The numerical solution of Fredholm integral equations of the second kind

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THE NUMERICAL SOLUTION
OF FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

by

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CONTENTS

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Some application areas for Fredholm integral equations of the second kind, reprinted from "The Application and Numerical Solution of Integral Equations" (eds. R.S. Anderssen, F.R. de Hoog, and M.A. Lukas), Sijthoff and Noordhoff, 1980.

FOREWORD

This thesis is dedicated to all those who tolerated bizarre mixtures of euphoria and demented paranoia and always gave me shelter from the storm. It is especially dedicated to Mary (Why don't you just look the answer up in the back of the book?) Tierney and Chris (You must be getting pretty near the end now) Kowal who lived with graceful acceptance through the birth pangs of this thesis (and came home to tell the tale).

ACKNOWLEDGEMENTS

The majority of the work of this thesis was prepared under the direction of Professor Ian Sloan. I shall always be indebted to him for teaching me many things, and particularly, for helping me understand the balance between theory and practice. His enthusiasm and encouragement have kindled many rushes of inspiration during the time that we have worked together. More personally, I thank him for steadying my (at first) shaky confidence, and for helping me settle in to life at the University of New South Wales when I first arrived. The results of Chapter 2 of this thesis were obtained jointly with Professor Sloan and were published in [26].

During the year 1979 this work was directed temporarily by Professor W.E. Smith, during which time the material contained in Chapter 3 was developed. I appreciate greatly the patience and support which Professor Smith gave me during this difficult time, and thank him for several enlightening discussions concerning the function spaces used in Chapter 3.

The work of this thesis has been greatly enhanced by friendships with several other people. Most particularly, I have had many great conversations over the past few years with Dr. Graeme Chandler. Those which concerned mathematics were a powerful catalyst for the development of a numerical analysis of weakly singular equations. The Nikol'skii space approach to singularities was first proposed by Graeme, and I thank him for sending me the manuscript of [12] and for drawing my attention to [14].

In addition, I am indebted to several members of the School of Mathematics at the University of New South Wales for useful discussions. In particular, I thank Dr. Jeff Sanders for generously giving of his time on many occasions.

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PUBLICATION DETAILS

The publication details of this thesis are given in references [26], [27], [28] and [29] .

ABSTRACT

This thesis tackles some problems encountered in the numerical solution of Fredholm integral equations of the second kind. We are concerned specifically with the applicability and numerical performance of algorithms for these equations, and are guided by the existence of the following problems.

(i) Theoretically, the applicability of many algorithms often depends on certain highly abstract assumptions being satisfied. These assumptions are often difficult to verify in practice.

(ii) Error analyses for certain algorithms have tended to assume that the given information and the solution are smooth, and hence predict a higher order of convergence than that obtained in practice (where there are usually singularities present).

In Chapter 2 we develop practical methods for deciding whether a given integral operator is compact as an operator between certain spaces of functions. This solves a problem of type (i), since compactness is an abstract assumption used in the analysis of many algorithms for integral equations. In Chapter 3 we look at a class of weakly singular convolution type equations (typical of many that arise in practice), and answer the question: What kind of singularities arise in the solutions to such equations? In Chapter 4, the results of Chapter 3 are used to give a realistic error analysis (i.e. one which takes account of the singularities in kernel and solution) for Galerkin type methods for the class of equations introduced in Chapter 3, hence solving a problem of type (ii) for that class. The results of Chapters 3 and 4 concern only one dimensional integral equations. An analysis of collocation methods for two dimensional equations is given

in Chapter 5. Convergence rates are obtained for the cases of equations with both smooth kernels and weakly singular kernels. The analysis in the latter case depends on a characterisation of the properties of the solution to a typical two dimensional weakly singular equation. This characterisation is also given in Chapter 5. The methods proposed in Chapter 5 are illustrated in Chapter 6 by the numerical solution of a two dimensional equation arising in electrical engineering.

NOTATION

Throughout this thesis, $\mathbb{N}$ will denote the set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. C will denote a generic constant which will be allowed to vary from instance to instance. In proofs we shall mention the variables which C is independent of only when it is necessary to do so. The distributional derivative of a function ϕ will be denoted by $D\phi$ or ϕ'. If ϕ depends on more than one variable, we shall write $D_t\phi$ for the distributional derivative of ϕ with respect to the variable t. The notation for higher order derivatives is explained on p.111. Unless otherwise stated Ω will be a domain (i.e. an open connected set) which is bounded in $\mathbb{R}^n$, and $\overline{\Omega}$ will denote its closure.

In each of Chapters 1, 2, and 6 the equations are numbered consecutively within that chapter, so that, for example, they run from (2.1) to (2.21) in Chapter 2. In Chapters 3, 4, and 5 the equations are numbered consecutively within each section, so that, for example, the equation numbered (5.2.3) is the third equation in Section 2 of Chapter 5.

Function Spaces and Classes

Space/class	Norm	Page
$L_p(\overline{\Omega})$, $1 \leq p \leq \infty$	$\Vert\cdot\Vert_p$	9
$C(\overline{\Omega})$	$\Vert\cdot\Vert_\infty$	9
$M_p(\overline{\Omega})$, $1 \leq p \leq \infty$	class	12
$W_p^m[a,b]$, $1 \leq p \leq \infty$, $m \in \mathbb{N}_0$	$\Vert\cdot\Vert_{m,p}$	32

Space/class	Norm	Page
$N_p^\alpha(\mathbb{R})$, $1 \le p \le \infty$, $\alpha > 0$	$\Vert\cdot\Vert_{\alpha,p,\mathbb{R}}$	39
$N_p^\alpha[a,b]$, $1 \le p \le \infty$, $\alpha > 0$	$\Vert\cdot\Vert_{\alpha,p,[a,b]}$	39
$\Lambda(\alpha,p,q,\mathbb{R})$, $B^{\alpha,p,q}(\mathbb{R})$; $1 \le p,q \le \infty$, $\alpha > 0$	Banach Spaces but norms not required	41, 154
$S_r^\nu(\Pi_n,[a,b])$, $r \in \mathbb{N}$, $\nu \in \mathbb{N}_0$, $\nu < r$, Π_n a mesh on $[a,b]$	class of splines on $[a,b]$	76
$C^m[a,b]$, $C^m(\overline{\Omega})$, $C^m(\overline{\Omega\times\Omega})$, $Lip_\beta(\overline{\Omega})$, $Lip_\beta(\overline{\Omega\times\Omega})$, $W_2^1(\overline{\Omega})$; $m \in \mathbb{N}_0$, $0 < \beta \le 1$, Ω a bounded domain in $\mathbb{R}^2$.	Banach spaces but norms not required	111
$S_r^{r-1}(\Pi_{N(\tau)},\overline{\Omega})$, $r \in \mathbb{N}$, Ω a bounded domain in $\mathbb{R}^2$, $\{\Pi_{N(\tau)}: \tau \in (0,1]\}$ a family of M.S. meshes on $\overline{\Omega}$ (p.130).	class of splines on $\overline{\Omega}$.	132

CHAPTER 1.

INTRODUCTION

This thesis tackles some problems encountered in the numerical solution of Fredholm integral equations of the second kind. All the integral equations which we shall consider here will be of the general form

$$y(t) = f(t) + \lambda \cdot \int_{\overline{\Omega}} k(t,s)\, y(s)ds\ , \qquad t \in \overline{\Omega}\ , \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^n$ $(n = 1 \text{ or } 2)$ is a domain (i.e. an open connected set) which is bounded, and $\overline{\Omega}$ denotes its closure. The kernel k, and the inhomogeneous term f, will be given functions on $\overline{\Omega} \times \overline{\Omega}$ and $\overline{\Omega}$ respectively, λ will be a given scalar, and our task will be to determine, by numerical approximation, the unknown solution y.

We abbreviate (1.1), using operator notation, by

$$y = f + \lambda K y\ , \tag{1.2}$$

where K is the integral operator given by

$$Ky(t) = \int_{\overline{\Omega}} k(t,s)\, y(s)ds\ , \qquad t \in \overline{\Omega}\ . \tag{1.3}$$

The main body of the work in this thesis is split into five chapters - Chapters 2 to 6 inclusive. Chapters 2 and 3 consist of some new developments in the theoretical analysis of (1.1). In Chapters 4 and 5 we then use this theoretical analysis to construct and analyse the convergence of various numerical methods for solving (1.1). In Chapter 6, we illustrate the uses of our theoretical and numerical analysis with the numerical solution of an equation of the form (1.1) which arises in electrical engineering. At the end of the thesis there is an Appendix, in which we give the proofs of some of the more technical results appearing in Chapters 2 to 6. The review [27] is submitted as supporting work.

Both the theoretical analysis of Chapters 2 and 3, and the numerical analysis of Chapters 4 and 5, will be applicable to a wide class of integral equations which arise in practice. In order to demonstrate our practical motivation, let us first look briefly at a few examples of integral equations of the form (1.1) which arise in applications.

Example 1 [34]. The equation

$$y(t) = f(t) + \lambda \int_a^b |t-s|^{\alpha-1} y(s)ds, \quad t \in [a,b] , \tag{1.4}$$

where f is a function on $[a,b]$, and λ is a scalar, is the Kirkwood-Riseman equation, which arises in certain problems of polymer physics.

Example 2 [27]. The two dimensional integral equation

$$y(t_1,t_2) = C_0 + \lambda \int_{\bar{\Omega}} \ln(|(t_1,t_2)-(s_1,s_2)|)y(s_1,s_2)ds_1ds_2, \ (t_1,t_2) \in \bar{\Omega}, \tag{1.5}$$

where $\bar{\Omega}$ is a simply-connected closed plane region, C_0 and λ are scalars, and $|(x_1,x_2)| = \sqrt{x_1^2 + x_2^2}$, for $(x_1,x_2) \in \bar{\Omega}$, arises in the mathematical formulation of the problem of determining the skin effect produced when an alternating current flows in a conducting bar of cross section $\bar{\Omega}$.

Example 3 [4]. Atkinson considers the Dirichlet problem

$$\Delta u(\underset{\sim}{r}) - P(r^2)\, u(\underset{\sim}{r}) = 0 , \qquad \underset{\sim}{r} \in D ,$$
$$u(\underset{\sim}{r}) = f(\underset{\sim}{r}) , \qquad \underset{\sim}{r} \in \Gamma ,$$

where D is a plane region with boundary Γ, D and Γ satisfy suitable topological and regularity requirements, and $P(r^2) \geq 0$, where $r^2 := x^2 + y^2$, for each $\underset{\sim}{r} = (x,y) \in D$. Atkinson shows that this problem may be solved by a technique involving the numerical solution of

$$\mu(t) = -\frac{1}{\pi} f(t) - \frac{1}{\pi} \int_0^A k(t,s)\, \mu(s)ds, \quad 0 \leq t \leq A < \infty \qquad (1.6)$$

with

$$k(t,s) = k_1(t,s) \ln|\underset{\sim}{r}(t) - \underset{\sim}{r}(s)| + k_2(t,s)$$

where k_1 is continuous, k_2 is bounded and continuous except for $s = t$, and the parametrisation $\underset{\sim}{r}(t)$ is chosen so that, as t runs from 0 to A, $\underset{\sim}{r}(t)$ travels around the boundary Γ of D.

The three equations (1.4), (1.5) and (1.6) are all of the form (1.1). In each case the kernel function contains at least one term which has a "weakly singular" factor, i.e., as is the case in (1.4) and (1.5), a factor of the form $\psi(|t-s|)$, or, as is the case in (1.6), a factor of the form $\psi(|r(t) - r(s)|)$, where, in all three cases, ψ is a scalar-valued function which has an infinite singularity at the origin, but is integrable over any finite interval containing the origin. Such "weakly singular" kernels are a common feature of many cases of (1.1) which occur in applications.

We shall be concerned with the numerical solution of (1.1), and we shall be especially interested in practical integral equations of the type given in the three examples above. All of the numerical methods which we shall consider can be grouped under the general heading of *projection methods*. Before we define these methods, we usually make the assumption that (1.1), or equivalently (1.2), has a unique solution y, in some Banach space $B(\bar{\Omega})$, say, of functions defined on $\bar{\Omega}$.

Then, for $n \in \mathbb{N}$, we seek an approximation y_n^I to y of the form

$$y_n^I = \sum_{i=1}^{n} a_i u_i \quad .$$

Here $\{u_1,\ldots,u_n\} \subseteq B(\bar{\Omega})$ is a set of linearly independent basis functions, which are chosen for their suitability in approximating the unknown solution y. To find the scalar coefficients $\{a_1,\ldots,a_n\}$, we demand that

$$y_n^I = P_n y_n^I = P_n(f + \lambda K y_n^I) , \qquad (1.7)$$

where P_n is a projection (i.e. a linear idempotent operator) from $B(\bar{\Omega})$ onto $U_n := \text{span}\{u_1,u_2,\ldots,u_n\}$. The equation (1.7) holds in the n-dimensional vector space U_n, and hence is equivalent to an $n \times n$ linear system with solution set $\{a_1,\ldots,a_n\}$. This system may be solved on a computer.

Once y_n^I has been found, we may also define another approximation to y, which we denote by y_n^{II}, via the "natural iteration":

$$y_n^{II} = f + \lambda K y_n^I \quad .$$

When P_n is an orthogonal projection, y_n^I and y_n^{II} are usually called the Galerkin and iterated Galerkin solutions respectively. When P_n is an interpolatory projection, y_n^I and y_n^{II} are usually called the collocation and iterated collocation solutions respectively.

General theories for projection methods are well documented, and, in particular, the existence and rate of convergence of the first approximation, y_n^I, has been extensively studied [5], [7], [25], [31], [35], [44], [59]. More recently, in the work of Sloan [63], [57], [58] and [64], and Chandler [9], [10] and [11], a theory for the second approximation, y_n^{II}, has also been developed.

The main thrust of the work of this thesis will be towards developing analyses of the error committed when projection methods are used to solve practical integral equations of the type described in Examples 1, 2 and 3.

Most error analyses for projection methods assume that the integral operator K, given by (1.3), is compact on the Banach space $B(\overline{\Omega})$. Compactness is a property which, if possessed by K, ensures that K has some "nice" properties. For example, if K is compact on $B(\overline{\Omega})$, then the Fredholm alternative [33, p.497] allows us to make deductions concerning the existence, uniqueness, and properties of the solution y of (1.1). Moreover, compactness features crucially in the proofs of the convergence of any of the projection methods described above. However, compactness is an abstract mathematical concept which is often very difficult for the practical person to verify. Chapter 2 is devoted to the development of sufficient and also (in some cases) necessary conditions for K to be compact as an operator from a certain Banach space to another.

These conditions are designed to be simple enough as to be easily verified practically and are particularly easy to apply to the operators of the type contained in equations (1.4),(1.5) and (1.6).

Until relatively recently, error analyses of projection methods for the solution of (1.1), although often allowing k to be weakly singular, have tended to assume that y is smooth. In practice, y is rarely smooth, and the assumption of a smooth y has led to the prediction of theoretical orders of convergence that are generally higher than those achieved when weakly singular equations are solved in practice. The key to obtaining error analyses that are accurate for the weakly singular case lies in the careful characterisation of the true nature of the solution in such a case. This characterisation is obtained for a class of weakly singular equations in Chapter 3.

Then, in Chapter 4, we consider the numerical solution of the class of equations analysed in Chapter 3. Using the analysis given there, we derive order of convergence estimates for Galerkin and iterated Galerkin methods, which take into account the natural singularities which will be contained in the solution y. The numerical methods of Chapter 4 use a space of spline functions as their underlying approximating subspace.

The results of Chapters 3 and 4 are valid only for one dimensional integral equations defined over finite intervals. In Chapter 5 we consider the case when (1.1) is defined over a closed region $\bar{\Omega}$, of two dimensional space. For this case, very little information is known about the convergence of projection methods, even when the kernel and solution are smooth.

In the first two sections of Chapter 5 we introduce and prove the basic convergence properties of a class of collocation and iterated collocation methods for the solution of the two dimensional version of (1.1). This time, the underlying approximating space, which we denote by U_N, is a certain space of piecewise constant functions (i.e. splines of degree 0) defined on $\overline{\Omega}$. (The use of N instead of n here is merely a notational device to distinguish between one and two dimensional analyses.)

In Section 5.3, order of convergence estimates for these collocation methods are obtained for the case when the kernel and solution are smooth. Section 5.5 is devoted to proving the analogues of the results of Section 5.3 for a class of two dimensional weakly singular equations. The analysis depends, as in the one dimensional case, on an accurate characterisation of the smoothness properties of the solution to a typical two dimensional weakly singular equation. These properties are proved in Section 5.4.

In Chapter 6, we use the methods introduced in Chapter 5 to solve the equation (1.5) numerically. The numerical results obtained are used to check the accuracy of the order of convergence estimates derived in Chapter 5.

Each of the chapters 2, 3, and 4 have an introduction in which the leading literature on the problem to be considered is surveyed, and the main results to be proved in that particular chapter are stated. In Chapter 5 this function is performed by the first two sections. It is worth pointing out at this stage, however, that one of the main themes of the thesis, namely the characterisation of weakly singular behaviour in integral equations, and the construction of numerical methods which are best geared to cope with that behaviour, was also being investigated by several other authors while this work was progressing. The most notable of these authors are Chandler [11], [12] (see also Acknowledgements) and Schneider [53],[54], [55]. A complete survey of the recent explosion of work on weakly singular equations is contained at relevant points in Chapters 3 and 4.

The applications review [27] is included as supporting work for this thesis. As well as describing the physical theory behind equation (1.5), it also describes an important class of second kind Fredholm integral equations which arise in applications-namely those which are reformulations of boundary value problems for differential equations. Although such equations are usually not strictly of the form (1.1), they do have some of the characteristics of the equations discussed in this thesis - e.g. they have weakly singular kernels. Since [27] was written, work has progressed on the numerical solution of boundary value problems using integral equation methods on a number of fronts. Specifically, we mention the recent paper of Atkinson [6] and the continuing interest in the Boundary Integral Method e.g. [15], [23]. A very useful review of integral equation methods for boundary value problems which came to hand after [27] was written, is contained in [14].

CHAPTER 2.

THE COMPACTNESS OF INTEGRAL OPERATORS

INTRODUCTION

In this chapter we consider the linear integral operator K, defined by

$$Ky(t) = \int_{\overline{\Omega}} k(t,s)y(s)ds , \qquad (2.1)$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\overline{\Omega}$ denotes its closure, and k and y are real-valued or complex-valued functions defined on $\overline{\Omega} \times \overline{\Omega}$ and $\overline{\Omega}$ respectively. Defining, for each $t \in \overline{\Omega}$, the function k_t as

$$k_t(s) = k(t,s) , \qquad s \in \overline{\Omega} ,$$

we can rewrite (1) more concisely as

$$Ky(t) = \int_{\overline{\Omega}} k_t(s)y(s)ds .$$

We shall assume throughout the chapter that y and k_t, for each $t \in \overline{\Omega}$, are Lebesgue measurable functions, so that (1) is well defined.

We introduce the space $L_p(\overline{\Omega})$, defined for $1 \leq p \leq \infty$, to be the space of all scalar valued measurable functions on $\overline{\Omega}$ with the property that

$$\|\phi\|_p := \left\{ \int_{\overline{\Omega}} |\phi(s)|^p \, ds \right\}^{1/p} < \infty , \qquad 1 \leq p < \infty ,$$

$$\|\phi\|_\infty := \operatorname*{ess\,sup}_{s\in\overline{\Omega}} |\phi(s)| < \infty \qquad p = \infty .$$

We note that $L_p(\overline{\Omega})$ is a Banach space under the norm $\|\cdot\|_p$. We also introduce $C(\overline{\Omega})$, the space of scalar-valued functions, which are bounded and uniformly continuous on Ω. $C(\overline{\Omega})$ is a Banach space under the norm

$$\|\phi\|_\infty = \sup_{s\in\Omega} |\phi(s)| .$$

Every function in $C(\Omega)$ may be uniquely extended to all of $\overline{\Omega}$, and we shall henceforth consider functions in $C(\overline{\Omega})$ to be defined on $\overline{\Omega}$ via this extension.

We shall refer to K as the integral operator induced by the kernel k, and consider it as an operator from $L_q(\overline{\Omega})$ to $C(\overline{\Omega})$, or as an operator from $C(\overline{\Omega})$ to $C(\overline{\Omega})$, where $1 \leq q \leq \infty$.

We recall that a linear operator K is compact (or completely continuous) if it is bounded, and if the image under K of any bounded set has compact closure.

We will be concerned with the development of sufficient conditions for the integral operator induced by k to be a compact operator from $L_q(\overline{\Omega})$ to $C(\overline{\Omega})$. Of course, if k does induce such a compact operator for some q in the range $1 \leq q \leq \infty$, then for all r in $q \leq r \leq \infty$, it follows from the inclusions

$$C(\overline{\Omega}) \subseteq L_r(\overline{\Omega}) \subseteq L_q(\overline{\Omega})$$

(which are valid because $\overline{\Omega}$ is compact), that k also induces a compact operator from $L_r(\overline{\Omega})$ to $C(\overline{\Omega})$, and from $C(\overline{\Omega})$ to $C(\overline{\Omega})$. The latter is often the most important case for applications.

This work is motivated by both abstract and practical considerations.

The abstract study of compact operators has long been an important part of functional analysis, these operators being in a sense the natural extension of linear transformations in a finite-dimensional space. Similarly, the well developed spectral theory for compact operators can be seen as an elegant generalisation of the classical eigenvalue theory for matrices.

On the other hand, practical applications of this chapter arise both within and outside the present thesis. In both Chapter 4, and, more particularly, in Chapter 5, where we consider the solution of equations of the form

$$y = f + \lambda Ky , \tag{2.2}$$

the compactness of K plays a vital role in the convergence theory of numerical methods. In addition, the work of this chapter has already found applications in a broader context, for the compactness of K is found to be equivalent to two conditions on the kernel k (see Theorem 2.1), and these conditions are an important ingredient in the theory of a much wider range of numerical methods for (2.2) than those considered in this thesis, see [60], [61], and [62].

From either the theoretical or practical viewpoint, it is clear that the easy recognition of compact operators is a useful goal, and the purpose of this chapter is to make that recognition easier.

A convenient starting point is a necessary and sufficient condition for compactness, contained in Theorem 2.1 below. The theorem is based on results attributed to Radon [47]. (For a summary of Radon's results, see [70, pp.90-91].) Related results are also given by Krasnosel'skii et. al. [36].

Throughout this chapter, we use $\int f$ as an abbreviation for the integral

$$\int_\Omega f(s)ds \quad .$$

Two numbers p,q which satisfy $1 \leq p \leq \infty$ and $1/p + 1/q = 1$ (implying that q also lies in the range $1 \leq q \leq \infty$) will be referred to as *conjugate indices*. In this definition we use the convention

$$\frac{1}{\infty} = 0 \quad ,$$

and this convention will also be used elsewhere in the thesis without further comment.

THEOREM 2.1 A NECESSARY AND SUFFICIENT CONDITION FOR COMPACTNESS.

Let p,q be any pair of conjugate indices, $1 \leq p \leq \infty$. Then the integral operator K given by (1) is compact as an operator from $L_r(\bar{\Omega})$ to $C(\bar{\Omega})$ for all r in the range $q \leq r \leq \infty$ if and only if k satisfies

$$\sup_{t\in\bar{\Omega}} \|k_t\|_p < \infty \tag{2.3}$$

and

$$\lim_{t\to\tau} \|k_t - k_\tau\|_p = 0 , \quad \text{for all} \quad \tau \in \bar{\Omega} . \tag{2.4}$$

The theorem is proved in Section 2.1.

The two conditions in this theorem occupy a central place in the work of this thesis. It is therefore convenient to introduce the following definition.

DEFINITION. A kernel function k which satisfies both (2.3) and (2.4) will be said to belong to the class $M_p(\bar{\Omega})$.

For the particular case $p = 1$, Theorem 2.1 asserts that the two conditions

$$\sup_{t\in\bar{\Omega}} \int_{\Omega} |k(t,s)|ds < \infty$$

and

$$\lim_{t\to\tau} \int_{\Omega} |k(t,s) - k(\tau,s)|ds = 0 , \qquad \tau \in \bar{\Omega} ,$$

are necessary and sufficient for K to be a compact operator from $L_\infty(\bar{\Omega})$ to $C(\bar{\Omega})$, and hence are sufficient for K to be a compact operator from $C(\bar{\Omega})$ to $C(\bar{\Omega})$. These conditions, or similar ones, are often cited in papers on the numerical solution of integral equations (for example [44], [5, p.25]).

It may be noticed, however, that verification of the conditions (2.3) and (2.4) of Theorem 2.1 (and especially of the latter) is not necessarily a trivial task, even if $p = 1$. It is true that many of the commonly occurring kernels are of so-called potential type (see for example [36, p.144]), for which the compactness question has been well studied. However, more complicated kernels may present problems. Consider, for example, the kernel given by

$$k(t,s) = \cos(ts)|t - s|^{-\frac{1}{4}} \ln|t + s| (1 - s^2)^{-\frac{1}{2}}, \tag{2.5}$$

with $\overline{\Omega} = [-1,1] \subseteq \mathbb{R}$. In this case the verification of (2.3), for appropriate values of p, is straightforward, but the direct verification of (2.4) involves much tedious analysis.

The problem is further complicated if the underlying space is of more than one dimension. Consider, for example, the difficulty of analysing the two-dimensional analogue of (2.5),

$$k(t,s) = \cos(t\cdot s) \|t - s\|^{-\frac{1}{4}} \ln\|t + s\| (1 - \|s\|^2)^{-\frac{1}{2}} \tag{2.6}$$

where $t,s \in \overline{\Omega} \subseteq \mathbb{R}^2$, $t\cdot s$ is the inner product of t and s, and $\|\cdot\|$ denotes, say, the Euclidean norm in $\mathbb{R}^2$.

It is clear from these examples that the practical value of Theorem 2.1 depends on the development of useable tests for determining when (2.3) and (2.4) are satisfied, i.e. for determining values of p for which $k \in M_p(\overline{\Omega})$.

The first such test, expressed in Theorem 2.2 below, is based on the recognition that the kernels occurring in practice often consist, as in (2.5) and (2.6) above, of the product of a finite number of more or less simple factors. (They may also, of course, consist of a sum of such products. However, the handling of sums is in practice trivial, since the sum of two compact operators is compact.)

The purpose of the theorem is to show that if k is a product of factors k_i, $i = 1,\ldots,m$, and if each factor k_i satisfies the conditions (2.3) and (2.4) of Theorem 2.1, with p replaced by p_i, then k itself also satisfies the conditions for a certain value of p.

THEOREM 2.2 KERNEL FUNCTIONS OF PRODUCT TYPE.

Let $k(t,s) = k^{(1)}(t,s)\, k^{(2)}(t,s)\ldots k^{(m)}(t,s)$, where $k^{(i)} \in M_{p_i}(\bar{\Omega})$, $1 \leq i \leq m$, with $1 \leq p_i \leq \infty$, and let the numbers $p_1,\ldots,p_m$ be such that

$$\frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_m} = \frac{1}{p} \leq 1 \quad .$$

Then $k \in M_p(\bar{\Omega})$.

This theorem is proved in Section 2.2.

To make use of Theorem 2.2, one should be able to determine, for each factor k_i in the kernel, the values of p_i for which $k_i \in M_{p_i}(\bar{\Omega})$. Two special cases of importance are dealt with in Theorem 2.3. Between them, they appear to cover the great majority of cases likely to be encountered in practice.

The first part of Theorem 2.3 deals with the case of continuous kernels, for which the result is especially simple.

THEOREM 2.3(i) CONTINUOUS KERNELS.

If k is continuous on $\bar{\Omega} \times \bar{\Omega}$, then $k \in M_p(\bar{\Omega})$ for all p in the range $1 \leq p \leq \infty$.

(Note that it is not necessary to specify a norm on the space $\bar{\Omega} \times \bar{\Omega}$, because all norms on a finite-dimensional space are equivalent.)

The second part of Theorem 2.3 is designed to handle kernels (or factors within a kernel) of the difference form $k(t,s) = \psi(s-t)$, or other similar forms such as $\psi(s+t)$, or even just $\psi(s)$. More generally, we consider $k(t,s) = \psi(s-g(t))$ where g is a continuous function from $\bar{\Omega}$ to $\mathbb{R}^n$. The set $\bar{\Omega}^*$ in the theorem is simply the set of all values of the argument of ψ as s and t range over $\bar{\Omega}$.

THEOREM 2.3(ii) DIFFERENCE-TYPE KERNELS.

Let the kernel function k be given by

$$k(t,s) = \psi(s-g(t)), \qquad s,t \in \bar{\Omega} \ ,$$

where g is a continuous function from $\bar{\Omega}$ to $\mathbb{R}^n$. Moreover, let $\psi \in L_p(\bar{\Omega}^*)$ for some p in the range $1 \leq p < \infty$, where $\bar{\Omega}^* = \{s - g(t) : s,t \in \bar{\Omega}\}$. Then $k \in M_p(\bar{\Omega})$.

EXAMPLE. If $\bar{\Omega} = [-1,1] \subseteq \mathbb{R}$, and $k(t,s) = |t - s|^{-1/\alpha}$ with $1 < \alpha < \infty$, then the theorem can be applied with $g(t) = t$, $\psi(x) = |x|^{-1/\alpha}$ and $\bar{\Omega}^* = [-2,2]$. Since $\psi \in L_p(\bar{\Omega}^*)$ if $1 \leq p < \alpha$, it follows that $k \in M_p(\bar{\Omega})$ for all p in the range $1 \leq p < \alpha$.

Theorems 2.3(i) and 2.3(ii) are proved in Section 2.3.

Taken together, Theorems 2.2 and 2.3 give a method for determining, in most cases, whether a given kernel function satisfies (2.3) and (2.4). If it does, then Theorem 2.1 gives a range of values of r for which K is compact from $L_r(\bar{\Omega})$ to $C(\bar{\Omega})$.

A bonus from Theorem 2.1 is that, since (2.3) and (2.4) are both necessary and sufficient, their necessity can be used, in principle and often also in practice, to determine the range of values of r for which K is not compact. However, the remaining theorems, 2.2 and 2.3 express merely sufficient conditions, and so cannot be used to prove non-compactness.

The following three sections are devoted to the proofs of the theorems stated above. In the final section, Section 2.4, we discuss an example to illustrate the way the results can be used in practice.

2.1. A NECESSARY AND SUFFICIENT CONDITION FOR COMPACTNESS.

The main result of this section is the proof of Theorem 2.1, which is stated in the Introduction to the chapter. The proof follows easily from three results, Theorems R1-R3 below, which are attributed to Radon (as described in the Introduction).

THEOREM R1. Let p,q be conjugate indices, $1 \leq p \leq \infty$, and let K be the integral operator defined by (2.1). Then K operates from $L_q(\overline{\Omega})$ to $C(\overline{\Omega})$ if and only if

(i) $$\sup_{t\in\Omega} \|k_t\|_p < \infty ,$$

and

(ii) for all measurable subsets D of $\overline{\Omega}$ and for any $\tau \in \overline{\Omega}$, we have

$$\lim_{t\to\tau} \int_D k_t(s)ds = \int_D k_\tau(s)ds .$$

PROOF. Suppose K operates from $L_q(\overline{\Omega})$ to $C(\overline{\Omega})$, with

$$Ky(t) = \int_\Omega k_t(s)y(s)ds = \int k_t y , \quad y \in L_q(\overline{\Omega}) ,$$

for some q in the range $1 \leq q \leq \infty$. We first prove (ii). Suppose D is a measurable subset of $\overline{\Omega}$, and let χ_D denote the characteristic function on $\overline{\Omega}$ of the set D . Now $\chi_D \in L_q(\overline{\Omega})$, since $\overline{\Omega}$ is compact and hence has finite measure, and it follows from the assumption that $K\chi_D \in C(\overline{\Omega})$. Since

$$K\chi_D(t) = \int_D k_t(s)ds ,$$

(ii) then follows.

To prove (i), we first observe from the assumption that $k_t y$ is integrable for all $t \in \overline{\Omega}$ and all $y \in L_q(\overline{\Omega})$, from which it follows that $k_t y \in L_1(\overline{\Omega})$. Hence we can assert that $k_t \in L_p(\overline{\Omega})$ – for $1 < q < \infty$ a proof is indicated in [30, p.232, (15.14) (b)] , and for $q = 1$ in [30, p.348] . For $q = \infty$ the result follows easily by considering Kz , where z is the function on $\overline{\Omega}$ which is identically 1 .

Now for each $t \in \overline{\Omega}$ define Φ_t on $L_q(\overline{\Omega})$ by

$$\Phi_t(y) = \int k_t y \quad . \tag{2.7}$$

It is clear from Holder's inequality that Φ_t is a continuous linear functional on $L_q(\overline{\Omega})$, and that

$$\|\Phi_t\| \leq \|k_t\|_p \quad .$$

We now demonstrate, using standard methods, that in fact

$$\|\Phi_t\| = \|k_t\|_p \quad . \tag{2.8}$$

Consider first p in the range $1 \leq p < \infty$. If $\|k_t\|_p = 0$, then (2.8) follows trivially. If $\|k_t\|_p > 0$, let

$y = |k_t|^{p-1}\,\overline{\mathrm{sgn}}(k_t)/\|k_t\|_p^{p/q}$, where $\overline{\mathrm{sgn}}(w)$ is zero if w is zero and is $\bar{w}/|w|$ if w is non-zero. It then follows that $y \in L_q(\bar{\Omega})$, $\|y\|_q = 1$, and

$$|\Phi_t(y)| = \|k_t\|_p \quad ,$$

from which (2.8) follows. For $p = \infty$, either $\|k_t\|_\infty = 0$, in which case (2.8) is trivially satisfied, or $\|k_t\|_\infty > 0$. In the latter case, let $\varepsilon > 0$ and $E = \{s \in \bar{\Omega}\colon |k_t(s)| > \|k_t\|_\infty - \varepsilon\}$. It is clear that $0 < \mu(E) \leq \mu(\bar{\Omega}) < \infty$, where for any measurable set A, the measure $\mu(A)$ is given by

$$\mu(A) = \int \chi_A \ .$$

If we define y by

$$y = \frac{1}{\mu(E)}\,\chi_E\,\overline{\mathrm{sgn}}\,(k_t) \ ,$$

then it follows that $\|y\|_1 = 1$, and that

$$|\Phi_t(y)| = \frac{1}{\mu(E)} \int_E |k_t(s)|\,ds \geq \|k_t\|_\infty - \varepsilon \ .$$

Since this is true for arbitrary $\varepsilon > 0$, (2.8) is satisfied.

Now since K operates into $C(\bar{\Omega})$ and since $\bar{\Omega}$ is compact, it follows that, for all $y \in L_q(\bar{\Omega})$,

$$\sup_{t\in\bar{\Omega}} |\Phi_t(y)| \leq N_y \quad ,$$

where N_y is a positive number which may depend on y . It follows by the principle of uniform boundedness [50, p.103], applied to the Banach space $L_q(\bar{\Omega})$, that

$$\sup_{t\in\bar{\Omega}} \|\Phi_t\| < \infty \ .$$

Then, on using (2.8), (i) follows.

Conversely, let conditions (i) and (ii) hold. First we consider p in the range $1 < p \leq \infty$, so that q lies in the range $1 \leq q < \infty$. Let $y \in L_q(\overline{\Omega})$, and let $\varepsilon > 0$. Since the simple (step) functions are dense in $L_q(\overline{\Omega})$, there exists a simple function g such that

$$\|y - g\|_q < \varepsilon \quad . \tag{2.9}$$

Now fix τ in $\overline{\Omega}$. From the triangle inequality and Holder's inequality it follows that for all $t \in \overline{\Omega}$

$$|Ky(t) - Ky(\tau)| \leq |Ky(t) - Kg(t)| + |Kg(t) - Kg(\tau)| + |Kg(\tau) - Ky(\tau)|$$

$$\leq \int |k_t(y-g)| + |Kg(t) - Kg(\tau)| + \int |k_\tau(g-y)|$$

$$\leq \|k_t\|_p \|y-g\|_q + |Kg(t) - Kg(\tau)| + \|k_\tau\|_p \|g-y\|_q \tag{2.10}$$

Now since g is a simple function, it follows from (ii) that there exists $\delta > 0$ such that, for all $t \in \overline{\Omega}$ satisfying $|t - \tau| < \delta$, we have

$$|Kg(t) - Kg(\tau)| < \varepsilon \quad ,$$

and hence from (2.9) and (2.10),

$$|Ky(t) - Ky(\tau)| \leq (2 \sup_{t\in\overline{\Omega}} \|k_t\|_p + 1)\varepsilon$$

This implies, with the aid of (i), that $Ky \in C(\overline{\Omega})$ as required.

For the case $p = 1$, refer to [21, p.291]. From this source it follows that if k satisfies conditions (i) and (ii), then k_t converges weakly (in the sense of Dunford and Schwartz [21, p.67] to k_τ in $L_1(\overline{\Omega})$ as $t \to \tau$, for all $\tau \in \overline{\Omega}$. Hence by the known results on the representation of linear functionals on $L_1(\Omega)$ [50, p.136], it then follows that, for all $y \in L_\infty(\overline{\Omega})$ and for all $\tau \in \overline{\Omega}$,

$$\lim_{t\to\tau} \int k_t y = \int k_\tau y \quad .$$

Thus, since $Ky(t) = \int k_t y$, it follows that $Ky \in C(\overline{\Omega})$. This completes the proof of Theorem R1.

