_w(b_{ij}))(1 \otimes \varphi(g_{ij}))\|_{A \times_\delta(G/G)} \cdot \|\xi\|^2
\end{equation}

\begin{equation}
\leq \zeta_{w,F} \cdot \| \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(\delta_w(b_{ij}))(1 \otimes g_{ij})\|_{A \times_\delta G} \cdot \|\xi\|^2
\end{equation}

(by lemma 5.4, where } \zeta_{w,F} \text{ is the constant of that lemma)

\begin{equation}
= \zeta_{w,F} \cdot \|x\|^2_{A \times_\delta G} \cdot \|\xi\|^2
\end{equation}

Hence } E \otimes \mathcal{P} \text{ is dense in } \mathcal{D} \otimes \mathcal{P}. \text{ Before proceeding we need to observe that if } a, b \in A, \ f \in C_c(G) \text{ and } u, v \in A_c(G), \text{ then }

\begin{equation}
\psi(\delta(\delta_u(a))(1 \otimes f)\delta(\delta_v(b))) = \delta(\delta_u(a))(1 \otimes \varphi(f))\delta(\delta_v(b)).
\end{equation}

To see this, let } y \in \mathcal{D}. \text{ Then by lemma 5.6 }

\begin{equation}
\psi(\delta(\delta_u(a))(1 \otimes f)\delta(\delta_v(b)))y = \int_H \delta_h(\delta(\delta_u(a))(1 \otimes f)\delta(\delta_v(b)))y \, dh
\end{equation}

- 112 -
\[= \delta(\delta_u(a)) \int_H (1 \otimes f^h) \, dh \delta(\delta_u(b)) y \]
\[= \delta(\delta_u(a))(1 \otimes \varphi(f)) \delta(\delta_u(b)) y. \]

Now suppose that \( \sum_{i=1}^n (1 \otimes f_i)\delta(a_i) \) and \( \sum_{j=1}^m (1 \otimes g_j)\delta(b_j) \in E \) and \( \xi, \eta \in \mathcal{P}. \)

Then

\[
\left\langle \sum_{i=1}^n ((1 \otimes f_i)\delta(a_i)) \otimes \xi, \sum_{j=1}^m ((1 \otimes g_j)\delta(b_j)) \otimes \eta \right\rangle_{\mathcal{P} \otimes \mathcal{P}}
= \left\langle \left\{ (\varphi \times 1) \circ \Gamma^{-1} \left( \left\langle \sum_{i=1}^n (1 \otimes f_i)\delta(a_i), \sum_{j=1}^m (1 \otimes g_j)\delta(b_j) \right\rangle_{\mathcal{P}} \right) \right\} \right\rangle_{\mathcal{P}}
= \sum_{i,j} \left\langle \left\{ (\varphi \times 1) \circ \Gamma^{-1} \circ \Psi \left( \delta(b_j^*) (1 \otimes (\bar{g}_j \cdot f_i)) \delta(a_i) \right) \right\} \right\rangle_{\mathcal{P}}
= \sum_{i,j} \left\langle \left\{ (\varphi \times 1)(\Gamma^{-1} (\delta(b_j^*) (1 \otimes \varphi(\bar{g}_j \cdot f_i)) \delta(a_i))) \right\} \right\rangle_{\mathcal{P}}
\]
(by the above)

\[
= \sum_{i,j} \int_G \overline{g_j(s)} f_i(s) \, ds \left\langle \left\{ (\varphi \times 1)(\delta(\bar{g}_j^* a_i)) \right\} \right\rangle_{\mathcal{P}}
\]
(since \( \varphi(\bar{g}_j \cdot f_i) \) is the constant function with value \( \int_G \overline{g_j(s)} f_i(s) \, ds \))

\[
= \sum_{i,j} \int_G \overline{g_j(s)} f_i(s) \, ds \left\langle \left\{ \varphi(a_i) \right\} \otimes f_i \right\rangle_{\mathcal{P}}
= \sum_{i,j} \int_G \left\langle \left\{ \varphi(a_i) \right\} (\xi) \otimes f_i, \left\{ \varphi(b_j) \right\} (\eta) \otimes g_j \right\rangle_{\mathcal{P} \otimes L^2(G)} \, ds
= \left\langle \sum_{i=1}^n \left\{ \varphi(a_i) \right\} (\xi) \otimes f_i, \sum_{j=1}^m \left\{ \varphi(b_j) \right\} (\eta) \otimes g_j \right\rangle_{\mathcal{P} \otimes L^2(G)}. \tag{5}
\]

Hence \( \| \sum_{i=1}^n (1 \otimes f_i)\delta(a_i) \otimes \xi \|_{\mathcal{P} \otimes \mathcal{P}} = \| \sum_{i=1}^n \left\{ \varphi(a_i) \right\} (\xi) \otimes f_i \|_{\mathcal{P} \otimes L^2(G)} \) and the map

\[V : E \otimes \mathcal{P} \to \mathcal{P} \otimes L^2(G) \]
defined by

\[
\sum_{i=1}^n (1 \otimes f_i)\delta(a_i) \otimes \xi \to \sum_{i=1}^n \left\{ \varphi(a_i) \right\} (\xi) \otimes f_i.
\]
is well-defined. We shall establish that $V$ extends to a unitary operator from the Hilbert space of the induced representation $\text{ind}_{\mathcal{A} \times G}(G) \mathcal{P}$ onto $\mathcal{P} \otimes L^2(G)$, which intertwines the actions of $\mathcal{D} \text{-ind}_{\mathcal{A} \times G}(G) \theta \times 1$ and $((\rho \otimes i) \circ \delta) \times (1 \otimes M_G)$.

(i) It is clear from § 5 that $V$ preserves the inner products.

(ii) $V$ maps onto a dense subspace of $\mathcal{P} \otimes L^2(G)$. To see this note that

$$A = \{\delta_u(a) : a \in A, \ u \in A_c(G)\}$$

is dense in $A$ (as in theorem 4.12). Hence, since $\rho$ is a (non-degenerate) representation $\{\rho(A)\}(\mathcal{P}) \otimes C_c(G)$ is dense in $\mathcal{P} \otimes L^2(G)$.

(iii) $V$ intertwines the actions. Suppose $\sum_{i=1}^n ((1 \otimes (u * g_i))(b_i)) \otimes \xi \in E \otimes \mathcal{P}$, $(1 \otimes f)\delta(a) \in \mathcal{D}$ and $\eta \otimes h \in \mathcal{P} \otimes L^2(G)$. Then

$$\left\langle V\left(\text{ind}_{\mathcal{A} \times G}(G)\right)((1 \otimes f)\delta(a)) \left(\sum_{i=1}^n ((1 \otimes (u * g_i))(b_i)) \otimes \xi\right), \eta \otimes h \right\rangle_{\mathcal{P} \otimes L^2(G)}$$

$$= \sum_{i=1}^n \left\langle V\left(((1 \otimes f)\delta(a))(1 \otimes (u * g_i))\delta(b_i)\right)(\otimes \xi), \eta \otimes h \right\rangle_{\mathcal{P} \otimes L^2(G)}$$

$$= \sum_{i=1}^n \left\langle V\left((1 \otimes f) \int_G (1 \otimes (g_i)_s)\delta(\delta_u(a)(s) \otimes \xi), \eta \otimes h \right\rangle_{\mathcal{P} \otimes L^2(G)}$$

(by proposition 4.1)

$$= \sum_{i=1}^n \int_G \left\langle V\left(((1 \otimes (f \cdot (g_i)_s))\delta(\delta_u(a)b_i)\right)(\otimes \xi), \eta \otimes h \right\rangle_{\mathcal{P} \otimes L^2(G)} ds$$

$$= \sum_{i=1}^n \int_G \left\langle \rho(\delta_u(a)b_i)\otimes (f \cdot (g_i)_s), \eta \otimes h \right\rangle_{\mathcal{P} \otimes L^2(G)} ds$$

$$= \sum_{i=1}^n \int_G \int_G f(t)g_i(s^{-1}t)\overline{h(t)}\langle \rho(\delta_u(a)b_i)\otimes \xi, \eta \rangle_{\mathcal{P}} dt ds$$

Now writing $\delta(a) = \lim_{n} \sum_{j=1}^{n} a_j \otimes \lambda_G(\gamma_{j})$ for some $a_j \in A$ and $\gamma_{j} \in C_c(G)$, so that $\delta_u(a) = \lim_{n} \sum_{j=1}^{n} \int_G u(s^{-1}r)\gamma_{j}(r) \ dr a_j$, we obtain that the above equals...
\[
\int_G \int_G \lim_{k \to \infty} \sum_{i,j} \int_G u(s^{-1}r) \gamma_{jk}(r) \, dr \, f(t) g_i(s^{-1}t) \overline{h(t)} \, dt \, \left< \{ \varrho(a_{jk} b_i) \}(\xi), \eta \right>_\mathcal{P} \, ds.
\]

Now we have that
\[
\left| \sum_{i=1}^n f(t) g_i(s^{-1}t) \overline{h(t)} \left< \{ \varrho \left( \sum_{j=1}^{n_j} \int_G u(s^{-1}r) \gamma_{jk}(r) \, dr \, a_{jk} b_i \right) \}(\xi), \eta \right>_\mathcal{P} \right|
\leq \sum_{i=1}^n \| g_i \|_{C_c(G)} \cdot |f(t)\overline{h(t)}| \left( \| \varrho(\delta_{u_i}(a) b_i) \|_{A(G)} \cdot \| a \|_A \cdot \| b_i \|_A \cdot \| \xi \| \cdot \| \eta \| + 1 \right) \chi_F(s)
\]

