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Ready to Preorder: The Case of Weak Process Semantics*

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Abstract

Recently, Aceto, Fokkink & Ingólfsdóttir proposed an algorithm to turn any sound and ground-complete axiomatisation of any preorder listed in the linear time – branching time spectrum at least as coarse as the ready simulation preorder, into a sound and ground-complete axiomatisation of the corresponding equivalence—its kernel. Moreover, if the former axiomatisation is ω-complete, so is the latter. Subsequently, de Frutos Escrig, Gregorio Rodríguez & Palomino generalised this result, so that the algorithm is applicable to any preorder at least as coarse as the ready simulation preorder, provided it is initials preserving. The current paper shows that the same algorithm applies equally well to weak semantics: the proviso of initials preserving can be replaced by other conditions, such as weak initials preserving and satisfying the second τ-law. This makes it applicable to all 87 preorders surveyed in "the linear time – branching time spectrum II" that are at least as coarse as the ready simulation preorder. We also extend the scope of the algorithm to infinite processes, by adding recursion constants. As an application of both extensions, we provide a ground-complete axiomatisation of the CSP failures equivalence for BCCS processes with divergence.

1. Introduction

The lack of consensus on what constitutes an appropriate notion of observable behaviour for reactive systems has led to a large number of proposals for behavioural equivalences and preorders for concurrent processes. These have been surveyed in the *linear time-branching time spectrum*, for concrete semantics [13], and for weak semantics that take into account the internal action τ [11]. Typically, a given semantical notion induces both a preorder and an equivalence, where the equivalence is the kernel of the corresponding preorder, meaning that two processes are considered equivalent if, and only if, each is a refinement of the other with respect to the preorder.

For equational reasoning about processes expressed in some process algebra, an axiomatisation of the semantics under consideration (both for the preorder and the equivalence) is required. This axiomatisation should be *sound*, and preferably also *ground-complete*, for the process algebra modulo the semantics at hand, meaning that all equivalent closed terms can be equated. Ideally, such an axiomatisation is also ω-*complete*, meaning that whenever all closed instances of an equation can be derived from it, then so can the equation itself. [3, 6, 14] offer positive and negative results on the existence of ω-complete, sound and ground-complete finite axiomatisations for several concrete behavioural equivalences and preorders in the spectrum from [13], over BCCSP. This process algebra contains only the basic operators from CCS and CSP, but is sufficiently powerful to express all finite synchronisation trees.

Such positive and negative axiomatisability results were always proved separately for a preorder and the corresponding equivalence. Aceto, Fokkink & Ingólfsdóttir [1] showed that for BCCSP such double effort can be avoided, by presenting an algorithm to turn a sound and ground-complete axiomatisation of any preorder in the linear time – branching time spectrum at least as coarse as the ready simulation preorder, into a sound and ground-complete axiomatisation of the corresponding equivalence.[1] Moreover, if the former axiomatisation is ω-complete, so is the latter. The requirement that the preorder is at least as coarse as ready simulation is essential; in [5] it was shown that for impossible futures semantics (which does not satisfy this

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[1] Another way to avoid the double effort is by deriving axiomatisations of preorders from those of the corresponding equivalences. This line of research is explored in [7].

requirement), there is a finite axiomatisation for the preorder, but not for the equivalence.

A serious drawback of the work reported in [1] is that their algorithm requires several properties to hold for the preorders to which it is applied, which have to be checked for each preorder separately. Especially their variable cancellation property is usually rather hard to prove, see [2]. Subsequently, de Frutos Escrig, Gregorio Rodríguez & Palomino [8, 9] improved upon this result, so that the algorithm is applicable not only to those preorders specifically mentioned in "the linear time – branching time spectrum" but to any preorder at least as coarse as the ready simulation preorder, provided it is *initials preserving*, meaning that a preorder relation ($p \sqsubseteq q$) implies inclusion of initial action sets ($I(p) \subseteq I(q)$). This condition is needed to guarantee soundness of the generated axiomatisation.

The current paper stems from an effort to apply the algorithm to weak semantics, which take into account internal activity τ. The results in [8, 9] do not suffice in this setting, because weak semantics tend to violate the initials preserving condition. So a new round of generalisation is needed. To this end, we show that in the setting of BCCS (BCCSP extended with τ), the algorithm originally proposed in [1] applies equally well to weak semantics; the proviso of initials preserving can be replaced by other conditions. We give three sufficient conditions on the preorder $\sqsubseteq$ and its corresponding equivalence $\equiv$: either (1) $p \equiv \tau p$ for all closed terms p; or (2) $p \equiv \tau p$ for all p with $I(p) \neq \emptyset$, and $p \sqsubseteq q$ with $I(p) \neq \emptyset$ implies $I(q) \neq \emptyset$; or (3) $\tau p \equiv \tau p + p$ for all closed terms p, and $\sqsubseteq$ is *weak* initials preserving.

This makes the algorithm applicable to all 87 preorders surveyed in "the linear time – branching time spectrum II" [11] that are coarser than the (strong) ready simulation preorder. That is, each of these preorders satisfies either the original initials preserving condition from [8, 9], or one of our three new conditions.

Moreover, we extend the scope of the algorithm to *infinite* processes, by adding recursion constants to BCCS. As an application of both extensions, we provide a ground-complete axiomatisation of the CSP failures equivalence, also known as must-testing equivalence, for BCCS processes with divergence.

