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Line Bundles and Curves on a del Pezzo Order

by

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Abstract

Orders on surfaces provided a rich source of examples of noncommutative surfaces. In [HS05] the authors prove the existence of the analogue of the Picard scheme for orders and in [CK11] the Picard scheme is explicitly computed for an order on $\mathbb{P}^2$ ramified on a smooth quartic. In this paper, we continue this line of work, by studying the Picard and Hilbert schemes for an order on $\mathbb{P}^2$ ramified on a union of two conics. Our main result is that, upon carefully selecting the right Chern classes, the Hilbert scheme is a ruled surface over a genus two curve. Furthermore, this genus two curve is, in itself, the Picard scheme of the order.

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Chapter 1

Introduction

Throughout the thesis we assume all objects and maps are defined over an algebraically closed field k of characteristic zero. All rings have an identity element. We denote the dimension of any cohomology group over k by the name of the group written with a non-capital letter for e.g. $\mathrm{ext}^i_A(M, N) := \dim_k \mathrm{Ext}^i_A(M, N)$ and similarly for h^i and hom.

1.1 Overview

Generalising algebraic geometry to a noncommutative setting has been a topic of much interest in the last few decades. This new field brings together techniques from algebraic geometry, ring theory, representation theory and category theory to make sense of, and study, noncommutative schemes. Naturally enough, the first place to begin is by studying noncommutative projective curves. These have now been completely described and as it turns out, essentially, there are no strictly noncommutative projective curves. See [SvdB01] for further details on this. Our attention thus turns to noncommutative projective surfaces where the theory is very rich. One particularly nice class of projective surfaces is that of orders on surfaces, which we now define.

Definition 1.1.1. *Let X be a normal integral surface. An* **order** *A on X is a coherent torsion free sheaf of $\mathcal{O}_X$-algebras such that $k(A) := A \otimes_X k(X)$ is a central simple $k(X)$-algebra. X is called the* **centre** *of A.*

For example, if X is as above and V is a vector bundle on X, then $\mathcal{E}nd_{\mathcal{O}_X} V$ is an order on X. More generally any Azumaya algebra on X is also an order on X. Furthermore, any Azumaya algebra on X is in fact a **maximal order** in the sense that it is not properly included in any other order, see Proposition 1.8.2 of [AdJ] for a proof of this.

Since orders are finite over their centres they are in some sense only mildly noncommutative and many classical geometric techniques can be used to study them. Orders should be thought of as degenerations of Azumaya algebras since given an order A on X, there exists a dense open subset $U \subset X$ such that $A|_U$ is Azumaya on U. The closed locus of points where A is not Azumaya is called the **ramification locus** of A and is an important invariant of the order.

In [CI05] the authors develop a minimal model program for orders on projective surfaces and show that there are only three possibilities:

- the order has a unique minimal model up to Morita equivalence,
- the order is ruled, or
- it is del Pezzo.

Thus del Pezzo orders play an important part in the general theory of noncommutative surfaces, which motivates their study. The precise definition of a del Pezzo order is given in Definition 1.4.1. For now, it suffices to say that a particularly nice aspect about del Pezzo orders is the fact that the moduli space of A-line bundles, defined in the case where $k(A)$ is a division ring, as locally projective A-modules of rank one, is smooth. The simplest interesting examples of such orders turn out to be those on $\mathbb{P}^2$ ramified on either a cubic or a quartic. Motivated by the results in [CK11] where the authors, Chan and Kulkarni, study the moduli space of an order on $\mathbb{P}^2$ ramified on a smooth quartic, we chose to follow a similar path but to study one ramified on a union of two conics.

To enable us to begin our project, we use the noncommutative cyclic covering trick, described in Chapter 1.2.1, to construct our order on $\mathbb{P}^2$. The key ingredient to this construction, is a double cover $Y := \mathbb{P}^1 \times \mathbb{P}^1 \to Z := \mathbb{P}^2$, a line bundle $L \in \text{Pic } Y$ and a morphism $\phi : L_\sigma^{\otimes 2} \to \mathcal{O}_Y$ where σ is the covering involution. Using this data one constructs a sheaf of algebras A on Y which is an order on Z.

The main tool we use for studying A-modules is the simple observation that any such module is also naturally an $\mathcal{O}_Y$-module. In particular, this allows us to talk about the Chern classes and semistability of A-modules when viewed as $\mathcal{O}_Y$-modules. Furthermore, since any A-line bundle is a rank two vector bundle on Y, their study is rather different from the study of the Picard scheme of Y and much closer related to the study of rank two vector bundles. The main points of difference are that, first of all, A-line bundles do not form a group for they are only left A-modules and so their moduli space is not naturally a group scheme. Furthermore, the second Chern class, which is zero when one looks at line bundles in the usual setting, plays a crucial role in their study, as do semistability considerations. More precisely, we are interested in studying those A-line bundles which have minimal second Chern class.

It is certainly not obvious that one can place a bound on the second Chern class of A-line bundles and hence talk about those A-line bundles with "minimal second Chern class". For Chan and Kulkarni, this was achieved easily from the fact that for them, ϕ was an isomorphism which implied (Proposition 3.8 in [CK11]) that any A-line bundle was automatically μ-semistable and so by invoking Bogomolov's inequality, this aim was achieved. The authors used the μ-semistability property further by noting by simply forgetting the extra A-module structure, one obtains the map

$$\left\{\begin{array}{c}\text{moduli space of}\\ A\text{-line bundles}\\ \text{with minimal } c_2\end{array}\right\} \to \left\{\begin{array}{c}\text{moduli space of}\\ \mu\text{-semistable rank}\\ \text{two vector bundles on } Y\end{array}\right\}.$$

It is the careful analysis of this map that allowed Chan and Kulkarni to prove that their moduli space was a genus two curve.

In our case, ϕ will not be an isomorphism, and even though we will be able to deduce a lower bound for the second Chern class (Proposition 2.1.3), the above map of moduli spaces will not be available for us, simply because A-modules will turn out to be not μ-semistable in general. Thus we will use a totally different approach.

Having bound the second Chern class we will show that it suffices to consider only two possible first Chern classes. It is well known that $\operatorname{Pic} Y = \mathbb{Z} \oplus \mathbb{Z}$ and from the exponential sequence one easily obtains that $\operatorname{Pic} Y \simeq H^2(Y, \mathbb{Z})$. Since the first Chern class of a coherent sheaf on Y is an element of $H^2(Y, \mathbb{Z})$, this implies we may also view the first Chern class as element of the Picard group, which we will often do. We shall see that that the two cases that need to be considered are $c_1 = \mathcal{O}_Y(-1,-1)$ with corresponding minimal $c_2 = 0$ and $c_1 = \mathcal{O}_Y(-2,-2)$ with corresponding minimal $c_2 = 2$. The former case will be rather simple and we will prove that the moduli space in that case is just one point. The latter case will be far more interesting and will be the prime focus of this thesis. We will prove, in Theorem 2.4.4, that for any A-line bundle M with this set of Chern classes, we have the following exact sequence

$$0 \longrightarrow M \longrightarrow A \longrightarrow Q \longrightarrow 0$$

for some A-module Q which depends on the choice of embedding $M \hookrightarrow A$. In fact, we will see that the number of ways M embeds in A is parametrised by $\mathbb{P}^1$. This establishes a connection between the moduli space of line bundles with minimal second Chern class and the Hilbert scheme of A which parametrises quotients of A with specified Chern classes. We will explore this connection in depth and ultimately prove:

Theorem 1.1.2. *Let* **Pic** A *be the moduli space of A-line bundles with* $c_1 = \mathcal{O}_Y(-2,-2)$ *and* $c_2 = 2$ *and* **Hilb** A *– the Hilbert scheme of A, parameterising quotients of A with* $c_1 = \mathcal{O}_Y(1,1)$ *and* $c_2 = 2$. *Then* **Pic** A *is a smooth genus* 2 *curve.* **Hilb** A *is a smooth ruled surface over* **Pic** A. *Furthermore,* **Hilb** A *exhibits an* $8:1$ *cover of* $\mathbb{P}^2$, *ramified on a union of* 2 *conics and their* 4 *bitangents.*

In their paper, Chan and Kulkarni had a remarkably similar result concerning the moduli of line bundles with minimal c_2. They also reduced the study of their moduli space of line bundles with minimal second Chern class to two possible first Chern classes. In the first case, the moduli space was a point and in the second case, also a genus two curve.

1.2 Orders on surfaces

For a great reference on orders, see [Cha12], it will make understanding certain proofs far easier.

We have already defined the notion of an order on a surface. We will now describe the aforementioned noncommutative cyclic covering trick which we will later use to construct the order whose moduli spaces we will be studying. This "trick" was introduced by Chan in [Cha05] and the reader is advised to look there, in particular Sections 2 and 3 for all the relevant details and proofs.

1.2.1 Noncommutative cyclic covering trick

The setup is as follows: Let W be a normal integral Cohen-Macaulay scheme and $\sigma \in \text{Aut } W$ with $\sigma^e = \text{id}$ for some minimal $e \in \mathbb{Z}^+$. Further, assume that $X := W/\langle\sigma\rangle$ is a scheme. Given any $L \in \text{Pic } W$, we can form the $\mathcal{O}_W$-bimodule L_σ such that ${}_{\mathcal{O}_W}L_\sigma \simeq L$ and $(L_\sigma)_{\mathcal{O}_W} \simeq \sigma^* L$. Given 2 such bimodules N_σ and M_τ with $N, M \in \text{Pic } W$ and $\sigma, \tau \in \text{Aut } W$ we have

$$N_\sigma \otimes_W M_\tau \simeq (N \otimes_W \sigma^* M)_{\sigma\tau}.$$

For a full treatment of bimodules, see [AVdB90] Section 2. Suppose we have an effective Cartier divisor D and an $L \in \text{Pic } W$ such that there exists a non-zero map of $\mathcal{O}_W$-bimodules $\phi\colon L_\sigma^{\otimes e} \xrightarrow{\sim} \mathcal{O}_W(-D) \hookrightarrow \mathcal{O}_W$ satisfying the

overlap condition; namely that the two maps $1 \otimes \phi$ and $\phi \otimes 1$ are equal on $L_\sigma \otimes_W L_\sigma^{\otimes(e-1)} \otimes_W L_\sigma$. Then

$$A = \mathcal{O}_W \oplus L_\sigma \oplus \cdots \oplus L_\sigma^{\otimes(e-1)}$$

is an order on X with multiplication given by:

$$L_\sigma^i \otimes_W L_\sigma^j \longrightarrow \begin{cases} L_\sigma^{\otimes(i+j)}, & i+j<e \\ L_\sigma^{\otimes(i+j)} \overset{1\otimes\phi\otimes 1}{\longrightarrow} L_\sigma^{\otimes(i+j-e)}, & i+j \geq e \end{cases}$$

which is independent of any choice that needs to be made when applying the map $1 \otimes \phi \otimes 1$ due to the overlap condition. Orders constructed in this manner are called **cyclic orders**. We will almost always regard A as an $\mathcal{O}_W$-bimodule on W, in which case we pay special consideration to the fact that it is not $\mathcal{O}_W$-central.

Note that if we want to use this method to construct an order on a specific scheme X we also need a way of finding a scheme W and an automorphism $\sigma \in \text{Aut } W$ such that $W/\langle\sigma\rangle = X$. We can do so, using the classical cyclic covering construction.

Construction 1.2.1. Let X be a normal integral scheme, let $E \geq 0$ be an effective divisor and $N \in \text{Pic } X$ such that $N^{\otimes e} \simeq \mathcal{O}_X(-E)$. Then

$$\pi \colon W := Spec_X(\mathcal{O}_X \oplus N \oplus \cdots \oplus N^{\otimes(e-1)}) \to X$$

is a cyclic cover of X. See Chapter 1, Section 17 of [BPVdV84] for more details. Note that if σ is the generator of $\text{Gal}\,(W/X)$ then $W/\langle\sigma\rangle = X$. To construct an order on X using the noncommutative cyclic covering trick, let

$E' \geq 0$ be another effective divisor on X and let $D = \pi^* E'$. Find an $L \in \operatorname{Pic} W$ and a non-zero morphism (if one exists) $\phi \colon L_\sigma^{\otimes e} \to \mathcal{O}_W(-D)$ satisfying the overlap condition. Then as described above, we can construct an order on X which we will denote by $A(W/X; \sigma, L, \phi)$. This order is ramified on $E \cup E'$, see [Cha05] Theorem 3.6 for a proof of this. We suppress E, E' and D from the notation.

1.3 The order we wish to study

In this chapter we will use the noncommutative cyclic covering trick to construct a del Pezzo order on $\mathbb{P}^2$ ramified on a union of two conics. It is the moduli space and Hilbert scheme of this order that we will be investigating for the remainder of the thesis.

Construction 1.3.1. Let $Z = \mathbb{P}^2$ and $\pi \colon Y \to Z$ be a double cover ramified on a smooth conic $E \subset Z$ and let σ be the covering involution. It is well known that $Y \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\operatorname{Pic} Y = \mathbb{Z} \oplus \mathbb{Z}$ and that $\sigma^*(\mathcal{O}_Y(n,m)) = \mathcal{O}_Y(m,n)$. Let H be the inverse image of a general line in Z. It is a $(1,1)$-divisor and is ample. Let $E' \subset Z$ be a second smooth conic, intersecting E in 4 distinct points, let $D = \pi^* E'$ which is a smooth $(2,2)$-divisor, let $L = \mathcal{O}_Y(-1,-1) \in \operatorname{Pic} Y$ and fix once and for all a morphism $\phi \colon L_\sigma^{\otimes 2} \xrightarrow{\sim} \mathcal{O}_Y(-D) \hookrightarrow \mathcal{O}_Y$.

Proposition 1.3.2. *ϕ satisfies the overlap condition.*

Proof. Corollary 3.4 in [Cha05] states that it is sufficient to check the overlap condition on the open set $Y' := Y - D$. Letting $L' := L|_{Y'}$ we see that $L_\sigma'^{\otimes 2} \simeq \mathcal{O}_{Y'}$ and so by Proposition 4.1 in [Cha05] ϕ will satisfy the overlap condition provided $\mathcal{O}(Y')^* = k^*$. To see that this is true, suppose $f \in \mathcal{O}(Y')^*$. Then div $f = D' - nD$ for some $n \geq 0$, where we can assume, since D is irreducible, that D and D' have no common components. However, $f^{-1} \in \mathcal{O}(Y')^*$ with div $f^{-1} = nD - D'$ showing that $D' = 0$ and hence $D = 0$. Thus $f \in k^*$. $\square$

Thus $A := A(Y/Z; \sigma, L, \phi)$ is an order on Z ramified on $E \cup E'$, which is a singular quartic. The following proposition gives us some useful properties of A.

