Non-tracking solar concentrators

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Publication Date: 1980

DOI: https://doi.org/10.26190/unsworks/4098

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THE UNIVERSITY OF NEW SOUTH WALES
SCHOOL OF PHYSICS

NON-TRACKING SOLAR CONCENTRATORS

Thesis submitted for the degree of
DOCTOR OF PHILOSOPHY

David R. Mills
May, 1980
This is to certify that the work embodied in this thesis has not previously been submitted for the award of a degree in any institution.

D.R. Mills
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ABSTRACT

This thesis is concerned with the development of cost-effective non-tracking concentrating collectors for solar energy purposes. As a result of this work, certain fundamental limits of concentrator performance have been identified, and practical configurations which approach these performance limits have been suggested.

Limits of radiation flux concentration are derived from the Second Law of Thermodynamics for two and three dimensional cases. Asymmetrical ideal concentrators are put forward as a generalization of ideal symmetrical concentrators in the literature, an ideal concentrator being one which accepts all radiation within its design acceptance angle and concentrates that radiation to the maximum degree permitted. The edge-ray principle is shown to be synonymous with ideality in a concentrator. The concept of symmetry of acceptance angle with respect to entrance aperture is established and peak performances of concentrators of differing symmetry are derived. The relationship between extreme asymmetrical concentrators and the solar motion is established, and advantages of this relationship are described. Non-ideal, maximally concentrating concentrators are postulated.

A number of configurations are developed to practically exploit the above concepts. These all represent non-tracking linear solar concentrators mounted along an East-West axis,
and none require any movement (powered tracking of the sun) throughout the day. These are:

1. A generalized mirror design method for ideal concentrators and a practical extreme asymmetrical version for thermal collection.
2. A photovoltaic concentrator family based upon totally internally reflecting prisms.
3. Maximally concentrating high performance thermal collectors using two-stage concentration.
4. Two-stage photovoltaic concentrators of close to maximal concentration.
5. Tilting concentrators of less than maximal performance and minimum mirror area.
ACKNOWLEDGEMENTS

I would like to express my thanks to all who have given me help towards the completion of this work. In particular, I would like to thank my supervisor, Professor J.E. Giutronich, for many hours of help and guidance to an unusually confused student. Also, the help of Dr. E. Harting and Dr. I. Bassett is gratefully acknowledged. In addition, the amazing lack of sense of humour of our lab staff, Mssrs. W. Cellich, A. Morton and I. Walker, and the equally outstanding cryptographic abilities of Mrs. E. Butcher, who typed this thesis, must be mentioned.

Finally, I would like to thank Miss J. Overton, without whom none of this could have occurred, Miss C. Carmichael for much good advice and encouragement, and my parents.
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DEDICATION

TO MY PARENTS.
CHAPTER I
INTRODUCTION

1.1 RESEARCH GOAL AND APPROACH

The goal of this research is to devise simple, practical and cost-effective means of concentrating and collecting solar radiation. There are two distinct reasons why concentrators might be used for the collection of solar energy. These are:

(a) To provide higher output temperatures than can be achieved by conventional stationary flat plate systems. By intercepting and redirecting solar radiation to a smaller absorber, thermal losses, which are proportional to absorber area, are reduced.

(b) To reduce the area of expensive receiver (e.g. copper plate or photovoltaic) used. In a concentrating collector, absorber area is largely replaced by less expensive mirror or refractive material.

The approach used in this research is to devise concentrating systems which do not require diurnal tracking, or tracking during the day. Such 'non-tracking' concentrators promise substantial performance and/or cost advantages over flat plate collectors without sacrificing the simplicity and reliability which have come to be associated with the latter. Although concentration performance of non-tracking concentrators is usually inferior to conventional focussing concentrators, cost-effectiveness should be considerably better for most industrial thermal and photovoltaic applications. This is important to the future of solar
energy utilization, for in spite of great efforts to produce an economic example, focussing concentrators remain too expensive for large scale deployment. Almost as important, the technical back-up to maintain tracking collectors is absent in many areas of the world, particularly the rapidly growing Third World.

The concentrators evolved by the author are unusual in that they are mostly asymmetrical in form. The performance limits and theory of operation of a variety of novel linear concentrators are developed in this thesis, and the analysis suggests that devices such as these may be the optimal economic solution in many solar collection situations.

Three broad groupings of concentrator are advocated by the author. The first group includes a range of predominantly asymmetrical mirror concentrators intended to achieve thermal output temperatures in the 80-250 °C range. The second consists of novel prism shaped asymmetrical concentrators intended for photovoltaics (and low temperature heat collection in hybrid form). The third comprises a family of two-stage concentrators in which a primary condenser throws radiation upon a secondary concentrator which usually belongs to the first or second group of concentrators. The latter would appear to be economic for large scale electricity or heat production at current prices in many locations.

The research area covered in this thesis is a broad one, and any attempt to provide a comprehensive review of the state of the art could result in too long a dissertation.
3.

This thesis will, therefore, be restricted to theoretical aspects of the author's work with references to the state of the art elsewhere being kept to a minimum. Some areas of innovation are:

1. The derivation of performance limits for linear concentrators (2.2).
2. Proof that the edge-ray principle is a sufficient condition for ideality (2.6).
3. The exposition of the advantages of variable concentrator response to solar collection (Chapter 3).
4. A more general profile design for ideal mirror-type concentrators (4.3).
5. The invention of prism concentrators and their variants (Chapter 5).
6. The invention of cost-effective two-stage maximal performance concentrators (Chapter 6).
7. A description of minimum mirror area single-stage concentrators of both symmetrical and asymmetrical form (6.5).

It is hoped that the above will be sufficient for consideration as a thesis submission by the examiners. Detailed experimental confirmation of the principles contained in this thesis has been neglected due to time limitations and the near impossibility of following up every avenue put forward. It is likely that such detailed experimental confirmation will be carried out by various groups in the coming years.
1.2 BACKGROUND

In many respects, our planet is a less than ideal place for the collection of solar energy. To put this in perspective, let us imagine a planet perfectly arranged for solar energy utilization. This best of all possible solar worlds would have three characteristics. First, it would be relatively cloud-free. Second, it would revolve around its sun in a circular orbit to minimize seasonal changes. Third, it would rotate about a North-South axis which was exactly perpendicular to the plane of the planetary orbit, or ecliptic.

The resultant apparent solar motion as seen from any point on such a planet would be a great circle which would be unchanging all year - the sun would never change in elevation from season to season because there are no seasons in such a situation. Because the atmosphere is also quite cloud-free, for any of their thermal or photovoltaic requirements, the inhabitants of such a planet would need merely to fix a linear focussing concentrator system securely in place and it would remain in focus throughout the year. Such a system would ensure cheap, reliable and inexhaustable energy.

Back on Earth, we are not quite so fortunate. Because the axis of rotation of our world is tilted with respect to its ecliptic, the shape and position of the sun's apparent path through the sky changes continually throughout the year. In fact, not only may a focussing collector need to be tilted to follow the seasonal movements of the sun, but elevation changes during the day will necessitate continuous tracking throughout most days of the year, even in a linear east-west
FIG. 11(a)

CORNOR MIRROR

FIG. 11(b)

SIMPLE "V" TROUGH
concentrator. The necessity for costly continuous tracking is a direct result of using a focusing optical system which has a very small acceptance angle for radiation. Attempts to devise non-tracking focusing concentrators with large enough acceptance angles to allow for yearly, or even daily apparent solar motion have proved unsuccessful because of unacceptable aberrations for off-axis rays [1]. A further disadvantage of focusing systems, is that such concentrators accept very little scattered radiation, again because of their small acceptance angles.

In order to approach as closely as possible the ideal of an inexpensive, stationary concentrating system, it is clear that conventional imaging or focusing optics must be abandoned, and a different non-focusing or non-imaging optics be devised. Just such a branch of optics has recently been investigated by a number of workers.

**Non-Focusing Optics**

The principles of conventional focusing optics have been researched since classical times and are well known. Radiation concentrators which do not form focal points or lines have a long history, but have been neglected in a theoretical sense. The simplest form of a non-focusing solar concentrator must be the flat corner mirror, sometimes used to augment the performance of conventional flat plate collectors. An example is shown in Fig. 1(a). Such mirrors have been used with flat plate collectors by Reynolds et al [2]. If two corner mirrors are used, the result is a "V"-groove concentrator (Fig. 1(b)), which has been analyzed by many authors, [3,4,5,6]. Although
A FOCON OR CPC CONE, NEARLY ALL RADIATION FROM WITHIN THE CONE-LIKE ACCEPTANCE $\theta_{\text{max}}$ IS ACCEPTED.

FIG. 1.2
the concentration levels achieved by such devices are low (1(b)), they can accept light from a large solid angle of the sky and are relatively inexpensive to install.

With focussing concentrators, the cost-effectiveness of a unit is often more dependent upon other factors (such as tracking mechanism cost) than concentration performance. As will be described later in this thesis, a non-focussing concentrator - even in ideal form - only achieves a modest concentration performance (1.5 - 15X) if it is not equipped to track the sun continuously. At such low concentration levels, concentration performance is critical to cost-effectiveness, particularly with expensive solar cell receivers. This makes it important to find the highest (non-tracking) concentration performance possible, and to devise practical concentrators which approach these performance limits as closely as possible. It can be easily shown [6] that V-channel and cone concentrators do not approach these performance limits.

As has been the case many times in the advancement of science and technology, optical devices which achieve close to the maximal concentration performance possible for a given acceptance angle were originated almost simultaneously by several workers in the mid-1960's.

The first to publicly disclose both such concentrators and their possible applications to solar energy conversion was V.K. Baranov (1965, 1966) in the Soviet Union [7,8]. Also, in 1966, Hinterberger and Winston [9] discussed similar concentrators for radiation detection work, while in 1967 Ploke [10]
Symmetrical (Flat receiver)  
(After Winston, Baranov)

Symmetrical (Cylindrical receiver)  
(After Winston and Hinterberger)

FIG. 1.3 (a)  
FIG. 1.3 (b)
described variants for use in photometric applications. All these concentrators were axially symmetric, formed by revolution of a parabolic surface around an axis as shown in Figure 1.2, although Ploke also included refractive elements in some designs. As Welford and Winston have pointed out [1(c)], these concentrators, strictly speaking, do not achieve maximal concentration performance, but approach that performance very closely. The example in Figure 2 is referred to as a FOCON by Baranov [8] and a Compound Parabolic Concentrator or CPC by Winston and Hinterberger [12,13]. Because of the relative wealth of Western publications on this subject, the term "CPC" will be adopted for use in this thesis.

Linear versions of the CPC (Fig. 1.3(a)) were first described by Baranov [11] in 1967 and later, independently, by Winston [12] in 1974. These were called 2-D CPC's by Winston and FOCLINES by Baranov [14]. Unlike the 3-D parabolotoroidal versions, the linear CPC's were shown not only to achieve maximal concentration levels but to accept all light crossing the entrance aperture within the nominal acceptance angle. For this reason, they were called "ideal" concentrators by Winston [12]. Additional variants of the 2-D CPC suitable for an absorber tube of any convex cross-section (Fig. 1.3(b)) were outlined by Winston and Hinterberger [13] in 1975. Various generalizations of this type of concentrator have since followed. In particular, linear asymmetrical versions (Fig. 1.4) are possible and were independently evolved by Rabl [15], Bassett and Derrick [18], Smith [19] and Mills and Giutronich [20]. A generalized method of mirror profile construction for all
Extreme asymmetrical, flat receiver (Smith, Rabl, Mills & Giutronich)  

![Fig. 14(a)](image)

Extreme asymmetrical, cylindrical receiver (Mills & Giutronich)  

![Fig. 14(b)](image)
ideal two-dimensional concentrators has been suggested by Mills and Giutronich [21, see also section 4.3].

The solar concentrators in this thesis are required to remain fixed throughout the day without tracking. Because of this, only cylindrical concentrators will be investigated because it is only such concentrators which have the "long and narrow" acceptance angle which will keep the sun in view for the entire day. This fact also suggests an East-West orientation for such concentrators, although with a very low concentration, wide acceptance angle concentrator, a North-South orientation is possible.

This thesis will deal with performance limitations, configurations, and philosophy of use of a variety of linear ideal and near-ideal solar concentrators. Most of these will be asymmetrical in form, for reasons which are described in the relevant sections. However, important exceptions are the symmetrical two-stage concentrators of Chapter 6, which are composed of two asymmetrical concentrators "fused" together back to back. Two-stage maximally concentrating concentrators, which are logical extensions of the work in the preceding chapters, may be used in either asymmetrical or symmetrical form according to the practical requirements of a particular installation.

1.3 DEFINITIONS

This research deals with concentration of radiation, but although the concept of concentration (or increasing the flux density) of radiation may seem clear, there are in fact
several distinct applications of the word. These will now be given in order to prevent later confusion.

1) Geometrical Concentration
   The area of the entrance aperture divided by the area of the exit aperture.

2) Instantaneous Concentration
   The ratio of the effective entrance aperture area seen by radiation coming from the source direction divided by the exit aperture area. It may be seen that this definition may only be used with distant sources of small apparent size (such as the solar disk) since it is only for such sources that a source direction may be defined. Instantaneous concentration may or may not be corrected for optical losses, but will be assumed to be uncorrected in this thesis unless otherwise indicated.

3) Peak Concentration
   The maximum possible instantaneous concentration for a given concentrator.

4) Time-Averaged Concentration
   The time-average of the instantaneous concentration over a specific time interval.

5) Maximally Concentrating Concentrator
   A concentrator has achieved maximal concentration if its exit aperture is isotropically illuminated at the brightness of the source when the source is uniform and fills the concentrator acceptance angle completely.

6) Ideal Concentrator
   A maximally concentrating concentrator which accepts all rays from within its nominal acceptance angle and none from outside that angle.
7) Ideal Transformer

Similar to an ideal concentrator, but having a limited angle for radiation from the exit aperture. It cannot maximally concentrate.
CHAPTER II
THE LIMITS OF LINEAR CONCENTRATOR PERFORMANCE

2.1 BACKGROUND

An ideal concentrator may be defined as one which accepts all radiation within a given acceptance angle and none outside that angle. A ray is considered 'accepted' by the concentrator if it arrives at the exit aperture. The CPC linear and two-dimensional concentrators described by Baranov and Winston and Hinterberger [14,13] are ideal, and have been shown to achieve a concentration of \( [\sin(\theta_{max}/2)]^{-1} \), where \( \theta_{max} \) is the entire acceptance angle. As will be shown in Sections 2.2 and 2.3, this corresponds to the maximal flux concentration possible for a symmetrical linear concentrator if both the entrance and exit apertures are in media of similar refractive index. The details of mirror profile design which allow this performance to be achieved are examined in Chapter 4.

In this chapter, the limits of performance of both ideal and non-ideal concentrators are derived from first principles for uniform sources. Non-uniform sources are examined briefly in this chapter, followed by more detailed examination in the solar context in Chapters 3 and 6.

2.2 IDEAL AND NEAR-IDEAL TWO-DIMENSIONAL CONCENTRATORS

The purpose of the section is to extend the theory of non-imaging ideal concentrators to asymmetrical forms. It will be demonstrated that higher peak concentrations are possible than those described by Winston [12] for sources which approximate distant point sources, and that the peak concentration of a symmetrical concentrator is, in fact, the
lowest attainable from an ideal concentrator. It will also be shown that symmetrical and asymmetrical ideal concentrators give a similar average concentration for diffuse radiation (or a distant point source of constant apparent angular velocity) when concentration is averaged over the entire angle of acceptance, but different average concentrations for sources of changing apparent angular velocity.

Rabl and Winston [16] have also examined the more general case of symmetrical radiation transformers, which accept radiation through the entrance aperture within a certain angular range $\theta_{\text{max}}$. Asymmetrical transformers will be treated as the most general case in the following section, and the performance of symmetrical and asymmetrical concentrators will be discussed as special cases. In principle, the method below is developed from that outlined by Winston [24] for symmetrical concentrators.

**The General Asymmetrical Linear Transformer**

For the most general case of a cylindrical transformer, cylindrical coordinates are convenient. In addition, only the two-dimensional case will be examined, since all rays may be projected and analyzed upon a two-dimensional cross-section of the cylindrical concentrator. The position of a ray may be specified by a radius $r$, centred at 0 in Fig. 2.1, and an angle $\phi$.

If $f(\phi)$ is a single-valued function which specifies a light ray such as the extreme ray $M_1$ in Fig. 2.1, Fermat's principle may be expressed
AN ASYMMETRICAL CONCENTRATOR
where \( \dot{r} = \frac{dr}{d\phi} \) and \( n \) is the refractive index.

The generalized momentum conjugate to \( r \) is given by

\[
\mathbf{p}_r = \frac{\partial}{\partial \dot{r}} \left[ n(\dot{r}^2 + r^2)^{1/2} \right] = \frac{n \dot{r}}{(\dot{r}^2 + r^2)^{1/2}}
\]  

(2.2)

For \( M_1 \) at the entrance aperture \( d_1 \), it can be seen from the inset in Fig. 2.1 that

\[
r = -\dot{r} \tan \alpha,
\]

(2.3)

and, upon substitution in Eq. 2.2, that

\[
\mathbf{p}_r = n \cos \alpha.
\]

(2.4)

For rays in the plane of \( r \) and \( \phi \), conservation of phase space area is expressed as

\[
\delta \int_{\phi_1}^{\phi_2} n(\dot{r}^2 + r^2)^{1/2} d\phi = 0
\]

(2.1)

For the transformer shown in Fig. 2.1 an angular range of \( \theta_\text{max} \) is assumed for light rays crossing the entrance aperture, and of \( \theta'_\text{max} \) for rays emerging from the exit aperture.

For the case of \( \phi = \phi_\text{in} \), 2.2 and 2.5 are combined to obtain

\[
\int_{\phi_\text{in}} \mathbf{dr}_r \mathbf{d} \mathbf{p}_r = 2n_1 d_1 \sin(\alpha + \theta_\text{max}/2) \sin(\theta_\text{max}/2).
\]

(2.6)

For \( \phi = \phi_\text{out} \),

\[
\int_{\phi_\text{out}} \mathbf{dr}_r \mathbf{d} \mathbf{p}_r = 2n_2 d_2 \sin(\alpha' + \theta'_\text{max}/2) \sin(\theta'_\text{max}/2).
\]

(2.7)
19.

At this point four different cases must be considered.

Case 1:

If \( \alpha < 90 \) or \( \alpha \geq 90 - \theta_{\text{max}} \)
and \( \alpha' < 90 \) or \( \alpha' \geq 90 - \theta_{\text{max}}' \),

the peak concentration is given by

\[
X_{\text{max}} = \frac{d_1}{d_2} = \frac{n_2 \sin(\alpha' + \theta_{\text{max}}')/2 \sin(\theta_{\text{max}}'/2)}{n_1 \sin(\alpha + \theta_{\text{max}})/2 \sin(\theta_{\text{max}}/2)},
\]  

(2.8)

where \( X_{\text{max}} \) is the ratio of irradiance at the entrance aperture to irradiance at the exit aperture for an infinitely distant point source.

Case 2:

If \( \alpha > 90 \) or \( \alpha \leq 90 - \theta_{\text{max}} \)
and \( \alpha' < 90 \) or \( \alpha' \geq 90 - \theta_{\text{max}}' \),

\[
X_{\text{max}} = \frac{d_1}{d_2},
\]  

(2.9)

where \( d_1 = d_1 \sin(\alpha + \theta_{\text{max}}) \) if \( \sin \alpha \leq \sin(\alpha + \theta_{\text{max}}) \) or \( d_1 = d_1 \sin \alpha \) if \( \sin \alpha \geq \sin(\alpha + \theta_{\text{max}}) \).

Case 3:

If \( \alpha < 90 \) or \( \alpha \geq 90 - \theta_{\text{max}} \)
and \( \alpha' > 90 \) or \( \alpha' \leq 90 - \theta_{\text{max}}' \),

\[
X_{\text{max}} = \frac{d_1}{d_2'},
\]

where \( d_2 = d_2 \sin(\alpha' + \theta_{\text{max}}') \) if \( \sin \alpha' \leq \sin(\alpha' + \theta_{\text{max}}') \) or \( d_2 = d_2 \sin \alpha' \) if \( \sin \alpha' \geq \sin(\alpha' + \theta_{\text{max}}') \). It can be seen that this case is impossible, however, since \( d_1/d_2' \) exceeds \( d_1/d_2 \),
the ratio of the true apertures.
Case 4:

\[
\text{If } \alpha > 90 \text{ or } \alpha < 90 - \theta_{\text{max}} \text{ and } \alpha' > 90 \text{ or } \alpha' > 90 - \theta_{\text{max}} ,
\]

\[
X_{\text{max}} = \frac{d_1}{d_2}.
\]

Case 4 is restricted in concentration because \( \theta_{\text{max}}' \) must be less than 90°. Hence only cases 1 and 2 represent useful transformers.

The numerator in Eqs. (2.8) and (2.9) may be maximized by allowing \( \theta_{\text{max}}' = 180° \), and the ideal transformer may now be called an ideal concentrator. If the form of the concentrator is symmetrical,

\[
\int \text{dr dp} = 2n_2d_2, \quad \phi = \phi_{\text{out}}
\]

\[\alpha = 90 - \theta_{\text{max}}/2,\]

and

\[\phi = 0.\]

This gives

\[
X_{\text{max}} = \frac{d_1}{d_2} = n_2/[n_1\sin(\theta_{\text{max}}/2)],
\]

agreeing with the Winston result [12] for compound parabolic (CPC) concentrators.

In the other limit, in which the ideal concentrator is so asymmetrical that one reflecting wall vanishes completely,

\[
\int \text{dr dp} = 2n_2d_2, \quad \phi = \phi_{\text{out}}
\]
Equations (2.8) and (2.9) may now be used to determine the maximum possible concentration for an ideal concentrator in terms of the parameter $\alpha$. Note that values of $\alpha$ such that $180 - \theta_{\text{max}} < \alpha$ or $\alpha < 0$ are not meaningful because they correspond to cases where incoming radiation within $\theta_{\text{max}}$ cannot enter the aperture. Figure 2.2 shows that peak concentration reaches a maximum for the case corresponding to Eq. 2.9 with $\alpha = 0$ or $\alpha = 180^\circ - \theta_{\text{max}}$. Hence the highest possible peak concentration for an ideal concentrator of $\theta_{\text{max}}$ is that given by Eq. 2.11, the most asymmetrical case. The lowest peak concentration is seen to correspond to the symmetrical (CPC) case, described by Eq. 2.1. 

Concentrators and transformers using receivers which are not flat will require different reflector shapes to achieve ideality, but will be subject to the same fundamental limits of concentration. Techniques for their construction have been described by Winston and Hinterberger [13], Rabl [22] and Mills and Giutronich [21].

As Rabl and Winston [16] have noted for the case of flat receivers, use of a limited $\theta_{\text{max}}$ transformer may be more desirable in solar energy applications where surface reflections from the receiver are a problem; this is also true where solar cells, which are more efficient for normal incidence,
are used. When using cylindrical absorber pipes, a concentrator of $\theta_{\text{max}}' = 180^\circ$ may be changed to a nearly ideal transformer of any $\theta_{\text{max}}'$ by simply increasing the pipe diameter by $r_2/r_1 = 1/\sin(\theta_{\text{max}}'/2)$. As an example, an increase in diameter of only 6.4% restricts the angles of incidence of most rays upon the pipe surface to $70^\circ$ or less. In this case the transformer is not truly ideal because some rays will directly strike the pipe at angles outside of $\theta_{\text{max}}'$ before having a chance to be properly directed by the reflector. However, the number of such rays will be relatively few and if such rays are reflected from the oversized pipe, they will be returned to the pipe within $\theta_{\text{max}}'$ if they happen to strike an involute reflector section.

**Time Averaged Concentration**

For an infinitely distant point source of radiation moving across the acceptance angle with a constant angular velocity $\omega$,

$$X_{\text{avg}} = \frac{X_{\text{max}}}{t_2-t_1} \int_{t_1}^{t_2} \cos \omega t \, dt$$

$$= \frac{X_{\text{max}}}{\omega(t_2-t_1)} (\sin \omega t_2 - \sin \omega t_1)$$

$$= \frac{X_{\text{max}}}{\theta_2-\theta_1} (\sin \theta_2 - \sin \theta_1). \quad (2.12)$$

If the time average is taken over the whole acceptance angle, this becomes

$$X_{\text{avg}} = \frac{X_{\text{max}}}{\theta_{\text{max}}'} (\sin \theta_2 - \sin \theta_1). \quad (2.13)$$
For uniformly diffuse radiation, the input is time independent, and the resultant concentration is also given by Eq. (2.13).

This result when applied to ideal concentrators, is of interest. For ideal concentrators obeying Eq. (2.8)

\[ X_{\text{max}} = \frac{n_2}{[n_1 \sin(\alpha + \theta_{\text{max}}/2) \sin(\theta_{\text{max}}/2)]}, \]

\[ \theta_1 = 90 - \alpha, \]

\[ \theta_2 = 90 - (\alpha + \theta_{\text{max}}), \]

giving an average concentration of

\[ X_{\text{avg}} = \frac{n_2}{n_1 \theta_{\text{max}}} \frac{\sin(90 - \alpha - \theta_{\text{max}}) \cdot \sin(90 - \alpha)}{\sin(\alpha + \theta_{\text{max}}/2) \sin(\theta_{\text{max}}/2)} \]

\[ = \frac{2n_2}{n_1 \theta_{\text{max}}}, \quad (2.14) \]

a result which is independent of concentrator symmetry. For concentrators obeying Eq. 2.9, \( X_{\text{max}} \) in Eq. 2.13 must be changed to \( X_{\text{max}}/\sin \alpha \) or \( X_{\text{max}}/\sin(\alpha + \theta_{\text{max}}) \), whichever is appropriate, to take account of the fact that \( X_{\text{max}} \) does not occur when incident radiation is perpendicular to the concentrator aperture. When this is done, the result of Eq. 2.14 is again obtained, illustrating that the average concentration of an ideal concentrator for a distant, constant velocity (or uniformly diffuse) source is independent of concentrator symmetry. This is also true for ideal transformers, but the actual value of \( X_{\text{avg}} \) will be less than that given by Eq. 2.14.

