Some existence theorems in the theory of doctrines

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SOME EXISTENCE THEOREMS IN THE THEORY OF DOCTRINES

by

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Abstract

This thesis is primarily concerned with a notion of an algebra which is of sufficient generality to have as examples algebras for a (pointed) endofunctor, algebras for a monad, lax-algebras for a 2-monad, and monoids in a monoidal category. To this end we introduce the notion of a polyad X on a 2-category A and define the 2-category $X\text{-}\mathcal{A\ell g}_*$ of algebras for the polyad X together with a forgetful 2-functor $V\colon X\text{-}\mathcal{A\ell g}_* \to A$.

The problem to which this thesis addresses itself is that of giving sufficient conditions for V to be 2-monadic. We show that in the case that A is complete the 2-monadicity of V is equivalent to the existence, in the 2-category $\mathit{Mon}\text{-}2\text{-}\mathit{CAT}$ of monoidal 2-categories, of the (lax) left Kan extension of a certain monoidal 2-functor $X\colon M \to [A,A]$ along the monoidal 2-functor $!\colon M \to \mathbb{1}$. We then give sufficient conditions for the (lax) left Kan extension of $X\colon M \to E$ along $!\colon M \to \mathbb{1}$ to exist in $\mathit{Mon}\text{-}2\text{-}\mathit{CAT}$ for an arbitrary monoidal 2-category E and a small monoidal 2-category M. Using these sufficient conditions we show that for a cocomplete A the required left Kan extension exists provided $X\colon M \to [A,A]$ factors through $[A,A]_*$ the monoidal 2-category of ranked endo-2-functors of A.

We therefore conclude that for a complete and cocomplete 2-category A the 2-functor $V\colon X\text{-}\mathcal{A\ell g}_* \to A$ is 2-monadic provided the polyad X has a rank, by which we mean that the appropriate $X\colon M \to [A,A]$ factors through $[A,A]_*$.

We are, moreover, able to show that the 2-monad in question has a rank and that the 2-category $X\text{-}\mathcal{A}\ell g_*$ is cocomplete. This result includes many well-known results, it shows that the free monad on an endofunctor R exists if R has a rank, it shows that the category of algebras for a ranked monad is cocomplete, and it shows that if $\mathcal{A}$ is a monoidal category the free monoid exists on each $A \in \mathcal{A}$ provided the functor $\otimes: \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ has a rank in each variable.

Contents

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Statement of Originality

I certify that this thesis does not incorporate any material previously submitted for a degree or diploma in any university; and to the best of my knowledge and belief it does not contain any material previously published or written by another person except where due reference is made in the text. I also certify that, with the above qualification, the material in this thesis is my own work with the exception of the material of sections 4 and 5 of Chapter 1 which was done jointly with my supervisor Professor G.M. Kelly.

INTRODUCTION

The work in this thesis originated in the following two questions, raised by G.M. Kelly in [12]. Firstly, if D is a doctrine (=2-monad) on a 2-category $\mathcal{A}$, give sufficient conditions for the 2-category Lax-D-$\mathcal{A}\ell g_*$ (the 2-category of lax-D-algebras and strict D-morphisms) to be 2-monadic over $\mathcal{A}$. Secondly, give conditions on $\mathcal{A}$ and on the doctrines D and D' so that the 2-category of algebras and strict morphisms for the pseudo distributive law (D,D',p,π) is 2-monadic over $\mathcal{A}$.

Rather than solve these problems directly we pose and solve a much more general question. The first step towards posing this more general problem is the observation that both of the original examples are instances of the following general situation. Consider a 2-category $\mathcal{A}$ which is equipped with a set of endo-2-functors, a set of 2-natural transformations between composites of the given endo-2-functors, and a set of modifications between composites of the given 2-natural transformations; all of the data being subject to a set of relations in the form of equations between composites of the data. An algebra for such a situation is an object A of $\mathcal{A}$ together with an action a_E: EA $\to$ E for each given E: $\mathcal{A} \to \mathcal{A}$ (and which we extend to all derived endo-2-functors by the equation $a_{T.S} = a_T.Ta_S$) and an action a_ρ

$$\begin{tikzcd}
TA \arrow[dr, "a_T"] \arrow[d, "\rho A"'] \arrow[dr, phantom, shift right=4, "\Downarrow a_\rho"] & \\
T'A \arrow[r, "a_{T'}"'] & A
\end{tikzcd}$$

for each given 2-natural transformation ρ (which we extend to derived 2-natural transformations in the obvious way) where these actions satisfy various axioms of their own as well as respecting the given relations. Finally we are given two sets X_1 and X_2 of (derived) 2-natural transformations and we require that a_σ be an identity if $\sigma \in X_1$ and an isomorphism if $\sigma \in X_2$.

The next step is the recognition that the data described above are nothing but a strict monoidal 2-functor $X\colon M \to [A,A]$ from a small strict monoidal 2-category M to the monoidal 2-category $[A,A]$ of endo-2-functors of A; the description above merely provides generators for M in the form of the data and relations in the form of the axioms. The classes X_1 and X_2 are then thought of as subcategories of the underlying category of M. An algebra is then an object A of A together with actions $a_t\colon X(t)A \to A$ for each object t of M and actions $a_\rho\colon a_t \Rightarrow a_{t'}.X(\rho)A$ for each $\rho\colon t \to t'$ in M which are to satisfy a certain "unit" and "associativity" axiom, and such that a_ρ is an identity if ρ is in X_1 and is an isomorphism if ρ is in X_2. If we write $X = (X, X_1, X_2)$ and denote by $X\text{-}\mathcal{A\ell g}_*$ the 2-category of algebras then the problem we wish to solve is that of the 2-monadicity of $X\text{-}\mathcal{A\ell g}_*$.

Finally if we define a polyad X to be a triple $X = (X, X_1, X_2)$ where X is a monoidal 2-functor from a small strict monoidal 2-category M to $[A,A]$ and where X_1 and X_2 are sub-categories of M; and if we define $X\text{-}\mathcal{A\ell g}_*$ to be the

2-category of X-algebras as defined above, then our general problem is to find sufficient conditions on a polyad X and a 2-category A so that $X\text{-}\mathcal{A}\ell g_*$ is 2-monadic over A.

We now briefly outline our method of solving this general problem. For simplicity however we treat (in this outline) the case where both M and A are categories not 2-categories and where X_1 and X_2 are empty. In this case algebras only have the actions a_t but not the actions a_ρ.

The first step towards giving sufficient conditions for the 2-monadicity of $X\text{-}\mathcal{A}\ell g_*$ is to change the nature of the problem. The technique we use to do this dates back, at least in principle, to the work of Dubuc [6] and Barr [2] on the existence of the free monad on an endofunctor. If S is any doctrine on A we show that there is a bijection χ between 2-functors $\Psi\colon S\text{-}\mathcal{A}\ell g_* \to X\text{-}\mathcal{A}\ell g_*$ satisfying $U^S = V\Psi$ and monoidal natural transformation σ as in

$$\begin{tikzcd} M \arrow[rr, "X"] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\sigma"] & {[A,A]} \\ & \mathbb{1} \arrow[ur, "S"'] & \end{tikzcd} .$$

We recall that a doctrine on A is just a monoid in $[A,A]$, which is precisely a monoidal functor $\mathbb{1} \to [A,A]$, and that $k\colon S \Rightarrow S'$ is a morphism of doctrines precisely when $k : S \Rightarrow S' : \mathbb{1} \to [A,A]$ is a monoidal natural transformation.

If the 2-category $X\text{-}\mathcal{A}\ell g_*$ is 2-monadic so that $\Lambda\colon T\text{-}\mathcal{A}\ell g_* \xrightarrow{\cong} X\text{-}\mathcal{A}\ell g_*$ and if τ is $\chi(\Lambda)$, then (T,τ) has the following universal property: for any other pair (S,σ) as above there exists a unique morphism $k\colon T \Rightarrow S$ of doctrines such that $\sigma = k!.\tau$. The crucial point now is that if A is complete, then this universal property of T is a sufficient as well as a necessary condition for $X\text{-}\mathcal{A}\ell g_*$ to be $T\text{-}\mathcal{A}\ell g_*$. The proof of this involves the functor $\{A,B\}\colon A \to A$ which is the right Kan extension of $\ulcorner B \urcorner\colon \mathbb{1} \to A$ along $\ulcorner A \urcorner\colon \mathbf{1} \to A$ and the resulting bijection θ between morphism $a\colon RA \to B$ and natural transformation $\alpha\colon R \to \{A,B\}$; for we show that (A,α_S) is an X-algebra if and only if $\theta(\alpha_S)\colon S \to \{A,A\}$ constitutes a monoidal natural transformation

$$
\begin{tikzcd}
M \arrow[rr, "X"] \arrow[dr, "!"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow \theta(\alpha)"] & & {[A,A]} \\
& \mathbb{1} \arrow[ur, "{\{A,A\}}"'] &
\end{tikzcd}
.
$$

Since the universal property of (T,τ) is that of the left Kan extension (in the 2-category *Mon-CAT* of strict monoidal categories) of X along $!\colon X \to \mathbb{1}$ (the unique morphism into the terminal object in *Mon-CAT*), we may by analogy with the classical definition of colimit call T <u>the colimit of</u> X <u>in</u> *Mon-CAT*, and call τ the colimit-cone of X in *Mon-CAT*. Thus our problem becomes that of giving conditions on X and A so that the colimit of X in *Mon-CAT* exists.

Rather than attack the problem as stated we first generalise it. Instead of working in the 2-category *Mon-CAT* we work in D-*CAT*, where D is a doctrine on *CAT* under which *Cat* is stable; and instead of looking for the existence of individual colimits we look for sufficient conditions for a D-category B = (B,b) to be cocomplete in D-*CAT* (that is, to admit all small colimits in D-*CAT*).

The sufficient conditions we give are stated in terms of the category D[B] = D-*CAT*($\mathbb{1}$,B) of D-oids in B and the forgetful functor U: D[B] $\to$ B; they are (i) that the category D[B] be cocomplete, and (ii) that the functor U: D[B] $\to$ B have a left adjoint F. We also show that a strict D-morphism H = (h,id):(B,b) $\to$ (C,c) preserves colimits in D-*CAT* if (iii) the functor D[H]: D[B] $\to$ D[C] preserves colimits, and (iv) if the functor $B \xrightarrow{F} D[B] \xrightarrow{D[H]} D[C]$ is the partial left adjoint of U: D[C] $\to$ C relative to h: B $\to$ C. We use these conditions to show that if *A* is cocomplete, then the monoidal category $[A,A]_*$ of ranked endofunctors of *A* is cocomplete in *Mon-CAT* and that the strict monoidal inclusion I_*: $[A,A]_* \to [A,A]$ preserves colimits in *Mon-CAT*. From this we conclude that, if *A* is complete and cocomplete, then $X\text{-}\mathcal{A}\ell g_*$ is 2-monadic over *A* provided *X* has a rank, by which we mean that X: *M* $\to$ [*A*,*A*] factors through I_*: $[A,A]_* \to [A,A]$. (The 2-monadicity result is exactly the same when *A* and *M* are 2-categories and when the term polyad is used in the corresponding wider sense.)

In the case that the doctrine D on CAT has a rank as well as preserving smallness it turns out that the conditions (i) and (ii) are also necessary. The proof of the necessity of these conditions is considerably harder than the proof of their sufficiency in that it requires a detailed study of the inclusion $J\colon \text{D-}\mathcal{A}\ell g_* \to \text{D-}\mathcal{A}\ell g$. This analysis, which occupies all of Chapter 1, involves constructing a left adjoint Φ to the 2-functor J and investigating some deeper properties of this adjunction. As an example of these deeper properties it turns out that if η and ε are the unit and counit of the adjunction $\Phi \dashv J$, then there exists a 2-cell $\alpha\colon \eta A.\varepsilon A \Rightarrow 1$ in $\text{D-}\mathcal{A}\ell g$ which, together with the equality $\varepsilon A.\eta A = 1$, exhibits εA as left adjoint to ηA in the 2-category $\text{D-}\mathcal{A}\ell g$. (As a final remark we observe that the results of Chapter 1 remain valid if we replace the 2-category $\text{D-}CAT$ by the 2-category $\text{D-}CAT_0$ of D-categories and pseudo D-functors . In this case the 2-cell α is an isomorphism).

The body of this thesis consists of four chapters. The first, called Chapter 0, is merely a chapter of preliminaries where we collect together various facts and definitions from the works of other authors that will be referred to in the text; it is recommended that the reader pass directly to Chapter 1 and only refer to Chapter 0 when necessary. As already mentioned Chapter 1, the first chapter of the thesis proper, is concerned with the inclusion $J\colon \text{D-}\mathcal{A}\ell g_* \to \text{D-}\mathcal{A}\ell g$. In Chapter 2 we are concerned with the concept of colimit in $\text{D-}CAT$ and it is in this chapter that we prove the sufficiency of condition (i), (ii),(iii) and (iv). Also in this chapter we consider a concept of colimit in *Mon*-2-*CAT* (the 3-category of monoidal 2-categories) that is appropriate to the question of the 2-monadicity of V: X-$A\ell g \to A$ when A is a 2-category. Finally in Chapter 3 we define polyads X on a 2-category A and the 2-category X-$A\ell g_*$, and we use the results of Chapter 2 to give sufficient conditions for the 2-monadicity of X-$A\ell g_*$. We also investigate the question of describing polyads in terms of generators on relations, and give some examples of polyads defined in this manner.

CHAPTER 0.

1. We work in ZF set theory with the extra axioms that arbitarily large inaccessibles exist , or equivalently that every set belongs to some universe. A set is small if it lies in some chosen universe which will not be referred to explicitly and which is usually regarded as fixed, but which may of course be changed if desired.

By a category we mean any model of the theory of categories; thus the set of objects and the set of morphisms can be any size - but are always sets. A category A is said to be small if its set of objects and its set of morphisms are small, and is said to be locally small if each set A(a,b) is small. For any category A at all there is some bigger universe with respect to which A is small; we write *SET* for the category of sets in such a bigger universe which is not usually thought of as fixed but which is large enough for the problem at hand, and in particular large enough to render *Set* small relative to it.

For a symmetric monoidal closed category V a V-category can have any set of objects but its hom-objects are in V; we write V-*Cat* for the 2-category of V-categories whose set of objects is in *Set* and V-*CAT* for the 2-category of those V-categories whose set of objects is in *SET*.

We write *Cat* for *Set*-*Cat* - which is the 2-category of small categories, and we write *CAT* for *SET*-*CAT*; we give no particular symbol to *Set*-*CAT* the 2-category of locally small categories. We write 2-*Cat* for *Cat*-*Cat* and 2-*CAT* for *CAT*-*CAT*, each of which is a cartesian closed 3-category. Except for the above we use the prefix "2-" as equivalent to the prefix "*CAT*-" by recalling that a *Cat*-category is of necessity a *CAT*-category. This fixes the notions of 2-functor, 2-natural transformation, 2-adjunction, 2-colimit, etc.

We adopt the convention that the prefixes "2-", "3-" (which is equivalent to "2-*CAT*-"), or generally "V-" will usually be omitted since the context will always indicate what situation we are in, and since we will not mix enrichments without being very explicit. Thus if we say that the V-functor U: $A \to B$ has a left adjoint, we always mean that it has a V-left adjoint, similarly if we say that a certain colimit exists in a V-category we always mean that it is a V-colimit. Finally if we say a 2-category A is cocomplete we always mean that it is *CAT*-cocomplete in the sense of Day-Kelly [5] and Borceux-Kelly [4].

$\underline{\underline{2}}$. If $U: \mathcal{B} \to \mathcal{A}$ is a functor (or a 2-functor or even a $\mathcal{V}$-functor) and if $J: \mathcal{A}' \to \mathcal{A}$ is also a functor, then we say that $F: \mathcal{A}' \to \mathcal{B}$ is the partial left adjoint of U relative to J, written $U \underset{J}{\dashv} F$, if for all $A \in \mathcal{A}'$ and $B \in \mathcal{B}$ there exists an isomorphism

$$\mathcal{B}(FA,B) \cong \mathcal{A}(JA,UB)$$

which is natural (or 2-natural or $\mathcal{V}$-natural) in $A \in \mathcal{A}'$ and $B \in \mathcal{B}$. In the category, or 2-category, case we can express this in terms of the universal property of the unit. We say that $F \underset{J}{\dashv} U$ if for each $A \in \mathcal{A}'$ there exists a morphism $\eta_A: JA \to UFA$ in $\mathcal{A}$ such that for any other $t: JA \to UB$ in $\mathcal{A}$ there exists a unique morphism $s: FA \to B$ in $\mathcal{B}$ such that $Us.\eta A = t$. For partial 2-adjoints ηA must also have the corresponding universal property for 2-cells $\alpha; t \Rightarrow t': JA \to UB$ in $\mathcal{A}$.

When $\mathcal{A}' = \mathbf{1}$ so that J is actually the name of an object A of $\mathcal{A}$, we say that FA is the free object on A relative to U, or that FA is the left adjoint, at A, to U. The morphism $\eta_A: A \to UFA$ is still called the unit.

3. If F,G: $\mathcal{A} \to \mathcal{B}$ are 2-functors, a lax-natural transformation α: F $\rightsquigarrow$ G assigns to each A $\in \mathcal{A}$ a morphism αA: FA $\to$ GA in $\mathcal{B}$, and to each morphism u: A $\to$ A' in $\mathcal{A}$ a 2-cell α_u in $\mathcal{B}$ as in

$$\begin{tikzcd}
FA \arrow[r, "Fu"] \arrow[d, "\alpha A"'] \arrow[dr, phantom, "\Downarrow \alpha_u"] & FA' \arrow[d, "\alpha A'"] \\
GA \arrow[r, "Gu"'] & GA' .
\end{tikzcd}$$

This data is to satisfy the axioms

$$\alpha_{1_A} = 1_{\alpha_A} , \quad \alpha_{u.v} = \alpha_u \cdot \alpha_v$$

and, for all γ: u $\Rightarrow$ u': A $\to$ A' in $\mathcal{A}$, the equation

$$\begin{tikzcd}
FA \arrow[r, "Fu"] \arrow[d, "\alpha A"'] \arrow[dr, phantom, "\Downarrow \alpha_u"] & FA' \arrow[d, "\alpha A'"] \\
GA \arrow[r, bend left, "Gu"] \arrow[r, bend right, "Gv"'] \arrow[r, phantom, "\Downarrow G\gamma"] & GA'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, bend left, "Fu"] \arrow[r, bend right, "Fv"'] \arrow[r, phantom, "\Downarrow F\gamma"] \arrow[d, "\alpha A"'] \arrow[dr, phantom, "\Downarrow \alpha_v"] & FA' \arrow[d, "\alpha A'"] \\
GA \arrow[r, "Gv"'] & GA' .
\end{tikzcd}$$

A 2-natural transformation α: F $\Rightarrow$ G can be thought of as a lax-natural transformation in which α_u is an identity 2-cell for each 1-cell u in $\mathcal{A}$. An op-lax-natural transformation is defined by reversing the direction of the 2-cells α_u in the above definition and by making the obvious corresponding changes in the axioms. We call α pseudo-natural if each α_u is an isomorphism.

If α and β are lax-natural transformations from F to G, a modification $\theta: \alpha \to \beta$ assigns to each $A \in \mathcal{A}$ a 2-cell in $\mathcal{B}$ of the form

$$\begin{tikzcd} FA \arrow[r, bend left, "\alpha A"] \arrow[r, bend right, "\beta A"'] \arrow[r, phantom, "\Downarrow \theta A"] & GA \end{tikzcd}$$

such that for every morphism $u: A \to A'$ in $\mathcal{A}$

$$\beta_u . \theta A \quad = \quad \theta A' . \alpha_u \quad .$$

It should be clear how to define modifications between op-lax-natural transformations.

We denote by $\mathcal{F}un(\mathcal{A},\mathcal{B})$ the 2-category of 2-functors from $\mathcal{A}$ to $\mathcal{B}$, lax-natural transformations, and modifications; and we denote by $[\![\mathcal{A},\mathcal{B}]\!]$ the 2-category with the same objects, but with op-lax-natural transformations as 1-cells and modifications of them as 2-cells. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are sub-categories of the underlying category of $\mathcal{A}$, then we denote by $\mathcal{F}un(\mathcal{A}_1;\mathcal{A}_2;\mathcal{A},\mathcal{B})$ the sub-2-category of $\mathcal{F}un(\mathcal{A},\mathcal{B})$ retaining only those lax-natural transformationsthat are 2-natural when restricted to $\mathcal{A}_1$ and pseudo-natural when restricted to $\mathcal{A}_2$. A 1-cell in $\mathcal{F}un(\mathcal{A}_1;\mathcal{A}_2;\mathcal{A},\mathcal{B})$ is called an $\{\mathcal{A}_1;\mathcal{A}_2\}$-lax-natural transformation.

For further details we refer the reader to Kelly [10] and Gray [7] and [8] (in the former Gray uses the name 2-natural for what we call lax-natural, while in the latter he uses the term quasi-natural).

4. If $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{C} \to \mathcal{B}$ are 2-functors, the lax-comma 2-category $F/\!\!/G$ (cf. Kelly [10] and Gray [7] and [8] where it is called $[F,G]$) has as objects triples (A,f,C) where $A \in \mathcal{A}$, $C \in \mathcal{C}$, and where $f: FA \to GC$ is a morphism in $\mathcal{B}$. A morphism in $F/\!\!/G$ from (A,f,C) to (A',f',C') is a triple (h,γ,k) where $h: A \to A'$ is a morphism in $\mathcal{A}$, where $k: C \to C'$ is a morphism in $\mathcal{C}$, and where γ is a 2-cell in $\mathcal{B}$ as in

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\Downarrow\gamma"] & GA \arrow[d, "Gk"] \\ FA' \arrow[r, "f'"'] & GA' \end{tikzcd} \quad .$$

A 2-cell in $F/\!\!/G$ from (h,γ,k) to (h',γ',k') is a pair (α_0,α_1) where $\alpha_0: h \Rightarrow h'$ is a 2-cell in $\mathcal{A}$, and where $\alpha_1: k \Rightarrow k'$ is a 2-cell in $\mathcal{C}$ such that

$$\gamma.F\alpha_0 = G\alpha_1.\gamma' \quad .$$

There are obvious projection 2-functors $\partial_0: F/\!\!/G \to \mathcal{A}$ and $\partial_1: F/\!\!/G \to \mathcal{C}$ sending (A,f,C) to A and C respectively. There is also a lax-natural transformation

$\delta\colon F\partial_0 \rightsquigarrow G\partial_1$ with components

$$\delta(A,f,B) = f$$

and

$$\delta_{(h,\gamma,k)} = \gamma \quad .$$

Putting this information in diagramatic form, we have:

$$\begin{tikzcd}
F/\!\!/G \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\rightsquigarrow\delta"] & A \arrow[d, "F"] \\
C \arrow[r, "G"'] & B
\end{tikzcd} .$$

The 2-category $F/\!\!/G$ has a universal property with respect to lax-natural transformations. If

$$\begin{tikzcd}
E \arrow[r, "D_0"] \arrow[d, "D_1"'] \arrow[dr, phantom, "\rightsquigarrow\varepsilon"] & A \arrow[d, "F"] \\
C \arrow[r, "G"'] & B
\end{tikzcd}$$

is a lax-natural tranformation then there exists a unique 2-functor $V\colon E \to F/\!\!/G$ such that $\partial_0 V = D_0$, $\partial_1 V = D_1$, and $\delta V = \varepsilon$. Furthermore if V and V' are 2-functors from E to $F/\!\!/G$ corresponding to ε and ε' respectively then lax-natural transformations $\alpha\colon V \rightsquigarrow V'$ are in bijection with triples $(\alpha_0,\alpha_1,\sigma)$ where α_0 and α_1 are lax-natural transformations as in

$$\begin{tikzcd} E \arrow[r, bend left, "D_0"] \arrow[r, bend right, "D'_0"'] \arrow[r, phantom, "\Downarrow \alpha_0"] & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} E \arrow[r, bend left, "D_1"] \arrow[r, bend right, "D'_1"'] \arrow[r, phantom, "\Downarrow \alpha_1"] & C \end{tikzcd}$$

and where σ is a modification from the lax-natural transformation $\alpha_1 G.\varepsilon$ to the lax-natural transformation $\varepsilon.\alpha_0 F$. The bijection is given by $\alpha_0 = D_0\alpha$, $\alpha_1 = D_1\alpha$ and $\sigma_E = \varepsilon_{\alpha E}$.

If we denote the category containing two objects 0 and 1, and one non-identity arrow, called x, by $\mathit{2}$ then there are evident functors $\partial_0, \partial_1 \colon \mathbb{1} \to \mathit{2}$ and $! \colon \mathit{2} \to \mathbb{1}$, where $\mathbb{1}$ is the terminal category, given by $\partial_0(1) = 0$ and $\partial_1(1) = 1$. It is easy to check that

$$\begin{tikzcd} {[\![\mathit{2}, A]\!]} \arrow[r, "{[\![\partial_0, 1]\!]}"] \arrow[d, "{[\![\partial_1, 1]\!]}"'] \arrow[dr, phantom, "\Downarrow \lambda"] & A \arrow[d, "1"] \\ A \arrow[r, "1"'] & A \end{tikzcd}$$

is a lax-comma object where λ has components $\lambda F = F(x)$ and $\lambda_\alpha = \alpha_x$. We use this fact later in this chapter and again in Chapter 1 to identify the objects of $[\![\mathit{2}, A]\!]$.

For further details we again refer the reader to Gray [7] and [8] and Kelly [10].

5. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are subcategories of the 2-category $\mathcal{A}$, then the $\{\mathcal{A}_1;\mathcal{A}_2\}$-lax-colimit of the 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is the object of $\mathcal{B}$ that is the left adjoint, at F, to the inclusion

$$(5.1) \qquad \mathcal{B} \xrightarrow{\Delta} \mathit{Fun}(\mathcal{A}_1;\mathcal{A}_2;\mathcal{A},\mathcal{B}) \quad .$$

That is, if we write $\mathcal{X} = \mathit{Fun}(\mathcal{A}_1;\mathcal{A}_2;\mathcal{A},\mathcal{B})$, there is a 2-natural isomorphism of 2-categories

$$\mathcal{X}(F,\Delta B) \cong \mathcal{B}(\text{lax-colim}F,B).$$

We observe that ΔB is the 2-functor $\mathcal{A} \xrightarrow{!} \mathbb{1} \xrightarrow{\ulcorner B \urcorner} \mathcal{B}$, so that the unit of the above isomorphism is of the form

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "F"] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\alpha"] & \mathcal{B} \\
& \mathbb{1} \arrow[ur, "\text{lax-colim } F"'] &
\end{tikzcd}$$

and is called the $\{\mathcal{A}_1;\mathcal{A}_2\}$-lax-colimit-cone of F. If $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{A}$ then $\{\mathcal{A}_1;\mathcal{A}_2\}$-colimits are just ordinary 2-colimits, while if $\mathcal{A}_1 = \mathcal{A}_2 = \phi$ they are what Gray [8] calls cartesian-quasi-colimits.

We say that a 2-category $\mathcal{B}$ is lax-cocomplete if for all small $\mathcal{A}$ and all subcategories $\mathcal{A}_1$ and $\mathcal{A}_2$ of $\mathcal{A}$ the Δ of (5.1) has a left adjoint.

Proposition 5.1: (Gray [8], Street [16]). A 2-category $\mathcal{B}$ is lax-cocomplete if and only if it is cocomplete as a $\mathcal{CAT}$-category in the sense of Day-Kelly [5]. □

For examples of lax-colimits we refer the reader to Gray [8] and Street [16]. We will, however, give one example of particular interest in this present work. Let $\mathcal{A}$ be the 2-category represented by the diagram

$$\begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ 2 & & 0 \end{tikzcd},$$

and let $\mathcal{A}_2 = \phi$ and $\mathcal{A}_1$ be the subcategory $1 \to 0$. We leave it to the reader to check that a 2-functor $F\colon \mathcal{A} \to \mathcal{B}$ is precisely a diagram

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "g"'] & B' \\ B'' & \end{tikzcd}$$

in $\mathcal{B}$ and that the $\{\mathcal{A}_1;\mathcal{A}_2\}$-lax-colimit of F is an object $f*g$ together with morphisms d_0, d_1 and a 2-cell λ as in

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \lambda"] & B' \arrow[d, "d_0"] \\ B'' \arrow[r, "d_1"'] & f*g \end{tikzcd}.$$

The universal property exhibited by $f*g$ is the following. If μ is any 2-cell of the form

(5.2)
$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\Downarrow \mu"] & B' \arrow[d, "q"] \\ B'' \arrow[r, "p"'] & C \end{tikzcd},$$

then there is a unique 1-cell $k\colon f*g \to C$ in $\mathcal{B}$ such that $kd_1 = p$, $kd_0 = q$, and $k\lambda = \mu$. Furthermore, if μ', q' and p' is another triple as in (5.2) and if $k'\colon f*g \to C$ is the corresponding 1-cell, then 2-cells $\alpha\colon k \Rightarrow k'$ are in bijection with pairs of morphisms $\beta_0\colon p \to p'$ and $\beta_1\colon q \to q'$ such that $g\beta_0.\mu = \mu'.f\beta_1$. The bijection being given by the equations $\beta_0 = \alpha d_0$ and $\beta_1 = \alpha d_1$. We call $f*g$ the op-comma object of f and g.

6. If $F\colon \mathcal{A} \to \mathcal{B}$ and $U\colon \mathcal{B} \to \mathcal{A}$ are 2-functors an op-quasi-adjunction between F and U, with F left-quasi-adjoint to U, consists of op-lax-natural transformations

$$\eta\colon 1 \rightsquigarrow UF \quad , \quad \varepsilon\colon FU \rightsquigarrow 1$$

and modifications

$$\begin{tikzcd} U \arrow[r, squiggly, "\eta U"] \arrow[dr, "1"'] & UFU \arrow[d, squiggly, "U\varepsilon"] \arrow[dl, phantom, "\overset{t}{\Leftarrow}"] \\ & U \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, squiggly, "F\eta"] \arrow[dr, "1"'] & FUF \arrow[d, squiggly, "\varepsilon F"] \arrow[dl, phantom, "\overset{s}{\Rightarrow}"] \\ & F \end{tikzcd}$$

satifying the following two axioms:

$$
\begin{tikzcd}
1 \arrow[r, squiggly, "\eta"] \arrow[d, squiggly, "\eta"'] \arrow[dr, phantom, "\eta_\eta"] & UF \arrow[d, squiggly, "UF\eta"] \arrow[ddr, bend left, "1"] & \\
UF \arrow[r, squiggly, "\eta UF"'] \arrow[drr, bend right, "1"'] & UFUF \arrow[dr, squiggly, "U\varepsilon F"] \arrow[r, phantom, "\swarrow Us"] \arrow[d, phantom, "\Downarrow tF"] & \\
& & UF
\end{tikzcd}
\quad = \quad \mathrm{id}
$$

and

$$
\begin{tikzcd}
FU \arrow[dr, squiggly, "F\eta U"] \arrow[drr, bend left, "1"] \arrow[ddr, bend right, "1"'] & & \\
& FUFU \arrow[r, squiggly, "FU\varepsilon"'] \arrow[d, squiggly, "\varepsilon FU"'] \arrow[dr, phantom, "\varepsilon_\varepsilon"] \arrow[u, phantom, "\nearrow Ft"] \arrow[l, phantom, "\nearrow sU"] & FU \arrow[d, squiggly, "\varepsilon"] \\
& FU \arrow[r, squiggly, "\varepsilon"'] & 1
\end{tikzcd}
\quad = \quad \mathrm{id} .
$$

When the context makes clear what the data η, ε, t, and s are to be, we will often write $F \rightsquigarrow\!\!| \; U$ to mean that there is an op-quasi-adjunction between F and U. Also, all op-quasi-adjunctions considered in this thesis have identity modifications for t and s, have a 2-natural transformation for η, and have an ε satisfying $\varepsilon_\varepsilon = \mathrm{id}$.

Proposition 6.1. (Gray [8], Butler [3]). *If* $F: \mathcal{A} \to \mathcal{B}$ *and* $U: \mathcal{B} \to \mathcal{A}$ *are* 2-*functors and if* $(U,F,\eta,\varepsilon,t,s)$ *is an op-quasi-adjunction, then for each* A *in* $\mathcal{A}$ *and* B *in* $\mathcal{B}$ *the functor*

$$\mathcal{B}(FA,B) \xrightarrow{\mathcal{A}(\eta A,1).U} \mathcal{A}(A,UB)$$

is the left adjoint of

$$\mathcal{A}(A,UB) \xrightarrow{\mathcal{B}(1,\varepsilon B).F} \mathcal{B}(FA,B) \quad .$$

Moreover the unit ν *and counit* σ *of this adjunction are given by the equations*

$$\nu_f = \varepsilon_f.s$$

$$\sigma_g = t.\eta_g$$

for $f \in \mathcal{B}(FA,B)$ *and* $g \in \mathcal{A}(A,UB)$. □

7. If $\mathcal{V}$ is a symmetric monoidal category the concepts of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations have been discussed by many authors, we therefore give no details of these concepts in this thesis but take for granted that the reader is familiar with $\mathcal{V}$-category theory. We do however wish to review some facts about $\mathcal{V}$-graphs.

A $\mathcal{V}$-__graph__ G consists of a set of objects $|G| \in SET$ together with, for all $A,B \in |G|$, an object $G(A,B)$ of $\mathcal{V}$.