Proposition 1.3.3. *A is a maximal and terminal order. Furthermore, $k(A)$ is a division ring.*

Proof. (Sketch, see [Cha12] for full details) By Theorem 1.5 in [AG60] to check maximality it is sufficient to check that A is reflexive and that A_η is maximal for every codimension one point $\eta \in Z^1$. Since A is locally free it is reflexive, and so to check maximality we need to check that A_η is a maximal $\mathcal{O}_{Z,\eta}$-order. If η is not the generic point of either E nor E' then A_η is in fact Azumaya, and is therefore maximal. Thus we only need to check maximality at E and E'.

By Theorem 2.3 in [AG60], A_η is maximal if and only its Jacobson radical is a maximal ideal (i.e. A_η is a local order) and that it is hereditary. Theorem 3.6 in [Cha05] states that A is normal, in particular this implies (Definition 2.3 in [CI05]) A_η is hereditary.

We now check that A_η is local when η is the generic point of E'. Suppose η is the generic point of E'. Let $I := \mathcal{O}_Y(-D) \oplus L_\sigma$, which is a 2-sided ideal of A. We claim that $I_\eta = \text{rad } A_\eta$. To see this, note first of all that $\mathfrak{m} := \mathcal{O}_Z(-E')_\eta$ is the unique maximal ideal of $\mathcal{O}_{Z,\eta}$. Furthermore, $I^2 = A \otimes_Y \mathcal{O}_Y(-D) = A \otimes_Z \mathcal{O}_Z(-E')$ and so $I_\eta^2 = \mathfrak{m}A_\eta$. Thus $I_\eta \subseteq \text{rad } A_\eta$. Finally, since $A_\eta / I_\eta = k(D)$ we see that $I_\eta = \text{rad } A_\eta$ and so A_η is local and hence maximal.

Now let η be the generic point of E. By Theorem 3.6 in [Cha05] the ramification index of A_η is 2 (see Section 2.2 of [CI05] for an explanation). By the Artin-Mumford sequence (Theorem 1, Section 3 of [AM72]), secondary ramification of A must cancel and so $Z(A_\eta/\text{rad } A_\eta)$ must be cyclic field extension of $k(E)$ (as opposed to $k(E) \times k(E)$ which are the only two possibilities by Proposition 2.4 of [CI05]). Thus A_η is maximal.

The above calculation shows that the ramification data of A at E' is given by the $2:1$ cover $D \to E'$ and so we can see from Definition 2.5 [CI05] that A is terminal.

Since A is maximal, of rank 4, and $k(A) \neq 0$ in Br $k(Z)$ the Artin-Mumford sequence implies $k(A)$ is a division ring. $\square$

We will only need the fact that A is terminal, for the proof of Proposition 2.0.2. Aside from that, this technical condition can be ignored by the reader for the purposes of this thesis. However, its importance can not be underestimated - see [CI05] and [Cha12] for more information.

As mentioned previously, in [CK11] the authors also consider a maximal order on $\mathbb{P}^2$ ramified, in their case, on a smooth quartic. More importantly, the relation used in their construction was of the form $L_\sigma^{\otimes 2} \simeq \mathcal{O}_W$. As we shall see this small difference makes their techniques for the study of the moduli space of A, unusable in our case.

1.4 The canonical bimodule

To finish off the introduction we would like to explain in what sense our order A is del Pezzo. We begin with the definition of the canonical bimodule which is the analogue of the canonical sheaf on a scheme.

Definition 1.4.1. *Let X be a normal integral scheme and A an order on X. The canonical bimodule of A is defined to be*

$$\omega_A := \mathcal{H}om_{\mathcal{O}_X}(A, \omega_X).$$

Mimicking the commutative definition, we say that A is del Pezzo if $\omega_A^* := \mathcal{H}om_A(\omega_X, A)$ is ample. For more details see [CK03] Section 3.

If X is Gorenstein, then $\omega_A = \mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X) \otimes_X \omega_X$. Using the reduced trace map, we can identify $\mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X)$ as an A-subbimodule of $k(A)$ and so ω_A can be identified as an A-subbimodule of $k(A) \otimes_X \omega_X$. The next theorem allows us to determine, in the case where A is constructed using Construction 1.2.1, precisely what this subbimodule is. Knowledge of ω_A will be very valuable to us in the future for various homological computations.

Theorem 1.4.2. *Let X be a normal integral Gorenstein scheme. Let $A := A(W/X; \sigma, L, \phi)$ be an order on X as described in Construction 1.2.1 and let $R \subset W$ be the reduced pullback of E to W.*

Then

$$\begin{aligned}\omega_A &= A \otimes_W L_\sigma \otimes_W \mathcal{O}_W((e-1)R + D) \otimes_X \omega_X \\ &= A \otimes_W L_\sigma \otimes_W \mathcal{O}_W(D) \otimes_W \omega_W\end{aligned}$$

in $k(A) \otimes_W \omega_W$.

Proof. From Lemma 17.1 of [BPVdV84] and the adjunction formula we know that

$$\omega_W = \pi^*\omega_X \otimes_W \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_X) = \pi^*\omega_X \otimes_W \mathcal{O}_W((e-1)\,R).$$

Thus, using the reduced trace map we have:

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X) &= \{f \in k(A) \mid \mathrm{tr}\,(fA) \subseteq \mathcal{O}_X\} \subseteq k(A)\\ &= D_0 \oplus D_1 \oplus \cdots \oplus D_{e-1}\end{aligned}$$

where

$$\begin{aligned}D_0 &= \{f \in k(W) \mid \mathrm{tr}\,(f\mathcal{O}_W) \subseteq \mathcal{O}_X\} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_W, \mathcal{O}_X) = \mathcal{O}_W\left((e-1)R\right)\\ D_1 &= \left\{f \in L_\sigma \otimes_W k(W) \mid \mathrm{tr}\,(fL_\sigma^{\otimes(e-1)}) \subseteq \mathcal{O}_X\right\} = L_\sigma \otimes_W \mathcal{O}_W\left((e-1)R + D\right)\\ &\vdots\\ D_{e-1} &= \left\{f \in L_\sigma^{\otimes(e-1)} \otimes_W k(W) \mid \mathrm{tr}\,(fL_\sigma) \subseteq \mathcal{O}_X\right\} = L_\sigma^{\otimes(e-1)} \otimes_W \mathcal{O}_W\left((e-1)R + D\right)\end{aligned}$$

and so:

$$\mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X) = A \otimes_W L_\sigma \otimes_W \mathcal{O}_W\left((e-1)R + D\right).$$

Thus:

$$\begin{aligned}\omega_A &:= \mathcal{H}om_{\mathcal{O}_X}(A, \omega_X)\\ &= \mathcal{H}om_{\mathcal{O}_X}(A, \mathcal{O}_X) \otimes_X \omega_X\\ &= A \otimes_W L_\sigma \otimes_W \mathcal{O}_W((e-1)R + D) \otimes_W \pi^*\omega_X\\ &= A \otimes_W L_\sigma \otimes_W \mathcal{O}_W(D) \otimes_W \omega_W.\end{aligned}$$

□

Applying this theorem to our specific order A we get:

Proposition 1.4.3. *Let $A := A(Y/Z; \sigma, L, \phi)$ be as in Construction 1.3.1. Then $\omega_A = A \otimes_Y \mathcal{O}_Y(-H)$. In particular, A is del Pezzo.*

Proof. We simply apply Theorem 1.4.2 and use the well known fact that $\omega_Y = \mathcal{O}_Y(-2, -2)$. □

1.5 Outline of thesis

The rest of the thesis is primarily devoted to making sense of, and proving Theorem 1.1.2. In Chapter 2 we will define what we mean by the moduli space parameterising line bundles on an order in a division ring. We will then explore some properties of this moduli space of our A. We will quickly discover why the methods used in [CK11] will not work for our order. In the next chapter we will introduce the Hilbert scheme of A, which should be thought of as a scheme parametrising noncommutative curves on the order - we will prove its existence, compute its dimension and prove that it is smooth. It is here that we will also explore the bizarre covering of $\mathbb{P}^2$ that it exhibits and study its ramification. In the last chapter, we will prove that the Hilbert scheme is in fact a ruled surface over the moduli space. Finally using the map to $\mathbb{P}^2$ we will be able to compute the self intersection of the canonical divisor of the Hilbert scheme which will allow us to compute the genus of the moduli space.

Chapter 2

The Moduli Space of A-Line Bundles

In this chapter we will study line bundles on A, where A is the order on $\mathbb{P}^2$ constructed in Construction 1.3.1.

First we need to define what it means to have a line bundle on A.

Definition 2.0.1. *Let X be a normal integral scheme and A an order in a division ring $k(A)$ on X. Let M be a sheaf of left A-modules. We say M is a* **line bundle** *on A if M is locally projective as an A-module and* $\dim_{k(A)} k(A) \otimes_A M = 1$. *The set of isomorphism classes of A-line bundles will be denoted by* Pic A.

Note that the above definition makes sense since $k(A)$ is a division ring. Also, note that Pic A is not a group since the modules are only one sided and so can not be tensored together. This definition may at first appear hard to use since checking whether an A-module is locally projective is not easy. However, the following proposition gives an easy criterion for checking this for orders constructed using the noncommutative cyclic covering trick.

Proposition 2.0.2. *Let $A := A(Y/Z; \sigma, L, \phi)$ be the order constructed in Construction 1.3.1. Then $M \in \text{Pic } A$ if and only if M is an A-module such that ${}_Y M$ is a rank two locally free sheaf on Y.*

Proof. Suppose $M \in \text{Pic } A$. Since A is rank two over Y, so is M. Furthermore, M is locally projective over A and since A is locally free over Y it implies M is locally projective over Y. However, over a commutative ring, locally projective implies locally free and so M is locally free on Y.

Conversely, from Proposition 1.3.3 we know that A is terminal and so by Theorem 2.12 in [CI05] A_p has global dimension 2 for every closed point p. The result then follows from the section "Finite Global Dimension" in [Cha12]. □

Example 2.0.3. Let A be as above and $N \in \text{Pic } Y$. Then $A \otimes_Y N$ is an A-line bundle since it is clearly an A-module and is locally free of rank two over Y.

This result is our first look at how the underlying $\mathcal{O}_Y$-module structure of A-modules helps with their study. We can further use the geometry of Y to give us some information about the possible Chern classes of A-line bundles when viewed as rank two vector bundles on Y. We analyse this in the following chapter.

2.1 Chern classes of A-line bundles

In this chapter we study the possible Chern classes of line bundles on our order A constructed in Construction 1.3.1. Recall that whenever we speak of Chern classes for any $M \in \text{Pic } A$ we imply that we are talking about the $\mathcal{O}_Y$-module ${}_Y M$.

The first natural question to ask about any A-line bundle is what could be its first Chern class. We answer this in the following proposition. As it turns out, the possibilities are fairly limited.

Proposition 2.1.1. *Let $M \in \operatorname{Pic} A$. Then $c_1(M) = \mathcal{O}_Y(n,n)$ for some $n \in \mathbb{Z}$. Conversely, given any such n, $A \otimes_Y \mathcal{O}_Y(0, n+1) \in \operatorname{Pic} A$ with $c_1 = \mathcal{O}_Y(n,n)$.*

Proof. First note that we have a chain of $\mathcal{O}_Y$-submodules $M(-D) := L_\sigma^{\otimes 2} \otimes_Y M < L_\sigma \otimes_Y M < M$ which means

$$0 \to \frac{L_\sigma \otimes_Y M}{L_\sigma^{\otimes 2} \otimes_Y M} \to \frac{M}{M(-D)} \to \frac{M}{L_\sigma \otimes_Y M} \to 0$$

is an exact sequence. Let $Q = M/(L_\sigma \otimes_Y M)$. The above then becomes:

$$0 \to L_\sigma \otimes_Y Q \to M|_D \to Q \to 0.$$

Now $M|_D$ is a locally free sheaf on D of rank 2, so $L_\sigma \otimes_Y Q$ is locally free (since we are on a smooth curve, torsion free means locally free) of rank 1 on D and thus Q must also be a line bundle on D. Lemma 1 Chapter 2 of [Fri98] then assures that $c_1(Q) = D$. Thus $c_1(M) = c_1(L_\sigma \otimes M) + D$ and so $c_1(M) = \sigma^* c_1(M)$ and so the result follows.

To see the converse, first note that by Example 2.0.3 we know that $M := A \otimes_Y \mathcal{O}_Y(0, n+1)$ is indeed an A-line bundle. Furthermore, $c_1(M) = c_1(\mathcal{O}_Y(0, n+1) \oplus \mathcal{O}_Y(n, -1)) = \mathcal{O}_Y(n,n)$. ∎

Having classified all the possible first Chern classes of A-line bundles, we move on to see what can be said about the second Chern class. As we shall see, the second Chern class has a strict lower bound analogous to Bogomolov's inequality, which we now recall.

Let X be a smooth projective surface and $\mathcal{F}$ a torsion free coherent sheaf on X with Chern classes c_1, c_2 and rank r. Fix an ample divisor H on X. The **slope** of $\mathcal{F}$ is defined to be

$$\mu(\mathcal{F}) := \frac{c_1.H}{r}.$$

$\mathcal{F}$ is said to be μ-semistable if for any subsheaf $\mathcal{F}' \subset \mathcal{F}$ we have $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$. Bogomolov's inequality (Theorem 12.1.1 in [LP97]) states, that if $\mathcal{F}$ is semistable then

$$\Delta(\mathcal{F}) := 4c_2 - c_1^2 \geq 0.$$

Thus, if considering any class of semistable sheaves on X with a fixed first Chern class, the second Chern class is bounded from below.

In [CK11] the authors were able to show to that for their cyclic order A, any A-line bundle was automatically μ-semistable as a sheaf on Y and they could thus bound the second Chern class using Bogomolov's inequality.

We modify their proof and achieve a slightly weaker result for our order.

Proposition 2.1.2. *Let $M \in \operatorname{Pic} A$ and let $N < M$ be an $\mathcal{O}_Y$-subsheaf. Then*

$$\mu(N) \leq \mu(M) + 1. \tag{1}$$

Proof. (Heavily inspired by the proof of Proposition 3.6 in [CK11]) Note that M is locally free of rank 2 over Y.

Thus the result is clear if rank N= 2 and so we assume rank $N = 1$. Observe that

$$c_1(L_\sigma \otimes N).H = c_1(L).H + \sigma^* c_1(N).H = -2 + c_1(N).H.$$

Now

$$\begin{aligned}
\mu(N \oplus L_\sigma \otimes_Y N) &= \frac{c_1(N).H + c_1(L_\sigma \otimes_Y N).H}{2} \\
&= \frac{2c_1(N).H - 2}{2} \\
&= c_1(N).H - 1 \\
&= \mu(N) - 1
\end{aligned}$$

and so $\mu(N) = \mu(N \oplus L_\sigma \otimes_Y N) + 1 \leq \mu(M) + 1$. $\square$

It is easy to see that this inequality is tight. For example the A-line bundle A has slope $\mu(A) = -1$ and an $\mathcal{O}_Y$-submodule $\mathcal{O}_Y < A$ with $\mu(\mathcal{O}_Y) = 0$. Thus A-line bundles are in general not μ-semistable and so we can not apply Bogomolov's inequality to give a lower bound for the second Chern class.