However, for a source which displays apparent accelerating motion across the acceptance angle, such as the sun, the symmetry of the transformer or concentrator assumes considerable
importance. If the motion of the accelerating source is represented by $f(t)$,

$$X_{\text{avg}} = \frac{x_{\text{max}}}{t_2 - t_1} \int_{t_1}^{t_2} \cos f(t) \, dt,$$

(2.15)

which is not purely dependent upon $\theta_2$ and $\theta_1$.

A numerical example comparing the concentration of an EAC (an extreme asymmetrical concentrators satisfying Eq. 2.11) and a CPC solar concentrator of identical acceptance angle at solstice is shown in Fig. 2.3. There are two possible orientations available at solstice for the extreme asymmetrical unit, and one of these gives superior performance to the comparable symmetrical CPC. It is always possible to find one orientation for which an asymmetrical transformer will give superior time-averaged performance over a symmetrical transformer of similar $\theta_{\text{max}}$ and $\theta_{\text{max}}'$, so long as the average position of the source is biased to one side of the angle of acceptance. This is also true for diffuse radiation if the latter is not uniform over the angle of acceptance.

If the source of radiation makes no apparent movement across the acceptance angle (as is the case in Fig. 2.3 for an E-W linear solar concentrator at equinox), the asymmetrical transformer exceeds the performance of its symmetrical counterpart by the ratio of their peak concentrations.

2.3 LIMITS OF CONCENTRATION PERFORMANCE IN THREE DIMENSIONS

In the last section, a two-dimensional analysis of an ideal concentrator was put forward. This was based upon the principle of conservation of phase space of a bundle of rays passing from the entrance to the exit aperture.
Concentration vs hour of day for ideal solar concentrators of $\theta_{\text{max}} = 25^\circ$, equinox and solstice.
In this section, the concentration problem is examined in three dimensions from a somewhat different point of view. The limiting principle in this case is the Second Law of Thermodynamics, expressed in the form that the temperature of a temperature sink may not exceed the temperature of the source at equilibrium. This method uses the empirical Second Law as an alternative foundation to the phase space limitation of the last section, which did not consider acceptance angles which may vary across the aperture.

**Uniform Distant Blackbody Source**

Fig. 2.4 shows the entrance aperture $A_1$ of a concentrator of general shape and acceptance angle. $A_1$ is taken to be planar, although $A_2$, the exit aperture, need not be. Every element of area, $dA_1$, of $A_1$ accepts radiation from an acceptance angle $\omega$, where $\omega$ is a function of the coordinates of $dA_1$. All radiation "accepted" by the concentrator is defined as reaching $A_2$ if the passage between $A_1$ and $A_2$ is free of losses. All other radiation crossing the plane of $A_1$ will be reflected out and lost. The solid acceptance angle $\omega$ may be discontinuous, having two or more sections. In addition, pockets of angular range in which radiation is not accepted may exist within the closed perimeter of $\omega$. These pockets, however, are not to be considered part of $\omega$ because radiation from these directions does not reach $A_2$. Hence, all radiation arriving within $\omega$ is accepted by the aperture area $dA_1$, and none outside $dA_1$.

In this section, it is assumed that $A_1$ is illuminated
CONCENTRATOR UNDER A DISTANT LUMINOUS HEMISPHERE

FIG. 2.4
uniformly by the sky or a distant hemispherical blackbody source at some arbitrary temperature $T_s$, so that an incident power $P_{\text{in}} = dA_1 \sigma T_s^4$ crosses the plane of $dA_1$. Although the brightness of the source appears the same in every direction, the power supplied to $dA_1$ from directions within $\omega$ depends upon the effective aperture available to each $d\omega$. This power, $P_{dA_1} d\omega$, is given by

$$P_{dA_1} d\omega = \frac{\sigma T_s^4 \cos \alpha dA_1 d\omega}{2\pi} = \frac{\sigma T_s^4 \cos \alpha dA_1 d\omega}{\pi}$$  \hspace{1cm} (2.16)$$

where $\alpha$ is the angle between $d\omega$ and the normal to $dA_1$.

The power, $P_{dA_1}$, which crosses $dA_1$ from directions within $\omega$, is given by

$$P_{dA_1} = dA_1 \frac{\sigma T_s^4}{\pi} \int_{\omega} \cos \alpha d\omega$$  \hspace{1cm} (2.17)$$

where $\frac{\sigma T_s^4}{\pi}$ may be taken to be the brightness of the source.

This can be integrated over $A_1$ to find $P_{\text{in}}$, the total power accepted by $A_1$. This is given by

$$P_{\text{in}} = \frac{\sigma T_s^4}{\pi} \int_{A_1} \int_{\omega} \cos \alpha d\omega \, dA_1$$  \hspace{1cm} (2.18)$$

At this point, we may invoke the Second Law in one of two forms. These are:

(a) an observer at $A_2$ may not see a brightness greater than $\frac{\sigma T_s^4}{\pi}$ without violating the Second Law.

(b) a perfectly insulated blackbody covering $A_2$ may not achieve a temperature greater than $T_s$ at equilibrium.
Using (a), the exit aperture of a concentrator with no internal losses "sees" a maximum transmitted power of

\[ P_{in} = A_2 \sigma T_S^4 \]  \hspace{1cm} (2.19)

if the entire available \( 2\pi \) steradians at \( A_2 \) are filled with the source at \( T_S \). This corresponds to maximum illumination of \( A_2 \) at the temperature of the source.

Using (b), the maximum power which can be re-radiated up the concentrator channel by the blackbody covering \( A_2 \) is \( A_2 \sigma T_S^4 \). At equilibrium this is equal to \( P_{in} \), and Eq. 2.19 is satisfied.

From Equations 2.18 and 2.19, the maximum ratio of real apertures is found to be

\[ \frac{A_1}{A_2} = \frac{\pi}{Q} \]  \hspace{1cm} (2.20)

where

\[ Q = \frac{1}{A_1} \int_{A_1} \int_{\omega} \cos \alpha d\omega dA_1. \]

Eq. 2.20 may be simplified if the assumption is made that \( \omega \) is everywhere uniform over \( A_1 \). In this case, which corresponds to the "ideal" concentrator definition,

\[ P_{in} = A_1 \frac{\sigma T_S^4}{\pi} \int_{\omega} \cos \alpha d\omega, \]

giving

\[ \frac{A_1}{A_2} = \pi \left[ \int_{\omega} \cos \alpha d\omega \right]^{-1} \]  \hspace{1cm} (2.21)

For a uniform source, there is no a priori reason why a constant acceptance angle (ideal concentrator) should be
preferred over a variable acceptance angle, since both will achieve the temperature of the source at \( A_2 \). For a distant localized source, however, it would appear that, in principle, the uniform acceptance of an ideal concentrator should be more desirable since all rays crossing the aperture from the source are accepted. This allows a greater amount of radiation to be collected for a given aperture size - in solar terms, a higher collection efficiency.

In practice, however, a variable acceptance angle may be preferred because freedom from the ideality constraint tends to result in a concentrator of lower mirror area and cost, as shown in Chapter 6.

The 3-dimensional analysis would at first appear to be superfluous to the discussion of linear concentrators. However, linear concentrators are not analytically identical to two-dimensional concentrators when refractive materials are involved, as will be shown in Chapter 5.

2.4 WHAT IS IDEAL FOR A NON-UNIFORM SOURCE?

Is an ideal concentrator for a uniform source also ideal for a non-uniform source?

Imagine a concentrator open to a sky of some specified brightness (or, more strictly, luminosity) distribution \( B(d\omega) \), where \( d\omega \) indicates some direction within the solid acceptance angle \( d\omega \). Since \( \omega \) may, in the most (not necessarily ideal) general case, vary from place to place on the entrance
aperture $A_1$, it is taken to be a function of the coordinates of $dA_1$. As before $\omega$ is strictly defined by the condition that rays from any direction within $\omega$ will strike $A_2$, while rays from outside $\omega$ will never reach $A_2$. Any shape whatsoever is assumed possible for $\omega$.

If the concentrator is considered loss free, the input power $P_{in}$ reaching $A_2$ is

$$P_{in} = \int_{A_1} \int_{\omega} B(d\omega) \cos \alpha (d\omega) \, d\omega \, dA_1$$  \hspace{1cm} (2.22)

At this point, an equivalent uniform blackbody source temperature, $T_x'$, may be defined such that for a given concentrator and brightness distribution

$$P_{in} = \frac{\sigma T_x'^4}{\pi} \int_{A_1} \int_{\omega} \cos \alpha (d\omega) \, d\omega \, dA_1 = \int_{A_1} \int_{\omega} B(d\omega) \cos \alpha (d\omega) \, d\omega \, dA_1$$  \hspace{1cm} (2.23)

The left side of Eq. 2.23 is the expression giving the power accepted from a uniform blackbody source at $T_x'$, arriving through the same acceptance angle as does the input power from $B(d\omega)$ on the right side of the same equation. It is as if we removed the concentrator in question from under the non-uniform sky distribution $B$, and placed it under a uniform sky supplying the same power $P_{in}$ to $A_2$. Consequently, $T_x'$ must be the maximum equilibrium temperature which can possibly be achieved by a perfectly insulated blackbody covering $A_2$. Therefore, at equilibrium, the best concentrator will achieve
As in the previous section, the ratio of apertures is again

\[
\frac{A_1}{A_2} = \frac{\pi}{Q}.
\]

In Eq. 2.24, to achieve \( T_X \) at \( A_2 \), there is no necessity that the concentrator be ideal; that is, to have a constant \( \omega \) across \( A_1 \). Indeed, the very value of \( T_X \) itself is a function of both the shape and orientation of \( \omega(dA_1) \) if \( B(\omega) \) is non-uniform.

If one could design an ideal concentrator with a widely adaptable concentration response instead of the simple "linear" responses shown in Fig. 2.5 (which are derived from Eq. 2.8), there would be a case for saying that an ideal concentrator would always be an optimal concentrator for a non-uniform source. This is because not only would a response be designed to maximize \( T_X \) for a given shape and size of acceptance angle, but the concentrator would accept all rays from the source which cross the entrance aperture.

Unfortunately, such a concentrator cannot be designed; any variations in response other than the range shown in Fig. 2.5 must be achieved by using a variable acceptance angle. However, it is known that Eq. 2.24 can also be satisfied by non-ideal variable response maximal concentration concentrators such as those described in Chapter 6. It is likely that, for many applications, these represent the
CONCENTRATION vs RADIATION INCIDENCE ANGLE FOR IDEAL CONCENTRATORS OF THREE SYMMETRIES ($\theta_{\text{max}} = 60^\circ$)
optimal concentrator *in principle* for a complex sky source or an apparently accelerating localized source such as the solar disk.

The conclusion is that, given a predetermined acceptance angle restriction, the highest concentration levels for a non-uniform source will in many cases be achieved by a maximally concentrating, "non-ideal" concentrator.

### 2.5 TIME-VARYING SOURCE

In the solar problem, the sky varies in brightness as a function of time. Over some time interval $\Delta t$, one may define an average brightness or luminosity distribution $\bar{B}(d\omega)$ such that

$$\bar{B}(d\omega) \cos \alpha \, d\omega = \frac{1}{\Delta t} \int_{\Delta t} B(d\omega, t) \cos \alpha (d\omega) \, d\omega \, dt \quad (2.25)$$

For $\bar{B}$ an optimal concentrator may be designed as in the stable non-uniform source case. In thermal concentrators, this may be complicated by taking into account a "threshold illumination" under which the concentrator does not provide useful heat or other output.

### 2.6 THE "EDGE-RAY" PRINCIPLE AS A DESIGN RULE FOR IDEAL CONCENTRATORS

Winston [1(c)] has noted that to construct a symmetrical ideal concentrator of constant $\omega$ everywhere over $A_1$, it is necessary to focus rays from directions on the boundary of
\( \omega \) upon the edge of the exit aperture \( A_2 \). This necessary condition will now be shown to be a sufficient condition for any ideal concentrator, and hence, any optimal concentrator for any source.

Let us assume the most general case, that \( \omega \) is a function of the coordinates of \( dA_1 \). From our definition of \( \omega \), rays outside \( \omega \) will not reach \( A_2 \). However, a ray from a direction infinitesimally outside \( \omega \) is almost coincident with a ray just inside \( \omega \). Since the optics of the concentrator are nearly identical for both rays, it is clear that the non-accepted ray just misses the edge of \( A_2 \), while the accepted ray from the boundary of \( \omega \) strikes the edge of \( A_2 \). This argument, which is due to Welford and Winston [1(c)], therefore shows that any concentrator which satisfies Eq. 2.20 also satisfies the edge-ray principle.

To prove the "sufficiency" argument (that is, to prove that any concentrator which satisfies the edge-ray principle is ideal), imagine a concentrator which satisfies the edge-ray principle and is not ideal. Either

a) rays reach \( A_2 \) from outside \( \omega \)

b) rays miss \( A_2 \) from inside \( \omega \)

c) only rays from inside \( \omega \) reach \( A_2 \).

Conditions (a) and (b) are impossible because they contradict the original definition of \( \omega \). For a concentrator in which ray paths are reversible, satisfaction of (c) means that there will be no path by which a ray could leave \( A_2 \).
and emerge from $A_1$ outside $\omega(dA_1)$. If, therefore, $\omega(dA_1)$ is filled with a uniform source at $T_S$, an observer at $A_2$ will see $T_S$ everywhere; that is, he is in a constant temperature enclosure at $T_S$. For such an enclosure, the radiation incident upon $A_2$ is just $A_2oT_S^*$. Hence, the concentrator satisfies Eq. 2.24 and is ideal:

Therefore, any concentrator which satisfies the edge-ray principle must be ideal (condition (c)) or impossible (conditions (a) and (b)). The edges of any non-acceptance regions within a closed boundary of $\omega$ must also be focussed upon the edge of $A_2$ to satisfy the edge-ray principle, since such regions are excluded from $\omega$.

It is clear then, that any geometrical method of satisfying the edge-ray principle should result in an ideal concentrator. For a solar concentrator, $\omega$ is constant everywhere on $A_1$ because a distant source is being used. This means that the concentrator optics will be such as to redirect light from certain extreme directions to the edge of $A_2$. A geometrical method of achieving this with mirrors is outlined in Section 4.3.

For close, extended sources, $\omega$ need not be constant over $A_1$ and the requirement is merely that rays from the edge of the source be focussed upon the edge of $A_2$. Such concentrators have been described by Rabl and Winston [16].
2.7 CONCENTRATORS FOR LIMITED EXIT ANGLES

Concentrators exist [16] in which the exit angle through which radiation may leave a given element \( dA_2 \) of \( A_2 \) is less than \( 2\pi \) steradians. Such a concentrator cannot be ideal since Eq. 2.20 is not obeyed.

To deal with this situation, a new acceptance angle \( \omega \) may be defined such that all radiation entering \( A_1 \) within \( \omega \) finds its way out of \( A_2 \) within a solid angle \( \omega' \), where \( \omega' \) is a function of the coordinates of \( dA_2 \). No radiation from outside \( \omega \) may emerge from inside \( \omega' \). Some radiation originating outside \( \omega \) may emerge from \( A_2 \) outside \( \omega' \). However, if a source is selected which exactly fills \( \omega \), no such radiation is collected, and this will be assumed the case in the following.

As before, the power accepted is

\[
P_{\text{in}} = \int_{A_1} \int_{\omega} B(\omega) \cos \alpha(\omega) \, d\omega = A_1 \frac{\sigma T_x^b}{\pi} \int_{\omega} \cos \alpha(\omega) \, d\omega \tag{2.26}
\]

which is also equal to

\[
= \frac{\sigma T_x^b}{\pi} \int_{A_2} \int_{\omega'} \cos \alpha'(\omega') \, d\omega' \, dA_2 \tag{2.27}
\]

where the primed variables refer to the exit aperture.

At equilibrium,

\[
P_{\text{out}} = A_2 \sigma T_x^b = P_{\text{in}} \tag{2.28}
\]

and

\[
\frac{A_1}{A_2} = \left( \frac{T_x^2}{T_x} \right)^b \cdot \frac{\pi}{A_1} \left[ \int_{A_1} \int_{\omega} \cos \alpha(\omega) \, d\omega \right]^{-1} \tag{2.29}
\]
Note that \( T_2 \leq T_x \) and only equals \( T_x \) when

\[
\frac{1}{A_2} \int_{\Omega} \int_{\Omega'} \cos \alpha' (d\omega') \, d\omega' \, dA_2 = \pi. \tag{2.30}
\]

i.e. when \( \omega' = 2\pi \). Hence, a loss-free concentrator of limited solid angle \( \omega' < 2\pi \) cannot achieve \( T_x \) at \( A_2 \); it does not achieve maximal concentration.

For a distant source, one may consider such a concentrator to be "ideal" when \( \omega \) is constant across \( A_1 \).

In this case,

\[
\frac{A_1}{A_2} = \left( \frac{T_2}{T_x} \right)^{-1} \cdot \pi \cdot \left[ \int_{\omega} \cos \alpha \, d\omega \right]^{-1} = \frac{\int_{\Omega'} \cos \alpha' \, d\omega' \, dA_2}{\int_{\omega} \cos \alpha \, d\omega} \tag{2.31}
\]

In the very important case where \( \omega' \) is constant over \( A_2 \), this becomes

\[
\frac{A_1}{A_2} = \frac{\int_{\Omega'} \cos \alpha' \, d\omega'}{\int_{\omega} \cos \alpha \, d\omega} \tag{2.32}
\]

Concentrators satisfying Eq. 2.32 have been called "Ideal Transformers" by Rabl and Winston [16]. Such concentrators achieve the highest possible \( T_2 \) at \( A_2 \) consistent with the restricted exit angle requirement, while accepting all radiation within \( \omega \).

If the exit aperture is in a material of different refractive index then the entrance aperture, the Lagrange-Helmholtz invariant [18] must be conserved between the entrance and exit aperture for any pencil of rays which travels between one and the other. This is expressed as,
\[ n_1^2 \cos \alpha \, d\omega \, dA_1 = n_2^2 \cos \alpha' \, d\omega' \, dA_2. \]  

(2.33)

This allows Eq. 2.32 to be expressed in the more general form,

\[ \frac{A_1}{A_2} = \left( \frac{n_2}{n_1} \right)^2 \frac{\int_{\omega'} \cos \alpha' \, d\omega'}{\int_{\omega} \cos \alpha \, d\omega} \]  

(2.34)

Dielectric concentrators of symmetrical form have been demonstrated by Winston [25]. In Chapter 5, a novel refractive transformer, the Prism Concentrator, is shown also to satisfy Eq. 2.34.

It is clear that since ideal transformers are not capable of satisfying Eq. 2.23, they cannot satisfy the edge-ray principle. Nevertheless, the performance of such concentrators is often quite close to that of Eq. 2.27, particularly if the restriction in \( \omega' \) is predominantly in directions having a high value of \( \alpha' \). In practice, the efficiency of radiation collection of a limited \( \omega' \) concentrator may exceed that of an ideal concentrator because glancing angle reflections from the exit aperture receiver may be reduced.

2.8 NON-REVERSIBLE OPTICS

Concentrating systems which cannot be adequately described by the arguments in previous sections can be put forward. When dichroic mirrors are used or when light from a specific direction of the sky experiences partial internal reflection within a dielectric segment of the concentrator, it is clear that a crisp boundary between directions of acceptance and non-acceptance may not be defined for
the entire entrance aperture. The receiver of such a concentrator cannot come into thermal equilibrium with the source. Hence, such a concentrator cannot be ideal.

Concentrators which operate by total internal reflection cannot be ideal because they exhibit partial acceptance for some directions. They may be considered ideal transformers for acceptance angles which match their total internal reflection property, but are not ideal transformers outside the T.I.R. acceptance angle. Further discussion of this will be carried out in Chapter 5.

2.9 TRUNCATED IDEAL CONCENTRATORS

As Winston and Hinterberger [13] have shown, ideal concentrators such as the CPC may be made more practical if their inherently large mirror area is cut down substantially. As is shown in Fig. 2.6, entrance aperture area suffers relatively little for a fairly large mirror removal. Such a concentrator cannot maximally concentrate, and cannot be ideal because the removal of upper mirror sections allows in a certain amount of radiation outside the acceptance angle (i.e. direction \( \mathbf{k}_3 \) in Fig. 2.6, which lies outside the nominal acceptance \( \mathbf{k}_1 \leftrightarrow \mathbf{k}_2 \)).

Mirror area requirements for symmetrical CPC's in truncated form are, however, still significantly higher than for many maximal concentration collectors, as will be demonstrated in Chapter 6.
A TRUNCATED CPC. A LARGE MIRROR REDUCTION IS ACHIEVED WITHOUT EXCESSIVE APERTURE LOSS.

FIG. 26
CHAPTER III

EXTREME ASYMMETRICAL CYLINDRICAL CONCENTRATORS
AS SOLAR COLLECTORS

3.1 INTRODUCTION

In Section 2.2, it was demonstrated that various two-dimensional ideal concentrators may exhibit different "tilts" of their acceptance angles with respect to their entrance apertures. The tilt of the acceptance angle affects the performance of the concentrator, and can be used to quantify a property called the "symmetry" of the device. This chapter will show that asymmetrical cylindrical ideal troughs based upon these two dimensional ideal concentrators are often to be preferred to symmetrical concentrators such as the CPC in solar applications.

Figure 3.1 shows three possible conditions of ideal trough symmetry, along with the peak concentrations (derived in Section 2.2) attainable from concentrators of these symmetries.

Note that the expression for intermediate asymmetry peak concentration becomes

\[ X_{\text{max}} = \frac{\sin(a + \theta_{\text{max}})}{[\sin(a + \theta_{\text{max}}/2)\sin\theta_{\text{max}}/2]} \]

if \( a < 90 - \theta_{\text{max}} \).

Table 3.1 below shows some typical values of peak concentration for various values of \( \theta_{\text{max}} \) and \( a \). Note that the
A cross-sectional view showing three tilts of $\theta_{\text{max}}$ with respect to the aperture of an ideal concentrator.
EAC gives the highest peak concentration in each case.

**TABLE 3.1**

[Peak concentration as a function of $\theta_{\text{max}}$ and $\alpha$ in an ideal concentrator.]

<table>
<thead>
<tr>
<th>$\theta_{\text{max}}$</th>
<th>$\alpha = 0$ (EAC)</th>
<th>$\alpha = 20$</th>
<th>$\alpha = 40$</th>
<th>CPC</th>
</tr>
</thead>
<tbody>
<tr>
<td>15</td>
<td>15.9</td>
<td>9.52</td>
<td>8.51</td>
<td>7.44</td>
</tr>
<tr>
<td>30</td>
<td>7.46</td>
<td>5.16</td>
<td>4.43</td>
<td>3.86</td>
</tr>
<tr>
<td>45</td>
<td>4.83</td>
<td>3.51</td>
<td>2.93</td>
<td>2.61</td>
</tr>
<tr>
<td>60</td>
<td>3.46</td>
<td>2.57</td>
<td>2.13</td>
<td>2.13</td>
</tr>
</tbody>
</table>

Using the above and information from Fig. 2.5, three operational situations can be suggested where extreme asymmetrical solar concentrators may offer advantages over symmetrical systems.

**3.2 FREQUENTLY TILTED SYSTEMS AND THE SOLAR MOTION**

In this section it will be assumed that the linear concentrator in question is adjusted to an optimum tilt once a day. This enables a relatively small acceptance angle to be used. Although daily tilting is more frequent than would be required in a practical solar case, it is mathematically convenient for the discussion to follow.

For a symmetrical non-imaging trough concentrator with an acceptance angle of the form indicated by Fig. 3.1(a), instantaneous concentration within the design acceptance angle is relatively constant if one assumes source directions exhibiting only small departures from the perpendicular to
the entrance aperture. In its east-west motion (during the day), however, the solar disk departs substantially from the normal to the entrance aperture. This deviation from the normal direction $Z$ may be quantified by the angle $\psi(t)$ shown in Fig. 3.2, which is measured in the plane ABCD. There may also be a smaller departure $\theta(t)$ in the north-south plane ADE, as shown in the same diagram. In consequence, the projected area of the collector aperture perpendicular to the incident radiation must be taken into account, and the instantaneous concentration, $X$, expressed as

$$X = X_{\text{max}} \cos \psi(t) \cos \theta(t).$$

where $X_{\text{max}}$ is the concentration of $\psi = \theta = 0$ and $t$ is the number of hours from noon (symmetrical concentrator case). The dependence of $X$ upon $t$ for a typical Winston-Baranov CPC trough concentrator of symmetrical form is shown in Fig. 2.5.