If G and L are $\mathcal{V}$-graphs a morphism $M\colon G \to L$ consists of a set function

$$M\colon |G| \to |L|$$

together with, for each A,B in $|G|$, a morphism

$$M_{A,B}\colon G(A,B) \to L(MA,MB)$$

in $\mathcal{V}$. We denote by $\mathcal{V}$-$GRAPH$ the category of $\mathcal{V}$-graphs and their morphisms, and by $\mathcal{V}$-Graph the category of __small__ $\mathcal{V}$-graphs. There is an evident forgetful functor

$$W_{\mathcal{V}}\colon \mathcal{V}\text{-}CAT \to \mathcal{V}\text{-}\mathit{Graph}\,.$$

__Proposition 7.1__. (Wolff [18]). __If__ $\mathcal{V}$ __is a cocomplete monoidal category, then the forgetful functor__ $W_{\mathcal{V}}$ __is monadic__. $\Box$

Since CAT has colimits of diagramsas big as objects of SET it is easily seen that Wolff's proof shows us that

$$U_1 = W_{SET}\colon CAT \to GRAPH$$

is monadic with a left adjoint denoted by F_1.

It is well known that any monoidal functor $V\colon \mathcal{V} \to \mathcal{V}'$ induces a 2-functor

$$\mathrm{V}\text{-}\mathit{CAT}\colon \mathcal{V}\text{-}\mathit{CAT} \to \mathcal{V}'\text{-}\mathit{CAT}$$

and similarly for monoidal natural transformations and 2-natural transformations. It is just as easy to see that any <u>functor</u> $V\colon \mathcal{V} \to \mathcal{V}'$ induces a <u>functor</u>

$$\mathrm{V}\text{-}\mathit{GRAPH}\colon \mathcal{V}\text{-}\mathit{GRAPH} \to \mathcal{V}'\text{-}\mathit{GRAPH} \quad ,$$

that a natural transformation $\alpha\colon V \Rightarrow V'$ induces a natural transformation

$$\alpha\text{-}\mathit{GRAPH}\colon \mathrm{V}\text{-}\mathit{GRAPH} \Rightarrow \mathrm{V}'\text{-}\mathit{GRAPH} \;,$$

and that $(-)$-*GRAPH* is functorial. Thus if the functor $U\colon \mathcal{V} \to \mathcal{V}'$ has a left adjoint $F\colon \mathcal{V}' \to \mathcal{V}$ then F-*GRAPH* is the left adjoint of U-*GRAPH*.

It is well known that *GRAPH* is a cartesian closed category, and that 2-*GRAPH* = *GRAPH*-*GRAPH* is also cartesian closed, so that we have the category 3-*GRAPH* = (2-*GRAPH*)-*GRAPH*.

Since $U_1\colon \mathit{CAT} \to \mathit{GRAPH}$ has a left adjoint F_1 it then follows immediately by Proposition 7.1 that the functor

$$U_2 \;=\; 2\text{-}\mathit{CAT} \xrightarrow{W_{\mathit{CAT}}} \mathit{CAT}\text{-}\mathit{GRAPH} \xrightarrow{U_1\text{-}\mathit{GRAPH}} 2\text{-}\mathit{GRAPH}$$

has a left adjoint which we call F_2. A similar argument shows that the functor

$$U_3 = \mathit{3\text{-}CAT} \xrightarrow{W_{\mathit{2\text{-}CAT}}} \mathit{(2\text{-}CAT)\text{-}GRAPH} \xrightarrow{U_{2}\text{-}\mathit{GRAPH}} \mathit{3\text{-}GRAPH}$$

has a left adjoint called F_2.

$\underline{\underline{8}}$. Let $\mathcal{A}$ be a 2-graph and $\mathcal{B}$ a 2-category and let $F,G:\mathcal{A} \to U_2\mathcal{B}$ be morphisms of 2-graphs.

A <u>lax-natural transformation</u> of 2-graphs $\alpha\colon F \rightsquigarrow G$ assigns to each object A of $\mathcal{A}$ a morphism $\alpha A\colon FA \to GA$ in $\mathcal{B}$ and to each morphism $U\colon A \to A'$ in $\mathcal{A}$ a 2-cell α_u in $\mathcal{B}$ as in

$$\begin{tikzcd}
FA \arrow[r, "Fu"] \arrow[d, "\alpha A"'] & FA' \arrow[d, "\alpha A'"] \\
GA \arrow[r, "Gu"'] \arrow[ur, phantom, "\overset{\alpha_u}{\Rightarrow}"] & GA'
\end{tikzcd} .$$

This data is subject to the following axioms. For each $\gamma\colon u \Rightarrow v$ in $\mathcal{A}$ we have the equality

$$\begin{tikzcd}
FA \arrow[r, bend left, "Fu"] \arrow[d, "\alpha A"'] \arrow[dr, phantom, "\Downarrow\alpha_u"] & FA' \arrow[d, "\alpha A'"] \\
GA \arrow[r, bend left, "Gu"] \arrow[r, bend right, "Gv"'] \arrow[r, phantom, "\Downarrow G\gamma"] & GA'
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[r, bend left, "Fu"] \arrow[r, bend right, "Fv"'] \arrow[r, phantom, "\Downarrow F\gamma"] \arrow[d, "\alpha A"'] \arrow[dr, phantom, "\Downarrow\alpha_v"] & FA' \arrow[d, "\alpha A'"] \\
GA \arrow[r, bend right, "Gv"'] & GA'
\end{tikzcd} .$$

If α and β are lax-natural transformations of 2-graphs, a modification θ: $\alpha \to \beta$ assigns to each $A \in \mathcal{A}$ a 2-cell in $\mathcal{B}$ of the form

$$\begin{tikzcd} FA \arrow[r, bend left, "\alpha A"] \arrow[r, bend right, "\beta A"'] \arrow[r, phantom, "\Downarrow \theta A"] & GA \end{tikzcd}$$

such that for every morphism u: $A \to A'$ in $\mathcal{A}$

$$\beta_u . \theta A \quad = \quad \theta A' . \alpha_u \ .$$

If we compare these definitions with those of lax-natural transformations and modifications of 2-categories as given in section 3, we will observe that the data involved in each case are the same, the only difference is that in section 3 we required certain axioms to hold which specified how the data was to interact with the composition in $\mathcal{A}$.

If $\mathcal{A}$ and $\mathcal{C}$ are 2-graphs and $\mathcal{B}$ is a 2-category and if F: $\mathcal{A} \to U_2\mathcal{B}$ and G: $\mathcal{C} \to U_2\mathcal{B}$ are morphisms of 2-graphs, then we define the 2-graph $F /\!\!/ G$ as follows. The objects of $F /\!\!/ G$ are triples (A,f,C) where $A \in \mathcal{A}$, $C \in \mathcal{C}$ and where f: $FA \to GC$ is a 1-cell in $\mathcal{B}$; the morphisms in $F /\!\!/ G$ from (A,f,C) to (A',f',C') are triples (h,γ, k) where h: $A \to A'$ is a 1-cell in $\mathcal{A}$, where k: $C \to C'$ is a 1-cell in $\mathcal{C}$, and where γ is a 2-cell in $\mathcal{B}$ as in

$$\begin{tikzcd} FA \arrow[r, "f"] \arrow[d, "Fh"'] \arrow[dr, phantom, "\Downarrow\gamma"] & GA \arrow[d, "Gk"] \\ FA' \arrow[r, "f'"'] & GA' \end{tikzcd} .$$

A 2-cell in $F/\!\!/G$ from (h,γ,k) to (h',γ',k') is a pair (α_0,α_1) of 2-cells $\alpha_0\colon k \Rightarrow k'$ in A and $\alpha_1\colon h \Rightarrow h'$ in C such that

$$\gamma . F\alpha_0 = G\alpha_1 . \gamma' .$$

We point out that $F/\!\!/G$ is defined here exactly as it was defined in section 3, except that now F and G are not 2-functors, so that it is clear how to define ∂_0, ∂_1 and δ as in

$$\begin{tikzcd} F/\!\!/G \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\rightsquigarrow\delta"] & A \arrow[d, "F"] \\ C \arrow[r, "G"'] & U_2B \end{tikzcd} .$$

This time however ∂_0 and ∂_1 are only morphisms of 2-graphs and δ is only a lax-natural transformation of 2-graphs. It is not surprising to find that $F/\!\!/G$ has a universal property with respect to lax-natural transformations and modifications of 2-graphs; this universal property is given by the following easy result.

Lemma 8.1. *If* $\mathcal{E}$ *is a* 2-*graph then triples* (D_0,ε,D_1) *as in*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "D_0"] \arrow[d, "D_1"'] \arrow[dr, phantom, "\wr\varepsilon"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{C} \arrow[r, "G"'] & U_2\mathcal{B} \end{tikzcd}$$

are in bijection with morphisms $V\colon \mathcal{E} \to F/\!\!/G$ *of* 2-*graphs. The bijection is given by* $\partial_0 V = D_0, \partial_1 V = D_1$ *and* $\delta V = \varepsilon$.

Moreover if V *and* V' *are morphism from* $\mathcal{E}$ *to* $F/\!\!/G$ *corresponding to* (D_0,ε,D_1) *and* (D_0',ε',D_1'), *then lax-natural transformation* $\alpha\colon V \rightsquigarrow V'$ *are in bijection with triples* $(\alpha_0,\sigma,\alpha_1)$ *where* α_0 *and* α_1 *are lax-natural transformations of* 2-*graphs* $\alpha_0\colon D_0 \rightsquigarrow D_0'$ *and* $\alpha_1\colon D_1 \rightsquigarrow D_1'$, *and where* σ *is a modification from* $\alpha_1 G.\varepsilon$ *to* $\varepsilon.\alpha_0 F$. *The bijection is given by* $\alpha_0 = D_0\alpha$, $\alpha_1 = D_1\alpha$, *and* $\sigma_E = \varepsilon_{\alpha_E}$. □

As an immediate consequence of this result we have:

Lemma 8.2. *If* $\mathcal{A}$ *is a* 2-*graph and* $\mathcal{B}$ *is a* 2-*category then for every lax-natural transformation of* 2-*graphs*

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\wr\alpha"] & U_2\mathcal{B} \end{tikzcd}$$

there exists a unique lax-natural transformation of 2-*categories*

$$\begin{tikzcd} F_2\mathcal{A} \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\wr\beta"] & \mathcal{B} \end{tikzcd}$$

such that $U_2\beta \cdot \eta_2 A = \alpha$. *Moreover if* α *and* α' *are a pair of lax-natural transformations of* 2-*graphs from* F *to* G *and if* $\sigma: \alpha \to \alpha'$ *is a modification, then there exists a unique modification of* 2-*categories* $\pi: \beta \to \beta': F' \Rightarrow G'$ *such that* $U_2\pi \cdot \eta_2 A = \sigma$.

Proof. Let

$$\begin{tikzcd} \mathcal{B}/\!\!/\mathcal{B} \arrow[r, "\partial_0"] \arrow[d, "\partial_1"'] \arrow[dr, phantom, "\Downarrow\delta"] & U_2\mathcal{B} \arrow[d, "1"] \\ U_2\mathcal{B} \arrow[r, "1"'] & U_2\mathcal{B} \end{tikzcd}$$

be the lax comma object as in the previous lemma, with F = 1 and G = 1. The lax-natural transformations α and α' induce unique morphisms L and L' from A to $\mathcal{B}/\!\!/\mathcal{B}$ with $\delta L = \alpha$ and $\delta L' = \alpha'$. From L and L' we get unique 2-functors $P, P': F_2 A \to \mathcal{B}/\!\!/\mathcal{B}$, since $\mathcal{B}/\!\!/\mathcal{B}$ is automatically a 2-category, and from these we get unique 2-cells β and β' as required, since $\mathcal{B}/\!\!/\mathcal{B}$ is also the lax-comma object described in section 3.

From the triple $(1_F, \sigma, 1_G)$ we get a unique lax-natural transformation $\lambda: L \rightsquigarrow L'$, so that by the first part of the lemma we have a unique lax-natural transformation $\mu: P \rightsquigarrow P'$ which in turn induces a unique modification π as required. $\square$

It is obvious that a straightforward imitation of the above gives the analogous result for the functors

$$U_3\colon \mathit{3\text{-}CAT} \to \mathit{3\text{-}GRAPH}$$

and

$$F_3\colon \mathit{3\text{-}GRAPH} \to \mathit{3\text{-}CAT} \quad ,$$

once the notions of lax-natural transformation and modifications of 3-graphs and 3-categories have been defined in the obvious way.

9. A doctrine on a 2-category K consists of a 2-functor $D\colon K \to K$, and 2-natural transformations $i\colon 1 \to D$ and $m\colon D^2 \to D$ such that

(9.1) $m.Di = m.iD = 1$ and $m.Dm = m.mD$.

It is clear that a doctrine is just a 2-monad on the 2-category A.

A D-algebra is a pair (A,a) where $A \in K$ and where $a\colon DA \to A$ is a morphism in K such that

(9.2) $$a.iA = 1$$

and

(9.3) $$a.Da = a.mA \quad .$$

A D-<u>morphism</u> $F\colon (A,a) \to (B,b)$ is a pair $(f,\bar{f})$ where $f\colon A \to B$ is a morphism in $\mathcal{A}$ and where $\bar{f}$ is a 2-cell in K as in

$$\begin{tikzcd} DA \arrow[r, "Df"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & DB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

such that

$$\bar{f}.iA \;=\; id$$

and

$$\bar{f}.mA \;=\; \bar{f}.D\bar{f} \quad .$$

We call a D-morphism <u>strict</u> when $\bar{f}$ is an identity 2-cell.

A D-2-<u>cell</u> $\alpha\colon F \Rightarrow G\colon (A,a) \to (B,b)$ is a 2-cell $\alpha\colon f \Rightarrow g$ in K such that

$$\begin{tikzcd} DA \arrow[r, bend left, "Df"] \arrow[r, bend right, "Dg"'] \arrow[r, phantom, "\Downarrow D\alpha"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{g}"] & DB \arrow[d, "b"] \\ A \arrow[r, bend right, "g"'] & B \end{tikzcd} \quad = \quad \begin{tikzcd} DA \arrow[r, bend left, "Df"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & DB \arrow[d, "b"] \\ A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & B \end{tikzcd} \quad .$$

We denote by D-$\mathcal{A}\ell g$ the 2-category of D-algebras, D-morphisms, and D-2-cells; while D-$\mathcal{A}\ell g_*$ is the sub-2-category which retains only the strict D-morphisms. We denote the inclusion of D-$\mathcal{A}\ell g_*$ into D-$\mathcal{A}\ell g$ by J: D-$\mathcal{A}\ell g_* \to$ D-$\mathcal{A}\ell g$. There is an evident forgetful 2-functor U^D: D-$\mathcal{A}\ell g \to K$ which takes (A,a) to A and $(f,\bar{f})$ to f. Since D-$\mathcal{A}\ell g_*$ is nothing more than the 2-category of Eilenberg-Moore algebras for the 2-monad D it is well known that the forgetful 2-functor U^DJ: D-$\mathcal{A}\ell g_* \to K$ has a left adjoint F^D: $K \to$ D-$\mathcal{A}\ell g_*$.

Let K' be the 2-category $[\![2,K]\!]$ defined in section 3, and let D' be the doctrine on K' given by $D' = [\![2,D]\!]$, $i' = [\![2,i]\!]$, and $m' = [\![2,m]\!]$ so that if we use the elementary description of $[\![2,K]\!]$ given in section 4, then the action of D',i', and m' are as follows:

$$D'(A,A \xrightarrow{f} B,B) = (DA,DA \xrightarrow{Df} DB,DB),$$

$$i'(A,A \xrightarrow{f} B,B) = (iA,id,iB)$$

and

$$m'(A,A \xrightarrow{f} B,B) = (mA,id,mB).$$

It is then clear that a D'-algebra consists of an object $(A,A \xrightarrow{f} B,B)$ of K' together with an action of D' on f as in

(9.4)
$$\begin{tikzcd} DA \arrow[r, "a"] \arrow[d, "Df"'] & A \arrow[d, "f"] \\ DB \arrow[r, "b"'] \arrow[ur, phantom, "\overset{\bar{f}}{\Longrightarrow}"] & B \end{tikzcd},$$

which is to satisfy the unit and associativity axioms. It is easy to see that the axioms required for (9.4) to be a D'-algebra are precisely the axioms required to make (A,a) and (B,b) D-algebras and F = $(f,\bar{f})$ a D-morphism. It is infact possible to describe D'-morphisms and D'-2-cells in terms of D; the following result (the proof of which can be found in Kelly [12]) does this for us.

Proposition 9.1. *A D'-algebra is precisely a pair of D-algebras and a D-morphism between them.*

A D'-morphism from F: A → B to G: C → E is precisely a pair of D-morphisms V: A → C and W: B → E and a D-2-cell α as in

$$\begin{tikzcd} A \arrow[r, "V"] \arrow[d, "F"'] & C \arrow[d, "G"] \\ B \arrow[r, "W"'] \arrow[ur, phantom, "\overset{\alpha}{\Longrightarrow}"] & E \end{tikzcd};$$

the D'-morphism is strict if and only if V and W are strict D-morphisms.

A D'-2-*cell from* (V,α,W) to (V',α',W') *is a pair of* D-2-*cells* β_0 *and* β_1 *where* β_0: V → V' *and* β_1: W → W' *such that*

$$
\begin{tikzcd}
A \arrow[rr, bend left, "V'"] \arrow[d, "F"'] \arrow[rr, phantom, shift right=4ex, "\overset{\alpha'}{\Rightarrow}"] & & C \arrow[d, "G"] \\
B \arrow[rr, bend left, "W'"] \arrow[rr, bend right, "W"'] \arrow[rr, phantom, "\Uparrow \beta_1"] & & E
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[rr, bend left, "V"] \arrow[rr, bend right, "V'"'] \arrow[rr, phantom, "\Uparrow \beta"] \arrow[d, "F"'] & & C \arrow[d, "G"] \\
B \arrow[rr, bend right, "W'"'] \arrow[rr, phantom, shift left=4ex, "\overset{\alpha}{\Rightarrow}"] & & E
\end{tikzcd}
\quad . \ \square
$$

As well as the 2-categories D-$\mathcal{A\ell g}$ and D-$\mathcal{A\ell g}_*$ we can also define the 2-categories $\mathcal{Lax}$-D-$\mathcal{A\ell g}$ and $\mathcal{Lax}$-D-$\mathcal{A\ell g}_*$ of lax-D-algebras, D-morphisms (resp. strict D-morphisms), and D-2-cells. A lax-D-algebra is an object A of $\mathcal{K}$ together with a morphism a: DA → A in $\mathcal{K}$ and 2-cells

$$
\begin{tikzcd}
A \arrow[r, "iA"] \arrow[dr, "1"'] & DA \arrow[d, "a"] \arrow[dl, phantom, shift left=2ex, "\overset{\alpha_0}{\Rightarrow}"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
D^2A \arrow[r, "mA"] \arrow[d, "Da"'] \arrow[dr, phantom, "\overset{\bar{\alpha}}{\Rightarrow}"] & DA \arrow[d, "a"] \\
DA \arrow[r, "a"'] & A
\end{tikzcd}
$$

which are to satisfy various axioms that may be found in Kelly [12], where may also be found the definitions of lax-D-morphisms of such things. A strict D-morphism of lax-D-algebras is just a morphims f: A → B such that b.Df = f.a, f.α_0 = β_0.f, and $\bar{\beta}$.D^2f = f.$\bar{\alpha}$.

If D and D' are any doctrines on the <u>same</u> 2-category K we mean by a lax-morphism of doctrines $H\colon D \to D'$ a triple $H = (h, h_0, \bar{h})$ where $h\colon D \Rightarrow D'$ is a 2-natural transformation and where h_0 and $\bar{h}$ are modifications as in

$$\begin{tikzcd} 1 \arrow[r, "i"] \arrow[dr, "i'"'] \arrow[dr, phantom, shift left=3ex, "\overset{h_0}{\Rightarrow}"] & D \arrow[d, "h"] \\ & D' \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} D^2 \arrow[r, "m"] \arrow[d, "h.h"'] \arrow[dr, phantom, "\overset{\bar{h}}{\Rightarrow}"] & D \arrow[d, "h"] \\ D'^2 \arrow[r, "m'"'] & D' \end{tikzcd} .$$

This data is to satisfy the two unit and one associativity axiom

$$(\bar{h}.Di).(m'.hD'.Dh_0) = id$$

$$(\bar{h}.iD).(m'.h_0D'.h) = id$$

and

$$(\bar{h}.Dm).(m'.hD'.D\bar{h}) = (\bar{h}.mD).(m'.\bar{h}D'.D^2h)$$

which may be found drawn more explicitly in Kelly [12] . The lax-morphism of doctrines $H = (h, h_0, \bar{h})$ is called a <u>strict morphism of doctrines</u>, or just a <u>morphism of doctrines</u> when h_0 and $\bar{h}$ are identity modifications.

Since morphisms of doctrines are just morphisms of 2-monads in the V-category sense, we have the expected correspondence between doctrine morphisms and algebraic 2-functors. That is, from a doctrine morphism $h\colon D \Rightarrow D'$ we get a 2-functor $h\text{-}\mathcal{A}\ell g_*\colon D'\text{-}\mathcal{A}\ell g_* \to D\text{-}\mathcal{A}\ell g_*$, such that

$$U^D . h\text{-}\mathcal{A}\ell g_* \;=\; U^{D'},$$

given by

$$h\text{-}\mathcal{A}\ell g_*(A,a) \;=\; (A, DA \xrightarrow{hA} D'A \xrightarrow{a} A).$$

Moreover any 2-functor $\Psi\colon D'\text{-}\mathcal{A}\ell g_* \to D\text{-}\mathcal{A}\ell g$ such that $U^D\Psi = U^{D'}$ is of necessity $h\text{-}\mathcal{A}\ell g_*$ for some unique doctrine morphism $h\colon D \Rightarrow D'$.

A 2-functor $U\colon \mathcal{B} \to \mathcal{A}$ is said to be 2-<u>monadic</u> or <u>doctrinal</u> if there exists a doctrine D on $\mathcal{A}$ and an isomorphism $\Sigma\colon D\text{-}\mathcal{A}\ell g_* \to \mathcal{B}$ of 2-categories such that

$$U\Sigma \;=\; U^D.$$

As in the case of monads on categories we can give necessary and sufficient conditions for a 2-functor to be 2-monadic, and also as in the case of monads on categories these conditions involve the notion of a U-split pair. A pair of morphisms $f,g\colon A \to B$ in $\mathcal{B}$ are a U-split pair if there exists an object C in $\mathcal{A}$ and morphisms

$$\begin{tikzcd} UA & UB \arrow[l, bend left, "d_1"] \arrow[r, "p"] & Z \arrow[l, bend left, "d_0"] \end{tikzcd}$$

such that

$$pUf = pUg,\; pd_0 = 1,\; d_0p = Ug.d_1, \text{ and } Uf.d_1 = 1.$$

<u>Proposition 9.2</u>. <u>A</u> 2-<u>functor</u> U: $\mathcal{B} \to \mathcal{A}$ <u>is</u> 2-<u>monadic if and only if</u> (i) U <u>has a left adjoint, and</u> (ii) U <u>creates coequalisers of</u> U <u>split pairs</u>.

<u>Proof</u>. A direct imitation of the corresponding well known result for monads on categories. □

Let $\mathcal{A}$ be a complete 2-category and let A and B be objects of $\mathcal{A}$; then we denote by $\{A,B\}: \mathcal{A} \to \mathcal{A}$ the right Kan extension of $\ulcorner B \urcorner : \mathbb{1} \to \mathcal{A}$ along $\ulcorner A \urcorner : \mathbb{1} \to \mathcal{A}$. It is well known that $\{A,B\}$ is characterised by the existence, for every 2-functor $R: \mathcal{A} \to \mathcal{A}$, of a 2-natural bijection θ between morphisms $a: RA \to B$ and 2-natural transformations $\alpha: R \to \{A,B\}$. We denote by $e: \{A,B\}(A) \to B$ the "evaluation" morphism which is actually $\theta(1_{\{A,B\}})$.

It is easy to see (cf. Kelly [12]) that the 2-natural transformations

$$\theta^{-1}(1_A): 1 \to \{A,A\}$$

and

$$m: \{A,A\}\circ\{A,A\} \to \{A,A\},$$

where m is θ^{-1} of the composite

$$\{A,A\}\circ\{A,A\}(A) \xrightarrow{1\circ e} \{A,A\}\circ A \xrightarrow{e} A \quad ,$$

give $\{A,A\}$ the structure of a doctrine. For any $f\colon A \to A'$ and $g\colon B \to B'$ in A we let

$$\begin{tikzcd}
{[f,g]} \arrow[r, "d_1"] \arrow[d, "d_0"'] & \{B,A\} \arrow[d, "\{1,f\}"] \\
\{B',A'\} \arrow[r, "\{g,1\}"'] & \{B,A'\}
\end{tikzcd}$$

be a pull back, let

$$\begin{tikzcd}
\langle f,g\rangle \arrow[r, "d_1"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Longrightarrow}"] & \{B,A\} \arrow[d, "\{1,f\}"] \\
\{B',A'\} \arrow[r, "\{g,1\}"'] & \{B,A'\}
\end{tikzcd}$$

be a comma object, and denote by $\varepsilon\colon [f,g] \to \langle f,g\rangle$ the obvious canonical map. Finally if $\gamma\colon f \Rightarrow f'$ and $\gamma'\colon g \Rightarrow g'$ are 2-cells in A we let

$$\begin{tikzcd}
{[\![\gamma,\gamma']\!]} \arrow[r, "d_1"] \arrow[d, "d_0"'] & \langle g,f\rangle \arrow[d, "\langle 1,\gamma\rangle"] \\
\langle g',f'\rangle \arrow[r, "\langle \gamma',1\rangle"'] & \langle g,f'\rangle
\end{tikzcd}$$

be a pull back. Once again easy formal arguments show that $[f,f]$, $\langle f,f\rangle$, and $[\![\gamma,\gamma]\!]$ are doctrines on A and that d_0 and d_1 are morphisms of doctrines. Further details of the above constructions, together with the proof of the following proposition may be found in Kelly [12].

Proposition 9.3. (i) The morphism $a: DA \to A$ is a D-algebra if and only if $\theta(a): D \to \{A,A\}$ is a morphism of doctrines.

(ii) The morphism $f: A \to B$ is a strict D-morphism from (A,a) to (B,b) if and only if there exists a unique morphism of doctrines $k: D \to [f,f]$ such that $d_0k = \theta(a)$ and $d_1k = \theta(b)$ in which case we denote k by $\theta(f)$.

(iii) The 2-cell $\rho: f \Rightarrow g$ is a D-2-cell of strict D-morphisms if and only if there exists a unique morphism of doctrines $k: D \to [\![\sigma,\sigma]\!]$ such that $d_0k = \varepsilon.\theta(f)$ and $d_1k = \varepsilon.\theta(g)$. $\square$

10. If α is a cardinal number (a small cardinal in the sense that it is a cardinal in *Set*) and A is a category we say that A is α-filtered if (cf. Schubert [17])

a) for every family $(A_\nu)_{\nu\in I}$ of objects in A with $\mathrm{card}(I) < \alpha$ there is an object $A \in A$ and a family of morphisms $(A_\nu \to A)_{\nu\in I}$

b) for every family $(\xi_\lambda: A_0 \to A_1)_{\lambda\in L}$ of morphisms in A with $\mathrm{card}(L) < \alpha$ there is a morphism $\zeta: A_1 \to A_2$ such that $\zeta\xi_\lambda = \zeta\xi_\mu$ for all $\lambda,\mu \in L$.

If γ is an ordinal number we say that γ is an α-filtered ordinal if the well ordered set γ is α-filtered when considered as a category. If we write γ for both the ordinal γ and for the ordered set considered as a category, then by a γ-sequence in a category A we mean a functor $K: \gamma \to A$.

We identify the cardinal numbers with the initial ordinals, so that if α is a cardinal we may mean either the cardinal number of the corresponding initial ordinal. We observe that regular ordinals are also cardinals so that in the definition that follows it does not matter whether α is an ordinal or cardinal.

If T is an endofunctor of a category A and α is a regular ordinal, then we say that T has rank $\leq \alpha$ if T preserves the colimits of γ-sequences for all α-filtered ordinals γ. We say that T has rank if there exists a regular α such that T has rank $\leq \alpha$. If T has rank $\leq \alpha$ then T at least preserves colimits of α-sequences since α is an α-filtered ordinal, also if α and β are regular with $\alpha < \beta$ then T has rank $\leq \beta$ whenever T has rank $\leq \alpha$.

CHAPTER 1

1. In this chapter we consider a doctrine D = (D,i,m) on a 2-category $\mathcal{K}$; we contemplate the inclusion 2-functor J: $\mathcal{D}_* \to \mathcal{D}$, where $\mathcal{D}$ = D-$\mathcal{Alg}$ and $\mathcal{D}_*$ = D-$\mathcal{Alg}_*$. Our aim is to prove the following two theorems; which besides being applied in the rest of this thesis, are of independent interest in the theory of algebras for a doctrine.

Theorem 1.1. *If the 2-category $\mathcal{K}$ is cocomplete and the 2-functor D has a rank, then the 2-functor J: $\mathcal{D}_* \to \mathcal{D}$ has a left adjoint Φ: $\mathcal{D} \to \mathcal{D}_*$.*

We write the adjunction isomorphism as

$$\pi: \quad \mathcal{D}(A,JB) \cong \mathcal{D}_*(\Phi A,B) \tag{1.1}$$

with unit η and co-unit ε as in

$$\eta: \quad 1 \Rightarrow J\Phi, \quad \varepsilon: \quad \Phi J \Rightarrow 1. \tag{1.2}$$

Theorem 1.2. *Let $\mathcal{K}$ be cocomplete and admit comma objects, and let D have a rank. Let U: $\mathcal{D} \to \mathcal{C}$ be a 2-functor such that the 2-functor UJ: $\mathcal{D}_* \to \mathcal{C}$ has a left adjoint F: $\mathcal{C} \to \mathcal{D}_*$ with unit j, counit n and adjunction isomorphism γ. Then the full inclusion*

$$J: \quad \mathcal{D}_*(FX,B) \to \mathcal{D}(JFX,JB) \tag{1.3}$$

is the left adjoint of the functor W, where W is the composite

$$(1.4) \qquad \begin{tikzcd} \mathcal{D}(JFX,JB) \arrow[r, "U"] \arrow[ddr, "W"'] & \mathcal{C}(UJFX,UJB) \arrow[d, "\mathcal{C}(jX,1)"] \\ & \mathcal{C}(X,UJB) \arrow[d, "\gamma", "\cong"'] \\ & \mathcal{D}_*(FX,B) \end{tikzcd} \quad .$$

We prove Theorem 1.1 in two stages. The first stage consists in embedding $\mathcal{D}_*$ (as a full sub-2-category) in the comma 2-category $D/\mathcal{K}$, and showing that $\mathcal{D}(A,B)$ is isomorphic, naturally in $B \in \mathcal{D}_*$, to $D/\mathcal{K}(X,B)$, for a certain $X \in D/\mathcal{K}$ constructed from the D-algebra (A,a) by the formation of certain colimits. (These are indexed colimits in the sense of Street [16] and $\mathcal{V}$-colimits in the sense of Borceux-Kelly [4]). This is the content of section 2 and 3 of this chapter.

The second stage consists in proving that, for cocomplete $\mathcal{K}$ and ranked D, the full sub-2-category $\mathcal{D}_*$ is reflective in $D/\mathcal{K}$; this occupies section 4, which sets up the machinery for a transfinite induction argument, and section 5 which uses the rank of D to complete the construction of the reflection R.

The two stages are now combined to complete the proof of Theorem 1.1 by setting $\Phi A = RX$ and noting the isomorphism

$$\mathcal{D}(A,B) \cong D/\mathcal{K}(X,B) \cong \mathcal{D}_*(RX,B).$$

To obtain Theorem 1.2 we extend the adjunction of Theorem 1.1 to something richer. Consider the unit η and co-unit ε as in (1.2) of the adjunction (1.1). The natural transformation η has arbitrary D-morphisms for components and moreover is natural for arbitrary D-morphisms. The natural transformation ε on the other hand has strict D-morphisms for components and is natural only for strict D-morphisms. We may ask how ε behaves in relation to arbitrary D-morphisms. It turns out that ε "behaves like an op-lax-natural transformation" with respect to such D-morphisms. More precisely there is an op-lax-natural transformation $\rho: J\Phi \rightsquigarrow 1$ with the property that $\rho J = J\varepsilon$; so that the object-components ρB of ρ are just the εB and the morphism-components ρF of ρ are identities when F is strict. It further turns out that $\eta: 1 \Rightarrow J\Phi$ and $\rho: J\Phi \rightsquigarrow 1$ satisfy the equation $\rho.\eta = \mathrm{id}$.

In order to obtain ρ we extend, in sections 6 and 7, the results of Theorem 1.1 from the doctrine D on K to the doctrine $D' = [\![2,D]\!]$ on $K' = [\![2,K]\!]$. We identify K with a sub-2-category of K' by sending $A \in K$ to the object $(A,1_A: A \to A,A)$ in K'; then the inclusion $I_0: K \to K'$ induces (in an obvious notation) inclusions $I: \mathcal{D} \to \mathcal{D}'$ and $I_*: \mathcal{D} \to \mathcal{D}_*'$. It does not seem to be known (the author has discussed the matter with Professors J.W. Gray and R.H. Street) whether K' is cocomplete when K is; still less how far D' would preserve sequential colimits in K'; but we can get away without this knowledge. If we assume that K

has comma objects a few formal arguments allow us to deduce that $J'\colon \mathcal{D}_*' \to \mathcal{D}'$ has a left adjoint Φ', and hence the existence of an isomorphism

$$(1.5) \qquad \pi'\colon \quad \mathcal{D}'(F,J'G) \cong \mathcal{D}_*'(\Phi'F,G).$$

We use this isomorphism to define, in section 8, the op-lax-natural transformation ρ.

