On the dictionary between ergodic transformations, Krieger factors and ergodic flows

Author:
Wong, Sing Yan Robert

Publication Date:
1986

DOI:
https://doi.org/10.26190/unsworks/11025

License:
https://creativecommons.org/licenses/by-nc-nd/3.0/au/
Link to license to see what you are allowed to do with this resource.

Downloaded from http://hdl.handle.net/1959.4/66300 in https://unsworks.unsw.edu.au on 2023-10-14
ON THE DICTIONARY BETWEEN
ERGODIC TRANSFORMATIONS,
KRIEGER FACTORS AND ERGODIC FLOWS

by

Sing Yan, Robert Wong

A thesis submitted for the degree of
Doctor of Philosophy
at the University of New South Wales.

I hereby certify that the work contained in this thesis has not been submitted for a higher degree to any other university or institution.

(Signature) 8/3/86
(Date)
CONTENTS

Acknowledgements. ii

Abstract. iii

Notation. v

CHAPTER 0 Introduction and Prelimaries. 1

CHAPTER 1 The inverse stable range map. 31

CHAPTER 2 Finite invariant measures on flows. 41

CHAPTER 3 II_1 extensions of an equivalence relation. 48

REFERENCES. 73
ACKNOWLEDGEMENTS

I would like to thank my supervisors, Professor Gavin Brown and Dr. Colin Sutherland for their expert supervision; without which this thesis would have been impossible. Dr. Sutherland, with his wide scope of knowledge in Mathematics, has not only impressed me as one of the best Mathematicians I have ever met; but also with his patience and caring, has demonstrated to be an exemplary supervisor. Professor Brown has also helped me with obtaining the scholarship and postdoctoral work. I hope this thesis would represent some of my deep gratitude to them.

I would like to thank Helen, Linda and Stephen for their valuable help with the production of the thesis.

I would like to thank Alex, Hang Fai, Ivan, Kevin, Milan, Peter, and all my friends in Australia, whose friendship and hospitality have made life smooth and pleasurable.

The work of the thesis was supported by the Dean's Scholarship of the University of N.S.W.

This thesis is dedicated to my mother, my brother Chung Yan and Helen.
ABSTRACT

On the dictionary between ergodic transformations, Krieger factors and ergodic flows.

Krieger's theorem sets up a one to one correspondence between orbit equivalence classes of ergodic transformations, isomorphism classes of Krieger factors and conjugacy classes of ergodic flows. The theorem suggests the possibility of finding a correspondence between the properties of the three categories of objects. This thesis is concerned with this task, which is analogous to compiling a dictionary between the objects involved.

The correspondence of Krieger is set up by maps between the categories, one of which (from transformations to flows) is the stable range map. In Chapter 1, the inverse of the stable range map is constructed. This provides two way communication between the category of transformations and the category of flows.

Chapter 2 investigates the entry "invariant measure" on flows. It is found that this corresponds to the property of a Krieger factor containing a $\text{III}_1$ subfactor which is the range of a faithful normal conditional expectation.

In chapter 3 the notion of a $\text{II}_1$-pair is defined on the category of equivalence relations. The corresponding entries in the other categories are found; for flows, it corresponds to one flow being a factor of the other, with a kind of "relatively invariant" measure between them.
The proofs of the results require the knowledge of ergodic theory and operator algebras. One conclusion of the work is that research in this area is promising and that more knowledge in this area will contribute to the advancement of the fields both of ergodic theory and operator algebras.
### NOTATIONS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_\phi$</td>
<td>Modular automorphism group of the weight $\phi$.</td>
</tr>
<tr>
<td>$H_\phi$</td>
<td>Centralizer of the weight $\phi$.</td>
</tr>
<tr>
<td>$d\mu/d\nu$</td>
<td>Radon-Nikodym derivative.</td>
</tr>
<tr>
<td>$\phi^*\nu$</td>
<td>Transport of the measure $\nu$ by $\phi$.</td>
</tr>
<tr>
<td>$[D\psi:D\phi]_t$</td>
<td>Connes Radon-Nikodym cocycle.</td>
</tr>
<tr>
<td>$\Delta, /$</td>
<td>Symmetric difference and difference of sets.</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>The set of real numbers.</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>The set of natural numbers.</td>
</tr>
<tr>
<td>$\mathbb{Z}$</td>
<td>The set of integers.</td>
</tr>
<tr>
<td>$\mathbb{C}$</td>
<td>The set of complex numbers.</td>
</tr>
<tr>
<td>$ds$</td>
<td>Lebesgue measure.</td>
</tr>
<tr>
<td>$\epsilon$</td>
<td>Is an element of.</td>
</tr>
<tr>
<td>$\circ$</td>
<td>Composition of functions.</td>
</tr>
<tr>
<td>$\text{Re}$</td>
<td>Real part of a complex number.</td>
</tr>
<tr>
<td>$\mathbb{R}^{(0)}$</td>
<td>Unit space of $\mathbb{R}$.</td>
</tr>
<tr>
<td>$\mathbb{R}</td>
<td>A$</td>
</tr>
<tr>
<td>$s,r$</td>
<td>Source and range maps.</td>
</tr>
<tr>
<td>$\mathbb{R}_G$</td>
<td>Equivalence relation generated by $G$.</td>
</tr>
<tr>
<td>$X_n$</td>
<td>Infinite product of the cyclic group of order $n$.</td>
</tr>
<tr>
<td>$\text{Ad}_U$</td>
<td>Automorphism induced by the unitary $U$.</td>
</tr>
<tr>
<td>$\ell^2(\mathbb{Z},\mathbb{H})$</td>
<td>Hilbert space of square integrable functions from $\mathbb{Z}$ to the Hilbert space $\mathbb{H}$.</td>
</tr>
<tr>
<td>~</td>
<td>Equivalence of measures.</td>
</tr>
<tr>
<td>$\simeq$</td>
<td>Is isomorphic to.</td>
</tr>
<tr>
<td>$\otimes$</td>
<td>Tensor product.</td>
</tr>
<tr>
<td>$H^2(\mathbb{R},\mathbb{T})$</td>
<td>Second cohomology group of $\mathbb{R}$ with coefficients in the circle group $\mathbb{T}$.</td>
</tr>
</tbody>
</table>
CHAPTER 0
Introduction and Preliminaries

This chapter is divided into two parts, an introduction and a preliminary part. In the introduction, a brief summary of each chapter is given. The preliminaries provide a more detailed explanation of and references to the non-elementary materials used in the thesis. However, because of the complexity of some of the mathematical structures, there is a limit as to how far a preliminaries can get. In the case that the preliminary knowledge is contained in some other work, the reader is sometimes referred to that work. The preliminary part is divided into sections which are numbered according to the chapter they refer to.

(A) Introduction

The Krieger theorem [0.2.3] asserts that in the type $\text{III}_0$ case, there is a one to one correspondence between orbit equivalence classes of ergodic transformations, algebraic isomorphism classes of Krieger factors and conjugate isomorphism classes of non-transitive ergodic flows. This correspondence is established through explicit maps between the objects, namely the stable range map, the crossed product and the flow of weights. Since this theorem was proved in the early 70's, very few results have been obtained in the direction of setting up a correspondence between the properties of the objects in these three categories, or to put it metaphorically, of compiling a dictionary of the three languages. The work of Connes and Woods on the equivalence of ITPFL factors and the A.T. property of flows [13] is perhaps the only major achievement in this direction. In this thesis,
we are concerned with expanding the present dictionary. One result obtained is the establishment of a very explicit inverse map of the stable range map. This will facilitate future research in this area.

This is accomplished in chapter one, where the following question is asked: Given a flow (not necessarily ergodic) on a standard measure space, how can we find a dynamical system whose stable range is the flow? Krieger’s theorem tells us that orbit equivalent transformations give rise to conjugate flows under the stable range, so that the choice of the dynamical system is not unique. However, by using some kind of skew product with the odometer, it is possible to choose a “natural” dynamical system which satisfies the requirement (Theorem 1.4).

We note that for the case the flow admits an invariant measure, such a map has been found in Hamachi and Osikawa [20] but for a general flow the problem is more difficult.

This result strengthens the link between the category of transformations and the category of flows, thus making the compilation of our dictionary easier. This will be seen when the result is applied in chapter 2 and chapter 3.

The stable range map is in fact a functor from the category of transformations with orbit transporting isomorphisms to the category of flows with conjugations. Theorem 1.6 gives the proof that the inverse stable range construction is a functor and that it is a right inverse functor to the stable range functor. It should be mentioned that a theorem of Bezuglyi and Golodet [3] concerning the surjectivity
of the 'mod' map is a direct consequence of Theorem 1.6.

In chapter 2 we characterize those Krieger factors whose flow of weights admit a finite invariant measure, thus creating a new entry in the dictionary. We see that they are factors which contain a III₁ subfactor which is the range of a faithful normal conditional expectation. This III₁ subfactor, being injective, is hyperfinite by a fundamental theorem of Connes [8] A theorem of Haagerup [19] then says it is unique up to isomorphisms.

It is not known which Krieger factors have their flows of weights admitting an infinite invariant measure. The question seems much harder to answer.

Chapter 3 gives one possible generalization of the ideas of chapter 2. We consider a new kind of extension of a flow, which we call a II₁ extension. Let \( G_t \) be a flow on a standard measure space \((\Omega,\nu,\mathcal{F})\). A flow \((F_t,\Omega,\nu')\) is said to be a II₁ extension of \((G_t,\Omega,\nu)\) if there exists finite measures \(\nu,\nu'\) equivalent to \(\nu,\nu'\), a surjective map \(\pi: \Omega \rightarrow \Omega\) which transports the measure \(\nu\) to \(\nu'\) and such that

\[
\begin{align*}
(1) \quad \pi_0 F_t &= G_t \circ \pi, \text{ for all } t \quad \text{and} \\
(2) \quad \frac{d\nu}{d\nu'}(w) &= \frac{d\nu}{d\nu'}(w), \quad \text{a.e. } w \in \Omega.
\end{align*}
\]

We shall say \((F_t,G_t)\) is a II₁ pair and investigate the properties corresponding to II₁ extensions in the category of equivalence relations and the category of Krieger factors. We use equivalence relations rather than transformations because they correspond to orbit equivalence classes of transformations, and are in more direct association with the category of flows and Krieger factors. We shall
find out that this notion of a II$_1$ pair corresponds to an analogous notion in the category of equivalence relations, which we also call a II$_1$ pair. This in turn corresponds to one Krieger factor being contained in the other in a nice way, with a faithful normal conditional expectation between them and some relation between their Cartan subalgebras. (Theorem 3.18)

Observe that a flow admits a finite invariant measure if and only if it is a II$_1$ extension of the trivial flow on a one point space. This provides the link between chapter 2 and chapter 3.
0.1. Basic definitions.

In this thesis, the term "standard measure spaces" will mean non-atomic standard measure spaces, or what is the same, Lebesgue measure spaces.

Two measure spaces \((X, \mu), (Y, \nu)\) are isomorphic if there exists a bimeasurable map (i.e., a measurable map which has a measurable inverse) \(\phi: X \rightarrow Y\) such that \(\phi^\ast \mu = \nu\), where \(\phi^\ast \mu\) is the transport of the measure \(\mu\) by \(\phi\), defined by \(\phi^\ast \mu(A) = \mu(\phi^{-1}(A))\) for \(A\) measurable in \(Y\). \(\phi\) is called an isomorphism between \((X, \mu)\) and \((Y, \nu)\). If the measure spaces are the same, such a map is also called an automorphism of the measure space.

A standard measure space, or a Lebesgue space, is a measure space isomorphic to the interval \([0,1)\) with the Lebesgue measure. We will write \((X, \mu)\) for such a space, suppressing the reference to the \(\sigma\)-algebra of measurable sets concerned.

A measurable transformation \(T\) on a standard measure space \((X, \mu)\) is nonsingular if for all measurable sets \(A\), \(\mu(A) = 0\) implies \(\mu(T^{-1}(A)) = 0\). We also say that \(\mu\) is quasi-invariant for \(T\), and in the case that \(T^\ast \mu = \mu\), \(\mu\) is called an invariant measure.

All transformations in this thesis will be automorphisms. We will use \((T, X, \mu)\) to denote an automorphism \(T\) on the standard measure space \((X, \mu)\), and call it a dynamical system.
A measurable set $A$ is invariant under $T$ if $\mu(A \Delta T(A)) = 0$.

$T$ is said to be ergodic if for all invariant sets $A$, either $\mu(A) = 0$ or $\mu(X/A) = 0$.

By a flow we mean a nonsingular Borel action of the real line on a standard measure space, i.e. if the space is $(\Omega, \nu)$, a one parameter group of automorphisms $F_t$ of $(\Omega, \nu)$ such that the map $(t, w) \mapsto F_t w$ from $\mathbb{R} \times \Omega \rightarrow \Omega$ is measurable.

Two flows $(F_t, \Omega, \nu), (G_t, \Omega', \nu')$ are conjugate if there exists an isomorphism $\pi: (\Omega, \nu) \rightarrow (\Omega', \nu')$ such that $\pi \circ F_t = G_t \circ \pi$ for all $t$. $\pi$ is called a conjugation, or a conjugate isomorphism between the flows.

Two transformations $(T, X, \mu), (S, Y, \nu)$ are orbit equivalent if there exists an isomorphism $\phi: (X, \mu) \rightarrow (Y, \nu)$ such that

$$\phi(0_T x) = 0_S (\phi(x)) \quad \mu \text{ a.e. } x$$

where $0_T x = \{T^m x : m \in \mathbb{Z}\}$, and similarly for $0_S$. Such an isomorphism will be called an orbit transporting isomorphism.

Let $(T, X, \mu)$ be a dynamical system. The full group of $T$, denoted by $[T]$, is the group of automorphisms $\alpha$ of $(X, \mu)$ such that there exists an integer valued measurable function $n$ on $X$, with

$$\alpha(x) = T^n(x)(x) \quad \mu \text{ a.e. } x \in X.$$ 

Finally a note on quantification. We often use the phrase "for all $t \in \mathbb{R}$, a.e. $x \in X". This is understood to mean "for each $t \in \mathbb{R}$, there is a conull set in $X$, and for $x$ belonging to this conull set".
We do not mean a single conull set works for all $t$. The order of quantification would have been reversed if this were the case.

0.1.1. Classification of ergodic transformations.

The following materials can be found in [20] or [30].

Ergodic transformations on standard measure spaces are classified up to orbit equivalence according to their Krieger ratio set $r(T)$. For $(T, X, \mu)$, it is defined by

$$r(T) = \{ \lambda > 0 : \text{for all non-null measurable sets } A, \text{ and all } \varepsilon > 0, \text{ there exists an integer } n \text{ and a non-null measurable set } B \subseteq A \text{ with } T^n(B) \subseteq A \text{ and}$$

$$|\frac{d\mu_T^n}{d\mu}(x) - \lambda| < \varepsilon \quad \text{a.e. } x \text{ in } B\}$$

$r(T) \setminus \{0\}$ can be shown to be a closed subgroup of $\mathbb{R}^+$. Hence it is either

(a) $\mathbb{R}^+$, or
(b) $\{\lambda^m : m \in \mathbb{Z}\}$ for some $0 < \lambda < 1$, or
(c) $\{1\}$.

$T$ is said to be of type II if $r(T) = \{1\}$. This is equivalent to the existence of a measure equivalent to $\mu$, which is invariant under $T$. If this invariant measure is finite, $T$ is said to be of type $II_1$, otherwise it is infinite and $T$ is of type $II_\infty$.