Once it is accepted that there is a strong dependence of $X$ upon $t$, it may be asked whether in certain circumstances a different $X(t)$ might be preferable to that offered by the symmetrical trough concentrators. In this section, it will be shown that a comparable asymmetrical non-imaging trough concentrator with a useful flexibility in $X(t)$ is possible. Whether such a concentrator would be used in practice is a question which depends upon concentrator economics.
THE DIRECTION OF THE SUN AS SEEN BY AN EAST-WEST LINEAR CONCENTRATOR.
Symmetrical Case

Using the geometry of the celestial sphere, it can easily be shown that

$$\cos \psi = (1 - \cos^2 \delta \sin^2 t)^{\frac{1}{2}} \tag{3.2}$$

where $\phi$ is the latitude and $\delta$ the declination angle. The declination lies within the limits $23.5^\circ > \delta > -23.5^\circ$ and is taken to be positive in summer for either hemisphere. From Eq. 3.1, it follows that, for a symmetrical concentrator

$$X = X_{\text{max}} \cos \theta(t) (1 - \cos^2 \theta \sin^2 t)^{\frac{1}{2}}.$$

If adjusted once daily, $\cos \theta(t) = 1$. From Winston [12] we have $X_{\text{max}} = 1/\sin(\theta_{\text{max}}/2)$ where $\theta_{\text{max}}$ is the acceptance angle of the concentrator in the north-south plane. This results in

$$X = (1 - \cos^2 \delta \sin^2 t)^{\frac{1}{2}} / \sin(\theta_{\text{max}}/2) \tag{3.3}$$

for less frequent tilting. Eq. 3.3 is symmetrical about equinox and gives the same results for winter and summer solstice.

Asymmetrical Case

From Fig. 2.3 and Eq. 2.8, it may be seen that the response of an asymmetrical concentrator in the north-south plane is given by

$$X(t) = X_{\text{max}} \frac{\cos \theta(t)}{\cos \theta_{\text{min}}} \cdot (1 - \cos^2 \delta \cos^2 t)^{\frac{1}{2}} \tag{3.4}$$

$$= \frac{(1 - \cos^2 \delta \cos^2 t)^{\frac{1}{2}} \cos \theta(t)}{\sin(a + \theta_{\text{min}}/2) \sin(\theta_{\text{max}}/2) \cos \theta_{\text{min}}} \tag{3.5}$$
From Fig. 3.3, it may be seen that

\[
\cos \theta_{\text{min}} = \cos(90 - \alpha - \theta_{\text{max}}) = \sin(\alpha + \theta_{\text{max}}).
\]

Cos \( \theta(t) \) may be determined as follows:

It is now necessary to determine \( \cos \theta(t) \). To do this, note that two orientations are possible for an asymmetrical concentrator covering a given angular range of the sky. The first, called the "horizontal" orientation (the work "horizontal" assumes a latitude of roughly 30°-50°) is indicated by Fig. 3.3(a). The second, called the "vertical" orientation, is indicated by Fig. 3.3(b). For the horizontal orientation,

\[
\theta(t) = |\alpha(\text{noon}) - \alpha(t)| = |90 - \delta - \tan^{-1}[\cos t \cdot \tan(90 - \delta)]| \tag{3.6}
\]

where the modulus indicates that the concentrator is oriented so that \( \theta \) is positive. Fig. 3.3(b) shows the "vertical" orientation, where the concentrator is set so that its acceptance angle limit is at an angle \( \gamma \) to the noon position of the sun. Here, the expression for \( \theta(t) \) is given by

\[
\theta(t) = \gamma - |90 - \delta - \tan^{-1}[\cos t \cdot \tan(90 - \delta)]| \tag{3.7}
\]

where \( \gamma \) is always taken to be positive.

Using the above, a final expression for loss-free concentration is achieved for an asymmetrical ideal linear concentrator. This is,
FIG. 3.3

The two possible orientations for an E-W linear solar concentrator.
In the most extreme case, the extreme asymmetrical concentrator (EAC), $\alpha = 0$ and $X_{\text{max}} = \frac{2}{\tan(\theta_{\text{max}}/2)}$ giving,

$$X = \frac{2(1-\cos^2 \delta \cos^2 t)^{\frac{1}{2}} \cos \theta(t)}{\tan(\theta_{\text{max}}/2)^2 \sin(\theta_{\text{max}}/2) \cos(\theta_{\text{max}}/2)}$$

$$= \frac{(1-\cos^2 \delta \cos^2 t)^{\frac{1}{2}} \cos \theta(t)}{\sin^2(\theta_{\text{max}}/2)}. \quad (3.9)$$

It is clear from Eqns. 3.8 and 3.9 that the EAC will outperform the symmetrical concentrator of equal $\theta_{\text{max}}$ for the few hours around noon. This results in a time-averaged concentration which is greater for the asymmetrical concentrator. The greater the asymmetry of the concentrator, the greater is the performance increase for a given $\theta_{\text{max}}$. However, it should be noted that in order to adequately exploit this effect, (a) $\theta_{\text{max}}$ should be as small as possible; (b) tilting must be carried out every few days and (c) the asymmetrical concentrator must be inverted at equinox because it changes from the "horizontal" model of behaviour to the "vertical" unless this inversion takes place. For maximum time-averaged concentration, the "horizontal" mode of operation is to be preferred.

Assuming it is aligned once daily, the optimum tilt of the "horizontal" concentrator depends only upon the noon position of the sun for that day. In the "vertical" orientation, however, there is an additional dependence upon $\gamma$. 

$$X = \frac{(1-\cos^2 \delta \cos^2 t)^{\frac{1}{2}} \cos \theta(t)}{\sin(\alpha+\theta_{\text{max}}/2) \sin(\theta_{\text{max}}/2) \sin(\alpha+\theta_{\text{max}})} \quad (3.8)$$
and the optimal setting may depend upon other factors such as available energy storage, schedule of energy demand, weather conditions, etc.

To illustrate this further, calculations for EAC concentrators of $\theta_{\text{max}} = 12^\circ$ were carried out using equations 3.6 - 3.9. These are shown in Figures 2.5 and 3.5. Fig. 3.4 shows the concentration at solstice for an EAC in the "horizontal" mode with $\gamma = 12^\circ$. This actually exhibits increased concentration where the solar disk is more remote from the noon position. Cut-off occurs when the sun passes the acceptance angle limit shown by the dashed lines. Curve shows the same concentrator at solstice with $\gamma$ limited to $6^\circ$. This gives higher concentration for a shorter period of collection. Note that the higher time-averaged concentration for this smaller acceptance angle tends to compensate for the shorter operating time between cut-off limits, so that both values of $\gamma$ give about the same total energy collection. For times other than solstice, the $\gamma = 6^\circ$ concentrator would be superior because the $\psi$ excursion during the day would be less.

The qualifications of small acceptance angles (leading to lengthy mirrors), frequent tilting, and inversion at equinox have prevented the establishment of practical ideal systems utilizing the daily solar motion as a means of significantly increasing time-averaged performance. However, the effects described can display themselves in a very similar manner with the maximally concentrating systems described in
FIG. 3.4

CONCENTRATION vs. HOUR OF DAY FOR AN EXTREME ASYMMETRICAL 10X CONCENTRATOR.
Chapter 6. The response functions in these concentrators are somewhat more complex than those of ideal systems, and will be discussed in that Chapter.

3.3 OCCASIONALLY TILTING SYSTEMS

For practical reasons, an acceptance angle may be chosen which is larger than is necessary to accommodate the daily apparent motion of the solar disk, but is too small to accept radiation from the sun for the entire year without tilting. Nevertheless, an asymmetrical concentrator - especially the EAC type - can be shown to outperform a symmetrical concentrator of similar $\theta_{\text{max}}$ under these circumstances if it is allowed to tilt more frequently. Tilting a conventional symmetrical concentrator has relatively little effect upon performance provided the solar disk remains within the acceptance angle. By tilting an EAC frequently enough, however, we can keep the solar disk within the high concentration region of the acceptance angle as shown in Fig. 3.5. This amounts to substitution of labour for capital costs, an important consideration in both developing countries and in industrial situations where tilt adjustments could be combined with maintenance functions at relatively little increase in costs. Indeed, in most cases, more frequent adjustments than once a month would seldom be necessary, on average.

Apart from increased concentration performance, a second characteristic of such a system is that performance, to a limited extent, is a function of tilt frequency. The consequence of this is an operational flexibility unrivalled
FIG. 35

TILT ADJUSTMENTS FOR AN EAC CONCENTRATOR.

EAC IDEAL CONCENTRATION

8:30 AM / 3:30 P.M.

TILT ADJUSTMENT

SUMMER
SOLSTICE

NOON DECLINATION $\delta$

EQUINOX

WINTER
SOLSTICE

$\delta^\approx 23.5^\circ$

$\delta^\approx 0^\circ$

$\delta^\approx -23.5^\circ$
in conventional solar collectors. Provided there is enough capacity in the system, unexpected variations in demand or weather, dirty mirrors, ageing components, etc., can be compensated for to some degree by changing the adjustment frequency.

Fig. 3.5 shows the calculated ideal performance of a 5X EAC mirror system which is tilted only in winter. Note that the scale on the horizontal axis is linear with declination and not time of year. Fig. 5.21 shows the loss-corrected performance of an EAC refractive prism concentrator of higher concentration, adjusted so that performance is kept to 95% of the once-a-day tilting regime. Notice that only 18 tilts are required to achieve this performance, a level which is consistent with likely maintenance schedules. Prism systems are described in detail in Chapter 5.

In addition to the advantages associated with prism occasionally tilting concentrators for photovoltaics, thermal EAC mirror-type concentrators should achieve running temperatures over 100°C using relatively unsophisticated absorber tubes. This may allow developing countries to make use of such systems. The design of such mirrors is examined in Section 4.3.

Once again, maximal concentration systems may be designed with a concentrator response suitable for this type of tilting regime. These are discussed in Chapter 6.
3.4 FIXED ASYMMETRICAL SYSTEMS AND YEARLY OUTPUT MODULATIONS

Completely stationary or fixed concentrator systems may be used where a tilting concentrator is either too expensive to use, too dangerous or inconvenient to adjust, or is intended for unattended remote operation. In order to collect radiation for the entire year, the concentration must be relatively low (≈ 1.5-3). However, once a stationary concentrating system has been decided upon, a new aspect of asymmetrical concentrator behaviour manifests itself in the possibility of yearly output modulation, or modification of the yearly output response to suit demand.

The solid lines in Fig. 3.6 show an ideal concentration response at noon and at 3.30 p.m. for an EAC of acceptance angle 72°. Fig. 3.4 is linear with declination and not time of year. Practical profiles which approach the behaviour are OAB in Fig. 3.7, which we have designed using the Winston-Hinterberger curvature [13], certain designs by Rabl [15], and the profiles shown in Fig. 4.3 after Smith [19] and Mills and Giutronich [20,21]. As before, there are two possible orientations available to cover a given section of the sky. The solid curves in Fig. 3.6 describe a concentrator which has been oriented so that maximum performance occurs near winter solstice - this is the "vertical" orientation. The "horizontal" orientation, indicated by the dashed lines, covers exactly the same solid angle of sky and exhibits enhanced performance in summer. The dotted lines indicate the performance of a symmetrical concentrator of the same acceptance angle. It is important to note that
FIG. 3.6

YEARLY OUTPUT VARIATIONS IN A 72° EAC.

WINTER BIASED EAC
SUMMER BIASED EAC
SYMmetrical EAC SYSTEM

8:30 AM
3:30 PM

NOON DECLINATION $\delta$
Figure 3.7: Extreme asymmetrical mirror profiles

- A: $X_{\text{max}} = 2.00$
- B: $X_{\text{max}} = 2.73$
- C: $X_{\text{max}} = 4.78$

Angles:
- $\theta_{\text{max}} = 45^\circ$
- $\theta_{\text{max}} = 72^\circ$
- $\theta_{\text{max}} = 90^\circ$
they also give the performance of a concentrator system using equal numbers of identical EAC's in both the "horizontal" and "vertical" orientations. Two lessons can be learned from this graph, therefore. The first is that we can bias yearly energy output to one solstice or the other without changing total yearly output greatly, for the equal areas under the curves in this graph are also very nearly equal for a graph having a horizontal axis linear with time of year (as demonstrated in Fig. 5.16 for EAC prism arrays).

The second is that a symmetrical ideal concentrator system may be constructed of asymmetrical components. The concept can be taken further than this, however. Any system symmetry between the symmetrical and the extreme asymmetrical can be synthesized by altering the ratio of identical component EAC's in the two orientations.

This is illustrated in Fig. 3.8, which shows an intermediate asymmetrical system composed of identical ideal EAC modules in two different orientations, here called the A and B orientations. The same acceptance angle $\theta_{\text{max}}$ is covered by every EAC component in the system. To be an ideal concentrator the system must achieve an instantaneous concentration of (see Eq. 2.8)

$$X_\beta = \frac{\sin(\alpha+\beta)}{\sin(\alpha+\theta_{\text{max}}/2)\sin(\theta_{\text{max}}/2)}$$

(3.10)

for radiation incident within $\theta_{\text{max}}$ at some arbitrary angle $\beta$ from one of the two limits of the angle of acceptance. The system is assumed to be of refractive index equal to unity.
AN INTERMEDIATE ASYMMETRICAL SYSTEM BASED UPON EAC COMPONENTS.

FIG. 3.8
they also give the performance of a concentrator system using equal numbers of identical EAC's in both the "horizontal" and "vertical" orientations. Two lessons can be learned from this graph, therefore. The first is that we can bias yearly energy output to one solstice or the other without changing total yearly output greatly, for the equal areas under the curves in this graph are also very nearly equal for a graph having a horizontal axis linear with time of year (as demonstrated in Fig. 5.16 for EAC prism arrays).

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$$X_\beta = \frac{\sin(a+\beta)}{\sin(a+\theta_{\text{max}}/2)\sin(\theta_{\text{max}}/2)}$$  \hspace{1cm} (3.10)

for radiation incident within $\theta_{\text{max}}$ at some arbitrary angle $\beta$ from one of the two limits of the angle of acceptance. The system is assumed to be of refractive index equal to unity.
The instantaneous concentration of a component EAC in the A orientation can be shown (see section 2.2) to equal

$$X_A = \frac{\sin(\theta_{\text{max}} - \beta)}{\sin^2(\theta_{\text{max}}/2)} \quad (3.11)$$

and for the B orientation

$$X_B = \frac{\sin \beta}{\sin^2(\theta_{\text{max}}/2)} \quad (3.12)$$

For the whole system, the average instantaneous concentration becomes,

$$X_s = \frac{N_A \sin(\theta_{\text{max}} - \beta) + N_B \sin \beta}{(N_A + N_B)[\sin^2(\theta_{\text{max}}/2)]} \quad (3.13)$$

where $N_A$ and $N_B$ are the number of EAC units in the A and B orientations respectively. $X_s$ can be shown to equal $X_B$ if use is made of the geometrical identity

$$\cos \alpha = [N_B - N_A \cos \theta_{\text{max}}][N_A^2 + N_B^2 - 2N_A N_B \cos \theta_{\text{max}}]^{-1/2}$$

Of course, there is no necessity that a V-type configuration be adopted for a practical system, and it is highly likely that some separation between A and B modules - which could themselves be in several separate sections - would be adopted in practice.

The advantage of yearly output modulation using symmetry variation are immediately apparent. For the first time we can "phase match" yearly output to approximate yearly demand cycles in a completely non-moving system. This is a different procedure from biasing, for example, a flat plate toward a particular solstice where higher demand exists, because
yearly clear sky output is significantly reduced by such a procedure. Clear sky output is not, however, significantly affected by symmetry alterations using different ratios of EAC's. Of course, if maximum yearly output is biased toward a time of year with increased cloud, yearly output will suffer. But other situations exist, where yearly output would be increased over that of a symmetrical system by biasing toward the sunny summer months. Whatever the locale, symmetry variation provides the means by which the yearly performance of the collector system can be optimized to reduce storage requirements to a minimum. By providing system output when required, and minimizing output in periods of low demand, the combined cost of collector and storage can be reduced to a minimum. This modulation of a completely fixed system output by the apparent solar motion can only be achieved by use of asymmetrical component concentrators, with EAC's giving the most complete flexibility. Long term alterations in demand can be accommodated by adding a few more EAC modules in the appropriate orientation, or by switching modules from one orientation to another.

Two general groups of EAC's may be used in this way. In Chapter 4, mirror type concentrators suitable for thermal work are investigated. In Chapter 5, prism ideal extreme asymmetrical transformers are described which result in simple, lightweight, efficient concentrator for photovoltaics.
4.1 BACKGROUND

The Compound Parabolic Concentrator

The "edge-ray principle" described in Section 2.6 has been shown to be synonymous with ideality in a two-dimensional concentrator. A concentrator satisfies the edge-ray principle if rays from the extreme directions of the acceptance angle are directed to the edge of the cylindrical receiver of radiation. So long as the orientation of the entrance aperture with respect to the acceptance angle and the receiver cross section are known, a concentrator profile may be derived from the edge-ray principle. Note that this profile is not unique, for the more general concentrators in Section 4.3 also satisfy the edge-ray principle.

The most well known case of profile construction is for a flat absorber parallel to the entrance aperture and an acceptance angle symmetrical about the normal to both. This is the "2-D CPC", originally put forward by Baranov, and Winston and Hinterberger, and it is shown in Figure 4.1. The edge-ray principle is expressed simply as a parabolic curve which focusses rays from the extreme ray directions of the acceptance angle upon the opposite end of the absorber. The parabolic curve has a focal length such that the curve intersects the near edge of the absorber, thus preventing rays from passing between the reflector and receiver. All rays within the design acceptance angle $\theta_{\text{max}}$ are, therefore, accepted and strike the receiver after either hitting the
A compound parabolic concentrator

FIG. 4.1
receiver directly or after one or more reflections from the mirror. The ratio of apertures is determined to be $1/\sin(\theta_{\text{max}}/2)$ from the geometry of the figure. This is, as was found in Section 2.2, the ideal geometrical concentration for an acceptance angle symmetrical about the normal to the entrance aperture.

In 1975, Winston and Hinterberger [13] showed that a symmetrical two-dimensional CPC could be constructed for any convex or flat receiver using a combination of involute curvature and a curvature based upon the edge-ray principle. In section 4.3, a more generalized method of ideal mirror profile construction originated by the author is described.

4.2 EARLY WORK ON ASYMMETRICAL MIRROR CONCENTRATORS

Some work was done in the early part of the author's research program on mirrors which could function as EAC's. These mirrors all used a combination of Winston-Hinterberger [13] and involute curvature to achieve concentration. A good deal of this work is described in the attached reference [20] which is contained in Appendix I(a). Figures 1.4 and 3.7 also show examples of these designs. Similar designs by Rabl [15] and Smith [19] were also independently evolved either at the same time or prior to the author's work without his knowledge.

A problem that arose in the development of all these EAC designs was the relatively high number of mirror reflections required to achieve concentration. These numerous reflections led to high mirror absorption losses, particularly...
when mirrors of reflectivity \( r < 85\% \) were used.

In order to attack this problem, a generalized method of generating ideal mirror profiles was developed by the author. Using this design philosophy, a new series of EAC profiles was produced which effectively rendered obsolete the designs mentioned above. For this reason, discussion of the older designs is abbreviated and contained in the reference in Appendix I(a). Discussion of the new, generalized mirror profile construction method follows in the next section, with the resultant stationary EAC derivative described in Section 4.4. In Section 4.5, minimum mirror area tilting concentrators of general symmetry are discussed.

4.3 GENERALIZED IDEAL MIRROR PROFILE DEVELOPMENT

Winston and Hinterberger [13] have shown how to construct ideal two-dimensional linear concentrators for any convex two-dimensional or cylindrical receiver. Such concentrators obey the edge-ray principle of Section 2.6 because the mirror profiles are designed so that all rays from boundary directions of the acceptance angle — such rays are called extreme rays by Winston [12] — are reflected tangentially to the receiver surface.

Rabl and Winston [16] have also described ideal concentrators for close extended sources of radiation. These are constructed using the condition that rays from the boundary of the source are reflected tangentially to the receiver. The latter condition is expressed in Eq. 4.2 of this section, while the former Winston-Hinterberger condition is described by Eq. 4.3.
Both these constraints may be used to develop an additional, more general family of both ideal and nearly ideal concentrators for distant sources of radiation. This method of mirror profile development owes much to papers by Winston and Hinterberger [13] and Rabl [22], and attempts to follow their format as closely as possible. All previous ideal mirror profile designs by these and other authors may be shown to be special cases of the generalized profile to be described below.

**General Profile Description**

Fig. 4.2 shows an asymmetrical two-dimensional concentrator which accepts radiation from an acceptance angle $\hat{k} \rightarrow \hat{k}' = \theta_{\text{max}}$ and distributes it upon the convex receiver shown. The mirror profile for the right side of the concentrator is composed of three sections. The first, an involute, extends from a point $S_0$ on the receiver surface to a point $P_1$ on the profile. This curved mirror segment, which will be called $S_0P_1$, is intended to reflect all rays crossing $S_1P_1$ either directly toward the receiver or indirectly toward the receiver after striking $S_0P_1$ one or more times. The length of arc from $S_0$ to a point $P$ in $S_0P_1$ is given by

$$S - S_0 = (\hat{R} - \hat{P}) \cdot \hat{t}$$

(4.1)

where $\hat{t}$ is a unit vector tangent to the absorber perimeter at $S$, and $\hat{R}$ and $\hat{P}$ are vectors from $S_0$ to $S$ and $P$, respectively.

At $S_1$, therefore,

$$S_1 - S_0 = (\hat{R}_1 - \hat{P}_1) \cdot \hat{t} = r_1$$
Asymmetrical Ideal Concentrator

![Diagram of an Asymmetrical Ideal Concentrator with various labeled points and lines indicating symmetry and equations (Eq. 4.1, Eq. 4.2, Eq. 4.3)]
where for a given $S$, $r$ is the distance from a point $P$ on the mirror profile to the point $S$ on the receiver perimeter along the tangent direction $\hat{t}$. The left profile side may be described in a similar manner using primed variables, giving

$$S'_1 - S'_0 = (\hat{R}'_1 - \hat{P}'_1) \cdot \hat{t}'_1 = r'_1.$$

In the second section of the mirror profile, between $P_1$ and $P_2$, a new constraint is introduced. This will be called the "quasi-elliptical" curvature, and is described by

$$ds = d(\ell + r)$$

where $s$ is the distance around the perimeter of the receiver measured from $S'_0$ and $\ell$ is the distance from $P'_3$ to $P$. This ensures that all rays which cross the concentrator aperture $A$ and strike $P_1P_2$ will either strike the receiver directly after reflection or cross $S_1P_1$. If Eq. 4.2 is integrated from $S_1$ to $S_2$, the result is

$$S_2 - S_1 = \int_{P_1}^{P_2} d(\ell + r)$$

for the left profile side, and

$$S'_2 - S'_1 = (\ell'_2 + r'_2) - (\ell'_1 + r'_1)$$

for the right profile side.

Between $P_2$ and $P_3$, the profile uses the Winston-Hinterberger curvature. This will be also referred to as "quasi-parabolic" curvature. This mirror segment has the property that all rays from the direction $\hat{k}$ incident upon
$P_2P_3$ are reflected tangent to the receiver. It is described by

$$\frac{d\vec{P}}{ds} \cdot \hat{t} = \frac{d\vec{P}}{ds} \cdot \hat{k}$$

(4.3)

The direction $\hat{k}$ may be taken as one limiting direction within the angle of acceptance because all rays crossing $A$ which strike $P_2P_3$ from directions anti-clockwise to $\hat{k}$ will miss the receiver and be reflected away.

From Fig. 4.2,

$$\vec{P} = \vec{R} - r \hat{t} \text{ where } \hat{t} = \frac{d\vec{R}}{ds}$$

and

$$\frac{d\vec{P}}{ds} \cdot \hat{t} = 1 - \frac{dr}{ds} = \frac{d\vec{P}}{ds} \cdot \hat{k}$$

Integrating from $S_2$ to $S_3$,

$$(S_3 - S_2) - (r_3 - r_2) = (\vec{P}_3 - \vec{P}_2) \cdot \hat{k}.$$  

(4.4)

The perimeter of the receiver, $S_p$, may now be determined in terms of the concentrator dimensions, and it is found that

$$S_p = (S_1 - S_0) + (S_2 - S_1) + (S_3 - S_2) + (S_0 - S_3)$$

$$= r_1 + (\ell_2 + r_2) - (\ell_1 + r_1) + (r_3 - r_2) + (\vec{P}_3 - \vec{P}_2) \cdot \hat{k} + r_1'$$

$$= \ell_2 - \ell_1 + r_3 + r_1' + (\vec{P}_3 - \vec{P}_2) \cdot \hat{k} \quad \text{(right side)}$$

$$= \ell'_2 - \ell'_1 + r'_3 + r_1' (\vec{P}_3' - \vec{P}_2') \cdot \hat{k}' \quad \text{(left side)}$$

Since $\ell_1' = r_1 + r_3'$ and $\ell_1' = r_1' + r_3$, it follows that

$$S_p = \frac{1}{2} [\ell_2 + \ell_2' + (\vec{P}_3 - \vec{P}_2) \cdot \hat{k} + (\vec{P}_3' - \vec{P}_2') \cdot \hat{k}']$$  

(4.5)

The directions $\hat{k}$ and $\hat{k}'$ are taken to make angles of $\phi$
and \( \phi' \) respectively with the aperture \( A \), as shown in Fig. 4.2. Therefore, from the geometry of the figure,

\[
\ell_2 = A \cos \phi - (\hat{P}_3 - \hat{P}_2) \cdot \hat{k}
\]

\[
\ell_2' = A \cos \phi' - (\hat{P}_3' - \hat{P}_2') \cdot \hat{k}'
\]

and, after substitution into Eqn. 4.5,

\[
S_p = \frac{1}{2} [A(\cos \phi + \cos \phi')].
\]

The ratio of entrance aperture width to receiver perimeter is, therefore

\[
A/S_p = 2/(\cos \phi + \cos \phi').
\]

In Section 2.2, it was shown that, in order to be ideal, a concentrator must possess an entrance to exit aperture ratio of \( A_1/A_2 \) such that

\[
A_1/A_2 = [\sin(\theta_{\text{max}}/2) \sin(\phi' + \theta_{\text{max}}/2)]^{-1}
\]

\[
= 2/[\cos \phi' - \cos(\phi' + \theta_{\text{max}})]
\]

From Fig. 4.2, \( \phi = \pi - \phi' - \theta_{\text{max}} \), giving

\[
A_1/A_2 = 2/(\cos \phi + \cos \phi').
\]

Hence, the concentrator is ideal. It should be noted, however, that in Fig. 4.2 no radiation is permitted to strike the receiver directly from outside \( \hat{k} \leftrightarrow \hat{k}' \); only radiation within \( \theta_{\text{max}} \) may be accepted by an ideal concentrator. The construction of the concentrator must fulfill this requirement.
Note that this method results in a concentrator in which rays do not cross over from one mirror half to the other before reaching the receiver. The author believes that the method described gives the most general method possible of designing a two-dimensional (or linear) non-crossover ideal concentrator profile for a given convex receiver. This can be shown as follows:

A given mirror element will either see the acceptance angle limit in Fig. 4.2 or be shaded by the opposite mirror rim. If the rim intrudes into the design acceptance angle, the best the mirror element can do is "accept" all light incident upon it. This it does if rays from the opposite rim are focussed upon the edge of the receiver as with Eq. 4.2, for it cannot focus rays from just below the top edge of the opposite rim without being a cross-over concentrator. If the acceptance angle is not intruded upon by the opposite rim, the mirror element will accept light from only within the design acceptance angle only if rays from the acceptance angle limit are focussed upon the edge of the receiver as dictated by the edge-ray principle. This is the case described by Eq. 4.3. Hence, the design procedure given is sufficient to describe the most general non-crossover ideal mirror profile.