Also in section 8 we use ρ to show that, for any F and U as in Theorem 1.2, there exists an op-lax-natural transformation

$$(1.6) \qquad \kappa\colon JF\mathcal{U} \rightsquigarrow 1_{\mathcal{D}}$$

satisfying

$$\kappa J = Jn$$

$$(1.7) \qquad \mathcal{U}\kappa . j\mathcal{U} = \text{id}.$$

We then show that j and κ exhibit $JF\colon \mathcal{C} \to \mathcal{D}$ as an op-quasi-left adjoint to $U\colon \mathcal{D} \to \mathcal{C}$; Theorem 1.2 follows directly from this result.

2. Recall from Chapter 0 the definition of comma object; we denote by D/K the comma object of $D\colon K \to K$ and $1_K\colon K \to K$ in the 2-category 2-CAT. We observe that an object of D/K is a triple (X_0,x,X_1) where X_0 and X_1 are objects of K and $x\colon DX_0 \to X_1$ is a morphism of K. Morphisms in D/K from $X = (X_0,x,X_1)$ to $Y = (Y_0,y,Y_1)$ are pairs (f_0,f_1) where $f_0\colon X_0 \to Y_0$ and $f_1\colon X_1 \to Y_1$ are morphisms in K satisfying

$$(2.1)\qquad \begin{tikzcd} DX_0 \arrow[r, dotted, "Df_0"] \arrow[d, "x"'] & DY_0 \arrow[d, "y"] \\ X_1 \arrow[r, dotted, "f_1"'] & Y_1 \end{tikzcd} .$$

The 2-cells of D/K from (f_0,f_1) to (g_0,g_1) are pairs (α_0,α_1) of 2-cells in K with α_0: $f_0 \Rightarrow g_0$ and α_1: $f_1 \Rightarrow g_1$ satisfying

$$(2.2)\qquad y.D\alpha_0 = \alpha_1.x$$

Consider the 2-functor L: $\mathcal{D}_* \to D/K$ which takes the D-algebra A = (A,a) to the object (A,a,A) of D/K, the strict D-morphism f to the morphism (f,f) in D/K, and the D-2-cell α to the 2-cell (α,α) in D/K. We now show that L is full and faithful.

Lemma. 2.1. If $(\alpha_0,\alpha_1):(f_0,f_1) \to (g_0,g_1):X \to LB$ is a 2-cell in D/K for B = (B,b) $\in \mathcal{D}_*$ then

$$(2.3)\qquad \begin{aligned} f_0 &= f_1 . x . iX_0 , \\ g_0 &= g_1 . x . iX_0 , \\ \alpha_0 &= \alpha_1 . x . iX_0 . \end{aligned}$$

<u>Proof</u>. The diagram

(2.4)

$$\begin{tikzcd}
X_0 \arrow[r, bend left, "f_0"] \arrow[r, bend right, "g_0"'] \arrow[r, phantom, "\Downarrow\alpha_0"] \arrow[d, "iX_0"'] & B \arrow[d, "iB"] \arrow[ddr, bend left, "1"] & \\
DX_0 \arrow[r, bend left, "Df_0"] \arrow[r, bend right, "Dg_0"'] \arrow[r, phantom, "\Downarrow D\alpha_0"] \arrow[d, "x"'] & DB \arrow[d, "b"] & \\
X_1 \arrow[r, bend left, "f_1"] \arrow[r, bend right, "g_1"'] \arrow[r, phantom, "\Downarrow\alpha_1"] & B \arrow[r, "1"'] & B
\end{tikzcd}$$

commutes; the top cylinder by the 2-naturality of i, the bottom cylinder by the definition of 2-cells in $D/\mathcal{K}$, and the triangle by the unit axiom for the D-algebra (B,b). □

<u>Corollary 2.2</u>. <u>The 2-functor</u> L <u>is fully faithful</u>.

<u>Proof</u>. If in Lemma 2.1 we let X = LA, for a D-algebra A = (A,a), then using the fact that a.iA = 1 we get $f_0 = f_1$, $g_0 = g_1$ and $\alpha_0 = \alpha_1$. The conditions (2.1) and (2.2) reduce, in this case, to the definitions of 1-cells and 2-cells of $\mathcal{D}_*$. □

Henceforth we use L to identify $\mathcal{D}_*$ with a full sub-2-category of D/$\mathcal{K}$.

3. If A = (A,a) is a D-algebra and X an object of D/$\mathcal{K}$ we shall have occasion below to consider triples (u,δ,v) where u: $A \to X_0$ and v: $A \to X_1$ are morphisms in $\mathcal{K}$ and δ is a 2-cell in $\mathcal{K}$ as in

(3.1)
$$\begin{tikzcd} DA \arrow[r, "Du"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\delta"] & DX_0 \arrow[d, "x"] \\ A \arrow[r, "v"'] & X_1 \end{tikzcd} .$$

We refer, somewhat loosely, to "the diagram (3.1)" when what we really mean is the corresponding triple. Among these diagrams are those giving the data for a D-morphism

(3.2)
$$\begin{tikzcd} DA \arrow[r, "Df"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\bar{f}"] & DB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} ;$$

of course these data have to satisfy two axioms to be a D-morphism.

From a diagram of the form (3.1) and a morphism g: X → B, where B = (B,b) is a D-algebra, we get, by pasting, a new diagram, namely,

(3.3)
$$\begin{tikzcd} DA \arrow[r, "Du"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\delta"] & DX_0 \arrow[r, "Dg_0"] \arrow[d, "x"] & DB \arrow[d, "b"] \\ A \arrow[r, "v"'] & X_1 \arrow[r, "g_1"'] & B \end{tikzcd}$$

which we call the <u>composite</u> of (3.1) and g. If g_0u coincides with g_1v the diagram (3.3) has the form (3.2) for $f = g_1v$ and $\bar{f} = g_1\delta$; it will therefore be a D-morphism if it satisfies the appropriate axioms.

This section is given to the proof of:

<u>Proposition 3.1</u>. <u>Let</u> K <u>be a cocomplete</u> 2-<u>category and let</u> $A = (A,a)$ <u>be a</u> D-<u>algebra</u>. <u>Then there exists an object</u> $X = (X_0,x,X_1)$ <u>of</u> D/K, <u>morphisms</u> $u\colon A \to X_0$ <u>and</u> $v\colon A \to X_1$ <u>in</u> K, <u>and a</u> 2-<u>cell</u> δ <u>in</u> K, <u>of the form</u> (3.1), <u>such that for every</u> $B \in \mathcal{D}$ <u>composition with</u> (3.1) <u>induces an isomorphism of categories</u>

(3.4) $$\theta\colon\ D/K(X,B) \cong \mathcal{D}(A,B) \quad .$$

<u>Proof</u>. The proof divides into three sections. First, starting with A and a, we construct the diagram (3.1) by forming certain (indexed) colimits in K. Next we show that the result of pasting (3.1) onto a morphism $g\colon X \to B$ is a D-morphism $(f,\bar{f})\colon\ A \to B$. Finally we show that every D-morphism $(f,\bar{f})$ is of this form for a unique $g\colon\ X \to B$; this establishes the isomorphism (3.4) at the level of 1-cells. Since K is cocomplete <u>as a</u> 2-<u>category</u>, the colimits we form have a universal property at the level of

2-cells as well as at the level of 1-cells; it is an easy matter, using this, to show that pasting with (3.1) induces the isomorphism (3.4) at the level of 2-cells as well as at the level of 1-cells. The extension to 2-cells, while being an easy imitation of the case for 1-cells, is tedious to write out; hence we leave it to the reader and give the details for the 1-cell level only.

We construct X_0 as the terminus of the universal (that is, initial) diagram in K of the form

(3.5)
$$\begin{tikzcd} DA \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\gamma"] \arrow[r, "n"] & X_0 \\ A \arrow[ur, "u"'] & \end{tikzcd}$$

subject to the requirements that

(3.6) $$u.a.iA = n.iA$$

and

(3.7) $$\gamma.iA = id.$$

By this we mean that any diagram of the form

(3.8)
$$\begin{tikzcd} DA \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\zeta"] \arrow[r, "z"] & Y \\ A \arrow[ur, "w"'] & \end{tikzcd}$$

satisfying $\zeta.iA = id$ is of the form $y\gamma$ for a unique 1-cell $y\colon X_0 \to Y$.

To get (3.5) from more familiar colimit-notions we have only to form the op-comma-object

(3.9)
$$\begin{tikzcd} DA \arrow[r, "1"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\lambda"] & DA \arrow[d, "h"] \\ A \arrow[r, "k"'] & H \end{tikzcd}$$

of a and 1_{DA}, and then compose with the co-identifier $r: H \to X_0$ of the 2-cell $\lambda.iA$. Note that since A = (A,a) is a D-algebra (3.6) gives

(3.10)
$$u = n.iA.$$

Consider the diagrams

(3.11)
$$\begin{tikzcd} D^2A \arrow[d, "mA"'] & & \\ DA \arrow[d, "a"'] \arrow[dr, "n"] \arrow[dr, phantom, "\Downarrow\gamma", near start, shift right=3ex] & & \\ A \arrow[r, "u"'] & X_0 \arrow[r, "iX_0"'] & DX_0 \end{tikzcd}$$

and

(3.12)
$$\begin{tikzcd} D^2A \arrow[d, "Da"'] \arrow[dr, "Dn"] \arrow[dr, phantom, "\Downarrow D\gamma", near start, shift right=3ex] & & \\ DA \arrow[r, "Du"'] \arrow[d, "a"'] \arrow[dr, "n"] \arrow[dr, phantom, "\Downarrow\gamma", near start, shift right=3ex] & DX_0 \arrow[dr, "1"] & \\ A \arrow[r, "u"'] & X_0 \arrow[r, "iX_0"'] & DX_0 \end{tikzcd}$$
;

these have the forms

(3.11)'
$$\begin{tikzcd}[column sep=huge] D^2A \arrow[r, bend left=50, "p"{name=P}] \arrow[r, bend right=50, "q"'{name=Q}] \arrow[Rightarrow, from=P, to=Q, phantom, "\Downarrow\rho"] & DX_0 \end{tikzcd}$$

and

(3.12)'
$$\begin{tikzcd}[column sep=huge] D^2A \arrow[r, bend left=70, "p'"] \arrow[r, bend left=20, "\ell"] \arrow[r, bend right=20, "\ell'"'] \arrow[r, bend right=70, "q"'] \arrow[r, phantom, shift left=4ex, "\Downarrow\sigma"] \arrow[r, phantom, shift right=4ex, "\Downarrow\tau"] & DX_0 \end{tikzcd}$$

respectively. We take for $x\colon DX_0 \to X_1$ the universal arrow out of DX_0 satisfying

(3.13) $$x\ell = x\ell'$$

(3.14) $$xp = xp'$$

and

(3.15) $$x\rho = x\tau . x\sigma \quad ;$$

the composite $x\tau . x\sigma$ making sense by (3.13). To give x in terms of more familiar colimit-operations we first take $s\colon DX_0 \to K$ to be the coequaliser of ℓ and ℓ', then take $t\colon K \to X_1$ to be the coequaliser of the two morphisms $2 \otimes D^2A \to K$ representing the 2-cells $s\rho$ and $s\tau . s\sigma$, finally setting $x = t.s$.

Define (3.1) to be

$$
(3.16)\qquad
\begin{tikzcd}
DA \arrow[rrr, "Du"] \arrow[d, "a"'] \arrow[dr, "n"] \arrow[d, phantom, shift left=3, "\Downarrow\gamma"] & & & DX_0 \arrow[d, "x"] \\
A \arrow[r, "u"'] & X_0 \arrow[r, "iX_0"'] & DX_0 \arrow[r, "x"'] & X_1
\end{tikzcd}
\quad ,
$$

observing that the right hand region commutes since $x\ell = x\ell'$.

Observe that from (3.7) we have

$$
(3.17)\qquad
\begin{tikzcd}
A \arrow[r, "iA"] & DA \arrow[r, "Du"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\delta"] & DX_0 \arrow[d, "x"] \\
& A \arrow[r, "v"'] & X_1
\end{tikzcd}
\quad = \quad id \quad ,
$$

and that by the definition of x and by (3.16) we have

$$
(3.18)\qquad
\begin{tikzcd}
D^2A \arrow[d, "mA"'] & \\
DA \arrow[r, "Du"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\delta"] & DX_0 \arrow[d, "x"] \\
A \arrow[r, "v"'] & X_1
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
D^2A \arrow[d, "Da"'] \arrow[dr, "Dn"] \arrow[d, phantom, shift left=3, "\Downarrow D\gamma"] & \\
DA \arrow[r, "Du"'] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow\delta"] & DX_0 \arrow[d, "x"] \\
A \arrow[r, "v"'] & X_1
\end{tikzcd}
\quad .
$$

Now let B = (B,b) be a D-algebra and g: X → B be a morphism in D/K which we write as

(3.19)
$$\begin{tikzcd} DX_0 \arrow[r, "Dg_0"] \arrow[d, "x"'] & DB \arrow[d, "b"] \\ X_1 \arrow[r, "g_1"'] & B \end{tikzcd} .$$

Write the composite (3.3) as

(3.20)
$$\begin{tikzcd} DA \arrow[r, "Df'"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & DB \arrow[d, "b"] \\ A \arrow[r, "f"'] & B \end{tikzcd} .$$

We wish to show that f = f' and that $(f,\bar{f})$ satisfies the unit and associativity laws for a D-morphism.

From (3.17) and the definition of (3.20), we have f.a.iA = b.Df'.iA; the latter is b.iB.f' by the naturality of i; but a.iA = 1 and b.iB = 1 since (A,a) and (B,b) are D-algebras; hence f = f' as required.

Again using (3.17) and the definition of (3.20) we have $\bar{f}$.iA = id, which is the unit law for a D-morphism.

To get the associativity law consider the composite

(3.21)
$$\begin{tikzcd} DA \arrow[r, "Du"] \arrow[d, "n"'] & DX_0 \arrow[r, "Dg_0"] \arrow[d, "x"] & DB \arrow[d, "b"] \\ X_0 \arrow[r, "x.iX_0"'] & X_1 \arrow[r, "g_1"'] & B \end{tikzcd}$$

of (3.19) with the commuting region in (3.16). We have $g_0u = f'$ by the definition of (3.20), so that $g_0u = f$. By the commutativity of (3.19) we have $g_1.x.iX_0 = b.Dg_0.iX_0$; by the naturality of i the latter is $b.iB.g_0$; which is g_0 since (B,b) is a D-algebra. We record this as

(3.22) $$g_0 = b.Dg_0.iX_0 \quad .$$

Thus the commutative diagram (3.21) may be written as

(3.23)
$$\begin{tikzcd}
DA \arrow[r, "Df"] \arrow[d, "n"'] & DB \arrow[d, "b"] \\
X_0 \arrow[r, "g_0"'] & B
\end{tikzcd} \quad .$$

Pasting (3.19) onto (3.18) and using D of (3.23) gives the desired associativity axiom in the form

(3.24)
$$\begin{tikzcd}
D^2A \arrow[d, "mA"'] & \\
DA \arrow[r, "Df"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & DB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
=
\begin{tikzcd}
D^2A \arrow[rr, "D^2f"] \arrow[d, "Da"'] \arrow[dr, "Dn"] \arrow[d, phantom, shift left=4, "\Downarrow D\gamma"] & & D^2B \arrow[d, "Db"] \\
DA \arrow[r, "Du"'] \arrow[d, "a"'] \arrow[drr, phantom, "\Downarrow \bar{f}"] & DX_0 \arrow[r, "Dg_0"'] & DB \arrow[d, "b"] \\
A \arrow[rr, "f"'] & & B
\end{tikzcd} \quad .$$

It remains to show that any D-morphism $(f,\bar{f})\colon A \to B$ is of the form (3.3), with δ defined by (3.16), for a unique $g\colon X \to B$. Using (2.3), observe that such a g <u>must</u> satisfy

(3.25)
$$
\begin{tikzcd}
DA \arrow[r, "Df"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{f}"] & DB \arrow[d, "b"] \\
A \arrow[r, "f"'] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
DA \arrow[d, "a"'] \arrow[dr, "n"] \arrow[dr, phantom, "\Downarrow \gamma"', near start] & & \\
B \arrow[r, "u"'] & X_0 \arrow[r, "g_0"'] & B
\end{tikzcd}
\;;
$$

but, because $\bar{f}.iA = id$, there is a unique g_0 satisfying (3.25). Using (3.25) and (3.22) we can rewrite the associativity law as

$$
\begin{tikzcd}
D^2A \arrow[d, "mA"'] & & & \\
DA \arrow[d, "a"'] \arrow[dr, "n"] \arrow[dr, phantom, "\Downarrow \gamma"', near start] & & & \\
A \arrow[r, "u"'] & X_0 \arrow[r, "iX_0"'] & DX_0 \arrow[r, "b.Dg_0"'] & B
\end{tikzcd}
$$

equals

$$
\begin{tikzcd}
D^2A \arrow[d, "Da"'] \arrow[dr, "Dn"] \arrow[dr, phantom, "\Downarrow D\gamma"', near start] & & & \\
DA \arrow[r, "Du"] \arrow[d, "a"'] \arrow[dr, "n"] \arrow[dr, phantom, "\Downarrow \gamma"', near start] & DX_0 \arrow[dr, "1"] & & \\
A \arrow[r, "u"'] & X_0 \arrow[r, "iX_0"'] & DX_0 \arrow[r, "b.Dg_0"'] & B
\end{tikzcd}
\;;
$$

so that by the definition of $x\colon DX_0 \to X_1$ there is a unique morphism $g_1\colon X_1 \to B$ satisfying (3.19). Moreover by (2.3) we have $g_0 = g_1.x.iX_0$; so that the composite of (3.19) with (3.16) is, by (3.25), indeed equal to $(f,\bar{f})$. □

4. In preparation for the proof in section 4 that $\mathcal{D}_*$ is reflective in D/K when K is cocomplete and D has a rank, we set up, in this section, the transfinite-induction machinery that will allow us to use the rank of D.

Let θ be a limit ordinal; fixed for the remainder of this section. Write Ord for the ordered set of ordinal numbers strictly less than θ considered as a category (and hence as a 2-category). Write $S\colon \mathrm{Ord} \to \mathrm{Ord}$ for the successor functor sending α to $\alpha + 1$, and $\sigma\colon 1 \Rightarrow S$ for the natural transformation whose component $\sigma_\alpha\colon \alpha \to \alpha+1$ is the unique map in Ord. Observe that $S\sigma = \sigma S$.

By a D-sequence we mean a pair (G,g) where $G\colon \mathrm{Ord} \to K$ is a functor and where $g\colon DG \to GS$ is a natural transformation satisfying

(4.1)
$$\begin{tikzcd} G \arrow[r, "iG"] \arrow[dr, "G\sigma"'] & DG \arrow[d, "g"] \\ & GS \end{tikzcd}$$

and

(4.2)
$$\begin{tikzcd} D^2G \arrow[r, "mG"] \arrow[d, "Dg"'] & DG \arrow[d, "g"] \\ DGS \arrow[dr, "gS"'] & GS \arrow[d, "G\sigma S"] \\ & GS^2 \end{tikzcd}$$

Note that (4.1) allows us to rewrite (4.2) as

(4.3)
$$\begin{tikzcd} D^2G \arrow[r, "mG"] \arrow[d, "Dg"'] & DG \arrow[d, "g"] \\ DGS \arrow[d, "gS"'] & GS \arrow[l, "iGS"] \\ GS^2 & \end{tikzcd}$$

If we write the value of G at the object α as G_α and its value at the morphism $\beta \to \alpha$ in Ord as $G_\beta{}^\alpha$, and if we write g_α: $DG_\alpha \to G_{\alpha+1}$ for the α-th component of g, we see that a D-sequence is a kind of "approximate D-algebra", with g as an "approximate action" and with (4.1) and (4.2) as "approximate unit and associativity axioms". A morphism $(G,g) \to (H,h)$ of D-sequences is accordingly defined to be a natural transformation $f: G \Rightarrow H$ such that

(4.4)
$$\begin{tikzcd} DG \arrow[r, "Df"] \arrow[d, "g"'] & DH \arrow[d, "h"] \\ GS \arrow[r, "fS"'] & HS \end{tikzcd} ,$$

while a D-sequence-2-cell is a modification $\rho\colon f \to k$ such that

(4.5)
$$\begin{tikzcd} DG \arrow[r, bend left, "Df"] \arrow[r, bend right, "Dk"'] \arrow[r, phantom, "\Downarrow D\rho"] \arrow[d, "g"'] & DH \arrow[d, "h"] \\ GS \arrow[r, bend left, "fS"] \arrow[r, bend right, "kS"'] \arrow[r, phantom, "\Downarrow \rho S"] & HS \end{tikzcd} .$$

Thus we have defined a 2-category D-$\mathcal{Seq}$ (depending on the chosen limit ordinal θ).

There is a forgetful 2-functor Z: D-$\mathcal{Seq} \to D/K$ sending (G,g) to (G_0,g_0,G_1), sending f to (f_0,f_1) and sending ρ to (ρ_0,ρ_1). The purpose of this section is to prove:

Proposition 4.1. *If K is cocomplete, the 2-functor Z: D-$\mathcal{Seq} \to D/K$ has a left adjoint V which satisfies $ZV = 1$. Moreover the unit $1 \Rightarrow ZV$ of the adjunction is the identity.*

Since the proof constructs the data G_α, $G_\alpha{}^\beta$ and g_α for a D-sequence (G,g) by transfinite induction starting with G_0,G_1 and g_0, we record some facts about the component-versions of the axioms for a D-sequence. The functoriality of G is expressed by

(4.6) $$G_\alpha^{\ \alpha} = 1; \quad G_\alpha^{\ \beta} G_\beta^{\ \gamma} = G_\alpha^{\ \gamma} \text{ for all } \alpha \leqslant \beta \leqslant \gamma.$$

The naturality of g is expressed by

(4.7)
$$\begin{tikzcd} DG_\beta \arrow[r, "DG_\beta^{\ \alpha}"] \arrow[d, "g_\beta"'] & DG_\alpha \arrow[d, "g_\alpha"] \\ G_{\beta+1} \arrow[r, "G_{\beta+1}^{\alpha+1}"'] & G_{\alpha+1} \end{tikzcd} \quad \text{for} \quad \beta \leqslant \alpha.$$

In terms of components (4.1) and (4.3) become

(4.8)
$$\begin{tikzcd} G_\alpha \arrow[r, "iG_\alpha"] \arrow[dr, "G_\alpha^{\alpha+1}"'] & DG_\alpha \arrow[d, "g_\alpha"] \\ & G_{\alpha+1} \end{tikzcd}$$

and

(4.9)
$$\begin{tikzcd} D^2G_\alpha \arrow[r, "Dg_\alpha"] \arrow[d, "mG_\alpha"'] & DG_{\alpha+1} \arrow[r, "g_{\alpha+1}"] & G_{\alpha+2} \\ DG_\alpha \arrow[r, "g_\alpha"'] & G_{\alpha+1} \arrow[u, "iG_{\alpha+1}"'] & \end{tikzcd}$$

respectively. In the inductive construction, (4.8) forces the value $G_\alpha^{\alpha+1}$ once we have G_α, $G_{\alpha+1}$ and g_α, and then (4.6) forces the value of $G_\beta^{\alpha+1}$ for <u>all</u> $\beta \leq \alpha + 1$. Thus in our inductive construction the only $G_\beta{}^\alpha$ we have to construct explicitly are those for α a limit ordinal. In all other cases the value of $G_\beta{}^\alpha$ is forced, by (4.8) and (4.6), from the knowledge of the g_γ. The forced value $G_\alpha^{\alpha+1}\ G_{\beta+1}^{\alpha}$ for $G_{\beta+1}^{\alpha+1}$ in (4.7) with the forced value of $G_\alpha^{\alpha+1}$ from (4.8) shows that the only instances of (4.7) that do not follow automatically are

(4.11)
$$\begin{tikzcd}
DG_\beta \arrow[r, "DG_\beta{}^\alpha"] \arrow[d, "g_\beta"'] & DG_\alpha \arrow[r, "g_\alpha"] & G_{\alpha+1} \\
G_{\beta+1} \arrow[r, "G_{\beta+1}^\alpha"'] & G_\alpha \arrow[u, "iG_\alpha"'] &
\end{tikzcd}$$
for a limit ordinal α and all $\beta<\alpha$,

and

(4.12)
$$\begin{tikzcd}
DG_\alpha \arrow[r, "DG_\alpha^{\alpha+1}"] \arrow[d, "g_\alpha"'] & DG_{\alpha+1} \arrow[r, "g_{\alpha+1}"] & G_{\alpha+2} \\
G_{\alpha+1} \arrow[ur, "iG_{\alpha+1}"'] & &
\end{tikzcd}$$
for all α.

<u>Proof of Proposition 4.1</u>.

Given $X = (X_0,x,X_1)$ in D/K we define by transfinite induction a D-sequence (G,g) that shall be VX. We begin by setting $G_0 = X_0$ and $G_1 = X_1$ and by taking $g_0\colon DG_0 \to G_1$ to be x.

Suppose that δ is an ordinal with $2 \leq \delta < \theta$, and that we have defined G_α for $\alpha < \delta$, $G_\beta{}^\alpha$ for $\beta \leq \alpha < \delta$ and $g_\alpha\colon DG_\alpha \to G_{\alpha+1}$ for $\alpha + 1 < \delta$, satisfying (4.8) - (4.12) as far as they make sense. We now show how to define the object G_δ, and the attendant data.

If δ is a limit ordinal α, we define G_α as the colimit

(4.13) $$G_\alpha = \operatorname*{colim}_{\beta<\alpha} G_\beta \quad ,$$

with the connecting morphisms $G_\gamma{}^\beta\colon G_\gamma \to G_\beta$ understood. This ensures (4.6).

If δ is $\alpha + 1$ for a limit ordinal α, we define $g_\alpha\colon DG_\alpha \to G_{\alpha+1}$ to be the simultaneous coequaliser of the left-hand squares of (4.11) for all $\beta < \alpha$, and take for $G_\alpha^{\alpha+1}$ the value forced by (4.8).

If $\delta = \alpha + 2$ for any ordinal α, we define $g_{\alpha+1}\colon DG_{\alpha+1} \to G_{\alpha+2}$ to be the simultaneous coequaliser of the left-hand squares of (4.9) and (4.12), and take for $G_{\alpha+1}^{\alpha+2}$ the value forced by (4.8). This completes the construction of (G,g). We set $VX = (G,g)$ and observe that $Z(G,g) = X$.

To complete the proof we have only to show that, given a D-sequence (H,h), each morphism $(f_0,f_1)\colon X \to ZH$ in D/K extends uniquely to a morphism $f\colon (G,g) \to (H,h)$ of D-sequences; that is, that there is a unique f with $Zf = (f_0,f_1)$. We shall define inductively the components

$f_\alpha : G_\alpha \to H_\alpha$ of f for $2 \leq \alpha < \theta$. (We leave to the reader the essentially identical verification at the level of 2-cells; once again the point is that the colimits in K are CAT-colimits).

For simplicity we write the axioms on f in terms of components. Thus (4.4) becomes

(4.14)
$$\begin{tikzcd} DG_\alpha \arrow[r, "g_\alpha"] \arrow[d, "Df_\alpha"'] & G_{\alpha+1} \arrow[d, "f_{\alpha+1}"] \\ DH_\alpha \arrow[r, "h_\alpha"'] & H_{\alpha+1} \end{tikzcd}$$

and the naturality of f is expressed by

(4.15)
$$\begin{tikzcd} G_\beta \arrow[r, "G_\beta^\alpha"] \arrow[d, "f_\beta"'] & G_\alpha \arrow[d, "f_\alpha"] \\ H_\beta \arrow[r, "H_\beta^\alpha"'] & H_\alpha \end{tikzcd} .$$

However composing (4.14) with $iG_\alpha : G_\alpha \to DG_\alpha$, using the naturality of i, and using (4.8), we get (4.15) automatically in the case that $\alpha = \beta + 1$. Thus the only case when (4.15) does not follow automatically is when α is a limit ordinal and $\beta < \alpha$.

Suppose that f_β is defined for $\beta < \delta$, where $2 \leqslant \delta < \theta$, satisfying (4.14) and (4.15) as far as they make sense, and with f_0 and f_1 being the given morphisms. We have only to define f_δ satisfying (4.14) and (4.15), and show it is unique.

If δ is a limit ordinal α, it is clear that

$$G_\beta \xrightarrow{f_\beta} H_\beta \xrightarrow{H^\alpha_\beta} H_\alpha$$

is a cone over $(G_\beta)_{\beta<\alpha}$, so that by (4.13) there is a unique f_α satisfying (4.15).

If δ is $\alpha + 1$ for some limit ordinal α, the morphism

$$DG_\alpha \xrightarrow[Df_\alpha]{} DH_\alpha \xrightarrow[h_\alpha]{} H_{\alpha+1}$$

coequalises the left-hand squares of (4.11) for all $\beta < \alpha$, because of the axioms satisfied by f_γ for $\gamma \leqslant \alpha$ and because the analogue of (4.11) is satisfied by (H,h). Hence by the definition of g_α there is a unique $f_{\alpha+1}\colon G_{\alpha+1} \to H_{\alpha+1}$ satisfying (4.14).

A precisely similar argument works in the case where $\delta = \alpha + 2$ for some ordinal α. This completes the proof. $\square$

Since the unit of the adjunction is the identity, we have:

Corollary 4.2. *The 2-functor* $V\colon D/K \to D\text{-}\mathcal{S}eq$ *is fully faithful*. □

We now define a 2-functor $P\colon \mathcal{D}_* \to D\text{-}\mathcal{S}eq$. If (A,a) is a D-algebra then the D-sequence $P(A,a) = (G,g)$ where G is the functor constant at A, and where $g_\alpha\colon DG_\alpha \to G_{\alpha+1}$ is $a\colon DA \to A$ for every α in Ord. If $f\colon (A,a) \to (B,b)$ is a strict D-morphism, Pf is the morphism of D-sequences whose every component is f; and P is similarly defined on 2-cells.

Proposition 4.3. *The following diagram of 2-functors commutes*.

(4.16)
$$\begin{tikzcd} \mathcal{D}_* \arrow[r, "L"] \arrow[dr, "P"'] & D/K \arrow[d, "V"] \\ & D\text{-}\mathcal{S}eq \end{tikzcd} .$$

Proof. We refer to the proof of Proposition 3.1 and examine the construction of $(G,g) = VX$ in the case when $X = L(A,a)$ for a D-algebra $A = (A,a)$. It is a matter of showing that each G_α is A, each $G_\alpha{}^\beta$ is 1 and each g_α is a. We have this for G_0, G_1 and g_0 by the way the construction starts; (4.8) gives $G_0{}^1 = 1$ by the unit axiom for a D-algebra. Suppose inductively that we have the result for all indices less that δ. When δ is a limit ordinal α, (4.13) gives $G_\alpha = A$ and $G_\beta{}^\alpha = 1$. For the other two cases we observe that, by the inductive hypothesis, the left-hand square of (4.9) becomes

(4.17)
$$\begin{tikzcd} D^2A \arrow[r, "Da"] \arrow[d, "mA"'] & DA \\ DA \arrow[r, "a"'] & A \arrow[u, "iA"'] \end{tikzcd}$$

and the left-hand squares of (4.11) and (4.12) both become

(4.18)
$$\begin{tikzcd} DA \arrow[r, "1"] \arrow[d, "a"'] & DA \\ A \arrow[r, "1"'] & A \arrow[u, "iA"'] \end{tikzcd} .$$

But a is the coequaliser of (4.18) as a.iA = 1; and is well known to be the coequaliser of mA and Da, hence of (4.17); thus a: DA → A is the simultaneous coequaliser of (4.17) and (4.18). □

5. In this section we use the results of §4 to help us prove:

Proposition 5.1. Let K be cocomplete and let D have a rank. Then the full inclusion 2-functor L: $\mathcal{D}_*$ → D/K has a left adjoint R.

This then gives us:

<u>Proof of Theorem 1.1</u>.

Let A = (A,a) be a D-algebra. From Proposition (3.1) we have an object $X \in D/K$ and an isomorphism (writing in the inclusion functors)

$$(5.1) \qquad \theta\colon \; D/K(X,LB) \cong \mathcal{D}(A,JB);$$

by the description of θ in Proposition 3.1, it is clear that it is 2-natural in $B \in \mathcal{D}_*$. But by Proposition 5.1 we also have a 2-natural isomorphism

$$(5.2) \qquad \mathcal{D}_*(RX,B) \cong D/K(X,LB).$$

Putting together (5.1) and (5.2) and writing ΦA for RX we get an isomorphism

$$(5.3) \qquad \pi\colon \; \mathcal{D}(A,JB) \cong \mathcal{D}_*(\Phi A,B)$$

which is 2-natural in $B \in \mathcal{D}_*$. Hence Φ extends to a 2-functor making (5.3) 2-natural in both variables, and provides the desired adjoint to J. □

Proposition 5.1 also gives:

<u>Proposition 5.2</u>. $\mathcal{D}_*$ <u>is a cocomplete</u> 2-<u>category</u>.

