Representations of quivers over finite fields

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REPRESENTATIONS OF QUIVERS
OVER FINITE FIELDS

by

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Finally, I would like to thank the School of Mathematics, the University of New South Wales for providing necessary research facilities for this thesis.
The main purpose of this thesis is to obtain surprising identities by counting the representations of quivers over finite fields. A classical result states that the dimension vectors of the absolutely indecomposable representations of a quiver \( \Gamma \) are in one-to-one correspondence with the positive roots of a root system \( \Delta \), which is infinite in general. For a given dimension vector \( \alpha \in \Delta^+ \), the number \( A_{\Gamma}(\alpha, q) \), which counts the isomorphism classes of the absolutely indecomposable representations of \( \Gamma \) of dimension \( \alpha \) over the finite field \( \mathbb{F}_q \), turns out to be a polynomial in \( q \) with integer coefficients, which have been conjectured to be non-negative by Kac.

The main result of this thesis is a multi-variable formal identity which expresses an infinite series as a formal product indexed by \( \Delta^+ \) which has the coefficients of various polynomials \( A_{\Gamma}(\alpha, q) \) as exponents. This identity turns out to be a \( q \)-analogue of the remarkable Weyl-Macdonald-Kac denominator identity modulus a conjecture of Kac, which asserts that the multiplicity of \( \alpha \) is equal to the constant term of \( A_{\Gamma}(\alpha, q) \). An equivalent form of this conjecture is established and a partial solution is obtained. A new proof of the integrality of \( A_{\Gamma}(\alpha, q) \) is given. Three Maple programs have been included which enable one to calculate the polynomials \( A_{\Gamma}(\alpha, q) \) for quivers with at most three nodes. All sample out-prints are consistence with Kac’s conjectures.

Another result of this thesis is as follows. Let \( A \) be a finite dimensional algebra over a perfect field \( \mathbb{K} \), \( M \) be a finitely generated indecomposable module over \( A \otimes_{\mathbb{K}} \mathbb{K} \). Then there exists a unique indecomposable module \( M^\dagger \) over \( A \) such that \( M \) is a direct summand of \( M^\dagger \otimes_{\mathbb{K}} \mathbb{K} \), and there exists a positive integer \( s \) such that \( M^s = M \oplus \cdots \oplus M \) (\( s \) copies) has a unique minimal field of definition which is isomorphic to the centre of \( \frac{\text{End}_A(M^\dagger)}{\text{rad}(\text{End}_A(M^\dagger))} \). If \( \mathbb{K} \) is a finite field, then \( s \) can be taken to be 1.
Chapter 1

Introduction

After the work of Gabriel [G1] [G2], it has become clear that a wide variety of problems of linear algebras can be formulated and studied in a uniform way in the context of representations of quivers (oriented graphs). In his paper [G1], Gabriel discovered a remarkable connection between the indecomposable representations of a quiver of finite type (i.e., admitting only a finite number of indecomposable representations) and the positive roots of the corresponding finite-dimensional simple Lie algebra. This was proved directly by Bernstein, Gel’fand and Ponomarev in [BGP] by making the use of the so-called reflection functors, which allow one to construct all the indecomposable representations of a quiver of finite type from the simplest ones in the same way as the Weyl group produces all positive roots from the simple ones. In the subsequent works [DF] [N] [DR], all representations of “tame” quivers (i.e., those for which the problem of classification of representations does not include the classification problem of pairs of matrices up to conjugation) have been classified. The obtained classification shows that the dimension vectors of indecomposable representations of a tame quiver correspond to the positive roots of certain infinite-dimensional Lie algebra (alternatively known as a contragredient Lie algebra or Cartan matrix Lie algebra) of finite Gel’fand-Kirillov dimension. All quivers remained unclassified are called “wild” quivers. The problem of classification of representations of wild quivers is extremely difficult, thus it is generally believed as a “hopeless” problem. Nevertheless, we can still say something about the dimension vectors of indecomposable representations. In his paper [K2], Kac proved that the dimension vectors of indecomposable representations of a wild quiver are in one-to-one correspondence with the positive roots of the corresponding Lie algebra known as a Kac-Moody algebra.

The main purpose of this thesis is to obtain surprising identities by counting
the numbers of classes of representations of quivers (indecomposables or absolutely indecomposables) over finite fields of given dimensions. These numbers are finite because there are only finitely many matrices of a given size over a finite field. Thus these numbers are integer-valued functions of \( q \), the number of elements in the finite field \( \mathbb{F}_q \). As the representations of quivers of finite type and tame type have been classified, we know all their representations and their dimensions, the counting problem for them may seem uninteresting. On the contrary, we can still get some interesting identities by counting them.

The really interesting counting problems in fact come from wild quivers, a better understanding of the polynomials \( A_{\Gamma}(\alpha, q) \) (\( \alpha \in \Delta^+ \)) would shed some light on the problem of classifying the representations of wild quivers. The original motivation of this research is to investigate the Kac conjecture regarding the positivity of the polynomials \( A_{\Gamma}(\alpha, q) \). In [K4], Kac conjectured that the coefficients of these polynomials are non-negative integers. This suggests that these polynomials may have some deep connections with the geometric properties of the variety formed by the representations (cohomology etc.). This problem has also been considered by Kraft-Riedtman [KR] and Le Bruyn [Le].

The key idea of making the counting possible is using the Frobenius operation and the concept of minimal field of definition for representations of algebras over finite fields. Let \( A \) be a finite dimensional algebra over a field \( \mathbb{K} \), \( M \) a finitely generated \( A \otimes_{\mathbb{K}} \bar{\mathbb{K}} \)-module, where \( \bar{\mathbb{K}} \) denotes the algebraic closure of \( \mathbb{K} \). If there exists a module \( N \) over \( A \otimes_{\mathbb{K}} E \), where \( E \) is a finite extension of \( \mathbb{K} \), such that \( M \cong N \otimes_{E} \bar{\mathbb{K}} \), then \( M \) is said to be defined over \( E \) and \( E \) is called a field of definition of \( M \). An alternative definition of this in terms of matrices is introduced in Chapter 2. Indeed when talking about fields of definition for modules, it is convenient to interpret modules as matrix representations rather than as vector spaces because the Galois groups act naturally on matrices. If \( \mathbb{K} \) is a finite field, Lang’s theorem (Theorem 2.2.2) implies that the minimal fields of definition are always unique. It is also proved that the Auslander-Reiten transformation preserves the minimal fields of definition. Over arbitrary fields, the minimal fields of definition are not unique. A counterexample is included in Section 2.2. However, if \( \mathbb{K} \) is a perfect field (e.g., fields
of characteristic zero), then for each finitely generated indecomposable $A \otimes_{K} \overline{K}$-module $M$, there exists a unique indecomposable module $M^\dagger$ over $A$ such that $M$ is a direct summand of $M^\dagger \otimes_{K} \overline{K}$, and there exists a positive integer $s$ such that $M^s = \bigoplus_{\text{s times}} \underbrace{M \oplus \cdots \oplus M}_s$ has a unique minimal field of definition $E$, which is isomorphic to the centre of $\frac{\text{End}_F(M^\dagger)}{\text{rad}(\text{End}_F(M^\dagger))}$ (see Theorem 5.7). Moreover, $M^s$ is indecomposable over $E$.

Let us have a quick look at Kac’s results in his papers [K2], [K3] and [K4]. Let $\mathbb{F}_q$ be the finite field of $q$ elements, $q$ a prime power. Let $\mathbb{Z}$ be the set of rational integers, $\mathbb{N}$ be the set of non-negative integers. Let $\Gamma$ be a connected graph with $n$ vertices labelled by $1, 2, \cdots, n$ (edge-loops are allowed), $a_{ij}$ denote the number of edges connecting vertices $i$ and $j$. In his paper [K4], Kac introduced a bilinear form $\langle \ , \ \rangle_\Gamma$ on $\mathbb{Z}^n$ associated with $\Gamma$, and defined a root system $\Delta \subset \mathbb{Z}^n$ which is infinite in general. Choose an arbitrary orientation on $\Gamma$ making $\Gamma$ into a quiver. For a fixed dimension $\alpha \in \mathbb{N}^n \setminus \{0\}$, we let $A_\Gamma(\alpha, q)$ denote the number of classes of absolutely indecomposable representations of $\Gamma$ over $\mathbb{F}_q$ of dimension $\alpha$. It is shown in [K2] that there exists an indecomposable representation of $\Gamma$ over $\overline{\mathbb{F}}_q$, the algebraic closure of $\mathbb{F}_q$, if and only if $\alpha$ is a positive root in $\Delta$. Consequently, $A_\Gamma(\alpha, q) \neq 0$ if and only if $\alpha \in \Delta^+$. In [K4], Kac proved that $A_\Gamma(\alpha, q)$ is a polynomial in $q$ with integer coefficients which are independent of the orientations on $\Gamma$. The integrality of $A_\Gamma(\alpha, q)$ is rediscovered in this thesis. In the same paper, Kac conjectured that these coefficients are non-negative. It is proved in [K3] that $A_\Gamma(\alpha, q)$ is monic of degree $u_\alpha = 1 - \langle \alpha, \alpha \rangle_\Gamma$. Thus, we may assume that $A_\Gamma(\alpha, q) = \sum_{i=0}^{u_\alpha} t_i^\alpha q^i$ with $t_i^\alpha \in \mathbb{Z}$.

Two important special cases are known. The first is when the quadratic form $\langle \ , \ \rangle_\Gamma$ is positive definite. This is known to occur only for the Dynkin diagrams $A_n \ (n \geq 1)$, $D_n \ (n \geq 4)$, $E_6$, $E_7$, $E_8$. Here the solutions of $\langle \alpha, \alpha \rangle = 1$ correspond to the positive roots of the corresponding Lie algebra and are finite in number. For example there are 120 such positive roots for $E_8$. The polynomials $A_\Gamma(\alpha, q)$ are then all 1. There are no non-zero solutions of $\langle \alpha, \alpha \rangle = 0$.

The second special case is when the quadratic form $\langle \ , \ \rangle_\Gamma$ is positive semi-definite. This is known to occur only for the extended Dynkin diagrams $\tilde{A}_n \ (n \geq
Here the solutions of $\langle \alpha, \alpha \rangle = 0$ are all of the form $n\delta$ for $n \in \mathbb{Z}\setminus\{0\}$ and for a unique fundamental solution $\delta$. The solutions of $\langle \alpha, \alpha \rangle = 1$ are then all of the form $\alpha_i + n\delta$ for $n \in \mathbb{Z}$, $\delta$ as before, and the $\alpha_i$ ($1 \leq i \leq N$) correspond to the roots (positive or negative) of the corresponding Lie algebra. For example $N = 240$ for $\tilde{E}_8$. The polynomials $A_\Gamma(\alpha_i + n\delta, q)$ are then all 1 while $A_\Gamma(n\delta, q)$ are then all of the form $n\delta$ for $n \in \mathbb{Z}\setminus\{0\}$ and for a unique fundamental solution $\delta$. The solutions of $\langle \alpha, \alpha \rangle = 1$ are then all of the form $\alpha_i + n\delta$ for $n \in \mathbb{Z}$, $\delta$ as before, and the $\alpha_i$ ($1 \leq i \leq N$) correspond to the roots (positive or negative) of the corresponding Lie algebra. For example $N = 240$ for $\tilde{E}_8$. The polynomials $A_\Gamma(\alpha_i + n\delta, q)$ are then all 1 while $A_\Gamma(n\delta, q)$ are then 1 while $A_\Gamma(n\delta, q) = q + t_0$, where the constant $t_0$ is the subscript used in the classification. For example $t_0 = 8$ for $\tilde{E}_8$.

In order to state the main result of this thesis, we have to introduce some notations first. A partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a finite sequence $\lambda_1 \geq \lambda_2 \geq \cdots$ of non-negative integers. The $\lambda_i$’s are called the parts of $\lambda$. The integer $|\lambda| = \lambda_1 + \lambda_2 + \cdots$ is called the weight of $\lambda$, and $\lambda$ is called a partition of $|\lambda|$. Let $P$ denote the set of all partitions including the unique partition of 0. For $\lambda = (\lambda_1, \lambda_2, \cdots) \in P$, we let $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ denote the partition conjugate to $\lambda$, i.e., $\lambda'_i$ is equal to the number of parts no less than $i$ in $\lambda$. For any $\lambda, \mu \in P$, we define $\langle \lambda, \mu \rangle = \sum_{i \geq 1} \lambda'_i \mu'_i$. For example, if $\lambda = (4 4 3 1)$ and $\mu = (5 3 2 2 1)$, then $\lambda' = (4 3 3 2)$ and $\mu' = (5 4 2 1 1)$, and so $\langle \lambda, \mu \rangle = 4 \times 5 + 3 \times 4 + 3 \times 2 + 2 \times 1 + 0 \times 1 = 40$. For any non-negative integer $r$, we define $\varphi_r(q) = (1 - q)/(1 - q^2) \cdots (1 - q^r)$, with $\varphi_0(q) = 1$ by convention. Any partition $\lambda = (\lambda_1, \lambda_2, \cdots)$ can be written in the form $(1^{n_1} 2^{n_2} \cdots)$ which means that there are exactly $n_i$ parts equals to $i$ in $\lambda$. Following Macdonald [M1], we define $b_\lambda(q) = \prod_{i \geq 1} \varphi_{n_i}(q)$ for $\lambda = (1^{n_1} 2^{n_2} \cdots) \in P$. Let $X_1, \cdots, X_n$ be $n$ independent commuting variables, and for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}^n$, we set $X^\alpha = X_1^{\alpha_1} \cdots X_n^{\alpha_n}$.

The main result of this thesis can be stated as the following formal identity:

\begin{equation}
\sum_{\lambda_1, \cdots, \lambda_n \in P} \prod_{1 \leq i \leq j \leq n} q^{a_{ij}(\lambda_i, \lambda_j)} \prod_{1 \leq i \leq n} q^{(\lambda_i, \lambda_i)b_{\lambda_i}(q^{-1})} X_1^{\lambda_1} \cdots X_n^{\lambda_n} = \prod_{\alpha \in \Delta^+} \prod_{i=0}^{\infty} \prod_{j=0}^{u_\alpha}(1 - q^{i+j} X^\alpha)^{t_{ij}}.
\end{equation}

Note that the exponents $t_{ij}^\alpha$ appeared in the product are the coefficients of the polynomials $A_\Gamma(\alpha, q)$, and $a_{ij}$ is the number of edges connecting vertices $i$ and $j$ in $\Gamma$. Thus the polynomials $A_\Gamma(\alpha, q)$ are of great importance.

If $\Gamma$ does not contain edge-loops, then there is a symmetric Cartan matrix associated with $\Gamma$, and a Kac-Moody algebra $\mathfrak{g}$ is thus defined (see [K2], page 62), and $\Delta$ is the root system accompanied with $\mathfrak{g}$. For each $\alpha \in \Delta^+$, there is an important positive integer $\text{mult}(\alpha)$, called the multiplicity of $\alpha$, which is defined as...
the dimension of the root space corresponding to $\alpha$. In [K4], Kac conjectured that
the multiplicity of $\alpha$ is equal to the constant term of the polynomial $A_{\Gamma}(\alpha, q)$, i.e.,
mult($\alpha$) = $A_{\Gamma}(\alpha, 0)$. This can be verified if $\Gamma$ is of finite type or tame type. For
wild graphs, it is still open. It is known that mult($\alpha$) are uniquely determined by
the following so-called Weyl-Macdonald-Kac denominator identity:

$$\sum_{w \in W} (-1)^{l(w)} X^{s(w)} = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{\text{mult}(\alpha)},$$

where $W$ is the Weyl group of $\Delta$ and $l$ is the length function on $W$, and where $s(w)$
equals the sum of positive roots mapped into negative roots by $w^{-1}$.

Now, suppose that the above Kac conjecture is true. We let $q \to 0$ in identity
(1.1), the right hand side thus becomes:

$$\prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{A_{\Gamma}(\alpha, 0)} = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{\text{mult}(\alpha)}.$$

Thus identity (1.1) can be thought as a $q$-analogue of the Weyl-Macdonald-Kac
denominator identity. As an application of identity (1.1), an equivalent form of
this conjecture was obtained in Section 4.2, and a partial solution of this equivalent
form is obtained.

If $\Gamma$ is of finite representation type, then $A_{\Gamma}(\alpha, q) = 1$ for all $\alpha \in \Delta^+$, thus
identity (1.1) amounts to:

$$\sum_{\lambda_1, \ldots, \lambda_n \in \mathcal{P}} \frac{\prod_{1 \leq i \leq j \leq n} q^{a_{ij}(\lambda_i, \lambda_j)}}{\prod_{1 \leq i \leq n} q^{b_{\lambda_i} (q^{-1})}} X_1^{\lambda_1} \cdots X_n^{\lambda_n} = \prod_{\alpha \in \Delta^+} \prod_{i=0}^{\infty} (1 - q^i X^\alpha).$$

This is already a new identity. If $\Gamma$ is the trivial graph, i.e., $\Gamma$ consists only one
node and no edges, then $\Gamma$ has only one absolutely indecomposable representation
over $\mathbb{F}_q$ whose dimension is 1, thus the corresponding identity for $\Gamma$ is the following:

$$\sum_{\lambda \in \mathcal{P}} \frac{X^{\lambda}}{q^{b_{\lambda}} b_{\lambda} (q^{-1})} = \prod_{i=0}^{\infty} (1 - q^i X).$$

This is a nontrivial identity. It is equivalent to a formula which is essentially due
to Macdonald ([M2], page 117). As an application of this identity, it is shown in
Chapter 4 that the number of nilpotent $r \times r$ matrices over $\mathbb{F}_q$ is equal to $q^{r(r-1)}$. 
If $Γ$ is of tame representation type, other explicit formulae follow. The example of $\tilde{A}_1$ is given below:

$$\sum_{\lambda, \mu \in \mathcal{P}} \frac{q^{2(\lambda, \mu)}}{q^{(\lambda, \lambda)+\langle \mu, \mu \rangle} b_\lambda (q-1) b_\mu (q-1)} X^{[\lambda]} Y^{[\mu]}$$

$$= \prod_{i=0}^{\infty} \prod_{j=1}^{\infty} (1 - q^i X^j Y^{j-1})(1 - q^i X^{j-1} Y^j)(1 - q^i X^j Y^j)(1 - q^{i+1} X^j Y^j).$$

As $q \to 0$, the right hand side of the above identity appears in the following Jacobi triple product identity:

$$\prod_{i=1}^{\infty} (1 - X^i Y^{i-1})(1 - X^{i-1} Y^i)(1 - X^i Y^i) = \sum_{i \in \mathbb{Z}} (−1)^i X^{i(i−1)/2} Y^{i(i+1)/2}.$$

Thus identity (1.2) can be considered as a $q$-analogue of the Jacobi triple product identity.

Two other families of polynomials $M_Γ(\alpha, q)$ and $I_Γ(\alpha, q)$ also fall into our consideration. Here $M_Γ(\alpha, q)$ is the number of classes of representations of $Γ$ over $\mathbb{F}_q$ of dimension $\alpha$, and $I_Γ(\alpha, q)$ is the number of classes of indecomposable representations of $Γ$ over $\mathbb{F}_q$ of dimension $\alpha$. Various relations among $M_Γ(\alpha, q)$, $I_Γ(\alpha, q)$ and $A_Γ(\alpha, q)$ have been discussed in this thesis. As a corollary of Kac’s theorem mentioned in the beginning of this introduction, $I_Γ(\alpha, q)$ is nonzero if and only if $\alpha$ is a positive root. The Krull-Schmidt theorem implies that $M_Γ(\alpha, q)$ and $I_Γ(\alpha, q)$ are related by the following formal identity:

$$\sum_{\alpha \in \mathbb{N}^n} M_Γ(\alpha, q) X^{\alpha} = \prod_{\alpha \in \Delta^+} (1 - X^{\alpha})^{-I_Γ(\alpha, q)}.$$

A formula which links $I_Γ(\alpha, q)$ and $A_Γ(\alpha, q)$ was obtained in [K4]. It can be shown that $I_Γ(\alpha, q)$ and $A_Γ(\alpha, q)$ are also related by the following symmetrical identity:

$$\sum_{d | \tilde{\alpha}} \frac{1}{d} I_Γ(\frac{\alpha}{d}, q) = \sum_{d | \tilde{\alpha}} \frac{1}{d} A_Γ(\frac{\alpha}{d}, q^d),$$

where $\tilde{\alpha} = \gcd(\alpha_1, \cdots, \alpha_n)$ if $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \setminus \{0\}$. This identity can be translated into the following formal identity:

$$\prod_{\alpha \in \Delta^+} (1 - X^{\alpha})^{I_Γ(\alpha, q)} = \prod_{\alpha \in \Delta^+} \prod_{i=0}^{u_\alpha} (1 - q^i X^{\alpha}) t^{\alpha}.$$
Thus, the relations among $A_\Gamma(\alpha, q)$, $I_\Gamma(\alpha, q)$ and $M_\Gamma(\alpha, q)$ can be summarised in the following:

$$
(1.3) \quad \sum_{\alpha \in \mathbb{N}^n} M_\Gamma(\alpha, q)X^\alpha = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{-I_\Gamma(\alpha, q)} = \prod_{\alpha \in \Delta^+} \prod_{i=0}^{u_\alpha} (1 - q^i X^\alpha)^{-t_\alpha^i}.
$$

Thus, in principal, if one knows one of the three families of the polynomials $M_\Gamma(\alpha, q)$, $I_\Gamma(\alpha, q)$ and $A_\Gamma(\alpha, q)$, then one knows the other two families. In fact the $A_\Gamma(\alpha, q)$ are the most important polynomials to us. They are all known for graphs of finite type and tame type. Unfortunately, there is no single graph of wild type whose polynomials $A_\Gamma(\alpha, q)$ are known. Thus, it would be nice to have an algorithm to calculate these polynomials. Appendix A which contains three Maple programs serves this purpose, some sample outputs are also included, all of them are consistence with Kac’s conjectures. Of course, the theory which supports the algorithms is more important than the algorithms themselves. The theory behind the programs in Appendix A is outlined here. First, let $P_\Gamma(X_1, \cdots, X_n, q)$ denote the left hand side of identity (1.1), then define $H_\Gamma(\alpha, q)$ which are rational functions in $q$ by the following:

$$
\log \left( P_\Gamma(X_1, \cdots, X_n, q) \right) = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} H_\Gamma(\alpha, q)X^\alpha / \bar{\alpha},
$$

It will be shown in Chapter 4 that

$$
(1.4) \quad A_\Gamma(\alpha, q) = \frac{q - 1}{\bar{\alpha}} \sum_{d | \bar{\alpha}} \mu(d)H_\Gamma\left( \frac{\alpha}{d}, q^d \right),
$$

where $\mu$ is the classical Möbius function. This formula is surprisingly simple. Thus the $A_\Gamma(\alpha, q)$ can be computed from the above formula and $M_\Gamma(\alpha, q)$, $I_\Gamma(\alpha, q)$ can be deduced from equation (1.3).

In fact, this thesis grew out of the paper [H] of the author, which was published in the Journal of Combinatorial Theory (Series A) in July 1997. Appendix B is the original version of [H]. In [H] we introduced two $q$-analogues for the partition function $p$, and showed that they satisfy similar recursion formula as $p$ does. The general setting of [H] is considering the classes of $g$-tuples of matrices over a finite field under simultaneous conjugation. This corresponds to the representations of
quivers with one node and \( g \)-arrows. The corresponding formula of (1.4) was obtained in [H] as Theorem 7. The result of [H] was a consequence of Theorem 7 by specialising \( g = 1 \), in which case the Jordan Normal Form Theorem implies that \( A_\Gamma(n, q) = 1 \) for all \( n \geq 1 \). As identity (1.1) was not realised when [H] was written, this thesis is a nontrivial generalisation of [H].

The structure of this thesis is as follows.

In Chapter 2, we introduced the concept of minimal fields of definition for representation of algebras over finite fields. By using a theorem of Lang, it is shown that the minimal fields of definition are unique. We then proved a key theorem (Theorem 2.2.8) which is fundamental for counting the representations of algebras over finite fields.

In Chapter 3, the results of Chapter 2 were generalised to the context of representations of quivers over finite fields, two centraliser counting formulae were obtained which are necessary when counting \( M_\Gamma(\alpha, q) \) by Molien-Burnside orbit counting formula.

The main results of this thesis are achieved in Chapter 4, a few examples of identity (1.1) will be examined. The relation between identity (1.1) and Kac’s denominator identity is discussed.

Chapter 5, which contains some input from my supervisor Dr. Peter Donovan, discusses the minimal fields of definition for representations of algebras over perfect fields. It is likely to be submitted in the form of [DH].

Appendix A contains three Maple programs which generate the polynomials \( A_\Gamma(\alpha, q) \), \( M_\Gamma(\alpha, q) \) and \( I_\Gamma(\alpha, q) \) for quivers with at most three nodes when \( \alpha \) is not too large.

Appendix B is the original version of [H] with minor changes.
Chapter 2

Representations of algebras over finite fields

Let $\Gamma$ be a finite dimensional algebra over $\mathbb{F}_q$, the field of $q$ elements. By using a theorem of Lang, it is shown that any finite dimensional representation of $\Gamma$ over $\overline{\mathbb{F}}_q$, the algebraic closure of $\mathbb{F}_q$, has a unique minimal field of definition. It is shown in this chapter that if $M$ is an indecomposable representation of $\Gamma$ over $\mathbb{F}_q$ such that the endomorphism algebra of $M$ modulo its radical is isomorphic to $\mathbb{F}_q^r$, then $M \otimes_{\mathbb{F}_q} \mathbb{F}_q^r$ is isomorphic to a direct sum of $r$ distinct absolutely indecomposable representations whose minimal field of definition is $\mathbb{F}_q^r$, with these summands forming a single orbit under the Frobenious operation. Thus an important formula which links the numbers of isomorphism classes of indecomposable and absolutely indecomposable representations of $\Gamma$ over $\mathbb{F}_q$ is obtained. It is also shown that the Auslander-Reiten transformation commutes with the Frobenius operation and thus preserves the minimal fields of definition.

2.1 Matrix representations

Let $\Gamma$ be an associative algebra over a field $\mathbb{K}$ with vector space basis $E_1, \cdots, E_d$. Let $a_{ijk} \in \mathbb{K}$ ($1 \leq i, j, k \leq d$) be the structure constants of $\Gamma$, that is the constants in the following equations:

$$E_i E_j = \sum_{k=1}^d a_{ijk} E_k, \quad 1 \leq i, j \leq d.$$  

As $(E_i E_j) E_k = E_i (E_j E_k)$, the structure constants satisfy additional conditions as follows:

$$\sum_{s=1}^d a_{ij s} a_{sk t} = \sum_{s=1}^d a_{ist} a_{jks}, \quad 1 \leq t \leq d.$$  

If $\Gamma$ has an identity, it will be taken to be $E_1$ and thus $a_{1jj} = 1$, $a_{1jk} = 0$ if $k \neq j$.  

A matrix representation of $\Gamma$ over $\mathbb{K}$ is a $d$-tuple of square matrices $(M_1, \cdots, M_d)$ whose entries are taken from $\mathbb{K}$, which satisfy the following equations:

$$M_i M_j = \sum_{k=1}^{d} a_{ijk} M_k, \quad 1 \leq i, j \leq d.$$ 

The common size of $M_1, \cdots, M_d$ is called the degree of this representation. If $\Gamma$ has an identity, we insist $M_1 = I$, the identity matrix.

If $M = (M_1, \cdots, M_d)$ is a matrix representation of $\Gamma$ over $\mathbb{K}$ of degree $n$, then the following map

$$\mu : \Gamma \rightarrow \mathcal{M}_n(\mathbb{K})$$

$$E_i \mapsto M_i$$

obviously defines a homomorphism between algebras, where $\mathcal{M}_n(\mathbb{K})$ denotes the full matrix algebra over $\mathbb{K}$ of degree $n$. If we let $\tilde{M} = \mathbb{K}^n$ and define $\gamma v = \mu(\gamma)v$ for all $\gamma \in \Gamma$ and $v \in \tilde{M}$, then $\tilde{M}$ becomes a left $\Gamma$-module of dimension $n$, which is called the module affording $M$. In this way, every matrix representation of $\Gamma$ over $\mathbb{K}$ gives rise to a left $\Gamma$-module. Conversely, if we have a left $\Gamma$-module $\tilde{M}$ of dimension $n$ with a fixed basis, then $\text{End}_\mathbb{K}(\tilde{M})$ is canonically isomorphic to $\mathcal{M}_n(\mathbb{K})$.

We define $\mu : \Gamma \rightarrow \text{End}_\mathbb{K}(\tilde{M})$ by $\mu(\gamma)(v) = \gamma v$ for all $\gamma \in \Gamma$ and $v \in \tilde{M}$. Thus $M = (\mu(E_1), \cdots, \mu(E_d))$ becomes a matrix representation of $\Gamma$ over $\mathbb{K}$ of degree $n$, which is called the matrix representation afforded by $\tilde{M}$.

If $M = (M_1, \cdots, M_d)$ and $N = (N_1, \cdots, N_d)$ are two matrix representations of $\Gamma$ over $\mathbb{K}$ of degrees $s$ and $t$ respectively, then a homomorphism from $M$ to $N$ is an $s \times t$ matrix over $\mathbb{K}$ which satisfies $(M_1 X, \cdots, M_d X) = (X N_1, \cdots, X N_d)$. If $X$ is invertible, then $M$ and $N$ are said to be isomorphic over $\mathbb{K}$, denoted by $M \cong N$. The space of homomorphisms from $M$ to $N$ is denoted by $\text{Hom}_\mathbb{K}(M, N)$. A homomorphism from $M$ to itself is called an endomorphism. The endomorphisms of $M$ form an algebra under matrix multiplication, which is denoted by $\text{End}_\mathbb{K}(M)$. Caution should be taken here as the classical notation for the endomorphism algebra of a $\Gamma$-module $M$ is $\text{End}_\Gamma(M)$. If $M = (M_1, \cdots, M_d)$ and $N = (N_1, \cdots, N_d)$ are two matrix representations of $\Gamma$ over $\mathbb{K}$, then the following $d$-tuple

$$\left( \left( \begin{array}{ccc} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_d \end{array} \right), \left( \begin{array}{ccc} 0 & N_1 & \cdots & 0 \\ 0 & 0 & \cdots & N_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{array} \right) \right)$$
is obviously a matrix representation of $\Gamma$ over $K$, called direct sum of $M$ and $N$, denoted by $M \oplus N$.

If $E$ is an extension field of $K$, then $\Gamma \otimes_K E$ is an algebra over $E$ with basis $E_1 \otimes 1, \ldots, E_d \otimes 1$. Thus, $\Gamma \otimes_K E$ and $\Gamma$ have the same structure constants $a_{ijk}$ ($1 \leq i, j, k \leq d$). If $M$ is a matrix representation of $\Gamma$ over $K$, then $M$ can be naturally considered as a matrix representation of $\Gamma \otimes_K E$ over $E$. This representation is commonly denoted by $M \otimes_K E$. A matrix representation $M$ of $\Gamma$ over $K$ is said to be indecomposable over $K$ if it is non-zero and not isomorphic to a direct sum of two non-zero matrix representations of $\Gamma$ over $K$. $M$ is said to be absolutely indecomposable if for every finite extension field $E$ of $K$, $M \otimes_K E$ is indecomposable over $E$. A classical theorem asserts that a matrix representation $M$ of $\Gamma$ over $K$ is indecomposable over $K$ if and only if $\text{End}_K(M)$ is local, i.e., $\text{End}_K(M)/\text{rad}(\text{End}_K(M))$ is a division algebra; here $\text{rad}(\text{End}_K(M))$ is the Jacobson radical of $\text{End}_K(M)$.

In what follows, when we say that $M$ is a matrix representation of $\Gamma$ over $E$, an extension field of $K$, we understand that $M$ is a matrix representation of $\Gamma \otimes_K E$ over $E$ with respect to the basis $E_1 \otimes 1, \ldots, E_d \otimes 1$.

It is clear that matrix representations of $\Gamma$ over $K$ and homomorphisms between them form a category, which is equivalent to the category of finitely generated left $\Gamma$-modules. A different choice of $K$ basis of $\Gamma$ will give an equivalent theory. In what follows, representations always mean matrix representations unless otherwise specified.

Remark. Although Lie algebras are not associative, this material of sections 2.1 to 2.3 could be applied to the representation theory of Lie algebras over finite fields as well.

### 2.2 Minimal fields of definition

From now on, $\Gamma$ will be a fixed algebra over $\mathbb{F}_q$ of dimension $d$. We are interested in the representations of $\Gamma$ over $\mathbb{F}_q$ and representations over $\overline{\mathbb{F}}_q$. Let $\Phi$ be the Frobenius automorphism on $\overline{\mathbb{F}}_q$ defined by $\Phi(x) = x^q$. Note that the fixed field of $\Phi^r$ is the unique extension of $\mathbb{F}_q$ of degree $r$, which coincides with $\mathbb{F}_{q^r}$. In other
words, \( x \in \mathbb{F}_{q^r} \) if and only if \( x = x^{q^r} \).

The Frobenius operation \( \Phi \) can be extended canonically to act on matrices and hence on representations. In fact the algebra automorphism \( \Phi \) of \( \mathcal{M}_n(\bar{\mathbb{F}}_q) \) can be defined by \( \Phi : (x_{ij}) \mapsto (x_{ij}^{q^r}) \). In this paper, \( \Phi(X) \) is denoted by \( X^{[q]} \). If \( M = (M_1, \ldots, M_d) \) is a representations of \( \Gamma \) over \( \bar{\mathbb{F}}_q \), then a trivial argument shows that \( M^{[q]} = (M_1^{[q]}, \ldots, M_d^{[q]}) \) is also a representation of \( \Gamma \) over \( \bar{\mathbb{F}}_q \). Moreover, \( M \cong N \) over \( \bar{\mathbb{F}}_q \) if and only if \( M^{[q]} \cong N^{[q]} \) over \( \bar{\mathbb{F}}_q \).

Every representation \( M \) of \( \Gamma \) over \( \bar{\mathbb{F}}_q \) is specified by finitely many matrices with entries in \( \bar{\mathbb{F}}_q \), and every matrix consists of finitely many entries. Thus, there exists a finite field \( \mathbb{F}_{q^r} \) such that all matrix entries of \( M \) lie in \( \mathbb{F}_{q^r} \), and we say that \( M \) is defined over the field \( \mathbb{F}_{q^r} \).

**Definition 2.2.1.** Given a representation \( M \) of \( \Gamma \) over \( \bar{\mathbb{F}}_q \), we say that \( M \) is defined over \( \mathbb{F}_{q^r} \) for some positive integer \( r \), if \( M \) is isomorphic over \( \bar{\mathbb{F}}_q \) to a representation whose matrix entries all lie in \( \mathbb{F}_{q^r} \). And we call \( \mathbb{F}_{q^r} \) a field of definition of \( M \).

The field of definition for representations of algebras over arbitrary fields can be defined analogously. From the above definition, every representation defined over \( \mathbb{F}_{q^r} \) can be realized over \( \mathbb{F}_{q^r} \). In the language of modules, the concept of fields of definition for modules can be defined in the following manner. Let \( M \) be a \( \Gamma \otimes_{\bar{\mathbb{F}}_q} \mathbb{F}_{q^r} \)-module. If there exists a \( \Gamma \otimes_{\bar{\mathbb{F}}_q} \mathbb{F}_{q^r} \)-module \( N \) such that \( N \otimes_{\mathbb{F}_{q^r}} \bar{\mathbb{F}}_q \cong M \), then \( M \) is said to be defined over \( \mathbb{F}_{q^r} \). The following theorem is crucial to our discussion.

**Theorem 2.2.2** (Lang [La1]). Let \( n \) be a positive integer, \( q \) a prime power. Then any \( X \in GL(n, \bar{\mathbb{F}}_q) \) can be written as \( Y^{-1} Y^{[q]} \) for some \( Y \in GL(n, \bar{\mathbb{F}}_q) \).

**Theorem 2.2.3.** Let \( M \) be a representation of \( \Gamma \) over \( \bar{\mathbb{F}}_q \). Then \( M \) is defined over \( \mathbb{F}_{q^r} \) for some positive integer \( r \) if and only if \( M \cong M^{[q^r]} \) over \( \bar{\mathbb{F}}_q \).

**Proof.** Let \( M = (M_1, \ldots, M_d) \), and let the degree of \( M \) be \( n \). First suppose \( M \cong M^{[q^r]} \) over \( \bar{\mathbb{F}}_q \). We prove that \( M \) is defined over \( \mathbb{F}_{q^r} \). Since \( M \cong M^{[q^r]} \), there exists \( X \in GL(n, \bar{\mathbb{F}}_q) \) such that

\[
(M_1, \ldots, M_d) = \left( X M_1^{[q^r]} X^{-1}, \ldots, X M_d^{[q^r]} X^{-1} \right).
\]
By Lang's theorem, there exists $Y \in GL(n, \overline{\mathbb{F}}_q)$ such that $X = Y^{-1}Y^{[q^r]}$. Thus,

$$(YM_1Y^{-1}, \cdots, YM_dY^{-1}) = \left( YXM_1^{[q^r]}X^{-1}Y^{-1}, \cdots, YXM_d^{[q^r]}X^{-1}Y^{-1} \right)$$

$$= \left( Y^{[q^r]}M_1^{[q^r]}Y^{-[q^r]}, \cdots, Y^{[q^r]}M_d^{[q^r]}Y^{-[q^r]} \right)$$

$$= \left( (YM_1Y^{-1})^{[q^r]}, \cdots, (YM_dY^{-1})^{[q^r]} \right).$$

This shows $YM_iY^{-1} = (YM_iY^{-1})^{[q^r]}$. Thus $YM_iY^{-1} \in \mathcal{M}_n(\mathbb{F}_{q^r})$. Note that $M \cong (YM_1Y^{-1}, \cdots, YM_dY^{-1})$ over $\overline{\mathbb{F}}_q$. Therefore, $M$ is defined over $\mathbb{F}_{q^r}$.

Conversely, suppose that $M$ is defined over $\mathbb{F}_{q^r}$. Then there exists a representation $N$ whose entries all lie in $\mathbb{F}_{q^r}$ such that $M \cong N$ over $\overline{\mathbb{F}}_q$. This implies that $M^{[q^r]} \cong N^{[q^r]}$ over $\overline{\mathbb{F}}_q$. But $N = N^{[q^r]}$. Thus $M \cong M^{[q^r]}$ over $\overline{\mathbb{F}}_q$.

Thus, by the previous theorem, there exists a smallest positive integer $r$ which satisfies $M \cong M^{[q^r]}$ over $\overline{\mathbb{F}}_q$, and we call $\mathbb{F}_{q^r}$ the minimal field of definition of $M$. Obviously, the minimal field of definition of $M$ is unique. In view of Theorem 2.2.3, if the number of elements in the orbit in which $M$ lives under the Frobenius operation is $r$, then the minimal field of definition of $M$ is $\mathbb{F}_{q^r}$.

**Corollary 2.2.4.** If the minimal field of definition of a representation $M$ is $\mathbb{F}_{q^r}$, then $M$ is defined over $\mathbb{F}_{q^r}$ if and only if $r$ is a divisor of $s$, that is $\mathbb{F}_{q^r} \subset \mathbb{F}_{q^s}$.

**Lemma 2.2.5.** Let $M$ be a representation of $\Gamma$ over $\mathbb{F}_q$. Then there is an algebra isomorphism

$$\frac{\text{End}_{\mathbb{F}_{q^r}}(M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r})}{\text{rad}(\text{End}_{\mathbb{F}_{q^r}}(M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}))} \cong \frac{\text{End}_{\mathbb{F}_q}(M)}{\text{rad}(\text{End}_{\mathbb{F}_q}(M))} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}.$$

**Proof.** Since $\mathbb{F}_{q^r}$ is a finite separable extension of $\mathbb{F}_q$, this is a special case of Theorem 7.9 on page 146 of [CR].

For a fixed degree $n$, all representations of $\Gamma$ over $\overline{\mathbb{F}}_q$ form an affine variety which is defined over $\mathbb{F}_q$. The linear algebraic group $GL(n, \overline{\mathbb{F}}_q)$ acts on this variety as follows. If $M = (M_1, \cdots, M_d)$ is a representation of $\Gamma$ over $\overline{\mathbb{F}}_q$ of degree $n$, $X \in GL(n, \overline{\mathbb{F}}_q)$, then $X \cdot M$ is defined as $(X^{-1}M_1X, \cdots, X^{-1}M_dX)$. Note that this action is compatible with the Frobenius operation, i.e., $(X \cdot M)^{[q]} = X^{[q]} \cdot M^{[q]}$. It is clear that the isomorphism classes of representations of $\Gamma$ over $\overline{\mathbb{F}}_q$ of degree $n$ are in one-to-one correspondence with the orbits under this action.
Lemma 2.2.6. Let $M$ and $N$ be two representations of $\Gamma$ over $\mathbb{F}_q$. If $M \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ and $N \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ are isomorphic over $\overline{\mathbb{F}_q}$, then $M$ and $N$ are isomorphic over $\mathbb{F}_q$.

Proof. Suppose that $M$ has degree $n$. Let $G = \{ X \in GL(n, \overline{\mathbb{F}_q}) \mid X \cdot M = M \}$. Thus $G$ is a linear algebraic group defined over $\mathbb{F}_q$. Moreover, $G$ is connected, because $G$ is a subset of $\text{End}_{\overline{\mathbb{F}_q}}(M)$ consisting of invertible elements and $\text{End}_{\overline{\mathbb{F}_q}}(M)$ is an affine space. Since $M \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ and $N \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ are isomorphic over $\overline{\mathbb{F}_q}$, there exists an $X \in GL(n, \overline{\mathbb{F}_q})$ such that $X \cdot M = N$. Thus,

$$X \cdot M = N = N^{[q]} = (X \cdot M)^{[q]} = X^{[q]} \cdot M^{[q]} = X^{[q]} \cdot M.$$

This shows $X^{-1}X^{[q]} \in G$. Since $G$ is a connected algebraic group, Lang's theorem can be applied on $G$ (see [C], page 32). So, there exists $Y \in G$ such that $X^{-1}X^{[q]} = Y^{-1}Y^{[q]}$. Thus $XY^{-1} = (XY^{-1})^{[q]}$. This implies $XY^{-1} \in GL(n, \mathbb{F}_q)$. Since $XY^{-1} \cdot M = X \cdot M = N$, thus $M$ and $N$ are isomorphic over $\mathbb{F}_q$.

The above lemma is a special case of the Noether-Deuring Theorem given as an exercise on page 139 of [CR]. An alternative proof exists in the style of Theorem 5.3 of Chapter 5.

Theorem 2.2.7. Let $M$ be a representation of $\Gamma$ over $\mathbb{F}_q$. Then $M$ is absolutely indecomposable if and only if $\text{End}_{\mathbb{F}_q}(M)/\text{rad}(\text{End}_{\mathbb{F}_q}(M)) \cong \mathbb{F}_q$.

Proof. First suppose that $M$ is absolutely indecomposable over $\mathbb{F}_q$. We have to show that $\text{End}_{\mathbb{F}_q}(M)/\text{rad}(\text{End}_{\mathbb{F}_q}(M)) \cong \mathbb{F}_q$. Since $M$ is indecomposable over $\mathbb{F}_q$, it follows that $\text{End}_{\mathbb{F}_q}(M)/\text{rad}(\text{End}_{\mathbb{F}_q}(M))$ is a division algebra over $\mathbb{F}_q$ of finite dimension. Since every finite division ring is commutative, $\text{End}_{\mathbb{F}_q}(M)/\text{rad}(\text{End}_{\mathbb{F}_q}(M))$ must be a finite field containing $\mathbb{F}_q$. Thus $\text{End}_{\mathbb{F}_q}(M)/\text{rad}(\text{End}_{\mathbb{F}_q}(M)) \cong \mathbb{F}_q^r$ for some positive integer $r$. As $M$ is absolutely indecomposable, $M \otimes_{\mathbb{F}_q} \mathbb{F}_q^r$ is indecomposable over $\mathbb{F}_q^r$. Thus, by Lemma 2.2.5, there would be an algebra isomorphism

$$\frac{\text{End}_{\mathbb{F}_q^r}(M \otimes_{\mathbb{F}_q} \mathbb{F}_q^r)}{\text{rad}(\text{End}_{\mathbb{F}_q^r}(M \otimes_{\mathbb{F}_q} \mathbb{F}_q^r))} \cong \frac{\text{End}_{\mathbb{F}_q}(M)}{\text{rad}(\text{End}_{\mathbb{F}_q}(M))} \otimes_{\mathbb{F}_q} \mathbb{F}_q^r \cong \mathbb{F}_q^r \otimes_{\mathbb{F}_q} \mathbb{F}_q^r.$$

Now $\mathbb{F}_q^r = \mathbb{F}_q[t]/(\phi(t))$, where $\phi(t)$ is an irreducible monic polynomial in $\mathbb{F}_q[t]$.
Hence
\[
\mathbb{F}_{q^r} \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} = \mathbb{F}_{q^r} \otimes_{\mathbb{F}_q} \mathbb{F}_q[t]/(\phi(t)) \cong \mathbb{F}_{q^r}[t]/(\phi(t)) \\
\cong \mathbb{F}_{q^r} \oplus \cdots \oplus \mathbb{F}_{q^r}.
\]

The last identification follows from the Chinese Remainder Theorem. If \( r \geq 2 \), then there would be a non-trivial idempotent in \( \mathbb{F}_{q^r} \oplus \cdots \oplus \mathbb{F}_{q^r} \). This contradicts the indecomposability of \( M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} \) over \( \mathbb{F}_{q^r} \). Therefore, \( r = 1 \), and \( \text{End}_{\mathbb{F}_q}(M)/\text{rad}(\text{End}_{\mathbb{F}_q}(M)) \cong \mathbb{F}_q \).