$T$ is said to be of type III if it is not of type II. It is of type $III_1$, $III_\lambda$ or $III_0$ according respectively to which of the cases (a), (b) or (c) $r(T) \setminus \{0\}$ belongs.
The type of a transformation is an invariant of orbit equivalence classes. There is only one orbit equivalence class for each of the \( \text{II}_1 \), \( \text{II}_\infty \) cases [15], and \( \text{III}_1 \), \( \text{III}_\lambda \) (\( 0 < \lambda < 1 \)) [21], [22] [23]. For the \( \text{III}_0 \) case, there is an uncountable number of classes [1].

**0.1.2. Point realization.**

We define a flow \( \alpha \) on an abelian von Neumann algebra \( A \) to be a continuous action of \( \mathbb{R} \) on \( A \), i.e. a homomorphism \( \alpha: \mathbb{R} \rightarrow \text{Aut}(A) \), the automorphism group of \( A \), such that, writing \( \alpha_t \) for \( \alpha(t) \), we have for any \( x \in A \), the map \( t \rightarrow \alpha_t(x) \) is \( \sigma \)-weakly continuous. (See 0.2.0 for some basic definitions about von Neumann algebras.)

We write \( (\alpha_t, A) \) for a flow on a von Neumann algebra. The flow is said to be ergodic if for a \( a \in A \), \( \alpha_t(a) = a \) for all \( t \) implies that \( a \) is a multiple of the identity. We denote by \( \mathcal{A} \) the category of ergodic flows on non-atomic abelian von Neumann algebras with separable preduals, with conjugations (i.e. isomorphisms which intertwine with the flows) as morphisms.

Let \( A, B \) be non-atomic abelian von Neumann algebras with separable preduals. Let \( \Phi: A \rightarrow B \) be an isomorphism. Suppose there are standard measure spaces \( (X, \mu), (Y, \nu) \), an isomorphism \( \hat{\Phi}: (X, \mu) \rightarrow (Y, \nu) \) and isomorphisms \( \sigma_1: A \rightarrow L^\infty(X, \mu), \sigma_2: B \rightarrow L^\infty(Y, \nu) \) such that

\[
(\sigma_2 \Phi)(a) = \sigma_1(a) \circ \hat{\Phi}^{-1}
\]

for all \( a \in A \). Then \( \hat{\Phi} \) is called a point realization of the map \( \Phi \) via \( \sigma_1 \) and \( \sigma_2 \).
Similarly if \((\alpha_t, A)\) belongs to \(\mathcal{A}\), and there exists a flow 
\((F_t, Q, \nu)\) and an isomorphism \(\sigma: A \rightarrow L^\infty(Q, \nu)\) such that \(F_t\) is a point 
realization of \(\alpha_t\) via \(\sigma\) for all \(t\), then \((F_t, Q, \nu)\) is called the point 
realization of \(\alpha_t\) via \(\sigma\). The reason we speak of the point realization 
is the following:

**Theorem.** Let \(\phi, A, B\) be as above, \((X, \mu), (Y, \nu)\) be standard measure 
spaces, and \(\sigma_1: A \rightarrow L^\infty(X, \mu), \sigma_2: B \rightarrow L^\infty(Y, \nu)\) be any two 
isomorphisms. Then there exists a unique point realization of \(\phi\) via 
\(\sigma_1\) and \(\sigma_2\).

**Proof.** See Mackey [26].

**Theorem.** Let \((\alpha_t, A)\) be as above, and let \((Q, \nu)\) be any standard measure 
space and \(\sigma: A \rightarrow L^\infty(Q, \nu)\) be any isomorphism. Then there exists a 
unique point realization of \((\alpha_t, A)\) via \(\sigma\).

**Proof.** This is a corollary of [26].

Note that any two point realizations of \((\alpha_t, A)\) are conjugate 
isosomorphic. If \(\phi: A \rightarrow B, \psi: B \rightarrow C\) are morphisms in \(\mathcal{A}\), and \(\hat{\phi}, \hat{\psi}\) are 
point realizations of \(\phi\) and \(\psi\) via some fixed isomorphisms, then \(\hat{\phi} \circ \hat{\psi}\) 
is the point realization of \(\phi \circ \psi\) via these isomorphisms. Thus point 
realization preserves commutative diagrams.

### 0.1.3. The stable range map.

Let \((T, X, \mu)\) be a dynamical system. We can construct a flow, 
called the stable range, or the Poincaré flow, or the associated flow 
of the transformation as follows. Let \(ds\) be the Lebesgue measure on 
\(\mathbb{R}\). Define \(\tilde{T}\) on \((X \times \mathbb{R}, \mu \times e^{-s} ds)\) by
\[ T(x,r) = (Tx, r + \log \frac{d\mu \circ T}{d\mu}(x)) \]

\( T \) is in general non-ergodic even though \( T \) may be ergodic. There is an action \( \alpha_T \) on \( L^\infty(\mathbb{X} \times \mathbb{R}) \) induced from \( T \):

\[ \alpha_T(f)(x,r) = f(T^{-1}(x,r)), \quad f \in L^\infty(\mathbb{X} \times \mathbb{R}). \]

We consider the fixed point subalgebra of \( L^\infty(\mathbb{X} \times \mathbb{R}) \) under \( \alpha_T \), i.e. the algebra \( L^\infty(\mathbb{X} \times \mathbb{R})^T \) of functions \( f \) such that \( \alpha_T(f) = f \). There exists an action \( F \) of \( \mathbb{R} \) on \( L^\infty(\mathbb{X} \times \mathbb{R}) \) given by

\[ F_t(f)(x,r) = f(x,r+t), \quad t \in \mathbb{R}, \]

This action commutes with \( \alpha_T \), i.e. \( F_t \circ \alpha_T^{-1} = \alpha_T^{-1} \circ F_t \) for all \( t \), and therefore leaves \( L^\infty(\mathbb{X} \times \mathbb{R})^T \) invariant. The stable range of \( (T,X,\mu) \) is any point realization of the restriction of \( F_t \) to \( L^\infty(\mathbb{X} \times \mathbb{R})^T \). As has been noted in [1.2], it is unique up to conjugations. It is ergodic if and only if \( (T,X,\mu) \) is ergodic [20]. For some other views of the stable range map, see [20] or [28].

The stable range can be thought of as a conjugacy class of flows, but we will choose a representative for each ergodic dynamical system in the following.

The stable range map is in fact a functor from the category of ergodic dynamical systems with orbit transporting transformations to the category of ergodic flows with conjugations. First we define a functor \( S_1 \) from \( \mathcal{A} \) to \( \mathcal{F} \). Fix a standard measure space \((I,m)\). For \((\alpha_t,A)\) in \( \mathcal{A} \), choose an isomorphism \( \sigma : A \rightarrow L^\infty(I,m) \), and define \( S_1(\alpha_t,A) \) to be the point realization of \((\alpha_t,A)\) on \((I,m)\) via \( \sigma \).
Let $\phi: (\alpha_t, A) \longrightarrow (\beta_t, B)$ be a conjugation. $S_1(\phi)$ is defined to be the automorphism on $(I, m)$ which is the point realization of $\phi$ via $\sigma_A, \sigma_B$. It is clear that $S_1(\phi)$ conjugates between $S_1(\alpha_t, A)$ and $S_1(\beta_t, B)$ and that it is a functor.

Let $(T, X, \mu)$ be an ergodic transformation, then by the construction of the stable range, we obtain a flow $\alpha_t$ on the abelian von Neumann algebra $L^\infty(X \times \mathbb{R})^T$. This association is a functor $S_2$ from $\mathcal{X}$ to $\mathcal{A}$. For if $\phi: (T, X, \mu) \longrightarrow (S, Y, \nu)$ is orbit transporting, then

$$\tilde{\phi}: (X \times \mathbb{R}) \longrightarrow (Y \times \mathbb{R})$$

$$\tilde{\phi}(x, r) = (\phi(x), r + \log \frac{d\nu}{d\phi^*\nu}((\phi(x))))$$

is also orbit transporting from $\tilde{T}$ to $\tilde{S}$. Hence it restricts to the fixed point subalgebras. Clearly it conjugates between the flows so that $S_2$ is a functor. The stable range functor $\Phi$ is defined to be $S_1 \circ S_2$.

0.1.4. The odometer.

We will be dealing only with 3 point or 5 point odometers, but here we describe a general $n$ point odometer. Let $X_n = \bigoplus_{l=1}^{n} \mathbb{Z}$, where $\mathbb{Z} = \langle 0, 1, 2, \ldots, n-1 \rangle, \triangleright$ is the cyclic group of order $n$. For $x \in X_n$, define $T_x$ inductively by

$$(T_x)_1 = x_1 + 1,$$

and for $k > 1$, 

\[ (Tx)_k = \begin{cases} x_{k+1} & \text{if } x_{k-1} = n-1 \text{ and } (Tx)_{k-1} = 0 \\ x_k & \text{otherwise.} \end{cases} \]

T is called the odometer transformation (with obvious reference to the car odometer). Let \( q = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \) be a probability measure on \( \mathbb{Z}_n \) which assigns mass \( \alpha_i \) to point \( i \). The measure

\[
p = \prod_{i=1}^{\infty} q_i
\]
on \( \mathbb{X}_n \) is then quasi-invariant and ergodic \([27]\). We are interested in the case when \( \log(\alpha_i/\alpha_{i-1}) \), \( i = 1, 2, \ldots, n-1 \) are rationally independent. Then the odometer is of type III because its ratio set contains the multiplicatively rationally independent numbers \( \alpha_i/\alpha_{i-1}, i=1,2,\ldots,n-1 \) \([20]\). The three point and the five point odometers will be useful in constructing the inverse stable range map.

0.1.5. Dinkin's theorem.

We state this elementary theorem of measure theory:

Theorem. Let \( (\mathbb{X}, \mathcal{B}, \mu) \) be a measure space, and let \( \mathcal{G} \) be a collection of sets in \( \mathcal{B} \) such that

(I) \( \sigma(\mathcal{G}) \), the \( \sigma \)-algebra generated by \( \mathcal{G} \), is equal to \( \mathcal{B} \).

(II) \( \mathcal{G} \) is closed under finite intersections.

Suppose \( \nu \) and \( \mu \) are measures such that \( \nu(A) = \mu(A) \) for all \( A \in \mathcal{G} \), then \( \nu = \mu \).

Proof. see \([4]\).

0.1.6. Hewitt Savage 0-1 law.
Let $X = \prod_{0}^{n-1}$ be the infinite product of an $n$ point space. Let $\mu$ be the infinite product measure of a fixed measure on \{0,1,..n-1\}. Then we have

Theorem. The natural action of the group of finite permutations on $(X,\mu)$ is ergodic.

Note. If $\tau$ is a finite permutation, then the natural action is given by $(\tau(x))_k = x_{\tau^{-1}(k)}$.

Proof. See [6].

0.1.7. Natural isomorphism of functors.

Let $S, T : \mathcal{C} \rightarrow \mathcal{D}$ be covariant functors. A natural transformation $n : S \rightarrow T$ is a map which assigns to every object $A$ in $\mathcal{C}$ a morphism $n_A : S(A) \rightarrow T(A)$ such that for any morphism $f : A \rightarrow B$ in $\mathcal{C}$, we have

$$T(f) \circ n_A = n_B \circ S(f).$$

$S$ and $T$ are said to be naturally isomorphic if such $n$ exists and every $n_A$ is an isomorphism.
0.2.0. Basic definitions

The following materials can be found in [31], [29] and [14].

An involution on a Banach algebra $A$ is a map $x \mapsto x^*$ such that

\begin{align*}
(1) \quad (x^*)^* &= x \\
(2) \quad (x+y)^* &= x^* + y^* \\
(3) \quad (\alpha x)^* &= \overline{\alpha} x^* \\
(4) \quad (xy)^* &= y^* x^* \\
(5) \quad \|x^*\| &= \|x\|
\end{align*}

for all $\alpha \in \mathbb{C}$, $x$, $y \in A$. With an involution, $A$ is called a Banach $\ast$-algebra, or simply a $\ast$-algebra. $A$ is called a C$^*$-algebra if the extra condition

\begin{align*}
(6) \quad \|x^* x\| &= \|x\|^2
\end{align*}

is satisfied. If in addition, $A$ is the dual space of a Banach space $F$, then $A$ is called a von Neumann algebra. $F$ is unique and called the predual of $A$. A von Neumann algebra $A$ always contains the unit 1. The center of $A$ is denoted by $Z(A)$. If $Z(A) = \mathbb{C}1$, $A$ is called a factor.

A positive element of a C$^*$-algebra is an element of the form $x^* x$. The set of positive elements of $A$ is denoted by $A_+$. 

Let $A$, $B$ be C$^*$ algebras. A homomorphism $\phi : A \longrightarrow B$ is called a $\ast$-homomorphism if
\[ \phi(x^*) = \phi(x)^* \]

for all \( x \) in \( A \). If \( \phi \) is a \( * \)-homomorphism which is also an isomorphism, then it is called a \( * \)-isomorphism, or \( * \)-automorphism if \( A = B \). We write \( A \cong B \) if a \( * \)-isomorphism exists between them.

Note that \( B(H) \), the set of all bounded operators on a Hilbert space \( H \) is a \( C^* \)-algebra. A \( * \)-homomorphism \( \pi \) of \( A \) into \( B(H) \) is called a representation of \( A \). \( \pi \) is said to be faithful if \( \pi(x) = 0 \) implies \( x = 0 \).

Let \( Y \) be a subset of a \( C^* \)-algebra. We will write

\[ Y^* = \{ y^* : y \in Y \} \]

Let \( S \) be a subset of \( B(H) \) such that \( S = S^* \). The commutant of \( S \) in \( B(H) \) is defined to be the set of all operators in \( B(H) \) which commute with all elements of \( S \). It is always a von Neumann algebra. The double commutant of \( S \) is the commutant of the commutant of \( S \). If \( 1 \in S \), then the double commutant of \( S \) is the smallest von Neumann algebra containing \( S \).

It can be shown that \( A \) is a von Neumann algebra if and only if there exists a faithful representation \( \pi \) of \( A \) into \( B(H) \) for some \( H \) such that the double commutant of \( \pi(A) \) is itself \([31]\). Thus a von Neumann algebra can be defined as a \( * \)-algebra of operators on some Hilbert space, which is equal to its own double commutant.

If \( A \) has a predual \( F \), then the \( \sigma(A,F) \) topology, or the weak-* topology on \( A \) is called the \( \sigma \)-weak topology.
Let $A$, $B$ be von Neumann algebras on the Hilbert spaces $H$, $K$ respectively. The operators on $H \otimes K$ of the form

$$\sum_{i=1}^{n} x_i \otimes y_i$$

with $x_i \in A$, $y_i \in B$ form a $\ast$-algebra. The von Neumann algebra generated by them is called the tensor product of $A$ and $B$, denoted by $A \otimes B$. [see 14]

$A$ is called properly infinite if $A \otimes B(H) = A$ for any separable Hilbert space $H$.

A weight on a von Neumann algebra $M$ is a linear functional defined on the positive part of $M$ (i.e. the set $M_+$ of positive elements of $M$), taking values in the extended positive real numbers. A weight $\phi$ is called

- faithful if $\phi(x^* x) = 0$ implies $x = 0$;
- semifinite if $\{x : \phi(x^* x) < \infty\}$ is $\sigma$-weakly dense;
- normal if for each increasing net of positive elements $x_\alpha$ with limit $x$, $\lim_{\alpha} \phi(x_\alpha) = \phi(x)$.