<u>Proof</u>. In view of Proposition 5.1 it is enough to show that D/K is a cocomplete 2-category; by Street [16] it suffices to show that D/K admits small colimits and

tensoring with $\mathcal{2}$. For colimits let $\mathcal{M}$ be a small category and $H: \mathcal{M} \to D/\mathcal{K}$ a functor; that is, a pair of functors $H_0, H_1: \mathcal{M} \to \mathcal{K}$ and a natural transformation $h: DH_0 \to H_1$. Let the colimit of H_0 be $\phi_0: H_0 \to X_0$ and let the colimit of H_1 be $\phi_1: H_1 \to X_1$. Let the colimit of DH_0 be $\psi_0: DH_0 \to Z_0$, and let the comparison map colim $DH_0 \to D$ colim H_0 be $k: Z_0 \to DX_0$. The natural transformation $h: DH_0 \to H_1$ induces a morphism $\bar{h}: Z_0 \to X_1$ of the colimits. Form the pushout

(5.4)
$$\begin{tikzcd} Z_0 \arrow[r, "\bar{h}"] \arrow[d, "k"'] & X_1 \arrow[d, "t"] \\ DX_0 \arrow[r, "y"'] & Y_1 \end{tikzcd} .$$

It is easy to verify that (X_0, y, Y_1), with the evident cone, is the colimit of F (as a $\mathcal{CAT}$-colimit). We leave to the reader the very similar construction of $\mathcal{2} \otimes X$ for $X \in D/\mathcal{K}$. □

As the first stage in the proof of Proposition 5.1 we prove:

Proposition 5.3. Let $\mathcal{K}$ be cocomplete and let D have a rank θ. If D-$\mathcal{Seq}$ is the 2-category of section 4 corresponding to this limit-ordinal θ, then the 2-functor $P: \mathcal{D}_* \to$ D-$\mathcal{Seq}$ has a left adjoint Q.

Proof. For $(G,g) \in$ D-$\mathcal{Seq}$ we define $(A,a) = Q(G,g)$ as follows. First set

(5.5)
$$A = \operatorname*{colim}_{\alpha<\theta} G_\alpha ,$$

with colimit cone

(5.6) $$u_\alpha : G_\alpha \longrightarrow A \ ;$$

the connecting morphisms $G_\gamma{}^\beta$ are understood in (5.5), so that we have

(5.7) $$u_\alpha G_\beta{}^\alpha \ = \ u_\beta \qquad \text{for all } \beta \leqslant \alpha < \theta$$

as the expression of the fact that u_α is a cone. The hypothesis that D has rank $\leqslant \theta$ tells us that

(5.8) $$Du_\alpha : DG_\alpha \longrightarrow DA$$

and

(5.9) $$D^2u_\alpha : D^2G_\alpha \longrightarrow D^2A$$

are both colimit-cones. We now observe that

(5.10) $$DG_\alpha \xrightarrow[g_\alpha]{} G_{\alpha+1} \xrightarrow[u_{\alpha+1}]{} A$$

is a cone over DG_α and hence induces a unique morphism $a : DA \to A$ such that

(5.11)
$$\begin{tikzcd} DG_\alpha \arrow[r, "Du_\alpha"] \arrow[d, "g_\alpha"'] & DA \arrow[d, "a"] \\ G_{\alpha+1} \arrow[r, "u_{\alpha+1}"'] & A \end{tikzcd} \qquad \text{for all } \alpha < \theta.$$

From (5.11), the naturality of i, and (4.8), we get $a.iA.u_\alpha = u_{\alpha+1}.G_\alpha{}^{\alpha+1}$; which is u_α since u is a cone.

Because u is a colimit cone we can conclude that $a.iA = 1$, which is the unit axiom for a D-algebra. To get the associativity axiom we notice that

$$\begin{aligned} a.mA.D^2u_\alpha &= a.Du_\alpha.mG_\alpha && \text{by naturality of } m \\ &= u_{\alpha+1}.g_\alpha.mG_\alpha && \text{by (5.11)} \\ &= u_{\alpha+2}.G_{\alpha+1}^{\alpha+2}.g_\alpha.mG_\alpha && \text{since } u \text{ is a cone} \\ &= u_{\alpha+2}.g_{\alpha+1}.Dg_\alpha && \text{by (4.9) and (4.8)} \\ &= a.Du_{\alpha+1}.Dg_\alpha && \text{by (5.11)} \\ &= a.Da.D^2u_\alpha && \text{by (5.11);} \end{aligned}$$

whence the desired result, since D^2u_α is a colimit cone. So $(A,a) = Q(G,g)$ is indeed a D-algebra.

Clearly by (5.11) the u_α are the components of a morphism of D-sequences $u: G \to PA$. To show that Q is the left adjoint of P it remains to verify that for every D-algebra $B = (B,b)$ every morphism of D-sequences $f: G \to PB$ is given by

$$f_\alpha = ku_\alpha \tag{5.12}$$

for a unique strict D-morphism $k: A \to B$. It is clear that $f_\alpha: G_\alpha \to B$ is a cone over $(G_\alpha)_{\alpha<\theta}$, so that there is a unique morphism $k: A \to B$ such that $f_\alpha = ku_\alpha$; it remains only to show k is a strict D-morphism. Notice that

$$
\begin{aligned}
b.Dk.Du_\alpha &= b.Df_\alpha && \text{by (5.12)} \\
&= f_{\alpha+1}.g_\alpha && \text{as } f \text{ is in } D\text{-}\mathcal{S}eq \\
&= k.u_{\alpha+1}.g_\alpha && \text{by (5.12)} \\
&= k.a.Du_\alpha && \text{by (5.11)} \quad ;
\end{aligned}
$$

hence $b.Dk = k.a$ as Du_α is a colimit cone; that is, k is a strict D-morphism. □

We now have:

Proof of Proposition 5.1.

By Proposition 4.3 we have $P = VL$; by Proposition 4.1 we have $ZV = 1$; hence $L = ZP$. As P has a left adjoint Q by Proposition 5.3, and Z has a left adjoint V by Proposition 4.1, it follows that QV is the left adjoint of L. □

6. The isomorphism π of (1.1) asserts that, for any D-morphisms $U,V: A \to B$ and any D-2-cell $\alpha: V \to U$ there is a unique D-2-cell $\beta: \pi(V) \to \pi(U)$ such that $\beta.\eta A = \alpha$ as in the diagram

$$
(6.1) \qquad
\begin{tikzcd}
A \arrow[r, bend left, "V"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\Downarrow\alpha"] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
A \arrow[r, "\eta A"'] & \Phi A \arrow[r, bend left, "\pi V"] \arrow[r, bend right, "\pi U"'] \arrow[r, phantom, "\Downarrow\beta"] & B
\end{tikzcd}
\quad ,
$$

namely $\beta = \pi(\alpha)$. From this it easily follows that, if $f\colon B \to C$ is a strict D-morphism, and $U\colon A \to B$ and $V\colon A \to C$ are arbitary D-morphisms, there is a bijection between D-2-cells $\alpha\colon V \to f.U$ and D-2-cells $\beta\colon \pi V \to f.\pi U$ such that

(6.2)
$$\begin{tikzcd} A \arrow[r, "U"] \arrow[dr, "V"'] & B \arrow[d, "f"] \arrow[dl, phantom, "\overset{\alpha}{\Rightarrow}"] \\ & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "\eta A"] & \Phi A \arrow[r, "\pi U"] \arrow[dr, "\pi V"'] & B \arrow[d, "f"] \arrow[dl, phantom, "\overset{\beta}{\Rightarrow}"] \\ & & C \end{tikzcd} \; ;$$

again $\beta = \pi(\alpha)$ as $f.\pi U = \pi(f.U)$.

The main purpose of this section is to show that composition with ηA still induces a bijection as in (6.2) when the strict D-morphism f is replaced by an arbitary D-morphism F; provided the 2-category K admits comma objects. (It is possible to establish this result without the last hypothesis, but the proof is then much less direct.) The essential tool for this is the following:

Proposition 6.1. *Let comma objects exist in* K. *Then for a morphism* $F\colon B \to C$ *in* $\mathcal{D}$ *the comma object*

(6.3)
$$\begin{tikzcd} X \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & B \arrow[d, "F"] \\ C \arrow[r, "1"'] & C \end{tikzcd}$$

<u>of</u> 1_C <u>and</u> F <u>in</u> $\mathcal{D}$ <u>exists</u>. <u>Moreover</u> ∂_0 <u>and</u> ∂_1 <u>are strict</u> D-<u>morphisms, and the</u> D-<u>morphism</u> G: C → X <u>is strict if and only if both</u> $\partial_0 G$ <u>and</u> $\partial_1 G$ <u>are strict</u>.

<u>Proof</u>. Let $F = (f,\bar{f})\colon B \to C$ be the given D-morphism. To get the underlying object of the D-algebra X = (X,x) we form the comma object

(6.4)
$$\begin{tikzcd} X \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & B \arrow[d, "f"] \\ C \arrow[r, "1"'] & C \end{tikzcd}$$

of 1_C and f in $\mathcal{K}$. By the universal property of λ there is a unique 1-cell $x\colon DX \to X$ in $\mathcal{K}$ such that

(6.5)
$$\begin{tikzcd} DX \arrow[r, "D\partial_1"] \arrow[d, "D\partial_0"'] \arrow[dr, phantom, "\overset{D\lambda}{\Rightarrow}"] & DB \arrow[r, "b"] \arrow[d, "Df"] \arrow[dr, phantom, "\overset{\bar{f}}{\Rightarrow}"] & B \arrow[d, "f"] \\ DC \arrow[r, "1"'] & DC \arrow[r, "c"'] & C \end{tikzcd} \quad = \quad \begin{tikzcd} DX \arrow[r, "x"] & X \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & B \arrow[d, "f"] \\ & C \arrow[r, "1"'] & C \end{tikzcd} ,$$

where $b\colon DB \to B$ and $c\colon DC \to C$ are the algebra-structures for B and C. We have now to verify that (X,x) is a D-algebra; we will however only show that x satisfies the unit law, leaving the equally simple associativity axiom to the reader. By the naturality of i we get that the composite of the left-hand side of (6.5) with $iX\colon X \to DX$ is equal to

$$\begin{tikzcd} X \arrow[r, "\partial_1"] \arrow[d, "\partial_0"'] \arrow[dr, phantom, "\overset{\lambda}{\Rightarrow}"] & B \arrow[r, "iB"] \arrow[d, "f"] & DB \arrow[r, "b"] \arrow[d, "Df"'] \arrow[dr, phantom, "\overset{\bar{f}}{\Rightarrow}"] & B \arrow[d, "f"] \\ C \arrow[r, "1"'] & C \arrow[r, "iC"'] & DC \arrow[r, "c"'] & C \end{tikzcd},$$

which is just λ by the unit law for $(f,\bar{f})$. We have, therefore, the required equation $x.iX = 1$. Equation (6.5) now tells us that ∂_0 and ∂_1 are strict D-morphisms and that λ is a D-2-cell from ∂_0 to $F\partial_1$.

We have now to verify that (6.3) is indeed the comma object in $\mathcal{D}$. Suppose that we have D-morphisms $U = (u,\bar{u})$ and $V = (v,\bar{v})$ and a D-2-cell α as in

(6.6)
$$\begin{tikzcd} A \arrow[r, "U"] \arrow[d, "V"'] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "F"] \\ C \arrow[r, "1"'] & C \end{tikzcd}.$$

The axiom for α to be a D-2-cell can be expressed by the equality of the 2-cells (ignore for the moment the broken arrows)

(6.7)
$$\begin{tikzcd} DA \arrow[r, "a"] \arrow[d, dashed, "Dv"'] \arrow[dr, phantom, "\overset{\bar{v}}{\Longrightarrow}"] & A \arrow[r, "u"] \arrow[d, "v"] \arrow[dr, phantom, "\overset{\alpha}{\Rightarrow}"] & B \arrow[d, "f"] \\ DC \arrow[r, dashed, "c"'] & C \arrow[r, "1"'] & C \end{tikzcd}$$

and

(6.8)
$$\begin{tikzcd}
DA \arrow[rr, dashed, "a"] \arrow[dr, "Du"] \arrow[dd, "Dv"'] \arrow[drrr, phantom, "\bar{u} \Rightarrow"] \arrow[dddr, phantom, "D\alpha \Rightarrow"] & & A \arrow[dr, dashed, "u"] & \\
 & DB \arrow[rr, "b"] \arrow[dd, "Df"] \arrow[ddrr, phantom, "\bar{f} \Rightarrow"] & & B \arrow[dd, "f"] \\
DC \arrow[dr, "1"'] & & & \\
 & DC \arrow[rr, "c"'] & & C
\end{tikzcd}$$
.

By the universal property of (6.4) there is a unique $w\colon A \to X$ in $\mathcal{K}$ such that

(6.9)
$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[dr, "v"'] \arrow[dr, phantom, shift left=3, "\alpha \Rightarrow"] & B \arrow[d, "f"] \\
 & C
\end{tikzcd}
=
\begin{tikzcd}
A \arrow[r, "w"] & X \arrow[r, "\partial_1"] \arrow[dr, "\partial_0"'] \arrow[dr, phantom, shift left=3, "\lambda \Rightarrow"] & B \arrow[d, "f"] \\
 & & C
\end{tikzcd}$$
.

It is easily verified that the unbroken part of (6.7) and (6.8) are $\lambda.w.a$ and $\lambda.x.Dw$ respectively; hence, by the universal property of λ for 2-cells, there is a unique 2-cell $\bar{w}$ as in

(6.10)
$$\begin{tikzcd}
DA \arrow[r, "Dw"] \arrow[d, "a"'] \arrow[dr, phantom, "\Downarrow \bar{w}"] & DX \arrow[d, "x"] \\
A \arrow[r, "w"'] & X
\end{tikzcd}$$

whose composite with λ is the common value of (6.7) and (6.8). An easy calculation shows that $W = (w, \bar{w})$ is a D-morphism from A to X; the statement that the composite of (6.10) with λ is the common value of (6.7) and (6.8) says exactly that

(6.11)
$$\begin{tikzcd} A \arrow[r, "U"] \arrow[dr, "V"'] & B \arrow[d, "F"] \arrow[dl, phantom, "\overset{\alpha}{\Rightarrow}"] \\ & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "W"] & X \arrow[r, "\partial_1"] \arrow[dr, "\partial_0"'] & B \arrow[d, "F"] \arrow[dl, phantom, "\overset{\lambda}{\Rightarrow}"] \\ & & C \end{tikzcd} .$$

We leave to the reader the task of checking the universal property of (6.3) on 2-cells (which is, of course, unnecessary if $\mathcal{K}$ is complete). This completes the proof that (6.3) is the comma object in $\mathcal{D}$.

Finally, if $\bar{u}$ and $\bar{v}$ are identities, the uniqueness-part of the universal property of (6.4) at the level of 2-cells gives at once that $\bar{w} = \mathrm{id}$; that is, W is strict if U and V are. Clearly $U = \partial_0 W$ and $V = \partial_1 W$ are strict if W is. $\square$

<u>Theorem 6.2</u>. *Suppose that $\mathcal{K}$ is cocomplete and admits comma objects, and that* D *has a rank. Let* π *be the isomorphism of Theorem 1.1. Let* $U\colon A \to B$, $F\colon B \to C$ *and* $V\colon A \to C$ *be D-morphisms. Then every* D-2-*cell* $\alpha\colon V \to F.U$ *is of the form*

(6.12)
$$\begin{tikzcd} A \arrow[r, "U"] \arrow[d, "V"'] & B \arrow[d, "F"] \\ C \arrow[r, "1"'] \arrow[ur, phantom, "\overset{\alpha}{\Rightarrow}"] & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "\eta A"] & \Phi A \arrow[r, "\pi U"] \arrow[dr, "\pi V"'] & B \arrow[d, "F"] \arrow[dl, phantom, "\overset{\beta}{\Rightarrow}"] \\ & & C \end{tikzcd}$$

for a unique D-2-cell β.

Proof. Let (6.3) be the comma object in $\mathcal{D}$ of 1_C and F; then every 2-cell α as in (6.12) is of the form $\lambda.W$ as in (6.11) for a unique $W: A \to X$ in $\mathcal{D}$ with $\partial_0 W = V$ and $\partial_1 W = U$. Furthermore every β as in (6.12) is $\lambda.g$ for a unique $g: \Phi A \to X$ and moreover g is strict as πU and πV are strict. Finally, by Theorem 1.1, W is $g.\eta A$ for a unique strict D-morphism $g: \Phi A \to X$. □

7. The most convenient way of getting op-lax-natural transformations ρ and τ as described in section 1 is to extend the result of Theorem 1.1 from the 2-category K to the 2-category $K' = [\![2, K]\!]$.

From the doctrine D = (D,i,m) on K we get a doctrine D' = (D',i',m') on K' by setting

$$
\begin{aligned}
(7.1) \qquad\qquad D' &= [\![2, D]\!] \\
i' &= [\![2, i]\!] \\
m' &= [\![2, m]\!] \quad .
\end{aligned}
$$

We embed K in K' as a (non-full) sub-2-category by the 2-functor $I_0: K \to K'$ which sends the object A of K to the object $(A, 1_A: A \to A, A)$ of K', which sends the morphism f in K to the morphism (f,id,f) in K', and which sends the 2-cell α in K to the 2-cell (α,α) in K'. It is clear that K is stable under the doctrine D' and that the restriction of D' to K is precisely D. In consequence the 2-functor I_0

induces 2-functors I: $\mathcal{D} \to \mathcal{D}'$ and I_*: $\mathcal{D}_* \to \mathcal{D}_*'$ where $\mathcal{D}_*'$ and $\mathcal{D}'$ are the analogues for D' of $\mathcal{D}_*$ and $\mathcal{D}$ for D; we have commutativity in

(7.2)
$$\begin{tikzcd} \mathcal{D}_* \arrow[r, "J"] \arrow[d, "I_*"'] & \mathcal{D} \arrow[d, "I"] \\ \mathcal{D}_*' \arrow[r, "J'"'] & \mathcal{D}' \end{tikzcd}$$

where J': $\mathcal{D}_*' \to \mathcal{D}'$ is the analogue of J: $\mathcal{D}_* \to \mathcal{D}$.

The point of the passage from D to D' is that a D'-algebra is a triple (A,G: A → E,E) where A and E are D-algebras and G is a D-morphism, while a D-morphism from G: A → E to F: B → C is a triple (U,α,V) where U and V are D-morphisms and α: FU → VG is a D-α-cell. (see Chapter 0 section 9).

The main result of this section is:

Theorem 7.1. If K is cocomplete and admits comma objects, and if D has rank, then the 2-functor J': $\mathcal{D}_*' \to \mathcal{D}'$ has a left adjoint Φ' whose value $\Phi'G$ at the object G: A → E of $\mathcal{D}'$ is the object ΦG: $\Phi A \to \Phi E$ of $\mathcal{D}_*'$. The unit η' of the adjunction has components $\eta' G$ given by

(7.3)
$$\begin{tikzcd} A \arrow[r, "\eta A"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{id}{\Rightarrow}"] & \Phi A \arrow[d, "\Phi G"] \\ E \arrow[r, "\eta E"'] & \Phi E \end{tikzcd} .$$

If we denote the adjunction isomorphism by

$$\pi': \mathcal{D}'(F, J'G) \cong \mathcal{D}_*(\Phi'F, G)$$

then $\pi'(U_0, \bar{U}, U_1)$ *has the form* $(V_0, \bar{V}, V_1)$ *where* $V_0 = \pi U_0$ *and* $V_1 = \pi U_1$.

Proof. It suffices to show that every morphism $U = (U_0, \bar{U}, U_1)$ in $\mathcal{D}'$ from $G: A \to E$ to $F: B \to C$ factorises as

$$\begin{tikzcd} A \arrow[r, "U_0"] \arrow[d, "G"'] \arrow[dr, phantom, "\overset{\bar{U}}{\Rightarrow}"] & B \arrow[d, "F"] \\ E \arrow[r, "U_1"'] & C \end{tikzcd} \quad = \quad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "G"'] & \Phi A \arrow[r, "V_0"] \arrow[d, "\Phi G"] \arrow[dr, phantom, "\overset{\bar{V}}{\Rightarrow}"] & B \arrow[d, "F"] \\ E \arrow[r, "\eta_E"'] & \Phi E \arrow[r, "V_1"'] & C \end{tikzcd}$$

for a unique morphism $V = (V_0, \bar{V}, V_1)$ in $\mathcal{D}_*'$ (V being strict means exactly that V_0 and V_1 are strict D-morphisms.). By Theorem 1.1 we do have unique V_0 and V_1, namely πU_0 and πU_1. Since $\eta E.G = \Phi G.\eta A$ by the naturality of η, the existence of the unique $\bar{V}$ follows from Theorem 6.2. The corresponding property on 2-cells follows from the uniqueness clause in Theorem 6.2. □

8. In this section we prove:

Theorem 8.1. Let $\mathcal{K}$ be a cocomplete 2-category which admits comma objects, and let D have a rank. Let U: $\mathcal{D} \to \mathcal{C}$ be a 2-functor such that the 2-functor UJ: $\mathcal{D}_* \to \mathcal{C}$ has a left adjoint F with unit j, counit n and adjunction isomorphism γ. Then there exists an op-lax-natural transformation κ: JFU $\rightsquigarrow 1_{\mathcal{D}}$ such that

$$(8.1) \qquad \kappa J = Jn,$$

and such that j and κ exhibit JF as an op-quasi-left adjoint to U.

We thus have:

Proof of Theorem 1.2. Since (JF,U,j,κ) is an op-quasi-adjunction we know that the functor

$$(8.2) \qquad \mathcal{C}(X,UJB) \xrightarrow{\underset{=}{\tilde{\gamma}}} \mathcal{D}_*(FX,B) \xrightarrow{J} \mathcal{D}(JFX,B),$$

which is equal to

$$(8.3) \qquad \mathcal{C}(X,UJB) \xrightarrow{JF} \mathcal{D}(JFX,JFUJB) \xrightarrow{\mathcal{D}(1,\kappa JB)} \mathcal{D}(JFX,JB).$$

is the left adjoint of (see Chapter 0 section 6)

$$(8.4) \qquad \mathcal{D}(JFX,JB) \xrightarrow{U} \mathcal{C}(UJFX,UJB) \xrightarrow{\mathcal{C}(jX,1)} \mathcal{C}(X,UJB).$$

Thus the required result follows immediately. □

The first step in the proof of Theorem 8.1 is:

Proposition 8.2. *There is an op-lax-natural transformation* $\rho\colon J\Phi \rightsquigarrow 1_{\mathcal{D}}$ *such that*

$$\rho J \;=\; J\varepsilon \tag{8.5}$$

and

$$\rho.\eta \;=\; \mathrm{id}. \tag{8.6}$$

Proof. The component $\varepsilon' F$ of the counit of the adjunction of Theorem 7.1 is the unique D'-morphism satisfying $\varepsilon' F.\eta' F = 1$; by Theorem 7.1 it has the form

(8.7)
$$\begin{tikzcd}
\Phi A \arrow[r, "\varepsilon A"] \arrow[d, "\Phi F"'] \arrow[dr, phantom, "\overset{\overline{\varepsilon' F}}{\Longrightarrow}"] & A \arrow[d, "F"] \\
\Phi B \arrow[r, "\varepsilon B"'] & B
\end{tikzcd}$$

where ε is the counit of the adjunction of Theorem 1.1.

We define the op-lax-natural transformation $\rho\colon J\Phi \rightsquigarrow 1$ by setting

(8.8)
$$\begin{tikzcd}
\Phi A \arrow[r, "\rho A"] \arrow[d, "\Phi F"'] \arrow[dr, phantom, "\rho F \Longrightarrow"] & A \arrow[d, "F"] \\
\Phi B \arrow[r, "\rho B"'] & B
\end{tikzcd}$$

equal to (8.7) for all D-algebras A and B and all D-morphisms $F: A \to B$. The part of the lax-naturality of ρ relating to identities and composition is now immediate from the universal property of η'; the part relating to 2-cells is immediate from the naturality of ε'. Clearly by the above definition we have

$$(8.9) \qquad \rho F . \eta A = \mathrm{id}.$$

Further if F is strict the exterior of (8.7) commutes by the naturality of ε; hence by the universal property of η' we have

$$\overline{\varepsilon' F} = \mathrm{id} ;$$

that is

$$(8.10) \qquad \rho F = \mathrm{id} .$$

From these considerations we obtain the equations

$$(8.11) \qquad \rho J = J\varepsilon$$

and

$$(8.13) \qquad \rho . \eta = \mathrm{id} . \quad \square$$

The second step in the proof of Theorem 8.1 is:

Proposition 8.3. *If F: $\mathcal{C} \to \mathcal{D}_*$ and U: $\mathcal{D} \to \mathcal{C}$ are 2-functors as in the hypotheses of Theorem 8.1, then there exists an op-lax-natural transformation* κ: $JFU \rightsquigarrow 1_{\mathcal{D}}$ *such that*

$$(8.14) \qquad \kappa J = Jn$$

and

$$(8.15) \qquad U\kappa . jU = \mathrm{id} \; .$$

Proof. We define κ to be the op-lax-natural transformation

(8.16)
$$\begin{tikzcd}
 & & & \mathcal{C} \arrow[ddr, "F"] & & \\
 & & \mathcal{D} \arrow[ur, "U"] \arrow[r, phantom, "\Downarrow n"] & {} & & \\
 & \mathcal{D}_* \arrow[ur, "J"] \arrow[rrr] \arrow[l, phantom, "\Downarrow\eta"] \arrow[drr, phantom, "\rho"] & & & \mathcal{D}_* \arrow[dr, "J"] & \\
\mathcal{D} \arrow[ur, "\Phi"] \arrow[uurr, bend left, "1"] \arrow[rrrrr] & & & & & \mathcal{D}
\end{tikzcd}$$

By putting J on the bottom left-hand corner of (8.16), and by using (8.5) and the triangle equation $J\varepsilon . \eta J = \mathrm{id}$, we get equation (8.14) as required.

Pasting j on to the right hand side of (8.16) gives

(8.17)
$$\begin{tikzcd}
& & \mathcal{C} \arrow[rr] \arrow[ddr, "F"] \arrow[rr, phantom, bend right=20, "\Downarrow j"] & & \mathcal{C} \\
& \mathcal{D} \arrow[ur, "U"] \arrow[drr, phantom, "\Downarrow n"] & & & \\
& \mathcal{D}_* \arrow[u, "J"] \arrow[rr] \arrow[drr, phantom, "\rightsquigarrow \rho"] & & \mathcal{D}_* \arrow[uur, "UJ"] \arrow[d, "J"] & \\
\mathcal{D} \arrow[ur, "\Phi"] \arrow[uur, bend left, "1"] \arrow[uur, phantom, "\Downarrow \eta"] \arrow[rrr] & & & \mathcal{D} \arrow[uuur, "U"'] &
\end{tikzcd}$$

which is the composite $U\kappa.jU$. Using the naturality of j,η and n to change the order of composition allows us to apply the triangle equation $UJn.jUJ = \mathrm{id}$ to get (8.17) equal to

(8.18)
$$\begin{tikzcd}
\mathcal{D} \arrow[r, "\Phi"] \arrow[rr, bend left=50, "1"] \arrow[rr, bend right=50, "1"'] & \mathcal{D}_* \arrow[r, "J"] \arrow[u, phantom, "\Downarrow \eta"] \arrow[d, phantom, "\rightsquigarrow \rho"] & \mathcal{D} \arrow[r, "U"] & \mathcal{C}
\end{tikzcd}$$

But by (8.6) the op-lax-natural transformation (8.18) is equal to id; that is, we have (8.15). □

We now complete the proof of Theorem 8.1 by proving:

<u>Proposition 8.4</u>. <u>Let</u> $F\colon \mathcal{C} \to \mathcal{D}_*$ <u>and</u> $U\colon \mathcal{D} \to \mathcal{C}$ <u>be</u> 2-<u>functors such that</u> $F \dashv UJ$ <u>with unit</u> j <u>and co-unit</u> n. <u>Let</u> $\kappa\colon JFU \rightsquigarrow 1_{\mathcal{D}}$ <u>be an op-lax-natural transformation such that</u>

(8.19) $$\kappa J = Jn$$

and

(8.20) $$U\kappa . jU = 1.$$

Then j *and* κ *exhibit* JF *as an op-quasi-left adjoint to* U.

Proof. Recall from Chapter 0 that we have only to show that the two triangle-equations are satisfied and that both j_j and κ_κ are identities.

The first triangle-equation is precisely (8.20), while the second is given by

$$\begin{aligned} \kappa JF . JFj &= JnF . JFj \\ &= J(nF . Fj) \\ &= 1. \end{aligned}$$

Since j is a proper natural transformation we have $j_j = \mathrm{id}$; while the chain of equalities

$$\begin{aligned} \kappa_{\kappa B} &= \kappa_{\kappa JB} & \\ &= \kappa_{JnB} & \text{by (8.19)} \\ &= Jn_{nB} & \text{by (8.19)} \\ &= \mathrm{id} & \text{as } n \text{ is 2-natural} \end{aligned}$$

gives $\kappa_\kappa = \mathrm{id}$. $\square$

Before leaving this Chapter we consider two special cases of Theorem 1.2 that will be of interest in Chapter 2.

Examples 8.5.

1. From Proposition 5.2 we know that under the hypothesis of Theorem 1.2 the 2-category $\mathcal{D}_*$ is cocomplete as a 2-category; thus, $\mathcal{D}_*$ is a tensored CAT-category, by which we mean that for all $A \in \mathcal{D}_*$ the 2-functor $\mathcal{D}_*(A,-)\colon \mathcal{D}_* \to CAT$ has the left adjoint $-\otimes A\colon CAT \to \mathcal{D}_*$. From the isomorphism π of (1.1) we see, therefore, that the 2-functor $\mathcal{D}(A,J-)\colon \mathcal{D}_* \to CAT$ has the left adjoint $-\otimes\Phi A\colon CAT \to \mathcal{D}_*$ giving a natural isomorphism

$$\chi\colon CAT(C,\mathcal{D}(A,JB)) \cong \mathcal{D}_*(C\otimes\Phi A,B),$$

the unit and counit of which are

$$\nu\colon 1 \Rightarrow \mathcal{D}(A,J(A,J(-\otimes\Phi A))$$

and

$$\sigma\colon \mathcal{D}(A,J-)\otimes\Phi A \Rightarrow 1$$

respectively.

Putting $F = -\otimes\Phi A$, $U = \mathcal{D}(A,-)$, $j = \nu$ and $n = \sigma$ in Theorem 1.2 we find that the functor

$$J\colon \mathcal{D}_*(C\otimes\Phi A,B) \to \mathcal{D}(J(C\otimes\Phi A),JB)$$

is the left adjoint of the functor W, where W is the composite

$$
\begin{tikzcd}
\mathcal{D}(J(C\otimes\Phi A),JB) \arrow[r, "\mathcal{D}(A,-)"] \arrow[rdd, "W"'] & \mathit{CAT}(\mathcal{D}(A,J(C\otimes\Phi A)),\mathcal{D}(A,JB)) \arrow[d, "\mathit{CAT}(\nu C,1)"] \\
& \mathit{CAT}(C,\mathcal{D}(A,JB)) \arrow[d, "\chi", "\cong"'] \\
& \mathcal{D}_*(C\otimes\Phi A,B)
\end{tikzcd}
$$

2. Let C be the 2-category K and let F be the free-algebra 2-functor F^D while U is the forgetful 2-functor $U^D\colon \mathcal{D} \to K$. It is well known that $F \dashv UJ$; since this is the usual Eilenberg-Moore adjunction. If we make the observation that $j = i$, then Theorem 1.2 gives that the functor

$$J\colon \mathcal{D}_*(FX,B) \to \mathcal{D}(FX,B)$$

is the left adjoint of the functor W, where W is the composite

$$
\begin{tikzcd}
\mathcal{D}(FX,B) \arrow[r, "U"] \arrow[rdd, "W"'] & K(UFX,UB) \arrow[d, "K(iX,1)"] \\
& K(X,UB) \arrow[d, "\gamma", "\cong"'] \\
& \mathcal{D}_*(FX,B)
\end{tikzcd}
\quad .
$$

CHAPTER 2

1. In any 2-category $\mathcal{E}$ which is equipped with a notion of small object, and which has a terminal object $\mathbb{1}$, we can imitate the classical notion of a cocomplete object that we have in $\mathcal{CAT}$. That is to say, we call $A \in \mathcal{E}$ cocomplete in $\mathcal{E}$ if A has all small colimits, by which we mean that for each small $X \in \mathcal{E}$ the functor:

$$(1.1) \qquad \mathcal{E}(\mathbb{1},A) \xrightarrow{\mathcal{E}(!,A)} \mathcal{E}(X,A)$$

has a left adjoint L. We then call LF: $\mathbf{1} \to A$ the colimit of F, and the component $F \Rightarrow (LF)!$ of the unit we call the colimit-cone.

Such a definition of cocompleteness is of no use at all in many good 2-categories; it gives a perfectly trivial notion of cocompleteness if applied to the 2-category of additive categories. In fact it has long been recognised (see Day-Kelly [5]) that cocompleteness in the 2-category $\mathcal{V}$-$\mathcal{CAT}$ of categories enriched over a symmetric monoidal closed $\mathcal{V}$ should be defined quite differently (and of course it is this definition of cocompleteness that we have been using and will continue to use for 2-categories). Only recently has a sufficiently general notion of "colimit" in $\mathcal{V}$-$\mathcal{CAT}$ been given, for which cocompleteness in the Day-Kelly sense means "admits all small colimits" (see Borceux-Kelly [4], Auderset [1]).

In spite of this the primitive definition of cocompleteness in terms of a left adjoint to (1.1) turns out to have considerable significance for the 2-category D-*CAT* of algebras for a doctrine D on *CAT*; and it is this definition of cocomplete object in D-*CAT* that we use in this chapter. In fact the special case of this where D is the doctrine whose algebras are monoidal categories was the impulse for much of the work in this thesis; for it turns out, as we shall see in Chapter 3, that many important questions of monadicity reduce to questions of the existence of colimits of 1-cells in *Mon*-*CAT*. Although our principal applications are with *Mon*-*CAT*, there is nothing special about it, and it is just as easy to work with D-*CAT* for a ranked doctrine D. Of course the terminal object in D-*CAT* is just the unit category $\mathbb{1}$ with its unique D-structure; and a D-algebra is small if its underlying category is small.