The author has been able to develop ideal mirror concentrators where crossover of rays from one mirror to the other occurs, but these are of doubtful utility since they are invariably longer than the non-crossover type for a given acceptance angle and symmetry. On the other hand, a non-mirror type of nearly ideal crossover transformer, the prism
concentrator, is described in Chapter 5.

Ideal transformers, in which the exit angle for rays at the exit aperture is restricted, and nearly ideal concentrators, in which some rays may strike the receiver from outside $\theta_{\text{max}}$, may both be constructed using the new method with appropriate modifications.

4.4 CONSTRUCTION OF EXTREME ASYMMETRICAL CONCENTRATORS

A special case of the new construction method occurs if the quasi-parabolic curve section is eliminated on one side of the concentrator. If $P_1P'_1$ is eliminated, in order for the concentrator to remain ideal, $P_1P'_2$ must be extended to meet the shadow line $\hat{k}$ to prevent rays from outside $\theta_{\text{max}}$ striking the absorber directly. Using methods similar to those in the last section, such a concentrator may easily be proved to be ideal. The acceptance angle for the new concentrator will be the angle between $\hat{k}$ and the aperture $A$. Because one extreme ray direction runs parallel to the aperture, the concentrator may be described as an Extreme Asymmetrical Concentrator (EAC), as described in Section 3.1.

A practical example of how the new technique may lead to a more desirable concentrator is illustrated in Fig. 4.3. Fig. 4.3(a) shows an ideal EAC after a design by Smith [19]. The properties and advantages of such EAC concentrators for Solar Energy collection have been discussed by Smith [19], Rabl [15] and Mills [27]. However, previous EAC designs in the literature, such as that in Fig. 4.3(a), are difficult to cast or form in one piece, leading to high manufacturing
FIG. 4.3
THREE EAC CONCENTRATORS.
costs. In addition (and perhaps more seriously) they have suffered from a high average number of reflections and, consequently, higher absorption losses in the mirror surface.

The design in Fig. 4.3(b) uses the new technique to create a nearly ideal concentrator in which the receiver tube is fully exposed to all directions within the acceptance angle. In addition, the mirror is kept relatively close to the receiver, so that the latter occupies a relatively larger solid angle from the point of view of rays being reflected from the mirror. Both considerations tend to reduce the average number of reflections.

Note that the profile 4.3(b) is designed for use with a cover tube. Instead of eliminating the mirror profile in the region between the cover tube and receiver tube, the profile has been designed so that the involutes coincide at the cover tube surface (i.e. $S_0 \neq S'_0$). This allows an increase in real aperture area which compensates for the loss in thermal output due to rays passing through the gap between receiver and mirror. This technique was independently originated by Winston [29], and will be utilized in the following Section 4.5.

The right side of Fig. 4.3(b) uses only involute plus Winston-Hinterberger curvature, while the left side uses only an involute plus a section described by Eq. 4.3. Considerable freedom of symmetry and absorber pipe location may be obtained by using various combinations of these
curvatures, however.

The profile Fig. 4.3(c) is similar to 4.3(b), but uses an additional plane section of mirror to reduce both cover plate area and reflection losses from the cover. Although this increases mirror area, it would be a cost-effective measure in a concentrator composed of plastic reflective film bonded to an inexpensive cast substrate and protected by a glass cover plate, and acceptance at low angles of incidence is enhanced. Unlike profile (a), profiles (b) and (c) may be moulded either singly or in nested modules.

Fig. 4.4 shows calculated net flux concentration corrected for absorption losses in a mirror of reflectivity equal to 0.9 as a function of radiation entrance angle. Profile 4.3(c) gives the best performance for low angles of incidence.

Reflection losses from the cover plate and cover tube are not included. A decrease in reflectivity would depress all the performance curves, but increase the performance advantage of the new designs.

4.5 DERIVATION OF REFLECTORS FOR AN EAC

The vector equations of Section 4.3 may be used to construct a reflector for any convex cross-sectional cylindrical receiver. As an example the most obvious and useful particular case to investigate in detail is that of a circular cylindrical pipe, with or without a transparent cover tube.
OUTPUT RESPONSE OF CONCENTRATORS IN FIG. 4.3.

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Instantaneous concentration less mirror absorption losses

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Angle of incidence $\theta$
To obtain a generalized mirror after Section 4.3, each mirror side may be composed of either an involute, an involute plus quasi-parabola, an involute plus quasi-ellipse or an involute, quasi-ellipse, and quasi-parabola. There is no necessity that the curves shown begin at the same point on the circumference of the tube: In fact, in the case shown in figure 4.5, the right and left reflectors, which mathematically begin at points A and A’ shown, do not actually start until they meet at point F, in order to allow room for a cover tube.

For simplicity, and because we wish to form an EAC, the right reflector will be composed of an involute and quasi-parabolic section, which gives a definite rim co-ordinate C(p,q) shown. The left side will be composed of an involute plus quasi-ellipse, the latter using point C(p,q) as a focus. As described in Section 4.3, this results in an EAC.

For the right reflector profile F B C, the dividing point between the involute and quasi-parabolic sections occurs at point B, where a ray from the boundary of $\theta_{\text{max}}$ tangent to the cylindrical receiver intersects the mirror. The lower section AB is defined in the normal involute manner, i.e. any normal to this curve is tangent to the cylindrical receiver. The upper section BC is defined such that a ray at the appropriate boundary of $\theta_{\text{max}}$ which strikes the mirror surface at any point between B and C will be reflected tangentially to the cylindrical absorber. Hence, all rays striking FC from within $\theta_{\text{max}}$ will eventually strike the receiver
after one or more reflections, with the exception of some rays lost through the gap AF.

For the left reflector, the lower section A'B' is an involute as for the right side, but beyond B' a quasi-ellipse curve is constructed such that a ray from $\theta_{\text{max}}$ which both grazes point C and strikes B'C' will be reflected tangentially to the receiver. The acceptance angle limit $\hat{k}$ is determined by the position of C'. Here, the termination point C' has been chosen to make the angle $y$ and $\hat{k} = \theta_{\text{max}}/2$, half the acceptance angle. The result is a concentrator which accepts all rays between $\hat{k}$ and $\hat{k}'$, which has an acceptance angle $\theta_{\text{max}}$.

The variables to be used in the following curve derivations are as follows.

- $R$ = radius of cylindrical receiver
- $\psi$ = angle between R and the negative direction of the y axis measured from the bottom of the receiver
- $P(x,y)$ = coordinates of a point on ABC
- $P'(x',y')$ = coordinates of a point on A'B'C'
- $r$ = distance between P and the point where the tangential ray touches the receiver
- $\psi_A$ = value of $\psi$ at A
- $\psi'_A$ = value of $\psi$ at A'
- $\theta_{\text{max}}$ = angle between $\hat{k}$ and $\hat{k}'$
- $G$ = point where BC' is tangent to the cylindrical receiver
- $A$ = origin of involute AFB
- $A'$ = origin of involute A'FB'.
The coordinates of the mirror may be determined in terms of \( \psi \), \( R \) and \( r \) to be

\[
x(r, \psi) = R \sin \psi - r \cos \psi \quad (4.7(a))
\]
\[
y(r, \psi) = -R \cos \psi - r \sin \psi \quad (4.7(b))
\]

Differentiating with respect to \( \psi \)

\[
\frac{dx}{d\psi} = R \cos \psi + r \sin \psi - \frac{3r}{3\psi} \cos \psi \quad (4.8(a))
\]
\[
\frac{dy}{d\psi} = R \sin \psi - r \cos \psi - \frac{3r}{3\psi} \sin \psi \quad (4.8(b))
\]

Now we take 4.3(a) and 4.3(b), multiply by \( \cos \psi \) and \( \sin \psi \) respectively and add to obtain

\[
\frac{dx}{d\psi} \cos \psi + \frac{dy}{d\psi} \sin \psi = R - \frac{3r}{3\psi} \quad (4.9)
\]

also noting that

\[
\hat{r} = \hat{i} \cos \psi + \hat{j} \sin \psi \quad (4.10)
\]

and

\[
\hat{t} = [\hat{i} \frac{dx}{d\psi} + \hat{j} \frac{dy}{d\psi}] \frac{ds}{d\psi} \quad (4.11)
\]

where \( \hat{t} \) is unit tangent vector to the mirror at \( P \).

**Solution for Involute Profile Section AB**

For the involute, \( \hat{r} \) is normal to \( \hat{t} \), so that the scalar product

\[
(\hat{r} \cdot \hat{t}) = 0
\]

giving

\[
\frac{dx}{d\psi} \cos \psi + \frac{dy}{d\psi} \sin \psi = 0,
\]

\[
R - \frac{3r}{3\psi} = 0,
\]

and

\[
r = R \cdot (\psi - \psi_A) \quad (4.12)
\]
where $\psi_A$ is the $\psi$ value for point A, the point where the mirror would contact the receiver if AF existed.

From 4.7 and 4.12, we obtain

\[ x = R \sin \psi - R \cdot (\psi - \psi_A) \cos \psi \]  \hspace{1cm} (4.13(a))

\[ y = -R \cos \psi - R \cdot (\psi - \psi_A) \sin \psi \]  \hspace{1cm} (4.13(b))

The limits of curve FB in Fig. 1 are

\[ \psi_F \geq \psi \geq (\pi/2 + \theta_{\text{max}}/2) \]

where $\psi_F$ is the value of $\psi$ such that $P(x,y)$ coincides with F.

**Solution for Quasi-Parabolic Profile Section BC**

In this section of the mirror, a ray from the boundary of $\theta_{\text{max}}$ (the direction $\hat{k}$) makes equal angles at the mirror surface with $\hat{r}$. Hence,

\[ (\hat{t} \cdot \hat{k}) = (\hat{t} \cdot \hat{r}) \]  \hspace{1cm} (4.14)

Now from 4.9, 4.10 and 4.11,

\[ \left( \frac{ds}{d\psi} \right) (\hat{t} \cdot \hat{r}) = \frac{dx}{d\psi} \cos \psi + \frac{dy}{d\psi} \sin \psi = R - \frac{3r}{\theta_{\text{max}}} \]  \hspace{1cm} (4.15)

and $\hat{k} = \hat{i} \sin \theta_{\text{max}}/2 - \hat{j} \cos \theta_{\text{max}}/2$

\[ \left( \frac{ds}{d\psi} \right) \left( \hat{t} \cdot \hat{k} \right) = \frac{dx}{d\psi} \sin \theta_{\text{max}}/2 - \frac{dy}{d\psi} \cos \theta_{\text{max}}/2 \]  \hspace{1cm} (4.16)

Combining 4.14, 4.15 and 4.16, and integrating,

\[ x \sin \theta_{\text{max}}/2 - y \cos \theta_{\text{max}}/2 + b = R \psi - r \]

where $b$ is the constant of integration.
Substituting 4.1 and 4.2 into 4.7

\[
R(\sin \psi \sin \theta_{\text{max}}/2 + \cos \psi \cos \theta_{\text{max}}/2) \\
+ r(\sin \psi \cos \theta_{\text{max}}/2 - \cos \psi \sin \theta_{\text{max}}/2) + b = R \psi - r
\]

\[
R \cos(\psi-\theta_{\text{max}}/2) + r \sin(\psi-\theta_{\text{max}}/2) + b = R \psi - r \quad (4.17)
\]

To obtain the integration constant \(b\),

\[
\psi_B = \pi/2 + \theta_{\text{max}}/2
\]

\[
r_B = R \cdot (\psi_B - \psi_A) \quad (4.18)
\]

giving

\[
R \cos \pi/2 + r \sin \pi/2 + b = R(\pi/2 + \theta_{\text{max}}) - r \cdot (\psi_B - \psi_A)
\]

\[
b = R (\pi/2 + \theta_{\text{max}}/2) - 2r
\]

\[
= R (\pi/2 + \theta_{\text{max}}/2) - 2R (\pi/2 + \theta_{\text{max}}/2 - \psi_A)
\]

\[
b = -R (\pi/2 + \theta_{\text{max}}/2 - 2 \psi_A) \quad (4.19)
\]

Substituting 4.19 back into 4.17

\[
r[1 + \sin(\psi - \theta_{\text{max}}/2)] = R[\pi/2 + \theta_{\text{max}}/2 - 2 \psi_A + \psi - \cos(\psi - \theta_{\text{max}}/2)]
\]

and finally

\[
r = R[\pi/2 + \theta_{\text{max}}/2 - 2 \psi_A + \psi - \cos(\psi - \theta_{\text{max}}/2)] \\
\frac{[1 + \sin(\psi - \theta_{\text{max}}/2)]}{[1 + \sin(\psi - \theta_{\text{max}}/2)]} \quad (4.20)
\]

This value of \(r\) may be substituted back into 4.6 to obtain the x and y coordinates of the required reflector profile.
Solution for Quasi-Ellipse Profile

In this section, the right mirror profile is taken to be of quasi-elliptical curvature for convenience. This may be changed to a left profile by changing the sign of each x coordinate obtained from Eq. 4.22 of this section.

In Fig. 4.6, the quasi-ellipse profile is extended from B to C instead of the quasi-parabola in Fig. 4.5. Eq. 4.2 insures that the length C'BSD will be conserved with DS being measured around the perimeter of the receiver. This constant conserved length will be called "k" in the following, but the selection of point D is arbitrary and unimportant as long as DS is always greater than zero.

The wish to have $\theta_{\text{max}}$ lying symmetrically about the y axis introduces another constraint on the curve, namely that C will finish such that CC' meets the y axis at the angle $\theta_{\text{max}}/2$ as shown. Thus, if C(p',q') is known (i.e. if the location of the tip of the opposite reflector is known), ABC may be determined uniquely. By varying the position of C' or A, a series of profiles of similar orientation and magnitude of $\theta_{\text{max}}$ may be built up. These may be compared for mirror length and number of reflections using ray-tracing techniques.

In Fig. 4.6, the coordinates of point D are (-R,0). The length of k is chosen such that the ellipse and involute curves coincide at point B shown.
FIG. 4.6

DIAGRAM FOR CONSTRUCTION OF RIGHT HAND QUASI-ELLIPSE CURVE
We may now examine a point P(x,y) on BC associated with an angle ψ. From Fig. 4.6,

Let \[ C'P + PS + DS = k \]
\[ C'P = \sqrt{(p'-x)^2 + (q'-y)^2} \]
\[ PS = r \]
\[ DS = R \left( \frac{270-\psi}{180} \right) \pi \]
\[ = R\epsilon \text{ where } \epsilon = \frac{1}{2}\pi - \psi \text{ (radius)}. \]

Now \( (C'P)^2 = (k - r - DS)^2 \)
\[ p'^2 - 2p'x + x^2 + q'^2 - 2q'y + y^2 = (k - r - R\epsilon)^2 \]
\[ = k^2 - 2kr - 2kR\epsilon + 2rR\epsilon + r^2 + R^2\epsilon^2 \]
\[ x = R\sin\psi - r\cos\psi \]
\[ y = -R\cos\psi - r\sin\psi \]
\[ p'x = p'R\sin\psi - p'r\cos\psi \]
\[ q'y = -q'R\cos\psi - q'r\sin\psi \]
The \( r^2 \) terms cancel, giving

\[
r[2p'\sin \psi + 2q'\sin \psi - 2(R\epsilon - k)] = k^2 + R^2(\epsilon^2 - 1) - p'^2 - q'^2
+ R(2p' \sin \psi + 2q' \cos \psi - 2k\epsilon)
\]

and

\[
r = \frac{k^2 - p'^2 - q'^2 + R^2(\epsilon^2 - 1)}{p' \cos \psi + q' \sin \psi - R\epsilon + k}
\] (4.22)

Eq. 4.22, as in the quasi-parabolic case, may be substituted into 4.6 to obtain the \( x \) and \( y \) coordinates of the required reflector profile for a given value of \( \psi \). If the curve is meant to be joined to the involute at point \( B \), the values of \( r_B \) and \( \psi_B \) at \( B \) may be obtained from the involute equation. Substitution into 4.22 then yields the correct "k", which is then used to generate other values of \( r \) by varying \( \psi \).
5.1 BACKGROUND OF DIELECTRIC-FILLED CONCENTRATORS

In 1976, Winston [25] described an example of a CPC filled with dielectric material. It was not only found that concentration performance was enhanced by filling the CPC with refractive material, but that total internal reflection (T.I.R.) could be used instead of normal mirror reflection at the CPC walls. Since T.I.R. has no absorption losses associated with it, efficiency of light transmission through the CPC was enhanced.

Fig. 5.1(a) shows the two-dimensional increase in acceptance angle gained by use of refractive material in a typical CPC, while Fig. 5.1(b) shows the three-dimensional acceptance of a typical wide-angle dielectric linear CPC. The latter is interesting because skew rays outside the meridional plane are accepted through wider angles than those within that plane. This widened skew ray acceptance is extremely useful when such a concentrator is used as a stationary solar concentrator, because the apparent solar path tends to follow the acceptance limit boundary fairly well. This increases the number of hours of acceptance for a given magnitude of solid acceptance angle.

In principle, the dielectric CPC represents an extremely attractive possibility as a concentrator for photovoltaics; it is highly efficient, and achieves relatively high concentration levels. In practice, however there
INCREASE IN ACCEPTANCE GAINED BY FILLING CPC WITH REFRACTIVE MEDIUM

FIG. 5.1(a)

DIELECTRIC CPC AND ACCEPTANCE.

FIG. 5.1 (b)

PATH OF SUN, SUMMER SOLSTICE

DIELECTRIC CPC ACCEPTANCE

HOURS FROM NOON

-6 -3 3 6

ELEVATION ANGLE

-90° -45° 45°

W. SOLSTICE

(after Welford and Winston, [1], page 138)
are practical considerations which mitigate against its use. The first is that, even in heavily truncated form, a very large mass of refractive material is required unless the solar cell used is very small. Workers using such concentrators [28] have attempted to use extremely narrow solar cells in order to reduce the mass of modules using these concentrators, but this opposes the trend towards ever larger solar cells, which is in itself based on the need to decrease photovoltaic production costs. The second practical problem is that absorption in the refractive material will be significant unless the concentrator is (again) kept small. This is because all rays must pass through the entire length of the concentrator before reaching the photovoltaic receiver.

At about the same time as Winston published his 1976 paper on dielectric CPC's, the author developed an alternative family of dielectric EAC concentrators for photovoltaics which appear to have certain practical advantages over dielectric CPC's. In the intervening years this approach has grown into a diverse family of configurations which will be described in the remainder of this chapter.

5.2 THE MIRRORLESS PRISM CONCENTRATOR

In this section, the principles of a new family of nearly ideal concentrators will be described which make use of total internal reflection within a wedge-shaped body of transparent refractive material to achieve concentration.
The new concentrators called Prism Concentrators, are generally used with a mirror to augment concentration. The prism may be used as a primary concentrator, or a second-stage concentrator in combination with a primary condenser. Both photovoltaic and thermal versions of the prism are possible.

Fig. 5.2 shows a two-dimensional schematic of a simple mirrorless prism concentrator. Let the top surface, AB, be taken to be the entrance aperture of the device and the short face BC be called the exit aperture or interface.

The operation of the device is as follows. Light from a distant source strikes AB at a glancing angle \( h_1 \). The ray then refracts at an angle of depression \( h_2 \) from AB. Upon encountering the bottom surface of the prism, however, two possibilities are evident. Either the ray will be transmitted through AC and lost, or, if the original angle \( h_1 \) was sufficiently small, be totally internally reflected away from AC back toward AB at a glancing angle of \( h_4 \) as shown in Fig. 5.2. If total internal reflection (T.I.R.) occurs at the bottom surface on the first pass, all subsequent encounters with AB and AC will result in T.I.R. Thus, the ray is effectively trapped between the two plane surfaces AB and AC and cannot leave the prism without encountering the exit aperture BC. It is here that the receiver of radiation is placed.
Prism concentrator in two dimensions
Total internal reflection will occur at AC if
\( h_3 \leq \cos^{-1} \left( \frac{1}{n} \right) \), where \( n \) is the refractive index of the transparent prism material. This immediately restricts the range of directions of complete ray collection to
\[ \theta < h_1 < h_{1\text{max}} \]

where
\[ h_{1\text{max}} = \cos^{-1}\{n \cos[\cos^{-1}(\frac{1}{n}) + \alpha]\} \quad (5.1) \]

The radiation within this range of angles of incidence all arrives at the exit aperture, save for reflection and absorption losses. Such radiation may therefore be interpreted as having been accepted by the prism, and \( h_{1\text{max}} \) may be called the acceptance angle of the two-dimensional prism concentrator.

It follows that instantaneous concentration, \( X \), in a loss-free prism is given by \( \frac{AD}{BC} \), or
\[ X = \sin h_1 / \sin \alpha \quad (5.2) \]

Peak concentration is, therefore
\[ X_{\text{max}} = \sin h_1 / \sin \alpha \quad (5.3) \]
providing that \( h_{1\text{max}} \leq 90^\circ \). If \( h_{1\text{max}} > 90^\circ \) (\( h_{1\text{max}} \) may be defined as exceeding 90° when incident rays are travelling toward the prism apex) peak concentration occurs when \( h_1 = 90^\circ \), giving
\[ X_{\text{max}} = 1 / \sin \alpha \quad (5.4) \]
A 6° STATIONARY MIRRORLESS PRISM CONCENTRATOR
The prism as described is certainly not an ideal concentrator because it does not satisfy the edge-ray principle and because it accepts light from outside the nominal acceptance angle [see Section 2.6]. However, the prism will be shown to satisfy Eq. 2.34 in both two and three dimensions, and it consequently qualifies as an ideal concentrator of limited exit angle [see Section 2.7]. If the exit angle is relatively large and lies so that glancing angle exit rays are preferentially prohibited, performance will be close to that of the hypothetical ideal concentrator. It turns out that this is the case. Proofs of the satisfaction of Eq. 2.34 are described in Section 5.3 and Section 5.4 using the "air gap mirror" concentrator variant as the case in point. Similar proofs may be derived for the mirrorless case in this section if required.

It was stated in the last paragraph that some radiation is accepted outside $h_{1\text{max}}$ in a prism concentrator. There are, in fact, two mechanisms for this. The first is the phenomenon of partial internal reflection, which allows a fraction of incoming radiation from angles of $h_1$ slightly greater than $h_{1\text{max}}$ to be accepted (it was shown in Section 2.8 that this is incompatible with ideality). The second is the acceptance of radiation which strikes the exit aperture either directly without being subject to T.I.R. at any prism surface. This extra acceptance of radiation will be illustrated more clearly using the example of a prism with flat mirror (Section 5.3).
FIG. 5.4

YEARLY OUTPUT OF PRISM IN FIG. 5.3 COMPARED TO YEARLY UNCONCENTRATED CELL OUTPUT (CORRECTED FOR LOSSES)
For solar work, it is generally advantageous to augment the mirrorless prism with an auxiliary mirror in one of the configurations to be described in the following sections. Nevertheless, a stationary solar concentrator such as depicted in Figure 5.3 would give a useful performance for a photovoltaic receiver. This could be useful in situations where higher performance at solstices than equinox is required, for example in a continental climate with heavy heating and air conditioning loads in alternate solstices. However, it is probable that a design of this sort would suffer from problems such as severe dust collection and the necessity for small photovoltaic sizes. The yearly response of this concentrator has been investigated with the help of Dr. Harting, and a typical yearly performance curve adjusted for optical losses is shown in Fig. 5.4.

Mirrorless prisms may be used perhaps more advantageously as secondary concentrators in two-stage designs such as the maximally concentrating designs in Chapter 6.