(since \( \sum_j \int_G u(s^{-1}r) \gamma_{jk}(r) \, dr \, a_{jk} b_i \to \delta_{u_i}(a) b_i \) strongly and where \( F \) is the compact set \((\text{supp} f)(\text{supp} g_i)^{-1}\))
\[
= \sum_{i=1}^n \| g_i \|_{C_c(G)} \cdot |f(t)\overline{h(t)}| \left( \| u \|_{A(G)} \cdot \| a \|_A \cdot \| b_i \|_A \cdot \| \xi \| \cdot \| \eta \| + 1 \right) \chi_F(s),
\]
which is clearly integrable as a function of \( s \) and \( t \). So the dominated convergence theorem applies to give that
\[
\int_G \int_G \lim_{k \to \infty} \sum_{i,j} \int_G u(s^{-1}r) \gamma_{jk}(r) \, dr \, f(t) g_i(s^{-1}t) \overline{h(t)} \, dt \, \left< \{ \varrho(a_{jk} b_i) \}(\xi), \eta \right>_\mathcal{P} \, ds
\]
\[
= \lim_{k \to \infty} \sum_{i,j} \int_G \int_G \gamma_{jk}(r) u(s^{-1}r) g_i(s^{-1}t) f(t) \overline{h(t)} \, dr \, dt \, ds \, \left< \{ \varrho(a_{jk} b_i) \}(\xi), \eta \right>_\mathcal{P}
\]
\[
= \lim_{k \to \infty} \sum_{i,j} \int_G \{ \gamma_{jk} \ast \bar{u} \ast g_i \}(t) f(t) \overline{h(t)} \, dt \, \left< \{ \varrho(a_{jk} b_i) \}(\xi), \eta \right>_\mathcal{P}
\]
(by Fubini's theorem)
\[
= \lim_{k \to \infty} \sum_{i,j} \left< \{ \varrho(a_{jk} b_i) \}(\xi) \otimes ((\gamma_{jk} \ast \bar{u} \ast g_i) \cdot f), \eta \otimes h \right>_{\mathcal{P} \otimes L^2(G)}
\]
\[
= \lim_{k \to \infty} \sum_{i,j} \left< (1 \otimes M_G(f)) \{ \varrho(a_{jk} b_i) \}(\xi) \otimes (\gamma_{jk} \ast \bar{u} \ast g_i) \right>_{\mathcal{P} \otimes L^2(G)}
\]
\[
= \lim_{k \to \infty} \sum_{i,j} \left< (1 \otimes M_G(f))(\varrho(a_{jk}) \otimes \lambda_G(\gamma_{jk}))(\varrho(b_i))(\xi) \otimes (\bar{u} \ast g_i), \eta \otimes h \right>_{\mathcal{P} \otimes L^2(G)}
\]

- 115 -
\[ \sum_{i=1}^{n} \left\langle \left\{ (1 \otimes M_G(f)) (\varphi \otimes \delta(a)) \right\} \left\{ \varphi(b_i) \right\} (\xi \otimes (\bar{u} * g_i)), \eta \otimes h \right\rangle_{P \otimes L^2(G)} \]

\[ = \left\langle \left\{ (\varphi \otimes \delta) \times (1 \otimes M_G)[V((1 \otimes f)\delta(a))] \right\} \left( \sum_{i=1}^{n} ((1 \otimes (\bar{u} * g_i))\delta(b_i)) \otimes \xi \right), \eta \otimes h \right\rangle. \]

Since elements of the form \( \sum_{i=1}^{n} (1 \otimes (\bar{u} * g_i))\delta(b_i) \) are dense in \( A \times \delta G \) the proposition follows. \( \square \)
Chapter 6. The Imprimitivity Theorem.

In this chapter we determine the imprimitivity algebra for the $D_H$-rigged space $D$ (or $X$, since by lemma 1.11 they are the same). With the imprimitivity algebra identified we interpret Rieffel's imprimitivity theorem [22 thm. 6.29] and obtain criteria that enable one to determine those representations of $A \times_\delta G$ which can be constructed from representations of $A \times_\delta (G/H)$ by the induction process of chapter 5, that is, we present an imprimitivity theorem for this process. Then to finish we briefly investigate the continuity of the induction and restriction processes.

Throughout this chapter $H$ will be a closed normal amenable subgroup of $G$, $\delta$ will be a non-degenerate coaction of $G$ on $A$ and $\pi$ will be a fixed faithful representation of $A$ on the Hilbert space $H$.

In what follows we will often be integrating operator valued functions. These functions will be norm continuous and compactly supported so that lemmas 1.1-1.7 may be (and will be) used freely, without reference.

Proposition 6.1 Suppose $\xi \in I_H$ and $x \in D$. Then

\[ \xi \cdot x = \int_H \sqrt{\Delta h} \xi(h)\hat{\delta}_h(x) \, dh \]

defines a left action of $I_H$ on $D$ such that $D$ becomes an $I_H$-$D_H$ bimodule.

Proof (i) Firstly we show that $\xi \cdot x$ does in fact belong to $D$. By assumption there exist compact subsets $E$ and $F$ of $G$ and $u, v \in A_c(G)$ such that $x$ is $(v, F)$ and $\xi(h)$ is $(u, E)$ for all $h \in H$. By lemma 4.14 there exists a compact subset $D$ of $G$ such that $\hat{\delta}_h(x)$ is $(v, D)$ for all $h \in \text{supp}\xi$. By lemma 4.10 there exists $\omega \in A_c(G)$ such that $\xi(h)\hat{\delta}_h(x)$ is $(\omega, D)$ for all $h \in \text{supp}\xi$ and hence for all $h \in H$. That is: $h \rightarrow \sqrt{\Delta h} \xi(h)\hat{\delta}_h(x)$ is an element of $I_H$. So by lemma 4.13 $\xi \cdot x \in D$ as required.

- 117 -
(ii) Let $\xi \in \mathcal{I}_H$, $x, y \in \mathcal{D}$. Then
\[\xi \cdot (x + y) = \int_H \sqrt{\Delta h} \xi(h) \hat{\delta}_h(x + y) \, dh = \xi \cdot x + \xi \cdot y.\]

(iii) Let $\xi, \gamma \in \mathcal{I}_H$, $x \in \mathcal{D}$. Then
\[\left(\xi + \gamma\right) \cdot x = \int_H \sqrt{\Delta h} \left(\xi + \gamma\right)(h) \hat{\delta}_h(x) \, dh = \xi \cdot x + \gamma \cdot x.\]

(iv) Let $\xi, \gamma \in \mathcal{I}_H$, $x \in \mathcal{D}$. Then
\[
\xi \cdot (\gamma \cdot x) = \int_H \sqrt{\Delta h} \xi(h) \hat{\delta}_h \left(\int_H \sqrt{\Delta r} \gamma(r) \hat{\delta}_r(x) \, dr\right) \, dh \\
= \int_H \int_H \sqrt{\Delta (hr)} \xi(h) \hat{\delta}_h(\gamma(h^{-1}r)) \hat{\delta}_{hr}(x) \, dr \, dh \\
= \int_H \int_H \sqrt{\Delta r} \xi(h) \hat{\delta}_h(\gamma(h^{-1}r)) \hat{\delta}_{r}(x) \, dr \, dh \\
= \int_H \sqrt{\Delta r} \left(\int_H \xi(h) \hat{\delta}_h(\gamma(h^{-1}r)) \, dh\right) \hat{\delta}_{r}(x) \, dr \\
= \int_H \sqrt{\Delta r} (\xi \ast \gamma)(r) \hat{\delta}_{r}(x) \, dr \\
= (\xi \ast \gamma) \cdot x. \quad (1)
\]

It remains to show $\mathcal{D}$ is an $\mathcal{I}_H \cdot \mathcal{D}_H$ bimodule. Let $\xi \in \mathcal{I}_H$, $x \in \mathcal{D}$ and $z \in \mathcal{D}_H$. Firstly we need to show that $\hat{\delta}_h(z) = z$. By assumption
\[z = \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(a_{ij})(1 \otimes \varphi(f_{ij}))\]
for some $a_{ij} \in A$ and $f_{ij} \in C_c(G)$. Thus
\[\hat{\delta}_h(z) = \lim_{i \to \infty} \sum_{j=1}^{n_i} \hat{\delta}_h(\delta(a_{ij})(1 \otimes \varphi(f_{ij})))\]
(by the continuity of $\hat{\delta}_h$)
\[
\lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(a_{ij})(1 \otimes \varphi(f_{ij})^h)
\]

(by \S\ 4.5 and \S\ 4.6)

\[
\lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(a_{ij})(1 \otimes \varphi(f_{ij})^h)
\]

\[
= \lim_{i \to \infty} \sum_{j=1}^{n_i} \delta(a_{ij})(1 \otimes \varphi(f_{ij}))
\]

\[
= z.
\]

So we have that

\[
\xi \cdot (x \cdot z) = \int_H \sqrt{\Delta h} \xi(h) \hat{\delta}_h(xz) \, dh
\]

\[
= \int_H \sqrt{\Delta h} \xi(h) \hat{\delta}_h(x) \, dh \, z
\]

\[
= (\xi \cdot x) \cdot z,
\]

as required. \hfill \Box

**Proposition 6.2** \(D\) is a possibly degenerate pre-Hermitian \(D_H\)-rigged \(I_H\) module.

**Proof** By proposition 6.1 \(D\) is an \(I_H\)-\(D_H\) bimodule. It remains to show that

(i) \[\langle \xi \cdot x , y \rangle_D = \langle x , \xi^* \cdot y \rangle_D\]

(ii) \[\langle \xi \cdot x , \xi \cdot x \rangle_D \leq \|\xi\|_1^2 \cdot \langle x , x \rangle_D\]

for all \(x, y \in D\) and \(\xi \in I_H\).