2. Preliminaries

BCCS is a basic process algebra for expressing finite process behaviour. Its signature consists of the constant $\mathbf{0}$, the binary operator $\_+\_$, and unary prefix operators $\tau\_$ and $a\_$, where a is taken from a nonempty set A of *visible actions*, called the *alphabet*. We assume $\tau \notin A$, τ being the *hidden action*, and write A_τ for $A \cup \{\tau\}$, ranged over by α, β.

$$t ::= 0 \mid \alpha t \mid t + t \mid x$$

Closed BCCS terms, ranged over by p, q, r, represent finite process behaviours, where $\mathbf{0}$ does not exhibit any behaviour, $p + q$ offers a choice between the behaviours of p and q, and αp executes action α to transform into p. This intuition is captured by the transition rules below. They give rise to A_τ-labelled transitions between closed BCCS terms.

$$\frac{}{\alpha x \xrightarrow{\alpha} x} \qquad \frac{x \xrightarrow{\alpha} x'}{x + y \xrightarrow{\alpha} x'} \qquad \frac{y \xrightarrow{\alpha} y'}{x + y \xrightarrow{\alpha} y'}$$

We assume a countably infinite set V of variables; w, x, y, z denote elements of V. Open BCCS terms, denoted by t, u, v, may contain variables from V. A (closed) substitution, typically denoted by σ, maps variables in V to (closed) terms. For open terms t and u, and a preorder $\sqsubseteq$ (or equivalence $\equiv$) on closed terms, we define $t \sqsubseteq u$ (or $t \equiv u$) if $\sigma(t) \sqsubseteq \sigma(u)$ (resp. $\sigma(t) \equiv \sigma(u)$) for all closed substitutions σ. A preorder $\sqsubseteq$ is called a *precongruence* (for BCCS) if $p \sqsubseteq q$ and $p' \sqsubseteq q'$ implies that $p + p' \sqsubseteq q + q'$ and $\alpha p \sqsubseteq \alpha q$ for $\alpha \in A_\tau$. The *kernel* of a preorder $\sqsubseteq$ is $\sqsubseteq \cap \sqsubseteq^{-1}$.

An *axiomatisation* is a collection of equations $t \approx u$ or of inequations $t \preccurlyeq u$. The (in)equations in an axiomatisation E are referred to as *axioms*. If E is an equational axiomatisation, we write $E \vdash t \approx u$ if the equation $t \approx u$ is derivable from the axioms in E using the rules of equational logic (reflexivity, symmetry, transitivity, substitution, and closure under BCCS contexts). For the derivation of an inequation $t \preccurlyeq u$ from an inequational axiomatisation E, denoted by $E \vdash t \preccurlyeq u$, the rule for symmetry is omitted. We will also allow equations $t \approx u$ in inequational axiomatisations, as an abbreviation of $t \preccurlyeq u$ and $u \preccurlyeq t$.

An axiomatisation E is *sound* modulo a precongruence $\sqsubseteq$ (or congruence $\equiv$) if for all terms t, u, from $E \vdash t \preccurlyeq u$ (or $E \vdash t \approx u$) it follows that $t \sqsubseteq u$ (or $t \equiv u$). E is *ground-complete* for $\sqsubseteq$ (or $\equiv$) if for all closed terms p, q, $p \sqsubseteq q$ (or $p \equiv q$) implies $E \vdash p \preccurlyeq q$ (or $E \vdash p \approx q$). And E is *ω-complete* if for all terms t, u with $E \vdash \sigma(t) \preccurlyeq \sigma(u)$ (or $E \vdash \sigma(t) \approx \sigma(u)$) for all closed substitutions σ, we have $E \vdash t \preccurlyeq u$ (or $E \vdash t \approx u$).

Bisimilarity is the largest equivalence relation $\leftrightarrows$ such that $p \leftrightarrows q$ and $p \xrightarrow{\alpha} p'$ implies $\exists q' : q \xrightarrow{\alpha} q'$ and $p' \leftrightarrows q'$. It is completely axiomatised by the following axioms:

$$\begin{array}{rcll} x + y & \approx & y + x & \text{(A1)} \\ (x + y) + z & \approx & x + (y + z) & \text{(A2)} \\ x + x & \approx & x & \text{(A3)} \\ x + \mathbf{0} & \approx & x & \text{(A4)} \end{array}$$

Summation $\sum_{i \in \{1,\ldots,n\}} t_i$ denotes $t_1 + \cdots + t_n$, where summation over the empty set denotes $\mathbf{0}$. Binary choice $\_+\_$ and summation bind weaker than $\alpha\_$. For each closed BCCS term p there exists a finite set $\{\alpha_i p_i \mid i \in I\}$ of closed terms such that $p \leftrightarrows \sum_{i \in I} \alpha_i p_i$ and hence A1–4 $\vdash p \approx \sum_{i \in I} \alpha_i p_i$. The $\alpha_i p_i$ are called the *summands* of p.

We write $p \Rightarrow q$ if there is a (possibly empty) sequence of τ-transitions $p \xrightarrow{\tau} \cdots \xrightarrow{\tau} q$; furthermore $p \xrightarrow{\alpha}$ denotes that there is a term q with $p \xrightarrow{\alpha} q$, and likewise $p \Rightarrow\xrightarrow{\alpha}$ denotes that there are a terms q, r with $p \Rightarrow q \xrightarrow{\alpha} r$.

Definition 1 (Initial actions) For any closed term p, the set $I(p)$ of *strongly initial actions* is $I(p) = \{\alpha \in A_\tau \mid p \xrightarrow{\alpha}\}$, whereas the set $\mathcal{I}_\tau(p)$ of *weakly initial actions* is $\mathcal{I}_\tau(p) = \{\alpha \in A_\tau \mid p \Rightarrow\xrightarrow{\alpha}\}$.

A preorder $\sqsubseteq$ is *(strong) initials preserving* if $p \sqsubseteq q$ implies $I(p) \subseteq I(q)$ for all p and q; it is *weak initials preserving* if $p \sqsubseteq q$ implies $\mathcal{I}_\tau(p) \subseteq \mathcal{I}_\tau(q)$. With $\mathcal{I}(p)$ we denote $\mathcal{I}_\tau(p) \cap A$, the weakly initial *visible* actions of p.