THEOREM R2. Let K be the integral operator defined by (2.1), and let q lie in the range $1 \leq q \leq \infty$. If K operates from $L_q(\overline{\Omega})$ to $C(\overline{\Omega})$, then K is bounded.

PROOF. The proof follows immediately from Theorem R1, with the aid of Holder's inequality. Let $y \in L_q(\overline{\Omega})$ with $\|y\|_q \leq 1$, and consider the uniform norm of Ky in $C(\overline{\Omega})$. Then

$$\|Ky\| = \sup_{t\in\Omega} \left| \int_{\overline{\Omega}} k_t(s)\ y(s)\ ds \right|$$

$$\leq \sup_{t\in\Omega} \|k_t\|_p \|y\|_q \leq \sup_{t\in\Omega} \|k_t\|_p < N ,$$

where N is some positive number independent of y . So K is bounded with $\|K\| \leq N$.

THEOREM R3. Let p,q be conjugate indices, $1 \leq p \leq \infty$, and let K be the integral operator defined by (2.1). Suppose K operates from $L_q(\overline{\Omega})$ to $C(\overline{\Omega})$. Then K is compact if and only if,

$$\lim_{t\to\tau} \|k_t - k_\tau\|_p = 0 , \qquad \text{for all} \quad \tau \in \overline{\Omega} .$$

PROOF. Suppose K is compact as an operator from $L_q(\overline{\Omega})$ to $C(\overline{\Omega})$ for some q in the range $1 \leq q \leq \infty$. Then conditions (i) and (ii) of Theorem R1 hold for the kernel k .

For $t \in \overline{\Omega}$, let Φ_t denote the linear functional on $L_q(\overline{\Omega})$ defined by (2.7). Then by an argument similar to that used in proving (2.8), it follows that for all $t, \tau \in \overline{\Omega}$, $\Phi_t - \Phi_\tau$ is also a linear functional on $L_q(\overline{\Omega})$, and satisfies

$$\|\Phi_t - \Phi_\tau\| = \|k_t - k_\tau\|_p \quad . \tag{2.11}$$

But we also have, by definition,

$$\|\Phi_t - \Phi_\tau\| = \sup_{y \in B_q} \left| \int (k_t - k_\tau) y \right| = \sup_{y \in B_q} |Ky(t) - Ky(\tau)| \ , \tag{2.12}$$

where B_q denotes the closed unit ball in $L_q(\overline{\Omega})$.

Since K is compact, the Ascoli-Arzela theorem [21, p.266] implies that the set $K\,B_q$ must be equicontinuous, hence it follows from (2.12) that

$$\lim_{t \to \tau} \|\Phi_t - \Phi_\tau\| = 0 \ , \qquad \text{for all} \quad \tau \in \overline{\Omega} \quad .$$

Hence, using (2.11), it follows that

$$\lim_{t \to \tau} \|k_t - k_\tau\|_p = 0 \ , \qquad \text{for all} \quad \tau \in \overline{\Omega} \quad ,$$

as required.

Conversely, suppose

$$\lim_{t \to \tau} \|k_t - k_\tau\|_p = 0 \qquad \text{for all} \quad \tau \in \overline{\Omega} \quad .$$

This implies that the mapping $t \to k_t$ which, by Theorem R1, maps $\overline{\Omega}$ into $L_p(\overline{\Omega})$, is continuous. Hence, since $t \to k_t$ is a continuous mapping from a compact metric space to another metric space, it follows (see, for example [21, p.24]) that this mapping is also uniformly continuous.

To prove that K is compact, we must show that the closure of KB_q is compact as a subset of $C(\overline{\Omega})$. We do this by showing that KB_q is bounded and equicontinuous, and evoking the Ascoli-Arzela theorem. It follows easily from Holder's inequality that, for all $t, \tau \in \overline{\Omega}$ and all $y \in B_q$, we have

$$|Ky(t) - Ky(\tau)| \leq \|k_t - k_\tau\|_p \quad . \tag{2.13}$$

Now fix $\varepsilon > 0$. The uniform continuity of the mapping $t \to k_t$ then implies the existence of $\delta > 0$ with the property that

$$\|k_t - k_\tau\|_p < \varepsilon \; ,$$

for all t, τ in $\overline{\Omega}$ satisfying $|t - \tau| < \delta$. Thus it follows from (13) that

$$|Ky(t) - Ky(\tau)| < \varepsilon \quad , \tag{2.14}$$

for all t, τ in $\overline{\Omega}$ satisfying $|t - \tau| < \delta$, and all $y \in B_q$. Hence KB_q is an equicontinuous subset of $C(\overline{\Omega})$. Also, Theorem R2 implies that K is bounded, so KB_q is also a bounded set. The Ascoli-Arzela theorem then implies that the closure of KB_q is compact, and this completes the proof of the compactness of K.

We now prove the main result of this section.

PROOF OF THEOREM 2.1. Suppose K is compact as an operator from $L_r(\Omega)$ to $C(\overline{\Omega})$ for all r in the range $q \leq r \leq \infty$. Then, using the specific case of $r = q$, we have, by Theorem R1

$$\sup_{t \in \Omega} \|k_t\|_p < \infty \quad ,$$

and, by Theorem R3,

$$\lim_{t\to\tau} \|k_t - k_\tau\|_p = 0 , \qquad \text{for all} \quad \tau \in \overline{\Omega} .$$

Thus k satisfies conditions (2.3) and (2.4).

Conversely, suppose $k \in M_p(\overline{\Omega})$; that is, k satisfies (2.3) and (2.4). Let D be any measurable subset of $\overline{\Omega}$, and let $\tau \in \overline{\Omega}$. Then

$$\left|\int_D k_t(s)ds - \int_D k_\tau(s)ds\right| \le \int_D |k_t(s) - k_\tau(s)|ds \le \int_\Omega |k_t(s)-k_\tau(s)|ds$$

$$\le \|k_t - k_\tau\|_p \, (\mu(\overline{\Omega}))^{1/q} \to 0 \qquad \text{as} \quad t \to \tau \quad ,$$

since $k \in M_p(\overline{\Omega})$.

Then we deduce from Theorems R1 and R2 that K is a bounded operator from $L_q(\overline{\Omega})$ to $C(\overline{\Omega})$, and in turn, from Theorem R3, that K is compact as an operator from $L_q(\overline{\Omega})$ to $C(\overline{\Omega})$. If r is any number in the range $q \le r \le \infty$, it then follows trivially, as discussed in the Introduction, that K is compact as an operator from $L_r(\overline{\Omega})$ to $C(\overline{\Omega})$. Thus the proof of Theorem 2.1 is complete.

2.2 KERNEL FUNCTIONS OF PRODUCT TYPE.

The main result of this section is Theorem 2.2, which is stated in the Introduction. The proof follows easily once some preliminary results have been established. We will require the following simple consequence of the Hölder inequality, stated without proof.

PROPOSITION. Suppose p, p_1, p_2 satisfy

$$1 \le p \le \infty , \; 1 \le p_1 \le \infty , \; 1 \le p_2 \le \infty , \quad \text{and} \quad 1/p_1 + 1/p_2 = 1/p. \qquad (2.15)$$

Moreover, suppose X is any measure space and let $f \in L_{p_1}(X)$, $g \in L_{p_2}(X)$. Then $fg \in L_p(X)$ and

$$\|fg\|_p \leq \|f\|_{p_1} \|g\|_{p_2} \quad .$$

Then we have the

Lemma Suppose $k(t,s) = k^{(1)}(t,s)\, k^{(2)}(t,s)$, for all $(t,s) \in \overline{\Omega} \times \overline{\Omega}$, and let p, p_1, p_2 be numbers satisfying (2.15), such that $k^{(1)} \in M_{p_1}(\overline{\Omega})$ and $k^{(2)} \in M_{p_2}(\overline{\Omega})$. Then it follows that $k \in M_p(\overline{\Omega})$.

Proof. Suppose the hypotheses are satisfied, and let $t \in \overline{\Omega}$. Then

$$\|k_t\|_p = \|k_t^{(1)}\, k_t^{(2)}\|_p$$

$$\leq \|k_t^{(1)}\|_{p_1} \|k_t^{(2)}\|_{p_2} \quad ,$$

which implies

$$\sup_{t\in\Omega} \|k_t\|_p \leq \sup_{t\in\Omega} \|k_t^{(1)}\|_{p_1} \sup_{t\in\Omega} \|k_t^{(2)}\|_{p_2} < \infty \quad ,$$

and therefore (2.3) is established.

Next, let $t, \tau \in \overline{\Omega}$, and consider

$$\|k_t - k_\tau\|_p = \|k_t^{(1)}\, k_t^{(2)} - k_\tau^{(1)}\, k_\tau^{(2)}\|_p$$

$$= \|k_t^{(1)}\, k_t^{(2)} - k_t^{(1)} k_\tau^{(2)} + k_t^{(1)} k_\tau^{(2)} - k_\tau^{(1)} k_\tau^{(2)}\|_p$$

$$\leq \|k_t^{(1)}(k_t^{(2)} - k_\tau^{(2)})\|_p + \|(k_t^{(1)} - k_\tau^{(1)})\, k_\tau^{(2)}\|_p$$

$$\leq \|k_t^{(1)}\|_{p_1} \|k_t^{(2)} - k_\tau^{(2)}\|_{p_2} + \|k_t^{(1)} - k_\tau^{(1)}\|_{p_1} \|k_\tau^{(2)}\|_{p_2}$$

$$\to 0 \ , \quad \text{as} \quad t \to \tau \ .$$

Thus (2.4) is satisfied and $k \in M_p(\bar{\Omega})$.

PROOF OF THEOREM 2.2. The theorem follows easily from the Lemma by induction on m , where m is the number of factors in the product.

2.3 TWO IMPORTANT SPECIAL CASES.

This section is devoted to the proof of Theorems 2.3 (i) and (2.3) (ii), stated in the Introduction. The first is the simpler result and is already known [22, p.657], but a proof is included for completeness.

PROOF OF THEOREM 2.3(1) . Since $\bar{\Omega}$ is compact, $\bar{\Omega} \times \bar{\Omega}$ is compact and k , since it is continuous on $\bar{\Omega} \times \bar{\Omega}$, must also be bounded there. Thus, by a trivial argument, we have, for any p in the range $1 \leq p \leq \infty$,

$$\sup_{t \in \Omega} \|k_t\|_p < \infty \ .$$

Also, the function $(t,s) \to k(t,s)$ is continuous, and hence uniformly continuous on $\bar{\Omega} \times \bar{\Omega}$ with the uniform topology. Thus, choosing $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|k(t,s) - k(\tau,s)| < \varepsilon \ ,$$

for all $s \in \bar{\Omega}$, and all $t,\tau \in \bar{\Omega}$ satisfying

$$|t - \tau| < \delta \ .$$

It follows easily from this that, for any p in the range $1 \leq p \leq \infty$,

$$\lim_{t\to\tau} \|k_t - k_\tau\|_p = 0 \quad , \quad \text{for all} \quad \tau \in \overline{\Omega} \, ,$$

and the theorem is proved.

PROOF OF THEOREM 2.3(ii) Let $\| \ \|_p^*$ denote the p norm in the space $L_p(\overline{\Omega}^*)$, and for all $t \in \overline{\Omega}$ let $\psi_t(s) = \psi(s - g(t))$. It then follows that

$$\begin{aligned} \|k_t\|_p &= \|\psi_t\|_p \\ &\leq \|\psi\|_p^* < \infty \, , \quad \text{for all} \quad t \in \overline{\Omega} \, , \end{aligned}$$

where the first inequality is achieved merely by extending the domain of integration from $\overline{\Omega}$ to $\overline{\Omega}^*$. Thus (2.3) follows.

To prove (2.4), let $\varepsilon > 0$ be given. Since $1 \leq p < \infty$ and since $\psi \in L_p(\overline{\Omega}^*)$, it follows [50, p.71] that there exists $F \in C(\overline{\Omega}^*)$ such that

$$\|\psi - F\|_p^* < \varepsilon/3 \, . \tag{2.16}$$

For all $t \in \overline{\Omega}$ let $F_t(s) = F(s - g(t))$. Then, fixing $\tau \in \overline{\Omega}$, we can write, for all $t \in \overline{\Omega}$,

$$\begin{aligned} \|k_t - k_\tau\|_p &= \|\psi_t - \psi_\tau\|_p \\ &= \|\psi_t - F_t + F_t - F_\tau + F_\tau - \psi_\tau\|_p \\ &\leq \|\psi_t - F_t\|_p + \|F_t - F_\tau\|_p + \|F_\tau - \psi_\tau\|_p \quad . \end{aligned} \tag{2.17}$$

Now, for any $t \in \overline{\Omega}$ we have

$$\|\psi_t - F_t\|_p = \left\{ \int_{\overline{\Omega}} |\psi(s - g(t)) - F(s - g(t))|^p ds \right\}^{1/p}$$

$$\leq \left\{ \int_{\overline{\Omega}^*} |\psi(x) - F(x)|^p dx \right\}^{1/p}$$

$$= \|\psi - F\|_p^* < \varepsilon/3 , \tag{2.18}$$

by (2.16) .

Also, $\overline{\Omega}^*$ is compact, since it is the image of the compact set $\overline{\Omega} \times \overline{\Omega}$ under the continuous mapping $(s,t) \to s - g(t)$, and it follows that F , being continuous on $\overline{\Omega}^*$, must also be uniformly continuous there. Hence we can find $\delta > 0$ such that

$$|F(s - g(t)) - F(s - g(\tau))| < \varepsilon/3(\mu(\overline{\Omega}))^{1/p} , \tag{2.19}$$

for all $s \in \overline{\Omega}$ and all $t \in \overline{\Omega}$ satisfying

$$|g(t) - g(\tau)| < \delta \quad .$$

By the continuity of g we can find $\delta' > 0$ such that (2.19) holds for all $s \in \overline{\Omega}$ and all $t \in \overline{\Omega}$ satisfying

$$|t - \tau| < \delta' .$$

Thus, if $|t - \tau| < \delta'$, we have

$$\|F_t - F_\tau\|_p = \left\{ \int_{\overline{\Omega}} |F(s - g(t)) - F(s - g(\tau))|^p ds \right\}^{1/p}$$

$$< (\varepsilon/3(\mu(\overline{\Omega}))^{1/p}) \left\{ \int_{\overline{\Omega}} ds \right\}^{1/p} = \varepsilon/3 . \tag{2.20}$$

Using (2.18) and (2.20) in (2.17), it follows that

$$\|k_t - k_\tau\|_p < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon ,$$

for all $t \in \overline{\Omega}$ satisfying

$$|t - \tau| < \delta' .$$

Since ε was chosen arbitrarily, (2.4) follows, and the theorem is proved.

2.4 A PRACTICAL ILLUSTRATION.

We now show how to apply Theorems 2.1 - 2.3 to find a range of values of p for which the induced operator of the kernel function

$$k(t,s) = \cos(ts)\ |t - s|^{-\frac{1}{4}}\ \ell n|t + s|\ (1 - s^2)^{-\frac{1}{2}}$$

is compact as an operator from $L_p(\overline{\Omega})$ to $C(\overline{\Omega})$, where $\overline{\Omega} = [-1,1] \subseteq \mathbb{R}$.

We adopt the following notation:

$$k^{(1)}(t,s) = \cos(ts) ,$$

$$k^{(2)}(t,s) = |t - s|^{-\frac{1}{4}} ,$$

$$k^{(3)}(t,s) = \ell n\ |t + s| ,$$

$$k^{(4)}(t,s) = (1 - s^2)^{-\frac{1}{2}} .$$

Our first step is to investigate the ranges of p for which each of the above functions is in $M_p(\overline{\Omega})$. To do this we use the following easily verified facts:

Let $0 < b < \infty$. Then,

(F1) If $\psi(x) = x^{-1/\alpha}$ on the interval $(0,b)$, where $1 < \alpha < \infty$, then $\psi \in L_p[0,b]$ for all p in the range $1 \leq p < \alpha$.

(F2) If $\psi(x) = \ell n\,|x|$ on the interval $(0,b)$, then $\psi \in L_p[0,b]$ for all p in the range $1 \leq p < \infty$.

Now, with the aid of these facts, we apply the results of Theorem 2.3.

(1) $k^{(1)}$ is continuous on $\overline{\Omega} \times \overline{\Omega}$ and thus, by Theorem 2.3(i), $k^{(1)} \in M_p(\overline{\Omega})$, for all p in the range $1 \leq p \leq \infty$.

(2) $k^{(2)}(t,s) = \psi(\, s - g(t))$ where $g(t) = t$ and $\psi(x) = |x|^{-\frac{1}{4}}$, and employing (F1) and Theorem 2.3(ii) we can infer that $K^{(2)} \in M_p(\overline{\Omega})$ for all p in the range $1 \leq p < 4$.

(3) $k^{(3)}(t,s) = \psi(s - g(t))$ where $g(t) = -t$ and $\psi(x) = \ell n\,|x|$. Using (F2) and Theorem 2.3 (ii), it follows that $k^{(3)} \in M_p(\overline{\Omega})$ for $1 \leq p < \infty$.

(4) $k^{(4)}(t,s) = \psi(s - g(t))$ where $g(t) = 0$ and $\psi(x) = (1-x^2)^{-\frac{1}{2}}$. It is easy to verify that $\psi \in L_p(\overline{\Omega}^*)$ for p in the range $1 \leq p < 2$, which in turn implies that $k^{(4)} \in M_p(\overline{\Omega})$ for p in $1 \leq p < 2$. (Note that in this case $\overline{\Omega} = \overline{\Omega}^*$.)

The next step is to collect results (1), (2), (3) and (4) above, and use Theorem 2.2 to infer that $k \in M_p(\overline{\Omega})$ where p is any number strictly less than the number P given by

$$\begin{aligned} \frac{1}{P} &= \frac{1}{\infty} + \frac{1}{4} + \frac{1}{\infty} + \frac{1}{2} \\ &= \frac{3}{4}\ . \end{aligned}$$

So $k \in M_p(\overline{\Omega})$ for any p in the range

$$1 \leq p < \frac{4}{3} \quad .$$

The final step is to employ Theorem 2.1 to assert (because the conjugate index of 4/3 is 4) that K is compact from $L_r(\overline{\Omega})$ to $C(\overline{\Omega})$ (and hence also from $C(\overline{\Omega})$ to $C(\overline{\Omega})$) , where r is any number in the range

$$4 < r \leq \infty \quad . \tag{2.21}$$

In this particular example we can also use Theorem 2.1 directly to show that K is not compact from $L_r(\overline{\Omega})$ to $C(\overline{\Omega})$ if r is any number outside the range (2.21). To see this, let $t = 1$ and consider

$$\|k_1\|_p^p = \int_{-1}^{1} |\cos(s)(1-s)^{-\frac{1}{4}} \ell n(1+s)(1-s^2)^{-\frac{1}{2}}|^p \, ds$$

$$= \int_{-1}^{1} \left|\frac{\cos(s)\ \ell n(1+s)}{(1-s)^{3/4}(1+s)^{1/2}}\right|^p ds \quad ,$$

which is clearly infinite if $p \geq 4/3$. So $k \notin M_p(\overline{\Omega})$ if $p \geq 4/3$, and Theorem 2.1 implies that K is not compact from $L_r(\overline{\Omega})$ to $C(\overline{\Omega})$ for any r outside the range (2.21).

CHAPTER 3

SINGULARITY EXPANSIONS FOR THE SOLUTIONS OF WEAKLY SINGULAR EQUATIONS

3.1 INTRODUCTION

In this chapter we will be concerned with integral equations of the form

$$y(t) = f(t) + \lambda \int_a^b k(t-s)y(s)ds, \qquad t \in [a,b], \qquad (3.1.1)$$

where $-\infty < a < b < \infty$, and $\lambda \in \mathbb{C}$. The kernel, k, and the inhomogeneous term, f are given real or complex-valued functions on $[a-b, b-a]$ and $[a,b]$ respectively, and y is the solution to be determined.

Throughout the chapter, we shall abbreviate (3.1.1) by

$$y = f + K_\lambda y ,$$

where $K_\lambda = \lambda K$, and K denotes the linear integral operator given by:

$$Ky(t) = \int_a^b k(t-s)y(s)ds , \qquad t \in [a,b] . \qquad (3.1.2)$$

It is obvious that, if a solution y of (3.1.1) exists (and conditions sufficient to ensure this will be assumed), then y will inherit its properties from the given information k, f, and λ. However, the more intimate connections between the given information and the induced solution are not yet fully understood.

It is the aim of this chapter to investigate these connections, with emphasis, in particular, on the case when k is weakly singular.

This case is exemplified by the prototype equations,

$$y(t) = f(t) + \lambda \int_a^b |t-s|^{\alpha-1} y(s)ds , \quad t \in [a,b], \ 0 < \alpha < 1, \tag{3.1.3}$$

and

$$y(t) = f(t) + \lambda \int_a^b \ell n|t-s| \, y(s)ds, \quad t \in [a,b] . \tag{3.1.4}$$

Equations of this form often arise in practical applications. For examples, see [27], [54], [55], and the references given there.

Before we can state the main results of the chapter, some explanation of notation is necessary.

For any interval $[a,b]$, and any $\phi \in L_1[a,b]$, we define the indefinite integral

$$I_{[a,b]}\phi(t) = \int_a^t \phi(x)dx , \quad t \in [a,b] , \tag{3.1.5}$$

and abbreviate this by $I\phi(t)$ when the interval $[a,b]$ is unambiguous.

We introduce the class of Sobolev spaces $W_p^m[a,b]$, which are defined for $m \in \mathbb{N}_0$, and $1 \leq p \leq \infty$ by

$$W_p^m[a,b] = \{\phi \in L_p[a,b] : \phi^{(i)} \in L_p[a,b], \ i = 0,\dots,m\} , \tag{3.1.6}$$

where the derivatives of ϕ are calculated in the domain (a,b). For any $m \in \mathbb{N}_0$, $W_p^m[a,b]$ is a Banach space under the norm

$$\|\phi\|_{m,p} = \sum_{i=0}^{m} \|\phi^{(i)}\|_p .$$

Note that $W_p^m[a,b] \subseteq C^{m-1}[a,b]$, $m \in \mathbb{N}$.

The main result of the chapter (Theorem 3,9 of Section 3.3) will show that, if $f \in W_1^r[a,b]$ for some $r \in \mathbb{N}_0$, and if k is weakly singular (a notion we shall make precise below), then the solution y of (3.1.1) may be expanded as a linear combination of (known) singular terms plus a smoother (unknown) remainder function. We shall refer to expansions of this type as "Singularity Expansions".

To be more explicit, Theorem3.9 will show that, if $f \in W_1^r[a,b]$, for some $r \in \mathbb{N}_0$, then for any $m \in \mathbb{N}_0$, we have

$$\left.\begin{aligned} y &= f \\ &+ \sum_{j=r}^{m} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell (DK_\lambda^n)^{j-r} K_\lambda f^{(r)} && \text{(a)} \\ &+ \sum_{i=1}^{m} \sum_{j=i}^{m} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell (DK_\lambda^n)^{i-1} k_{j-i} && \text{(b)} \\ &+ \phi \end{aligned}\right\} \qquad (3.1.7)$$

where $\phi \in W_1^{m+1}[a,b]$, $I = I_{[a,b]}$,

$$k_{j-i}(t) = c_{j-i}k(t-a) - d_{j-i}k(t-b), \quad t \in [a,b], \; i=1,\dots,m, \; j=i,\dots,m,$$

and the constants c_{j-i}, d_{j-i} and positive integer n will be identified in terms of known quantities in Section 3.3. The value of n will depend on the strength of the singularity in k. The expansion (3.1.7) is written for general $r \in \mathbb{N}_0$, and $m \in \mathbb{N}_0$, with the convention that, when $r > m$, (3.1.7) (a) is void and when $m = 0$, (3.1.7)(b) is void.

Note that $f \in W_1^r[a,b]$ implies that $f \in W_1^{r'}[a,b]$, for any $r' \leq r$, and so the dominant singularities in y may often be written in terms of known functions in more than one way. However, if we choose r to be the largest possible integer such that $f \in W_1^r[a,b]$, then we shall minimise the number of functions in (3.1.7) (a), and the singularity expansion (3.1.7) will be in its simplest possible form.

In particular, if $r > m$, the summation (3.1.7)(a) gives no contribution at all and (3.1.7) takes a much simpler form, all the singular terms in this case being induced by the kernel k, and contained in the summation (3.1.7)(b).

On the other hand, if we take the trivial case $m = 0$, then (3.1.7)(b) gives no contribution at all, and (3.1.7)(a) only gives a contribution if $r = 0$.

As an example, consider the case of (3.1.1) with

$$\left.\begin{aligned} &[a,b] = [0,1] \,, \\ &k(x) = |x|^{-\frac{1}{2}} \,, \qquad x \in [-1,1] \\ \text{and}\quad &f(t) = e^t \,, \qquad t \in [0,1] \,. \end{aligned}\right\} \tag{3.1.8}$$

In this example, f is infinitely continuously differentiable on $[0,1]$, and so, for any $m \in \mathbb{N}_0$, we can always find r with $r > m$, and $f \in W_1^r[a,b]$. The summation (3.1.7)(a) can thus be neglected in this case. Also, we shall show in Section 3.3 that, for this kernel, $n = 2$. Hence, for any $m \in \mathbb{N}_0$, y has the singularity expansion

$$y = f + \sum_{i=1}^{m} \sum_{j=i}^{m} \sum_{\ell=0}^{1} I^j K_\lambda^\ell (DK_\lambda^2)^{i-1} k_{j-i} + \phi \,, \tag{3.1.9}$$

where

$$I = I_{[0,1]} ,$$

$$k_{j-i}(t) = c_{j-i}\, t^{-\frac{1}{2}} - d_{j-i}(1-t)^{-\frac{1}{2}} , \quad t \in [0,1] ,$$

for some constants c_{j-i} and d_{j-i}, and

$$\phi \in W_1^{m+1}[a,b] .$$

If we choose $m = 0$ the summation in (3.1.9) gives no contribution, and we merely have

$$y = f + \phi ,$$

where $\phi \in W_1^1[a,b]$.

The practical value of the expression (3.1.7) clearly hinges on whether its terms (which are all obtained as the images of known functions under various combinations of the operators K, I and D) can be evaluated explicitly. Illustrations of practical methods for calculating these singular terms for given k and f are given in Section 3.4. In most cases these terms are integrals which, although they do not have a closed form, may be expanded explicitly in terms of known singular terms using fairly simple techniques.

In particular, the singularity expansion (3.1.9) for example (3.1.8) will be shown to have the specific form

$$\begin{aligned} y(t) = e^t + \Bigg\{ & \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} t^j (\ell n t)^i (t^{\frac{1}{2}} + t \ell n t) \\ & + \sum_{i=0}^{m-1} \sum_{j=i}^{m-1} (1-t)^j (\ell n(1-t))^i ((1-t)^{\frac{1}{2}} + (1-t)\ell n(1-t)) \Bigg\} \\ & + \phi(t) , \quad t \in [0,1] , \end{aligned} \tag{3.1.10}$$

where $\phi \in W_1^{m+1}[0,1]$, and $\{a(t) + b(t) + \ldots + z(t)\}$ denotes a linear combination of the functions $a(t), b(t), \ldots,$ and $z(t)$.

In order to complete this programme, the necessary theoretical details are first proved in Section 3.2.

There has been interest in equations of the form (3.1.1) for quite some time. For example, in [39] , an asymptotic expansion for the solution of (3.1.4) was obtained.

Singularity expansions of the type described in this chapter were first introduced by Richter [49] . Richter's technique, based on the smoothing properties of K , was shown to be valid in the particular cases of (3.1.4) and (3.1.3) $(\frac{1}{2} < \alpha < 1)$, given sufficient differentiability of f .

The results given here allow us to obtain singularity expansions of arbitrary length for the solution of (3.1.1) when k is any weakly singular kernel, and allow f to be (in the worst case) merely an $L_1[a,b]$ function, and so encompass the results of Richter as a special case.

Related regularity results are contained in the recent work of Chandler [11] , [12] , and Schneider [54] . Both these authors obtain results about the general smoothness properties of the generalisation of (3.1.1) obtained by replacing $k(t-s)$ by $k(t-s)m(t,s)$ where k is weakly singular and m smooth. The results given here, which lead to singularity expansions, are dependent on the explicit difference-type kernel, and extensions to more general kernels do not appear to follow easily. Here we present much more detailed information for the less general case.

Results of the type given here have important numerical consequences. Until relatively recently, error analyses for numerical methods for solving (3.1.1) have tended to assume that the solution is smooth, and hence are somewhat inapplicable to practical situations. Knowledge of the smoothness of the solution will enable accurate and practical error predictions. It will often be the case that convergence rates for existing methods will be considerably slower for a non smooth solution, than for a smooth solution. However, a judicious modification of existing methods to take account of singular behaviour in the solution will speed convergence considerably. Explicit knowledge of singularities will obviously be an important pre-requisite for the optimal modification of methods.

In Chapter 4, we shall use the results given to analyse the convergence of the Galerkin and iterated Galerkin methods for weakly singular equations. We shall also show how to obtain better convergence rates by taking into account the (now known) singularity of the solution.

Similar programmes for the product integration method have been carried out by Chandler [11], [12] and Schneider [55]. From an application point of view, the success of such an approach had been demonstrated previously by Noble [43].

Finally, we note the work of Sloan [60] and Mayers [40], where it is pointed out that misleadingly high convergence rates have often been attained for the numerical solution of (3.1.1) by using as a test example, a special case in which the solution is contrived to be smooth. Now that it is known what the singularities in the solution of (3.1.1) are in general, realistic testing of numerical methods in the manner suggested by Sloan should follow.

3.2. THEORETICAL BASIS

In this section we develop some theoretical results concerning the properties of the integral operator K given by (3.1.2), when the kernel k is weakly singular.

The correct choice of function space setting is crucial to the theory, and we shall see that, for the prototype equations (3.1.3) and (3.1.4), the usual L_p setting is somewhat inappropriate. This is because, when f is sufficiently smooth, the solutions of (3.1.3) and (3.1.4) have first derivatives which behave, respectively, like $(t - a)^{\alpha-1}$ and $\ell n(t - a)$ near $t = a$, and have equivalent singularities near $t = b$. Now, when $0 < \alpha < 1$, the function $(t - a)^{\alpha-1}$ certainly belongs to the space $L_p[a,b]$ for any p in the range $1 \leq p < \frac{1}{1-\alpha}$, and $\ell n(t - a)$ is in the space $L_p[a,b]$ for any p in the range $1 \leq p < \infty$. However, since these functions are also smooth (except at one point), it is inappropriate to cast them in some $L_p[a,b]$, since such a space also contains many non-smooth functions. A more appropriate setting is provided by spaces of functions with fractional derivatives. A setting of this nature has been suggested, in slightly different ways, in each of [11], [49] and [54], and we adopt here the setting suggested by Chandler, [11].

We introduce the following notation.

For any $\alpha > 0$, let a and α_0 denote numbers such that

$$\left.\begin{aligned} &[\alpha] \in \mathbb{N}_0 \ , \qquad 0 < \alpha_0 \le 1 \ , \\ \text{and} \\ &\alpha = [\alpha] + \alpha_0 \ . \end{aligned}\right\} \tag{3.2.1}$$

Note that $[\alpha]$ denotes the largest integer less than α, and *not* the integer part of α. For $h \in \mathbb{R}$, and any function $\phi : \mathbb{R} \to \mathbb{C}$, let Δ_h denote the usual forward difference operator:

$$\Delta_h \phi(t) = \phi(t + h) - \phi(t) \ .$$

Then the Nikol'skii space $N_p^\alpha(\mathbb{R})$, defined for $1 \le p \le \infty$, by

$$N_p^\alpha(\mathbb{R}) = \left\{\phi \in L_p(\mathbb{R}) \ : \ |\phi|_{\alpha,p,\mathbb{R}} : \sup_{h\neq 0} \frac{\|\Delta_h^2 \phi^{[\alpha]}\|_{L_p(\mathbb{R})}}{|h|^{\alpha_0}} < \infty\right\} ,$$

is a Banach space under the norm

$$\|\phi\|_{\alpha,p,\mathbb{R}} = \|\phi\|_{L_p(\mathbb{R})} + |\phi|_{\alpha,p,\mathbb{R}} \ .$$

For any interval $[a,b]$, the space $N_p^\alpha[a,b]$ is defined by

$$N_p^\alpha[a,b] = \left\{\phi \in L_p[a,b] : |\phi|_{\alpha,p,[a,b]} : = \sup_{h\neq 0} \frac{\|\Delta_h^2 \phi^{[\alpha]}\|_{L_p[a,b]_{2h}}}{|h|^{\alpha_0}} < \infty\right\} ,$$

where, for any $\epsilon \in \mathbb{R}$,

$$[a,b]_\epsilon = \{t \in [a,b] : t + \epsilon \in [a,b]\} , \tag{3.2.2}$$

and is a Banach space under the norm

$$\|\phi\|_{\alpha,p,[a,b]} = \|\phi\|_{L_p[a,b]} + |\phi|_{\alpha,p,[a,b]} \ .$$

We abbreviate these norms by $\| \ \|_{\alpha,p}$ when no confusion can occur.

Remark. This is the definition of $N_p^\alpha[a,b]$ which is given by Nikol'skii [42, p.160]. The device (3.2.2) ensures that the norm is taken over an interval for which $\Delta_h^2\,\phi^{[\alpha]}$ is well defined. Equivalent definitions of $N_p^\alpha[a,b]$ which employ some variant of (3.2.2) are given in [42]. One of these equivalent definitions was used in [28].

Example. Consider the function $t^{\alpha-1}$ $(0 < \alpha < 1)$ defined for $t \in [0,1]$. Then for $0 < h \leq 1/2$

$$\|\Delta_h^2(t^{\alpha-1})\|_{L_1[0,1]_{2h}} = \|(t+2h)^{\alpha-1} - 2(t+h)^{\alpha-1} + t^{\alpha-1}\|_{L_1[0,1]_{2h}}$$

$$\leq \|(t+2h)^{\alpha-1} - (t+h)^{\alpha-1}\|_{L_1[0,1]_{2h}} + \|(t+h)^{\alpha-1} - t^{\alpha-1}\|_{L_1[0,1]_{2h}}$$

$$= \int_0^{1-2h} ((t+h)^{\alpha-1} - (t+2h)^{\alpha-1})\,dt + \int_0^{1-2h} (t^{\alpha-1} - (t+h)^{\alpha-1})\,dt$$

(since $t^{\alpha-1}$ is decreasing)

$$= \int_0^{1-2h} (t^{\alpha-1} - (t+2h)^{\alpha-1})\,dt$$

$$= \frac{1}{\alpha}\,[t^\alpha - (t+2h)^\alpha]_0^{1-sh} \leq Ch^\alpha ,$$

where C is a constant. This argument can be used to obtain a similar result for $-\tfrac{1}{2} \leq h < 0$. Since, for $|h| > 1/2$ we have $[0,1]_{2h} = \emptyset$, it follows that $t^{\alpha-1} \in N_1^\alpha[0,1]$.

In fact, it can be shown that $t^{\alpha-1} \notin N_1^\gamma[0,1]$, for any $\gamma > \alpha$, and also [71, p.73] that $\ln t \in N_1^1[0,1]$, but $\ln t \notin N_1^\gamma[0,1]$ for any $\gamma > 1$, this last fact being the motivation for the use of the second difference in the definitions of $N_1^\alpha[0,1]$.

We have the following continuous imbeddings [37, p.p.383-384, p.p.389-391]

$$\left.\begin{array}{ll} & N_p^{m+\epsilon}[a,b] \subseteq W_p^m[a,b] \subseteq N_p^m[a,b] \subseteq N_p^{m-\epsilon}[a,b], \\ \text{for} & m \in \mathbb{N}\ ,\ 0 < \epsilon < 1\ ,\quad \text{and} \quad 1 \leq p \leq \infty\ , \\ \text{and} & \\ & N_p^{\alpha}[a,b] \subseteq N_q^{\beta}[a,b], \\ \text{for} & \alpha > 0\ ,\quad 1 \leq p \leq q \leq \infty\ ,\quad \text{and} \\ & \beta = \alpha - (\frac{1}{p} - \frac{1}{q}) > 0\ . \end{array}\right\} \tag{3.2.3}$$

The first chain of imbeddings demonstrates the fact that the Sobolev spaces $W_p^m[a,b]$ are naturally immersed in the continuum of Nikol'skii spaces $N_p^{\alpha}[a,b]$, while the second imbedding shows that, given a function in a certain Nikol'skii space, we may trade in some of its differentiability to obtain some stronger integrability properties. For further details of the properties of Nikol'skii spaces, see [11], [37], [42] and [69] .