We will use s.n.f to stand for "semifinite normal faithful".

A trace $\text{Tr}$ is a weight satisfying the property $\text{Tr}(x^* x) = \text{Tr}(xx^*)$ for all $x$. On $B(H)$ for $H$ separable, there exists a s.n.f trace, unique up to scalar multiples, which we denote by $\text{Trace}$.

Since the advent of the Tomita-Takesaki modular theory of weights on a von Neumann algebra, there have been introduced a great number of non-elementary concepts which we will encounter in chapters 2 and 3. We will not attempt to define them, but be contented with
characterizing them.

Let $\phi$ be a s.n.f. weight on a von Neumann algebra $M$ and put

$$\eta_\phi = \{x \in M : \phi(x^*x) < \infty\}$$

$$\eta_\phi^* = \{y \in M : y^* \in \eta_\phi\}$$

$$m_\phi = \eta_\phi^* \eta_\phi$$

the set of finite sums of elements of the form $x^*y$, where $x, y \in \eta_\phi$.

Then every element in $m_\phi$ is a linear combination of elements $x \in M_+$

with $\phi(x) < \infty$. Hence $\phi$ can be extended to a complex valued function on $m_\phi$.

The modular automorphism group $\{\sigma_t^\phi\}_{t \in \mathbb{R}}$ of $\phi$ is the unique group of $\ast$-automorphisms of $M$ such that

1. $\phi \sigma_t^\phi = \phi$ for all $t$

2. For any $x, y \in \eta_\phi \cap \eta_\phi^*$, there exists a bounded continuous function $f$ defined on the strip

$$I = \{z \in \mathbb{C} : 0 < \Re z < 1\}$$

and analytic in the interior such that

$$f(it) = \phi(x \sigma_t^\phi(y))$$

$$f(1 + it) = \phi(\sigma_t^\phi(y)x).$$

Condition (2) is called the KMS-condition.

The centralizer of $\phi$ is the von Neumann subalgebra
$M_\phi = \{ x : \sigma_t^\phi(x) = x \text{ for all } t \}$

$M_\phi$ always contains $Z(M)$, the center of $M$.

Let $\psi$ be another s.n.f. weight on $M$. There exists a unique strongly continuous mapping $t \rightarrow u_t$, $t \in \mathbb{R}$, such that

1. $u_t^* u_t = u_t u_t^* = 1$
2. $u_{t+s} = u_t^\phi(u_s)$
3. $u_{-t} = \sigma_t^\phi(u_t^*)$
4. $\sigma_t^\psi(x) = u_t \sigma_t^\phi(x) u_t^*$
5. For every $x \in \eta_\phi \cap \eta_\psi^*$, $y \in \eta_\phi \cap \eta_\psi^*$, there exists a bounded continuous function $f$ defined on the strip $I$ and analytic in the interior such that

$$f(it) = \psi(x u_t \sigma_t^\phi(y))$$
$$f(1 + it) = \psi(\sigma_t^\phi(y) u_t x)$$

See [p.47, 29] for details. This mapping is called Connes Radon-Nikodym cocycle associated with the weights $\phi$ and $\phi$ and is denoted by $[D\phi : D\phi]_t$.

Let $\phi_k$ be s.n.f. weights on the von Neumann algebras $M_k$, $k = 1, 2$.

There exists a unique n.s.f. weight $\phi = \phi_1 \Phi \phi_2$ on the algebra $M = M_1 \otimes M_2$ such that

1. $x_1 \in M_{\phi_1}, x_2 \in M_{\phi_2}$ implies $x_1 \otimes x_2 \in M_{\phi}$ and $\phi(x_1 \otimes x_2) = \phi_1(x_1)\phi_2(x_2)$
(2) $\phi_t = \phi_1 \otimes \phi_2$

$\phi$ is called the tensor product of the weights $\phi_1$ and $\phi_2$.

Now we describe an important construction which gave some of the first examples of non-trivial von Neumann algebras, a construction of Murray-von Neumann:

Let $T$ be a transformation on a standard measure space $(X, \mu)$. We will denote by $\mathcal{W}^*(T, X, \mu)$ the von Neumann algebra on $H = L^2(\mathbb{Z}, L^2(X, \mu))$ generated by the following elements:

(1) $\{\pi_T(a) : a \in L^\infty(X, \mu)\}$,

(2) $\{U_T\}$,

where

$\pi_T(a)\xi(m) = a_{\mu}T^m\xi(m),$

and

$U_T(\xi)(m) = \xi(m-1), \quad \xi \in H$

This von Neumann algebra is called the crossed product of $(T, X, \mu)$. Every element in it can be represented uniquely in the form

$x = \sum_{m \in \mathbb{Z}} \pi_T(a_m) U^m,$

where the sum is taken in the $\sigma$-weak topology [31].

The algebra $\mathcal{W}^*(T, X, \mu)$ is a factor if and only if $(T, X, \mu)$ is ergodic. Factors which can be obtained this way are called Krieger factors. (to be exact, non-type 1 Krieger factors)
A factor $M$ with separable predual is called hyperfinite if it contains an increasing sequence of subfactors $M_k$, each isomorphic to some full matrix algebra, such that $\bigcup_{k=1}^{\infty} M_k$ is $\sigma$-weakly dense in $M$.

Hyperfinite factors are important objects to study as they occur quite frequently in analysis. It is known that the Krieger factors are exactly the hyperfinite factors. (Connes [7], [8] for the $\text{III}_\lambda$, $\lambda \neq 1$ and the $\text{II}_\infty$ cases, Haagerup [19] for the $\text{III}_1$ case and von Neumann [see 14] for the $\text{II}_1$ case)

0.2.1 Classification of factors.

Factors can be classified into types I, II, and III. Type I factors are either matrix algebras or isomorphic to $B(H)$ for some Hilbert space $H$. Type II factors are factors $M$ admitting a n.s.f. trace $\tau$ whose values on the set of projections form a nondiscrete set (A projection is an element $e$ such that $e = e^2 = e^*$. Such a trace $\tau$ is unique up to a constant; if $\tau(1)^{<\omega}$, $M$ is of type $\text{II}_1$; otherwise $\tau(1) = \infty$ and $M$ is of type $\text{II}_\infty$.

Type III factors are those having no s.n.f trace. The Connes invariant $S(M)$ [9] further classifies them into $\text{III}_0$, $\text{III}_\lambda$ ($0 < \lambda < 1$) and $\text{III}_1$ factors.

By Krieger's theorem [0.2.3], two ergodic transformations are orbit equivalent if and only if their crossed products are isomorphic. Since the type of a transformation is an invariant of orbit equivalence, associated to each isomorphism class $C$ of Krieger factors is the type of the transformations whose crossed products belong to $C$. 
This gives rise to a classification of Krieger factors, which we will call the classification by Krieger ratio sets.

Theorem. For Krieger factors, the classification by Krieger ratio sets coincides with the classification by $S(M)$.

Proof. See [9].

0.2.2. Flow of weights.

From a properly infinite von Neumann algebra $M$, a flow can be constructed, called the flow of weights of $M$, whose conjugacy class depends only on the isomorphism class of $M$. There are many ways of constructing the flow of weights; here we describe a concrete construction of this flow for a type III $M$, which is contained in [13].

Let $H$ be the Hilbert space on which $M$ acts faithfully and let $\phi$ be a faithful normal state or a s.n.f. weight on $M$. We define on $L^2(R)$

$$(V_s f)(t) = f(t-s), \quad \text{for } s \in \mathbb{R}, f \in L^2(\mathbb{R}),$$

and

$$(\rho f)(t) = e^{t}f(t) \quad \text{for } f \in L^2(\mathbb{R}).$$

On $B(L^2(\mathbb{R}))$ define a s.n.f. weight by

$$\omega(x) = \text{Trace } (\rho x).$$

Let

$$\tilde{M} = M \otimes B(L^2(\mathbb{R})).$$
\[ \tilde{\phi} = \phi \circ \omega, \]
\[ \theta_s = \text{Ad} (1 \circ \psi_s), \]
\[ N = \tilde{M}_\psi. \]

Then \( \theta_s \) leaves \( N \) invariant globally. The flow of weights of \( M \) is the point realization of the restriction of \( \theta_s \) to the center of \( N \).

The flow of weights construction is a functor \( F \) from the category \( \mathcal{M} \) of Krieger factors with isomorphisms to the category \( \mathcal{S} \) [12].

### 0.2.3. Krieger's theorem

Let \( \mathcal{X}_0 \) denote the subcategory of \( \mathcal{X} \) consisting of \( \text{III}_0 \) ergodic systems, \( \mathcal{X}_0 \) the subcategory of \( \mathcal{M} \) consisting of \( \text{III}_0 \) Krieger factors; and \( \mathcal{S}_0 \) the subcategory of \( \mathcal{S} \) consisting of nontransitive ergodic flows.

**Theorem.** The restriction of each of the functors

\[ \phi: \mathcal{X} \longrightarrow \mathcal{S}, \]
\[ \omega^*: \mathcal{X} \longrightarrow \mathcal{M}, \]
\[ F: \mathcal{M} \longrightarrow \mathcal{S} \]

to \( \mathcal{X}_0, \mathcal{X}_0, \mathcal{M}_0 \) takes values in \( \mathcal{S}_0, \mathcal{M}_0, \mathcal{S}_0 \) respectively. Denoting their restrictions by the same symbols, we have

\[ \phi = F \circ \omega^*. \]

Each map induces a bijection between the sets of isomorphism classes of objects.

**Proof.** See [24].
0.2.4. Conditional expectations and Takesaki's theorem.

Let \( A \) be a \( C^* \) algebra, and \( B \subseteq A \) be a unital subalgebra.

A linear map \( E : A \rightarrow B \) is called a conditional expectation if

(i) \( E(b^*) = b \),

(ii) \( E(x^* x) \geq 0 \),

(iii) \( E(axb) = aE(x)b \),

for all \( a, x \in A \), \( b \in B \).

E is said to be faithful if for all \( x \in A \), \( E(x^* x) = 0 \) implies \( x = 0 \).

In Chapter 2, we will consider conditional expectations on von Neumann algebras. In this case, \( E \) is said to be normal if it is \( \sigma \)-weakly continuous.

A linear map which satisfies condition (i) above is called a projection. A projection has norm \( \geq 1 \).

Proposition. (Tomiyama's Theorem) Every projection of norm 1 of a \( C^* \) algebra onto a subalgebra is a conditional expectation.

Proof. See p.116, [29].

We state a corollary of a theorem of Takesaki which will be used frequently in chapter 2 and 3.

Theorem. Let \( E : M \rightarrow N \) be a faithful normal conditional expectation of a von Neumann algebra \( M \) onto a unital subalgebra \( N \). If \( \psi \) is a s.n.f. weight on \( N \), then \( \phi = \psi \circ E \) is also a s.n.f. weight. \( \sigma_\phi \) leaves \( N \) invariant and
\( \phi_t \circ \phi = E \circ \phi_t \).

If \( \phi_1, \phi_2 \) are s.n.f. weights on \( N \) and \( \phi_1 = \phi_1 \circ E, \phi_2 = \phi_2 \circ E \) then

\[
[D\phi_2 : D\phi_1]_t = [D\phi_2 : D\phi_1]_t \quad \text{for all } t \in \mathbb{R}.
\]

Proof. See [29].
0.3.0. Basic definitions.

The following materials can be found in Feldman and Moore [16].

An equivalence relation $R$ on a standard Borel space $X$ is called standard Borel if it is a Borel subset of $X^2$. All equivalence relations to be considered will be standard Borel. $X$ is called the unit space of $R$ and is denoted by $R^{(0)}$. $R^{(0)}$ is canonically Borel isomorphic to the diagonal of $R$, and hence is usually considered as a subset of $R$.

The range and source maps $r, s : R \to X$ are defined by

$$r(y,x) = y,$$
$$s(y,x) = x.$$

$r$ and $s$ are Borel maps. We let

$$R^{(2)} = \{(y,\beta) : s(y) = r(\beta)\}$$

$R^{(2)}$ is a Borel subset of $R \times R$, being the subset on which two Borel maps to a standard Borel space agree. Let $\gamma, \beta$ denote elements in $R$, and let and $\gamma \beta = (r(\gamma), s(\beta))$; then the maps

$$\gamma \to \gamma^{-1}, \quad R \to R;$$

$$(\gamma, \beta) \to \gamma \beta, \quad R^{(2)} \to R$$

are Borel maps, where $\gamma^{-1} = (x,y)$ if $\gamma = (y,x)$. 
Let $R, S$ be standard Borel equivalence relations. A homomorphism $\phi: R \to S$ is a Borel map satisfying

1. $\phi(\gamma^{-1}) = \phi(\gamma)^{-1}$ and
2. $\phi(\gamma \beta) = \phi(\gamma) \phi(\beta)$ for all $(\gamma, \beta) \in R^2$.

A homomorphism $\phi$ always takes $R^0$ to $S^0$. $\phi$ is called an epimorphism if it is surjective and an isomorphism if it is bijective.

A standard Borel equivalence relation $R$ is said to be countable if every equivalence class is countable. All the equivalence relations in this thesis will be countable. Assuming this, the maps $r, s$ are countable to one; a theorem of Kuratowski [25] shows that they send Borel sets to Borel sets. Hence if $A$ is a Borel set in $R^0$, the saturation of $A$ in $R$, defined by

$$\overline{A} = s(r^{-1}(A))$$

is Borel.

Let $A$ be a saturated Borel set in $R^0$, i.e. $A = \overline{A}$. The restriction of $R$ on $A$, denoted by $R|A$ is defined to be $R \cap A^2 A$. $R|A$ is an equivalence relation with unit space $A$.

A $\sigma$-finite measure $\mu$ defined on $R^0$ is called nonsingular if the saturation of a $\mu$-null set is $\mu$-null.

We will be interested in equivalence relations with a nonsingular measure and write $(R, \mu)$ for such a pair. It is understood that $\mu$ is defined on $R^0$, not $R$ itself. Often we write $R$ for $(R, \mu)$ and call it a measured equivalence relation.
For a measured equivalence relation \((R, \mu)\), a measure \(v_\lambda\) can be defined on \(R\) with the \(\sigma\)-algebra of Borel sets:

\[
v_\lambda(C) = \int \#(r^{-1}(x) \cap C) \, d\mu(x), \quad \text{for } C \text{ Borel in } R,
\]

where \(\#\) denotes the cardinality of the set. This measure is \(\sigma\)-finite and is called the left counting measure of \((R, \mu)\). Similarly \(v_\sigma\), the right counting measure can be defined by substituting \(s\) for \(r\) in the integrand. \(v_\sigma\) and \(v_\lambda\) are equivalent measures on \(R\) \[\|E\], and a set \(A\) in \(R\) is null if and only if \(\mu(r(A)) = 0\) \[\|E\]. When we say a.e in \(R\), we will always mean a.e. with respect to either of the counting measures.

The Radon-Nikodym derivative of \(v_\lambda\) with respect to \(v_\sigma\) will be denoted by \(c\):

\[
c(y,x) = \frac{dv_\lambda}{dv_\sigma}(y,x).
\]

Because it is a \(1\)-cocycle, \(c\) will also be called the Radon-Nikodym cocycle.