(i) Let \(w \in D\). Then

\[
\langle \xi \cdot x , y \rangle_D^w = \int_H \delta_h((\xi \cdot x)^* y) w \, dh
\]

(by lemma 5.6)
\[
\begin{align*}
&= \int_H \hat{\delta}_h \left( \left( \int_H \sqrt{\Delta r} \xi(r) \hat{\delta}_r(x) \, dr \right)^* y \right) w \, dh \\
&= \int_H \int_H \sqrt{\Delta r} \delta_h(\hat{\delta}_r(x^*) \xi(r) y) w \, dr \, dh \\
&= \int_H \int_H \left( \frac{1}{\sqrt{\Delta r}} \right)^3 \delta_h \{\delta_{r-1}(x^*) \xi(r^{-1}) y \} w \, dr \, dh \\
&= \int_H \int_H \frac{1}{\sqrt{\Delta r}} \delta_{hr} \{\delta_{r-1}(x^*) \xi(r^{-1}) y \} w \, dr \, dh \\
&= \int_H \int_H \frac{1}{\sqrt{\Delta r}} \delta_h \{x^* \delta_r \{\xi(r^{-1}) y \} \} w \, dr \, dh \\
&= \int_H \delta_h(x^* \left( \int_H \sqrt{\Delta r} \left( \frac{1}{\Delta r} \delta_r(\xi(r^{-1}) y) \right) \, dr \right)) w \, dh \\
&= \int_H \delta_h(x^* \left( \int_H \sqrt{\Delta r} \xi^*(r) \delta_r(y) \, dr \right)) w \, dh \\
&= \int_H \delta_h(x^* (\xi^* \cdot y)) w \, dh \\
&= \langle x , \xi^* \cdot y \rangle_D w .
\end{align*}
\]

Letting \( w \) run over the net of lemma 5.8 (ii) we see that \( \langle \xi \cdot x , y \rangle_D = \langle x , \xi^* \cdot y \rangle_D \) as required.

(ii) Define an action of \( H \) on \( \mathcal{D} \) by

\[
h \cdot x = \sqrt{\Delta h} \hat{\delta}_h(x) \quad h \in H \ x \in \mathcal{D} .
\]

Let \( w \in \mathcal{D} \). Then

\[
w \langle h \cdot x , h \cdot x \rangle_D = \int_H \Delta h \omega \hat{\delta}_{r,h}(x^* x) \, dr \quad \text{(by lemma 5.6)}
\]

\[
= \int_H w \hat{\delta}_r(x^* x) \, dr \\
= w \langle x , x \rangle_D .
\]

Now letting \( w \) run over the net \( (x_j)_{j \in I \times \xi} \) of lemma 5.8 (ii) we have that \( \langle h \cdot w , h \cdot w \rangle_D = \langle w , w \rangle_D \) as required. \( \quad (2) \)
Let $p$ be a state on $A \times_\delta (G/H)$. Then $p((\cdot, \cdot)_\mathcal{D})$ is a scalar valued pre-inner product on $\mathcal{D}$. Suppose $\mathcal{D}_p$ is the Hilbert space obtained by taking the quotient of $\mathcal{D}$ by vectors of length zero and completing and $\| \|_p$ is the norm on $\mathcal{D}_p$. Then

$$
\|\xi \cdot x\|_p = \left\| \int_H \sqrt{\Delta h} \xi(h)\delta_h(x) \, dh \right\|_p
$$

$$
\leq \int_H \|\sqrt{\Delta h} \xi(h)\delta_h(x)\|_p \, dh
$$

$$
\leq \int_H \|\xi(h)\|_{A \times G} \, dh \cdot \|x\|_p
$$

$$
\leq \|\xi\|_1 \cdot \|x\|_p,
$$

since

$$
\|\xi(h)\delta_h(x)\|_p = p((\sqrt{\Delta h} \xi(h)\delta_h(x), \sqrt{\Delta h} \xi(h)\delta_h(x))_\mathcal{D})^{\frac{1}{2}}
$$

$$
\leq \|\xi(h)\|_{A \times G} \cdot p((h \cdot x, h \cdot x)_\mathcal{D})^{\frac{1}{2}} \quad \text{(by \S 5.4)}
$$

$$
= \|\xi(h)\|_{A \times G} \cdot p((x, x)_\mathcal{D})^{\frac{1}{2}} \quad \text{(by \S 2)}
$$

$$
= \|\xi(h)\|_{A \times G} \cdot \|x\|_p.
$$

So we have that

$$
p((\xi \cdot x, \xi \cdot x)_\mathcal{D}) \leq \|\xi\|_1^2 \cdot p((x, x)_\mathcal{D})
$$

for every state on $A \times_\delta (G/H)$. Hence

$$
(\xi \cdot x, \xi \cdot x)_\mathcal{D} \leq \|\xi\|_1^2 \cdot (x, x)_\mathcal{D}.
$$

\square

Corollary 6.3 The map

$$
: \xi \to [\theta_\xi] : I_H \to \mathcal{L}(\mathcal{D})/J,
$$

where $[\theta_\xi]$ is the equivalence class of the operator defined by

$$
\theta_\xi(y) = \xi \cdot x \ \forall \ x \in \mathcal{D},
$$

- 121 -
is a \(-\)-homomorphism which is norm-decreasing for the \(C^*\)-norm on \((A \times_\delta G) \times_\delta H\) and hence extends to a \(*\)-homomorphism

\[ \Theta : (A \times_\delta G) \times_\delta H \to L(D). \]

**Proof** By lemma 1.12 the map : \( \xi \to [\theta_\xi] \) is a \(*\)-homomorphism which is norm decreasing for the \(L^1\)-norm on \(L^1(H, A \times_\delta G)\). Hence, by the universality property of the enveloping \(C^*\)-algebra, it is also norm decreasing for the \(C^*\)-norm and thus extends to give the required map. \( \square \)

We now wish to show the linear span \( \mathcal{O} \) of the maps

\[ \gamma_{x,y}(h) = \frac{1}{\sqrt{\Delta h}} x^\delta h(y^*) \quad x, y \in D \]

is dense in \((A \times_\delta G) \times_\delta H\). To do this we need to introduce the inductive limit topology on \(C_c(H, A \times_\delta G)\) and establish the following lemma. Recall that a net \((f_k)\) in \(C_c(H, A \times_\delta G)\) converges to \(f\) in the inductive limit topology if and only if the \(f_k\) converge to \(f\) uniformly (in the norm of \(A \times_\delta G\)) and there exists a compact subset \(F\) of \(G\) such that \(\text{supp} f_k \subseteq F\) for all \(k\) sufficiently large.

**Lemma 6.4** \( \mathcal{O} \) contains a net \((z_\alpha)_{\alpha \in A}\) such that if \(\xi \in \mathcal{I}_H\) is of the form \(\xi(h) = \eta(h)(1 \otimes (\bar{u} * f))\delta(a)\), where \(\eta \in C_c(H)\), \(f \in C_c(G)\), \(u \in A_c(G)\) and \(a = \delta_v(b)\), for some \(b \in A\) and \(v \in A_c(G)\), then \(z_\alpha * \xi \to \xi\) in the inductive limit topology.

**Proof** The above mentioned net will be indexed by quadruples \((N, C, \epsilon, i)\) where \(N\) runs over the relatively compact neighbourhoods of the identity in \(H\) contained in some fixed compact neighbourhood \(E\) of the identity, \(C\) runs over the compact subsets of \(G\), \(\epsilon \in \mathbb{R}^+\) and \(i\) runs over the index set of an approximate identity \((e_i)\) of \(A\). The net is directed by

\[(N, C, \epsilon, i) \preceq (N', C', \epsilon', i') \iff N \supseteq N', \ C \subseteq C', \ \epsilon \geq \epsilon', \ i \leq i'.\]
Let \((N, C, \epsilon, i)\) be given. We begin the construction of the net by choosing an open cover \((U_j)_{j=1}^n\) of \(C\) such that
\[
\{ h \in H : U_j \cdot h \cap U_j \neq \emptyset \} \subset N . \tag{3}
\]
This is possible by the first lemma of [25]. Now we choose \(f_j \in C_c^+(U_j)\) such that
\[
\sum_{j=1}^n f_j \equiv 1 \quad \text{on} \quad C \tag{4}
\]
and is between zero and one elsewhere. Choose \(g_j \in C_c^+(U_j)\) such that
\[
|f_j(t) - g_j(t) \int_H \frac{1}{\sqrt{\Delta h}} g_j(th) \, dh| < \epsilon/n \quad \forall \, t \in G .
\]
This can be done by the second lemma of [25] and we have that
\[
\left\| \sum_{j=1}^n f_j - \sum_{j=1}^n g_j \cdot \int_H \frac{1}{\sqrt{\Delta h}} g_j^h \, dh \right\|_{C_\sigma(G)} < \epsilon . \tag{5}
\]
By lemma 5.7 (ii) we can choose a fixed element \(\omega\) of \(A_c(G)\) such that \((\delta_{\omega}(e_i))_{i \in I}\) is an approximate identity of \(A\). Let \(d_i = \delta_{\omega}(e_i)\). Then the required net is given by
\[
z_\alpha(h) = \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^n \delta(d_i)(1 \otimes g_j)\delta_h(\delta(d_i)(1 \otimes g_j))^* \\
= \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^n \delta(d_i)(1 \otimes (g_j \cdot g_j^h)\delta(d_i) .
\]
To show that \(z_\alpha \ast \xi \to \xi\) in the inductive limit topology we will show that for all \(\sigma > 0\) there exists \(\alpha_\sigma \in \mathcal{A}\) such that each of the three quantities (i), (ii) and (iii) on the right hand side of the following inequality are less than \(\sigma/3\) for all \(r \in F = E \cdot \text{supp} \xi\) (note that \(\text{supp} z_\alpha \subset N \subset E\) by \(\dagger 3\), so \(\text{supp}(z_\alpha \ast \xi) \subset F\)).