Luckily, due to a theorem by Langer in [Lan04] the result of Proposition 2.1.2 is good enough to achieve a lower bound on c_2.

Proposition 2.1.3. *Let $M \in \operatorname{Pic} A$ with Chern classes c_1 and c_2. Then*

$$\Delta(M) = 4c_2 - c_1^2 \geq -2.$$

Proof. Follows immediately from Theorem 5.1 of [Lan04] with $D_1 = H$ and Proposition 2.1.2. $\square$

Remark 2.1.4. The above theorem can also be proven using rather elementary facts, without needing the generality of [Lan04].

Chan and Kulkarni then went to on to prove that for their order A, for any choice of c_1 and c_2, provided they satisfy $4c_2 - c_1^2 > 0$, there exists an A-line bundle with these Chern classes. We were also able to prove the same result, assuming of course $4c_2 - c_1^2 \geq -2$, but we need techniques we haven't yet developed and so we defer the proof until Chapter 3.2.3.

Having shown that for a fixed first Chern class, the second Chern class of any A-line bundle is bounded from below, it is natural to begin studying those line bundles, with minimal second second Chern class. In particular, we would like to determine the moduli space of such bundles.

It is now also clear why the methods used by Chan and Kulkarni in [CK11] for the purposes of studying the moduli space of line bundles with minimal second Chern class will not work for us: as mention previously, the key ingredient that they used was the map

$$\left\{\begin{array}{c}\text{moduli space of}\\ A\text{-line bundles}\\ \text{with minimal } c_2\end{array}\right\} \to \left\{\begin{array}{c}\text{moduli space of}\\ \mu\text{-semistable rank}\\ \text{two vector bundles on } Y\end{array}\right\}.$$

Since for us, A-line bundles are not in general semistable, the above map does not exist for us, and so we must find an alternate technique.

We begin by making all these notions precise.

2.2 Existence of the coarse moduli scheme

We begin by making precise the notion of a flat family of A-line bundles. Then we will go on to explore the question of the existence of the moduli space parametrising all such bundles. The main result we will need is Theorem 2.4 in [HS05] where the authors prove the existence of a coarse moduli scheme parametrising generically simple torsion free A-modules, where A is an order on a surface. Using this result, we will prove that for our order A, there exists a coarse moduli scheme parameterising A-line bundles with minimal second Chern class.

Definition 2.2.1 (Flat family of A-line bundles.)**.** *Let X be a smooth projective variety, and A an order in a division ring on X. Fix a polynomial P and a scheme S. A* **flat family of A-line bundles on** *S, is a coherent sheaf $\mathcal{F}$ on $X \times_k S$ of left A_S-modules where A_S is the pull back of A to $X \times_k S$ such that:*

1. *$\mathcal{F}$ is flat over S.*
2. *For every $p \in S$, $\mathcal{F}_{k(p)}$ is an $A_{k(p)}$-line bundle with Hilbert polynomial P.*

where $k(p)$ is the residue field at p and $\mathcal{F}_{k(p)}$ (respectively $A_{k(p)}$) is the pull back of $\mathcal{F}$ (respectively A_S) to $X \times_k \operatorname{Spec} k(p)$.

We denote the corresponding moduli functor by

$$\mathcal{M} : (\text{Schemes}/k)^{\text{op}} \longrightarrow (\text{Sets})$$

where for any scheme S

$$\mathcal{M}(S) := \{\text{set of isomorphism classes of flat families of } A\text{-line bundles over } S\}\,.$$

We do not include P anywhere in the notation, always assuming, when we talk about the moduli space of line bundles, that we have fixed a Hilbert polynomial or a set of Chern classes.

The only difference between this approach and that of Hoffmann and Stuhler in [HS05] is that they consider generically simple modules over an order. However, when the order is in a division ring, which is what we assume, the notions clearly coincide.

Theorem 2.2.2 (Existence of Coarse Moduli Scheme)**.** *There exists a quasi-projective coarse moduli scheme* $\mathbf{M}$ *for the functor* $\mathcal{M}$*.*

Proof. Follows from Theorem 2.4 [HS05], which is more general, and the remarks at the beginning of Section 4 of that paper. □

Note that $\mathbf{M}$ need not be projective. In order to compactify the space Hoffmann and Stuhler are forced to drop the locally projective assumption and settle with just torsion free. The locally projective locus then corresponds to an open subscheme of this larger moduli space, as they point out in Section 4.

However, as we are about to see, if we consider only those components which correspond to line bundles with minimal second Chen class, then the moduli scheme is projective. To prove this we need to note the following:

Lemma 2.2.3. *Let* X *be a smooth projective surface. Let* $\mathcal{F}$ *be a torsion free coherent sheaf on* X *which is not locally free. Then:*

1. $\mathcal{F}^{**}$ *is locally free.*
2. $c_1(\mathcal{F}^{**}) = c_1(\mathcal{F})$.
3. $c_2(\mathcal{F}^{**}) < c_2(\mathcal{F})$.

Proof. (1) From Corollary 1.2 of [Har80] $\mathcal{F}^{**}$ is reflexive and hence from Corollary 1.4 of [Har80] it is locally free. (2) Follows from the fact that $c_1(\mathcal{F}^*) = -c_1(\mathcal{F})$ (see [Fri98] Chapter 2). (3) Since $\mathcal{F}$ is torsion free we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{F}^{**} \to \mathcal{G} \to 0$$

for some coherent sheaf $\mathcal{G}$. $\mathcal{G} \neq 0$ since $\mathcal{F}$ is not locally free. From the proof of Proposition 1.3 of [Har80] we know that $\mathcal{G}$ is supported on a finite number of points. Thus $c_1(\mathcal{G}) = 0$ and $c_2(\mathcal{G}) < 0$ (see [Fri98] Chapter 2) and so we are done. □

The above proposition together with Proposition 2.0.2 put together tell us the following: Let $\mathcal{F}$ be a coherent torsion free A-module. Then $\mathcal{F}^{**}$ is a locally projective A-module with the same first Chern class and a smaller second Chern class than $\mathcal{F}$. Thus if we only consider rank one torsion free A-modules with a minimal second Chern class, which is possible by Proposition 2.1.3, the moduli space parameterising these will be projective.

Putting all this together, we have:

Theorem 2.2.4. *There exists a coarse moduli space parametrising A-line bundles with first Chern class c_1 and second Chern class $\lceil (c_1^2 - 2)/4 \rceil$. Furthermore, it is projective and smooth.*

Proof. We have already shown everything apart from smoothness. This follows from the proof of Proposition 4.1 in [CK11] and is essentially due to the fact that A is del Pezzo. □

Remark 2.2.5. Since the functor $\mathcal{O}_Y(nH) \otimes_Y -$ is a category autoequivalence of A-Mod, it induces an automorphism of the moduli space of A-line bundles. Note that for any $M \in \text{Pic } A$, $c_1(\mathcal{O}_Y(nH) \otimes_Y M) = 2nH + c_1(M)$. Since by the previous proposition, $c_1(M) = mH$ for some $m \in \mathbb{Z}$ we may assume that $c_1(M) = \mathcal{O}_Y(-1,-1)$ or $c_1(M) = \mathcal{O}_Y(-2,-2)$. It will turn out that the first case is rather simple, whilst the study of the second case will be the goal of most of the thesis.

Before we finish off this chapter and continue with the study of the moduli space, we need to examine the inequality (1) we met in Proposition 2.1.2 a little further.

Definition 2.2.6. *Let X be a surface and V a vector bundle on X. We say V is* **almost semistable** *if for any subbundle $V' \subset V$ we have $\mu(V') \leq \mu(V) + 1$.*

Proposition 2.2.7. *Let X be a surface and V a vector bundle on X.*

1. *V is almost semistable if and only if $V \otimes_X N$ is almost semistable for all $N \in \text{Pic } X$.*

2. *If V is rank 2 and almost semistable, then so is V^*.*

Proof.

1. Suppose V is almost semistable and $V' \subseteq V \otimes_X N$. Then $V' \otimes_X N^{-1} \subseteq V$ and so $\mu(V' \otimes_X N^{-1}) \leq \mu(V) + 1$ thus $\mu(V') - c_1(N).H \leq \mu(V) + 1$ and so $\mu(V') \leq (V \otimes_X N) + 1$. To see the converse simply let $N = \mathcal{O}_X$.

2. Follows from (1) and the fact that $V^* \simeq V \otimes_X (\det V)^{-1}$.

$\square$

As we have seen in Proposition 2.1.2, A-line bundles are almost semistable. We will use the above proposition later on for proving various properties regarding line bundles on A.

2.3 Case 1: $c_1 = \mathcal{O}_Y(-1,-1)$

As mentioned in Remark 2.2.5 the problem of studying the moduli space of A-line bundles with minimal c_2 naturally breaks up into two parts $c_1 = \mathcal{O}_Y(-1,-1)$ or $\mathcal{O}_Y(-2,-2)$. In this chapter we examine the former case. By Theorem 2.2.4 the minimal $c_2 = 0$ and this corresponds to $\Delta = -2$ which, by Proposition 2.1.3 is the smallest value it can take. It is easy to see that the moduli space of A-line bundles with these Chern classes isn't empty for clearly A itself, regarded as a left A-module, has the desired Chern classes. As it turns out, this is in fact the only such A-line bundle.

Theorem 2.3.1. *Let $M \in \text{Pic } A$ with $c_1 = \mathcal{O}_Y(-1,-1)$ and $c_2 = 0$. Then $M \simeq A$. In particular, the coarse moduli space of A-line bundles with these Chern classes is a reduced point.*

This theorem is analogous to Proposition 6.1 in [CK11] and uses techniques from the proof of Proposition 5.2 in the same paper.

Proof. By the Riemann-Roch theorem

$$\chi(M) = 2 + \frac{1}{2}c_1.(c_1 - K_Y) - c_2$$

and so $\chi(M) = 1 > 0$. On the other hand $h^2(M) = h^0(\omega_Y \otimes_Y M^*)$ and $c_1(\omega_Y \otimes_Y M^*) = \mathcal{O}_Y(-3,-3)$. As we saw in Proposition 2.1.2, M is almost semistable, and so by Proposition 2.2.7, $\omega_Y \otimes_Y M^*$ is also almost semistable and so $h^2(M) = 0$ for otherwise $\mathcal{O}_Y \hookrightarrow \omega_Y \otimes_Y M^*$ which is impossible since $\mu(\mathcal{O}_Y) = 0$ whilst $\mu(\omega_Y \otimes_Y M^*) = -3$. Thus $h^0(M) \neq 0$ and so $\mathcal{O}_Y \hookrightarrow M$ which gives an injection of A-modules $A \otimes_Y \mathcal{O}_Y = A \hookrightarrow M$. Since their first

Chern classes are equal their slopes are equal and so by Lemma 3 Chapter 4 in [Fri98] the map must be an isomorphism.

Finally

$$\mathrm{Ext}^1_A(A, A) = \mathrm{Ext}^1_Y(\mathcal{O}_Y, \mathcal{O}_Y \oplus \mathcal{O}_Y(-1,-1)) = H^1(Y, \mathcal{O}_Y \oplus \mathcal{O}_Y(-1,-1)) = 0.$$

where the first equality follows from Proposition 2.6 of [CK11] which asserts that there is a natural isomorphism of functors $\mathrm{Ext}^i_A(A \otimes_Y N, -) \simeq \mathrm{Ext}^i_Y(N, -)$ for any $N \in \mathrm{Pic}\, Y$. See Chapter 3, Exercise 5.6 of [Har77] for the cohomology of $\mathbb{P}^1 \times \mathbb{P}^1$. Thus the tangent space at the point corresponding to the A-line bundle A is 0-dimensional and so the moduli space is just a reduced point. □

2.4 Case 2: $c_1 = \mathcal{O}_Y(-2,-2)$

We now study the second case mentioned in Remark 2.2.5: the case where $c_1 = \mathcal{O}_Y(-2,-2)$. By Theorem 2.2.4 the minimal $c_2 = 2$ which corresponds to $\Delta = 0$ which is its second smallest value for clearly Δ must be even and $\Delta \geq -2$ by Proposition 2.1.3. Note that $A \otimes_Y \mathcal{O}_Y(-1,0)$ is an A-line bundle by Example 2.0.3 and has the desired Chern classes. Thus the moduli space of such A-line bundles is not empty.

From now on **Pic** A will denote the moduli space of A-line bundles with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$. We first establish all the possible $\mathcal{O}_Y$-module structures that such A-line bundles can have.

Theorem 2.4.1 ($\mathcal{O}_Y$-module structure). *Let $M \in \mathrm{Pic}\ A$ with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$. Then either $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ as an $\mathcal{O}_Y$-module or $M \simeq A \otimes_Y \mathcal{O}_Y(-F)$ as A-modules where F is either a $(1,0)$ or a $(0,1)$-divisor.*

Proof. The beginning of this proof is very similar to the proof of Theorem 2.3.1 so we skip some details which we have already explained there. Let $M_1 = \mathcal{O}_Y(1,1) \otimes_Y M$. Then $c_1(M_1) = 2c_1(\mathcal{O}_Y(1,1)) + c_1(M) = 0$ and $c_2(M_1) = c_2(M) + c_1(M).c_1(\mathcal{O}_Y(1,1)) + c_1(\mathcal{O}_Y(1,1))^2 = 2 - 4 + 2 = 0$. Thus by the Riemann-Roch theorem $\chi(M_1) = 2 > 0$. However, by duality $h^2(M_1) = h^0(\omega_Y \otimes_Y M_1^*)$ whilst $\omega_Y \otimes_Y M_1^*$ is almost semistable with slope -4 and so $h^0(\omega_Y \otimes_Y M_1^*) = h^2(M_1) = 0$ and so $h^0(M_1) \neq 0$. Thus we know $\mathcal{O}_Y(-1,-1) \hookrightarrow M$. Now if there exists a bigger $\mathcal{O}_Y$-line bundle (ordered by inclusion) which embeds into M then $\mathcal{O}_Y(-F)$ embeds into M where F is either a $(1,0)$ or a $(0,1)$-divisor. This extends to an embedding $A \otimes_Y \mathcal{O}_Y(-F) \hookrightarrow M$ of A-line bundles and so comparison of the first Chern classes guarantees that $M \simeq A \otimes_Y \mathcal{O}_Y(-F)$. Suppose on the other hand that $\mathcal{O}_Y(-1,-1)$ is the biggest line bundle which embeds into M. Let the quotient be Q. We claim that Q is torsion free. To see this, suppose $0 \neq T \subset Q$ is the torsion subsheaf and $f : M \to Q$ is the quotient map. Then $\mathcal{O}_Y(-1,-1) \subset f^{-1}(T)$. We then have an exact sequence

$$0 \longrightarrow f^{-1}(T) \longrightarrow M \longrightarrow \frac{M}{f^{-1}(T)} \simeq Q/T \longrightarrow 0$$

where Q/T is torsion free. By Proposition 1.1 of [Har80] $f^{-1}(T)$ is reflexive and hence locally free of rank 1 which contradicts the maximality of $\mathcal{O}_Y(-1,-1)$. Thus Q is torsion free. By Proposition 5 (ii) in [Fri98]

$Q = L' \otimes_Y \mathcal{I}_Z$ for some $L' \in \operatorname{Pic} Y$ and $\mathcal{I}_Z$ being the ideal sheaf of some 0-dimensional subscheme. In summary, we have the following short exact sequence:

$$0 \longrightarrow \mathcal{O}_Y(-1,-1) \longrightarrow M \longrightarrow L' \otimes_Y \mathcal{I}_Z \longrightarrow 0.$$

Equation (2.9) of Chapter 2 in [Fri98] states that

$$\begin{aligned} c_1(M) &= c_1(\mathcal{O}_Y(-1,-1)) + c_1(L') \\ c_2(M) &= c_1(\mathcal{O}_Y(-1,-1)).c_1(L') + l(Z) \end{aligned}$$

where $l(Z) \geq 0$ and $l(Z) = 0$ precisely when $Z = 0$. Thus $L' = \mathcal{O}_Y(-1,-1)$ and $l(Z) = 0$ i.e. $Z = 0$. Finally, $\operatorname{Ext}^1_Y(\mathcal{O}_Y(-1,-1), \mathcal{O}_Y(-1,-1)) = H^1(Y, \mathcal{O}_Y) = 0$ and so we see that as an $\mathcal{O}_Y$-module $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$. $\square$

This result is very different to what Chan and Kulkarni encountered in [CK11]. In their example if an A-module was split as an $\mathcal{O}_Y$-module then they prove that the module must be of the form $A \otimes_Y N$ for some $N \in \operatorname{Pic} Y$. Furthermore, any rank two vector bundle on Y could be given at most two A-module structures. In our case, as the above theorem at least suggests, the $\mathcal{O}_Y$-vector bundle $\mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ can be given an infinite number of non-isomorphic A-module structures. In the following proposition, we prove that this is indeed the case.