5.3 Two Dimensional Prisms Faced with One Flat Mirror; The Air Gap Mirror Prism.

The bottom surface of the prism in Fig. 5.2 may be faced by a conventional mirror either very close to or optically bonded to the face AC. In such a system, T.I.R. is not required at the bottom surface during the first pass of a ray across the prism because the flat mirror will return the ray anyway. Total internal reflection is still required upon the return to the top surface AB, however. The consequence of this is that a new acceptance angle $h_{\text{max}}$ may be defined
Ideal peak concentration vs prism apex angle

$\alpha = 1.5$

FIG. 5.5
for the flat mirror concentrator such that

\[ h_{\text{max}} = \cos^{-1}(n \cos[\cos^{-1}\left(\frac{1}{n}\right) + 2 \alpha]) \]  

(5.5)

which is greater than the corresponding mirrorless prism acceptance angle. Peak concentration is also improved, but is still given by Eq. 5.3. Peak concentration vs. prism apex angle is plotted in Fig. 5.5.

All mirrors which rely on physical effects other than T.I.R. to produce specular reflection incur some absorption loss, which may range from 4-30% for the range of materials considered suitable for mirror coatings. Prism concentrators exhibit a relatively high number of reflections compared to other types of stationary concentrator, as is demonstrated in Section 5.4. They are, consequently, vulnerable to bottom surface mirror absorption, particularly in higher concentration, narrow apex angle variants.

A feature of this prism design which reduces such mirror loss is the "air gap mirror", shown in Fig. 5.6. Separation of the mirror from the prism absorptive reflection will be required. All subsequent top and bottom surface reflections will be of the T.I.R. type. In addition, for incident angles of \( h_1 \) less than the mirrorless acceptance angle, i.e.

\[ h_1 \leq \cos^{-1}(n \cos[\cos^{-1}\left(\frac{1}{n}\right) + \alpha]), \]

T.I.R. will take place at the bottom surface on the first pass, and no mirror absorption will occur. Concentration loss due to enlargement of the exit aperture by the air gap can be minimized by keeping the gap as small as possible -
typically, 0.5 mm or less.

**Ideality of Prism Concentrator**

We may easily prove that the prism concentrator satisfies the requirements of an ideal concentrator of limited exit angle. For this to be the case, the prism must satisfy the relationship derived in Section 2.2,

\[
X_{\text{max}} = \frac{n \sin(\theta_{\text{max}}/2)}{\sin(\theta_{\text{max}}/2)} \text{ if } \theta_{\text{max}} \leq 90^\circ \quad (5.6(a))
\]

or

\[
X_{\text{max}} = \frac{n \sin(\theta'_{\text{max}}/2)}{\sin(\theta_{\text{max}}/2)} \text{ if } \theta_{\text{max}} \geq 90^\circ \quad (5.6(b))
\]

where \(\theta_{\text{max}}\) is the acceptance angle for radiation at the entrance aperture and \(\theta'_{\text{max}}\) the exit angle for radiation at the exit aperture.

Now \(\theta_{\text{max}} = h_{\text{max}} = \cos^{-1}\left\{n \cos[\cos^{-1}(\frac{1}{n}) + 2\alpha]\right\}\) and it can be determined from Fig. 5.2 that

\[
\theta'_{\text{max}} = h'_{\text{max}} = \cos^{-1}\left(\frac{1}{n}\right) + \alpha. \quad (5.7)
\]

Comparing equations 5.3, 5.4, 5.6 and 5.7, it becomes clear that there remains only to prove that

\[
\frac{n \sin h_{\text{max}}}{\sin^2(h_{\text{max}}/2)} = \frac{1}{\sin \alpha}
\]

L.H.S. =

\[
= \frac{n \sin(h_{\text{max}})}{1-\cos(h_{\text{max}})}
\]

\[
= \frac{2n \sin[\cos^{-1}(\frac{1}{n})+\alpha]}{1-n[\cos^{-1}(\frac{1}{n})+\alpha]-\sin[\cos^{-1}(\frac{1}{n})+\alpha]\sin \alpha}
\]

\[
= \frac{2[(n^2-1)^{3/2}\cos \alpha + \sin \alpha]}{1+2(n^2-1)^{3/2}\sin \alpha \cos \alpha + \sin^2 \alpha - \cos^2 \alpha}
\]

\[
= \frac{1}{\sin \alpha} = \text{R.H.S.}
\]
Hence, a two-dimensional prism is an ideal concentrator for a limited exit angle. Using Eq. 2.34 and computer ray-tracing, this is also shown in Section 5.4 to be true in the three-dimensional case.

As with the mirrorless prism, radiation is also accepted outside $h_{\text{max}}$ by partial internal reflection and by striking the receiver directly before encountering the bottom surface of the prism. In air-gap mirror prisms, however, some additional radiation is also accepted outside $h_{\text{max}}$ by reflections from the bottom mirror directly striking the receiver. This is shown in Fig. 5.7, in which a 5° prism with losses calculated by computer is compared with two prototypes, one of which had shape distortions at the surface and one which had an unsatisfactorily polished surface. All three accept significant radiation outside $h_{\text{max}}$, and it can be demonstrated by full three-dimensional computer calculations (see next section) that such radiation may improve yearly energy collection significantly.

It is clear from a study of Section 2.2 that the greater the exit angle, the more closely the ideal transformer prism comes to satisfying the requirements for a true ideal concentrator. Can the performance of the two-dimensional prism, therefore, be improved further?

The answer is yes in principle, but only slightly in practice. If one were to add a second stage CPC (Fig. 5.8) mirror concentrator to the exit aperture within the medium,
FIG. 5.7.
Concentration vs radiation incidence angle for transverse case, 5° prism.

- prediction
- rough surface
- distorted surface

Inclination due to finite source size and diffuse + circumsolar radiation

Excess due to shape distortion of surface

Region of partial acceptance

Critical angle

Output

Degrees from critical angle
an additional concentration factor, $X_{\text{add}}$, could be achieved where

$$X_{\text{add}} = \frac{1}{\sin h_{4,\text{max}}}.$$ 

Since $h_{4,\text{max}}$ lies typically in the range $60^\circ \leq 53^\circ$, for a two-dimensional concentrator, the value of $X_{\text{add}}$ ranges from about 8% to 25% with figures of 12% and 23% being typical for concentrators of stationary tilting configurations, respectively. Although this seems worthwhile at first glance, the concentration increase is achieved largely at glancing angles of incidence and reflection losses from the receiver surface (usually a photovoltaic) may be severe. This has the effect of reducing overall module efficiency. Further, the performance increase is achieved at the cost of a curved refractive segment of quite significant extra mass and requires curved metallized mirrors with reflection losses of perhaps 6–15%. Third, and most important, the exit angle $h_{4,\text{max}}$ is not constant for rays entering a three-dimensional, linear prism at different skew angles. In fact, $h_{4,\text{max}}$ tends to increase for skew rays, so that a second stage CPC designed for noon would reject rays at other times of the day which would have been accepted by the basic prism. In practice, the increases in concentration obtained by this method are not considered by the author to be worth the extra complication, material, and overall efficiency losses incurred.
FIG. 5.8

PRISM WITH SECOND-STAGE CPC CONCENTRATOR

\[ 2h_{4_{\text{max}}} = \text{CPC acceptance} \]
Radiation Distribution at Exit Aperture

For photovoltaic applications, the distribution of radiation at the exit aperture of a prism concentrator is particularly suitable. Not only is the radiation incident angle to the solar cell surface restricted to angles where reflection from the solar cell surface is not serious, but the intensity distribution of radiation across the solar cell surface is very uniform, a factor which can be important with some photovoltaics. This uniformity is due both to the cross-over nature of light transmission through the prism and the fact that radiation from any given direction of the sky enters the prism along the entire distance from the exit aperture to the prism apex. Two-dimensional distribution for a 21° prism were calculated for two different incident angles \( h_1 \) and are shown in Fig. 5.9. These would be smoothed out in practice due to the finite solar disk size and prism irregularities. These may be compared with the rather more extreme intensity variations experienced by CPC concentrators [27].

The prevention of "hot spots" at the exit aperture is particularly important at high concentration levels, and is of some advantage when the prism is used as a second-stage concentrator as in Chapter 6.
IDEAL TRANSVERSE RADIATION DISTRIBUTION ACROSS A 21° PRISM FOR TWO INCIDENT ANGLES, \( h_i = 45° \) AND \( h_i = 90° \).
5.4 THREE-DIMENSIONAL PRISM PERFORMANCE CALCULATIONS

The air gap mirror prism has been extensively analyzed in three dimensions, and a number of computer programs have been written to model expected prism performance. These require detailed ray-tracing subroutines.

In order to trace a ray path between two prism surfaces A and B as shown in Fig. 5.10, consider a ray leaving surface A within the prism. Such a ray can be parameterized by an elevation angle $h_A$ and an azimuth angle $A_A$ as shown. This ray may be projected upon a two-dimensional surface normal to the concentrator linear axis. The ray projection may also be parameterized by an elevation angle $h_p$ such that

$$h_p = \tan^{-1}(-\tan h_A / \cos A_A) \text{ if this is } \geq 0 \quad (5.8(a))$$

or

$$h_p = \tan^{-1}(-\tan h_A / \cos A_A) + \pi \text{ otherwise } \quad (5.8(b))$$

Using the above equation 5.8, the azimuth and elevation angles $A_B$ and $h_B$ of the reflected ray leaving surface B may be found to be

$$A_B = \tan^{-1}[-|\cos h_B \cdot \tan A_A| / \cos (h_p - \alpha)] \quad (5.9(a))$$

and

$$h_B = \tan^{-1}[|\tan (h_p - \alpha) \cdot \cos A_A|] \quad (5.9(b))$$

where $\alpha$ is the rotational angle between the planes A and B and is equal in magnitude to the prism apex angle, being positive or negative as the sense of the problem demands.
FIG. 5.10

RAY TRACING DIAGRAM FOR SECTION 5.4.

RAY PATH

PROJECTON OF RAY PATH

PLANE A

PLANE B
Equations 5.9(a) and 5.9(b) may be used as a subroutine in a computer program to trace rays through any prism, provided care is taken to avoid division by zero which can happen for certain values of ray direction.

In order to begin the process of ray-tracing, the direction of the initial ray striking the prism top surface must be determined first. Fig. 5.11(a) shows a prism tilted at an angle $\beta$ to the horizontal beneath the celestial hemisphere. Using the cosine rule of spherical geometry,

$$\cos(90-h_s) = \cos(90-\phi) + \cos(90-\delta) + \sin(90-\phi) \sin(90-\delta) \cos t$$

giving

$$\sin h_s = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t . \quad (5.10(a))$$

Also, using the sin rule,

$$\frac{\sin(90-\delta)}{\sin A_s} = \frac{\sin(90-h_s)}{\sin t}$$

$$\sin A_s = \frac{\cos \delta \sin t}{\cos h_s} \quad (5.10(b))$$

where $h_s$ and $A_s$ are the elevation and azimuth of the incident ray

$h_1$ and $A_1$ are the elevation and azimuth angles of the incident ray with respect to the prism top surface

$\delta$ is the declination angle

$\phi$ is the latitude angle

$t$ is the hour angle, equal to the number of hours from noon times 15°.
FIG. 5.11(a)

Celestial sphere construction to calculate true elevation and azimuth from latitude, declination, and time of day.

FIG. 5.11(b)

Construction to find $h_1$ and $A_1$ in a reference frame tilted by $\beta$. 
Since equations 5.10(a) and 5.10(b) are developed using the true elevation and azimuth, it is necessary to rotate the frame of reference in order to determine the azimuth and elevation of an incident ray with respect to the prism top surface. In Fig. 5.11(b), such a rotation has been carried out and it is found, again by using spherical geometry, that

\[ \sin h_1 = \sin(\phi+\delta) \sin \phi + \cos(\phi+\delta) \cos \phi \cos \phi \cos t \] (5.11(a))

and

\[ \sin A_1 = \frac{\cos \delta \sin t}{\cos h_1} \] (5.11(b))

where \( \delta \) is the angle of tilt of the prism to the horizontal.

Using Snell's law, the elevation and azimuth angles of the path of the refracted ray having the top surface within the prism is given by

\[ h_1 = \cos^{-1}\left[\frac{1}{n} \cos h_1\right] \] (5.12)

and

\[ A_2 = A_1 \] (5.13)

The subsequent path of the ray may be determined by using \( h_2 \) and \( A_2 \) as \( h_A \) and \( A_A \) in Eq. 5.8.

**Prism Path Length**

The path length through a prism may be determined using the geometrical construction in Fig. 5.12, which is similar to constructions used by Williamson [3] for the analysis of "V"-trough concentrators. In this construction, mirror images of the prism are successively placed next to each other in the manner shown so that the ray path may be
FIG. 5.12

MIRROR IMAGE RAY-PATH CONSTRUCTION
represented as a straight line in place of the normal zig-zag path.

In the diagram in Fig. 5.12, a ray refracts from the prism top surface at point P. After travelling a distance PQ, it encounters the exit interface at Q. The two-dimensional projection shown is adequate to determine the true three-dimensional path length because $h_2$ and $A_2$ are constant along the three-dimensional path; hence, the 3-D path is greater than the 2-D by the constant factor $\left[ \cos h_p / (\cos h_2 \cos A_2) \right]$.

The coordinates of points $B_1$, $B_2$, $B_3$, ... $B_n$

$B_1 = [0,0]$  
$B_2 = [2L \sin^2 \alpha, -2L \sin \alpha \cos \alpha]$  
$B_3 = [2L \sin \alpha (\sin \alpha + \sin 3\alpha), -2L \sin \alpha (\cos \alpha + \cos 3\alpha)]$  
$B_n = [2L \sin \alpha (\sin \alpha + ... + \sin (2n-3)\alpha), -2L \sin \alpha (\cos \alpha + ... + \cos (2n-3)\alpha)]$  
where $L = AB_1$, the prism top surface length.

The equation of $B_1B_2$  
\[ y = -x \cot \alpha \]

of $B_2B_3$:  
\[ y = -\cot 3\alpha (x - 2L \sin^2 \alpha) - 2L \sin \alpha \cos \alpha \]

of $B_3B_4$:  
\[ y = -\cot 5\alpha [x - 2L \sin \alpha (\sin \alpha + \sin 3\alpha)] - 2L \sin \alpha (\cos \alpha + \cos 3\alpha) \]

and of $B_{n-1}B_n$:  
\[ y = -\cot [(2n-3)\alpha] [x - 2L \sin \alpha (\sin \alpha + ... + \sin (2n-3)\alpha)] - 2L \sin \alpha (\cos \alpha + ... + \cos (2n-3)\alpha) \]

(5.14)
FIG. 5.13

Path length through a 12.5° prism at equinox as a function of time of day and top surface entry distance from apex.

Path length (Bottom surface length=1.0)

Ray entrance distance from apex. (Top surface length=1.0)
The equation of \( PQ \) is given by

\[
y = \tan h_p (x - B_1P) \tag{5.15}
\]

Hence, if \( B_1P \) and \( h_p \) are known, \( Q \) may be determined by equating equations (5.14) and (5.15). However, the situation is somewhat more complicated than this in practice, because the correct value of \( n \) must be determined in Eq. (5.14). This may be done (while computing) by providing appropriate limits on the intersections allowed. Since both \( Q \) and \( P \) are known, the length of \( PQ \) is easily found.

The path length may be calculated by computer for rays entering at different points on the prism top surface for various times of day and year. A typical result is shown in Fig. 5.13, which shows the path length through a 12° prism at equinox as a function of top surface ray entry position. The interesting "quantization" of the curve is caused by the integral number of reflections; as is evident from Fig. 5.12, only certain values of path length gradient in Fig. 5.13 are possible for any given incident ray angle. Fig. 5.14 shows the average path length through the same prism as a function of time of day at equinox.

Prism Acceptance in Three Dimensions

Initial programs written by the author led to the performance graphs given in the paper "Ideal Prism Solar Concentrators" contained in Appendix I(b). These have now been supplanted by much more sophisticated programs written by
FIG 5.14

AVERAGE PATH LENGTH vs. TIME OF DAY
FOR A 12° PRISM AT EQUINOX.
Dr. Harting and the author. The results of these programs will be referred to using graphical presentation throughout the remainder of this Chapter, and performance calculations for a number of prisms is contained in Appendix II.

Both the acceptance and exit angles of a prism concentrator are complex in three dimensions. As in the case of dielectric CPC's, acceptance is enlarged for skew rays and this leads to a larger average number of hours of solar collection. This shows up in Section 5.5, where an extended mirror prism, which does not share this advantage, exhibits a lower yearly average energy collection than an air gap mirror prism of the same two-dimensional acceptance.

In Fig. 5.15, the acceptance and exit angles of a 7° air gap prism are presented on a polar diagram.

The curvatures shown are a function of prism apex angle and refractive index. The boundaries shown are based upon the Total Internal Reflection property, so that some partial acceptance occurs outside these limits.

**Conservation of Étendue**

For the prism to be an ideal concentrator in three dimensions, it must obey Eq. 2.34. To test this, a program was prepared which followed a large number of rays through the prism. These rays, which were homogeneously distributed throughout the acceptance angle and across the entrance aperture, were found to be homogeneously distributed through-
FIG. 5.15

Entrance and exit solid angles for 7° prism
out the exit solid angle though they were mixed thoroughly on the way through the prism. Since there were nearly 3000 rays involved for each prism tested, this is a convincing demonstration of conservation of étendue, and, indeed, the Second Law of Thermodynamics. Since the airgap mirror prism obeys Eq. (2.34), it must be considered to be an ideal concentrator for a limited solid angle in three dimensions. This would also be true for the mirrorless case.

Complete Prism Acceptance Array

For practical performance calculations, partial internal reflection and the possibility of direct illumination of the receiver by rays from outside $h_{\text{max}}$ must be taken into account. The response of a prism for uniform hemispherical illumination has been calculated and one result, that for a $5^\circ$ air gap mirror prism, is presented in Fig. 5.7. This figure takes into account a practical absorption coefficient of .992 cm$^{-1}$ in the prism material and .9 mirror absorption.

In order to simulate the response of a prism to the solar motion, for each prism to be ray-traced and this was stored in computer memory. A "source motion" could then be simulated by using a separate program to assess the parts (directions) of the array which were "active" at any one time. All the detailed ray-tracing calculations in the remainder of this thesis were carried out in this manner.
Stationary Prism Solar Concentrator Performance

Clear sky performance for an air gap mirror prism which is fixed for the entire year can be calculated using a program developed by Dr. Harting and the author. The tilt of the prism is first selected by the computer as the optimum at one solstice. Then, for any day of the year the hourly variation in output compared to a standard encapsulated flat cell array is calculated. Next, the program calculates the daily output of the prism corrected for surface reflection losses and absorption losses, taking into account attenuation caused by air mass and variations in earth-sun distance. The yearly output advantage of the concentrator compared to the standard flat photovoltaic standard is then determined.

Fig. 5.16 shows a typical output for a 16° air gap prism for several times of day. This is in terms of the noon equinox output of a flat solar cell encapsulated in a material of refractive index 1.5, and inclined at the latitude angle \( \phi \) from the azimuth. Fig. 5.17 shows the same thing, except the prism has been placed in the "vertical" orientation instead of the "horizontal" orientation of Fig. 5.16. A system composed of both orientations will have a performance which is a linear combination of the two. This is shown in Fig. 5.18, which compares the total daily prism output for both orientations and a typical linear combination of them.
Figure 5.16

16° Stationary prism concentrator

"horizontal orientation"

Hourly clear sky output corrected for losses
n = 1.47 air gap mirror reflection = 90%
cell size 2 cm
FIG. 5.17

16° Stationary prism concentrator

"vertical orientation"

hourly clear sky output corrected for losses
n = 1.47 air gap mirror reflection = 90°
cell size 2 cm

Relative corrected clear sky output

Day of year

0 100 200 300 400 500
Summer solstice Winter solstice Summer solstice
16° STATIONARY PRISM CONCENTRATOR

FIG. 5.18

"Horizontal" orientation

"Vertical" orientation

System of 2N "Vertical" and N "Horizontal" modules.

Flat array output

Total daily clear sky output corrected for losses, air mass and earth-sun distance.

$n = 1.47$ air gap mirror reflection $= 90\%$

Cell size $2\,\text{cm}$

Day of year
As the program is written, the prism tilt, if computer selected, is optimized on the basis of maximum daily energy output at one solstice. Note that the automatic tilt selection does not necessarily give the best overall yearly performance, because if the acceptance angle is too small, the concentrator (optimized on one solstice) runs out to zero performance before the opposite solstice is reached. However, if the tilt is present so that the first solstice occurs in the partial acceptance region outside $h_{\text{max}}$, collection may be extended to the opposite solstice, thereby increasing total yearly energy collection. This is demonstrated in Fig. 5.19, and shows the possible value of the partial acceptance region. The effect of incorrect tilt is shown in the $\beta = 170^\circ$ case as a premature drop in performance at summer solstice.

The yearly performance of stationary prism concentrators is relatively insensitive to apex angle, as is shown in Fig. 5.20. However, the yearly performance distribution is significantly affected by prism apex angle. A factor also to consider is that the wider angle prisms use less material. The yearly performance distributions of a number of stationary prisms are given in Appendix II.

**Tilting Prisms**

As in mirror concentrators, the average concentration performance of prism concentrators can be increased by occasional tilt adjustments. A prism program was set up to
FIG. 5.19

13° Stationary prism concentrator
yearly output for three different tilts
horizontal setting, cell size 2 cm
n=1.47 mirror reflection = 90°
FIG. 5.20

Acceptance angle becoming too small

90% tilt criterion
"Horizontal" stationary
"Vertical" stationary

Yearly performance of wide angle prisms vs prism apex angle for 2 cm solar cell width clear sky only

Encapsulated flat array
'tilt' a given prism concentrator according to a predetermined criterion. The criterion used in computing the following results was that the tilt would occur when total daily performance dropped to 90% of the maximum daily output possible on that day. This ensures that about 95% of possible performance is obtained on average. The number of tilts per year required to maintain this performance level is presented in Fig. 5.21.

The yearly output from a given number of cells mounted in prisms of varying apex angle is shown in Fig. 5.22, alongside the output of the same solar cells mounted in a conventional stationary flat array. The effect of absorption in the larger (6 cm) prism is apparent.

Fig. 5.23 and 5.24 show the hourly loss-corrected output of a 7° tilting prism in the two possible orientations, while the yearly outputs for the same prism are shown in Fig. 5.25. Note that by choosing the appropriate orientation, a choice of a flat or 'sinusoidal' yearly clear sky output is possible, although it is possible that seasonal weather changes could obscure this pattern in practice. The tilts, of course, need not be carried out on a specific day; it is only the tilt frequency that is important. For this reason, the concentrators in a large array could be tilted sequentially, allowing the labour required for such an operation to be better utilized. In such a situation, the "serrated" output waveform would also be averaged out to a
FIG. 5.21

Tilt adjustment vs apex angle

$n = 1.47 , r = 90°$
clear sky only, 2cm solar cell
FIG. 5.22

Yearly performance of small angle prisms vs prism apex angle
Tilt criterion: 90% of best performance clear sky only

Relative output

2 cm cell

6 cm cell

Encapsulated flat array

Prism apex angle

5° 6° 7° 8°
Hourly clear sky output, corrected for all losses, air mass and earth-sun distance.

- Cell size = 2 cm
- Air gap mirror reflection = 90%
- Tilt criterion = 10% below maximum possible output.
7° TILTING PRISM CONCENTRATOR
"VERTICAL" ORIENTATION

Tilt times

Hourly clear sky output, corrected for all losses, air mass and earth-sun distance.

n = 1.47 air gap mirror reflection = 90% cell size = 2 cm
Tilt criterion = 10% below maximum possible output.
FIG. 5.25
7° TILTING PRISM CONCENTRATOR

Total daily clear sky output corrected for all losses, air mass and earth-sun distance.

- $n = 1.47$ air gap mirror reflection = 90%
- Tilt criterion 10% below maximum possible output.

Conventional flat array
smooth total array response. The computer calculated performance of other tilting prisms is given in Appendix III.

5.5 THE EXTENDED MIRROR PRISM

Fig. 5.26 shows another prism configuration which uses a mirror. Unlike the air gap configuration, however, which uses a mirror to extend the acceptance angle and peak concentration of a right-angled mirrorless prism, this new prism arrangement uses a mirror to extend the aperture of an isosceles mirrorless prism. This is done by using the bottom side of the prism for light collection as well as the top. Here radiation from the region [AD] is reflected upon the bottom surface by the mirror shown, so that all of this radiation arrives at the bottom surface within the mirrorless prism acceptance angle. Thus, the acceptance angle is unchanged by addition of the mirror, but the aperture and concentration are doubled. The dashed line in Fig. 5.26 shows how an identically performing air gap mirror prism compares in shape and size with the extended mirror prism (EMP).

We need not define the mirror shape in order to determine the relationship between the two prism elements. Let the prism element in the EMP have an acceptance angle given by

\[ h_{\text{max}} = \cos^{-1} \left( n \cos \left[ \cos^{-1} \left( \frac{1}{n} \right) + \alpha_1 \right] \right), \]

(where \( \alpha_1 \) is the mirrorless prism apex angle).

The mirror is defined as having the property that all light from within \( h_{\text{max}} \) which crosses AE will be accepted by the
prism bottom surface. Hence, \( h_{\text{max}} \) is the acceptance angle right across the total aperture.

For the corresponding air gap mirror prism (dashed outline)

\[
h_{\text{max}} = \cos^{-1}\left\{ n \cos \left[ \cos^{-1}\left(\frac{1}{n}\right) + 2\alpha_2 \right] \right\}
\]

where \( \alpha_2 \) is the air gap mirror prism apex angle.

This immediately gives,

\[
\alpha_2 = \alpha_1/2,
\]

i.e. the apex angle of the extended mirror prism element is twice that of the air gap prism it replaces.