Let \(G\) be a group acting on a standard Borel space \(X\), i.e. \(G\) acts in such a way that the map

\[(g, x) \mapsto gx \quad G \times X \mapsto X\]

is Borel. Then

\[R_G = \{(gx, x) : x \in X, g \in G\}\]

is a equivalence relation on \(X\), called the equivalence relation generated by \(G\). Similarly we write \(R_S\) for the equivalence relation generated by a single transformation \(S\) acting on \(R^{(0)}\).
A measured equivalence relation \( R \) is called hyperfinite if there is a conull saturated set \( A \) in \( R^{(0)} \) such that \( R|A \) is equal to \( R_S \) for some \( S \) acting on \( A \); or equivalently, \( R|A = \bigcup_{n=1}^{\infty} R_n \), where \( R_n \) are Borel equivalence relations with finite equivalence classes and \( R_n \subseteq R_{n+1} \) for all \( n \). A hyperfinite \((R, \mu)\) corresponds to an orbit equivalence class of transformations on \( (R^{(0)}, \mu) \) and will be our main object of study in Chapter 3.

\((R, \mu)\) is called ergodic if every saturated null set is either null or conull.

0.3.1. Souslin's lemma.

We state here only a weak version of Souslin's lemma:

Theorem. Let \( X, Y \) be standard Borel space and let \( f \) be a 1-1 Borel map of \( X \) into \( Y \). Then \( f(X) \) is a Borel set and \( f:X \rightarrow f(X) \) is a Borel isomorphism.

Proof. See Arveson [2].

0.3.2. Stable range of \((R, \mu)\).

Given \((R, \mu)\), we let \( X = R^{(0)} \) and define \( \tilde{R} \) on \( (X \times \mathbb{R}, \mu \times e^{-s} ds) \) by

\[ ((y,s),(x,r)) \in \tilde{R} \text{ if } (y,x) \in R \text{ and } c(y,x) = s - r, \]

where \( c \) is the \( R.N \) cocycle of \( R \). \( \tilde{R} \) is in general not ergodic even though \( R \) may be ergodic. We abbreviate \( X \times \mathbb{R} \) by \( X \). On
\[ L^\infty(R) = \{ f \in L^\infty(X): f(y,s) = f(x,r) \text{ for a.e. } ((y,s),(x,r)) \in R \} \]

define an action \( \alpha_t \) of \( R \) on \( L^\infty(X) \) by
\[ \alpha_t(f)(x,r) = f(x,r-t). \]

This action leaves \( A \) invariant because it takes equivalence classes to equivalence classes. The stable range is any point realization of \( (\alpha_t,A) \).

If \( R = R_S \) for some transformation \( S \) on \( (X,\mu) \), then the stable range is the same as the stable range of \( (S,X,\mu) \) as defined in Chapter 1. Hence it is a direct extension of the concept to equivalence relations.

0.3.3. A point realization theorem.

Theorem. Let \( i: M \rightarrow N \) be a \(*\)-monomorphism between abelian von Neumann algebras with separable preduals. For any standard measure spaces (atomic spaces allowed) \( (X,\mu), (Y,\nu) \) and isomorphisms
\[ \sigma_1: M \rightarrow L^\infty(X,\mu) \quad \text{and} \quad \sigma_2: N \rightarrow L^\infty(Y,\nu), \]

there exists a surjective Borel map \( \pi: Y \rightarrow X \) such that \( \pi^* \nu = \mu \) and
\[ \sigma_\pi \circ \sigma_1 = \sigma_2 \circ i, \]

where \( \sigma_\pi: L^\infty(X,\mu) \rightarrow L^\infty(Y,\nu) \) is the map induced by \( \pi \).

\( \pi \) is called the point realization of \( i \) via \( \sigma_1 \) and \( \sigma_2 \).

Proof. (Sketch) By the point realization theorem of [0.1.2], it is sufficient to set up one point realization of \( i \). Let A and B be
separable C*-algebras $\sigma$-weakly dense in $M$ and $N$ respectively, containing $1$ and with $i(A) \subseteq B$. Let

$$s_1 : A \rightarrow C(X),$$

$$s_2 : B \rightarrow C(Y)$$

be the Gelfand transforms from $A, B$ onto the algebras of continuous functions with compact support on their spectra $X, Y$. Note that $X$ and $Y$ are Polish spaces. Let $\pi$ be the map $Y \rightarrow X$ conjugate to $i$ and

$\alpha_{\pi} : C(X) \rightarrow C(Y)$

be the map induced by $\pi$. Then we have

$$s_2 \circ i = \alpha_{\pi} \circ s_1.$$

Fix a faithful normal state $\phi$ on $M$. $\phi|_A, \phi|_B$ induce measures $\mu, \nu$ on $X, Y$ respectively with $\pi^* \mu = \nu$. $s_1, s_2$ extend uniquely to normal isomorphisms

$$\sigma_1 : M \rightarrow L^\infty(X, \mu)$$

and

$$\sigma_2 : N \rightarrow L^\infty(Y, \nu).$$

Denote also by $\alpha_{\pi} : L^\infty(X, \mu) \rightarrow L^\infty(Y, \nu)$ the induced map of $\pi$. Since $i$, $\sigma_1$, $\sigma_2$ and $\alpha_{\pi}$ are all normal maps, we deduce that

$$\sigma_2 \circ i = \alpha_{\pi} \circ \sigma_1.$$
CHAPTER 1.
THE INVERSE STABLE RANGE MAP

Let $\mathbb{Z}_5$ be the cyclic group of 5 elements, and let

$$\mathbb{X} = \prod_{i=1}^{5}$$

Let $\beta_1, i = 1, 2, 3, 4$ be positive real numbers such that

$\log \beta_1, \log \beta_2, \log \beta_3$ and $\log \beta_4$ are rationally independent. Define a probability measure $p$ on $\mathbb{X}$ by

$$p = \prod_{i=1}^{\infty} q$$

where $q = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5)$ is the probability measure on $\mathbb{Z}_5$ which gives mass $\alpha_i$ to point $i$, with ratios

$$\frac{\alpha_i}{\alpha_{i+1}} = \beta_i \quad i = 1, 2, 3, 4$$

We let $T$ be the usual odometer transformation on $\mathbb{X}$ (see 0.1.4).

Lemma 1.1 There exist integer-valued Borel functions $n_i(x), i = 1, 2, 3, 4$ such that

$$\frac{d \text{po}_T}{dp}(x) = \beta_1^{n_1(x)} \beta_2^{n_2(x)} \beta_3^{n_3(x)} \beta_4^{n_4(x)} \quad \text{a.e. } x \in \mathbb{X}$$

This set of $n_i$ is uniquely determined by the equation, and is the same regardless of the choice of $\beta_1$, subject to the rational independence condition mentioned above.

Proof. Delete the point of null measure $(4, 4, \ldots)$ from $\mathbb{X}$. Then the collection $\mathcal{B}$ of sets of the form

$$(x_1, x_2, \ldots, x_n) \times \prod_{n+1}^{\infty} \mathbb{Z}_5$$
with \( n > 1 \) and not all of the \( x_i, i = 1, 2, \ldots, n \) equal to 4 generate the product Borel structure on \( X \). For \( A \in \mathcal{B} \) and \( x \in A \), define \( n_i(x) \), \( i = 1, 2, 3, 4 \) by the formulae

\[
\frac{p(T(A))}{p(A)} = \prod_{i=1}^{4} \beta_i(x)
\]

It is clear that \( n_i(x) \) is well defined. Let \( \chi_A \) be the characteristic function of \( A \); we have

\[
\int \chi_A(x) \frac{dp}{d\mu} \, dp(x) = \int \chi_A(x) \, d\mu \, dp(x)
\]

\[
= \int \chi_A(T^{-1}x) \, dp(x)
\]

\[
= \int \chi_A(T(A)) \, dp(x)
\]

\[
= p(T(A)) - p(A) \prod_{i=1}^{4} \beta_i(x)
\]

\[
= \int \chi_A(x) \prod_{i=1}^{4} \beta_i(x) \, dp(x).
\]

Since \( B \) is closed under finite intersections, a standard theorem in measure theory [4] gives the formula for \( \frac{dp}{d\mu} \).

To prove the rest of the lemma, we note that the \( \beta_i \) play only an auxiliary role in the definition of \( n_i \). Therefore \( n_i \) are defined independent of the \( \beta_i \), and they are intrinsic to the odometer.

Uniqueness of the \( n_i \) is obvious. \( \square \)

We shall denote by \( P \) the group of finite permutations, i.e the subgroup of the group of permutations on \( N \), consisting of permutations which change only a finite number of elements.
We note that $P$ acts on $X$ in a canonical way, and $P \subseteq [T]$, the full group of $T$, and for $\tau \in P, \frac{d\rho_T}{dp} = 1$. This action is ergodic by the Hewitt-Savage 0-1 law [0.1.6].

Lemma 1.2. Let $x \in X$. If $k$ is a positive integer such that $T^k(x)$ is a finite permutation of $x$, then

$$\sum_{j=0}^{k-1} n_1(T^j(x)) = 0 \quad \text{for } i=1,2,3,4.$$

Proof: Let $\tau \in P$ with $\tau(x) = T^k(x)$. For $j=0$ to $k$, $T^j x$ differs from $x$ in only a finite number of coordinates, say the first $n$ coordinates.

We let

$$A_0 = (x_1,x_2,\ldots,x_n) \times \prod_{n+1}^{\infty} \mathbb{Z}_5$$

$$A_j = T^j(A_0) \quad j=0,1,2,\ldots k.$$

Then

$$\frac{p(A_{j+1})}{p(A_j)} = \prod_{i=1}^{4} \frac{n_1(T^j x)}{\beta_i} \quad j=0,1,2,\ldots k-1.$$

Multiplying these equations, we have

$$1 = \frac{p(A_k)}{p(A_0)} = \prod_{i=1}^{4} \frac{\sum_{j=0}^{k} n_1(T^j x)}{\beta_i}.$$

Since all $\beta_i$ are multiplicatively independent, we have the result.

Lemma 1.3. Let $(\mathbb{F}, \mathbb{Q}, \nu)$ be a flow [0.1.0]. There exists a Borel map $p(\omega,t)$ from $\Omega \times \mathbb{R}$ to $\mathbb{R}^+$ such that
\[ \frac{d\nu_F}{d\nu} t(\omega) = p(\omega, t) \quad \text{for all } t, \text{ a.e. } \omega. \]

Proof: See Varadarajan [32].

Notation. We shall use the same symbol \( \frac{d\nu_F}{d\nu} t(\omega) \) to denote \( p(\omega, t) \). We shall also write

\[
a(x) = n_1(x) \log \beta_1 + n_2(x) \log \beta_2, \\
b(x) = n_3(x) \log \beta_3 + n_4(x) \log \beta_4.
\]

Theorem 1.4. Let \( \{ F_t, \Omega, \nu \} \) be a flow on a standard measure space. Let \( S \) be the transformation on \( \{ X \times \Omega \times \mathbb{R}, p \times \nu \times e^{-s} ds \} \) defined by

\[
S(x, \omega, r) = (T_x F_a(x)(\omega), r + \log \frac{d\nu_F}{d\nu}(\omega) + b(x));
\]

Then the stable range of \( S \) is the flow \( \{ F_t, \Omega, \nu \} \). The transformation is ergodic if and only if the flow is ergodic.

Proof. We denote the measure \( p \times \nu \times e^{-s} ds \) by \( \mu \). In the construction of the stable range of \( S \) [0.1.3], \( S \) is defined on \( (X \times \Omega \times \mathbb{R} \times \mathbb{R}, \mu \times dt) \) by

\[
S(x, \omega, r, t) = (S(x, \omega, r), t + \log \frac{d\mu_S}{d\mu}(x, \omega, r));
\]

We calculate \( \frac{d\mu_S}{d\mu} \). Let \( A \in \mathcal{B} \), where \( \mathcal{B} \) is as in 1.1. Then there exist \( a, b \in \mathbb{R} \) such that \( a(x) = a \) and \( b(x) = b \) for all \( x \in A \). Write

\[
Z_{ab}(\omega, r) = (F_a(\omega), r + \log \frac{d\nu_F}{d\nu} a(\omega) + b),
\]

and

\[
\mu^r = \nu x e^{-s} ds.
\]

\( Z_{ab} \) is the composition of a measure preserving transformation and a
translation which shrinks the measure by $e^{-b}$. So

$$\mu'(Z_{ab}(C)) = \mu'(C) e^{-b},$$

for $C$ Borel in $\Omega \times \mathbb{R}$. Now for such $C$,

$$\int_{A^x} \frac{d\mu_{\text{os}}(y)}{d\mu}(y) \, d\mu(y)$$

$$= \mu_{\text{os}}(A^x) = \mu(T(A)^x Z_{ab}(C))$$

$$= \mu(T(A)) \mu'(C) e^{-b} = e^a \mu(T(A)) \mu'(C)$$

$$= e^a \mu(A^x) = \int_{A^x} e^a(x) \, d\mu(y),$$

where $x$ is the projection of $y$ to $A$. Since such sets $A^x$ are closed under finite intersections and generate the $\sigma$-algebra of $\mathbb{R} \times \Omega$, we see that, by [0.1.5],

$$\frac{d\mu_{\text{os}}(y)}{d\mu}(y) = e^a(x).$$

Hence

$$S(x, \omega, r, t) = (S(x, \omega, r) + t, a(x)).$$

Let $\tau \in P$, and let $k(x)$ be the integer valued function such that

$$\tau(x) = T^k(x) x,$$

for all $x \in \mathbb{R}$. Then if $x$ is such that $k(x) > 0$, and abbreviating $\sum_{i=0}^{j} a(T^i x)$ to $s(j)$, we have

$$\int_{k(x)} S(x, \omega, r, t)$$

$$= (S^k(x)(x, \omega, r, t), t + s(k(x) - 1))$$

$$= \left( T^k(x), F(s(k(x) - 1)), r^+ \right)$$

$$+ \log \left( \prod_{j=0}^{k(x)-1} \frac{dF}{d\mu} \circ \mu \right)$$

$$= (\tau x, \omega, r, t) \quad \text{a.e. } (x, \omega, r, t),$$

by lemma 1.2. By considering $\tau^{-1}$ applied on $\tau(x)$, we see that this is
also valid for $x$ with $k(x) < 0$.

Let $A$ be an $S$ invariant set in $\mathbb{X} \times \Omega \times \mathbb{R} \times \mathbb{R}$. Then $A$ is $P \times \lambda \times \lambda \times \lambda$ invariant. As $P$ is ergodic on $\mathbb{X} [0.1.6]$, $A = \mathbb{X} \times B$ for some measurable $B \subset \Omega \times \mathbb{R} \times \mathbb{R}$. $B$ is invariant under

$$(\omega, r, t) \mapsto (F_{\alpha}(\omega), r + \log \frac{d\mu_0 F}{d\mu}(\omega) + b(x), t + a(x))$$

for all $x \in \mathbb{X}$. Note that each of the following sets has positive measure:

- $\{x_1 : a(x_1) = -\log \beta_1 \text{ and } b(x_1) = 0\}$,
- $\{x_2 : a(x_2) = -\log \beta_2 \text{ and } b(x_2) = 0\}$,
- $\{x_3 : a(x_3) = 0 \text{ and } b(x_3) = -\log \beta_3\}$, and
- $\{x_4 : a(x_4) = 0 \text{ and } b(x_4) = -\log \beta_4\}$.