- 123 -
\[ \| (z_\alpha * \xi)(r) - \xi(r) \| \]
\[ \leq \| (z_\alpha * \xi)(r) - \delta(d_i) \int_G \left( 1 \otimes \left( \sum_{j=1}^{n} \int_h \frac{1}{\sqrt{\Delta h}} g_j g_j^h \text{dh} \right) \right) \delta(\delta_u(\eta)) \text{ds} \delta(a) \]
\[ + \| \delta(d_i) \int_G \left( 1 \otimes \left( \sum_{j=1}^{n} \int_h \frac{1}{\sqrt{\Delta h}} g_j g_j^h \text{dh} \right) \right) \delta(\delta_u(\eta)) \text{ds} \delta(a) - \delta(d_i^2) \xi(r) \]
\[ + \| \delta(d_i^2) \xi(r) - \xi(r) \| \]
\[ = (i) + (ii) + (iii). \]

Let \( \sigma > 0 \). Choose \( N_\sigma \) such that
\[
\| \eta(h^{-1}r) f_s^h - \eta(r) f_s \|_{C_\sigma(G)} < \sigma/(6 \cdot \kappa \cdot \|B(G)\| \cdot \|a\|_A) \] (6)
for all \( h \in N, r \in H \) and \( s \in G \), where \( \kappa = \mu_G((\text{supp} \omega) \cdot (\text{supp} u)^{-1}) \). Note that this is possible since \( \eta \) and \( f \) are uniformly continuous and since
\[
| (\eta(h^{-1}r) f_s^h - \eta(r) f_s)(t) | \leq | \eta(h^{-1}r) - \eta(r) | \cdot \|f\|_{C_\sigma(G)} + \| \eta \|_{C_\sigma(G)} \cdot | f(th) - f(t) | .
\]
Choose \( C_\sigma \) such that
\[
C_\sigma \supset (\text{supp} \xi) \cdot (\text{supp} \omega) \cdot (\text{supp} u)^{-1} \cdot (\text{supp} f) .
\] (7)

Choose \( \epsilon_\sigma = \min \left( 1 + \sigma(3 \cdot \kappa \cdot \|\eta\|_{C_\sigma(G)} \cdot \|f\|_{C_\sigma(G)} \cdot \|u\|_{B(G)} \cdot \|a\|_A) \right) \). (8)

Now \( (d_i)_{i \in I} \) is an approximate identity of \( A \). Hence so is \( (d_i^2)_{i \in I} \). Also by lemma 5.7 (i), with \( H \) the trivial subgroup, we have that \( \delta(d_i^2) \to 1 \) strictly in \( M(A \times_\delta G) \).

This and the fact that \( F \) is compact enable us to choose \( i_o \in I \) such that \( i \geq i_o \) implies
\[
\| \delta(d_i^2) \xi(r) - \xi(r) \| \leq \sigma/3 \quad \forall r \in F ,
\]

- 124 -
i.e., (iii) \( < \sigma/3 \) for all \( r \in F \).

Now we show that for \( \alpha \geq \alpha_0 \) (i) \( < \sigma/3 \) for all \( r \in F \). Firstly note that

\[
(z_{\alpha} \ast \xi)(r) = \int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} \delta(d_i)(1 \otimes (g_j \cdot g_j^h))\delta(d_i)\tilde{h}(\xi(h^{-1}r)) \, dh
\]

\[
= \int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} \delta(d_i)(1 \otimes (g_j \cdot g_j^h))\delta(d_i)(1 \otimes (\tilde{u} \ast f)^h)\delta(a)\eta(h^{-1}r) \, dh
\]

\[
= \int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} \delta(d_i)(1 \otimes (g_j \cdot g_j^h))\delta(d_i)(1 \otimes (\tilde{u} \ast f^h))\delta(a)\eta(h^{-1}r) \, dh
\]

\[
= \int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} \delta(d_i)(1 \otimes (g_j \cdot g_j^h)) \int_G (1 \otimes f_s^h)\delta(\delta_{u_s}(d_i)) \, ds \delta(a)\eta(h^{-1}r) \, dh
\]

(by lemma 4.1, with \( H \) the trivial subgroup)

\[
= \int_H \int_G \delta(d_i) (1 \otimes (\frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} (g_j \cdot g_j^h \cdot f_s^h \eta(h^{-1}r))) \delta(\delta_{u_s}(d_i)) \delta(a) \, ds \, dh
\]

Now the fact that

\[
\|\delta(\delta_{u_s}(d_i)) - \delta(\delta_{u_s}(d_i))\| = \|\delta_{u_s \cdot u}(e_i) - \delta_{u \cdot u}(e_i)\|
\]

\[
= \|\delta_{(u_s \cdot u) \cdot u}(e_i)\|
\]

\[
\leq \|u_s \cdot u\|_{B(\mathcal{G})} \cdot \|\omega\|_{B(\mathcal{G})}
\]

implies: \( s \rightarrow \delta(\delta_{u_s}(d_i)) : G \rightarrow B(\mathcal{H} \otimes L^2(\mathcal{G})) \) is norm continuous. (11)

Also

\[
\left\| \sum_{j=1}^{n} g_j \cdot g_j^h \cdot f_s^h \eta(h^{-1}r) - \sum_{j=1}^{n} g_j \cdot g_j \cdot f \eta(r) \right\|_{C_s(\mathcal{G})}
\]

\[
\leq \sum_{j=1}^{n} \|g_j\|_{C_s(\mathcal{G})} \cdot \|g_j^h - g_j\|_{C_s(\mathcal{G})} \cdot \|f\|_{C_s(\mathcal{G})} \cdot \|\eta\|_{C_s(\mathcal{G})}
\]

\[
+ \sum_{j=1}^{n} \|g_j\|_{C_s(\mathcal{G})} \cdot \|f_s^h - f_s\|_{C_s(\mathcal{G})} \cdot \|\eta\|_{C_s(\mathcal{G})}
\]

- 125 -
implies

\[ \text{(h, s)} \rightarrow \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} g_j \cdot g_j^h \cdot f^h_s \eta(h^{-1} r) : H \times G \to C_0(G) \]

is continuous. Hence

\[ \text{(h, s)} \rightarrow 1 \otimes \left( \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} g_j \cdot g_j^h \cdot f^h_s \eta(h^{-1} r) \right) : H \times G \to B(\mathcal{H} \otimes L^2(G)) \]

is continuous. This and \( \text{b 10} \) imply the integrand of \( \text{b 9} \), namely

\[ \text{(h, s)} \rightarrow \delta(d_i) \left( 1 \otimes \left( \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} g_j \cdot g_j^h \cdot f^h_s \eta(h^{-1} r) \right) \right) \delta(\delta_{u_s}(d_i)) \delta(a) \]

\[ : H \times G \to B(\mathcal{H} \otimes L^2(G)) \]

is norm continuous. By lemma 1.23 \( \delta_{u_s}(d_i) = \delta_{u_s}(\delta_{\omega}(e_i)) = \delta_{u_s \cdot \omega}(e_i) \) so the s-support of the integrand is contained in \( (\supp \omega) \cdot (\supp u)^{-1} \). The h-support is contained in \( E \) since by \( \text{b 3} \)

\[ \text{supp}(g_j \cdot g_j^h) \subset \{ h \in H : U_j \cdot h \cap U_j \neq \phi \} \subset N_o \subset E . \]

Hence the integrand is continuous and compactly supported. So by lemma 1.8 we can change the order of integration in \( \text{b 9} \) to obtain

\[ \text{b 9} = \int_G \delta(d_i) \int_H \left( 1 \otimes \left( \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} g_j \cdot g_j^h \cdot f^h_s \eta(h^{-1} r) \right) \right) dh \delta(\delta_{u_s}(d_i)) \delta(a) ds \]

\[ = \int_G \delta(d_i) \left( \int_H \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} g_j \cdot g_j^h \cdot f^h_s \eta(h^{-1} r) \right) dh \right) \delta(\delta_{u_s}(d_i)) \delta(a) ds \]
(by lemma 1.5 with \( \gamma : B(L^2(G)) \to B(\mathcal{H} \otimes L^2(G)) \) defined by \( \gamma(z) = 1 \otimes z \)).