Proposition 2.4.2. *The tangent space to* **Pic** A *at the point corresponding to* $A \otimes_Y \mathcal{O}_Y(0,-1)$ *and* $A \otimes_Y \mathcal{O}_Y(-1,0)$ *has dimension* 1.

Proof. The dimension of the tangent space is given by:

$$\begin{aligned}&\operatorname{ext}^1_A\left(A\otimes_Y\mathcal{O}_Y(-1,0),A\otimes_Y\mathcal{O}_Y(-1,0)\right)\\&=\operatorname{ext}^1_Y\left(\mathcal{O}_Y(-1,0),\mathcal{O}_Y(-1,0)\oplus\mathcal{O}_Y(-1,-2)\right)=1.\end{aligned}$$

The other case is identical. $\square$

Thus at least one connected component of this moduli space is a smooth curve with all, except at most 2 points, corresponding to A-modules with the underlying $\mathcal{O}_Y$-module structure being $\mathcal{O}_Y(-1,-1)\oplus\mathcal{O}_Y(-1,-1)$.

We finish off the chapter with an algebraic description of the A-line bundles.

Proposition 2.4.3. *Let $M\in\operatorname{Pic}A$ with $c_1=\mathcal{O}_Y(-2,-2)$ and $c_2=2$. Then* $\operatorname{Hom}_A(M,A)=2$. *Further, if $0\neq\varphi\in\operatorname{Hom}_A(M,A)$ then φ is injective.*

Proof. We consider all the possibilities from Theorem 2.4.1. If $M\simeq A\otimes_Y\mathcal{O}_Y(-F)$ then

$$\begin{aligned}\hom_A(M,A)&=\hom_A\left(A\otimes_Y\mathcal{O}_Y(-F),A\right)\\&=\hom_Y\left(\mathcal{O}_Y(-F),\mathcal{O}_Y\oplus\mathcal{O}_Y(-1,-1)\right)=2.\end{aligned}$$

If, on the other hand, $M\simeq\mathcal{O}_Y(-1,-1)\oplus\mathcal{O}_Y(-1,-1)$ as an $\mathcal{O}_Y$-module then:

$$\begin{aligned}\hom_A(M, A) &= \operatorname{ext}_A^2(A, \omega_A \otimes_A M)^* \\ &= \operatorname{ext}_Y^2(\mathcal{O}_Y, \mathcal{O}_Y(-H) \otimes_Y M)^* \\ &= h^2(Y, \mathcal{O}_Y(-2,-2) \oplus \mathcal{O}_Y(-2,-2)) \\ &= h^0(Y, \mathcal{O}_Y \oplus \mathcal{O}_Y) = 2.\end{aligned}$$

Since M and A are torsion free, any non zero map $M \to A$ must be injective.

□

To understand better how M sits inside A we need to understand the all the possible cokernels. We do so, in the next theorem.

Theorem 2.4.4. *Let $M \in \operatorname{Pic} A$ with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$. Then for any $0 \neq \varphi \in \hom_A(M, A)$ there exists an exact sequence of A-modules*

$$0 \longrightarrow M \xrightarrow{\varphi} A \longrightarrow Q \longrightarrow 0$$

where:

1. *if $M \simeq A \otimes_Y \mathcal{O}_Y(-1,0)$ (respectively $M \simeq A \otimes_Y \mathcal{O}_Y(0,-1)$) then $Q \simeq A \otimes_Y \mathcal{O}_F$ where F is a $(1,0)$ (respectively $(0,1)$) divisor;*
2. *if $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ then $Q \simeq \mathcal{O}_C$ as an $\mathcal{O}_Y$-module, where C is a $(1,1)$-divisor.*

Proof. From the previous proposition, we know $\varphi\colon M \to A$ is injective. Let us compute the cokernel.

1. We prove only the case where $M \simeq A \otimes_Y \mathcal{O}_Y(-1,0)$ because the other is similar. Note that $\hom_Y(\mathcal{O}_Y(-1,0), \mathcal{O}_Y) = 2 = \hom_A(M, A)$

and so all A-module morphisms arise from an $\mathcal{O}_Y$-module morphism $\mathcal{O}_Y(-1,0) \to \mathcal{O}_Y$ via $A \otimes_Y -$. Since any non zero morphism $\mathcal{O}_Y(-1,0) \to \mathcal{O}_Y$ gives rise to the following exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-1,0) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_F \longrightarrow 0$$

for some $(1,0)$-divisor F and because A is flat over Y, the result follows.

2. Note that with respect to the $\mathcal{O}_Y$-module decomposition

$$\begin{aligned} M &= \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1) \\ A &= \mathcal{O}_Y \oplus \mathcal{O}_Y(-1,-1) \end{aligned}$$

we have

$$\begin{aligned} \mathrm{Hom}_Y(M,A) &= \mathrm{Hom}_Y\left(\mathcal{O}_Y(-1,-1)) \oplus \mathcal{O}_Y(-1,-1), \mathcal{O}_Y \oplus \mathcal{O}_Y(-1,-1)\right) \\ &= \begin{pmatrix} H^0(Y, \mathcal{O}_Y(-1,-1)^*) & \mathrm{End}_Y(\mathcal{O}_Y(-1,-1)) \\ H^0(Y, \mathcal{O}_Y(-1,-1)^*) & \mathrm{End}_Y(\mathcal{O}_Y(-1,-1)) \end{pmatrix}. \end{aligned}$$

Thus any $\mathcal{O}_Y$-module morphism $\varphi \colon M \to A$ is given by $X = \begin{pmatrix} \varphi_1 & \lambda_1 \\ \varphi_2 & \lambda_2 \end{pmatrix}$ where $\varphi_1, \varphi_2 \in \mathcal{O}_Y(-1,-1)^*$ and $\lambda_1, \lambda_2 \in \mathrm{End}_Y(\mathcal{O}_Y(-1,-1)) = k$ which acts as right multiplication on the row vector $\mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$. For this to be in fact an A-module morphism further conditions on X need to be imposed. In particular φ needs to be injective and so λ_1, λ_2 are not both zero.

We claim that

$$Q = \frac{\mathcal{O}_Y}{\mathrm{im}(\lambda_2\varphi_1 - \lambda_1\varphi_2)}$$

and that we have the following exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi} A \xrightarrow{\psi} Q \longrightarrow 0$$

with $\psi \colon A \to Q$ given by right multiplication by

$$\begin{cases} \begin{pmatrix} \lambda_1 + \lambda_2 \\ -(\varphi_1 + \varphi_2) \end{pmatrix} & \text{if } \lambda_1 + \lambda_2 \neq 0 \\ \begin{pmatrix} \lambda_1 \\ -\varphi_1 \end{pmatrix} & \text{if } \lambda_1 + \lambda_2 = 0. \end{cases}$$

Since $M \to A$ must be injective, $\mathrm{im}(\lambda_2\varphi_1 - \lambda_1\varphi_2) \neq 0$ and so, Q is isomorphic, as an $\mathcal{O}_Y$-module, to $\mathcal{O}_C$ for some $(1,1)$-divisor C. It suffices to check this claim locally which we do in the following lemma:

Lemma 2.4.5. *Let $k \subset R$ be a commutative ring, N an R-module $\varphi_1, \varphi_2 \in \mathrm{Hom}_R(N, R)$ and $\lambda_1, \lambda_2 \in k$ not both zero. Then the following is an exact sequence of R-modules.*

$$N \oplus N \xrightarrow{\varphi} R \oplus N \xrightarrow{\psi} \frac{R}{\mathrm{im}\,(\lambda_2\varphi_1 - \lambda_1\varphi_2)} \longrightarrow 0$$

where

$$\varphi(n_1, n_2) := (\varphi_1(n_1) + \varphi_2(n_2), \lambda_1 n_1 + \lambda_2 n_2)$$

$$\psi(r, n) := \begin{cases} (\lambda_1 + \lambda_2)r - (\varphi_1 + \varphi_2)(n) & \text{if } \lambda_1 + \lambda_2 \neq 0 \\ \lambda_1 r - \varphi_1(n) & \text{if } \lambda_1 + \lambda_2 = 0. \end{cases}$$

Proof. Suppose $\lambda_1 + \lambda_2 \neq 0$. To show $\text{im}\,\varphi \subseteq \ker\psi$ note that

$$\begin{aligned} &(\lambda_1 + \lambda_2)(\varphi_1(n_1) + \varphi_2(n_2)) - (\varphi_1 + \varphi_2)(\lambda_1 n_1 + \lambda_2 n_2) \\ &= \lambda_2\varphi_1(n_1) + \lambda_1\varphi_2(n_2) - \varphi_1(\lambda_2 n_2) - \varphi_2(\lambda_1 n_1) \\ &= \lambda_2\varphi_1(n_1 - n_2) - \lambda_1\varphi_2(n_1 - n_2) \in \text{im}\,(\lambda_2\varphi_1 - \lambda_1\varphi_2)\,. \end{aligned}$$

To show $\ker\psi \subseteq \text{im}\,\phi$: let $(r, n) \in \ker\psi$. Then there exists $m \in N$ such that

$$\lambda_1 r + \lambda_2 r - \varphi_1(n) - \varphi_2(n) = \lambda_2\varphi_1(m) - \lambda_1\varphi_2(m). \tag{$*$}$$

To show $(r, n) \in \text{im}\,\varphi$ we need to show that there exist $n_1, n_2 \in N$ such that

$$r = \varphi_1(n_1) + \varphi_2(n_2) \tag{1}$$

$$n = \lambda_1 n_1 + \lambda_2 n_2. \tag{2}$$

From $(*)$ we know that

$$r = \varphi_1\left(\frac{\lambda_2 m + n}{\lambda_1 + \lambda_2}\right) + \varphi_2\left(\frac{n - \lambda_1 m}{\lambda_1 + \lambda_2}\right).$$

Letting $n_1 = \frac{\lambda_2 m + n}{\lambda_1 + \lambda_2}$ and $n_2 = \frac{n - \lambda_1 m}{\lambda_1 + \lambda_2}$ we get a solution to Equations (1) and (2) above and we are done.

Suppose $\lambda_1 + \lambda_2 = 0$. Note that in this case im $(\lambda_2 \varphi_1 - \lambda_1 \varphi_2) =$ im $(\varphi_1 + \varphi_2)$. To show im $\varphi \subseteq \ker \psi$ note that

$$
\begin{aligned}
&\lambda_1(\varphi_1(n_1) + \varphi_2(n_2)) - \varphi_1(\lambda n_1 - \lambda n_2) \\
&= \varphi_1(\lambda_1 n_2) + \varphi_2(\lambda n_2) \in \text{im } (\varphi_1 + \varphi_2)\,.
\end{aligned}
$$

To show $\ker \psi \subseteq$ im φ: let $(r, n) \in \ker \psi$. Then there exists $m \in N$ such that

$$\lambda_1 r - \varphi_1(n) = \varphi_1(m) + \varphi_2(m)$$

and we require $n_1, n_2 \in N$ such that

$$
\begin{aligned}
r &= \varphi_1(n_1) + \varphi_2(n_2) \\
n &= \lambda_1 n_1 - \lambda_1 n_2.
\end{aligned}
$$

Clearly $n_1 = (n + m)/\lambda_1$ and $n_2 = m/\lambda_1$ work. □

This completes the proof of the theorem. □

The above theorem suggests that we should study quotients of A. In particular, we should try to better understand the component(s) of the Hilbert scheme of A containing the A-modules whose underlying $\mathcal{O}_Y$-module structure is $\mathcal{O}_C$ where C is a $(1,1)$-divisor. We do this in the following chapter.

Chapter 3

The Hilbert Scheme of A

In this chapter we will study the Hilbert scheme of A - the moduli space of left sided quotients of A. Mimicking the commutative case, one should think of a quotient of A, which is supported on a curve on Y, as a noncommutative curve lying on A. As mentioned at the end of the last chapter, we are primarily interested in those quotients of A which are supported on a $(1,1)$-divisor on Y. In this chapter we shall see that the moduli space of such quotients is a smooth projective surface which exhibits an $8:1$ cover of $(\mathbb{P}^2)^\vee$ ramified on a union of 2 conics and their four bitangents. We shall then describe this ramification in detail.

As we have done previously for the moduli of line bundles, we begin by making the notion of a Hilbert scheme precise, and prove its existence.

Definition 3.0.6. *Let X be a smooth projective surface and A an order on X. Let S be a scheme and fix a polynomial $P \in \mathbb{Q}[z]$. A* **flat family of quotients of** A **on** S*, with Hilbert polynomial P, is a coherent sheaf $\mathcal{F}$ on $X \times_k S$ of left A_S-modules where A_S is the pull back of A to $X \times_k S$, together* *with a surjective morphism*

$$A_S \longrightarrow \mathcal{F} \longrightarrow 0$$

such that $\mathcal{F}$ is flat over S and for each $p \in S$, $\mathcal{F}_{k(p)}$ has Hilbert polynomial P.