This implies $B$ is invariant under

$$(\omega, r, t) \mapsto (F_{\alpha}(\omega), r + \log \frac{d\mu_0 F}{d\mu}(\alpha) + b, t + a),$$

where $\alpha \in G_1$, $b \in G_2$, and $G_1, G_2$ are the groups generated by $\log \beta_1, \log \beta_2$ and $\log \beta_3, \log \beta_4$ respectively. Since these groups are dense in $\mathbb{R}$, we have

$$B = \{((\omega, r, t) : r \in \mathbb{R}, (\omega, t) \in C\}$$

for some Borel set $C$ of $\Omega \times \mathbb{R}$, and $C$ is invariant under

$\alpha : (\omega, t) \mapsto (F_{\alpha}(\omega), t + a) \quad \text{for all } \alpha \in G_1$.

Since the action of $\mathbb{R}$ on $L^\infty(\Omega)$ induced by $F_t$ is $\sigma$-strongly continuous, i.e. $t_n \to t$ implies $f \circ F_{t_n} \to f \circ F_t$ $\sigma$-strongly for $f \in L^\infty(\Omega)$, the action of $\mathbb{R}$ on $L^\infty(\Omega \times \mathbb{R})$ induced by $\alpha_a$, $\alpha \in \mathbb{R}$ is continuous. Thus $C$ is invariant under $\alpha_a$ for all $\alpha \in \mathbb{R}$. Such invariant sets are in one to one correspondence with the Borel subsets of $\Omega$, via the map
(ω,t) ----> F_ t(ω)

Hence

\[ L^\infty(X \times \Omega \times R \times R)^S \simeq L^\infty(\Omega), \]

via the isomorphism

\[ \sigma: f ----> g \in L^\infty(\Omega), \]

where \( g \) is the function which satisfies the equation

\[ g(F_ t(\omega)) = f(x,\omega,r,t) \quad \text{a.e.}(x,\omega,r,t). \]

The point realization of the canonical \( \mathbb{R} \) action [0,1,2] is then \( F_s \), as required.

The proof of the ergodicity part is standard (see for e.g. [20]), and we will omit it. \( \square \)

Remark. In case the flow we start with is measure preserving, the construction can be achieved with a three point odometer. We describe the construction below, as it will be used in later chapters.

Theorem 1.5. Suppose that \((F_t, \Omega, \nu)\) is measure preserving, i.e.

\[ \nu F_t = \nu \text{ for all } t \in \mathbb{R}. \]

Let \( X_3 = \prod_1^\infty \mathbb{Z}_3 \) and \( p = \prod_1^\infty \frac{1}{2,3,6} \). Then the transformation \( S \) on \( X_3 \times \Omega \) defined by

\[ S(x,\omega) = (Tx,F_a(x)\omega), \]

where \( a(x) = \log \frac{d\nu T}{dp}(x) \), has stable range equal to \((F_t, \Omega, \nu)\).

Proof. The proof is analogous to but simpler than the proof of the main theorem. The reader can also refer to [20] for a proof. \( \square \)
In the following theorem, the stable range functor \([0.1.3]\) is denoted by \(\Phi\).

**Theorem 1.6.** The construction of the transformation from the flow in Theorem 1.4 is a functor \(\Psi\) from the category of ergodic flow with conjugations to the category of ergodic transformations with orbit transporting isomorphisms. \(\Psi\) is a right inverse functor to the stable range functor, i.e. \(\Phi \circ \Psi\) is naturally isomorphic to the identity functor.

**Proof.** Suppose we have two flows \(F = (F_t, \Omega, \nu)\), \(G = (G_t, \Omega_1, \nu_1)\) and \(\phi : \Omega \to \Omega_1\) a conjugation between them. Let \((S, Y, \mu), (S_1, Y_1, \mu_1)\) be respectively \(\Psi(F), \Psi(G)\). Define

\[
\Psi(\phi) : \Psi(F) \times \Psi(G) \to \Psi(F) \times \Psi(G)
\]

where

\[
p(\omega^\prime) = \log \frac{d \nu_1}{d \phi^* \nu}(\omega^\prime)
\]

Then \(\Psi(\phi)\) is an isomorphism, and

\[
\Psi(\phi) \circ S(x, \omega, r)
\]

\[
= \Psi(\phi)(Tx, F_a(x)(\omega), r + b(x) + \log \frac{d \nu_0 F_a(x)(\omega)}{d \nu})
\]

\[
= (Tx, \phi F_a(x)(\omega), r + b(x) + \log \frac{d \nu_0 F_a(x)(\omega)}{d \nu} + p(\phi(F_a(x)(\omega))))
\]

\[
= (Tx, G_a(x)(\phi(\omega), r + b(x) + \log \frac{d \nu_1 G_a(x)(\phi(\omega))}{d \nu_1} + p(\phi(\omega))))
\]

\[
= S_1 \circ \Psi(\phi)(x, \omega, r)
\]

a.e. \((x, \omega, r) \in X \times \Omega \times \mathbb{R}\).

Hence \(\Psi(\phi)\) not only preserves the orbit of \(S\), but conjugates between \(S\)
and $S_1$ as well.

Given two composable conjugations $\Phi$, $\Psi$ between flows, it is easy to see that $\Psi(\Phi_1 \circ \Phi_2) = \Psi(\Phi_1) \Psi(\Phi_2)$. Hence $\Psi$ is a functor.

To prove the last statement, let $(F_t, Q, \nu)$ be in $S$. Notations as above, we have $\Psi(F) = (S, Y, \nu)$. Let $A = L(\mu^R, \mu^\nu)$. Then there exists isomorphisms

$$\sigma_A : A \rightarrow L(I, m) \quad \text{and}$$

$$\sigma : A \rightarrow L(Q, \nu),$$

where $\sigma_A$ is as in [0.1.3] and $\sigma$ is defined in the last part of Theorem 1.4. Since $F$ and $\Psi(F)$ are the point realizations of the flow on $A$ via $\sigma$ and $\sigma_A$ respectively, the point realization of $\sigma_A \sigma^{-1}$ via the identity map is a conjugation between the flows $F$ and $\Psi(F)$. Denote it by $n_F$. We prove that it is a natural isomorphism between the identity functor and $\Psi$.

Let $\Phi : (F_t, Q, \nu) \rightarrow (G_t, Q, \nu_1)$ be a conjugation. Notation as before. Let $B = L(\mu_1^R, \mu_1^\nu)$. Define $\widetilde{\Psi}(\Phi)$ as in the construction of $\Phi$ [0.1.3]. $\widetilde{\Psi}(\Phi)$ induces a map $\alpha : A \rightarrow B$ by $\alpha(f) = f \circ \widetilde{\Psi}(\Phi)^{-1}$. Let $\sigma_1 : B \rightarrow L(Q_1, \nu_1)$ be the canonical identification map similar to $\sigma$. Then

$$(\sigma_1 \sigma^{-1})(\sigma \sigma^{-1})_A$$

$$= \sigma_1 \sigma^{-1}$$

$$= \sigma_1 \sigma^{-1}$$
Hence the following diagram commutes

\[
\begin{array}{ccc}
L(I,m) & \xrightarrow{\sigma_0} & L(I,m) \\
\downarrow{\sigma_0} & & \downarrow{\sigma_0} \\
L(Q_1,\nu) & \xrightarrow{\sigma_0} & L(Q_1,\nu)
\end{array}
\]

The point realization of \( \sigma_0 \) via the identity maps is \( \Phi \)(\( \phi \)), while it is easy to see that the point realization of \( \sigma_0 \) via the identity map is \( \phi \) itself. By taking point realization of the diagram, we get

\[
\phi \circ n_F = n_G \circ \Phi(\phi).
\]

It is not known, but likely that \( \Phi \) is naturally isomorphic to the identity functor on \( \mathcal{K} \). The problem is of course with the morphisms, since Krieger's theorem proves that the objects \( \Phi(T) \) and \( T \) are isomorphic.
CHAPTER 2
FINITE INVARIANT MEASURES ON FLOWS

Let \((F_t, \Omega, \nu)\) be a flow on a standard measure space.

Definition 2.1. \((F_t, \Omega, \nu)\) admits an invariant measure if there exists a \(\nu' \sim \nu\) such that

\[ \nu'(F_t(A)) = \nu'(A) \]

for all measurable set \(A\) in \(\Omega\), and for all \(t \in \mathbb{R}\). \(\nu'\) is called an invariant measure for the flow.

Proposition 2.2. If an ergodic flow admits an invariant measure, then it is unique up to a scalar multiple.

Proof. Let \(\nu_1, \nu_2\) be two invariant measures for the flow \(F_t\). We have \(\nu_1 \sim \nu_2\), and

\[ \frac{d\nu_1}{d\nu_2} = \frac{d\nu_1}{d\nu_2} \cdot \frac{d\nu_2}{d\nu_2} = \frac{d\nu_2}{d\nu_2} = \frac{d\nu_1}{d\nu_2} \]

So \(\frac{d\nu_1}{d\nu_2}\) is an \(F_t\) invariant function on \(\Omega\), and must be equal to a constant by ergodicity.

By the proposition, the cases that

(i) \(F_t\) does not admit an invariant measure;
(ii) \(F_t\) admits a finite invariant measure;
(iii) \(F_t\) admits an infinite invariant measure,

are mutually exclusive.

Theorem 2.3. Let \(M\) be a \(\text{III}_0\) Kreiger factor [0.2.0]. The flow of weights of \(M\) admits a finite invariant measure iff there exists a \(\text{III}_1\)
subfactor of $M$ which is the range of a faithful normal conditional
expectation.

Proof. (If part) Let $(F_t, Q, \nu)$ be the flow of weights of $M$. By
changing to an equivalent measure if necessary, we can assume that $\nu$ is
the invariant measure, and that $\nu(Q) = 1$. By theorem 1.5., the system
$(S, X, \times Q; p \times \nu)$ where

$$S(x, \omega) = (T_x F_a(x)(\omega)) \quad (x, \omega) \in Q,$$

and

$$a(x) = \log \frac{d\mu_T(x)}{dp}(x)$$

has the flow as its stable range. By Krieger’s theorem [0.2.3],

$$M = \mathcal{W}^*(S, X, Q, p^\times \nu).$$

Let $M_1 = \mathcal{W}^*(T, X, p)$. $M_1$ is a III$_1$ factor because $T$ is a III$_1$
transformation. We shall construct an embedding $i$ of $M_1$ into $M$ and a
faithful normal conditional expectation $E : M \rightarrow i(M_1)$. For
convenience, we shall identify the spaces
$L^2(X \times Z), L^2(Z, L^2(X)), L^2(X) \otimes L^2(Z)$ etc. in the canonical
way.

The map $i$:

$$B(L^2(X_3 \times Z)) \rightarrow B(L^2(X_3 \times Z) \otimes B(L^2(Q)) = B(L^2(X_3 \times Q \times Z))$$
given by

$$i(x) = x \otimes 1$$

is a $*$-monomorphism. $M_1$ is generated by elements of the form $\pi_T(a)$, a
$\in \mathcal{L}(X_3)$, and a single unitary $U_T$ (See [0.3.0]). For such a
$\pi_T(a)$ and $\xi \in L^2(X_3 \times Q \times Z)$ we have
\[ i(\pi_T(a)) \xi(m) \]
\[ = (\pi_T(a) \circ 1) \xi(m) \]
\[ = (a \circ T^{-m} \circ 1_Q) \xi(m) \]
\[ = [\pi_S(a \circ 1_Q) \xi](m) \]
for all \( m \in \mathbb{Z} \). Also

\[ i(U_T) \xi(m) \]
\[ = (U_T \circ 1) \xi(m) \]
\[ = \xi(m-1) \]
\[ = (U_S \xi)(m). \]

Hence

\[ i(\pi_T(a)) = \pi_S(a \circ 1) \]

and

\[ i(U_T) = U_S. \]

So \( i \) embeds \( M_1 \) into \( M \) unitally. Let \( 1_Q \) denote the constant function 1 on \( Q \). Since \( v \) is finite, \( 1_Q \in L^2(Q) \). There is an isometry \( v : L^2(X_3 \times Z) \rightarrow L^2(X_3 \times Z) \) given by

\[ v \xi = \xi \circ 1_Q. \]

The conditional expectation is defined from \( M \) to \( i(M_1) \) by

\[ E(x) = i(v^* x v), \quad x \in M. \]

We show that \( E \) is indeed a conditional expectation. For \( x \in M_1 \), and \( \xi \in L^2(X_3 \times Z) \) we have
\begin{align*}
    v^* i(x) v (\xi) \\
    = v^* (x \otimes 1) (\xi \otimes 1) \\
    = \int x(\xi) \otimes 1 d\nu \\
    = x(\xi).
\end{align*}

Hence \( E(x) = x \) for all \( x \in i(M_1) \). In particular \( E(U_S) = i(U_T) = U_S \).

Also for \( a \in L^m(x_3^x \Omega) \) and \( \xi \in L^2(x_3^x \Omega) \) we have

\begin{align*}
    (v^* \pi_S(a) v) \xi(x,m) \\
    = \int (\pi_S(a) v \xi)(x,m,\omega)d\nu(\omega) \\
    = \int a \circ S^{-m}(x,\omega) \xi(x,m) d\nu(\omega) \\
    = (\int a(x,\omega) d\nu(\omega)) \circ T^{-m} \xi(x,m) \\
    = \pi_T(b) \xi(x,m)
\end{align*}

for all \( m \in \mathbb{Z} \), a.e \( x \in x_3 \)

where \( b \in L^2(x_3), b(x) = \int a(x,\omega) d\nu(\omega) \). Hence \( (\pi_S(a)) = i\pi_T(b) \in i(M_1) \). Using the normality of \( E \), we see that \( E \) is a projection from \( M \) onto \( M_1 \). From

\[ |E(x)| = |v^* x v| < |x| \]

and

\[ E(1) = v^* v \otimes 1 = 1, \]

we conclude that \( E \) is a projection of norm 1. By proposition 2.3., \( E \) is a conditional expectation.
To show that $E$ is faithful, let $A = \pi_S(L^\infty(X_3^{x_Q}))$ and $F : M \rightarrow A$ be the canonical conditional expectation; i.e., if $x$ has the representation

$$x = \sum_{n \in \mathbb{Z}} \pi_S(a_n) u^n$$

then $F(x) = \pi_S(a_0)$. $F$ is normal and faithful. By a calculation similar to the one above, we get $E(\pi_S(a) u^n) = E(\pi_S(a)) u^n$ for all $a \in L^\infty(X_3^{x_Q})$ and $n \in \mathbb{Z}$. Using the normality of $E$ and $F$, we see that $E \circ F = F \circ E$. Let $E(xx^*) = 0$ for some $x \in M$. Then $EF(xx^*) = FE(xx^*) = 0$. Since $E$ is obviously faithful on $A$, $F(xx^*) = 0$ and hence $x = 0$.

Now we prove the converse of the theorem. As in [0.2.2], we let

$$\hat{M} = M \circ F_\infty,$$

$$\hat{M}_1 = M_1 \circ F_\infty,$$

$$\hat{E} = E \circ 1,$$

$$\hat{\Theta}_S = \text{Ad}(1 \circ V_s),$$

where $F_\infty = B(L^2(R))$ and $(V_s \xi)(t) = \xi(t-s)$. We have for $x \circ y \in M \circ F_\infty$,

$$\hat{E} \circ \hat{\Theta}_S(x \circ y)$$

$$= \hat{E}(x \circ \text{Ad}(V_s)(y))$$
\[ E(x) \odot \text{Ad}_{\mathcal{V}_s}(y) = (1 \odot \text{Ad}_{\mathcal{V}_s})(E(x) \odot y) = \theta_s^E(x \odot y). \]

Hence \( \tilde{E} \) is a faithful normal conditional expectation which commutes with \( \theta_s \). Let \( \phi_1 \) be a f.n.s. weight on \( M_1 \) and let \( \phi = \phi_1 \circ E \). We write
\[ \tilde{\phi} = \phi \circ \omega, \]
\[ \tilde{\phi}_1 = \phi_1 \circ \omega, \]
\[ N = M_{\tilde{\phi}}, \]
\[ N_1 = M_{\tilde{\phi}_1}. \]

By theorem [0.2.4], \( \sigma_{1}^\phi \big| M_1 = \sigma_{1}^\phi \), so \( N_1 \) is a unital subalgebra of \( N \).