3. The algorithm for producing equational axiomatisations

In Aceto, Fokkink & Ingólfsdóttir [1] an algorithm is presented which takes as input a sound and ground-complete inequational axiomatisation E for BCCS modulo a precongruence in the linear time – branching time spectrum that contains the ready simulation preorder, and generates as output an equational axiomatisation $\mathcal{A}(E)$ which is sound and ground-complete for BCCS modulo the corresponding congruence. Moreover, if the original axiomatisation E is ω-complete, so is the resulting axiomatisation. The axiomatisation $\mathcal{A}(E)$ generated by the algorithm from E contains the axioms A1–4 as well as the axioms:

(RS$_\equiv$): $\alpha(\beta x + z) + \alpha(\beta x + \beta y + z) \approx \alpha(\beta x + \beta y + z)$

for $\alpha, \beta \in A_\tau$, that are valid in ready simulation semantics, together with the following equations, for each inequational axiom $t \preccurlyeq u$ in E:

(1) $t + u \approx u$; and

(2) $\alpha(t + x) + \alpha(u + x) \approx \alpha(u + x)$ (for each $\alpha \in A_\tau$, and some variable x that does not occur in $t + u$).

Instead of explicitly adding the axioms RS$_\equiv$ one can equivalently add the axioms

(RS): $\beta x \preccurlyeq \beta x + \beta y$ for $\beta \in A_\tau$

to E prior to invoking steps (1) and (2) above. Moreover, as observed in [8], the conversion from E to $\mathcal{A}(E)$ can be factored into two steps:

- Given an inequational axiomatisation E, its *BCCS-context closure* $\overline{E}$ is

  $E \cup \{\alpha(t+x) \preccurlyeq \alpha(u+x) \mid \alpha \in A_\tau \wedge t \preccurlyeq u \in E\} \cup \{\text{RS}\}$

  where x is a variable not occurring in E.
- Now $\mathcal{A}(E) = \{t + u \approx u \mid t \preccurlyeq u \in \overline{E}\} \cup$ (A1–4).

In [1] the correctness of this algorithm was shown for all precongruences listed in the linear time – branching time spectrum of [13] that are included between trace inclusion and the ready simulation preorder. The proof contained a few arguments that had to be checked for each of these preorders separately. Subsequently, in de Frutos Escrig, Gregorio & Palomino [8] the following more general result was obtained:

Theorem 1 Let $\sqsubseteq$ be an initials preserving precongruence that contains the ready simulation preorder $\sqsubseteq_{\mathrm{RS}}$, and let E be a sound and ground-complete axiomatisation of $\sqsubseteq$. Then $\mathcal{A}(E)$ is a sound and ground-complete axiomatisation of the kernel of $\sqsubseteq$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$.

As all preorders in the linear time – branching time spectrum of [13] between trace inclusion and ready simulation are initials preserving, the above theorem strengthens the result of [1].

4. Correctness proof of the algorithm

Below we recreate the proof of Theorem 1. Lemma 1 and Proposition 1 constitute the completeness argument, and are taken directly from [8]. However, the proofs below are significantly simpler—in the case of Lemma 1 employing ideas from the completeness proof in [1]. The essence of Lemma 2 and its proof come from [8] as well; this is the soundness argument. Our rewording of Lemma 2 allows it to be reused in Sections 5 and 7.

Lemma 1 Let E be an inequational axiomatisation. Then for any $t \preccurlyeq u \in E$ and any context $\mathcal{C}[\cdot]$ we have $\mathcal{A}(E) \vdash \mathcal{C}(t) + \mathcal{C}(u) \approx \mathcal{C}(u)$.

Proof: By structural induction on the context $\mathcal{C}[\cdot]$.

In case of the trivial context $\mathcal{C}[\cdot] = \cdot$ we have to show $\mathcal{A}(E) \vdash t + u \approx u$, which follows immediately from step (1) in the construction of $\mathcal{A}(E)$.

For a context $\alpha(\,\cdot + v)$ we have to show $\mathcal{A}(E) \vdash \alpha(t + v) + \alpha(u + v) \approx \alpha(u + v)$, which follows from step (2) in the construction of $\mathcal{A}(E)$, substituting the term v for the variable x.

Now let the result be obtained for a context $\mathcal{D}[\cdot]$ and let $\mathcal{C}[\cdot]$ be of the form $\mathcal{D}[\cdot] + v$, where v is an arbitrary term, possibly $\mathbf{0}$. We have to show that $\mathcal{A}(E) \vdash \mathcal{D}(t) + v + \mathcal{D}(u) + v \approx \mathcal{D}(u) + v$. This follows immediately from the induction hypothesis.

Finally, let the result be obtained for a context $\beta\mathcal{D}[\cdot]$ and let $\mathcal{C}[\cdot]$ be of the form $\alpha(\beta\mathcal{D}[\cdot] + v)$. We have to obtain

$$\mathcal{A}(E) \vdash \alpha(\beta\mathcal{D}(t) + v) + \alpha(\beta\mathcal{D}(u) + v) \approx \alpha(\beta\mathcal{D}(u) + v).$$

By the induction hypothesis we have $\mathcal{A}(E) \vdash \beta\mathcal{D}(t) + \beta\mathcal{D}(u) \approx \beta\mathcal{D}(u)$, so it suffices to obtain

$$\begin{aligned}\mathcal{A}(E) \vdash \quad & \alpha(\beta\mathcal{D}(t) + v) + \alpha(\beta\mathcal{D}(t) + \beta\mathcal{D}(u) + v) \\ & \approx \alpha(\beta\mathcal{D}(t) + \beta\mathcal{D}(u) + v).\end{aligned}$$

This is an instance of the axiom $\mathrm{RS}_{\equiv}$. $\square$

Proposition 1 Let E be an inequational axiomatisation. Then whenever $E \vdash t \preccurlyeq u$ we also have $\mathcal{A}(E) \vdash t+u \approx u$.