The practical advantages of such a configuration are self-evident. First, only about half of incoming rays ever strike the mirror at all, reducing the sensitivity to mirror absorption. Second, refractive material required is much reduced. Third, path length through the refractive material is halved, reducing absorption losses. A major disadvantage of the configuration is the requirement for a curved mirror, but the curvature required is not severe as can be seen in Fig.

The curvature developed by the author for the EMP is as follows. From A to B in Fig. 5.26 the curvature is parabolic with focus at E and optic axis parallel to the acceptance
angle limit of the prism bottom surface. This ensures that all rays which cross BE are accepted. Between C and D the curvature is also parabolic, again with focus at E but now with an optic axis parallel to the bottom surface acceptance angle. This ensures that all rays from within the top surface acceptance angle which strike CD are reflected through BE, and are therefore accepted. The mirror section BC may be flat and all rays striking it from within the top surface acceptance angle will cross BE and be accepted. Hence, all rays from within the top surface acceptance angle are accepted; the EMP has the same acceptance angle as the dashed prism in Fig. 5.26. From Fig. 5.26, the aperture ratio of the EMP may be shown to be

\[ \frac{1}{2} \frac{\sin \alpha_1}{\sin \alpha_2} \left( \text{contribution from prism top face} \right) + \frac{1}{2} \frac{\sin \alpha_1}{\sin \alpha_2} \left( \text{contribution from AE} \right) = \left[ \frac{\sin \alpha_1/2}{\sin \alpha_2} \right]^{-1} = \left[ \sin \alpha_2 \right]^{-1} \]

which is also identical to the aperture ratio of the air gap mirror prism equivalent in Fig. 5.26.

There is no reason why the mirror should be stopped at point A, of course. At the suggestion of Dr. Giutronich, the investigation of prisms with mirrors extended beyond A has been carried out. The mirror section AG in Fig. 5.26 would be parabolic with focus at F if reflection losses from the top surface of the prism were to be disregarded. This would give the widest possible aperture consistent with the requirement that rays from the direction of \( h_{\max} \) be incident upon EF within the acceptance angle limit. However, in practice,
a focus at E may lead to lower reflection losses. Extension of the mirror beyond A reduces acceptance angle as well as increasing concentration.

The loss-corrected performance of EMP's of varying mirror length and prism apex angle have been calculated using computer techniques similar to those employed in Section 5.4. In Figures 5.27 and 5.28, the performance of EMP's is compared to that of air gap mirror prisms of the same nominal acceptance. It can be seen that performance is substantially the same, with the EMP having slightly lower performance at solstice due to its smaller acceptance for skew rays.

An interesting point to come out of investigation of comparisons of EMP performance is that performance is dependent upon mirror length but *almost independent of prism apex angle* (see Fig. 5.29). In extreme examples using very short prisms, the prism mainly functions as a change of refractive index rather than a geometrical concentrating device in the normal sense. This discovery could have strong economic implications, since the prism is more costly to construct than curved mirror. However, the existence of highly performing maximally concentrating collectors of minimal mirror area (Chapter 6) would appear to restrict the use of EMP's to stationary concentrator and second-stage concentrator applications, where EMP mirror area is comparatively low.
COMPARISON OF SIMPLE AND EMP PRISM OUTPUT

Ref. coeff. = 88%

16° SIMPLE PRISM

32° EMP

DAYS FROM SUMMER SOLSTICE

FIG. 5.27
FIG. 5.28

COMPARISON OF OUTPUTS OF TILING SIMPLE AND EXTENDED MIRROR PRISMS

TILT CRITERION = 0.9 OF MAXIMUM PERFORMANCE ON THAT DAY

REFLECTION COEFFICIENT = 0.88
MIRROR AREA VS YEARLY OUTPUT FOR EMP PHOTOVOLTAIC CONCENTRATORS

FIG. 5.29
5.6 ADVANTAGES OF PRISM CONCENTRATORS FOR SOLAR ENERGY COLLECTION

Prism Concentrators exhibit a number of unique characteristics compared with conventional imaging and non-imaging concentrators for solar energy. Many of those characteristics may be turned to economic or operational advantage. A number of these characteristics are described below.

I. Low Material Usage and Low Module Weight

Prism concentrators accept radiation from the side and trap it within a relatively thin piece of refractive material. This is quite unlike the behaviour of normal trough concentrators, in which radiation must proceed from the entrance aperture through the complete height of the trough to the receiver at the exit aperture. A prism module is only required to be slightly thicker than the solar cell width used. To illustrate this point, a prism and a truncated dielectric CPC are compared in Fig. 5.30. Both achieve about the same (tilting) concentration performance in loss-free form. It can be seen that the dielectric material required has been cut more than five times by use of the prism. In this case, the prism would have to be tilted
SIZE COMPARISON OF DIELECTRIC CPC AND PRISM OF SIMILAR PERFORMANCE

FIG. 5.30
more often than the CPC, but never more than 14 times a year.

II. Manufacturing Simplicity

The flat sides of prisms lend themselves to many different forms of manufacture. Prisms may be extruded or moulded from acrylic, or may be constructed of inexpensive oil confined by glass or acrylic plates. High temperature versions are possible using silicone oil and glass, and are discussed in Chapter 6.

III. Low Absorption Losses

Because the average path length through the prime is only about 0.6 times the prism length, path length is much less than in a corresponding dielectric trough (Fig. 5.30) and large solar cells may be accommodated without undue absorption.

IV. Flexibility of Operation in Tilting Mode

As shown in Section 3.3, output of a prism is a function of tilting frequency. This allows some compensation for excessive demand or poor weather.

V. Flexibility of Operation in Stationary Applications

As shown in Fig. 3.8 and Section 3.3, the yearly output of a prism array may be tailored to match likely demand by using two distinct orientations, and dividing the total number of concentrators between the two orientations appropriately. In applications such as remote microwave transmitting
PRISM ARRAYS ON BUILDINGS.

FIG. 5.31
facilities, this could reduce the battery storage requirement very substantially. Fig. 5.31(a) shows how prisms might be arranged in two orientations on a house roof. Fig. 5.31(b) demonstrates that a stationary prism system using both the walls and the roof of a conventional building is possible at some latitudes, usually about 45°.
CHAPTER VI

PRACTICAL CONCENTRATORS ATTAINING MAXIMAL CONCENTRATION

6.1 INTRODUCTION

In this chapter, a new class of radiation concentrators is described which achieves maximal concentration of radiation from a uniform source. Unlike ideal concentrators, which accept all radiation within a given acceptance angle and none outside, the new maximal concentration collectors may reject some radiation from within the nominal acceptance angle. However, the new concentrators offer very small mirror or refractor area, very high practical concentration levels (unlike ideal designs, which must be truncated), and an adaptable concentration response vs. radiation incident angle.

In Section 6.2 of the following, the broad characteristics of two-stage maximally concentrating collectors are described. Sections 6.3 and 6.4 describe thermal and photovoltaic versions of these concentrators, respectively. Section 6.5 describes a procedure for constructing minimum area truncated maximal concentration collectors with only a single stage of concentration.

Maximally Concentrating Collectors

If a uniform, stationary radiation source completely fills the acceptance angle of a concentrator, and if the brightness $B$, seen from any direction anywhere on the exit
is equal to the brightness of the source, the concentrator may be said to have achieved maximal concentration. A special class of maximally performing concentrators are the "ideal" concentrators described in previous chapters, which accept all radiation crossing the entrance aperture within the acceptance angle. However, maximal concentration may be also achieved by a class of two-stage concentrators to be described in this chapter even though they reject some radiation within the nominal acceptance angle. The concentrators would appear to be more practical than ideal concentrators in many cases.

A maximally performing two-dimensional concentrator will satisfy the equation

\[ B \int_{\theta_1}^{\theta_2} f(\theta) A_1 \cos \theta \, d\theta \]

\[ = B \int_{-\pi/2}^{\pi/2} A_2 \cos \theta' \, d\theta' = 2B A_2 \]  

(6.1)

where \( \theta \) is the angle of incidence to the entrance aperture \( A_1 \),

\( \theta' \) is the angle of incidence to the exit aperture \( A_2 \),

\( \theta_1, \theta_2 \) are the limits of the acceptance angle, \( \theta_{\text{max}} \)

\( \theta_1 - \theta_2 = \theta_{\text{max}} \)

and \( f(\theta) \) is the fraction of \( A_1 \) which accepts radiation from the direction \( \theta \) and is a geometrical attribute of the concentrator.
Re-arranging Eq. 6.1, the ratio of aperture is obtained as

$$\frac{A_1}{A_2} = 2\int_{\theta_1}^{\theta_2} f(\theta) \cos \theta \, d\theta$$  \hspace{1cm} (6.2)

It can be seen that the smaller the average value of $f(\theta)$, the larger is $A_1$ to compensate. This reduces the efficiency of radiation collection. However, the extra mirror required may be quite insignificant compared to the mirror saving gained by using such concentrators in place of ideal concentrators.

To continue, if the uniform source is now replaced by a distant point source of radiation which traverses $\theta_{\text{max}}$ with a constant angular velocity from $\theta_1$ to $\theta_2$, the time-averaged instantaneous flux concentration at $A_2$ may be expressed as

$$X_{\text{avg}} = \frac{1}{t} \int_{t} \frac{A_1}{A_2} f[\theta(t)] \cos \theta(t) \, dt$$  \hspace{1cm} (6.3)

$$= \frac{A_1}{A_2 \theta_{\text{max}}} \int_{\theta_2}^{\theta_1} f(\theta) \cos \theta \, d\theta$$

since $\theta$ is a linear function of $t$.

Combining Eqs. 6.2 and 6.3, the average concentration which results is
which is independent of \( f(\theta) \), \( \theta_1 \) and \( \theta_2 \). If the exit aperture has a limited exit angle \( \theta'_{\text{max}} = \theta_2' - \theta_1' \), and \( A_1 \) and \( A_2 \) are situated in media of refractive indices \( n_1 \) and \( n_2 \) respectively, Eq. 6.4 may be generalized (see Section 2.2) to

\[
X_{\text{avg}} = \frac{n_2}{n_1} (\sin \theta_2' - \sin \theta_1')
\] (6.5)

The significance of Eq. 6.4 is that, in principle, a maximally performing concentrator may be designed with any concentration characteristic \( f(\theta) \) and acceptance limits \( \theta_1 \) and \( \theta_2 \) so long as it satisfies Eq. 6.1. It is true that this has little importance if a uniform or constant angular velocity source is used. For a non-uniform or apparently accelerating source, however, the value of \( X_{\text{avg}} \) can be increased substantially by matching \( f(\theta) \), \( \theta_1 \) and \( \theta_2 \) to the source characteristics so that the concentrator presents maximum effective aperture to directions of the sky where energy input is greatest. Can such a concentrator be designed in practice?

In Chapter 3 it was shown that \( \theta_1 \) and \( \theta_2 \) may be altered in ideal concentrators to provide some optimization of input energy or to tailor output to projected demand in solar energy applications.

\* This becomes \( X_{\text{avg}} = \pi/\Omega \) in three dimensions, where \( \Omega \) is the solid acceptance angle.
FIG. 6.1 A TWO-STAGE CONCENTRATOR.
6.2 TWO-STAGE CONCENTRATORS

In this section, a new family of two-stage maximally concentrating collectors is described which achieve great flexibility in \( f(\theta) \), \( \theta_1 \) and \( \theta_2 \) yet require much less primary mirror material than ideal concentrators or previous maximal concentration single-stage designs described by Winston [29].

Figure 6.1 shows an arbitrary ideal linear concentrator used as a secondary concentrator for a primary mirror AB (configurations using a primary refracting condenser are also possible). The dotted circle in the figure is a locus such that, at any point upon it, the secondary concentrator entrance aperture CD appears to occupy a solid angle equal to \( \theta_{\text{max}} \). This solid angle \( \theta_{\text{max}} \) will also be taken to be the acceptance angle of the whole two-stage concentrator.

If any part of the primary mirror lies within the \( \theta_{\text{max}} \) locus, rays from outside \( \theta_{\text{max}} \) must be accepted into CD because CD subtends a greater solid angle than \( \theta_{\text{max}} \). A concentrator with a primary mirror partly inside the locus may not, therefore, achieve maximal performance because such a concentrator accepts no radiation from directions outside the acceptance angle. For maximal performance the two-stage concentrator must, consequently have no part of its primary mirror inside the \( \theta_{\text{max}} \) locus. A second restriction is that the mirror must occupy the entire acceptance angle, \( \phi \), of the secondary concentrator stage, otherwise rays from outside both the mirror and \( \theta_{\text{max}} \) may enter CE. Finally, the
FIG. 6.2
RESPONSES OF THREE TWO-STAGE CONCENTRATORS
OF $12^{1/2}^\circ$ ACCEPTANCE.

$X_{\text{avg}} = 9.18X$

$\theta_{\text{max}} = 12.5^\circ$
mirror slope at all points must be such that CD is completely covered by the "reflected" $\theta_{\text{max}}$ or part thereof. However, many primary mirror profiles (both parabolic and non-parabolic) are able to satisfy these restrictions for a given $\theta_{\text{max}}$.

The result of this design procedure is a family of concentrators in which the entire acceptance of CD is filled by reflected rays from within $\theta_{\text{max}}$. Since the secondary concentrator is ideal (or at least maximal) the exit aperture EF "sees" nothing but rays from within $\theta_{\text{max}}$. Hence, EF is fully illuminated at the brightness B and the concentrator has achieved maximal performance.

As previously suggested, the new designs offer greatly increased flexibility of concentration response for a given $\theta_{\text{max}}$. Figure 6.2 shows three of many possible concentration response curves for an acceptance angle of 12.5°. It has previously been shown in Chapter 3 that solar concentrators need not have a flat response curve, and that a skewed response may be preferable in certain applications.

Two-stage high-performance concentrators have been suggested previously [30] but strictly as systems in which a secondary ideal concentrator upgrades the performance of a conventional imaging condenser. Wide acceptance angle two-stage concentrators were considered by Tabor [31] as far back as 1958, but the required maximally performing second stage did not exist at that time.
Compared to CPC and other ideal designs, the mirror material required in the new concentrators is extremely low, typically about 1.2-1.6 times $A_1$, as little as $\frac{1}{6}$th of the material of a comparable ideal concentrator. Examples of this new family of concentrators are described in following sections.

6.3 TWO-STAGE THERMAL SOLAR CONCENTRATOR

Profile Design

Fig. 6.3 shows an asymmetrical, nearly maximally concentrating two-stage concentrator designed for a round absorber tube. The primary mirror is a simple parabolic mirror with focus at $F$ and optic axis along $k'$. The second-stage concentrator uses quasi-ellipse curvature to create an almost ideal concentrator [16] for its close source, which is the primary mirror. The top mirror is composed of an involute section DE and a curved section EF obeying Eq. 4.2 with reference point at primary rim point B. The bottom mirror CD obeys Eq. 4.2, with a reference point at the primary mirror rim A. The acceptance of the resultant secondary concentrator mirror exactly matches the primary mirror position. The secondary concentrator shown is slightly "non-ideal" to reduce secondary mirror area, so that a few rays may strike the tube directly from outside $\theta_{\text{max}}$. Performance of this section is quite close to maximal, however, and an average flux concentration of 9 is obtained compared with the theoretical maximum of 9.57 for a $12^\circ$ acceptance angle.
9X TWO STAGE CONCENTRATOR FOR CYLINDRICAL RECEIVER
Fig. 6.4 shows the response of this concentrator as a function of radiation incidence angle. The response is asymmetrical, but quite flat responses are possible from other designs. An asymmetrical or peaked response has the advantage that at times of year near equinox the solar disk can be confined to high concentration of regions of the acceptance angle by appropriate tilting.

Mirror area of the design shown, including secondary mirrors, is 1.6 times full aperture. This figure is about one fifth the area of a $12^\circ$ acceptance CPC truncated to a similar concentration [32].

In a two-stage concentrator, the secondary concentrator may be designed to accommodate not only the absorber tube, but the clear cover tube as well. This may be done in the same manner as described in Section 4.4, except that quasi-ellipse curvature is used in place of quasi-parabolic. In fact, a "distant source" secondary concentrator could be used with such a system, but would require more primary mirror area for a given average flux concentration. For two-stage concentrators designed for cover tubes, a larger mirror area is required because some rays from within $\theta_{\text{max}}$ pass through the gap. Maximal flux concentration may still be attained, however.
FIG. 6.4

FLUX CONCENTRATION vs ANGLE OF INCIDENCE FOR THE CONCENTRATOR IN FIG. 6.3

\( \theta_{\text{max}} = 12^\circ \)

\( \chi_{\text{avg.}} = 9.2 \)
6.4 A PHOTOVOLTAIC TWO-STAGE CONCENTRATOR WITH POSSIBLE THERMAL OUTPUT

Fig. 6.5 shows a two-stage concentrator for photovoltaics which uses a mirrorless prism (see Section 5.2) as the secondary concentrator. Because the solar cells are immersed in a medium of refractive index of about 1.5, a similar increase in concentration is possible in principle. For example, a concentration (averaged over the acceptance angle) of 14.35X should be possible for a 12° acceptance angle and \( n = 1.5 \), compared with 9.57X for \( n = 1 \). In practice, concentrations of about 12X are achieved. This drop in concentration is due to non-ideality of the prism second stage and to limitations on permitted primary mirror area. This is, in fact, a convenient practical limit in another sense, for above this concentration level specialized concentrator solar cells are required. Below 12X concentration, normal (much cheaper) non-concentration photovoltaics may be used with relatively minor grid modifications.

The prism is constructed of a transparent mineral oil confined by glass plates. In the prototype which has been built, this liquid doubles as a heat transfer fluid to cool the cells (active cooling is required at concentration levels above 6X for silicon photovoltaics). An alternative method of construction could involve a solid prism with the photovoltaic cooled from behind.
FIG. 6.5

A TWO-STAGE PRISM CONCENTRATOR.

- SOLAR CELLS
- TRANSPARENT LIQUID
- GLASS/PLATE
- PRIMARY MIRROR
The prism in Fig. 6.5 is slightly unusual in that it has slightly curved long faces. This method of construction, originated by Dr. Giutronich, enables the prism acceptance to vary slightly along the prism face to compensate for the varying apparent solid angle occupied by the primary mirror. The prism, therefore, is a concentrator for a close source and equivalent to the non-refractive second stage of Section 6.3.

The design in Fig. 6.5 reflects away only 6% of light from within the nominal acceptance angle (12°). Mirror area is only 1.08 times maximum aperture, including the dead area of aperture occupied by the prism back, and 1.24 times the aperture not including the prism back. The latter figure is extremely low and allows the economic use of high quality glass mirrors. Estimated optical efficiency of the design is 86% (not including solar cell absorptivity).

Response of the concentrator as a function of radiation angle is given in Fig. 6.6. An effort has been made to keep this response as flat as possible to prevent high concentrations of flux on the ordinary silicon cells used. Specially made concentrator cells, if made inexpensively enough, could allow a relaxation of this requirement.

**Heat Production**

Cooling the cells actively allows the extraction of heat from the concentrator in Fig. 6.5 as well as electricity. The heat obtained is in the temperature range of only 30-50°C
FIG. 6.6

FLUX CONCENTRATION AS A FUNCTION OF RADIATION INCIDENCE ANGLE FOR THE CONCENTRATOR IN FIG. 6.5

FLUX CONCENTRATION

ANGLE (θ)

ONE SIDE ONLY

AVERAGE

BOTH SIDES
because silicon solar cells rapidly deteriorate in output above 50°C. However, this thermal output could well be used for space heating or pre-heating of high temperature collectors. If the thermal output can be used, the value of the concentrator is enhanced considerably because the thermal output at present is approximately three times that of the electrical.

6.5 MINIMUM MIRROR AREA, SINGLE STAGE SOLAR CONCENTRATORS USING WINSTON-HINTERBERGER CURVATURE

McIntyre [33] has described in detail the performance of truncated ideal concentrators, and the author has done similar work (not included in this thesis) on asymmetrical ideal concentrators. The reason for truncation is that ideal concentrators which follow the edge ray principle have inherently large mirror area. As shown in Fig. 2.6, this mirror area can be reduced substantially by truncation without very much loss in collecting aperture. With ideal EAC concentrators of the type described in Section 4.5, mirror economy is achieved by redesign, but the effect is the same.

One may now ask if a process similar to truncation can be successfully applied to maximally concentrating, non-ideal collectors. Such a process is outlined in the remainder of this section.

In the following, only the procedure for the construction of a minimum mirror area concentrator of 5X concentration is
described, but the evolution of minimum mirror area concentrators of other concentrations is similar. The result is compared to a 5X truncated CPC.

It was well demonstrated by McIntyre [33] that a narrow acceptance angle tends to "flatten" the mirror near the absorber tube. For a non-tracking concentrator, a minimum acceptance angle of $12^\circ$ will give seven hours of collection at solstice and longer collection at other times of year. This acceptance angle will be used as the design acceptance angle of the new concentrator.

Winston [29] has described a method whereby a cover tube around the absorber tube may be accommodated by beginning the curved mirror profile at the cover tube as shown in Fig. 4.5, and not at the actual absorber tube. This method results in a larger aperture than required for an ideal concentrator, but maximal flux concentration is maintained at the absorber surface. There is no reason why, however, the mirror profile must begin at the cover tube.

Fig. 6.7 shows three 5X concentrators with cusp points at varying distances from the centre of the tube. The entrance apertures of the profiles differ in size because they have been chosen to achieve an average of 5X concentration in spite of some rays from within the acceptance angle missing the absorber tube. The lower profiles, which are farther from the tube, tend to have more of such rays, and,
FIG. 6.7

THREE 5X CONCENTRATORS FOR A ROUND ABSORBER TUBE WITH TRANSPARENT COVER TUBE.

MINIMUM AREA CONFIGURATION

\[ R = 8.15 \]

MINIMUM DEPTH CONFIGURATION

\[ R = 9.34 \]
hence, must have a wider aperture to maintain the required flux concentration.

In Fig. 6.8, mirror area is plotted against the distance of the cusp from the centre of the tube. It is clear that a definite minimum exists. This occurs because there are two competing mechanisms for mirror length increase. One is the increasing height of trough walls for the profiles closer to the tube. The other is the increasing horizontal size for profiles farther away from the tube, caused by compensation for a greater number of rays from within $\theta_{\text{max}}$ missing the tube. The minimum occurs at approximately when the cusp is at a distance 6-9 $r$ from the tube centre, where $r$ is the tube radius. The response of a concentrator with a cusp at 8.15$r$ plotted as a function of incident radiation angle is shown in Fig. 6.9, and compared to that of a 5X truncated CPC concentrator designed for the same tube. Both concentrators are pictured in Fig. 6.7.

It is clear that the new concentrator has lower mirror area, less depth, and requires mirrors of shallower curvature. It also has less than one reflection, on average. The new method also can be used to construct concentrators with highly asymmetrical response functions, and these can be used in the manner described in Section 3 to increase time-averaged output. Such concentrators are slightly asymmetrical in form.
FIG. 6.8

AVG. CONC. = 5X
ACC. ANGLE = 12°

MIRROR AREA vs CUSP LOCATION
FOR WINSTON-HINTERBERGER CURVATURE

MIRROR AREA

CUSP DISTANCE FROM PIPE CNTR. (RADII)
FIG. 6.9

MINIMUM AREA CURVE IN FIG. 6.7

CPC IN FIG. 6.7

FLUX CONCENTRATION vs ANGLE FOR CONCENTRATORS OF 12° ACCEPTANCE

FLUX CONCENTRATION vs ANGLE FROM SYSTEM OPTIC AXIS
It is possible that, for example, concentrator depth could be important and the profile chosen on that basis. Concentrator depth as a function of cusp point position is shown in Fig. 6.10 and also has a definite minimum. The minimum depth concentrator is also shown in Fig. 6.7.

Another situation which could arise is that concentrator tilting frequency or diffuse radiation acceptance could be important enough to warrant an increase in acceptance angle. This would lead to high mirror areas, but a minimum mirror configuration may also exist for wider acceptance angles.

Although minimum mirror area configurations may occupy slightly more of a land area than a truncated CPC, this will not be economically significant in most cases.
FIG. 6.10

\[ \theta_{\text{max}} = 12^\circ \]

Concentrator depth as a function of mirror position.

Cusp distance from pipe centre.
CHAPTER VII
CONCLUSIONS

There are good reasons to believe that the goal of this research, the creation of cost-effective solar concentrating systems, is close to being achieved.

For heating purposes, the maximally concentrating system in Section 6.2 is representative of a family of collectors which are inexpensive, require little material, durable (glass and metal are used), and achieve output temperatures of up to 300°C. Because cheaper high temperature storage can be used, storage costs may be low enough that very little backup will be required. Preliminary costing indicate lifetime energy costs of 1-3¢/kWh over a 30 year lifetime, and variants of these devices should be useful and economically competitive anywhere in the world up to latitudes of about 55°. Where tilting is inconvenient, stationary concentrators of either the CPC type or the EAC type could provide thermal energy up to 200°C at a somewhat higher cost, the higher cost being a result of increased absorber tube and plumbing costs. Such concentrators are very flexible in design, however, and as shown in Chapter 3 can use the apparent yearly solar motion to modulate seasonal output. An intermediate range of tilting, minimum mirror area concentrators of about 3-5X concentration, operating at up to 250°C, was also described in Section 6.5. These have a wider diffuse radiation acceptance than the two stage type,
can be easily mounted on a vertical wall, and can tilt around fixed plumbing and vacuum tubing. It is clear that there are now a variety of potentially cost-effective methods of supplying solar thermal energy to residences and industry. When it is realized that up to 50% of the energy use of a nation can be for thermal energy up to 300°C, the implications are clear.