Conversely, suppose \( \text{End}_{\mathbb{F}_q}(M)/\text{rad}(\text{End}_{\mathbb{F}_q}(M)) \cong \mathbb{F}_q \). As \( \text{End}_{\mathbb{F}_q}(M) \) is local, \( M \) is indecomposable over \( \mathbb{F}_q \). For any positive integer \( r \), we have
\[
\frac{\text{End}_{\mathbb{F}_q}(M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r})}{\text{rad}(\text{End}_{\mathbb{F}_q}(M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}))} \cong \frac{\text{End}_{\mathbb{F}_q}(M)/\text{rad}(\text{End}_{\mathbb{F}_q}(M))}{\otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}} \cong \mathbb{F}_q \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} \cong \mathbb{F}_{q^r}.
\]
This shows that \( \text{End}_{\mathbb{F}_q}(M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}) \) is local. Thus \( M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} \) is indecomposable over \( \mathbb{F}_{q^r} \). Consequently, \( M \) is absolutely indecomposable.

The following theorem is a generalisation of Lemma 3.4 of [K3], which is proved in the context of representations of quivers over finite fields.

**Theorem 2.2.8.** Let \( M \) be an indecomposable representation of \( \Gamma \) over \( \mathbb{F}_q \), and suppose \( \text{End}_{\mathbb{F}_q}(M)/\text{rad}(\text{End}_{\mathbb{F}_q}(M)) \cong \mathbb{F}_{q^r} \). Then there exists an absolutely indecomposable representation \( N \) whose minimal field of definition is \( \mathbb{F}_{q^r} \) such that
\[
M \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r} \cong N \oplus N[q^1] \oplus \cdots \oplus N[q^{r-1}].
\]
Such a decomposition is unique up to cyclic order. Conversely, for any absolutely indecomposable representation \( N \) whose minimal field of definition is \( \mathbb{F}_{q^r} \), set
\[
M = N \oplus N[q^1] \oplus \cdots \oplus N[q^{r-1}].
\]
Then \( M \) is defined over \( \mathbb{F}_q \) and indecomposable over \( \mathbb{F}_q \).

**Proof.** By the Krull-Schmidt Theorem, every representation can be written as a direct sum of absolutely indecomposable representations over \( \mathbb{F}_q \). Thus we can assume that over \( \mathbb{F}_q \) there holds
\[
M \otimes_{\mathbb{F}_q} \mathbb{F}_q \cong N_1 \oplus \cdots \oplus N_s,
\]
where $N_i$ are absolutely indecomposable representations. Theorem 7.5 and 7.9 on page 146 of [CR] imply that

$$\frac{\text{End}_{\bar{F}_q}(M \otimes_{F_q} \bar{F}_q)}{\text{rad}(\text{End}_{\bar{F}_q}(M \otimes_{F_q} \bar{F}_q))} \cong \frac{\text{End}_{F_q}(M)}{\text{rad}(\text{End}_{F_q}(M))} \otimes_{F_q} \bar{F}_q \cong F_q^r \otimes_{F_q} \bar{F}_q \cong F_q \oplus \cdots \oplus F_q.$$

It follows that $r = s$, and $N_1, \cdots, N_r$ are mutually non-isomorphic. Obviously,

$$(M \otimes_{F_q} \bar{F}_q)^{[q]} = M^{[q]} \otimes_{F_q} \bar{F}_q = M \otimes_{F_q} \bar{F}_q.$$

Thus

$$N_1^{[q]} \oplus \cdots \oplus N_r^{[q]} \cong N_1 \oplus \cdots \oplus N_r.$$

This implies that the set $\{N_1, N_2, \cdots, N_r\}$ is invariant under the Frobenius operation. We claim that $\{N_1, N_2, \cdots, N_r\}$ consists of exactly one orbit. Suppose that there are $m$ orbits $\mathcal{O}_1, \cdots, \mathcal{O}_m$ and $m \geq 2$. Let

$$T_i = \oplus_{X \in \mathcal{O}_i} X, \quad 1 \leq i \leq m.$$

Thus, we have

$$M \otimes_{F_q} \bar{F}_q \cong T_1 \oplus \cdots \oplus T_m.$$

Every $T_i$ is defined over $F_q$ because

$$T_i^{[q]} = \oplus_{X \in \mathcal{O}_i} X^{[q]} \cong \oplus_{X \in \mathcal{O}_i} X \cong T_i.$$

As both $M \otimes_{F_q} \bar{F}_q$ and $T_1 \oplus \cdots \oplus T_m$ are defined over $F_q$ and they are isomorphic over $\bar{F}_q$, they are isomorphic over $F_q$ by Lemma 2.2.6. This is impossible because $M$ is indecomposable over $F_q$. This forces $m = 1$. Let $N$ be any member of this unique orbit. Thus, $N \cong N^{[q]}$, but $N \not\cong N^{[q^s]}$ for $s < r$. Theorem 2.2.3 implies that $N$ has minimal field of definition $q^r$. The above discussion shows that

$$M \otimes_{F_q} \bar{F}_q \cong N \oplus N^{[q]} \oplus \cdots \oplus N^{[q^{r-1}]}.$$

We may assume that all entries of $N$ lie in $F_{q^r}$, otherwise as $N$ is defined over $F_{q^r}$ we can replace $N$ by $N'$ whose entries lie in $F_{q^r}$ and $N' \cong N$ over $\bar{F}_q$. Thus, Lemma 2.2.6 implies that we have the following isomorphism over $F_{q^r}$:

$$M \otimes_{F_q} F_{q^r} \cong N \oplus N^{[q]} \oplus \cdots \oplus N^{[q^{r-1}]}.$$
This proved the first part of this theorem. The second part follows easily from the first.

The field \( F_{q^r} \) appeared in above theorem is called the **minimal splitting field** of \( M \). It will be noted that \( r \) divides the degree of \( M \).

**Remark.** The above theory uses the elementary theorem that the Galois group of a finite extension \( F_{q^r} \) of the finite field \( F_q \) is the cyclic group generated by the Frobenius automorphism. It also uses the elementary theorem that every finite division algebra over a finite field is commutative, that is the Brauer group is trivial. It uses Lang’s Theorem 2.2.2 and the separability of finite fields. While the following question is not relevant to the sequel, it is worth considering briefly here. Let \( \Gamma \) be an algebra over the rational field \( \mathbb{Q} \) of dimension \( d \). Let \( M = (M_1, \ldots, M_d) \) be a fixed matrix representation of \( \Gamma \) over \( \bar{\mathbb{Q}} \) of degree \( n \). Thus, for any \( S \in \text{GL}(n, \bar{\mathbb{Q}}) \), we get a representation \( S^{-1}MS = (S^{-1}M_1S, \ldots, S^{-1}M_dS) \) which is isomorphic to \( M \). Since \( S^{-1}MS \) has only finitely many matrix entries, they generate a finite extension field of \( \mathbb{Q} \), which is denoted by \( F(S) \). We may ask whether the minimal field of definition for \( M \) is unique. This is equivalent to asking whether there is an \( X \in \text{GL}(n, \bar{\mathbb{Q}}) \) such that \( F(X) = \cap_{S \in \text{GL}(n, \bar{\mathbb{Q}})} F(S) \). A negative answer is unavoidable because of the following counterexample.

**A counter-example.** Let \( H \) be the rational quaternion algebra. Thus \( H \) has a \( \mathbb{Q} \)-basis \( \{1, j_1, j_2, j_3\} \) which satisfy the relations: \( j_1^2 = j_2^2 = -1, j_1j_2 = j_3 = -j_2j_1 \). Then \( H \) is a central division algebra having \( \mathbb{Q}[j_1] \) as a maximal subfield (see [CR], page 160). Thus \( \mathbb{Q}[j_1] \) splits \( H \), i.e.,

\[
H \otimes_{\mathbb{Q}} \mathbb{Q}[j_1] \cong M_2(\mathbb{Q}[j_1]).
\]

It follows that

\[
H \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \cong M_2(\bar{\mathbb{Q}}).
\]

Thus there exists only one indecomposable \( H \otimes_{\mathbb{Q}} \bar{\mathbb{Q}} \)-module \( M \) which is \( \bar{\mathbb{Q}}^2 \). For any \( a \in \mathbb{Q} \), let \( b = \sqrt{-(a^2 + 1)} \in \bar{\mathbb{Q}} \). Then the quadratic field \( \mathbb{Q}[b] \) splits \( H \). Indeed the isomorphism from \( H \otimes_{\mathbb{Q}} \mathbb{Q}[b] \) to \( M_2(\mathbb{Q}[b]) \) can be defined explicitly as follows:

\[
1 \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad j_1 \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad j_2 \rightarrow \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \quad j_3 \rightarrow \begin{pmatrix} -b & a \\ a & b \end{pmatrix}.
\]
This shows that the $H \otimes \overline{\mathbb{Q}}$-module $\overline{\mathbb{Q}}^2$ is defined over $\mathbb{Q}[b]$. As there is only one indecomposable $H$-module which is $H$ itself, which is of dimension 4, $\overline{\mathbb{Q}}^2$ can not be defined over $\mathbb{Q}$. Thus the $\mathbb{Q}[b]$ are minimal fields of definition of $\overline{\mathbb{Q}}^2$. Since there are infinitely many choices for $a$, the minimal field of definition is not unique in this case. However it is $M^2 = M \oplus M$ that has a unique minimal field of definition $\mathbb{Q}$. This matter is investigated in greater depth later as Chapter 5. [See Note 1 on page 122].

2.3 Numbers of representations

Keeping the notations as in Section 2.2, we are now interested in the numbers of isomorphism classes of representations (indecomposables and absolutely indecomposables) of $\Gamma$ over $\mathbb{F}_q$ of given degrees. As there are only finitely many matrices of a given size over a finite field, these numbers are finite.

First, we introduce the following notations:

$$M_{\Gamma}(n, q) = \text{the number of isomorphism classes of representations of } \Gamma \text{ over } \mathbb{F}_q \text{ of degree } n.$$

$$I_{\Gamma}(n, q) = \text{the number of isomorphism classes of indecomposable representations of } \Gamma \text{ over } \mathbb{F}_q \text{ of degree } n.$$

$$A_{\Gamma}(n, q) = \text{the number of isomorphism classes of absolutely indecomposable representations of } \Gamma \text{ over } \mathbb{F}_q \text{ of degree } n.$$

Note that $M_{\Gamma}(n, q)$ is equal to the number of isomorphism classes of representations of $\Gamma$ over $\overline{\mathbb{F}}_q$ of degree $n$ which are defined over $\mathbb{F}_q$, this is a simple consequence of Lemma 2.2.6. Similarly, $A_{\Gamma}(n, q)$ is equal to the number of isomorphism classes of absolutely indecomposable representations of $\Gamma$ over $\overline{\mathbb{F}}_q$ of degree $n$ which are defined over $\mathbb{F}_q$. It is conceivable that for a large classes of algebras (e.g. group algebras, path algebras), $M_{\Gamma}(n, q)$, $A_{\Gamma}(n, q)$ are polynomials in $q$ with integer coefficients, and $I_{\Gamma}(n, q)$ are polynomials in $q$ with rational coefficients. See the example in the next section.

The Krull-Schmidt Theorem shows that every representation of $\Gamma$ over a field can be written as a direct sum of indecomposable representations in a unique way.
up to order. This implies the following formal identity:

\[ (*) \quad 1 + \sum_{n=1}^{\infty} M_{\Gamma}(n, q)X^n = \prod_{n=1}^{\infty} (1 - X^n)^{-I_{\Gamma}(n, q)}. \]

Let us recall the Möbius Inversion Formula (Theorem 3.24 on page 83 of [LN]) which will be frequently used in Chapter 4: Let \( f \) and \( g \) be two functions from \( \mathbb{N}^+ = \mathbb{N}\setminus\{0\} \) to \( \mathbb{C} \). Then

\[ f(n) = \sum_{d \mid n} g(d) \quad \text{for all } n \in \mathbb{N}^+, \]

if and only if

\[ g(n) = \sum_{d \mid n} \mu\left(\frac{n}{d}\right) f(d) \quad \text{for all } n \in \mathbb{N}^+, \]

where \( \mu \) is the classical Möbius function.

The following theorem generalises a formula on page 91 of [K4]. A special case of Kac’s formula is also proved in [Le] on page 153.

**Theorem 2.3.1.** For any positive integer \( n \), the following formulas are valid:

\[ I_{\Gamma}(n, q) = \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu\left(\frac{d}{r}\right) A_{\Gamma}\left(\frac{n}{d}, q^r\right), \]

\[ A_{\Gamma}(n, q) = \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu(r) I_{\Gamma}\left(\frac{n}{d}, q^r\right). \]

**Proof.** Let \( K_{\Gamma}(n, q^r) \) denote the number of isomorphism classes of absolutely indecomposable representations of \( \Gamma \) over \( \overline{\mathbb{F}_q} \) of degree \( n \) with minimal field of definition \( \mathbb{F}_{q^r} \). Clearly,

\[ A_{\Gamma}(n, q^d) = \sum_{r \mid d} K_{\Gamma}(n, q^r). \]

The Möbius inversion of this shows that

\[ K_{\Gamma}(n, q^d) = \sum_{r \mid d} \mu\left(\frac{d}{r}\right) A_{\Gamma}(n, q^r). \]

It follows from Theorem 2.2.8 that

\[ I_{\Gamma}(n, q) = \sum_{d \mid n} \frac{1}{d} K_{\Gamma}\left(\frac{n}{d}, q^d\right). \]
Substituting $K_{\Gamma}\left(\frac{n}{d}, q^d\right)$ by $\sum_{r \mid d} \mu\left(\frac{d}{r}\right) A_{\Gamma}\left(\frac{n}{d}, q^r\right)$ in the above equation thus gives

$$I_{\Gamma}(n, q) = \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu\left(\frac{d}{r}\right) A_{\Gamma}\left(\frac{n}{d}, q^r\right).$$

The Möbius inversion of the above formula now amounts to the following:

$$A_{\Gamma}(n, q) = \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu(r) I_{\Gamma}\left(\frac{n}{d}, q^r\right).$$

It is routine to show that if $A_{\Gamma}(n, q)$ are polynomials in $q$ for all $n \geq 1$, then $I_{\Gamma}(n, q)$ and $M_{\Gamma}(n, q)$ are also polynomials in $q$, moreover $A_{\Gamma}(n, q)$ and $I_{\Gamma}(n, q)$ have the same constant term. Thus if $\Gamma$ is of finite representation type, i.e., $\Gamma$ has only finitely many non-isomorphic absolutely indecomposable representations, then every indecomposable representation of $\Gamma$ over a finite field is absolutely indecomposable and defined over the prime field.

The following simple identity is needed below in dealing with double summations over divisors of integers. The proof is elementary and so is omitted.

$$\sum_{d \mid n} \sum_{r \mid n/d} f(d, r) = \sum_{r \mid n} \sum_{d \mid n/r} f(d, r) = \sum_{r \mid n} \sum_{d \mid r} f\left(d, \frac{n}{r}\right),$$

where $f$ is any function from $\mathbb{N}^+ \times \mathbb{N}^+$ to $\mathbb{C}$. The above identity will be frequently used in Chapter 4.

**Corollary 2.3.2.** With notations as above, we have the following identity:

$$\sum_{d \mid n} \frac{1}{d} I_{\Gamma}\left(\frac{n}{d}, q^d\right) = \sum_{d \mid n} \frac{1}{d} A_{\Gamma}\left(\frac{n}{d}, q^d\right).$$

**Proof.** In fact, by the proof of the previous theorem, we have

$$\sum_{d \mid n} \frac{1}{d} A_{\Gamma}\left(\frac{n}{d}, q^d\right) = \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} K_{\Gamma}\left(\frac{n}{d}, q^r\right).$$
By making obvious substitutions and altering the order of summations, we get

\[
\sum_{d|n} \frac{1}{d} A_{\Gamma}(\frac{n}{d}, q^d) = \sum_{d|n} \frac{d}{n} \sum_{r|n/d} K_{\Gamma}(d, q^r)
\]

\[
= \sum_{r|n} \sum_{d|n/r} \frac{d}{n} K_{\Gamma}(d, q^r)
\]

\[
= \sum_{r|n} \sum_{d|n/r} \frac{1}{d} K_{\Gamma}(\frac{n}{dr}, q^r)
\]

\[
= \sum_{d|n} \sum_{r|n/d} \frac{1}{r} K_{\Gamma}(\frac{n}{dr}, q^r)
\]

Again by the proof of Theorem 2.3.1, we conclude that

\[
\sum_{d|n} \frac{1}{d} A_{\Gamma}(\frac{n}{d}, q^d) = \sum_{d|n} \frac{1}{d} I_{\Gamma}(\frac{n}{d}, q).
\]

**Corollary 2.3.3.** Suppose that $A_{\Gamma}(n, q)$ are polynomials in $q$ for all $n \geq 1$, and let $A_{\Gamma}(n, q) = \sum_{i=0}^{u_n} t_n q^i$, where $u_n = \deg A_{\Gamma}(n, q)$. Then we have the following formal identity:

\[
\prod_{n=1}^{\infty} (1 - X^n)^{I_{\Gamma}(n, q)} = \prod_{n=1}^{\infty} \prod_{i=0}^{u_n} (1 - q^i X^n)^{t_{ni}}.
\]

**Proof.** We take logarithms on both sides, and then compare the coefficients of the corresponding terms. In fact,

\[
\log \prod_{n=1}^{\infty} (1 - X^n)^{I_{\Gamma}(n, q)} = \sum_{n=1}^{\infty} I_{\Gamma}(n, q) \log(1 - X^n)
\]

\[
= -\sum_{n=1}^{\infty} I_{\Gamma}(n, q) \sum_{i=1}^{\infty} \frac{1}{i} X^{ni}
\]

\[
= -\sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \frac{1}{i} I_{\Gamma}(n, q) X^{ni}
\]

Thus, the coefficient of $X^n$ in the above series is

\[-\sum_{d|n} \frac{1}{d} I_{\Gamma}(\frac{n}{d}, q).\]
On the other hand,

$$\log \prod_{n=1}^{\infty} \prod_{i=0}^{u_n} (1 - q^i X^n)^{t_{ni}} = \sum_{n=1}^{\infty} \sum_{i=0}^{u_n} t_{ni} \log(1 - q^i X^n)$$

$$= - \sum_{n=1}^{\infty} \sum_{i=0}^{u_n} t_{ni} \sum_{j=1}^{\infty} \frac{1}{j} q^{ij} X^{nj}$$

$$= - \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} t_{nj} q^{ij} X^{nj}$$

$$= - \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} A_{\Gamma}(n, q^j) X^{nj}$$

Thus, the coefficient of $X^n$ in the above series is

$$- \sum_{d|n} \frac{1}{d} A_{\Gamma}(\frac{n}{d}, q^d).$$

The validity of this corollary is now followed from the previous corollary.

Thus, if we further assume that $A_{\Gamma}(n, q)$ are polynomials in $q$ with integer coefficients for all $n \geq 1$, then it follows from the above corollary and identity (*) that $M_{\Gamma}(n, q)$ are also polynomials in $q$ with integer coefficients.

### 2.4 The Gel’fand-Ponamarev example

The following example is based on a paper of I. Gel’fand and V. Ponomarev [GP]. Although their paper is written in terms of the field of complex numbers, it generalises immediately to arbitrary fields. The algebra $\Gamma$ under consideration is $\mathbb{K}\langle X, Y \rangle / \langle XY = YX = X^s = Y^t = 0 \rangle$, where $s$ and $t$ are integers both greater than 1 and $\mathbb{K}$ is an arbitrary field. In order to describe the indecomposable modules of $\Gamma$ more explicitly, we have to introduce the concepts of open words and closed words. An **open word** is a word in $X$ and $Y^\#$ arranged along a straight line, in which $X^s$ and $(Y^\#)^t$ are prohibited. A **closed word** is a word in $X$ and $Y^\#$ arranged around a circle, in which $X^s$ and $(Y^\#)^t$ are prohibited. For instance, if $s = 4$ and $t = 3$, then $X X Y^\# Y^\# X Y^\# Y^\#$ is an open word of length 8, while $\hat{X} X Y^\# X Y^\# \hat{X}$ gives rise to a closed word of length 5, in which the dotted characters are identified.
Note that a cyclic permutation of a closed word gives the same closed word, thus $\hat{Y}^# XY^# XX \hat{Y}^#$ is the same as $\hat{X} XY^# Y^# \hat{X}$.

Roughly speaking, the indecomposable modules of $\Gamma$ are parametrised by words and indecomposable invertible matrices. More precisely, given an open word $w$ of length $n$, we can construct a $\Gamma$-module $D(w)$ of dimension $n + 1$. This construction goes as follows. First, every open words $w$ of length $n$ can be specified by $n + 1$ vertices labelled by $1, 2, \cdots, n + 1$ and $n$ arrows labelled by $X$ or $Y$ in a natural way. We may assume that each $X$ represents an arrow going north-east, and that each $Y$ represents an arrow going north-west. For example, $w = XY^# Y^# XX$ is associated with the following diagram:

```
X
\o e_1
\o e_2
Y
\o e_3
\o e_4
X
\o e_5
Y
\o e_6
```

Each vertex $i$ is associated with a basis element $e_i$. The module structure on $D(w)$ is defined according to the rules: (1) $Ye_1 = Xe_{n+1} = 0$. (2) If there is an arrow $\o \xrightarrow{X} \circ_{i+1}$, then $Xe_i = e_{i+1}$ and $Ye_{i+1} = 0$. (3) If there is an arrow $\o \xrightarrow{Y} \circ_{i-1}$, then $Ye_i = e_{i-1}$ and $Xe_{i-1} = 0$. For example, if $w = XY^# Y^# XX$ as before, then the module structure on $D(w)$ is defined as follows:

- $Xe_1 = e_2$, $Ye_1 = 0$
- $Xe_2 = 0$, $Ye_2 = 0$
- $Xe_3 = 0$, $Ye_3 = e_2$
- $Xe_4 = e_5$, $Ye_4 = e_3$
- $Xe_5 = e_6$, $Ye_5 = 0$
- $Xe_6 = 0$, $Ye_6 = 0$

Gel’fand and Ponomarev show that the module $D(w)$ so constructed is indecomposable and that different open words give non-isomorphic modules.
Given any closed word $w$ of length $n$ and any square matrix $M$ over $K$ of size $d$, we can construct a $\Gamma$-module $C(w, M)$ of dimension $nd$. First, $w$ can be represented by a regular $n$-gon in an obvious way, with vertices labelled by $1, 2, \ldots, n$ in clockwise direction, and edges labelled by $X$ or $Y$. We may assume that $X$ always goes in the clockwise direction, and that $Y$ goes in the anti-clockwise direction. We associate each vertex $i$ with a $d$-dimensional vector space $V_i$, each $V_i = \mathbb{K}^n$. As a vector space $C(w, M) = V_1 \oplus V_2 \oplus \cdots \oplus V_n$. The module structure of $C(w, M)$ is defined according to the following rules: (1) If there is an arrow $\circ_i \xrightarrow{X} \circ_{i+1}$, then $X$ maps $V_i$ onto $V_{i+1}$ as an identity map, while $Y$ acts on $V_{i+1}$ as 0 map. (2) If there is an arrow $\circ_i \xrightarrow{Y} \circ_{i-1}$, then $Y$ maps $V_i$ onto $V_{i-1}$ as identity, and $X$ acts on $V_{i-1}$ as 0. (3) The arrow between 1 and $n$ is either $\circ_n \xrightarrow{X} \circ_1$ or $\circ_1 \xrightarrow{Y} \circ_n$. If the first case happens, then $X$ maps $V_n$ onto $V_1$ through the matrix $M$, and $Y$ acts on $V_1$ as 0. If the second case happens then $Y$ maps $V_1$ onto $V_n$ through the matrix $M$, and $X$ acts on $V_n$ as 0.

It is easy to verify that the above procedure determines a unique $\Gamma$-module structure on $C(w, M)$. If $w = \dot{X}XYXYY\dot{X}$, then $C(w, M)$ can be described by the following diagram:

\[
\begin{array}{cccc}
V_1 & \xrightarrow{X} & I & \xleftarrow{X} & V_2 \\
& \Downarrow{Y = M} & & \Downarrow{X = I} & \\
V_6 & \xrightarrow{Y = I} & \circ & \xleftarrow{Y = I} & V_3 \\
& \Downarrow{X = I} & & \Downarrow{X = I} & \\
V_5 & & \circ & \xleftarrow{X = I} & V_4 \\
\end{array}
\]

In fact, Gel’fand and Ponomarev proved that $C(w, M)$ is indecomposable if and only if $w$ is a closed word which is not a power of words of smaller length, and $M$ is an indecomposable matrix. Further more, if $w$ and $w'$ are closed words which are not powers of words of smaller length, and $M, M'$ are two indecomposable matrices, then $C(w, M) \cong C(w', M')$ if and only if $w'$ is a cyclic permutation of $w$ and $M$ is similar to $M'$ over $K$. 
Recall from [H] that for a given irreducible monic polynomial $f$ over $\mathbb{K}$ the companion matrix of $f$ is denoted by $J(f)$. For each positive integer $m$, $J_m(f)$ is the Jordan block matrix consisting of $m^2$ block $\deg(f) \times \deg(f)$ matrices with $J(f)$ in each diagonal block. A classical result shows that every indecomposable matrix $M$ over $\mathbb{K}$ is similar to some $J_m(f)$.

The classification of indecomposable $\Gamma$-modules amounts to

1. Each indecomposable $\Gamma$-module is either “discrete” or “continuous”, but not both;
2. Each discrete indecomposable module is of the form $D(w)$, where $w$ is an open word. It is absolutely indecomposable and defined over the prime field;
3. Each continuous indecomposable module is of the form $C(w, J_m(f))$, where $w$ is a closed word which is not a power of a smaller word, and $m$ is a positive integer, and where $f$ is a monic irreducible polynomial in $t$ over $\mathbb{K}$ other than $t$.

If $w$ is a closed word which is not a power of a smaller word, and $M$ is an indecomposable singular matrix over $\mathbb{K}$, in other words, $M$ is of the form $J_m(f)$ where $f = t$, we can still construct a $\Gamma$-module $C(w, M)$, but it turns out to be isomorphic to a discrete module.

Let $a_n$ denote the number of open words of length $n - 1$, $b_n$ denote the number of closed words of length $n$ which are not powers of words of smaller length. Note that cyclic permutation of a closed word gives the same closed word. If we take $\mathbb{K}$ to be $\mathbb{F}_q$, then it follows from the above classification theorem that

$$A_\Gamma(n, q) = a_n + (q - 1) \sum_{r \mid n} b_r,$$

$$I_\Gamma(n, q) = a_n + \sum_{r \mid n} b_r \sum_{d \mid n/r} \phi_d(q),$$

where $\phi_d(q)$ is the number of monic irreducible polynomials in $t$ of degree $d$ over $\mathbb{F}_q$ with $t$ excluded. It is a routine exercise to check that $A_\Gamma(n, q)$ and $I_\Gamma(n, q)$ satisfy the reciprocal relationship described in Theorem 2.3.1. Thus, the $A_\Gamma(n, q)$ are polynomials in $q$ with integer coefficients, while the $I_\Gamma(n, q)$ are polynomials with rational coefficients. It follows from identity (*) on page 19 and Corollary 2.3.3 that
the \( M_\Gamma(n, q) \) are also polynomials in \( q \) with integer coefficients. It can be easily shown that the polynomials \( A_\Gamma(n, q) \) have only non-negative integer coefficients for all \( n \).

There are various other algebras of tame representation type whose representations have been classified, thus similar calculations could be made for them. A good source for representations of tame algebras is [Ri1].

**Remark.** If the field \( \mathbb{F}_q \) is replaced by \( \mathbb{Q} \) in the Gel’fand-Ponomarev example, the minimal field of definition of an indecomposable \( \mathbb{Q} \)-module exists and is seen to be \( \mathbb{Q} \) for discrete modules, \( \mathbb{Q}[\lambda] \) for continuous module \( C(w, J_m(\lambda)) \), where \( J_m(\lambda) \) is the Jordan block matrix with eigenvalue \( \lambda \) of size \( m \).

### 2.5 Frobenius operation and Auslander-Reiten sequences

\( \Gamma \) is again a finite dimensional algebra over \( \mathbb{F}_q \). In this section, we prove that the Auslander-Reiten transformation commutes with the Frobenius operation, and thus preserves the minimal fields of definition. All representations considered in this section are matrix representations of \( \Gamma \) over \( \overline{\mathbb{F}}_q \).

Let \( M, N, L \) be representations of \( \Gamma \) over \( \overline{\mathbb{F}}_q \) of degrees \( m, n \) and \( m+n \) respectively, and let \( \phi \in \text{Hom}_{\overline{\mathbb{F}}_q}(M, L) \), \( \psi \in \text{Hom}_{\overline{\mathbb{F}}_q}(L, N) \). Then the following sequence

\[
0 \longrightarrow M \overset{\phi}{\longrightarrow} L \overset{\psi}{\longrightarrow} N \longrightarrow 0
\]

is called an exact sequence if \( \phi\psi = 0 \) and \( \text{rank}(\phi) = m \), \( \text{rank}(\psi) = n \). A homomorphism \( \phi \) from \( M \) to \( L \) is called a split monomorphism if there exists \( \xi \in \text{Hom}_{\overline{\mathbb{F}}_q}(L, M) \) such that \( \phi\xi \) is equal to the identity matrix. An exact sequence \( 0 \longrightarrow M \overset{\phi}{\longrightarrow} L \overset{\psi}{\longrightarrow} N \longrightarrow 0 \) is called split if \( \phi \) is a split monomorphism.

A non-split exact sequence \( 0 \longrightarrow M \overset{\phi}{\longrightarrow} L \overset{\psi}{\longrightarrow} N \longrightarrow 0 \) is called an Auslander-Reiten sequence if the following conditions are satisfied:

1. \( M \) and \( N \) are indecomposable over \( \overline{\mathbb{F}}_q \);
2. For any representation \( X \) of \( \Gamma \) over \( \overline{\mathbb{F}}_q \) and any \( \gamma \in \text{Hom}_{\overline{\mathbb{F}}_q}(M, X) \) which is not a split monomorphism, there exists \( \xi \in \text{Hom}_{\overline{\mathbb{F}}_q}(L, X) \) such that \( \gamma = \phi\xi \).

For any \( \phi = (a_{ij}) \), a matrix over \( \overline{\mathbb{F}}_q \), we use \( \phi^{[q]} \) to denote the matrix \( (a_{ij}^q) \). Thus if \( \phi \in \text{Hom}_{\overline{\mathbb{F}}_q}(M, N) \), then \( \phi^{[q]} \in \text{Hom}_{\overline{\mathbb{F}}_q}(M^{[q]}, N^{[q]}) \).
A theorem of Auslander and Reiten asserts that for any indecomposable non-injective representation $M$ of $\Gamma$ over $\mathbb{F}_q$, there exists a unique Auslander-Reiten sequence starting from $M$, and for any indecomposable non-projective representation $N$, there exists a unique Auslander-Reiten sequence ending with $N$. For a proof of this theorem, we refer to page 72 of [Ri1]. If $0 \rightarrow M \rightarrow L \rightarrow N \rightarrow 0$ is an Auslander-Reiten sequence, we define $\tau(N) = M$. Here $\tau$ is called the Auslander-Reiten transformation.

**Lemma 2.5.1.** If $0 \rightarrow M \xrightarrow{\phi} L \xrightarrow{\psi} N \rightarrow 0$ is an Auslander-Reiten sequence, then so is $0 \rightarrow M^{[q]} \xrightarrow{\phi^{[q]}} L^{[q]} \xrightarrow{\psi^{[q]}} N^{[q]} \rightarrow 0$.

**Proof.** It is obvious that $0 \rightarrow M \xrightarrow{\phi} L \xrightarrow{\psi} N \rightarrow 0$ is an exact sequence and that $M^{[q]}$ and $N^{[q]}$ are indecomposable. For any representation $X$ of $\Gamma$ over $\mathbb{F}_q$ and any $\gamma \in \text{Hom}_{\mathbb{F}_q}(M^{[q]}, X)$ which is not a split monomorphism, there exists a positive integer $r$, such that all entries of $M$, $N$, $L$, $X$, $\phi$, $\psi$ and $\gamma$ lie in $\mathbb{F}_{q^r}$. Since $\gamma \in \text{Hom}_{\mathbb{F}_q}(M^{[q]}, X)$, it follows that $\gamma^{[q^{-1}]} \in \text{Hom}_{\mathbb{F}_q}(M^{[q^{-1}]}, X^{[q^{-1}]}) = \text{Hom}_{\mathbb{F}_q}(M, X^{[q^{-1}]})$. $\gamma^{[q^{-1}]}$ cannot be a split monomorphism otherwise $\gamma$ is a split monomorphism. Similarly, $\phi^{[q]}$ is not a split monomorphism because $\phi$ is not. As $0 \rightarrow M \xrightarrow{\phi} L \xrightarrow{\psi} N \rightarrow 0$ is an Auslander-Reiten sequence, there exists $\xi \in \text{Hom}_{\mathbb{F}_q}(L, X^{[q^{-1}]})$ such that $\gamma^{[q^{-1}]} = \phi \xi$. If follows that $\gamma = \gamma^{[q]} = \phi^{[q]} \xi^{[q]}$. This shows that $0 \rightarrow M^{[q]} \xrightarrow{\phi^{[q]}} L^{[q]} \xrightarrow{\psi^{[q]}} N^{[q]} \rightarrow 0$ is an Auslander-Reiten sequence.

**Theorem 2.5.2.** If $0 \rightarrow M \xrightarrow{\phi} L \xrightarrow{\psi} N \rightarrow 0$ is an Auslander-Reiten sequence, then $M$ and $N$ have the same minimal field of definition.

**Proof.** It is sufficient to show that if $M$ is defined over $\mathbb{F}_{q^r}$ then $N$ is also defined over $\mathbb{F}_{q^r}$ and vice versa. Now, suppose that $M$ is defined over $\mathbb{F}_{q^r}$. Thus Theorem 2.2.3 implies that $M \cong M^{[q^r]}$ over $\mathbb{F}_q$. The above lemma shows that $0 \rightarrow M^{[q^r]} \xrightarrow{\phi^{[q^r]}} L^{[q^r]} \xrightarrow{\psi^{[q^r]}} N^{[q^r]} \rightarrow 0$ is an Auslander-Reiten sequence. By the uniqueness of the existence of Auslander-Reiten sequence starting from $M$, it follows that $N \cong N^{[q^r]}$ over $\mathbb{F}_q$. Again by Theorem 2.2.3, $N$ is defined over $\mathbb{F}_{q^r}$. If $N$ is defined over $\mathbb{F}_{q^r}$ then a similar argument shows that $N$ is also defined over $\mathbb{F}_{q^r}$. 


Chapter 3  
Representations of quivers over finite fields

The theory of representations of quivers is very rich. The quivers of finite representation type and tame representation type have been classified, and all their representations are known. A complete list of references can be found in [Ri1]. Here we are interested in getting information on the numbers of classes of representations of quivers over finite fields for given dimensions. This produces various new identities. This chapter serves as a preparation of the next chapter which contains the main results of this thesis.

3.1 Basic definitions

Recall that a quiver $\Gamma = (\Gamma_0, \Gamma_1, s, e)$ consists of a set of vertices $\Gamma_0$, and a set of arrows $\Gamma_1$, and two maps $s, e$ from $\Gamma_1$ to $\Gamma_0$. For any $\gamma \in \Gamma_1$, $s(\gamma)$ is called the starting point of $\gamma$ and $e(\gamma)$ the end point of $\gamma$. If $\gamma \in \Gamma_1$ and $s(\gamma) = i$, $e(\gamma) = j$, then the arrow $\gamma$ is commonly presented by $\circ_i \gamma \rightarrow \circ_j$. In this thesis, we only consider finite quivers, i.e., $|\Gamma_0| < \infty$ and $|\Gamma_1| < \infty$.

Let $\mathbb{K}$ be a field, $\Gamma = (\Gamma_0, \Gamma_1, s, e)$ be a fixed finite quiver. Suppose $\Gamma_0 = \{1, 2, \cdots, n\}$. A representation $V$ of $\Gamma$ over $\mathbb{K}$ is a set $\{V_i\}_{i \in \Gamma_0}$ of finite dimensional vector spaces over $\mathbb{K}$ together with a set $\{V_\gamma\}_{\gamma \in \Gamma_1}$ of $\mathbb{K}$-linear transformations $V_\gamma : V_{s(\gamma)} \rightarrow V_{e(\gamma)}$. The vector $(\dim V_1, \cdots, \dim V_n) \in \mathbb{Z}_+^n$ is called the dimension of this representation. A homomorphism from a representation $V$ to a representation $W$ is a collection $\{g_i\}_{i \in \Gamma_0}$ of $\mathbb{K}$-linear maps $g_i : V_i \rightarrow W_i$, such that $W_\gamma g_j = g_i V_\gamma$ for all $\circ_i \gamma \rightarrow \circ_j$ in $\Gamma_1$. If every $g_i$ is invertible, $V$ and $W$ are said to be isomorphic. If $V$ is a representation of $\Gamma$ over $\mathbb{K}$, and we fix a basis for each vector space $V_i$, then every $\mathbb{K}$-linear map $V_\gamma$ is specified by a matrix over $\mathbb{K}$. This motivates us making the following definition, which is more convenient when counting the numbers of
representations of quivers over finite fields.

A matrix representation $M$ of a quiver $\Gamma$ over $K$ of dimension $(d_1, d_2, \ldots, d_n) \in \mathbb{N}^n$ is a collection $\{ M_\gamma \}_{\gamma \in \Gamma_1}$ of $K$-matrices such that $M_\gamma$ has size $d_i \times d_j$ if there is an arrow $\circ_i \xrightarrow{\gamma} \circ_j$ in $\Gamma_1$. For example, the following diagram gives a representation of the underlying quiver over $\mathbb{Q}$ of dimension $(3, 2, 2, 2, 2, 3)$.

If $M$ and $N$ are matrix representations of $\Gamma$ over $K$ of dimensions $(d_1, d_2, \ldots, d_n)$ and $(d'_1, d'_2, \ldots, d'_n)$ respectively, then a homomorphism from $M$ to $N$ is an $n$-tuple of $K$-matrices $(X_1, \ldots, X_n)$ such that $X_i$ has size $d_i \times d'_i$ for all $i$, and for each arrow $\circ_i \xrightarrow{\gamma} \circ_j$ in $\Gamma_1$, there holds $X_i N_\gamma = M_\gamma X_j$, i.e., the following diagram commutes.

If all $X_i$ are invertible, then $M$ and $N$ are called isomorphic over $K$. The space of homomorphisms from $M$ to $N$ is denoted by $\text{Hom}_K(M, N)$. A homomorphism from $M$ to itself is called an endomorphism of $M$. The endomorphisms of $M$ form an algebra under the matrix multiplication, which is denoted by $\text{End}_K(M)$. If $M$ and $N$ are matrix representations of $\Gamma$ over $K$, then the direct sum of $M$ and $N$, which is denoted by $M \oplus N$, is defined by $(M \oplus N)_\gamma = \begin{pmatrix} M_\gamma & 0 \\ 0 & N_\gamma \end{pmatrix}$ for all $\gamma \in \Gamma_1$.

The matrix representations of $\Gamma$ over $K$ and homomorphisms between matrix representations form a category which is equivalent to the category of representations of the path algebra associated with $\Gamma$. 
If $M$ is a matrix representation of $\Gamma$ over $\mathbb{K}$, and $E$ is an extension field of $\mathbb{K}$, then $M$ can be naturally regarded as a matrix representation of $\Gamma$ over $E$, we denote this representation by $M \otimes_{\mathbb{K}} E$. $M$ is said to be \textit{indecomposable} over $\mathbb{K}$ if it is non-zero and not isomorphic to a direct sum of two non-zero matrix representations of $\Gamma$ over $\mathbb{K}$. $M$ is said to be \textit{absolutely indecomposable} if for any finite extension field $E$ of $\mathbb{K}$, $M \otimes_{\mathbb{K}} E$ is indecomposable over $E$. As before, $M$ is indecomposable over $\mathbb{K}$ if and only if $\text{End}_{\mathbb{K}}(M)/\text{rad}(\text{End}_{\mathbb{K}}(M))$ is a division algebra.

For any dimension $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, let $R_{\Gamma}(\alpha, \mathbb{K})$ be the set of all matrix representations of $\Gamma$ over $\mathbb{K}$ of dimension $\alpha$. Thus $R_{\Gamma}(\alpha, \mathbb{K})$ is an affine variety defined over $\mathbb{K}$. The algebraic group $GL(\alpha, \mathbb{K}) = GL(\alpha_1, \mathbb{K}) \times \cdots \times GL(\alpha_n, \mathbb{K})$ acts on $R_{\Gamma}(\alpha, \mathbb{K})$ as follows. If $M$ is a matrix representation of $\Gamma$ over $\mathbb{K}$ of dimension $\alpha$, $(X_1, \cdots, X_n)$ be an arbitrary element of $GL(\alpha, \mathbb{K})$, then

$$(\gamma)_{\alpha}^\sharp(X_1, \cdots, X_n) \cdot M = X_i^{-1}M\gamma X_j,$$

for any arrow $\circ \xrightarrow{\gamma} \circ$ in $\Gamma_1$, This action is illustrated by the following diagram:

$$\begin{array}{ccc}
\circ & \xrightarrow{M\gamma} & \circ \\
X_i & & X_j \\
\circ & \xrightarrow{X_i^{-1}M\gamma X_j} & \circ
\end{array}$$

It is obvious that the isomorphism classes of representation of $\Gamma$ over $\mathbb{K}$ of dimension $\alpha$ are in one-to-one correspondence with the orbits in $R_{\Gamma}(\alpha, \mathbb{K})$ under the action of $GL(\alpha, \mathbb{K})$.

### 3.2 Minimal fields of definition

In what follows, we take $\mathbb{K}$ to be the algebraically closed field $\overline{\mathbb{F}}_q$. The concept of a \textit{field of definition} can be defined similarly. If $M$ is a matrix representation of $\Gamma$ over $\overline{\mathbb{F}}_q$, then $M$ is said to be defined over $\mathbb{F}_q$ if $M$ is isomorphic to a matrix representation of $\Gamma$ over $\overline{\mathbb{F}}_q$ whose matrix entries all belong to $\mathbb{F}_q$. The Frobenius operation can be defined similarly, it raises every entry of a matrix representation to its $q$-th power, the image of $M$ under the Frobenius operation is still denoted
by $M^{[q]}$. Since Lang’s theorem is still valid for the direct product of finitely many linear algebraic groups over $\overline{F}_q$, the arguments used in Section 2.2 show that every matrix representation of $\Gamma$ over $\overline{F}_q$ has a unique minimal field of definition. The following theorems can also be proved by using the same methods.

**Theorem 3.2.1.** Let $M$ be a representation of $\Gamma$ over $\overline{F}_q$. Then $M$ is defined over $F_qr$ for some positive integer $r$ if and only if $M \cong M^{[q^r]}$ over $\overline{F}_q$.

**Theorem 3.2.2.** Let $M$ be a representation of $\Gamma$ over $F_q$. Then $M$ is absolutely indecomposable if and only if $\text{End}_{F_q}(M)/\text{rad}(\text{End}_{F_q}(M)) \cong F_q$.