So for \( \alpha \geq \alpha_0 \) we have that

\[
\| (z_\alpha \ast \xi)(r) - \delta(d_i) \int_G (1 \otimes \left( \sum_{j=1}^{n} \int_{H} \frac{1}{\sqrt{\Delta h}} g_j \cdot g_j^h \, dh \, f_s \eta(r) \right)) \delta(\delta_{u_0}(d_i)) \, ds \, \delta(a) \|
\]

\[
\leq \| \delta(d_i) (1 \otimes \left( \int_{H} \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} g_j g_j^h (f_s^h \eta(h^{-1} r) - f_s \eta(r)) \, dh \right)) \delta(\delta_{u_0}(d_i)) \, ds \, \|u\|_{B(G)} \cdot \|\delta(a)\|
\]

\[
(\text{since } \|\delta(\delta_{u_0}(d_i))\| = \| \delta_{u_0}(d_i) \|_A \leq \|u\|_{B(G)} \cdot \|d_i\|_A \leq \|u\|_{B(G)} \text{ and } \|\delta(d_i)\| = \|d_i\|_A \leq 1)
\]

\[
\leq \int_G \left| \int_{H} \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} g_j \cdot g_j^h \cdot (f_s^h \eta(h^{-1} r) - f_s \eta(r)) \, dh \right| \|u\|_{B(G)} \cdot \|\delta(a)\|
\]

\[
(\text{since } \|1 \otimes M_G(\gamma)\| = \|\gamma\|_{C_s(G)})
\]

\[
\leq \int_G \left| \int_{H} \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} g_j \cdot g_j^h \cdot (f_s^h \eta(h^{-1} r) - f_s \eta(r)) \, dh \right| \|u\|_{B(G)} \cdot \|a\|_A
\]

\[
\leq \kappa \cdot 2 \cdot \sigma/(6 \cdot \kappa \cdot \|u\|_{B(G)} \cdot \|a\|_A) \cdot \|u\|_{B(G)} \cdot \|a\|_A = \sigma/3
\]

The last inequality follows from the fact that if \( t \in G \), then

\[
\left| \int_{H} \frac{1}{\sqrt{\Delta h}} \sum_{j=1}^{n} g_j(t) g_j(t h) \left( f(s^{-1} t h) \eta(h^{-1} r) - f(s^{-1} t) \eta(r) \right) \, dh \right|
\]

\[
\leq \sum_{j=1}^{n} g_j(t) \int_{H} \frac{1}{\sqrt{\Delta h}} g_j(t h) |f(s^{-1} t h) \eta(h^{-1} r) - f(s^{-1} t) \eta(r)| \, dh
\]

\[
\leq \sum_{j=1}^{n} g_j(t) \int_{H} \frac{1}{\sqrt{\Delta h}} g_j(t h) \, dh \cdot \sigma/(6 \cdot \kappa \cdot \|u\|_{B(G)} \cdot \|a\|_A)
\]

(by § 6 and since the \( h \)-support of the integrand is contained in \( N_0 \))
\[
\leq \left(1 + \sum_{j=1}^{n} f_j(t)\right) \cdot \sigma / (6 \cdot \kappa \cdot \|u\|_{B(G)} \cdot \|a\|_A)
\]

\[
\leq 2 \cdot \sigma / (6 \cdot \kappa \cdot \|u\|_{B(G)} \cdot \|a\|_A) \quad \text{(by \(\S 4\) and \(\S 8\))}
\]

So we have that \(\alpha \geq \alpha_0\) implies (i) \(\leq \sigma / 3\) for all \(r \in F\).

Now we show that for \(\alpha \geq \alpha_0\) (ii) \(\leq \sigma / 3\) for all \(r \in F\). Since

\[
\delta(d_i^2)\xi(r) = \eta(r)\delta(d_i^2)(1 \otimes (\bar{u} * f))\delta(a) = \eta(r)\delta(d_i) \int_G (1 \otimes f_s)\delta(\delta_{u_s}(d_i)) \, ds \, \delta(a)
\]

we have that for all \(\alpha \geq \alpha_0\)

\[
\|\delta(d_i) \int_G (1 \otimes \left(\sum_{j=1}^{n} \int_H \frac{1}{\sqrt{\Delta h}} g_j \cdot g_j^h \, dh \cdot f_s \eta(r)\right))\delta(\delta_{u_s}(d_i)) \, ds \, \delta(a) - \delta(d_i^2)\xi(r)\|
\]

\[
= \|\eta(r)\delta(d_i) \int_G (1 \otimes \left(\sum_{j=1}^{n} \int_H \frac{1}{\sqrt{\Delta h}} g_j \cdot g_j^h \, dh - \sum_{j=1}^{n} f_j\right) \cdot f_s)\delta(\delta_{u_s}(d_i)) \, ds \, \delta(a)\|
\]

(since \(\sum f_j\) is identically one on \(C_0\) which by \(\S 7\) contains \(\text{supp} f_s\) for all \(s\) in the \(s\)-support of the integrand)

\[
\leq \|\eta\| \cdot \int_G \left\|\sum_{j=1}^{n} \int_H \frac{1}{\sqrt{\Delta h}} g_j \cdot g_j^h \, dh - \sum_{j=1}^{n} f_j\right\|_G \cdot \|f_s\|_{C_s(G)} \cdot \|u_s\|_{B(G)} \cdot \|a\|_A
\]

\[
\leq \frac{\|\eta\|_{C_s(G)} \cdot \kappa \cdot \|f\|_{C_s(G)} \cdot \|u\|_{B(G)} \cdot \|a\|_A}{3 \cdot \|\eta\|_{C_s(G)} \cdot \kappa \cdot \|f\|_{C_s(G)} \cdot \|u\|_{B(G)} \cdot \|a\|_A} \quad \text{(by \(\S 5\) and \(\S 8\))}
\]

\[
= \sigma / 3
\]

So we have that \(\alpha \geq \alpha_0\) implies (ii) \(\leq \sigma / 3\) for all \(r \in F\), showing that \(z_\alpha * \xi \to \xi\) uniformly on \(F\) and hence on \(G\) since the support of the \(z_\alpha * \xi\) is contained in \(F\) for all \(\alpha\).

Lemma 6.5 The linear span \(\mathcal{O}\) of the maps

\[
\gamma_{x,y}(h) = \frac{1}{\sqrt{\Delta h}} x\delta_h(y^*) \quad x, y \in D
\]

- 128 -
is dense for the inductive limit topology on $C_c(H, A \times \delta G)$ and hence is dense in $L^1(H, A \times \delta G)$.

**Proof** Firstly we observe that if $\xi \in I_H$, then we have

$$
(\gamma_{z,y} \ast \xi)(h) = \int_H \gamma_{z,y}(r) \delta_r(\xi(r^{-1} h)) \, dr
$$

$$
= \int_H \frac{1}{\sqrt{\Delta r}} x \delta_r(\xi^*(r^{-1} h)) \, dr
$$

$$
= \int_H \frac{1}{\sqrt{\Delta(h r)}} x \delta_{hr}(\xi^*(r^{-1})) \, dr
$$

$$
= \frac{1}{\sqrt{\Delta h}} \int_H x \delta_h \left( \delta_r \left( \frac{1}{\sqrt{\Delta r}} \xi(\xi^{*-1})^*(y) \right) \right) \, dr
$$

$$
= \frac{1}{\sqrt{\Delta h}} \int_H x \delta_h \left( \left( \int H \sqrt{\Delta r} \left( \frac{1}{\Delta r} \delta_r(\xi(r^{-1})) \right) \delta_r(\xi^*(r)) \, dr \right)^* \right)
$$

$$
= \frac{1}{\sqrt{\Delta h}} x \delta_h \left( \left( \int H \sqrt{\Delta r} \xi^*(r) \delta_r(\xi) \, dr \right)^* \right)
$$

$$
= \frac{1}{\sqrt{\Delta h}} x \delta_h \left( (\xi^* \ast y)^* \right)
$$

$$
= \gamma_{z,y} \ast \xi \ast y(h).
$$

Now let $(z_\alpha)_{\alpha \in A}$ be the net of lemma 6.4. Note that $z_\alpha = \gamma_{x_\alpha, z_\alpha}$, where $x_\alpha = \delta(\delta_\omega(e_j))(1 \otimes g_i)$. So by the above

$$
z_\alpha \ast \xi = \gamma_{x_\alpha, z_\alpha} \ast \xi = \gamma_{x_\alpha, \xi \ast \xi_\alpha} \in \mathcal{O}.
$$

Let $\xi$ be of the form $\xi(r) = \eta(r)(1 \otimes (\tilde{u} \ast f))\delta(a)$, where $\eta \in C_c(H)$, $f \in C_c(G)$, $u \in A_c(G)$ and $a = \delta_v(b)$, for some $b \in A$ and $v \in A_c(G)$. Then by lemma 6.4 we have that $\gamma_{x_\alpha, \xi \ast \xi_\alpha} \to \xi$ in the inductive limit topology, that is, there exists a fixed compact set $F$ such that $\text{supp} \gamma_{x_\alpha, \xi \ast \xi_\alpha} \subset F$ and $\gamma_{x_\alpha, \xi \ast \xi_\alpha} \to \xi$ uniformly on $F$. To see that $\gamma_{x_\alpha, \xi \ast \xi_\alpha} \to \xi$ in $L^1(H, A \times \delta G)$, let $\epsilon > 0$ and choose $\alpha_1 \in A$ such that for all $\alpha > \alpha_1$

$$
\|\gamma_{x_\alpha, \xi \ast \xi_\alpha}(r) - \xi(r)\| < \epsilon / \mu_H(F) \quad \forall r \in F.
$$