We shall use the results of Taibleson [67] concerning certain Lipschitz spaces of functions, $\Lambda(\alpha,p,q,\mathbb{R}^n)$, which are defined for $\alpha > 0$, $\infty \geq p \geq 1$, $\infty \geq q \geq 1$, and $n \geq 1$. We show in Theorem A1 that, in fact,

$$N_1^{\alpha}(\mathbb{R}) = \Lambda(\alpha,1,\infty,\mathbb{R}) \tag{3.2.4}$$

We shall make use of relation (3.2.4) in the proof of Theorem 3.3 below. The proof requires the following lemma.

<u>Lemma 3.1</u> Let $k \in L_1[a-b,\ b-a]$, and let $y \in W_1^1[a,b]$. Then $Ky \in W_1^1[a,b]$, and

$$(Ky)'(t) = y(a)k(t-a) - y(b)k(t-b) + Ky'(t),\quad \text{for almost all} \quad t \in [a,b].$$

<u>Proof</u> A proof is given in Theorem A3 .

We shall refer to the following assumptions:

A1. $k \in N_1^{\alpha}[a-b,\ b-a]$, for some α in the range $0 < \alpha < 1$.

A2. The homogeneous version of (3.1.1),

$$y(t) = \lambda \int_a^b k(t-s)y(s)ds \quad ,$$

has no non-trivial solutions in $L_1[a,b]$.

Assumption A1 ensures that k is "weakly singular", while A2 will allow us to invoke the Fredholm Alternative.

<u>Note.</u> The function $k(x) = \ln|x|$ is in $N_1^1[a-b,\ b-a]$, and hence satisfies A1 for all α in the range $0 < \alpha < 1$.

<u>Theorem 3.3</u> Let A1 be satisfied. Then

(i) $K : N_1^{\gamma}[a,b] \to N_1^{\alpha+\gamma}[a,b]$, $0 < \gamma < 1$,

(ii) $K : N_1^{1}[a,b] \to N_1^{\alpha+\gamma}[a,b]$, $0 < \gamma < 1$,

(iii) $K : N_1^{\gamma}[a,b] \to N_1^{\alpha+1}[a,b]$, $\gamma > 1$,

and the mappings (i), (ii) and (iii) are bounded.

The proof of Theorem 3.3 will be given below; the key ingredient is the observation that Ky is simply the restriction to $[a,b]$ of the convolution

$$k_e * y_e(t) = \int_{-\infty}^{\infty} k_e(t-s)\ y_e(s)ds \ , \qquad t \in \mathbb{R} \ , \qquad (3.2.5)$$

where y_e equals y on $[a,b]$ and zero elsewhere, and k_e is the analogous extension of k from $[a-b,\ b-a]$ to $\mathbb{R}$.

With this observation, the proof is obtained by utilising the results of Taibleson [67 II, Lemma 1] on convolution in the space $\Lambda(\alpha,1,\infty,\mathbb{R})$ (i.e. $N_1^{\alpha}(\mathbb{R})$, by (3.2.4)). The success of the argument depends on the properties of the above extensions of y to y_e and k to k_e. The following lemma studies the properties of such extensions.

<u>Lemma 3.2</u> Let $\phi \in N_1^{\gamma}[u,v]$, where $-\infty < u < v < \infty$, and $0 < \gamma < 1$, and define ϕ_e on $\mathbb{R}$ by

$$\phi_e(t) = \phi(t), \qquad t \in [u,v],$$

and

$$\phi_e(t) = 0, \qquad t \in \mathbb{R}\setminus [u,v].$$

Then the extension map: $\phi \to \phi_e$ is a continuous linear operator from $N_1^{\gamma}[u,v]$ to $N_1^{\gamma}(\mathbb{R})$.

<u>Proof.</u> It will be sufficient to prove this result for $[u,v] = [0,1]$. For, suppose the result holds for $[0,1]$, and let $\phi \in N_1^{\gamma}[u,v]$, where $0 < \gamma < 1$, and $[u,v]$ is any interval. Then we may define $\Phi \in N_1^{\gamma}[0,1]$ with

$$\|\Phi\|_{\gamma,1,[0,1]} \leq C\, \|\phi\|_{\gamma,1,[u,v]}, \tag{3.2.6}$$

by

$$\Phi(t) = \phi((v-u)t+u), \qquad t \in [0,1],$$

and we may extend Φ to $\Phi_e \in N_1^{\gamma}(\mathbb{R})$, where

$$\|\Phi_e\|_{\gamma,1,\mathbb{R}} \leq C\|\Phi\|_{\gamma,1,[0,1]}. \tag{3.2.7}$$

Now ϕ_e, the extension of ϕ from $[u,v]$ to $\mathbb{R}$, satisfies

$$\phi_e(x) = \Phi_e\left(\frac{x-u}{v-u}\right), \qquad x \in \mathbb{R},$$

and it follows that $\phi_e \in N_1^\gamma(\mathbb{R})$,

and that

$$\|\phi_e\|_{\gamma,1,\mathbb{R}} \leq C\|\phi_e\|_{\gamma,1,\mathbb{R}} \quad . \tag{3.2.8}$$

Since the constants C in (3.2.6), (3.2.7) and (3.2.8) are independent of ϕ, the continuity of the mapping $\phi \to \phi_e$ then follows.

To prove the result on $[0,1]$, let $\phi \in N_1^\gamma[0,1]$ for some $0 < \gamma < 1$. Then, for any $h > 0$, $\Delta_h^2 \phi_e$ will be zero outside $[-2h, 1]$, and so

$$\int_{-\infty}^{\infty} |\Delta_h^2 \phi_e(t)|dt = \int_{-2h}^{1} |\Delta_h^2 \phi_e(t)|dt \; . \tag{3.2.9}$$

For $1/4 \geq h > 0$, (3.2.9) gives

$$\int_{-\infty}^{\infty} |\Delta_h^2\phi_e(t)|dt = \int_{-2h}^{0} |\Delta_h^2\phi_e(t)|dt + \int_{1-2h}^{1} |\Delta_h^2\phi_e(t)|dt + \int_{0}^{1-2h} |\Delta_h^2\phi_e(t)|dt \; . \tag{3.2.10}$$

By definition of ϕ_e , and, since $\phi \in N_1^\gamma[0,1]$, we have

$$\begin{aligned}\int_{-2h}^{0} |\Delta_h^2\phi_e(t)|dt &= \int_{-2h}^{0} |\phi_e(t+2h) - 2\phi_e(t+h) + \phi_e(t)|dt \\ &\leq \int_{0}^{2h} |\phi(t)|dt + 2 \int_{0}^{h} |\phi(t)|dt \\ &\leq C\, h^\gamma \|\phi\|_{\gamma,1,[0,1]} \; ,\end{aligned} \tag{3.2.11}$$

where the final inequality follows from [11,p.72]. Similarly, we can show that

$$\int_{1-2h}^{1} |\Delta_h^2\phi_e(t)|dt \leq C\, h^\gamma \|\phi\|_{\gamma,1,[0,1]} \; , \tag{3.2.12}$$

and, since $\phi \in N_1^{\gamma}[0,1]$, we have

$$\int_0^{1-2h} |\Delta_h^2\phi_e(t)|dt = \int_0^{1-2h} |\Delta_h^2\phi(t)|dt \le h^{\gamma}|\phi|_{\gamma,1,[0,1]} \le h^{\gamma}\|\phi\|_{\gamma,1,[0,1]}. \quad (3.2.13)$$

It follows, on substitution of (3.2.11), (3.2.12) and (3.2.13) into (3.2.10), that, for $\frac{1}{4} \ge h > 0$,

$$\int_{-\infty}^{\infty} |\Delta_h^2\phi_e(t)|dt \le C\, h^{\gamma}\, \|\phi\|_{\gamma,1,[0,1]} \quad . \qquad (3.2.14)$$

For $1 \ge h > \frac{1}{4}$, (3.2.9) gives,

$$\begin{aligned}\int_{-\infty}^{\infty} |\Delta_h^2\phi_e(t)|dt &= \int_{-2h}^{2h}|\Delta_h^2\phi_e(t)|dt + \int_{2h}^{1}|\Delta_h^2\phi_e(t)|dt \\ &\le \int_{-2h}^{2h}|\Delta_h^2\phi_e(t)|dt + \int_{1-2h}^{1}|\Delta_h^2\phi_e(t)|dt \\ &\le C\, h^{\gamma}\, \|\phi\|_{\gamma,1,[0,1]} \quad , \qquad (3.2.15)\end{aligned}$$

where the final inequality is achieved similarly to (3.2.11) and (3.2.12).

For $h > 1$, (3.2.9) gives

$$\begin{aligned}\int_{-\infty}^{\infty} |\Delta_h^2\phi_e(t)|dt &\le \int_{-2h}^{1}|\phi_e(t+2h)|dt + 2\int_{-2h}^{1}|\phi_e(t+h)|dt + \int_{-2h}^{1}|\phi_e(t)|dt \\ &\le 4\, \|\phi\|_{L_1[0,1]} \\ &\le C\, h^{\gamma}\, \|\phi\|_{L_1[0,1]} \\ &\le C\, h^{\gamma}\, \|\phi\|_{\gamma,1,[0,1]} \qquad (3.2.16)\end{aligned}$$

Results similar to (3.2.14), (3.2.15) and (3.2.16) may be proved for $h < 0$, leading to, finally,

$$\int_{-\infty}^{\infty} |\Delta_h^2\phi_e(t)|dt \le C\, |h|^{\gamma}\, \|\phi\|_{\gamma,1,[0,1]} \quad , \qquad \text{for all} \quad h \ne 0 \ ,$$

from which it follows that

$$|\phi_e|_{\gamma,1,\mathbb{R}} \le C \, \|\phi\|_{\gamma,1,[0,1]} ,$$

and hence that

$$\|\phi_e\|_{\gamma,1,\mathbb{R}} \le \|\phi_e\|_{L_1(\mathbb{R})} + C \, \|\phi\|_{\gamma,1,[0,1]}$$

$$= \|\phi\|_{L_1[0,1]} + C \, \|\phi\|_{\gamma,1,[0,1]}$$

$$\le (1 + C) \, \|\phi\|_{\gamma,1,[0,1]} ,$$

giving the required result.

Proof of Theorem 3.3

(i) Suppose $0 < \gamma < 1$, and let $y \in N_1^\gamma[a,b]$. Using Lemma 3.2 and (3.2.4) we can continuously extend y to $y_e \in N_1^\gamma(\mathbb{R}) = \Lambda(\gamma,1,\infty,\mathbb{R})$, and k to $k_e \in N_1^\alpha(\mathbb{R}) = \Lambda(\alpha,1,\infty,\mathbb{R})$. Thus, from [67 II, Lemma 1] and (3.2.4) again,

$$k_e * y_e \in \Lambda(\alpha+\gamma, 1, \infty, \mathbb{R}) = N_1^{\alpha+\gamma}(\mathbb{R}),$$

where $k_e * y_e$ is defined by (3.2.5).

On restriction of $k_e * y_e$ to $[a,b]$, it follows that $Ky \in N_1^{\alpha+\gamma}[a,b]$, and [69, p.310] that

$$\|Ky\|_{\alpha+\gamma,1,[a,b]} \le \|k_e * y_e\|_{\alpha+\gamma,1,\mathbb{R}} .$$

Hence [67 II, Lemma 1], we have

$$\|Ky\|_{\alpha+\gamma,1,[a,b]} \le C \, \|k_e\|_{\alpha,1,\mathbb{R}} \|y_e\|_{\gamma,1,\mathbb{R}} ,$$

and, by the continuity of the extension $y \to y_e$,

$$\|Ky\|_{\alpha+\gamma,1,[a,b]} \le C \, \|y\|_{\gamma,1,[a,b]} ,$$

where C is independent of y, proving the required result.

(ii) By (3.2.3), and (i), the composition

$$N_1^1[a,b] \xrightarrow[\text{Inclusion}]{} N_1^\gamma[a,b] \xrightarrow[K]{} N_1^{\alpha+\gamma}[a,b]$$

for any $0 < \gamma < 1$ is continuous, and the result follows.

(iii) It is shown in [11] that

$$K : W_1^1[a,b] \to N_1^{\alpha+1}[a,b]$$

is continuous. Thus for any $\gamma > 1$, it follows from (3.2.3) that the composition

$$N_1^\gamma[a,b] \xrightarrow[\text{Inclusion}]{} W_1^1[a,b] \xrightarrow[K]{} N_1^{\alpha+1}[a,b]$$

is continuous, and the result follows.

<u>Corollary 3.4</u> Let A1 be satisfied, and let $n = \left[\frac{1}{\alpha}\right] + 1$. Then the following maps are continuous

(i) $K \quad : \quad L_1[a,b] \to N_1^\alpha[a,b]$,

(ii) $K^n \quad : \quad N_1^p[a,b] \to W_1^1[a,b]$, $\quad 0 < p \leq \alpha$,

(iii) $DK^n : \quad N_1^p[a,b] \to N_1^q[a,b]$, $\quad 0 < q < p \leq \alpha$.

<u>Proof</u>. A proof of (i) can be found in [11].

Note that, by definition of n, see (3.2.1), we have

$$(n - 1)\alpha < 1 \leq n\alpha ,$$

and let

$$0 < p \leq \alpha .$$

Then, Theorem 3.3 and (3.2.3) imply that, for any p' in the range

$$1 < p' < \min \{n\alpha + p, \alpha + 1\} \tag{3.2.17}$$

the composition

$$N_1^p[a,b] \xrightarrow[K^n]{} N_1^{p'}[a,b] \xrightarrow[\text{Inclusion}]{} W_1^1[a,b] \tag{3.2.18}$$

is continuous, and (ii) follows.

Now, if $0 < q < p$, then $q + 1$ is in the range (3.2.17), and it then follows, by (3.2.18), and using the interpolation theorem of Chandler [11, p.74], that the composition

$$N_1^p[a,b] \xrightarrow[K^n]{} N_1^{q+1}[a,b] \xrightarrow[D]{} N_1^q[a,b]$$

is continuous, and (iii) follows.

The next theorem states some results on the compactness of K, (i) is a standard result, see [70, p.321], while the proof of (ii) follows from the results of [11].

<u>Theorem 3.5</u> Let A1 be satisfied. Then K is compact as an operator on either of the spaces

(i) $L_1[a,b]$,

or

(ii) $W_1^1[a,b]$.

<u>Corollary 3.6</u> Let A1 and A2 be satisfied.

(i) If $f \in L_1[a,b]$, then (3.1.1) has a unique solution y in $L_1[a,b]$.

(ii) If $f \in W_1^1[a,b]$, then (3.1.1) has a unique solution $y \in W_1^1[a,b]$, and y' satisfies the integral equation

$$y'(t) = f'(t) + \lambda y(a)k(t-a) - \lambda y(b)k(t-b) + K_\lambda y'(t) ,$$

for almost all $t \in [a,b]$.

Proof. The proof of (i) follows immediately from Theorem 3.5 and the Fredholm Alternative [33, p.497] and (ii) follows similarly, with the integral equation satisfied by y' being obtained using Lemma 3.1.

Remark. In $L_1[a,b]$ or in $W_1^1[a,b]$, the uniqueness of the solution y means that any other solution must coincide with y except, possibly, on a set of measure zero. Throughout Section 3.3, when an integral equation is shown to have a solution in $L_1[a,b]$ or $W_1^1[a,b]$, it will be assumed, without further notification, that the equation is satisfied for almost all $t \in [a,b]$.

3.3 THE MAIN RESULT

In this section we obtain the singularity expansion (3.1.7) for the solution of (3.1.1). The proof of the main result, Theorem 3.9 below, depends on the intermediate Lemmas, 3.7 and 3.8.

Consider (3.1.1) and suppose that k satisfies A1 and A2. The singularity expansion, valid for $f \in W_1^r[a,b]$, for any $r \in \mathbb{N}_0$, will be obtained by first defining inductively, for any $m \in \mathbb{N}_0$, a sequence of integral equations $\{(3.3.1)_j\}_{j=0}^m$, and an associated sequence of functions $\{\omega_j\}_{j=0}^m$. The important properties of these two sequences in the general case, $r \in \mathbb{N}_0$ and $m \in \mathbb{N}_0$, are proved in Lemma 3.7. The proof has rather a lot of technical detail, and so to illustrate the method, we consider first the particular case $r = 2$.

We first define $(3.3.1)_0$ to be the same as (3.1.1), denote its solution by y_0

$$y_0 = f + K_\lambda y_0 , \qquad (3.3.1)_0$$

and set

$$\omega_0 = 0. \qquad (3.3.2)_0$$

Then, since $f \in W_1^2[a,b] \subseteq W_1^1[a,b]$, it follows, by Corollary 3.6, that $y_0 \in W_1^1[a,b]$, and that y_0' satisfies the integral equation:

$$y_0' = f' + \omega_1 + K_\lambda y_0' , \qquad (3.3.1)_0'$$

where

$$\omega_1(t) = \lambda y_0(a)k(t-a) - \lambda y_0(b)k(t-b), \text{ almost all } t \in [a,b]. \qquad (3.3.2)_1$$

Since $\omega_1 \in N_1^\alpha[a,b]$ and may be infinite on some subset of $[a,b]$ of measure zero, $(3.3.1)_0'$ is understood to hold in the $L_1[a,b]$ sense (see Remark following Corollary 3.6), and so we cannot use Corollary 3.6 to obtain any information about y_0'' . However, setting $n = \left[\frac{1}{\alpha}\right] + 1$, defining a new function y_1 by

$$y_1 = y_0' - \sum_{\ell=0}^{n-1} K_\lambda^\ell \omega_1 ,$$

and substituting for y_0' in $(3.3.1)_0'$, we obtain an integral equation for y_1 , which we take as the next equation in our sequence:

$$y_1 = f' + K_\lambda^n \omega_1 + K_\lambda y_1 . \qquad (3.3.1)_1$$

It follows then, by Corollary 3.4, that $(3.3.1)_1$ has inhomogeneous term in $W_1^1[a,b]$. Thus, by Corollary 3.6, $y_1 \in W_1^1[a,b]$ and y_1' satisfies

$$y_1'(t) = f''(t) + (DK_\lambda^n)\omega_1(t) + \lambda y_1(a)k(t-a) - \lambda y_1(b)k(t-b) + K_\lambda y_1'(t) \qquad (3.3.1)_1'$$

for almost all $t \in [a,b]$.

Since y_1 was obtained by subtracting the singular (i.e. non-$W_1^1[a,b]$) terms away from y_0', our aim now is to define y_2 from y_1' in the same way, and obtain an integral equation $(3.3.1)_2$ for y_2. Since $f \in W_1^2[a,b]$, it follows that $f'' \in L_1[a,b]$, and hence, y_1' may contain non-$W_1^1[a,b]$ terms induced by f''.

If we first subtract f'' from y_1' and rewrite $(3.3.1)_1'$ as

$$(y_1' - f'') = \omega_2 + K_\lambda(y_1' - f'') \,, \qquad (3.3.1)_1'$$

where

$$\omega_2(t) = K_\lambda f''(t) + (DK_\lambda^n)\omega_1(t) + \lambda y_1(a)k(t-a) - \lambda y_1(b)k(t-b) \,,$$

$$\text{almost all } t \in [a,b] \,, \qquad (3.3.2)_2$$

it then follows, from Corollary 3.4, that $\omega_2 \in N_1^q[a,b]$ for all $q < \alpha$.

Then, setting

$$y_2 = (y_1' - f'') - \sum_{\ell=0}^{n-1} K_\lambda^\ell \omega_2 \,,$$

thus subtracting the rest of the non-$W_1^1[a,b]$ terms away from $y_1' - f''$, and substituting for $y_1' - f''$ in $(3.3.1)_1'$, we obtain the equation

$$y_2 = K_\lambda^n \omega_2 + K_\lambda y_2 \,, \qquad (3.3.1)_2$$

which has inhomogeneous term in $W_1^1[a,b]$. Thus, by Corollary 3.6, $y_2 \in W_1^1[a,b]$, and

$$y_2' = \omega_3 + K_\lambda y_2' , \qquad (3.3.1)_2'$$

where

$$\omega_3(t) = (DK_\lambda^n)\omega_2(t) + \lambda y_2(a)k(t-a) - \lambda y_2(b)k(t-b), \text{ almost all } t \in [a,b]. \qquad (3.3.2)_3$$

Now by Corollary 3.4, $\omega_3 \in N_1^q[a,b]$ for all $q < \alpha$, and so, defining y_3 by

$$y_3 = y_2' - \sum_{\ell=0}^{n-1} K_\lambda^\ell \, \omega_3 ,$$

and substituting for y_2' in $(3.3.1)_2'$, we obtain,

$$y_3 = K_\lambda^n \, \omega_3 + K_\lambda \, y_3 , \qquad (3.3.1)_3$$

which has inhomogeneous term in $W_1^1[a,b]$.

By mimicking the transition from $(3.3.1)_2$ to $(3.3.1)_3$, we can continue this process indefinitely. Thus for any $m \in \mathbb{N}_0$ we have the sequences $\{(3.3.1)_j\}_{j=0}^m$, and $\{\omega_j\}_{j=0}^m$. For each $j = 0,1,\ldots,m$, ω_j will be in $N_1^q[a,b]$ for all $q < \alpha$, and $(3.3.1)_j$ will have inhomogeneous term, and hence solution, y_j, in $W_1^1[a,b]$. Moreover, for each $j = 1,\ldots,m$, y_j will be obtained by subtracting the non-$W_1^1[a,b]$ components away from y_{j-1}'.

In this illustration, the integral equations $(3.3.1)_0$ and $(3.3.1)_1$ both contain explicitly the inhomogeneous term f (or its derivative), while the equations $(3.3.1)_j$, for $j = 2,\ldots,m$ do not. This is because we have assumed that $f \in W_1^2[a,b]$ and hence we "run out" of $W_1^1[a,b]$ derivatives of f at the point $j = 2$ in the sequence $\{(3.3.1)_j\}_{j=0}^m$.

When $f \in W_1^r[a,b]$, for any $r \in \mathbb{N}_0$, an analogous process to that described above will define sequences $\{(3.3.1)_j\}_{j=0}^m$ and $\{\omega_j\}_{j=0}^m$. In this general case, we shall "run out" of $W_1^1[a,b]$ derivatives of f at the point $j = r$.

In the particular case $r = 0$, we have $f \in L_1[a,b]$, and so (3.1.1) has a non-$W_1^1[a,b]$ inhomogeneous term, and so we cannot simply adopt (3.1.1) as $(3.3.1)_0$. Instead, we modify (3.1.1), using a method analogous to that used to define $(3.3.1)_2$ in the case $r = 2$ above, to obtain an equation which has inhomogeneous term in $W_1^1[a,b]$, and which we then take to be $(3.3.1)_0$.

To do this, we rewrite (3.1.1) as

$$(y - f) = K_\lambda f + K_\lambda(y - f) . \tag{3.1.1}$$

Then we define ω_0 by

$$\omega_0 = K_\lambda f ,$$

and set

$$y_0 = (y - f)' - \sum_{\ell=0}^{n-1} K_\lambda \omega_0 .$$

Then, substitution for $y - f$ in (3.1.1) yields

$$y_0 = K_\lambda^n \omega_0 + K_\lambda y_0 ,$$

which we take to be $(3.3.1)_0$, the first equation in our sequence. The rest of the sequence is then defined by mimicking the transition from $(3.3.1)_2$ to $(3.3.1)_3$ in the case $r = 2$ above.

The general result is now given in the following lemma.

<u>Lemma 3.7.</u> Let A1 and A2 be satisfied, let $f \in W_1^r[a,b]$, for some $r \in \mathbb{N}_0$, and set $n = \left[\frac{1}{\alpha}\right] + 1$.

(i) For any $m \in \mathbb{N}_0$, define inductively the sequence of integral equations $\{(3.3.1)_j\}_{j=0}^m$ by

$$y_j = \xi_{j,r} f^{(j)} + K_\lambda^n \omega_j + K_\lambda y_j \,, \quad j = 0,1,\ldots,m \,, \tag{3.3.1$_j$}$$

where

$$\xi_{j,r} = 1 \,, \qquad 0 \le j < r \,,$$

and

$$\xi_{j,r} = 0 \,, \qquad r \le j \le m \,,$$

and the sequence of functions $\{\omega_j\}_{j=0}^m$ is defined by

$$\omega_j(t) = \delta_{j,r} K_\lambda f^{(j)}(t) + DK_\lambda^n \omega_{j-1}(t) + \lambda y_{j-1}(a) k(t-a) - \lambda y_{j-1}(b) k(t-b), \tag{3.3.2$_j$}$$

for almost all $t \in [a,b]$, and $j = 0,\ldots,m$, where

$$\omega_{-1} := 0 \,,$$

$$y_{-1} := 0 \,,$$

and δ denotes Kronecker's delta. Then, for $j = 0,1,\ldots,m$, ω_j is well defined in $N_1^q[a,b]$ for all $q < \alpha$, and y_j (the solution of $(3.3.1)_j$) exists and is unique in $W_1^1[a,b]$.

(ii)
$$y_0 = y - \delta_{0,r} f - \sum_{\ell=0}^{n-1} K_\lambda^\ell \omega_0 \,, \tag{3.3.3$_0$}$$

where y is the solution of (3.1.1), and,

$$y_j = y'_{j-1} - \delta_{j,r} f^{(j)} - \sum_{\ell=0}^{n-1} K_\lambda^\ell \omega_j \,, \quad j = 1,\ldots,m \,. \tag{3.3.3$_j$}$$

Proof. We prove (i) by induction on j. First, consider the case $j = 0$. Then,

$$\omega_0 = \left\{ \begin{array}{ll} 0 , & r > 0 , \\ & \\ K_\lambda f , & r = 0 , \end{array} \right\} \tag{3.3.2$_0$}$$

and so in either case (see Corollary 3.4),

$$\omega_0 \in N_1^\alpha [a,b] \subseteq N_1^q [a,b] \ , \quad \text{for all} \quad q < \alpha \ .$$

The equation $(3.3.1)_0$ is then

$$\left. \begin{array}{ll} y_0 = f + K_\lambda y_0 , & r > 0 , \\ \text{or} & \\ y_0 = K_\lambda^{n+1} f + K_\lambda y_0 , & r = 0 , \end{array} \right\} \tag{3.3.1$_0$}$$

and so, in either case, by Corollaries 3.4 and 3.6, y_0 exists and is unique in $W_1^1 [a,b]$.

Now, suppose (i) is true for $j - 1$, where $j \in \{1,\ldots,m\}$. Then, since $y_{j-1} \in W_1^j [a,b]$, it follows that ω_j is well-defined, and by Corollary 3.4, we have $\omega_j \in N_1^q [a,b]$, for all $q < \alpha$. Hence, by Corollary 3.4, $(3.3.1)_j$ has inhomogeneous term in $W_1^1 [a,b]$, and thus, by Corollary 3.6, y_j exists and is unique in $W_1^1 [a,b]$.

The proof of (i) then follows by induction.

(ii) We first prove $(3.3.3)_0$. Since $f \in W_1^r [a,b]$, for some $r \in \mathbb{N}_0$, we know, by Corollary 3.6, that the solution y of (3.1.1) exists and is unique in $L_1 [a,b]$ (and y is in $W_1^1 [a,b]$ if $r > 0$).

Set

$$Y = y - \delta_{0,r}f - \sum_{\ell=0}^{n-1} K_\lambda^\ell \omega_0 ,$$

and substitute for y in (3.1.1) to obtain

$$\delta_{0,r}f + \sum_{\ell=0}^{n-1} K_\lambda^\ell \omega_0 + Y = f + \delta_{0,r}K_\lambda f + \sum_{\ell=0}^{n-1} K_\lambda^{\ell+1} \omega_0 + K_\lambda Y ,$$

which is equivalent, via $(3.3.2)_0$ to

$$Y = (1 - \delta_{0,r})f + K_\lambda^n \omega_0 + K_\lambda Y ,$$

which, since

$$(1 - \delta_{0,r}) = \xi_{0,r} ,$$

is the same as $(3.3.1)_0$. Thus, by the existence and uniqueness of the solution to $(3.3.1)_0$ we have $Y = y_0$ and $(3.3.3)_0$ follows.

Now, let $j \in \{1,...,m\}$, and use part (i) of this Lemma and Corollary 6, to differentiate $(3.3.1)_{j-1}$, obtaining, via $(3.3.2)_j$,

$$y_{j-1}' = \xi_{j-1,r}f^{(j)} - \delta_{j,r}K_\lambda f^{(j)} + \omega_j + K_\lambda y_{j-1}' . \qquad (3.3.1)_{j-1}'$$

If we now set

$$Y = y_{j-1}' - \delta_{j,r} f^{(j)} - \sum_{\ell=0}^{n-1} K_\lambda^\ell \omega_j ,$$

and substitute for y_{j-1}' in $(3.3.1)_{j-1}'$, we obtain,

$$\delta_{j,r}f^{(j)} + \sum_{\ell=0}^{n-1} K_\lambda^\ell \omega_j + Y = \xi_{j-1,r}f^{(j)} + \omega_j + \sum_{\ell=0}^{n-1} K_\lambda^{\ell+1}\omega_j + K_\lambda Y ,$$

and thus,

$$Y = (\xi_{j-1,r} - \delta_{j,r})f^{(j)} + K_\lambda^n \omega_j + K_\lambda Y .$$

Now, since

$$\xi_{j-1,r} - \delta_{j,r} = \xi_{j,r} , \quad j = 1,\ldots,m , \tag{3.3.4}$$

it follows, by the existence and uniqueness of the solution to $(3.3.1)_j$ that $Y = y_j$, and $(3.3.3)_j$ follows, completing the proof of Lemma 3.7.

We now use the sequence $\{y_j\}_{j=0}^{m}$ to obtain a singularity expansion for y. Starting with y_m, which we know satisfies $(3.3.1)_m$ we may "unravel" the singular terms in y by a process of applying $(3.3.3)_j$, $j = m, m-1, m-2, \ldots, 0$, and integrating, to obtain successively, an expression for $y_{m-1}, y_{m-2}, \ldots, y_1, y_0$, and, finally, y.

The general expression for the p-th step in this unravelling process is proved in Lemma 3.8.

<u>Lemma 3.8.</u> Suppose the conditions of Lemma 3.7 are satisfied, and, for $j \in \{0,1,\ldots,m\}$, let y_j be the solution of $(3.3.1)_j$. Then

$$y_m = \xi_{m,r} f^{(m)} + \phi_m , \qquad (3.3.5)_m$$

and

$$y_{m-p} = \xi_{m-p,r} f^{(m-p)} + \sum_{j=1}^{p} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell \omega_{m-p+j} + \phi_{m-p}, \quad p=1,\ldots,m, \qquad (3.3.5)_{m-p}$$

where $I = I_{[a,b]}$ as defined by (3.1.5), and $\phi_{m-p} \in W_1^{p+1}[a,b]$, for $p = 0, 1, \ldots, m$.

<u>Proof.</u> We shall prove this result by induction on p.

First consider the case $p = 0$. By Lemma 3.7 (i) we know that y_m satisfies $(3.3.1)_m$, and that $y_m \in W_1^1[a,b]$, and $\omega_m \in N_1^q[a,b]$, for all $q < \alpha$. By Corollary 3.4 and Theorem 3.5, $K_\lambda^n \omega_m$ and $K_\lambda y_m$ are both in $W_1^1[a,b]$, and so by $(3.3.1)_m$,

$$y_m = \xi_{m,r} f^{(m)} + \phi_m \quad ,$$

where $\phi_m \in W_1^1[a,b]$ and we have proved $(3.3.5)_m$.

Utilising $(3.3.3)_m$ and integrating using I, we obtain

$$y_{m-1} = (\xi_{m,r} + \delta_{m,r}) f^{(m-1)} + \sum_{\ell=0}^{n-1} IK_\lambda^\ell \omega_m + \phi_{m-1} \quad ,$$

where

$$\phi_{m-1} = I\phi_m + \text{constant of integration,}$$

and so $\phi_{m-1} \in W_1^2[a,b]$. Utilising (3.3.4), we then complete the proof of $(3.3.5)_{m-1}$.

Suppose now that $(3.3.5)_{m-(p-1)}$ is true for some $p \in \{2,\dots,m\}$. Then, using $(3.3.3)_{m-(p-1)}$, and integrating using I, we obtain

$$y_{m-p} = (\xi_{m-(p-1),r} + \delta_{m-(p-1),r}) f^{(m-p)} + \sum_{j=1}^{p-1} \sum_{\ell=0}^{n-1} I^{j+1} K_\lambda^\ell \omega_{m-(p-1)+j}$$

$$+ \sum_{\ell=0}^{n-1} IK_\lambda^\ell \omega_{m-(p-1)} + \phi_{m-p} \quad , \tag{3.3.6}$$

where $\phi_{m-p} = I\,\phi_{m-(p-1)}$+constant of integration, and hence $\phi_{m-p} \in W_1^{p+1}[a,b]$.

Using (3.3.4), and collecting terms on the right hand side of (3.3.6), we obtain

$$y_{m-p} = \xi_{m-p,r}\, f^{(m-p)} + \sum_{j=1}^{p} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell \omega_{m-p+j} + \phi_{m-p} \quad ,$$

and thus we have proved $(3.3.5)_{m-p}$. Induction completes the proof of $(3.3.5)_{m-p}$ for all $p = 0,1,\ldots,m$.

<u>Theorem 3.9</u>. Suppose A1 and A2 are satisfied, let $f \in W_1^r[a,b]$ for some $r \in \mathbb{N}_0$, and set $n = \left[\frac{1}{\alpha}\right] + 1$. Then, for any $m \in \mathbb{N}_0$ the solution y of (3.1.1) has the singularity expansion (3.1.7) with

$$\left.\begin{aligned} c_{j-i} &= \lambda y_{j-i}(a) \\ d_{j-i} &= \lambda y_{j-i}(b) \end{aligned}\right\} \qquad \begin{aligned} i &= 1,\ldots,m \ , \\ j &= i,\ldots,m \ , \end{aligned}$$

where, for each $j \in \{0,1,\ldots,m\}$, y_j is the solution of $(3.3.1)_j$.

<u>Proof</u>. Using Lemma 3.8 with $p = m$, we have

$$y_0 = \xi_{0,r} f + \sum_{j=1}^{m} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell \omega_j + \phi_0 \ , \tag{3.3.5$_0$}$$

where $\phi_0 \in W_1^{m+1}[a,b]$, and, using $(3.3.3)_0$, we obtain

$$y = (\xi_{0,r} + \delta_{0,r})f + \sum_{j=1}^{m} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell \omega_j + \sum_{\ell=0}^{n-1} K_\lambda^\ell \omega_0 + \phi_0$$

$$= f + \sum_{j=0}^{m} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell \omega_j + \phi_0 \ , \tag{3.3.7}$$

since, for any $r \in \mathbb{N}_0$

$$\xi_{0,r} + \delta_{0,r} = 1 \ .$$

Now, from $(3.3.2)_0$, we have

$$\omega_0 = \begin{cases} 0 & r > 0 \\ K_\lambda f \ , & r = 0 \end{cases} \ ,$$

and, by induction on the relations $(3.3.2)_j$ for $j = 1,\dots,m$, it can be shown that

$$\omega_j = \sum_{i=1}^{j} (DK_\lambda^n)^{i-1} k_{j-i} \ , \qquad j = 1,2,\dots,r-1 \ ,$$

and

$$\omega_j = (DK_\lambda^n)^{j-r} K_\lambda f^{(r)} + \sum_{i=1}^{j} (DK_\lambda^n)^{i-1} k_{j-i} \ , \quad j = r,\dots,m \ ,$$

where

$$k_{j-i}(t) = \lambda y_{j-i}(a)k(t-a) - \lambda y_{j-i}(b)k(t-b), \quad \begin{cases} i = 1,\dots,j, \\ j = 1,\dots,m. \end{cases}$$

Substitution for ω_j in (3.3.7) then yields

$$y = f + \sum_{j=r}^{m} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell (DK_\lambda^n)^{j-r} K_\lambda f^{(r)} + \sum_{j=1}^{m} \sum_{\ell=0}^{n-1} \sum_{i=1}^{j} I^j K_\lambda^\ell (DK_\lambda^n)^{i-1} k_{j-i} + \phi_0$$

which, on rearrangement of the order of summation in the last term, yields

$$y = f + \sum_{j=r}^{m} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell (DK_\lambda^n)^{j-r} K_\lambda f^{(r)} + \sum_{i=1}^{m} \sum_{j=i}^{m} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell (DK_\lambda^n)^{i-1} k_{j-i} + \phi \ ,$$

where $\phi = \phi_0 \in W_1^{m+1}[a,b]$, and

$$k_{j-i}(t) = \lambda y_{j-i}(a)k(t-a) - \lambda y_{j-i}(b)k(t-b), \quad i=1,\dots,m, \ j=i,\dots,m \ ,$$

as required.

3.4. APPLICATIONS

In this section, we shall use (3.1.7) to obtain explicit singularity expansions for the solution of the equation

$$y(t) = f(t) + \lambda \int_0^1 |t-s|^{\alpha-1} y(s)ds \ , \quad t \in [0,1] \ , \quad 0 < \alpha < 1$$

in the following four cases:

In Example	α is	and f is
1	irrational	infinitely continuously differentiable on $[0,1]$.
2	rational	infinitely continuously differentiable on $[0,1]$.
3	$\frac{1}{2}$	given by $f(t) = e^t$, $t \in [0,1]$.
4	$\frac{1}{4}$	given by $f(t) = t^{-1/3} + (1-t)^{-1/3}$, $t \in [0,1]$.