**Theorem 3.2.3.** Let $M$ be an indecomposable representation of $\Gamma$ over $F_q$, and suppose $\text{End}_{F_q}(M)/\text{rad}(\text{End}_{F_q}(M)) \cong F_q^r$. Then there exists an absolutely indecomposable representation $N$ whose minimal field of definition is $F_{q^r}$ such that

$$M \otimes_{F_q} F_{q^r} \cong N \oplus N^{[q]} \oplus \ldots \oplus N^{[q^{r-1}]}.$$ 

Such a decomposition is unique up to cyclic order. Conversely, for any absolutely indecomposable representation $N$ whose minimal field of definition is $F_{q^r}$, set

$$M = N \oplus N^{[q]} \oplus \ldots \oplus N^{[q^{r-1}]}.$$ 

Then $M$ is defined over $F_q$ and indecomposable over $F_q$.

In this thesis we are interested in the numbers of isomorphism classes of representations (indecomposables and absolutely indecomposables) of $\Gamma$ over $F_q$ of given dimensions and the relations between them. For any $\alpha \in \mathbb{N}^n$, we make the following notations:

$$M_\Gamma(\alpha, q) = \text{ the number of isomorphism classes of matrix representations of } \Gamma \text{ over } F_q \text{ of dimension } \alpha.$$ 

$$I_\Gamma(\alpha, q) = \text{ the number of isomorphism classes of indecomposable matrix representations of } \Gamma \text{ over } F_q \text{ of dimension } \alpha.$$ 

$$A_\Gamma(\alpha, q) = \text{ the number of isomorphism classes of absolutely indecomposable matrix representations of } \Gamma \text{ over } F_q \text{ of dimension } \alpha.$$ 

Lemma 2.2.6 implies that $M_\Gamma(\alpha, q)$ is equal to the number of isomorphism classes of matrix representations of $\Gamma$ over $\overline{F}_q$ of dimension $\alpha$ which are defined over $F_q$,
and that $A_\Gamma(\alpha, q)$ is equal to the number of isomorphism classes of absolutely indecomposable representations of $\Gamma$ over $\bar{F}_q$ of dimension $\alpha$ which are defined over $F_q$.

As already noted, the finite group $GL(\alpha, F_q)$ acts on the finite set $R_\Gamma(\alpha, F_q)$, and $M_\Gamma(\alpha, q)$ is equal to the number of orbits in $R_\Gamma(\alpha, F_q)$. Thus the Molien-Burnside orbit counting formula becomes:

$$M_\Gamma(\alpha, q) = \frac{1}{|GL(\alpha, F_q)|} \sum_{g \in GL(\alpha, F_q)} |X_g| = \sum_{g \in Cl(\alpha, F_q)} |X_g| |Z_g|,$$

where $X_g = \{V \in R_\Gamma(\alpha, F_q) | g \cdot V = V\}$, $Z_g = \{x \in GL(\alpha, F_q) | gx = xg\}$, and where $Cl(\alpha, F_q)$ is a subset of $GL(\alpha, F_q)$ which consists of complete representatives of conjugacy classes of $GL(\alpha, F_q)$.

The Krull-Schmidt Theorem implies that every representation of $\Gamma$ over a field can be written as a direct sum of indecomposable representations in a unique way up to order. This implies the following formal identity:

$$\sum_{\alpha \in \mathbb{N}^n} M_\Gamma(\alpha, q) X^\alpha = \prod_{\alpha \in \mathbb{N}^n \{0\}} (1 - X^\alpha)^{-I_\Gamma(\alpha, q)}.$$

For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n \{0\}$, we let $\bar{\alpha} = \gcd(\alpha_1, \cdots, \alpha_n)$.

**Theorem 3.2.4.** For any $\alpha \in \mathbb{N}^n \{0\}$, the following formulas are valid:

$$I_\Gamma(\alpha, q) = \sum_{d | \bar{\alpha}} \frac{1}{d} \sum_{r | d} \mu\left(\frac{d}{r}\right) A_\Gamma\left(\frac{\alpha}{d}, q^r\right),$$

$$A_\Gamma(\alpha, q) = \sum_{d | \bar{\alpha}} \frac{1}{d} \sum_{r | d} \mu(r) I_\Gamma\left(\frac{\alpha}{d}, q^r\right).$$

**Proof.** Let $K_\Gamma(\alpha, q^r)$ denote the number of isomorphism classes of absolutely indecomposable representations of $\Gamma$ of dimension $\alpha$ with minimal field of definition $F_{q^r}$. Clearly,

$$A_\Gamma(\alpha, q^d) = \sum_{r | d} K_\Gamma(\alpha, q^r).$$

The Möbius inversion formula shows that

$$K_\Gamma(\alpha, q^d) = \sum_{r | d} \mu\left(\frac{d}{r}\right) A_\Gamma(\alpha, q^r).$$
It follows from Theorem 3.2.3 that

\[ I_{\Gamma}(\alpha, q) = \sum_{d \mid \bar{\alpha}} \frac{1}{d} K_{\Gamma}\left(\frac{\alpha}{d}, q^d\right). \]

Substituting \( K_{\Gamma}(\alpha/d, q^d) \) by \( \sum_{r \mid d} \mu(d/r) A_{\Gamma}(\alpha/d, q^r) \) thus gives

\[ I_{\Gamma}(\alpha, q) = \sum_{d \mid \bar{\alpha}} \frac{1}{d} \sum_{r \mid d} \mu\left(\frac{d}{r}\right) A_{\Gamma}\left(\frac{\alpha}{d}, q^r\right). \]

The Möbius inversion of the above formula now amounts to the following:

\[ A_{\Gamma}(\alpha, q) = \sum_{d \mid \bar{\alpha}} \frac{1}{d} \sum_{r \mid d} \mu(r) I_{\Gamma}\left(\frac{\alpha}{d}, q^r\right). \]

**Corollary 3.2.5.** With the above notations, we have the following identity:

\[ \sum_{d \mid \bar{\alpha}} \frac{1}{d} I_{\Gamma}\left(\frac{\alpha}{d}, q^d\right) = \sum_{d \mid \bar{\alpha}} \frac{1}{d} A_{\Gamma}\left(\frac{\alpha}{d}, q^d\right). \]

**Corollary 3.2.6.** Suppose that \( A_{\Gamma}(\alpha, q) \) are polynomials in \( q \) with integer coefficients for all \( \alpha \in \mathbb{N}^n \), and let \( A_{\Gamma}(\alpha, q) = \sum_{i=0}^{u_\alpha} t_i^\alpha q^i \), where \( u_\alpha = \deg A_{\Gamma}(\alpha, q) \). Then we have the following formal identity:

\[ \prod_{\alpha \in \mathbb{N}^n} (1 - X^\alpha)^{I_{\Gamma}(\alpha, q)} = \prod_{\alpha \in \mathbb{N}^n} \prod_{i=0}^{u_\alpha} (1 - q^i X^\alpha)^{t_i^\alpha}. \]

**Remark.** It will be shown in the next chapter that the \( A_{\Gamma}(\alpha, q) \) are indeed polynomials in \( q \) with integer coefficients. This Corollary 3.2.6 shows that the individual coefficients \( t_i^\alpha \) occur as exponents in a formal product.

### 3.3 Root systems associated with graphs

We have already noted that representations of quivers are closely related to root systems. The dimension vectors of absolutely indecomposable representations of a quiver are precisely the positive roots of the root system determined by the underlying graph of the quiver. In this section, we present a combinatorial construction of root systems associated with given graphs which is due to Kac [K4].
Let $\Gamma$ be a connected graph with vertices $\{1, 2, \cdots, n\}$ (edge-loops are allowed), and let $a_{ij}$ denote the number of edges connecting vertices $i$ and $j$. We introduce the associated root system $\Delta$ as a subset of $\mathbb{Z}^n$ as follows. Let $\alpha_i = (\delta_{i1}, \cdots, \delta_{in})$, $i = 1, \cdots, n$, be the standard basis of $\mathbb{Z}^n$. Introduce a bilinear form $\langle , \rangle$ on $\mathbb{Z}^n$ by:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 1 - a_{ii} & \text{if } i = j, \\ -\frac{1}{2}a_{ij} & \text{otherwise.} \end{cases}$$

Let $q_\Gamma(\alpha) = \langle \alpha, \alpha \rangle$ be the associated quadratic form. It is clear that this is a $\mathbb{Z}$-valued form. The element $\alpha_i$ is called a fundamental root if there are no edge-loops at the vertex $i$. Denote by $\Pi$ the set of fundamental roots. For a fundamental root $\alpha$ define the fundamental reflection $r_\alpha \in \text{Aut}(\mathbb{Z}^n)$ by

$$r_\alpha(\lambda) = \lambda - 2(\lambda, \alpha)\alpha \text{ for } \lambda \in \mathbb{Z}^n.$$ 

This is a reflection since $\langle \alpha, \alpha \rangle = 1$ and hence $r_\alpha(\alpha) = -\alpha$, and also $r_\alpha(\lambda) = \lambda$ if $\langle \lambda, \alpha \rangle = 0$. In particular, $\langle r_\alpha(\lambda), r_\alpha(\lambda) \rangle = \langle \lambda, \lambda \rangle$. It is clear that if $\alpha_i$ is a fundamental root then $r_{\alpha_i} \in \text{Aut}(\mathbb{Z}^n)$ can be defined as follows:

$$r_{\alpha_i}(\alpha_j) = \begin{cases} -\alpha_i & \text{if } i = j, \\ \alpha_j + a_{ij}\alpha_i & \text{otherwise.} \end{cases}$$

The group $W \subset \text{Aut}(\mathbb{Z}^n)$ generated by all fundamental reflections is called the Weyl group of the graph $\Gamma$. Note that the bilinear form $\langle , \rangle$ is $W$-invariant. Define the set of real roots $\Delta_{re}$ by

$$\Delta_{re} = \bigcup_{w \in W} w(\Pi).$$

For an element $\alpha = \sum_{i=1}^nk_i\alpha_i \in \mathbb{Z}^n$, we define the height of $\alpha$, denoted by $ht(\alpha)$, to be the number $\sum_{i=1}^nk_i$. We call the support of $\alpha$ (written by $\text{supp} \alpha$) the subgraph of $\Gamma$ consisting of those vertices $i$ for which $k_i \neq 0$ and all the edges joining these vertices. We define the fundamental set $M \subset \mathbb{Z}^n$ by

$$M = \{\alpha \in \mathbb{Z}_+^n \setminus \{0\} \mid \langle \alpha, \alpha_i \rangle \leq 0, \forall \alpha_i \in \Pi, \text{ and } \text{supp} \alpha \text{ is connected}\}.$$ 

Likewise we define the set of imaginary roots $\Delta_{im}$ by

$$\Delta_{im} = \bigcup_{w \in W} w(M \cup -M).$$
Then the root system $\Delta$ is defined as

$$\Delta = \Delta_{re} \cup \Delta_{im}.$$ 

An element $\alpha \in \Delta \cap \mathbb{Z}_n^+$ is called a positive root. Denote by $\Delta^+$ (resp. $\Delta^+_{re}$ or $\Delta^+_{im}$) the set of all positive (resp. real or imaginary) roots. It is obvious that $\langle \alpha, \alpha \rangle = 1$ if $\alpha \in \Delta_{re}$. On the other hand $\langle \alpha, \alpha \rangle \leq 0$ if $\alpha \in \Delta_{im}$. Hence

$$\Delta_{re} \cap \Delta_{im} = \emptyset.$$ 

Furthermore, it can be proved by representation theory that

$$\Delta = \Delta^+ \sqcup -\Delta^+.$$ 

Recall that if $\alpha = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$, then $\bar{\alpha} = \gcd(k_1, k_2, \cdots, k_n)$. As each fundamental reflection preserves $\bar{\alpha}$, $W$ preserves $\bar{\alpha}$. In other words, $w(\bar{\alpha}) = \bar{\alpha}$, for $w \in W$. As a consequence, $\bar{\alpha} = 1$ if $\alpha$ is a real root.

**Proposition 3.3.1.** Let $\Gamma$ be a connected graph, $\Delta$ its associated root system. Then

1. If $\alpha$ is a real root, then the only multiples of $\alpha$ which are roots are $\pm \alpha$;
2. If $\alpha$ is an imaginary root, then $\frac{1}{\bar{\alpha}} \alpha$ is an imaginary root, and any multiple of $\alpha$ is an imaginary root.

**Proof.** Let us prove (2) first. Without loss of generality, we may assume that $\alpha \in \Delta^+_{im}$. Thus there exists $w \in W$ such that $w(\alpha) \in M$. Obviously, $\frac{1}{\bar{\alpha}} w(\alpha) \in \mathbb{Z}_n^+$. By the definition of $M$, it is easy to see that $\frac{1}{\bar{\alpha}} w(\alpha) \in M$. As $w(\frac{1}{\bar{\alpha}} \alpha) = \frac{1}{\bar{\alpha}} w(\alpha)$, thus $\frac{1}{\bar{\alpha}} \alpha$ is an imaginary positive root. A similar argument shows that any multiple of $\alpha$ is an imaginary root.

Now, suppose that $\alpha$ is a real root and $d\alpha$ is a root for some $d \in \mathbb{Z}$. $d\alpha$ cannot be an imaginary root, otherwise (2) implies that $\alpha$ is an imaginary root. Thus $d\alpha$ must be a real root. And so $1 = \gcd(d\alpha) = |d| \gcd(\alpha)$. As $\alpha$ is a real root, $\gcd(\alpha) = 1$. Thus $|d| = 1$, and so $d = \pm 1$. This proves (1).

According as the bilinear form $\langle \ , \ \rangle$ is positive definite, positive semidefinite or indefinite the graph is called a graph of finite, tame and wild type respectively. The complete list of graphs of finite type is as follows:
The complete list of graphs of tame type is given in the following:

\[ A_n \ (n \geq 1) \]

\[ D_n \ (n \geq 4) \]

\[ E_6 \]

\[ E_7 \]

\[ E_8 \]
The subscript in the notation of a tame graph is the number of its vertices minus 1. The kernel of the bilinear form \( \langle \cdot, \cdot \rangle \) is \( \mathbb{Z}\delta \), where \( \delta = \sum_i a_i\alpha_i \), \( a_i \) being the label by the vertices.

A graph of wild type is called hyperbolic if every one of its proper connected subgraphs is of finite or tame type. In the case of a finite, tame or hyperbolic graph, there is a simple description of the root system \( \Delta \), which can be found on page 78 of [K4].

**Theorem 3.3.2.** If a graph \( \Gamma \) is of finite, tame or hyperbolic type, then

\[
\Delta = \{ \alpha \in \mathbb{Z}^n \setminus \{0\} \mid \langle \alpha, \alpha \rangle \leq 1 \}.
\]

In particular, if \( \Gamma \) is of finite type, then

\[
\Delta = \{ \alpha \in \mathbb{Z}^n \mid \langle \alpha, \alpha \rangle = 1 \}.
\]

And if \( \Gamma \) is of tame type, then

\[
\Delta_{re} = \{ \alpha \in \mathbb{Z}^n \mid \langle \alpha, \alpha \rangle = 1 \}, \quad \Delta_{im} = \{ n\delta \mid n \in \mathbb{Z}, n \neq 0 \}.
\]

where \( \delta = \sum a_i\alpha_i \), \( a_i \) being the labels by the vertices in the classification table.

**Example.** Let \( \Gamma \) be the graph \( \circ\longrightarrow\longrightarrow\longrightarrow\circ \). Then for any \( \alpha = (x, y) \in \mathbb{Z}^2 \), we have

\[
\langle \alpha, \alpha \rangle = x^2 + y^2 - 2xy = (x - y)^2.
\]

Thus \( \alpha \) is a real root if and only if \( |x - y| = 1 \), while \( \alpha \) is an imaginary root if and only if \( x = y \). Thus we have

\[
\Delta_{re} = \{ (x, y) \in \mathbb{Z}^2 \mid x - y = \pm 1 \}, \quad \Delta_{im} = \{ (x, x) \in \mathbb{Z}^2 \mid x \neq 0 \}.
\]
The representations of quivers are closely related to the root systems associated with their underlying graphs. The following theorem is fundamental in representation theory of quivers which is due to Gabriel [G1] in finite cases, to Donovan and Freislich [DF] and independently to Nazarova [N] in tame cases, and to Kac [K2] in general cases.

**Theorem 3.3.3 (Fundamental Theorem of Representations of Quivers).** Let \( \Gamma \) be a connected quiver, \( \Delta \) be the root system associated with \( \Gamma \). Then the following statements are valid:

1. The polynomials \( A_\Gamma(\alpha, q) \) are independent of the orientations of the quiver;
2. There exists an absolutely indecomposable representation of \( \Gamma \) over \( \mathbb{F}_q \) of dimension \( \alpha \) if and only if \( \alpha \) is a positive root in \( \Delta \), that is, \( A_\Gamma(\alpha, q) \neq 0 \) if and only if \( \alpha \in \Delta^+ \);
3. \( A_\Gamma(\alpha, q) = 1 \) if and only if \( \alpha \) is a real positive root;
4. For any \( w \in W \), \( A_\Gamma(w(\alpha), q) = A_\Gamma(\alpha, q) \) provided \( \alpha \) and \( w(\alpha) \) are positive roots.

**Corollary 3.3.4.** \( \Gamma \) and \( \Delta \) as before. \( I_\Gamma(\alpha, q) \neq 0 \) if and only if \( \alpha \in \Delta^+ \). Moreover, \( I_\Gamma(\alpha, q) = 1 \) if and only if \( \alpha \in \Delta^+_r \).

**Proof.** Suppose that \( \alpha \in \Delta^+ \). Then \( I_\Gamma(\alpha, q) \neq 0 \) because \( I_\Gamma(\alpha, q) \geq A_\Gamma(\alpha, q) \) and \( A_\Gamma(\alpha, q) \neq 0 \) by the previous theorem. Now suppose that \( \alpha \notin \Delta^+ \). If \( \bar{\alpha} = 1 \), then Theorem 3.2.4 shows that \( I_\Gamma(\alpha, q) = A_\Gamma(\alpha, q) \). As \( A_\Gamma(\alpha, q) = 0 \) by the above theorem, \( I_\Gamma(\alpha, q) = 0 \). Suppose \( \bar{\alpha} > 1 \) and \( I_\Gamma(\alpha, q) \neq 0 \). Theorem 3.2.4 implies that there exists \( d \mid \bar{\alpha} \) such that \( A_\Gamma(\frac{1}{d}\alpha, q) \neq 0 \). Thus the above theorem shows that \( \frac{1}{d}\alpha \in \Delta^+ \). Proposition 3.3.1 shows that \( \frac{1}{d}\alpha \) cannot be an imaginary root. Thus \( \frac{1}{d}\alpha \in \Delta^+_r \). And so \( 1 = \gcd(\frac{1}{d}\alpha) = \frac{1}{d} \gcd(\alpha) = \frac{1}{d}\bar{\alpha} \), and hence \( d = \bar{\alpha} \). It follows from Theorem 3.2.4 that

\[
I_\Gamma(\alpha, q) = \sum_{d \mid \bar{\alpha}} \sum_{r \mid d} \mu\left(\frac{d}{r}\right) A_\Gamma\left(\frac{\alpha}{d}, q^r\right)
= \frac{1}{\bar{\alpha}} \sum_{r \mid \bar{\alpha}} \mu\left(\frac{\bar{\alpha}}{r}\right) A_\Gamma\left(\frac{\alpha}{\bar{\alpha}}, q^r\right)
= \frac{1}{\bar{\alpha}} \sum_{r \mid \bar{\alpha}} \mu\left(\frac{\bar{\alpha}}{r}\right).
\]
As $\bar{\alpha} > 1$, $I_\Gamma(\alpha, q) = 0$. This is a contradiction. Thus $I_\Gamma(\alpha, q) = 0$ if $\alpha \notin \Delta^+$. 

Now, suppose that $\alpha \in \Delta_{re}^+$. Then $\bar{\alpha} = 1$. The above theorem shows that $A_\Gamma(\alpha, q) = 1$. Thus Theorem 3.2.4 implies that $I_\Gamma(\alpha, q) = A_\Gamma(\alpha, q) = 1$. Conversely, suppose $I_\Gamma(\alpha, q) = 1$. The previous argument shows that $\alpha \in \Delta^+$ and so $A_\Gamma(\alpha, q) \neq 0$. As $A_\Gamma(\alpha, q) \leq I_\Gamma(\alpha, q)$, $A_\Gamma(\alpha, q) = 1$. Again by the previous theorem, $\alpha \in \Delta_{re}^+$. 

### 3.4 Conjugacy classes in $GL(n, \mathbb{F}_q)$ 

In order to get useful formulae, we have to recall some results on the conjugacy classes in $GL(n, \mathbb{F}_q)$ and introduce some notations. More details can be found on page 138 of [M2].

Let $\Phi$ be the set of all monic irreducible polynomials in the indeterminant $t$ over $\mathbb{F}_q$ with $t$ excluded. Each $g \in GL(n, \mathbb{F}_q)$ acts on the vector space $\mathbb{F}_q^n$ and hence defines a $\mathbb{F}_q[t]$-module structure on $\mathbb{F}_q^n$ with the property that $t \cdot v = g v$ for all $v \in \mathbb{F}_q^n$. We shall denote this module by $V_g$. Two such modules $V_g$ and $V_h$ are isomorphic if and only if $g$ and $h$ are conjugate. The conjugacy classes of $GL(n, \mathbb{F}_q)$ are thus in one-to-one correspondence with the isomorphism classes of $\mathbb{F}_q[t]$-modules $V$ of dimension $n$ such that (i) $\dim V = n$, (ii) $t \cdot v = 0$ implies $v = 0$.

Because $\mathbb{F}_q[t]$ is a principle ideal domain any finite dimensional module is of the form 

$$V \cong \bigoplus_i \mathbb{F}_q[t]/(f_i)^{m_i}$$

for some $m_i \geq 1$ and irreducible polynomials $f_i$. Therefore, $V$ defines a partition valued function

$$\mu : \Phi \longrightarrow \mathcal{P}.$$ 

If we denote $\mu(f) = (\mu_1(f), \mu_2(f), \cdots) \in \mathcal{P}$, then

$$V \cong \bigoplus_{f, i} \left( \mathbb{F}_q[t]/(f)^{\mu_i(f)} \right).$$

In this thesis, we use $d(f)$ to denote the degree of the polynomial $f$. Since $\dim(V) = n$, $\mu$ must satisfy

$$||\mu|| := \sum_{f \in \Phi} d(f) |\mu(f)| = n.$$
In this way we find that there is a one-to-one correspondence between conjugacy classes in $GL(n, \mathbb{F}_q)$ and functions $\mu$ from $\Phi$ to $\mathcal{P}$ which satisfy $||\mu|| = n$. Recall that for each $f = t^d - \sum_{i=1}^{d} a_i t^{i-1} \in \Phi$, the companion matrix $J(f)$ of $f$ is defined as follows:

$$
J(f) = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
a_1 & a_2 & a_3 & \ldots & a_d
\end{pmatrix}.
$$

For each integer $m \geq 1$, we define

$$
J_m(f) = \begin{pmatrix}
J(f) & I_d & 0 & \ldots & 0 \\
0 & J(f) & I_d & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & I_d \\
0 & 0 & 0 & \ldots & J(f)
\end{pmatrix}
$$

with $m$ diagonal blocks $J(f)$, where $I_d$ means the identity matrix of degree $d$. For any $f \in \Phi$, $\pi = (n_1, n_2, \ldots) \in \mathcal{P}$, let

$$
J(f, \pi) = J_{n_1}(f) \oplus J_{n_2}(f) \oplus \cdots.
$$

This is a diagonal block matrix with $J_{n_i}(f)$ ($i \geq 1$) in the diagonal. Then any element of $GL(n, \mathbb{F}_q)$ has the Jordan canonical form

$$
J(f_1, \pi_1) \oplus J(f_2, \pi_2) \oplus \cdots \oplus J(f_k, \pi_k),
$$

with $\sum_{i=1}^{k} d(f_i)|\pi_i| = n$, where $f_1, \ldots, f_k$ are distinct polynomials from $\Phi$, and where $\pi_1, \ldots, \pi_k \in \mathcal{P}$, $k$ is some positive integer. The Jordan canonical form corresponding to the function $\mu$ is thus $\bigoplus_{f \in \Phi} J(f, \mu(f))$.

Recall from Section 2.4 that $\phi_d(q)$ is the number of monic irreducible polynomials in $t$ over $\mathbb{F}_q$ with $t$ excluded. The following is a known formula modified in the case $n = 1$ (see page 84 of [LN]):

$$
\phi_n(q) = \frac{1}{n} \sum_{d|n} \mu(d) \left( q^{\frac{n}{d}} - 1 \right).
$$

The following lemma is an easy consequence of the above formula.
Lemma 3.4.1. We have the following formal identity:

\[ \frac{1 - X}{1 - qX} = \prod_{d=1}^{\infty} (1 - X^d)^{-\phi_d(q)}. \]

Proposition 3.4.2. Let \( c_n(q) \) be the number of conjugacy classes in \( GL(n, \mathbb{F}_q) \). Then the following identity holds:

\[ 1 + \sum_{n=1}^{\infty} c_n(q)X^n = \prod_{n=1}^{\infty} \frac{1 - X^n}{1 - qX^n}. \]

Thus, the \( c_n(q) \) are polynomials in \( q \) with integer coefficients.

Proof. From the above discussion, we have the following factorisation:

\[ 1 + \sum_{n=1}^{\infty} c_n(q)X^n = \prod_{d=1}^{\infty} (1 + p(1)X^d + p(2)X^{2d} + \cdots + p(n)X^{nd} + \cdots)^{\phi_d(q)}, \]

where \( p(n) \) is the partition function. It is well known that

\[ 1 + \sum_{n=1}^{\infty} p(n)X^n = \prod_{n=1}^{\infty} (1 - X^n)^{-1}. \]

Thus,

\[ 1 + \sum_{n=1}^{\infty} c_n(q)X^n = \prod_{d=1}^{\infty} \prod_{n=1}^{\infty} (1 - X^{nd})^{-\phi_d(q)} = \prod_{n=1}^{\infty} \prod_{d=1}^{\infty} (1 - X^{nd})^{-\phi_d(q)}. \]

It follows from the previous lemma that

\[ 1 + \sum_{n=1}^{\infty} c_n(q)X^n = \prod_{n=1}^{\infty} \frac{1 - X^n}{1 - qX^n}. \]

The above proposition simplifies an identity on page 408 of [Gr1], Green’s paper appears not to note that the coefficients of \( c_n(q) \) are integers.

The first six values of \( c_n(q) \) are listed below:

\[
\begin{align*}
    c_1(q) &= q - 1 \\
    c_2(q) &= q^2 - 1 = (q - 1)(q + 1) \\
    c_3(q) &= q^3 - q = (q - 1)(q + 1)q \\
    c_4(q) &= q^4 - q = (q - 1)(q^2 + q + 1)q \\
    c_5(q) &= q^5 - q^2 - q + 1 = (q - 1)(q + 1)(q^3 + q - 1) \\
    c_6(q) &= q^6 - q^2 = (q - 1)(q + 1)(q^2 + 1)q^2. 
\end{align*}
\]
The fact that \( c_n(q) \) is divisible by \( q - 1 \) easily follows from the above proposition by specialising \( q = 1 \) on both sides. These calculations suggest the following identity:

\[
\lim_{q \to 1} \frac{c_n(q)}{(q - 1)} = \text{the number of divisors of } n.
\]

In fact, this can be proved as follows:

Taking logarithms on both sides of the identity in Proposition 3.4.2, we have

\[
\sum_{n=1}^{\infty} \log(1 - X^n) - \sum_{n=1}^{\infty} \log(1 - qX^n) = \log \left( \sum_{n=0}^{\infty} c_n(q)X^n \right).
\]

Differentiating with respect to \( q \) gives

\[
\sum_{n=1}^{\infty} \frac{X^n}{1 - qX^n} = \sum_{n=0}^{\infty} c'_n(q)X^n.
\]

Thus, there holds

\[
\left( \sum_{n=1}^{\infty} \frac{X^n}{1 - qX^n} \right) \left( \sum_{n=0}^{\infty} c_n(q)X^n \right) = \sum_{n=0}^{\infty} c'_n(q)X^n.
\]

Let \( q \to 1 \), we have

\[
\sum_{n=0}^{\infty} c'_n(1)X^n = \sum_{n=1}^{\infty} \frac{X^n}{1 - X^n} = \sum_{n=1}^{\infty} X^n \sum_{i=0}^{\infty} X^{ni} = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} X^{ni}.
\]

It follows that

\[
\lim_{q \to 1} \frac{c_n(q)}{(q - 1)} = c'_n(1) = \text{the number of divisors of } n.
\]

### 3.5 Numbers of centralizers

In order to count the numbers of isomorphism classes of representations of a quiver by using the Molien-Burnside orbit counting formula described on page 32, we have to deduce effective formulas for the numbers of centralizers of a given conjugacy class in \( GL(\alpha, \mathbb{F}_q) \) and the number of representations fixed by it. We will sort out this problem in the current section.

First let us recall the definition of \( \langle \lambda, \mu \rangle \), where \( \lambda, \mu \in \mathcal{P} \). If we use \( \lambda' = (\lambda'_1, \lambda'_2, \cdots) \) and \( \mu' = (\mu'_1, \mu'_2, \cdots) \) to denote the conjugate partitions of \( \lambda \) and \( \mu \) respectively, then \( \langle \lambda, \mu \rangle = \sum_{i \geq 1} \lambda'_i \mu'_i \), where \( \lambda'_1, \lambda'_2, \cdots \) and \( \mu'_1, \mu'_2, \cdots \) are arranged in non-increasing order respectively.

The following lemma gives an alternative way to calculate \( \langle \lambda, \mu \rangle \).
Lemma 3.5.1. For any $\mu = (1^{m_1}2^{m_2} \cdots), \nu = (1^{n_1}2^{n_2} \cdots) \in \mathcal{P}$, we have

$$\langle \mu, \nu \rangle = \sum_{i \geq 1} \sum_{j \geq 1} \min(i, j)m_in_j.$$

Proof. Let $\mu' = (\mu_1', \mu_2', \cdots)$ and $\nu' = (\nu_1', \nu_2', \cdots)$ be the partitions conjugate to $\mu$ and $\nu$ respectively. Then $\mu_k' = \sum_{i \geq k} m_i, \nu_k' = \sum_{i \geq k} n_i$ for $k \geq 1$. By the definition of $\langle \mu, \nu \rangle$, we have the following:

$$\langle \mu, \nu \rangle = \mu_1'\nu_1' + \mu_2'\nu_2' + \mu_3'\nu_3' + \cdots$$

$$= (m_1 + m_2 + m_3 + \cdots)(n_1 + n_2 + n_3 + \cdots) +$$

$$= \sum_{j \geq 1} n_j \left( \sum_{j \geq 1} n_j + \sum_{j \geq 2} n_j \right) +$$

$$= \sum_{i \geq 1} \sum_{j \geq 1} \min(i, j)m_in_j.$$

Now for any $f, g \in \Phi, \lambda, \mu \in \mathcal{P}, (J(f, \lambda), J(g, \mu))$ represents a conjugacy class of the group $GL(m, \mathbb{F}_q) \times GL(n, \mathbb{F}_q)$, where $m = d(f)|\lambda|$ and $n = d(g)|\mu|$. We define $X_{(J(f, \lambda), J(g, \mu))} = \{ M \in \mathcal{M}_{m \times n}(\mathbb{F}_q) \mid J(f, \mu)M = MJ(g, \mu) \}$.

Lemma 3.5.2. For any $f, g \in \Phi, \lambda, \mu \in \mathcal{P}$, we have $|X_{(J(f, \lambda), J(g, \mu))}| = q^{d(f)\langle \lambda, \mu \rangle}$ if $f = g$, otherwise $|X_{(J(f, \lambda), J(g, \mu))}| = 1$.

Proof. Suppose $\lambda = (1^{m_1}2^{m_2} \cdots), \mu = (1^{n_1}2^{n_2} \cdots)$, and $d = \deg(f), e = \deg(g)$. Let $A = \mathbb{F}_q[x]$, and $\tilde{A} = A \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Given any $m \times m$ matrix $\alpha$ over $\mathbb{F}_q$, we can define an $A$-module structure on $\mathbb{F}_q^m$ by setting $x \cdot v = \alpha v$ for $v \in \mathbb{F}_q^m$. Let $V_\alpha$ denote this module, and define $\tilde{V}_\alpha = V_\alpha \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. By the definition of $X_{(J(f, \lambda), J(g, \mu))}$, it is easy to see that $X_{(J(f, \lambda), J(g, \mu))} = \text{Hom}_A(V_{J(f, \lambda)}, V_{J(g, \lambda)})$. Note that $\text{Hom}_A(V_{J(f, \lambda)}, V_{J(g, \lambda)})$ is a finite dimensional vector space over $\mathbb{F}_q$, which has the same dimension as $\text{Hom}_{\tilde{A}}(\tilde{V}_{J(f, \lambda)}, \tilde{V}_{J(g, \lambda)})$ over $\bar{\mathbb{F}}_q$. This reduces the calculation to the corresponding calculation over the field $\bar{\mathbb{F}}_q$. Since $J(f, \mu) = J_1(f)^{m_1} \oplus$
\( J_2(f)^{m_2} \oplus \cdots \), it follows that \( \tilde{V}_{J(f, \mu)} \cong \tilde{V}_{J_1(f)}^{m_1} \oplus \tilde{V}_{J_2(f)}^{m_2} \oplus \cdots \). Similarly, \( \tilde{V}_{J(g, \mu)} \cong \tilde{V}_{J_1(g)}^{n_1} \oplus \tilde{V}_{J_2(g)}^{n_2} \oplus \cdots \). Therefore,

\[
\text{Hom}_A\left( \tilde{V}_{J(f, \lambda)}, \tilde{V}_{J(g, \lambda)} \right) \cong \oplus_{i=1}^d \oplus_{j=1}^e \text{Hom}_A\left( \tilde{V}_{J_i(f)}, \tilde{V}_{J_j(g)} \right) \\
\cong \oplus_{i=1}^d \oplus_{j=1}^e \text{Hom}_A\left( \tilde{V}_{J_i(f)}, \tilde{V}_{J_j(g)} \right)^{m_i n_j}.
\]

The classical theorem that any finite field is separable implies that for any irreducible monic polynomial \( f(t) \) in \( \mathbb{F}_q[t] \) there is an invertible matrix \( X \) with entries in \( \overline{\mathbb{F}}_q \) such that \( X J(f) X^{-1} \) is diagonal with distinct diagonal entries \( \xi_1, \ldots, \xi_d \). Here \( f(t) = \prod_{i=1}^d (t - \xi_i) \). Thus \( J_i(f) \) is similar to \( J(\xi_1, i) \oplus \cdots \oplus J(\xi_d, i) \) over \( \overline{\mathbb{F}}_q \), where \( J(\xi, i) \) represents the Jordan block matrix of size \( i \) with eigenvalue \( \xi \). Similarly, \( J_i(g) \) is similar to \( J(\eta_1, i) \oplus \cdots \oplus J(\eta_e, i) \) over \( \overline{\mathbb{F}}_q \), where \( \eta_1, \ldots, \eta_e \) are the roots of \( g \) over \( \overline{\mathbb{F}}_q \). Therefore,

\[
\text{Hom}_A\left( \tilde{V}_{J_i(f)}, \tilde{V}_{J_j(g)} \right) \cong \oplus_{s=1}^d \oplus_{t=1}^e \text{Hom}_A\left( \tilde{V}_{J_s(i)}, \tilde{V}_{J_t(j)} \right).
\]

Note that every \( \tilde{V}_{J(\delta, i)} \) is an indecomposable \( A \)-module whose sole composition factor is \( \tilde{V}_{J(\delta, 1)} \). Thus \( \text{Hom}_A\left( \tilde{V}_{J(\xi, i)}, \tilde{V}_{J(\eta, j)} \right) = 0 \) if and only if \( \xi = \eta \). Also note that \( f \) and \( g \) have a common root if and only if \( f = g \). Thus if \( f \neq g \), then for all \( \xi_s, \eta_t \) \((1 \leq s \leq d, 1 \leq t \leq e)\), \( \text{Hom}_A\left( \tilde{V}_{J(\xi, i)}, \tilde{V}_{J(\eta, j)} \right) = 0 \), and so

\[
\text{Hom}_A\left( \tilde{V}_{J(\lambda, \lambda)}, \tilde{V}_{J(\mu, \mu)} \right) = 0,
\]

which implies that \(| X_{(J(\lambda, \lambda), J(\mu, \mu))} | = 1 \). Now, suppose that \( f = g \). Therefore,

\[
\text{Hom}_A\left( \tilde{V}_{J_i(f)}, \tilde{V}_{J_j(g)} \right) \cong \oplus_{s=1}^d \text{Hom}_A\left( \tilde{V}_{J_s(i)}, \tilde{V}_{J_s(i)} \right).
\]

An elementary calculation with matrices reveals that

\[
\dim_{\mathbb{F}_q} \text{Hom}_A\left( \tilde{V}_{J(\xi, i)}, \tilde{V}_{J(\xi, j)} \right) = \min(i, j).
\]

Thus, we have

\[
\dim_{\mathbb{F}_q} X_{(J(\lambda, \lambda), J(\mu, \mu))} = \dim_{\overline{\mathbb{F}}_q} \text{Hom}_A\left( \tilde{V}_{J(\lambda, \lambda)}, \tilde{V}_{J(\mu, \mu)} \right) \\
= \sum_{i \geq 1} \sum_{j \geq 1} m_i n_j \dim_{\overline{\mathbb{F}}_q} \text{Hom}_A\left( \tilde{V}_{J_i(f)}, \tilde{V}_{J_j(g)} \right) \\
= \sum_{i \geq 1} \sum_{j \geq 1} m_i n_j \min(i, j) d \\
= d \langle \lambda, \mu \rangle \quad \text{(by Lemma 3.5.1)}.
\]
Finally, $|X_{(J(f,\lambda),J(g,\mu))}| = q^{d(\lambda,\mu)}$. Note that these formulae depend only on the degrees of $f$ and $g$.

We assume that $b_{ij}$ is the number of arrows from vertex $i$ to vertex $j$ in $\Gamma$, and $a_{ij}$ is the number of edges between $i$ and $j$.

**Theorem 3.5.3.** Let $g = (J(f,\pi_1), \cdots, J(f,\pi_n))$, where $f \in \Phi$, $\deg(f) = d$, and $\pi_1, \cdots, \pi_n \in \mathcal{P}$. Then there hold:

$$|Z_g| = \prod_{i=1}^{n} q^{d(\pi_i,\pi_i)} b_{\pi_i}(q^{-d}),$$

$$|X_g| = \prod_{i=1}^{n} \prod_{j=1}^{n} q^{db_{ij}(\pi_i,\pi_j)}.$$  

**Proof.** First note that $|Z_g| = \prod_{i=1}^{n} |Z_{J(f,\pi_i)}|$. Since for any $\lambda \in \mathcal{P}$, $|Z_{J(f,\lambda)}|$ is just the number of centralisers of $J(f,\lambda)$ in $GL(n, \mathbb{F}_q)$, where $n = d|\lambda|$, formula (2.6) on page 139 of [M2] implies that $|Z_{J(f,\lambda)}| = q^{d(|\lambda|+2n(\lambda))b_{\lambda}(q^{-d})}$, where $n(\lambda) = \sum_{i \geq 1} (i-1)\lambda_i$ if $\lambda = (\lambda_1, \lambda_2, \cdots) \in \mathcal{P}$. It follows from Lemma 1 of [H] and Lemma 3.5.1 that $|\lambda| + 2n(\lambda) = \langle \lambda, \lambda \rangle$. Therefore, $|Z_{J(f,\pi_i)}| = q^{d(\pi_i,\pi_i)b_{\pi_i}(q^{-d})}$. And so,

$$|Z_g| = \prod_{i=1}^{n} q^{d(\pi_i,\pi_i)b_{\pi_i}(q^{-d})}.$$  

It is easy to see that $|X_g| = \prod_{i=1}^{n} \prod_{j=1}^{n} |X_{(J(f,\pi_i),J(f,\pi_j))}|^{b_{ij}}$. It follows from the previous lemma that

$$|X_g| = \prod_{i=1}^{n} \prod_{j=1}^{n} q^{db_{ij}(\pi_i,\pi_j)}.$$  

**Remark.** Since $\langle \lambda, \mu \rangle = \langle \mu, \lambda \rangle$, for any $\lambda, \mu \in \mathcal{P}$, and $a_{ii} = b_{ii}, a_{ij} = b_{ij} + b_{ji}$ if $i \neq j$, we have

$$|X_g| = \prod_{i=1}^{n} \prod_{j=1}^{n} q^{db_{ij}(\pi_i,\pi_j)} = \prod_{1 \leq i \leq j \leq n} q^{da_{ij}(\pi_i,\pi_j)}.$$  

Therefore, $|X_g|$ is independent of the orientations on $\Gamma$. It is this property that ensures that the polynomials $A_{\Gamma}(\alpha, q)$, $I_{\Gamma}(\alpha, q)$ and $M_{\Gamma}(\alpha, q)$ are independent of the orientations on $\Gamma$. This fact will be clarified in Section 4.1.
Chapter 4
Generating functions and formal identities

4.1 Formal identities associated with graphs

In this chapter, \( \Gamma \) will be a fixed connected graph with vertices \( \{1, 2, \cdots, n\} \). Fix an orientation on \( \Gamma \), the resulted quiver is still denoted by \( \Gamma \). Let \( a_{ij} \) denote the number of edges connecting the vertices \( i \) and \( j \), and let \( b_{ij} \) be the number of arrows from \( i \) to \( j \). Thus \( a_{ii} = b_{ii} \), \( a_{ij} = b_{ij} + b_{ji} \) if \( i \neq j \).

Recall that \( R_{\Gamma}(\alpha, \mathbb{F}_q) \) is the set of all matrix representations of \( \Gamma \) over \( \mathbb{F}_q \) of dimension \( \alpha \) with the finite group \( GL(\alpha, \mathbb{F}_q) = GL(\alpha_1, \mathbb{F}_q) \times \cdots \times GL(\alpha_n, \mathbb{F}_n) \) acting on it. Obviously, \( |R_{\Gamma}(\alpha, \mathbb{F}_q)| = \prod_{1 \leq i, j \leq n} q^{b_{ij}\alpha_i\alpha_j} = \prod_{1 \leq i \leq j \leq n} q^{a_{ij}\alpha_i\alpha_j} \).

The basic feature of this chapter is treating infinite sequences by their generating functions. To avoid convergence problems, all generating functions are treated as formal power series. Let \( \mathbb{R} \) be an arbitrary ring with identity. The ring of formal power series over \( \mathbb{R} \) in one variable \( X \) is denoted by \( \mathbb{R}[[X]] \). Thus each element of \( \mathbb{R}[[X]] \) can be written formally as \( \sum_{i \geq 0} a_i X^i \). The addition and multiplication in \( \mathbb{R}[[X]] \) are defined in an obvious way. There are four important series in the ring \( \mathbb{R}[[X]] \) which are defined as follows:

\[
e^X = 1 + X + \frac{1}{2!} X^2 + \frac{1}{3!} X^3 + \frac{1}{4!} X^4 + \cdots
\]

\[
\log(1 - X) = -X - \frac{1}{2} X^2 - \frac{1}{3} X^3 - \frac{1}{4} X^4 - \cdots
\]

\[
\frac{1}{1 - X} = 1 + X + X^2 + X^3 + X^4 + \cdots
\]

\[
(1 - X)^{-a} = 1 + aX + \frac{a(a + 1)}{2!} X^2 + \frac{a(a + 1)(a + 2)}{3!} X^3 + \cdots \quad \text{for all } a \in \mathbb{R}.
\]

These satisfy the evident functional equations.

**Definition 4.1.1.** Let \( X_1, \cdots, X_n \) be \( n \) independent commuting variables. We
define
\[ P_{\Gamma}(X_1, \ldots, X_n, q) = \sum_{\lambda_1, \ldots, \lambda_n \in \mathcal{P}} \prod_{1 \leq i \leq j \leq n} q^{a_{ij}(\lambda_i, \lambda_j)} X_1^{\lambda_1} \cdots X_n^{\lambda_n}. \]

Thus \( P_{\Gamma}(X_1, \ldots, X_n, q) \) is a formal power series in \( X_1, \ldots, X_n \) with coefficients in \( \mathbb{Q}(q) \). Note that the constant term of \( P_{\Gamma}(X_1, \ldots, X_n, q) \) is equal to 1. At a later stage, it will be clear that the coefficients of \( P_{\Gamma}(X_1, \ldots, X_n, q) \) can also be considered as formal power series in \( q \). Thus, \( P_{\Gamma}(X_1, \ldots, X_n, q) \) can be considered as a formal power series in \( n + 1 \) variables \( q, X_1, \ldots, X_n \).