One feature that the above notions of cocompleteness have in common is that they all demand the existence of certain left Kan extensions. The definition we are using calls $A \in \mathcal{E}$ cocomplete if every morphism $F: X \to A$ from a small X admits a left Kan extension along $!: X \to \mathbb{1}$, the unique morphism into the terminal object. On the other hand the Day-Kelly [5] definition of cocompleteness in $\mathcal{V}$-*CAT* amounts (see Borceux-Kelly [4]) to demanding the existence of the <u>pointwise</u> left Kan extension of any $F: X \to A$ from a small X, along any morphism $G: X \to B$. A difficulty in comparing these two definitions is the lack, in a general 2-category $\mathcal{E}$, of a notion of <u>pointwiseness</u> for Kan extensions.

$\underline{\underline{2}}$. Let D = (D,i,m) be a doctrine on *CAT* and let *Cat* be stable under D (that is, the category DX is small whenever X is small); furthermore let D have a small rank. In this chapter we will be concerned entirely with doctrines of this type.

As usual we denote the 2-categories of D-algebras by $\mathcal{D}_*$ and $\mathcal{D}$; if at any time we need to refer to <u>small</u> D-algebras we denote the respective 2-categories of <u>small</u> D-algebras by D-$\mathcal{C}at_*$ and D-$\mathcal{C}at$. We will use the terms D-algebra and D-category interchangeably; similarly with D-morphism and D-functor, and D-2-cell and D-natural transformation.

A D-category A = (A,a) is said to admit the colimit in $\mathcal{D}$ of the D-functor G: X → A if there is in $\mathcal{D}$ a universal diagram of the form

$$\begin{tikzcd} X \arrow[rr, "G"] \arrow[dr, "!"'] & \Downarrow \alpha & A \\ & \mathbb{1} \arrow[ur, "Y"'] & \end{tikzcd} ;$$

that is if there is a free object over G relative to the functor

(2.1) $$\mathcal{D}(\mathbb{1},A) \xrightarrow{\mathcal{D}(!,A)} \mathcal{D}(X,A).$$

If such a free object exists over every G: X → A with X small, that is if (2.1) has a left adjoint for every small

D-category X, we say that A = (A,a) is <u>cocomplete</u> in $\mathcal{D}$ or D-cocomplete; or that A = (A,a) admits all D-colimits.

The category $\mathcal{D}(\mathbf{1},A)$ will play an important role in the work of this chapter; we therefore give this category a special name. If we consider the case when D is the doctrine for monoidal categories, we observe that a monoidal functor $\mathbb{1} \to A$ is just a <u>monoid</u> in the monoidal category A (see Mac Lane [14] page 166); consequently we call a D-functor $\mathbb{1} \to A$ a D-<u>oid</u> in A, and call the category $\mathcal{D}(\mathbb{1},A)$ the <u>category</u> of D-<u>oids in</u> A, denoting it by D[A].

From the forgetful 2-functor U^D: $\mathcal{D} \to CAT$ we get a forgetful functor $U = U_A$: D[A] $\to$ A which is equal to

$$(2.2) \qquad \mathcal{D}(\mathbb{1},A) \xrightarrow{U^D} CAT(U^D\mathbf{1},U^DA) = CAT(\mathbb{1},A) \cong A.$$

We have already mentioned that if $\mathcal{D}$ = *Mon-CAT* then the objects of D[A] are precisely the monoids in A; it is in fact true that D[A] is the <u>category</u> of monoids and <u>monoid-morphisms</u> in A, which is called Mon(A) by Dubuc [6]. If D = $\Delta \times -$, where Δ is the simplicial category, it is well known (see Kelly [9]) that the algebras for D are categories equipped with a monad. Then if (A,T) is a D-algebra it is easy to check that D[A] = A^T, the category of Eilenberg-Moore algebras for the monad T, and that U is the usual forgetful functor for such algebras.

3. Since CAT is cocomplete as a CAT-category and hence <u>a fortiori</u> as a $\mathcal{C}at$-category, since further $\mathcal{C}at$ is cocomplete as a $\mathcal{C}at$-category, and since moreover D has a <u>small</u> rank, all the results of Chapter 1 apply both to D and to the restriction of D to $\mathcal{C}at$.

We observe that the constructions in Propositions 3.1, 4.1, 5.1 and 5.3, from which the adjoint Φ of J: $\mathcal{D}_* \to \mathcal{D}$ was obtained, only used the construction of colimits in K of size not exceeding θ; the rank of D. It follows, therefore, that <u>smallness</u> is stable under all of these constructions. In particular this gives:

<u>Lemma 3.1</u>. (i) <u>The</u> D-<u>category</u> ΦA <u>is small whenever the</u> D-<u>category</u> A <u>is small</u>.

(ii) <u>The</u> D-<u>category</u> C$\otimes$A <u>is small whenever the</u> D-<u>category</u> A <u>and the category</u> C <u>are both small</u>. □

4. In this section we give a characterization of those D-categories B = (B,b) that are cocomplete in $\mathcal{D}$; in terms of the cocompleteness in CAT of D[B] and the existence of a left adjoint to the functor U: D[B] $\to$ B. Of equal importance for our applications, however, is the question of the <u>preservation</u> of D-colimits by a strict D-functor H: B $\to$ C; here we give only <u>sufficient</u> conditions in terms of the preservation by D[H] = $\mathcal{D}(\mathbb{1},H)$: D[B] $\to$ D[C] of colimits in CAT, and of the preservation by D[H] of <u>free objects</u> relative to U. In our applications it will not in general be the case that C is cocomplete in $\mathcal{D}$, and our only concern

is with colimits of those D-functors $X \to C$ which factor through $H: B \to C$. To avoid repetition we collect into one theorem the results on the existence of D-colimits and those on their preservation. Observe that our proofs of sufficiency for the conditions we give are quite elementary; while our proof of necessity, as regards existence, requires the results of Chapter 1.

Theorem 4.1. Let D be a doctrine on *CAT* under which *Cat* is stable and which has a small rank. Then a D-category $B = (B,b)$ is cocomplete in $\mathcal{D}$ if and only if the following two conditions are satisfied:

(i) the functor $U_B: D[B] \to B$ has a left adjoint F;

(ii) the category $D[B]$ is cocomplete in *CAT*.

Let $H = (h, \mathrm{id})$ be a strict D-functor from B to C where B satisfies the conditions (i) and (ii) above. Then $H: B \to C$ preserves colimits in $\mathcal{D}$ provided that:

(iii) if $\eta_x: x \to U_B Fx$ is the x-component of the unit of $F \dashv U_B$ then $h(\eta_x) : h(x) \to h(U_B Fx) = U_C\, D[H](Fx)$ is the unit of the free object over $h(x)$ relative to U_C. (An equivalent statement is that $D[H] . F : B \to D[C]$ is the partial left adjoint of U_C relative to $h: B \to C$);

(iv) $D[H] : D[B] \to D[C]$ preserves colimits in *CAT*.

Proof. We first show the necessity of (i) and (ii) for the cocompleteness in $\mathcal{D}$ of $B = (B,b)$. Consider the diagram

$$(4.1)\qquad \begin{tikzcd} CAT(\mathbb{1},B) & \mathcal{D}(F^D\mathbb{1},B) \arrow[l, "\Psi"'] \\ & \mathcal{D}(\mathbb{1},B) \arrow[ul, "U^D_{\mathbb{1},B}"] \arrow[u, "\mathcal{D}(!,B)"'] \end{tikzcd}$$

where F^D and U^D are as in Examples 8.5 (ii) of Chapter 1 and where Ψ is the composite

$$\begin{tikzcd} \mathcal{D}(F^D\mathbb{1},B) \arrow[r, "U^D"] \arrow[dr, "\Psi"'] & CAT(U^DF^D\mathbb{1},U^DB) \arrow[d, "CAT(i\mathbb{1},1)"] \\ & CAT(\mathbb{1},B) \end{tikzcd} .$$

It is clear that diagram (4.1) commutes since $\mathbb{1}$ is terminal. Since $F^D\mathbf{1} = D\mathbf{1}$ is small and since $B = (B,b)$ is cocomplete in $\mathcal{D}$ by hypothesis, $\mathcal{D}(!,B)$ has a left adjoint; and from Chapter 1 Examples 8.5 (ii), Ψ has the left adjoint

$$CAT(\mathbb{1},B) \cong \mathcal{D}_*(F^D\mathbb{1},B) \xrightarrow{J} \mathcal{D}(F^D\mathbb{1},B).$$

Hence $U^D_{\mathbb{1},B}$ has a left adjoint; thus by the definition of $U\colon D[B] \to B$ it too has a left adjoint, proving (i).

Consider the diagram

(4.2)

$$\begin{tikzcd}
CAT(C,\mathcal{D}(\mathbb{1},B)) & & \mathcal{D}(C\otimes\Phi\mathbb{1},B) \arrow[ll, "CAT(\nu{,}1).\mathcal{D}(\mathbf{1}{,}-)"'] \\
CAT(\mathbb{1},\mathcal{D}(\mathbb{1},B)) \arrow[u, "CAT(!{,}\mathcal{D}(\mathbb{1}{,}B))"] \arrow[rr, phantom, "\cong"] & & \mathcal{D}(\mathbb{1},B) \arrow[u, "\mathcal{D}(!{,}B)"']
\end{tikzcd}$$

where ν is the 2-cell in Chapter 1 Examples 8.5 (i). The diagram (4.2) commutes; for as $\mathcal{D}(\mathbb{1},\mathbb{1}) = \mathbb{1}$, evaluating the top and bottom paths at $G\colon \mathbb{1} \to B$ gives the top and bottom lines of

$$\begin{tikzcd}
C \arrow[r, "\nu\mathbb{1}"] \arrow[rrd, "!"'] & \mathcal{D}(\mathbb{1},C\otimes\Phi\mathbb{1}) \arrow[r, "\mathcal{D}(\mathbb{1}{,}!)"] & \mathcal{D}(\mathbb{1},\mathbb{1}) \arrow[r, "\mathcal{D}(\mathbf{1}{,}G)"] \arrow[d, "U", "\cong"'] & \mathcal{D}(\mathbb{1},B) \\
 & & \mathbb{1} \arrow[ru, "\ulcorner G\urcorner"'] &
\end{tikzcd}$$

which are clearly equal. If C is small then $C\otimes\Phi\mathbb{1}$ is also small by Lemma 3.1, and hence $\mathcal{D}(!,B)$ in (4.2) has a left adjoint since B is cocomplete in $\mathcal{D}$ by hypothesis; moreover from Chapter 1 Examples 8.5 (i) the functor $CAT(\nu,1).\mathcal{D}(\mathbb{1},-)$ has a left adjoint. Thus we have shown that $CAT(!,\mathcal{D}(\mathbb{1},B))$ has a left adjoint whenever C is small, or that $\mathcal{D}(\mathbb{1},B)$ is cocomplete in CAT; proving (ii).

We now prove simultaneously the sufficiency of (i) and (ii) and that of (iii) and (iv). We suppose that (i), (ii), (iii) and (iv) are satisfied, and we construct the colimit in $\mathcal{D}$ of the composite

$$X \xrightarrow{G} B \xrightarrow{H} C$$

of the D-functors $G = (g,\bar{g})$ and $H = (h,\mathrm{id})$; the sufficiency of (i) and (ii) will follow by taking $H = 1_B: B \to B$; while the sufficiency of (iii) and (iv) will follow by observing that the colimit-cone of H.G is precisely the colimit-cone of G composed with H.

Let F be the left adjoint of U_B as given in hypothesis (i) and further let $\eta: 1 \to U_B F$ be the unit of this adjunction; then by hypothesis (iii) and the universal property of the unit there is a natural bijection θ_H between 2-cells $\alpha: h.g \Rightarrow U_C k!$ in *CAT* and 2-cells $\beta: D[H].F.g \Rightarrow k!$ in *CAT* where $\alpha = \theta_H^{-1}(\beta)$ is the composite

(4.4)
$$\begin{tikzcd}
X \arrow[r, "g"] \arrow[dd, "!"'] \arrow[ddr, phantom, "\Downarrow\beta"] & B \arrow[d, "F"'] \arrow[dr, "1"] \arrow[dr, phantom, "\Downarrow\eta"'] & \\
& D[B] \arrow[r, "U_B"'] \arrow[d, "D[H]"] & B \arrow[d, "h"] \\
1\!\!1 \arrow[r, "k"'] & D[C] \arrow[r, "U_C"'] & C
\end{tikzcd}$$

If θ_1 is the bijection between 2-cells $\alpha: g \Rightarrow U_B k'!$ and 2-cells $\beta: F.g \Rightarrow k'!$ (that is the θ corresponding to the case when $H = 1_B$) then if $\alpha: h.g \Rightarrow U_C k!$ is of the form $h.\alpha'$ for some $\alpha' : g \Rightarrow U_B k!$ the 2-cell $\theta_H(\alpha)$ is clearly equal to the composite $D[H]\theta_1(\alpha')$; and similarly if $\beta: D[H].F.g \Rightarrow k!$ is of the form $D[H].\beta'$ for some $\beta': F.g \Rightarrow k'!$ then $\theta_H^{-1}(\beta)$

is equal to $h.\theta_1^{-1}(\beta')$. Note that if the value of k at the unique object of $\mathbb{1}$ is $Y = (y,\bar{y})$, then $k = \ulcorner Y \urcorner$ and $y = U_C k$, allowing us to write α in the form

(4.5)
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "!"'] & B \arrow[r, "h"] \arrow[d, phantom, "\Downarrow\alpha"] & C \\ & \mathbb{1} \arrow[ur, "y"'] & \end{tikzcd} .$$

Let $X = (X,x)$ be a small D-category, let $G = (g,\bar{g})$ be a D-functor from X to B, let $Y = (y,\bar{y})$ be an object of D[C], and consider D-2-cells of the form

(4.6)
$$\begin{tikzcd} X \arrow[r, "G"] \arrow[dr, "!"'] & B \arrow[r, "H"] \arrow[d, phantom, "\Downarrow\alpha"] & C \\ & \mathbb{1} \arrow[ur, "Y"'] & \end{tikzcd} .$$

A D-2-cell as in (4.6) is just a 2-cell in *CAT* as in (4.5), satisfying the D-naturality condition for D-2-cells; however to give a 2-cell α as in (4.5) is just to give a 2-cell $\beta = \theta_H(\alpha)$ as in (4.4). If we write

(4.7)
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "!"'] & B \arrow[r, "F"] \arrow[d, phantom, "\Downarrow\lambda"] & D[B] \\ & \mathbb{1} \arrow[ur] & \end{tikzcd} \qquad \ell = \ulcorner Z \urcorner = \ulcorner (z,\bar{z}) \urcorner$$

for the colimit-cone of F.g in *CAT*, which exists by hypothesis (ii), then by hypothesis (iv)

(4.8)
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "!"'] & B \arrow[r, "F"] \arrow[d, phantom, "\Downarrow\lambda"] & D[B] \arrow[r, "D[H]"] & D[C] \\ & \mathbb{1} \arrow[ur, "\ell"'] & & \end{tikzcd}$$

is the colimit-cone of D[H].F.g in *CAT*. We see therefore that every 2-cell β as in (4.4) is the result of pasting (4.8) onto

$$\begin{tikzcd} \mathbb{1} \arrow[rr, bend left, "D[H].\ell"] \arrow[rr, bend right, "k"'] \arrow[rr, phantom, "\Downarrow\ulcorner\gamma\urcorner"] & & D[C] \end{tikzcd}$$

for a unique morphism γ: D[H](Z) $\to$ Y in D[C]. If we now apply θ_H^{-1} to (4.8) we see that the result is $h.\theta_1^{-1}(\lambda)$ which we write as

(4.9)
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "!"'] & B \arrow[r, "h"] \arrow[d, phantom, "\Downarrow\mu"] & C \\ & \mathbb{1} \arrow[ur, "z"'] & \end{tikzcd}$$

using μ for $\theta_1^{-1}(\lambda)$. Thus we see that every 2-cell α as in (4.5) is the result of pasting (4.9) onto

$$\begin{tikzcd} \mathbb{1} \arrow[rr, bend left, "h.z"] \arrow[rr, bend right, "y"'] \arrow[rr, phantom, "\Downarrow\gamma"] & & B \end{tikzcd}$$

where γ is the underlying natural transformation of a unique morphism γ: D[H](Z) $\to$ Y in D[C].

We now give conditions on γ that are equivalent to the D-naturality condition for α; that is, equivalent to the equality of

(4.10)

$$\begin{tikzcd}
DX \arrow[r, "Dg"] \arrow[d, "x"'] \arrow[dr, phantom, "\Downarrow\bar{g}"] & DB \arrow[r, "Dh"] \arrow[d, "b"] & DC \arrow[d, "c"] \\
X \arrow[r, "g"] \arrow[dr, "!"'] \arrow[d, phantom, "\Downarrow\mu"', xshift=4em] & B \arrow[r, "h"] & C \\
& \mathbb{1} \arrow[u, "z"'] \arrow[ur, bend right, "y"'] \arrow[ur, phantom, "\Downarrow\gamma"] &
\end{tikzcd}$$

and

(4.11)

$$\begin{tikzcd}
DX \arrow[rr, "Dg"] \arrow[dr, "D!"'] \arrow[rr, phantom, "\Downarrow D\mu"', yshift=-1.5em] & & DB \arrow[r, "Dh"] \arrow[r, phantom, "\Downarrow D\gamma"', yshift=-1.5em] & DC \arrow[d, "c"] \\
& D\mathbb{1} \arrow[ur, "Dz"'] \arrow[urr, "Dy"] \arrow[d, "!"'] \arrow[rr, phantom, "\Downarrow\bar{y}"] & & C \\
& \mathbb{1} \arrow[urr, "y"'] & &
\end{tikzcd}$$

.

Since γ is a D-2-cell, (4.11) may be rewritten as

(4.12)

$$\begin{tikzcd}
DX \arrow[rr, "Dg"] \arrow[dr, "D!"'] \arrow[rr, phantom, "\Downarrow D\mu"', yshift=-1.5em] & & DB \arrow[r, "Dh"] \arrow[dd, "b"] & DC \arrow[dd, "c"] \\
& D\mathbb{1} \arrow[ur, "Dz"'] \arrow[dd, "!"'] \arrow[dr, phantom, "\Downarrow\bar{z}"] & & \\
& & B \arrow[r, "h"] & C \\
& \mathbb{1} \arrow[ur, "z"] \arrow[urr, bend right, "y"'] \arrow[urr, phantom, "\Downarrow\gamma"] & &
\end{tikzcd}$$

;

so that the D-naturality condition for α is equivalent to the equality

(4.13)
$$\begin{tikzcd} DX \arrow[r, "b.Dg"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\rho"] & B \arrow[r, "h"] \arrow[d, phantom, "\Downarrow\gamma"] & C \\ \mathbb{1} \arrow[ur, "z"] \arrow[urr, "y"'] & & \end{tikzcd} = \begin{tikzcd} DX \arrow[r, "b.Dg"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\sigma"] & B \arrow[r, "h"] \arrow[dr, phantom, "\Downarrow\gamma"] & C \\ \mathbb{1} \arrow[ur, "z"'] \arrow[urr, bend right, "y"'] & & \end{tikzcd}$$

where ρ is the composite of μ and $\bar{g}$ in (4.10) and σ is the composite of $\bar{z}$ and $D\mu$ in (4.12). Applying θ_H (with DX replacing X) to $h.\rho$ and $h.\sigma$, recalling that $\theta_H(h.\rho) = D[H].\theta_1(\rho)$ and $\theta_H(h.\sigma) = D[H].\theta_1(\sigma)$, we see that (4.13) is equivalent to the equality of

(4.14)
$$\begin{tikzcd} DX \arrow[r, "F.b.Dg"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\theta_1(\rho)"] & D[B] \arrow[r, "D{[H]}"] \arrow[d, phantom, "\Downarrow\ulcorner\gamma\urcorner"] & D[C] \\ \mathbb{1} \arrow[ur, "\ell"'] \arrow[urr, "k"'] & & \end{tikzcd}$$

and

(4.15)
$$\begin{tikzcd} DX \arrow[r, dash, "F.b.Dg"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\theta_1(\sigma)"] & D[B] \arrow[r, dash, "D{[H]}"] \arrow[d, phantom, "\Downarrow\ulcorner\gamma\urcorner"] & D[C] \\ \mathbb{1} \arrow[ur, "\ell"'] \arrow[urr, "k"'] & & \end{tikzcd}$$

Consider now the colimit in *CAT* of the functor $F.b.Dg\colon DX \to D[B]$; it exists because D[B] is cocomplete by hypothesis (ii) and because DX is small. Let the colimit be

(4.16)
$$\begin{tikzcd} DX \arrow[rr, "F.b.Dg"] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow \nu"] & D[B] \\ & \mathbb{1} \arrow[ur, "n"'] & \end{tikzcd} \qquad n = \ulcorner W \urcorner = \ulcorner (w,\bar{w}) \urcorner \; ;$$

then $\theta_1(\rho)$ and $\theta_1(\sigma)$ are the result of pasting (4.16) onto uniquely-determined 2-cells

(4.17)
$$\begin{tikzcd} \mathbb{1} \arrow[rr, bend left, "n"] \arrow[rr, bend right, "\ell"'] \arrow[rr, phantom, "\Downarrow \ulcorner \rho_* \urcorner"] & & D[B] \end{tikzcd} \; , \quad \begin{tikzcd} \mathbb{1} \arrow[rr, bend left, "n"] \arrow[rr, bend right, "\ell"'] \arrow[rr, phantom, "\Downarrow \ulcorner \sigma_* \urcorner"] & & D[B] \end{tikzcd}$$

in *CAT*, corresponding to morphisms $\rho_*, \sigma_*: W \to Z$ in $D[B]$. Let $\tau: Z \to V$ be the coequaliser of ρ_* and σ_* in $D[B]$ which exists by hypothesis (ii), then $D[H]\tau : D[H]Z \to D[H]V$ is the coequaliser of $D[H]\rho_*$ and $D[H]\sigma_*$ in $D[C]$. It follows, therefore, that $\gamma: D[H]Z \to Y$ renders equal (4.14) and (4.15), so making the corresponding α in (4.6) a D-2-cell, if and only if γ factors through $D[H]\tau$.

If we now define δ to be the 2-cell in *CAT* given by

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow \mu"] & B \\ & \mathbb{1} \arrow[ur, bend left, "z"] \arrow[ur, bend right, "v"'] \arrow[ur, phantom, "\Downarrow \tau"] & \end{tikzcd}$$

it is clear that δ is a D-2-cell

$$\begin{tikzcd} X \arrow[rr, "G"] \arrow[dr, "!"'] \arrow[rr, phantom, shift right=4, "\Downarrow\delta"] & & B \\ & \mathbb{1} \arrow[ur, "V"'] & \end{tikzcd},$$

and the V is the colimit-cone in $\mathcal{D}$ for G: X $\rightarrow$ B, while

$$\begin{tikzcd} X \arrow[rr, "G"] \arrow[dr, "!"'] \arrow[rr, phantom, shift right=4, "\Downarrow\delta"] & & B \arrow[rr, "H"] & & C \\ & \mathbb{1} \arrow[ur, "V"'] & & & \end{tikzcd}$$

is the colimit-cone $\mathcal{D}$ for HG: X $\rightarrow$ C. □

<u>Examples 4.2</u>. Returning to the examples given in §2 we see that a monoidal category B is cocomplete in *Mon-CAT* if and only if (i) the category Mon(A) of monoids in A is a co-complete category, and (ii) the forgetful functor U: Mon(A) $\rightarrow$ A has a left adjoint.

In the case when D = Δ ×- we see that a D-algebra (A,T) (that is, a category A with a monad T) is cocomplete in $\mathcal{D}$ if and only if A^T is a cocomplete category; for in this case U always has a left adjoint.

Since we know sufficient conditions (cf. Schubert [17], Barr [2], and Proposition 5.2 of Chapter 1) under which a category of algebras for a monad is cocomplete, verification of the sufficient conditions (i) and (ii) of Theorem 4.1 is assisted by:

Proposition 4.3. If $\mathrm{B} = (B,b)$ is a D-category the functor $U\colon D[B] \to B$ is monadic if and only if it has a left adjoint.

Proof. It suffices, because of (2.2), to show that the functor $U^{D}_{\mathbb{1},B}\colon \mathcal{D}(\mathbb{1},B) \to \mathcal{CAT}(\mathbb{1},B)$ creates coequalisers of $U^{D}_{\mathbb{1},B}$-split pairs. What we show is that $U^{D}_{A,B}\colon \mathcal{D}(A,B) \to \mathcal{CAT}(A,B)$ creates coequalisers of $U^{D}_{A,B}$-split pairs.

We write $\mathcal{K}$ for $\mathcal{CAT}$ and as in Chapter 1 section 7 we consider the doctrine $D' = [\![\mathcal{2},D]\!]$ on $\mathcal{K}' = [\![\mathcal{2},\mathcal{K}]\!]$ with its 2-category $\mathcal{D}'_*$ of algebras and strict morphisms. We ignore here the 2-cells of $\mathcal{K}'$, so that $\mathcal{D}'_*$ is the category of algebras for the monad D' on the category $\mathcal{K}'$. Now let $\mathcal{D}''$ be the subcategory of $\mathcal{D}'$ in which we retain all the objects, but only the morphisms of the form

$$\begin{tikzcd} A \arrow[r, "1"] \arrow[d, "X"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & A \arrow[d, "Y"] \\ B \arrow[r, "1"'] & B \end{tikzcd} \quad ;$$

we define similarly the subcategory $\mathcal{K}''$ of $\mathcal{K}'$. Thus $\mathcal{D}''(X,Y) = \mathcal{D}(A,B)(X,Y)$.

A $U^{D}_{A,B}$-split pair α,β in $\mathcal{D}(A,B)$ is clearly a $U^{D'}$-split pair in $\mathcal{D}'$ which lies in $\mathcal{D}''$. The splitting in $\mathcal{K}(A,B)$ is moreover a splitting in $\mathcal{K}'$ which lies in $\mathcal{K}''$; whence the coequalizer γ in $\mathcal{D}'$ created by the monadic $U^{D'}$ necessarily lies in $\mathcal{D}''$. Further γ is clearly the coequalizer in $\mathcal{D}''$, and is a coequalizer of α and β in $\mathcal{D}(A,B)$ created by $U^{D}_{A,B}$. □

We shall show in Chapter 3 that when B is complete and cocomplete in *CAT*, the left adjoint F of U: D[B] $\to$ B does indeed exist and that the monad UF on B has a rank, provided that the action has a certain "smallness" property; we call this smallness property <u>having rank</u>. For such a D-algebra B = (B,b), Proposition 4.2 tells us that D[B] is the category of algebras for a ranked monad on B; thus D[B] is cocomplete in *CAT* by Proposition 5.2 of Chapter 1, and so B = (B,b) is cocomplete in $\mathcal{D}$ by Theorem 4.1.

To carry out this proof, however, we shall need to know that the monoidal category $E_* = \mathit{End}_*B$ of ranked endofunctors of a cocomplete category B is cocomplete in *Mon-CAT* and that the strict monoidal inclusion $I_*: E_* \to E$, where E = *End* B is the monoidal category of <u>all</u> endofunctors of B, preserves *Mon-CAT*-colimits. We devote the following section to a direct proof, using Theorem 4.1, of this fact.

5. Throughout this section let B be a cocomplete category, and denote by E the strict monoidal category *End* B = [B,B] of endofunctors of B. For each small regular ordinal θ the endofunctors of B with rank $\leqslant \theta$ constitute a full strict monoidal subcategory E_θ of E. We have full strict monoidal inclusions $I_\theta^{\theta'}: E_\theta \to E_{\theta'}$ for $\theta \leqslant \theta'$; the union E_* of the E_θ for all small regular ordinals θ is itself a full monoidal subcategory (the subcategory of <u>ranked</u> endofunctors) of E; and the full inclusions $I_\theta^*: E_\theta \to E_*$ are again strict monoidal. Finally we have the strict monoidal inclusions $I_\theta: E_\theta \to E$ and $I_*: E_* \to E$.

Since colimits in E are computed pointwise, and since colimits commute with colimits, it is immediate that each E_θ is closed under colimits in E. Thus each E_θ is cocomplete, as is E_*; and the inclusions $I_\theta^{\theta'}: E_\theta \to E_{\theta'}$, $I_\theta: E_\theta \to E$, $I_\theta^*: E_\theta \to E_*$ and $I_*: E_* \to E$ preserve colimits.

We recall that when D is the doctrine for strict monoidal categories, so that $\mathcal{D}$ = *Mon-CAT*, the category D[M] is the category Mon(M) of monoids in the monoidal category M. Thus Mon(E) is the category of <u>monads</u> on B. A monad on B is said to have rank $\leq \theta$, or to have rank, precisely when the underlying endofunctor has rank $\leq \theta$, or has rank; so that Mon(E_θ) is the category of monads on B with rank $\leq \theta$, while Mon(E_*) is the category of ranked monads on B.

It is known (Dubuc [6], Barr [2]) that if R is a ranked endofunctor of B, then the free monad T on R exists; that is, there is a monad T and a natural transformation $\eta R: R \to T$ such that if S is a monad and $\rho: T \to S$ a natural transformation, then there is a unique morphism of monads $k: T \to S$ such that $\rho = k.\eta R$. That is, there is a functor $F_*: E_* \to \mathrm{Mon}(E)$ which is the partial left adjoint of $U: \mathrm{Mon}(E) \to E$ relative to $I_*: E_* \to E$.

In fact rather more is true; the free monad T on R exists <u>pointwise</u> in a sense made precise in a forthcoming paper by Kelly and Wolff [13]; the facts are essentially in Barr [2] without the nomenclature. The point is that, if we define an R-algebra to be a pair (X,x) where X is an object of B and $x: RX \to X$ is a morphism in B, then the forgetful functor $V: R\text{-}\mathcal{A}\ell g \to B$ is monadic when R has a rank; and the monad in question is then the desired free monad T on R (Barr [2] Theorem 5.5).

(An alternative and somewhat more general proof is to appear in Kelly-Wolff [13], using a modification of the comma-category construction used in Chapter 1, §4 and §5, above. Replace K by B, replace D by the <u>free pointed</u> endofunctor 1 + R on R, and repeat our considerations at the level of categories rather than 2-categories, omitting all reference to the multiplication $m: D^2 \to D$ which is now lacking, but keeping the unit $i: 1 \to D$ which we do have. As we found before that $\mathcal{D}_*$ is reflective in D/K, so we now find that $R\text{-}\mathcal{A}\ell g$ is reflective in D/B; whence the forgetful $V: R\text{-}\mathcal{A}\ell g \to B$ has a left adjoint since the forgetful $D/B \to B$ has a trivial left adjoint. An easy argument (cf. Barr [2]) shows that V is monadic whenever it has a left adjoint.)

<u>Lemma 5.1</u>. <u>Whenever the endofunctor</u> R <u>of</u> B <u>has rank</u> $\leq \theta$ <u>so has the free monad</u> T <u>on</u> R.

<u>Proof</u>. The left adjoint to $V: R\text{-}\mathcal{A}\ell g \to B$ preserves all colimits; so we have only to show that V itself preserves colimits of γ-sequences for all θ-filtered ordinals γ. If we have a γ-sequence $(X_\beta)_{\beta<\gamma}$ of R-algebras $X_\beta = (X_\beta, x_\beta)$ we have only to take the colimit Y of the sequence $(X_\beta)_{\beta<\gamma}$ of the underlying objects, and observe that RY is the colimit of the sequence $(RX_\beta)_{\beta<\gamma}$; so that there is an action $y: RY \to Y$ induced by the actions $x_\beta: RX_\beta \to X_\beta$; and finally

observe that (Y,y) is clearly the colimit in R-$\mathcal{A}\ell g$ of the original sequence. □

Hence the functor $F_*\colon E_* \to \mathrm{Mon}(E)$ actually lands in $\mathrm{Mon}(E_*)$; we henceforth consider the functor F_* as having codomain $\mathrm{Mon}(E_*)$ so that $F_*\colon E_* \to \mathrm{Mon}(E_*)$. Thus we have:

Proposition 5.2. The forgetful functor $U_*\colon \mathrm{Mon}(E_*) \to E_*$ has the left adjoint F_*, and the functor

$$E_* \xrightarrow{F_*} \mathrm{Mon}(E_*) \xrightarrow{\mathrm{Mon}(I_*)} \mathrm{Mon}(E)$$

is the partial left adjoint of $U\colon \mathrm{Mon}(E) \to E$ relative to $I_*\colon E_* \to E$. □

It follows further that the restriction $U_\theta\colon \mathrm{Mon}(E_\theta) \to E_\theta$ of U_* has a left adjoint the restriction $F_\theta\colon E_\theta \to \mathrm{Mon}(E_\theta)$ of F_*; thus we have by Proposition 4.2:

Proposition 5.3. The forgetful functors $U_*\colon \mathrm{Mon}(E_*) \to E_*$ and $U_\theta\colon \mathrm{Mon}(E_\theta) \to E_\theta$ are monadic. □

Proposition 5.4. The monad $U_\theta F_\theta$ on the category E_θ has rank $\leqslant \theta$.

Proof. Since F preserves all colimits it suffices to show that $U_\theta\colon \mathrm{Mon}(E_\theta) \to E_\theta$ preserves colimits of γ-sequences for all θ-filtered ordinals γ.