Proof: If $E \vdash t \preccurlyeq u$ then there is a chain of terms $t_0, \ldots, t_n$ for $n \geq 0$ with $t_0 = t$ and $t_n = u$ such that for $0 \leq i < n$ the inequation $t_i \preccurlyeq t_{i+1}$ is provable from E by one application of an axiom. We now prove the claim by induction on n. The case $n = 0$ is an instance of axiom A3, and the case $n = 1$ is an immediate consequence of Lemma 1, by applying substitution.

Now for the general case, let v be t_i for some $0 < i < n$. By induction we have $\mathcal{A}(E) \vdash t + v \approx v$ and $\mathcal{A}(E) \vdash v + u \approx u$. Applying once again A3 this yields $\mathcal{A}(E) \vdash t + u \approx t + v + u \approx v + u \approx u$. $\square$

Lemma 2 Let $\sqsubseteq$ be a precongruence containing $\sqsubseteq_{RS}$ and $\equiv$ be its kernel. Let p, q be closed terms with $p \sqsubseteq q$ and $I(p) \subseteq I(q)$. Then $p + q \equiv q$.

Proof: As $p \sqsubseteq q$ and $\sqsubseteq$ is a precongruence for choice, we have $p + q \sqsubseteq q + q \sqsubseteq q$. To show that $q \sqsubseteq p + q$, let $p \leftrightarroweq \sum_{i\in I} \alpha_i p_i$ and $q \leftrightarroweq \sum_{j\in J} \beta_j q_j$. It is well known [13] that $p \leftrightarroweq q$ implies $I(p) = I(q)$ as well as $p \sqsubseteq_{RS} q$ and hence $p \sqsubseteq q$. Writing $p|_\beta$ for $\sum_{\alpha_i=\beta} \alpha_i p_i$, the collection of β-summands of p, and likewise $q|_\beta = \sum_{\beta_j:=\beta} \beta_j q_j$, we have $p \leftrightarroweq \sum_{\beta\in I(p)} p|_\beta$ and $q \leftrightarroweq \sum_{\beta\in I(q)} q|_\beta$. Using that $I(p) \subseteq I(q)$, and that $\sqsubseteq$ is a precongruence for the choice operator $+$, it suffices to show that $q|_\beta \sqsubseteq p|_\beta + q|_\beta$ for all $\beta \in A_\tau$. This is an immediate consequence of the axiom RS, which is sound for $\sqsubseteq_{RS}$ and hence for $\sqsubseteq$. $\square$

Proof of Theorem 1: As $\sqsubseteq$ is a precongruence contained in the ready simulation preorder, all inequations in the BCCS-context closure $\overline{E}$ of E are sound w.r.t. $\sqsubseteq$. Considering that the soundness of an (in)equation is tantamount to the soundness of its closed substitution instances, the soundness of $\mathcal{A}(E)$ now follows from Lemma 2.

Ground-completeness and ω-completeness follow directly from Proposition 1: If $t \equiv u$, that is $t \sqsubseteq u$ and $u \sqsubseteq t$, we have $E \vdash t \preccurlyeq u$ and $E \vdash u \preccurlyeq t$ by the completeness of E. So Proposition 1 yields $\mathcal{A}(E) \vdash t \approx t + u \approx u$. $\square$

5. Applying the algorithm to weak semantics

The results of [1, 8] were obtained for the language BCCSP, containing the basic operators of CCS and CSP. This language is obtained from BCCS as presented above by omitting the unary operator τ. Naturally, as shown above, these results generalise smoothly to BCCS by treating τ just like any visible action from A. Preorders or equivalences that do so are called *strong*. The main purpose of the present paper is to apply the same ideas to *weak* preorders: those that in some way abstract from internal activity, by treating τ differently from visible actions.

When reading Theorem 1 in the context of weak process semantics, it helps to remember that $\sqsubseteq_{RS}$ is the *strong* ready simulation preorder, and "initials preserving" refers to preservation of the *strongly* initial actions. Theorem 1 directly applies to the rooted variants of the η-simulation surveyed in [11], for these preorders are coarser than the strong ready simulation preorder and strong initials preserving. However, most weak semantics are not strong initials preserving (for instance, typically $\tau x \preccurlyeq x$ is sound), and consequently Theorem 1 fails to apply to them.

The precondition of being initials preserving is in fact nowhere used in the completeness proof in [8], or its recreation in Section 4. Hence, this condition applies to the soundness claim only. Therefore, in order to apply the algorithm to weak semantics, all we need is to find another way of guaranteeing the soundness of the generated axioms.

Given that we deal with preorders containing the ready simulation preorder, the axiom $\mathrm{RS}_{\equiv}$ will always be sound. Moreover, the axioms generated by step (2) in the construction of $\mathcal{A}(E)$ are guaranteed to be sound by Lemma 2, for we have $\alpha(t + x) \sqsubseteq \alpha(u + x)$ and $I(\alpha(t + x)) = I(\alpha(u + x)) = \{\alpha\}$. One way to guarantee soundness of the remaining axioms, is to check this for each of them explicitly:

Theorem 2 Let $\sqsubseteq$ be a precongruence that contains the ready simulation preorder $\sqsubseteq_{\mathrm{RS}}$, and let E be a sound and ground-complete axiomatisation of $\sqsubseteq$, such that for each axiom $t \preccurlyeq u$ in E the law $t + u \approx u$ is sound as well. Then $\mathcal{A}(E)$ is a sound and ground-complete axiomatisation of the kernel of $\sqsubseteq$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$. $\square$

Note that for the axioms stemming from $t \preccurlyeq u$ with $I(\sigma(t)) \subseteq I(\sigma(u))$ for any closed substitution σ, no check is needed, by Lemma 2. Next we present three other conditions that guarantee soundness of $\mathcal{A}(E)$.