**Theorem 4.1.2.** With the notations as above, the following formal identity holds:
\[
\sum_{\alpha \in \mathbb{N}^n} M_{\Gamma}(\alpha, q) X^\alpha = \prod_{d=1}^{\infty} \left( \sum_{g \in CL(\alpha, F_q)} \frac{|X_g|}{|Z_g|} \right)^{\phi_d(q)}.
\]

**Proof.** It follows from the Molien-Burnside formula that for any \( \alpha \in \mathbb{N}^n \),
\[
M_{\Gamma}(\alpha, q) = \frac{1}{|GL(\alpha, F_q)|} \sum_{g \in GL(\alpha, F_q)} |X_g| = \sum_{g \in CL(\alpha, F_q)} \frac{|X_g|}{|Z_g|},
\]
where \( X_g, Z_g \) and \( CL(\alpha, q) \) are defined as on page 32. Thus,
\[
\sum_{\alpha \in \mathbb{N}^n} M_{\Gamma}(\alpha, q) X^\alpha = \sum_{\alpha \in \mathbb{N}^n} \left( \sum_{g \in CL(\alpha, F_q)} \frac{|X_g|}{|Z_g|} \right) X^\alpha.
\]
The discussion of Section 3.3 shows that for a given \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), each conjugacy class of \( GL(\alpha, F_q) \) is a direct sum of finitely many elements of the form \( (J(f_1, \pi_{a_1}), J(f_1, \pi_{a_2}), \ldots, J(f_1, \pi_{a_n})) \), \( i \geq 1 \), where \( f_i \in \Phi, \pi_{a_i} \in \mathcal{P} \), subject to the conditions:

(i) the \( f_i \) (\( i \geq 1 \)) are mutually distinct;

(ii) for a fixed \( k \), \( 1 \leq k \leq n \), \( \sum_{i \geq 1} d(f_i) |\pi_{a_i}| = \alpha_k \).

It follows from Lemma 3.5.2 that, for any \( f, g \in \Phi \) with \( f \neq g \), and for any \( \lambda_i, \mu_i \in \mathcal{P} \) (\( i = 1, \ldots, n \)), the following formula holds:
\[
|X(J(f, \lambda_1) \oplus J(g, \mu_1), \ldots, f, \lambda_n) \oplus J(g, \mu_n))| = |X(J(f, \lambda_1), \ldots, J(f, \lambda_n))| |X(J(g, \mu_1), \ldots, J(g, \mu_n))|.
\]

Therefore, we have the following factorisation:
\[
\sum_{\alpha \in \mathbb{N}^n} \left( \sum_{g \in CL(\alpha, F_q)} \frac{|X_g|}{|Z_g|} \right) X^\alpha
= \prod_{f \in \Phi} \left( \sum_{\pi_1, \ldots, \pi_n \in \mathcal{P}} \frac{|X(J(f, \pi_1), \ldots, J(f, \pi_n))|}{Z(J(f, \pi_1), \ldots, J(f, \pi_n))} X^{d(f)|\pi_1|} \cdots X^{d(f)|\pi_n|},
\]
Note that in the above product, each factor is a formal power series with constant term 1. Thus the above product does make sense. It follows from Theorem 3.5.3 that

\[
\prod_{f \in \Phi} \left( \sum_{\pi_1, \ldots, \pi_n \in P} \frac{X(J(f, \pi_1), \ldots, J(f, \pi_n))}{Z(J(f, \pi_1), \ldots, J(f, \pi_n))} X_1^{d(f)|\pi_1|} \cdots X_n^{d(f)|\pi_n|} \right)
\]

= \prod_{f \in \Phi} \left( \sum_{\pi_1, \ldots, \pi_n \in P} \prod_{1 \leq i \leq j \leq n} q^{d(f)\alpha_{ij} \langle \pi_i, \pi_j \rangle} \frac{X_1^{d(f)|\pi_1|} \cdots X_n^{d(f)|\pi_n|}}{\prod_{1 \leq i \leq n} q^{d(f)\alpha_{ii} \langle \pi_i, \pi_i \rangle} b_{\pi_i}(q-d(f))} \right)

= \prod_{f \in \Phi} P_\Gamma \left( X_1^{d(f)}, \ldots, X_n^{d(f)}, q^{d(f)} \right) \quad \text{(by Definition 4.1.1)}

= \prod_{d=1}^\infty \left( P_\Gamma \left( X_1^{d}, \ldots, X_n^{d}, q^d \right) \right) \phi_d(q).

This finishes the proof.

The Krull-Schmidt Theorem implies that every representation of \( \Gamma \) over a field can be written as a direct sum of indecomposable representations in a unique way up to order. This implies the following formal identity:

\[
\sum_{\alpha \in \mathbb{N}^n} M_\Gamma(\alpha, q) X^\alpha = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{-I_\Gamma(\alpha, q)}.
\]

For \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \setminus \{0\} \), we let \( \bar{\alpha} = \gcd(\alpha_1, \ldots, \alpha_n) \).

**Definition 4.1.3.** For any \( \alpha \in \mathbb{N}^n \setminus \{0\} \), let \( E_\Gamma(\alpha, q) \) and \( H_\Gamma(\alpha, q) \) be the rational functions in \( q \) determined by the following:

\[
\log \left( \sum_{\alpha \in \mathbb{N}^n} M_\Gamma(\alpha, q) X^\alpha \right) = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} E_\Gamma(\alpha, q) X^\alpha / \bar{\alpha},
\]

\[
\log \left( P_\Gamma(X_1, \ldots, X_n, q) \right) = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} H_\Gamma(\alpha, q) X^\alpha / \bar{\alpha}.
\]

Theorem 4.1.2 thus implies that

\[
E_\Gamma(\alpha, q) = \sum_{d \mid \bar{\alpha}} d \phi_d(q) H_\Gamma \left( \frac{\alpha}{d}, q^d \right).
\]
Lemma 4.1.4. With notations as above, we have

\[ I_\Gamma(\alpha, q) = \frac{1}{\alpha} \sum_{d \mid \alpha} \mu(d) E_\Gamma\left(\frac{\alpha}{d}, q\right). \]

Proof. It follows from identity (4.1.1) and Definition 4.1.3 that

\[
\sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} E_\Gamma(\alpha, q) X^\alpha/\bar{\alpha} = \log \left( \prod_{\alpha \in \mathbb{N}^n \setminus \{0\}} (1 - X^\alpha)^{-I_\Gamma(\alpha, q)} \right)
\]

\[
= \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} I_\Gamma(\alpha, q) \log \frac{1}{1 - X^\alpha}
\]

\[
= \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} I_\Gamma(\alpha, q) \sum_{i=1}^{\infty} \frac{1}{i} X^{i\alpha}
\]

\[
= \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \sum_{i=1}^{\infty} \frac{1}{i} I_\Gamma(\alpha, q) X^{i\alpha}.
\]

By comparing the coefficients of \(X^\alpha\) on both sides, we have

\[
\frac{1}{\alpha} E_\Gamma(\alpha, q) = \sum_{d \mid \alpha} \frac{1}{d} I_\Gamma\left(\frac{\alpha}{d}, q\right).
\]

Thus,

\[ E_\Gamma(\alpha, q) = \sum_{d \mid \alpha} \frac{\bar{\alpha}}{d} I_\Gamma\left(\frac{\alpha}{d}, q\right). \]

The Möbius inversion of this shows that

\[ I_\Gamma(\alpha, q) = \sum_{d \mid \alpha} \mu(d) E_\Gamma\left(\frac{\alpha}{d}, q\right). \]

Lemma 4.1.5. With notations as above, we have

\[ A_\Gamma(\alpha, q) = \frac{1}{\alpha} \sum_{d \mid \alpha} \mu(d) E_\Gamma\left(\frac{\alpha}{d}, q^d\right), \]

\[ E_\Gamma(\alpha, q) = \sum_{d \mid \alpha} \frac{\bar{\alpha}}{d} A_\Gamma\left(\frac{\alpha}{d}, q^d\right). \]
Proof. We only need to prove the first identity, as the second is just the Möbius inversion of the first. In fact,

\[ A_\Gamma(\alpha, q) = \sum_{d \mid \bar{\alpha}} \frac{1}{d} \sum_{r \mid d} \mu(r) I_\Gamma \left( \frac{\alpha}{d}, q^r \right) \]  

(by Theorem 3.2.4)

\[ = \sum_{d \mid \bar{\alpha}} \frac{1}{d} \sum_{r \mid d} \mu(r) \frac{d}{\bar{\alpha}} \sum_{s \mid \bar{\alpha}/d} \mu(s) E_\Gamma \left( \frac{\alpha}{ds}, q^r \right) \]  

(by Lemma 4.1.4)

\[ = \frac{1}{\bar{\alpha}} \sum_{d \mid \bar{\alpha}} \sum_{r \mid d} \sum_{s \mid \bar{\alpha}/d} \mu(r) \mu \left( \frac{d}{s} \right) E_\Gamma \left( \frac{s\alpha}{\bar{\alpha}}, q^r \right) \]  

\[ = \frac{1}{\bar{\alpha}} \sum_{r \mid \bar{\alpha}} \sum_{d \mid \bar{\alpha}/r} \sum_{s \mid \bar{\alpha}/d} \mu(r) \mu \left( \frac{d}{s} \right) E_\Gamma \left( \frac{s\alpha}{\bar{\alpha}}, q^r \right) \]  

\[ = \frac{1}{\bar{\alpha}} \sum_{r \mid \bar{\alpha}} \sum_{s \mid \bar{\alpha}/r} \sum_{d \mid \bar{\alpha}/sr} \mu(r) \mu \left( \frac{\bar{\alpha}}{dsr} \right) E_\Gamma \left( \frac{s\alpha}{\bar{\alpha}}, q^r \right) \]  

Note that the Möbius function \( \mu \) has the following property:

\[ \sum_{d \mid n} \mu(d) = \begin{cases} 
1 & \text{if } n = 1, \\
0 & \text{otherwise}. 
\end{cases} \]

Thus we have

\[ A_\Gamma(\alpha, q) = \frac{1}{\bar{\alpha}} \sum_{r \mid \bar{\alpha}} \mu(r) E_\Gamma \left( \frac{\alpha}{r}, q^r \right). \]

This finishes the proof.

Theorem 4.1.6. With notations as before, the following identities hold

\[ A_\Gamma(\alpha, q) = \frac{q-1}{\bar{\alpha}} \sum_{d \mid \bar{\alpha}} \mu(d) H_\Gamma \left( \frac{\alpha}{d^q}, q^d \right), \]

\[ H_\Gamma(\alpha, q) = \sum_{d \mid d^q} \frac{1}{q^d - 1} A_\Gamma \left( \frac{\alpha}{d}, q^d \right). \]
Proof. We only need to prove the first formula since the second one is just the Möbius inverse of the first. In fact,

\[ A_{\Gamma}(\alpha, q) = \frac{1}{\bar{\alpha}} \sum_{d | \bar{\alpha}} \mu(d) E_{\Gamma}\left(\frac{\alpha}{d}, q^{d}\right) \]  

(by Lemma 4.1.5)

\[ = \frac{1}{\bar{\alpha}} \sum_{d | \bar{\alpha}} \mu(d) \sum_{r | \bar{\alpha}/d} r \phi_{\alpha/d}(q^{d}) H_{\Gamma}\left(\frac{r \alpha}{\bar{\alpha}}, q^{r} \frac{\bar{\alpha}}{r}\right) \]  

(by (4.1.2))

\[ = \sum_{d | \bar{\alpha}} \sum_{r | \bar{\alpha}/d} \mu(d) \frac{1}{d r} \phi_{\alpha/d}(q^{d}) H_{\Gamma}\left(\frac{r \alpha}{\bar{\alpha}}, q^{r} \frac{\bar{\alpha}}{r}\right) \]

\[ = \sum_{r | \bar{\alpha}} \sum_{d | r} \mu(d) \frac{r}{d \bar{\alpha}} \phi_{r/d}(q^{d}) H_{\Gamma}\left(\frac{\alpha}{r}, q^{r}\right) \]

\[ = \frac{1}{\bar{\alpha}} \sum_{r | \bar{\alpha}} H_{\Gamma}\left(\frac{\alpha}{r}, q^{r}\right) \sum_{d | r} \mu(d) \frac{r}{d} \phi_{r/d}(q^{d}) . \]

Recall from Section 3.4 that the following formula holds:

\[ \phi_{n}(q) = \frac{1}{n} \sum_{d | n} \mu(d)(q^{\frac{n}{d}} - 1). \]

The Möbius inverse of this amounts to the following:

\[ \sum_{d | n} \frac{n}{d} \mu(d) \phi_{n/d}(q^{d}) = \mu(n)(q - 1). \]

Thus, we have

\[ A_{\Gamma}(\alpha, q) = \frac{q - 1}{\bar{\alpha}} \sum_{r | \bar{\alpha}} \mu(r) H_{\Gamma}\left(\frac{\alpha}{r}, q^{r}\right) . \]

As the \( H_{\Gamma}(\alpha, q) \) are rational functions in \( q \), so are the \( A_{\Gamma}(\alpha, q) \). As \( A_{\Gamma}(\alpha, q) \) takes integer values for prime powers, \( A_{\Gamma}(\alpha, q) \) must be polynomials in \( q \) with rational coefficients. Therefore, \( I_{\Gamma}(\alpha, q) \) and \( M_{\Gamma}(\alpha, q) \) are all polynomials in \( q \) with rational coefficients. It is proved in [K3] that \( A_{\Gamma}(\alpha, q) \) is monic of degree \( u_{\alpha} = 1 - \langle \alpha, \alpha \rangle \).

Recall that \( A_{\Gamma}(\alpha, q) = \sum_{j=0}^{u_{\alpha}} t_{\alpha}^{j} q^{j} \). As the formal power series \( P_{\Gamma}(X_{1}, \ldots, X_{n}, q) \) is independent of the orientations on \( \Gamma \), the functions \( E_{\Gamma}(\alpha, q) \) and \( H_{\Gamma}(\alpha, q) \), and hence the polynomials \( A_{\Gamma}(\alpha, q) \), \( I_{\Gamma}(\alpha, q) \) and \( M_{\Gamma}(\alpha, q) \) are all independent of the orientations on \( \Gamma \).
Theorem 4.1.7. The following identity is valid in the ring $\mathbb{Q}[[q, X_1, \cdots, X_n]]$:

$$\sum_{\alpha \in \mathbb{N}^n} M_\Gamma(\alpha, q) X^\alpha = \prod_{\alpha \in \Delta^+} \prod_{j=0}^{u_\alpha} (1 - q^j X^\alpha)^{-t_j^\alpha},$$

Proof. Lemma 4.1.5 shows that

$$E_\Gamma(\alpha, q) = \sum_{d \mid \bar{\alpha}} \bar{\alpha} \frac{1}{d} A_\Gamma\left(\frac{\alpha}{d}, q^d\right).$$

It follows that

$$\log \left( \sum_{\alpha \in \mathbb{N}^n} M_\Gamma(\alpha, q) X^\alpha \right) = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} E_\Gamma(\alpha, q) X^\alpha / \bar{\alpha}$$

$$= \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \sum_{d \mid \bar{\alpha}} \frac{1}{d} A_\Gamma\left(\frac{\alpha}{d}, q^d\right) X^\alpha$$

$$= \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \sum_{i=1}^{\infty} \frac{1}{i} A_\Gamma(\alpha, q^i) X^{i\alpha}$$

By Theorem 3.3.3, $A_\Gamma(\alpha, q) \neq 0$ if and only if $\alpha \in \Delta^+$. Thus,

$$\log \left( \sum_{\alpha \in \mathbb{N}^n} M_\Gamma(\alpha, q) X^\alpha \right) = \sum_{\alpha \in \Delta^+} \sum_{i=1}^{\infty} \frac{1}{i} A_\Gamma(\alpha, q^i) X^{i\alpha}$$

$$= \sum_{\alpha \in \Delta^+} \sum_{i=1}^{\infty} \frac{1}{i} \sum_{j=0}^{u_\alpha} t_j^\alpha q^{ij} X^{i\alpha}$$

$$= \sum_{\alpha \in \Delta^+} \sum_{j=0}^{u_\alpha} t_j^\alpha \sum_{i=1}^{\infty} \frac{1}{i} q^{ij} X^{i\alpha}$$

$$= \sum_{\alpha \in \Delta^+} \sum_{j=0}^{u_\alpha} t_j^\alpha \log \left( 1 - q^j X^\alpha \right)^{-1}$$

$$= \log \prod_{\alpha \in \Delta^+} \prod_{j=0}^{u_\alpha} \left( 1 - q^j X^\alpha \right)^{-t_j^\alpha}.$$

This implies this theorem.

Lemma 4.1.8. The following identity is valid in the ring $\mathbb{Q}[[q, X]]$:

$$\sum_{i=1}^{\infty} \frac{1}{i} \frac{X^i}{q^i - 1} = \log \prod_{i=0}^{\infty} (1 - q^i X).$$
Proof. In fact,
\[
\log \prod_{i=0}^{\infty} (1 - q^i X) = \sum_{i=0}^{\infty} \log(1 - q^i X)
\]
\[
= - \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} (q^i X)^j
\]
\[
= - \sum_{j=1}^{\infty} X^j \sum_{i=0}^{\infty} q^{ij}
\]
\[
= \sum_{j=1}^{\infty} \frac{1}{j} \frac{X^j}{q^j - 1}.
\]

For any \( \alpha \in \mathbb{N}^n \setminus \{0\} \), define \( B_{\Gamma} (\alpha, q) \in \mathbb{Q}(q) \) by the following:

\[
\log (P_{\Gamma}(X_1, \cdots, X_n, q)) = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \frac{1}{\bar{\alpha}} \frac{B_{\Gamma}(\alpha, q)}{q^{\alpha} - 1} X^\alpha.
\]

Thus,

\[
B_{\Gamma}(\alpha, q) = (q^{\bar{\alpha}} - 1) H_{\Gamma}(\alpha, q).
\]

The following corollary is an easy consequence of Theorem 4.1.6.

**Corollary 4.1.9.** The following formulae are valid:

\[
B_{\Gamma}(\alpha, q) = \sum_{d \mid \bar{\alpha}} \frac{\bar{\alpha} q^{\bar{\alpha}} - 1}{d} A_{\Gamma}\left(\frac{\alpha}{d}, q^d\right),
\]

\[
A_{\Gamma}(\alpha, q) = \frac{1}{\bar{\alpha}} \frac{q - 1}{q^{\alpha} - 1} \sum_{d \mid \bar{\alpha}} \mu(d) B_{\Gamma}\left(\frac{\alpha}{d}, q^d\right).
\]

It follows from the first formula in the above corollary that \( B_{\Gamma}(\alpha, q) \) is a polynomial in \( q \) with rational coefficients. Note that, \( (1 - q)^{-1} = 1 + q + q^2 + q^3 + \cdots \) as a formal power series. Thus \( \log (P_{\Gamma}(X_1, \cdots, X_n, q)) \) can be considered as a formal power series in \( q, X_1, \cdots, X_n \), i.e., \( \log (P_{\Gamma}(X_1, \cdots, X_n, q)) \in \mathbb{Q}[[q, X_1, \cdots, X_n]] \).

This in turn shows that \( P_{\Gamma}(X_1, \cdots, X_n, q) \in \mathbb{Q}[[q, X_1, \cdots, X_n]] \).

The following theorem is one of the main results of this thesis.

**Theorem 4.1.10.** The following identity is valid in the ring \( \mathbb{Q}[[q, X_1, \cdots, X_n]] \):

\[
\sum_{\lambda_1, \cdots, \lambda_n \in \mathbb{N}} \frac{\prod_{1 \leq i \leq j \leq n} q^{\alpha_{ij} \langle \lambda_i, \lambda_j \rangle}}{\prod_{1 \leq i \leq n} q^{\lambda_i, \lambda_i} b_{\lambda_i} (q - 1)} X_1^{\lambda_1} \cdots X_n^{\lambda_n} = \prod_{\alpha \in \Delta^+} \prod_{i=0}^{\infty} \prod_{j=0}^{t_{ij}^\alpha} (1 - q^{i+j} X^\alpha)^{t_{ij}^\alpha}.
\]
Proof. Theorem 4.1.6 shows that
\[ H_{\Gamma}(\alpha, q) = \sum_{d \mid \bar{\alpha}} \frac{1}{d} \frac{q^{\alpha}}{q^{d} - 1} A_{\Gamma}(\frac{\alpha}{d}, q^d). \]

Thus, we have
\[
\log(P_{\Gamma}(X_1, \cdots, X_n, q)) = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} H_{\Gamma}(\alpha, q) X^\alpha / \bar{\alpha} \\
= \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \sum_{d \mid \alpha} \frac{1}{d} \frac{q^{\alpha}}{q^{d} - 1} A_{\Gamma}(\frac{\alpha}{d}, q^d) X^\alpha \\
= \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} \sum_{i=1}^{\infty} \frac{1}{i(q^i - 1)} A_{\Gamma}(\alpha, q^i) X^{i\alpha}
\]

By Theorem 3.3.3, \( A_{\Gamma}(\alpha, q) \neq 0 \) if and only if \( \alpha \in \Delta^+ \). Thus,
\[
\log(P_{\Gamma}(X_1, \cdots, X_n, q)) = \sum_{\alpha \in \Delta^+} \sum_{i=1}^{\infty} \frac{1}{i(q^i - 1)} A_{\Gamma}(\alpha, q^i) X^{i\alpha} \\
= \sum_{\alpha \in \Delta^+} \sum_{i=1}^{\infty} \frac{1}{i(q^i - 1)} \sum_{j=0}^{u_\alpha} t_j^\alpha q^{ij} X^{i\alpha} \\
= \sum_{\alpha \in \Delta^+} \sum_{j=0}^{u_\alpha} t_j^\alpha \sum_{i=1}^{\infty} \frac{(q^j X^\alpha)^i}{i(q^i - 1)} \\
= \sum_{\alpha \in \Delta^+} \sum_{j=0}^{u_\alpha} t_j^\alpha \sum_{i=1}^{\infty} \log(1 - q^i q^j X^\alpha) \quad (\text{by Lemma 4.1.8}) \\
= \log \left( \prod_{\alpha \in \Delta^+} \prod_{i=0}^{\infty} \prod_{j=0}^{u_\alpha} (1 - q^{i+j} X^\alpha)^{t_j^\alpha} \right).
\]

This implies that
\[ P_{\Gamma}(X_1, \cdots, X_n, q) = \prod_{\alpha \in \Delta^+} \prod_{i=0}^{\infty} \prod_{j=0}^{u_\alpha} (1 - q^{i+j} X^\alpha)^{t_j^\alpha}. \]

The theorem is now followed from the definition of \( P_{\Gamma}(X_1, \cdots, X_n, q) \).

Now, we are in a situation to prove one of the most fascinating properties of the polynomials \( A_{\Gamma}(\alpha, q) \).

**Theorem 4.1.11.** For each \( \alpha \in \Delta^+ \), the coefficients of \( A_{\Gamma}(\alpha, q) \) are all integers, i.e., \( A_{\Gamma}(\alpha, q) \in \mathbb{Z}[q] \).

**Proof.** Suppose that the assertion of this theorem is not true. Then there must exist \( \alpha_0 \in \Delta^+ \) such that \( A_{\Gamma}(\alpha_0, q) \notin \mathbb{Z}[q] \). Furthermore we can assume that
$\alpha_0$ has minimal height, i.e., for any $\alpha \in \Delta^+$ with $ht(\alpha) < ht(\alpha_0)$, there holds $A_{\Gamma}(\alpha, q) \in \mathbb{Z}[q]$. Let $k = ht(\alpha_0)$ and let $t_j^{\alpha_0}$ be a coefficient of $A_{\Gamma}(\alpha_0, q) \notin \mathbb{Z}[q]$ which is not an integer. Let

$$U = \prod_{\alpha \in \Delta^+} \prod_{ht(\alpha) < k} \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1 - q^{i+j}X^{\alpha})^{t_j^\alpha},$$

and let $u$ be the coefficient of $q^jX^{\alpha_0}$ in $U$. Let us consider the coefficient of $q^jX^{\alpha_0}$ in the product

$$\prod_{\alpha \in \Delta^+} \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1 - q^{i+j}X^{\alpha})^{t_j^\alpha}.$$ We denote this coefficient by $w$. It can be seen that this coefficient is the same as the coefficient of $q^jX^{\alpha_0}$ in the following product:

$$U (1 - q^j X^{\alpha_0})^{t_j^\alpha}.$$ 

Recall that the following identity holds in the ring $\mathbb{Q}[[X]]$:

$$(1 - X)^{-a} = 1 + aX + \frac{a(a + 1)}{2!}X^2 + \frac{a(a + 1)(a + 2)}{3!}X^3 + \cdots$$ for all $a \in \mathbb{Q}$. Thus, the coefficient of $q^jX^{\alpha_0}$ in $(1 - q^j X^{\alpha_0})^{t_j^\alpha}$ is $-t_j^\alpha$, which is not an integer, and $w = u - t_j^\alpha$. By the assumption on $\alpha_0$, $t_j^\alpha$ are integers for all $\alpha \in \Delta^+$ with $ht(\alpha) < k$, thus $u$ must be an integer. Consequently, $w$ is not an integer.

Now, let $P_{\Gamma}(X_1, \ldots, X_n, q) = \sum_{\alpha \in \mathbb{N}_n} C^\Gamma_{\alpha}(\alpha, q)X^{\alpha}$ with $C^\Gamma_{\alpha}(\alpha, q)$ being rational functions in $q$. As already noted, $C^\Gamma_{\alpha}(\alpha, q)$ is a formal power series in $q$ with rational coefficients. We claim that they are indeed in $\mathbb{Z}[[q]]$, and thus we get a contradiction with $w$ being a strictly fractional number. It follows from the definition of $P_{\Gamma}(X_1, \ldots, X_n, q)$ that if $\alpha = (k_1, \ldots, k_n) \in \mathbb{N}^n$ then

$$C^\Gamma_{\alpha}(\alpha, q) = \sum_{\lambda_1 \vdash k_1} \cdots \sum_{\lambda_n \vdash k_n} \prod_{1 \leq i \leq j \leq n} q^{a_{ij}(\lambda_i, \lambda_j)} \prod_{1 \leq i \leq n} q^{(\lambda_i, \lambda_i)} b_{\lambda_i}(q^1).$$

From the definition of $b_{\lambda}(q)$, it is easily to see that $C^\Gamma_{\alpha}(\alpha, q)$ can be written in the following form:

$$C^\Gamma_{\alpha}(\alpha, q) = \frac{f(q)}{q^r \prod_{i=1}^{m} (q^i - 1)^{r_i}},$$
where \( f(q) \in \mathbb{Z}[q] \), and \( m, r, r_i \) are nonnegative integers, they all depend only on \( \alpha \) and \( \Gamma \). Since \( C_\Gamma(\alpha, q) \) is a formal power series in \( q \) with rational coefficients, thus as \( q \to 0 \), the limit of \( C_\Gamma(\alpha, q) \) must exist. This shows \( q^r \mid f(q) \). Let \( f(q) = q^r g(q) \). Then \( f(q) \in \mathbb{Z}[q] \). Thus

\[
C_\Gamma(\alpha, q) = \frac{g(q)}{\prod_{i=1}^m (q^{r_i} - 1)^{r_i}}.
\]

And so \( C_\Gamma(\alpha, q) \) are formal power series in \( q \) with integer coefficients, i.e., \( C_\Gamma(\alpha, q) \in \mathbb{Z}[\llbracket q \rrbracket] \). This finishes the proof.

What is really more fascinating about the polynomials \( A_\Gamma(\alpha, q) \) is the following remarkable Kac conjecture.

**Conjecture 1 (K4).** The coefficients of the polynomials \( A_\Gamma(\alpha, q) \) (\( \alpha \in \Delta^+ \)) are all non-negative.

It follows from Theorem 4.1.7 that if the above Kac conjecture is true then all coefficients of \( M_\Gamma(\alpha, q) \) (\( \alpha \in \mathbb{N}^n \)) are nonnegative. Thus, it is natural to make following conjecture:

**Conjecture 2.** The coefficients of the polynomials \( M_\Gamma(\alpha, q) \) (\( \alpha \in \mathbb{N}^n \)) are all non-negative.

The author has done a lot of calculations on the polynomials \( A_\Gamma(\alpha, q) \), \( M_\Gamma(\alpha, q) \) and \( I_\Gamma(\alpha, q) \) in various cases. All results obtained are consistent with the above conjectures.

Here, we list a few examples which are carried out by Maple. More examples can be found in Appendix A. If \( \Gamma \) is the hyperbolic graph \( \circ \overset{1}{\longrightarrow} \overset{2}{\longrightarrow} \overset{3}{\longrightarrow} \overset{4}{\longrightarrow} \overset{5}{\longrightarrow} \circ \) which consists of two nodes connected by three edges, then we have the following:

\[
\begin{align*}
A_\Gamma((1,1),q) &= q^2 + q + 1 \\
A_\Gamma((2,3),q) &= q^6 + q^5 + 3q^4 + 4q^3 + 5q^2 + 3q + 2 \\
A_\Gamma((3,6),q) &= q^{10} + q^9 + 3q^8 + 5q^7 + 8q^6 + 12q^5 + 17q^4 + 16q^3 + 14q^2 + 7q + 3 \\
M_\Gamma((1,1),q) &= q^2 + q + 2 \\
M_\Gamma((2,3),q) &= q^6 + 2q^5 + 6q^4 + 10q^3 + 16q^2 + 12q + 10 \\
M_\Gamma((3,6),q) &= 2q^{12} + 2q^{11} + 8q^{10} + 12q^9 + 26q^8 + 44q^7 + 83q^6 + 124q^5 + 180q^4 + 194q^3 + 182q^2 + 104q + 45
\end{align*}
\]
If $\Gamma$ is the hyperbolic graph $\xymatrix@C=1em{\bullet \ar@{-}[r] & \bullet \ar@{-}[r] & \bullet}$, then we have the following:

\begin{align*}
I_\Gamma((1,1), q) &= q^2 + q + 1 \\
I_\Gamma((2,3), q)) &= q^6 + q^5 + 3q^4 + 4q^3 + 5q^2 + 3q + 2 \\
I_\Gamma((3,6), q) &= q^{10} + q^9 + 3q^8 + 5q^7 + \frac{25}{6}q^6 + 12q^5 + 17q^4 + \frac{49}{3}q^3 + \frac{41}{3}q^2 + \frac{20}{3}
\end{align*}

If $\Gamma$ is the wild quiver $\xymatrix@C=1em{\bullet \ar@{-}[r] & \bullet \ar@{-}[r] & \bullet}$, then there hold:

\begin{align*}
A_\Gamma((1), q) &= q^2 \\
A_\Gamma((2), q) &= q^5 + q^3 \\
A_\Gamma((3), q) &= q^{10} + q^8 + q^7 + q^6 + q^5 + q^4 \\
A_\Gamma((4), q) &= q^{17} + q^{15} + q^{14} + 2q^{13} + q^{12} + 3q^{11} + 2q^{10} + 4q^9 + 2q^8 + 3q^7 + q^6 + q^5 \\
I_\Gamma((1), q) &= q^2 \\
I_\Gamma((2), q) &= q^5 + \frac{1}{2}q^4 + q^3 - \frac{1}{2}q^2 \\
I_\Gamma((3), q) &= q^{10} + q^8 + q^7 + \frac{1}{3}q^6 + q^5 + q^4 - \frac{1}{7}q^2 \\
I_\Gamma((4), q) &= q^{17} + q^{15} + q^{14} + 2q^{13} + q^{12} + 3q^{11} + \frac{5}{2}q^{10} + 4q^9 + \frac{2}{7}q^8 + 3q^7 + \frac{3}{2}q^6 + \\
&\quad + \frac{1}{2}q^5 - \frac{1}{4}q^4 - \frac{1}{2}q^3 \\
M_\Gamma((1), q) &= q^2 \\
M_\Gamma((2), q) &= q^5 + q^4 + q^3 \\
M_\Gamma((3), q) &= q^{10} + q^8 + 2q^7 + 2q^6 + 2q^5 + q^4 \\
M_\Gamma((4), q) &= q^{17} + q^{15} + q^{14} + 2q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 6q^9 + 5q^8 + 5q^7 + 3q^6 + q^5
\end{align*}

The above examples show that $I_\Gamma(\alpha, q)$ may have strictly fractional coefficients, but they also suggest the following conjecture:

\textbf{Conjecture 3.} If $\Gamma$ does not contain edge-loops, then the coefficients of the polynomials $I_\Gamma(\alpha, q)$ ($\alpha \in \Delta^+$) are all non-negative.
The following theorem ensures that the factorisation in Theorem 4.1.10 is unique.

**Theorem 4.1.12 (Unique Factorisation Theorem).** Let $\Delta, \bar{\Delta}$ be two subsets of $\mathbb{N}^n \setminus \{0\}$. Suppose that for each $\alpha \in \Delta$, there is $U_\alpha(q) = \sum_{j=0}^{\infty} t_\alpha^j q^j \in \mathbb{Z}[q]$, and for each $\alpha \in \bar{\Delta}$, there is $\bar{U}_\alpha(q) = \sum_{j=0}^{\infty} \bar{t}_\alpha^j q^j \in \mathbb{Z}[q]$. Furthermore suppose that the following formal identity holds:

\[
\prod_{\alpha \in \Delta} \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1 - q^{i+j} X^\alpha)^{t_\alpha^j} = \prod_{\alpha \in \bar{\Delta}} \prod_{i=0}^{\infty} \prod_{j=0}^{\infty} (1 - q^{i+j} X^\alpha)^{\bar{t}_\alpha^j}.
\]

Then $\Delta = \bar{\Delta}$, and $t_\alpha^j = \bar{t}_\alpha^j$ for all $\alpha \in \Delta, j \geq 0$.

**Proof.** Let $S_\alpha(k) = \sum_{j=0}^{k} t_\alpha^j$, and $\bar{S}_\alpha(k) = \sum_{j=0}^{k} \bar{t}_\alpha^j$. The identity in the hypothesis thus gives:

\[
(*) \quad \prod_{\alpha \in \Delta} \prod_{k=0}^{\infty} (1 - q^k X^\alpha)^{S_\alpha(k)} = \prod_{\alpha \in \bar{\Delta}} \prod_{k=0}^{\infty} (1 - q^k X^\alpha)^{\bar{S}_\alpha(k)}.
\]

Choose an $\alpha_0 \in \Delta$ such that $ht(\alpha_0)$ is minimal, and then take logarithms on $(*)$, thus we have

\[
(**) \quad \sum_{\alpha \in \Delta} \sum_{k=0}^{\infty} S_\alpha(k) \sum_{i=1}^{\infty} \frac{1}{i} q^{ik} X^{i\alpha} = \sum_{\alpha \in \bar{\Delta}} \sum_{k=0}^{\infty} \bar{S}_\alpha(k) \sum_{i=1}^{\infty} \frac{1}{i} q^{ik} X^{i\alpha}.
\]

A simple argument shows that $\alpha_0 \in \Delta$, and $ht(\alpha_0)$ is minimal in $\Delta$. Comparing the coefficients of $X^{\alpha_0}$ on both sides of $(**)$ gives

\[
\sum_{k=0}^{\infty} S_{\alpha_0}(k) q^k = \sum_{k=0}^{\infty} \bar{S}_{\alpha_0}(k) q^k.
\]

Thus, $S_{\alpha_0}(k) = \bar{S}_{\alpha_0}(k)$. A simple induction shows that $t_\alpha^0 = \bar{t}_\alpha^0$ for all $j \geq 0$.

Next, divide both sides of identity $(*)$ by $\prod_{k=0}^{\infty} (1 - q^k X^{\alpha_0})^{S_{\alpha_0}}$, we have

\[
\prod_{\alpha \in \Delta \setminus \{\alpha_0\}} \prod_{k=0}^{\infty} (1 - q^k X^\alpha)^{S_\alpha(k)} = \prod_{\alpha \in \bar{\Delta} \setminus \{\alpha_0\}} \prod_{k=0}^{\infty} (1 - q^k X^\alpha)^{\bar{S}_\alpha(k)}.
\]

Now, choose an $\alpha_1 \in \Delta \setminus \{\alpha_0\}$ such that $ht(\alpha_1)$ is minimal, and proceed the previous argument, we have $t_{\alpha_1}^0 = \bar{t}_{\alpha_1}^0$ for all $j \geq 0$. An inductive argument finishes the proof.
4.2 Examples and applications

Example I. The trivial graph

The simplest graph is a single point, it is denoted by $A_1$. A representation of $A_1$ is just a vector space. Thus every representation of dimension bigger than one is decomposable. Consequently, $A_{A_1}(1,q) = 1$ and $A_{A_1}(n,q) = 0$ for $n > 1$. Thus Theorem 4.1.10 implies the following identity:

\[
\sum_{\lambda \in \mathcal{P}} \frac{X^{\lambda}}{q^{(\lambda, \lambda)} b_{\lambda}(q^{-1})} = \prod_{i=0}^{\infty} (1 - q^i X).
\]

Let us recall the Euler identity whose proof can be found on page 19 of [M2]:

\[
\sum_{i=0}^{\infty} \frac{q^{i(i-1)/2}}{\varphi_i(q)} X^i = \prod_{i=0}^{\infty} (1 + q^i X),
\]

If we define

\[
\tau_n(q) = \frac{q^{n(n-1)/2}}{(q - 1)(q^2 - 1) \cdots (q^n - 1)},
\]

and substituting $X$ by $-X$ in the Euler identity, we get

\[
1 + \sum_{n=1}^{\infty} \tau_n(q) X^n = \prod_{i=0}^{\infty} (1 - q^i X).
\]

Thus (4.2.1) and (4.2.2) imply the following formula:

\[
\sum_{\lambda \vdash n} q^{-\langle \lambda, \lambda \rangle} b_{\lambda}(q^{-1}) = \tau_n(q).
\]

This is equivalent to the following formula due to Macdonald:

\[
\sum_{\lambda \vdash n} q^{\langle \lambda, \lambda \rangle} b_{\lambda}(q) = \frac{q^n}{\varphi_n(q)}.
\]

This formula does not appear explicitly in Macdonald’s book, but it is an easy consequence of the identity in Example 1 on page 117 of [M2].

Formula (4.2.3) has a good application for counting the numbers of nilpotent matrices over finite fields.
Proposition 4.2.1. The number of nilpotent $n \times n$ matrices over $\mathbb{F}_q$ of is equal to $q^{n(n-1)}$.

Proof. Let $X$ be the set of nilpotent matrices over $\mathbb{F}_q$ of degree $n$. Then the group $G = GL(n, \mathbb{F}_q)$ acts on $X$ by conjugation. We claim that the orbits in $X$ are in one-to-one correspondence with the partitions of $n$. Let $J(0)$ be the Jordan block matrix of size $i$ with 0 as its eigenvalue. The Jordan Normal Theorem implies that every $M \in X$ is conjugate over $\bar{\mathbb{F}}_q$ to a matrix of the form $J_{\lambda_1}(0) \oplus J_{\lambda_2}(0) \oplus \cdots \oplus J_{\lambda_k}(0)$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. A permutation of $\lambda_1, \lambda_2, \cdots, \lambda_k$ will result a similar matrix. Thus $\lambda_1, \lambda_2, \cdots, \lambda_k$ give rise to a unique partition of $n$. Since all entries of $J_{\lambda_1}(0) \oplus J_{\lambda_2}(0) \oplus \cdots \oplus J_{\lambda_k}(0)$ live in $\mathbb{F}_q$, Lang’s theorem implies that $M$ and $J_{\lambda_1}(0) \oplus J_{\lambda_2}(0) \oplus \cdots \oplus J_{\lambda_k}(0)$ are indeed conjugate over $\mathbb{F}_q$. For a partition $\lambda = (\lambda_1, \lambda_2, \cdots) \vdash n$, we use $J_\lambda$ to denote the matrix $J_{\lambda_1}(0) \oplus J_{\lambda_2}(0) \oplus \cdots$. Let $C_G(J_\lambda)$ be the centraliser of $J_\lambda$, i.e., $C_G(J_\lambda) = \{ g \in G \mid g^{-1} J_\lambda g = J_\lambda \}$. From the proof of Lemma 2.4.9, we know that $|C_G(J_\lambda)| = q^{(\lambda, \lambda)} b(\lambda(q^{-1})$. And so,

$$|X| = \sum_{\lambda \vdash n} \frac{|G|}{|C_G(J_\lambda)|} = |G| \sum_{\lambda \vdash n} \frac{1}{q^{(\lambda, \lambda)} b(\lambda(q^{-1})}.$$ 

It follows from (4.2.3) that

$$|X| = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1}) \tau_n(q) = q^{n(n-1)}.$$

Example II. The Jordan graph

The Jordan graph is the simplest graph of tame type. It consists of a single node plus a single loop, it is denoted by $\tilde{A}_0$. It follows from the Jordan Normal Form Theorem, that $A_{\tilde{A}_0}(n, q) = q$ for $n \geq 1$. Thus Theorem 4.1.10 shows that

$$\sum_{\lambda \in \mathcal{P}} \frac{X^{\lambda}}{b(\lambda(q^{-1})} = \prod_{i=1}^\infty \prod_{j=1}^\infty (1 - q^i X^j).$$

And Theorem 4.1.7 shows that

$$1 + \sum_{i=1}^\infty M_{\tilde{A}_0}(i, q) X^i = \prod_{i=1}^\infty (1 - q X^i)^{-1}.$$ 

It follows that

$$M_{\tilde{A}_0}(n, q) = \sum_{\lambda \vdash n} q^{l(\lambda)},$$

where $l(\lambda)$ is the length of $\lambda$, i.e., $l(\lambda) = \sum_{i \geq 0} n_i$ if $\lambda = (1^{n_1} 2^{n_2} 3^{n_3} \cdots)$. This result was obtained in an earlier paper of the author, see Appendix B.
Proposition 4.2.2. The following formal identity holds:

\[
\prod_{i=1}^{\infty} (1 - qX^i) = \prod_{i=1}^{\infty} (1 - X^i)^{G(i,q)},
\]

where \(G(n, q) = \frac{1}{n} \sum_{d \mid n} \phi\left(\frac{n}{d}\right) q^d\), and where \(\phi\) is the classical Euler function, i.e., \(\phi(n)\) is the number of numbers up to \(n\) which are relatively prime to \(n\).

Proof. In fact,

\[
\log \left( \prod_{i=1}^{\infty} (1 - qX^i) \right) = -\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{k} (qX^i)^k = -\sum_{n=1}^{\infty} \left( \sum_{r \mid n} \frac{1}{r} q^r \right) X^n.
\]

Recall that the Euler function \(\phi\) has the following property:

\[
\sum_{d \mid n} \phi(d) = n.
\]

Thus, we have

\[
\log \left( \prod_{i=1}^{\infty} (1 - X^i)^{G(i,q)} \right) = -\sum_{i=1}^{\infty} G(i, q) \sum_{k=1}^{\infty} \frac{1}{k} X^{ik}
\]

\[
= -\sum_{n=1}^{\infty} \left( \sum_{d \mid n} \frac{1}{d} G\left(\frac{n}{d}, q\right) \right) X^n
\]

\[
= -\sum_{n=1}^{\infty} \left( \sum_{d \mid n} \frac{1}{d} \sum_{r \mid n/d} \phi\left(\frac{n}{dr}\right) q^r \right) X^n
\]

\[
= -\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{r \mid n} \sum_{d \mid n/r} \phi\left(\frac{n}{dr}\right) q^r \right) X^n
\]

\[
= -\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{r \mid n} q^r \right) X^n
\]

\[
= -\sum_{n=1}^{\infty} \left( \sum_{r \mid n} \frac{1}{r} q^r \right) X^n.
\]

Thus,

\[
\log \left( \prod_{i=1}^{\infty} (1 - qX^i) \right) = \log \left( \prod_{i=1}^{\infty} (1 - X^i)^{G(i,q)} \right).
\]

This proves the proposition.
Now equation (4.2.5) and the above proposition imply that

\[ I_{\tilde{A}_0}(n, q) = \frac{1}{n} \sum_{d \mid n} \phi\left(\frac{n}{d}\right) q^d. \]

Thus, the \( I_{\tilde{A}_0}(n, q) \) are polynomials in \( q \) with nonnegative rational coefficients.