Physical motivation for these examples can be found in the Kirkwood Riseman theory of intrinsic viscosities [34]. (See Example 1 of Chapter 1.) The results given in these examples depend on the technical Lemma A4, proved in the Appendix.

Recall that the notation $\{a(t) + b(t) + \ldots + z(t)\}$ denotes some linear combination of the functions $a(t)$, $b(t), \ldots,$ and $z(t)$.

Example 1. Throughout this example, let $m \in \mathbb{N}_0$, and let ϕ denote an unknown $W_1^{m+1}[0,1]$ function. No two instances of ϕ will necessarily be equal. Since f is infinitely differentiable, we may choose r as large as we wish, and (3.1.7) gives

$$y = f + \sum_{i=1}^{m} \sum_{j=i}^{m} \sum_{\ell=0}^{n-1} I^j K_\lambda^\ell (DK_\lambda^n)^{i-1} k_{j-i} + \phi \ ,$$

where

$$I = I_{[0,1]},$$

$$n = \left[\frac{1}{\alpha}\right] + 1$$

and

$$k_{j-i}(t) = \{t^{\alpha-1} + (1-t)^{\alpha-1}\}, \qquad t \in [0,1], \quad \begin{cases} i=1,\dots,m, \\ j=i,\dots,m. \end{cases}$$

Since $K_\lambda = \lambda K$, we can write this, using the notation for linear combinations, as

$$y(t) = f(t) + \left\{ \sum_{i=1}^{m} \sum_{j=i}^{m} \sum_{\ell=0}^{n-1} I^j K^\ell (DK^n)^{i-1} (t^{\alpha-1} + (1-t)^{\alpha-1}) \right\} + \phi(t), \qquad t \in [0,1]. \tag{3.4.1}$$

Now, from Lemma A4,

$$K(t^{\alpha-1} + (1-t)^{\alpha-1}) = \left\{ t^{2\alpha-1} + (1-t)^{2\alpha-1} + \sum_{j=0}^{m-1} (t^{\alpha+j} + (1-t)^{\alpha+j}) \right\} + \phi(t),$$

$$\vdots$$

$$K^{n-1}(t^{\alpha-1} + (1-t)^{\alpha-1}) = \left\{ t^{n\alpha-1} + (1-t)^{n\alpha-1} + \sum_{j=0}^{m-1} \sum_{\ell=1}^{n-1} (t^{\ell\alpha+j} + (1-t)^{\ell\alpha+j}) \right\} + \phi(t),$$

and

$$K^n(t^{\alpha-1} + (1-t)^{\alpha-1}) = \left\{ t^{(n+1)\alpha-1} + (1-t)^{(n+1)\alpha-1} + \sum_{j=0}^{m-1} \sum_{\ell=1}^{n} (t^{\ell\alpha+j} + (1-t)^{\ell\alpha+j}) \right\} + \phi(t).$$

Thus

$$\sum_{j=1}^{m} \sum_{\ell=0}^{n-1} I^j K^\ell (t^{\alpha-1} + (1-t)^{\alpha-1}) = \left\{ \sum_{j=0}^{m-1} \sum_{\ell=1}^{n} (t^{\ell\alpha+j} + (1-t)^{\ell\alpha+j}) \right\} + \phi(t) \tag{3.4.2}$$

and, similarly,

$$\sum_{j=2}^{m}\sum_{\ell=0}^{n-1} I^jK^\ell(DK^n)(t^{\alpha-1}+(1-t)^{\alpha-1}) = \left\{\sum_{j=0}^{m-2}\sum_{\ell=1}^{n}(t^{(n+\ell)\alpha+j}+(1-t)^{(n+\ell)\alpha+j})\right\}$$

$$+ \text{ terms already in (3.4.2)}$$

$$+ \phi(t) .$$

Continuing this process, we obtain

$$\sum_{i=1}^{m}\sum_{j=i}^{m}\sum_{\ell=0}^{n-1} I^jK^\ell(DK^n)^{i-1}(t^{\alpha-1}+(1-t)^{\alpha-1})$$

$$= \left\{\sum_{i=0}^{m-1}\sum_{j=0}^{m-i-1}\sum_{\ell=1}^{n}(t^{(in+\ell)\alpha+j}+(1-t)^{(in+\ell)\alpha+j})\right\} + \phi(t) ,$$

and substitution into (3.4.1) yields the singularity expansion for y:

$$y(t) = f(t) + \left\{\sum_{i=0}^{m-1}\sum_{j=0}^{m-i-1}\sum_{\ell=1}^{n}(t^{(in+\ell)\alpha+j}+(1-t)^{(in+\ell)\alpha+j})\right\} + \phi(t), \quad t\in[0,1] ,$$

where $\phi \in W_1^{m+1}[0,1]$.

Example 2. Let $\alpha = p/q$, where p,q are comprime and $p < q$. Following Example 1, y has expansion (3.4.1) for any $m \in \mathbb{N}_0$, and $n = \left[\frac{q}{p}\right] + 1$.

Note first that this implies that $q \leq np$. Now let ρ be the smallest integer such that $q \leq n\rho$, i.e.

$$\left.\begin{aligned} &q = n(\rho-1) + \sigma , \\ \text{where} \\ &0 < \sigma \leq n , \quad \text{and} \quad \rho \in \mathbb{N} . \end{aligned}\right\} \qquad (3.4.3)$$

It then follows that $\rho \leq p$.

It will be convenient to rewrite (3.4.1) with m replaced by $m\rho$, and with ϕ denoting some unknown $W_1^{m\rho+1}[0,1]$ function:

$$y(t) = f(t) + \left\{ \sum_{i=1}^{m\rho} \sum_{j=i}^{m\rho} \sum_{\ell=0}^{n-1} I^j K^\ell (DK^n)^{i-1} (t^{p/p-1} + (1-t)^{p/q-1}) \right\} + \phi(t),$$

$$t \in [0,1] . \quad (3.4.4)$$

Then, as in Example 1,

$$\sum_{i=1}^{\rho-1} \sum_{j=i}^{m\rho} \sum_{\ell=0}^{n-1} I^j K^\ell (DK^n)^{i-1} (t^{p/q-1} + (1-t)^{p/q-1})$$

$$= \left\{ \sum_{i=0}^{\rho-2} \sum_{j=0}^{m\rho-i-1} \sum_{\ell=1}^{n} (t^{(in+\ell)p/q+j} + (1-t)^{(in+\ell)p/q+j}) \right\} + \phi(t) ,$$

$$= \{t^{p/q} + \dots + t^{np/q}\}\{1 + \dots + t^{m\rho-1}\}$$

$$+ \{t^{(n+1)p/q} + \dots + t^{2np/q}\}\{1 + \dots + t^{m\rho-2}\}$$

$$+ \dots$$

$$+ \{t^{((\rho-2)n+1)p/q} + \dots + t^{(\rho-1)np/q}\}\{1 + \dots + t^{m\rho-\rho+1}\}$$

$$+ \{(1-t)^{p/q} + \dots + (1-t)^{np/q}\}\{ 1 + \dots + (1-t)^{m\rho-1}\}$$

$$+ \{(1-t)^{(n+1)p/q} + \dots + (1-t)^{2np/q}\}\{ 1 + \dots + (1-t)^{m\rho-2}\}$$

$$+ \dots$$

$$+ \{(1-t)^{((\rho-2)n+1)p/q} + \dots + (1-t)^{(\rho-1)np/q}\}$$

$$\{1 + \dots + (1-t)^{m\rho-\rho+1}\}$$

$$+ \phi(t) . \quad (3.4.5)$$

However, by (3.4.3) the integer q $(= (\rho-1)n+\sigma)$ lies in the set $\{(\rho-1)n+1,\dots,\rho n\}$, and hence, by Lemma A4, we generate a log term in the singularity expansion when the index $((\rho-1)n + \sigma)p/q$ is attained. To be precise,

$$\sum_{j=\rho}^{m\rho}\sum_{\ell=0}^{n-1} I^j K^\ell (DK^n)^{\rho-1}(t^{p/q-1} + (1-t)^{p/q-1})$$

$$= \{t^{((\rho-1)n+1)p/q} + \ldots + t^{(q-1)p/q} + t^p \ell nt\}\{1 + \ldots + t^{(m-1)\rho}\}$$

$$+ \{t^p \ell nt\}\{t^{p/q} + \ldots + t^{\rho np/q-p}\}\{1 + \ldots + t^{(m-1)\rho}\}$$

$$+ \{(1-t)^{((\rho-1)n+1)p/q} + \ldots + (1-t)^{(q-1)p/q} + (1-t)^p \ell n(1-t)\}$$

$$\{1 + \ldots + (1-t)^{(m-1)\rho}\}$$

$$+ \{(1-t)^p \ell n(1-t)\}\{(1-t)^{p/q} + \ldots + (1-t)^{\rho np/q-p}\}\{1 + \ldots + (1-t)^{(m-1)\rho}\}$$

$+$ terms already in (3.4.5)

$$+ \phi(t) . \tag{3.4.6}$$

Combining (3.4.5) and (3.4.6), and generalising to obtain the summation in (3.4.4), we obtain

$$y(t) = f(t)$$

$$+ \left\{ \sum_{i=0}^{m-1} (t^p \ell nt)^i \Big((t^{p/q} + \ldots + t^{np/q})(1 + \ldots + t^{(m-i)\rho-1})\right.$$

$$+ \ldots\ldots$$

$$+ \ldots\ldots$$

$$+ (t^{((\rho-1)n+1)p/q} + \ldots + t^p \ell nt)(1 + \ldots + t^{(m-i-1)\rho})\Big)$$

$$+ \sum_{i=0}^{m-1} ((1-t)^p \ell n(1-t))^i \Big(((1-t)^{p/q} + \ldots + (1-t)^{np/q})(1 + \ldots + (1-t)^{(m-i)\rho-1})$$

$$+ \ldots\ldots$$

$$+ \ldots\ldots$$

$$+ ((1-t)^{((\rho-1)n+1)p/q} + \ldots + (1-t)^p \ell n(1-t))$$

$$\left.(1 + \ldots + (1-t)^{(m-i-1)\rho})\Big)\right\}$$

$$+ \phi(t), \qquad t \in [0,1] , \tag{3.4.7}$$

where $\phi \in W_1^{m\rho+1}[0,1]$.

Example 3. This is the example (3.1.8), given in Section 3.1. We merely apply the results of Example 2 with $p = 1, q = 2$. We have $n = [2] + 1 = 2$, and hence $\rho = 1$ and $\sigma = 2$. With these values of the indices and with $f(t) = e^t$, for any $m \in \mathbb{N}_0$, (3.4.7) becomes

$$
\begin{aligned}
y(t) &= e^t \\
&+ \left\{ \sum_{i=0}^{m-1} (t\ell n t)^i (t^{\frac{1}{2}} + t\ell n t)(1 + \dots + t^{m-i-1}) \right. \\
&+ \left. \sum_{i=0}^{m-1} ((1-t)\ell n(1-t))^i ((1-t)^{\frac{1}{2}} + (1-t)\ell n(1-t))(1+\dots+(1-t)^{m-i-1}) \right\} \\
&+ \phi(t), \quad t \in [0,1],
\end{aligned}
$$

where $\phi \in W_1^{m+1}[0,1]$, which gives us (3.1.10).

Example 4. Since, in this case, $f \in L_1[0,1] = W_1^0[0,1]$, we must calculate (3.1.7) with $r = 0$. Let $m \in \mathbb{N}_0$, then with ϕ denoting an unknown $W_1^{m+1}[0,1]$ function,

$$
\left.
\begin{aligned}
y(t) &= t^{-1/3} + (1-t)^{-1/3} \\
&+ \left\{ \sum_{j=0}^{m} \sum_{\ell=0}^{n-1} I^j K^\ell (DK^n)^j K(t^{-1/3} + (1-t)^{-1/3}) \right\} \quad \text{(a)} \\
&+ \left\{ \sum_{i=1}^{m} \sum_{j=i}^{m} \sum_{\ell=0}^{n-1} I^j K^\ell (DK^n)^{i-1} (t^{-3/4} + (1-t)^{-3/4}) \right\} \quad \text{(b)} \\
&+ \phi(t), \quad t \in [0,1].
\end{aligned}
\right\} \quad (3.4.8)
$$

Now with $p = 1, q = 4, n = 4, \rho = 1$ and $\sigma = 4$, the results of Example 2 may be applied directly to obtain (3.4.8) (b). We must now calculate (3.4.8) (a), which equals

$$\left\{ \sum_{j=0}^{m} \sum_{\ell=0}^{3} I^j K \, (DK^4)^j K(t^{-1/3} + (1-t)^{-1/3}) \right\} \quad .$$

By Lemma A4, with $\alpha = \frac{1}{4}$, we have

$$K(t^{-1/3} + (1-t)^{-1/3}) = \{t^{-1/12} + (1-t)^{-1/12} + \sum_{j=0}^{m-1} (t^{\alpha+j} + (1-t)^{\alpha+j})\} + \phi(t),$$

$$K^2(t^{-1/3} + (1-t)^{-1/3}) = \{t^{1/6} + (1-t)^{1/6} + \sum_{j=0}^{m-1} \sum_{\ell=1}^{2} (t^{\ell\alpha+j} + (1-t)^{\ell\alpha+j})\} + \phi(t) \ ,$$

$$K^3(t^{-1/3} + (1-t)^{-1/3}) = \{t^{5/12} + (1-t)^{5/12} + \sum_{j=0}^{m-1} \sum_{\ell=1}^{3} (t^{\ell\alpha+j} + (1-t)^{\ell\alpha+j})\} + \phi(t) \ ,$$

$$K^4(t^{-1/3} + (1-t)^{-1/3}) = \{t^{2/3} + (1-t)^{2/3} + \sum_{j=0}^{m-1} \sum_{\ell=1}^{4} (t^{\ell\alpha+j} + (1-t)^{\ell\alpha+j})\} + \phi(t)\},$$

$$K^5(t^{-1/3} + (1-t)^{-1/3}) = \{t^{11/12} + (1-t)^{11/12} + \sum_{j=0}^{m-1} \sum_{\ell=1}^{5} (t^{\ell\alpha+j} + (1-t)^{\ell\alpha+j})\} + \phi(t) \ ,$$

so that

$$\begin{aligned}\sum_{\ell=0}^{3} K^{\ell} K(t^{-1/3} + (1-t)^{-1/3}) = {} & \{t^{-1/12} + t^{1/6} + t^{5/12} + t^{2/3}\} \\ & + \{(1-t)^{-1/12} + (1-t)^{1/6} + (1-t)^{5/12} + (1-t)^{2/3}\} \\ & + \text{terms already in (3.4.8)(b)} \\ & + \ \phi(t) \ .\end{aligned}$$

Similarly

$$\begin{aligned}\sum_{\ell=0}^{3} IK^{\ell}(DK^4)K(t^{-1/3} + (1-t)^{-1/3}) = {} & \{t^{-1/12} + t^{1/6} + t^{5/12} + t^{2/3}\}\{t\} \\ & + \{(1-t)^{-1/12} + (1-t)^{1/6} + (1-t)^{5/12} + (1-t)^{2/3}\}\{(1-t)\} \\ & + \text{terms already in (3.4.8)(b)} \\ & + \phi(t) \ ,\end{aligned}$$

and continuing this process, we obtain

$$
\begin{aligned}
(3.4.8)(a) = {} & \{t^{-1/12} + t^{1/6} + t^{5/12} + t^{2/3}\}\{1 + t + \dots + t^m\} \\
& + \{(1-t)^{-1/12} + (1-t)^{1/6} + (1-t)^{5/12} + (1-t)^{2/3}\} \\
& \qquad \{1 + (1-t) + \dots + (1-t)^m\} \\
& + \text{terms already in } (3.4.8)(b) \\
& + \phi(t) .
\end{aligned}
$$

Combining this with (3.4.8)(b), obtainable from Example 2, we have

$$
\begin{aligned}
y(t) = {} & t^{-1/3} + (1-t)^{-1/3} \\
& + \{t^{-1/12} + t^{1/6} + t^{5/12} + t^{2/3}\}\{1 + \dots + t^m\} \\
& + \{(1-t)^{-1/12} + (1-t)^{1/6} + (1-t)^{5/12} + (1-t)^{2/3}\}\{1 + \dots + (1-t)^m\} \\
& + \Big\{ \sum_{i=0}^{m-1} (t\ln t)^i (t^{1/4} + t^{1/2} + t^{3/4} + t\ln t)(1 + \dots + t^{m-i-1}) \\
& + \sum_{i=0}^{m-1} ((1-t)\ln(1-t))^i ((1-t)^{1/4} + (1-t)^{1/2} + (1-t)^{3/4} + (1-t)\ln(1-t)) \\
& \qquad (1 + \dots + (1-t)^{m-i-1}) \Big\} \\
& + \phi(t), \qquad t \in [0,1] ,
\end{aligned}
$$

where $\phi \in W_1^{m+1}[0,1]$.

CHAPTER 4

GALERKIN METHODS FOR EQUATIONS WITH SINGULARITIES

4.1 INTRODUCTION

In this chapter we shall discuss the numerical solution of equations of the form

$$y(t) = f(t) + \lambda \int_a^b k(t,s)\, y(s)ds , \qquad t \in [a,b] ,$$

where k and f are given functions on $[a,b] \times [a,b]$, and $[a,b]$ respectively, λ is a given scalar, and y is the solution to be determined.

Without loss of generality, we may assume that $\lambda = 1$, $[a,b] = [0,1]$, and so in this chapter we simplify the treatment by considering only the equation:

$$y(t) = f(t) + \int_0^1 k(t,s)\, y(s)ds, \qquad t \in [0,1]. \tag{4.1.1}$$

We abbreviate (4.1.1) by

$$y = f + Ky ,$$

where K is the integral operator given by

$$Ky(t) = \int_0^1 k(t,s)\, y(s)ds . \tag{4.1.2}$$

The Galerkin and iterated Galerkin methods are well-established numerical algorithms for the approximate solution of (4.1.1).

It has been shown by Sloan et. al., [63], [57], [58], that the iterated Galerkin method provides, in general, a more accurate approximation to y than does the Galerkin method.

Accurate quantitative estimates for this improvement in order (or "Superconvergence") have been obtained by Chandler, [9], [10], [11], for the case when the underlying approximating subspace is a space of splines, and when both the kernel, k, and the inhomogeneous term, f, are suitably smooth.

The aim of this chapter will be to obtain such quantitative estimates, again when splines are used as approximating functions, in the case when k is of weakly singular convolution type, and also when f may have a low order of smoothness.

Our main quantitative order of convergence result is Theorem 4.8 of Section 4.4. To illustrate the results of this rather general theorem, consider the particular equation

$$y(t) = t^{\beta-1} + \int_0^1 |t-s|^{\alpha-1} y(s)ds, \quad t \in [0,1] , \tag{4.1.3}$$

where $1 > \alpha > 0$, and $2 > \beta > 1$. Suppose our approximating subspace is the space of splines of order $r \in \mathbb{N}$ defined on a uniform mesh over $[0,1]$, and let y_n^I and y_n^{II} denote, respectively, the Galerkin and iterated Galerkin approximants to y. Then, Theorem 4.8 predicts that

$$\|y - y^I\|_\infty = 0\left(\frac{1}{n^\gamma}\right) , \tag{4.1.4}$$

and

$$\|y - y_n^{II}\|_\infty = 0\left(\frac{1}{n^{\gamma+\delta}}\right) , \tag{4.1.5}$$

where

$$\gamma = \min\{r, \alpha, \beta-1\} ,$$

and

$$\delta = \min\{r, \alpha\} , \tag{4.1.6}$$

and $n+1$ is the number of points in the mesh.

Much more general error estimates for the numerical solution of weakly singular equations are given in Theorem 4.8. However, the illustration given here highlights two important points which are also true in the general case.

(i) The order of the improvement obtained by using y_n^{II} instead of y_n^I is $0\left(\frac{1}{n^\delta}\right)$, with δ given by (4.1.6).

(ii) If either α or β is small, then both y_n^I and y_n^{II} may converge rather slowly to y, regardless of how large r is.

The reason for the phenomenon (ii) is of course that, as demonstrated in Chapter 3, any weakly singular convolution integral equation, such as (4.1.3), will, in general, have a non-smooth solution, and the order of approximation of such a solution using splines on a uniform (or arbitrary) mesh will, in general, be rather low.

This order may be improved, however, if we use a mesh which takes account of the singularities in the solution. In Section 4.5, we consider equation (4.1.3), and demonstrate how to improve convergence by using an appropriate non uniform mesh. In particular, we show that with the correct mesh, (4.1.4) and (4.1.5) may be improved to (in the particular case $r = 1$)

$$\|y - y_n^I\|_\infty = 0\left(\frac{1}{n}\right) \tag{4.1.7}$$

amd

$$\|y - y_n^{II}\|_\infty = 0\left(\frac{1}{n^{1+\delta}}\right) \tag{4.1.8}$$

For any $r \geq 1$, we also show that, provided we use a suitable mesh we may obtain the high order convergence estimates in the $L_2[0,1]$ norm:

$$\|y - y_n^I\|_2 = O\left(\frac{1}{n^r}\right) \quad , \tag{4.1.9}$$

and

$$\|y - y_n^{II}\|_2 = o\left(\frac{1}{n^r}\right) \quad . \tag{4.1.10}$$

The main convergence results are contained in Sections 4.4 and 4.5. The remainder of the chapter is organised as follows. In Section 4.2, we define the Galerkin and iterated Galerkin algorithms and give a résumé of existing convergence results. In Section 4.3 we present some necessary theoretical tools which we shall use to prove our order of convergence estimates. It is at this point that we restrict attention to the case when the kernel k of (4.1.2) is of convolution type. In Theorem 4.3 we prove (with the aid of the analysis of Chapter 3) two results which describe how the smoothness of k and f affects the properties of K and y. In Theorem 4.4 we prove some spline approximation properties of typical weakly singular functions. Finally, the order of convergence estimates contained in Sections 4.4 and 4.5, are illustrated in Section 4.6 by some numerical calculations.

The numerical solution of weakly singular integral equations has recently been the subject of much research activity. For example, Chandler [11], [12], and Schneider [55] have studied product integration using graded meshes to obtain good convergence rates. Spence [65] and Lin Qun [38] have considered the use of extrapolation methods to improve the rates of convergence of product integration and iterated collocation methods respectively. Anselone and Krabs [3] have used a double approximation scheme based on replacing singular functions by bounded approximations, while Anselone [2] has given a theoretical basis for the popular practical technique [52] of subtracting out the singularity from the solution. Delves, Abd-Elal and Hendry [17] have studied ways of making the Galerkin method for weakly singular equations more economical.

Finally, we note the extensive treatment in Baker's book [7, Sections 5.3-5.8], where the performance of most of the standard methods, as applied to the numerical solution of weakly singular equations is discussed. Many numerical examples are given.

4.2 METHODS AND BACKGROUND.

In this section we introduce the Galerkin and iterated Galerkin methods for (4.1.1), and describe some of the recent progress which has been made in research on these methods.

For each $n \in \mathbb{N}$, let U_n denote a finite dimensional subspace of $L_2[0,1]$, and let P_n denote the orthogonal projection of $L_2[0,1]$ onto U_n.

Let us assume, for the moment, that K is a compact operator on $L_2[0,1]$, that $(I - K)^{-1}$ is well defined on $L_2[0,1]$, and that $f \in L_2[0,1]$. (Conditions sufficient to ensure this are formally stated in Section 4.3), and let us assume also that the spaces U_n are constructed to have the property that

$$\|\phi - P_n\phi\|_2 \to 0 , \quad \text{as} \quad n \to \infty ,$$

for every $\phi \in L_2[0,1]$.

The Galerkin solution of (4.1.1), y_n^I, is then defined by the equation

$$y_n^I = P_n f + P_n K y_n^I , \tag{4.2.1}$$

and the iterated Galerkin solution, y_n^{II} is obtained by the natural iteration:

$$y_n^{II} = f + K y_n^I . \tag{4.2.2}$$

For the details of the practical computation of these approximate solutions, see Sloan et. al.[63].

Applying the operator P_n to each side of (4.2.2), and comparing with (4.2.1), it follows that

$$P_n y_n^{II} = y_n^I \quad ,$$

which on substitution into (4.2.2) gives

$$y_n^{II} = f + KP_n y_n^{II} \quad . \tag{4.2.3}$$

A proof of Theorem 4.1 below can be found in the seminal paper of Sloan [57].

<u>Theorem 4.1</u> (Sloan) For sufficiently large n, y_n^{II} is well defined, and

$$\|y - y_n^{II}\|_2 \leq C\|Ky - KP_n y\|_2 \leq \epsilon_n \|y - P_n y\|_2 \quad ,$$

where $\epsilon_n \to 0$, as $n \to \infty$.

Since it is also known [57], that y_n^I is also well defined, for sufficiently large n , and

$$\|y - P_n y\|_2 \leq \|y - y_n^I\|_2 \leq (1 + \epsilon_n')\|y - P_n y\|_2 \quad , \tag{4.2.4}$$

where $\epsilon_n' \to 0$, as $n \to \infty$, it may be deduced that $\|y - y_n^I\|_2$ approaches zero with an order of convergence that is asymptotically the same as that of $\|y - P_n y\|_2$, while $\|y - y_n^{II}\|_2$ approaches zero more quickly (by a factor of $O(\epsilon_n)$) than $\|y - P_n y\|_2$.

This "improvement by iteration" has particular practical significance since the calculation of y_n^{II} requires roughly the same amount of computation time as the calculation of y_n^I [63].

The obviously interesting mathematical problem, therefore, is: What is the order of the improvement in accuracy obtained by using y_n^{II} as an approximation to y rather than y_n^I ? We shall consider this problem for the particular case when U_n is a certain space of spline functions, which we now define.

For any interval $[a,b]$, and any $n \in \mathbb{N}$, let Π_n denote the mesh (partition) given by

$$\Pi_n : a = x_0 < x_1 < x_2 < \dots < x_n = b \ .$$

For $r \in \mathbb{N}$ and $\nu \in \mathbb{N}_0$, with $\nu < r$, we shall let $S_r^\nu(\Pi_n, [a,b])$ denote the space of splines on $[a,b]$ which have order r, continuity ν, and knots Π_n. Thus

$u \in S_r^\nu(\Pi_n, [a,b])$ if $u \in C^{\nu-1}[a,b]$, and u is a polynomial of degree not greater than $r - 1$ on each $(x_{i-1}, x_i]$, for $i = 1,\dots,n$. When $\nu = 0$ the splines are possibly discontinuous at the knot points, $x_0,\dots,x_n$, but, to ensure that they are well defined, we assume left continuity at each knot, and right continuity at a. We shall abbreviate $S_r^\nu(\Pi_n,[0,1])$ by $S_r^\nu(\Pi_n)$.

Throughout the remainder of this chapter y_n^I and y_n^{II} will denote the approximations to y defined by (4.2.1) and (4.2.3), where,

$$\left.\begin{array}{l} U_n = S_r^\nu(\Pi_n) \ , \qquad n \in \mathbb{N} \ , \\ \text{for some fixed } r \in \mathbb{N} \text{ and } \nu \in \mathbb{N}_0 \ , \text{ with } \nu < r \ . \end{array}\right\} \quad (4.2.5)$$

We shall give our order of convergence estimates in terms of the maximum mesh spacing h, defined by

$$h = \max_{i=1,\dots,n} (h_i) ,$$

where

$$h_i = (x_i - x_{i-1}) , \quad i = 1,\dots,n . \tag{4.2.6}$$

Note that, for a *uniform* mesh we have

$$h = \frac{1}{n} .$$

Then the following quantitative estimates have been derived by Chandler .

<u>Theorem 4.2</u> (Chandler). If k and f are sufficiently smooth (for precise requirements see [10] or [11]) , and if, as n varies the meshes Π_n satisfy a certain quasiuniformity condition (see Section 4.4), then

$$\left.\begin{aligned} \|y - y_n^I\|_\infty &= O(h^r) , \\ \text{and} \qquad \|y - y_n^{II}\|_\infty &= O(h^{2r}) . \end{aligned}\right\} \tag{4.2.7}$$

<u>Remarks.</u> (i) The estimates (4.2.7) demonstrate the great improvement of y_n^{II} over y_n^I when all our given information is sufficiently smooth. The startling fact that y_n^{II} converges to y with $O(h^{2r})$, when the best approximation to y from splines of order r is generally only $O(h^r)$, is generally referred to as <u>superconvergence</u>. Since the estimates (4.2.7) are in the infinity

norm, they demonstrate the global nature of the superconvergence, and automatically imply estimates of the same order in $L_p[0,1]$, for any $1 \leq p < \infty$. However, if weak singularities are present in k or f, the regularity requirements of Theorem 4.2 will not be satisfied, see, for example [10, p.106], and estimates of $\|y - y_n^I\|_\infty$ and $\|y - y_n^{II}\|_\infty$ are not yet available. Such estimates will be obtained in Sections 4.4 and 4.5 of this thesis.

(ii) An elegant overview of Superconvergence phenomena, which includes the results reviewed above, may be found in Chatelin [13].

4.3 REGULARITY AND APPROXIMATION

In the first part of this section, we give a result describing the properties of the integral operator K, and the solution y of (4.1.1), in the case when the kernel, k, or the inhomogeneous term, f, may be weakly singular.

First, we introduce the assumptions:

B1. The kernel k of (4.1.2) has the specific convolution form

$$k(t,s) = k(t-s), \qquad t, s \in [0,1],$$

with $k \in N_1^{\alpha}[-1,1]$ for some $\alpha > 0$.

B2. The homogeneous equation

$$y(t) = \int_0^1 k(t-s)\, y(s)ds, \quad t \in [0,1],$$

has no non trivial solutions in $L_1[0,1]$.

Remark. While the results of this chapter hold only for pure convolution equations with kernels of the type described by B1 and B2, the methods used to obtain these results are readily generalised to deal with kernels of the form

$$k(t,s) = \kappa(t-s)\, m(t,s)$$

where κ satisfies B1 and B2 and m is suitably smooth.

Theorem 4.3 Suppose B1 and B2 are satisfied.

(i) If $f \in C[0,1]$, then

$$y = (I-K)^{-1} f$$

is well defined in $C[0,1]$.

(ii) If $f \in N_1^{\beta}[0,1]$, for some $\beta > 1$,

then

$$y \in N_1^{\min\{\alpha+1,\beta\}}[0,1].$$

Proof. Since $N_1^{\alpha}[-1,1] \subseteq L_1[-1,1]$, it follows from Theorem 2.3 that $k \in M_1[0,1]$, and hence that K is compact from $L_\infty[0,1]$ to $C[0,1]$, and hence also from $C[0,1]$ to $C[0,1]$. The proof of (i) follows by the Fredholm Alternative. To obtain (ii), refer to Chapter 3, and note that by (3.2.3), A1 is satisfied with $[a,b] = [0,1]$. Since B2 implies A2 with $[a,b] = [0,1]$, we may apply Theorem 3.9. The required result then follows, since Theorem 3.9 implies that $y(t)$ is a linear combination of $f(t)$, terms of the form $\int_0^t k(x)dx$ and $\int_0^t k(x-1)dx$, plus smoother functions.

The spline approximation properties of the space $N_p^{\alpha}[a,b]$ are proved in Theorem 4.4 below. The proof involves the $L_p[a,b]$ *mth order modulus of smoothness,* $\omega_m(\phi,h)_{L_p[a,b]}$, which is defined for arbitrary $\phi \in L_p[a,b]$, $m \in \mathbb{N}_0$, $h > 0$, and $1 \leq p \leq \infty$, by

$$\omega_m(\phi,h)_{L_p[a,b]} = \sup_{0<|\epsilon|\leq h} \|\Delta_\epsilon^m \phi\|_{L_p[a,b]_{m\epsilon}},$$

with $[a,b]_{m\epsilon}$ given by (3.2.2). We abbreviate this by $\omega_m(\phi,h)_p$, when $[a,b]$ is unambiguous.

Some important properties of the modulus of smoothness are collected in Lemma A5.

Recalling the notation of Section 4.2, we shall let Π_n denote a family of meshes on $[0,1]$, with h denoting the maximum mesh space of Π_n.

<u>Theorem 4.4</u> Let $r \in \mathbb{N}$, $\nu \in \mathbb{N}_0$ be fixed with $\nu < r$.

(i) Let B1 be satisfied. Then, for each $t \in [0,1]$, there exists a spline $u_t \in S_r^{\nu}(\Pi_n)$, such that

$$\int_0^1 |k(t-s) - u_t(s)|ds \le \begin{cases} Ch^{\delta} & r \neq \alpha, \\ Ch^{\delta}\ell n(\frac{1}{h}), & r = \alpha, \end{cases}$$

where $\delta = \min\{r,\alpha\}$, and C is independent of t and h.

(ii) Let $\phi \in N_{\infty}^{\eta}[0,1] \cap C^{[\eta]}[0,1]$, for some $\eta > 0$, where $[\eta]$ is given by (3.2.1). Then there exists a spline $v \in S_r^{\nu}(\Pi_n)$, such that

$$\|\phi - v\|_{\infty} \le \begin{cases} Ch^{\gamma} & r \neq \eta, \\ Ch^{\gamma}\ell n(\frac{1}{h}), & r = \eta \end{cases}$$

where $\gamma = \min\{r,\eta\}$, and C is independent of h.

<u>Proof</u> (i) It follows from De Vore [18, Theorem 4.1], that, for $t \in [0,1]$ there exists $u_t \in S_r^{\nu}(\Pi_n)$, such that

$$\int_0^1 |k(t-s) - u_t(s)|ds \le C\omega_r(k_t,h)_1 ,$$

where C depends only on r, and k_t is the function

$$k_t(s) = k(t-s), \quad s \in [0,1]. \tag{4.3.1}$$

The required estimates then follow from Lemma A5 (ii) and (i) .

(ii) It follows from De Vore [18, Theorem 4.1] that there exists a spline $v \in S_r^{\nu}(\Pi_n)$, such that

$$\|\phi - v\|_\infty \leq \omega_r(\phi, h)_\infty \quad ,$$

and the required result follows directly from Lemma A5(i) .

4.4 ORDER OF CONVERGENCE ESTIMATES.

In this section we derive global order of convergence estimates for the Galerkin and iterated Galerkin approximants to the solution y of (4.1.1), in the case when the kernel k satisfies B1 and B2. The first step in proving the required estimates is given in Theorem 4.5, and consists of transforming the original convergence theory (Theorem 4.1, and its sequel), from its $L_2[0,1]$ setting into a $C[0,1]$ setting. An analogous global convergence theory for equations with smooth kernels and solutions was first given by Chandler [11].

Recall that y_n^I and y_n^{II} are defined by (4.2.1) and (4.2.3), where the approximating subspaces U_n are defined for fixed $r \in \mathbb{N}$, $\nu \in \mathbb{N}_0$, $\nu < r$ by (4.2.5). As indicated in Theorem 4.2, we shall assume that, as n varies, the partitions Π_n, used in the defintions of the splines, remain quasi uniform as n varies, i.e. there exists a constant C with the property that

$$\frac{\max\limits_{i=1,\ldots,n} (h_i)}{\min\limits_{i=1,\ldots,n} (h_i)} \leq C , \tag{4.4.1}$$

for each partition Π_n, where h_i is given by (4.2.6).

One important consequence of (4.4.1), is that it implies that $h \to 0$ as $n \to \infty$, where h is the maximum grid spacing. Another important consequence of (4.4.1), which is well known in the finite element literature [8], [20], is the fact that P_n is bounded when considered as an operator on $L_\infty[0,1]$, and, in fact, there exists a constant C, independent of n, such that

$$\|P_n\|_\infty \le C\ , \qquad n \in \mathbb{N}\ . \tag{4.4.2}$$

Note that it follows that P_n is also bounded as an operator from $C[0,1]$ to $L_\infty[0,1]$, with norm also satisfying (4.4.2)

Then we can prove the following theorem.

Theorem 4.5 Let B1, B2 be satisfied, let $f \in C[0,1]$, and suppose that

$$\|K - KP_n\|_{C[0,1]} \to 0\ , \qquad \text{as} \quad n \to \infty\ . \tag{4.4.3}$$

Then, for sufficiently large n, y_n^I, y_n^{II} are well defined, $y_n^I \in L_\infty[0,1]$, $y_n^{II} \in C[0,1]$,

$$C_1\|y - P_ny\|_\infty \le \|y - y_n^I\|_\infty \le C_2\|y - P_ny\|_\infty\ , \tag{4.4.4}$$

and

$$\|y - y_n^{II}\|_\infty \le C\|Ky - KP_ny\|_\infty\ , \tag{4.4.5}$$

with C_1, C_2 and C independent of n.

Proof. We consider first y_n^{II}, and aim to apply the Collectively Compact Operator Approximation Theory of Anselone [1, Theorem 1.6].

Note first that, when proving Theorem 4.3 (i), we showed that $k \in M_1[0,1]$, and that K is compact on $C[0,1]$. Since K is also compact from $L_\infty[0,1]$ to $C[0,1]$, it follows that KP_n is compact on $C[0,1]$.