By the same theorem, we have
\[ E \circ \sigma_{1}^\phi = \sigma_{1}^\phi \circ E. \]

Hence
\[ \tilde{E} \circ \sigma_{1}^\phi = (E \circ 1) \circ (\sigma_{1}^\phi \circ 1) = (\sigma_{1}^\phi \circ E) \circ 1 = \sigma_{1}^\phi \circ \tilde{E}, \]

so that \( \tilde{E}(N) = N_1 \). Now we again let \( E \) denote the restriction of \( \tilde{E} \) to \( C = Z(N) \). Then \( E \) commutes with \( \theta_s \), and for \( f \in C \), \( y \in N_1 \), we have
\[ yE(f) = E(yf) = E(fy) = E(f)y. \]

So \( E(f) \in Z(N_1) = \mathbb{C}I. \) Hence E is a faithful normal functional which is invariant under the flow of weights of \( M, \) which is \( \theta_s \mid C. \) This means that E is the finite invariant measure for the flow. \( \square \)

Remark. In the proof of the "only if" part of the theorem (i.e. if \( M \) admits a \( III_1 \) subfactor which is the range of a faithful normal conditional expectation, then the flow of weights of \( M \) admits a finite invariant measure), we did not use the fact that \( M \) is a Krieger factor, nor even that it is a factor. Hence half of the theorem is true for all \( III_0 \) von Neumann algebras \( M. \)

The above result for finite invariant measure suggests that a similar result should be valid in the case of infinite invariant measures. Naturally we should consider the notion of an "unbounded" conditional expectation, analogous to the extension of a state to a weight. This has been considered in [18], where operator valued weight is defined. Using the machinery from [10], it is possible to prove that, if the flow of weights of a Krieger factor \( M \) admits an infinite invariant measure, then there exists a normal faithful semifinite operator valued weight from \( M \) onto a subfactor of type \( III_1. \) However, the converse, though possibly true, cannot be proved by the same method as above. The difficulty is due to the fact that the restriction of a semifinite operator valued weight to a subalgebra need not be semifinite.
CHAPTER 3

II\textsubscript{1} EXTENSIONS OF AN EQUIVALENCE RELATION

In this chapter, all the equivalence relations will be countable measured equivalence relations, i.e. a pair \((R, \mu)\), where \(R\) is a standard Borel equivalence relation, and \(\mu\) is a nonsingular \(\sigma\)-finite measure on \(R^{(0)}\). When there is no confusion with the measure, we will often omit \(\mu\) and write \(R\) for \((R, \mu)\). The counting measures have been defined in [0.3.0]. When we say a.e.(almost everywhere) in \(R\), we shall always mean a.e. with respect to either of the counting measures.

Definition 3.1. \((R, \mu)\) and \((R_1, \mu_1)\) are isomorphic if there exist saturated conull subsets \(A, A_1\) of \(R^{(0)}\), \(R_1^{(0)}\) respectively, and an isomorphism \(\phi: A \to A_1\) such that

\[(i)\phi^* \mu \sim \mu_1,\]

\[(ii)(\phi \times \phi)(R|A) = R_1|A_1.\]

In this case, we write \((R, \mu) \cong (R_1, \mu_1)\).

Note. If \(R\) and \(R_1\) are regarded as groupoids, then the above definition is the same as using the definition of isomorphism of groupoids.

In the following, it is sometimes convenient to identify isomorphic equivalence relations. We will sometimes make such identifications without mentioning them explicitly.

Definition 3.2. Let \((R, \mu), (R_1, \mu_1)\) be equivalence relations with \(\mu\) and \(\mu_1\) finite. A covering homomorphism \(\phi\) of \((R, \mu)\) onto \((R_1, \mu_1)\) is a homomorphism \(\phi: R \to R_1\) which is
(i) surjective and such that \( \phi^* \mu = \mu_1 \),

(ii) For all \( Y = (y,x) \in R_1 \) and \( w \in \phi^{-1}(x) \), there exists a unique \( z \in \phi^{-1}(y) \) such that \( \phi(z,w) = (y,x) \).

Note. We consider \( R^{(0)} \) to be a subset of \( R \). Since \( \phi \) is an epimorphism, \( \phi(R^{(0)}) = R_1^{(0)} \), and hence giving the meaning of \( \phi^* \mu = \mu_1 \).

Proposition 3.3. Let \( \phi \) be a covering homomorphism from \((R,\mu)\) to \((R_1,\mu_1)\) and let \( \nu, \nu_1 \) be counting measures on \( R \) with respect to \( \mu \) and \( \mu_1 \) respectively, then \( \phi^* \nu = \nu_1 \).

Proof. We can assume \( \nu, \nu_1 \) are left counting measures. Let \( A \) be a Borel subset of \( R_1 \) and \( Y \in R \) with \( \phi(Y) = Y_1 \in A \); then

\[
\phi r(Y) = r \phi(Y) = r(Y_1)
\]

so that \( r \phi^{-1}(A) \subseteq \phi^{-1} r(A) \). Let \( z \in R^{(0)} \) and \( \phi(z) \in r(A) \), and choose \( Y_1 \in A \) such that \( \phi(z) = r(Y_1) \). By 3.2.(ii), there exists a \( w \in \phi^{-1}(s(Y_1)) \) such that \( (z,w) \in R \) and \( \phi(z,w) = Y_1 \). Hence \( z \in r \phi^{-1}(A) \) and we have

\[
r \phi^{-1}(A) = \phi^{-1} r(A).
\]

Now

\[
\nu_1(A) = 0
\]

\[\iff\] \( \mu_1(r(A)) = 0 \) by [0.3.0]

\[\iff\] \( \mu(r \phi^{-1}(A)) = 0 \)

\[\iff\] \( \mu(r \phi^{-1}(A)) = 0 \)
\[ \forall (\phi^{-1}(A)) = 0. \]

Hence \( \Phi \cdot \nu = \nu_1. \square \)

In view of the above proposition, we can make the following definition:

**Definition 3.4.** Let \((R, \mu), (R_1, \mu_1)\) be equivalence relations with \(\mu, \mu_1\) finite. A \(\text{II}_1\) covering homomorphism \(\phi: R \rightarrow R_1\) is a covering homomorphism which satisfies

\[ (iii) \ c(Y) = c_1(\phi(Y)) \quad \text{a.e.} \ \gamma \in R. \]

where \(c\) and \(c_1\) are the Radon Nikodym cocycles of \(\mu\) and \(\mu_1\) respectively.

**Definition 3.5.** Let \((S, \nu), (S_1, \nu_1)\) be equivalence relations. \(((S, \nu), (S_1, \nu_1))\), or briefly \((S, S_1)\), is a \(\text{II}_1\) pair if there exist equivalence relations \((R, \mu), (R_1, \mu_1)\) and \(\phi\) satisfying definition 3.4., such that \((S, \nu) = (R, \mu)\), and \((S_1, \nu_1) = (R_1, \mu_1)\)

Note that the property of being a \(\text{II}_1\) pair depends only on the isomorphism classes of the equivalence relations concerned.

Imitating [10], we make the following definitions:

**Definition 3.6.** \(R_{+, 1}\) is the category with objects \((X, \mu)\), where \(X\) is a standard Borel space and \(\mu\) is a probability measure on \(X\). The morphisms are measure preserving isomorphisms, i.e. \(\phi: (X, \mu) \rightarrow (Y, \nu)\) such that \(\nu(\phi(A)) = \mu(A)\), for all measurable sets \(A\) in \(X\).

Note. Atomic measure spaces are allowed in the definition.
Viewing an equivalence relation $R$ as a category, in which the set of objects is $R^{(0)}$, and the morphisms are the elements of $R$, we make the following definitions:

**Definition 3.7.** A measurable functor $F$ from $R$ to $R_{+,1}$ is a functor, together with a standard Borel structure on $Y = \bigcup_{x \in R^{(0)}} F_{x}$ in which the following maps are Borel:

1. The natural projection $\pi: Y \rightarrow R^{(0)}$.
2. The canonical embedding $F_{x} \rightarrow Y$, for all $x$.
3. The map from $A = \{(Y,y) \in R^{x}Y: \pi(y) = s(Y)\}$ to $Y$ given by $(Y,y) \mapsto F_{y}(y)$.
4. For each Borel subset $B$ of $Y$, the map $x \mapsto \mu^{x}(B)$, where $\mu^{x}$ is the measure on $F_{x}$.

Note. There is some abuse of notations in (2). We have written $(F_{x}, \mu^{x})$ for $F_{x}$.

**Definition 3.8.** Let $F$ be a measurable functor from $R$ to $R_{+,1}$. $R \ast F$ is the equivalence relation on $Y$ defined by

$$(z,w) \in R \ast F \iff \gamma = (\pi(z), \pi(w)) \in R \text{ and } F_{\gamma}(w) = z$$

**Proposition 3.9.** $R \ast F$ is a standard Borel equivalence relation.

**Proof.** The map $\phi: A \rightarrow R^{x}Y$, $A$ as in definition 3.7., given by

$$\phi(Y,y) = (F_{\gamma}(y), y)$$

is Borel by 3.7.(3). It is also an injection. Hence $R \ast F = \text{Im } \phi$ is standard Borel. □
Proposition 3.10. If \( \nu \) is a nonsingular measure on \( R \), then the measure on \( Y = (R^4F)^{(0)} \) defined by the integration [5]

\[
\mu = \int_{Y} \nu(y)
\]

is nonsingular with respect to \( R^4F \).

Proof. Denote the saturation of a set \( E \) in \( R^{(0)} \) by \( \overline{E} \). Let \( A \) be a Borel set in \( Y \) such that \( \mu(A) = 0 \). Let

\[
C = \{ y \in R^{(0)} : \mu_Y(A) \neq 0 \},
\]

\[
D = \{ y \in R^{(0)} : \mu_Y(A) > 0 \}.
\]

Suppose that \( y \) does not belong to \( D \); then

\[
\mu_Y(A) = \mu_Y\left( \bigcup_{x \sim y} F(y,x) (A \cap \pi^{-1}x) \right)
\]

\[
= \sum_{x \sim y} \mu_Y(F(y,x)(A \cap \pi^{-1}x)) = 0
\]

So \( y \) does not belong to \( C \). Hence \( C \subseteq D \). Since \( \nu(D) = 0 \) and \( \nu \) is nonsingular, \( \nu(D) = 0 \). So \( \nu(C) < \nu(D) = 0 \). Hence \( \mu(A) = 0 \).

Notation. Let \( G \) be a group acting on a measure space \((X, \mu)\). For \( g \in G \), \( g\mu \) and \( \mu g^{-1} \) will denote the same measure on \( X \), given by \( g\mu(A) = \mu(g^{-1}(A)) \), for \( A \) measurable in \( X \).

Lemma 3.11. Let \((X, \mu), (Y, \nu)\) be standard measure spaces with \( \mu, \nu \) being finite measures. Let \( G \) be a countable discrete group which acts on both \((X, \mu)\) and \((Y, \nu)\). Suppose that there exists a surjective map \( \phi : X \rightarrow Y \) which is \( G \) equivariant, and \( \phi^* \mu = \nu \). Let
be the disintegration [10] of \( \mu \) over \( \nu \). Suppose also that the action of \( G \) is nonsingular on \((Y,\nu)\). Then the action of \( G \) is nonsingular on \((X,\mu)\) if and only if

\[
\mu^Y = g^{-1} \mu^Y \quad \text{for } \nu \text{ a.e. } y \in Y
\]

In this case, we have

\[
\frac{d\mu_{og}}{d\mu}(x) = \frac{d\nu_{og}}{d\nu}(y) \frac{d\mu^Y_{og}}{d\mu^Y}(x),
\]

where \( y = \Phi(x) \), for all \( g \in G \), a.e. \( x \in X \).

Proof. (only if) We disintegrate \( g\mu \) over \( \nu \): For a measurable function \( f \) on \( X \), we have

\[
\int_{X} f d\mu_{og} = \int_{X} f(gx) \ d\mu(x)
\]

\[
= \int_{X} \int_{Y} f(gx) \ d\mu^Y(x) d\nu(y)
\]

\[
= \int_{Y} \int_{X} f(x) \ d\mu^Y(x) \ d\nu(y)
\]

\[
= \int_{Y} \int_{X} f(x) \ d\mu^Y(x) \ d\nu(y)
\]

On the other hand,

\[
\int_{X} f(x) d\mu_{og}(x) = \int_{X} \frac{d\mu_{og}}{d\mu}(x) d\mu^Y(x) \ d\nu(y)
\]

By uniqueness of decomposition of \( g\mu \) over \( \nu \) [10], we have

\[
\frac{d\mu_{og}}{d\mu_{og}}(x) \mu^Y = \frac{d\nu_{og}}{d\nu}(y) \ g^{-1} \mu^Y
\]

for \( \nu \text{ a.e. } y \). Hence
\[ \frac{d g \mu}{d \mu}(x) = \frac{d g \nu}{d \nu}(y) \frac{d g \mu g^{-1}}{d \mu}(x), \]

where \( \phi(x) = y \) for \( \mu \) a.e. \( x \). Since \( G \) is countable, the above equation is true for all \( g, \mu \) a.e. \( x \).

Now we prove the converse. Suppose that \( g \mu Y = \mu g Y \) a.e. \( y \). Let \( A \) be a measurable subset of \( X \). We have

\[
\begin{align*}
\int_A g \mu(x) &= \int_A \chi_A(x) d \mu Y(x) d \nu(y) \\
&= \int_A \chi_A(x) d \mu Y(g^{-1}x) d \nu(y) \\
&= \int_A \chi_A(x) d g \mu Y d \mu g Y(x) d \nu(y) \\
&= \int_A \chi_A(x) d g \mu -1 Y d \mu Y(x) d \nu(y) d \nu(y),
\end{align*}
\]

so that

\[ g \mu(A) = 0 \]

\[ \iff \int_A \chi_A(x) d g \mu -1 Y d \mu Y(x) = 0 \]

\[ \iff \mu Y(A) = 0, \quad \nu \text{ a.e.} y, \]

\[ \iff \mu(A) = 0, \]

and so \( \mu \sim g \mu \). \( \square \)

Theorem 3.12. Let \( R = (R, \mu) \) and \( S = (S, \nu) \) be equivalence relations with \( \mu \) and \( \nu \) finite. Then \( R \) is isomorphic to \( S + F \) for some measurable functor \( F \) from \( S \) to \( R^+ \) if and only if there exists a \( II_1 \) covering homomorphism \( \phi : R \to S \).
Proof. Suppose \( R = S \ast F \) as in 3.8., then \( \phi = (\pi \times \pi) | R \) is a covering homomorphism. Since \( S \) is countable, there exist a countable discrete group \( G \) together with an action on \( S^{(0)} \) such that \( S \) is the equivalence relation generated by the action [16]. Define an action of \( G \) on \( R^{(0)} \) by \( g | F_x = F_{(gx, x)} \). This is a Borel action by assumption (3) of 3.7. By the above lemma, this action is nonsingular. We have \( R = R_G \), the equivalence relation generated by the action of \( G \), and

\[
\frac{d\mu_{\phi}}{d\mu}(w) = \frac{d\nu_{\phi}}{d\nu}(x) \frac{d\mu^x_{\phi}}{d\mu^x}(w) = \frac{d\nu_{\phi}}{d\nu}(x),
\]

where \( \pi(w) = x \) and \( \mu = \int \mu^x \, d\nu(x) \). Then for almost all \( Y \in R \), if \( Y = (gw, w) \)

\[
c(Y) = \frac{d\mu_{\phi}}{d\mu}(w) = \frac{d\nu_{\phi}}{d\nu}(x) = c_1(\phi(Y)).
\]

(see [16])

To prove the converse, suppose there exists a II\(_1\) covering homomorphism \( \phi : R \rightarrow S \). We define a functor \( F : S \rightarrow R_{+1} \) as follows:

For \( x \in S^{(0)} \), \( F_x \) is the space \( \phi^{-1}(x) \) with the measure \( \mu^x \) coming from the disintegration \( \mu = \int \mu^x \, d\nu(x) \). For \( Y = (y, x) \in S \) and \( w \in F_x \), \( F_Y(w) = z \) is defined by the conditions

(i) \((z, w) \in R\),

(ii) \( \phi(z, w) = Y \).