We denote the corresponding Hilbert functor by

$$Hilb_A^P : (\text{Schemes}/k)^{\text{op}} \longrightarrow (\text{Sets})$$

where for any scheme S

$$Hilb_A^P(S) := \left\{ \begin{matrix} \text{set of isomorphism classes of flat families of quotients of} \\ A \text{ on } S \text{ with Hilbert polynomial } P \end{matrix} \right\}.$$

Note that two quotients $\mathcal{F}$ and $\mathcal{F}'$ on S are isomorphic if there exists an isomorphism $\phi \colon \mathcal{F} \to \mathcal{F}'$ such that

$$\begin{tikzcd} A_S \arrow[r] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d, "\phi"'] & 0 \\ A_S \arrow[r] & \mathcal{F}' \arrow[r] & 0 \end{tikzcd}$$

is commutative. We now prove the existence of the Hilbert scheme of A. As we shall see in the proof, it is a closed subscheme of the more familiar quotient scheme of A.

Theorem 3.0.7 (Existence of Hilbert Scheme)**.** *There exists a fine moduli moduli scheme $\mathbf{Hilb}^P A$ for the functor $Hilb_A^P$. In fact, $\mathbf{Hilb}^P A$ is a projective scheme.*

We will suppress the P from the notation from now on.

Proof. Theorem 3.2 in [Gro95] proves the existence of the Quot scheme of A - the scheme parameterising $\mathcal{O}_X$-quotients of A. Furthermore, the theorem proves it is projective. We claim that the $\mathcal{O}_X$-quotients of A which are also A-modules form a closed subscheme of this Quot scheme. To see this, let S be a scheme and let $\mathcal{F}$ be a flat family over S of $\mathcal{O}_X$-quotients of A. We have the following exact sequence of $\mathcal{O}_{X\times S}$-modules:

$$0 \longrightarrow \mathcal{I} \longrightarrow A_S \longrightarrow \mathcal{F} \longrightarrow 0.$$

From Proposition 17.1(ii) in [CN08] the locus in S where $A_S \otimes_{X\times S} \mathcal{I} \to \mathcal{F}$ is zero is closed. This map being zero is precisely the condition for $\mathcal{I}$ to be a left ideal of A. $\square$

For our purposes it will be more useful to talk about the Hilbert scheme parametrising quotients of A with certain fixed Chern classes. Since the Chern classes of a coherent sheaf on a smooth projective surface uniquely determine its Hilbert polynomial, this Hilbert scheme exists by the above theorem. See Chapter 9 of [LP97] for more details on this.

3.1 Properties of Hilb A

Having established the existence of the Hilbert scheme of orders, we return to the study of the Hilbert scheme of our specific order we constructed in Construction 1.3.1.

Recall that in Theorem 2.4.4 we saw a link between the moduli space of A-line bundles with with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$ and quotients of A, or noncommutative curves on A, with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$.

Proposition 3.1.1. *Let S be a scheme. Let $\mathcal{F}$ be a flat family of quotients of A on S with Chern classes $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$. Let $I := \ker(A_S \to \mathcal{F})$. Then I is a flat family of A-line bundles on S with Chern classes $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$.*

Proof. We have the following exact sequence

$$0 \longrightarrow I \longrightarrow A_S \longrightarrow \mathcal{F} \longrightarrow 0.$$

I is flat over S because A_S and $\mathcal{F}$ are. Restricting to the fibre above any $p \in S$ we get a sequence

$$0 \longrightarrow I_{k(p)} \longrightarrow A \longrightarrow \mathcal{F}_{k(p)} \longrightarrow 0$$

of A-modules which is exact because $\mathcal{F}$ is flat over S and so $\mathcal{T}or^1_{\mathcal{O}_S}(\mathcal{F}, k(p)) = 0$. Since

$$c_1(A) = \mathcal{O}_Y(-1,-1), \quad c_2(A) = 0, \quad c_1(\mathcal{F}_{k(p)}) = \mathcal{O}_Y(1,1), \quad c_2(\mathcal{F}_{k(p)}) = 2$$

we see that $c_1(I_{k(p)}) = \mathcal{O}_Y(-2,-2)$ and $0 = c_2(A) = c_1(I_{k(p)}).c_1(\mathcal{F}_{k(p)}) + c_2(I_{k(p)}) + c_2(\mathcal{F}_{k(p)})$ which implies $c_2(I_{k(p)}) = 2$. $I_{k(p)}$ is torsion free because it is a submodule of A which is torsion free and thus must also be rank 1 over A. Thus $I^{**}_{k(p)} \in \operatorname{Pic} A$ because it is reflexive and hence locally free over Y and hence by Proposition 2.0.2 locally projective over A. By Lemma 2.2.3 and Proposition 2.1.3 we have $c_2(I^{**}_{k(p)}) = 2$ and so $I^{**}_{k(p)} = I_{k(p)}$. □

Having established a relationship between flat families of A-line bundles and flat families of quotients of A, we now use Theorem 2.4.1 to classify all the possible $\mathcal{O}_Y$-module structures that quotients of A may possess. As we shall see some (and, as we shall later see, most) must all also be quotients of $\mathcal{O}_Y$.

Corollary 3.1.2. *Let Q be a quotient of A with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$. Then either:*

- *$Q \simeq A \otimes_F \mathcal{O}_F$ (as an A-module) where F is either a $(1,0)$ or $(0,1)$-divisor; or*
- *$Q \simeq \mathcal{O}_C$ (as an $\mathcal{O}_Y$-module) for some σ-invariant $(1,1)$-divisor $C \subset Y$.*

Proof. The above proposition asserts that the kernel of $A \to Q$ is an A-line bundle with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$. We have already classified all such line bundles and their respective cokernels in Proposition 2.4.1 and Theorem 2.4.4. The fact that C must be σ-invariant follows from the fact that in order to be an A-module there must be a non-zero map $L_\sigma \otimes_Y \mathcal{O}_C \to \mathcal{O}_C$ which is only possible if $\sigma^* C = C$. □

Corollary 3.1.3. *Let Q be a quotient of A with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$. If the support Q is smooth (i.e. its the support is $\mathbb{P}^1$) then Q is also quotient of $\mathcal{O}_Y$.*

Proof. Obvious from the previous Corollary because the support of $A \otimes_Y \mathcal{O}_F$ is not smooth. □

From now on **Hilb** A will denote the Hilbert scheme of A corresponding to quotients of A with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$. We now proceed to study its properties.

Proposition 3.1.4. *The dimension of* **Hilb** A *at the point corresponding to* $A \otimes_Y \mathcal{O}_F$*, where* F *is a* $(1,0)$ *or* $(0,1)$*-divisor, is* 2*.*

Proof. We have

$$0 \longrightarrow A \otimes_Y \mathcal{O}_Y(-F) \longrightarrow A \longrightarrow A \otimes_Y \mathcal{O}_F \longrightarrow 0.$$

Let $F' = \sigma^* F$. The dimension of the tangent space is given by:

$$\begin{aligned}
\hom_A\left(A \otimes_Y \mathcal{O}_Y(-F), A \otimes_Y \mathcal{O}_F\right) &= \hom_Y\left(\mathcal{O}_Y(-F), A \otimes_Y \mathcal{O}_F\right) \\
&= \hom_Y\left(\mathcal{O}_Y(-F), \mathcal{O}_F \oplus \mathcal{O}_{F'}(-1)\right) \\
&= h^0(Y, \mathcal{O}_F \oplus \mathcal{O}_{F'}) \\
&= 2.
\end{aligned}$$

□

Unfortunately, we were unable to compute the dimension of the tangent space at any other points as directly as in the above proposition. We thus proceed by first showing that **Hilb** A is smooth and later, after a considerable amount of work, that it is connected. This will of course prove that **Hilb** A is a smooth projective surface.

Theorem 3.1.5. **Hilb** A *is smooth.*

Proof. Let Q be a quotient of A corresponding to some point $p \in$ **Hilb** A. Let M the kernel of $A \to Q$. We have an exact sequence

$$0 \longrightarrow M \longrightarrow A \longrightarrow Q \longrightarrow 0 \qquad (*)$$

where by Proposition 3.1.1 $M \in \operatorname{Pic} A$. Obstruction to smoothness at p is in $\operatorname{Ext}^1_A(M, Q)$ which we now compute. From Corollary 3.1.2 there are only three cases to consider:

- $M \simeq A \otimes_Y \mathcal{O}_Y(-1,0)$ and $Q \simeq A \otimes_Y \mathcal{O}_F$ where F is a $(1,0)$ divisor. Let $F' = \sigma^* F$ which is a $(0,1)$-divisor.

$$\begin{aligned} \operatorname{ext}^1_A(A \otimes_Y \mathcal{O}_Y(-1,0), A \otimes_Y \mathcal{O}_F) &= \operatorname{ext}^1_Y(\mathcal{O}_Y(-1,0), \mathcal{O}_F \oplus \mathcal{O}_{F'}(-1)) \\ &= h^1(Y, \mathcal{O}_F \oplus \mathcal{O}_{F'}) = 0. \end{aligned}$$

- $M \simeq A \otimes_Y \mathcal{O}_Y(0,-1)$ and $Q \simeq A \otimes_Y \mathcal{O}_F$ where F is a $(0,1)$ divisor. The proof is the same as in the case above.

- $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ as an $\mathcal{O}_Y$-module and $Q \simeq \mathcal{O}_C$ as an $\mathcal{O}_Y$-module for some $(1,1)$-divisor C. Using Serre duality, we have:

$$\operatorname{ext}^1_A(M, \mathcal{O}_C) = \operatorname{ext}^1_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M).$$

  Using the local-global spectral sequence we have

$$\begin{aligned} 0 \to H^1(Y, \mathcal{H}om_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M)) &\to \operatorname{Ext}^1_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M) \\ \to H^0(Y, \mathcal{E}xt^1_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M)). \end{aligned}$$

  $\mathcal{H}om_A(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M) = 0$ since $\mathcal{O}_C$ is a torsion sheaf. Furthermore, $(*)$ is a locally projective A-module resolution of $\mathcal{O}_C$ and so we get

$$0 \to \mathcal{H}om_A\,(A, \mathcal{O}_Y(-H) \otimes_Y M) \to \mathcal{H}om_A\,(M, \mathcal{O}_Y(-H) \otimes_Y M)$$
$$\to \mathcal{E}xt^1_A\,(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M) \to 0.$$

Finally, since

$$H^0(\mathcal{H}om_A\,(M, \mathcal{O}_Y(-H) \otimes_Y M)) = 0$$

and

$$H^1(\mathcal{H}om_A\,(A, \mathcal{O}_Y(-H) \otimes_Y M)) = H^1(Y, \mathcal{O}_Y(-H) \otimes_Y M) = 0$$

we see that

$$H^0(Y, \mathcal{E}xt^1_A\,(\mathcal{O}_C, \mathcal{O}_Y(-H) \otimes_Y M)) = 0$$

and so the result follows.

□

Thus, so far we know that at least one connected component of **Hilb** A is a smooth projective surface. As mentioned earlier, in the next chapter we will see that in fact **Hilb** A is connected, which will prove that this must be its only component.

Corollary 3.1.2 says that some quotients of A are in fact also quotients of $\mathcal{O}_Y$. In particular, they are isomorphic to $\mathcal{O}_C$ where C is a σ-invariant $(1,1)$-divisor. Furthermore, the support of $A \otimes_Y \mathcal{O}_F$ is $F \cup \sigma^* F$ which is also a σ-invariant $(1,1)$-divisor.

Furthermore, all σ-invariant $(1,1)$-divisors are equal to $\pi^* l$ where l is a line on Z. Since lines on Z are parameterised by $(\mathbb{P}^2)^\vee \simeq \mathbb{P}^2$, can view $(\mathbb{P}^2)^\vee$ as the parameter space of σ-invariant $(1,1)$-divisors. Thus we have:

Theorem 3.1.6. *Let $\mathcal{F}$ be the universal family of quotients of A on $Y \times_k \mathbf{Hilb}\ A$. There exists a regular map*

$$\begin{aligned} \Psi\colon \mathbf{Hilb}\ A &\longrightarrow (\mathbb{P}^2)^\vee \\ p &\longmapsto \operatorname{supp} \mathcal{F}_{k(p)} \end{aligned}$$

Proof. The sheaf $\mathcal{O}_{\operatorname{supp} \mathcal{F}}$ is a family of $\mathcal{O}_Y$-quotients on $\mathbf{Hilb}\ A$ for as we saw, the support of every quotient of A is a σ-invariant $(1,1)$-divisor. Furthermore, it is flat over $\mathbf{Hilb}\ A$ since every fibre above $\mathbf{Hilb}\ A$ has the same Chern class and hence the same Hilbert polynomial. This gives us the map $\mathbf{Hilb}\ A \to \mathbf{Hilb}\ Y$ with the image being the subscheme of $\mathbf{Hilb}\ Y$ parameterising σ-invariant $(1,1)$-divisors. As discussed, this subscheme is just $(\mathbb{P}^2)^\vee$ and so the result follows. $\square$

In summary, the map Ψ does the following: every closed point on $\mathbf{Hilb}\ A$ corresponds to some quotient of A. There are two possibilities: either

(i) it is also a quotient of $\mathcal{O}_Y$, in which case as an $\mathcal{O}_Y$-module it is isomorphic to $\mathcal{O}_C$, where C is a σ-invariant $(1,1)$-divisor, or

(ii) it is not a quotient of $\mathcal{O}_Y$, then it is isomorphic, as an A-module, to $A \otimes_Y \mathcal{O}_F$ where F is either a $(0,1)$ or $(1,0)$-divisor.

The crucial point is that the support of $A \otimes_Y \mathcal{O}_F$ is also a σ-invariant $(1,1)$-divisor. Thus to every closed point on $\mathbf{Hilb}\ A$ one can associate a σ-invariant $(1,1)$-divisor. Since σ-invariant $(1,1)$-divisors are parameterised by $(\mathbb{P}^2)^\vee$, we get a natural set-theoretic map from (closed points of $\mathbf{Hilb}\ A$) $\to$ (closed points of $(\mathbb{P}^2)^\vee$). The above theorem proves that this map is in fact a morphism of schemes.

3.2 The ramification of $\Psi\colon \mathbf{Hilb}\ A \to (\mathbb{P}^2)^\vee$

We want to study the map Ψ, in particular we want to understand its ramification for then we will be able to compute $(K_{\mathbf{Hilb}\ A})^2$ in the next chapter. This amounts to computing the number of quotients of A which have support a σ-invariant $(1,1)$-divisor and $c_2 = 2$. Corollary 3.1.2 implies that this question will be answered provided we can understand the number of A-module structures that $\mathcal{O}_C$ can be given, where C is a σ-invariant $(1,1)$-divisor.