By virtue of (4.4.3), it follows directly that $KP_n \to K$ pointwise on $C[0,1]$, and also, less directly, that the set $\{KP_n\}$ is collectively compact [1, p.5]. We prove this last assertion, by showing, using the Ascoli-Arzela Theorem, that the set

$$S: = \{KP_n\phi : n \in \mathbb{N} , \phi \in C[0,1], \|\phi\|_\infty \leq 1\}$$

has compact closure in $C[0,1]$. Firstly, for $\phi \in C[0,1]$, with $\|\phi\|_\infty \leq 1$, and $n \in \mathbb{N}$, we have, using (4.4.2),

$$\begin{aligned} \|KP_n\phi\|_\infty &\leq \|K\|_\infty \|P_n \phi\|_\infty \\ &\leq C\|K\|_\infty\|\phi\|_\infty \leq C\|K\|_\infty , \end{aligned}$$

where $\|K\|_\infty$ denotes the norm of K operating from $L_\infty[0,1]$ to $C[0,1]$, and thus S is bounded. Secondly, for $\phi \in C[0,1]$, with $\|\phi\|_\infty \leq 1$, $n \in \mathbb{N}$, $t\ \tau \in [0,1]$, we have, using (4.4.2) and Hölder's inequality,

$$\begin{aligned} |KP_n\phi(t) - KP_n\phi(\tau)| &= \left|\int_0^1 (k(t-s) - k(\tau-s))P_n\phi(s)ds\right| \\ &\leq \|k_t - k_\tau\|_1 \|P_n\phi\|_\infty \\ &\leq C\|k_t - k_\tau\|_\infty \to 0 , \quad \text{as} \quad t \to \tau , \end{aligned}$$

since $k \in M_1[0,1]$, and hence S is an equicontinuous set in $C[0,1]$. Thus, by the Ascoli-Arzela Theorem, [46, p.82], it follows that S has compact closure in $C[0,1]$, and hence that the set $\{KP_n\}$ is collectively compact in $C[0,1]$.

Then, since, by Theorem 4.3 (i), $(I-K)^{-1}$ exists on $C[0,1]$, it follows that $(I-KP_n)^{-1}$ exists on $C[0,1]$, for sufficiently large n, and is uniformly bounded in n. Using (4.2.3), then, it follows that y_n^{II} exists for large enough n, with

$$y_n^{II} = (I-KP_n)^{-1} f ,$$

and by Theorem 4.3 (i), we have,

$$\begin{aligned} y - y_n^{II} &= [(I-K)^{-1} - (I-KP_n)^{-1}]f \\ &= (I-KP_n)^{-1} (K-KP_n)y , \end{aligned} \tag{4.4.6}$$

from which (4.4.5) follows, on taking the infinity norm.

Now return to y_n^I, the existence of which is ensured, for sufficiently large n, by the fact that

$$P_n y_n^{II} = y_n^I . \tag{4.4.7}$$

To obtain the bounds (4.4.4), we use (4.4.6) and (4.4.7) to write

$$\begin{aligned} y - y_n^I = y - P_n y_n^{II} &= (y-P_n y) + P_n(y-y_n^{II}) \\ &= (y-P_n y) + P_n(I-KP_n)^{-1} (K-KP_n)y . \end{aligned}$$

Then, since K is bounded as an operator from $L_\infty[0,1]$ to $C[0,1]$, $(I-KP_n)^{-1}$ is uniformly bounded on $C[0,1]$, and P_n is uniformly bounded as an operator from $C[0,1]$ into $L_\infty[0,1]$, we have

$$\begin{aligned} \|y-y_n^I\|_\infty &\le \|y-P_n y\|_\infty + C\|(I-KP_n)^{-1} (K-KP_n)y\|_\infty \\ &\le \|y-P_n y\|_\infty + C\|(K-KP_n)y\|_\infty \\ &\le \|y-P_n y\|_\infty + C\|K\|_\infty \|y-P_n y\|_\infty \\ &\le C\|y-P_n y\|_\infty , \end{aligned} \tag{4.4.8}$$

and also, in view of (4.2.1), we may write

$$(I-P_nK)(y-y_n^I) = y - P_nKy - P_nf$$
$$= y - P_ny \quad ,$$

from which it follows that

$$\|y-P_ny\|_\infty \le \|I-P_nK\|_\infty \|y-y_n^I\|_\infty$$
$$\le (1 + C\|K\|_\infty) \|y-y_n^I\|_\infty \tag{4.4.9}$$

The result (4.4.4) then follows from (4.4.8) and (4.4.9), and this completes the proof.

It is clear that in order to satisfy (4.4.3), and to estimate the order of the right hand side of (4.4.5), we must estimate

$$\|K\phi - KP_n\phi\|_\infty \quad ,$$

for any $\phi \in C[0,1]$. This is the purpose of the next theorem. In fact, it turns out that (4.4.3) is a redundant assumption, being automatically satisfied by B1 .

Theorem 4.6 Let B1 be satisfied. Then, for $\phi \in C[0,1]$, we have

$$\|(K-KP_n)\phi\|_\infty \le \begin{cases} Ch^\delta\|\phi-P_n\phi\|_\infty \ , & r \neq \alpha \ , \\ Ch^\delta \ln(\frac{1}{h})\|\phi-P_n\phi\|_\infty \ , & r = \alpha \ , \end{cases}$$

where

$$\delta = \min\{r, \alpha\} \quad .$$

<u>Proof</u> For $t \in [0,1]$, $n \in \mathbb{N}$, and $\phi \in C[0,1]$, we have, using Theorem 4.4 (i) and the duality arguments from the mathematical theory of the finite element method [11],

$$\begin{aligned} |(K-KP_n)\phi(t)| &= \left| \int_0^1 k(t-s)(\phi(s) - P_n\phi(s))ds \right| \\ &= \left| \int_0^1 (k(t-s) - u_t(s))(\phi(s) - P_n\phi(s))ds \right| , \end{aligned}$$

and hence one application of Hölder's inequality yields,

$$|(K-KP_n)\phi(t)| \leq \int_0^1 |k(t-s) - u_t(s)|ds \quad \|\phi - P_n\phi\|_\infty ,$$

from which the required estimate follows via Theorem 4.4 (i) .

This result leads immediately to the following corollary.

<u>Corollary 4.7.</u> Let B1, B2 be satisfied, and let $f \in C[0,1]$. Then, for sufficiently large n, y_n^I and y_n^{II} are well defined, (4.4.4) holds, and

$$\|y - y_n^{II}\|_\infty \leq Ch^\delta \|y - P_n y\|_\infty , \qquad r \neq \alpha ,$$

$$\|y - y_n^{II}\|_\infty \leq Ch^\delta \ln(\tfrac{1}{h}) \|y - P_n y\|_\infty , \; r = \alpha ,$$

where $\delta = \min\{r,\alpha\}$.

<u>Proof.</u> It follows by Theorem 4.6 and (4.4.2) that for $\phi \in C[0,1]$,

$$\|(K-KP_n)\phi\|_\infty \leq \begin{cases} Ch^\delta \|\phi\|_\infty , & r \neq \alpha , \\ \\ Ch^\delta \ln(\frac{1}{h}) \|\phi\|_\infty , & r = \alpha , \end{cases}$$

where $\delta = \min\{r,\alpha\}$. Hence (4.4.3) holds, and the estimates (4.4.4) and (4.4.5) follow. The required estimates for $\|y-y_n^{II}\|_\infty$ then are obtained by applying the results of Theorem 4.6 to (4.4.5).

Remark. It has been shown by Chandler (see Theorem 4.2 above) that, if k is sufficiently smooth then $\|y-y_n^{II}\|_\infty$ converges faster to zero than $\|y-y_n^{I}\|_\infty$, the order of the improvement being $O(h^r)$, if splines of order r are used as approximating functions. The results obtained here show that, even if k is weakly singular, $\|y-y_n^{II}\|_\infty$ still converges faster than $\|y-y_n^{I}\|_\infty$. However, the order of improvement may be drastically reduced, and indeed, may not be enhanced by the employment of higher order splines.

The final theorem of this section estimates the rates of convergence of y_n^I and y_n^{II} to y, given certain smoothness properties of f and k. This is the main theorem of the chapter, and the results of Corollary 4.7 are included in it.

Theorem 4.8 Let B1, B2 be satisfied, and let $f \in N_1^\beta[0,1]$, for some $\beta > 1$. Then, the conclusions of Corollary 4.7 hold, and

$$\left.\begin{aligned} \|y - y_n^I\|_\infty &= O(h^\gamma) \\ \|y - y_n^{II}\|_\infty &= O(h^{\gamma+\delta}) \end{aligned}\right\} \begin{aligned} &r \neq \alpha \ , \\ &r \neq \min\{\alpha,\beta-1\} \ , \end{aligned}$$

$$\left.\begin{aligned} \|y - y_n^I\|_\infty &= O(h^\gamma \ln(\tfrac{1}{h})) \\ \|y - y_n^{II}\|_\infty &= O(h^{\gamma+\delta} \ln(\tfrac{1}{h})) \end{aligned}\right\} \begin{aligned} &r \neq \alpha \ , \\ &r = \min\{\alpha,\beta-1\} \ , \end{aligned}$$

$$\left.\begin{aligned} \|y - y_n^I\|_\infty &= O(h^\gamma) \\ \|y - y_n^{II}\|_\infty &= O(h^{\gamma+\delta}\ln(\tfrac{1}{h})) \end{aligned}\right\} \begin{aligned} & r = \alpha \quad , \\ & r \neq \min\{\alpha,\beta-1\} \end{aligned} \quad ,$$

$$\left.\begin{aligned} \|y - y_n^I\|_\infty &= O(h^\gamma \ln(\tfrac{1}{h})) \\ \|y - y_n^{II}\|_\infty &= O(h^{\gamma+\delta}\ln^2(\tfrac{1}{h})) \end{aligned}\right\} \begin{aligned} & r = \alpha \quad , \\ & r = \min\{\alpha,\beta-1\} \end{aligned} \quad .$$

with $\gamma = \min\{r, \alpha, \beta-1\}$,

and $\delta = \min\{r,\alpha\}$.

<u>Proof.</u> By (3.2.3) , we have

$$f \in N_1^\beta[0,1] \subseteq W_1^{[\beta]}[0,1] \subseteq C[0,1] \; ,$$

and thus the conclusions of Corollary 4.7 hold. The required estimates are obtained by estimating $\|y - P_n y\|_\infty$.

Note that, by Theorem 4.3 (ii), and (3.2.3), it follows that

$$y \in N_1^{\min\{\alpha+1,\beta\}}[0,1] \subseteq W_1^{[\min\{\alpha+1,\beta\}]}[0,1] \subseteq C^{[\min\{\alpha,\beta-1\}]}[0,1],$$

and

$$y \in N_1^{\min\{\alpha+1,\beta\}}[0,1] \subseteq N_\infty^{\min\{\alpha,\beta-1\}}[0,1] \; .$$

Hence, using (4.4.2), we have, for any $\xi_n \in S_r^\nu(\Pi_n)$,

$$\begin{aligned} \|y - P_n y\|_\infty &= \|(I-P_n)y\|_\infty = \|(I-P_n)(y-\xi_n)\|_\infty \\ &\leq (1+C)\|y - \xi_n\|_\infty \quad , \end{aligned}$$

and thus, by Theorem 4.4 (ii),

$$\|y - P_n y\|_\infty \le \begin{cases} Ch^\gamma , & r \neq \min\{\alpha,\beta-1\} , \\ Ch^\gamma \ln(\frac{1}{h}) , & r = \min\{\alpha,\beta-1\} , \end{cases}$$

with $\delta = \min\{r, \alpha, \beta-1\}$,

and the required estimates follow.

4.5 A GRADED MESH

The results of Section 4.4 demonstrate that Galerkin methods for equations with singularities may sometimes possess rather poor rates of convergence. It may be remarked, however, that these poor rates arise partly as a result of our (rather naïve) approach of using splines defined on arbitrary (quasiuniform) meshes, and that much better results may be obtained by using meshes specially chosen to take account of the singularities in the solution, y. Recently, such an approach has been employed by Chandler [12] and Schneider [55] to improve the performance of product integration methods for weakly singular equations.

We shall illustrate some methods that may be used to improve Galerkin methods, by referring to equation (4.1.3).

Note that B1 is satisfied by the integral operator of (4.1.3). We shall assume throughout that B2 is also satisfied.

It follows from Theorem 3.9 that the solution y of (4.1.3) is of the type $\{\gamma, r, \{0,1\}\}$, see Rice [48], for any $r \in \mathbb{N}$, where $\gamma = \min\{\alpha, \beta-1\}$. Suppose we consider the solution of (4.1.3) using splines from $S_r^0(\Pi_n)$, where, for $n \in \mathbb{N}$, the mesh Π_n is no longer arbitrary and quasiuniform, but is given (see Rice [48]) by

$$\begin{aligned} x_i &= \left(\frac{i}{n}\right)^q & i &= 0,\dots,\ell \\ &= 1 - \left(\frac{n-i}{n}\right)^q & i &= \ell+1,\dots,n \end{aligned}$$

where

$$\ell = n/2 \qquad \text{(n even)}$$

$$\ell = \frac{n-1}{2} \qquad \text{(n odd)}$$

and

$$q = r/\gamma \quad .$$

Note that the knots of this mesh are "bunched up" near the end points 0 and 1 (where y behaves badly), and "spread out" in the interior of the interval $[0,1]$ (where y is well-behaved).

It is shown by Rice that, for each $n \in \mathbb{N}$, there exists a spline $\xi_n \in S_r^0(\Pi_n)$ such that

$$\|y - \xi_n\|_\infty = O\left(\frac{1}{n^r}\right) \quad . \tag{4.5.1}$$

The mesh Π_n does not, however, satisfy the quasiuniformity requirement (4.4.1), and so, with P_n denoting the orthogonal projection of $L_2[0,1]$ onto $S_r^0(\Pi_n)$, we do not necessarily have, for general $r \in \mathbb{N}$,

$$\|P_n\|_\infty \leq C , \qquad n \in \mathbb{N} . \tag{4.5.2}$$

However, when $r = 1$, we have, for $\phi \in L_\infty[0,1]$

$$P_n\phi = \sum_{i=1}^{n} \frac{(\phi, u_i)}{(u_i, u_i)} u_i \quad ,$$

where, for each $i = 1,\ldots,n$, u_i is the function on $[0,1]$ defined by the relations

$$\left.\begin{array}{lll} u_1(t) = 1\,, & t \in [0,\, x_1] \\ u_1(t) = 0\,, & t \in (x_1,\, 1] \end{array}\right\} i = 1\,,$$

$$\left.\begin{array}{lll} u_i(t) = 1\,, & t \in (x_{i-1},\, x_i] \\ u_i(t) = 0\,, & t \in [0,1]\setminus(x_{i-1},x_i] \end{array}\right\} i \neq 1\,.$$

Hence,

$$\begin{aligned} \|P_n\phi\|_\infty &\leq \sup_{i=1,\ldots,n} \frac{|(\phi,u_i)|}{|(u_i,u_i)|} \\ &\leq \sup_{i=1,\ldots,n} \frac{\|u_i\|_1}{\|u_i\|_2^2} \quad \|\phi\|_\infty \\ &= \sup_{i=1,\ldots,n} \frac{\int_{x_{i-1}}^{x_i} dt}{\int_{x_{i-1}}^{x_i} dt} \quad \|\phi\|_\infty = \|\phi\|_\infty \end{aligned}$$

and so (4.5.2) is satisfied in this case.

Thus, using the space $S_1^0(\Pi_n)$ as our approximating subspace, the estimates of Corollary 4.7 are true for $r = 1$. Using (4.5.2), we also have, for any $\xi_n \in S_1^0(\Pi_n)$,

$$\begin{aligned} \|y-P_ny\|_\infty &= \|(I-P_n)(y-\xi_n)\|_\infty \\ &\leq C\|y-\xi_n\|_\infty \,, \end{aligned}$$

and, using (4.5.1) with $r = 1$, and the estimates of Corollary 4.7, we then obtain the improved estimates (4.1.7) and (4.1.8) for $\|y-y_n^I\|_\infty$ and $\|y-y_n^{II}\|_\infty$.

If we wish to use higher order splines (i.e. $r \geq 2$) defined on the non-uniform mesh Π_n , then (4.5.2) is not known to hold, and so we may not use the results of Corollary 4.7 to estimate $\|y-y_n^I\|_\infty$ and $\|y-y_n^{II}\|_\infty$. However, if we are willing to accept estimates in the $L_2[0,1]$ norm, then we may appeal to the initial convergence results (Theorem 4.1 and (4.2.4)), to obtain the following result.

<u>Theorem 4.9</u> Let y_n^I , y_n^{II} be the approximate solutions to (4.1.3) defined by (4.2.1), (4.2.3), with

$$U_n = S_r^0(\Pi_n) , \qquad n \in \mathbb{N} ,$$

for some $r \in \mathbb{N}$, where Π_n is the graded mesh introduced in this section. Then estimates (4.1.9) and (4.1.10) hold.

<u>Proof.</u> Note that (4.1.3) is of the form (4.1.1), with $f \in L_2[0,1]$, and [70, p.321], K compact on $L_2[0,1]$. Since we have assumed B2 , it follows that $(I-K)^{-1}$ is well defined on $L_2[0,1]$. Also [18] we have

$$\|\phi - P_n\phi\|_2 \to 0 , \qquad \text{as} \quad n \to \infty ,$$

for every $\phi \in L_2[0,1]$. Thus, Theorem 4.1 and (4.2.4) hold.

Now, by (4.5.1), we have, for $n \in \mathbb{N}$,

$$\begin{aligned} \|y-P_n y\|_2 &= \|(I-P_n)(y-\xi_n)\|_2 \\ &\leq 2\|y-\xi_n\|_2 \leq 2\|y-\xi_n\|_\infty \\ &\leq C\frac{1}{n^r} \quad , \end{aligned}$$

and, using this in Theorem 4.1 and (4.2.4), we obtain (4.1.9) and (4.1.10), completing the proof of the theorem.

4.6 NUMERICAL EXAMPLES.

In all four examples given below, splines of order 1 (i.e. piecewise constant functions) were used as approximating subspaces. In Examples 1, 2, and 3 a uniform mesh was used, while in Example 4 a graded mesh, as discussed in Section 4.5, was used.

To obtain reliable estimates of the orders of convergence for our numerical calculations, we choose to solve equations which have known solutions. Thus in each of our examples the inhomogeneous term, f, is chosen specially so that y has a particularly simple closed form. To obtain theoretical convergence rates, in the case of Examples 1, 2 and 3, we use the known properties of y to estimate the order of $\|y-P_n y\|_\infty$, and then we use Corollary 4.7 to estimate $\|y-y_n^I\|_\infty$ and $\|y-y_n^{II}\|_\infty$. In the case of Example 4, theoretical convergence rates are given directly by (4.1.7) and (4.1.8). Theorem 4.8 is not applicable to these examples, since it employs the full singularity analysis for a general f given in Chapter 3, and hence is inappropriate when f is specially chosen.

Although the solution in Examples 1 and 2 is smooth, in Examples 3 and 4 it is singular, and so these examples do constitute a representative sample of the type of problems encountered in practice.

In Tables 1 to 4 the estimated order of convergence, EOC, of the quantity e_n, say, was calculated using the formula

$$\mathrm{EOC} = \frac{\ln(e_n/e_{2n})}{\ln 2} .$$

In all examples we assume that B2 is satisfied.

Example 1.

$$y(t) = f(t) + \int_0^1 |t-s|^{-\frac{1}{2}} y(s)ds, \qquad t \in [0,1] ,$$

where f was chosen so that $y(t) = t$. Note that B1 is satisfied with $\alpha = \frac{1}{2}$. Since the solution is contrived to be smooth,

$$\|y - P_n y\|_\infty = O(\frac{1}{n}) ,$$

and so Corollary 4.7 gives

$$\|y - y_n^I\|_\infty = O(\frac{1}{n}) ,$$

and

$$\|y - y_n^{II}\|_\infty = O\left(\frac{1}{n^{3/2}}\right) .$$

The results are shown in Table 1.

Example 2.

$$y(t) = f(t) + \int_0^1 |t-s|^{\frac{1}{2}} y(s)ds , \qquad t \in [0,1] ,$$

where f was chosen so that $y(t) = t$. Note that B1 is satisfied with $\alpha = 3/2$. Corollary 4.7 predicts

$$\|y - y_n^I\|_\infty = O(\frac{1}{n})$$

$$\|y - y_n^{II}\|_\infty = O(\frac{1}{n^2}) .$$

The results are shown in Table 2.

n	$\|y-y_n^{I}\|_\infty$	EOC (Theory predicts 1.0)	$\|y-y_n^{II}\|_\infty$	EOC (Theory predicts 1.5)
2	0.47		0.32	
		1.31		1.54
4	0.19		0.11	
		1.27		1.78
8	0.79(-1)		0.32(-1)	
		1.13		1.71
16	0.36(-1)		0.98(-2)	
		1.08		1.71
32	0.17(-1)		0.30(-2)	

Table 1.

n	$\|y-y_n^{I}\|_\infty$	EOC (Theory predicts 1.0)	$\|y-y_n^{II}\|_\infty$	EOC (Theory predicts 2.0)
2	0.26		0.17(-1)	
		1.00		1.95
4	0.13		0.44(-2)	
		1.05		2.00
8	0.63(-1)		0.11(-2)	
		1.02		1.92
16	0.31(-1)		0.29(-3)	
		0.95		2.01
32	0.16(-1)		0.72(-4)	

Table 2.

Example 3.

$$y(t) = f(t) + \frac{2}{3} \int_{-1}^{1} |t-s|^{-\frac{1}{2}} y(s)ds, \qquad t \in [-1,1] \quad ,$$

where f was chosen so that $y(t) = (1-t^2)^{3/4}$. B1 is satisfied with $\alpha = 1/2$. This example has been considered by several authors, see, for example, Baker [7], Phillips [44], Spence [65] and Schneider [55]. In this case, the solution is not smooth, and in fact $y \in N_1^{7/4}[-1,1] \subseteq N_\infty^{3/4}[-1,1] \cap C[-1,1]$, and so Theorem 4.4 (ii) implies that

$$\|y - P_n y\|_\infty = O\left\{\frac{1}{n^{3/4}}\right\} \quad .$$

Thus Corollary 4.7 predicts

$$\|y - y_n^{I}\|_\infty = O\left\{\frac{1}{n^{3/4}}\right\} \quad ,$$

and

$$\|y - y_n^{II}\|_\infty = O\left\{\frac{1}{n^{5/4}}\right\} \quad .$$

The results are shown in Table 3.

Example 4.

We consider the same equation as in Example 3, but this time we use a graded mesh as described in Section 4.5. Then, the predictions

$$\|y - y_n^{I}\|_\infty = O(\tfrac{1}{n}) \quad ,$$

and

$$\|y - y_n^{II}\|_\infty = O\left\{\frac{1}{n^{3/2}}\right\}$$

follow from (4.1.7) and (4.1.8). The results are shown in Table 4.

n	$\|y-y_n^I\|_\infty$	EOC (Theory predicts 0.75)	$\|y-y_n^{II}\|_\infty$	EOC (Theory predicts 1.25)
2	0.75		0.40	
		0.35		0.57
4	0.59		0.27	
		0.71		1.17
8	0.36		0.12	
		0.78		1.29
16	0.21		0.49(-1)	
		0.81		1.29
32	0.12		0.20(-1)	

Table 3

n	$\|y-y_n^I\|_\infty$	EOC (Theory predicts 1.0)	$\|y-y_n^{II}\|_\infty$	EOC (Theory predicts 1.5)
2	0.75		0.40	
		0.50		0.80
4	0.53		0.23	
		0.92		1.31
8	0.28		0.93(-1)	
		1.00		1.54
16	0.14		0.32(-1)	
		1.11		1.68
32	0.65(-1)		0.10(-1)	

Table 4

CHAPTER 5

COLLOCATION METHODS FOR TWO-DIMENSIONAL PROBLEMS

5.1 THE METHODS.

In this chapter, we will be concerned, in general, with the numerical solution of the equation

$$y(t) = f(t) + \lambda \int_{\overline{\Omega}} k(t,s)y(s)ds, \qquad t \in \overline{\Omega} \,,$$

where $\Omega \subseteq \mathbb{R}^2$ is a domain (i.e. an open connected set) which is bounded, and $\overline{\Omega}$ denotes its closure. The functions k and f are given on $\overline{\Omega} \times \overline{\Omega}$ and $\overline{\Omega}$ respectively, λ is a given scalar, and y is the solution to be determined.

Without loss of generality, we may simplify matters by considering the equation

$$y(t) = f(t) + \int_{\overline{\Omega}} k(t,s)y(s)ds, \qquad t \in \overline{\Omega} \,. \tag{5.1.1}$$

Equations of this form are important in applications (see Chapter 6 and [27]) . We abbreviate (5.1.1) in the usual way by

$$y = f + Ky \,,$$

where

$$Ky(t) = \int_{\overline{\Omega}} k(t,s)y(s)ds, \qquad t \in \overline{\Omega} \,. \tag{5.1.2}$$

In order to analyse the numerical methods which will be devised for this equation, we introduce the following basic assumptions.

C1. k is in the class $M_1(\overline{\Omega})$.

C2. The homogeneous version of (5.1.1) ,

$$y(t) = \int_{\overline{\Omega}} k(t,s)y(s)ds, \qquad t \in \overline{\Omega} \,,$$

has no non-trivial solutions in $L_1(\overline{\Omega})$.

C3. $f \in C(\bar{\Omega})$.

It then follows, by C1 and Theorem 2.1 that K is compact as an operator from $L_\infty(\bar{\Omega})$ to $C(\bar{\Omega})$, and hence, also from $C(\bar{\Omega})$ to $C(\bar{\Omega})$. Hence it follows from C2, C3, and the Fredholm alternative, that

$$y = (I-K)^{-1} f$$

is well defined in $C(\bar{\Omega})$.

We shall use the methods of collocation and iterated collocation to define two different approximations, y_N^I and y_N^{II} , to y . Specifically, we shall seek y_N^I in the form

$$y_N^I = \sum_{i=1}^{N} a_i u_i , \tag{5.1.3}$$

where $\{u_1,\ldots,u_N\}$ is a certain set of piecewise constant basis functions defined on $\bar{\Omega}$, and the coefficients $\{a_1,\ldots,a_N\}$ are the solution set of the $N \times N$ linear system obtained by demanding that

$$y_N^I(t_j) = f(t_j) + Ky_N^I(t_j) , \qquad j = 1,\ldots,N , \tag{5.1.4}$$

where $\{t_1,\ldots,t_N\} \subseteq \bar{\Omega}$ is some predetermined set of collocation points.

We then define y_N^{II} by the natural iteration,

$$y_N^{II} = f + Ky_N^I = f + \sum_{i=1}^{N} a_i Ku_i . \tag{5.1.5}$$

The basis set $\{u_1,\ldots,u_N\}$ and the collocation points $\{t_1,\ldots,t_N\}$ are defined as follows.

For each $N \in \mathbb{N}$ we introduce a mesh (partition) Π_N of $\bar{\Omega}$, consisting of N open, simply-connected, pairwise-disjoint subsets of $\bar{\Omega}$, $\{\Omega_i: i = 1,\dots,N\}$, with the property that each Ω_i contains its centroid, and

$$\bar{\Omega} = \bigcup_{i=1}^{N} \bar{\Omega}_i .$$

For $i = 1,\dots,N$, we then define u_i to be the function on $\bar{\Omega}$ which takes the value 1 on Ω_i, and 0 elsewhere. We assume that

$$\|\Pi_N\|_\infty \to 0 , \qquad \text{as} \quad N \to \infty , \tag{5.1.6}$$

where

$$\|\Pi_N\|_\infty = \max_{i=1,\dots,N} \sup_{t,t'\in\Omega_i} \|t - t'\|_\infty ,$$

and we also assume that

$$t_i \in \Omega_i ,$$

for $i = 1,\dots,N$.

It may be noticed immediately from (5.1.4), (5.1.5), C3 and the fact that K maps $L_\infty(\bar{\Omega})$ into $C(\bar{\Omega})$, that y_N^{II} is a continuous function which coincides with y_N^{I} at each of the collocation points, and hence is a kind of natural continuous interpolation for y_N^{I}. The main theme of this chapter will be to compare the numerical performances of y_N^{I} and y_N^{II}. In fact we shall show that y_N^{II} generally has better convergence properties than y_N^{I}, provided the collocation points are appropriately chosen. A summary of the main results which we shall obtain is given at the end of Section 5.2. My two-dimensional results given here were first reported in [64] ; where analogous methods for one dimensional equations were discussed.

5.2 THEORETICAL FRAMEWORK.

In order to analyse the convergence of the approximations y_N^I and y_N^{II} defined in Section 5.1, we need to cast equations (5.1.4) and (5.1.5) (which define our approximations) in some suitable operator theoretic setting. With this in mind, we define a projection P_N from $C(\overline{\Omega})$ onto $\mathrm{Span}\{u_1,\ldots u_N\} \subseteq L_\infty(\overline{\Omega})$ by

$$P_N\phi = \sum_{i=1}^{N} \phi(t_i)u_i \quad . \tag{5.2.1}$$

It is easy to show that P_N is bounded as an operator from $C(\overline{\Omega})$ to $L_\infty(\overline{\Omega})$, with the operator norm satisfying

$$\|P_N\| \leq 1 \ , \qquad N \in \mathbb{N} \ . \tag{5.2.2}$$

Then, noting that conditions C1, C2 and C3 ensure that f, y and Ky are all in $C(\overline{\Omega})$, we may rewrite (5.1.4) as

$$y_N^I = P_N f + P_N K y_N^I \quad . \tag{5.2.3}$$

It follows then, from (5.1.5) and (5.2.3), that

$$P_N y_N^{II} = y_N^I \ , \tag{5.2.4}$$

and hence, on substitution of this relation into (5.1.5), that

$$y_N^{II} = f + KP_N y_N^{II} \quad . \tag{5.2.5}$$

In order to prove that y_N^I and y_N^{II} exist and converge to y we shall use some standard arguments. Our method is analogous to that used to prove Theorem 4.5. We first consider y_N^{II}, which is defined by equation (5.2.5). Aiming to apply the Collectively Compact Operator Approximation Theory of Anselone, we show that the hypotheses of [1, Theorem 1.6] are satisfied in the space $C(\overline{\Omega})$.

Firstly, K is compact as an operator on $C(\bar{\Omega})$ (see Section 5.1). Secondly, K is also compact as an operator from $L_\infty(\bar{\Omega})$ to $C(\bar{\Omega})$, and since P_N is bounded as an operator from $C(\bar{\Omega})$ to $L_\infty(\bar{\Omega})$, it follows that KP_N is compact, and hence bounded as an operator on $C(\bar{\Omega})$.

Thirdly, for $\phi \in C(\bar{\Omega})$, we have,

$$\begin{aligned} \|(K-KP_N)\phi\|_\infty &= \|K(\phi-P_N\phi)\|_\infty \\ &\le \|K\|_\infty \|\phi - P_N\phi\|_\infty \\ &\le \|K\|_\infty\, \omega(\phi, \|\Pi_N\|_\infty) \to 0 , \quad \text{as} \quad N \to \infty , \end{aligned} \tag{5.2.6}$$

where ω is the two-dimensional modulus of continuity, defined [68, p.111], by

$$\omega(\phi,\epsilon) = \sup_{\substack{t,t'\in\bar{\Omega} \\ \|t-t'\|_\infty \le \epsilon}} |\phi(t) - \phi(t')| , \tag{5.2.7}$$

for $\epsilon > 0$, and $\phi \in C(\bar{\Omega})$.

Thus,

$$KP_N\phi \to K\phi ,$$

as $N \to \infty$, in $C(\bar{\Omega})$, for each $\phi \in C(\bar{\Omega})$.

Fourthly, we show that the set

$$S = \{KP_N\phi : N \in \mathbb{N} , \phi \in C(\bar{\Omega}) , \|\phi\|_\infty \le 1\}$$

has compact closure in $C(\bar{\Omega})$. We do this using the Ascoli-Arzela theorem [46, p.82]. When $\phi \in C(\bar{\Omega})$ with $\|\phi\|_\infty \le 1$, we have, using (5.2.2),

$$\begin{aligned} \|KP_N\phi\|_\infty &\le \|K\|_\infty \|P_N\phi\|_\infty \\ &\le \|K\|_\infty \|\phi\|_\infty \le \|K\|_\infty , \end{aligned}$$

and it follows that the set S is bounded. Also, for t, $\tau \in \overline{\Omega}$, $\phi \in C(\overline{\Omega})$, $\|\phi\|_\infty \leq 1$, we have, by Hölder's inequality and (5.2.2),

$$\begin{aligned} |KP_N\phi(t) - KP_N\phi(\tau)| &= \left| \int_{\overline{\Omega}} (k(t,s) - k(\tau,s))\, P_N\phi(s)ds \right| \\ &\leq \|k_t - k_\tau\|_1 \, \|P_N\phi\|_\infty \\ &\leq \|k_t - k_\tau\|_1 \to 0 , \quad \text{as} \quad t \to \tau , \end{aligned}$$

by C1. Hence the set S is equicontinuous. We may now use the Ascoli-Arzela theorem to conclude that S has compact closure in $C(\overline{\Omega})$.

Since $(I-K)^{-1}$ is bounded on $C(\overline{\Omega})$, it now follows [1, Theorem 1.6] that, for sufficiently large N, the operators $(I-KP_N)^{-1}$ exist on $C(\overline{\Omega})$, and are uniformly bounded in N.

Returning to (5.2.5), it follows that y_N^{II} exists in $C(\overline{\Omega})$, for sufficiently large N, and

$$y_N^{II} = (I- KP_N)^{-1} f .$$

Hence

$$\begin{aligned} y - y_N^{II} &= [(I - K)^{-1} - (I - KP_N)^{-1}]f \\ &= (I - KP_N)^{-1} (K - KP_N)y , \end{aligned}$$

and thus

$$\|y-y_N^{II}\|_\infty \leq C\|Ky - KP_Ny\|_\infty ,$$

with C independent of N.

Now, let us return to y_N^I. The existence of y_N^I, for large enough N, is guaranteed by (5.2.4). It may also be shown, using an argument analogous to that used to prove (4.4.4), that

$$C_1\|y-P_Ny\|_\infty \leq \|y-y_N^I\|_\infty \leq C_2\|y-P_Ny\|_\infty ,$$

with C_1 and C_2 independent of N. Since $y \in C(\overline{\Omega})$, we have, cf. (5.2.6),

$$\|y - P_N y\|_\infty \to 0 \text{, as } N \to \infty \text{ ,}$$

and thus we have proved the following theorem.

Theorem 5.1 Let C1, C2 and C3 be satisfied. Then, for sufficiently large N, y_N^I exists in $L_\infty(\bar{\Omega})$, y_N^{II} exists in $C(\bar{\Omega})$, y_N^I, y_N^{II} converge to y, and

$$\|y - y_N^I\|_\infty = O(\|y - P_N y\|_\infty)$$

and

$$\|y - y_N^{II}\|_\infty = O(\|Ky - KP_N y\|_\infty) \text{ .}$$

Our task in the remainder of this chapter is to use Theorem 5.1 to analyse the rates of convergence of y_N^I and y_N^{II} to y. In Section 5.3 we analyse the case when the kernel k, the inhomogeneous term f, and the solution y are suitably smooth. In such case we obtain the estimates

$$\|y - y_N^I\|_\infty = O(\|\Pi_N\|_\infty) \text{ ,} \tag{5.2.8}$$

and

$$\|y - y_N^{II}\|_\infty = O(\|\Pi_N\|_\infty^2) \text{ ,} \tag{5.2.9}$$

the latter result being dependent on the correct choice of collocation points $\{t_1,\ldots,t_N\}$. These estimates are proved in Theorem 5.3.

It is often (indeed usually) the case that the properties of k and f are known, but that the properties of y are unknown. In Theorem 5.4 we give conditions on k and f which ensure that y has the smoothness properties needed for (5.2.8) and (5.2.9) to hold.

The analysis of Section 5.3 uses Taylor's series methods, and hence requires that both k and y be fairly smooth. If k is weakly singular then not all the conditions of Theorem 5.3 are satisfied. The analogues of (5.2.8) and (5.2.9) for the weakly singular case are proved in Section 5.5, using approximation theoretic arguments which are more sensitive to the regularity of both k and y than the Taylor's series methods. We restrict ourselves to the case when $\overline{\Omega}$ is a rectangle and k has a weak singularity along $t = s$.