By (ii) of definition 3.2., this \( z \) is uniquely determined by \( Y \) and \( w \).

We check that \( F \) is a Borel functor. (1),(2),(4) of 3.7. are trivially verified (\( Y = R^{(0)} \) is equipped with the given Borel structure). To see (3), the map \( F : (Y, w) \rightarrow F_Y(w) \) is equal to \( r \circ h \),
where \( h(\gamma, w) = (F_\gamma(w), w) \). But \( h \) has an inverse given by

\[
h^{-1}(z, w) = (\phi(z, w), w),
\]

and \( h^{-1} \) is Borel. Thus the Souslin lemma \([0.3.1]\) implies that \( h \) is Borel, and so is \( F \).

Choose a countable discrete group \( G \), together with an action of \( G \) on \( S(O) \) which generates \( S \). \( G \) also acts on \( R(O) \) via

\[
g \cdot F_x = F(gx, x).
\]

This action generates \( R \) and is Borel because \( F \) is Borel. So if \( \mu(E) = 0 \) for some \( E \subseteq R(O) \), then \( \overline{\mu(E)} = 0 \) because \( R \) is nonsingular.

But \( E = \bigcup g(E) \), so \( \mu(g(E)) = 0 \) for all \( g \in G \) and \( G \) is nonsingular.

By lemma 3.10, \( \mu^x = \mu^{gx} \) for a.e. \( x \in S \), and for all \( g \) and almost all \( w \),

\[
\frac{d\mu^{g_\pi}(w)}{d\mu}(w) = \frac{d\nu^{og}(x)}{d\mu}(x) \cdot \frac{d\mu^{gx}}{d\mu^x}(w),
\]

where \( \pi(w) = x \). But for almost all \( \gamma = (gw, w) \in S \),

\[
\frac{d\mu^{g_\pi}(w)}{d\mu}(w) = c(\gamma) = c_1(\phi(\gamma)) = \frac{d\nu^{og}(x)}{d\nu}(x)
\]

Hence

\[
\frac{d\mu^{gx}}{d\mu^x}(gx, x)(w) = \frac{d\mu^{gx}}{d\mu^x}og(w) = 1,
\]

and \( F(gx, x) \) is measure preserving. It is clear that \( R = R_G = S^*F \).

The concepts of a II\(_1\) covering homomorphism and the product \( R^*F \) in definition 3.8, as well as the idea in Theorem 3.12 originate from C. Sutherland, in an unpublished paper of his. I thank him for
allowing me to include them here.

Definition 3.13. Let \((\alpha_t, A), (\beta_t, B)\) be flows on abelian von Neumann algebras \(A, B\) with separable preduals. \((\alpha_t, \beta_t)\) is a II\(_1\) pair if there exists a *-monomorphism \(i : B \rightarrow A\) and normal states \(\phi, \phi_1\) on \(A, B\) respectively, such that

\[
\begin{align*}
(1) & \; \circ \phi_t = \alpha_t \circ i \\
(2) & \; \phi_t = \phi_1, \\
(3) & \; [D\phi_t : D\phi_1]_t = i[D\phi_t : D\phi_1]_t 
\end{align*}
\]

for all \(s, t \in \mathbb{R}\).

Lemma 3.14. \(((\alpha_t, A), (\beta_t, A_1))\) is a II\(_1\) pair of flows if and only if there exists \((Q, \nu), (Q_1, \nu_1) \in \mathbb{R}^+, 1\) and a surjection \(\pi : Q \rightarrow Q_1\) such that

\[
\begin{align*}
(1) & \; \pi^* \nu = \nu_1 \\
(2) & \; A_t = L(\pi(Q), \nu), A_1 = L(\pi(Q_1), \nu_1) \text{ in such a way that } \alpha_t, \beta_t \text{ have point realizations as flows } (F_t, \nu) \text{ and } (G_t, \nu_1) \text{ satisfying} \\
(3) & \; \sigma^t_t \circ G_t = G_t \circ \sigma, \quad \frac{d\nu}{d\nu} \circ F_t = \frac{d\nu_1}{d\nu_1} \circ G_t \\
(4) & \; \frac{d\nu}{d\nu} (\pi(w)) = \frac{d\nu_1}{d\nu_1} (\pi(w)) \text{ for all } t, \text{ for a.e. } w \in Q.
\end{align*}
\]

Proof. Let \((Q, \nu), (Q_1, \nu_1) \in \mathbb{R}^+, 1\) and let

\[
\sigma : A \rightarrow L^\infty(Q, \nu) \\
\sigma_1 : A_1 \rightarrow L^\infty(Q_1, \nu_1)
\]

be isomorphisms. By [0.3.3] there exists a surjective Borel map

\[
\pi : Q \rightarrow Q_1, \pi^* \nu = \nu_1 \text{ which is the point realization of } i \text{ via } \sigma \text{ and } \sigma_1.
\]

We can assume that \(\nu, \nu_1\) are the measures induced by \(\phi, \phi_1\) via \(\sigma, \sigma_1\). Condition 3.13 (ii) implies that \(\pi^* \nu = \nu_1\). If we let \(F_t, G_t\) be the point realizations of \(\alpha_t, \beta_t\) via \(\sigma, \sigma_1\), we have \(\sigma^t_t \circ G_t = G_t \circ \sigma\) by 3.13.
(i). To prove (ii)(b), we choose \( p(t,w) \) to be a Borel function on \( \mathbb{R}^\Omega \) such that \( p(t,w) = \frac{d\omega F_t}{d\nu}(w) \) for all \( t \), a.e. \( w \), as in Lemma 1.3. Similarly we choose \( p_1(t,w_1) \). For each \( t,s \in \mathbb{R} \), we have

\[
[D\phi_0 F_t : D\phi]_s(w) = p(t,w)^{is}, \quad \text{see [9].}
\]

\[
- \frac{i}{2\pi} [D\phi_1 \circ G_t : D\phi_1]_s(w) = p_1(t,\pi(w))^{is} \quad \text{a.e. } w
\]

by 3.13(iii), where we have identified \( A, A_1 \) with \( L^\infty(Q,\nu), L^\infty(Q_1,\nu_1) \) via \( \sigma, \sigma_1 \). Suppose for some \( t \), there exists a set \( C \subseteq \Omega \) of positive measure such that \( p(t,w) \neq p_1(t,\pi(w)) \) for \( w \in C \). We can assume that

\[
\left[ \frac{p(t,w)}{p_1(t,\pi(w))} \right]^{2\pi i} = 1 \quad \text{on } C.
\]

On some subset \( C' \) of \( C \) of positive measure, \( p_1(t,\pi(w)) = e^k p(t,w) \) for some integer \( k \neq 0 \). Letting \( s = \pi/k \), we have

\[
p(t,w)^{i\pi/k} = p(t,w)^{i\pi/k} e^{i\pi}
\]

for a.a \( w \in C' \), which is a contradiction. Hence \( p(t,w) = p_1(t,\pi(w)) \) for all \( t \), a.a. \( w \).

For the converse, let \( A = L^\infty(Q,\nu), A_1 = L^\infty(Q_1,\nu_1) \), \( \phi \) and \( \phi_1 \) be the normal states on \( A \) and \( A_1 \) induced by \( \nu \) and \( \nu_1 \). Define \( i: A_1 \rightarrow A \) by \( i(f)(\omega) = f(\pi(\omega)) \). Then (i), (ii) of 3.12 are satisfied, and

\[
[D\phi_0 F_t : D\phi]_s(w) = p(t,w)^{is}
\]

\[
= p_1(t,\pi(w))^{is} = [D\phi_1 \circ G_t : D\phi_1]_s(\pi(w))
\]

\[
= i [D\phi_1 \circ G_t : D\phi_1]_s(w)
\]

for a.a \( w \in \Omega \). \( \square \)
We now define flows \((F_t, \mathcal{O}, \nu^-), (G_t, \mathcal{O}_1, \nu_1^-)\) to be a \(\mathbb{II}_1\) pair if their induced flows on the \(L^\infty\) algebras form a \(\mathbb{II}_1\) pair. By lemma 3.14, this is equivalent to the existence of probability measures \(\nu = \nu^-, \nu_1 = \nu_1^-\) and a surjection \(\pi\) satisfying the conditions (i), (ii)a, (ii)b of the lemma. Note that as for equivalence relations, the property of \(\mathbb{II}_1\) pair of flows depends only on the conjugacy classes of the flows.

Theorem 3.15. Let \((R, \mu), (R_1, \mu_1)\) be equivalence relations whose stable ranges are \((F_t, \mathcal{O}, \nu), (G_t, \mathcal{O}_1, \nu_1)\). If \((R, R_1)\) is a \(\mathbb{II}_1\) pair, then \((F_t, G_t)\) is a \(\mathbb{II}_1\) pair. If \(R, R_1\) are hyperfinite, then the converse is also true.

Proof. Suppose that \((R, R_1)\) is a \(\mathbb{II}_1\) pair. By Theorem 3.12, without loss of generality, we can assume that \(R = R_1 \ast F\) and that \(\mu_1\) is finite. We use the notations of Definition 3.7 and let \(X = R(0), X_1 = R_1(0)\). There exists an injection \(i : L(\infty)(X_1, \mu_1) \rightarrow L(\infty)(X, \mu)\) given by

\[
i(f)(w) = f(\pi(w)) \quad w \in X
\]

From the disintegration \(\mu = \int \mu y \, d \mu_1(y)\), there exists a conditional expectation \(E : L(\infty)(X, \mu) \rightarrow L(\infty)(X_1, \mu_1)\) given by

\[
E(f)(y) = \int f \, d \mu_y.
\]

This conditional expectation is clearly faithful, and normal by the Lebesgue monotone convergence theorem, which can be applied since the \(\sigma\)-weak topology is metrizable on bounded sets in \(L(\infty)(X, \mu), (X, \mu)\) being standard. Using the construction of the stable range of [0.1.3], let

\[
\tilde{X} = X^\mathbb{R}, \quad \tilde{\mu} = \mu \rho;
\]
\[ \bar{X}_1 = X_1 \times \mathbb{R} \quad \bar{\mu}_1 = \mu_1 \times \rho, \]

where \( \rho \) is a probability measure on \( \mathbb{R} \), equivalent to the Lebesgue measure. \( L^\infty(X_1, \bar{\mu}_1) \) can be embedded in \( L^\infty(X, \bar{\mu}) \) via \( \bar{i} = i \circ l \), where \( l \) is the identity map on \( L^\infty(R) \) (We have identified, for example, \( L^\infty(X, \mu) \) with \( L^\infty(X, \mu) \circ L^\infty(R, \rho) \)). The conditional expectation \( \bar{E} = E \circ l: L^\infty(\bar{X}, \bar{\mu}) \rightarrow L^\infty(X_1, \bar{\mu}_1) \) is then faithful and normal. An equivalence relation \( \bar{R} \) can be defined on \((\bar{X}, \bar{\mu})\) by

\[
((z, s), (w, t)) \in \bar{R} \text{ if } (z, w) \in R \text{ and } c(z, w) = s - t,
\]

where \( c \) is the Radon-Nikodym cocycle of \((R, \mu)\). Similarly we define \( \bar{R}_1 \) on \((\bar{X}_1, \bar{\mu}_1)\) and \( c_1 \). Let

\[ L^\infty(\bar{X})^\bar{R} = \{ f \in L^\infty(\bar{X}) : f(z, t) = f(w, s), \text{ for a.e. } ((z, t), ((w, s)) \in \bar{R} \} \]

Similarly, \( L^\infty(X_1)\) \( \bar{R}_1 \) is defined. Let \( f \in L^\infty(X_1) \) \( \bar{R}_1 \) and

\[
((w, s), (z, t)) \in \bar{R}. \text{ Then because } ((\pi(w), s), (\pi(z), t)) \in \bar{R}_1 \text{ by } 3.4(iii),
\]

\[
\bar{i}(f)(w, s) = f(\pi(w), s)
\]

\[
= f(\pi(z), t) = \bar{i}(f)(z, t).
\]

So \( \bar{i}: L^\infty(X_1) \rightarrow L^\infty(\bar{X}) \) \( \bar{R}_1 \) injectively. Let \( h \in L^\infty(\bar{X}) \) \( \bar{R} \) and

\[
((x, s), (y, t)) \in \bar{R}_1. \text{ Then because } F(x, y) \text{ is measure preserving and } c(F(x, y)z, z) = c_1(x, y),
\]

\[
\bar{E}(h)(x, s) = \int h(w, s) d\mu^X(w)
\]

\[
= \int h(F(x, y)z, s) d\mu^X(F(x, y)z)
\]

\[
= \int h(z, s - c(F(x, y)z, z)) d\mu^Y(z)
\]
\[ E(h)(y,t) = \int h(z,t) d\mu^y(z) \]

\[ = E(h)(y,t). \]

Hence \( E: L^\infty(X) \to L^\infty(X_1) \). The stable range of \( R \), with respect to \( c \)

is the flow \( F_t \) on \( A = L^\infty(X)^R \) given by

\[ F_t(h)(w,s) = h(w,s-t). \]

We then have \( E_\circ F_t = G \circ E \). Now let \( \phi, \phi_1 \) be the normal states on \( A \) and

\( A_1 = L^\infty(X_1) \) induced by \( \tilde{\mu} \) and \( \tilde{\mu}_1 \). We have

\[ \phi = \phi_1 \circ E, \]

and for all \( t, s \in \mathbb{R}, \)

\[ [D_\phi \circ F_t : D_\phi]_s = [D_\phi_1 \circ E \circ F_t : D_\phi_1 \circ E]_s \]

\[ = [D_\phi_1 \circ G \circ E : D_\phi_1 \circ E]_s = [D_\phi_1 \circ G : D_\phi_1]_s \]

by Takesaki's theorem [0.2.4].