Theorem 3 Let $\sqsubseteq$ be a precongruence that contains the strong ready simulation preorder $\sqsubseteq_{\mathrm{RS}}$, such that $p \equiv \tau p$, with $\equiv$ the kernel of $\sqsubseteq$, for all processes p. Let E be a sound and ground-complete axiomatisation of $\sqsubseteq$. Then $\mathcal{A}(E)$ is a sound and ground-complete axiomatisation of $\equiv$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$.

Proof: It suffices to show that $p \sqsubseteq q$ implies $p + q \equiv q$. So assume $p \sqsubseteq q$. Let $p' := \tau p$ and $q' := \tau q$. By assumption we have $p \equiv p'$ and $q \equiv q'$, and therefore $p' \sqsubseteq q'$. As

$I(p') = I(q') = \{\tau\}$, Lemma 2 yields $p' + q' \equiv q'$, which implies $p + q \equiv q$. $\square$

Theorem 4 Let $\sqsubseteq$ be a precongruence that contains the strong ready simulation preorder $\sqsubseteq_{\mathrm{RS}}$, such that $p \equiv \tau p$, with $\equiv$ the kernel of $\sqsubseteq$, for all processes p with $I(p) \neq \emptyset$, and such that $p \sqsubseteq q$ implies that if $I(p) \neq \emptyset$ then $I(q) \neq \emptyset$. Let E be a sound and ground-complete axiomatisation of $\sqsubseteq$. Then $\mathcal{A}(E)$ is a sound and ground-complete axiomatisation of $\equiv$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$.

Proof: Again it suffices to show that $p \sqsubseteq q$ implies $p+q \equiv q$. So assume $p \sqsubseteq q$. If $I(p) = \emptyset$ then trivially $I(p) \subseteq I(q)$ and the result follows from Lemma 2. Otherwise, we have $p \equiv \tau p$ and $q \equiv \tau q$ and the result follows as in the previous proof. $\square$

Let T2 be the second τ-law of CCS [15]: $\quad \tau x \approx \tau x + x$.

Theorem 5 Let $\sqsubseteq$ be a weak initials preserving precongruence that contains the strong ready simulation preorder $\sqsubseteq_{\mathrm{RS}}$ and satisfies T2, and let E be a sound and ground-complete axiomatisation of $\sqsubseteq$. Then $\mathcal{A}(E)$ is a sound and ground-complete axiomatisation of the kernel of $\sqsubseteq$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$.

Proof: A straightforward induction on the length of a path $p \Rightarrow p'$, using the soundness of T2, yields that if $p \Rightarrow p' \xrightarrow{\alpha} p''$ then $p \equiv p + \alpha p''$, where $\equiv$ is the kernel of $\sqsubseteq$. Hence for any closed term p there is a closed term p' such that $p \equiv p'$ and $\mathcal{I}_\tau(p) = I(p')$. Using this, the soundness claim follows from Lemma 2, reasoning as in the proof of Theorem 3. $\square$

Note that $\mathcal{I}_\tau(p) = \mathcal{I}(p) \cup \{\tau \mid p \xrightarrow{\tau}\}$ (see Definition 1). Thus, the precondition of Theorem 5 is that $p \sqsubseteq q$ implies that $\mathcal{I}(p) \subseteq \mathcal{I}(q)$ and that if $p \xrightarrow{\tau}$ then $q \xrightarrow{\tau}$.

So far, Theorem 2 applies to the widest selection of preorders, but it comes with the need to check the soundness of some of the generated axioms separately. We can go even further in this direction by observing that also the precondition of containing the ready simulation preorder is not used anywhere in the completeness proof:

Theorem 6 Let $\sqsubseteq$ be any precongruence, and let E be a ground-complete axiomatisation of $\sqsubseteq$. Then $\mathcal{A}(E)$ is a ground-complete axiomatisation of the kernel of $\sqsubseteq$. Moreover, if E is ω-complete, then so is $\mathcal{A}(E)$. $\square$

Note that this theorem makes no statement on the soundness of $\mathcal{A}(E)$. Hence an application of this theorem to achieve a sound and ground-complete axiomatisation involves checking the soundness of all axioms generated by both step (1) and step (2) of the algorithm explicitly, as well as the soundness of the axioms A1–4 and $\mathrm{RS}_\equiv$. As A1–4 and $\mathrm{RS}_\equiv$ constitute a sound and ground-complete axiomatisation of strong ready simulation equivalence, checking the soundness of these axioms is naturally done by checking that the kernel of $\sqsubseteq$ contains strong ready simulation equivalence. As we shall illustrate in the next section, this is a meaningful improvement over the precondition of Theorem 2 that $\sqsubseteq$ contains the strong ready simulation preorder. The price to be paid for this improvement is that also the soundness of the axioms generated by step (2) of the algorithm has to be checked separately. This is because the proof of Lemma 2 uses that $\sqsubseteq$ contains the strong ready simulation preorder.

In [11] 155 weak preorders are reviewed. Most of them fail to be congruences for the choice operator of BCCS. Axiomatisations are typically proposed for the *congruence closures* of these preorders: the coarsest congruence contained in them. All preorders $\sqsubseteq$ in [11] and their congruence closures satisfy the property that if $p \sqsubseteq q$ then $\mathcal{I}(p) \subseteq \mathcal{I}(q)$. [2]

Of the 155 preorders surveyed in [11], 87 contain the strong ready simulation preorder. We can partition this collection into four classes.

6 preorders are variants of trace inclusion and the simulation preorder. They are precongruences for BCCS and satisfy the axiom $x \approx \tau x$. Consequently, they fall in the scope of Theorem 3.