We write $\otimes$ for the tensor product of the monoidal category E_θ; the tensor product is actaully composition. If $X \in E_\theta$ then it is clear that $-\otimes X\colon E_\theta \to E_\theta$ preserves all colimits since colimits in E are computed pointwise, while the rank of X shows that $X\otimes -\colon E_\theta \to E_\theta$ preserves colimits of all θ-filtered sequences; that is $X\otimes -$ has rank $\leqslant \theta$. Therefore if $(M_\beta)_{\beta<\gamma}$ is a γ-sequence in $\mathrm{Mon}(E_\theta)$ for some θ-filtered ordinal γ and if Y is the colimit of the γ-sequences $(U_\alpha M_\beta)_{\beta<\gamma}$ with colimit-cone $\mu_\beta\colon U_\alpha M_\beta \to Y$, then $\mu_\beta\otimes\mu_\beta\colon U_\alpha M_\beta\otimes U_\alpha M_\beta \to Y\otimes Y$ is the colimit-cone of the sequence $(U_\alpha M_\beta\otimes U_\alpha M_\beta)_{\beta<\gamma}$. It is now clear that the monoid structure on each M_β induces a monoid structure on Y in such a way that $\mu_\beta\colon M_\beta{}^M \to Y$ is the colimit-cone in $\mathrm{Mon}(E_\theta)$. □

We now prove a lemma which will be used in our next proposition.

Lemma 5.5. *Let* $A = (A,a)$ *be a* $U_\theta F_\theta$*-algebra, that is an object of* $\mathrm{Mon}(E_\theta)$, *and let* B *be an object of* $\mathrm{Mon}(E)$, *then* $g\colon A \to UB$ *in* E *is a morphism of monoids from* A *to* B *if and only if the composite* $g.a$ *is a morphism of monoids from* $F_\theta A$ *to* B.

Proof. For the duration of this proof we write $D = U_\theta F_\theta$. If g is a morphism of monoids then g.a certainly is since $a\colon DA \to A$ is always a morphism of monoids.

Since both U_θ: $\mathrm{Mon}(E_\theta) \to E_\theta$ and U: $\mathrm{Mon}(E) \to E$ create coequalisers of U_θ (resp. U)-split pairs, the diagram

$$(5.1) \qquad D^2A \underset{Da}{\overset{mA}{\rightrightarrows}} DA \xrightarrow{a} A$$

is a coequaliser in E_θ, E, $\mathrm{Mon}(E_\theta)$ and Mon(E), so that from the commutativity of

$$D^2A \underset{Da}{\overset{mA}{\rightrightarrows}} DA \xrightarrow{g.a} B$$

we have a unique morphism k: $A \to B$ in Mon(E) such that k.a = g.a. However, since (5.1) is a coequaliser in E we have k = g so that g is a monoid morphism as required. □

We observe that the proof of this lemma is of wider validity than the statement of the lemma indicates since U could equally well be any functor U: $B \to A$ which creates coequalisers of U-split pairs and creates limits and F_θ could be a partial left adjoint to U relative to some full subcategory A_θ of A.

<u>Proposition 5.6</u>. <u>The category</u> $\mathrm{Mon}(E_\theta)$ <u>is cocomplete and the inclusion</u> $\mathrm{Mon}(I_\theta)$: $\mathrm{Mon}(E_\theta) \to \mathrm{Mon}(E)$ <u>preserves colimits</u>.

<u>Proof</u>. To see that $\mathrm{Mon}(E_\theta)$ is cocomplete we invoke Proposition 5.2 of Chapter 1 noticing that $\mathrm{Mon}(E_\theta)$ is the category of algebras for the ranked monad $U_\theta F_\theta$ on the cocomplete category E_θ.

To see that $\mathrm{Mon}(I_\theta)$ preserves colimits we reconsider the construction of colimits in $\mathrm{Mon}(E_\theta)$ as represented by the proof of Proposition 5.2 of Chapter 1. We write $\mathcal{K}$ for the <u>category</u> E_θ, write D for the <u>monad</u> $U_\theta F_\theta$ and write U_θ: D-$\mathcal{A}\ell g_*$ $\to \mathcal{K}$ for the forgetful <u>functor</u> U_θ: $\mathrm{Mon}(E_\theta) \to E_\theta$. If f: A $\to$ UB is a morphism in E for which $A \in E_\theta$ we write $\hat{f}$: DA $\to$ B for the unique morphism in Mon(E) satisfying $\hat{f}$,iA = f where i is the unit of the monad D. We observe that if $A \in \mathrm{Mon}(E_\theta)$ and $B \in \mathrm{Mon}(E)$, then Lemma 5.5 tells us that f: A $\to$ B in E is a morphism of monoids if and only if $\hat{f} = f.a$ where a: DA $\to$ A is the D-action for the D-algebra A.

Let H: M $\to$ D/$\mathcal{K}$ be a functor with small domain which factors through the inclusion L: $\mathrm{Mon}(E_\theta) \to$ D/$\mathcal{K}$; recall that Mon(E) = D-$\mathcal{A}\ell g_*$. Further let H_0, H_1, X_0, X_1, ϕ_0, ϕ_1, ψ_0, Z_0 and k be as in the proof of Proposition 5.2 of Chapter 1. Since H factors through L we have $H_0 = H_1$ so that $X_0 = X_1$ and $\phi_0 = \phi_1$, we write H, X and ϕ for the common values. Finally let μ: H $\to$ L be a cone in Mon(E).

The cone $\hat{\mu}$: DH $\to$ L induces a unique morphism ℓ: $Z_0 \to$ L such that $\ell\psi_0 = \hat{\mu}$; moreover it is clear that $\ell = \hat{p}_0 k$ where p_0: X $\to$ L is the unique morphism such that $\mu = p_0\phi$. The cone $\mu.h$: DH $\to$ L induces a unique morphism n: $Z_0 \to$ L such that $n\psi_0 = \mu.h$; clearly $n = p_0.\bar{h}$ where $\bar{h}$ is as described in the proof of Proposition 5.2 of Chapter 1.

Since each component of μ is a morphism of monoids we have $\hat{\mu} = \mu.h$ so that we have the equation

$$\hat{p}_0 k = p_0 \bar{h} \quad ,$$

from which we have, since (5.4) of Chapter 1 is a pushout, a unique morphism $p_1\colon Y_1 \to L$ such that $p_1 t = p_0$ and $p_1 y = \hat{p}_0$ where t and y are defined by (5.4) of Chapter 1.

If $V(X_0,x,Y_1) = (G,g)$ then the pair of morphisms (p_0,p_1) induce for each $\alpha \in \mathcal{O}rd$ a unique morphism $p_\alpha\colon G_\alpha \to L$ such that $p_{\alpha+1}G_\alpha^{\alpha+1} = p_\alpha$ and $p_{\alpha+1}g_\alpha = \hat{p}_\alpha$. The proof of this follows easily from the definition of $V\colon D/K \to D\text{-}\mathcal{S}eq$ by transfinite-induction; it is clear what to do at the α-th step of the induction if α is a limit ordinal; if $\alpha = \beta+1$ for some ordinal β it is easy to see that the morphism $\hat{p}_\beta\colon G_\beta \to L$ coequalises the diagrams required to induce a unique $p_{\beta+1}$ with the desired properties.

From the definition of $Q(G,g) = (A,a)$ we see that the family of morphisms p_α induce a unique morphism $p\colon A \to L$ in E such that $\hat{p} = p.a$; that is a unique map $p\colon A \to L$ in Mon(E). Since $(A,a) = Q(G.g)$ is the colimit, in $\mathrm{Mon}(E_\theta)$, of the functor H we have shown every cone $\mu\colon H \to L$ factors uniquely through the colimit-cone $H \to A$. □

<u>Corollary 5.7</u>. <u>The category</u> $\mathrm{Mon}(E_*)$ <u>is cocomplete and the inclusion</u> $\mathrm{Mon}(I_*)\colon \mathrm{Mon}(E_*) \to \mathrm{Mon}(E)$ <u>preserves colimits</u>.

<u>Proof</u>. Any small diagram in $\mathrm{Mon}(E_*)$ actually lands in $\mathrm{Mon}(E_\theta)$ for some θ so that its colimit can be formed in $\mathrm{Mon}(E_\theta)$. □

Theorem 5.8. If B is a cocomplete category then the monoidal category $E_* = End_* B$ is cocomplete in *Mon-CAT* and the inclusion $I_*\colon E_* \to E$ preserves *Mon-CAT*-colimits. $\square$

6. In section 1 we remarked that in Chapter 3 we would see that many questions of monadicity could be reduced to the question of the existence of colimits of monoidal functors. It will in fact turn out, again in Chapter 3, that an even larger class of monadicity questions can be answered by using a notion of "lax-colimit" of monoidal 2-functors. It is our purpose in the remainder of this chapter to give the definition of "lax-colimit" of monoidal 2-functors and to give an existence and preservation theorem for such colimits.

A <u>monoidal</u> <u>2-category</u> consists of a 2-category A, a strictly associative 2-functor $\otimes\colon A\times A \to A$, and a distinguished object $I\colon \mathbb{1} \to A$ which is a strict left and right identity for $\otimes$.

A <u>monoidal</u> <u>2-functor</u> $G\colon A \to B$ is a triple (g, g_0, g_1) where $g\colon A \to B$ is a 2-functor and where g_1 and g_0 are 2-natural transformations as in

$$\begin{tikzcd} A\times A \arrow[r, "g\times g"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow g_1"] & B\times B \arrow[d, "\otimes"] \\ A \arrow[r, "g"'] & B \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{1} \arrow[r, "1"] \arrow[d, "I"'] \arrow[dr, phantom, "\Downarrow g_0"] & \mathbb{1} \arrow[d, "I"] \\ A \arrow[r, "g"'] & B \end{tikzcd}$$

satisfying the axioms

(6.2)

$$
\begin{tikzcd}
{} \arrow[r, "1\times g\times g"] \arrow[d, "1\times\otimes"'] \arrow[dr, phantom, "\Downarrow 1\times g_1"] & {} \arrow[r, "g\times 1\times 1"] \arrow[d, "1\times\otimes"] & {} \arrow[d, "1\times\otimes"] \\
{} \arrow[r, "1\times g"'] \arrow[d, "\otimes"'] \arrow[drr, phantom, "\Downarrow g_1"] & {} \arrow[r, "g\times 1"] & {} \arrow[d, "\otimes"] \\
{} \arrow[rr, "g"'] & & {}
\end{tikzcd}
=
\begin{tikzcd}
{} \arrow[r, "g\times g\times 1"] \arrow[d, "\otimes\times 1"'] \arrow[dr, phantom, "\Downarrow g_1\times 1"] & {} \arrow[r, "1\times 1\times g"] \arrow[d, "\otimes\times 1"] & {} \arrow[d, "\otimes\times 1"] \\
{} \arrow[r, "g\times 1"'] \arrow[d, "\otimes"'] \arrow[drr, phantom, "\Downarrow g_1"] & {} \arrow[r, "1\times g"] & {} \arrow[d, "\otimes"] \\
{} \arrow[rr, "g"'] & & {}
\end{tikzcd}
$$

(6.3)

$$
\begin{tikzcd}
{} \arrow[r] \arrow[d, "I\times 1"'] \arrow[dr, phantom, "\Downarrow g_0\times 1"] & {} \arrow[r] \arrow[d, "I\times 1"] & {} \arrow[d, "I\times 1"] \\
{} \arrow[r] \arrow[d, "\otimes"'] \arrow[drr, phantom, "\Downarrow g_1"] & {} \arrow[r] & {} \arrow[d, "\otimes"] \\
{} \arrow[rr, "g"'] & & {}
\end{tikzcd}
=
\begin{tikzcd}
{} \arrow[r] \arrow[d, "1\times I"'] \arrow[dr, phantom, "\Downarrow 1\times g_0"] & {} \arrow[r] \arrow[d, "1\times I"] & {} \arrow[d, "1\times I"] \\
{} \arrow[r] \arrow[d, "\otimes"'] \arrow[drr, phantom, "\Downarrow g_1"] & {} \arrow[r] & {} \arrow[d, "\otimes"] \\
{} \arrow[rr] & & {}
\end{tikzcd}
= \quad \mathrm{id}
$$

A monoidal lax-natural transformation α: G $\rightsquigarrow$ H is a lax-natural transformation α: g $\rightsquigarrow$ h satisfying the axioms

(6.4)
$$
\begin{tikzcd}
\mathbb{1} \arrow[r, dash, "1"] \arrow[d, "I"'] \arrow[dr, phantom, "\Downarrow g_0"] & \mathbb{1} \arrow[d, "I"] \\
A \arrow[r, bend left, "g"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Downarrow \alpha"] & B
\end{tikzcd}
=
\begin{tikzcd}
\mathbb{1} \arrow[r, dash, "1"] \arrow[d, "I"'] \arrow[dr, phantom, "\Downarrow h_0"] & \mathbb{1} \arrow[d, "I"] \\
A \arrow[r, bend right, "h"'] & B
\end{tikzcd}
$$

and

$$
(6.5)\qquad
\begin{tikzcd}[column sep=huge, row sep=huge]
A\times A \arrow[r, bend left=50, "g\times g"] \arrow[r, bend right=30, "h\times h"'] \arrow[r, phantom, "\wr\,\alpha\times\alpha"] \arrow[d, "\otimes"'] & B\times B \arrow[d, "\otimes"] \\
A \arrow[r, bend right=30, "h"'] \arrow[ur, phantom, "\Downarrow h_1"] & B
\end{tikzcd}
\quad = \quad
\begin{tikzcd}[column sep=huge, row sep=huge]
A\times A \arrow[r, bend left=30, "g\times g"] \arrow[d, "\otimes"'] \arrow[dr, phantom, "\Downarrow g_1"] & B\times B \arrow[d, "\otimes"] \\
A \arrow[r, bend left=30, "g"] \arrow[r, bend right=30, "h"'] \arrow[r, phantom, "\wr\,\alpha"] & B
\end{tikzcd}
\; .
$$

A <u>monoidal modification</u> $\pi\colon \alpha \to \beta$ between monoidal lax-natural transformations is a modification $\pi\colon \alpha \to \beta$ such that the analogue of (6.4) and (6.5) hold when we replace $\wr\,\alpha$ in (6.4) and (6.5) by $(\alpha \wr \overset{\pi}{\to} \wr\,\beta)$.

We denote by M (resp. M_*) the 3-category of monoidal 2-categories, monoidal (resp. strict monoidal) 2-functors, monoidal 2-natural transformations (not lax!), and monoidal modifications.

We observe that there is a 3-monad $D = (D,i,m)$ on the 3-category 2-CAT for which M_* is the 3-category of Eilenberg-Moore algebras and for which M is the bigger 3-category containing also the non-strict D-morphisms; these correspond in the 2-categorical situation to $\mathcal{D}_*$ and $\mathcal{D}$. The 3-functor D is essentially what Kelly [9] calls $\mathbb{N}^{o}$-; that is, if A is a 2-category then DA is the 2-category with objects of the form $n[A_1,\ldots,A_n]$ for $n \in \mathbb{N}$ and $A_i \in A$, with 1-cells of the form

$$n[f_1,\ldots,f_n] \; : \; n[A_1,\ldots,A_n] \longrightarrow n[A_1',\ldots,A_n']$$

for $f_i\colon A_i \to A_i'$ in A, and with 2-cells defined similarly. It should be clear how to define D on 2-functors, 2-natural transformations, and modifications. We notice that if $\alpha\colon g \rightsquigarrow h$ is a lax-natural transformation we can define a lax-natural transformation $D\alpha\colon Dg \rightsquigarrow Dh$ by the equations

$$(D\alpha)_{n[A_1 \ldots A_n]} \quad = \quad n[\alpha_{A_1} \ldots \alpha_{A_n}]$$

and

$$(D\alpha)_{n[f_1 \ldots f_n]} \quad = \quad n[\alpha_{f_1} \ldots \alpha_{f_n}] \quad .$$

Furthermore if $\pi\colon \alpha \to \beta$ is a modification between lax-natural transformations we can define a modification $D\pi\colon D\alpha \to D\beta$ by the equation

$$(D\pi)_{n[A_1 \ldots A_n]} \quad = \quad n[\pi_{A_1} \ldots \pi_{A_n}] \quad .$$

The 3-natural transformations i and m are such that their A-th components are given on objects by

$$(iA)(A) \quad = \quad 1[A]$$

and

$$(mA)(k[n_1[A_{11} \ldots A_{1n_1}] \; \ldots \; n_k[A_{k1} \ldots A_{km_k}]])$$

$$= \quad (\textstyle\sum n_i)[A_{11} \ldots A_{1n_1} \ldots A_{kn_k}] \quad .$$

Finally we observe that if $G = (g,\bar{g})$ and $H = (h,\bar{h})$ are D-morphisms (that is, monoidal 2-functors) from $\mathcal{A} = (A,a)$ to $\mathcal{B} = (B,b)$, then $\alpha\colon g \rightsquigarrow h$ is a monoidal lax-natural transformation from G to H if and only if

(6.6)
$$\begin{tikzcd}
D\mathcal{A} \arrow[rr, bend left, "Dg"] \arrow[rr, bend right, "Dh"'] \arrow[rr, phantom, "\wr\, D\alpha"] \arrow[d, "a"'] \arrow[drr, phantom, "\Downarrow \bar{h}"] & & D\mathcal{B} \arrow[d, "b"] \\
\mathcal{A} \arrow[rr, bend right, "h"'] & & \mathcal{B}
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
D\mathcal{A} \arrow[rr, bend left, "Dg"] \arrow[d, "a"'] \arrow[drr, phantom, "\Downarrow \bar{g}"] & & D\mathcal{B} \arrow[d, "b"] \\
\mathcal{A} \arrow[rr, bend left, "g"] \arrow[rr, bend right, "h"'] \arrow[rr, phantom, "\wr\, \alpha"] & & \mathcal{B}
\end{tikzcd}$$

while $\pi\colon \alpha \to \beta$ is a D-modification (that is, monoidal modification) between D-lax-natural transformations if and only if the analogoue of (6.6) holds when we replace α in (6.6) by the modification π.

If $\mathcal{A} = (\mathcal{A},a)$ and $\mathcal{B} = (\mathcal{B},b)$ are monoidal 2-categories and if $\mathcal{X}_1$ and $\mathcal{X}_2$ are subcategories of the underlying category of $\mathcal{A}$, then we mean by $\mathcal{M}[\![\mathcal{X}_1;\mathcal{X}_2;\mathcal{A},\mathcal{B}]\!]$ the 2-category of monoidal 2-functors, monoidal lax-natural transformations that are 2-natural when restricted to $\mathcal{X}_1$ and pseudo natural when restricted to $\mathcal{X}_2$, and monoidal modifications. We mean by $\mathcal{M}_*[\![\mathcal{X}_1;\mathcal{X}_2;\mathcal{A},\mathcal{B}]\!]$ the analogous 2-category in which the objects are the <u>strict</u> monoidal 2-functors.

The usual enriched hom-functors for $\mathcal{M}$ and $\mathcal{M}_*$ are $\mathcal{M}(-,-)$ and $\mathcal{M}_*(-,-)$ respectively. We observe that for any monoidal 2-categories $\mathcal{A}$ and $\mathcal{B}$ and any subcategories $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathcal{A}$ there are inclusion 2-functors

$$(6.7) \qquad \mathcal{M}_*(\mathcal{A},\mathcal{B}) \hookrightarrow \mathcal{M}_*[\![\mathcal{X}_1;\mathcal{X}_2;\mathcal{A},\mathcal{B}]\!]$$

and

$$(6.8) \qquad M(A,B) \hookrightarrow M[\![X_1;X_2;A,B]\!].$$

The latter inclusion, together with the observation that $M[\![\mathbb{1},B]\!] = M(\mathbb{1},B)$ gives us a "diagonal" 2-functor

$$(6.9) \qquad M[\![\mathbb{1},B]\!] \xrightarrow{\ d\ } M[\![X_1;X_2;A,B]\!]$$

which is equal to the composite

$$M(\mathbb{1},B) \xrightarrow{M(!,B)} M(A,B) \hookrightarrow M[\![X_1;X_2;A,B]\!] \ .$$

We call a monoidal 2-category $B = (B,b)$ <u>lax-cocomplete</u> in $M = \mathit{Mon}\text{-}2\text{-}\mathit{CAT}$ if for all small monoidal 2-categories $A = (A,a)$ and all subcategories X_1 and X_2 of A the diagonal 2-functor (6.9) has a left 2-adjoint L. If $G\colon A \to B$ is a monoidal 2-functor we call $LG\colon \mathbb{1} \to B$ the $\{X_1;X_2\}$-lax-colimit of G and we call the component $G \rightsquigarrow (LG)!$ of the unit the $\{X_1;X_2\}$-lax-colimit-cone of G. More generally if $G\colon A \to B$ is a monoidal 2-functor we call the monoidal 2-functor $L\colon \mathbb{1} \to B$ the $\{X_1;X_2\}$-lax-colimit of G in $\mathit{Mon}\text{-}2\text{-}\mathit{CAT}$, and we call $\alpha\colon G \rightsquigarrow L!$ the $\{X_1;X_2\}$-lax-colimit-cone of G if $\alpha\colon G \rightsquigarrow L!$ is the unit of the free object on G relative to the 2-functor d of (6.9). That is, if $C = M(\mathbb{1},B)$ and $E = M[\![X_1;X_2;A,B]\!]$ then for every K in C there is an isomorphism

$$E(G,d(K)) \cong C(L,K)$$

which is 2-natural in K. Finally we observe that d(K) is the 2-functor $A \xrightarrow{!} \mathbb{1} \xrightarrow{K} B$ so that we may write K! for the value of d(K).

We denote the 2-category $M(\mathbf{1},B)$ by $M[B]$ or $Mon(B)$ and observe that it is the 2-category of monoids, strict monoid morphisms and 2-cells of monoid morphisms. The obvious forgetful 3-functor $U : M \to 2\text{-}CAT$ gives a forgetful 2-functor $U_B\colon M[B] \to B$.

Theorem 6.1. A monoidal 2-category $B = (B,b)$ is lax-cocomplete in Mon-2-CAT if the following 2-conditions are satisfied:

(i) the 2-functor $U_B\colon M[B] \to B$ has a left adjoint

(ii) the 2-category $M[B]$ is cocomplete (as a 2-category).

Let $H = (h,id)$ be a strict monoidal 2-functor from B to C where B satisfies conditions (i) and (ii) above and where C is any monoidal 2-category. Then H preserves lax-colimits in Mon-2-CAT if the following two conditions are satisfied:

(iii) the 2-functor $B \xrightarrow{F} M[B] \xrightarrow{M[H]} M[C]$ is the partial left adjoint to the 2-functor $U_C\colon M[C] \to C$ relative to $h\colon B \to C$.

(iv) the 2-functor $M[H]\colon M[B] \to M[C]$ is 2-cocontinuous.

Proof. Except for two minor variations the proof is the same as the proof of Theorem 4.1. The first point of variation is that the pasting-on of η, the unit of the adjunction $F \dashv U_B$, gives bijections θ_H and θ_1 between

<u>lax-natural</u> transformations as in

$$\alpha\colon h.g \rightsquigarrow U_{C}k! \overset{\theta_H}{\leftrightarrow} \beta\colon M[H].F.g \rightsquigarrow k!$$

and

$$\alpha'\colon g \rightsquigarrow U_{B}k'! \overset{\theta_1}{\leftrightarrow} \beta'\colon F.g \rightsquigarrow k'! \quad ;$$

the formula $\theta_H(h.\alpha') = M[H]\ \theta_1(\alpha')$ still provides the connection between θ_H and θ_1.

The second variation is that instead of taking λ to be the colimit-cone of F.g we let λ as in

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[dr, "!"'] & B \arrow[r, "F"] \arrow[d, phantom, "\lambda"] & M[B] \\ & \mathbb{1} \arrow[ur, "\ell"'] & \end{tikzcd}$$

be the $\{X_1;X_2\}$-lax-colimit-cone of F.g in 2-CAT, which exists by the cocompleteness of $M[B]$ (cf. Chapter 0). The proof then proceeds in exactly the same manner as the proof of Theorem 4.1. □

As with Theorem 4.1, it will be of assistance in applying Theorem 6.1 to have:

<u>Proposition 6.2</u>. <u>The</u> 2-<u>functor</u> $U_B\colon M[B] \to B$ <u>is</u> 2-<u>monadic</u> <u>if and only if it has a left adjoint</u>.

Proof. The proof is a direct imitation of Proposition 4.2 using the 2-monadicity theorem given in Chapter 0. □

As mentioned at the beginning of this section, the reason for considering lax-colimits in $\mathcal{Mon}$-2-$\mathcal{CAT}$ in the first place is that we will apply this concept to certain monadicity problems. The result we need for this application is:

Theorem 6.3. If $\mathcal{B}$ is a cocomplete 2-category then the monoidal 2-category $\mathcal{E}_*$ of ranked endo-2-functors of $\mathcal{B}$ is lax-cocomplete in $\mathcal{Mon}$-2-$\mathcal{CAT}$ and the strict monoidal 2-functor I_*: $\mathcal{E}_* \to \mathcal{E}$, where $\mathcal{E}$ is the monoidal 2-category of all endo-2-functors of $\mathcal{B}$, preserves lax-colimits in $\mathcal{Mon}$-2-$\mathcal{CAT}$.

Proof. The proof is just an imitation, at the level of 2-categories rather than categories, of the results of section 5. The only comment that need be made concerns the existence of F: $\mathcal{E}_* \to$ Mon($\mathcal{E}$). If R is an endo-2-functor of $\mathcal{B}$ such that R-$\mathcal{A}\ell g$ = T-$\mathcal{A}\ell g_*$ for some 2-monad T, then it is easy to see that T is the free object on R relative to U: $\mathcal{Mon}(\mathcal{E}) \to \mathcal{E}$ so that FR = T, however the universal property at the level of 2-cells, required for F to be a 2-left adjoint, does not appear to follow from the pointwise existence of T. We can overcome this problem in two ways; the first is to assume that $\mathcal{B}$ is complete, then it follows automatically that any left adjoint to the underlying functor of U of necessity enriches to a 2-left adjoint to U (since U preserves cotensors). The second way to overcome the

problem is to either observe that Dubuc's ([6])
construction of the free monad automatically gives the 2-left adjoint, or to prove directly from the construction of the left adjoint to V: R-$\mathcal{A}\ell g \to \mathcal{B}$ (as described in section 5 as the variation of the transfinite construction of Chapter 1) that T is universal at the level of 2-cells. There will be no loss of generality, so far as our applications are concerned, if we assume that $\mathcal{B}$ is complete since we will need to make this assumption for other reasons. □

CHAPTER 3

1. In this chapter we are concerned with a class of structures, called polyads, on a 2-category A and with the 2-category of algebras for a polyad. Our aim is to develop a formalism of sufficient generality to include a large class of examples and to give conditions under which the 2-category of algebras for a polyad is 2-monadic. Typical of the kind of examples we have in mind are algebras for an endofunctor, algebras for a pointed endofunctor, algebras for a monad, lax-algebras for a doctrine (see Kelly [12]), and algebras for a pseudo distributive law (see Kelly [12]).

The chapter is divided into three parts; in the first part, which comprises sections 2 and 3, we define polyads and their algebras and give sufficient conditions for the 2-category of algebras to be 2-monadic. The second part, comprising sections 4 and 5, deals with the question of giving a polyad in terms of generators and relations, and gives a description of the algebras for a polyad in terms of the generators and relations only; we also give the sufficient conditions, for the 2-category of algebras to be 2-monadic, in terms of the generators. Finally, in part three, we examine some applications; one of these is the investigation of the category D[A] that was foreshadowed in Chapter 2; we show that if A is cocomplete then the category D[A] is the category of algebras for a polyad on A, and that if moreover A is complete and the action $a\colon DA \to A$ "has a rank" then D[A] is monadic over A and the induced monad has a rank.

2. By a <u>type</u> $\mathcal{T}$ we mean a small 3-category with only one object $* \in \mathcal{T}$, and by a <u>model</u> of the type $\mathcal{T}$ we mean a 3-functor $X\colon \mathcal{T} \to 2\text{-}\mathcal{CAT}$. Equivalently a type is a small strict monoidal 2-category $\mathcal{M}$; namely the 2-category of 1-cells, 2-cells, and 3-cells of $\mathcal{T}$. Then a model X of $\mathcal{T}$ is just a 2-category $X(*) = \mathcal{A}$ together with a strict monoidal 2-functor (which we still call X)

$$(2.1) \qquad X\colon \mathcal{M} \to 2\text{-}\mathcal{CAT}(\mathcal{A},\mathcal{A}) = [\mathcal{A},\mathcal{A}].$$

Moreover to give X as in (2.1) is the same as to give an <u>action</u>

$$(2.2) \qquad x\colon \mathcal{M} \times \mathcal{A} \to \mathcal{A}$$

of the 3-monad $\mathcal{M} \times -$ on $\mathcal{A}$, where X and x are mates under the cartesian adjunction on $2\text{-}\mathcal{CAT}$ and are connected by the equation

$$(2.3) \qquad \begin{tikzcd} \mathcal{M}\times\mathcal{A} \arrow[d, "X\times 1"'] \arrow[dr, "x"] & \\ {[\mathcal{A},\mathcal{A}]\times\mathcal{A}} \arrow[r, "ev"'] & \mathcal{A} \end{tikzcd} .$$

Thus for any $A \in \mathcal{A}$ we have commutativity in the diagram

(2.4)
$$\begin{tikzcd}
\mathcal{M} = \mathcal{M}\times\mathbb{1} \arrow[rr, "\mathcal{M}\times\ulcorner A\urcorner"] \arrow[d, "X"'] & & \mathcal{M}\times\mathcal{A} \arrow[d, "x"] \\
{[\mathcal{A},\mathcal{A}]} \arrow[rr, "\mathrm{ev}_A"'] & & \mathcal{A}
\end{tikzcd} .$$

A <u>polyad</u> $\mathcal{X}$ on $\mathcal{A}$ of type T is an ordered triple $\mathcal{X} = (X, \mathcal{X}_1, \mathcal{X}_2)$ where X is a model of the type T such that $X(*) = \mathcal{A}$, and where $\mathcal{X}_1$ and $\mathcal{X}_2$ are subcategories of the underlying category of $\mathcal{M}$.

An $\mathcal{X}$-algebra is a pair (A,α) where $A \in \mathcal{A}$ and where α is an $\{\mathcal{X}_1;\mathcal{X}_2\}$-lax natural transformation as in

(2.5)
$$\begin{tikzcd}
\mathcal{M}\times\mathbb{1} \arrow[rr, "\mathcal{M}\times\ulcorner A\urcorner"] \arrow[d, "!"'] & & \mathcal{M}\times\mathcal{A} \arrow[d, "x"] \\
\mathbb{1} \arrow[rr, dash, "\ulcorner A\urcorner"'] \arrow[urr, phantom, "\Downarrow\alpha"] & & \mathcal{A}
\end{tikzcd}$$

such that

(2.6)
$$\begin{tikzcd}
\mathbb{1} = \mathbb{1}\times\mathbb{1} \arrow[r, "j\times 1"] & \mathcal{M}\times\mathbb{1} \arrow[r, "\mathcal{M}\times\ulcorner A\urcorner"] \arrow[d, "!"'] & \mathcal{M}\times\mathcal{A} \arrow[d, "x"] \\
& \mathbb{1} \arrow[r, "\ulcorner A\urcorner"'] \arrow[ur, phantom, "\Downarrow\alpha"] & \mathcal{A}
\end{tikzcd} \quad = \quad \mathrm{id}$$

and

$$
(2.7)\qquad
\begin{tikzcd}
M\times M\times \mathbb{1} \arrow[d, "n\times !"'] & \\
M\times \mathbb{1} \arrow[r, "M\times \ulcorner A\urcorner"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow \alpha"] & M\times A \arrow[d, "x"] \\
\mathbb{1} \arrow[r, "\ulcorner A\urcorner"'] & A
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
M\times M\times \mathbb{1} \arrow[r, "M\times M\times \ulcorner A\urcorner"] \arrow[d, "M\times !"'] \arrow[dr, phantom, "\Downarrow M\times\alpha"] & M\times M\times A \arrow[d, "M\times x"] \\
M\times \mathbb{1} \arrow[r, "M\times \ulcorner A\urcorner"'] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow \alpha"] & M\times A \arrow[d, "x"] \\
\mathbb{1} \arrow[r, "\ulcorner A\urcorner"'] & A
\end{tikzcd}
$$

where $j\colon \mathbb{1} \to M$ and $n\colon M\times M \to M$ are the unit and multiplication of the monoidal 2-category M.

If $A = (A,\alpha)$ and $B = (B,\beta)$ are X-algebras, then an X-<u>morphism</u> (resp. X-2-cell) from A to B consists of a 1-cell $f\colon A \to B$ (resp. a 2-cell $\rho\colon f \Rightarrow g\colon A \to B$) in A such that

$$
(2.8)\qquad \begin{aligned} \beta.(M\times f) &= f.\alpha \\ (\text{resp. } \beta.(M\times \rho) &= \rho.\alpha\ ) \end{aligned}
$$

where these are the evident pasting-composites.

We denote by $X\text{-}\mathcal{A}\ell g_*$ the 2-category of X-algebras, X-morphisms, and X-2-cells. (Observe that we are again,for uniformity, using the subscript $*$ to mean "strict" morphisms; we do not give a definition of non-strict X-morphisms.) There is an evident forgetful 2-functor $V\colon X\text{-}\mathcal{A}\ell g_* \to A$ which sends (A,α) to A.