- 129 -
Then for all $\alpha \geq \alpha_0$

$$\|\gamma_{z,\alpha} \xi^* x - \xi\|_{L^1(H, A \times \delta G)} = \int_H \|\gamma_{z,\alpha} \xi^* x\xi - \xi(r)\| \, dr$$

$$\leq \mu_H(F) \epsilon / \mu_H(F)$$

$$= \epsilon,$$

(since $\text{supp} \gamma_{z,\alpha} \xi^* x \subset F$ for all $\alpha$ sufficiently large). So $\gamma_{z,\alpha} \xi^* x \rightarrow \xi$ for all $\xi$ of the above special form. Since $D$ is dense in $A \times \delta G$ (Theorem 4.12) the linear span of such $\xi$ are dense in $L^1(H, A \times \delta G)$ and hence $O$ is dense in $L^1(H, A \times \delta G)$. \(\square\)

**Proposition 6.6** The map $\Theta : (A \times \delta G) \times \delta H \rightarrow L(D)$ of Corollary 6.3 is an injective $\ast$-homomorphism of $(A \times \delta G) \times \delta H$ onto $K(D)$.

**Proof** Let $z \in D$. Then

$$\{\Theta(\gamma_{z,y})(z) = \int_H \sqrt{\Delta r} \gamma_{z,y}(r) \delta_r(z) \, dr$$

$$= \int_H x \delta_r(y^* z) \, dr$$

$$= x \langle y , z \rangle_D,$$  \hspace{1cm} (by lemma 5.6)

that is, $\Theta(\gamma_{z,y})$ is the generator $T_{z,y}$ of $K(D)$. By Lemma 6.5 the linear span of the $\gamma_{z,y}$ is dense in $L^1(H, A \times \delta G)$, and hence in $(A \times \delta G) \times \delta H$. Also the linear span of the $T_{z,y}$ is dense in $K(D)$. Hence $\Theta$ maps $(A \times \delta G) \times \delta H$ onto $K(D)$.

Now we proceed to show that $\Theta$ is injective. Let $i$ be the representation of $A \times \delta G$ and $A \times \delta G/H$ on $\mathcal{H} \otimes L^2(G)$ defined by $i(z) = z$. Let $A$ be the representation of $K(D)$ on $Z$ induced from $i$, where $Z$ is $D \otimes \mathcal{H} \otimes L^2(G)$ factored and completed with respect to the pre-inner product

$$\langle x \otimes \xi , y \otimes \eta \rangle_{D \otimes \mathcal{H} \otimes L^2(G)} = \langle \{1 \otimes \rho_H\{\langle y , x \rangle_D\}\}(\xi) , \eta \rangle_{\mathcal{H} \otimes L^2(G)}.$$
Let \( i \) be the representation of \( A \times_\delta G \) on \( L^2(H, \mathcal{H} \otimes L^2(G)) \) defined by
\[
\{\{i(x)\}(\zeta)\}(h) = \{\delta_h(x)\}(\zeta(h)) \quad h \in H, \ \zeta \in C_c(H, \mathcal{H} \otimes L^2(G)).
\]

Let \( 1 \otimes \rho_H \) be the unitary representation of \( H \) on \( L^2(H, \mathcal{H} \otimes L^2(G)) \) defined by
\[
\{\{1 \otimes \rho_H(r)\}(\zeta)\}(h) = \sqrt{\Delta r} \zeta(hr).
\]

Now
\[
\{\{(1 \otimes \rho_H(h))i(x)(1 \otimes \rho_H(h)^*)\}(\zeta)\}(r) = \sqrt{\Delta r} \{\delta_{rh}(x)\}\{\{1 \otimes \rho_H^*(h)\}(\zeta)\}(rh)
\]
\[
= \{\delta_r(\delta_h(x))\}(\zeta(r))
\]
\[
= \{\{i(\delta_h(x))\}(\zeta)\}(r).
\]

Hence \((i, 1 \otimes \rho_H)\) is a covariant representation (the right regular representation) of \((A \times_\delta G, H, \delta)\). Since \( H \) is amenable \((i, 1 \otimes \rho_H)\) is faithful \([20 \S 7.7.5]\). We will show that \( i \times (1 \otimes \rho_H) \) is unitarily equivalent to \( \Lambda \circ \Theta \). Hence \( \Lambda \circ \Theta \) is also faithful and \( \Theta \) must be injective.

To see that \( i \times (1 \otimes \rho_H) \) is unitarily equivalent to \( \Lambda \circ \Theta \). Let
\[
\mathcal{F} = \{x(\xi) : x \in D, \ \xi \in \mathcal{H} \otimes L^2(G)\}.
\]

By lemma 5.8 (ii) \( D \) contains a net which converges to 1 strongly in \( B(\mathcal{H} \otimes L^2(G)) \).

Hence \( \mathcal{F} \) is dense in \( \mathcal{H} \otimes L^2(G) \). Define a map
\[
V : D \otimes \mathcal{F} \to L^2(H, \mathcal{H} \otimes L^2(G)) \quad \text{by} \quad \{V(x \otimes \xi)\}(h) = \{(\delta_h(x))\}(\xi).
\]

Firstly note that if \( \zeta = w(\xi) \in \mathcal{F} \) for \( w \in D \) and \( \xi \in \mathcal{H} \otimes L^2(G) \), then
\[
\{V(x \otimes \zeta)\}(h) = \{\delta_h(x)w\}(\xi),
\]

which shows that \( V(x \otimes \xi) \) is continuous and compactly supported and hence in \( L^2(H, \mathcal{H} \otimes L^2(G)) \) (since by lemma 5.6 the map : \( h \to \delta_h(x)w \) is continuous and compactly supported).
(i) Let $\epsilon > 0$. Let $g$ be the element of $C_c(H, F)$ defined by $g(h) = \alpha(h)\omega(\xi)$, where $\alpha \in C_c(H)$, $\omega \in D$ and $\xi \in \mathcal{H} \otimes L^2(G)$. By lemma 6.5 there exists a compact subset $F$ of $H$ and $\gamma_{y_{ij},x_{ij}}$ for some $x_{ij}, y_{ij} \in D$ with $\text{supp} \gamma_{y_{ij},x_{ij}} \subset F$ for all $i$ and $j = 1, \ldots, n_i$ such that

$$\sum_{j=1}^{n_i} \gamma_{y_{ij},x_{ij}} \to \left( h \to \sqrt{\Delta h} g^*(h) \right)$$

uniformly. So choose $i_0$ such that $i \geq i_0$ implies

$$\left\| \sum_{j=1}^{n_i} \gamma_{y_{ij},x_{ij}}(h) - \sqrt{\Delta h} g^*(h) \right\|_{A \times \mathcal{L}^2(G)}^2 < \frac{\epsilon^2}{\kappa^2 \cdot \|\xi\|^2} \cdot \mu_H(F) \quad \forall h \in F,$$

where $\kappa$ is the maximum value of $\sqrt{\Delta h}$ on $F$. Then

$$\left\| \left\{ \sum_{j=1}^{n_i} x_{ij} \otimes (y_{ij}(\xi)) \right\}(h) - g(h) \right\|_{L^2(H, \mathcal{H} \otimes L^2(G))}

= \int_H \left\| \sum_{j=1}^{n_i} \{ \delta_h(x_{ij}) \} (y_{ij}(\xi)) - \alpha(h)w(\xi) \right\|_{A \otimes \mathcal{L}^2(G)}^2 \, dh

\leq \int_H \left\| \sum_{j=1}^{n_i} \delta_h(x_{ij})y_{ij} - \alpha(h)w \right\|_{B(A \otimes \mathcal{L}^2(G))}^2 \, dh \cdot \|\xi\|^2

= \int_H \left\| \sum_{j=1}^{n_i} y_{ij}^* \delta_h(x_{ij}^*) - \overline{\alpha(h)}w^* \right\|_{B(A \otimes \mathcal{L}^2(G))}^2 \, dh \cdot \|\xi\|^2

\leq \int_H \left\| \sum_{j=1}^{n_i} \gamma_{y_{ij},x_{ij}}(h) - \frac{1}{\sqrt{\Delta h}} \overline{\alpha(h)}w^* \right\|_{A \times \mathcal{L}^2(G)} \, dh \cdot \kappa^2 \cdot \|\xi\|^2

\leq \int_H \left\| \sum_{j=1}^{n_i} \gamma_{y_{ij},x_{ij}}(h) - \sqrt{\Delta h} g^*(h) \right\|_{A \times \mathcal{L}^2(G)} \, dh \cdot \kappa^2 \cdot \|\xi\|^2

< \mu_H(F) \cdot \left( \epsilon^2 / \kappa^2 \cdot \|\xi\|^2 \cdot \mu_H(F) \right) \cdot \kappa^2 \cdot \|\xi\|^2

= \epsilon.$$

- 132 -
Now \( C_c(H) \otimes \mathcal{F} \) is dense in \( L^2(H, \mathcal{H} \otimes L^2(G)) \). Hence \( V \) maps onto a dense subspace.