**Example III. The graph \( \circ\cdots\circ \)**

The graph \( \circ\cdots\circ \) is denoted by \( A_2 \). The root system associated with \( A_2 \) is \( \{ \pm(1, 0), \pm(0, 1), \pm(1, 1) \} \). It follows from Theorem 3.3.3 and Theorem 4.1.7 that

\[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} M_{A_2}((i, j), q) X^i Y^j = (1 - X)^{-1}(1 - Y)^{-1}(1 - XY)^{-1}, \]

This implies that \( M_{A_2}((i, j), q) = \min(i, j) \). Theorem 4.1.10 implies that

\[ \sum_{\lambda, \mu \in P} q^{\langle \lambda, \mu \rangle} X^{\|\lambda\|} Y^{\|\mu\|} \frac{b_\lambda(q^{-1}) b_\mu(q^{-1})}{b_{\lambda + (\mu, \mu)}(q^{-1})} = \prod_{i=0}^{\infty} (1 - q^i X)(1 - q^i Y)(1 - q^i XY). \]

If \( q \) is substituted by \( XY \) in the second identity, then the right hand side appears in the Jacobi triple product identity. So an apparently new formula is obtained:

\[ \sum_{\lambda, \mu \in P} \frac{(XY)^{\langle \lambda, \mu \rangle - \langle \lambda, \lambda \rangle - \langle \mu, \mu \rangle}}{b_\lambda(X^{-1}Y^{-1}) b_\mu(X^{-1}Y^{-1})} X^{\|\lambda\|} Y^{\|\mu\|} = \sum_{m \in \mathbb{Z}} (-1)^m X^{m(m-1)/2} Y^{-m(m+1)/2}. \]

Now, (4.2.7) and (4.2.2) imply that

\[ \sum_{\lambda=\mu=\cdots=n} \frac{q^{\langle \lambda, \mu \rangle - \langle \lambda, \lambda \rangle - \langle \mu, \mu \rangle}}{b_\lambda(q^{-1}) b_\mu(q^{-1})} = \sum_{i=0}^{\min(m,n)} \tau_i(q) \tau_{m-i}(q) \tau_{n-i}(q). \]

**Example IV. The Kronecker graph**

The graph \( \circ\cdots\circ \) is called the Kronecker graph, it is denoted by \( \tilde{A}_1 \). Recall from Section 3.3 that the real roots of the root system associated with \( \tilde{A}_1 \) are of the forms \( (i, i+1) \) and \( (i, i-1) \) \( (i \in \mathbb{Z}) \), while the imaginary roots are of the forms \( (i, i) \) \( (i \in \mathbb{Z}, i \neq 0) \). It follows from the classification theorem of representations of tame quivers that \( A_{\tilde{A}_1}(i, i) = q + 1 \). Thus Theorem 4.1.7 and Theorem 4.1.10 amount to the following:

\[ \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} M_{\tilde{A}_1}((i, j), q) X^i Y^j = \prod_{j=1}^{\infty} (1 - X^j Y^{j-1})^{-1}(1 - X^{j-1} Y^j)^{-1}(1 - X^j Y^j)^{-1}(1 - q X^j Y^j)^{-1}, \]
\[
\sum_{\lambda, \mu \in P} q^{2\langle \lambda, \mu \rangle} X^{[\lambda]} Y^{[\mu]} \\
= \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} (1 - q^{i-1} X^j Y^{j-1})(1 - q^{i-1} X^{j-1} Y^j)(1 - q^{i-1} X^j Y^j)(1 - q^i X^j Y^j).
\]

Let \( q \to 0 \) in the last identity, then the right hand side appears in the Jacobi triple identity. Thus the following holds:

\[
\lim_{q \to 0} \sum_{\lambda, \mu \in P} q^{2\langle \lambda, \mu \rangle} X^{[\lambda]} Y^{[\mu]} = \sum_{m \in \mathbb{Z}} (-1)^m X^{m(m-1)/2} Y^{m(m+1)/2}.
\]

Now equation (4.2.9) and Proposition 4.2.2 imply that

\[
\prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{-I_{\tilde{A}_1}((\alpha, q)}
= \prod_{j=1}^{\infty} (1 - X^j Y^{j-1})^{-1}(1 - X^{j-1} Y^j)^{-1}(1 - X^j Y^j)^{-1}(1 - qX^j Y^j)^{-1}
= \prod_{j=1}^{\infty} (1 - X^j Y^{j-1})^{-1}(1 - X^{j-1} Y^j)^{-1}(1 - X^j Y^j)^{-G(j, q)-1}
\]

It follows that

\[
I_{\tilde{A}_1}((m, n), q) = \begin{cases} 1, & \text{if } |m-n| = 1, \\ G(m, q) + 1, & \text{if } |m-n| = 0, \\ 0, & \text{if } |m-n| > 1. \end{cases}
\]

Thus, the \( I_{\tilde{A}_1}((m, n), q) \) are polynomials in \( q \) with nonnegative rational coefficients.

### 4.3 Generalised Weyl-Macdonald-Kac denominator identity

First, let us review a little history on denominator identities. The denominator formula was first proved by H. Weyl in the 1930s, which is a consequence of specialising the Weyl character formula upon the trivial module. Let \( \mathfrak{g} \) be a finite dimensional complex simple Lie algebra, \( \Delta \) be the corresponding reduced root system, \( W \) its Weyl group. Then the Weyl denominator formula can be written as follows (see page 46, [M2]):

\[
\sum_{w \in W} (-1)^{l(w)} e^{\rho - w(\rho)} = \prod_{\alpha \in \Delta^+} (1 - e^{\alpha}),
\]
where $l$ is the length function on $W$, and $\rho$ is the half sum of positive roots in $\Delta$, and the $e$'s are formal exponentials. Let $\{\alpha_1, \cdots, \alpha_n\}$ be a simple system in $\Delta$. For any $i$, $1 \leq i \leq n$, we set $e^{\alpha_i} = X_i$ and for any $\alpha = k_1\alpha_1 + \cdots + k_n\alpha_n \in \Delta$, we set $X^\alpha = X_1^{k_1} \cdots X_n^{k_n}$. Then the Weyl denominator formula can be written in the following polynomial form:

$$
\sum_{w \in W} (-1)^{l(w)} X^{\rho - w(\rho)} = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{\text{mult}(\alpha)}.
$$

It can be shown that, for any $w \in W$, $\rho - w(\rho)$ is equal to the sum of positive roots mapped into negative roots by $w^{-1}$. This summation is denoted by $s(w)$ in this thesis. The example of the Weyl denominator formula for $A_2$ is as follows:

$$
1 - x - y + x^2 y + xy^2 - x^2 y^2 = (1 - x)(1 - y)(1 - xy).
$$

In 1972, Macdonald introduced affine root systems and obtained a family of complicated identities in a remarkable paper [M1]. The simplest example of Macdonald’s identities which is for $\tilde{A}_1$ is the Jacobi triple product identity. Soon after, Kac and Moody independently recognised that Macdonald’s identities are the precise analogs of “Weyl’s denominator formulas” for the infinite dimensional contragredient Lie algebra of finite growth. Kac also obtained the character formula and denominator identity for general Kac-Moody algebras in [K1].

In this section we suppose that $\Gamma$ does not contain edge-loops. Thus $a_{ii} = 0$. Therefore, $\Gamma$ defines a symmetric generalised Cartan matrix $(c_{ij})$ by setting $c_{ii} = 2$ and $c_{ij} = -a_{ij}$. A Kac-Moody algebra is thus defined in term of this matrix by the usual Serre relations, for details see [K5]. Thus, $g$ admits a root space decomposition, i.e., $g = (\bigoplus_{\alpha \in \Delta} g_{\alpha}) \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the Cartan sub-algebra of $g$ and $\Delta$ is the root system associated with $\Gamma$ introduced in Section 3.3. The multiplicity of $\alpha$, denoted by $\text{mult}(\alpha)$, is defined as the dimension of the root space $g_{\alpha}$ which corresponds to $\alpha$. The Kac denominator identity can be written as follows (see [K2]):

$$
(4.3.1) \quad \sum_{w \in W} (-1)^{l(w)} X^{s(w)} = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{\text{mult}(\alpha)},
$$

where $s(w)$ equals the sum of positive roots mapped into negative roots by $w^{-1}$. 
Now we deduce a formula for \( s(w) \) and show that \( s(w) \) satisfies certain quadratic equation. Recall that \( r_{\alpha_i} \) is a fundamental reflection defined in Section 3.3. For simplicity, \( r_{\alpha_i} \) is denoted by \( r_i \) temporarily.

**Lemma 4.3.1.** Let \( w = r_{i_1} r_{i_2} \cdots r_{i_k} \in W \) be a reduced expression of \( w \). Then we have \( s(w) = \alpha_{i_1} + r_{i_1}(\alpha_{i_2}) + r_{i_1} r_{i_2}(\alpha_{i_3}) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_{k-1}}(\alpha_{i_k}) \).

**Proof.** Indeed, we claim that

\[
w(\Delta^-) \cap \Delta^+ = \{\alpha_{i_1}, r_{i_1}(\alpha_{i_2}), r_{i_1} r_{i_2}(\alpha_{i_3}), \ldots, r_{i_1} r_{i_2} \cdots r_{i_{k-1}}(\alpha_{i_k})\},
\]

and the lemma follows.

Suppose \( \beta \in w(\Delta^-) \cap \Delta^+ \), then \( w^{-1}(\beta) < 0 \). Thus there exists \( t, 1 \leq t \leq k \), such that \( r_{i_t} \cdots r_{i_2} r_{i_1}(\beta) > 0 \) for all \( s, 1 \leq s \leq t - 1 \), and \( r_{i_t} \cdots r_{i_2} r_{i_1}(\beta) < 0 \). Since the only positive root which is mapped into negative root by \( r_{i_t} \) is \( \alpha_{i_t} \), thus \( r_{i_{t-1}} \cdots r_{i_2} r_{i_1}(\beta) = \alpha_{i_t} \). This implies that \( \beta = r_{i_1} r_{i_2} \cdots r_{i_{t-1}}(\alpha_{i_t}) \).

Lemma 3.11 of [K5] asserts that if \( w = r_{i_1} r_{i_2} \cdots r_{i_k} \in W \) is a reduced expression of \( w \), then \( w(\alpha_{i_k}) < 0 \). Since \( w^{-1} = r_{i_k} \cdots r_{i_2} r_{i_1} \) is also in reduced expression, thus \( w^{-1}(\alpha_{i_1}) < 0 \), and so \( \alpha_{i_1} \in w(\Delta^-) \cap \Delta^+ \). Since \( w^{-1} r_{i_1} = r_{i_k} \cdots r_{i_3} r_{i_2} \) is in reduced expression, thus \( w^{-1} r_{i_1}(\alpha_{i_2}) < 0 \). This implies that \( r_{i_1}(\alpha_{i_2}) \in w(\Delta^-) \cap \Delta^+ \). A trivial induction shows that \( r_{i_1} r_{i_2} \cdots r_{i_{j-1}}(\alpha_{i_j}) \in w(\Delta^-) \cap \Delta^+ \) for all \( j, 1 \leq j \leq k \). It still needs to show that if \( s \neq t \) then \( r_{i_1} \cdots r_{i_s}(\alpha_{i_{s+1}}) \neq r_{i_1} \cdots r_{i_t}(\alpha_{i_{t+1}}) \). Suppose this is not true and we can assume that \( 1 \leq s < t \leq k - 1 \). It follows that \( \alpha_{i_{s+1}} = r_{i_{s+1}} \cdots r_{i_t}(\alpha_{i_{t+1}}) \), and hence, \( r_{i_{s+2}} \cdots r_{i_t} r_{i_{t+1}}(\alpha_{i_{t+1}}) = \alpha_{i_{s+1}} < 0 \). This contradicts the expression of \( r_{i_{s+2}} \cdots r_{i_t} r_{i_{t+1}} \) being reduced.

**Corollary 4.3.2.** The following statements are true:

1. \( s(r_i w) = \alpha_i + r_i s(w) \), for any \( w \in W \), \( 1 \leq i \leq n \);
2. For any \( w = r_{i_1} r_{i_2} \cdots r_{i_k} \in W \), not necessarily in reduced expression, there holds \( s(w) = \alpha_{i_1} + r_{i_1}(\alpha_{i_2}) + r_{i_1} r_{i_2}(\alpha_{i_3}) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_{k-1}}(\alpha_{i_k}) \).

Let \( Q_\Gamma \) be the quadratic form defined by:

\[
Q_\Gamma(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2 - \sum_{i<j} a_{ij} x_i x_j - \sum_{i=1}^{n} x_i.
\]

Notice that \( Q_\Gamma(\alpha) = 0 \) if and only if \( \langle \alpha, \alpha \rangle = \text{ht}(\alpha) \).
Theorem 4.3.3. If $\alpha = s(w)$ for some $w \in W$, then $Q_\Gamma(\alpha) = 0$.

Proof. We prove this by induction on the length of $w$. The case of $w = 1$ is trivial. Suppose that $\alpha = s(w) = \sum_{i=1}^{n} a_i \alpha_i$ satisfying $Q_\Gamma(\alpha) = 0$. Thus $\langle \alpha, \alpha \rangle = ht(\alpha) = \sum_{i=1}^{n} a_i$. Let $\beta = s(r_\alpha, w)$. By Corollary 4.3.2, $\beta = \alpha_i + r_i s(w) = \alpha_i + r_i(\alpha)$. Since

$$r_i(\alpha) = r_i \left( \sum_{i=1}^{n} a_i \alpha_i \right) = \sum_{j \neq i} a_j \alpha_j + \left( -a_i + \sum_{j \neq i} a_j b_{ij} \right) \alpha_i,$$

thus,

$$\langle \beta, \beta \rangle = \langle \alpha_i + r_i(\alpha), \alpha_i + r_i(\alpha) \rangle$$

$$= \langle \alpha_i, \alpha_i \rangle + 2 \langle \alpha_i, r_i(\alpha) \rangle + \langle r_i(\alpha), r_\alpha(\alpha) \rangle$$

$$= 1 + 2 \langle \alpha_i, \sum_{j \neq i} a_j \alpha_j + \left( -a_i + \sum_{j \neq i} a_j b_{ij} \right) \alpha_i \rangle + \langle \alpha, \alpha \rangle$$

$$= 1 - \sum_{j \neq i} a_j b_{ij} + 2 \left( -a_i + \sum_{j \neq i} a_j b_{ij} \right) + \sum_{j=1}^{n} a_j$$

$$= 1 - a_i + \sum_{j \neq i} a_j b_{ij} + \sum_{j \neq i} a_j$$

$$= ht(\beta).$$

This finishes the proof.

Conjecture 4 ([K4]). Let $\Gamma$ be a connected graph without edge-loops, $\Delta$ its associated root system. Then for any $\alpha \in \Delta^+$ the multiplicity of $\alpha$ is equal to the constant term of $A_\Gamma(\alpha, q)$, that is $\text{mult}(\alpha) = A_\Gamma(\alpha, 0)$.

In case of $\Gamma$ is of finite or tame type, the above Kac conjecture can be easily verified by using the classification theorems. Now, let $q$ approach $0$ in the identity of Theorem 4.1.10. Then we have

$$\lim_{q \to 0} \left( \sum_{\lambda_1, \ldots, \lambda_n \in \mathcal{P}} \prod_{1 \leq i \leq j \leq n} q^{a_{ij}(\lambda_i, \lambda_j)} \frac{\prod_{1 \leq i \leq n} q^{b_{\lambda_i}(\lambda_i)} (q^{-1})^X_{1}^{\lambda_1} \ldots X_{n}^{\lambda_n}}{\prod_{1 \leq i \leq n} q^{b_{\lambda_i}(\lambda_i)} (q^{-1})^X_{1}^{\lambda_1} \ldots X_{n}^{\lambda_n}} \right) = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{A_\Gamma(\alpha, 0)}.$$

Thus, if the above Kac Conjecture is true then the identity of Theorem 4.1.10 is indeed a $q$-analogue of the Kac denominator identity. In view of Kac’s denominator identity, Conjecture 4 is equivalent to the following identity:

$$\lim_{q \to 0} \left( \sum_{\lambda_1, \ldots, \lambda_n \in \mathcal{P}} \prod_{1 \leq i \leq j \leq n} q^{a_{ij}(\lambda_i, \lambda_j)} \frac{\prod_{1 \leq i \leq n} q^{b_{\lambda_i}(\lambda_i)} (q^{-1})^X_{1}^{\lambda_1} \ldots X_{n}^{\lambda_n}}{\prod_{1 \leq i \leq n} q^{b_{\lambda_i}(\lambda_i)} (q^{-1})^X_{1}^{\lambda_1} \ldots X_{n}^{\lambda_n}} \right) = \sum_{w \in W} (-1)^{(w)} X^{s(w)}.$$
The function $K$ (named in honour of Kostant) is defined by the following identity (see [K5], page 122):

$$\prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{-\text{mult}(\alpha)} = \sum_{\alpha \in \mathbb{N}^n} K(\alpha)X^\alpha.$$ 

Suppose that the above Kac conjecture is true. If we let $q = 0$ in the identity of Theorem 4.1.7, then we have the following:

$$\sum_{\alpha \in \mathbb{N}^n} M_\Gamma(\alpha, 0)X^\alpha = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{-t_0^\alpha} = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{-\text{mult}(\alpha)}.$$

Thus $M_\Gamma(\alpha, q)$ can be thought as a $q$-analogue of the Kostant function $K$ at least in the case of finite type or tame type.

For any $\alpha \in \mathbb{Z}^n$, we define $D_\Gamma(\alpha) \in \mathbb{Z}$ by the following:

$$\prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{A_\Gamma(\alpha, 0)} = \sum_{\alpha \in \mathbb{Z}^n_+} D_\Gamma(\alpha)X^\alpha.$$ 

Thus, $A_\Gamma(\alpha, 0) = \text{mult}(\alpha)$ for all $\alpha \in \Delta^+$ if and only if the following identity holds:

$$\sum_{\alpha \in \mathbb{Z}^n_+} D_\Gamma(\alpha)X^\alpha = \sum_{w \in W} (-1)^{l(w)}X^{s(w)}.$$ 

The following proposition gives a partial solution to Conjecture 4.

**Proposition 4.3.4.** Let $\Gamma$, $\Delta$ and $W$ be as above. Then $D_\Gamma(\alpha) = (-1)^{l(w)}$ if $\alpha = s(w)$ for some $w \in W$.

**Proof.** First, let us define an action of $W$ on the ring $\mathbb{Z}[X_1^\pm, \ldots, X_n^\pm]$ by setting $w \cdot X^\alpha = X^{w(\alpha)}$. Theorem 3.3.3 shows that for any $w \in W$, $A(\alpha, q) = A(w(\alpha), q)$ as long as $w(\alpha)$ is a positive root. As $r_\alpha$ maps $\alpha_i$ into $-\alpha_i$ and permutes other positive roots, it follows that

$$(1 - X_i^{-1}) \cdot \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{A_\Gamma(\alpha, 0)}$$

is invariant under $r_\alpha$. Thus,

$$(1 - X_i^{-1}) \left( \sum_{\alpha \in \mathbb{Z}^n_+} D_\Gamma(\alpha)X^\alpha \right)$$
is invariant under $r_{\alpha_i}$. So we have

$$
(1 - X_i) \left( \sum_{\alpha \in \mathbb{Z}_n^+} D_{\Gamma}(\alpha) X^{r_{\alpha_i}(\alpha)} \right) = (1 - X_i^{-1}) \left( \sum_{\alpha \in \mathbb{Z}_n^+} D_{\Gamma}(\alpha) X^\alpha \right).
$$

It follows that

$$
\sum_{\alpha \in \mathbb{Z}_n^+} D_{\Gamma}(\alpha) X^{r_{\alpha_i}(\alpha) + \alpha_i} = - \sum_{\alpha \in \mathbb{Z}_n^+} D_{\Gamma}(\alpha) X^\alpha.
$$

Comparing the coefficients of $X^\alpha$ on both sides, we get

$$
D_{\Gamma}(\alpha) = -D_{\Gamma}(r_{\alpha_i}(\alpha) + \alpha_i).
$$

The Weyl group $W$ acts naturally on the space $\mathbb{Z}^n$, we define a new action of $W$ on $\mathbb{Z}^n$. For any $w \in W$, $\alpha \in \mathbb{Z}^n$, the new action of $w$ on $\alpha$ is denoted by $w(\alpha)$. Note that $W$ is generated by simple reflections $r_{\alpha_1}, \ldots, r_{\alpha_n}$, we define $r_{\alpha_i}(\alpha) = r_{\alpha_i}(\alpha) + \alpha_i$. It is easy to verify that this actually defines an action of $W$ on $\mathbb{Z}^n$. Thus the previous formula can be written as

$$
D_{\Gamma}(\alpha) = -D_{\Gamma}(r_{\alpha_i}(\alpha)).
$$

By induction on the length of $w$, we have the following:

(*)

$$
D_{\Gamma}(\alpha) = (-1)^{l(w)} D_{\Gamma}(w(\alpha)).
$$

This shows that $D_{\Gamma}(\alpha)$ is constant on each orbit up to a sign. By using Lemma 4.3.2, an inductive argument implies that

$$
w(\alpha) = w(\alpha) + s(w).
$$

Note that $w$ is a linear transformation on $\mathbb{Z}$. Replacing $\alpha$ by 0 in the above identity shows that $s(w) = w(0)$. Thus identity (*) shows that

$$
D_{\Gamma}(0) = (-1)^{l(w)} D_{\Gamma}(w(0)) = (-1)^{l(w)} D_{\Gamma}(s(w)).
$$

Since $D_{\Gamma}(0) = 1$, $D_{\Gamma}(s(w)) = (-1)^{l(w)}$. This finishes the proof.
Chapter 5

Fields of definition for representations of algebras over perfect fields

For finite fields this chapter provides proofs not using Lang’s theorem of the theory of Section 2.2. However, the fact that finite division algebras are commutative is still needed. As seen in the example of Section 2.2, the rational quaternion algebra provides a counter-example to strong theorems about the existence of unique minimal fields of definition in characteristic 0. The following result is proved as Corollary 5.8.

**Corollary 5.8.** Let $\mathbb{K}$ be a number field (that is a finite extension of $\mathbb{Q}$) contained in a fixed algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. Let $\Gamma$ be a $\mathbb{K}$-algebra of finite dimension as a vector space and let $M$ be a finitely generated indecomposable $\Gamma \otimes \mathbb{K} \overline{\mathbb{Q}}$-module. Let $M^\dagger$ be the indecomposable $\Gamma$-module such that $M^\dagger \otimes \mathbb{K} \overline{\mathbb{Q}}$ contains $M$ as a direct summand and let $D = \frac{\text{End}_\Gamma(M^\dagger)}{\text{rad}(\text{End}_\Gamma(M^\dagger))}$ with centre $\Lambda$. Then $M^s = M \oplus \cdots \oplus M$ has a unique minimal field of definition $\Lambda_1 \subset \overline{\mathbb{Q}}$ which is isomorphic to $\Lambda$, where $s = \sqrt{\dim_\Lambda D}$.

**The $*$--operator.** Let $\mathbb{K}$ be a perfect field, $\overline{\mathbb{K}}$ be a fixed algebraic closure of $\mathbb{K}$. Let $\Gamma$ be an algebra of finite dimension over $\mathbb{K}$ with vector space basis $\{E_1, \cdots, E_d\}$, which satisfies

$$E_i E_j = \sum_{k=1}^d a_{ijk} E_k, \quad a_{ijk} \in \mathbb{K}, \quad 1 \leq i, j, k \leq d.$$ 

Recall that a field $\mathbb{K}$ is said to be perfect if every irreducible polynomial in $\mathbb{K}[x]$ is separable. Thus, all fields of characteristic zero and finite fields are perfect. Let $\mathbb{F}$ be any finite extension of $\mathbb{K}$. Theorem 4.25 on page 279 of [J] shows that every finite separable extension of a field has a primitive element. Thus $\mathbb{F} = \mathbb{K}[c]$, where
\( \epsilon \) is a root of the monic irreducible polynomial
\[
\phi(x) = x^r + c_1 x^{r-1} + \cdots + c_r, \quad c_i \in K.
\]

Thus there is an embedding \( \iota : F \to \mathcal{M}_r(K) \) of \( K \)-algebras induced by \( \epsilon \to J(\phi) \), where \( J(\phi) \) is the companion matrix of \( \phi \) defined in Section 3.4. Now we can assign to each \( m \times m \) matrix \( N \) with entries in \( F \) the \( mr \times mr \) matrix \( N^* \) with entries in \( K \) which is made up of \( r^2 \) square blocks, each of which is the image under \( \iota \) of the corresponding element of \( F \). For example, let \( K = \mathbb{Q}, F = \mathbb{Q}[\sqrt{2}], \) then \( \phi(x) = x^2 - 2 \)
and \( J(\phi) = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \).

If \( N = \begin{pmatrix} 2 + \sqrt{2} & -2 \\ 4 - \sqrt{2} & 2 \end{pmatrix} \), then \( N^* = \begin{pmatrix} 2 & 1 & 3 & -2 \\ 2 & 2 & -4 & 3 \\ 4 & -1 & 2 & -2 \\ -2 & 4 & -4 & 2 \end{pmatrix} \).

Let \( M = (M_1, \ldots, M_d) \) be a matrix representation of \( \Gamma \) over \( F \) of degree \( m \), that means that \( M_1, \ldots, M_d \in \mathcal{M}_m(F) \) and satisfy the equations:
\[
M_i M_j = \sum_{k=1}^{d} a_{ijk} M_k, \quad 1 \leq i, j \leq d.
\]
The assignment \( M_i \to M_i^* \) thus induces a representation \( M^* = (M_1^*, \ldots, M_d^*) \) of \( \Gamma \) over \( K \) which is of degree \( mr \).

Let \( \phi(x) = (x - \epsilon_1)(x - \epsilon_2) \cdots (x - \epsilon_r) \) in \( \overline{K} \) with \( \epsilon_1 = \epsilon \). Since \( K \) is a perfect field, \( f(x) \) is separable and hence the \( \epsilon_i \)'s are mutually distinct. Thus there exists an invertible matrix \( X \) with entries in \( \overline{K} \) (or rather in \( \overline{F} \), the Galois closure of \( F \) contained in \( \overline{K} \)) such that \( X^{-1}J(\phi)X = \text{diag} (\epsilon_1, \ldots, \epsilon_n) \). Let \( N = (a_{ij}) \) be an \( n \times n \) matrix with entries in \( F \). As \( F = \mathbb{K}[\epsilon] \), each \( a_{ij} \) can be written as \( f_{ij}(\epsilon) \), where \( f_{ij} \) is a polynomial over \( K \). Thus \( a_{ij}^* = \iota(a_{ij}) = f_{ij}(J(\phi)) \).
And so,
\[
X^{-1} a_{ij}^* X = X^{-1} f_{ij}(J(\phi)) X = f_{ij}(X^{-1} J(\phi) X) = f_{ij}(\text{diag} (\epsilon_1, \ldots, \epsilon_n)) = \text{diag} (f_{ij}(\epsilon_1), \ldots, f_{ij}(\epsilon_n)).
\]
Let $A_{ij} = \text{diag}(f_{ij}(\epsilon_1), \cdots, f_{ij}(\epsilon_n))$, $1 \leq i, j \leq n$. Then we have

$$
\begin{pmatrix}
X^{-1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & X^{-1}
\end{pmatrix}
\begin{pmatrix}
a_{11}^* & \cdots & a_{1n}^* \\
\vdots & \ddots & \vdots \\
a_{n1}^* & \cdots & a_{nn}^*
\end{pmatrix}
\begin{pmatrix}
X & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & X
\end{pmatrix}
= \begin{pmatrix}
X^{-1}a_{11}^*X & \cdots & X^{-1}a_{1n}^*X \\
\vdots & \ddots & \vdots \\
X^{-1}a_{n1}^*X & \cdots & X^{-1}a_{nn}^*X
\end{pmatrix}
= \begin{pmatrix}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{pmatrix},
$$

which is similar to the following matrix:

$$
\begin{pmatrix}
f_{11}(\epsilon_1) & \cdots & f_{1n}(\epsilon_1) \\
\vdots & \ddots & \vdots \\
f_{n1}(\epsilon_1) & \cdots & f_{nn}(\epsilon_1)
\end{pmatrix}
\oplus \cdots \oplus
\begin{pmatrix}
f_{11}(\epsilon_r) & \cdots & f_{1n}(\epsilon_r) \\
\vdots & \ddots & \vdots \\
f_{n1}(\epsilon_r) & \cdots & f_{nn}(\epsilon_r)
\end{pmatrix}.
$$

Since $N^* = \begin{pmatrix} a_{11}^* & \cdots & a_{1n}^* \\
\vdots & \ddots & \vdots \\
a_{n1}^* & \cdots & a_{nn}^*
\end{pmatrix}$ and $\epsilon_1 = \epsilon$, $N^* \otimes_K \bar{K}$ contains $N$ as a direct summand. It follows that if $M$ is a matrix representation of $\Gamma$ over $F$ then $M^* \otimes_K \bar{K}$ contains $M$ as a direct summand. Thus we have proved:

**Theorem 5.1.** Any representation $M$ of $\Gamma$ over $\bar{K}$ is a direct summand of a representation of the form $M^* \otimes_K \bar{K}$, where $M^*$ is a representation of $\Gamma$ over $K$.

If $F$ is not separable over $K$ then Theorem 5.1 is not valid any more. Here is a counterexample. Let $K = F_2(a)$ be the transcendental extension in one variable $a$ of $F_2$, $F = K[b]$ where $b^2 = a$, and let $\Gamma = K[x]/(x^2 - a)$. Thus $x \rightarrow b$ induces a one-dimensional representation $\tilde{b}$ of $\Gamma$ over $F$. As $b$ has minimal polynomial $f(x) = x^2 - a$ over $K$, $\tilde{b}^* = J(f) = \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}$. Since

$$
\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}
\begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix}
\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}
= \begin{pmatrix} -b & 1 \\ a - b^2 & b \end{pmatrix}
= \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix},
$$

$\tilde{b}$ cannot be a direct summand of $\tilde{b}^*$. This example also shows that if $M$ is a simple module of $\Gamma$ then $M \otimes_K F$ is not necessarily a semi-simple module of $\Gamma \otimes_K F$.

If in Theorem 5.1 above $M$ is indecomposable, then the Krull-Schmidt theorem implies that there exists an indecomposable direct summand $M^\dagger$ of $M^*$ such that $M$ is a direct summand of $M^\dagger \otimes_K \bar{K}$. So we have the following:

**Theorem 5.2.** Any indecomposable representation $M$ of $\Gamma$ over $\bar{K}$ is a direct summand of the form $M^\dagger \otimes_K \bar{K}$, where $M^\dagger$ is an indecomposable representation of $\Gamma$ over $K$. 
The uniqueness of $M^{\dagger}$ in the above theorem is guaranteed by the following theorem.

**Theorem 5.3.** Let $E$ be an extension field of $K$, $M$ be a finitely generated $\Gamma \otimes_K E$-module. Let $N$, $N'$ be indecomposable finitely generated $\Gamma$-modules such that $M$ is a direct summand of both $N \otimes_K E$ and $N' \otimes_K E$. Then $N \cong N'$ as $\Gamma$-modules.

**Proof.** Define $\Phi : N \otimes_K E \to N' \otimes_K E$ be the identity on one summand isomorphic to $M$ and zero on the complementary summand. Define $\Psi : N' \otimes_K E \to N \otimes_K E$ analogously. Then $(\Psi \Phi)^n = \Psi \Phi \neq 0$ for all $n$. Now let $\{\phi_1, \cdots, \phi_s\}$ be a $K$-basis of $\text{Hom}_\Gamma(N, N')$ and so an $E$-basis of $\text{Hom}_{\Gamma \otimes_K E}(N \otimes_K E, N' \otimes_K E)$. As $\Psi \neq 0$ is in this vector space, $s > 0$. Likewise let $\{\psi_1, \cdots, \psi_t\}$ be a $K$-basis of $\text{Hom}_\Gamma(N', N)$: again $t > 0$. The radical $\text{rad}(\text{Hom}_\Gamma(N, N'))$ is nilpotent and so has its $n^{\text{th}}$ power $0$ for some $n$. Hence, if $\psi_i \phi_j \in \text{rad}(\text{Hom}_\Gamma(N, N'))$ for all $i, j$ then $(\Psi \Phi)^n = 0$, contradicting the above. So for some pair $i, j$ there is an invertible $\sigma : N \to N$ with $\psi_i \phi_j \sigma = 1_N$. Thus, $N \cong N'$.

**Theorem 5.4.** Let $E$ be an extension field of $K$, $M$ a representation of $\Gamma$ over $E$ and $D = \frac{\text{End}_E(M)}{\text{rad}(\text{End}_E(M))}$. Then $M$ is absolutely indecomposable if and only if $D \cong E$.

**Proof.** First suppose that $D \cong E$. Thus Theorem 7.5 and 7.9 on page 146 of [CR] imply that

$$\frac{\text{End}_E(M \otimes_E \bar{K})}{\text{rad}(\text{End}_E(M \otimes_E \bar{K}))} \cong \frac{\text{End}_E(M)}{\text{rad}(\text{End}_E(M))} \otimes_E \bar{K} \cong D \otimes_E \bar{K} \cong E \otimes_E \bar{K} \cong \bar{K}.$$ 

Thus, $M \otimes_E \bar{K}$ is indecomposable over $\bar{K}$, and so $M$ is absolutely indecomposable.

Conversely, suppose that $M$ is absolutely indecomposable. Then $D$ is a division algebra over $E$. Since $M \otimes_E \bar{K}$ is indecomposable over $\bar{K}$, $\frac{\text{End}_E(M \otimes_E \bar{K})}{\text{rad}(\text{End}_E(M \otimes_E \bar{K}))} \cong D \otimes_E \bar{K}$ is a division algebra over $\bar{K}$. Since $\bar{K}$ is algebraically closed, the only division algebra over $\bar{K}$ is $\bar{K}$ itself. Thus $D \otimes_E \bar{K} \cong \bar{K}$. This implies that $D$ is commutative and hence a field of finite extension over $E$. Suppose that $D = E[x]/(f(x))$ with $f$ irreducible and monic over $E$. Let $f(x) = (x - \alpha_1) \cdots (x - \alpha_r)$ with $\alpha_i \in \bar{K}$, and let
\[ \hat{E} = E[\alpha_1, \ldots, \alpha_r]. \] Again by Theorem 7.9 of [CR], we have

\[
\frac{\text{End}_{\hat{E}}(M \otimes_{E} \hat{E})}{\text{rad}(\text{End}_{\hat{E}}(M \otimes_{E} \hat{E})))} \cong D \otimes_{E} \hat{E} \cong E[x]/(f(x)) \cong \hat{E}[x]/(f(x))
\]

\[
\cong \hat{E}[x]/(x - \alpha_1) \oplus \cdots \oplus \hat{E}[x]/(x - \alpha_r)
\]

\[
\cong \hat{E} \oplus \cdots \oplus \hat{E} \ (r \text{ copies}).
\]

The second last identification follows from the Chinese Remainder Theorem. If \( r > 1 \) then \( M \otimes_{E} \Lambda \) would be decomposable. Thus \( r = 1 \) and hence \( D \cong E \).

**Lemma 5.5.** Let \( M \) be a finite dimensional indecomposable representation of \( \Gamma \) over \( K \), \( \Lambda \) be any finite Galois extension of \( K \) contained in \( \bar{K} \). Then

\[
M \otimes_{K} \Lambda \cong N_1^n \oplus N_2^n \oplus \cdots \oplus N_s^n,
\]

in which \( N_1, \ldots, N_s \) are pairwise non-isomorphic indecomposable representation of \( \Gamma \) over \( \Lambda \) which satisfy \( \deg N_i = \deg N_j \) and \( \text{End}_{\Lambda}(N_i) = \text{End}_{\Lambda}(N_j) \). In other words, all indecomposable direct summands of \( M \otimes_{K} \Lambda \) have the same degree, the same multiplicity, and the same endomorphism algebra.

**Proof.** Let \( \alpha \) be a primitive element of \( \Lambda \), i.e., \( \Lambda = K[\alpha] \). This \( \alpha \) has a minimal polynomial \( f(x) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_r) \in K[x] \), \( \alpha_i \in \Lambda \) with \( \alpha_1 = \alpha \). Let \( N \) be an indecomposable direct summand of \( M \otimes_{K} \Lambda \). Thus \( N \) can be written in the form

\[
N = \left( \left( f_{ij}^{(1)}(\alpha) \right), \left( f_{ij}^{(2)}(\alpha) \right), \cdots, \left( f_{ij}^{(d)}(\alpha) \right) \right),
\]

where the \( f_{ij}^{(s)} \) are polynomials in \( K[x] \) and the \( \left( f_{ij}^{(s)}(\alpha) \right) \) are square matrices of the same size.

Now we define \( N_i = \left( \left( f_{ij}^{(1)}(\alpha_i) \right), \left( f_{ij}^{(2)}(\alpha_i) \right), \cdots, \left( f_{ij}^{(d)}(\alpha_i) \right) \right) \). As \( \alpha \rightarrow \alpha_i \) induces an automorphism of \( \Lambda \), it is easy to show that \( \text{End}_{\Lambda}(N) \cong \text{End}_{\Lambda}(N_i) \). Thus each \( N_i \) is indecomposable over \( \Lambda \). The discussion in pages 71 and 72 shows that

\[
N^* \otimes_{K} \Lambda \cong N_1 \oplus N_2 \cdots \oplus N_r.
\]

By Theorem 5.3, \( N^* \cong M \) over \( K \). Thus, \( M \otimes_{K} \Lambda \) is a direct summand of \( N^* \otimes_{\Gamma} K\Lambda \).

Without loss of generality, we assume that

\[
M \otimes_{K} \Lambda \cong N_1^{a_1} \oplus N_2^{a_2} \oplus \cdots \oplus N_s^{a_s},
\]
where \( N_1, \ldots, N_s \) are mutually non-isomorphic over \( \Lambda \). So there is an \( X \in GL(n, \Lambda) \) such that

\[
X^{-1}MX = N_1^{a_1} \oplus N_2^{a_2} \oplus \cdots \oplus N_s^{a_s}.
\]

We can also assume that \( a_p = \max(a_1, \ldots, a_s) \) and \( a_s = \min(a_1, \ldots, a_s) > 0 \). Let \( \sigma \) be the automorphism of \( \Lambda \) defined by \( \alpha_p \to \alpha_s \). For a matrix \( N = (a_{ij}) \) with \( a_{ij} \in \Lambda \), we let \( N^\sigma = (\sigma(a_{ij})) \). Then we have

\[
(X^{-1}MX)^\sigma = (N_1^\sigma)^{a_1} \oplus \cdots \oplus (N_s^\sigma)^{a_s}.
\]

As \( (X^{-1}MX)^\sigma = (X^\sigma)^{-1}M^\sigma X^\sigma = (X^\sigma)^{-1}MX^\sigma \) and \( N_p^\sigma = N_s \), the multiplicity of \( N_s \) in the decomposition of \( M \otimes_{\mathbb{K}} \Lambda \) is at least \( a_p \). And so \( a_s \geq a_p \). This forces \( a_1 = a_2 = \cdots = a_s \). This finishes the proof.

**Theorem 5.6.** Let \( M \) be an indecomposable representation of \( \Gamma \) over \( \mathbb{K} \), \( D \) be the division algebra \( \frac{\text{End}_{\mathbb{K}}(M)}{\text{rad}(\text{End}_{\mathbb{K}}(M))} \). Let \( \Lambda \) be the centre of \( D \) and let \( r = \dim_{\mathbb{K}} \Lambda \), \( s = \sqrt{\dim_{\Lambda} D} \). Let \( \Lambda_1, \ldots, \Lambda_r \) denote the images of the \( r \) \( \mathbb{K} \)-embeddings of \( \Lambda \) into \( \bar{\mathbb{K}} \). Then

\[
M \otimes_{\mathbb{K}} \bar{\mathbb{K}} = N_1^s \oplus \cdots \oplus N_r^s,
\]

where \( N_1, \ldots, N_s \) are pairwise non-isomorphic indecomposable representations of \( \Gamma \) over \( \bar{\mathbb{K}} \). Furthermore, for each \( i \), \( N_i^s \) is defined over \( \Lambda_i \) and indecomposable over \( \Lambda_i \).

**Proof.** As \( \Lambda \) is a finite extension of \( \mathbb{K} \), \( \Lambda \cong \mathbb{K}[x]/(f(x)) \), where \( f(x) \) is a monic irreducible polynomial over \( \mathbb{K} \). Suppose \( f(x) = (x - \alpha_1) \cdots (x - \alpha_r) \) with \( \alpha_i \in \bar{\mathbb{K}} \). Thus we may assume that \( \Lambda_i = \mathbb{K}[\alpha_i] \). Let \( \tilde{\Lambda} = \mathbb{K}[\alpha_1, \ldots, \alpha_r] \) be the splitting field of \( f(x) \). Then \( \tilde{\Lambda} \) is Galois over \( \mathbb{K} \).

Suppose that \( f(x) = h_1(x)h_2(x) \cdots h_p(x) \) with \( h_i(x) \) irreducible over \( \Lambda_1 \) and \( h_1(x) = x - \alpha_1 \). \( D \otimes_{\mathbb{K}} \Lambda_1 \) has centre \( \Lambda \otimes_{\mathbb{K}} \Lambda_1 \cong \Lambda_1[x]/(f(x)) \). By the Chinese Remainder Theorem, we have

\[
\Lambda_1[x]/(f(x)) \cong \Lambda_1[x]/(h_1(x)) \oplus \cdots \oplus \Lambda_1[x]/(h_p(x)).
\]

Now, let \( M \otimes_{\mathbb{K}} \Lambda_1 = U_1^{a_1} \oplus \cdots \oplus U_p^{a_p} \), where \( U_1, \ldots, U_p \) are pairwise non-isomorphic indecomposable representations of \( \Gamma \) over \( \Lambda_1 \), and let \( D_i = \frac{\text{End}_{\Lambda_1}(U_i)}{\text{rad}(\text{End}_{\Lambda_1}(U_i))} \). Then

\[
D \otimes_{\mathbb{K}} \Lambda_1 \cong \mathcal{M}_{a_1}(D_1) \oplus \cdots \oplus \mathcal{M}_{a_p}(D_p).
\]
For a ring $R$, we use $C(R)$ to denote the centre of $R$. Thus we have

$$
C(D \otimes_K \Lambda_1) \cong C(D_1) \oplus \cdots \oplus C(D_p) \cong \Lambda_1[x]/(h_1(x)) \oplus \cdots \oplus \Lambda_1[x]/(h_p(x)).
$$

Without loss of generality, we may assume that $C(D_1) \cong \Lambda_1[x]/(h_1(x))$, which is isomorphic to $\Lambda_1$ because $h_1(x) = x - \alpha_1$. And so $D_1$ is a central division algebra. Thus $D_1 \otimes_{\Lambda_1} \tilde{\Lambda}$ is a central simple algebra. So there is an indecomposable representation $V_i$ of $\Gamma$ defined over $\Lambda_1$ such that $U_i \otimes_{\Lambda_1} \tilde{\Lambda} \cong V_i^b$. As $D \otimes_K \tilde{\Lambda}$ has centre $\Lambda \otimes_K \tilde{\Lambda}$, which is isomorphic to $\tilde{\Lambda} \oplus \cdots \oplus \tilde{\Lambda}$ (r copies), by Lemma 5.5, we may assume that

$$
M \otimes_K \tilde{\Lambda} \cong V_1^n \oplus \cdots \oplus V_r^n.
$$

It is easy to show that $n = a_1 b$. Thus $\deg(V_1) = \deg(M)/nr = m/na_1 r$ and therefore $\deg(U_1) = b \deg(V_1) = m/a_1 r$. As $U_1$ is defined over $\Lambda_1$ and $U_1^*$ contains $M$ as a direct summand, $\deg(U_1^*) \geq \deg(M)$, so $(m/a_1 r)r \geq m$, thus $a_1 = 1$.