For the production of electricity, the two-stage concentrator of Section 6.3 is the forerunner of a series of concentrators which could have a significant impact on developing nations. Simple and sturdy, they can provide electricity at a cost which is well below that of flat photovoltaic arrays, and do it in a village setting. Waste heat may be used for crop-drying if the array is large enough. In Western countries, present two-stage concentrators are unlikely to be economic except in off-grid applications, but future developments with up to half electrical and half medium temperature (~130°C) thermal output are feasible. The latter could be economically viable in Western nations within 5-10 years. For remote applications, the prism concentrators of Chapter 5 provide a simple and less expensive alternative to flat photovoltaic arrays, and fixed versions can be used to reduce battery storage substantially by yearly output modulation using the seasonal changes in solar position.
These, then, are the implications of the solar concentrators which have been evolved. However, this is a thesis in physics as well as solar energy, and it is hoped that the areas of investigation in optics covered - the limits of concentration performance, the general concept of ideal asymmetrical concentrators, the work on maximally concentrating, non-ideal collectors and other work - will prove useful in not only solar energy, but other fields of research.
REFERENCES


1(b) ibid., p. 48.
1(c) ibid., p. 51.


26. Ref. 1, Section 8.5.

27. Ref. 1, Section 7.4, 8.4.


32. Ref. 1, pp. 84.

APPENDIX I

PUBLISHED PAPERS

(a) ASYMMETRICAL NON-IMAGING CYLINDRICAL SOLAR CONCENTRATORS
(b) IDEAL PRISM SOLAR CONCENTRATORS
(c) SYMMETRICAL AND ASYMMETRICAL IDEAL CYLINDRICAL RADIATION TRANSFORMERS AND CONCENTRATORS
(d) NEW IDEAL CONCENTRATORS FOR DISTANT RADIATION SOURCES
(e) THE PLACE OF EXTREME ASYMMETRICAL AND NON-FOCUSING CONCENTRATORS IN SOLAR ENERGY UTILIZATION
(f) PRACTICAL CONCENTRATORS ATTAINING MAXIMAL CONCENTRATION
ASYMMETRICAL NON-IMAGING CYLINDRICAL SOLAR CONCENTRATORS

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(Received 5 November 1976, in revised form 22 March 1977)

Abstract—Asymmetrical Non-Imaging Cylindrical Concentrators have been proposed recently by Smith[5] and Rabl[6] which are related to the Symmetrical Non-Imaging Cylindrical Concentrators described by Winston[1, 2] and Baranov[3]. In this paper, both Parabolic and Non-Parabolic Asymmetrical Concentrators are examined and compared with symmetrical designs. Among the advantages of asymmetrical systems are (1) a concentration vs time-of-day relationship which can be designed to compensate for projected solar area fall-off in early morning and late afternoon, allowing more uniform output where this is desirable, (2) greater operational flexibility for accommodating unexpected fluctuations in demand, and (3) easier adaptation to vacuum-insulated receivers in one configuration, (4) the possibility of substantially increased concentration and energy collection per unit of mirror area for systems with receivers which can make use of large daily variations in energy input. A higher tracking frequency may be required, but a tilting adjustment every few days should be the maximum required for practical systems.

1 INTRODUCTION

Winston[1, 2] and Baranov[3] have described a family of what may be called Non-Imaging Trough Concentrators which achieve moderate concentrations yet require tracking only once every few days at most for practical systems. These concentrators are symmetrical about their axial plane, and depending upon the shape of receiver used, may or may not have true parabolic curvature.

For a symmetrical non-imaging trough concentrator along the E-W axis, concentration within its designed angle of acceptance is relatively constant only if one assumes small departures from X, the perpendicular to the plane of maximum collector aperture. In the E-W direction, however, the sun departs substantially from X by an angle \( \phi(t) \) during the day as shown in Fig 1. There may also be a smaller departure \( \theta(t) \) in the N-S direction as shown in the same diagram. In consequence, the projected area of the collector aperture perpendicular to the incident radiation must be taken into account, and the concentration expressed as

\[
X = X_{\text{max}} \cos \phi(t) \cos \theta(t) \tag{1}
\]

where \( X_{\text{max}} \) is the concentration at \( \phi = \theta = 0 \) and \( t \) the number of hours from noon. The dependence of concentration on \( t \) for typical Winston–Baranov concentrator at solstice is shown in Fig 6(d).

Once it is accepted that there is a strong dependence of \( X \) upon \( t \), it may be asked whether in certain circumstances a different \( X(t) \) might be preferable to that obtainable from the symmetrical trough. The following is intended to show that a comparable asymmetrical non-imaging trough concentrator with a useful flexibility in \( X(t) \) is possible, and in many cases desirable.

In the following, initial comparisons will be made between the Compound-Parabolic Symmetrical Trough for a Fin Absorber, or SPF concentrator, and the closely related Semi-Parabolic Asymmetrical Trough for a Fin Absorber, or SPF concentrator. Later, comparisons will be extended to other configurations, both parabolic and non-parabolic.

2 THE SEMI-PARABOLIC CONCENTRATOR

Figure 2 shows a parabolic arc of arbitrary length \( EB \) concentrating incident radiation on an absorber \( EF \) which is a fin, either insulated on one side or having a photo-voltaic strip on one side only. \( EB \) is aligned with the axial plane of the cylindrical semi-parabola, which is of focal length \( f \). The E-W deviation of the sun, \( \phi(t) \), is neglected for the moment and the case is considered of incident radiation parallel to \( AF \) meeting the axial plane at an angle \( \theta \). Only radiation passing through \( AB \) will arrive at \( EF \); so the concentration of radiation from this direction may be written as

\[
X_{\text{SP}} = AB/EF,
\]

where \( X_{\text{SP}} \) is defined as the concentration of a semi-parabolic trough.

From the geometry of Fig 2, it can be shown that

\[
X_{\text{SP}} = AB/EF = y_{\text{max}} \cos \theta f \left[ 1 - (y_{\text{max}}/4f - f/y_{\text{max}}) \tan \theta \right]
\]

and since

\[
X_{\text{SP, max}} = y_{\text{max}}/f = DB/EF,
\]

then

\[
X_{\text{SP}} = X_{\text{SP, max}} \cos \theta \left[ 1 - (X_{\text{SP, max}}/4 - 1/X_{\text{SP, max}}) \tan \theta \right] \tag{2}
\]

Inspection of eqn (2) shows that \( X_{\text{SP}} \) is at a maximum when \( \theta = 0 \) and decreases with increasing \( \theta \) until \( X_{\text{SP}} = 0 \) when \( (X_{\text{SP, max}}/4 - 1/X_{\text{SP, max}}) \tan \theta = 1 \). At this point...
connection with non-parabolic asymmetrical concentrators

As is the case with CP concentrators[2], it may often be advantageous to truncate an SP concentrator to reduce reflector cost. The latter may be considered untruncated for a given $\theta_{\text{max}}$ if the reflector is just long enough to intersect a line drawn from the focal line at an angle $\theta_{\text{max}}$ to the axial plane of the parabola. In Fig. 2, $EB$ is untruncated for the $\theta_{\text{max}}$ indicated. Lengthening $EB$ would reduce angle of acceptance, while truncation would reduce concentration. Figure 3 shows $XSP$ vs $\theta$ for several truncations of an SP concentrator, neglecting

\[ \theta = \theta_{\text{max}} \] and values of $\theta$ greater than this correspond to situations where the absorber is shaded by the reflector. Similarly, negative values of $\theta$ result in zero concentration because the reflected radiation passes above $EF$, leaving the receiver in shadow. $EB$, therefore, defines a non-imaging trough concentrator having a definite angle of acceptance $\theta_{\text{max}}$ and a concentration $XSP$ which decreases from $XSP_{\text{max}}$ to zero as $\theta$ increases from zero to $\theta_{\text{max}}$.

The SPF concentrator concept may be extended to any cylindrical absorber cross-section by using an appropriate cusp or involute reflector as outlined by Derrick[4]. No losses are incurred by this method with the exception of some absorption by the cusp reflector surface. For a round pipe absorber, Smith[5] has used both single and double-sided cusps. An example of the SP concentrator cusped for a round pipe is shown in Fig. 12(a). Cusping will later be described in more detail in

Fig 1 The direction of the sun relative to the collector. The angle $\phi$ is the direction of the sun relative to the N-S vertical plane. The angle $\theta$ is the angle to the E-W plane which contains the perpendicular to the plane of maximum collector aperture.

Fig 2 A semi-parabolic concentrator

Fig 3 Concentration vs angle of deviation from the axial plane for semi-parabolic concentrators of five different truncations. For a No 18 truncation, $X_{\text{max}} = 18$ etc.

Fig 4 (a) Superimposition of related SPF and CPF concentrators. (b) Some possible truncations of an SPF concentrator.
φ(φ) Figure 4(b) indicates the relative lengths of some of those truncations. The operation of truncation on an SP concentrator is equivalent to using only the higher concentration regions of an SP concentrator of greater \( \theta_{\text{max}} \). For example, if a 12° acceptance angle is required, the full, untruncated curve No 18 shown in Fig 3 may be chosen. For this curve the acceptance angle is slightly larger than 12° and the concentration varies from 18 to 0 across this \( \theta_{\text{max}} \). As an alternative, the very much truncated (and cheaper) reflector No 6 of \( \theta_{\text{max}} = 37° \) might be chosen to collect radiation only between \( \theta = 0° \) and 12°, giving a XSP which varies between 6 and 4.

At this stage two characteristics of the SP concentrators may be appreciated. First, a considerable variation both in the form and magnitude of XSP(θ) is possible through truncation. Secondly, maintaining an SP concentrator in the high XSP region of \( \theta \) can increase concentration for periods of high demand, leading to greater operational flexibility than that possible with CP units.

3 RELATIONSHIP BETWEEN SEMI-PARABOLIC AND COMPOUND-PARABOLIC CONCENTRATORS

Except for possible cusping in the area of the receiver, it can be seen from Fig 4(a) that a CPF concentrator is essentially made up of two SPF reflecting surfaces facing one another with their respective optic axes inclined at an angle equal to the desired angle of acceptance Winston[1, 2] used \( \theta_{\text{max}} \) to designate this angle of acceptance, but in this paper \( \theta_{\text{max}} \) will be defined as the whole angle of acceptance of both SP and CP concentrators. The relationship between the two types of concentrator is most clearly apparent in the fin case, and Fig 4(a) shows an SPF concentrator superimposed on a comparable CPF concentrator, both having been adapted for fin receivers of similar height. Although the reflector area of the SPF concentrator is half that of the CPF concentrator, the SPF fin is insulated on one side so that the ratio of mirror to absorber area is the same for both types. As can be seen from the diagram, the angle of acceptance is also identical. Using the geometry of Fig 4(a), the maximum concentrations of the SPF and CPF concentration occur along directions M and X respectively, and can be related by

\[
XSP_{\text{max}} = 2 \times XCP_{\text{max}} \cos(\theta/2) \tag{3}
\]

where \( XCP \) is the concentration of the CP concentrator. Winston[1] has shown that \( XCP_{\text{max}} = 1/\sin(\theta_{\text{max}}/2) \), so that eqn (3) becomes

\[
XSP_{\text{max}} = 2/\tan(\theta_{\text{max}}/2) \tag{4}
\]

so that for small \( \theta_{\text{max}} \),

\[
XSP_{\text{max}}/XCP_{\text{max}} \approx 2
\]

It can also be seen that for radiation along the \( -X \) direction

\[
XSP = XCP_{\text{max}}
\]

and for radiation along the \( N \) direction

\[
XSP = 0
\]

Note that if the SPF reflector were positioned to follow contour PA with the absorber turned around and aligned with FP, radiation from direction N would then be concentrated by 2 XCPmax cos(\( \theta_{\text{max}}/2 \)) and the concentration gradient XSP(\( \theta \)) would be reversed. If the movement of the sun across the range of \( \theta_{\text{max}} \), were at a uniform rate, there would be little to choose between the two orientations and each would give, when averaged over the day, a concentration about equal to XCPmax. In reality, the solar disk spends more time near the noon end of the \( \theta \) range. If the SPF concentrator is, therefore, aligned so that its axial plane is directed towards the noon position of the sun, a higher average concentration (and hence energy collection) per unit of mirror area compared to the CPF concentrator is possible for most days of the year. Furthermore, since the CPF concentrator must have its two “SP axes” aligned for the worst daily situation of the year (i.e. solstice), its maximum concentration is limited for the entire year to the level achievable at solstice. In contrast, the SPF concentrator needs only use the higher concentration regions of its \( \theta_{\text{max}} \) at other periods of the year. The extreme case of this is at equinox, when \( \theta = 0° \) for the entire day. The concentrator can be aligned to give a concentration of

\[
XSP_{\phi} = 2 XCP_{\phi} \cos(\theta_{\text{max}}/2)
\]

for the whole day, where both SPF and CPF concentrators are untruncated. For practical truncated concentrators the difference is smaller, but still appreciable as shown in Fig 7. Neglecting ψ, the concentration of a CPF concentrator of any truncation may be determined from its SPF components using Fig 3 and the equation

\[
XCP_{\phi} = XSP_{\phi} \theta_{\text{max}}/2 \cos \theta
\]

where \( \theta \) for the CP case would be the angle of deviation from X as in Fig 4(a).

The following are the expressions for the concentration of CP and SP concentrators in terms of the hour angle from noon, latitude \( \phi \), and declination \( \delta \). The declination lies within the limits 23.5° > \( \delta > -23.5° \) and is taken to be positive in summer for either hemisphere

**CP Concentrator**

Using the geometry of the celestial sphere, it can be found that

\[
\cos \psi = (1 - \cos^2 \delta \sin^2 t)^{1/2} \tag{5}
\]

From eqn (1), it follows that, assuming once-daily tracking

\[
XCP = XCP_{\text{max}} \cos \theta(t) (1 - \cos^2 \delta \sin^2 t)^{1/2}
\]

From Winston[1, 2], we have \( XCP_{\text{max}} = 1/\sin(\theta_{\text{max}}/2) \) so that

\[
XCP = (1 - \cos^2 \delta \sin^2 t)^{1/2} \cos \theta(t)/\sin(\theta_{\text{max}}/2) \tag{6}
\]
where \( \cos \theta(t) \approx 1 \) in most practical cases. This expression is symmetrical around equinox and gives, for example, similar results for winter and summer solstices.

**SP Concentrator**

For an SP concentrator, the effect of \( \theta \) has already been included in eqn 2. To determine \( \theta(t) \), the angle \( \alpha(t) \) which the sun makes to the E-W vertical plane must be determined first. This can be obtained from

\[
\tan(90 - \phi + \alpha(t)) = \cos t \tan(90 - \delta)
\]

At this point, it should be realized that an Asymmetrical Trough Concentrator such as the SP can be oriented towards the sun in two ways. Figure 5(a) shows such a concentrator with its axial plane pointed at the noon position of the sun. This will be referred to as the "\( \alpha \)" orientation, and the expression for \( \theta(t) \) is given by

\[
\theta(t) = \alpha(\text{Noon}) - \alpha(t)
\]

\[
= [90 - \delta - \tan^{-1}[\cos t \tan(90 - \delta)]]
\]

(7)

where the modulus indicates that the concentrator is oriented so that \( \theta \) is positive.

Figure 5(b) shows a second possible alignment in which the reflector is inverted with respect to the "\( \alpha \)" orientation and set so that its axial plane is at an angle \( \gamma \) from the noon position of the sun. This will be called the "\( \gamma \)" configuration, and the expression for \( \theta(t) \) is given by

\[
\theta(t) = \gamma - [90 - \delta - \tan^{-1}[\cos t \tan(90 - \delta)]]
\]

(8)

where \( \gamma \) is always taken to be positive.

From eqns (2) and (5) the complete expression for \( X_{SP} \), including the effect of the E-W deviation of the sun, \( \phi(t) \), is

\[
X_{SP} = X_{SP_{max}} \cos \theta(t) \left[ 1 - (X_{SP_{max}}/4 - 1/X_{SP_{max}}) \tan \theta(t) \right](1 - \cos^2 \delta \sin^2 t)^{1/2}
\]

where

\[
X_{SP_{max}} = \frac{2}{\tan(\theta_{max}/2)}
\]

Like eqn (6), this is symmetrical around equinox if either eqns (7) or (8) is consistently used. However, it should be noted that the orientation type and, hence, the expression for \( \theta(t) \) will change twice during the year if the reflector is only tilted, and not inverted. If this change in character is not desired, provision must be made to invert the reflector when the concentrator axial plane is tracked through equinox.

Assuming it is aligned once daily, the optimum tilt of an "\( \alpha \)" concentrator depends only on the noon position of the sun for that day. With the "\( \gamma \)" orientations, however, there is an additional dependence upon \( \gamma \) and the optimum setting may depend very much on such factors as available energy storage, schedule of energy demand, weather conditions and intended use for the collected energy.

Calculations for 12° No 10 truncation concentrators carried out using eqns (5-8) are shown in Figs 6 and 7.
Asymmetrical non-imaging cylindrical solar concentrators

Figure 6(a) shows the concentration at solstice for an SP No. 10 truncation concentrator in the "γ" configuration with γ = 12°. This exhibits increased concentration when the sun is more remote from the noon position. Cut-off occurs when the sun passes the axial plane of the concentrator. Curve 6(b) shows the same concentrator at solstice with γ = 6°. It can be seen that orientation 6(b) gives higher concentration for a shorter period of collection. For this truncation, the two values of γ give about the same total energy collection, this being proportional to the area under the XSP vs t curve. Although the 6° setting would seem preferable in many instances, the optimum value of γ could be quite different if the schedule of energy demand or other factors previously mentioned were taken into account. Figure 6(d) shows the comparable CPF concentrator at solstice. Figure 6(c) shows that the "α" configuration has a higher peak concentration and a longer collection day at solstice than either the SPF "γ" concentrators or the CPF concentrator of equivalent truncation. For a system which can accept the large daily fluctuations associated with this orientation (for example, solar cells with adequate battery storage) this would be the most economic system. Figure 7 shows CPF and SPF No. 10 truncation concentrators at equinox at which time the "α" and "γ" orientations are equivalent. The concentration of the SPF concentrator can be seen to be clearly superior for the whole day.

The curves in Figs 6 and 7 have not been corrected for atmospheric absorption, which would have its maximum effect in early morning and later afternoon.

4 GENERAL PARABOLIC ASYMMETRICAL CONCENTRATOR

General Parabolic Asymmetrical Concentrators (GPA) similar to that shown in Fig 8 have been outlined by Smith[5] and Rabl[6]. Rabl suggests that this type of concentrator has a maximum geometrical concentration of \[ \left( \sin \left( \frac{\theta_{\text{max}}}{2} \right) \right)^{-1} \] However, maximum concentrations considerably in excess of this are possible within the angle of acceptance. From Fig 8 the maximum concentration can be seen to equal \[ \frac{d_2}{d_1} \text{ when } ABH > 90° \] and \[ \frac{d_2}{d_1} \text{ when } ABH < 90° \] It may be shown geometrically that

\[ \frac{d_2}{d_1} = \frac{1 + \sin \sigma_2}{\tan \left( \frac{\theta_{\text{max}}}{2} \right)} - \cos \sigma_2 \] (9)
where \( \theta_{\text{max}} = \sigma_1 + \sigma_2 \) and \( \sigma_1, \sigma_2 \) are parameter angles measured from the perpendicular to the receiver fin \( d_4 \) such that \( \sigma_2 \) is positive clockwise and \( \sigma_1 \) is positive anti-clockwise. \( \sigma_1 \) and \( \sigma_2 \) may vary up to 90° or even go negative so long as their algebraic sum is \( \theta_{\text{max}} \). When either \( \sigma_1 \) or \( \sigma_2 \) is equal to 90°, the maximum concentration attains the limiting value of \( d_3/d_4 = 2\tan(\theta_{\text{max}}/2) \) and the GPA has degenerated into an SPF concentrator with only one reflective surface. If \( \sigma_2 = \sigma_1 \), the maximum concentration is given by

\[
\frac{d_3}{d_4} = \frac{d_2}{d_4} \left( \cos \left( \frac{\theta_{\text{max}}}{2} \right) \right)^{-1}.
\]

As with the SP concentrators, the GPA asymmetrical concentrators have two classes of orientation similar to the \( a \) and \( y \) orientations. In the GPA concentrators, however, the presence of \( k_{\text{max}} \) well within the angle of acceptance (instead of parallel to \( k_i \) or \( k_e \) as in the SP case) will lead to more complex output curves than those shown in Figs. 7 and 8.

5 NON-PARABOLIC ASYMMETRICAL CONCENTRATORS

The asymmetrical concentrators just described are of true parabolic section. For SP and GPA concentrators which are intended for use with non-fin receivers of finite cross-section (such as a round pipe surface such that the aperture of the involute is identical to the pipe circumference) this is, however, essentially joining a pipe receiver to a reflector idealized for a fin, and a discontinuity at the involute results in all GPA cases except the limiting SP. The dotted pipe receiver and involute in Fig. 8 show such discontinuities at \( F \) and \( G \).

The following will show that non-parabolic sections and shortened involutes can also be used to construct an asymmetrical non-imaging concentrator with no discontinuity in the reflector surface. Development will parallel that of previously described parabolic asymmetrical concentrators. This is natural because a parabolic non-imaging concentrator is merely the limiting case of a non-parabolic concentrator for an infinitely thin fin-type receiver.

6 SEMI-NONPARABOLIC CONCENTRATOR

Consider the case for a round absorber pipe as shown in Fig. 9. For the section of the reflector surface between \( A \) and \( B' \), it is desired that all incident rays crossing the boundary \( BB' \) be reflected to the absorber section \( ADB \). The condition for this, as has been shown by Winston[2], is \((dr/ds) \hat{P} = 0 \) where \( \hat{P} \) is a unit vector tangent to the absorber surface, \( \hat{r} \) is a vector from the cusp origin \( A \) to the reflector at \( R \), and \( S \) is the distance \( CR \). This describes an involute extended to \( B' \) such that \( S_p = b = ADB \).
From $B'$ to $A'$ it is sufficient that rays parallel to $\hat{k}_i$ are reflected tangentially to the pipe surface, for then any rays incident from directions between $\hat{k}_i$ and $\hat{k}_2$ will be reflected to the involute or absorber. The upper boundary for this section of the curve is $A'$, since, above this point, tangential rays may no longer reach the absorber surface beyond $A$ due to blocking by the involute. The condition for this section of the curve is $(dr/ds) \hat{P} = (dr/ds) \hat{k}_i$, as has been shown by Winston [2] for non-parabolic symmetrical concentrators. For a round pipe this may be expressed parametrically as

$$x = 2m(a \tan^{-1} m + b) + a$$
$$y = (m^2 - 1)(a \tan^{-1} m + b) + am$$

(12)

where $m$ is the slope of the curve at a given point in a coordinate system centred on 0 with $y$ axis parallel to $\hat{k}_i$ and $x$ axis perpendicular to $\hat{k}_i$.

Finally, the reflector section from $A'$ to $G$ must be parabolic with the axial plane parallel to $\hat{k}_i$ and focus at $A$. The focal length can be calculated from length $A'$ to $G$ and the angle $\gamma A'A'$. It can be seen from the diagram that if $A$ lies between $E$ and $B$, section $GB'$ is partly parabolic and partly non-parabolic, while if $A$ lies between $E$ and $D$, the curve is completely non-parabolic. Moving the involute origin changes the maximum concentration, which always occurs when radiation is parallel to $\hat{k}_i$. Figure 10 shows the effect of varying the origin of the involute for a concentrator of $\theta_{\max} = 45°$. The optimum angle $AOB$ in Fig. 9 to maximize concentration for this type of concentrator (which will be referred to as the Semi-Nonparabolic or SNP Type I) is found to be about $21.6°$, with $X_{\text{SNP}} = 45°$. The angle $AOB$ was chosen for mirror economy, structural simplicity and more uniform irradiation of the absorber.

Figure 11 shows a situation similar to that of the SNP Type I case, except that a second involute is allowed to extend beyond $A$ and terminate at any point $C$ between $A$ and $D$, the termination being parameterized by the angle $\phi$ shown. This configuration will be referred to as the SNP Type II. Termination of the second involute beyond $D$ would result in some shading of the main reflector by the involute mirror. If the involute termination occurs at point $C$ shown, $B'C'$ will be non-parabolic while $C'D'$ will be parabolic with focus at $C$ and an axial plane parallel to $\hat{k}_i$. If $A$ and $C$ are located such that $\hat{k}_2$ cannot intersect $C$, $B'D'$ is completely non-parabolic. An involute termination precisely at $D$ results in the entire length $B'D'$ being parabolic with focus at $D$.

The SP and SNP families of concentrators both exhi-
CONCENTRATOR

Parabolic Asymmetrical Concentrator (GNP A) for an Ideal non-imaging concentrator index equal to unity is possible ratio of entrance to exit aperture possible for any form. Furthermore, acceptance, actual truncation angular acceptance, \( \pi \) exhibit this process to symmetrical truncation concentrators, which has the effect of increasing angular acceptance. Furthermore, just as truncation in symmetrical concentrators changes concentration but not angular acceptance, variations in design such as those shown in Figs 10 and 12 have the same effect for SNP concentrators and may be seen to be the equivalent SNP process to symmetrical truncation. This is in contrast to actual truncation in single-surface asymmetrical concentrators, which has the effect of increasing angular acceptance.

Finally, since SP and optimised SNP concentrators exhibit the highest peak concentration for a given angular acceptance, it is probable that the highest possible ratio of entrance to exit aperture possible for any ideal non-imaging concentrator in a medium of refractive index equal to unity is \( 2 \tan (\theta_{\text{max}}/2) \).