(ii) Let \( w, x, y \in D \) and \( \xi, \eta, \zeta \in \mathcal{H} \otimes L^2(G) \) with \( \zeta = w(\xi) \). Then

\[
\langle x \otimes \zeta, y \otimes \eta \rangle_{D \otimes \mathcal{H} \otimes L^2(G)}
\]

\[
= \langle \langle y, x \rangle_D(\zeta), \eta \rangle_{\mathcal{H} \otimes L^2(G)}
\]

\[
= \langle \langle \{ y, x \rangle_D \rangle w(\xi), \eta \rangle_{\mathcal{H} \otimes L^2(G)}
\]

\[
= \langle \int_H \hat{\delta}_h(y \cdot x)w \, dh \rangle(\xi), \eta \rangle_{\mathcal{H} \otimes L^2(G)} \quad \text{ (by lemma 5.6)}
\]

\[
= \int_H \langle \{ \hat{\delta}_h(y \cdot x)w \rangle(\xi), \eta \rangle_{\mathcal{H} \otimes L^2(G)} \, dh
\]

\[
= \int_H \langle \{ \hat{\delta}_h(x) \}(w(\xi)), \{ \hat{\delta}_h(y) \}(\eta) \rangle_{\mathcal{H} \otimes L^2(G)} \, dh
\]

\[
= \int_H \langle \{ V(x \otimes \zeta) \}(h), \{ V(y \otimes \eta) \}(h) \rangle_{L^2(H, \mathcal{H} \otimes L^2(G))} \, dh
\]

\[
= \langle V(x \otimes \zeta), V(y \otimes \eta) \rangle_{L^2(H, \mathcal{H} \otimes L^2(G))}.
\]

So \( V \) preserves the pre-inner products.

(iii) Suppose \( \gamma \in I_H, x \otimes \zeta \in D \otimes \mathcal{F} \), with \( \zeta = w(\xi) \) and \( \eta \in C_c(H, \mathcal{H} \otimes L^2(G)) \).

Then

\[
\langle \{ V(\Lambda \circ \Theta(\gamma)) \}(x \otimes \zeta), \eta \rangle_{L^2(H, \mathcal{H} \otimes L^2(G))}
\]

\[
= \int_H \langle \{ V(\Lambda \circ \Theta(\gamma)) \}(x \otimes \zeta) \rangle(h), \eta(h) \rangle_{\mathcal{H} \otimes L^2(G)} \, dh
\]

\[
= \int_H \langle \{ V(\{ \Theta(\gamma) \}(x) \otimes \zeta) \} \rangle(h), \eta(h) \rangle_{\mathcal{H} \otimes L^2(G)} \, dh
\]

\[
= \int_H \langle \{ V((\gamma \cdot x) \otimes \zeta) \} \rangle(h), \eta(h) \rangle_{\mathcal{H} \otimes L^2(G)} \, dh
\]

\[
= \int_H \langle \hat{\delta}_h(\gamma \cdot x) \rangle(h), \eta(h) \rangle_{\mathcal{H} \otimes L^2(G)} \, dh
\]
\[ \int_{H} \langle \{\hat{\delta}_{h}\left(\int_{H} \sqrt{\Delta r} \gamma(r)\delta_{r}(x) \, dr\right)\}(\zeta), \eta(h) \rangle_{\mathcal{H} \otimes L^{2}(G)} \, dh \]

\[ = \int_{H} \langle \{\int_{H} \sqrt{\Delta r} \hat{\delta}_{h}(\gamma(r)\delta_{r}(x)) \, dr\}(\zeta), \eta(h) \rangle_{\mathcal{H} \otimes L^{2}(G)} \, dh \]

\[ = \int_{H} \int_{H} \langle \sqrt{\Delta r} \{\hat{\delta}_{h}(\gamma(r)\delta_{r}(x))w\}(\xi), \eta(h) \rangle_{\mathcal{H} \otimes L^{2}(G)} \, dr \, dh \]

Since \( \gamma \) is continuous with compact support, so is \( (h, r) \to \hat{\delta}_{h}(\gamma(r)\delta_{r}(x))w \), and since we are assuming that \( \eta \) is continuous with compact support, the integrand is clearly integrable. Thus we can change the order of integration to obtain,

\[ = \int_{H} \int_{H} \langle \{\hat{\delta}_{h}(\gamma(r))\}(\sqrt{\Delta r} \{V(x \otimes \zeta)\}(hr)), \eta(h) \rangle_{\mathcal{H} \otimes L^{2}(G)} \, dh \, dr \]

\[ = \int_{H} \int_{H} \langle \{\hat{\delta}_{h}(\gamma(r))\}(\{(1 \otimes \rho H)(r)\}(V(x \otimes \zeta)))(h)), \eta(h) \rangle_{\mathcal{H} \otimes L^{2}(G)} \, dh \, dr \]

\[ = \int_{H} \int_{H} \langle \{\hat{\delta}_{h}(\gamma(r))\}(\{V(x \otimes \zeta)\}(h)), \eta(h) \rangle_{\mathcal{H} \otimes L^{2}(G)} \, dh \, dr \]

\[ = \int_{H} \langle \{\hat{\delta}_{h}(\gamma(r))\}(V(x \otimes \zeta)), \eta \rangle_{L^{2}(\mathcal{H}, \mathcal{H} \otimes L^{2}(G))} \, dh \]

\[ = \langle \{\int_{H} \hat{\delta}_{h}(\gamma(r))(1 \otimes \rho H)(r) \, dr \}V(x \otimes \zeta), \eta \rangle_{L^{2}(\mathcal{H}, \mathcal{H} \otimes L^{2}(G))} \]

\[ = \langle \{\{i \times (1 \otimes \rho H)\}(\gamma)V\}(x \otimes \zeta), \eta \rangle_{L^{2}(\mathcal{H}, \mathcal{H} \otimes L^{2}(G))} \]

i.e., \( \langle \{V(\Lambda \circ \Theta)(\gamma)) - \{i \times (1 \otimes \rho H)\}(\gamma)V\}(x \otimes \zeta), \eta \rangle_{L^{2}(\mathcal{H}, \mathcal{H} \otimes L^{2}(G))} = 0 \).

By letting \( \eta = \{V(\Lambda \circ \Theta)(\gamma)) - \{i \times (1 \otimes \rho H)\}(\gamma)V\}(x \otimes \zeta) \) we see that

\[ V(\Lambda \circ \Theta)(\gamma)) = \{i \times (1 \otimes \rho H)\}(\gamma)V \]

That is \( V \) intertwines the \( \mathcal{I}_{H} \) actions.

So \( V \) extends to a unitary operator from \( Z \) onto \( L^{2}(\mathcal{H}, \mathcal{H} \otimes L^{2}(G)) \) which intertwines the actions of \( (A \times_{\delta} G) \times_{\delta} H \). Hence \( i \times (1 \otimes \rho H) \) is unitarily equivalent to \( \Lambda \circ \Theta \) as claimed.
Theorem 6.7 Suppose $\delta : A \to \tilde{M}(A \otimes C^*_r(G))$ is a non-degenerate coaction of a locally compact group $G$ on a $C^*$-algebra $A$, $H$ is a closed normal amenable subgroup of $G$, $\delta | : A \to \tilde{M}(A \otimes C^*_r(G/H))$ is the restriction of $\delta$ to $G/H$, as in lemma 2.4, and $\hat{\delta}$ is the dual action of $G$ on $A \times_{\hat{\delta}} G$. Then

$$(A \times_{\hat{\delta}} G) \times_{\hat{\delta}} H \approx A \times_{\delta l}(G/H).$$

Proof By theorem 5.10 $D$ is a right $D_H$-rigged space, so by theorem 1.10, $K(D)$ is strongly Morita equivalent to $A \times_{\delta} (G/H)$. But by proposition 6.6 and proposition 2.6 respectively, $K(D)$ is isomorphic to $(A \times_{\delta} G) \times_{\hat{\delta}} H$, and $A \times_{\delta} (G/H)$ is isomorphic to $A \times_{\delta l}(G/H)$, establishing the theorem.

Lemma 6.8 Let $\Theta$, $i_{A \times_{\delta} G}$ and $j_{A \times_{\delta} G}$ be as in corollary 6.3, proposition 1.15 and corollary 5.11, respectively. Then

$$\Theta \circ i_{A \times_{\delta} G} = j_{A \times_{\delta} G}.$$

Proof We shall consider $\Theta \circ i_{A \times_{\delta} G}$ and $j_{A \times_{\delta} G}$ as maps from $M((A \times_{\delta} G) \times_{\hat{\delta}} H) \supset A \times_{\delta} G$ to $M(K(D)) \subset L(D)$. If $\gamma_{w,z}$, $T_{w,z}$ and $\theta_z$ are as in lemma 6.5, 6.22 and lemma 1.12, respectively, then

$$\Theta \circ i_{A \times_{\delta} G}(y) \Theta(\gamma_{w,z}) = \Theta(i_{A \times_{\delta} G}(y) \gamma_{w,z})$$

$$= \Theta(\gamma_{w,z})$$

(since $\{i_{A \times_{\delta} G}(y) \gamma_{w,z}\}(h) = y(1/\sqrt{\Delta h})\omega \delta_k(z^*) = \gamma_{w,z}$)

$$= T_{w,z}$$

$$= \theta_y T_{w,z}$$

$$= \theta_y \Theta(\gamma_{w,z}).$$

So (as multipliers) $\Theta \circ i_{A \times_{\delta} G}(y) = \theta_y = j_{A \times_{\delta} G}(y)$ for all $y \in D$ and hence for all $y \in A \times_{\delta} G$. 