Now let $P_1 = U_1$. Then $P_1^* = M$. Since $P_1$ is defined over $\Lambda_1 = K[\alpha_1]$, $P_1$ can be written as $\left( (f_{ij}^{(1)}(\alpha_1)), (f_{ij}^{(2)}(\alpha_1)), \cdots, (f_{ij}^{(d)}(\alpha_1)) \right)$, where $f_{ij}^{(s)}$ are polynomials over $K$. Let $P_1 = \left( (f_{ij}^{(1)}(\alpha_1)), (f_{ij}^{(2)}(\alpha_1)), \cdots, (f_{ij}^{(d)}(\alpha_1)) \right)$. And so

$$
M \otimes_K \tilde{\Lambda} \cong P_1 \oplus P_2 \oplus \cdots \oplus P_r.
$$

As there is an automorphism of $\tilde{\Lambda}$ which sends $\alpha_i$ to $\alpha_1$, $P_1$ is an indecomposable representation of $\Gamma$ defined over $\Lambda_i$. Let $T_i = \frac{\text{End}_{\Lambda_1}(P_i)}{\text{rad}(\text{End}_{\Lambda_1}(P_i))}$. By the same reason, $\dim_{\Lambda_i}(T_i) = \dim_{\Lambda_1}(T_1)$. As

$$
D \otimes_K \tilde{\Lambda} \cong (T_1 \otimes_{\Lambda_1} \tilde{\Lambda}) \oplus \cdots \oplus (T_r \otimes_{\Lambda_1} \tilde{\Lambda}),
$$

and $\dim_{K} D = s^2 r$, each $T_i$ has dimension $s^2$. $T_1 = D_1$ is a central division algebra, it follows that $T_i$ is also a central division algebra. For each $T_i$ there exists a maximal subfield $E_i$ in $T_i$. It follows from Theorem 7.15 of [R] that $E_i$ splits $T_i$, i.e.,

$$
T_i \otimes_{\Lambda_i} E_i \cong \mathcal{M}_s(E_i).
$$

And so $P_i \otimes_{\Lambda_i} E_i \cong N_i^s$ for some $N_i$ indecomposable over $E_i$ with $\frac{\text{End}_{E_i}(N_i)}{\text{rad}(\text{End}_{E_i}(N_i))} \cong E_i$. It follows from Theorem 5.4 that $N_i$ is absolutely indecomposable. A trivial argument shows that

$$
M \otimes_K K = N_1^s \oplus \cdots \oplus N_r^s.
$$
Since the centre of $D \otimes_K \overline{K}$ is isomorphic to $\overline{K} \oplus \cdots \oplus \overline{K}$ ($r$ copies), there are exactly $r$ distinct indecomposable direct summands in the decomposition of $M \otimes_K \overline{K}$. Thus, $N_i \cong N_j$ if and only $i = j$. This finishes the proof.

**Theorem 5.7.** With the hypothesis as the above theorem, $\Lambda_i$ is the unique minimal field of definition of $N_i^s$, $1 \leq i \leq r$.

**Proof.** It is sufficient to prove that $\Lambda_1$ is the unique minimal field of definition of $N_1^s$. To do so it is sufficient to show that if $N_1^s$ is defined over a field $F$ then $\Lambda_1 \subset F$. Now suppose that $N_1^s$ is defined over $F$. Let $a$ be the smallest integer such that $N_1^a$ is defined over $F$. It follows that $N_1^a$ is indecomposable over $F$. We claim that $a | s$. Let $s = ab + c$ with $0 \leq c \leq a - 1$. Then over $\overline{K}$ there holds $N_1^s = N_1^{ab} \oplus N_1^c$. As both $N_1^s$ and $N_1^{ab}$ are defined over $F$, an easy exercise shows that $N_1^c$ is defined over $F$. As $a$ is minimal which satisfies such property, $c = 0$.

Let $M_1 = N_1^a$. Without loss of generality, we may assume that all matrix entries of $M_1$ lie in $F$. Since $M_1^\dagger$ and $M$ are indecomposable representation of $\Gamma$ over $K$, and after tensoring $\overline{K}$ they have a common direct summand $N_1$, Theorem 5.3 shows that $M_1^\dagger \cong M$ over $K$. It follows that $M_1$ is a direct summand of $M \otimes_K F$. Now suppose that

$$M \otimes_K F \cong M_1^{b_1} \oplus M_2^{b_2} \oplus \cdots \oplus M_t^{b_t},$$

in which the $M_i$'s are indecomposable and mutually non-isomorphic, and let $R_i = \frac{\text{End}_F(M_i)}{\text{rad}(\text{End}_F(M_i))}$. As $M \otimes_K \overline{K} \cong N_1^s \oplus \cdots \oplus N_r^s$, $M_1^{b_1} \otimes_F \overline{K} \cong N_1^s$. It follows that $b_1 = s/a$. Now since

$$D \otimes_K F = \frac{\text{End}_F(M \otimes_K F)}{\text{rad}(\text{End}_F(M \otimes_K F))} \cong M_{b_1}(R_1) \oplus \cdots \oplus M_{b_1}(R_t),$$

we have

$$C(R_1) \oplus \cdots \oplus C(R_t) \cong C(D \otimes_K F) \cong \Lambda \otimes_K F \cong F[x]/(f(x)) \cong F[x]/(f_1(x)) \oplus \cdots \oplus F[x]/(f_k(x)),$$

where $f_1(x), \cdots, f_k(x)$ are the monic irreducible polynomials over $F$ which satisfy $f_1(x) \cdots f_k(x) = f(x)$. After some reordering we may assume that $C(R_1) \cong F[x]/(f_1(x))$. Since $M_1 \otimes_F \overline{K} \cong N_1^s$, $R_1$ is central, that is $C(R_1) \cong F$. It follows that $f_1(x) = x - \alpha$, $\alpha$ is a root of $f(x)$. 

To finish the proof it is enough to show that $\alpha_1 \in F$. This can be proved as follows. Let $\tilde{M}$ be the $\Gamma$-module affording $M$, $\tilde{M}_i$ be the $\Gamma \otimes_K \bar{K}$-module affording $M_i$, and $\tilde{N}_i$ be the $\Gamma \otimes_K \bar{K}$-module affording $N_i$. Then the above discussion shows that $\tilde{M} \otimes_K F \cong \tilde{M}_1^{b_1} \oplus \cdots \oplus \tilde{M}_r^{b_r}$, $\tilde{M} \otimes_K \bar{K} \cong \tilde{N}_1^s \oplus \cdots \oplus \tilde{N}_r^s$, $\tilde{M}_1^{b_1} \otimes_F \bar{K} \cong \tilde{N}_1^s$, and 
\[
\frac{\text{End}_F(\tilde{M})}{\text{rad}(\text{End}_F(\tilde{M}))} \cong D.
\]

The following formula can be checked by a simple analysis of residues:
\[
\frac{1}{(X - \alpha_1) \cdots (X - \alpha_r)} = \sum_{i=1}^r \frac{f'(\alpha_i)^{-1}}{X - \alpha_i}.
\]

It follows that we have the following central idempotent decomposition of the identity in $D \otimes_K \bar{K}$:
\[
1 = e_1 + \cdots + e_r, \quad \text{where} \quad e_i = \frac{f(X)}{f'(\alpha_i)(X - \alpha_i)}.
\]

Therefore, after some reordering on $\alpha_i$, we have $\tilde{N}_i^s = e_i(\tilde{M} \otimes_K \bar{K})$. Now, let $S_i = \{ \text{the roots of } f_i \text{ in } \bar{K} \}$, and let $\pi_i = \sum_{\alpha_j \in S_i} e_j$. Then an easy exercise shows that $\pi_i \in K[X]/(f_i(X))$. Thus, we have the following central idempotent decomposition of the identity in $D \otimes_K F$:
\[
1 = \pi_1 + \cdots + \pi_t.
\]

Thus, after some permutation on $\tilde{M}_2, \cdots, \tilde{M}_t$, we have $\tilde{M}_i^{b_i} = \pi_i(\tilde{M} \otimes_K F)$. As $\tilde{M}_1^{b_1} \otimes_F \bar{K} \cong \tilde{N}_1^s$, $e_1 = \pi_1$. It follows that $f_1(X) = X - \alpha_1$, and so $\alpha_1 \in F$.

The following corollary is immediate.

**Corollary 5.8.** Let $K$ be a number field (that is a finite extension of $Q$) contained in a fixed algebraic closure $\bar{Q}$ of $Q$. Let $\Gamma$ be a $K$-algebra of finite dimension as a vector space and let $M$ be a finitely generated indecomposable $\Gamma \otimes_K \bar{Q}$-module. Let $M^{\dagger}$ be the indecomposable $\Gamma$-module such that $M^{\dagger} \otimes_K \bar{Q}$ contains $M$ as a direct summand and let $D = \frac{\text{End}_F(M^{\dagger})}{\text{rad}(\text{End}_F(M^{\dagger}))}$. Then $M^{\dagger}$ has a unique minimal field of definition $\Lambda_1 \subset \bar{K}$ which is isomorphic to the centre of $D$, where $s = \sqrt{\dim_\Lambda D}$.

**Corollary 5.9.** Let $D$ be a finite dimensional division algebra over $K$, $\Lambda$ be the centre of $D$. Let $r = \dim_K \Lambda$ and $s = \sqrt{\dim_\Lambda D}$. Let $\Lambda_1, \cdots, \Lambda_r$ denote the images of the $r$ $K$-embeddings of $\Lambda$ into $\bar{K}$ and let $\tilde{\Lambda}$ be the composite of $\Lambda_1, \cdots, \Lambda_r$. Then we have
\[
D \otimes_K \tilde{\Lambda} \cong (T_1 \otimes_{\Lambda_1} \tilde{\Lambda}) \oplus \cdots \oplus (T_r \otimes_{\Lambda_r} \tilde{\Lambda}),
\]
where for each \( i \), \( T_i \) is a central division algebra over \( \Lambda_i \) of dimension \( s^2 \).

Proof. \( D \) can be viewed as a left \( D \)-module. We denote this module by \( M \). As \( D \) is a simple algebra, \( M \) is semi-simple. As \( \text{End}_K(M) \cong D \) (see page 267, [AB]), \( M \) is simple. By the proof of the previous theorem, we have

\[
M \otimes K \tilde{\Lambda} = (P_1 \otimes_{\Lambda_1} \tilde{\Lambda}) \oplus \cdots \oplus (P_r \otimes_{\Lambda_r} \tilde{\Lambda}),
\]

where for each \( i \), \( P_i \) is an indecomposable representation of \( D \) over \( \Lambda_i \). Moreover \( P_i \otimes_{\Lambda_i} \tilde{\Lambda} \) and \( P_j \otimes_{\Lambda_j} \tilde{\Lambda} \) has no common indecomposable direct summands. As \( P_i \) is indecomposable and semi-simple, it is simple. Let \( T_i = \text{End}_{\Lambda_i}(P_i) \). The Schur’s Lemma shows that \( T_i \) is a division algebra and hence has zero radical. By the proof of the above theorem, \( T_i \) is a central division algebra over \( \Lambda_i \) of dimension \( s^2 \). Now,

\[
\text{End}_{\tilde{\Lambda}}(M \otimes K \tilde{\Lambda}) \cong \text{End}_{\tilde{\Lambda}}(P_1 \otimes_{\Lambda_1} \tilde{\Lambda}) \oplus \cdots \oplus \text{End}_{\tilde{\Lambda}}(P_r \otimes_{\Lambda_r} \tilde{\Lambda})
\]

\[
\cong (\text{End}_{\Lambda_1}(P_1) \otimes_{\Lambda_1} \tilde{\Lambda}) \oplus \cdots \oplus (\text{End}_{\Lambda_r}(P_r) \otimes_{\Lambda_r} \tilde{\Lambda})
\]

\[
\cong (T_1 \otimes_{\Lambda_1} \tilde{\Lambda}) \oplus \cdots \oplus (T_r \otimes_{\Lambda_r} \tilde{\Lambda}).
\]

As \( \text{End}_{\tilde{\Lambda}}(M \otimes K \tilde{\Lambda}) \cong \text{End}_K(M) \otimes K \tilde{\Lambda} \cong D \otimes K \tilde{\Lambda} \), this finishes the proof.

Remark. The antecedents of the \(*\)–operator in other contexts (Abelian varieties; modular representation theory) may be found in Chapter 8 of Lang [La2] and in Green’s well-known paper [Gr2]. The results obtained in this section are partially due to my thesis advisor Peter Donovan. [See Note 2 on page 122].
Appendix A

Counting representations by Maple

This appendix contains three programs written in the language of Maple which calculate various polynomials $A_\Gamma(\alpha, q)$, $M_\Gamma(\alpha, q)$ and $I_\Gamma(\alpha, q)$ for graphs of ranks no more than 3. To the best of my knowledge, there is no algorithm available in the literature to calculate these important polynomials. The programs in this appendix have all been run through on Maple, part of the output is included.

The theory which supports these programs goes as follows. Recall that The rational functions $H_\Gamma(\alpha, q)$ are defined by the following:

$$\log (P_\Gamma(X_1, \cdots, X_n, q)) = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} H_\Gamma(\alpha, q)X^\alpha/\bar{\alpha}.$$ 

Now, Theorem 4.1.9 shows that

$$A_\Gamma(\alpha, q) = \frac{q-1}{\bar{\alpha}} \sum_{d | \bar{\alpha}} \mu(d) H_\Gamma\left(\frac{\alpha}{d}, q^d\right).$$

(*)

Since $P_\Gamma(X_1, \cdots, X_n, q)$ can be computed without much difficulty, $H_\Gamma(\alpha, q)$ can be obtained easily. Thus $A_\Gamma(\alpha, q)$ can be computed from the formula (*). Once we have computed $A_\Gamma(\alpha, q)$ for those $\alpha$ whose heights are less than or equal to a positive integer $N$, then we can compute $M_\Gamma(\alpha, q)$ and $I_\Gamma(\alpha, q)$ for those $\alpha$ whose heights are less than or equal to $N$ from the following identity:

$$\sum_{\alpha \in \mathbb{N}^n} M_\Gamma(\alpha, q)X^\alpha = \prod_{\alpha \in \Delta^+} (1 - X^\alpha)^{-I_\Gamma(\alpha, q)} = \prod_{\alpha \in \Delta^+} \prod_{i=0}^{u_\alpha} (1 - q^i X^\alpha)^{-t_i^\alpha}.$$

All the obtained calculations are consistence with the conjectures mentioned in this thesis.

**Program I.** This program produces the polynomials $A_\Gamma(\alpha, q)$, $M_\Gamma(\alpha, q)$ and $I_\Gamma(\alpha, q)$ for quivers with one node and $g$ arrows. It should be noted that Diane Maclagan, a vacation visitor of Dr. Donovan from Christchurch, New Zealand, wrote
a Maple program in [Ma] which served the same purpose without using formula (*). She had obtained $A_\Gamma(n, q)$ and $I_\Gamma(n, q)$ for $n \leq 11$ when $g = 2$. Her results are the same as what we have here.

```maple
> with(numtheory): with(combinat):
> varphi:=proc(r,q) local i:
> if r=0 then 1 else product((1-q^i),i=1..r) fi
> end:
> g:=2: N:=16:
> P[1]:=[0]: t:=1:
> for n to N do
> for i to number(n) do
> U:=partition(n)[i]:
> r:=nops(U): t:=t+1:
> P[t]:=[seq(U[r+1-j],j=1..r)]
> od
> od:
> PX:=0:
> for i to t do
> U:=P[i]: a:=nops(U):
> UU:=sum(U[s]^2,s=1..a):
> wU:=sum(U[s],s=1..a):
> bU:=product(varphi(U[s]-U[s+1],1/q),s=1..a-1)*varphi(U[a],1/q):
> PX:=PX+simplify(q^((g-1)*UU)/bU)*X^wU
> od:
> logm:=taylor(log(PX),X,N+1):
> for i to N do H[i,q]:=simplify(coeff(logm,X,i)*i) od:
> for m to N do
> S:=0: DV:=divisors(m):
> for k to nops(DV) do
> d:=DV[k]:
S:=S+mobius(d)*subs(q=q^d,H[m/d,q])

A[m,q]:=sort(expand(simplify((q-1)/m*S))):

print(A[Gamma](m,q)="

\[A_1(q) = q^2\]

\[A_2(q) = q^5 + q^3\]

\[A_3(q) = q^{10} + q^8 + q^7 + q^6 + q^5 + q^4\]

\[A_4(q) = q^{17} + q^{15} + q^{14} + 2q^{13} + q^{12} + 3q^{11} + 2q^{10} + 4q^9 + 2q^8 + 3q^7 + q^6 + q^5\]

\[A_5(q) = q^{26} + q^{24} + 2q^{23} + 2q^{22} + 2q^{21} + 3q^{20} + 3q^{19} + 5q^{18} + 5q^{17} + 7q^{16} + 7q^{15} + 9q^{13} + 10q^{12} + 9q^{11} + 8q^{10} + 6q^9 + 4q^8 + 2q^7 + q^6\]

\[A_6(q) = q^{37} + q^{35} + q^{34} + 2q^{33} + 2q^{32} + 4q^{31} + 3q^{30} + 6q^{29} + 6q^{28} + 9q^{27} + 9q^{26} + 14q^{25} + 13q^{24} + 19q^{23} + 19q^{22} + 25q^{21} + 25q^{20} + 33q^{19} + 30q^{18} + 37q^{17} + 34q^{16} + 36q^{15} + 31q^{14} + 31q^{13} + 21q^{12} + 18q^{11} + 10q^{10} + 6q^9 + 2q^8 + q^7\]

\[A_7(q) = q^{50} + q^{48} + q^{47} + 2q^{46} + 2q^{45} + 4q^{44} + 4q^{43} + 6q^{42} + 7q^{41} + 10q^{40} + 11q^{39} + 16q^{38} + 17q^{37} + 23q^{36} + 26q^{35} + 33q^{34} + 37q^{33} + 46q^{32} + 51q^{31} + 62q^{30} + 69q^{29} + 81q^{28} + 89q^{27} + 103q^{26} + 111q^{25} + 124q^{24} + 131q^{23} + 141q^{22} + 144q^{21} + 148q^{20} + 144q^{19} + 139q^{18} + 126q^{17} + 111q^{16} + 91q^{15} + 70q^{14} + 49q^{13} + 31q^{12} + 17q^{11} + 8q^{10} + 3q^9 + q^8\]

\[A_8(q) = q^{65} + q^{63} + q^{62} + 2q^{61} + 2q^{60} + 4q^{59} + 4q^{58} + 7q^{57} + 7q^{56} + 11q^{55} + 12q^{54} + 18q^{53} + 19q^{52} + 27q^{51} + 30q^{50} + 40q^{49} + 44q^{48} + 58q^{47} + 64q^{46} + 82q^{45} + 90q^{44} + 112q^{43} + 124q^{42} + 152q^{41} + 166q^{40} + 200q^{39} + 219q^{38} + 259q^{37} + 281q^{36} + 328q^{35} + 353q^{34} + 406q^{33} + 432q^{32} + 487q^{31} + 513q^{30} + 566q^{29} + 584q^{28} + 629q^{27} + 635q^{26} + 663q^{25} + 648q^{24} + 651q^{23} + 609q^{22} + 581q^{21} + 511q^{20} + 455q^{19} + 368q^{18} + 298q^{17} + 213q^{16} + 151q^{15} + 90q^{14} + 53q^{13} + 24q^{12} + 11q^{11} + 3q^{10} + q^9\]

\[A_9(q) = q^{82} + q^{80} + q^{79} + 2q^{78} + 2q^{77} + 4q^{76} + 4q^{75} + 7q^{74} + 8q^{73} + 11q^{72} + 13q^{71} + 19q^{70} + 21q^{69} + 29q^{68} + 34q^{67} + 44q^{66} + 51q^{65} + 66q^{64} + 75q^{63} + 95q^{62} + 110q^{61} + 134q^{60} + 155q^{59} + 188q^{58} + 214q^{57} + 256q^{56} + 293q^{55} + 344q^{54} + 391q^{53} + 457q^{52} + 514q^{51} + 595q^{50} + 669q^{49} + 763q^{48} + 853q^{47} + 968q^{46} + 1071q^{45} + 1205q^{44} + 1329q^{43} + 1477q^{42} + 1617q^{41} + 1783q^{40} + 1931q^{39} + 2106q^{38} + 2262q^{37} + 2430q^{36} + 2579q^{35} + \]
2736q^{34} + 2855q^{33} + 2976q^{32} + 3054q^{31} + 3109q^{30} + 3118q^{29} + 3095q^{28} + 3007q^{27} + 2884q^{26} + 2701q^{25} + 2472q^{24} + 2201q^{23} + 1903q^{22} + 1580q^{21} + 1263q^{20} + 960q^{19} + 687q^{18} + 461q^{17} + 286q^{16} + 160q^{15} + 81q^{14} + 36q^{13} + 13q^{12} + 4q^{11} + q^{10}

A_\Gamma(10, q) = q^{101} + q^{99} + q^{98} + 2q^{97} + 2q^{96} + 4q^{95} + 4q^{94} + 7q^{93} + 8q^{92} + 12q^{91} + 13q^{90} + 20q^{89} + 22q^{88} + 31q^{87} + 36q^{86} + 48q^{85} + 55q^{84} + 73q^{83} + 83q^{82} + 107q^{81} + 123q^{80} + 154q^{79} + 177q^{78} + 220q^{77} + 251q^{76} + 306q^{75} + 350q^{74} + 421q^{73} + 479q^{72} + 572q^{71} + 647q^{70} + 764q^{69} + 864q^{68} + 1009q^{67} + 1136q^{66} + 1318q^{65} + 1477q^{64} + 1700q^{63} + 1899q^{62} + 2169q^{61} + 2411q^{60} + 2737q^{59} + 3029q^{58} + 3415q^{57} + 3765q^{56} + 4215q^{55} + 4624q^{54} + 5147q^{53} + 5617q^{52} + 6211q^{51} + 6743q^{50} + 7403q^{49} + 7990q^{48} + 8709q^{47} + 9337q^{46} + 10095q^{45} + 10743q^{44} + 11511q^{43} + 12143q^{42} + 12881q^{41} + 13448q^{40} + 14098q^{39} + 14543q^{38} + 15036q^{37} + 15286q^{36} + 15545q^{35} + 15523q^{34} + 15472q^{33} + 15112q^{32} + 14694q^{31} + 13962q^{30} + 13164q^{29} + 12080q^{28} + 10954q^{27} + 9614q^{26} + 8293q^{25} + 6871q^{24} + 5558q^{23} + 4271q^{22} + 3177q^{21} + 2208q^{20} + 1470q^{19} + 893q^{18} + 513q^{17} + 259q^{16} + 122q^{15} + 47q^{14} + 17q^{13} + 4q^{12} + q^{11}

\cdots \cdots \cdots \cdots

A_\Gamma(16, q) = q^{257} + q^{255} + q^{254} + 2q^{253} + 2q^{252} + 4q^{251} + 4q^{250} + 7q^{249} + 8q^{248} + 12q^{247} + 14q^{246} + 21q^{245} + 24q^{244} + 34q^{243} + 41q^{242} + 55q^{241} + 65q^{240} + 87q^{239} + 103q^{238} + 134q^{237} + 160q^{236} + 203q^{235} + 242q^{234} + 305q^{233} + 361q^{232} + 448q^{231} + 532q^{230} + 652q^{229} + 770q^{228} + 938q^{227} + 1103q^{226} + 1331q^{225} + 1563q^{224} + 1870q^{223} + 2189q^{222} + 2605q^{221} + 3036q^{220} + 3590q^{219} + 4176q^{218} + 4909q^{217} + 5689q^{216} + 6659q^{215} + 7692q^{214} + 8960q^{213} + 10323q^{212} + 11970q^{211} + 13750q^{210} + 15887q^{209} + 18193q^{208} + 20940q^{207} + 23922q^{206} + 27434q^{205} + 31254q^{204} + 35732q^{203} + 40601q^{202} + 46271q^{201} + 52451q^{200} + 59596q^{199} + 67392q^{198} + 76367q^{197} + 86149q^{196} + 97358q^{195} + 109593q^{194} + 123532q^{193} + 138746q^{192} + 156022q^{191} + 174862q^{190} + 196172q^{189} + 219415q^{188} + 245596q^{187} + 274142q^{186} + 306201q^{185} + 341117q^{184} + 380209q^{183} + 422777q^{182} + 470270q^{181} + 521948q^{180} + 579463q^{179} + 641978q^{178} + 711367q^{177} + 786739q^{176} + 870166q^{175} + 960712q^{174} + 1060707q^{173} + 1169106q^{172} + 1288544q^{171} + 1417919q^{170} + 1560123q^{169} + 1714000q^{168} + 1882801q^{167} + 2065245q^{166} + 2264968q^{165} + 2480626q^{164} + 2716211q^{163} + 2970324q^{162} + 3247403q^{161} + 3545908q^{160} + 3870788q^{159} + 4220429q^{158} + 4600229q^{157} + 5008509q^{156} + 5451248q^{155} + 5926608q^{154} + 6441186q^{153} + 6993057q^{152} + 7589403q^{151} + 8228219q^{150} +
\[ 8917383q^{149} + 9654695q^{148} + 10448809q^{147} + 11297405q^{146} + 12209833q^{145} + 13183654q^{144} + 14229068q^{143} + 15343388q^{142} + 16537702q^{141} + 17809146q^{140} + 19169644q^{139} + 20616140q^{138} + 22161508q^{137} + 23802341q^{136} + 25552538q^{135} + 27408404q^{134} + 29384735q^{133} + 31477514q^{132} + 33702587q^{131} + 36055419q^{130} + 38552912q^{129} + 41190040q^{128} + 43984652q^{127} + 46931167q^{126} + 50048444q^{125} + 53330105q^{124} + 56796036q^{123} + 60439053q^{122} + 64279849q^{121} + 68310311q^{120} + 72552010q^{119} + 76995591q^{118} + 81663395q^{117} + 86544750q^{116} + 91662524q^{115} + 97004538q^{114} + 102594105q^{113} + 108417142q^{112} + 114497266q^{111} + 120818322q^{110} + 127403813q^{109} + 134235217q^{108} + 141335786q^{107} + 148684138q^{106} + 156302938q^{105} + 164167664q^{104} + 172299906q^{103} + 180671605q^{102} + 189302915q^{101} + 198161638q^{100} + 207266078q^{99} + 216579505q^{98} + 226117689q^{97} + 235838882q^{96} + 245755891q^{95} + 255821298q^{94} + 266044353q^{93} + 276371537q^{92} + 286807871q^{91} + 297293278q^{90} + 307828035q^{89} + 318344970q^{88} + 328839165q^{87} + 339236108q^{86} + 349525143q^{85} + 359624288q^{84} + 369517003q^{83} + 379113829q^{82} + 388392348q^{81} + 397256087q^{80} + 405677046q^{79} + 413552589q^{78} + 420849943q^{77} + 427461609q^{76} + 433531458q^{75} + 438409199q^{74} + 442597311q^{73} + 445805538q^{72} + 447997875q^{71} + 449067743q^{70} + 448984403q^{69} + 447649617q^{68} + 445042579q^{67} + 441078963q^{66} + 435753400q^{65} + 429001758q^{64} + 420840250q^{63} + 411231684q^{62} + 400220110q^{61} + 387801896q^{60} + 374054725q^{59} + 359014148q^{58} + 342795782q^{57} + 325477922q^{56} + 307215723q^{55} + 288130367q^{54} + 268414051q^{53} + 248226188q^{52} + 22778221q^{51} + 207287281q^{50} + 186960157q^{49} + 167005000q^{48} + 147654216q^{47} + 129094991q^{46} + 111531525q^{45} + 95115425q^{44} + 79998399q^{43} + 66273760q^{42} + 54021253q^{41} + 43260821q^{40} + 33991404q^{39} + 26157475q^{38} + 19683249q^{37} + 14451298q^{36} + 10332585q^{35} + 7174856q^{34} + 4827657q^{33} + 3136817q^{32} + 1962882q^{31} + 1177731q^{30} + 675380q^{29} + 368022q^{28} + 189843q^{27} + 91948q^{26} + 41639q^{25} + 17405q^{24} + 6689q^{23} + 2308q^{22} + 715q^{21} + 188q^{20} + 43q^{19} + 7q^{18} + q^{17} 

> PH:=1;
> for n to N do
> S:=0: DV:= divisors(n):
> for k to nops(DV) do
> d:= DV[k]: Dd:=divisors(d):
> for i to nops(Dd) do
>   r:=Dd[i]:
>   S:=S+1/d*mobius(d/r)*subs(q=q^r,A[n/d,q])
> od:
> od:
> sort(expand(simplify(S))):
> PH:=PH*(1-X^n)^(-1):
> print(I[Gamma](n,q)="")
> od:

\[ I_\Gamma(1,q) = q^2 \]

\[ I_\Gamma(2,q) = q^5 + \frac{1}{2}q^4 + q^3 - \frac{1}{2}q^2 \]

\[ I_\Gamma(3,q) = q^{10} + q^8 + q^7 + \frac{4}{3}q^6 + q^5 + q^4 - \frac{1}{3}q^2 \]

\[ I_\Gamma(4,q) = q^{17} + q^{15} + q^{14} + 2q^{13} + q^{12} + 3q^{11} + \frac{5}{2}q^{10} + 4q^9 + \frac{9}{4}q^8 + 3q^7 + \frac{3}{2}q^6 + \frac{1}{2}q^5 - \frac{1}{4}q^4 - \frac{1}{2}q^3 \]

\[ I_\Gamma(5,q) = q^{26} + q^{24} + q^{23} + 2q^{22} + 2q^{21} + 3q^{20} + 3q^{19} + 5q^{18} + 5q^{17} + 7q^{16} + 7q^{15} + 9q^{14} + 9q^{13} + 10q^{12} + 9q^{11} + \frac{11}{5}q^{10} + 6q^9 + 4q^8 + 2q^7 + q^6 - \frac{1}{5}q^2 \]

\[ I_\Gamma(6,q) = q^{37} + q^{35} + q^{34} + 2q^{33} + 2q^{32} + 4q^{31} + 3q^{30} + 6q^{29} + 6q^{28} + 9q^{27} + 9q^{26} + 14q^{25} + 13q^{24} + 19q^{23} + 19q^{22} + 25q^{21} + \frac{51}{2}q^{20} + 33q^{19} + 30q^{18} + 37q^{17} + \frac{69}{2}q^{16} + \frac{109}{3}q^{15} + \frac{63}{2}q^{14} + 31q^{13} + \frac{65}{3}q^{12} + 18q^{11} + 10q^{10} + \frac{10}{3}q^9 + 2q^8 + \frac{1}{2}q^7 - \frac{2}{3}q^6 - \frac{5}{6}q^5 - \frac{2}{3}q^4 - \frac{1}{3}q^3 + \frac{1}{6}q^2 \]

\[ I_\Gamma(6,q) = q^{37} + q^{35} + q^{34} + 2q^{33} + 2q^{32} + 4q^{31} + 3q^{30} + 6q^{29} + 6q^{28} + 9q^{27} + 9q^{26} + 14q^{25} + 13q^{24} + 19q^{23} + 19q^{22} + 25q^{21} + \frac{51}{2}q^{20} + 33q^{19} + 30q^{18} + 37q^{17} + \frac{69}{2}q^{16} + \frac{109}{3}q^{15} + \frac{63}{2}q^{14} + 31q^{13} + \frac{65}{3}q^{12} + 18q^{11} + 10q^{10} + \frac{10}{3}q^9 + 2q^8 + \frac{1}{2}q^7 - \frac{2}{3}q^6 - \frac{5}{6}q^5 - \frac{2}{3}q^4 - \frac{1}{3}q^3 + \frac{1}{6}q^2 \]

\[ I_\Gamma(7,q) = q^{50} + q^{48} + q^{47} + 2q^{46} + 2q^{45} + 4q^{44} + 4q^{43} + 6q^{42} + 7q^{41} + 10q^{40} + 11q^{39} + 16q^{38} + 17q^{37} + 23q^{36} + 26q^{35} + 33q^{34} + 37q^{33} + 46q^{32} + 51q^{31} + 62q^{30} + 69q^{29} + 81q^{28} + 89q^{27} + 103q^{26} + 111q^{25} + 124q^{24} + 131q^{23} + 141q^{22} + 144q^{21} + 148q^{20} + 144q^{19} + 139q^{18} + 126q^{17} + 111q^{16} + 91q^{15} + \frac{49}{2}q^{14} + 49q^{13} + 31q^{12} + 17q^{11} + 8q^{10} + 3q^9 + q^8 - \frac{1}{2}q^7 \]

\[ I_\Gamma(8,q) = q^{65} + q^{63} + q^{62} + 2q^{61} + 2q^{60} + 4q^{59} + 4q^{58} + 7q^{57} + 7q^{56} + 11q^{55} + 12q^{54} + 18q^{53} + 19q^{52} + 27q^{51} + 30q^{50} + 40q^{49} + 44q^{48} + 58q^{47} + 64q^{46} + 82q^{45} + 90q^{44} + 112q^{43} + 124q^{42} + 152q^{41} + 166q^{40} + 200q^{39} + 219q^{38} + 259q^{37} + 281q^{36} + 328q^{35} + \]
\[
I_r(9, q) = q^{82} + q^{80} + q^{79} + 2q^{78} + 2q^{77} + 4q^{76} + 4q^{75} + 7q^{74} + 8q^{73} + 11q^{72} + 13q^{71} + 19q^{70} + 21q^{69} + 29q^{68} + 34q^{67} + 44q^{66} + 51q^{65} + 66q^{64} + 75q^{63} + 95q^{62} + 110q^{61} + 134q^{60} + 155q^{59} + 188q^{58} + 214q^{57} + 256q^{56} + 293q^{55} + 344q^{54} + 391q^{53} + 457q^{52} + 514q^{51} + 595q^{50} + 669q^{49} + 763q^{48} + 853q^{47} + 968q^{46} + 1071q^{45} + 1205q^{44} + 1329q^{43} + 1477q^{42} + 1617q^{41} + 1783q^{40} + 1931q^{39} + 2106q^{38} + 2262q^{37} + 2430q^{36} + 2579q^{35} + 2736q^{34} + 2855q^{33} + 2976q^{32} + 3054q^{31} + \frac{9328}{3}q^{30} + 3118q^{29} + 3095q^{28} + 3007q^{27} + 2884q^{26} + 2701q^{25} + \frac{7417}{3}q^{24} + 2201q^{23} + 1903q^{22} + \frac{4741}{3}q^{21} + 1263q^{20} + 960q^{19} + \frac{6187}{9}q^{18} + 461q^{17} + 286q^{16} + \frac{481}{3}q^{15} + 81q^{14} + 36q^{13} + \frac{40}{9}q^{12} + 4q^{11} + \frac{2}{3}q^{10} - \frac{1}{3}q^9 - \frac{4}{9}q^8 - \frac{3}{4}q^7 - \frac{3}{4}q^6 - \frac{1}{2}q^5
\]

\[
I_r(10, q) = q^{101} + q^{99} + q^{98} + 2q^{97} + 2q^{96} + 4q^{95} + 4q^{94} + 7q^{93} + 8q^{92} + 12q^{91} + 13q^{90} + 20q^{89} + 22q^{88} + 31q^{87} + 36q^{86} + 48q^{85} + 55q^{84} + 73q^{83} + 83q^{82} + 107q^{81} + 123q^{80} + 154q^{79} + 177q^{78} + 220q^{77} + 251q^{76} + 306q^{75} + 350q^{74} + 421q^{73} + 479q^{72} + 572q^{71} + 647q^{70} + 764q^{69} + 864q^{68} + 1009q^{67} + 1136q^{66} + 1318q^{65} + 1477q^{64} + 1700q^{63} + 1899q^{62} + 2169q^{61} + 2411q^{60} + 2737q^{59} + 3029q^{58} + 3415q^{57} + 3765q^{56} + 4215q^{55} + 4624q^{54} + 5147q^{53} + \frac{11232}{3}q^{52} + 6211q^{51} + 6743q^{50} + 7403q^{49} + \frac{15981}{2}q^{48} + 8709q^{47} + \frac{18675}{4}q^{46} + 10095q^{45} + 10744q^{44} + 11511q^{43} + 12144q^{42} + 12881q^{41} + \frac{26899}{2}q^{40} + 14098q^{39} + \frac{29089}{2}q^{38} + 15036q^{37} + \frac{30577}{2}q^{36} + 15545q^{35} + \frac{31051}{2}q^{34} + 15472q^{33} + \frac{30231}{2}q^{32} + 14694q^{31} + \frac{27931}{2}q^{30} + 13164q^{29} + \frac{24169}{2}q^{28} + 10954q^{27} + 9618q^{26} + \frac{41466}{5}q^{25} + \frac{13751}{2}q^{24} + \frac{11115}{2}q^{23} + \frac{8549}{2}q^{22} + 3176q^{21} + \frac{11035}{5}q^{20} + \frac{2937}{2}q^{19} + \frac{1787}{2}q^{18} + \frac{1021}{2}q^{17} + \frac{515}{2}q^{16} + \frac{1187}{10}q^{15} + \frac{87}{2}q^{14} + \frac{25}{2}q^{13} - \frac{1}{2}q^{12} - \frac{7}{3}q^{11} - \frac{41}{10}q^{10} - 3q^9 - 2q^8 - q^7 - \frac{1}{7}q^6 - \frac{1}{5}q^5 - \frac{1}{10}q^4 - \frac{1}{5}q^3 + \frac{1}{10}q^2
\]

\[
I_r(16, q) = q^{257} + q^{255} + q^{254} + 2q^{253} + 2q^{252} + 4q^{251} + 4q^{250} + 7q^{249} + 8q^{248} + 12q^{247} + 14q^{246} + 21q^{245} + 24q^{244} + 34q^{243} + 41q^{242} + 55q^{241} + 65q^{240} + 87q^{239} + 103q^{238} + 134q^{237} + 160q^{236} + 203q^{235} + 242q^{234} + 305q^{233} + 361q^{232} + 448q^{231} + 532q^{230} + 652q^{229} + 770q^{228} + 938q^{227} + 1103q^{226} + 1331q^{225} + 1563q^{224} + 1870q^{223} + 2189q^{222} + 2605q^{221} + 3036q^{220} + 3590q^{219} + 4176q^{218} + 4909q^{217} + 5689q^{216} + 6659q^{215} + 7692q^{214} + 8960q^{213} + 10323q^{212} + 11970q^{211} + 13750q^{210} + 15887q^{209} + 18193q^{208} + 20940q^{207} + 23922q^{206} + 27434q^{205} + 31254q^{204} + 35732q^{203} + 40601q^{202} + 46271q^{201} + 52451q^{200} + 59596q^{199} + 67392q^{198} + 76367q^{197} + 86149q^{196} + 97358q^{195} + 109593q^{194} + \ldots
\]
\[123532q^{193} + 138746q^{192} + 156022q^{191} + 174862q^{190} + 196172q^{189} + 219415q^{188} + 245596q^{187} + 274142q^{186} + 306201q^{185} + 341117q^{184} + 380209q^{183} + 422777q^{182} + 470270q^{181} + 521948q^{180} + 579463q^{179} + 641978q^{178} + 711367q^{177} + 786739q^{176} + 870166q^{175} + 960712q^{174} + 1060707q^{173} + 1169106q^{172} + 1288544q^{171} + 1417919q^{170} + 1560123q^{169} + 1714000q^{168} + 1882801q^{167} + 2065245q^{166} + 2264968q^{165} + 2480626q^{164} + 2716211q^{163} + 2970324q^{162} + 3247403q^{161} + 3545908q^{160} + 3870788q^{159} + 4220429q^{158} + 4600229q^{157} + 5008509q^{156} + 5451248q^{155} + 592608q^{154} + 6441186q^{153} + 6993057q^{152} + 7589403q^{151} + 8228219q^{150} + 891738q^{149} + 9654695q^{148} + 10448809q^{147} + 11297405q^{146} + 12209833q^{145} + 13183654q^{144} + 14229068q^{143} + 15343388q^{142} + 16537702q^{141} + 17809146q^{140} + 1916944q^{139} + 2061640q^{138} + 2216158q^{137} + 2380234q^{136} + 2555253q^{135} + 2740840q^{134} + 2938473q^{133} + 3147751q^{132} + 3370258q^{131} + 7211083q^{130} + 38552912q^{129} + 41190040q^{128} + 43984652q^{127} + 9386235q^{126} + 5004844q^{125} + \frac{106660211}{2}q^{124} + 56796036q^{123} + 60439054q^{122} + 64279849q^{121} + 68310312q^{120} + 72552010q^{119} + 76995593q^{118} + 81663395q^{117} + 86544752q^{116} + 91662524q^{115} + 194009083q^{114} + 102594105q^{113} + 216834291q^{112} + 114497266q^{111} + 241636655q^{110} + 127403813q^{109} + 134235223q^{108} + 141335786q^{107} + 148684147q^{106} + 156302938q^{105} + 328335347q^{104} + 172299906q^{103} + 361342337q^{102} + 189302915q^{101} + 198161653q^{100} + 207266078q^{99} + 21657925q^{98} + 22611768q^{97} + 23583890q^{96} + 24575589q^{95} + 25582132q^{94} + 26604435q^{93} + 27637159q^{92} + 28680787q^{91} + 29729331q^{90} + 30782803q^{89} + 31834501q^{88} + 32883916q^{87} + 33923616q^{86} + 34952514q^{85} + 35962435q^{84} + 36951700q^{83} + 37911390q^{82} + 38839234q^{81} + 39725617q^{80} + 40567704q^{79} + 41355268q^{78} + 42084994q^{77} + 8549233q^{76} + 43335148q^{75} + \frac{87681857}{2}q^{74} + 44259731q^{73} + \frac{89161357}{2}q^{72} + 44799785q^{71} + 44906790q^{70} + 44898440q^{69} + \frac{179059175}{4}q^{68} + 44504257q^{67} + 44107916q^{66} + 87506799q^{65} + 42900197q^{64} + \frac{84168049}{2}q^{63} + 41123192q^{62} + 40022010q^{61} + 15512086q^{60} + 37405472q^{59} + 35901442q^{58} + \frac{68559155}{2}q^{57} + \frac{1301912843}{4}q^{56} + \frac{61443143}{5}q^{55} + \frac{576261351}{2}q^{54} + 268414042q^{53} + \frac{496452993}{2}q^{52} + \frac{455576415}{2}q^{51} + 414575195q^{50} + 186960137q^{49} + \frac{668021209}{4}q^{48} + 147654187q^{47} + \frac{258190569}{4}q^{46} + 111531484q^{45} + \frac{390467241}{4}q^{44} + 79998343q^{43} + \frac{13254797q}{2}q^{42} + 54021177q^{41} + \frac{346087953}{8}q^{40} + 33991304q^{39} + 26157593q^{38} + \frac{39366239}{2}q^{37} + \frac{28902865}{2}q^{36} + 10332421q^{35} + \frac{28699313}{4}q^{34} + \ldots \]
\[
\begin{align*}
482745q^{33} & + \frac{50187329}{16} q^{32} + \frac{3925277}{2} q^{31} + \frac{4710199}{4} q^{30} + 675097q^{29} + \frac{735551}{2} q^{28} + \frac{379057}{2} q^{27} + \\
& + \frac{183313}{2} q^{26} + \frac{86215}{2} q^{25} + \frac{136745}{8} q^{24} + \frac{12727}{2} q^{23} + \frac{80333}{4} q^{22} + \frac{849}{2} q^{21} - \frac{531}{8} q^{20} - \frac{369}{2} q^{19} - \\
& - \frac{355}{2} q^{18} - 148q^{17} - \frac{1713}{16} q^{16} - \frac{151}{2} q^{15} - \frac{183}{4} q^{14} - \frac{53}{2} q^{13} - \frac{99}{8} q^{12} - \frac{11}{2} q^{11} - \frac{7}{4} q^{10} - \frac{1}{2} q^{9}
\end{align*}
\]

> PH:=taylor(PH,X,N+1):
> for n to N do
> M[n,q]:= sort(simplify(coeff(PH,X,n))):
> print(M[Gamma](n,q)=")
> od:
\[
\begin{align*}
M_\Gamma(1, q) &= q^2 \\
M_\Gamma(2, q) &= q^5 + q^4 + q^3 \\
M_\Gamma(3, q) &= q^{10} + q^8 + 2q^7 + 2q^6 + 2q^5 + q^4 \\
M_\Gamma(4, q) &= q^{17} + q^{15} + q^{14} + 2q^{13} + 2q^{12} + 3q^{11} + 4q^{10} + 6q^9 + 5q^8 + 5q^7 + 3q^6 + q^5 \\
M_\Gamma(5, q) &= q^{26} + q^{24} + q^{23} + 2q^{22} + 2q^{21} + 3q^{20} + 4q^{19} + 5q^{18} + 6q^{17} + 8q^{16} + 10q^{15} + \\
& + 11q^{14} + 14q^{13} + 15q^{12} + 17q^{11} + 15q^{10} + 13q^9 + 8q^8 + 4q^7 + q^6 \\
M_\Gamma(6, q) &= q^{37} + q^{35} + q^{34} + 2q^{33} + 2q^{32} + 4q^{31} + 3q^{30} + 3q^{29} + 7q^{28} + 9q^{27} + 10q^{26} + \\
& + 15q^{25} + 15q^{24} + 21q^{23} + 23q^{22} + 29q^{21} + 33q^{20} + 40q^{19} + 42q^{18} + 50q^{17} + 52q^{16} + \\
& + 56q^{15} + 56q^{14} + 55q^{13} + 46q^{12} + 37q^{11} + 24q^{10} + 13q^9 + 5q^8 + q^7 \\
M_\Gamma(7, q) &= q^{50} + q^{48} + q^{47} + 2q^{46} + 2q^{45} + 4q^{44} + 4q^{43} + 6q^{42} + 7q^{41} + 10q^{40} + 12q^{39} + \\
& + 16q^{38} + 18q^{37} + 24q^{36} + 28q^{35} + 35q^{34} + 41q^{33} + 49q^{32} + 58q^{31} + 69q^{30} + 80q^{29} + 92q^{28} + \\
& + 108q^{27} + 121q^{26} + 139q^{25} + 154q^{24} + 172q^{23} + 186q^{22} + 204q^{21} + 210q^{20} + 222q^{19} + \\
& + 220q^{18} + 217q^{17} + 201q^{16} + 181q^{15} + 146q^{14} + 111q^{13} + 72q^{12} + 41q^{11} + 18q^{10} + 6q^9 + q^8 \\
M_\Gamma(8, q) &= q^{65} + q^{63} + q^{62} + 2q^{61} + 2q^{60} + 4q^{59} + 4q^{58} + 7q^{57} + 7q^{56} + 11q^{55} + 12q^{54} + \\
& + 18q^{53} + 20q^{52} + 27q^{51} + 31q^{50} + 41q^{49} + 46q^{48} + 60q^{47} + 68q^{46} + 86q^{45} + 96q^{44} + \\
& + 119q^{43} + 135q^{42} + 164q^{41} + 184q^{40} + 219q^{39} + 246q^{38} + 290q^{37} + 323q^{36} + 374q^{35} + \\
& + 415q^{34} + 475q^{33} + 521q^{32} + 588q^{31} + 639q^{30} + 707q^{29} + 758q^{28} + 822q^{27} + 866q^{26} + \\
& + 916q^{25} + 941q^{24} + 966q^{23} + 957q^{22} + 940q^{21} + 887q^{20} + 821q^{19} + 721q^{18} + 609q^{17} + \\
& + 476q^{16} + 347q^{15} + 226q^{14} + 131q^{13} + 63q^{12} + 25q^{11} + 7q^{10} + q^9 \\
M_\Gamma(9, q) &= q^{82} + q^{80} + q^{79} + 2q^{78} + 2q^{77} + 4q^{76} + 4q^{75} + 7q^{74} + 8q^{73} + 11q^{72} + 13q^{71} + \\
& + 19q^{70} + 21q^{69} + 29q^{68} + 35q^{67} + 44q^{66} + 52q^{65} + 67q^{64} + 77q^{63} + 97q^{62} + 114q^{61} + \\
& + 128q^{60} + 143q^{59} + 158q^{58} + 173q^{57} + 188q^{56} + 203q^{55} + 218q^{54} + 233q^{53} + 248q^{52} + 263q^{51} + 278q^{50} + 293q^{49} + 308q^{48} + 323q^{47} + 338q^{46} + 353q^{45} + 368q^{44} + 383q^{43} + 398q^{42} + 413q^{41} + 428q^{40} + 443q^{39} + 458q^{38} + 473q^{37} + 488q^{36} + 503q^{35} + 518q^{34} + 533q^{33} + 548q^{32} + 563q^{31} + 578q^{30} + 593q^{29} + 608q^{28} + 623q^{27} + 638q^{26} + 653q^{25} + 668q^{24} + 683q^{23} + 698q^{22} + 713q^{21} + 728q^{20} + 743q^{19} + 758q^{18} + 773q^{17} + 788q^{16} + 803q^{15} + 818q^{14} + 833q^{13} + 848q^{12} + 863q^{11} + 878q^{10} + 893q^{9} + 908q^{8} + 923q^{7} + 938q^{6} + 953q^{5} + 968q^{4} + 983q^{3} + 998q^{2} + 1013q^{1} + 1028q^{0} 
\end{align*}
\]
\[ M_\Gamma (10, q) = q^{101} + q^{99} + q^{98} + 2q^{97} + 2q^{96} + 4q^{95} + 4q^{94} + 7q^{93} + 8q^{92} + 12q^{91} + 13q^{90} + 20q^{89} + 22q^{88} + 31q^{87} + 36q^{86} + 48q^{85} + 56q^{84} + 73q^{83} + 84q^{82} + 108q^{81} + 125q^{80} + 156q^{79} + 181q^{78} + 224q^{77} + 258q^{76} + 314q^{75} + 361q^{74} + 434q^{73} + 498q^{72} + 593q^{71} + 677q^{70} + 799q^{69} + 910q^{68} + 1062q^{67} + 1206q^{66} + 1398q^{65} + 1580q^{64} + 1820q^{63} + 2048q^{62} + 2342q^{61} + 2625q^{60} + 2981q^{59} + 3328q^{58} + 3760q^{57} + 4178q^{56} + 4689q^{55} + 5190q^{54} + 5789q^{53} + 6378q^{52} + 7076q^{51} + 7754q^{50} + 8547q^{49} + 9321q^{48} + 10202q^{47} + 11064q^{46} + 12031q^{45} + 12956q^{44} + 13979q^{43} + 14947q^{42} + 15985q^{41} + 16944q^{40} + 17946q^{39} + 18826q^{38} + 19713q^{37} + 20437q^{36} + 21101q^{35} + 21559q^{34} + 21896q^{33} + 21959q^{32} + 21851q^{31} + 21425q^{30} + 20770q^{29} + 19782q^{28} + 18552q^{27} + 17007q^{26} + 15265q^{25} + 13304q^{24} + 11260q^{23} + 9163q^{22} + 7151q^{21} + 5280q^{20} + 3672q^{19} + 2361q^{18} + 1391q^{17} + 732q^{16} + 339q^{15} + 131q^{14} + 41q^{13} + 9q^{12} + q^{11} \]

\[ \cdots \cdots \cdots \]

\[ M_\Gamma (16, q) = q^{257} + q^{255} + q^{254} + 2q^{253} + 2q^{252} + 4q^{251} + 4q^{250} + 7q^{249} + 8q^{248} + 12q^{247} + 14q^{246} + 21q^{245} + 24q^{244} + 34q^{243} + 41q^{242} + 55q^{241} + 65q^{240} + 87q^{239} + 103q^{238} + 134q^{237} + 160q^{236} + 203q^{235} + 242q^{234} + 305q^{233} + 361q^{232} + 448q^{231} + 532q^{230} + 652q^{229} + 771q^{228} + 938q^{227} + 1104q^{226} + 1332q^{225} + 1565q^{224} + 1872q^{223} + 2193q^{222} + 2609q^{221} + 3043q^{220} + 3598q^{219} + 4188q^{218} + 4923q^{217} + 5710q^{216} + 6683q^{215} + 7726q^{214} + 9001q^{213} + 10377q^{212} + 12035q^{211} + 13836q^{210} + 15989q^{209} + 18325q^{208} + 21098q^{207} + 24121q^{206} + 27672q^{205} + 31552q^{204} + 36085q^{203} + 41038q^{202} + 46790q^{201} + 53084q^{200} + 60344q^{199} + 68300q^{198} + 77434q^{197} + 87433q^{196} + 98866q^{195} + 111391q^{194} + 125637q^{193} + 141244q^{192} + 158932q^{191} + 178296q^{190} + 200165q^{189} + 224098q^{188} + 251022q^{187} + 280480q^{186} + 313518q^{185} + 349626q^{184} + 390009q^{183} + 434120q^{182} + 483296q^{181} + 536976q^{180} + 596663q^{179} + 661751q^{178} + 733946q^{177} + 812599q^{176} + 899617q^{175} + 994344q^{174} + 1098903q^{173} + 1212593q^{172} + 1337820q^{171} + 1473851q^{170} + 1623350q^{169} + 1785584q^{168} + 1963521q^{167} + 2156396q^{166} + 2367538q^{165} + \]
2596148q^{164} + 2845918q^{163} + 3116076q^{162} + 3410690q^{161} + 3728975q^{160} +
4075473q^{159} + 4449384q^{158} + 4855708q^{157} + 5293689q^{156} + 5768826q^{155} +
6280374q^{154} + 6834413q^{153} + 7430193q^{152} + 8074394q^{151} + 8766339q^{150} +
9513308q^{149} + 10314657q^{148} + 11178390q^{147} + 12103872q^{146} + 13099829q^{145} +
14165702q^{144} + 15310955q^{143} + 16535081q^{142} + 17848380q^{141} + 19250344q^{140} +
20752135q^{139} + 22353327q^{138} + 24065894q^{137} + 25889443q^{136} + 27836915q^{135} +
29907857q^{134} + 32116152q^{133} + 34461363q^{132} + 36958259q^{131} + 39606384q^{130} +
42421474q^{129} + 45402892q^{128} + 48567355q^{127} + 51914101q^{126} + 55460653q^{125} +
59206053q^{124} + 63168727q^{123} + 67347292q^{122} + 71761004q^{121} + 76408077q^{120} +
81308355q^{119} + 86459544q^{118} + 91882059q^{117} + 97572731q^{116} + 103552476q^{115} +
109817206q^{114} + 116387888q^{113} + 123259353q^{112} + 130452553q^{111} + 137960734q^{110} +
145804594q^{109} + 153975631q^{108} + 162493691q^{107} + 171348284q^{106} + 180551834q^{105} +
190110050q^{104} + 200021344q^{103} + 210275831q^{102} + 220888498q^{101} + 231839925q^{100} +
243142043q^{99} + 254771356q^{98} + 266736740q^{97} + 279009680q^{96} + 291594816q^{95} +
304458456q^{94} + 317600232q^{93} + 330980160q^{92} + 344592307q^{91} + 358389840q^{90} +
372359935q^{89} + 386448511q^{88} + 400635207q^{87} + 414857560q^{86} + 429087192q^{85} +
443252813q^{84} + 457316910q^{83} + 471199285q^{82} + 484852941q^{81} + 498188199q^{80} +
511148825q^{79} + 523635960q^{78} + 535583869q^{77} + 546885629q^{76} + 557466991q^{75} +
567213845q^{74} + 576045262q^{73} + 583842723q^{72} + 590520515q^{71} + 595958452q^{70} +
600070466q^{69} + 602738879q^{68} + 603882114q^{67} + 603390928q^{66} + 601194092q^{65} +
597198263q^{64} + 591350159q^{63} + 583580309q^{62} + 573862473q^{61} + 562160387q^{60} +
548483942q^{59} + 532839975q^{58} + 515283734q^{57} + 495873860q^{56} + 474719253q^{55} +
451937073q^{54} + 427694522q^{53} + 402170246q^{52} + 375589519q^{51} + 348188720q^{50} +
320243968q^{49} + 292037639q^{48} + 263880206q^{47} + 236078966q^{46} + 208952761q^{45} +
182803142q^{44} + 157923248q^{43} + 134571699q^{42} + 112977691q^{41} + 93319471q^{40} +
75727953q^{39} + 60271770q^{38} + 46962689q^{37} + 35748846q^{36} + 26524421q^{35} +
19131884q^{34} + 13376340q^{33} + 9034971q^{32} + 5873811q^{31} + 3659702q^{30} + 2174874q^{29} +
1225867q^{28} + 651244q^{27} + 323602q^{26} + 149094q^{25} + 62978q^{24} + 24066q^{23} + 8163q^{22} +
240q^{21} + 587q^{20} + 113q^{19} + 15q^{18} + q^{17}
Program II. This program produces the polynomials $A_Γ(α, q), M_Γ(α, q)$ and $I_Γ(α, q)$ for graphs with two vertices \{1, 2\}, in which $a$ and $b$ are the numbers of edge-loops of vertices 1 and 2 respectively, and $g$ is the number of edges between 1 and 2. Some sample outputs were carried out for the hyperbolic graph $\overset{1}{\square} ≡ \overset{2}{\square}$ for which $a = b = 0$ and $g = 3$.