Now we prove the converse. We can assume that \( (F_t, \mathcal{Q}, \nu) \) and

\( (G_t, \mathcal{Q}_1, \nu_1) \) satisfy the conditions of lemma 3.14. Let \( Y = X_3 \times \mathcal{Q} \times \mathbb{R} \) and \( \mu \)

be the probability measure \( p \times \nu \times \rho \) on \( Y \), where \( \nu \) is the measure on \( \mathcal{Q} \)
corresponding to \( \phi \); \( \rho \) is a probability measure on \( \mathbb{R} \) equivalent to the

Lebesgue measure, as before. Let \( S \) be the transformation on \( Y \) defined

as in Theorem 1.4. Then by that theorem and the assumption that \( R \) is

hyperfine, \( R = R \). Similarly we define \( Y_1, \mu_1 \) and \( S_1 \). Letting

\( \phi = 1 \times \pi \times 1: Y \to Y_1, \phi \mu = \mu_1 \) and since

\[ \frac{d\nu \circ G}{d\nu}(\pi(w)) = \frac{d\nu \circ F}{d\nu}(w) \]

for all \( t, \) a.e. \( w \in \mathcal{Q}, \)

we have
Let \( a(x) \) and \( b(x) \) be as in Theorem 1.4. It is easily checked that

\[
\frac{d\mu \circ \Phi}{d\mu}(y) = e^{a(x)+b(x)} \frac{d\nu \circ a(x)}{d\nu}(w) \frac{\rho^{-}(r+s)}{\rho^{-}(r)}
\]

where \( y = (x,w,r), s = b(x) + \log \frac{d\nu \circ a(x)}{d\nu}(w) \) and \( \rho^{-} = \frac{d\rho}{ds} \). Hence \( \Phi \) induces a \( II_1 \) covering homomorphism from \( R \) to \( R_1 \) and \( (R,R_1) \) is a \( II_1 \) pair.

Corollary 3.16. The stable range of a hyperfinite equivalence relation \( R \) admits a finite invariant measure if and only if \( (R,R_1) \) is a \( II_1 \) pair, where \( R_1 \) is the hyperfinite \( III_1 \) ergodic equivalence relation.

Proof. Let \( G_t \) be the trivial flow on a one point space and let \( \Phi \) denote the stable range map. We have \( \Phi(R_1) = G_t [12] \) and \( (R,R_1) \) is a \( II_1 \) pair

\[
\Leftrightarrow (\Phi(R),\Phi(R_1)) \text{ is a } II_1 \text{ pair}
\]

\[
\Leftrightarrow (\Phi(R),G_t) \text{ is a } II_1 \text{ pair}
\]

if and only if \( \Phi(R) \) admits a finite invariant measure.

Lemma 3.17. Let \( (R,\nu), (R_1,\nu_1) \) be equivalence relations with \( R_1 \) being hyperfinite and having a.e. infinite equivalence classes. \( (R,R_1) \) is a \( II_1 \) pair if and only if there are probability measures \( \mu \) and \( \mu_1 \) on \( X = R(0) \) and \( X_1 = R_1(0) \) equivalent to \( \nu, \nu_1 \) respectively, transformations \( S, S_1 \) acting freely on \( X, X_1 \), and a surjective map \( \pi: X \rightarrow X_1 \) such that

1. \( \pi^* \mu = \mu_1 \),
\textbf{Theorem 3.18.} Let $M$, $M_1$ be hyperfinite von Neumann algebras with no semifinite summand, and $F(M)$, $F(M_1)$ be their flows of weights. Then $(F(M), F(M_1))$ is a II_1 pair is equivalent to the following:

$M_1$ can be embedded unitally in $M$ in such a way that there exists Cartan subalgebras $A \supseteq A_1$ of $M$, $M_1$, and a faithful normal conditional expectation $E : M \to M_1$ such that $E(A) = A_1$. Moreover, the $\sigma$-algebra generated by $A$ is $\sigma$-finite and $\sigma$-complete.

Proof. We proceed in two steps. First, we show that $M_1$ can be embedded unitally in $M$ in such a way that there exists a Cartan subalgebra $A$ of $M$ containing $A_1$ and a faithful normal conditional expectation $E : M \to M_1$ such that $E(A) = A_1$. Then, we prove that the $\sigma$-algebra generated by $A$ is $\sigma$-finite and $\sigma$-complete.

\textbf{Notation.} Let $M$ be a von Neumann algebra, and $A$ be an abelian subalgebra. $N_M(A)$ will stand for the group of unitary normalizers of $A$ in $M$ (i.e., $u \in M$ is a unitary normalizer of $A$ in $M$ if $A$ is the range of a faithful normal expectation from $M$.)

As in [17], a *-subalgebra $A$ of $M$ is called a Cartan subalgebra if it is maximal abelian, $N_M(A)$ generates $M$, and $A$ is the range of a faithful normal expectation from $M$.

\textbf{Theorem 3.18.} Let $M$, $M_1$ be hyperfinite von Neumann algebras with no semifinite summand, and $F(M)$, $F(M_1)$ be their flows of weights. Then $(F(M), F(M_1))$ is a II_1 pair is equivalent to the following:

$M_1$ can be embedded unitally in $M$ in such a way that there exists Cartan subalgebras $A \supseteq A_1$ of $M$, $M_1$, and a faithful normal conditional expectation $E : M \to M_1$ such that $E(A) = A_1$. Moreover, the $\sigma$-algebra generated by $A$ is $\sigma$-finite and $\sigma$-complete.

Proof. We proceed in two steps. First, we show that $M_1$ can be embedded unitally in $M$ in such a way that there exists a Cartan subalgebra $A$ of $M$ containing $A_1$ and a faithful normal conditional expectation $E : M \to M_1$ such that $E(A) = A_1$. Then, we prove that the $\sigma$-algebra generated by $A$ is $\sigma$-finite and $\sigma$-complete.
expectation \( E : M \rightarrow M_1 \) such that

1. \( E(A) = A_1 \),
2. \( N_{M_1}(A_1) \subseteq N_M(A) \),
3. \( M = A \vee N_{M_1}(A_1) \).

Proof. (if part) Suppose the conditions are satisfied. We claim that there exists a \( u \in N_{M_1}(A_1) \) such that \( \text{Ad}(U) \) acts freely on \( A_1 \) and \( \{U\} \) is amenable. We sketch the proof of this. Since \( A_1 \) is a Cartan subalgebra of \( M_1 \), there exists an equivalence relation \((R_1, \mu_1)\) and a \( t \in H^2(R_1, T)\) such that \( M_1 = M(R_1, t) \), the left von Neumann algebra of \( R_1 \) and the 2-cocycle \( t \), via an isomorphism which maps \( A_1 \) to the diagonal algebra [Th.1, 17]. By Zimmer [33], \( R_1 \) is amenable and by Connes, Feldman and Weiss [11] \( R_1 \) is singularly generated. Hence \( t \) can be chosen to be 1 and \( R_1 \) is the equivalence relation generated by some transformation \( T \) on \( X_1 = R_1(0) \). Since \( M_1 \) has no semifinite summand, \( R_1 \) has a.e. infinite equivalence classes, and so \( T \) acts freely. In this case \( M(R_1, t) = M(R_1, 1) = W(T, X_1, \mu_1) \) [17], where the diagonal algebra is identified with \( \pi_T(L(X_1, \mu_1)) \). Let \( \sigma_1 : M_1 \rightarrow W(T, X_1, \mu_1) \) be the composition of these isomorphisms. Then \( U = \sigma_1^{-1}(U_1) \) is the required normalizer.

Let \( \omega_1 \) be the normal state which corresponds to \( \mu_1 \) through \( \sigma_1 \), and let \( \omega = \omega_1 \circ E \). By (i) \( \omega \) is a normal state on \( A \). Let \( \sigma : A \rightarrow L_\infty(X, \mu) \) be an isomorphism for some \((X, \mu)\) in \( R_+, 1 \). \( \text{Ad}(U) \) acts on \( A \) by (ii). Let \( S, T \) be the point realization of this action via \( \sigma \). By taking the point realization of the natural embedding \( i : A_1 \rightarrow A \), we see that there exists a surjective map \( \pi : X \rightarrow X_1 \), \( \pi \mu = \mu_1 \) such that \( \pi \circ \sigma = T \circ \sigma \). But because \( \phi_1 = \phi \circ i \), we can assume that \( \pi \mu = \mu_1 \).

Let \( R, R_1 \) be the equivalence relations generated by \( S, T \). Let \( G \) be a countable dense subgroup of \( N_M(A) \) containing \( U \). By Theorem 1 of [17] and its proof, there exists a \( s \in H^2(R, T) \) such that
where $R_G$ is the equivalence relation generated by the action of $G$ on $(X, \mu)$. Let $H = \{U^n : n \in \mathbb{Z}\}$. $M$ is generated by $A$ and $H$ by (iii). Letting $N = A$, the conditions of theorem 1.55.(c) of [9] are satisfied. Hence for every $g \in G$, there exist mutually orthogonal projections $e_k \in A, \Sigma e_k = 1$ and $s_k \in H$ such that $s_k(e_k) = ge_k g^*$ and
\[ g x g^* = \Sigma s_k(e_k x) s_k^* \quad \text{for all } x \in A \]

But this means that $G \subseteq [R_G] = [R]$, where $[\ ]$ denotes the full group. Hence $R_G \subseteq R$. Since $G \supseteq H$, $R_G \supseteq R$, and hence $R_G = R$.

Since $R$ is hyperfinite, $s$ can be taken to be 1, and $M = M(R, 1)$

We now prove that $(R, R_1)$ is a $II_1$ pair. Let $F: M \rightarrow A$ be a faithful normal conditional expectation, and let $F_1 = (EoF)|M_1$. $F_1$ is a faithful normal conditional expectation from $M_1$ to $A_1$. Letting $\phi = \omega o F$, $\phi_1 = \omega_1 o F$, we have
\[ \phi = \omega o F = \omega_1 o E o F \]
\[ = \omega_1 o F_1 o E = \phi_1 o E. \]

So
\[ [D(\phi Ad U) : D\phi]_t = [D(\phi_1 o E o Ad U) : D\phi_1]_t \]
\[ = [D(\phi_1 o Ad U o E) : D\phi_1 o E]_t = [D\phi_1 o Ad U : D\phi_1]_t. \]

Also
Similarly

$$[D\phi_0 \circ \text{AdU} : D\phi]_t = \left[\frac{d\omega_0 \circ S}{d\omega}ight]_t$$

Hence

$$\left[\frac{d\omega_0 \circ S}{d\mu} \right]_t = \left[\frac{d\omega_1 \circ T}{d\omega_1} \right]_t$$

for all $t \in \mathbb{R}$, and so

$$\frac{d\mu_0 \circ S}{d\mu}(x) = \frac{d\mu_1 \circ T}{d\mu_1}((\pi(x))) \quad \mu\text{-a.e. } x \in X.$$

Since $T$ acts freely, conditions (1) - (4) of lemma 3.17 are satisfied. We have then $(R, R_1)$ is a II$_1$ pair. By [0.2.3], $F(M)$, $F(M_1)$ are the stable ranges of $R$ and $R_1$. Theorem 3.15 then implies that $(F(M), F(M_1))$ is a II$_1$ pair.

(only if part) Suppose that $(F(M), F(M_1))$ is a II$_1$ pair. Let $R, R_1$ be the hyperfinite equivalence relations whose stable ranges are $F(M), F(M_1)$. By theorem 3.15, $(R, R_1)$ is a II$_1$ pair. By lemma 3.17., there exists $\pi: X \rightarrow X_1$ and $(S, X, \mu), (T, X_1, \mu_1)$ satisfying the conditions (1) - (4). Theorem 0.2.3 then implies that

$$\mathcal{N}^w = (S, X, \mu).$$
We let
\[ Y_n = \{ y \in X_1: \text{the dimension of } L^2(\pi^{-1}(y), \mu^Y) = n \}, \]
n = 1, 2, \ldots, \infty, \text{ and decompose}
\[ M_1 = \mathcal{W}^*(\mathcal{T}, X_1, \mu_1) = \sum_{n} \mathcal{W}^*(\mathcal{T}, Y_n, \mu_1) \]
Letting \( X_n = \pi^{-1}(Y_n) \), we have
\[ M = \sum_{n} \mathcal{W}^*(\mathcal{S}, X_n, \mu) \]
Each \( (X_n, \mu) \) is isomorphic to \( (Y_n, \mu_1 \times \nu_n) \) for some probability
measure space \( (Z_n, \nu_n) \). It thus suffices to construct the desired
conditional expectation and verify the relation of the Cartan
subalgebras for each summand individually.

So we consider the case that \( (\mathcal{S}, X, \mu) = (\mathcal{S}, X_1 \times Y, \mu_1 \times \nu) \) for some
measure space \( (Y, \nu) \) in \( R_{+1} \), and \( \pi \) the canonical projection onto \( X_1 \).

We have an injection
\[ i: B(L^2(X_1, \mu_1) \otimes \ell^2(Z)) \to B(L^2(X, \mu) \otimes \ell(Z)) \]
given by \( i(x) = x \otimes 1 \), where \( 1 \) is the unit element in \( B(L^2(Y, \nu)) \). As
before \( i(M_1) \) \( M \) is a unital embedding of \( M_1 \) into \( M \). We identify \( M_1 \)
with \( i(M_1) \). There exists an isometry
\[ j: L^2(X_1, \mu_1) \otimes \ell^2(Z) \to L^2(X, \mu) \otimes \ell(Z) \]
j(\( \xi \)) = \( \xi \otimes 1 \),
where \( 1 \) is the function identically equal to \( 1 \) on \( (Y, \nu) \). Define a map
E: \( M \rightarrow \mathcal{M}_1 \) by \( E(x) = j^* x j \), for \( x \in M \). Then \( E \) is a faithful normal conditional expectation, just as in Theorem 2.3. Let

\[
A = \pi_S(L^\infty(X, \mu)) \subseteq M
\]

\[
A_1 = \pi_T(L^\infty(X_1, \mu_1)) \subseteq M_1
\]

be the Cartan subalgebras. Let \( \mu = \int \nu^Y d\mu_1(y) \) be the disintegration of \( \mu \). For \( a \in L^\infty(X, \mu), \xi \in L^2(X \times Z), y \in X_1, \)

\[
(j^* \pi_S(a) j) \xi(y, m) = \int (\pi_S(a) j) \xi(x, m) d\nu^Y(x)
\]

\[
= \int a o s^m(x) (j^* \xi)(x, m) d\nu^Y(x)
\]

\[
= \int a o s^m(x) \xi(y, m) d\nu^Y(x)
\]

\[
= (a(x^\nu) o s^{-m}(x^\nu)) d\nu^Y(x)
\]

\[
= \int a(x^\nu) d\nu^T(x) \xi(y, m), \quad \text{since } \nu^v o s^{-m} = \nu^T a.e. \ y \text{ by 3.11 and 3.17(3)}
\]

\[
= b o T^m(y) \xi(y, m) = (b \xi)(y, m),
\]

where \( b(y) = \int a(x) d\nu^Y(x) \in L^\infty(X_1, \mu_1) \). Hence \( E(a) = b \in A_1 \) and \( E(A) = A_1 \). The conditions (ii) and (iii) are trivially verified. \( \square \)
BIBLIOGRAPHY


[28] C. C. Moore, Ergodic Theory and Von Neumann Algebras, 

[29] S. Stratila and L. Zsido, Lectures on Von Neumann Algebras, 

[30] C. E. Sutherland, Notes on orbit equivalence, Krieger’s theorem, 
Lecture note series No.23 Univ. i Oslo, (1976).

[31] M. Takesaki, Theory of Operator algebras 1, Springer-Verlag, 
(1979).


[33] R. J. Zimmer, Hyperfinite factors are amenable ergodic actions, 