16 preorders are variants of completed trace inclusion or the completed simulation preorder. Each of their congruence closures $\sqsubseteq$ has the property that $p \sqsubseteq q$ implies that if $I(p) \neq \emptyset$ then $I(q) \neq \emptyset$. Moreover, the kernels $\equiv$ of $\sqsubseteq$ have the property that $p \equiv \tau p$ for all processes p with $I(p) \neq \emptyset$. Consequently, these congruence closures fall in the scope of Theorem 4.

22 are variants of the η-simulation or the η-ready simulation. Their congruence closures are strong initials preserving, and hence fall under the scope of Theorem 1.

The congruence closures $\sqsubseteq$ of the remaining 43 preorders satisfy the property that $p \sqsubseteq q$ implies that if $p \xrightarrow{\tau}$ then $q \xrightarrow{\tau}$, and hence $\mathcal{I}_\tau(p) \subseteq \mathcal{I}_\tau(q)$. These precongruences therefore fall in the scope of Theorem 5.

Thus, the algorithm of [1] applies to all congruence closures of preorders in [11] coarser than the ready simulation preorder.

[2] In fact, most preorders in [11] are actually pairs of preorders, as for every semantics a *may* and a *must* preorder are proposed. Inspired by [10], there are two differences between the may and the must preorders. One is a different treatment of *divergence*—this has no effect when restricting attention to BCCS processes. The other is that the preorders are oriented in opposite directions. This entire paper, as well as [1, 8], has been written from the perspective of the may preorders. When dealing with must preorders $\sqsubseteq$ we have that if $p \sqsubseteq q$ then $\mathcal{I}(p) \supseteq \mathcal{I}(q)$. Moreover, none of these preorders contains $\sqsubseteq_{RS}$—at best their inverses have this property. Consequently, for preorders oriented in the must direction, the algorithm is to be applied in the reverse direction, where an inequational axiom $t \preccurlyeq u$ gives rise to equational axioms like $t \approx t + u$.

6. Applications

In De Nicola & Hennessy [10] three testing preorders are defined, and for each of them a sound and ground-complete axiomatisation over BCCS is provided. In fact the axiomatisations apply to all of CCS, enriched with a special constant Ω, and the semantics of processes involves, besides A_τ-labelled transitions, a *convergence* predicate. However, the completeness proofs remain valid when restricting attention to the sublanguage BCCS, and there the convergence predicate plays no rôle (for all processes are convergent). The *combined may- and must-testing preorder* is axiomatised by the laws A1–4 together with the axioms

$$\begin{array}{cl} \alpha x + \alpha y \approx \alpha(\tau x + \tau y) & \text{(N1)} \\ x + \tau y \preccurlyeq \tau(x+y) & \text{(N2)} \\ \alpha x + \tau(\alpha y + z) \approx \tau(\alpha x + \alpha y + z) & \text{(N3)} \\ \tau x \preccurlyeq x & \text{(N4)} \end{array}$$

where α ranges over A_τ. The *must preorder* has the additional axiom

$$\tau x + \tau y \preccurlyeq x \qquad \text{(E1)}$$

and the *may preorder* has the additional axiom

$$x \preccurlyeq \tau x + \tau y \qquad \text{(F1).}$$

Note that T2 follows from N2 and N4. We will now apply the algorithm to obtain sound and ground-complete axiomatisations of the three associated testing equivalences.

Beforehand, we mention a trivial simplification in applying the algorithm: if the inequational axiomatisation features an equation $t \approx u$, formally speaking this is an abbreviation for the two axioms $t \preccurlyeq u$ and $u \preccurlyeq t$. Thus, step (1) of the algorithm generates the equations $t + u \approx u$ and $u + t \approx t$. Together, these are equivalent to the original equation $t \approx u$. Moreover, in the presence of $t \approx u$ the two axioms generated by step (2) of the algorithm are redundant. Thus, we can simplify the algorithm by leaving equations untouched.

The may preorder. The may preorder of [10] coincides with weak trace inclusion, which is coarser than the ready simulation preorder. As remarked in [10], it is not hard to see that the axiomatisation above can be simplified to A1–4 together with

$$\begin{array}{c} \tau x \approx x \\ \alpha x + \alpha y \approx \alpha(x+y) \\ x \preccurlyeq x + y \end{array}$$

Applying Theorem 3 yields a sound and ground-complete axiomatisation of may-testing equivalence, which coincides with weak trace equivalence. It consists of A1–4, $\mathrm{RS}_\equiv$ and

$$\begin{array}{c} \tau x \approx x \\ \alpha x + \alpha y \approx \alpha(x+y) \\ x + x + y \approx x + y \\ \alpha(x+z) + \alpha(x+y+z) \approx \alpha(x+y+z) \end{array}$$

As $\mathrm{RS}_\equiv$ is an instance of the last axiom above, that last axioms follows from the second, and the third from A3, this axiomatisation can be simplified to A1–4 together with

$$\begin{array}{c} \tau x \approx x \\ \alpha x + \alpha y \approx \alpha(x+y) \end{array}$$

The must preorder. On BCCS, the must preorder of [10] coincides with the failures preorder of CSP [4]. Its inverse contains the ready simulation preorder and is weak initials preserving. Hence we can apply Theorem 5 to obtain a sound and ground-complete axiomatisation of must-testing equivalence. First we note that N4 is a simple consequence of E1 and thus can be omitted. Now Theorem 5 yields the axioms A1–4, $\mathrm{RS}_\equiv$ and

$$\begin{array}{cl} \alpha x + \alpha y \approx \alpha(\tau x + \tau y) & \text{(N1)} \\ x + \tau y \approx x + \tau y + \tau(x+y) & (\text{N2}_1) \\ \alpha(x{+}\tau y{+}z) \approx \alpha(x{+}\tau y{+}z) + \alpha(\tau(x{+}y){+}z) & (\text{N2}_2) \\ \alpha x + \tau(\alpha y + z) \approx \tau(\alpha x + \alpha y + z) & \text{(N3)} \\ \tau x + \tau y \approx \tau x + \tau y + x & (\text{E1}_1) \\ \alpha(\tau x + \tau y + z) \approx \alpha(\tau x + \tau y + z) + \alpha(x + z) & (\text{E1}_2) \end{array}$$