We now write axioms (2.6) and (2.7) in terms of components. If the unit of the monoidal 2-category M is I, so that $j(1) = I$, then the equation (2.6) is precisely the equation

$$(2.9) \qquad \alpha_I = 1_A .$$

If we write $K\colon R \to S$ and $K'\colon R' \to S'$ for the value of X at the morphisms $k\colon r \to s$ and $k'\colon r' \to s'$ in M, then in view of (2.4) equation (2.7) becomes the equality

$$(2.10) \qquad \begin{tikzcd} RR'A \arrow[rr, "KK'A"] \arrow[dr, "\alpha_{n(r,r')}"'] & \arrow[d, phantom, "\overset{\alpha_{n(k,k')}}{\Longrightarrow}"] & SS'A \arrow[dl, "\alpha_{n(s,s')}"] \\ & A & \end{tikzcd}$$

equals

$$\begin{tikzcd} RR'A \arrow[r, "RK'A"] \arrow[dr, "R\alpha_{r'}"'] & RS'A \arrow[r, "KS'A"] \arrow[d, "R\alpha_{s'}"] \arrow[dl, phantom, "\overset{R\alpha_{k'}}{\Longrightarrow}"] & SS'A \arrow[d, "S\alpha_{s'}"] \\ & RA \arrow[r, "KA"] \arrow[dr, "\alpha_r"'] & SA \arrow[d, "\alpha_s"] \arrow[dl, phantom, "\overset{\alpha_k}{\Longrightarrow}"] \\ & & A \end{tikzcd}$$

for all k and k' in M.

If $\rho\colon f \Rightarrow g\colon A \to B$ is a 2-cell in A then axiom (2.7), for ρ to be an X-2-cell, is precisely the equality

(2.11)

$$\begin{tikzcd}
RA \arrow[rr, "\alpha_r"] \arrow[dr, "KA"'] & \arrow[d, phantom, "\Downarrow \alpha_k"] & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \rho"] & B \\
& SA \arrow[ur, "\alpha_s"'] & &
\end{tikzcd}$$

equals

$$\begin{tikzcd}
RA \arrow[r, bend left, "Rf"] \arrow[r, bend right, "Rg"'] \arrow[r, phantom, "\Downarrow R\rho"] & RB \arrow[rr, "\beta_r"] \arrow[dr, "KB"'] & \arrow[d, phantom, "\Downarrow \beta_k"] & A \\
& & SA \arrow[ur, "\beta_s"'] &
\end{tikzcd}$$

for all $k\colon r \to s$ in $\mathcal{M}$, where again we write R,S, and K for X(r), X(s), and X(k).

Theorem 2.1. *If* $\mathcal{X} = (X, \mathcal{X}_1, \mathcal{X}_2)$ *is a polyad on* $\mathcal{A}$, *then the* 2-*functor* $V\colon \mathcal{X}\text{-}\mathcal{A}\ell g_* \to \mathcal{A}$ *is* 2-*monadic if and only if it has a left adjoint*.

Proof. Because of Proposition 8.1 of Chapter 0 we have only to show that V creates coequalisers of V-split pairs.

Let A = (A,α) and B = (B,β) be two $\mathcal{X}$-algebras, let f,g: A → B be a pair of $\mathcal{X}$-morphisms which are V-split, and let

(2.12)
$$\begin{tikzcd}
VA \arrow[r, shift left, "Vf"] \arrow[r, shift right, "Vg"'] & VB \arrow[r, "p"] & Z
\end{tikzcd}$$

be the coequaliser in A given by the splitting. It is well known that the coequaliser (2.12) is an absolute coequaliser; hence the rows of

$$\begin{tikzcd}
x.M\times\ulcorner VA\urcorner \arrow[r, shift left, "x.M\times\ulcorner Vf\urcorner"] \arrow[r, shift right, "x.M\times\ulcorner Vg\urcorner"'] \arrow[d, squiggly, "\alpha"'] & x.M\times\ulcorner VB\urcorner \arrow[r, "x.M\times\ulcorner p\urcorner"] \arrow[d, squiggly, "\beta"] & x.M\times\ulcorner Z\urcorner \\
\ulcorner VA\urcorner\,! \arrow[r, shift left, "\ulcorner Vf\urcorner\,!"] \arrow[r, shift right, "\ulcorner Vg\urcorner\,!"'] & \ulcorner VB\urcorner\,! \arrow[r, "\ulcorner p\urcorner\,!"] & \ulcorner Z\urcorner\,!
\end{tikzcd}$$

are coequalisers in the category $Fun(X_1;X_2;M,A)$, so that the two vertical arrows induce an arrow $\zeta\colon x.M\times\ulcorner Z\urcorner \rightsquigarrow \ulcorner Z\urcorner!$. It is an easy matter to show that (Z,ζ) is an X-algebra; then $p\colon B \to (Z,\zeta)$ is an X-morphism by the definition of ζ; it is also clear that $p\colon B \to Z$ is the coequaliser, in $X\text{-}\mathcal{A}\ell g$, of the pair (f,g). □

In spite of the above theorem we do not intend to prove, under suitable hypotheses, the 2-monadicity of $X\text{-}\mathcal{A}\ell g_*$ by constructing the left adjoint of $V\colon X\text{-}\mathcal{A}\ell g_* \to A$; rather we construct the 2-monad in question as the lax-colimit of a monoidal 2-functor. To this end we make the following definition.

3. A <u>monad</u> on X is a pair (T,τ) where $T = (T,t_0,t_2)$ is a 2-monad (= doctrine) on A (that is, a monoidal 2-functor $T\colon \mathbb{1} \to [A,A]$) and where τ is an $\{X_1;X_2\}$-monoidal-lax

natural transformation as in

(3.1)
$$\begin{tikzcd} M \arrow[rr, "X"] \arrow[dr, "!"'] \arrow[rr, phantom, shift right=6ex, "\tau"] & & {[A,A]} \\ & \mathbb{1} \arrow[ur, "T"'] & \end{tikzcd}$$
.

The category *Monad*(X) has as objects monads on X while the morphisms in *Monad*(X) from (T,τ) to (S,σ) are doctrine morphisms $k\colon T \to S$ such that $k!.\tau = \sigma$.

Given a τ as in (3.1) we construct as follows a 2-functor $\chi\tau = \Psi\colon T\text{-}\mathcal{A}\ell g_* \to X\text{-}\mathcal{A}\ell g_*$ such that $V\Psi = U^T$. For a T-algebra $A = (A,a)$ we define ΨA to be the X-algebra (A,α) where α is the $\{X_1;X_2\}$-lax natural transformation

(3.2)
$$\begin{tikzcd} M \arrow[r, "X"] \arrow[d, "!"'] \arrow[dr, phantom, shift right=2ex, "\tau"] & {[A,A]} \arrow[d, "ev_A"] \arrow[dl, phantom, shift right=2ex, "\Downarrow a"] \\ \mathbb{1} \arrow[ur, "T"] \arrow[r, "\ulcorner A \urcorner"'] & A \end{tikzcd}$$
;

observing from (2.4) that this is indeed of the form (2.5). The reader will easily verify that (A,α) satisfies the axioms for an X-algebra; as an example we verify the object-part of the associativity axiom. Let r and s be objects of M; evaluating the left-hand diagram of (2.7) at the pair (r,s) yields the 1-cell

$$RSA \xrightarrow[(\tau_{n(r,s)})A]{} TA \xrightarrow[a]{} A$$

while evaluating the right-hand diagram of (2.7) at (r,s) yields

$$\begin{tikzcd} RSA \arrow[r, "(\tau_r\cdot\tau_s)A"'] & T^2A \arrow[r, "t_2A"'] & TA \arrow[r, "a"'] & A \end{tikzcd} .$$

However these are equal since the monoidal axioms for τ give us the equation

$$\begin{tikzcd}[column sep=huge, row sep=large] RS \arrow[r, "\tau_r\cdot\tau_s"] \arrow[dr, "\tau_{n(r,s)}"'] & T^2 \arrow[d, "t_2"] \\ & T \end{tikzcd} .$$

We define Ψ to be the identity on 1-cells and 2-cells; we must verify that a strict T-morphism $f\colon (A,a) \to (B,b)$ is also an X-algebra morphism from ΨA to ΨB, as well as the corresponding result for 2-cells. We do it only for 1-cells, observing that the axiom (2.8) for an X-morphism may also be written as

$$\begin{tikzcd}[column sep=huge, row sep=large] M \arrow[r, "X"] \arrow[d, "!"'] \arrow[dr, phantom, "\beta"] & {[A,A]} \arrow[d, bend right=40, "ev_B"'] \arrow[d, bend left=40, "ev_A"] \arrow[d, phantom, "\overset{ev_f}{\Leftarrow}"] \\ \mathbb{1} \arrow[r, "\ulcorner B\urcorner"'] & A \end{tikzcd} = \begin{tikzcd}[column sep=huge, row sep=large] M \arrow[r, "X"] \arrow[d, "!"'] \arrow[dr, phantom, "\alpha"] & {[A,A]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, bend left=30, "\ulcorner A\urcorner"] \arrow[r, bend right=30, "\ulcorner B\urcorner"'] \arrow[r, phantom, "\Downarrow\ulcorner f\urcorner"] & A \end{tikzcd} .$$

Since the axiom for a strict T-morphism can clearly be written as

$$\begin{tikzcd} & {[A,A]} \arrow[d, bend right, "ev_B"'] \arrow[d, bend left, "ev_A"] \arrow[d, phantom, "\overset{ev_f}{\Leftarrow}"] \\ \mathbb{1} \arrow[ur, "T"] \arrow[r, bend right, "\ulcorner B\urcorner"'] \arrow[r, phantom, "\Downarrow b"] & A \end{tikzcd} \quad = \quad \begin{tikzcd} & {[A,A]} \arrow[d, bend left, "ev_A"] \\ \mathbb{1} \arrow[ur, bend left, "T"] \arrow[ur, phantom, "\Downarrow a"'] \arrow[r, bend left, "\ulcorner A\urcorner"] \arrow[r, bend right, "\ulcorner B\urcorner"'] \arrow[r, phantom, "\Downarrow \ulcorner f\urcorner"] & A \end{tikzcd}$$

the result is immediate.

<u>Proposition 3.1</u>. <u>For any doctrine</u> $T = (T,t_0,t_2)$ <u>on</u> $\mathcal{A}$, <u>the function</u> χ <u>is a bijection between monoidal</u> $\{X_1;X_2\}$-<u>lax-natural transformations</u> τ <u>as in</u> (3.1) <u>and</u> 2-<u>functors</u> $\Psi\colon T\text{-}\mathcal{A}\ell g_* \to X\text{-}\mathcal{A}\ell g_*$ <u>satisfying</u> $V\Psi = U^T$.

<u>Proof</u>. We have only to show that any Ψ as above is of the form $\chi\tau$ for a unique τ as in (3.1). Let (TA,t_2A) be the free T-algebra on A and let its image under Ψ be $(TA,\gamma A)$; then if $\Psi = \chi\tau$ for some τ we have

$$(3.4) \qquad \begin{tikzcd} \mathcal{M} \arrow[r, "X"] \arrow[d, "!"'] \arrow[dr, phantom, "\rightsquigarrow \gamma_A"] & {[A,A]} \arrow[d, "ev_{TA}"] \\ \mathbb{1} \arrow[r, "TA"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} \mathcal{M} \arrow[r, "X"] \arrow[d, "!"'] \arrow[dr, phantom, "\rightsquigarrow\tau", near start] & {[A,A]} \arrow[d, "ev_{TA}"] \\ \mathbb{1} \arrow[ur, "T"] \arrow[r, "TA"'] \arrow[dr, phantom] & A \arrow[ul, phantom, "\Downarrow t_2A", near start] \end{tikzcd} .$$

Pasting ev_{t_0A} on to the right hand side of the equation and using the equation $t_2A.t_0TA = 1_A$ yields

$$(3.5)\qquad \begin{tikzcd} M \arrow[r, "X"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\gamma_A"] & {[A,A]} \arrow[d, "ev_{TA}"'] \arrow[d, bend left=80, "ev_A"] \arrow[d, phantom, shift left=5, "\Leftarrow ev_{t_0A}"] \\ \mathbb{1} \arrow[r, "\ulcorner TA \urcorner"'] & A \end{tikzcd} \quad = \quad \begin{tikzcd} M \arrow[rr, "X"] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\tau"] & {[A,A]} \arrow[r, "ev_A"] & A \\ & \mathbb{1} \arrow[ur, "T"'] & & \end{tikzcd},$$

for which it follows immediately that if $\Psi = \check{\chi}\tau$ then τ is uniquely determined.

Thus if Ψ is any 2-functor with $V\Psi = U^T$, we are forced to define τ by equation (3.5), by which we mean that for any $r \in M$ we set

$$(3.6)\qquad (\tau_r)_A = (\gamma_A)_r . Rt_0A$$

and for any $k: r \to s$ in M we set

$$(3.7)\qquad (\tau_k)_A = (\gamma_A)_k . Rt_0A$$

where $R = X(r)$. To see that (τ_r) is 2-natural and that (τ_k) is a modification, we observe that for any $\rho: f \Rightarrow g : A \to B$ in A, the 2-cell $T\rho: Tf \Rightarrow Tg: TA \to TB$ is an X-2-cell from $(TA,\gamma A)$ to $(TB,\gamma B)$, which together with the 2-naturality of t_0 gives us the 2-naturality of τ_r and the modification property for τ_k. To see that τ is lax-natural we have only to observe that each $(\tau_-)_A$ is lax-natural.

Finally we must show that the τ defined above satisfies the equation $\Psi = \chi(\tau)$. In other words we must show that if, for a T-algebra (A,a), we write $\Psi(A,a) = (A,\alpha)$, then α is the lax-natural transformation of (3.2);or in terms of components we must show that

$$\alpha_r = a.\tau_r$$

for all r in M, and that

$$\alpha_k = a.\tau_k$$

for all $k: r \to s$ in M. Since $a: TA \to A$ is a T-morphism from (TA, t_2A) to (A,a) it is also an X-morphism from $(TA, \gamma A)$ to (A, α) so that we have

$$\alpha_r.Ra = a.(\gamma_A)_r$$

for all r in M and

$$\alpha_k.Ra = a.(\gamma_A)_k$$

for all $k: r \to s$ in M. Thus combining with (3.6) and (3.7) yields

$$\alpha_r.Ra.Rt_0A = a.(\gamma_A)_r.Rt_0A$$

and

$$\alpha_k.Ra.Rt_0A = a.(\gamma_A)_k.Rt_0A \quad ;$$

which since $a.t_0A = 1$ gives us

$$\alpha_r = a(\tau_r)_A$$

and

$$\alpha_k = a(\tau_k)_A,$$

as required. □

We also observe that χ is natural in the following sense:

Proposition 3.2. If (T,τ) and (S,σ) are monads on X and if $k\colon (T,\tau) \to (S,\sigma)$ is a morphism in $\mathit{Monad}(X)$ then $\chi(\sigma) = \chi(\tau).k\text{-}\mathcal{A}\ell g_*$. □

Proposition 3.3. The 2-functor $V\colon X\text{-}\mathcal{A}\ell g_* \to A$ is 2-monadic if and only if there exists a monad (T,τ) on X for which $\chi(\tau)$ is an isomorphism. If such a (T,τ) exists it is an initial object in $\mathit{Monad}(X)$.

Proof. Immediate from Proposition 3.1 and 3.2 and the definition of 2-monadicity. □

We call an initial object (T,τ) of $\mathit{Monad}(X)$ the free monad on X. If it has the further property that $\chi(\tau)$ is an isomorphism of 2-categories, we say that the free monad exists pointwise or we say that (T,τ) is the pointwise free monad on X.

Proposition 3.4. If

$$\begin{tikzcd} X \arrow[rr, "X"] \arrow[dr, "!"'] & \arrow[d, phantom, "\Downarrow\tau"] & {[A,A]} \\ & \mathbb{1} \arrow[ur, "T"'] & \end{tikzcd}$$

<u>is the</u> $\{X_1;X_2\}$-<u>lax-colimit of</u> X <u>in</u> *Mon*-2-*CAT*, <u>then</u> (T,τ) <u>is the free monad on</u> X.

<u>Proof</u>. Directly from the universal property of lax-colimits.□

Consider now the following three properties that the pair (T,τ) may possess:

a) it is the free monad on X,

b) it is the pointwise free monad on X,

c) it is the $\{X_1;X_2\}$-lax-colimit of X in *Mon*-2-*CAT*.

We have seen that b) implies a) and that c) implies a). As far as the author knows a) does not imply b) even in the special case of *A* being merely a category; indeed even when it is just a question of a free monad T on an endofunctor R it is not clear that a) implies b). We shall however show that a) and b) are equivalent when the 2-category *A* is <u>complete</u>.

In the case of *A* being a mere category it is evident that a) and c) are equivalent; for then the property of lax-colimits in *Mon*-2-*CAT* is exactly that of being initial. In the 2-category case, however, it is not clear to the author that a) implies c); for the universal property of the lax-colimit in *Mon*-2-*CAT* has a 2-cell element which is, on the face of it, stronger than being merely an initial object in *Monad*(*X*). However, when *A* has cotensors with the category *2*, so that in particular when *A* is complete, the free monad (T,τ) on *X* is also the $\{X_1;X_2\}$-lax colimit of X in *Mon-2-CAT* since the universal property at the level of 2-cells follows automatically from that at the level of 1-cells.

Proposition 3.5. *If* X *is a polyad on a complete* 2-*category* A, *then whenever the free monad on* X *exists, it is always the pointwise free monad on* X.

Proof. We refer the reader to Chapter 0 for a review of the definitions and properties of $\{A,B\}$, $[f,g]$, and $[\![\rho,\sigma]\!]$; these objects and their properties will be used in this proof.

Let $k\colon r \to s$ be a morphism in M and write $K\colon R \to S$ for its image under X. From the universal property of $\{A,A\}$ (see Chapter 0 section 9) we have, for all k in M and all A in A, a bijection between 2-cells α_k in A and 2-cells β_k in $[A,A]$ as in

$$\begin{tikzcd}
RA \arrow[ddrr, "\alpha_r"] \arrow[dddd, "KA"'] & & & & R \arrow[ddrr, "\beta_r"] \arrow[dddd, "K"'] & & \\
& & & & & & \\
& \arrow[r, phantom, "\Downarrow \alpha_k"] & A \arrow[rr, phantom, "\overset{\theta}{\longleftrightarrow}"] & & & \arrow[r, phantom, "\Downarrow \beta_k"] & \{A,A\} \\
& & & & & & \\
SA \arrow[uurr, "\alpha_s"'] & & & & S \arrow[uurr, "\beta_s"'] & &
\end{tikzcd} \quad ;$$

it is easy to see, since θ is a bijection, that α is an $\{X_1;X_2\}$-lax-natural transformation if and only if $\beta = \theta(\alpha)$ is an $\{X_1;X_2\}$-lax-natural transformation. It is an easy matter to show that $\beta = \theta(\alpha)$ is a monoidal lax-natural transformation as in

$$\begin{tikzcd}
M \arrow[rr, "X"] \arrow[dr, "!"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow \beta"] & & {[A,A]} \\
& \mathbb{1} \arrow[ur, "{\{A,A\}}"'] &
\end{tikzcd}$$

if and only if the pair (A,α) satisfy equations (2.6) and (2.7); one only need observe that (2.6) and (2.7) are just the axioms corresponding, under θ, to the monoidal axioms for β. Thus $\beta = \theta(\alpha)$ constitutes a monad on X if and only if α is an action for an X-algebra.

Thus if $(A,\alpha) \in X\text{-}\mathcal{A\ell g}_*$ then $(\{A,A\},\beta)$ is a monad on X so that β is the composite of τ with a unique monad map $k_\alpha\colon T \to \{A,A\}$, which corresponds to an action $a\colon TA \to A$. It follows immediately that (A,a) is the unique object of $T\text{-}\mathcal{A\ell g}_*$ whose image under $\chi(\tau)$ is (A,α).

Let $f\colon (A,\alpha) \to (A',\alpha')$ be an X-morphism. The equality (2.8) is equivalent to the commutativity for all $k\colon r \to s$ in M of

$$\begin{tikzcd}
& R \arrow[dl, "K"'] \arrow[rr, "\theta(\alpha)_r"] \arrow[dd, "\theta(\alpha')_r"] \arrow[rr, phantom, shift right=3ex, "\Swarrow \theta(\alpha)_k"] & & \{A,A\} \arrow[dd, "{\{1,f\}}"] \\
S \arrow[urrr, "\theta(\alpha)_s"'] \arrow[dr, "\theta(\alpha')_s"'] \arrow[r, phantom, "\theta(\alpha')_k \Swarrow"] & & & \\
& \{A',A'\} \arrow[rr, "{\{f,1\}}"'] & & \{A,A'\}
\end{tikzcd} .$$

This induces 1-cells ℓ_r and 2-cells ℓ_k as in

$$\begin{tikzcd} R \arrow[drr, "\ell_r"] \arrow[d, "k"'] & & \\ {} \arrow[r, phantom, "\ell_k \Downarrow"] & & {[f,f]} \\ S \arrow[urr, "\ell_s"'] & & \end{tikzcd}$$

such that $d_0\ell_k = \theta(\alpha)_k$ and $d_1\ell_k = \theta(\alpha')_k$ (where d_0 and d_1 are defined in section 9 of Chapter 0). Then it is easily checked that ℓ constitutes a monoidal $\{X_1;X_2\}$-lax-natural transformation as in

$$\begin{tikzcd} M \arrow[rr, "X"] \arrow[dr, "!"'] & {} \arrow[d, phantom, "\rightsquigarrow \ell"] & {[A,A]} \\ & \mathbb{1} \arrow[ur, "{[f,f]}"'] & \end{tikzcd} ,$$

so that ℓ is the composite of τ with a unique map $h_f\colon T \to [f,f]$ of doctrines satisfying

$$d_0 h_f! \;=\; \theta(\alpha) \text{ and } d_1 h_f! \;=\; \theta(\alpha') \;.$$

It follows at once that $f\colon (A,\alpha) \to (A',\alpha')$ is the image under $\chi(\tau)$ of a unique T-morphism $f\colon (A,a) \to (A',a')$. Similarly the bijectivity of $\chi(\tau)$ on 2-cells is proved by considering $[\![\rho,\rho]\!]$. $\square$

We say that the polyad $X = (X, X_1, X_2)$ has rank if the 2-functor $X\colon M \to [A,A]$ factors through the monoidal sub-2-category $[A,A]_*$ of ranked endo-2-functors of A. In view of Theorem 6.3 of Chapter 2 we have the following existence theorems:

Theorem 3.6. *If X is a polyad with rank on a cocomplete 2-category A, then the free monad (T,τ) on X exists, and moreover T has rank.* □

We combine this result with Theorem 3.5 and Corollary 3.4 to get:

Theorem 3.7. *If X is a ranked polyad on a complete and cocomplete 2-category A, then $V\colon X\text{-}\mathcal{A}\ell g_* \to A$ is 2-monadic, and moreover the 2-monad has rank.* □

4. We define a 3-category $\mathcal{A}ct$ whose set of objects $|\mathcal{A}ct|$ is equal to the set $|A|$ and where $\mathcal{A}ct(A,B)$ is the lax-comma 2-category of $ev_A\colon [A,A] \to A$ and $\ulcorner B \urcorner\colon \mathbb{1} \to A$ as in

(4.1)
$$\begin{tikzcd} \mathcal{A}ct(A,B) \arrow[r, "\partial_0"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\lambda"] & {[A,A]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, "\ulcorner B \urcorner"'] & A \end{tikzcd}$$
.

We refer the reader to Chapter 0 for an elementary description of lax-comma-2-categories which we now use to define the unit and composition of the 3-category $\mathcal{Act}$. The unit $j_A\colon \mathbb{1} \to \mathcal{Act}(A,A)$ is the 2-functor whose value at the unique object in $\mathbb{1}$ is

$$\begin{tikzcd} 1_A(A) \arrow[d, "1_A"] \\ A \end{tikzcd} ,$$

while the composition law, for 1-cells and 2-cells in $\mathcal{Act}$, is given by

$$\left(\begin{tikzcd} T \arrow[d, "f"'] \\ T' \end{tikzcd} , \begin{tikzcd} TA \arrow[d, "fA"'] \arrow[rr, "k"] & \arrow[d, phantom, "\Downarrow\alpha"] & B \\ T'A \arrow[urr, "k'"'] & & \end{tikzcd} \right) \circ \left(\begin{tikzcd} S \arrow[d, "g"'] \\ S' \end{tikzcd} \begin{tikzcd} SB \arrow[d, "gB"'] \arrow[rr, "h"] & \arrow[d, phantom, "\Downarrow\beta"] & C \\ S'B \arrow[urr, "h'"'] & & \end{tikzcd} \right)$$

equals

$$\left(\begin{tikzcd} ST \arrow[d, "g.f"'] \\ S'T' \end{tikzcd} , \begin{tikzcd} STA \arrow[d, "SfA"'] \arrow[r, "Sk"] \arrow[dr, phantom, "\Downarrow S\alpha"] & SB \arrow[d, "gB"'] \arrow[r, "h"] \arrow[dr, phantom, "\Downarrow\beta"] & B \\ ST'A \arrow[d, "gT'A"'] \arrow[ur, "Sk'"'] & S'B \arrow[ur, "h'"'] & \\ S'T'A \arrow[ur, "S'k'"'] & & \end{tikzcd} \right) ;$$

a similar definition gives the composition of 3-cells. We leave to the reader the easy, but tedious, verification of the required axioms. We observe that $\mathcal{A}$ may be identified as a sub-2-category of $\mathcal{A}ct$ since every $\rho: f \Rightarrow g : A \to B$ in $\mathcal{A}$ is in $\mathcal{A}ct$ in the form

$$\begin{tikzcd}
1_A(A) \arrow[dr, "f"] \arrow[dd, "1_A"'] & \\
{} \arrow[r, phantom, "\Downarrow \rho"] & B \\
1_A(A) \arrow[ur, "g"'] &
\end{tikzcd} \quad ;$$

it is clear that this inclusion is 2-functorial.

It is now automatic that the "endo 2-category" $\mathcal{A}ct(A,A)$ is a monoid in 2-$\mathcal{CAT}$; that is, $\mathcal{A}ct(A,A)$ is a strict monoidal 2-category; moreover, it is clear that ∂_0 and ∂_1 of (4.1) (with $B = A$) are monoidal 2-functors.

We recall that the universal property of $\mathcal{A}ct(A,B)$ gives a bijection ϕ between lax-natural transformations α as in (2. 5) and 2-functors $W: \mathcal{M} \to \mathcal{A}ct(A,A)$ with $\lambda W = \alpha$; we write $\phi(\alpha) = W$. Since we are concerned with $\mathcal{X}$-algebras we may well ask what the algebra axioms for α tell us about the corresponding $\phi(\alpha)$; this question is answered by:

Proposition 4.1. *The* $\{\mathcal{X}_1;\mathcal{X}_2\}$-*lax-natural transformation* α *as in* (2.5) *satisfies axioms* (2.6) *and* (2.7) *if and only if the* 2-*functor* $\phi(\alpha)$ *is strict monoidal*.

<u>Proof</u>. To see this write down, in terms of components, what it means for $\phi(\alpha)$ to be monoidal, and then observe these required axioms are precisely the component version of the algebra axioms for α as given in (2.9) and (2.10). To help the reader in this calcualtion we recall that for any α as in (2.5) the 2-functor $\phi(\alpha)$ is defined by

$$\begin{aligned}
\phi(\alpha)(t) &= (X(t),\ \alpha_t\colon X(t)A \to A) \\
\phi(\alpha)(f) &= (X(f),\ \alpha_f) \\
\phi(\alpha)(\rho) &= (X(\rho),\ 1) \quad . \qquad \square
\end{aligned}$$

Since a monoidal 2-functor $K\colon \mathcal{M} \to \mathcal{A}ct(A,A)$ is precisely a 3-functor $K\colon \mathcal{T} \to \mathcal{A}ct$ we see that (A,α) is an $\mathcal{X}$-algebra if and only if there is a 3-functor (necessarily unique) $K_\alpha\colon \mathcal{T} \to \mathcal{A}ct$ with $K_\alpha(*) = A$, such that $\phi(\alpha)$ is the monoidal 2-functor

$$K_\alpha\colon \mathcal{T}(*,*) \to \mathcal{A}ct(K_\alpha(*),K_\alpha(*)) \ .$$

The morphism $f\colon A \to B$ is an $\mathcal{X}$-morphism from (A,α) and (B,β) if and only if (2.8) is satisfied; but this is equivalent to the equality of

(4.2)
$$\begin{tikzcd}
\mathcal{M} \arrow[rr, "X"] \arrow[d, "!"'] \arrow[drr, phantom, "\Downarrow\alpha"] & & {[A,A]} \arrow[d, bend left, "ev_A"] \\
\mathbb{1} \arrow[rr, bend left, "\ulcorner A\urcorner"] \arrow[rr, bend right, "\ulcorner B\urcorner"'] \arrow[rr, phantom, "\Downarrow\ulcorner f\urcorner"] & & A
\end{tikzcd}$$

and

(4.3)
$$\begin{tikzcd}[column sep=huge, row sep=huge]
M \arrow[r, "X"] \arrow[d, "!"'] \arrow[dr, phantom, "\rightsquigarrow\beta"] & {[A,A]} \arrow[d, bend right=50, "ev_B"'] \arrow[d, bend left=50, "ev_A"] \arrow[d, phantom, "\overset{ev_f}{\Leftarrow}"] \\
\mathbb{1} \arrow[r, bend right, "B"'] & A
\end{tikzcd}$$
,

which in turn is equivalent to the commutativity of

(4.4)
$$\begin{tikzcd}[column sep=huge, row sep=huge]
M \arrow[r, "\phi(\alpha)"] \arrow[d, "\phi(\beta)"'] & \mathcal{Act}(A,A) \arrow[d, "\mathcal{Act}(1,f)"] \\
\mathcal{Act}(B,B) \arrow[r, "\mathcal{Act}(f,1)"'] & \mathcal{Act}(A,B)
\end{tikzcd}$$
.

We notice, however, that (4.4) is precisely the condition for f to constitute a 3-natural transformation $f:K_\alpha \Rightarrow K_\beta$. An analogous consideration with 2-cells ρ: $f \Rightarrow g$ in A will show that ρ is an X-2-cell from (A,α) to (B,β) if and only if

$$\begin{tikzcd}[column sep=huge]
T \arrow[r, bend left=50, "K_\alpha"] \arrow[r, bend right=50, "K_\beta"'] \arrow[r, phantom, "f\Downarrow \underset{\rho}{\rightarrow} \Downarrow g"] & \mathcal{Act}
\end{tikzcd}$$

is a modification of 3-natural transformations. We collect these results into:

Theorem 4.2. *Let* $\mathcal{X}$ *be a polyad on* $\mathcal{A}$, *let* $A \in \mathcal{A}$, *and let* α *be an* $\{\mathcal{X}_1;\mathcal{X}_2\}$*-lax natural transformation as in* (2.5). *Then* (A,α) *is an* $\mathcal{X}$*-algebra if and only if there exists a (unique)* 3-*functor* $K_\alpha\colon \mathcal{T} \to \mathcal{A}ct$ *with* $K_\alpha(*) = A$ *such that* $\phi(\alpha)$ *is the monoidal* 2-*functor*

$$K_\alpha\colon \mathcal{T}(*,*) \to \mathcal{A}ct(K_\alpha(*),K_\alpha(*)) .$$

If (A,α) *and* (B,β) *are* $\mathcal{X}$*-algebras and if* K_α *and* K_β *are the corresponding* 3-*functors, then* $\rho\colon f \Rightarrow g : A \to B$ *in* $\mathcal{A}$ *is an* $\mathcal{X}$-2-*cell from* (A,α) *to* (B,β) *if and only if* ρ *constitutes a modification of* 3-*natural transformations as in*

$$\begin{tikzcd} \mathcal{T} \arrow[rr, bend left=50, "K_\alpha"] \arrow[rr, bend right=50, "K_\beta"'] \arrow[rr, phantom, "f\Downarrow \xrightarrow{\rho} \Downarrow g"] & & \mathcal{A}ct \end{tikzcd} \quad . \quad \square$$

5. In many of our applications we shall not be dealing with polyads as such but rather with "presentations" of polyads; that is, polyads which are in some sense given by generators and relations. It is our purpose in this section to say precisely what *we* mean by generators and relations for a polyad $\mathcal{X}$, and moreover to see to what extent the 2-category $\mathcal{X}$-$\mathcal{A}\ell g_*$ can be described using only the generators and relations of $\mathcal{X}$.

Let $F\colon$ 3-$\mathcal{G}raph \to$ 3-$\mathcal{C}at$ be the left adjoint to the *functor* $U\colon$ 3-$\mathcal{C}at \to$ 3-$\mathcal{G}raph$ (the existence of F was discussed in Chapter 0) and let $\eta\colon 1 \to UF$ be the unit of the this adjunction.

A presentation of a type $\mathcal{T}$ consists of a pair of small 3-graphs $\mathcal{R}$ and $\mathcal{G}$ each with one object, and a pair of morphisms of 3-graphs

$$\mathcal{R} \underset{Q}{\overset{P}{\rightrightarrows}} F\mathcal{G}\ ,$$

together with a 3-functor $E\colon F\mathcal{G} \to \mathcal{T}$ such that

$$(5.1) \qquad F\mathcal{R} \underset{\bar{Q}}{\overset{\bar{P}}{\rightrightarrows}} F\mathcal{G} \xrightarrow{\ E\ } \mathcal{T}$$

is a coequaliser diagram in 3-$\mathcal{CAT}$; where the 3-functors $\bar{P}$ and $\bar{Q}$ are those generated by P and Q respectively.

It is clear that any 3-functor $X\colon \mathcal{T} \to \mathcal{B}$ is precisely a morphism $X\colon \mathcal{G} \to \mathcal{B}$ of 3-graphs such that

$$(5.2) \qquad \bar{X}P = \bar{X}Q\ .$$

In particlar any model X of the type $\mathcal{T}$ is a morphism $X\colon \mathcal{G} \to 2\text{-}\mathcal{CAT}$ of 3-graphs satisfying (5.2). Also, recall from section 3 that an X-algebra is just a 3-functor $K\colon \mathcal{T} \to \mathcal{A}ct$ (such that the corresponding α is an $\{X_1;X_2\}$-lax -natural transformation); again such 3-functors are just 3-graph morphisms $K\colon \mathcal{G} \to 2\text{-}\mathcal{CAT}$ such that $\bar{K}P = \bar{K}Q$.