The main result of Section 5.5 is Theorem 5.15. As an illustration of the kind of information contained in Theorem 5.15, consider the prototype equations,

$$y(t) = f(t) + \int_0^d \int_0^1 |t-s|^{\alpha-1} y(s)ds, \quad t \in [0,1]\times[0,d], \qquad (5.2.10)$$

with $0 < \alpha < 1$, and

$$y(t) = f(t) + \int_0^d \int_0^1 \ln|t-s| y(s)ds, \quad t \in [0,1]\times[0,d], \qquad (5.2.11)$$

where $|x|$ denotes the length of any vector $x \in \mathbb{R}^2$, and f is twice continuously differentiable on $[0,1] \times [0,d]$. Theorem 5.15 then predicts that, for these prototype equations,

$$\|y - y^{I}_{N(\tau)}\|_\infty = O(\|\Pi_{N(\tau)}\|_\infty),$$

and

$$\|y - y^{II}_{N(\tau)}\|_\infty = O(\|\Pi_{N(\tau)}\|_\infty^{\beta+1}),$$

where, in the case of equation (5.2.10), β is any number satisfying $0 < \beta < \alpha$, and, in the case of equation (5.2.11), β is any number satisfying $0 < \beta < 1$. Here $\Pi_{N(\tau)}$ is a family of rectangular meshes on $\overline{\Omega} = [0,1] \times [0,d]$, which depend on a parameter

τ in such a way that, as $\tau \to 0$, $N(\tau) \to \infty$, and, as $\tau \to 0$, the subsets of $\overline{\Omega}$ given by $\Pi_{N(\tau)}$ shrink in size in a suitably uniform manner. The precise way that $N(\tau)$ depends on τ will be explained in Section 5.5.

One of the crucial ingredients of the proof of Theorem 5.15 is an accurate characterisation of the regularity of the solution to (5.1.1), in the case when $\overline{\Omega}$ is a rectangle and when k has a weak singularity along $t = s$. The required regularity theory is given in Section 5.4.

5.3 EQUATIONS WITHOUT SINGULARITIES

Let $\Omega \subseteq \mathbb{R}^2$ be the domain introduced in Section 5.1. In the remainder of this chapter, we shall make use of the Banach spaces

$$C^m(\overline{\Omega\times\Omega}) \ , \quad C^m(\overline{\Omega}) \ , \quad C^m[a,b] \ , \quad Lip_\beta(\overline{\Omega\times\Omega}) \quad \text{and} \quad Lip_\beta(\overline{\Omega}) \ ,$$

where $m \in \mathbb{N}_0$, and $0 < \beta \leq 1$. A unified definition of these spaces may be given as follows.

Let D be any domain in $\mathbb{R}^n$. Then we let $C^m(\overline{D})$ be the space of all functions $\phi \in C(\overline{D})$, which have the property that

$$\frac{\partial^{|\gamma|}\phi}{\partial x_1^{\gamma_1} \cdots \partial x_n^{\gamma_n}} \in C(\overline{D}) \ ,$$

for all multiindices γ satisfying $|\gamma| \leq m$. (We use here the standard notation for multiindices, see [37, p.19].) Also, we let $Lip_\beta(\overline{D})$ denote the space of all functions $\phi \in C(\overline{D})$, which have the property that

$$\sup_{\substack{t\in D \\ t+h\in D}} |\phi(t+h) - \phi(t)| \leq C|\underset{\sim}{h}|^\beta \ ,$$

for all $h \in \mathbb{R}^n \backslash \{\underset{\sim}{0}\}$, with C independent of h. Both $C^m(\overline{D})$ and $Lip_\beta(\overline{D})$ become Banach spaces when equipped with an appropriate norm, [37, p.25], but the precise definition of the norm will not be required for what follows. We shall also refer to the Sobolev space $W_2^1(\overline{\Omega})$, the space of all functions $\phi \in L_2(\overline{\Omega})$ such that $\frac{\partial\phi}{\partial t_1} \in L_2(\overline{\Omega})$, and $\frac{\partial\phi}{\partial t_2} \in L_2(\overline{\Omega})$. This is also a Banach space under an appropriate norm [37, p.264].

When we define t to be any point in $\overline{\Omega}$, then, without further explanation, we shall assume that t has coordinates (t_1, t_2).

We shall consider the orders of convergence of y_N^I and y_N^{II} when the collocation points $\{t_i : i = 1, \ldots, N\}$ are chosen so that t_i is the centroid of Ω_i, for each $i = 1, \ldots, N$.

Using elementary calculus, it can be shown that, in this case $t_i = (t_{i1}, t_{i2})$, where

$$\left.\begin{aligned} t_{i1} &= \frac{1}{A_i} \int_{\Omega_i} s_1 \, ds_1 \, ds_2 , \\ &\text{and} \\ t_{i2} &= \frac{1}{A_i} \int_{\Omega_i} s_2 \, ds_1 \, ds_2 , \quad i = 1, \ldots, N , \end{aligned}\right\} \qquad (5.3.1)$$

and A_i denotes the area of Ω_i.

The choice of points (5.3.1) is crucial to the following analysis, the main motivation for this choice coming from the following lemma.

Lemma 5.2. Let $y \in C^1(\overline{\Omega})$ with $\frac{\partial y}{\partial t_1} \in \mathrm{Lip}_1(\overline{\Omega})$, $\frac{\partial y}{\partial t_2} \in \mathrm{Lip}_1(\overline{\Omega})$, and let the collocation points $\{t_1, \ldots, t_N\}$ be chosen according to (5.3.1). Then, for $i = 1, \ldots, N$,

$$\left| \int_{\Omega_i} (y(s) - y(t_i)) ds \right| \leq C \|\Pi_N\|_\infty^2 \int_{\Omega_i} ds ,$$

with C independent of i and N, and

$$\left| \int_{\overline{\Omega}} (y(s) - P_N y(s)) ds \right| \leq C \|\Pi_N\|_\infty^2 ,$$

with C independent of N.

Remark. This lemma shows that the function $y - P_N y$, although only $O(\|\Pi_N\|_\infty)$ in the supremum norm, has an integral which converges with order $\|\Pi_N\|_\infty^2$. This is the central fact which allows us to prove increased rates of convergence for $y - y_N^{II}$ in Theorem 5.3.

Proof of Lemma 5.2

For $i = 1,\ldots,N$, we have, using the two dimensional Taylor's theorem,

$$\int_{\Omega_i} (y(s) - y(t_i))ds = \frac{\partial y}{\partial t_1}(t_i) \int_{\Omega_i} (s_1 - t_{i1})ds + \frac{\partial y}{\partial t_2}(t_i) \int_{\Omega_i} (s_2 - t_{i2})ds$$

$$+ \int_{\Omega_i} (s_1 - t_{i1})\left(\frac{\partial y}{\partial t_1}(\xi_{i,s}) - \frac{\partial y}{\partial t_1}(t_i)\right)ds + \int_{\Omega_i} (s_2 - t_{i2})\left(\frac{\partial y}{\partial t_2}(\xi_{i,s}) - \frac{\partial y}{\partial t_2}(t_i)\right)ds \tag{5.3.2}$$

where $\xi_{i,s}$ denotes some point on the segment joining t_i and s.

By choice of collocation points (5.3.1), the first two terms of (5.3.2) vanish, and, by the hypotheses of the lemma, the integrands of the remaining terms are uniformly $O(\|\Pi_N\|_\infty^2)$. Thus

$$\left| \int_{\Omega_i} (y(s) - y(t_i))ds \right| \le C\|\Pi_N\|_\infty^2 \int_{\Omega_i} ds \quad , \tag{5.3.3}$$

with C independent of i and N, as required.

Since, by (5.2.1),

$$\int_{\bar{\Omega}} (y(s) - P_N y(s))ds = \sum_{i=1}^{N} \int_{\Omega_i} (y(s) - y(t_i))ds \quad ,$$

the second part of the lemma follows by summation of (5.3.3) over i.

Theorem 5.3 Let C1 , C2 and C3 be satisfied.

(i) If $y \in C^1(\bar{\Omega})$, then (5.2.8) holds.

(ii) If $y \in C^1(\bar{\Omega})$, $\frac{\partial y}{\partial t_1}$, $\frac{\partial y}{\partial t_2} \in \mathrm{Lip}_1(\bar{\Omega})$, $k \in \mathrm{Lip}_1(\overline{\Omega\times\Omega})$, and the collocation points are chosen according to (5.3.1), then (5.2.8) and (5.2.9) hold.

Proof. (i) Since $y \in C^1(\bar{\Omega})$, it follows simply from Taylor's theorem, that, for $t, t' \in \bar{\Omega}$, with $\|t-t'\|_\infty \leq \|\Pi_N\|_\infty$, we have

$$|y(t) - y(t')| \leq C\|\Pi_N\|_\infty \quad ,$$

with C independent of t and t' . Since

$$\|y - P_N y\|_\infty \leq \omega(y, \|\Pi_N\|_\infty) \quad ,$$

with ω defined by (5.2.7), estimate (5.2.8) now follows via Theorem 5.1.

(ii) Clearly (5.2.8) holds by the reasoning of part (i). To prove (5.2.9), we estimate $\|Ky - KP_N y\|_\infty$, and then apply Theorem 5.1. Note first that, for $t \in \bar{\Omega}$,

$$\begin{aligned} |Ky(t) - KP_N y(t)| &= \left| \int_{\bar{\Omega}} k(t,s)(y(s) - P_N y(s))ds \right| \\ &= \left| \sum_{i=1}^{N} \int_{\Omega_i} k(t,s)(y(s) - y(t_i))ds \right| , \end{aligned} \qquad (5.3.4)$$

using (5.2.1).

Then, since $k \in Lip_1(\overline{\Omega\times\Omega})$, we have

$$\left| \sum_{i=1}^{N} \int_{\Omega_i} k(t,s)(y(s) - y(t_i))ds \right|$$

$$= \left| \sum_{i=1}^{N} \left[k(t,t_i) \int_{\Omega_i} (y(s) - y(t_i))ds + \int_{\Omega_i} (k(t,s)-k(t,t_i))(y(s)-y(t_i))ds \right] \right|$$

$$\leq \|k\|_\infty \sum_{i=1}^{N} \left|\int_{\Omega_i}(y(s)-y(t_i))ds\right| + \sum_{i=1}^{N} \left(\int_{\Omega_i} |k(t,s)-k(t,t_i)||y(s)-y(t_i)|ds \right)$$

$$\leq C \|\Pi_N\|_\infty^2 \tag{5.3.5}$$

with C independent of N and t , where the final estimate is obtained by Lemma 5.2, and the fact that, since $k \in Lip_1(\overline{\Omega\times\Omega}) = Lip_1(\bar{\Omega}\times\bar{\Omega})$, and $y \in C^1(\bar{\Omega})$,

$$|k(t,s) - k(t,t_i)||y(s) - y(t_i)| \leq C\|\Pi_N\|_\infty^2 , \ s \in \Omega_i$$

with C independent of N , t and s .

The estimate (5.2.9) then follows via Theorem 5.1, after we have used (5.3.4) and (5.3.5) to show that

$$\|Ky - KP_N y\|_\infty \leq C\|\Pi_N\|_\infty^2 .$$

As remarked in Section 5.2, generally the properties of y are unknown, while the properties of k and f are known. Therefore the results given here will be a lot more relevant practically if we can present conditions on k and f which ensure that the regularity requirements of Theorem 5.3 are satisfied. Such is the purpose of the following theorem.

<u>Theorem 5.4</u> Let $k \in C^2(\overline{\Omega\times\Omega})$, $f \in C^2(\bar{\Omega})$ and let C2 be satisfied. Then $y \in C^2(\bar{\Omega})$, and, provided the collocation points are chosen according to (5.3.1), the estimates (5.2.8) and (5.2.9) hold.

<u>Proof</u>. It follows from Theorems 2.3, and 2.1, that k satisfies C1, and thus, by C2, y exists in $C(\bar{\Omega})$.

Now, letting D^2 be any differential operator of order two with respect to the multivariable (t_1,t_2) , and using Lemma A2 , we have

$$D^2y(t) = D^2f(t) + \int_\Omega D^2k(t,s)\; y(s)ds\;. \tag{5.3.6}$$

Since $D^2k(t,s) \in C(\overline{\Omega\times\Omega}) = C(\bar{\Omega}\times\bar{\Omega})$, it follows from Theorems 2.3 and 2.1 that the integral on the right hand side of (5.3.6) is continuous in t . Hence, since $f \in C^2(\bar{\Omega})$, it follows from (5.3.6) that $D^2y \in C(\bar{\Omega})$, and thus $y \in C^2(\bar{\Omega})$.

The estimates (5.2.8) and (5.2.9) then follow via Theorem 5.3.

5.4 REGULARITY RESULTS FOR WEAKLY SINGULAR EQUATIONS

We will be concerned in this section, with the equation (5.1.1), in the case when $\bar{\Omega}$ is a rectangle in $\mathbb{R}^2$, and k has a weak singularity along the diagonal $t = s$. Without loss of generality, we assume that $\bar{\Omega} = [0,1] \times [0,d]$, for some $d > 0$.

To clarify notation, when x is a scalar, $|x|$ will denote the absolute value of x. When $x = (x_1,x_2) \in \mathbb{R}^2$, $|x|$ will denote the length of x, $(x_1^2 + x_2^2)^{\frac{1}{2}}$. For the rectangle $\bar{\Omega} = [0,1] \times [0,d]$, we let $\bar{\Omega}^* = \{t-s : t,s \in \bar{\Omega}\} = [-1,1] \times [-d,d]$, and we define

$$|\bar{\Omega}^*| = \sup_{x\in\bar{\Omega}^*} |x| .$$

From now on we shall study integral equations of the form (5.1.1) which satisfy C1', C2, and C3', where C1' and C3' are new assumptions on k and f given as follows.

C1' $k(t,s) = \psi_\alpha(|t-s|)$ for some $0 < \alpha \leq 1$,

with

$$\psi_\alpha(x) = B(x)x^{\alpha-1} , \quad 0 < \alpha < 1 ,$$

and

$$\psi_1(x) = B(x) \ln x ,$$

where $B \in C^1[0,|\bar{\Omega}^*|]$.

C3' $f \in C^2(\bar{\Omega})$.

A kernel k which satisfies C1' is said to be weakly singular along the diagonal $t = s$.

It is easy to show, using the properties of B, and Lemma A6, that the function

$$x \to \psi_\alpha(|x|)$$

is in $L_2(\overline{\Omega^*})$, for all $0 < \alpha \leq 1$, and it follows by Theorem 2.3, that, if k satisfies C1', then $k \in M_2(\overline{\Omega})$. Hence $k \in M_1(\overline{\Omega})$, and so C1' implies C1. Since C3' trivially implies C3, it follows that any results which are true under C1, C2 and C3, are also true under C1', C2 and C3'.

In the next theorem we list the important properties of the integral operator K (given by (5.1.2)) when the kernel k is weakly singular.

Theorem 5.5 Let k satisfy C1'. Then the operator K has the following properties.

(i) $K : L_2(\overline{\Omega}) \to C(\overline{\Omega})$ is compact.

(ii) $K : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact.

(iii) K maps $L_2(\overline{\Omega})$ into $W_2^1(\overline{\Omega})$.

(iv) K maps $C(\overline{\Omega})$ into $\mathrm{Lip}_1(\overline{\Omega})$.

Proof. The proof of (i) follows from Theorem 2.1 (since $k \in M_2(\overline{\Omega})$), and (ii) follows immediately. Part (iii) follows from some results of Mikhlin (see the recent paper of Pitkäranta [45, Lemma 1]).

To prove (iv), we appeal to the results of Kantorovich and Akilov [33, p. 363]. Denoting the gradient with respect to the multi-variable t by ∇_t, we have, for $0 < \alpha \leq 1$,

$$\nabla_t \psi_\alpha(|t-s|) = \frac{\psi'_\alpha(|t-s|)}{|t-s|}\left[(t_1-s_1)\underset{\sim}{i} + (t_2-s_2)\underset{\sim}{j}\right] ,$$

and thus

$$| \nabla_t \psi_\alpha(|t-s|) | = |\psi'_\alpha(|t-s|)|$$
$$\leq C|t-s|^{\alpha-2} \quad , \qquad (5.4.1)$$

with C independent of t and s using $C1'$.

Thus, using Lemma A6, and (5.4.1), it follows that, for $0 < \alpha \leq 1$,

$$\int_{\bar{\Omega}} |\nabla_t \psi_\alpha(|t-s|)| ds \leq C \int_{\bar{\Omega}} |t-s|^{\alpha-2} ds$$
$$\leq C \int_{\bar{\Omega}^*} |s|^{\alpha-2} ds \leq C < \infty , \qquad (5.4.2)$$

with C independent of t .

Since, using $C1'$, we also have, for $0 < \alpha \leq 1$,

$$\int_{\bar{\Omega}} \frac{|\psi_\alpha(|t-s|)|}{|t-s|} ds \leq \|B\|_\infty \int_{\bar{\Omega}} |t-s|^{\delta-2} ds$$

where $\delta = \alpha$ if $0 < \alpha < 1$,

and δ is any number satisfying $0 < \delta < 1$, if $\alpha = 1$, it follows by the reasoning used to obtain (5.4.2), that for all $0 < \alpha \leq 1$,

$$\int_{\bar{\Omega}} \frac{|\psi_\alpha(|t-s|)|}{|t-s|} ds \leq C < \infty ,$$

with C independent of t .

The required result (iv) then follows from [33, p.363, Theorem 4], and on recalling that $C(\bar{\Omega}) \subseteq L_\infty(\bar{\Omega})$.

In order to motivate the next theorem, which investigates the properties of certain integral operators which are related to K , recall the methods of Chapter 3, where we analysed one dimensional weakly singular equations of the form (3.1.1). Our technique hinged on the fact that, provided y is sufficiently regular, we may differentiate the integral Ky given by(3.1.2) to obtain (cf. Lemma 3.1):

$$Ky'(t) = \frac{d}{dt}\int_a^b k(t-s)y(s)ds = y(a)k(t-a)-y(b)k(t-b)+\int_a^b k(t-s)y'(s)ds \ .$$

Consider the two dimensional equation (5.1.1), with operator of the form given by (5.1.2), and Cl':

$$Ky(t) = \int_{\overline{\Omega}} \psi_\alpha(|t-s|)y(s)ds = \int_0^d\int_0^1 \psi_\alpha(|(t_1,t_2) - (s_1,s_2)|)y(s_1,s_2)ds_1ds_2 \ .$$

A technique analogous to that used in Lemma 3.1 may be employed. Suppose $\frac{\partial y}{\partial t_1} \in L_2(\overline{\Omega})$. Then we may obtain, formally,

$$\frac{\partial}{\partial t_1}\int_0^d\int_0^1 \psi_\alpha(|(t_1,t_2) - (s_1,s_2)|)y(s_1,s_2)ds_1ds_2$$

$$= \int_0^d \frac{\partial}{\partial t_1}\int_0^1 \psi_\alpha(|(t_1,t_2) - (s_1,s_2)|)\ y(s_1,s_2)\ ds_1ds_2$$

$$= \int_0^d \psi_\alpha(|(t_1,t_2) - (0,s_2)|)\ y(0,s_2)\ ds_2$$

$$- \int_0^d \psi_\alpha(|(t_1,t_2) - (1,s_2)|)\ y(1,s_2)\ ds_2$$

$$+ \int_0^d\int_0^1 \psi_\alpha(|(t_1,t_2) - (s_1,s_2)|)\ \frac{\partial y}{\partial t_1}\ (s_1,s_2)\ ds_1ds_2 \ . \qquad (5.4.3)$$

We show in Theorem 5.7 that formula (5.4.3) is valid. But first, Theorem 5.6 investigates the properties of a class of integral operators which arise in such formulas for the differentiation of Ky .

For $z = (z_1, z_2) \in \bar{\Omega}$, define the integral operators S_{z_1}, T_{z_2} by

$$S_{z_1} y(t) = \int_0^d \psi_\alpha(|(t_1,t_2) - (z_1,s_2)|) y(z_1,s_2) ds_2, \quad t \in \bar{\Omega},$$

and,

$$T_{z_2} y(t) = \int_0^1 \psi_\alpha(|(t_1,t_2) - (s_1,z_2)|) y(s_1,z_2) ds_1, \quad t \in \bar{\Omega}.$$

The operators S_{z_1}, T_{z_2} arise naturally in the differentiation of the two dimensional weakly singular integrals, Ky. For example, the formula (5.4.3) can be written much more concisely as

$$\frac{\partial}{\partial t_1} Ky = S_0 y - S_1 y + K \frac{\partial y}{\partial t_1}.$$

Concerning the properties of T_{z_1}, S_{z_2}, we have the following theorem.

Theorem 5.6 Let $z = (z_1, z_2)$ be any point in $\bar{\Omega}$.

(i) S_{z_1}, T_{z_2} are compact on $C(\bar{\Omega})$.

(ii) S_{z_1}, T_{z_2} map $C(\bar{\Omega})$ into $Lip_\beta(\bar{\Omega})$, for any $0 < \beta < \alpha$.

(iii) If $y \in C(\bar{\Omega})$ with $\frac{\partial y}{\partial t_1} \in C(\bar{\Omega})$, then $\frac{\partial}{\partial t_1} T_{z_2} y \in L_2(\bar{\Omega})$ and

$$\frac{\partial}{\partial t_1} T_{z_2} y(t) = y(0,z_2)\psi_\alpha(|(t_1,t_2) - (0,z_2)|) - y(1,z_2)\psi_\alpha(|(t_1,t_2)-(1,z_2)|) + T_{z_2} \frac{\partial y}{\partial t_1}(t),$$

for all $t_2 \in [0,d]$, and almost all $t_1 \in [0,1]$.

(iv) If $y \in C(\bar{\Omega})$ with $\frac{\partial y}{\partial t_2} \in C(\bar{\Omega})$, then $\frac{\partial}{\partial t_2} S_{z_1} y \in L_2(\bar{\Omega})$ and

$$\frac{\partial}{\partial t_2} S_{z_1} y(t) = y(z_1,0)\psi_\alpha(|(t_1,t_2)-(z_1,0)|)-y(z_1,d)\psi_\alpha(|(t_1,t_2)-(z_1,d)|)$$

$$+ \ S_{z_1} \frac{\partial y}{\partial t_1}(t) \quad ,$$

for all $t_1 \in [0,1]$, and almost all $t_2 \in [0,d]$.

Proof (i) We prove the result for S_{z_1} only; the proof for T_{z_2} is analogous.

First, write

$$S_{z_1} y(t) = \int_{\overline{\Omega}} \psi_\alpha(|(t_1,t_2) - (z_1,s_2)|)\, y(z_1,s_2)ds \ ,$$

and note that $S_{z_1} = Q_2 Q_1$,

where

$$Q_1\, y(t_1,t_2) = y(z_1,t_2) \ , \quad (t_1,t_2) \in \overline{\Omega} \ ,$$

and

$$Q_2\, y(t_1,t_2) = \int_{\overline{\Omega}} \psi_\alpha(|(t_1,t_2) - (z_1,s_2)|) y(s_1,s_2)ds, \qquad (t_1,t_2) \in \overline{\Omega} \ .$$

Since Q_1 maps $C(\overline{\Omega})$ continuously into $C(\overline{\Omega})$, the proof will follow, provided we can show that Q_2 is a compact operator, on $C(\overline{\Omega})$. To prove the compactness of Q_2 , we show that its kernel is in $M_1(\overline{\Omega})$, and use Theorem 2.1. Note that Theorem 2.3 (ii) is not applicable to this case, so instead we must argue from first principles.

We first abbreviate the quantity $|(t_1,t_2) - (s_1,s_2)|$, for any $t,s \in \overline{\Omega}$ by $|t_1,t_2,s_1,s_2|$.

Then, for $t \in \overline{\Omega}$, and $\alpha \neq 1$ we have by C1' , and Lemma A7 ,

$$\int_{\overline{\Omega}} \psi_\alpha(|(t_1,t_2) - (z_1,s_2)|)ds \leq \|B\|_\infty \int_{\overline{\Omega}} |t_1,t_2,z_1,s_2|^{\alpha-1} ds \leq C. \quad (5.4.4)$$

with C independent of t, and a similar argument holds if $\alpha = 1$.

Also, for $t,t' \in \overline{\Omega}$, $\alpha \neq 1$, we have

$$\int_{\overline{\Omega}} |\psi_\alpha(|t_1,t_2,z_1,s_2|) - \psi_\alpha(|t_1',t_2',z_1,s_2|)|ds$$

$$= \int_{\overline{\Omega}} |(B(|t_1,t_2,z_1,s_2|) - B(|t_1',t_2',z_1,s_2|))|t_1,t_2,z_1,s_2|^{\alpha-1}$$

$$+ B(|t_1',t_2',z_1,s_2|)(|t_1,t_2,z_1,s_2|^{\alpha-1} - |t_1',t_2',z_1,s_2|^{\alpha-1})|ds$$

$$\leq C\left[\sup_{s_2\in[0,d]} |B(|t_1,t_2,z_1,s_2|) - B(|t_1',t_2',z_1,s_2|)| + \|B\|_\infty |t-t'|^\beta\right],$$

where C is independent of t and t', and $0 < \beta < \alpha < 1$, and the final inequality is obtained using Lemmas A7 and A8. Also, since, by C1', $B \in C^1[0,|\overline{\Omega}^*|]$, a simple application of the two dimensional Taylor's theorem shows that

$$\sup_{s_2\in[0,d]} |B(|t_1,t_2,z_1,s_2|) - B(|t_1',t_2',z_1,s_2|)| \leq C|t - t'|$$

with C independent of t,t', we have

$$\int_{\overline{\Omega}} |\psi_\alpha(|t_1,t_2,z_1,s_2|)-\psi_\alpha(|t_1',t_2',z_1,s_2|)|ds \leq C|t-t'|^\beta, \quad (5.4.5)$$

with $0 < \beta < \alpha < 1$. An analogous result may be proved if $\alpha = 1$.

It follows from (5.4.4) and (5.4.5) that the kernel of Q_2 is in $M_1(\overline{\Omega})$, and the required result follows.

(ii) Again we prove this result for S_{z_1} only ; the proof for T_{z_2} is analogous. The proof is simple, for, letting $y \in C(\bar{\Omega})$, and using the same abbreviations as in (i) we obtain, analogously to (5.4.5), that for $t, t+h \in \Omega$

$$|S_{z_1}y(t+h) - S_{z_1}y(t)|$$

$$= |\int_0^d (\psi_\alpha(|t_1+h_1,t_2+h_2,z_1,s_2|) - \psi_\alpha(|t_1,t_2,z_1,s_2|))y(z_1,s_2)ds_2|$$

$$\leq C|h|^\beta \quad ,$$

with β any number satisfying $0 < \beta < \alpha \leq 1$, and C independent of t and h. This completes the proof.

The proofs of (iii) and (iv) follow simply from Theorem A3. For example, letting

$$\kappa(x) = \psi_\alpha(\sqrt{x^2+(t_2-z_2)^2}) \quad , \qquad x \in [-1,1] \quad ,$$

$$\phi(x) = y(x,z_2) \quad , \qquad x \in [0,1] \; ,$$

then it is easy to show that $\kappa \in L_1[-1,1]$, and since, by (i), $T_{z_2}y$ exists for all $(t_1,t_2) \in \bar{\Omega}$, Theorem A3 may be applied to obtain the formula for $\frac{\partial}{\partial t_1} T_{z_1}$ given in (iii). The fact that $\frac{\partial}{\partial t_1} T_{z_2} y \in L_2(\bar{\Omega})$ follows from (i) and Lemma A6.

The result (iv) may be proved similarly.

The properties of the operators S_{z_1} and T_{z_2} proved in Theorem 5.6 now enable us to prove the following theorem, which is the natural generalisation to two variables of Lemma 3.1.

<u>Theorem 5.7</u> Let k satisfy C1', and let $y \in C(\bar{\Omega})$.

(i) If $\frac{\partial y}{\partial t_1} \in L_2(\bar{\Omega})$, then

$$\frac{\partial}{\partial t_1} Ky = S_0 y - S_1 y + K \frac{\partial y}{\partial t_1} .$$

(ii) If $\frac{\partial y}{\partial t_2} \in L_2(\bar{\Omega})$, then

$$\frac{\partial}{\partial t_2} Ky = T_0 y - T_1 y + K \frac{\partial y}{\partial t_1} .$$

<u>Proof</u>. We prove (i) only, (ii) is proved similarly. Fix $t_2 \in [0,d]$, and let

$$g(t_1,s_2) = \int_0^1 \psi_\alpha(|(t_1,t_2) - (s_1,s_2)|)\, y(s_1,s_2)ds_1 .$$

Then, by Theorem 5.6(iii) we have, for all $s_2 \in [0,d]$, and almost all $t_1 \in [0,1]$,

$$\begin{aligned}\frac{\partial g}{\partial t_1}(t_1,s_2) &= y(0,s_2)\psi_\alpha(|(t_1,t_2) - (0,s_2)|) \\ &\quad - y(1,s_2)\psi_\alpha(|(t_1,t_2) - (1,s_2)|) \\ &\quad + \int_0^1 \psi_\alpha(|(t_1,t_2) - (s_1,s_2)|)\frac{\partial y}{\partial t_1}(s_1,s_2)ds_1 \end{aligned} \qquad (5.4.6)$$

Now, by Theorem A2 (using the methods of Rudin [50, Chapter 7] to verify the required measurability conditions), it follows that

$$\begin{aligned}&\frac{d}{dt_1} \int_0^d \int_0^1 \psi_\alpha(|(t_1,t_2) - (s_1,s_2)|)y(s_1,s_2)ds_1ds_2 \\ &= \int_0^d \frac{d}{dt_1} \int_0^1 \psi_\alpha(|(t_1,t_2) - (s_1,s_2)|)y(s_1,s_2)ds_1ds_2 ,\end{aligned} \qquad (5.4.7)$$

and the required result follows on substitution of (5.4.6) into (5.4.7).

Recall the notation used in Chapter 3: The expression $\{a(t) + b(t) + \ldots + z(t)\}$ is used to denote some linear combination of the functions $a(t), b(t), \ldots,$ and $z(t)$.

We are now ready to state and prove the main result of this section.

<u>Theorem 5.8</u> Let C1', C2 and C3' be satisfied. Then, if y is the solution to (5.1.1),

(i) $y \in C^1(\overline{\Omega})$.

(ii) $\dfrac{\partial y}{\partial t_1} \in \mathrm{Lip}_\beta(\overline{\Omega})$ and $\dfrac{\partial y}{\partial t_2} \in \mathrm{Lip}_\beta(\overline{\Omega})$,

where β is any number in the range $0 < \beta < \alpha$.

(iii) $\dfrac{\partial^2 y}{\partial t_1 \partial t_2} \in L_2(\overline{\Omega})$, with

$$\frac{\partial^2 y}{\partial t_1 \partial t_2}(t) = \{\psi_\alpha(|(t_1,t_2)|) + \psi_\alpha(|(t_1,t_2) - (0,d)|)$$

$$+ \psi_\alpha(|(t_1,t_2) - (1,0)|) + \psi_\alpha(|(t_1,t_2) - (1,d)|)\}$$

$$+ \phi(t), \quad \text{for almost all } t \in \overline{\Omega},$$

where $\phi \in C(\overline{\Omega})$.

<u>Proof.</u> Since $f \in C^2(\overline{\Omega}) \subseteq C(\overline{\Omega})$, it follows from C2, Theorem 5.5 (ii), and the Fredholm alternative, that (5.1.1) has a unique solution $y \in C(\overline{\Omega})$. Thus, Theorem 5.5 (iii) $Ky \in W_2^1(\overline{\Omega})$, and by C3', it follows that the right hand side of (5.1.1) is in $W_2^1(\overline{\Omega})$, and hence $y \in W_2^1(\overline{\Omega})$.

We may then use Theorem 5.7 (i) to differentiate (5.1.1), obtaining

$$\frac{\partial y}{\partial t_1} = \frac{\partial f}{\partial t_1} + S_0 y - S_1 y + K\frac{\partial y}{\partial t_1} \quad . \tag{5.4.8}$$

By Theorem 5.5(i), $K\frac{\partial y}{\partial t_1} \in C(\overline{\Omega})$, and by Theorem 5.6 (i) $S_0 y \in C(\overline{\Omega})$ and $S_1 y \in C(\overline{\Omega})$. Hence, by C3', the right-hand side of (5.4.8) is in $C(\overline{\Omega})$, and thus $\frac{\partial y}{\partial t_1} \in C(\overline{\Omega})$. It may be proved similarly that $\frac{\partial y}{\partial t_2} \in C(\overline{\Omega})$, and we have proved (i).

It follows from Theorems 5.5 (iv) and 5.6 (ii), and since $f \in C^2(\overline{\Omega})$, that the right hand side of (5.4.8) is in $\mathrm{Lip}_\beta(\overline{\Omega})$ for any β satisfying $0 < \beta < \alpha$, and hence $\frac{\partial y}{\partial t_1} \in \mathrm{Lip}_\beta(\overline{\Omega})$, for $0 < \beta < \alpha$. It may be shown analogously that $\frac{\partial y}{\partial t_2} \in \mathrm{Lip}_\beta(\overline{\Omega})$, for $0 < \beta < \alpha$, and hence (ii) follows.

To prove (iii), note again that $\frac{\partial y}{\partial t_1} \in C(\overline{\Omega})$, and so, by Theorem 5.5 (iii) $K\frac{\partial y}{\partial t_1} \in W_2^1(\overline{\Omega})$, and also, by Theorem 5.6 (iv), $\frac{\partial}{\partial t_2} S_0 y \in L_2(\overline{\Omega})$ and $\frac{\partial}{\partial t_2} S_1 y \in L_2(\overline{\Omega})$. Since $f \in C^2(\overline{\Omega})$, the right hand side of (5.4.8) has a partial derivative with respect to t_2 which is in $L_2(\overline{\Omega})$. Hence $\frac{\partial^2 y}{\partial t_2 \partial t_1} \in L_2(\overline{\Omega})$, and we may use Theorems 5.6 (iv) and 5.7 (ii) to differentiate (5.4.8) with respect to t_2, to obtain:

$$\frac{\partial^2 y}{\partial t_2 \partial t_1}(t) = \{\psi_\alpha(|(t_1,t_2)|) + \psi_\alpha(|(t_1,t_2) - (0,d)|)$$
$$+ \psi_\alpha(|(t_1,t_2) - (1,0)|) + \psi_\alpha(|(t_1,t_2) - (1,d)|)\}$$
$$+ \frac{\partial^2 f}{\partial t_2 \partial t_1}(t)$$
$$+ S_0 \frac{\partial y}{\partial t_2}(t) - S_1 \frac{\partial y}{\partial t_2}(t)$$
$$+ T_0 \frac{\partial y}{\partial t_1}(t) - T_1 \frac{\partial y}{\partial t_1}(t)$$
$$+ K \frac{\partial^2 y}{\partial t_2 \partial t_1}(t) \quad . \tag{5.4.9}$$

By Theorems 5.5 (i) and 5.6 (i), and since $f \in C^2(\bar{\Omega})$, it follows that the last six terms on the right hand side of (5.4.9) are in $C(\bar{\Omega})$, and the result (iii) follows. This completes the proof of Theorem 5.8.

In this section we have investigated the regularity properties of the solution to a weakly singular two dimensional integral equation. It is clear from Theorem 5.8, that, in the case that C1', C2 and C3' are satisfied and $\bar{\Omega} = [0,1] \times [0,d]$, the conditions of Theorem 5.3 (i) are satisfied, but the conditions of Theorem 5.3 (ii) are not satisfied, either by y or k. The analogue of (5.2.9) for the weakly singular case will be proved in Theorem 5.15 using the regularity results of this section.

5.5 CONVERGENCE RESULTS FOR WEAKLY SINGULAR EQUATIONS.

In this section, we analyse the convergence of the numerical approximations to y defined in Section 5.1, in the case when the kernel of the integral equation (5.1.1) is weakly singular, and when $\bar{\Omega} = [0,1] \times [0,d]$.

Since the piecewise constant functions introduced in Section 5.1 are really just two dimensional splines (of order 1 or, equivalently, of degree 0), it is reasonable to expect that a tight analysis of these numerical methods will require some two-dimensional spline approximation theory. Appealing to Munteanu and Schumaker [41] for such a theory, we must first define a certain class of rectangular meshes on $\bar{\Omega}$.

Definition 5.9 For each $\tau \in (0,1]$, let there exist integers $p(\tau)$, $q(\tau)$, and meshes

$$\Pi_{p(\tau)} : 0 = x_0(\tau) < x_1(\tau) < \dots < x_{p(\tau)}(\tau) = 1 ,$$

and

$$\Pi_{q(\tau)} : 0 = y_0(\tau) < y_1(\tau) < \dots < y_{q(\tau)}(\tau) = d ,$$

with the property that, for some constants C_1, C_2,

$$C_1\tau \le \underline{\Delta}_i(\tau) \le \bar{\Delta}_i(\tau) \le C_2\tau , \quad i = 1,2, \quad \tau \in (0,1],$$

where

$$\underline{\Delta}_1(\tau) = \min_{j=1,\dots,p(\tau)} (x_j(\tau) - x_{j-1}(\tau)) ,$$

$$\bar{\Delta}_1(\tau) = \max_{j=1,\dots,p(\tau)} (x_j(\tau) - x_{j-1}(\tau)) ,$$

$$\underline{\Delta}_2(\tau) = \min_{\ell=1,\dots,q(\tau)} (y_\ell(\tau) - y_{\ell-1}(\tau)) ,$$

and

$$\bar{\Delta}_2(\tau) = \max_{\ell = 1,\ldots,q(\tau)} (y_\ell(\tau) - y_{\ell-1}(\tau)) .$$

In addition, suppose that, for $0 < \tau \leq \frac{1}{2}$,

$$\{x_j(2\tau) : j = 0,\ldots,p(2\tau)\} \subseteq \{x_j(\tau) : j = 0,\ldots,p(\tau)\} ,$$

and

$$\{y_\ell(2\tau) : \ell = 0,\ldots,q(2\tau)\} \subseteq \{y_\ell(\tau) : \ell = 0,\ldots,q(\tau)\} .$$

Then, with $N(\tau) = p(\tau)q(\tau)$, we have, for each $\tau \in (0,1]$, a mesh $\Pi_{N(\tau)}$ on $\bar{\Omega}$, given by

$$\Pi_{N(\tau)} = \{(x_{j-1}(\tau), x_j(\tau)) \times (y_{\ell-1}(\tau), y_\ell(\tau)) : j=1,\ldots,p(\tau); \ell=1,\ldots,q(\tau)\}.$$

We call such a family of meshes $\{\Pi_{N(\tau)} : \tau \in (0,1]\}$ an <u>M.S.Family of meshes on $\bar{\Omega}$</u>.