This axiomatisation can be simplified to A1–4 together with

$$\begin{array}{cl} \alpha x + \alpha y \approx \alpha(\tau x + \tau y) & \text{(N1)} \\ x + \tau y \approx \tau y + \tau(x+y) & (\text{N2}^*) \\ \alpha x + \tau(\alpha y + z) \approx \tau(\alpha x + \alpha y + z) & \text{(N3)} \end{array}$$

Namely, E1_1 implies T2 which allows us to reformulate N2_1 as N2^*. The latter axiom implies T2 (by taking $y = x$) and hence also N2_1 and E1_1. It remains to derive N2_2, E2_2 and $\mathrm{RS}_\equiv$. In all three cases, by N1 it suffices to derive the instance where $\alpha = \tau$. Substituting τy for y in N2^* and applying $\tau\tau y \approx \tau y$ (which follows from N1) and T2 gives $\tau(x + \tau y) \approx x + \tau y$. Now it is straightforward to derive N2_2^τ, E2_2^τ and $\mathrm{RS}_\equiv^\tau$.

This axiomatisation has been mentioned in [12], just like the axiomatisation of weak trace equivalence mentioned above. However, we have not found an actual proof of its ground-completeness (or the ground-completeness of any other axiomatisation of must-testing equivalence over BCCS) in the literature.

The combined may and must preorder. The combined may- and must-testing preorder is the intersection of the may- and the must-testing preorders. It is known that on BCCS the combined preorder has the same kernel as the must preorder, so that we can reuse the axiomatisation above. Nevertheless, obtaining a sound and complete axiomatisation of this kernel by means of the algorithm provides a useful illustration of some of the issues that play a rôle in this process. On BCCS, the combined preorder is contained in weak trace equivalence, and hence contains neither the strong ready simulation preorder, nor its inverse. Therefore, Theorems 2–5 are not applicable to it. However, its kernel does contain strong ready simulation equivalence, and with help of Theorem 6 we can obtain a sound and ground-complete axiomatisation of it. The algorithm yields the axioms A1–4, $RS_\equiv$ and

$$\alpha x + \alpha y \approx \alpha(\tau x + \tau y) \qquad (N1)$$
$$x + \tau y \approx x + \tau y + \tau(x + y) \qquad (N2_1)$$
$$\alpha(x+\tau y+z) \approx \alpha(x+\tau y+z) + \alpha(\tau(x+y)+z) \qquad (N2_2)$$
$$\alpha x + \tau(\alpha y + z) \approx \tau(\alpha x + \alpha y + z) \qquad (N3)$$
$$\tau x \approx \tau x + x \qquad (N4_1)$$
$$\alpha(\tau x + z) \approx \alpha(\tau x + z) + \alpha(x + z) \qquad (N4_2)$$

The soundness of these axioms follows from the fact that they are derivable both from the axioms for the may preorder and from the axioms for the must preorder.

As expected, the axiomatisation above is easily seen to be equivalent to the axiomatisation of must-testing equivalence.

7. A generalisation to infinite processes

The results in [1, 8] were obtained for finite processes only: processes that can be expressed in BCCSP. Hereby we extend these results to infinite processes that can be expressed by adding constants to BCCS. This is an easy way of dealing with recursion—an alternative to introducing recursion as a syntactic construct and requiring congruence properties for it. An infinite process can be defined by introducing one or more constants C together with axioms like $C \approx abC$; in this example, C represents a process that performs an infinite alternating sequence of a and b actions.

In order to obtain completeness of the axiomatisations $\mathcal{A}(E)$, any extension of BCCS with constants will do. Lemma 1, Proposition 1 and Theorem 6 remain valid in this setting. The only place where structural induction is used is in the proof of Lemma 1, and there constants do not bother us, as they cannot occur on a path from the root of a context, seen as a parse tree, to the hole.

In order to obtain soundness, we furthermore assume that for any constant C in the language there is a closed term $\sum_{i\in I} \alpha_i p_i$ in our extension of BCCS with constants—so I is finite—such that $C \leftrightarroweq \sum_{i\in I} \alpha_i p_i$. It then follows that any closed term is bisimulation equivalent to a closed term of the form $\sum_{i\in I} \alpha_i p_i$. With this assumption, all our results generalise to BCCS augmented with constants.

The proof of Lemma 2 goes through unaltered. The only proof that needs to be adapted is the one of Theorem 5.

Lemma 3 Let $\equiv$ be a congruence containing $\leftrightarroweq$ that satisfies T2. If $p \Rightarrow p' \xrightarrow{\alpha} p''$ then $p \equiv p + \alpha p''$.

Proof: By induction on the length of the path $p \Rightarrow p'$. In the base case $p = p' \leftrightarroweq \sum_{i\in I} \alpha_i p_i$, and by definition of $\leftrightarroweq$ there must be an $i \in I$ with $\alpha_i = \alpha$ and $p_i \leftrightarroweq p''$. It follows that $p \leftrightarroweq p + \alpha p''$ and hence $p \equiv p + \alpha p''$.
Now assume $p \xrightarrow{\tau} p' \Rightarrow\xrightarrow{\alpha} p''$. By induction, $p' \equiv p' + \alpha p''$. T2 yields $p \equiv p + \alpha p''$. $\square$