- 135 -
Now applying [22 thm. 6.29] to the Hermitian $A \times_\delta (G/H)$-rigged $A \times_\delta G$ module $X$, used in chapter 5 to induce representations from $A \times_{\delta 1} (G/H)$ to $A \times_\delta G$, we see that a representation $\mu$ of $A \times_\delta G$ on $Q$ is unitarily equivalent to a representation induced (via $X$) from a representation $\nu$ of $A \times_{\delta 1} (G/H)$ (more precisely, from the representation $\nu \circ \Gamma^{-1}$ of $A \times_\delta (G/H)$, where $\Gamma$ is as in proposition 2.6) if, and only if, there exists a representation $\psi$ of $K(X)$ on $Q$ such that

$$\{\mu(b)\}((\psi(T))(\xi)) = \{\psi(\alpha \circ j_{A \times_\delta G}(b)T)\}(\xi),$$

for all $b \in A \times_\delta G$, $T \in K(X)$ and $\xi \in Q$, where $\alpha : L(D) \to L(X)$ and $j_{A \times_\delta G} : A \times_\delta G \to L(D)$ are the maps of lemma 1.11 and corollary 5.11, respectively. Recognising $L(X)$ as $M(K(X))$ [6 lemma 16] this is if, and only if, there exists a representation $\psi$ of $K(X)$ on $Q$ such that

$$\mu(b) = \psi(\alpha \circ j_{A \times_\delta G}(b)) \quad \forall \ b \in A \times_\delta G,$$

where $\psi$ has been extended to $M(K(X))$. Since $\Theta : (A \times_\delta G) \times_\delta H \to K(D)$ is an isomorphism and $\Theta^{-1} \circ j_{A \times_\delta G} = i_{A \times_\delta G}$ (lemma 6.8), this is if, and only if, there exists a representation $\phi (= \psi \circ \alpha \circ \Theta)$ of $(A \times_\delta G) \times_\delta H$ on $Q$ such that

$$\mu(b) = \phi(i_{A \times_\delta G}(b)) \quad \forall \ b \in A \times_\delta G,$$

but this is if, and only if, there exists a unitary representation $U$ of $H$ on $Q$ such that $(\mu, U)$ is a covariant representation of $(A \times_\delta G, H, \delta)$, where $\delta$ is the dual action (1.31) of $H$ on $A \times_\delta G$.

So we have the following theorem:

**The Imprimitivity Theorem 6.9** A representation $\mu$ of $A \times_\delta G$ on $Q$ is induced (via $X$) from a representation $\nu$ of $A \times_{\delta 1} (G/H)$ if and only if there exists a unitary representation $U$ of $H$ on $Q$ such that

$$\mu(\delta h(b)) = U_h \mu(b) U_h^* \quad \forall \ h \in H, \ b \in A \times_\delta G.$$ 

- 136 -
Note that we could have stated the imprimitivity theorem for representations induced via $D$, since representations induced via $D$ are unitarily equivalent to those induced via $X$ (chapter 1 §4).

We now wish to investigate the continuity of the induction and restriction processes of theorem 5.10 and proposition 2.4 respectively.

Let $\Gamma$ be the isomorphism of proposition 2.6 and let $\mu$ be a representation of $A \times \delta G$. Then

$$\text{res}^{A \times \delta G}_{A \times \delta G(\text{G/H})} \mu = \mu \circ \Gamma,$$

where $\mu$ has been extended to $M(A \times \delta G) \supset A \times \delta (G/H)$. To see this let

$$(\mu \circ \delta, \mu \circ (1 \otimes M_G))$$

be the covariant representation of $(A, G, \delta)$ corresponding to $\mu$. Then by proposition 1.19 the covariant representation of $(A, G/H, \delta)$ corresponding to $\mu \circ \Gamma$ is

$$(\mu \circ \Gamma \circ \delta, \mu \circ \Gamma \circ (1 \otimes M_{G/H})) = (\mu \circ \delta, \mu \circ (1 \otimes M_G) \circ q) \quad = \text{res}^{A \times \delta G}_{A \times \delta G(\text{G/H})} (\mu \circ \delta, \mu \circ (1 \otimes M_G)).$$

Now we present a similar result for $\text{ind}^{A \times \delta G}_{A \times \delta G(\text{G/H})}$.

Let $X$ be as above. Then $X$ is a Hermitian $A \times \delta (G/H)$-rigged $K(X)$ module [22 prop. 6.14] and thus establishes the induction process

$$\text{ind}^{K(X)}_{A \times \delta (\text{G/H})} : \text{Rep}(A \times \delta (G/H)) \to \text{Rep}(K(X)).$$

We define the map

$$\text{ind}^{(A \times \delta G) \times \delta H}_{A \times \delta (\text{G/H})} : \text{Rep}(A \times \delta (G/H)) \to \text{Rep}((A \times \delta G) \times \delta H)$$

by

$$\text{ind}^{(A \times \delta G) \times \delta H}_{A \times \delta (\text{G/H})} \nu = (\text{ind}^{K(X)}_{A \times \delta (\text{G/H})} \nu) \circ \alpha \circ \Theta,$$

- 137 -
Then by Rieffel's imprimitivity theorem \[22 \text{ thm. 6.29}\]

\[
\text{ind}_{A \times_\delta (G/H)}^{A \times_\delta G} \nu = (\text{ind}_{A \times_\delta (G/H)}^{K(X)} \nu \circ \alpha \circ j_{A \times_\delta G} = (\text{ind}_{A \times_\delta (G/H)}^{(A \times_\delta G) \times \delta H} \nu \circ i_{A \times_\delta G} \] . \tag{12}

Suppose \( P : A \to M(B) \) is a \(*\)-homomorphism. Then we can define a map

\[
P^* : \text{Ideals}(B) \to \text{Ideals}(A)
\]

by

\[
P^*(I) = \{ a \in A : P(a)B \subseteq I \} ,
\]

where \( I \) is an ideal of \( B \). Green [6 prop. 9] shows that \( P^* \) preserves intersections and is continuous with respect to the (inner) hull-kernel topology (see chapter 1 §4). The relevance of \( P^* \) lies in the fact that if \( \psi : B \to B(Q) \) is a representation of \( B \) on \( Q \), then

\[
\ker(\psi \circ P) = \{ a \in A : P(a)B \subseteq I \} = P^*(\ker \psi) ,
\]

where \( \psi \) has been extended to \( M(B) \).

Now from \( \S 11 \), if \( \mu \) is a representation of \( A \times_\delta G \), then

\[
\text{res}_{A \times_\delta (G/H)}^{A \times_\delta G} \mu = \mu \circ \Gamma .
\]

Hence

\[
\ker(\text{res}_{A \times_\delta (G/H)}^{A \times_\delta G} \mu) = \Gamma^*(\ker \mu) ,
\]

which shows that weakly equivalent representations of \( A \times_\delta G \) are mapped to weakly equivalent representations of \( A \times_\delta (G/H) \). Hence the restriction process can be considered to be a map on ideals and as such is continuous, since \( \Gamma^* \) is.

Now from \( \S 13 \), if \( \nu \) is a representation of \( A \times_\delta (G/H) \), then

\[
\text{ind}_{A \times_\delta (G/H)}^{A \times_\delta G} \nu = (\text{ind}_{A \times_\delta (G/H)}^{(A \times_\delta G) \times \delta H} \nu \circ i_{A \times_\delta G} .
\]
So

\[ \ker(\text{ind}_{A \times \delta (G/H)}^{A \times \delta (G)}) \cap \ker(\text{ind}_{A \times \delta (G/H)}^{A \times \delta (G)}) = i_{A \times \delta (G)}^{\star}(\ker(\text{ind}_{A \times \delta (G/H)}^{A \times \delta (G)}) \cap \ker(\text{ind}_{A \times \delta (G/H)}^{A \times \delta (G)})) \]

\[ = i_{A \times \delta (G)}^{\star}(\ker(\text{ind}_{A \times \delta (G/H)}^{A \times \delta (G)})) \], \quad \text{(by \textsection 1.21)}

where \( h : \text{Ideals}(A \times \delta (G/H)) \rightarrow \text{Ideals}((A \times \delta G) \times \delta H) \) is the continuous bijection of \textsection 1.20 determined by the equivalence bimodule \( X \). This shows that the induction process \( \text{ind}_{A \times \delta (G/H)}^{A \times \delta (G)} \) maps weakly equivalent representations to weakly equivalent representations and thus can be considered a map on ideals. As such it is clearly continuous since both \( h \) and \( i_{A \times \delta (G)}^{\star} \) are.

So summing up we have the following proposition.

**Proposition 6.10** The maps

\[ \text{ind}_{A \times \delta (G/H)}^{A \times \delta (G)} : \text{Ideals}(A \times \delta (G/H)) \rightarrow \text{Ideals}(A \times \delta G) \]

\[ \text{res}_{A \times \delta (G/H)}^{A \times \delta (G)} : \text{Ideals}(A \times \delta G) \rightarrow \text{Ideals}(A \times \delta (G/H)) \]

are continuous with respect to the hull-kernel topologies.

Proposition 5.12 shows that (the first part of) this result is an extension of Gootman and Lazar's theorem 3.8 [5]. Theirs is the special case \( H = G \).
References