```
> with(numtheory): with(combinat): readlib(mtaylor):
> varphi:=proc(r,q) local i:
>     if r=0 then 1 else product((1-q^i),i=1..r) fi
> end:
> a:=0: b:=0: g:=3:
> N:=6: P[1]:=[0]: t:=1:
> for n to N do
>     for i to numbpart(n) do
>         U:=partition(n)[i]:
>         r:=nops(U): t:=t+1:
>         P[t]:=[seq(U[r+1-j],j=1..r)]
>     od
> od:
> PX:=0:
> for i to t do
>     for j to t do
>         U:=P[i]: V:=P[j]:
>         aa:=nops(U): bb:=nops(V): ab:=min(aa,bb):
>         UU:=sum(U[s]^2,s=1..aa):
>         VV:=sum(V[s]^2,s=1..bb):
>         UV:=sum(U[s]*V[s],s=1..ab):
>         wU:=sum(U[s],s=1..aa):  wV:=sum(V[s],s=1..bb):
>         bU:=product(varphi(U[s]-U[s+1],1/q),s=1..aa-1)*
>             varphi(U[aa],1/q):
>         bV:=product(varphi(V[s]-V[s+1],1/q),s=1..bb-1)*
>             varphi(V[bb],1/q):
```

\text{varphi}(V[bb],1/q): \\
\text{> } d:=(a-1)\timesUU+(b-1)\timesVV+g\timesUV: \\
\text{> } PX:=PX+\text{simplify}(q^{-d/bU/bV})\times X^{wU}\times Y^{wV} \\
\text{> } \text{od} \\
\text{> } \text{od}: \\
\text{logm:=mtaylor(log(PX),[X,Y],2*N+1):} \\
\text{PY:=collect(logm,X):} \\
\text{for i from 0 to N do} \\
\text{\hspace{1cm} for j from 0 to N do} \\
\text{\hspace{2cm} if i+j>0 then} \\
\text{\hspace{3cm} coeff(PY,X,i): collect("Y,:)} \\
\text{\hspace{3cm} H[i,j,q]:=simplify(coeff("Y,j)*igcd(i,j))} \\
\text{\hspace{2cm} fi} \\
\text{\hspace{1cm} od} \\
\text{\hspace{1cm} od:} \\
\text{for m from 0 to N do} \\
\text{\hspace{1cm} for n from 0 to N do} \\
\text{\hspace{2cm} if m+n>0 then} \\
\text{\hspace{3cm} S:=0: DV:=divisors(igcd(m,n)):} \\
\text{\hspace{3cm} for r to nops(DV) do} \\
\text{\hspace{4cm} d:=DV[r]:} \\
\text{\hspace{4cm} S:=S+mobius(d)*subs(q=q^d,H[m/d,n/d,q]):} \\
\text{\hspace{3cm} od:} \\
\text{\hspace{3cm} AA[m,n,q]:=sort(expand(simplify(S*(q-1)/u))):} \\
\text{\hspace{2cm} if not "=0 and m<=n then print(A[Gamma]((m,n),q)=") fi} \\
\text{\hspace{2cm} fi} \\
\text{\hspace{1cm} od} \\
\text{\hspace{1cm} od:} \\
A_{\Gamma}(0,1,q) = 1 \\
A_{\Gamma}(1,1,q) = q^2 + q + 1 \\
A_{\Gamma}(1,2,q) = q^2 + q + 1
\[ A_\Gamma(1, 3, q) = 1 \]
\[ A_\Gamma(2, 2, q) = q^5 + q^4 + 3q^3 + 3q + 1 \]
\[ A_\Gamma(2, 3, q) = q^6 + q^5 + 3q^4 + 4q^3 + 5q^2 + 3q + 2 \]
\[ A_\Gamma(2, 4, q) = q^5 + q^4 + 3q^3 + 3q^2 + 3q + 1 \]
\[ A_\Gamma(2, 5, q) = q^2 + q + 1 \]
\[ A_\Gamma(3, 3, q) = q^{10} + q^9 + 3q^8 + 5q^7 + 8q^6 + 12q^5 + 17q^4 + 16q^3 + 14q^2 + 7q + 3 \]
\[ A_\Gamma(3, 4, q) = q^{12} + q^{11} + 3q^{10} + 5q^9 + 13q^7 + 21q^6 + 26q^5 + 31q^4 + 28q^3 + 21q^2 + 10q + 4 \]
\[ A_\Gamma(3, 5, q) = q^{12} + q^{11} + 3q^{10} + 5q^9 + 13q^7 + 21q^6 + 26q^5 + 31q^4 + 28q^3 + 21q^2 + 10q + 4 \]
\[ A_\Gamma(3, 6, q) = q^{10} + q^9 + 3q^8 + 5q^7 + 8q^6 + 12q^5 + 17q^4 + 16q^3 + 14q^2 + 7q + 3 \]
\[ A_\Gamma(4, 4, q) = q^{17} + q^{16} + 3q^{15} + 5q^{14} + 10q^{13} + 14q^{12} + 25q^{11} + 35q^{10} + 55q^9 + 71q^8 + 97q^7 + 111q^6 + 123q^5 + 109q^4 + 85q^3 + 49q^2 + 22q + 6 \]
\[ A_\Gamma(4, 5, q) = q^{20} + q^{19} + 3q^{18} + 5q^{17} + 10q^{16} + 15q^{15} + 26q^{14} + 38q^{13} + 59q^{12} + 83q^{11} + 119q^{10} + 156q^9 + 201q^8 + 235q^7 + 259q^6 + 248q^5 + 211q^4 + 145q^3 + 82q^2 + 32q + 9 \]
\[ A_\Gamma(4, 6, q) = q^{21} + q^{20} + 3q^{19} + 5q^{18} + 10q^{17} + 15q^{16} + 27q^{15} + 38q^{14} + 61q^{13} + 85q^{12} + 125q^{11} + 164q^{10} + 222q^9 + 266q^8 + 315q^7 + 327q^6 + 314q^5 + 252q^4 + 175q^3 + 93q^2 + 37q + 9 \]
\[ A_\Gamma(5, 5, q) = q^{26} + q^{25} + 3q^{24} + 5q^{23} + 10q^{22} + 16q^{21} + 27q^{20} + 41q^{19} + 65q^{18} + 95q^{17} + 141q^{16} + 197q^{15} + 278q^{14} + 374q^{13} + 499q^{12} + 636q^{11} + 791q^{10} + 928q^9 + 1040q^8 + 1071q^7 + 1018q^6 + 850q^5 + 624q^4 + 377q^3 + 187q^2 + 66q + 16 \]
\[ A_\Gamma(5, 6, q) = q^{30} + q^{29} + 3q^{28} + 5q^{27} + 10q^{26} + 16q^{25} + 28q^{24} + 42q^{23} + 68q^{22} + 100q^{21} + 150q^{20} + 213q^{19} + 307q^{18} + 421q^{17} + 580q^{16} + 771q^{15} + 1017q^{14} + 1295q^{13} + 1623q^{12} + 1949q^{11} + 2269q^{10} + 2498q^9 + 2605q^8 + 2504q^7 + 2208q^6 + 1721q^5 + 1174q^4 + 665q^3 + 307q^2 + 101q + 23 \]
\[ A_\Gamma(6, 6, q) = q^{37} + q^{36} + 3q^{35} + 5q^{34} + 10q^{33} + 16q^{32} + 29q^{31} + 43q^{30} + 71q^{29} + 105q^{28} + 161q^{27} + 231q^{26} + 341q^{25} + 473q^{24} + 672q^{23} + 914q^{22} + 1252q^{21} + 1660q^{20} + 2212q^{19} + 2854q^{18} + 3685q^{17} + 4619q^{16} + 5742q^{15} + 6924q^{14} + 8218q^{13} + 9384q^{12} + 10423q^{11} + 11009q^{10} + 11094q^9 + 10424q^8 + 9108q^7 + 7190q^6 + 5088q^5 + 3114q^4 + 1613q^3 + 669q^2 + 205q + 39 \]
\[ \phi(n, q) := \text{proc}(n, q) \text{ local } s, \text{ DN}, i, d:\n\]
\[ \text{ s:=0: } \text{ DN:=divisors}(n): \n\]
\[ \text{ for } i \text{ to nops(DN) do } \n\]
\[ \text{ d:=DN[i]: } \n\]
\[ \text{ s:=s+\text{mobius}(d)\ast(q^{n/d}-1) } \n\]
\[ \text{ od: } \n\]
\[ \frac{1}{n}\ast\text{simplify(s)} \n\]
\[ \text{ end: } \n\]
\[ e(m, n, q) := \text{proc}(m, n, q) \text{ local } s, \text{ DN}, i, r, u:\n\]
\[ \text{ s:=0: } \text{ u:=igcd}(m, n): \text{ DN:=divisors}(u): \n\]
\[ \text{ for } i \text{ to nops(DN) do } \n\]
\[ \text{ r:=DN[i]: } \n\]
\[ \text{ s:=s+r\ast\phi(r, q)\ast\text{subs}(q=q^r, H[m/r, n/r, q]): } \n\]
\[ \text{ od: } \n\]
\[ \text{simplify(s) } \n\]
\[ \text{ end: } \n\]
\[ K:=1: \n\]
\[ \text{ for } m \text{ from 0 to N do } \n\]
\[ \text{ for } n \text{ from 0 to N do } \n\]
\[ \text{ if } m+n>0 \text{ then } \n\]
\[ \text{ S:=0: } \text{ u:=igcd}(m, n): \text{ DV:=divisors}(u): \n\]
\[ \text{ for } i \text{ to nops(DV) do } \n\]
\[ \text{ d:=DV[i]: } \n\]
\[ \text{ S:=S+\text{mobius}(d)\ast e(m/d, n/d, q) } \n\]
\[ \text{ od: } \n\]
\[ \text{ II:=sort(simplify(S/u)): } \text{ K:=K\ast(1-X^m\ast Y^n)^(-") : } \n\]
\[ \text{ if not II=0 and } m\leq n \text{ then print } (I[\text{Gamma}](m, n, q)=II): \n\]
\[ \text{ fi } \n\]
\[ \text{ od } \n\]
\[ \text{ od: } \n\]

\[ I_1(0, 1, q) = 1 \]
\[ I_\Gamma(1,1,q) = q^2 + q + 1 \]
\[ I_\Gamma(1,2,q) = q^2 + q + 1 \]
\[ I_\Gamma(1,3,q) = 1 \]
\[ I_\Gamma(2,2,q) = q^5 + \frac{3}{2} q^4 + 3 q^3 + 3 q^2 + \frac{5}{2} q + 1 \]
\[ I_\Gamma(2,3,q) = q^6 + q^5 + 3 q^4 + 4 q^3 + 5 q^2 + 3 q + 2 \]
\[ I_\Gamma(2,4,q) = q^5 + \frac{3}{2} q^4 + 3 q^3 + 3 q^2 + \frac{5}{2} q + 1 \]
\[ I_\Gamma(2,5,q) = q^2 + q + 1 \]
\[ I_\Gamma(3,3,q) = q^{10} + q^9 + 3 q^8 + 5 q^7 + \frac{25}{3} q^6 + 12 q^5 + 17 q^4 + \frac{49}{3} q^3 + \frac{41}{3} q^2 + \frac{20}{3} q + 3 \]
\[ I_\Gamma(3,4,q) = q^{12} + q^{11} + 3 q^{10} + 5 q^9 + 9 q^8 + 13 q^7 + 21 q^6 + 26 q^5 + 31 q^4 + 28 q^3 + 21 q^2 + 10 q + 4 \]
\[ I_\Gamma(3,5,q) = q^{12} + q^{11} + 3 q^{10} + 5 q^9 + 9 q^8 + 13 q^7 + 21 q^6 + 26 q^5 + 31 q^4 + 28 q^3 + 21 q^2 + 10 q + 4 \]
\[ I_\Gamma(3,6,q) = q^{10} + q^9 + 3 q^8 + 5 q^7 + \frac{25}{3} q^6 + 12 q^5 + 17 q^4 + \frac{49}{3} q^3 + \frac{41}{3} q^2 + \frac{20}{3} q + 3 \]
\[ I_\Gamma(4,4,q) = q^{17} + q^{16} + 3 q^{15} + 5 q^{14} + 10 q^{13} + 14 q^{12} + 25 q^{11} + \frac{71}{2} q^{10} + 55 q^9 + \frac{287}{4} q^8 + 97 q^7 + \frac{225}{2} q^6 + \frac{245}{2} q^5 + 110 q^4 + \frac{167}{2} q^3 + \frac{105}{4} q^2 + \frac{41}{2} q + 6 \]
\[ I_\Gamma(4,5,q) = q^{20} + q^{19} + 3 q^{18} + 5 q^{17} + 10 q^{16} + 15 q^{15} + 26 q^{14} + 38 q^{13} + 59 q^{12} + 83 q^{11} + 119 q^{10} + 156 q^9 + 201 q^8 + 235 q^7 + 259 q^6 + 248 q^5 + 211 q^4 + 145 q^3 + 82 q^2 + 32 q + 9 \]
\[ I_\Gamma(4,6,q) = q^{21} + q^{20} + 3 q^{19} + 5 q^{18} + 10 q^{17} + 15 q^{16} + 27 q^{15} + 38 q^{14} + 61 q^{13} + \frac{171}{2} q^{12} + 125 q^{11} + \frac{329}{2} q^{10} + 222 q^9 + \frac{535}{2} q^8 + 315 q^7 + \frac{657}{2} q^6 + \frac{627}{2} q^5 + 253 q^4 + 173 q^3 + 92 q^2 + \frac{71}{2} q + 9 \]
\[ I_\Gamma(5,5,q) = q^{26} + q^{25} + 3 q^{24} + 5 q^{23} + 10 q^{22} + 16 q^{21} + 27 q^{20} + 41 q^{19} + 65 q^{18} + 95 q^{17} + 141 q^{16} + 197 q^{15} + 278 q^{14} + 374 q^{13} + 499 q^{12} + 636 q^{11} + \frac{3956}{5} q^{10} + 928 q^9 + 1040 q^8 + 1071 q^7 + 1018 q^6 + \frac{4251}{5} q^5 + 624 q^4 + 377 q^3 + \frac{934}{5} q^2 + \frac{329}{5} q + 16 \]
\[ I_\Gamma(5,6,q) = q^{30} + q^{29} + 3 q^{28} + 5 q^{27} + 10 q^{26} + 16 q^{25} + 28 q^{24} + 42 q^{23} + 68 q^{22} + 100 q^{21} + 150 q^{20} + 213 q^{19} + 307 q^{18} + 421 q^{17} + 580 q^{16} + 771 q^{15} + 1017 q^{14} + 1295 q^{13} + 1623 q^{12} + 1949 q^{11} + 2269 q^{10} + 2498 q^9 + 2605 q^8 + 2504 q^7 + 2208 q^6 + 1721 q^5 + 1174 q^4 + 665 q^3 + 307 q^2 + 101 q + 23 \]
\[ I_\Gamma(6,6,q) = q^{37} + q^{36} + 3 q^{35} + 5 q^{34} + 10 q^{33} + 16 q^{32} + 29 q^{31} + 43 q^{30} + 71 q^{29} + 105 q^{28} + 161 q^{27} + 231 q^{26} + 341 q^{25} + 473 q^{24} + 672 q^{23} + 914 q^{22} + 1252 q^{21} + \frac{3321}{2} q^{20} + \]
\[2212q^{19} + \frac{5709}{2}q^{18} + 3685q^{17} + \frac{9241}{2}q^{16} + \frac{17227}{3}q^{15} + \frac{13853}{2}q^{14} + 8218q^{13} + \frac{18777}{2}q^{12} + 10423q^{11} + \frac{22029}{2}q^{10} + \frac{22189}{2}q^9 + 10431q^8 + \frac{15245}{3}q^7 + 7195q^6 + \frac{18211}{2}q^5 + 13853q^4 + \frac{9629}{6}q^3 + \frac{1329}{2}q^2 + 602q + 39\]

\[> \text{mtaylor}(K, [X,Y], 2*N+1): \quad KL:=\text{simplify}(");\]

\[> \text{for m from 0 to N do}\]

\[> \quad \text{for n from 0 to N do}\]

\[> \quad \quad \text{if m+n>0 then}\]

\[> \quad \quad \quad \text{collect}(KL, X):\]

\[> \quad \quad \quad \text{coeff}(\";X,m): \quad \text{collect}(\";Y):\]

\[> \quad \quad \quad \text{M}[m,n,q]: \quad \text{sort}(\text{coeff}(\";Y,n)):\]

\[> \quad \quad \quad \text{if m<=n then print(M[\Gamma](m,n,q)=") fi:}\]

\[> \quad \quad \text{fi}\]

\[> \quad \text{od}\]

\[> \text{od:}\]

\[M_\Gamma(0,1,q) = 1 \quad \quad \quad \quad \quad M_\Gamma(1,1,q) = q^2 + q + 2\]

\[M_\Gamma(0,2,q) = 1 \quad \quad \quad \quad \quad M_\Gamma(1,2,q) = 2q^2 + 2q + 3\]

\[M_\Gamma(0,3,q) = 1 \quad \quad \quad \quad \quad M_\Gamma(1,3,q) = 2q^2 + 2q + 4\]

\[M_\Gamma(0,4,q) = 1 \quad \quad \quad \quad \quad M_\Gamma(1,4,q) = 2q^2 + 2q + 4\]

\[M_\Gamma(0,5,q) = 1 \quad \quad \quad \quad \quad M_\Gamma(1,5,q) = 2q^2 + 2q + 4\]

\[M_\Gamma(0,6,q) = 1 \quad \quad \quad \quad \quad M_\Gamma(1,6,q) = 2q^2 + 2q + 4\]

\[M_\Gamma(2,2,q) = q^5 + 2q^4 + 4q^3 + 8q^2 + 7q + 6\]

\[M_\Gamma(2,3,q) = q^6 + 2q^5 + 6q^4 + 10q^3 + 16q^2 + 12q + 10\]

\[M_\Gamma(2,4,q) = q^6 + 3q^5 + 8q^4 + 14q^3 + 22q^2 + 17q + 13\]

\[M_\Gamma(2,5,q) = q^6 + 3q^5 + 8q^4 + 14q^3 + 24q^2 + 19q + 15\]

\[M_\Gamma(2,6,q) = q^6 + 3q^5 + 8q^4 + 14q^3 + 24q^2 + 19q + 16\]

\[M_\Gamma(3,3,q) = q^{10} + q^9 + 3q^8 + 6q^7 + 13q^6 + 21q^5 + 37q^4 + 45q^3 + 50q^2 + 31q + 20\]

\[M_\Gamma(3,4,q) = q^{12} + q^{11} + 4q^{10} + 6q^9 + 13q^8 + 22q^7 + 42q^6 + 63q^5 + 93q^4 + 102q^3 + \]


\[
99q^2 + 58q + 32
\]

\[
M_\Gamma(3, 5, q) = 2q^{12} + 2q^{11} + 7q^{10} + 11q^9 + 23q^8 + 38q^7 + 71q^6 + 105q^5 + 149q^4 + 159q^3 + 148q^2 + 85q + 44
\]

\[
M_\Gamma(3, 6, q) = 2q^{12} + 2q^{11} + 8q^{10} + 12q^9 + 26q^8 + 44q^7 + 83q^6 + 124q^5 + 180q^4 + 194q^3 + 182q^2 + 104q + 54
\]

\[
M_\Gamma(4, 4, q) = q^{17} + q^{16} + 3q^{15} + 5q^{14} + 10q^{13} + 17q^{12} + 29q^{11} + 48q^{10} + 77q^9 + 119q^8 + 177q^7 + 253q^6 + 321q^5 + 368q^4 + 339q^3 + 266q^2 + 138q + 62
\]

\[
M_\Gamma(4, 5, q) = q^{20} + q^{19} + 3q^{18} + 6q^{17} + 11q^{16} + 18q^{15} + 32q^{14} + 50q^{13} + 83q^{12} + 125q^{11} + 195q^{10} + 284q^9 + 412q^8 + 558q^7 + 728q^6 + 837q^5 + 869q^4 + 732q^3 + 523q^2 + 253q + 101
\]

\[
M_\Gamma(4, 6, q) = q^{21} + 2q^{20} + 4q^{19} + 8q^{18} + 16q^{17} + 26q^{16} + 45q^{15} + 72q^{14} + 115q^{13} + 179q^{12} + 269q^{11} + 401q^{10} + 576q^9 + 805q^8 + 1066q^7 + 1333q^6 + 1481q^5 + 1472q^4 + 1197q^3 + 818q^2 + 382q + 144
\]

\[
M_\Gamma(5, 5, q) = q^{26} + q^{25} + 3q^{24} + 5q^{23} + 10q^{22} + 16q^{21} + 29q^{20} + 44q^{19} + 73q^{18} + 111q^{17} + 171q^{16} + 249q^{15} + 371q^{14} + 526q^{13} + 754q^{12} + 1034q^{11} + 1415q^{10} + 1847q^9 + 2355q^8 + 2807q^7 + 3158q^6 + 3159q^5 + 2825q^4 + 2089q^3 + 1301q^2 + 568q + 194
\]

\[
M_\Gamma(5, 6, q) = q^{30} + q^{29} + 3q^{28} + 5q^{27} + 11q^{26} + 17q^{25} + 31q^{24} + 47q^{23} + 79q^{22} + 119q^{21} + 185q^{20} + 270q^{19} + 405q^{18} + 578q^{17} + 830q^{16} + 1155q^{15} + 1609q^{14} + 2176q^{13} + 2930q^{12} + 3819q^{11} + 4903q^{10} + 6039q^9 + 7180q^8 + 7996q^7 + 8314q^6 + 7720q^5 + 6374q^4 + 4388q^3 + 2526q^2 + 1032q + 321
\]

\[
M_\Gamma(6, 6, q) = q^{37} + q^{36} + 3q^{35} + 5q^{34} + 10q^{33} + 16q^{32} + 29q^{31} + 45q^{30} + 73q^{29} + 112q^{28} + 173q^{27} + 257q^{26} + 383q^{25} + 550q^{24} + 792q^{23} + 1118q^{22} + 1565q^{21} + 2154q^{20} + 2948q^{19} + 3975q^{18} + 5311q^{17} + 6997q^{16} + 9111q^{15} + 11694q^{14} + 14766q^{13} + 18299q^{12} + 22120q^{11} + 25999q^{10} + 29363q^9 + 31644q^8 + 31921q^7 + 29781q^6 + 24917q^5 + 18457q^4 + 11510q^3 + 5966q^2 + 2231q + 612
\]
Program III. This program produces the polynomials $A_\Gamma(\alpha, q)$, $M_\Gamma(\alpha, q)$ and $I_\Gamma(\alpha, q)$ for graphs with three vertices $\{1, 2, 3\}$, in which $aa$, $bb$ and $cc$ are the numbers of edge-loops at vertices 1, 2 and 3 respectively, and $ab$, $bc$, $ca$ are the numbers of edges connecting 1 and 2, 2 and 3, 3 and 1 respectively. The sample outputs are carried out for the hyperbolic graph $\circ 1 \overleftrightarrow{2} \circ 3 \circ$ for which $aa = bb = cc = 0$, $ab = 2$, $bc = 1$, and $ca = 0$.