To give a coherent sheaf $\mathcal{G}$ on Y an A-module structure amounts to giving a left $\mathcal{O}_Y$-module morphism $\varphi : A \otimes_Y \mathcal{G} \to \mathcal{G}$ satisfying the necessary associativity condition. Two such morphisms φ, φ' give rise to isomorphic A-modules provided there exists $\psi \in \mathrm{Aut}_Y \mathcal{G}$ such that

$$\begin{tikzcd} A \otimes_Y \mathcal{G} \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}\otimes\psi"', "\wr"] & \mathcal{G} \arrow[d, "\psi", "\wr"'] \\ A \otimes_Y \mathcal{G} \arrow[r, "\varphi'"'] & \mathcal{G} \end{tikzcd}$$

commutes. In general it may be rather difficult to determine whether such a ψ exists, and consequently, whether two seemingly different A-module structures are actually isomorphic. The problem becomes increasingly difficult as the size of $\mathrm{Aut}_Y \mathcal{G}$ increases. Luckily, in our case, this issue is easily manageable.

Example 3.2.1. We can illustrate of the above phenomenon with two (related) examples. Recall from Theorem 2.4.1 that an A-line bundle had two possible $\mathcal{O}_Y$-module structures: either it was $\mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1-1)$ or $A \otimes_Y \mathcal{O}_Y(-1,0) \stackrel{Y}{\simeq} \mathcal{O}_Y(-1,0) \oplus \mathcal{O}_Y(-1,-2)$. The former, as we later saw, had infinitely many non-isomorphic A-module structures whilst the latter, only had one. The fact that that $\mathcal{O}_Y(-1,0) \oplus \mathcal{O}_Y(-1,-2)$ has only one A-module structure is only clear, when it is written as $A \otimes_Y \mathcal{O}_Y(-1,0)$ and from the fact that:

Lemma 3.2.2. *Let $\mathcal{G}$ be a coherent sheaf on Y and let $\mathcal{F}$ be an A-module such that $A \otimes_Y \mathcal{G} \simeq \mathcal{F}$ as $\mathcal{O}_Y$-modules. If the natural A-module map $A \otimes_Y \mathcal{G} \to \mathcal{F}$ is injective, then they are isomorphic as A-modules.*

Proof. This is well known and follows from the fact that the assumption implies $A \otimes_Y \mathcal{G}$ and $\mathcal{F}$ have the same Hilbert polynomial and so each graded piece of their associated graded modules have the same dimension. Since an injective map of vector spaces of the same dimension must be an isomorphism, the result follows. $\square$

Hence, if one does not realise that $\mathcal{O}_Y(-1,0) \oplus \mathcal{O}_Y(-1,-2) \stackrel{Y}{\simeq} A \otimes_Y \mathcal{O}_Y(-1,0)$ then determining the fact that all possible A-module structures are isomorphic may be very hard indeed.

A similar phenomenon occurs for quotients of A. Let $Q := A \otimes_Y \mathcal{O}_F$ and forget the natural A-module structure, and ask: how many (non-isomorphic) A-module structures can Q have? If one does not realise that at least as an $\mathcal{O}_Y$-module $Q \simeq A \otimes_Y \mathcal{O}_F$ it will be difficult to prove that all the potentially different A-module structures are in fact isomorphic. Furthermore, as we are about to see, for most σ-invariant $(1,1)$-divisors C, $\mathcal{O}_C$ will have several, but finitely many, A-module structures.

The reason for the difference in the number of A-module structures is partly due to the size of the endomorphism ring of the modules. In the first example, $\dim_k \mathrm{End}_Y(\mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)) = 4$ whilst $\dim_k \mathrm{End}_Y(A \otimes_Y \mathcal{O}_Y(-1,0)) = 5$. A larger automorphism group means it is "easier" for two A-modules structures to be isomorphic.

We now study the number of A-module structures that $\mathcal{O}_C$ may possess. For any $p \in (\mathbb{P}^2)^\vee$ we will denote by l_p the corresponding line in $\mathbb{P}^2$ and we let $C_p := \pi^* l_p$ which is a σ-invariant $(1,1)$-divisor.

As we saw, for every $p \in (\mathbb{P}^2)^\vee$, giving $\mathcal{O}_{C_p}$ an A-module structure amounts to giving a left $\mathcal{O}_Y$-module map $A \otimes_Y \mathcal{O}_{C_p} \to \mathcal{O}_{C_p}$ satisfying the necessary associativity condition. In order to better understand this we first introduce some notation: we let $\bar{L} := L \otimes_Y \mathcal{O}_{C_p} = L|_{C_p}$ and $\bar{D} := D \cap C_p$. Then, since $A \otimes_Y \mathcal{O}_{C_p} = A/\mathcal{I}_{C_p} A = A|_{C_p}$ and because $L_\sigma^{\otimes 2} \simeq \mathcal{O}_Y(-D)$ this condition is equivalent to giving a map $m \colon \bar{L} \to \mathcal{O}_{C_p}$ such that

$$\mathcal{O}_{C_p}(-\bar{D}) \simeq \bar{L}_\sigma \otimes_{C_p} \bar{L}_\sigma \xrightarrow{1 \otimes m} \bar{L}_\sigma \xrightarrow{m} \mathcal{O}_{C_p}(-\bar{D})$$

is the identity. Note that given such a map m, the map $-m$ gives a different, non isomorphic A-module structure to $\mathcal{O}_{C_p}$. This observation gives us the following:

Proposition 3.2.3. *There exist an involution $\tau \colon$ **Hilb** $A \to$ **Hilb** A sending an A-module structure given by m to the one given by $-m$. The fixed points are those which correspond to quotients of A that are not quotients of $\mathcal{O}_Y$.*

Proof. If τ sends the A-module structure given by m to the one given by $-m$ then if the module is also a quotient of $\mathcal{O}_Y$ then as we just saw, these two

A-module structures are not isomorphic. If the module is not a quotient of $\mathcal{O}_Y$ then by Corollary 3.1.2 it must be of the form $A \otimes_Y \mathcal{O}_F$ and so it may only possess one A-module structure as we saw in Lemma 3.2.2. $\square$

Corollary 3.2.4. *The map* $\Psi \colon \mathbf{Hilb}\ A \to (\mathbb{P}^2)^\vee$ *factors through* $\mathbf{Hilb}\ A/\langle \tau \rangle$. *I.e. we have the following commutative diagram*

$$\begin{tikzcd}
\mathbf{Hilb}\ A \arrow[dd] \arrow[dr] & \\
& \mathbf{Hilb}\ A/\langle \tau \rangle \arrow[dl] \\
(\mathbb{P}^2)^\vee &
\end{tikzcd}$$

Proof. Clear from the above proposition and Theorem 3.1.6. $\square$

We can view m as an element of $H^0(C_p, \bar{L}^{-1})$ and, up to multiplication by ± 1, the associativity condition then simply says that we need $\operatorname{div} m + \operatorname{div} \sigma^* m = \bar{D}$, where each such m gives rise to two A-module structures. Since $\bar{D}$ is a finite number of points we have proved the following lemma, which also finishes off the proof of the Theorem 3.1.6:

Lemma 3.2.5. *The map* Ψ *is finite.*

Proof. For each closed point $p \in (\mathbb{P}^2)^\vee$, $\Psi^{-1}(p)$ consists of points corresponding to:

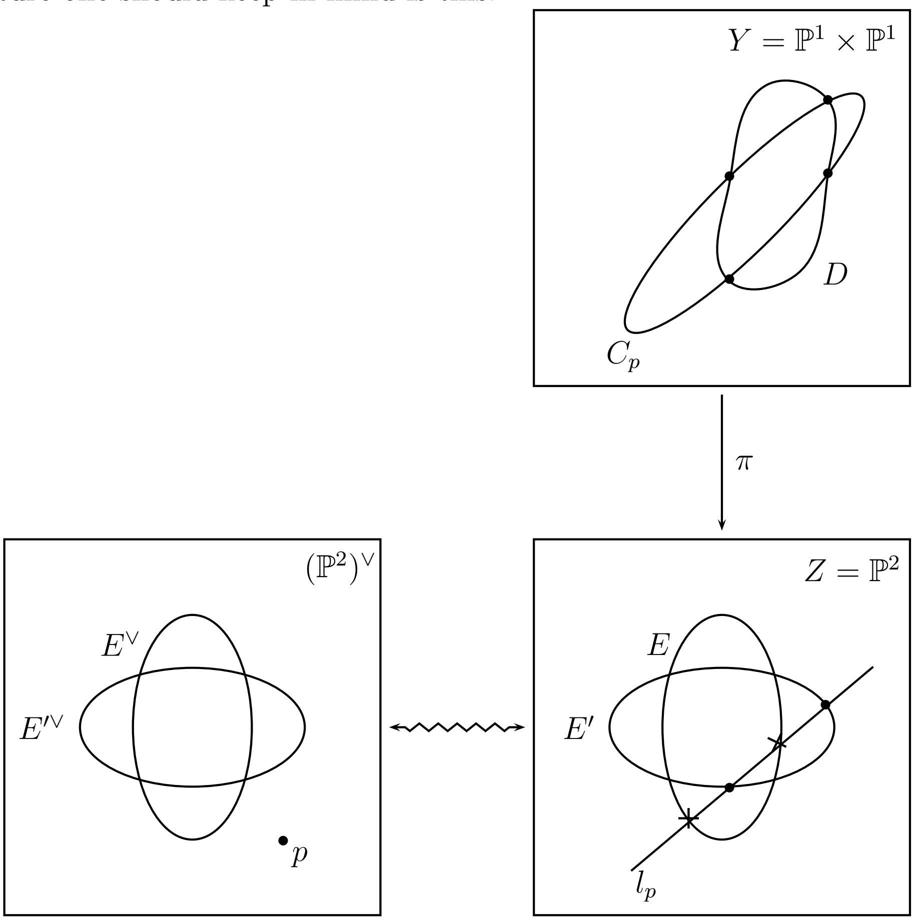
(i) All the possible (non-isomorphic) A-modules with underlying $\mathcal{O}_Y$-module structure $\mathcal{O}_{C_p}$. As we mentioned, $\bar{D}$ is finite and so there are only a finite number of choices for $\operatorname{div} m$ each giving rise to precisely two A-module structures on $\mathcal{O}_{C_p}$.

(ii) If $C_p = F_p + F'_p$, where F is $(1,0)$-divisor and $F' = \sigma^* F$, we get two more points in the fibre, corresponding to $A \otimes_Y \mathcal{O}_F$ and $A \otimes_Y \mathcal{O}_{F'}$.

$\square$

This way of thinking allows us to view the problem of giving $\mathcal{O}_{C_p}$ an A-module structure geometrically. As we are about to see, the number of A-module structures that $\mathcal{O}_{C_p}$ can be given depends primarily how many points l_p intersects with E and E'.

Note also that the dual of a smooth conic in $\mathbb{P}^2$ is another smooth conic in $(\mathbb{P}^2)^\vee$. We denote the duals of E and E' by $E^\vee$ and $E'^\vee$ respectively. The picture one should keep in mind is this:



We mark where l_p intersects E with a "×" and where l_p intersects E' with a "•". The problem of giving $\mathcal{O}_{C_p}$ an A-module structure breaks up into two cases:

1. l_p is not tangential to E. In this case we get $C_p \to l_p$ is a $2:1$ cover ramified at two points and hence $C_p \simeq \mathbb{P}^1$, in particular it is smooth. We analyse this case first, in Chapter 3.2.1.

2. l_p is tangential to E. In this case $C_p \to l_p$ is ramified at only one point and hence C_p is the union of two $\mathbb{P}^1$'s, in particular it is singular. We analyse this case second, in Chapter 3.2.2.

From now on, in all subsequent diagrams, any conic on Z whose major axis is vertical will be E, any conic whose major axis is horizontal will be E' and similarly with $E^\vee$ and $E'^\vee$ on $(\mathbb{P}^2)^\vee$ and hence will not longer be labelled.

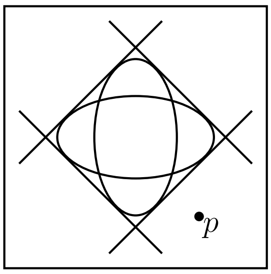
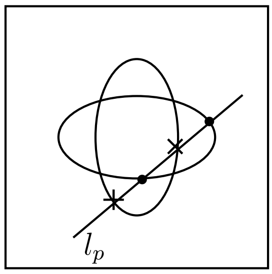
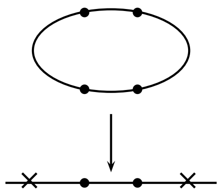
3.2.1 If C is smooth

As mentioned earlier, we begin by studying the first of the two cases mentioned above. Recall that C_p is smooth, in fact $C_p \simeq \mathbb{P}^1$, precisely when l_p is not tangential to E or, equivalently, when p doesn't lie on $E^\vee$. In this case, from Corollary 3.1.3 we know that all quotients of A with this support have their underlying $\mathcal{O}_Y$-module structure isomorphic to $\mathcal{O}_C$.

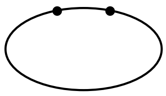
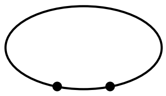
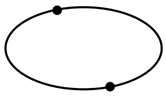
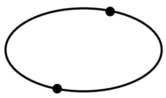
In this case, since Pic $C_p \simeq \mathbb{Z}$ we have $H^0(C_p, \bar{L}^{-1}) = H^0(C_p, \mathcal{O}_{C_p}(2))$ and so to give $\mathcal{O}_{C_p}$ an A-module structure corresponds to choosing two points $\bar{D}' \subseteq \bar{D} := C_p \cap D$ such that $\bar{D}' + \sigma^* \bar{D}' = \bar{D}$. As mentioned earlier, any such choice gives rise to precisely two A-module structures. There are several cases that need to be considered which are best summarised by the configuration of the "×" and "•" on l_p. Note that we are only concerned with which × and •'s overlap, and not their actual relative position on l_p.

Case 1: The first case we consider is when there is no overlap between the × and •'s. The configuration of "×" and "•" on l_p is as follows:

This will occur precisely when l_p is not tangent to either E or E' and does not pass through $E \cap E'$ or, equivalently, when p does not lie on either $E^\vee$ or $E'^\vee$ nor on any of the four bitangents to them and so we see that this is the generic case. In summary we have:

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

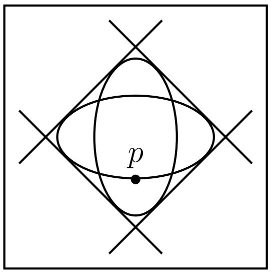
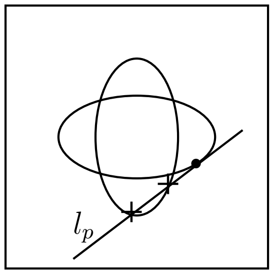
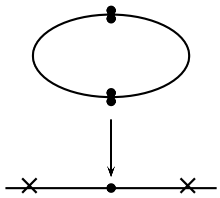
Thus there are 4 choices for $\bar{D}'$ which results in 8 different A-module structures on $\mathcal{O}_{C_p}$. In order for us to later study the ramification of Ψ we also include the column which shows which branch corresponds to which module structure.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
	$1a$ $1b$	8
	$2a$ $2b$	
	$3a$ $3b$	
	$4a$ $4b$	

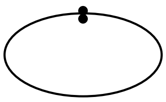
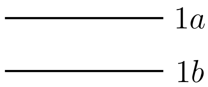
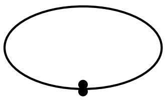
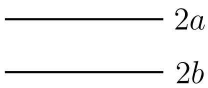
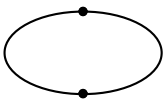
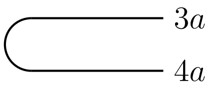
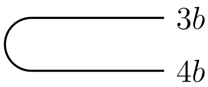
We may thus conclude that Ψ is an $8 : 1$ cover of $(\mathbb{P}^2)^\vee$. The other cases are used to study the ramification of this map.