We shall refer to the mesh $\Pi_{N(\tau)}$ as being made up of the subsets $\Omega_i(\tau)$ (or Ω_i when τ is understood), for $i = 1,\ldots,N(\tau)$, where, for definiteness, we adopt the indexation convention

$$\Omega_{(\ell-1)p(\tau)+j}(\tau) = (x_{j-1}(\tau), x_j(\tau)) \times (y_{\ell-1}(\tau), y_\ell(\tau)) ,$$

for $j = 1,\ldots,p(\tau)$, and $\ell = 1,\ldots,q(\tau)$.

<u>Remarks 5.10.</u> Let $\{\Pi_{N(\tau)} : \tau \in (0,1]\}$ be an M.S. family of meshes on $\bar{\Omega}$.

(i) For each $i = 1,\ldots,N(\tau)$, the collocation point t_i , defined by (5.3.1) then turns out to be the centre of the rectangle $\bar{\Omega}_i(\tau)$.

(ii) It is clear from the definition of $\Pi_{N(\tau)}$, that

$$N(\tau) \to \infty \quad \text{as} \quad \tau \to 0 ,$$

and

$$C_1\tau \le \|\Pi_{N(\tau)}\|_\infty \le C_2\tau \quad ,$$

so that

$$\|\Pi_{N(\tau)}\|_\infty \to 0 \ , \quad \text{as} \quad \tau \to 0 \ .$$

(iii) Four examples of M.S. families of meshes are given in [41]. A particularly simple example is the case where, for $\tau \in (0,1]$, $p(\tau)$ is chosen to be the integer which satisfies

$$\frac{1}{\tau} + \frac{1}{2} \ge p(\tau) > \frac{1}{\tau} - \frac{1}{2} \quad ,$$

and $q(\tau)$ is set equal to $p(\tau)$. Then

$$\Pi_{p(\tau)} : 0 = x_0(\tau) < x_1(\tau) < \ldots < x_{p(\tau)}(\tau) = 1$$

is given by

$$x_j(\tau) = j\tau \ , \quad j = 0,\ldots,p(\tau) - 1 \ ,$$

$$x_{p(\tau)}(\tau) = 1 \ ,$$

and

$$\Pi_{q(\tau)} : 0 = y_0(\tau) < y_1(\tau) < \ldots < y_{q(\tau)}(\tau) = d$$

is given by

$$y_\ell(\tau) = \ell\tau d \ , \qquad \ell = 0,\ldots,q(\tau) - 1 \quad ,$$

$$y_{q(\tau)}(\tau) = d \ .$$

(iv) A practically important subfamily of the family given in Remark 5.10 (iii) is

$$\{\Pi_{N(n^{-1})} : n = 1,2,\ldots\} \quad .$$

In this case $p(n^{-1}) = q(n^{-1}) = n$, and the mesh $\Pi_{N(n^{-1})}$ is just obtained simply by dividing $\overline{\Omega}$ into $N(n^{-1}) = n^2$ subrectangles, each of dimensions $\frac{1}{n}$ by $\frac{d}{n}$.

From now on we shall let $\{\Pi_{N(\tau)} : \tau \in (0,1]\}$ denote some fixed family of M.S. meshes on $\overline{\Omega}$. We shall let $P_{N(\tau)}$ denote the projection, analogous to (5.2.1), onto the space spanned by the set of piecewise constant functions defined on the mesh $\Pi_{N(\tau)}$, using the collocation points discussed in Remark 5.10 (i). Then, for any $\tau \in (0,1]$, $P_{N(\tau)}$ is a bounded operator from $C(\overline{\Omega})$ to $L_\infty(\overline{\Omega})$, with operator norm satisfying

$$\|P_{N(\tau)}\| \leq 1 , \qquad \tau \in (0,1] . \tag{5.5.1}$$

We define, for $r \in \mathbb{N}$, the two dimensional spline space $S_r^{r-1}(\Pi_{N(\tau)}, \overline{\Omega})$, by

$$S_r^{r-1}(\Pi_{N(\tau)},\overline{\Omega}) = \Big\{\xi : \xi(s_1,s_2) = \xi_1(s_1)\xi_2(s_2), \quad \text{for} \quad (s_1,s_2) \in \overline{\Omega} ,$$

$$\text{where} \quad \xi_1 \in S_r^{r-1}(\Pi_{p(\tau)}, [0,1]), \ \xi_2 \in S_r^{r-1}(\Pi_{q(\tau)}, [0,d]) \Big\},$$

and $S_r^{r-1}(\Pi_{p(\tau)}, [0,1])$ and $S_r^{r-1}(\Pi_{q(\tau)}, [0,1])$ are the one dimensional spline spaces defined in Section 4.2.

We describe some important approximation theoretic properties of this two dimensional spline space in the next two lemmas.

<u>Lemma 5.11</u> Let ψ_α be defined as in C1'. Then, for each $t \in \overline{\Omega}$, there exists a spline $u_{\alpha,t} \in S_1^0(\Pi_{N(\tau)},\overline{\Omega})$, such that

$$\int_{\overline{\Omega}} |\psi_\alpha(|t-s|) - u_{\alpha,t}(s)|ds \leq C\|\Pi_{N(\tau)}\|_\infty ,$$

with C independent of t and τ.

Proof. The proof follows from Munteanu and Schumaker [41, Lemma 5.5], where it is shown that there exists $u_{\alpha,t} \in S_1^0(\Pi_{N(\tau)},\bar{\Omega})$ such that

$$\int_{\bar{\Omega}} |\psi_\alpha(|t-s|) - u_{\alpha,t}(s)|ds$$

$$\leq \left[\tau \int_{\bar{\Omega}} |\psi_\alpha(|t-s|)|ds + \sup_{0\leq\|h\|_\infty\leq\tau} \int_{\bar{\Omega}_h} |\psi_\alpha(|t-s-h|) - \psi_\alpha(|t-s|)|ds\right], \quad (5.5.2)$$

with C independent of t and τ, and where for $\varepsilon \in \mathbb{R}^2$,

$$\bar{\Omega}_\varepsilon = \{s \in \bar{\Omega} : s+\varepsilon \in \bar{\Omega}\} . \quad (5.5.3)$$

Now, for $t \in \bar{\Omega}$, we have, using C1',

$$\int_{\bar{\Omega}}|\psi_\alpha(|t-s|)|ds \leq \left\{\begin{array}{ll} \|B\|_\infty \int_{\bar{\Omega}} |t-s|^{\alpha-1}\, ds \leq C_1, & 0 < \alpha < 1, \\ \\ \|B\|_\infty \int_{\bar{\Omega}} |\ln|t-s||ds \leq C_2 , & \alpha = 1 , \end{array}\right\} \quad (5.5.4)$$

with C_1, C_2 independent of t.

Also, arguing as in Theorem 5.6(i), we may show that, for $h \in \mathbb{R}^2$, $0 < \alpha < 1$,

$$\int_{\bar{\Omega}_h} |\psi_\alpha(|t-s-h|) - \psi_\alpha(|t-s|)|ds$$

$$\leq C\left[\sup_{s\in\bar{\Omega}_h} |B(|t_1-h_1, t_2-h_2,s_1,s_2 |) - B(|t_1,t_2,s_1,s_2|)| \right]$$

$$+ \quad \|B\|_\infty \int_{\bar{\Omega}_h} \left||t_1-h_1,t_2-h_2,s_1,s_2|^{\alpha-1} - |t_1,t_2,s_1,s_2|^{\alpha-1}\right|ds$$

where C is independent of t and h. It then follows, using Lemma A9 and the fact that $B \in C^1[0,|\bar{\Omega}^*|]$, that for $h \in \mathbb{R}^2$, $0 < \alpha < 1$,

$$\int_{\bar{\Omega}_h} |\psi_\alpha(|t-s-h|) - \psi_\alpha(|t-s|)|ds \leq C\, |h| , \quad (5.5.5)$$

with C independent of t and h, and an identical bound may be proved when $\alpha = 1$.

Combination of (5.5.2), (5.5.4) and (5.5.5) and noting that $|h| \le \sqrt{2}\,\|h\|_\infty$, yields

$$\int_{\Omega} |\psi_\alpha(|t-s|) - u_{\alpha,t}(s)|ds \le C\,\tau\ ,$$

with C independent of t and τ, and the result then follows from Remark 5.10(ii).

<u>Remark 5.12</u> Note that, by the triangle inequality, we have, from Lemma 5.11,

$$\begin{aligned}\int_{\overline{\Omega}} |u_{\alpha,t}(s)|ds &\le C\|\Pi_{N(\tau)}\|_\infty + \int_{\overline{\Omega}} |\psi_\alpha(|t-s|)|ds \\ &\le C\ ,\end{aligned}$$

for some C which is independent of t and τ, where the final inequality follows from (5.5.4), and the observation that

$$\|\Pi_{N(\tau)}\|_\infty \le (1 + d^2)^{\frac{1}{2}}\ , \qquad \tau \in (0,1]\ .$$

<u>Lemma 5.13</u> Let C1', C2 and C3' be satisfied, and let y be the solution of (5.1.1). Then there exists a spline $\xi \in S_2^1(\Pi_{N(\tau)},\overline{\Omega})$ such that

$$\|y - \xi\|_\infty \le C\|\Pi_{N(\tau)}\|_\infty^{\beta+1}\ ,$$

with C independent of τ.

<u>Proof.</u> Note that, by Theorem 5.8, $y \in C^1(\overline{\Omega})$ and $\frac{\partial y}{\partial t_1}, \frac{\partial y}{\partial t_2} \in \mathrm{Lip}_\beta(\overline{\Omega})$, for any β in the range $0 < \beta < \alpha$. It follows from Munteanu and Schumaker [41, Lemma 5.5], that there exists $\xi \in S_2^1(\Pi_{N(\tau)},\overline{\Omega})$ such that

$$\|y - \xi\|_\infty \le C[\tau^2\|y\|_\infty + \omega_2(y,\tau)]\ , \qquad (5.5.6)$$

with C independent of τ , where $\omega_2(y,\tau)$ is the two dimensional modulus of continuity given by

$$\omega_2(y,\tau) = \sup_{0\leq\|h\|_\infty\leq\tau} \sup_{t\in\overline{\Omega}_{2h}} |y(t + 2h) - 2y(t + h) + y(t)| ,$$

and $\overline{\Omega}_{2h}$ is defined by (5.5.3).

Now, it follows easily from the two-dimensional Taylor's theorem, and the known properties of y , that

$$\omega_2(y,\tau) \leq C\,\tau^{\beta+1} \tag{5.5.7}$$

with C independent of τ, and the required result follows on substitution of (5.5.7) into (5.5.6) and using Remark 5.10(ii) .

The next lemma highlights an important property of the choice of collocation points given in (5.3.1) and Remark 5.10(i) .

<u>Lemma 5.14</u> Let $\xi \in S_2^1(\Pi_{N(\tau)}, \overline{\Omega})$. Then

$$\int_{\overline{\Omega}_i(\tau)} (\xi(s) - \xi(t_i))ds = 0 , \qquad i = 1,\ldots,N(\tau) .$$

<u>Remark.</u> This result demonstrates the special role played by the points (5.3.1), and one consequence of it is the fact that the two dimensional approximate integration rule

$$\int_{\overline{\Omega}} \phi(s)ds \simeq \sum_{i=1}^{N(\tau)} \phi(t_i) \int_{\overline{\Omega}_i(\tau)} ds$$

turns out to be exact for functions in $S_2^1(\Pi_{N(\tau)},\overline{\Omega})$, i.e. for functions which reduce to bilinear functions almost everywhere on each of the subsets $\overline{\Omega}_i$. In other words, this approximate integration rule is the two dimensional analogue of the product mid-point rule.

Proof of Lemma 5.14. Note that, for all $s \in \Omega_i$, and thus, for almost all $s \in \overline{\Omega}_i$, we have $\xi(s) = \xi_1(s_1)\xi_2(s_2)$, where ξ_1 and ξ_2 are linear. Note also that, by Remark 5.10(i), t_i is the mid point of $\overline{\Omega}_i$. Thus, to prove this lemma, it would be sufficient to show that

$$\int_0^d \int_0^1 ((a_1s_1+b_1)(a_2s_2+b_2) - (a_1t_1+b_1)(a_2t_2+b_2))\, ds_1ds_2 = 0 ,$$

where a_1, b_1, a_2, b_2 are constants, and $t_1 = \frac{1}{2}, t_2 = \frac{d}{2}$.

Now,

$$\int_0^d \int_0^1 ((a_1s_1+ b_1)(a_2s_2+ b_2) - (a_1t_1+ b_1)(a_2t_2+ b_2))\, ds_1ds_2$$

$$= a_1a_2 \int_0^d \int_0^1 (s_1s_2- t_1t_2)ds_1ds_2$$

$$+ a_1b_2 \int_0^d \int_0^1 (s_1- t_1)ds_1ds_2$$

$$+ a_2b_1 \int_0^d \int_0^1 (s_2- t_2)ds_1ds_2 , \tag{5.5.8}$$

and

$$\int_0^d \int_0^1 (s_1- t_1)ds_1ds_2 = d \int_0^1 (s_1- t_1)ds_1 = 0 , \tag{5.5.9}$$

since $t_1 = \frac{1}{2}$, and, similarly,

$$\int_0^d \int_0^1 (s_2- t_2)ds_1ds_2 = 0 . \tag{5.5.10}$$

Also,

$$\int_0^d \int_0^1 (s_1s_2- t_1t_2)ds_1ds_2 = \int_0^d \left(\tfrac{1}{2}s_2 - t_1t_2\right)ds_2$$

$$= \frac{1}{4} d^2 - dt_1t_2 = 0 , \tag{5.5.11}$$

and the required result follows on combination of (5.5.9), (5.5.10) and (5.5.11) with (5.5.8).

Let $y^{I}_{N(\tau)}$, $y^{II}_{N(\tau)}$ denote the approximations to y defined by the collocation and iterated collocation methods respectively, using the M.S. family of meshes $\{\Pi_{N(\tau)} : \tau \in (0,1]\}$. Note that, in view of Remark 5.10(ii), Theorem 5.1 holds with N replaced by $N(\tau)$, provided that the phrase "N sufficiently large" is replaced by the phrase " τ sufficiently small" . This fact will be used in the proof of the following theorem, which is the main result of this section.

<u>Theorem 5.15</u> Let C1', C2 and C3' be satisfied. Then

(i) $$\|y - y^{I}_{N(\tau)}\|_\infty = O(\|\Pi_{N(\tau)}\|_\infty) ,$$

and

(ii) $$\|y - y^{II}_{N(\tau)}\|_\infty = O(\|\Pi_{N(\tau)}\|_\infty^{\beta+1}) ,$$

for any β in the range $0 < \beta < \alpha$.

<u>Proof</u>. By definition of the projection $P_{N(\tau)}$, it follows that

$$\|y - P_{N(\tau)}y\|_\infty \le \sup_{\substack{t,t' \in \bar{\Omega} \\ \|t-t'\|_\infty < \|\Pi_{N(\tau)}\|_\infty}} |y(t) - y(t')| ,$$

and since it was proved in Theorem 5.8 that $y \in C^1(\bar{\Omega})$, an easy application of Taylor's theorem yields

$$\|y - P_{N(\tau)}y\|_\infty \le C\|\Pi_{N(\tau)}\|_\infty , \tag{5.5.12}$$

with C independent of τ , and the result (i) follows from Theorem 5.1.

To obtain (ii), recall Lemma 5.11, and write, for $t \in \bar{\Omega}$,

$$Ky(t) - KP_{N(\tau)}y(t) = \int_{\overline{\Omega}} \psi_\alpha(|t-s|)(y(s) - P_{N(\tau)}y(s))ds$$

$$= \int_{\overline{\Omega}} (\psi_\alpha(|t-s|) - u_{\alpha,t}(s))(y(s) - P_{N(\tau)}y(s))ds$$

$$+ \int_{\overline{\Omega}} (u_{\alpha,t}(s))(y(s) - P_{N(\tau)}y(s))ds$$

$$= I_1(t) + I_2(t) , \tag{5.5.13}$$

say .

Now, using Hölder's inequality, we have for $t \in \overline{\Omega}$,

$$|I_1(t)| \leq \int_{\overline{\Omega}} |\psi_\alpha(|t-s|) - u_{\alpha,t}(s)|ds \, \|y - P_{N(\tau)}y\|_\infty ,$$

and it follows from Lemma 5.11, and (5.5.12), that

$$|I_1(t)| \leq C\|\Pi_{N(\tau)}\|_\infty^2 , \tag{5.5.14}$$

with C independent of t and τ .

Also, since $u_{\alpha,t} \in S_1^0(\Pi_{N(\tau)}, \overline{\Omega})$, we have, for some $c_1,\dots,c_n$ which are constant with respect to s ,

$$I_2(t) = \sum_{i=1}^{N(\tau)} c_i \int_{\overline{\Omega}_i} (y(s) - P_{N(\tau)}y(s))ds$$

$$= \sum_{i=1}^{N(\tau)} c_i \int_{\overline{\Omega}_i} (I - P_{N(\tau)})(y - \xi)(s)ds ,$$

where ξ is any function in $S_2^1(\Pi_{N(\tau)},\overline{\Omega})$, using Lemma 5.14. Thus, making use again of Hölder's inequality, we have, for $t \in \overline{\Omega}$,

$$\begin{aligned} |I_2(t)| &= \left| \int_{\bar{\Omega}} u_{\alpha,t}(s)\ (I - P_{N(\tau)})(y - \xi)(s)ds \right| \\ &\le \int_{\bar{\Omega}} |u_{\alpha,t}(s)|ds\ \|(I - P_{N(\tau)})(y - \xi)\|_\infty \\ &\le C\|y - \xi\|_\infty \quad , \end{aligned}$$

with C independent of t and τ, where we have used (5.5.1), and Remark 5.12. Thus, by Lemma 5.13 ,

$$|I_2(t)| \le C\|\Pi_{N(\tau)}\|_\infty^{\beta+1} \quad , \tag{5.5.15}$$

with C independent of t and τ, and β any number such that $0 < \beta < \alpha$.

Combining (5.5.14) and (5.5.15) with (5.5.13), we obtain

$$\|Ky - KP_{N(\tau)}y\|_\infty \le C\|\Pi_{N(\tau)}\|_\infty^{\beta+1} \quad ,$$

with C independent of τ, and (ii) then follows from Theorem 5.1.

CHAPTER 6

NUMERICAL SOLUTION OF A CURRENT DISTRIBUTION PROBLEM

In this chapter we discuss the numerical solution of a two dimensional integral equation which describes the distribution of sinusoidally varying current in an infinitely long conducting bar of rectangular cross section. In such a conductor, the alternating current induces a magnetic field, which in turn sets up eddy currents, causing the current to be displaced towards the surface of the conductor. In the case of a conductor with a circular cross section, an analytic solution to the problem is known [56], but in the case of a rectangular cross section the problem must be formulated as an integral equation, and solved numerically. The numerical solutions obtained are of interest to electrical engineers [16], [51].

The physics of this problem is discussed in [56] and [27, Section 4], where it is shown that, for a conductor with rectangular cross-section of length a and breadth b, the current distribution y may be found by solving the integral equation

$$\hat{y}(t_1,t_2) = C_0 + i\lambda \int_0^{b/a}\int_0^1 \ln\sqrt{(t_1-s_1)^2+(t_2-s_2)^2}\ \hat{y}(s_1,s_2)ds_1ds_2 \qquad (6.1)$$

over the scaled rectangle $[0,1] \times [0,b/a] \subseteq \mathbb{R}^2$, and retrieving the solution y over $[0,a] \times [0,b]$ using the relation

$$y(at_1,at_2) = \hat{y}(t_1,t_2),\ (t_1,t_2) \in [0,1] \times [0,b/a] \ . \qquad (6.2)$$

In (6.1), the parameter λ is given by

$$\lambda = \frac{a^2\mu g\omega}{2\pi} \ ,$$

where g is the conductivity of the material which the bar is made of, μ is the permeability of free space, and ω is the angular frequency of the alternating current. In all the examples given below, we consider a copper bar, which has conductivity given by

$$g = \frac{1}{2.83} \times 10^{8} ,$$

and we set

$$\mu = 4\pi \times 10^{-7} ,$$

and

$$\omega = 60 \times 2\pi$$

these quantities being given in RMKS units.

<u>Remark</u>. We point out a misprint in [27, p.99] where the value of ω given should read $60 \times 2\pi$, instead of 60.

As explained in [27], C_0 is a constant which may be chosen arbitrarily, and may be considered to be a scaling factor which determines the total amount of current flowing in the conductor. To understand this point more clearly, note that if we modify (6.1) by replacing C_0 by $2C_0$, then the solution to the equation thus obtained is merely twice the solution of (6.1).

Let us suppose that, for given $N \in \mathbb{N}$, we have a mesh $\{\Omega_i : i = 1,\dots,N\}$ of $\overline{\Omega}: = [0,1] \times [0,b/a]$, as described in Section 5.1, and let us denote, by $\hat{y}_N^I$ and $\hat{y}_N^{II}$, the approximations to the solution $\hat{y}$ of (6.1), defined by (5.1.4) and (5.1.5) respectively. The approximate solutions y_N^I and y_N^{II} of the unscaled current distribution problem are then retrieved using (6.2). We describe below the numerical results obtained for three different cases of this problem, all using rectangular meshes, with the collocation points chosen as the mid points of the rectangular subdivisions. In Examples 1 and 2 the scaling factor C_0 in (6.1) will be chosen so that

$$\int_0^{b/a}\int_0^1 \hat{y}_N^I(t)dt = b/a \quad , \tag{6.3}$$

or, equivalently,

$$\int_0^b \int_0^a y_N^I(t)dt = ab \ ,$$

i.e. the total current flowing, according to the first approximation y_N^I, is equal to the cross-sectional area of the conductor. In Example 3, the value of C_0 is kept constant as the meshes vary. (The actual choice of C_0 will be given below.) It is necessary to keep C_0 constant if we wish to compare our numerical results with the theoretical predictions of Section 5.5, since the theory assumes that the inhomogeneous term does not change as the mesh varies.

In order to determine the coefficients in the system of linear equations which arises from (5.1.4), and in order to compute the solution y_N^{II} at an arbitrary point, it is necessary to calculate the integrals

$$\int_{\Omega_i} \ell n\sqrt{(t_1-s_1)^2+(t_2-s_2)^2}\, ds_1 ds_2 \ , \qquad i = 1,\ldots,N$$

at an arbitrary point $t = (t_1,t_2)$. Fortunately these integrals are sufficiently simple (in the case of the rectangular meshes we describe below) to be calculated analytically. We remark however, that, in the case of a mesh with a more complicated geometry, or in the case of a less simple kernel, these integrals may have to be calculated by quadrature, a technique that would introduce more errors into our numerical scheme.

Example 1. We choose a rectangular bar with length $a = 0.1$, and breadth $b = 0.005$ (dimensions in meters), and we solve the problem using three different meshes

(a) 4 equal partitions lengthwise and 1 breadthwise.

(b) 8 equal partitions lengthwise and 1 breadthwise.

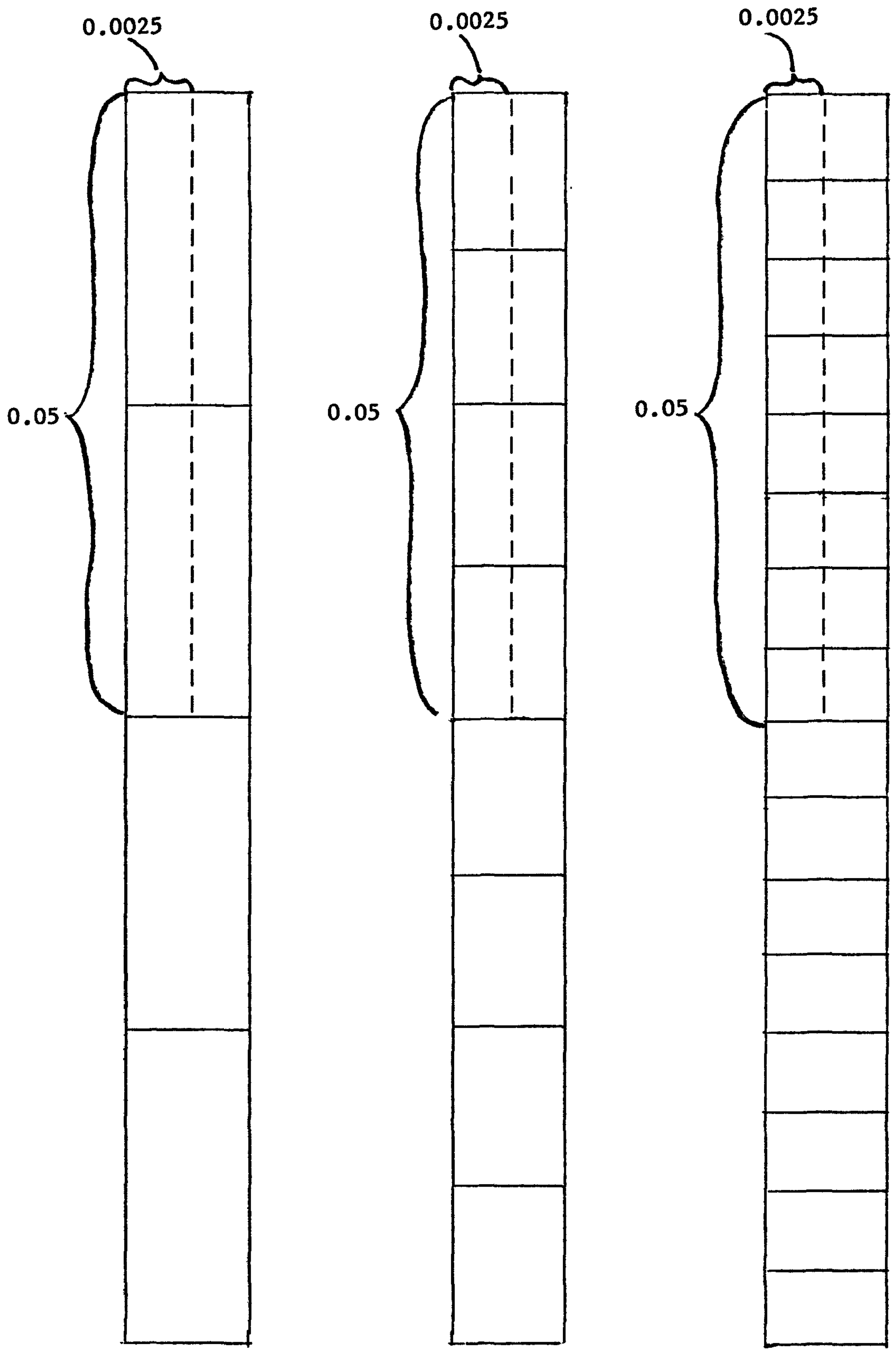
(c) 16 equal partitions lengthwise and 1 breadthwise.

These meshes are illustrated in Figure 1. In Tables 5 and 6 we give the results of the numerical solution of this problem. In Table 5 we give $|y_N^{II}|$, the approximation to $|y|$, the physical current flowing, while in Table 6 we give $\arg(y_N^{II})$ (defined in radians in the range $-\pi < \theta \leq \pi$), the approximation to $\arg(y)$ the phase angle of the current.

The results are given at a number of points along the line drawn lengthwise through the centre of the cross-section (the results along any line across the width varied only on the third significant figure). We mark with an asterisk the values of $|y_N^{II}|$, $\arg(y_N^{II})$ at the collocation points. Note that Tables 5 and 6 also include the values of $|y_N^I|$ and $\arg(y_N^I)$, since y_N^I is constant over each of the mesh subdivisions, and equal to y_N^{II} at each of the collocation points.

The results are given for the top half of the cross section only. The results for the bottom half will be the same by symmetry.

The values of $|y_N^I|$ and $|y_N^{II}|$ for the 20×1 case are given in [27]. The 20×1 case is also solved by Silvester [56], using a different method.



(a)
4 × 1 case

(b)
8 × 1 case

(c)
16 × 1 case

In each case y_N^{II} is calculated at points along the dotted line

Figure 1.

Values of $|y^{II}_{N}|$ for 0.1×0.005 bar

Distance along length \ Distance along breadth	0.0025	0.0025	0.0025
0.0	1.24	1.26	1.27
0.003125			1.22*
0.00625		1.17*	1.18
0.009375			1.13*
0.0125	1.09*	1.10	1.10
0.015625			1.06*
0.01875		1.04*	1.03
0.021875			1.01*
0.025	1.00	0.987	0.985
0.02815			0.965*
0.03125		0.949*	0.947
0.034375			0.933*
0.0375	0.931*	0.924	0.921
0.040625			0.912*
0.04375		0.907*	0.906
0.046875			0.902*
0.05	0.918	0.903	0.901
	(a) 4×1 case	(b) 8×1 case	(c) 16×1 case

* denotes the value at a collocation points.

Table 5.

Values of $\arg(y_N^{II})$ for 0.1×0.005 bar

Distance along breadth / Distance along length	0.0025	0.0025	0.0025
0.0	0.527	0.486	0.473
0.003125			0.342*
0.00625		0.254*	0.245
0.009375			0.167*
0.0125	0.134*	0.109	0.102
0.015625			0.0462*
0.01875		0.00419*	-0.00194
0.021875			-0.0435*
0.025	-0.0583	-0.0745	-0.0794
0.02815			-0.110*
0.03125		-0.131*	-0.136
0.034375			-0.157*
0.0375	-0.156*	-0.171	-0.175
0.040625			-0.188*
0.04375		-0.193*	-0.197
0.046875			-0.203*
0.05	-0.194	-0.202	-0.205
	(a) 4×1 case	(b) 8×1 case	(c) 16×1 case

* denotes the value at a collocation point.

Table 6.

Example 2. We choose a square bar with length $a = 0.1$, and breadth $b = 0.1$ (dimensions in meters), and we solve the problem using three different meshes

(a) 4 equal divisions lengthwise, and 8 equal divisions breathwise.

(b) 6 equal divisions lengthwise, and 8 equal divisions breadthwise.

(c) 8 equal divisions lengthwise, and 8 equal divisions breadthwise.

The numerical results are given in Tables 7(a), 7(b) and 7(c).

In this example we confine ourselves to displaying $|y_N^{II}|$, evaluated at a fixed grid of 36 points equally spaced within the square $[0,0.05] \times [0,0.05]$. Since the bar has a square cross-section, it is easy to see, by symmetry, that $|y_N^{II}|$ will have the same values as those given here at the corresponding points of the other three squares; $[0,0.05] \times [0.05, 0.1]$, $[0.05,0.1] \times [0,0.05]$, and $[0.05, 0.1] \times [0.05,0.1]$.

This problem was solved using the Galerkin and iterated Galerkin methods by Dewar [19], where the same meshes as in (a), (b) and (c) above were used. On comparison of the present results with those of [19], it turns out that the collocation methods converge a little faster at interior points of the cross section and a little slower at points on the edge of the cross section, than do the Galerkin methods.

Distance along breadth \ Distance along length	0.00	0.01	0.02	0.03	0.04	0.05
0.00	7.65	5.09	3.79	3.16	2.85	2.75
0.01	5.24	2.93	2.46	1.68	1.19	1.04
0.02	4.05	1.68	1.67	1.03	0.61	0.52
0.03	3.63	1.19	1.15	0.64	0.42	0.46
0.04	3.51	1.11	0.90	0.41	0.31	0.42
0.05	3.48	1.11	0.83	0.33	0.27	0.40

Values of $|y_N^{II}|$ for 0.1×0.1 bar

4 divisions lengthwise, 8 divisions breadthwise.

Table 7(a).

Distance along breadth \ Distance along length	0.00	0.01	0.02	0.03	0.04	0.05
0.00	7.22	4.87	3.63	3.01	2.83	2.79
0.01	4.87	3.07	2.08	1.43	1.19	1.13
0.02	3.61	1.93	1.29	0.84	0.54	0.44
0.03	3.11	1.35	0.81	0.54	0.31	0.24
0.04	2.95	1.16	0.56	0.35	0.21	0.22
0.05	2.91	1.13	0.50	0.27	0.19	0.22

Values of $|y_N^{II}|$ for 0.1×0.1 bar.

6 divisions lengthwise, 8 divisions breadthwise.

Table 7(b)

Distance along length / Distance along breadth	0.00	0.01	0.02	0.03	0.04	0.05
0.00	7.10	4.81	3.54	3.00	2.82	2.78
0.01	4.81	3.09	1.98	1.39	1.18	1.14
0.02	3.54	1.98	1.23	0.77	0.52	0.45
0.03	3.00	1.39	0.77	0.47	0.29	0.22
0.04	2.82	1.18	0.52	0.29	0.19	0.18
0.05	2.78	1.14	0.45	0.22	0.18	0.18

Values of $|y_N^{II}|$ for 0.1×0.1 bar.

8 divisions lengthwise, 8 divisions breadthwise.

Table 7(c).

Example 3. In this example we choose a bar with length $a = 0.1$, and breadth $b = 0.05$. We solve (6.1) by the collocation and iterated collocation methods, using the family of meshes

$$\{\Pi_{N(n^{-1})} : n = 2, 3, 4, 5, 6, 7, 8\} \quad \text{over} \quad [0,1] \times [0,b/a] ,$$

which were described in Remark 5.10 (iv). That is, for each $n = 2,3,\dots,8$, $\hat{y}^{I}_{N(n^{-1})}$ and $\hat{y}^{II}_{N(n^{-1})}$ were defined by (5.1.4) and (5.1.5) using the mesh obtained by dividing $[0,1] \times [0,b/a]$ into n^2 subrectangles, each of dimension $1/n$ by b/an. The solutions $y^{I}_{N(n^{-1})}$ and $y^{II}_{N(n^{-1})}$ over $[0,a] \times [0,b]$ were then retrieved using (6.2). In this example C_0 was kept constant as the meshes varied, and in fact was chosen so that $\hat{y}^{I}_{N(5^{-1})}$ satisfied (6.3). We shall display here only our results for $y^{II}_{N(n^{-1})}$ for $n = 2, 4, 6$ and 8, and we use them to obtain experimental rates of convergence of our approximate solutions, for comparison with the theoretical estimates of Theorem 5.15(ii). In Table 8 we give the values of $y^{II}_{N(n^{-1})}$ at the four points (0.0, 0.0), (0.05, 0.0), (0.0, 0.025) and (0.05, 0.025) of the 0.1×0.05 cross section, where the coordinates denote, respectively, the distance along the length and breadth of the cross section.

According to Theorem 5.15 (ii),

$$\|y - y^{II}_{N(n^{-1})}\|_\infty = \|\hat{y} - \hat{y}^{II}_{N(n^{-1})}\|_\infty = O\left(\|\Pi_{N(n^{-1})}\|_\infty^{\beta+1}\right)$$

$$= O\left(\frac{1}{n^{\beta+1}}\right) \qquad (6.4)$$

(since we have a uniform rectangular mesh), where β is any number satisfying $0 < \beta < 1$.

To estimate the experimental rate of convergence, we conjecture that

$$(y - y^{II}_{N(n^{-1})})(t) = c(t) \frac{1}{n^{\lambda}} \tag{6.5}$$

where the $c(t)$ are complex constants which depend on the point $t \in [0,a] \times [0,b]$, and $\lambda > 0$ is to be determined. Using the computed values of $y^{II}_{N(n^{-1})}$ for $n = 2,4,6$ and (6.5) we obtain three equations in the unknowns $y(t)$, $c(t)$ and λ. Eliminating $y(t)$ and $c(t)$, we obtain a non-linear equation in λ which we solve using the secant method. A second approximation to λ is obtained by the same procedure, using, this time, the numerical values of $y^{II}_{N(n^{-1})}$ for $n = 4, 6, 8$. The approximate values of λ obtained in these two approximations are displayed in Table 9. The values of λ obtained by the first approximation are rather erratic in comparison to (6.4). This is possibly because the asymptotic convergence rate proposed in (6.5) only holds true for sufficiently large values of n. The values of λ obtained by the second approximation conform more satisfactorily to the prediction (6.4), at least in the cases of the points (0.0, 0.0), (0.05, 0.0), and (0.0, 0.025). The exceptionally high value of λ at (0.05, 0.025) may be seen as evidence that the global prediction (6.4), although probably sharp on the edges of the domain (where the solution is singular), is likely to be pessimistic at points in the interior of the domain (where the solution is smooth). For each of the four points t, the second approximation to λ (given in Table 9) was used along with the values of $y^{II}_{N(6^{-1})}(t)$ and $y^{II}_{N(8^{-1})}(t)$ to calculate the constants $c(t)$ in (6.5). The value of $c(t)$ was then used to estimate the maximum absolute error in $y^{II}_{N(8^{-1})}$. The results are given in Table 10.

n	$y^{II}_{N(n^{-1})}$ at (0.0, 0.0)	$y^{II}_{N(n^{-1})}$ at (0.05, 0.0)	$y^{II}_{N(n^{-1})}$ at (0.0, 0.025)	$y^{II}_{N(n^{-1})}$ at (0.05, 0.025)
2	1.270+5.857i	0.9466+1.062i	1.140+3.931i	0.7277-2.186i
4	1.900+4.421i	1.026+1.244i	1.399+2.347i	-0.2349-0.4783i
6	2.188+4.055i	1.166+1.171i	1.680+1.840i	-0.1187-0.3580i
8	2.327+3.929i	1.211+1.134i	1.841+1.647i	-0.09286-0.3467i

Table 8.