```maple
> with(numtheory): with(combinat): readlib(mtaylor):
> varphi:= proc(r,q) local i:
> if r=0 then 1 else product((1-q^i),i=1..r) fi end:
> aa:=0: bb:=0: cc:=0: ab:=2: bc:=1: ca:=0:
> N:=6: P[1]:=[0]: t:=1:
> for n to N do
> for i to numbpart(n) do
> U:=partition(n)[i]:
> r:=nops(U): t:=t+1:
> P[t]:=[seq(U[r+1-j], j=1..r)]
> od
> od:
> PX:=0:
> for i to t do
> for j to t do
> A:=P[i]: B:=P[j]: C:=P[k]:
> ua:=nops(A): ub:=nops(B): uc:=nops(C):
> aB:=min(ua,ub): bC:=min(ub,uc): cA:=min(uc,ua):
> AA:=sum(A[s]^2,s=1..ua):
```
BB := sum(B[s]^2, s=1..ub):

CC := sum(C[s]^2, s=1..uc):

AB := sum(A[s]*B[s], s=1..AB):

BC := sum(B[s]*C[s], s=1..bC):

CA := sum(C[s]*A[s], s=1..cA):

wA := sum(A[s], s=1..ua):

wB := sum(B[s], s=1..ub):

wC := sum(C[s], s=1..uc):

bA := product(varphi(A[s]-A[s+1], 1/q), s=1..ua-1) * varphi(A[ua], 1/q):

bB := product(varphi(B[s]-B[s+1], 1/q), s=1..ub-1) * varphi(B[ub], 1/q):

bC := product(varphi(C[s]-C[s+1], 1/q), w=1..uc-1) * varphi(C[uc], 1/q):

d := (aa-1)*AA + (bb-1)*BB + (cc-1)*CC + ab*AB + bc*BC + ca*CA:

PX := PX + simplify(q^d/bA/bB/bC)*X^wA*Y^wB*Z^wC

logm := mtaylor(log(PX), [X,Y,Z], 3*N+1):

PY := collect(logm, X):

for i from 0 to N do
  for j from 0 to N do
    for k from 0 to N do
      if i+j+k>0 then
        coeff(PY, X, i): collect("Y, coeff("Y, j):
> H[i,j,k,q]:=simplify(coeff("Z,k)*igcd(i,j,k))
> fi
> od
> od
> od:
> for m from 0 to N do
> for n from 0 to N do
> for l from 0 to N do
> if m+n+l>0 then
> S:=0: DV:=divisors(igcd(m,n,l)):
> for r to nops(DV) do
> d:=DV[r]:
> S:=S+mobius(d)*subs(q=q^d,H[m/d,n/d,l/d,q]):
> od:
> od:
> AI[m,n,l,q]:=sort(expand(simplify(S*(q-1)/u))):
> if not "=0 then print(A[Gamma](m,n,l,q=") fi
> fi
> od
> od
> od:

\[
A_\Gamma(0,0,1,q) = 1 \quad A_\Gamma(0,1,0,q) = 1 \\
A_\Gamma(0,1,1,q) = 1 \quad A_\Gamma(1,0,0,q) = 1 \\
A_\Gamma(1,1,0,q) = q + 1 \quad A_\Gamma(1,1,1,q) = q + 1 \\
A_\Gamma(1,2,0,q) = 1 \quad A_\Gamma(1,2,1,q) = q + 1 \\
A_\Gamma(1,2,2,q) = 1 \quad A_\Gamma(2,1,0,q) = 1 \\
A_\Gamma(2,1,1,q) = 1 \quad A_\Gamma(2,2,0,q) = q + 1
\]
\[ A_\Gamma(2, 2, 1, q) = q^2 + 2q + 2 \]
\[ A_\Gamma(2, 2, 2, q) = q + 1 \]
\[ A_\Gamma(2, 3, 0, q) = 1 \]
\[ A_\Gamma(2, 3, 1, q) = q^2 + 2q + 2 \]
\[ A_\Gamma(2, 3, 2, q) = q^2 + 2q + 2 \]
\[ A_\Gamma(2, 3, 3, q) = 1 \]
\[ A_\Gamma(2, 4, 1, q) = 1 \]
\[ A_\Gamma(2, 4, 2, q) = q + 1 \]
\[ A_\Gamma(2, 4, 3, q) = 1 \]
\[ A_\Gamma(3, 2, 1, q) = q + 1 \]
\[ A_\Gamma(3, 2, 2, q) = 1 \]
\[ A_\Gamma(3, 3, 0, q) = q + 1 \]
\[ A_\Gamma(3, 3, 1, q) = q^3 + 2q^2 + 4q + 3 \]
\[ A_\Gamma(3, 3, 2, q) = q^3 + 2q^2 + 4q + 3 \]
\[ A_\Gamma(3, 3, 3, q) = q + 1 \]
\[ A_\Gamma(3, 4, 0, q) = 1 \]
\[ A_\Gamma(3, 4, 1, q) = q^3 + 2q^2 + 4q + 3 \]
\[ A_\Gamma(3, 4, 2, q) = q^4 + 2q^3 + 6q^2 + 7q + 5 \]
\[ A_\Gamma(3, 4, 3, q) = q^3 + 2q^2 + 4q + 3 \]
\[ A_\Gamma(3, 4, 4, q) = 1 \]
\[ A_\Gamma(4, 3, 1, q) = q^2 + 2q + 2 \]
\[ A_\Gamma(4, 3, 2, q) = q^2 + 2q + 2 \]
\[ A_\Gamma(4, 3, 3, q) = 1 \]
\[ A_\Gamma(4, 4, 1, q) = q^4 + 2q^3 + 5q^2 + 7q + 5 \]
\[ A_\Gamma(4, 4, 0, q) = q + 1 \]
\[ A_\Gamma(4, 4, 2, q) = q^5 + 2q^4 + 6q^3 + 10q^2 + 13q + 7 \]
\[ A_\Gamma(4, 4, 3, q) = q^4 + 2q^3 + 5q^2 + 7q + 5 \]
\[ A_\Gamma(4, 4, 4, q) = q + 1 \]

> PH:=1:
> for m from 0 to N do
> for n from 0 to N do
> for l from 0 to N do
> if m+n+l > 0 then
> S:=0: u:=igcd(m,n,l): DV:=divisors(u):
> for k to nops(DV) do
> d:= DV[k]: Dd:=divisors(d):
> for i to nops(Dd) do
> r:=Dd[i]:
> S:= S+1/d*mobius(d/r)*subs(q=q^r, A[m/d,n/d,l/d,q]):
> od:
> od:
> od:
> II[m,n,l,q]:=sort(expand(simplify(S))):
> if not "=0 then print(I[Gamma](m,n,l,q)=") fi:
> PH:=PH*(1-X*m*Y*n*Z^1)^(-II[m,n,l,q])
> fi
> od
> od
> od:

\[
\begin{align*}
I_\Gamma(0,0,1,q) &= 1 & I_\Gamma(0,1,0,q) &= 1 \\
I_\Gamma(0,1,1,q) &= 1 & I_\Gamma(1,0,0,q) &= 1 \\
I_\Gamma(1,1,0,q) &= q + 1 & I_\Gamma(1,1,1,q) &= q + 1 \\
I_\Gamma(1,2,0,q) &= 1 & I_\Gamma(1,2,1,q) &= q + 1 \\
I_\Gamma(1,2,2,q) &= 1 & I_\Gamma(2,1,0,q) &= 1 \\
I_\Gamma(2,1,1,q) &= 1 & I_\Gamma(2,2,0,q) &= \frac{1}{2}q^2 + \frac{1}{2}q + 1 \\
I_\Gamma(2,2,1,q) &= q^2 + 2q + 2 & I_\Gamma(2,2,2,q) &= \frac{1}{2}q^2 + \frac{1}{2}q + 1 \\
I_\Gamma(2,3,0,q) &= 1 & I_\Gamma(2,3,1,q) &= q^2 + 2q + 2 \\
I_\Gamma(2,3,2,q) &= q^2 + 2q + 2 & I_\Gamma(2,3,3,q) &= 1 \\
I_\Gamma(2,4,1,q) &= 1 & I_\Gamma(2,4,2,q) &= \frac{1}{2}q^2 + \frac{1}{2}q + 1 \\
I_\Gamma(2,4,3,q) &= 1 & I_\Gamma(3,2,0,q) &= 1 \\
I_\Gamma(3,2,1,q) &= q + 1 & I_\Gamma(3,2,2,q) &= 1 \\
I_\Gamma(3,3,0,q) &= \frac{1}{3}q^3 + \frac{2}{3}q + 1 & I_\Gamma(3,3,1,q) &= q^3 + 2q^2 + 4q + 3 \\
I_\Gamma(3,3,2,q) &= q^3 + 2q^2 + 4q + 3 & I_\Gamma(3,3,3,q) &= \frac{1}{3}q^3 + \frac{2}{3}q + 1 \\
I_\Gamma(3,4,0,q) &= 1 & I_\Gamma(3,4,1,q) &= q^3 + 2q^2 + 4q + 3 \\
I_\Gamma(3,4,2,q) &= q^4 + 2q^3 + 6q^2 + 7q + 5 & I_\Gamma(3,4,3,q) &= q^3 + 2q^2 + 4q + 3 \\
I_\Gamma(3,4,4,q) &= 1 & I_\Gamma(4,3,0,q) &= 1 \\
I_\Gamma(4,3,1,q) &= q^2 + 2q + 2 & I_\Gamma(4,3,2,q) &= q^2 + 2q + 2 \\
I_\Gamma(4,3,3,q) &= 1 & I_\Gamma(4,4,0,q) &= \frac{1}{4}q^4 + \frac{1}{4}q^2 + \frac{1}{2}q + 1 \\
I_\Gamma(4,4,1,q) &= q^4 + 2q^3 + 5q^2 + 7q + 5
\end{align*}
\]
\[ I_{\Gamma}(4,4,2,q) = q^{5} + \frac{5}{2}q^{4} + 6q^{3} + \frac{21}{4}q^{2} + 12q + 7 \]

\[ I_{\Gamma}(4,4,3,q) = q^{4} + 2q^{3} + 5q^{2} + 7q + 5 \]

\[ I_{\Gamma}(4,4,4,q) = \frac{1}{4}q^{4} + \frac{1}{4}q^{2} + \frac{1}{2}q + 1 \]

\[ \text{> PH:=mtaylor(PH,[X,Y,Z],3*N+1):} \]
\[ \text{> for i from 0 to N do} \]
\[ \text{> for j from 0 to N do} \]
\[ \text{> for k from 0 to N do} \]
\[ \text{> if i+j+k\(>\)0 then} \]
\[ \text{> coeff(PH,X,i): collect("Y): coeff("Y,j): coeff("Z,k):} \]
\[ \text{> M[i,j,k,q]:=sort(simplify(")):} \]
\[ \text{> if not " = 0 then print(M[Gamma](i,j,k,q=") fi} \]
\[ \text{> fi} \]
\[ \text{> od} \]
\[ \text{> od:} \]

\[ M_{\Gamma}(0,0,1,q) = 1 \quad M_{\Gamma}(0,0,2,q) = 1 \]

\[ M_{\Gamma}(0,0,3,q) = 1 \quad M_{\Gamma}(0,0,4,q) = 1 \]

\[ M_{\Gamma}(0,1,0,q) = 1 \quad M_{\Gamma}(0,1,1,q) = 2 \]

\[ M_{\Gamma}(0,1,2,q) = 2 \quad M_{\Gamma}(0,1,3,q) = 2 \]

\[ M_{\Gamma}(0,1,4,q) = 2 \quad M_{\Gamma}(0,2,0,q) = 1 \]

\[ M_{\Gamma}(0,2,1,q) = 2 \quad M_{\Gamma}(0,2,2,q) = 3 \]

\[ M_{\Gamma}(0,2,3,q) = 3 \quad M_{\Gamma}(0,2,4,q) = 3 \]

\[ M_{\Gamma}(0,3,0,q) = 1 \quad M_{\Gamma}(0,3,1,q) = 2 \]

\[ M_{\Gamma}(0,3,2,q) = 3 \quad M_{\Gamma}(0,3,3,q) = 4 \]

\[ M_{\Gamma}(0,3,4,q) = 4 \quad M_{\Gamma}(0,4,0,q) = 1 \]

\[ M_{\Gamma}(0,4,1,q) = 2 \quad M_{\Gamma}(0,4,2,q) = 3 \]

\[ M_{\Gamma}(0,4,3,q) = 4 \quad M_{\Gamma}(0,4,4,q) = 5 \]
\[
M_\Gamma(1, 0, 0, q) = 1 \quad M_\Gamma(1, 0, 1, q) = 1 \\
M_\Gamma(1, 0, 2, q) = 1 \quad M_\Gamma(1, 0, 3, q) = 1 \\
M_\Gamma(1, 0, 4, q) = 1 \quad M_\Gamma(1, 1, 0, q) = q + 2 \\
M_\Gamma(1, 1, 1, q) = 2q + 4 \quad M_\Gamma(1, 1, 2, q) = 2q + 4 \\
M_\Gamma(1, 1, 3, q) = 2q + 4 \quad M_\Gamma(1, 1, 4, q) = 2q + 4 \\
M_\Gamma(1, 2, 0, q) = q + 3 \quad M_\Gamma(1, 2, 1, q) = 4q + 7 \\
M_\Gamma(1, 2, 2, q) = 5q + 10 \quad M_\Gamma(1, 2, 3, q) = 5q + 10 \\
M_\Gamma(1, 2, 4, q) = 5q + 10 \quad M_\Gamma(1, 3, 0, q) = q + 3 \\
M_\Gamma(1, 3, 1, q) = 4q + 8 \quad M_\Gamma(1, 3, 2, q) = 7q + 13 \\
M_\Gamma(1, 3, 3, q) = 8q + 16 \quad M_\Gamma(1, 3, 4, q) = 8q + 16 \\
M_\Gamma(1, 4, 0, q) = q + 3 \quad M_\Gamma(1, 4, 1, q) = 4q + 8 \\
M_\Gamma(1, 4, 2, q) = 7q + 14 \quad M_\Gamma(1, 4, 3, q) = 10q + 19 \\
M_\Gamma(1, 4, 4, q) = 11q + 22 \quad M_\Gamma(2, 0, 0, q) = 1 \\
M_\Gamma(2, 0, 1, q) = 1 \quad M_\Gamma(2, 0, 2, q) = 1 \\
M_\Gamma(2, 0, 3, q) = 1 \quad M_\Gamma(2, 0, 4, q) = 1 \\
M_\Gamma(2, 1, 0, q) = q + 3 \quad M_\Gamma(2, 1, 1, q) = 2q + 6 \\
M_\Gamma(2, 1, 2, q) = 2q + 6 \quad M_\Gamma(2, 1, 3, q) = 2q + 6 \\
M_\Gamma(2, 1, 4, q) = 2q + 6 \quad M_\Gamma(2, 2, 0, q) = q^2 + 3q + 6 \\
M_\Gamma(2, 2, 1, q) = 3q^2 + 10q + 15 \quad M_\Gamma(2, 2, 2, q) = 4q^2 + 13q + 21 \\
M_\Gamma(2, 2, 3, q) = 4q^2 + 13q + 21 \quad M_\Gamma(2, 2, 4, q) = 4q^2 + 13q + 21 \\
M_\Gamma(2, 3, 0, q) = q^2 + 4q + 8 \quad M_\Gamma(2, 3, 1, q) = 6q^2 + 18q + 24 \\
M_\Gamma(2, 3, 2, q) = 11q^2 + 32q + 40 \quad M_\Gamma(2, 3, 3, q) = 12q^2 + 36q + 48 \\
M_\Gamma(2, 3, 4, q) = 12q^2 + 36q + 48 \quad M_\Gamma(2, 4, 0, q) = q^2 + 4q + 9 \\
M_\Gamma(2, 4, 1, q) = 6q^2 + 20q + 29 \quad M_\Gamma(2, 4, 2, q) = 15q^2 + 43q + 55 \\
M_\Gamma(2, 4, 3, q) = 20q^2 + 59q + 75 \quad M_\Gamma(2, 4, 4, q) = 21q^2 + 63q + 84 \\
M_\Gamma(3, 0, 0, q) = 1 \quad M_\Gamma(3, 0, 1, q) = 1 
\]
\[
\begin{align*}
M_G(3, 0, 2, q) &= 1 & M_G(3, 0, 3, q) &= 1 \\
M_G(3, 0, 4, q) &= 1 & M_G(3, 1, 0, q) &= q + 3 \\
M_G(3, 1, 1, q) &= 2q + 6 & M_G(3, 1, 2, q) &= 2q + 6 \\
M_G(3, 1, 3, q) &= 2q + 6 & M_G(3, 1, 4, q) &= 2q + 6 \\
M_G(3, 2, 0, q) &= q^2 + 4q + 8 & M_G(3, 2, 1, q) &= 3q^2 + 14q + 20 \\
M_G(3, 2, 2, q) &= 4q^2 + 18q + 28 & M_G(3, 2, 3, q) &= 4q^2 + 18q + 28 \\
M_G(3, 2, 4, q) &= 4q^2 + 18q + 28 & M_G(3, 3, 0, q) &= q^3 + 3q^2 + 9q + 14 \\
M_G(3, 3, 1, q) &= 4q^3 + 16q^2 + 40q + 44 & M_G(3, 3, 2, q) &= 7q^3 + 29q^2 + 71q + 74 \\
M_G(3, 3, 3, q) &= 8q^3 + 32q^2 + 80q + 88 & M_G(3, 3, 4, q) &= 8q^3 + 32q^2 + 80q + 88 \\
M_G(3, 4, 0, q) &= q^3 + 4q^2 + 12q + 19 & M_G(3, 4, 1, q) &= 8q^3 + 29q^2 + 66q + 68 \\
M_G(4, 0, 0, q) &= 1 & M_G(3, 4, 2, q) &= q^4 + 20q^3 + 74q^2 + 146q + 135 \\
M_G(4, 0, 1, q) &= 1 & M_G(3, 4, 3, q) &= q^4 + 27q^3 + 99q^2 + 200q + 184 \\
M_G(4, 0, 2, q) &= 1 & M_G(3, 4, 4, q) &= q^4 + 28q^3 + 103q^2 + 212q + 203 \\
M_G(4, 0, 3, q) &= 1 & M_G(4, 0, 4, q) &= 1 \\
M_G(4, 1, 0, q) &= q + 3 & M_G(4, 1, 1, q) &= 2q + 6 \\
M_G(4, 1, 2, q) &= 2q + 6 & M_G(4, 1, 3, q) &= 2q + 6 \\
M_G(4, 1, 4, q) &= 2q + 6 & M_G(4, 2, 0, q) &= q^2 + 4q + 9 \\
M_G(4, 2, 1, q) &= 3q^2 + 14q + 22 & M_G(4, 2, 2, q) &= 4q^2 + 18q + 31 \\
M_G(4, 2, 3, q) &= 4q^2 + 18q + 31 & M_G(4, 2, 4, q) &= 4q^2 + 18q + 31 \\
M_G(4, 3, 0, q) &= q^3 + 4q^2 + 12q + 19 & M_G(4, 3, 1, q) &= 4q^3 + 22q^2 + 54q + 60 \\
M_G(4, 3, 2, q) &= 7q^3 + 40q^2 + 96q + 101 \\
M_G(4, 3, 3, q) &= 8q^3 + 44q^2 + 108q + 120 \\
M_G(4, 3, 4, q) &= 8q^3 + 44q^2 + 108q + 120 \\
M_G(4, 4, 0, q) &= q^4 + 3q^3 + 10q^2 + 23q + 32 \\
M_G(4, 4, 1, q) &= 5q^4 + 22q^3 + 68q^2 + 128q + 119
\end{align*}
\]
\[ M_\Gamma(4, 4, 2, q) = q^5 + 13q^4 + 56q^3 + 170q^2 + 287q + 240 \]
\[ M_\Gamma(4, 4, 3, q) = q^5 + 17q^4 + 75q^3 + 228q^2 + 392q + 327 \]
\[ M_\Gamma(4, 4, 4, q) = q^5 + 18q^4 + 78q^3 + 238q^2 + 415q + 359 \]
Appendix B

The JCT Paper


Generalizing the Recursion Relationship
for the Partition Function

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The known recursion relationship for the partition function \( p(n) \) which represents the number of partitions of the positive integer \( n \) is exhibited as the limit as \( q \to \infty \) in one identity and as the case 1 substituted for \( q \) in a second formula that arise from a matrix problem over the field of \( q \) elements. Such identities can be further generalized.

1. INTRODUCTION

The partition function \( p(n) \) is determined by the following identity:

\[
1 + \sum_{n=1}^{\infty} p(n)t^n = \prod_{n=1}^{\infty} (1 - t^n)^{-1}.
\]

---

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Differentiation of this followed by multiplication by \( t \) yields

\[
p(n) = \frac{1}{n} \sum_{r=1}^{n} \sigma(r)p(n-r),
\]

where \( \sigma(r) \) represents the sum of divisors of \( r \).

In this paper we write \( \lambda \vdash n \) to denote “\( \lambda \) is a partition of \( n \)”, \( l(\lambda) \) to denote the largest part of \( \lambda \), and set \( n = |\lambda| \). For positive integer \( r \), let \( \varphi_r(q) \) denote the polynomial \((1 - q)(1 - q^2) \cdots (1 - q^r)\) and set \( \varphi_0(q) = 1 \). Given a partition \( \mu = (1^{\mu_1}2^{\mu_2} \cdots) \), we let \( b_{\mu}(q) = \prod_{i \geq 1} \varphi_{\mu_i}(q) \), and define

\[
P_1(n, q) = \sum_{\mu \vdash n} \frac{1}{b_{\mu}(q^{-1})}.
\]

Thus \( \lim_{q \to \infty} P_1(n, q) = p(n) \). Likewise we define

\[
H_1(n, q) = \sum_{d \mid n} n \frac{1}{d} \frac{1}{1 - q^{-d}}.
\]

thus \( \lim_{q \to \infty} H_1(n, q) = \sum_{d \mid n} \frac{n}{d} = \sigma(n) \).

Two generalizations of formula (1) are given in this paper. One is the following:

\[
P_1(n, q) = \frac{1}{n} \sum_{r=1}^{n} H_1(r, q)P_1(n-r, q),
\]

which turns out to be formula (1) when \( q \to \infty \). The other is

\[
M_1(n, q) = \frac{1}{n} \sum_{r=1}^{n} E_1(r, q)M_1(n-r, q),
\]

where \( M_1(n, q) = \sum_{\lambda \vdash n} q^{l(\lambda)} \), \( E_1(n, q) = \sum_{d \mid n} dq^{\frac{n}{d}} \). It turns out to be formula (1) when \( q \) is substituted by 1. More general functions \( P_g(n, q), H_g(n, q), M_g(n, q) \) and \( E_g(n, q) \) for \( g \geq 1 \) are introduced later, and satisfy similar formulae.

2. THE MATRIX PROBLEM

Let \( g \) be a fixed non-negative integer, \( q \) a prime power, \( \mathbb{F}_q \) the field of \( q \) elements, \( \bar{\mathbb{F}}_q \) the algebraic closure of \( \mathbb{F}_q \). Let \( n \) be any positive integer, \( G_n \) be the general linear group of degree \( n \) over \( \mathbb{F}_q \), and \( C_n \) be a subset of \( G_n \) containing one element
from each conjugacy class of $G_n$. Let $M_g(n, q)$ denote the number of classes of ordered $g$-tuples of $n \times n$ matrices over $\mathbb{F}_q$ up to simultaneous similarity. In this context the Molien-Burnside orbit counting formula becomes:

$$M_g(n, q) = \frac{1}{|G_n|} \sum_{\gamma \in G_n} |X_\gamma|^g = \sum_{\gamma \in G_n} \frac{|X_\gamma|^g}{|Z_\gamma|},$$

where $X_\gamma = \{x \in M_n(\mathbb{F}_q) \mid \gamma x \gamma^{-1} = x\}$, $Z_\gamma = \{x \in G_n \mid \gamma x \gamma^{-1} = x\}$.

Let $\Phi$ denote the set of all irreducible monic polynomials in $t$ over $\mathbb{F}_q$ other than $t$, and let $\mathcal{P}$ denote the set of partitions of positive integers. For each $f \in \Phi$, let $d(f)$ denote the degree of $f$, $J(f)$ denote the companion matrix of $f$ (see Macdonald [4], page 140), and for $m \geq 1$ let $J_m(f)$ denote the Jordan block matrix consisting of $m^2$ block $d(f) \times d(f)$ matrices with $J(f)$ in each diagonal block. For any partition $\pi = (1^{n_1}2^{n_2} \cdots) \in \mathcal{P}$, let

$$J(f, \pi) = J_1(f)^{n_1} \oplus J_2(f)^{n_2} \oplus \cdots.$$ 

This is a diagonal block matrix with $n_i$ copies of $J_i(f)$ ($i \geq 1$) in the diagonal. Then any element of $G_n$ has Jordan canonical form as follows:

$$J(f_1, \pi_1) \oplus J(f_2, \pi_2) \oplus \cdots \oplus J(f_k, \pi_k),$$

with $\sum_{i=1}^{k} d(f_i) |\pi_i| = n$, where $f_1, \cdots, f_k$ are distinct polynomials from $\Phi$, and where $\pi_1, \cdots, \pi_k \in \mathcal{P}$, $k$ is some positive integer.

Let $\mathbb{N}$ denote the set of positive integers. Given a partition $\mu = (\mu_1, \mu_2, \cdots)$ with $\mu_1 \geq \mu_2 \geq \cdots$, we define $n(\mu) = \sum_{i \geq 1} (i - 1) \mu_i$.

**Lemma 1.** Given a partition $\mu = (1^{n_1}2^{n_2} \cdots k^{n_k})$, $k \in \mathbb{N}$, there holds:

$$\sum_{i=1}^{k} \sum_{j=1}^{k} n_i n_j \min(i, j) = |\mu| + 2n(\mu).$$

**Proof.** Let $\mu' = (\mu'_1, \mu'_2, \cdots)$ be the conjugate partition of $\mu$, then $\mu'_i = \sum_{j \geq 1} n_j$. 


Moreover \( n(\mu) = \sum_{i \geq 1} \binom{\mu_i}{2} \) (Macdonald [4], page 3). Thus,

\[
2n(\mu) = 2 \left( \frac{n_1 + n_2 + \cdots + n_k}{2} \right) + 2 \left( \frac{n_2 + n_3 + \cdots + n_k}{2} \right) + \cdots + 2 \left( \frac{n_k}{2} \right)
\]

\[
= (n_1 + n_2 + \cdots + n_k)(n_1 + n_2 + \cdots + n_k - 1) + (n_2 + n_3 + \cdots + n_k)(n_2 + n_3 + \cdots + n_k - 1) + \cdots + n_k(n_k - 1)
\]

\[
= n_1(n_1 + n_2 + n_3 + \cdots + n_k) + n_2(n_1 + 2n_2 + 2n_3 + \cdots + 2n_k) + n_3(n_1 + 2n_2 + 3n_3 + \cdots + 3n_k) + \cdots + n_k(n_1 + 2n_2 + \cdots + kn_k) - (n_1 + 2n_2 + 3n_3 + \cdots + kn_k)
\]

\[
= \sum_{i=1}^{k} \sum_{j=1}^{k} n_in_j \min(i,j) - |\mu|.
\]

**Lemma 2.** For any \( f \in \Phi \) with \( d(f) = d, \mu = (1^{n_1}2^{n_2}\cdots) \in \mathcal{P} \), the following formulae hold:

\[
|Z_{J(f,\mu)}| = q^{d(|\mu| + 2n(\mu))} b_\mu(q^{-d}),
\]

\[
|X_{J(f,\mu)}| = q^{d(|\mu| + 2n(\mu))}.
\]

**Proof.** The first formula is given by Macdonald ([4], page 139). The second is proved as follows.

Let \( A = \mathbb{F}_q[x], \bar{A} = A \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \). Given any \( m \times m \) matrix \( \alpha \) over \( \mathbb{F}_q \), we define an \( \bar{A} \)-module structure on \( \mathbb{F}_q^m \) by \( x \cdot v = \alpha v \) for \( v \in \mathbb{F}_q^m \). Let \( V_\alpha \) denote this module, and define \( \bar{V}_\alpha = V_\alpha \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \). Obviously, \( X_{J(f,\mu)} = \text{End}_A(V_{J(f,\mu)}) \). Note that \( \text{End}_A(V_{J(f,\mu)}) \) is a finite dimensional vector space over \( \mathbb{F}_q \), which has the same dimension as \( \text{End}_{\bar{A}}(\bar{V}_{J(f,\mu)}) \) over \( \overline{\mathbb{F}_q} \). This reduces the calculation to the corresponding calculation over the field \( \overline{\mathbb{F}_q} \). Since \( J(f,\mu) = J_1(f)^{n_1} \oplus J_2(f)^{n_2} \oplus \cdots \), it follows that \( \bar{V}_{J(f,\mu)} \cong \bar{V}_{J_1(f)}^{n_1} \oplus \bar{V}_{J_2(f)}^{n_2} \oplus \cdots \). Therefore,

\[
\text{End}_{\bar{A}}(\bar{V}_{J(f,\mu)}) \cong \sum_{i,j \geq 1} \text{Hom}_{\bar{A}}(\bar{V}_{J_i(f)}^{n_i}, \bar{V}_{J_j(f)}^{n_j}) \cong \sum_{i,j \geq 1} \text{Hom}_{\bar{A}}(\bar{V}_{J_i(f)}, \bar{V}_{J_j(f)})^{n_in_j}.
\]

The classical theorem that any finite field is separable implies that for any irreducible monic polynomial \( f(t) \) in \( \mathbb{F}_q[t] \) there is an invertible matrix \( X \) with entries in \( \overline{\mathbb{F}_q} \) such that \( XJ(f)X^{-1} \) is diagonal with distinct diagonal entries \( \lambda_1, \ldots, \lambda_d \).
Here \( f(t) = \prod_{i=1}^{d} (t - \lambda_i) \). Thus \( J_i(f) \) is similar over \( \mathbb{F}_q \) to \( J(\lambda_1, i) + \cdots + J(\lambda_d, i) \), where \( J(\lambda, i) \) represents the Jordan normal form. Therefore,

\[
\text{Hom}_{\tilde{A}} \left( \tilde{V}_{J_1(f)}, \tilde{V}_{J_d(f)} \right) \cong \bigoplus_{s=1}^{d} \bigoplus_{t=1}^{d} \text{Hom}_{\tilde{A}} \left( \tilde{V}_{J(s, i)}, \tilde{V}_{J(t, j)} \right).
\]

Note that every \( \tilde{V}_{J(s, i)} \) (\( 1 \leq s \leq d, \ i \geq 1 \)) is an indecomposable \( \tilde{A} \)-module whose sole composition factor is \( \tilde{V}_{J(s, 1)} \). Thus \( \text{Hom}_{\tilde{A}} \left( \tilde{V}_{J(s, i)}, \tilde{V}_{J(t, j)} \right) \neq 0 \) if and only if \( s = t \). Therefore,

\[
\text{Hom}_{\tilde{A}} \left( \tilde{V}_{J_1(f)}, \tilde{V}_{J_d(f)} \right) \cong \bigoplus_{s=1}^{d} \text{Hom}_{\tilde{A}} \left( \tilde{V}_{J(s, i)}, \tilde{V}_{J(s, j)} \right).
\]

An elementary calculation with matrices reveals that

\[
\dim_{\mathbb{F}_q} \text{Hom}_{\tilde{A}} \left( \tilde{V}_{J(s, i)}, \tilde{V}_{J(t, j)} \right) = \min(i, j).
\]

Thus

\[
\dim_{\mathbb{F}_q} X_{J(f, \mu)} = \sum_{i \geq 1} \sum_{j \geq 1} n_i n_j \dim_{\mathbb{F}_q} \text{Hom}_{\tilde{A}} \left( \tilde{V}_{J_i(f)}, \tilde{V}_{J_j(f)} \right)
= \sum_{i \geq 1} \sum_{j \geq 1} n_i n_j \min(i, j) d
= d(|\mu| + 2n(\mu)) \quad \text{(by Lemma 1)}.
\]

Finally, \(| X_{J(f, \mu)} | = q^{d(|\mu| + 2n(\mu))} \). Note that these formulae depend only on the degree of \( f \).

Now we let \( P_g(0, q) = 1 \). For \( n \geq 1 \), we define

\[
P_g(n, q) = \sum_{\pi \vdash n} \frac{q^{g-1}(|\pi| + 2n(\pi))}{b_{\pi}(q^{-1})}.
\]

**Theorem 3.** The generating function of \( M_g(n, q) \) \((n \geq 0)\) can be factorized as follows:

\[
1 + \sum_{n=1}^{\infty} M_g(n, q) t^n = \prod_{d=1}^{\infty} \left( 1 + \sum_{j=1}^{\infty} P_g(j, q^d) t^{dj} \right)^{\phi_d(q)},
\]

where \( t \) is an indeterminate, and where \( \phi_d(q) \) denotes the number of polynomials in \( \Phi \) which have degree \( d \). Note that exceptionally \( \phi_1(q) = q - 1 \).
Proof.

\[ 1 + \sum_{n=1}^{\infty} M_g(n, q) t^n = 1 + \sum_{n=1}^{\infty} \sum_{\gamma \in C_n} \frac{|X_{\gamma}|^g}{|Z_{\gamma}|} t^n \]

\[ = 1 + \sum_{n=1}^{\infty} \sum_{f_i, \pi_i} \frac{|X_{J(f_1, \pi_1)} \oplus \cdots \oplus J(f_k, \pi_k)|^g}{|Z_{J(f_1, \pi_1)} \oplus \cdots \oplus Z_{J(f_k, \pi_k)}|} t^{d(f_1)|\pi_1| + \cdots + d(f_k)|\pi_k|} \]

where the summation is over \( f_i \in \Phi \), which are distinct, \( \pi_i \in \mathcal{P} \), \( 1 \leq i \leq k \), \( k \in \mathbb{N} \) such that \( \sum_{i=1}^{k} d(f_i)|\pi_i| = n \).

\[ = 1 + \sum_{f_i, \pi_i} |X_{J(f_1, \pi_1)}|^g \cdots |X_{J(f_k, \pi_k)}|^g t^{d(f_1)|\pi_1| + \cdots + d(f_k)|\pi_k|} \]

where the summation is over \( f_i \in \Phi \), which are distinct, \( \pi_i \in \mathcal{P} \), \( 1 \leq i \leq k \), \( k \in \mathbb{N} \).

\[ = \prod_{f \in \Phi} \left( 1 + \sum_{\pi \in \mathcal{P}} \frac{|X_{J(f, \pi)}|^g}{|Z_{J(f, \pi)}|} t^{d(f)|\pi|} \right) \]

\[ = \prod_{f \in \Phi} \left( 1 + \sum_{j=1}^{\infty} \sum_{\pi \vdash j} \frac{|X_{J(f, \pi)}|^g}{|Z_{J(f, \pi)}|} t^{d(f)j} \right) \]

\[ = \prod_{f \in \Phi} \left( 1 + \sum_{j=1}^{\infty} P_g(j, q^{d(f)}) t^{d(f)j} \right) \quad \text{(by Lemma 2)} \]

\[ = \prod_{d=1}^{\infty} \left( 1 + \sum_{j=1}^{\infty} P_g(j, q^d) t^{d(j)} \right)^{\phi_d(q)} \].

Let \( M = (M_1, \ldots, M_g) \) be an ordered \( g \)-tuple of \( n \times n \) matrices over \( \mathbb{F}_q \). Recall that \( M \) is said to be decomposable over \( \mathbb{F}_q \) if there exists an invertible \( n \times n \) matrix \( X \) with entries in \( \mathbb{F}_q \) such that

\[ (XM_1X^{-1}, \ldots, XM_gX^{-1}) = \left( \begin{pmatrix} A_1 & 0 \\ 0 & B_1 \end{pmatrix}, \ldots, \begin{pmatrix} A_g & 0 \\ 0 & B_g \end{pmatrix} \right), \]

and \( A_1, \ldots, A_g \) are square matrices with the same degree. Otherwise \( M \) is said to be indecomposable over \( \mathbb{F}_q \). Now, let \( I_g(n, q) \) denote the number of classes of indecomposable ordered \( g \)-tuples of \( n \times n \) matrices over \( \mathbb{F}_q \) up to simultaneous similarity. The Krull-Schmidt Theorem states that every \( g \)-tuple of \( n \times n \) matrices over a field can be written as a direct sum of indecomposable \( g \)-tuples in a unique
way up to order. Thus

\[ 1 + \sum_{n=1}^{\infty} M_g(n, q) t^n = \prod_{i=1}^{\infty} (1 - t^i)^{-I_g(i, q)}. \]

Then, by Theorem 3,

\[ \prod_{d=1}^{\infty} \left( 1 + \sum_{j=1}^{\infty} P_g(j, q^d) t^{dj} \right)^{\phi_d(q)} = \prod_{i=1}^{\infty} (1 - t^i)^{-I_g(i, q)}. \]

This implies upon taking logarithms:

\[ \sum_{d=1}^{\infty} \phi_d(q) \log \left( 1 + \sum_{j=1}^{\infty} P_g(j, q^d) t^{dj} \right) = \sum_{i=1}^{\infty} I_g(i, q) \log \frac{1}{1 - t^i}. \]

Now, we define

\[ \log_m g(q, t) = \log \left( 1 + \sum_{j=1}^{\infty} P_g(j, q) t^j \right). \]

Note that the constant term of \( \log_m g(q, t) \) is 0. Therefore,

\[ (2) \quad \log \left( 1 + \sum_{n=1}^{\infty} M_g(n, q) t^n \right) = \sum_{i=1}^{\infty} I_g(i, q) \log \frac{1}{1 - t^i} = \sum_{d=1}^{\infty} \phi_d(q) \log_m g(q^d, t^d). \]

For \( i \geq 1 \), we define \( H_g(i, q) \) by

\[ \log \left( 1 + \sum_{j=1}^{\infty} P_g(j, q) t^j \right) = \sum_{i=1}^{\infty} \frac{1}{i} H_g(i, q) t^i, \]

and define \( E_g(i, q) \) by

\[ \log \left( 1 + \sum_{j=1}^{\infty} M_g(j, q) t^j \right) = \sum_{i=1}^{\infty} \frac{1}{i} E_g(i, q) t^i. \]

Thus, identity (2) implies that

\[ E_g(n, q) = \sum_{r | n} r \phi_r(q) H_g \left( \frac{n}{r}, q^r \right). \]
Lemma 4.

\[ I_g(n, q) = \frac{1}{n} \sum_{d \mid n} \mu(d)E_g\left(\frac{n}{d}, q\right), \]

where \( \mu \) is the classical Möbius function.

Proof.

\[
\begin{align*}
\sum_{i=1}^{\infty} \frac{1}{i} E_g(i, q) t^i &= \sum_{i=1}^{\infty} I_g(i, q) \log \frac{1}{1-t^i} \\
&= \sum_{i=1}^{\infty} I_g(i, q) \sum_{j=1}^{\infty} \frac{1}{j} t^{ij} \\
&= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{j} I_g(i, q) t^{ij}.
\end{align*}
\]

By comparing the coefficients of \( t^n \) on both sides, we have

\[
\frac{1}{n} E_g(n, q) = \sum_{d \mid n} \frac{1}{d} I_g\left(\frac{n}{d}, q\right).
\]

It follows that

\[
E_g(n, q) = \sum_{d \mid n} \frac{n}{d} I_g\left(\frac{n}{d}, q\right) = \sum_{d \mid n} dI_g(d, q).
\]

Möbius inversion of this shows that

\[
I_g(n, q) = \frac{1}{n} \sum_{d \mid n} \mu(d)E_g\left(\frac{n}{d}, q\right).
\]

Let \( A_g(n, q) \) denote the number of classes of absolutely indecomposable ordered \( g \)-tuples of \( n \times n \) matrices over \( \mathbb{F}_q \) up to simultaneous similarity. Recall that an ordered \( g \)-tuple is said to be absolutely indecomposable if it remains indecomposable when the field is extended to \( \overline{\mathbb{F}_q} \).

Theorem 5. (Kac) \( I_g(n, q) \) and \( A_g(n, q) \) are linked by the following formulae:

\[
\begin{align*}
I_g(n, q) &= \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu\left(\frac{d}{r}\right) A_g\left(\frac{n}{d}, q^r\right), \\
A_g(n, q) &= \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu(r) I_g\left(\frac{n}{d}, q^r\right).
\end{align*}
\]
Proof. The first identity is given by Kac ([2], page 91), and can be found in Le Bruyn ([3] page 153). The second is the M"obius inverse of the first.

The following simple identity is needed below in dealing with double summations over divisors of integers:

$$\sum_{d \mid n} \sum_{r \mid n/d} f(d, r) = \sum_{r \mid n} \sum_{d \mid n/r} f(d, r) = \sum_{r \mid n} \sum_{d \mid r} f(d, n/r).$$

The following is a known formula modified in the case $n = 1$:

$$\phi_n(q) = \frac{1}{n} \sum_{d \mid n} \mu(d) \left( q^\frac{n}{d} - 1 \right).$$

The M"obius inverse of this amounts to the following formula:

$$(3) \quad \sum_{d \mid n} \frac{n}{d} \mu(d) \phi_{n/d}(q^d) = \mu(n)(q - 1).$$

Lemma 6.

$$A_g(n, q) = \frac{1}{n} \sum_{d \mid n} \mu(d) E_g \left( \frac{n}{d}, q^d \right).$$

Proof.

$$A_g(n, q) = \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu(r) I_g \left( \frac{n}{d}, q^r \right)$$

$$= \sum_{d \mid n} \frac{1}{d} \sum_{r \mid d} \mu(r) \frac{d}{n} \sum_{s \mid n/d} \mu \left( \frac{n}{ds} \right) E_g(s, q^r) \quad \text{(by Lemma 4)}$$

$$= \frac{1}{n} \sum_{d \mid n} \sum_{r \mid d} \sum_{s \mid n/d} \mu(r) \mu \left( \frac{n}{ds} \right) E_g(s, q^r)$$

$$= \frac{1}{n} \sum_{r \mid n} \sum_{s \mid n/r} \sum_{d \mid n/rs} \mu(r) \mu \left( \frac{n}{rds} \right) E_g(s, q^r)$$

$$= \frac{1}{n} \sum_{r \mid n} \sum_{s \mid n/r} E_g(s, q^r) \sum_{d \mid n/rs} \mu \left( \frac{n}{rds} \right)$$

$$= \frac{1}{n} \sum_{r \mid n} \mu(r) E_g \left( \frac{n}{r}, q^r \right).$$
Theorem 7.

\[ A_g(n, q) = \frac{q-1}{n} \sum_{d \mid n} \mu(d) H_g \left( \frac{n}{d}, q^d \right). \]

Proof.

\[ A_g(n, q) = \frac{1}{n} \sum_{d \mid n} \mu(d) E_g \left( \frac{n}{d}, q^{d^2} \right) \] (by Lemma 6)

\[ = \frac{1}{n} \sum_{d \mid n} \mu(d) \sum_{r \mid n/d} r \phi_r(q^d) H_g \left( \frac{n}{dr}, q^{dr} \right) \]

\[ = \frac{1}{n} \sum_{d \mid n} \mu(d) \sum_{r \mid n/d} \frac{n}{dr} \phi_{n/dr}(q^d) H_g(r, q^{r^2}) \]

\[ = \frac{1}{n} \sum_{r \mid n} \sum_{d \mid n/r} \mu(d) \frac{n}{dr} \phi_{n/dr}(q^d) H_g \left( \frac{n}{dr}, q^{r^2} \right) \]

\[ = \frac{1}{n} \sum_{r \mid n} \sum_{d \mid r} \mu(d) \frac{r}{d} \phi_{r/d}(q^d) H_g \left( \frac{n}{r}, q^{r^2} \right) \]

\[ = \frac{q-1}{n} \sum_{r \mid n} \mu(r) H_g \left( \frac{n}{r}, q^{r^2} \right) \] (by (3)).

Remark. For fixed \( n \) and \( g \) the functions \( M_g(n, q), I_g(n, q), A_g(n, q) \) of \( q \) are evidently rational functions of \( q \). As they take integer values for all integers \( q \) that are powers of primes, these functions are polynomial functions of \( q \) with rational coefficients. They have been calculated in various cases by Diane Maclagan and the author. The polynomials \( A_g(n, q) \) appear to have non-negative integer coefficients while \( I_g(n, q) \) do not. This can be viewed as extra support for Kac’s conjecture. More detail about Kac’s conjecture can be found in Le Bruyn [3].

3. RECURSION FORMULAE

In this section we prove the main results of this paper.

Proposition 8. \( P_g(n, q) \) and \( M_g(n, q) \) satisfy the following recursion formulae:

\[ P_g(n, q) = \frac{1}{n} \sum_{r=1}^{n} H_g(r, q) P_g(n - r, q), \]

\[ M_g(n, q) = \frac{1}{n} \sum_{r=1}^{n} E_g(r, q) M_g(n - r, q). \]
Proof. By the definition of \( H_g(r, q) \), we have the following identity:

\[
\log \left( \sum_{r=0}^{\infty} P_g(r, q)t^r \right) = \sum_{r=1}^{\infty} \frac{H_g(r, q)t^r}{r}.
\]

Differentiate both sides respecting to \( t \), we have

\[
\frac{\sum_{r=1}^{\infty} rP_g(r, q)t^{r-1}}{\sum_{r=0}^{\infty} P_g(r, q)t^r} = \sum_{r=1}^{\infty} \frac{H_g(r, q)t^{r-1}}{r}.
\]

Thus,

\[
\sum_{r=1}^{\infty} rP_g(r, q)t^{r-1} = \left( \sum_{r=1}^{\infty} \frac{H_g(r, q)t^{r-1}}{r} \right) \left( \sum_{r=0}^{\infty} P_g(r, q)t^r \right).
\]

By comparing the coefficients of \( t^{n-1} \) on both sides, we get

\[
nP_g(n, q) = \sum_{r=1}^{n} H_g(r, q)P_g(n - r, q).
\]

Thus the first recursion formula has been established.

As \( \log \left( \sum_{r \geq 0} M_g(r, q)t^r \right) = \sum_{r \geq 1} E_g(r, q)t^r/r \), the second identity can be proved similarly.

**Proposition 9.** The functions \( H_1(n, q) \), \( P_1(n, q) \), \( E_1(n, q) \) and \( M_1(n, q) \) have the forms specified in the introduction.

**Proof.** Let \( g = 1 \). By the Jordan Normal Form Theorem, the classes of \( n \times n \) of matrices over \( \tilde{\mathbb{F}}_q \) under conjugation are in one-one correspondence with Jordan normal forms \( J(\lambda, n) \), where \( \lambda \in \tilde{\mathbb{F}}_q \). As a consequence, \( A_1(n, q) = q \) for all \( n \geq 1 \).

The Möbius inverse of Theorem 7 with \( g = 1 \) now amounts to

\[
H_1(n, q) = \sum_{d \mid n} \frac{n}{d} \frac{q^d}{q^d - 1} = \sum_{d \mid n} \frac{n}{d} \frac{1}{1 - q^{-d}}.
\]

By the definition of \( P_g(n, q) \) in section 2, \( P_1(n, q) \) has the form as required.

Möbius inversion of Lemma 6 shows that \( E_g(n, q) = \sum_{d \mid n} d A_g(d, q^{\frac{1}{d}}) \), and so \( E_1(n, q) = \sum_{d \mid n} dq^{\frac{1}{d}} \).

Let \( R_1(n, q) = \sum_{\lambda \mid n} q^{l(\lambda)} \). Note that \( \sum_{\lambda \mid n} q^{l(\lambda)} = \sum_{\lambda \mid n} q^{l(\lambda')} \), where \( \lambda' \) is the partition conjugate to \( \lambda \). Also note that \( l(\lambda') \) is equal to the number of parts of \( \lambda \).
We claim that $M_1(n, q) = R_1(n, q)$. In fact,

$$1 + \sum_{n=1}^{\infty} R_1(n, q) t^n = 1 + \sum_{\lambda \in \mathcal{P}} q^{l(\lambda')} t^{|\lambda|}$$

$$= 1 + \sum_{(1^{n_1}2^{n_2}) \in \mathcal{P}} q^{(n_1+n_2+n_3\cdots)} t^{n_1+2n_2+3n_3\cdots}$$

$$= 1 + \sum_{(1^{n_1}2^{n_2}) \in \mathcal{P}} (qt^1)^{n_1} (qt^2)^{n_2} (qt^3)^{n_3} \cdots$$

$$= \prod_{i=1}^{\infty} \left(1 + \sum_{j=1}^{\infty} (qt^i)^j\right)$$

$$= \prod_{i=1}^{\infty} \frac{1}{1 - qt^i}.$$  

Thus, taking logarithms on both sides implies

$$\log \left(1 + \sum_{n=1}^{\infty} R_1(n, q) t^n\right) = \sum_{i=1}^{\infty} \log \frac{1}{1 - qt^i}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{1}{1} (qt^i)^j$$

$$= \sum_{n=1}^{\infty} \left(\sum_{d|n} \frac{d}{n} q^{\frac{n}{d}}\right) t^n$$

$$= \sum_{n=1}^{\infty} E_1(n, q) t^n/n.$$  

Therefore, $\log \left(1 + \sum_{n=1}^{\infty} R_1(n, q) t^n\right) = \log \left(1 + \sum_{n=1}^{\infty} M_1(n, q) t^n\right)$. It follows that $M_1(n, q) = R_1(n, q) = \sum_{\lambda \vdash n} q^{l(\lambda)}$, for all $n \geq 1$.

**Corollary 10.**

$$1 + \sum_{n=1}^{\infty} M_1(n, q) t^n = \prod_{n=1}^{\infty} \frac{1}{1 - qt^n}. $$

**Remark.** Le Bruyn mentioned analogous characteristic 0 results due to H. Kraft and D. Peterson in [3]. More precisely, if $\mathbb{C}$ is an algebraically closed field of characteristic 0, and $R^{iso}(S_1, n)(\lambda)$ denotes the isomorphism classes of $n$-dimensional representations of $\mathbb{C}[x]$ whose root-multiplicity-partition is conjugate to $\lambda$ (see Le Bruyn [3], page 144), then $R^{iso}(S_1, n)(\lambda)$ is in a natural way an affine space of dimension equal to $l(\lambda)$. This can be translated into the following identity:

$$1 + \sum_{n=1}^{\infty} \sum_{\lambda \vdash n} s_{\dim R^{iso}(S_1, n)(\lambda)} t^n = \prod_{n=1}^{\infty} \frac{1}{1 - st^n}.$$
The rest of this section is devoted to the algebraic interpretation of Corollary 10. Let $F$ be any field, $M_n(F)$ be the set of all $n \times n$ matrices over $F$, $GL_n(F)$ be the general linear group. $GL_n(F)$ acts on $M_n(F)$ by conjugation, the orbit space is denoted by $M_n(F)/GL_n(F)$. Let $f$ be a monic polynomial over $F$, say $f(t) = t^m - a_{m-1}t^{m-1} - \cdots - a_1 t - a_0$, recall that the companion matrix $J(f)$ of $f$ is defined by:

$$J(f) = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
a_0 & a_1 & a_2 & \cdots & a_{m-1}
\end{bmatrix}.$$ 

By Theorem 24 (Birkhoff-Mac Lane [1], page 332), we know that any $n \times n$ matrix $M$ over $F$ is similar over $F$ to one and only one direct sum of companion matrices $J(f_1) \oplus J(f_2) \oplus \cdots \oplus J(f_k)$, such that $f_{i+1} \mid f_i$ for all $i \leq k - 1$, where $f_i$ ($i \geq 1$) are monic polynomials over $F$. Note that $\sum_{i \geq 1} \deg f_i = n$, thus $\lambda = (\deg f_1, \deg f_2, \ldots, \deg f_k)$ forms a partition of $n$. We call $\lambda$ the rational partition afforded by $M$.

**Theorem 11.** Let $Q_\lambda$ denote the set of all similarity classes whose rational partitions are $\lambda$, then

$$M_n(F)/GL_n(F) = \bigcup_{\lambda \vdash n} Q_\lambda,$$

where the union is a disjoint union. $Q_\lambda$ is in a natural way an affine space of dimension equal to $l(\lambda)$.

**Proof.** The first statement follows from the Rational Canonical Form Theorem. Suppose $\lambda = (\lambda_1, \cdots, \lambda_k)$. Let $\mu_i = \lambda_i - \lambda_{i+1}$ for $i \leq k - 1$, and $\mu_k = \lambda_k$. By the previous discussion, every $M \in Q_\lambda$ is similar uniquely over $F$ to $J(f_1) \oplus J(f_2) \oplus \cdots \oplus J(f_k)$, where $f_i$ is some monic polynomial of degree $\lambda_i$ over $F$, such that $f_{i+1} \mid f_i$ for all $i \leq k - 1$. Now, let $g_i = f_i/f_{i+1}$ for $i \leq k - 1$, and $g_k = f_k$, then $g_i$ is a monic polynomial of degree $\mu_i$ over $F$ for all $i \geq 1$. Note that $g_i$ ($1 \leq i \leq k$) are uniquely determined by $M$, and vice versa. Thus the elements in $Q_\lambda$ are in one-one correspondence with the elements in the following set:

$$S = \{ (g_1, \cdots, g_k) \mid g_i \in \mathbb{F}[t] \text{ monic, } \deg g_i = \mu_i, 1 \leq i \leq k \}.$$
If we define \( S_i = \{ g \mid g \in \mathbb{F}[t] \text{ monic}, \deg g = \mu_i \} \), then \( S = \prod_{i=1}^{k} S_i \), where the product is the Cartesian product. It is trivial to prove that \( S_i \) is in a natural way isomorphic to the affine space \( \mathbb{F}^{\mu_i} \). Note that \( \mu_1 + \cdots + \mu_k = \lambda_1 = l(\lambda) \), thus \( Q_\lambda \) is in a natural way isomorphic to the affine space \( \mathbb{F}^{l(\lambda)} \). Here “natural” means compatible with change of field.

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REFERENCES


Bibliography


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1) I thank one of the examiners for pointing out that the results of Section 2.2 and 2.3 of Chapter 2 are indeed valid for $C_1$ fields, and that any central simple algebra which has different splitting fields will do the counterexample on page 17.

2) I thank one of the examiners for pointing out that the results of the first half of Chapter 5 were already available for group algebras in the literature, e.g., [Huppert and Blackburn, Finite Groups II, Springer-Verlag, Berlin, 1982].

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