Case 2: The configuration of "×" and "•" on l_p is as follows:

This will occur precisely when l_p is tangent to E' and intersects E at two other points or, equivalently, when p lies on $E'^\vee$ but not on $E^\vee$ nor on any of the four bitangents.

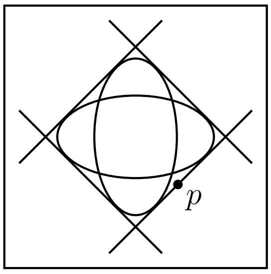
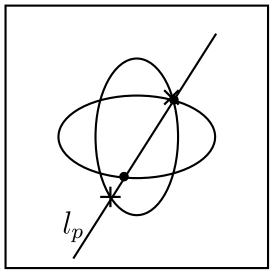
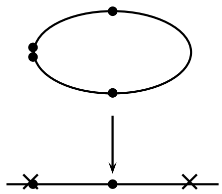
position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

There are now only 3 choices for $\bar{D}'$ as we see in the table below.

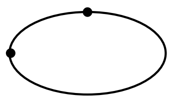
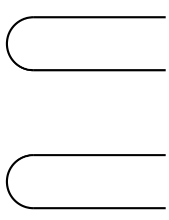
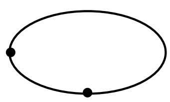
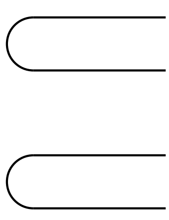
$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		
		6
	 	

Case 3: The configuration of "×" and "•" on l_p is as follows:

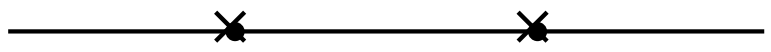
This will occur precisely when l_p passes through exactly one of the point of intersection of E and E' but is not tangent to either conic or, equivalently, when p lies on one exactly one of the four bitangents but not where they meet the conics.

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

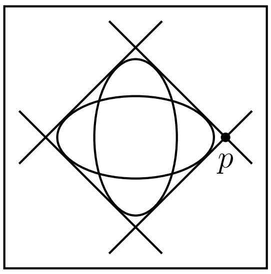
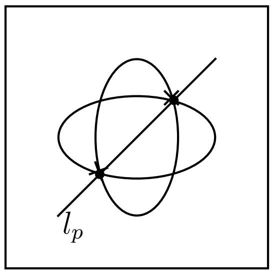
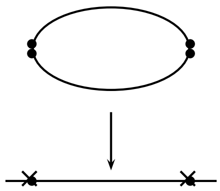
There are now only 2 choices for $\bar{D}'$ as we explain in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		4
		

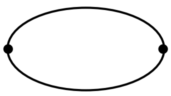
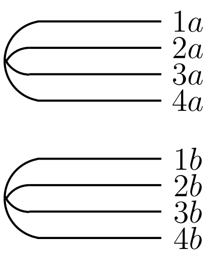
Case 4: The configuration of "×" and "•" on l_p is as follows:



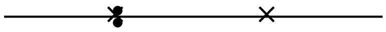
This will occur precisely when l_p passes through two of the four points intersection of E and E' or, equivalently, when p is chosen to be the point of intersection of two bitangents to $E^\vee$ and $E'^\vee$.

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

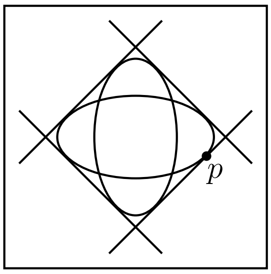
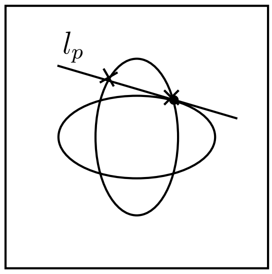
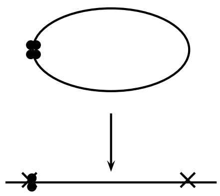
There is now only 1 choice for $\bar{D}'$ as we explain in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		2

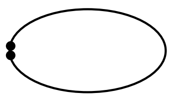
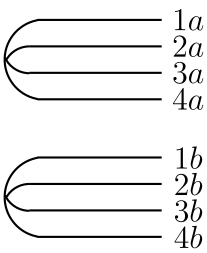
Case 5: The configuration of "×" and "•" on l_p is as follows:



This will occur precisely when l_p is tangent to E' at the point where E and E' intersect or, equivalently, when p lies on the intersection of one of the bitangents and $E'^{\vee}$.

position of $p \in (\mathbb{P}^2)^{\vee}$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

There is now only 1 choice for $\bar{D}'$ as we explain in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		2

3.2.2 If C is singular.

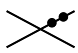
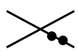
We now analyse the second case mentioned on page 49. Here C_p is singular, in fact it is the union of two $\mathbb{P}^1$'s crossing at one point. This occurs precisely when l_p is tangential to E or, equivalently, when p lies on $E^\vee$. Let $C_p = F_p + F_p'$ where F_p is a $(1,0)$-divisor and $F_p' = \sigma^* F_p$ which is a $(0,1)$-divisor.

Lemma 3.2.6. *Let C be the union of two $\mathbb{P}^1$'s crossing at one point. Then* $\operatorname{Pic} C \simeq \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Let $\Gamma \subset \mathbb{P}^2$ be the divisor consisting of two $\mathbb{P}^1$'s crossing at a single point. From Proposition 4.14 of [Rei97], $\operatorname{Pic} C \simeq \mathbb{Z} \oplus \mathbb{Z}$ provided $H^1(\mathbb{P}^2, \mathcal{O}_\Gamma) = 0$. To check this note that we have an exact sequence

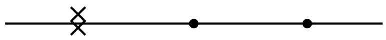
$$\cdots \to H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \longrightarrow H^1(\mathbb{P}^2, \mathcal{O}_\Gamma) \longrightarrow H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(-\Gamma)) \to \cdots$$

from which it is clear that $H^1(\mathbb{P}^2, \mathcal{O}_\Gamma) = 0$. $\square$

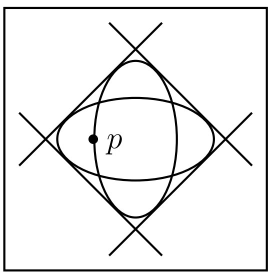
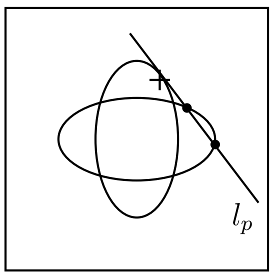
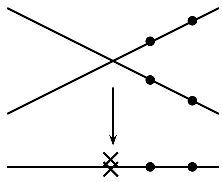
Thus $H^0(C_p, \bar{L}^{-1}) = H^0(C_p, \mathcal{O}_{C_p}(1,1))$ and so to give $\mathcal{O}_{C_p}$ an A-module structure corresponds to choosing two points $\bar{D}' \subseteq \bar{D} := C_p \cap D$ one lying on F_p the other on F_p' such that $\bar{D}' + \sigma^* \bar{D}' = \bar{D}$. As before, any such choice gives rise to precisely two A-module structures. Since we must choose one point from F_p and the other from F_p' (and can not choose both points to lie on F_p nor on F_p') implies that we have "lost" some quotients of A corresponding to p. From a geometric view point, this means that $\bar{D}' =$  and $\bar{D}' =$  do not correspond to A-module structures on $\mathcal{O}_{C_p}$.

However we are now in the case where Corollary 3.1.3 no longer applies, and so not all quotients of A have their underlying $\mathcal{O}_Y$-module structure equal to $\mathcal{O}_C$ for some $(1,1)$-divisor C. In fact from Corollary 3.1.2 we know that for every p lying on $E'^{\vee}$ there are two additional quotients of A (in the sense that they have no analogue in Cases 1-5 because they are not quotients of $\mathcal{O}_Y$) with support C_p and they are $A \otimes_Y \mathcal{O}_{F_p}$ and $A \otimes_Y \mathcal{O}_{F'_p}$. It is thus natural to think of the above two choices of $\bar{D}'$ as giving rise to these two quotients of A and so we make this association in our future analysis of Ψ.

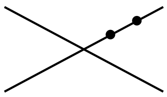
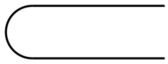
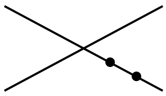
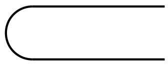
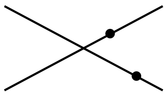
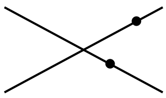
Case 6: The configuration of "×" and "•" on l_p is as follows:

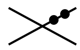
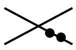


This will occur precisely when l_p is tangent to E but is not tangent to E' nor does it pass through any of the points of intersection of E and E' or, equivalently, when p lies on $E^{\vee}$ but not on $E'^{\vee}$ nor on any of the four bitangents.

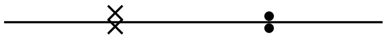
position of $p \in (\mathbb{P}^2)^{\vee}$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

There are now the full 4 choices for $\bar{D}'$, however they only gives rise to six quotients of A as we explain below.

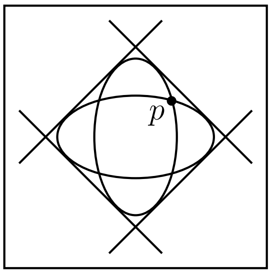
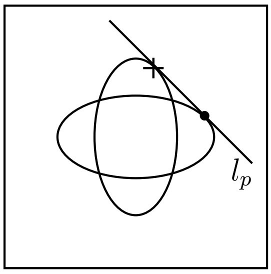
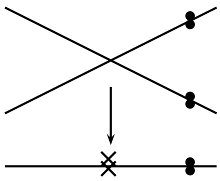
$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
	 $1a$ $1b$	
	 $2a$ $2b$	6
	 $3a$ $3b$	
	 $4a$ $4b$	

Let us explain further why branches $1a$ and $1b$ come together here and why this case is different to Case 1. Recall that to picking $\bar{D}' =$  and  we associate not a total of four A-module structure on $\mathcal{O}_{C_p}$ but the two quotients of A that are not quotients of $\mathcal{O}_Y$ with support C_p, namely $A \otimes_Y \mathcal{O}_F$ and $A \otimes_Y \mathcal{O}_{F'}$. We also saw that the involution τ from Proposition 3.2.3 fixes points of **Hilb** A corresponding to $A \otimes_Y \mathcal{O}_F$ and that by Corollary 3.2.4 the map Ψ factors through τ. Hence the branches $1a$ and $1b$ must intersect at precisely points corresponding to $A \otimes_Y \mathcal{O}_F$. The same argument applies to explain why the branches $2a$ and $2b$ also merge.

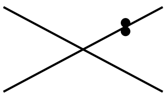
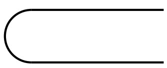
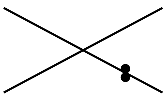
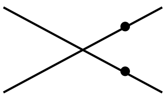
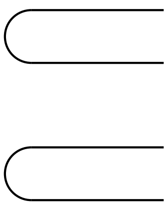
Case 7: The configuration of "×" and "•" on l_p is as follows:



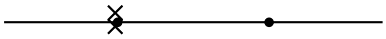
This occurs precisely when l_p is a bitangent to E and E' or, equivalently, when p lies on the intersection of $E^\vee$ and $E'^\vee$.

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

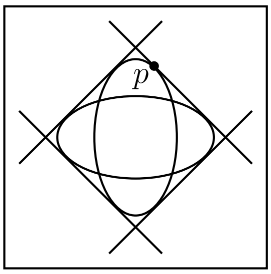
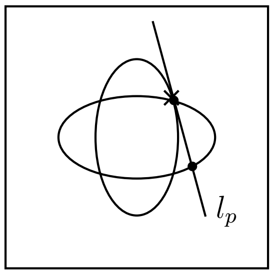
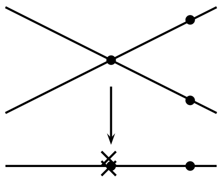
There are now only 4 choices for $\bar{D}'$ as we explain in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		
		4
		

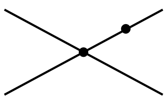
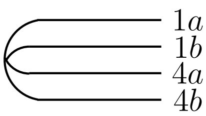
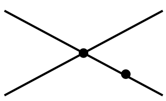
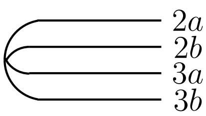
Case 8: The configuration of “$\times$” and “$\bullet$” on l_p is as follows:



This occurs precisely when l_p is a tangent to E at a point where E and E' intersect or, equivalently, when p is one of the points of intersection of the bitangents with $E^\vee$.

position of $p \in (\mathbb{P}^2)^\vee$	position of $l_p \subset \mathbb{P}^2$	$C_p \to l_p$
		

There are now only 2 choices for $\bar{D}'$ as we explain in the table below.

$\bar{D}'$	Branches above p corresponding to $\bar{D}'$	No. of A-quotients with support C_p
		2
		

Note that the two A-module structures with support $C_p = F + F'$ are $A \otimes_Y \mathcal{O}_F$ and $A \otimes_Y \mathcal{O}_{F'}$.

By carefully following which branch connects to which branch we can see that **Hilb** A is in fact connected and thus we may conclude that **Hilb** A is in fact a smooth projective surface.

3.2.3 Possible second Chern classes of A-line bundles

In this chapter we tie up one loose end that we have left from Chapter 2.1 and prove the existence of lines bundles with all possible combinations of Chern classes, provided they satisfy our Bogomolov-type inequality. We continue with the same notation as before.