Proof of Theorem 5: Suppose $p \sqsubseteq q$. We have to show that $p + q \equiv q$, where $\equiv$ is the kernel of $\sqsubseteq$. By the assumption above, $p \leftrightarroweq \sum_{i\in I} \alpha_i p_i$ for a finite index set I and closed terms $\alpha_i p_i$ in our extension of BCCS with constants. We have $\{\alpha_i \mid i \in I\} = I(p) \subseteq \mathcal{I}_\tau(p) \subseteq \mathcal{I}_\tau(q)$, so for every $i \in I$ there is a term q_i such that $q \Rightarrow\xrightarrow{\alpha_i} q_i$. Let $q' := q + \sum_{i\in I} \alpha_i q_i$. Applying Lemma 3 once for every $i \in I$ we obtain $q \equiv q'$. Now $I(p) \subseteq I(q)$, so Lemma 2 yields $p + q' \equiv q'$, which implies $p + q \equiv q$. $\square$

8. Applications (continued)

Adding divergence. In [10] a special constant Ω denoting *divergence* is considered, and the three ground-complete axiomatisations of the preorders mentioned in Section 6 extend to the presence of divergence by means of the extra axiom

$$\Omega \preccurlyeq x. \qquad (\Omega)$$

Although Ω is defined in terms of a convergence predicate, in all three testing preorders it is equivalent to a process engaging in an infinite τ-loop only. We could therefore equivalently think of Ω as the process generated by adding the transition rule $\Omega \xrightarrow{\tau} \Omega$ to BCCS. This way we obtain $\Omega \leftrightarroweq \tau\Omega$, thereby fulfilling the soundness requirement of Section 7. Note that $I_\tau(\Omega) = \mathcal{I}_\tau(\Omega) = \{\tau\}$.

Invoking Theorem 3 we obtain a ground-complete axiomatisation for may-testing equivalence by adding the extra axioms

$$\Omega + x \approx x$$
$$\alpha(\Omega + z) + \alpha(x + z) \approx \alpha(x + z)$$

to the ones mentioned in Section 6. The second one is derivable from the first and $\alpha x + \alpha y \approx \alpha(x + y)$. Using A4, the first one is equivalent to $\Omega \approx \mathbf{0}$.

As the must preorder $\sqsubseteq$ satisfies $\Omega \sqsubseteq a0$ for some $a \neq \tau$, it is not weak initials preserving (in either direction) and we may not apply Theorem 5, as we did in Section 6. In order to obtain a sound and ground-complete axiomatisation of must-testing equivalence, we therefore resort to Theorem 2. Applying the algorithm to the ground-complete axiomatisation of the must preorder yields the extra axioms

$$\Omega \approx \Omega + x \qquad (\Omega_1)$$
$$\alpha(\Omega + z) \approx \alpha(\Omega + z) + \alpha(x + z) \qquad (\Omega_2)$$

Theorem 2 requires us to explicitly check the soundness of $N2_1$, $E1_1$ and Ω_1. We may not use the soundness of $N2_1$

and $\mathrm{E1}_1$ obtained in Section 6, as it could have been invalidated by the addition of Ω to the language. The soundness of $\mathrm{N2}_1$ follows from Lemma 2, applying the remark right after Theorem 2. The soundness of $\mathrm{E1}_1$ follows because it is derivable from T2, which is derivable from N2 and N4. The soundness of Ω_1 follows because it is derivable from Ω, T2 and E1, as shown in [10].

E2 and T2 yield $\Omega \approx \Omega + \tau\Omega \approx \tau\Omega$. With N1 the axiom Ω_2 follows from its instance where $\alpha = \tau$, which follows from E2 and $\tau\Omega = \Omega$. Hence a sound and ground-complete axiomatisation of must-testing equivalence, also known as the failures equivalence of CSP, consists of (N1), (N2*), (N3) and Ω_1.

Applying the algorithm to the combined may and must preorder again yields the extra axioms Ω_1 and Ω_2, and using Theorem 6 we cannot assume soundness without establishing this separately. In the presence of Ω the kernels of the must preorder and the combined preorder do not coincide, and this time Ω_1 turns out not to be sound. This is an example where we cannot apply the algorithm to obtain a sound and ground-compete axiomatisation. We conjecture that such an axiomatisation exists nonetheless, namely consisting of N1, N2*, N3 and

$$\Omega + \tau x \approx \Omega + x \qquad \text{(D1)}$$
$$\Omega + \alpha x \approx \Omega + \alpha(\Omega + x) \qquad \text{(D3)}$$

In [10] the axioms D1 and D3 have been derived from N1–4, thereby establishing their soundness.

9. Concluding remark

In [8], de Frutos Escrig, Gregorio Rodríguez & Palomino also present a simplification of the algorithm of [1] for a large class of applications. The simplification consists in skipping step (2) in favour of a *constrained similarity axiom*

(NS$_\equiv$):$N(x,y) \Rightarrow \alpha x + \alpha(x+y) \approx \alpha(x+y)$ for $\alpha \in A_\tau$

Here $N(x,y)$ is a congruence relation on processes such that $N(p,q)$ is implied by $I(p) = I(q)$. The constrained similarity axiom is a conditional equation, but it can in several cases be recast in equational terms. In the special case where $N(p,q)$ holds iff $I(p) = I(q)$, NS$_\equiv$ is equivalent to RS$_\equiv$. They show that the simplified algorithm applies to preorders $\sqsubseteq$ satisfying

(NS): $\qquad N(x,y) \Rightarrow x \preceq x + y$

and such that $p \sqsubseteq q$ implies $N(p,q)$. In case $N(p,q) \Leftrightarrow I(p) = I(q)$ we have that NS is equivalent to RS.

In applying this algorithm to τ-free preorders in the linear time – branching time spectrum, they use three different constraints N, whose ranges of application match those of our Theorems 3, 4 and 5. Yet, we have not been able to apply the simplified algorithm to weak preorders, due to the fact that we would need an asymmetric precongruence N, whereas symmetry is used crucially in the proofs in [8]. The same applies to the generalisations of the constrained similarity approach investigated in [9].

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