Denote by $\mathcal{H}$ the 2-graph of 1-cells, 2-cells, and 3-cells of $\mathcal{G}$, and denote by $\mathcal{N}$ the monoidal 2-category of 1-cells, 2-cells, and 3-cells of $F\mathcal{G}$. We denote by $E\colon \mathcal{N} \to \mathcal{M}$ the action of $E\colon F\mathcal{G} \to \mathcal{T}$ on 1-cells, 2-cells, and 3-cells; and further we denote by $\eta\colon \mathcal{H} \to \mathcal{N}$ the action of $\eta\mathcal{G}\colon \mathcal{G} \to F\mathcal{G}$ on 1-cells, 2-cells, and 3-cells.

Since G has only one object the universal property of the free 3-category FG may be restated as:

Lemma 5.1. *If $\mathcal{B}$ is a monoidal 2-category, then the equation $\kappa(G) = K = G.\eta$ sets up a bijection κ between monoidal 2-functors*

$$G: \mathcal{N} \to \mathcal{B}$$

and morphism of 2-graphs

$$K: \mathcal{H} \to \mathcal{B} \quad . \quad \square$$

If we denote by $\mathcal{K}$ the 2-graph of 1-cells, 2-cells, and 3-cells of $\mathcal{R}$, and denote by $P,Q: \mathcal{K} \to \mathcal{N}$ the action of P and Q on 1-cells, 2-cells, and 3-cells, then the coequaliser property of $E: FG \to \mathcal{T}$ may be restated as:

Lemma 5.2. *The equation $G = X.E$ sets up a bijection between monoidal 2-functors*

$$X: \mathcal{M} \to \mathcal{B}$$

and monoidal 2-functors

$$G: \mathcal{N} \to \mathcal{B}$$

satisfying

$$GP = GQ \quad . \quad \square$$

Therefore, by combining Lemma 5.1 and 5.2, we have:

Lemma 5.3. The equation $K = \gamma(X) = X.E.\eta$ sets up a bijection between monoidal 2-functors

$$X\colon \mathcal{M} \to \mathcal{B}$$

and morphisms of 2-graphs

$$K\colon \mathcal{H} \to \mathcal{B}$$

satisfying

$$\kappa^{-1}(K)P = \kappa^{-1}(K)Q \quad . \qquad \square$$

Recall that, because (4.1) is a lax-comma object for lax-natural transformations of 2-graphs as well as for 2-categories (cf. Chapter 0), there is for all 2-graphs $\mathcal{Y}$ a bijection ψ between lax-natural transformations α of 2-graphs as in

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "Z"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\alpha"] & {[A,A]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, "\ulcorner B \urcorner"'] & A \end{tikzcd}$$

and morphisms of 2-graphs

$$W\colon \mathcal{Y} \to \mathcal{A}ct(A,B)$$

where $\alpha = \psi^{-1}(W) = W.\lambda$ (see (4.1) for the definition of λ). Thus we have:

Lemma 5.4. *The equation* $\beta = \nu(\alpha) = \alpha.\eta$ *sets up a bijection* ν *between lax-natural transformations* α *as in*

$$\begin{tikzcd} N \arrow[r, "G"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\alpha"] & {[A,A]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, "\ulcorner A \urcorner"'] & A \end{tikzcd}$$

satisfying the unit and associativity axioms corresponding to (2.6) and (2. 7), and lax-natural transformations β *of 2-graphs as in*

$$\begin{tikzcd} H \arrow[r, "\kappa(G)"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow\beta"] & {[A,A]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, "\ulcorner A \urcorner"'] & A \end{tikzcd} .$$

Proof. Define $\beta = \psi\kappa\psi^{-1}(\alpha)$ and observe that $\kappa\psi^{-1}(\alpha)$ makes sense if and only if $\psi^{-1}(\alpha)\colon N \to \mathcal{A}ct(A,A)$ is a strict monoidal 2-functor; however, this is equivalent to α satisfying the analogues of (2.6) and (2. 7). From the naturality of ψ and the definition of κ we see that $\beta = \psi\psi^{-1}(\alpha.\eta) = \alpha.\eta$. $\square$

<u>Lemma 5.5</u>. <u>The equation</u> $\beta = \mu(\alpha) = \alpha.E.\eta$ <u>sets up a bijection</u> μ <u>between lax-natural transformations</u> α <u>as in</u>

$$\begin{tikzcd} \mathcal{M} \arrow[r, "M"] \arrow[d, "!"'] \arrow[dr, phantom, "\alpha"] & {[\mathcal{A},\mathcal{A}]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, "\ulcorner A \urcorner"'] & \mathcal{A} \end{tikzcd}$$

<u>satisfying</u> (2.6) <u>and</u> (2.7) <u>and lax-natural transformations</u> β <u>of</u> 2-<u>graphs as in</u>

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\gamma(M)"] \arrow[d, "!"'] \arrow[dr, phantom, "\beta"] & {[\mathcal{A},\mathcal{A}]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, "\ulcorner A \urcorner"'] & \mathcal{A} \end{tikzcd}$$

<u>satisfying</u> $\nu^{-1}(\beta)P = \nu^{-1}(\beta)Q$.

<u>Proof</u>. Notice that $\beta = \alpha.E.\eta$ for some α satisfying (2.6) and (2.7) if and only if $\psi^{-1}(\beta): \mathcal{H} \to \mathcal{Act}(A,A)$ is equal to $\gamma(N)$ for some strict monoidal 2-functor $N: \mathcal{M} \to \mathcal{Act}(A,A)$. However, the latter is the case if and only if

$$\kappa^{-1}\psi^{-1}(\beta)P = \kappa^{-1}\psi^{-1}(\beta)Q ,$$

or equivalently

$$\psi\kappa^{-1}\psi^{-1}(\beta)P \;=\; \psi\kappa^{-1}\psi^{-1}(\beta)Q$$

which is precisely $\nu^{-1}(\beta)P = \nu^{-1}(\beta)Q$. □

A <u>presentation of a polyad</u> is a triple $\mathcal{L} = (L, \mathcal{L}_1, \mathcal{L}_2)$ where $L\colon \mathcal{H} \to [\mathcal{A},\mathcal{A}]$ is a 2-graph morphism such that

$$\kappa^{-1}(L)P \;=\; \kappa^{-1}(L)Q$$

and where $\mathcal{L}_1$ and $\mathcal{L}_2$ are subgraphs of the 2-graph $\mathcal{N}$.

An $\mathcal{L}$-<u>algebra</u> is a pair (A,α) where $A \in \mathcal{A}$ and α is a lax-natural transformation of 2-graphs as in

(5.3)
$$\begin{tikzcd} \mathcal{H} \arrow[r, "L"] \arrow[d, "!"'] \arrow[dr, phantom, "\rightsquigarrow\alpha"] & {[\mathcal{A},\mathcal{A}]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, "\ulcorner A \urcorner"'] & \mathcal{A} \end{tikzcd}$$

such that $\nu^{-1}(\alpha)$ is an $\{\mathcal{L}_1;\mathcal{L}_2\}$-lax-natural transformation and such that

(5.4) $$\nu^{-1}(\alpha)P \;=\; \nu^{-1}(\alpha)Q \;.$$

An L-morphism from (A,α) to (B,β) is a morphism $f\colon A \to B$ in A such that

(5.5)

$$\begin{tikzcd} H \arrow[r, "L"] \arrow[d, "!"'] \arrow[dr, phantom, "\alpha"] & {[A,A]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, bend left, "\ulcorner A \urcorner"] \arrow[r, bend right, "\ulcorner B \urcorner"'] \arrow[r, phantom, "\Downarrow \ulcorner f \urcorner"] & A \end{tikzcd} \quad = \quad \begin{tikzcd} H \arrow[r, "L"] \arrow[d, "!"'] \arrow[dr, phantom, "\beta"] & {[A,A]} \arrow[d, bend right, "ev_B"'] \arrow[d, bend left, "ev_A"] \arrow[d, phantom, "\Leftarrow ev_f"] \\ \mathbb{1} \arrow[r, bend right, "\ulcorner B \urcorner"'] & A \end{tikzcd} .$$

An L-2-cell from f to g is a 2-cell $\rho\colon f \Rightarrow g\colon A \to B$ in A such that the obvious analogue to (5.3) is satisfied, namely

$$(5.6) \qquad \rho! \,.\, \alpha = ev_f L.\beta \ .$$

These definitions clearly give us a 2-category $L\text{-}\mathcal{A}\ell g_*$ together with an evident forgetful 2-functor $V\colon L\text{-}\mathcal{A}\ell g_* \to A$.

If L is a presentation of a polyad we define the polyad $X = (X, X_1, X_2)$ by setting $X = \gamma^{-1}(L)$ and letting X_1 and X_2 be the smallest monoidal subcategories of M containing the images of

$$L_1 \hookrightarrow N \xrightarrow{E} M \quad \text{and} \quad L_2 \hookrightarrow N \xrightarrow{E} M$$

respectively; it is clear that X_1 and X_2 exist since M is small. We call X <u>the polyad generated by</u> L, or we say that L is <u>a presentation of the polyad</u> X.

Since we will in practice often have only an explicit description of the presentation L of a polyad X, and not an explicit description of X itself, it will be useful to think of the presentation L as <u>being</u> the polyad. Consequently whenever, in future, we refer to the polyad $X = (X, X_1, X_2)$ we mean either that X is a polyad as defined in section 2 or that $X = (X, X_1, X_2)$ is the presentation of a polyad as defined above. Furthermore when we speak of (A,α) being an X-algebra we mean that (A,α) is an X-algebra as in section 2 when X actually is a polyad as in section 2, but that (A,α) is an algebra for the presentation X when X is only a presentation of a polyad. The result we need to make this usage consistent is:

<u>Theorem 5.6</u>. *If $L = (L, L_1, L_2)$ is a presentation of the polyad $X = (X, X_1, X_2)$ on A, then there is an isomorphism of 2-categories $\Sigma \colon X\text{-}\mathcal{A}\ell g_* \xrightarrow{\cong} L\text{-}\mathcal{A}\ell g_*$ such that*

$$\begin{tikzcd} X\text{-}\mathcal{A}\ell g_* \arrow[rr, "\Sigma"] \arrow[dr, "V"'] & & L\text{-}\mathcal{A}\ell g_* \arrow[dl, "V"] \\ & A & \end{tikzcd}$$

commutes.

<u>Proof</u>. If (A,α) is an X-algebra we define ΣA to be $(A, \mu(\alpha))$ where μ is the bijection of Lemma 5.5. To show that this definition makes sense we must show that $(A, \mu(\alpha))$ is an

L-algebra whenever (A,α) is an X-algebra; what we in fact show is that $(A,\mu(\alpha))$ is an L-algebra if and only if (A,α) is an X-algebra; thus establishing that Σ is a bijection between the objects of $X\text{-}\mathcal{A}\ell g_*$ and those of $L\text{-}\mathcal{A}\ell g_*$.

Let C be the comma-object, in 2-CAT, of $\ulcorner A \urcorner\colon \mathbb{1} \to A$ and $ev_A\colon [A,A] \to A$ as in

$$\begin{tikzcd} C \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\Downarrow\lambda"] & {[A,A]} \arrow[d, "ev_A"] \\ \mathbb{1} \arrow[r, "\ulcorner A \urcorner"'] & A \end{tikzcd}.$$

From the universal property of the lax-comma object $\mathcal{Act}(A,A)$ we have a 2-functor $J\colon C \to \mathcal{Act}(A,A)$ which is in fact an inclusion of a non-full sub-2-category (as can be seen by considering an elementary description of the 2-category $C = ev_A/\ulcorner A \urcorner$). In fact we can easily see, again by the elementary description of C, that C is closed under the monoidal structure of $\mathcal{Act}(A,A)$; so that C is a monoidal 2-category and the inclusion J is a strict monoidal 2-functor.

From the universal property of the comma object C we see that the α of (A,α), is 2-natural when restricted to X_1 if and only if $X_1 \hookrightarrow M \xrightarrow{\phi(\alpha)} \mathcal{Act}(A,A)$ factors through the 2-functor $J\colon C \to \mathcal{Act}(A,A)$. On the other hand, since colimits in 2-CAT are really computed in 2-$GRAPH$ we see that (4.1) is a comma object in 2-$GRAPH$; so that $\mu(\alpha)$

is 2-natural when restricted to $\mathcal{L}_1$ if and only if the morphism

$$\mathcal{L}_1 \xrightarrow{E.L_1} \mathcal{M} \xrightarrow{\phi(\alpha)} \mathcal{A}\!\mathit{ct}(A,A)$$

of 2-graphs factors through the 2-functor $J\colon \mathcal{C} \to \mathcal{A}\!\mathit{ct}(A,A)$. Finally, because $\mathcal{X}_1$ is the smallest monoidal sub-category of $\mathcal{M}$ containing the image of $\mathcal{L}_1 \xrightarrow{L_1} \mathcal{N} \xrightarrow{E} \mathcal{M}$, we observe that, for <u>any</u> strict monoidal inclusive $J\colon \mathcal{C} \to \mathcal{B}$ and <u>any</u> strict monoidal 2-functor $G\colon \mathcal{M} \hookrightarrow \mathcal{B}$, $\mathcal{X}_1 \to \mathcal{M} \xrightarrow{G} \mathcal{B}$ factors through $J\colon \mathcal{C} \to \mathcal{B}$ if and only if $\mathcal{L} \xrightarrow{EL_1} \mathcal{M} \xrightarrow{G} \mathcal{B}$ factors through J.

To see that $\mu(\alpha)$ is pseudo on $\mathcal{L}_2$ if and only if α is pseudo on $\mathcal{X}_2$, use a similar argument with $\mathcal{C}$ replaced by the pseudo-comma object of $\ulcorner A \urcorner\colon \mathbb{1} \to A$ and $ev_A\colon [A,A] \to A$.

To define Σ on 1-cells and 2-cells we observe that $\rho\colon f \Rightarrow g : A \to B$ is an $\mathcal{X}$-2-cell from $A = (A,\alpha)$ to $B = (B,\beta)$ if and only if it is an $\mathcal{L}$-2-cell from ΣA to ΣB. For 1-cells we observe that $f\colon A \to B$ is an $\mathcal{X}$-1-cell if and only if f constitutes a 3-natural transformation from K_α to K_β (cf. section 4). However from the universal property of the free-3-category at the level of 3-natural transformations (cf. Chapter 0), this is equivalent to f being a 3-natural transformation of 3-graph morphisms as in

$$\begin{tikzcd} \mathcal{G} \arrow[rr, bend left, "K_\alpha.E.\eta G"] \arrow[rr, bend right, "K_\beta.E.\eta G"'] \arrow[rr, phantom, "\Downarrow f"] & & \mathit{2\text{-}CAT} \end{tikzcd} ,$$

which is clearly equivalent to the equality (5.5); just recall that $H = G(*,*)$. □

If we say that L has rank whenever $L\colon H \to [A,A]$ factors through $[A,A]_*$, then X has rank whenever L has rank. If A is complete and cocomplete we see that $V\colon L\text{-}A\ell g_* \to A$ is 2-monadic if L has rank; that is, Theorem 3.7 remains valid when we use our new and wider meaning of the term polyad. To stress this fact we restate Theorem 3.7 as:

Theorem 5.7. If X is a ranked polyad on a complete and cocomplete 2-category, then $V\colon X\text{-}A\ell g_* \to A$ is 2-monadic, and morevoer the 2-monad has a rank. □

Before leaving this section we remark that we could also perform an analysis of monads on X to determine what they are in terms of the presentation L. What we would find is that composition with $E.\eta\colon H \to M$ induces a bijection between monads (T,τ) on X and $\{L_1;L_2\}$-lax-natural transformations σ as in

$$\begin{tikzcd} H \arrow[rr, "L"] \arrow[dr, "!"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow\sigma"] & & {[A,A]} \\ & \mathbb{1} \arrow[ur, "T"'] & \end{tikzcd}$$

satisfying $\sigma^* P = \sigma^* Q$ (where σ^* is a lax-natural transformation

$$\begin{tikzcd} N \arrow[rr, "L"] \arrow[dr, "!"'] \arrow[rr, phantom, shift right=4ex, "\Downarrow \sigma^*"] & & {[A,A]} \\ & \mathbb{1} \arrow[ur, "T"'] & \end{tikzcd}$$

determined uniquely by σ). We refrain from giving the details of such an investigation since these results have no direct bearing on the question of the monadicity or the description of X-$A\ell g_*$.

6. In this section we consider three examples of polyads; two of them on a 2-category A and one of them on a category A (thought of as a 2-category).

1. Let the 3-category T be defined by putting M equal to the monoidal category Δ of finite ordinals and order preserving maps. Recall(cf. Mac Lane ([14] page 163) that a strict monoidal functor X: $\Delta \to [A,A]$ is just a monoid in $[A,A]$, or in other words a doctrine D on A. If we set $X_1 = X_2 = \phi$ we get a polyad $X = (X, X_1, X_2)$. We leave to the reader the calculation that shows that a monad (T,τ) on X is just a lax-morphism of doctrines from D to T, and that the free monad on X is precisely what Kelly ([12] page 311) calls (D^*,H).

It is in fact the case that X-$\mathcal{A}\ell g_*$ is the 2-category that Kelly calls *Lax*-D-$\mathcal{A}\ell g_*$. While we can show this using the above polyad X, in doing so we would have to make use of the fact that Δ is generated by the morphisms i: $0 \to 1$ and m: $2 \to 1$ together with the obvious axioms. We therefore define another polyad L which generates X.

Let G be the 3-graph, with one object, defined as follows. Write $*$ for the object of G; the 1-cells of G are $0: * \to *$, $D: * \to *$, and $D_2: * \to *$, while the 2-cells are $i: 0 \to D$ and $m: D_2 \to D$. The relations given by R, P, and Q are: $0 = id_*$, $D \cdot D = D_2$, $m.iD = 1$, $m.Di = 1$, and $m.mD = m.Dm$. (If $L = (L, L_1, L_2)$ is a polyad for which $L_1 = L_2 = \phi$ then we observe the following: in defining L the only choice we have is in the values we give to L(D), L(i) and L(m); so that if we write D, i, and m for these values it is easy to see that L is precisely a doctrine on $\mathcal{A}$.)

If (A,α) is an $\mathcal{L}$-algebra we observe that α is defined completely once values are given for $\alpha_0, \alpha_D, \alpha_{D_2}, \alpha_i$, and α_m. If we denote α_D by $a\colon DA \to A$, we see that α_{D_2} must be the composite $D^2A \xrightarrow{Da} DA \xrightarrow{a} A$, while α_0 must be $1\colon A \to A$. Next we observe that α_i and α_m are 2-cells in $\mathcal{A}$ of the form

$$\begin{tikzcd} A \arrow[r, "iA"] \arrow[dr, "1"'] \arrow[r, phantom, shift right=4, "\Rightarrow\alpha_i"] & DA \arrow[d, "a"] \\ & A \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D^2A \arrow[r, "mA"] \arrow[d, "Da"'] \arrow[dr, phantom, "\Rightarrow\alpha_m"] & DA \arrow[d, "a"] \\ DA \arrow[r, "a"'] & A \end{tikzcd}$$

respectively. Finally observe that since α must respect the relations we have that a, α_i, and α_m satisfy precisely the conditions necessary to make (A,a,α_i,α_m) a lax-D-algebra. From this point is is an easy calculation to show that $\mathcal{L}\text{-}\mathcal{A}\ell g_* = \text{Lax-D-}\mathcal{A}\ell g_*$; thus showing that if D has rank and if $\mathcal{A}$ is complete and cocomplete, then Lax-D-$\mathcal{A}\ell g_*$ is 2-monadic over $\mathcal{A}$.

2. This example is concerned with the pseudo distributive laws of Kelly ([12], §5). Let the 3-graph G on one object be defined as follows. The 1-cells of G are e, D, D_2, D', D_2', a, b, x, and y; the 2-cells are i: e $\to$ D, m: $D_2 \to$ D, i': e $\to$ D', m': $D_2' \to$ D', p: a $\to$ b, u: x $\to$ y, and v: x $\to$ y; and the only 3-cell of G is π: u $\to$ v. The relations represented by P and Q are: $e = 1_*$, $D_2 = D.D$, $D_2' = D'.D'$, a = D'.D, b = D.D', x = D'DD, y = DD', (D,i,m) satisfies the monad axioms, (D',i',m') satisfies the monad axioms, and D,D',p, and π satisfy the axioms for a pseudo distributive law as on pages 324-326 of Kelly [12]. If we set $L_2 = \phi$ and let L_1 be the graph consisting of i, m, i', and m' then a polyad $L = (L, L_1, L_2)$ is precisely what Kelly ([12] §5) called a pseudo distirbutive law (except that we do not require that π be an isomorphism). It is then an easy matter to show that L-$\mathcal{A}\ell g_*$ is what, in the notation of Kelly [12], would be called $\tilde{D}$-$\mathcal{A}\ell g_*$. Since L has a rank if and only if both D and D' have a rank we see that $\tilde{D}$-$\mathcal{A}\ell g_*$ is 2-monadic if (i) A is complete and cocomplete, and (ii) both D and D' have a rank.

In §5.4 of [12] Kelly introduced the notion of a map K from the pseudo distirbutive law (D,D',p,π) to a doctrine D^*. It turns out that a pair (D_*, K) is nothing more than a monad on L, and that the initial such thing is just the free monad on L, which for cocomplete A exists whenever D and D' have a rank.

3. If A is a category we define a polyad L on A as follows (consider A as a trivial 2-category).

The 3-graph G has one object $*$, four 1-cells e, T_1, T_2 and T_3, and five 2-cells η_1: $e \to T_1$, η_2, η_3: $T_1 \to T_2$, μ: $T_2 \to T_1$, and θ: $T_2 \to T_3$. The relations represented by P and Q are: $e = 1_*$, $T_3 = T_1 . T_1$, $\eta_2 . \eta_1 = \eta_3 . \eta_1$, $\mu\eta_2 = \mu\eta_3 = id_{T_1}$, $\theta\eta_2 = T_1 . \eta_1$, and $\theta\eta_3 = \eta_1 T_1$. Finally let $L_2 = \phi$ and $L_1 = \phi$ and recall that since A is a category all lax-naturals landing in A are actually proper natural transformations. If $L = (L, L_1, L_2)$ is a polyad with L_1 and L_2 as above, and if we denote the object of G and its image under L by the same symbol, then we see that the polyad L is just a septuple $(T_1, T_2, \eta_1, \eta_2, \eta_3, \mu, \theta)$ satisfying the axioms listed above.

An algebra for the polyad L is easily seen to be a pair (Y,y) where $Y \in A$ and where y: $T_1 Y \to Y$ is a morphism in A satisfying

$$y . \eta_1 Y = 1_Y$$

and

(6.1)
$$\begin{tikzcd}
T_2 Y \arrow[r, "\theta Y"] \arrow[d, "\mu Y"'] & T_1^2 Y \arrow[r, "T_1 Y"] & T_1 Y \arrow[d, "y"] \\
T_1 Y \arrow[rr, "y"'] & & Y
\end{tikzcd}$$
;

while it is further clear that f: $Y \to Y'$ is an L-morphism from (Y,y) to (Y',y') if and only if we have commutativity in

$$\begin{tikzcd} T_1Y \arrow[r, "Tf"] \arrow[d, "y"'] & T_1Y' \arrow[d, "y'"] \\ Y \arrow[r, "f"'] & Y' \end{tikzcd} .$$

We call a polyad of the above form a _dyad_ on A; the property of dyads, that makes them significant enough to warrant a special name, is the following result .

Proposition 6.1. _If_ D _is a doctrine on_ CAT _as in Chapter 2 (that is_, D _has rank and_ Cat _is stable under_ D), _and if_ $A = (A,a)$ _is any_ D-_category for which the category_ A _is cocomplete in_ CAT, _then there exists a dyad_ L _on_ A _and an isomorphism of categories_ $\Sigma\colon L\text{-}\mathcal{A}\ell g_* \xrightarrow{\cong} D[A]$ _such that_

(6.2)
$$\begin{tikzcd} L\text{-}\mathcal{A}\ell g_* \arrow[rr, "\Sigma", "\cong"'] \arrow[dr, "V"'] & & D[A] \arrow[dl, "U"] \\ & A & \end{tikzcd}$$

commutes.

<u>Proof</u>. If A is a category we denote by $\{A,A\}$ the endo-2-functor of CAT that is the right Kan extension of $\ulcorner A \urcorner : \mathbb{1} \to CAT$ along itself (see Chapter 0 section 9 for details). It is well known that in this case

$$\{A,A\}(-) = [[-,A],A]$$

where $[-,-]$ is the internal-hom of CAT, and that for any $a: DA \to A$ in CAT the corresponding 2-natural transformation $\theta(a): D \Rightarrow \{A,A\}$ is such that the C-th component $\theta(a)_C: DC \to [[C,A],A]$ corresponds under the cartesian adjunction of CAT to the morphism

$$DC\times[C,A] \xrightarrow{1\times D} DC\times[DC,DA] \xrightarrow{\text{eval}} DA \xrightarrow{a} A \ .$$

Notice therefore, that for any $X \in A$, the diagrams

(6.3)
$$\begin{tikzcd} D\mathbb{1} \arrow[r, "D\ulcorner X\urcorner"] \arrow[d, "\theta(a)_{\mathbb{1}}"'] & DA \arrow[d, "a"] \\ {[A,A]} \arrow[r, "ev_X"'] & A \end{tikzcd}$$

and

(6.4)
$$\begin{tikzcd} D\mathbb{1} \arrow[r, "D^2\ulcorner X\urcorner"] \arrow[d, "\theta(a)_{\mathbb{1}}.m\mathbb{1}"'] & D^2A \arrow[d, "a.Da \,=\, a.m"] \\ {[A,A]} \arrow[r, "ev_X"'] & A \end{tikzcd}$$

commutes.

We define T_1 to be the colimit of the functor $\theta(a)_{\mathbb{1}}: D\mathbb{1} \to [A,A]$ and T_2 to be the colimit of $\theta(a)_{\mathbb{1}}.m\mathbb{1}: D^2\mathbb{1} \to [A,A]$. We define the natural transformations η_1, η_2, η_3, and μ as follows:

η_1 is the comparison map

$\mathrm{Colim}(\theta(a)_{\mathbb{1}}.i\mathbb{1}) \to \mathrm{Colim}(\theta(a)_{\mathbb{1}})$,

η_2 is the comparison map

$\mathrm{Colim}(\theta(a)_{\mathbb{1}}.m\mathbb{1}.Di\mathbb{1}) \to \mathrm{Colim}(\theta(a)_{\mathbb{1}}.m\mathbb{1})$,

η_3 is the comparison map

$\mathrm{Colim}(\theta(a)_{\mathbb{1}}.m\mathbb{1}.iD\mathbb{1}) \to \mathrm{Colim}(\theta(a)_{\mathbb{1}})$,

while μ is the comparison map

$\mathrm{Colim}(\theta(a)_{\mathbb{1}}.m\mathbb{1}) \to \mathrm{Colim}(\theta(a)_{\mathbb{1}})$.

If $X \in A$ we observe that by (6.3) and (6.4) we have

$$T_1(X) = \mathrm{colim}(D\mathbb{1} \xrightarrow{D\ulcorner X\urcorner} DA \xrightarrow{a} A)$$

and

$$T_2(X) = \mathrm{colim}(D^2\mathbb{1} \xrightarrow{D^2\ulcorner X\urcorner} D^2A \xrightarrow{a.Da} A)$$

with the corresponding colimit-cones denoted by

(6.5)

$$\begin{tikzcd} D\mathbb{1} \arrow[r, "D\ulcorner X\urcorner"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow \alpha X"] & DA \arrow[d, "a"] \\ \mathbb{1} \arrow[r, "\ulcorner T_1(X)\urcorner"'] & A \end{tikzcd}$$

and

(6.6)

$$\begin{tikzcd} D^2\mathbb{1} \arrow[r, "D^2\ulcorner X\urcorner"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow \beta X"] & D^2A \arrow[d, "a.Da"] \\ \mathbb{1} \arrow[r, "\ulcorner T_2(X)\urcorner"'] & A \end{tikzcd} .$$

Thus for any morphism $f: X \to X'$ in A the morphisms T_1f and T_2f are the unique morphisms satisfying

(6.7) $$T_1f.\alpha X = \alpha X'.Df$$

and

(6.8) $$T_2f.\beta X = \beta X'.D^2f$$

respectively. Furthermore, from the definitions above we have the equations

(6.9) $$\alpha X.i\mathbb{1} = \eta_1X$$

(6.10) $$\beta X.iD\mathbb{1} = \eta_2X.\alpha X$$

(6.11) $$\beta X.iD\mathbb{1} = \eta_3X.\alpha X$$

(6.12) $$\alpha X.m\mathbb{1} = \mu X.\beta X \quad ;$$

while from (6.7) we have

$$(6.13) \qquad T_1\eta_1X.\alpha X \;=\; \alpha T_1X.D(\eta_1X).$$

We now define θ to have for its X-component $\theta X\colon T_2X \to T_1^2X$ the unique such morphism satisfying

$$(6.14) \qquad \alpha T_1X.D\alpha X \;=\; \theta X.\beta X$$

induced by the cone

$$\begin{tikzcd}
D^2\mathbb{1} \arrow[rr, "D^2\ulcorner X\urcorner"] \arrow[d, "D!"'] \arrow[drr, phantom, "\Downarrow D\alpha X"] & & D^2A \arrow[d, "Da"] \\
D\mathbb{1} \arrow[rr, "D\ulcorner T_1(X)\urcorner"'] \arrow[d, "!"'] \arrow[drr, phantom, "\Downarrow \alpha T_1X"] & & DA \arrow[d, "a"] \\
\mathbb{1} \arrow[rr, "\ulcorner T_1{}^2(X)\urcorner"'] & & A
\end{tikzcd} .$$

The naturality of $\theta\colon T_2 \to T_1^2$ is easily seen.

To see that $L = (T_1,T_2,\eta_1,\eta_2,\eta_3,\mu,\theta)$ is actually a dyad use equations (6.7) to (6.14), the doctrine axioms for D, and the fact that αX and βX are colimit-cones for all $X \in A$. For example, to get the equation $\theta\eta_3 = \eta_1.T_1$ put $iD\mathbb{1}$ onto (6.14) to get

$$(6.15) \qquad \theta X.\beta X.iD\mathbb{1} \;=\; \alpha T_1X.D\alpha X.iD\mathbb{1} \;.$$

Because of the definitions of T_1 and T_2 we see that both T_1 and T_2 have a rank if $\theta(a)\mathbb{1}$: $D\mathbb{1} \to [A,A]$ actually factors through $[A,A]_*$, so that in this case the polyad or dyad L also has a rank. Thus if we say the action a: $DA \to A$ of the D-algebra (A,a) has a rank whenever $\theta(a)\mathbb{1}$ factors through $[A,A]_*$, then we have:

Proposition 6.2. If A = (A,a) is a D-category, if A is complete and cocomplete, and if the action of A has a rank, then U: D[A] $\to$ A is monadic and the monad in question has a rank.

Proof. The monadicity of U follows immediately from Proposition 6.1 and Theorem 5.7, as does the rank of the monad. □

It is clear, therefore, that D[A] is a cocomplete category, so that by Theorem 4.1 of Chapter 2 we have

Theorem 6.3. If A = (A,a) is a D-category, if A is complete and cocomplete, and if the action of A has a rank, then A is cocomplete in $\mathcal{D}$. □

The above result is of special relevance when D = Ko- for some club over finite sets (see Kelly [9]) since in this case it is easy to show that the action a: KoA $\to$ A of the K-category A has a rank if for each T $\in K$ the functor T(...): $A^n \to A$ has a rank in each variable. An immediate consequence is that any closed K-category is is cocomplete in *K-CAT* provided its underlying category is complete and cocomplete in *CAT*; so that in particular complete and cocomplete biclosed monoidal categories are necessarily cocomplete in *Mon-CAT*.

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A few applications of the 2-naturality of i, together with equation (6.9), gives $\eta_1 T_1 X.\alpha X$ for the right hand side of (6.15), while (6.11) gives the value $\theta X.\eta_3 X.\alpha X$ for the left-hand side; then as αX is a colimit-cone we have $\theta X.\eta_3 X = \eta_1 T_1 X$ as required.

We now define the functor Σ. If $X = (X,x)$ is an algebra for the dyad defined above it is easy to see that the diagram

$$\begin{tikzcd}[column sep=huge, row sep=large]
D\mathbb{1} \arrow[r, bend left=40, "D\ulcorner X\urcorner"] \arrow[d, "!"'] \arrow[dr, phantom, "\Downarrow \alpha X"] & DA \arrow[d, "a"] \\
\mathbb{1} \arrow[r, bend left=30, "\ulcorner T_1 X\urcorner"] \arrow[r, bend right=30, "\ulcorner X\urcorner"'] \arrow[r, phantom, "\Downarrow \ulcorner x\urcorner"] & A
\end{tikzcd}$$

represents a D-morphism from $\mathbb{1}$ to $A = (A,a)$ (that is, $(\ulcorner X\urcorner, \ulcorner x\urcorner, \alpha X)$ is a D-oid in A) and it is this object of D[A] that we define to be ΣX. On morphisms we define Σ to be the identity. In fact we leave it to the reader to prove that $f: X \to X'$ is an L-morphism from (X,x) to (X',x') if and only if $f: X \to X'$ is a morphism of D-oids from ΣX to $\Sigma X'$, thus showing that Σ is full and faithful. In view of this, to show that Σ is an isomorphism we need only show that Σ is bijective on objects; however, this is clear from the definition of T_1 and T_2. $\square$

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