Theorem 3.2.7. *Let* $c_1 \in \operatorname{Pic} Y$ *and* $c_2 \in \mathbb{Z}$ *such that* $4c_2 - c_1^2 \geq -2$*. Then there exists an* $M \in \operatorname{Pic} A$ *with these Chern classes.*

Before we begin the proof, we need the following lemma:

Lemma 3.2.8. *Let* C *be a smooth,* σ*-invariant* $(1,1)$*-divisor on* Y *and* $N \in \operatorname{Pic} C$*. Endow* $\mathcal{O}_C$ *with an* A*-module structure, which we saw is always possible from Cases 1-5 previously. Then* N *inherits an* A*-module structure from* $\mathcal{O}_C$*.*

Proof. We need give an $\mathcal{O}_Y$-module morphism $A \otimes_C N \to N$ satisfying the required associativity condition. Suppose $\psi\colon A \otimes_C \mathcal{O}_C \to \mathcal{O}_C$ is the morphism which gives $\mathcal{O}_C$ its A-module structure. Then $A \otimes_C N \to A \otimes_Y \mathcal{O}_C \otimes_C N \xrightarrow{\psi \otimes 1} \mathcal{O}_C \otimes_C N \to N$ is the required morphism. $\square$

Proof of theorem. The discriminant of any rank two vector bundle M, defined to be the integer $4c_2(M) - c_1(M)^2$, is unchanged by tensoring with a line bundle (see Chapter 12.1 of [LP97]) and so as we saw before we can thus assume $c_1 = \mathcal{O}_Y(-1,-1)$ or $c_1 = \mathcal{O}_Y$. We deal with these two cases separately although the proofs will be very similar. Fix for the remainder of the proof a smooth σ-invariant $(1,1)$-divisor C and an A-module structure on $\mathcal{O}_C$.

We will now construct an A-line bundle with $c_1 = \mathcal{O}_Y$ and $c_2 = n$ for an arbitrary $n \geq 0$. Using Lemma 3.2.8 endow $\mathcal{O}_C(n+2)$ with an A-module structure. Note that

$$\begin{aligned} \mathrm{Hom}_A\left(A \otimes_Y \mathcal{O}_Y(1,1), \mathcal{O}_C(n+2)\right) &= \mathrm{Hom}_Y\left(\mathcal{O}_Y(1,1), \mathcal{O}_C(n+2)\right) \\ &= \mathrm{Hom}_C\left(\mathcal{O}_C, \mathcal{O}_C(n)\right) \\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) \neq 0. \end{aligned}$$

We claim that there is at least one morphism $\varphi\colon A \otimes_Y \mathcal{O}_Y(1,1) \to \mathcal{O}_C(n+2)$ which is surjective. From the above computation, we see that any A-module morphism $A \otimes_Y \mathcal{O}_Y(1,1) \to \mathcal{O}_C(n+2)$ arises from an $\mathcal{O}_Y$-module morphism $\phi\colon \mathcal{O}_C \to \mathcal{O}_C(n)$. Choose ϕ in such a way that coker $\phi = \oplus k_{p_i}$, where k_{p_i} is the skyscraper sheaf at p_i, with the p_i lying in the Azumaya locus of A. Then, since $A|_p = M_2(k)$, when we extend ϕ to a morphism $\varphi\colon A \otimes_Y \mathcal{O}_Y(1,1) \to \mathcal{O}_C(n+2)$ we must have coker $\varphi = 0$ for the simple representations of $M_2(k)$ are all two dimensional. Letting $M :=$ ker φ we have

$$0 \longrightarrow M \longrightarrow A \otimes_Y \mathcal{O}_Y(1,1) \longrightarrow \mathcal{O}_C(n+2) \longrightarrow 0. \qquad (*)$$

It is easy to check that $c_1(M) = \mathcal{O}_Y$ and $c_2(M) = n$. Furthermore, M is clearly torsion free. To see that it is reflexive note that upon taking reflexive hulls of $(*)$ we get $M^{**} \to A \otimes_Y \mathcal{O}_Y(1,1)$ which is injective because the kernel must be torsion. Thus M^{**}/M which is supported only on points must be a subsheaf of $\mathcal{O}_C(n+2)$ and hence must be zero. Thus $M = M^{**}$ and so M is reflexive and hence locally free over Y. Thus by Proposition 2.0.2 we have $M \in \operatorname{Pic} A$.

Constructing an A-line bundle with $c_1 = \mathcal{O}_Y(-1,-1)$ and $c_2 = n$ for an arbitrary $n \geq 0$ is an almost identical process where one finds a surjective morphism $\varphi : A \to \mathcal{O}_C(n)$ in the same manner as before and then proves that the kernel must be a line bundle. A simple computation shows that this kernel has the desired Chern classes. $\square$

Chapter 4

The Link

In this chapter we establish a link between the moduli space of A-line bundles with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$, which as before we denote by **Pic** A, and the Hilbert scheme of A, which parameterises quotients of A with $c_1 = \mathcal{O}_Y(1,1)$ and $c_2 = 2$, which as before is denoted by **Hilb** A. In particular we will show that **Hilb** A is a ruled surface over **Pic** A. Thus by using the map Ψ from the previous chapter, we will calculate $(K_{\mathbf{Hilb}\ A})^2$, which will allow us to determine the genus of **Pic** A.

We have already seen this link between line bundles on A and quotients of A. It is summarised with the following exact sequence

$$0 \longrightarrow M \longrightarrow A \longrightarrow Q \longrightarrow 0$$

where:

- $Q \simeq \mathcal{O}_C$ as an $\mathcal{O}_Y$-module, which occurs precisely when $M \in \text{Pic } A$ with $M \simeq \mathcal{O}_Y(-1,-1) \oplus \mathcal{O}_Y(-1,-1)$ as an $\mathcal{O}_Y$-module, or
- $Q \simeq A \otimes_Y \mathcal{O}_F$, where F is a $(1,0)$ (respectively $(0,1)$) divisor, which oc-

curs precisely when $M \simeq A \otimes_Y \mathcal{O}_Y(-1,0)$ (respectively $A \otimes_Y \mathcal{O}_Y(0,-1)$).

Furthermore, we saw in Proposition 2.4.3 that in both cases $\hom_A(M, A) = 2$ which suggests there is a map $\mathbf{Hilb}\ A \to \mathbf{Pic}\ A$ with $\mathbb{P}^1$ fibres. We prove this now.

Theorem 4.0.9. *$\mathbf{Hilb}\ A$ is a ruled surface over $\mathbf{Pic}\ A$.*

Proof. Let $\mathcal{F}$ be the universal family on $\mathbf{Hilb}\ A$. From Proposition 3.1.1 $\ker(A_{\mathbf{Hilb}\ A} \to \mathcal{F})$ is a flat family of A-line bundles on $\mathbf{Hilb}\ A$ and so we get a map $\Phi : \mathbf{Hilb}\ A \to \mathbf{Pic}\ A$. From Theorem 2.2.4 we know $\mathbf{Pic}\ A$ is smooth and from Proposition 2.4.2 we know at least one of its components is a curve. However, from the previous chapter we know that $\mathbf{Hilb}\ A$ is a smooth projective surface and thus $\mathbf{Pic}\ A$ must in fact be connected and hence must be a smooth projective curve. It thus suffice to show that every fibre of Φ is isomorphic to $\mathbb{P}^1$.

We do so by constructing, for every closed point in $\mathbf{Pic}\ A$, a flat family of quotients of A on $\mathbb{A}^2 - \{0\}$ which will subsequently give us a map

$$\begin{tikzcd} \mathbf{Hilb}\ A \arrow[r, "\Phi"] & \mathbf{Pic}\ A \\ \mathbb{P}^1 \arrow[u] & \end{tikzcd}$$

with the $\mathbb{P}^1$ contracting down to the chosen point.

Let $M \in \mathrm{Pic}\ A$ correspond to some point in $\mathbf{Pic}\ A$. Let $V := \mathrm{Hom}_A(M, A)$ which as we showed previously is a 2-dimensional vector space over k. We have the following morphism

$$M \otimes_k V \longrightarrow A$$
$$m \otimes v \longmapsto v(m)$$

which, via the tensor-hom adjunction, gives a morphism $M \to A \otimes_k V^*$. This extends to a map $M \to A \otimes_k S(V^*)$ where $S(V^*)$ is the symmetric algebra of V^*. This further extends to a map $M \otimes_k S(V^*) \to A \otimes_k S(V^*)$ and we let $\mathcal{G}$ denote the cokernel. We thus have an exact sequence of $A_{\mathrm{Spec}\, S(V^*)}$-modules on $Y \times \mathcal{O}_{\mathrm{Spec}\, S(V^*)}$

$$0 \longrightarrow M \otimes_k S(V^*) \longrightarrow A \otimes_k S(V^*) \longrightarrow \mathcal{G} \longrightarrow 0.$$

Since any non-zero A-module morphism $M \to A$ is injective we see that $\mathcal{G}$ is a flat family of A-quotients on $\mathbb{A}^2 - \{0\} = \mathrm{Spec}\, S(V^*) - \{0\}$. Note that we have a canonical isomorphism $V \simeq \mathrm{Spec}\, S(V^*)$ sending any closed point $v \in V$ to $\langle f \in V^* \mid f(v) = 0 \rangle$ and so if $v \neq 0$ we have the exact sequence

$$0 \longrightarrow M \xrightarrow{v} A \longrightarrow \mathcal{G}_{k(v)} \longrightarrow 0.$$

Since as quotients of A, $\mathcal{G}_{k(v)} \simeq \mathcal{G}_{k(\lambda v)}$ for all $\lambda \in k^*$ the corresponding map $(\mathbb{A}^2 - \{0\}) \to \mathbf{Hilb}\ A$ factors through $\mathrm{Proj}\, S(V^*) \simeq \mathbb{P}^1$. Furthermore, since $\mathrm{End}_A(A \otimes_Y \mathcal{O}_F) = \mathrm{End}_A \mathcal{O}_C = k$ we see that $\mathbb{P}^1$ in fact embeds into $\mathbf{Hilb}\ A$.

$\square$

Since **Hilb** A is a ruled surface over **Pic** A we can determine the genus of **Pic** A using Corollary 2.11 in Chapter 5 of [Har77] which states that

$$(K_{\mathbf{Hilb}\ A})^2 = 8(1 - g(\mathbf{Pic}\ A)).$$

Furthermore, we can determine $(K_{\mathbf{Hilb}\ A})^2$ using the map Ψ.

Theorem 4.0.10. *The moduli space parameterising A-line bundles with $c_1 = \mathcal{O}_Y(-2,-2)$ and $c_2 = 2$ is a smooth projective curve of genus 2.*

Proof. As discussed above, all that we need to do is compute $(K_{\mathbf{Hilb}\ A})^2$. Recall from the previous chapter we have an $8:1$ map $\Psi\colon \mathbf{Hilb}\ A \to (\mathbb{P}^2)^\vee$. Thus using Formula 19 of Section 16 in Chapter 1 of [BPVdV84] we have:

$$K_{\mathbf{Hilb}\ A} = \Psi^* K_{(\mathbb{P}^2)^\vee} + R$$

where R is the ramification divisor on **Hilb** A.

Let us describe R. Looking at Case 2 of Chapter 3.2.1 we define R_1 and U_1 to be the divisors such that $\Psi^* E'^\vee = 2R_1 + U_1$. Similarly looking at Case 6 in Chapter 3.2.2 we define R_2 and U_2 to be such that $\Psi^* E^\vee = 2R_2 + U_2$. Denote by $L_3, \cdots, L_6$ the four bitangents to $E^\vee \cup E'^\vee$. Looking at Case 3 of Chapter 3.2.1 we see that $\Psi^* L_i$ is two divisible and we let R_i be such that $\Psi^* L_i = 2R_i$. Thus $R = R_1 + R_2 + \cdots + R_6$.

We now compute $(K_{\mathbf{Hilb}\ A})^2 = (\Psi^*(K_{(\mathbb{P}^2)^\vee}) + R_1 + \cdots + R_6)^2$. Throughout this calculation K denotes $K_{(\mathbb{P}^2)^\vee}$.

- $(\Psi^* K)^2 = 8 \cdot (-3)^2 = 72$
- $(\Psi^* K).R_1 = K.(\Psi_* R_1) = 2K.E' = 2 \cdot (-6) = -12$. Similarly,

- $(\Psi^* K).R_2 = -12$.
- $(\Psi^* K).R_i = K.(\Psi_* R_i) = 4K.L_i = 4 \cdot (-3) = -12$ for $i = 3, \cdots, 6$.
- $R_1.R_2 = 0$ from Case 7 on page 59.
- $R_1.R_i = \frac{1}{2} R_1.(\Psi^* L_i) = \frac{1}{2}(\Psi_* R_1).L_i = \frac{1}{2} 2E'.L_i = 2$ for $i = 3, \cdots, 6$. Similarly,
- $R_2.R_i = 2$ for $i = 3, \cdots, 6$.
- $R_i.R_j = \frac{1}{2}(\Psi^* L_i).R_j = \frac{1}{2} L_i.(\Psi_* R_j) = \frac{1}{2} L_i.4L_j = 2L_i.L_j = 2$ for $i, j = 3, \cdots, 6$.

What remains is to compute R_1^2 and R_2^2.

We can see from Case 2, 5 and 7 that $\Psi|_{R_1} : R_1 \to E'$ is an étale double cover of E'. Thus $R_1 = R_1' + R_1''$ where both R_1' and R_1'' have genus zero. We now use the adjunction formula (Proposition 1.5 in Chapter V of [Har77]) to compute $R_1'^2$. We have

$$\begin{aligned} -2 &= R_1'.(2R_1' + \Psi^* K + R_2 + R_3 + \cdots + R_6) \\ &= 2R_1'^2 + E'.K + 0 + 4 \cdot R_1'.\frac{1}{2}(\Psi^* L_3) \\ &= 2R_1'^2 - 6 + 4 \cdot 1 = 2R_1'^2 - 2. \end{aligned}$$

Thus $R_1'^2 = 0$ and an identical computation shows $R_1''^2 = 0$. Thus $R_1^2 = 0$.

The same argument shows $R_2^2 = 0$ since $R_2 \to E$ is also an étale double cover.

Thus

$$\begin{aligned}
(K_{\mathbf{Hilb}\,A})^2 &= (\Psi^*K)^2 + R_1^2 + \cdots + R^6 + \\
&\quad + 2\left((\Psi^*K).R_1 + \cdots (\Psi^*K).R_6 + \sum R_i.R_j\right) \\
&= 72 + 0 + 0 + 4\cdot 2 + 2\,(6\cdot(-12)4\cdot 2 + 4\cdot 2 + 6\cdot 2) \\
&= -8
\end{aligned}$$

and so $g(\mathbf{Pic}\ A) = 2$. □

Note that at no stage did we use the fact that **Hilb** A is ruled in order to calculate $(K_{\mathbf{Hilb}\,A})^2$. In particular, we didn't use the fact that we knew in advance that $(K_{\mathbf{Hilb}\,A})^2$ is a multiple of eight. We could have simplified the computation above if we had done so, but it seemed nice to spend the extra work and get an independent confirmation that fact.

As we saw in the above proof R_2 is the union of two $\mathbb{P}^1$'s. These $\mathbb{P}^1$'s are fibres of $\Phi : \mathbf{Hilb}\ A \to \mathbf{Pic}\ A$ above the two very special points on $\mathbf{Pic}\ A$ corresponding to the A-line bundles $A \otimes_Y \mathcal{O}_Y(-1,0)$ and $A \otimes_Y \mathcal{O}_Y(0,-1)$. Since R_1 is also a union of two $\mathbb{P}^1$'s it would have been nice to find the two A-line bundles which they are fibres of, but unfortunately, we were unable to do so.

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