	At point (0.0, 0.0)	At point (0.05,0.0)	At point (0.0, 0.025)	At point (0.05, 0.025)
1st approx. to λ using $y^{II}_{N(2^{-1})}$, $y^{II}_{N(4^{-1})}$, $y^{II}_{N(6^{-1})}$	1.3	($\cong$ 0.0)	0.9	3.3
2nd approx. to λ using $y^{II}_{N(4^{-1})}$, $y^{II}_{N(6^{-1})}$, $y^{II}_{N(8^{-1})}$	1.6	1.9	1.4	4.0

Table 9.

	At point (0.0,0.0)	At point (0.05, 0.0)	At point (0.0, 0.025)	At point (0.05, 0.025)
Estimated error in $y^{II}_{N(8^{-1})}$	0.32	0.080	0.12	0.013

Table 10.

APPENDIX

In this appendix we present the proofs of some of the more technical results which are used in the main body of the thesis. Within each theorem the equations will be numbered consecutively starting from (1).

The first theorem proves the connection between the Lipschitz spaces of Taibleson [67], and the Nikol'skii space $N_1^\alpha(\mathbb{R})$, introduced in Chapter 3. This theorem pinpoints one connection between the Applied Analysis school of Besov-Nikol'skii et. al., in the U.S.S.R, and the Harmonic Analysis literature which was developed contemporaneously in the West. For a unified treatment of the results of these two schools see, respectively, Nikol'skii [42], and Stein [66].

<u>Theorem A1.</u> Let $\Lambda(\alpha,p,q,\mathbb{R}^n)$ be the Lipschitz space of Taibleson [67I, p.421], and let $N_1^\alpha(\mathbb{R})$ be the Nikol'skii space defined in Section 3.2. Then

$$N_1^\alpha(\mathbb{R}) = \Lambda(\alpha,1,\infty,\mathbb{R}) \quad ,$$

for all $\alpha > 0$.

<u>Proof</u>. We prove the more general result

$$\Lambda(\alpha,p,q,\mathbb{R}) = B^{\alpha,p,q}(\mathbb{R}) \ , \qquad 1 \le p,q \le \infty \ , \tag{1}$$

where $B^{\alpha,p,q}(\mathbb{R})$ is the Besov space defined by Kufner et. al. [37, p.388]. The required result then follows since [37, p.389]

$$B^{\alpha,p,\infty}(\mathbb{R}) = N_p^\alpha(\mathbb{R}) \ , \qquad 1 \le p \le \infty \ .$$

To prove (1), let $\alpha > 0$, $1 \le p \le \infty$, $1 \le q \le \infty$, and α be split into the sum of $[\alpha]$ and α_0 according to (3.2.1).

Now, by [67I, Theorem 10, p.444], it follows that $\phi \in \Lambda(\alpha,p,q,\mathbb{R})$ if and only if $\phi \in L_p(\mathbb{R})$, and $\phi^{[\alpha]} \in \Lambda(\alpha_0,p,q,\mathbb{R})$, and thus by [67I, Theorem 3, p.421], since $0 < \alpha_0 \le 1 < 2$, it follows that $\phi \in \Lambda(\alpha,p,q,\mathbb{R})$ if and only if $\phi \in L_p(\mathbb{R})$ and

$$\| y^{2-\alpha_0} \; u_{yy}(x,y) \|_{pq} < \infty \quad ,$$

where u is the Poisson Integral of $\phi^{[\alpha]}$, and we have adopted the mixed norm notation [67I, p.411].

Since $0 < \alpha_0 \le 1 < 2$, it follows [67I, Theorem 4, p.421] that $\phi \in \Lambda(\alpha,p,q,\mathbb{R})$ if and only if $\phi \in L_p(\mathbb{R})$, and

$$\| |h|^{-\alpha_0} \Delta_h^2 \phi^{[\alpha]}(x-h) \|_{pq} < \infty \quad . \tag{2}$$

Now, using the definition of mixed norms [67I, p.411], we have

$$\| |h|^{-\alpha_0} \Delta_h^2 \phi^{[\alpha]}(x-h) \|_{pq} = \left\{ \int_{\mathbb{R}} \| |h|^{-\alpha_0} \Delta_h^2 \phi^{[\alpha]}(x-h) \|_p^q \frac{dh}{|h|} \right\}^{1/q}, \quad q \neq \infty ,$$

and

$$\| |h|^{-\alpha_0} \Delta_h^2 \phi^{[\alpha]}(x-h) \|_{p\infty} = \operatorname*{ess\,sup}_{h \in \mathbb{R}} \| |h|^{-\alpha_0} \Delta_h^2 \phi^{[\alpha]}(x-h) \|_p \quad ,$$

and thus it follows, that for $q \neq \infty$,

$$\begin{aligned} \| |h|^{-\alpha_0} \Delta_h^2 \phi^{[\alpha]}(x-h) \|_{pq} &= \left\{ \int_{\mathbb{R}} |h|^{-1-\alpha_0 q} \| \Delta_h^2 \phi^{[\alpha]}(x-h) \|_p^q dh \right\}^{1/q} \\ &= 2 \left\{ \int_0^\infty h^{-1-\alpha_0 q} \| \Delta_h^2 \phi^{[\alpha]}(x-h) \|_p^q \, dh \right\}^{1/q} \\ &= 2 \left\{ \int_0^\infty h^{-1-\alpha_0 q} \| \Delta_h^2 \, \phi^{[\alpha]} \|_p^q \, dh \right\}^{1/q} \quad , \end{aligned} \tag{3}$$

and;

$$\| |h|^{-\alpha_0} \Delta_h^2 \phi^{[\alpha]}(x-h)\|_{p_\infty} = \operatorname*{ess\,sup}_{h\in\mathbb{R}} \frac{\|\Delta_h^2 \phi^{[\alpha]}\|_p}{|h|^{\alpha_0}}$$

$$= \sup_{h\neq 0} \frac{\|\Delta_h^2 \phi^{[\alpha]}\|_p}{|h|^{\alpha_0}} \tag{4}$$

It follows from (2), (3) and (4) and using the definition of $B^{\alpha,p,q}(\mathbb{R})$ given by Kufner et. al., [37, pp.388-389], that $\phi \in \Lambda(\alpha,p,q,\mathbb{R})$ if and only if $\phi \in B^{\alpha,p,q}(\mathbb{R})$, and the proof is complete.

The next two theorems consider some important properties of distributional derivatives. The first, Theorem A2, answers the question: If the integrand of some integral depends on a parameter, when may we differentiate under the integral sign to obtain the derivative of the integral? The second, Theorem A3, examines the distributional derivatives of convolution integrals over finite intervals.

Theorem A2. Suppose g is a Lebesgue measurable function of $(t,s) \in (a,b) \times (c,d)$, $\frac{\partial g}{\partial t}$ is Lebesgue measurable on $(a,b) \times (c,d)$, and the iterated integrals,

$$\int_a^b \int_c^d g(t,s)\,dsdt, \quad \text{and} \quad \int_a^b \int_c^d \frac{\partial g}{\partial t}(t,s)\,dsdt$$

exist. Then, for almost all $t \in (a,b)$

$$\frac{d}{dt} \int_c^d g(t,s)ds = \int_c^d \frac{\partial g}{\partial t}(t,s)ds \quad .$$

Proof. By definition of distributional derivatives [24, p.142] we have, for all $s \in (c,d)$, $\phi \in C^1(a,b)$, ϕ having compact support,

$$\int_a^b \phi'(t)\, g(t,s)dt = -\int_a^b \phi(t) \frac{\partial g}{\partial t}(t,s)dt \quad . \tag{1}$$

Thus, using Fubini's Theorem and (1), we have, for $\phi \in C^1(a,b)$, with ϕ having compact support,

$$\int_a^b \phi'(t) \int_c^d g(t,s)dsdt$$

$$= \int_c^d \int_a^b \phi'(t)\, g(t,s)dtds = -\int_c^d \int_a^b \phi(t)\frac{\partial g}{\partial t}(t,s)dtds$$

$$= -\int_a^b \phi(t) \int_c^d \frac{\partial g}{\partial t}(t,s)dsdt \quad ,$$

and the result follows.

Theorem A3. For any interval $[a,b]$, let $\kappa \in L_1[a-b, b-a]$, and let $\phi \in W_1^1[a,b]$. Then

$$\frac{d}{dt}\left\{\int_a^b \kappa(t-s)\phi(s)ds\right\} = \phi(a)\kappa(t-a) - \phi(b)\kappa(t-b) + \int_a^b \kappa(t-s)\phi'(s)ds \ ,$$

for almost all $t \in [a,b]$.

Proof. Since $W_1^1[a,b]$, we may integrate by parts to obtain

$$\int_a^b \kappa(t-s)\phi(s)ds = \left[\left\{-\int_{a-b}^{t-s} \kappa(x)\right\}\phi(s)\right]_a^b + \int_a^b \left\{\int_{a-b}^{t-s} \kappa(x)dx\right\} \phi(s)ds \quad .$$

Hence,

$$\frac{d}{dt}\left\{\int_a^b \kappa(t-s)\phi(s)ds\right\} = \phi(a)\kappa(t-a) - \phi(b)\kappa(t-b)$$

$$+ \frac{d}{dt}\int_a^b \left\{\int_{a-b}^{t-s} \kappa(x)dx\right\} \phi'(s)ds \ , \tag{1}$$

for almost all $t \in [a,b]$, and the final result follows on using Theorem A2. (The measurability conditions required may be verified, for example, using the methods of [30, p.396].)

In the following lemma we prove some technical results concerning the integral operators from the examples of Section 3.4. The method follows Richter [49].

<u>Lemma A4.</u> Let K be defined by

$$Ky(t) = \int_0^1 |t-s|^{\alpha-1} y(s)ds, \qquad t \in [0,1], \quad 0 < \alpha < 1,$$

and let $m \in \mathbb{N}_0$. Then

(i) For $\gamma > 0$, $\gamma + \alpha \notin \mathbb{N}$, $i \in \mathbb{N}_0$, we have

$$K(t^{\gamma-1}(\ell n t)^i + (1-t)^{\gamma-1}(\ell n(1-t))^i) = \{\sum_{j=0}^{i} (t^{\alpha+\gamma-1}(\ell n t)^j + (1-t)^{\alpha+\gamma-1}(\ell n(1-t))^j)\}$$
$$+ \{\sum_{j=0}^{m-1} (t^{\alpha+j} + (1-t)^{\alpha+j})\} + \phi(t),$$

where $\phi \in W_1^{m+1}[0,1]$.

(ii) For $\gamma > 0$, $\gamma + \alpha \in \mathbb{N}$, $i \in \mathbb{N}_0$, we have

$$K(t^{\gamma-1}(\ell n t)^i + (1-t)^{\gamma-1}(\ell n(1-t))^i) = \{\sum_{j=0}^{i+1} (t^{\alpha+\gamma-1}(\ell n t)^j + (1-t)^{\alpha+\gamma-1}(\ell n(1-t))^j\}$$
$$+ \{\sum_{j=0}^{m-1} (t^{\alpha+j} + (1-t)^{\alpha+j})\} + \phi(t),$$

where $\phi \in W_1^{m+1}[0,1]$.

<u>Proof</u>. We have for $\gamma > 0$, $i \in \mathbb{N}_0$,

$$K(t^{\gamma-1}(\ell nt)^i) = \int_0^1 |t-s|^{\alpha-1} s^{\gamma-1}(\ell ns)^i ds$$

$$= \int_0^t (t-s)^{\alpha-1}s^{\gamma-1}(\ell ns)^i ds + \int_t^1 (s-t)^{\alpha-1}s^{\gamma-1}(\ell ns)^i ds$$

$$= I_1(t) + I_2(t) \text{ , say.} \tag{1}$$

Now, using the transformation $s = ut$, we obtain

$$I_1(t) = t^{\alpha+\gamma-1} \int_0^1 (1-u)^{\alpha-1} u^{\gamma-1}(\ell nu + \ell nt)^i du$$

$$= \left\{ \sum_{j=0}^{i} t^{\alpha+\gamma-1}(\ell nt)^j \right\} \quad . \tag{2}$$

Now, let $1 > \delta > 0$. If $t \in [\delta,1]$, then

$$s^{\gamma-1}(\ell ns)^i = \sum_{j=0}^{\infty} a_j(1-s)^j \quad ,$$

for some scalars a_j , with uniform convergence for $s \in [t,1]$,

and thus we may integrate term by term to obtain:

$$I_2(t) = \sum_{j=0}^{\infty} a_j \int_t^1 (s-t)^{\alpha-1} (1-s)^j ds$$

$$= \sum_{j=0}^{\infty} a_j \left(\int_0^1 (1-u)^{\alpha-1} u^j du \right) (1-t)^{\alpha+j} \quad ,$$

where we have used the substitution

$$(1-t)u = 1 - s \quad .$$

Thus, for $t \in [\delta,1]$, $m \in \mathbb{N}_0$,

$$I_2(t) = \sum_{j=0}^{m-1} b_j(1-t)^{\alpha+j} + (1-t)^{m+\alpha} \sum_{j=0}^{\infty} b_{j+m}(1-t)^j \quad ,$$

for some scalars b_i, with the convention that the first term is void if $m = 0$, and so

$$I_2(t) = \{ \sum_{j=0}^{m-1} (1-t)^{\alpha+j} \} + \phi_1(t) , \tag{3}$$

where $\phi_1 \in W_1^{m+1}[\delta,1]$.

Now, if $t \in [0, 1-\delta]$, then

$$(s-t)^{\alpha-1} = s^{\alpha-1} \left(1 - \frac{t}{s} \right)^{\alpha}$$

$$= s^{\alpha-1} \sum_{j=0}^{\infty} a_j \, t^j \, s^{-j} \qquad ,$$

for some scalars a_j , and convergence is uniform provided $s > t$.

Thus, for $\epsilon > 0$, we have

$$I_2(t) = \int_t^{(1+\epsilon)t} (s-t)^{\alpha-1} s^{\gamma-1} (\ell n s)^i ds + \int_{(1+\epsilon)t}^1 (s-t)^{\alpha-1} s^{\gamma-1} (\ell n s)^i ds \tag{4}$$

with

$$\int_{(1+\epsilon)t}^1 (s-t)^{\alpha-1} s^{\gamma-1} (\ell n s)^i ds = \sum_{j=0}^{\infty} a_j t^j \int_{(1+\epsilon)t}^1 s^{\alpha+\gamma-2-j} (\ell n s)^i ds \quad .$$

Now, since, for $j \neq \alpha + \gamma - 1$, we have,

$$\int s^{\alpha+\gamma-2-j} (\ell n s)^i ds = \{ s^{\alpha+\gamma-1-j} ((\ell n s)^i + \ldots + (\ell n s) + 1) \} \qquad ,$$

and for $j = \alpha + \gamma - 1$, we have

$$\int s^{\alpha+\gamma-2-j} (\ell n s)^i ds = \{ (\ell n s)^{i+1} \} ,$$

thus, with $\alpha + \gamma \notin \mathbb{N}$, it follows that

$$\int_{(1+\epsilon)t}^1 (s-t)^{\alpha-1} s^{\gamma-1} (\ell n s)^i ds = \{ \sum_{j=0}^{i} t^{\alpha+\gamma-1} (\ell n t)^j \} + \phi_2(t) , \tag{5}$$

where $\phi_2 \in W_1^{m+1}[0,1-\delta]$,

and with $\alpha + \gamma \in \mathbb{N}$, it follows that

$$\int_{(1+\epsilon)t}^{1} (s-t)^{\alpha-1} s^{\gamma-1}(\ell ns)^i ds = \{ \sum_{j=0}^{i+1} t^{\alpha+\gamma-1}(\ell nt)^j \} + \phi_2(t), \tag{6}$$

where $\phi_2 \in W_1^{m+1}[0,1-\delta]$.

Also, using the change of variable $s = ut$, we obtain

$$\int_{t}^{(1+\epsilon)t} (s-t)^{\alpha-1} s^{\gamma-1}(\ell ns)^i ds = t^{\alpha+\gamma-1}\int_{1}^{1+\epsilon} (u-1)^{\alpha-1} u^{\gamma-1}(\ell nu+\ell nt)^i du$$

$$= \sum_{j=0}^{i} t^{\alpha+\gamma-1}(\ell nt)^j \quad . \tag{7}$$

Combining (4), (5), (6) and (7), we obtain

$$I_2(t) = \{ \sum_{j=0}^{i} t^{\alpha+\gamma-1}(\ell nt)^j \} + \phi_2(t), \qquad \alpha+\gamma \notin \mathbb{N} \quad ,$$

or

$$I_2(t) = \{ \sum_{j=0}^{i+1} t^{\alpha+\gamma-1}(\ell nt)^j \} + \phi_2(t) \ , \qquad \alpha+\gamma \in \mathbb{N} \quad ,$$

where, in either case, ϕ_2 is a generic $W_1^{m+1}[0,1-\delta]$ function, and it follows from (3), that

$$I_2(t) = \{ \sum_{j=0}^{m-1} (1-t)^{\alpha+j} \} + \{ \sum_{j=0}^{i} t^{\alpha+\gamma-1}(\ell nt)^j \} + \phi(t), \ \alpha+\gamma \notin \mathbb{N} \ ,$$

or

$$I_2(t) = \{ \sum_{j=0}^{m-1} (1-t)^{\alpha+j} \} + \{ \sum_{j=0}^{i+1} t^{\alpha+\gamma-1}(\ell nt)^j \} + \phi(t), \ \alpha+\gamma \in \mathbb{N} \ ,$$

where in either case ϕ is a generic $W_1^{m+1}[0,1]$ function, and it follows using (1) and (2), that

$$K(t^{\gamma-1}(\ell n t)^i) = \{ \sum_{j=0}^{m-1} (1-t)^{\alpha+j} \} + \{ \sum_{j=0}^{i} t^{\alpha+\gamma-1}(\ell n t)^j \} + \phi(t), \alpha+\gamma \notin \mathbb{N} ,$$

or

$$K(t^{\gamma-1}(\ell n t)^i) = \{ \sum_{j=0}^{m-1} (1-t)^{\alpha+j} \} + \{ \sum_{j=0}^{i+1} t^{\alpha+\gamma-1}(\ell n t)^j \} + \phi(t), \alpha+\gamma \in \mathbb{N} ,$$

where $\phi \in W_1^{m+1}[0,1]$.

Analogous results may be proved for $K((1-t)^{\gamma-1}(\ell n(1-t))^i)$, and the required result follows.

In the next lemma we investigate the properties of the m-th order $L_p[a,b]$ modulus of smoothness, introduced in Section 4.3. These results are used in the proof of Theorem 4.4.

Lemma A5. Let $r \in \mathbb{N}$.

(i) For $\eta > 0$ and $1 \leq p \leq \infty$, let

$$\phi \in N_p^{\eta}[a,b] \qquad (1 \leq p < \infty) ,$$

or

$$\phi \in N_\infty^{\eta}[a,b] \cap C^{[\eta]}[a,b] \qquad (p = \infty) .$$

Then, for $0 < h \leq 1$, $1 \leq p \leq \infty$, we have

$$\omega_r(\phi,h)_p \leq C h^{\gamma} , \qquad r \neq \eta ,$$

$$\omega_r(\phi,h)_p \leq C h^{\gamma} \ell n(\tfrac{1}{h}) , \quad r = \eta ,$$

where $\gamma = \min(r,\eta)$.

(ii) Let k satisfy B1 (Section 4.3), and define k_t by (4.3.1). Then, for $0 < h \le 1$, we have

$$\omega_r(k_t,h)_{L_1[0,1]} \le \omega_r(k,h)_{L_1[-1,1]} \quad .$$

<u>Proof</u>. Throughout this proof C will denote a generic constant, which is independent of h. Unless otherwise stated p will lie in the range $1 \le p \le \infty$.

To obtain (i) let $0 < h \le 1$, and consider three cases.

CASE I: $\eta < r$. Since $\phi \in N_p^\eta[a,b]$, it follows directly from Nikol'skii [42, p.159], that

$$\omega_r(\phi,h)_p \le C\, h^\eta \quad .$$

CASE II: $\eta > r$. In this case it follows, from (3.2.3), that $\phi \in W_p^r[a,b]$, and hence [32, Proposition 2.2],

$$\begin{aligned}\omega_r(\phi,h)_p &\le h^r \,\|\phi^{(r)}\|_p \\ &\le C\, h^r \quad .\end{aligned}$$

CASE III: $\eta = r$. In this case we can infer, via (3.2.1) and (3.2.3), that $\phi \in W_p^{[\eta]}[a,b]$, with

$$[\eta] = \eta - 1 = r - 1 \;, \qquad (1)$$

and hence [32, Proposition 2.2], there obtains

$$\omega_r(\phi,h)_p \le h^{[\eta]}\; \omega_1(\phi^{[\eta]},h)_p \quad . \qquad (2)$$

Now, it follows from (1) and the hypotheses of the lemma, that

$$\phi^{[\eta]} \in N_p^1[a,b] \qquad (1 \le p < \infty) \quad ,$$

and

$$\phi^{[\eta]} \in N_{\infty}^{1}[a,b] \cap C[a,b] \qquad (p = \infty) ,$$

and thus [42, p.159]

$$\omega_2(\phi^{[\eta]},h)_p \leq C\ h , \qquad 1 \leq p \leq \infty . \tag{3}$$

Then [32, Proposition 5.2], we may extend $\phi^{[\eta]}$ to a function $T\phi^{[\eta]}$ such that

$$T\phi^{[\eta]} \in L_p(\mathbb{R}) \qquad (1 \leq p < \infty) ,$$

and

$$T\phi^{[\eta]} \in C(\mathbb{R}) \qquad (p = \infty) ,$$

with $\qquad T\phi^{[\eta]} = \phi^{[\eta]} \qquad$ on $\quad [a,b]$,

and

$$\omega_2(T\phi^{[\eta]},h)_{L_p(\mathbb{R})} \leq C\left[h^2\|\phi^{[\eta]}\|_p + \omega_2(\phi^{[\eta]},h)_p\right] (1 \leq p < \infty),$$

$$\omega_2(T\phi^{[\eta]},h)_{L_p(\mathbb{R})} \leq C\ \omega_2(\phi^{[\eta]},h)_\infty \quad (p = \infty) ,$$

where the modulus of smoothness of $T\phi^{[\eta]}$ is defined on the whole of $\mathbb{R}$ in the usual way [32]. Thus we have, using (3),

$$\omega_2(T\phi^{[\eta]},h)_p \leq C\ h , \qquad 1 \leq p \leq \infty ,$$

from which it follows [68, p.107, 110], that

$$\omega_1(T\phi^{[\eta]},h)_p \leq C\ h\ \ell n\ \frac{1}{h} , \qquad 1 \leq p \leq \infty . \tag{4}$$

Now, since

$$\begin{aligned}\omega_1(\phi^{[\eta]},h)_p &= \omega_1(T\phi^{[\eta]}\big|_{[a,b]},h)_p \\ &\leq \omega_1(T\phi^{[\eta]},h)_p , \end{aligned} \tag{5}$$

it follows from (2), (4), and (5), that

$$\omega_r(\phi,h)_p \leq h^{\eta} \ln \frac{1}{h} \quad .$$

The required result (ii) then follows on collection of the results of the three cases I, II and III above.

To prove (ii), note that

$$\omega_r(k_t,h)_1 = \sup_{0<|\epsilon|\leq h} \| \Delta_\epsilon^r k_t \|_{L_1[0,1]_{r\epsilon}} \tag{6}$$

$$= \max \left\{ \sup_{0<\epsilon\leq h} \| \Delta_\epsilon^r k_t \|_{L_1[0,1]_{r\epsilon}} , \sup_{-h\leq\epsilon<0} \| \Delta_\epsilon^r k_t \|_{L_1[0,1]_{r\epsilon}} \right\}$$

Consider the case $0 < \epsilon \leq h$. Then

$$\| \Delta_\epsilon^r k_t \|_{L_1[0,1]_{r\epsilon}} = \int_0^{1-r\epsilon} \left| \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{r}{\ell} k(t-s-\ell\epsilon) \right| ds$$

$$= \int_{t-1+r\epsilon}^{t} \left| \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{r}{\ell} k(u-\ell\epsilon) \right| du$$

$$\leq \int_{-1+r\epsilon}^{1} \left| \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{r}{\ell} k(u-\ell\epsilon) \right| du , \quad \text{since} \quad t \in [0,1]$$

$$= \| \Delta_{-\epsilon}^r k \|_{L_1[-1,1]_{-r\epsilon}} \leq \omega_r(k,h)_{L_1[-1,1]} \quad , \tag{7}$$

and similarly it may be shown that for $-h \leq \epsilon < 0$,

$$\| \Delta_\epsilon^r k_t \|_{L_1[0,1]_{r\epsilon}} \leq \omega_r(k,h)_{L_1[-1,1]} \quad , \tag{8}$$

and the required result follows from (6), (7) and (8).

In the next four lemmas, we investigate the integrability properties of some weakly singular functions defined over two-dimensional regions. These results are used extensively in Sections 5.4 and 5.5.

Lemma A6 Let D be any compact subset of $\mathbb{R}^2$. Then, for $\delta > 0$,

$$\int_D |s|^{\delta-2}\, ds < \infty \quad ,$$

and for $i = 1,2$,

$$\int_D |\ell n^i |s||\, ds < \infty \quad .$$

Proof. Since D is compact, it must be closed and bounded. Let R be a disc, centred on the origin, with radius r_0, say, large enough to ensure that $D \subseteq R$. Then, transforming to polar coordinates, we have,

$$\int_D |s|^{\delta-2} ds \leq \int_R |s|^{\delta-2}\, ds = 2\pi \int_0^{r_0} r^{\delta-2}\, rdr$$

$$= 2\pi \int_0^{r_0} r^{\delta-1}\, dr < \infty \quad .$$

The second part of the lemma is proved similarly.

In Lemmas A7 to A9 we make use of the abbreviations introduced in Theorem 5.6.

Lemma A7. For all $t,\ z \in \overline{\Omega}$, and $0 < \alpha < 1$, we have

(i) $$\int_0^d |t_1,t_2,z_1,s_2|^{\alpha-1} ds_2 \leq C_1 \quad ,$$

(ii) $$\int_0^d |\ell n(|t_1,t_2,z_1,s_2|)| ds_2 \leq C_2 \quad ,$$

(iii) $$\int_0^1 |t_1,t_2,s_1,z_2|^{\alpha-1} ds_1 \leq C_3 \quad ,$$

and

(iv) $$\int_0^1 |\ell n(|t_1,t_2,s_1,z_2|)| ds_1 \leq C_4 \quad ,$$

with C_1, C_2, C_3 and C_4 independent of t and z.

Proof. We give the proof of (i) only. The proofs of (ii), (iii) and (iv) are similar.

We have, for all $t,z \in \overline{\Omega}$,

$$\int_0^d |t_1,t_2,z_1,s_2|^{\alpha-1} ds_2 = \int_0^d ((t_1-z_1)^2 + (t_2-s_2)^2)^{(\alpha-1)/2} ds_2$$

$$\leq \int_0^d |t_2-s_2|^{\alpha-1} ds_2 \leq \int_{-d}^d |x|^{\alpha-1} dx < \infty ,$$

and the result follows.

Lemma A8. Let $t, t', z \in \overline{\Omega}$.

Then,

(i) $$\int_0^d \left| |t_1,t_2,z_1,s_2|^{\alpha-1} - |t_1',t_2',z_1,s_2|^{\alpha-1} \right| ds_2 \leq C_1 |t-t'|^\beta ,$$

for any β satisfying $0 < \beta < \alpha < 1$,

(ii) $$\int_0^d \left| \ln|t_1,t_2,z_1,s_2| - \ln|t_1',t_2',z_1,s_2| \right| ds_2 \leq C_2 |t-t'|^\beta ,$$

for any β satisfying $0 < \beta < 1$,

(iii) $$\int_0^1 \left| |t_1,t_2,s_1,z_2|^{\alpha-1} - |t_1',t_2',s_1,z_2|^{\alpha-1} \right| ds_1 \leq C_3 |t-t'|^\beta ,$$

for any β satisfying $0 < \beta < \alpha < 1$,

and

(iv) $$\int_0^1 \left| \ln|t_1,t_2,s_1,z_2| - \ln|t_1',t_2',s_1,z_2| \right| ds_1 \leq C_4 |t-t'|^\beta ,$$

for any β satisfying $0 < \beta < 1$,

where C_1, C_2, C_3 and C_4 are independent of t,t' and z.

<u>Proof.</u> We give the proof of (i) only; (ii), (iii) and (iv) are proved in a similar way. The method used here follows Kantovovich and Akilov [33, p.363, Theorem 4].

We divide $[0,d]$ into two regions, $[0,d]_1$, and $[0,d]_2$ as follows.

$$[0,d]_1 := \{s_2 \in [0,d] : |(t_1,t_2) - (z_1,s_2)| < 2|t-t'|\} ,$$

$$[0,d]_2 := [0,d] \setminus [0,d]_1 .$$

Then, noting that, for $s_2 \in [0,d]_1$, we have

$$|t_1', t_2', z_1, s_2| \leq |t_1, t_2, z_1, s_2| + |t - t'| < 3|t-t'| ,$$

it follows that

$$\int_{[0,d]_1} \left| |t_1,t_2,z_1,s_2|^{\alpha-1} - |t_1',t_2',z_1,s_2|^{\alpha-1} \right| ds_2$$

$$\leq \int_{[0,d]_1} |t_1,t_2,z_1,s_2|^{\alpha-1} ds_2 + \int_{[0,d]_1} |t_1',t_2',z_1,s_2|^{\alpha-1} ds_2$$

$$\leq \int_{[0,d]_1} \frac{|t_1,t_2,z_1,s_2|^{\alpha-1}}{|t_1,t_2,z_1,s_2|^{\beta}} |t_1,t_2,z_1,s_2|^{\beta} ds_2 + \int_{[0,d]_1} \frac{|t_1',t_2',z_1,s_2|^{\alpha-1}}{|t_1',t_2',z_1,s_2|^{\beta}} |t_1',t_2',z_1,s_2|^{\beta} ds_2$$

$$\leq 2^{\beta}|t-t'|^{\beta} \int_{[0,d]_1} |t_1,t_2,z_1,s_2|^{\alpha-\beta-1} ds_2 + 3^{\beta}|t-t'|^{\beta} \int_{[0,d]_1} |t_1',t_2',z_1,s_2|^{\alpha-\beta-1} ds_2$$

$$\leq C|t-t'|^{\beta} , \qquad (1)$$

provided $0 < \beta < \alpha$, with C independent of t,t' and z, where the final inequality follows from Lemma A7.

Also,

$$\int_{[0,d]_2} \left| |t_1,t_2,z_1,s_2|^{\alpha-1} - |t_1',t_2',z_1,s_2|^{\alpha-1} \right| ds_2$$

$$= \int_{[0,d]_2} \left| \int_{t'}^{t} \nabla_\lambda(|\lambda_1,\lambda_2,z_1,s_2|^{\alpha-1}) . \ d\lambda \right| ds_2$$

$$\leq C \int_{[0,d]_2} \int_{t'}^{t} |\lambda_1,\lambda_2,z_1,s_2|^{\alpha-2} \ |d\lambda| ds_2 \tag{2}$$

Now, for $s_2 \in [0,d]_2$, λ in the straight line joining t and t', we have,

$$\begin{aligned} |\lambda_1,\lambda_2,z_1,s_2| &= |(\lambda_1,\lambda_2) - (z_1,s_2)| \\ &\geq |(t_1,t_2) - (z_1,s_2)| - |(t_1,t_2) - (\lambda_1,\lambda_2)| \\ &\geq |(t_1,t_2) - (z_1,s_2)| - |t-t'| \\ &\geq \tfrac{1}{2} |(t_1,t_2) - (z_1,s_2)| \quad , \end{aligned}$$

and so it follows that

$$1 \leq \frac{2|\lambda_1,\lambda_2,z_1,s_2|}{|t_1,t_2,z_1,s_2|} \quad ,$$

and raising this to the power of $1-\beta$, and substituting into (2), we have

$$\int_{[0,d]_2} \left| \ |t_1,t_2,z_1,s_2|^{\alpha-1} - |t_1',t_2',z_1,s_2|^{\alpha-1} \right| ds_2$$

$$\leq C \int_{[0,d]_2} \left\{ \int_{t'}^{t} |\lambda_1,\lambda_2,z_1,s_2|^{\alpha-\beta-1} |d\lambda| \right\} |t_1,t_2,z_1,s_2|^{\beta-1} ds_2$$

$$\leq C|t-t'|^{\beta-1} \int_{t'}^{t} \int_{[0,d]_2} |\lambda_1,\lambda_2,z_1,s_2|^{\alpha-\beta-1} ds_2 \ \ |d\lambda|$$

$$\leq C|t-t'|^{\beta-1} \int_{t'}^{t} |d\lambda| \ = \ C|t-t'|^{\beta} \quad . \tag{3}$$

The result then follows from (1) and (3)

<u>Lemma A9.</u> Let $\overline{\Omega} = [0,1] \times [0,d]$, $t,t' \in \overline{\Omega}$, $0 < \alpha < 1$.

Then

$$\text{(i)} \quad \int_{\Omega} \Big| |t_1,t_2,s_1,s_2|^{\alpha-1} - |t_1',t_2',s_1,s_2|^{\alpha-1} \Big| ds \le C_1 |t-t'| \quad ,$$

and

$$\text{(ii)} \quad \int_{\Omega} \Big| \ln|t_1,t_2,s_1,s_2| - \ln|t_1',t_2',s_1,s_2| \Big| ds \le C_2 |t-t'| \quad ,$$

with C_1 C_2 independent of t and t' .

<u>Proof.</u> Again we confine ourselves to proving (i).

The proof follows similar lines to Lemma A8 . Divide $\overline{\Omega}$ into two regions, $(\overline{\Omega})_1$ and $(\overline{\Omega})_2$ as follows.

$$(\overline{\Omega})_1 = \{s \in \overline{\Omega} : |t-s| < 2|t-t'|\} \quad ,$$

$$(\overline{\Omega})_2 = \overline{\Omega} \setminus (\overline{\Omega})_1 \quad .$$

Then,

$$\int_{(\overline{\Omega})_1} \Big| |t_1,t_2,s_1,s_2|^{\alpha-1} - |t_1',t_2',s_1,s_2|^{\alpha-1} \Big| \, ds$$

$$\le \int_{(\overline{\Omega})_1} |t_1,t_2,s_1,s_2|^{\alpha-1} ds + \int_{(\overline{\Omega})_1} |t_1',t_2',s_1,s_2|^{\alpha-1} \, ds$$

$$= \int_{(\overline{\Omega})_1} |t_1,t_2,s_1,s_2|^{\alpha-2} |t_1,t_2,s_1,s_2| ds + \int_{(\overline{\Omega})_1} |t_1',t_2',s_1,s_2|^{\alpha-2} |t_1',t_2',s_1,s_2| ds$$

$$< C|t-t'| \quad , \tag{1}$$

with C independent of t and t' , using the definition of $(\overline{\Omega})_1$ and Lemma A6 to obtain the final inequality.

Also (similarly to Lemma A8),

$$\int_{(\overline{\Omega})_2} \Big| |t_1,t_2,s_1,s_2|^{\alpha-1} - |t_1',t_2',s_1,s_2|^{\alpha-1}\Big| ds$$

$$= \int_{(\overline{\Omega})_2} \Big| \int_{t'}^{t} \nabla_\lambda \big(|\lambda_1,\lambda_2,s_1,s_2|^{\alpha-1}\big) .\ d\lambda \Big| ds$$

$$\leq C \int_{(\overline{\Omega})_2} \int_{t'}^{t} |\lambda_1,\lambda_2,s_1,s_2|^{\alpha-2} |d\lambda| ds$$

$$= C \int_{t'}^{t} \int_{(\overline{\Omega})_2} |\lambda_1,\lambda_2,s_1,s_2|^{\alpha-2}\, ds\, |d\lambda|$$

$$\leq C \int_{t'}^{t} |d\lambda| = C|t-t'| \quad , \tag{2}$$

where we have utilised Lemma A6 again, and the required result follows from (1) and (2).

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