7. GENERAL NON-PARABOIC ASYMMETRICAL CONCENTRATOR

The solid line in Fig. 13 shows a General Non-Parabolic Asymmetrical Concentrator (GNPA) for a round pipe receiver. This resembles the GPA concentrator in general appearance, but the reflecting sides are non-parabolic and obey eqn (12), each side taking as its axial plane the appropriate extreme ray \( k_1 \) or \( k_2 \). Elimination of one reflective side of a GNPA results in the SNP Type I concentrator. Substitution of an involute section for one reflecting GNPA side produces an SNP Type II Behaviour of the GNPA and SNP family of concentrators is generally similar to their GPA and SP counterparts, and they can be constructed for any cylindrical receiver of regular well-behaved cross-section. An "ideal" concentrator of this type would be represented by the form allowing no radiation outside the nominal angle of acceptance to strike the receiver. In Fig 13 this is represented by the solid line profile ANB. A more practical, but non-ideal (truncated) version allowing some light from outside \( k_1 \) to directly strike the pipe is shown as the solid line profile CND.

The solid profile shown in Fig 13 has an acceptance angle of 72° and would be suitable as a stationary low temperature water or space heating unit for domestic or industrial use. It could be positioned so that the maximum concentration occurred near winter solstice and the minimum near summer solstice. With glass across the concentrator aperture, such a unit could fit in well architecturally under a sloping roof in a domestic situation. Heat storage requirements should be minimised since the output more closely resembles the demand than in conventional symmetrical and flat plate concentrators.

8. ASYMMETRY BY UNEQUAL TRUNCATION

All symmetrical non-imaging concentrators may be rendered asymmetrical by truncating one reflector side more than the other. This can be used as an alternative to designing a special asymmetrical concentrator having sides of different curvature. The dotted line in Fig 13 shows a symmetrical non-parabolic concentrator of a type described by Winston[2]. If this is truncated along EG, an asymmetrical concentrator results with a performance similar to the GNPA unit indicated by the solid line. In this case the former unit uses less mirror area.
than its GNPA counterpart and would likely be preferred for this reason. However, if both the GNPA and unequally truncated designs are cut down to CD and HI respectively, both units then give about the same mirror economy and performance. In such cases a choice between the two asymmetrical units would have to be made on other grounds, such as ease of manufacture.

9. TRACKING REQUIREMENTS

For practical non-imaging concentrators, daily tracking is unnecessary because the daily change in the altitude of the sun at noon is small, about 0.1–0.3° per day. The optimum frequency for tracking depends upon many factors, including the time of year and how much variation is acceptable to the user. One criterion for the frequency of tilt adjustments for symmetrical concentrators can be a minimum collecting time per day, and Rabl[6] has used this as the basis of a table for frequency of adjustments for CP concentrators. Care should be taken with interpreting such a criterion, however. For example, Rabl[6] has shown that for a 12° θmax CP concentrator, tilt adjustments should be carried out every day at solstices. While this is strictly true when using a 7 hr per day minimum collecting time criterion, weekly adjustment at the solstices would lead to a very small loss in total energy collected.

For asymmetrical concentrators, the “a” configuration, the situation is complicated by the much larger variation in X vs θ during the day. A suitable criterion for this type of concentrator may be total energy collection which takes into account concentration and length of collecting day. To optimize the greater concentration potential of the asymmetrical “a” configuration near equinoxes, however, it may be desired to modify this criterion by additional adjustments around that time of the year. For industrial purposes a weekly adjustment routine throughout the year might be the best practical solution for a 12° concentrator, with “fine-tuning” adjustments occasionally to handle unexpected high-demand situations.

Tracking of asymmetrical concentrators in the “y” configuration is a more complex subject, and a good deal would have to be known about operating conditions before a tracking frequency table could be calculated. In general, however, it can be said that emphasis in this configuration would not be on total energy collection so much as maintenance of a specific absorber temperature range between certain hours of the day.

10 PROBLEMS WITH RECEIVER DESIGN

For fin-type receivers such as a flat solar cell array, overheating near the focal line can be a problem for non-imaging concentrators having parabolic sections at times when incident radiation is near the parabolic axial plane. This may be avoided by the use of reflector defocusing wells or cusps as illustrated in Fig. 14(a, b).

For a round pipe, overheating of the pipe surface at point A in Fig. 9 could occur. For this reason, configurations as shown in Fig. 14(c) or Fig. 10(c) could be more successful because radiation from the θ max direction is distributed much more evenly over the pipe surface.
For concentrators where vacuum is considered unnecessary, a flat fin absorber configuration may be more advantageous in thermal applications where glazing is still used. This is because a round pipe absorber would necessitate use of a reflecting involute section (inside a cylindrical glass envelope) which would be subject to possible degradation due to heat conduction and convection. Again, the problem could also be avoided by use of an SNP Type I reflector with involute approximation.

11. APPLICATIONS

Low concentration (=2–4) symmetrical and asymmetrical non-imaging concentrators have tracking requirements which are roughly seasonal so that they may be acceptable for domestic purposes. Smith[5] has already put forward "flat plate" and other low concentration configurations of the SP type for the heating of buildings and it should be possible for the concentration gradient in asymmetrical concentrators to compensate to some extent for changes in the yearly heating load, as both Smith[5] and Rabl[6] have suggested. With high concentration non-imaging concentrators, very frequent tracking and large mirror areas are required, so that these are probably not competitive with simple continuously-tracked cylindrical-parabolic units. However, moderate concentration non-imaging concentrators, tilted every few days at most, may be more economical than motorized systems for low-to-medium grade heat production. In developing areas, the possible utilization of local materials and simple construction may prove attractive, so that the viability of non-imaging concentrators may be accentuated in such circumstances.

12 SUMMARY

Asymmetrical Non-Imaging Concentrators have been shown to be a promising alternative to the Symmetrical Non-Imaging Concentrator and have the following possible relative advantages:

(1) Increased design flexibility. Asymmetrical concentrator can be tailored to compensate for a significant extent for cyclical climatological or demand variations

(2) Increased operational flexibility. If a sudden burst in demand were to require an increase in output, higher frequency tracking could be used in such emergency conditions to achieve substantially greater concentrations. This cannot be done with Symmetrical Non-Imaging Concentrators.

(3) A higher yearly average energy input per unit of mirror surface for systems in which large daily variations in concentration are not a problem.

(4) Increased practicability for use with vacuum-insulated absorbers in the SNP Type I case.

Possible relative disadvantages are

(1) The mode of operation of an Asymmetrical Concentrator changes from "γ" to "α" to "γ" each year. If this change is not wished, provision must be made so that the reflector can be inverted. This may be difficult to reconcile with involute reflector construction in certain instances. With the SNP Type I, however, and using the involute approximation described, reflector inversion becomes relatively easy.

(2) More frequent tracking is necessary, particularly at equinox. Adjustments would still be reasonably infrequent, however, taking place at periods of 3–4 days at most for practical concentrators.

Note added in Proof—In Section 6, we stated that geometrical considerations suggested a peak concentration of 2\(\tan(\theta_{\max}/2)\) as the highest possible for a refractive index of unity. This result, along with more general ones, has now been derived by the authors[8] from fundamental thermodynamic considerations.

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2. D. R. Mills and J. E. Gurnon, Symmetrical and asymmetrical ideal cylindrical radiation transformers and concentrators, *J Opt Soc Am* Accepted for publication

Resumen—Los concentradores asimétricos cilíndricos sin imagen han sido propuestos recientemente por Smith[5] y Rabi[6], quienes se relacionan con los simétricos descritos por Winston[1,2] y Baranov[3]. En este artículo se examinan los concentradores asimétricos tanto parabólicos como los que no lo son y se comparan con los diseños simétricos. Entre las ventajas de los sistemas asimétricos se encuentra (1) la relación entre la concentración y la hora solar, la cual puede ser diseñada para compensar las mermas tempranas matutinas y tarde vespertas para el área solar proyectada, permitiendo una salida más uniforme donde esto es deseable, (2) mayor flexibilidad operacional para acomodarse a fluctuaciones inesperadas de la demanda, (3) fácil adaptación a receptores aislados al vacío en una configuración (4) la posibilidad de incrementar substancialmente la concentración y la colectación de energía por unidad de área especular para sistemas con receptores que pueden hacer uso de variaciones grandes diarias de la entrada de energía. Puede ser requerida una frecuencia grande de seguimiento, pero el ajuste de la inclinación debe ser requerido cada pocos días como máximo para sistemas prácticos.

Résumé—Smith et Rabi ont proposé récemment des concentrateurs cylindriques asymétriques sans image qui s'apparentent aux concentrateurs cylindriques symétriques sans image décrits par Winston et Baranov. On étudie dans cet article des concentrateurs paraboliques asymétriques et les concentrateurs asymétriques non paraboliques. Parmi les avantages des systèmes asymétriques on peut citer (1) un rapport concentración/moment du jour que l'on peut concevoir afin de compenser les chutes correspondant au petit matin et fin d'après-midi pour des zones solaires projetées, ceci permettant un rendement plus uniforme lorsque cela est désirable (2) une flexibilité opérationnelle plus grande pour répondre aux fluctuations imprévues de la demande (3) une adaptation plus facile aux récepteurs isolés par le vide en une seule configuration (4) la possibilité d'une augmentation substantielle de la concentration et de la captation solaire par unité de surface de miroirs pour des systèmes utilisant des récepteurs qui peuvent faire usage de grandes variations journalières dans l'énergie d'entrée. On peut avoir besoin d'une plus haute fréquence de poursuite, mais un ajustage régulier de l'inclinaison devrait être le maximum requis pour des systèmes appliqués.
IDEAL PRISM SOLAR CONCENTRATORS

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(Received 18 July 1977; revision accepted 22 May 1978)

Abstract—Non-imaging solar concentrators using both symmetrical [1-3] and asymmetrical mirrors have recently been described by several workers [4-6]. In this paper, a completely separate but parallel family of non-imaging concentrators is proposed which uses the phenomenon of Total Internal Reflection within a material of high refractive index to achieve concentration. In both symmetrical and asymmetrical forms, the new concentrators satisfy the maximum concentration limits for ideal radiation transformers for a given acceptance angle [7]. Light radiating from the exit aperture is restricted in angle, but concentration performance is very good and irradiation of the exit aperture is much more uniform for a distant point source than in any other design. The new forms are easier to construct than CPC concentrators filled with refractive medium because, in most designs, only flat surfaces are required and far less refractive material is used for a given aperture. Stationary concentrators with acceptance angles up to that of a flat plate are possible.

1. SIMPLE ASYMMETRICAL PRISM CONCENTRATOR

(a) Two dimensional case

The basic unit of the entire family of concentrators to be described in this paper is the Simple Prism Concentrator, shown in Fig. 1. It is constructed of a suitable transparent material or combination of materials (solid or liquid) of high refractive index. The bottom surface of the prism is coated with a reflective layer to provide an effective front-surface mirror within the medium. The end of the concentrator opposite the prism apex angle \( \alpha \) is the exit aperture interface. Figure 1(b) is a perspective sketch of the prism.

In the following, only orientations in which the apex of the prism is aligned parallel to the E-W axis will be considered. There are two basic E-W orientations possible, one with the coated undersurface facing the equator as in Fig. 2(a), and one with the undersurface facing the pole of the hemisphere in which the prism is situated, as in Fig. 2(b). These will be called the "A" and "B" orientations respectively.

In Fig. 1(a), light is incident perpendicular to the E-W axis at a glancing angle of \( h_1 \) to the prism top surface. The ray shown refracts at an angle of depression \( h_2 \), reflects from the bottom mirror at a glancing angle of \( h_3 \), and totally internally reflects from the top surface at a glancing angle \( h_4 \) where

\[
h_3 \leq \cos^{-1} \left( \frac{1}{n} \right)
\]

(1)

and \( n \) is the refractive index of the prism material. Equation (1) restricts the value of \( h_1 \) such that \( h_1 \leq h_{1\text{max}} \) where

\[
h_{1\text{max}} = \cos^{-1} \left[ n \cos \left( \cos^{-1} \left( \frac{1}{n} \right) + 2\alpha \right) \right]
\]

(2)

Since light incident at values of \( h_1 \) less than \( h_{1\text{max}} \) will be accepted and concentrated (except for losses due to reflection at the top surface), \( h_{1\text{max}} \) will be called the angle of acceptance of a simple prism concentrator.

The concentration of the prism in Fig. 1 is given by

\[
X = \sin h_1 / \sin \alpha
\]

(3)

![Fig. 1. Schematic of a simple prism concentrator](image)
where $X$ is the instantaneous ideal flux concentration for a distant point source of radiation. This definition is a suitable one for dealing with the time-averaged performance of an asymmetrical ideal concentrator.\footnote{A suitable definition of asymmetry is outlined in an accompanying technical note.} In the case of the prism, $X$ is simply equal to the ratio of the effective entrance aperture, $AD$, to the exit aperture, $BC$.

The peak concentration for the wedge shown in Fig. 1 is then

$$X_{\text{max}} = \frac{\sin h_{\text{max}}}{\sin \alpha}$$

(4)

providing $h_{\text{max}} < 90^\circ$ if $h_{\text{max}} \geq 90^\circ$, the maximum concentration occurs when $h = 90^\circ$,\footnote{$h_{\text{max}}$ will be defined as exceeding 90° in cases where incident light is travelling toward the wedge apex.} so that

$$X_{\text{max}} = \frac{1}{\sin \alpha}.$$  

(5)

Peak concentration vs prism apex angle is plotted in Fig. 3, neglecting absorption and reflection losses.

An ideal concentrator accepts all the light within a given acceptance angle and concentrates it to the maximum degree possible\[1, 2\]. Strictly speaking, however, the prism belongs to the family of ideal radiation concentrators, but to a more general family of ideal radiation transformers\[7, 8\] which are characterised by a restricted range of exit angles for radiation from the exit aperture. It may be determined from Fig. 1 that the angular range of radiation incident upon the exit aperture is $2[\cos^{-1}(1/n)] \pm \alpha = 2h_{\text{max}}$, and using the requirements for ideality in an asymmetrical transformer put forward by Mills and Gutronch\[7\], it may be easily shown (see Appendix) that the prism concentrator satisfies those requirements. It should be noted, however, that as the acceptance angle $h_{\text{max}}$ increases past 90°, the concentrator becomes less and less asymmetrical in behaviour around the normal to the entrance aperture until, at $h_{\text{max}} = 180^\circ$ (i.e. $\alpha = \cos^{-1}(1/n)$), the prism behaves as a symmetrical unit.

It is commonly assumed that a flat plate collects radiation from an acceptance angle of 180°. In practice, however, flat plate collectors are often tilted toward the equinox or winter solstice sun by some angle $\epsilon$, so that unless reflection from the ground is significant (as in snowy areas), a prism concentrator of

![Fig. 3. Ideal peak concentration vs wedge apex angle for n = 1.5.](image)
ceptance angle is required, inspection of Fig 3 shows that an ideal concentration equal to the prism refractive index may be achieved.

(b) Three dimensional case

Up to this point, only incident radiation perpendicular to the E-W axis (i.e. the situation at noon) has been considered. For light incident from other directions, both refraction and reflection must be considered in the full three dimensional reference frame. The complete three dimensional calculations for concentration through the prism for various times of day and year are too tedious to repeat here, but with the aid of computer ray-tracing the performance of any prism, corrected for reflection and absorption losses, can be modelled. Note that the ideal expression for prism concentration still takes the form of eqn (3) because the angle of elevation is independent of the skew angle from the N-S direction.

A corrected expression for the net radiation flux concentration, \( X_s \), at the exit aperture is

\[
X_s = (1 - F) A_w A_r \sin h_s / \sin \alpha
\]

where \( A_w \) is absorption in the wedge body, \( A_r \) is absorption in the mirror coating and \( F \) is the Fresnel Reflection Factor, which may be expressed in terms of

\[
a = \frac{1}{2} \cos^{-1} \left[ \frac{1}{n} \cos(180 - \varepsilon) - \cos^{-1} \left( \frac{1}{n} \right) \right]
\]

Note that an ideal concentration equal to the prism refractive index may be achieved.

\[
h_1 \text{ and } h_2 \text{ as}
\]

\[
F = \frac{1}{2} \frac{\sin^2(h_1 - h_2) + \tan^2(h_1 + h_2)}{\sin^2(h_1 + h_2) + \tan^2(h_1 + h_2)}
\]

\( A \) depends upon the number of reflections from the bottom surface and is independent of prism size, while \( A_w \) is dependent upon the path length and is, therefore, a function of prism size. Figure 4 shows the path length through a stationary prism concentrator of \( \alpha = 12.5^\circ \) at equinox as a function of time of day and incident ray entrance position. Note the discontinuous nature of the curve. Each change in gradient corresponds to an additional top surface reflection. Figure 3 shows the average path length vs time of day for the same concentrator at equinox.

Figure 6 shows concentration vs time of year at noon and 9 a.m./d.p.m. for a low-iron content crown glass prism of \( \alpha = 22^\circ \) for both the ideal case of eqn (4) and the corrected case of eqn (6) applied in the B orientation for two prism sizes. The effect of increasing size is apparent and the maximum practical prism length in this case should be in the order of about 4 cm to keep total efficiency at reasonable levels. Use of more transparent materials than crown glass, such as perspex or low loss liquids confined by glass or perspex would result in increased maximum unit sizes.

Equation (6) takes no account of back reflections from the exit interface. These reflections can be minimized if low index of refraction boundaries (e.g. air gaps) are avoided in the exit aperture region.

One interesting aspect that has come out of computer ray-tracking of prism concentrators is that the effective acceptance angle is wider for skew rays—such rays are preferentially bent toward the exit aperture—than in non-refractive ideal concentrators. This means that a higher concentration factor is allowed for a given number of daylight hours of operation. A similar effect has been noted in TIR CPC concentrators[9, 10]. A completely stationary prism concentrator with an equinox noon concentration of 3 operating for 8 hr a day at solstice is possible with \( \alpha = 12.5^\circ \) and \( n = 1.5 \).

2. SYMMETRY VARIATION IN PRISM CONCENTRATORS

The simple prism concentrator thus far described is of extreme asymmetrical[7, 11] symmetry. All extreme asymmetrical concentrators, however, have the property that they may be combined in certain ways[11] to produce ideal concentrators of any other symmetry, including the symmetrical. Thus, the extreme asymmetrical is the most fundamental ideal concentrator symmetry and has a versatility in combination unmatched by other symmetries of concentrator.

For example, two identical simple prisms may be combined into a "V"-shaped symmetrical ideal transformer as in Fig. 7. In Appendix II it is shown that this concentrator satisfies the Rabl-Winston[8] requirements for an ideal "\( \theta_1 - \theta_5 \)" transformer. Intermediate symmetries of ideal transformer may be constructed by allowing the two constituent prisms to differ in size, apex angle or refractive index—so long as the acceptance...
Fig 5. Average path length vs time of day for a 12.5° stationary prism concentrator at equinox.

Fig 6. Ideal and corrected concentrations for a stationary crown glass prism of \( \alpha = 22° \) positioned normal to the winter solstice noon sun. The ideal concentration of a flat plate facing the equinox noon sun is shown for comparison (dotted lines).

Fig 7. A symmetrical V-shaped prism concentrator.
relative numbers are equal, the PAF system is a symmetrical transformer. If all the simple prism modules are put into one orientation only, the system is extreme asymmetrical[11]. Thus, we have a concentrator of great versatility which makes use of only one type of prism module.

A schematic of a stationary PAF system for the latitude of Sydney based upon a simple prism system of 22° is shown in Fig. 8(a). Variation of $A_A/A_B$, the relative areas of simple prism modules in the A and B orientations, respectively, changes the concentration gradient over the year. If $A_A = A_B$, the system performs symmetrically around equinox. Figure 9 shows the computed noon and 3.00 p.m. concentrations obtained for various ratios of $A_A/A_B$ using the stationary system of Fig. 8(b). The curves are corrected for absorption and top surface reflection (but not exit interface reflection) in a low-iron crown glass prism of 40 mm top length.

Notice that both the walls and roof of a structure can be used with the system shown in Fig. 8(b), in which 14° apex angle prism concentrators are used at a latitude of about 45°. Other latitudes and types of building may be accommodated using suitable prism apex angles.

3. STATIONARY PRISM CONCENTRATORS

Low concentration simple prism units may be combined into completely stationary, flat-top concentrators as shown in Fig. 10(a). The prisms have large apex angles and relatively few (1-2) absorptive reflections. Concentration vs time of year is shown in Fig. 6 for such a unit, which will give an ideal peak concentration of nearly 2.7 in winter and about 2 in summer. This exceeds the performance of a comparable reflective non-imaging concentrator, the GNPA[4], by a considerable margin. The flat-top construction enables these stationary concentrators to be formed from one large piece of transparent medium. Flat-top symmetrical and PAF designs are also possible.

A "symmetrical" system of equal numbers of simple asymmetrical flat-top concentrators in two different orientations, as described in the previous section, will be able to exceed the peak concentration of the flat-top symmetrical prism shown in Fig. 10(b). This is because the symmetrical prism in Fig. 10(b) must have a very large angle of acceptance and, hence, a low concentration to satisfy the flat-top requirement: i.e the overlap of the angles of acceptance of opposing prisms must be enough to accommodate the sun's movement for the entire year. Dimension "d" shown must also be kept very small to avoid losses.
4. HIGH CONCENTRATION PRISM CONCENTRATORS

High Concentration Prism Concentrators require very small apex angles, as is shown in Fig. 2. As the apex angle declines, however, $h_{\text{max}}$ declines and the acceptance angle of the concentrator falls increasingly into the range where top surface reflections are significant. For practical purposes, a simple prism of 12° useful acceptance angle would in fact require a minimum $h_{\text{max}}$ of about 37° to avoid undue reflection losses. This restricts $\alpha$ to a minimum of about 5° and, hence, $X_{\text{max}}$ to a maximum of approximately 7°. This is also about the practical limit of curved mirror non-imaging concentrators, which are restricted by the problem of excessive mirror length.

A problem inherent in high concentration prism designs is the higher number of absorptive bottom-surface reflections required. It can be shown that more than 10 such absorptive reflections may be required in a frequently tilted prism of $\alpha = 5°$, with the average at winter solstice varying from about 3 at noon to about 4 at 4 p.m. for the B orientation. Such losses are serious in a practical concentrator even if the mirror is of high reflectivity.

A solution to this problem again lies in the phenomenon of Total Internal Reflection, for if the mirror and bottom surface of the prism are separated by a tiny air gap, as in Fig. 10(c), a maximum of one absorptive reflection is required and all subsequent bottom surface reflections are of the T.I.R. type. Concentration loss due to the enlargement of the exit aperture by the air gap is minimal if the gap is kept small.

Path length is longer in high concentration prisms, although the average path length rarely exceeds the length of the prism top surface. Excessive absorption may be avoided by the use of small prism unit sizes, and low-loss refractive materials, such as transparent liquids confined in the appropriate shape.

High concentration prisms in Solar Energy applications must be tilted occasionally to keep the solar disk within $h_{\text{max}}$.

5. RECEIVER DESIGN FOR PRISM CONCENTRATORS

Concentrated solar radiation can be used either for direct conversion to electric power via solar cells or for heating. Design of a prism concentrator will vary a great deal according to the method of solar energy utilization.

(a) Solar cell receiver

Silicon solar cells may be moulded or bonded to the prism exit interface. The prism provides a layer of clear refractive material of $n = 1.5$ between the cell surface and the air. Since silicon cells are normally protected by a layer of glass or resin of $n = 1.4 - 1.5$, reflection from the cell surface is about the same in both cases, except that the prism, functioning as an ideal transformer, does not allow reduction of angle of incidence more than $h_{\text{max}}$ to strike the cell surface. It may also be shown that the more reflection an incident ray undergoes or the less the initial glancing angle $h_i$, the less is the angle of incidence at the exit aperture (solar cell) surface. Since solar cells operate more efficiently for uniform light of normal incidence, this exit aperture behaviour would seem to be particularly appropriate, especially as it is combined with a much more uniform radiation flux distribution across the exit aperture than is possible for other ideal photovoltaic concentrators. Flux intensity at the exit aperture for a distance point source is shown in Fig. 11 for a prism of $\alpha = 21°$, a relatively non-uniform example.

Figure 10(a) shows a stationary prism configuration in which the glass or plastic medium can be extruded or pressed in the manner of a Fresnel lens. The cost of such prism units may be low enough to reduce the cost of direct electrical conversion by as much as a factor of two.
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Fig. 12. A hybrid liquid prism concentrator. As an alternative to ducting, energy could be extracted from the liquid medium itself if it could be used as a transfer fluid for stationary systems and five for occasionally tilted systems at present day prices, and should be worthwhile even at much lower future cell costs.

(b) Heat absorbing receiver

In a system designed for heat absorption, the prism material must be a very good thermal insulator. Glass is unsatisfactory in this respect. Clear plastics are attractive from this point of view, but cannot withstand long-term exposure at temperatures over about 80°C. Prism units with an air or vacuum gap between the exit interface and the absorber are perhaps possible, but may suffer from excessive back-reflections at high values of \( h \). Solid prisms for heat recovery are, therefore, a doubtful possibility although they are not worse off in these respects than other concentrators filled with solid refractive media.

Liquids exist which have definite advantages over glass and perspex for prism construction, however. Dow-Corning Silicone Oil No. 200, for example, is an excellent thermal insulator. It is also stable, available in highly viscous forms, and opaque in convenient areas of the IR. It does not react with most materials, including plastics.

A major disadvantage of silicone oils for prism construction is their high cost. The possibility of using other, cheaper fluids is being investigated.

(c) Hybrid heat/electricity receivers

Liquid concentrators extracting both electricity and heat from incident solar radiation are possible, as shown in Fig. 12. Rabl[12] has stated that the proportions of electricity and heat energy recovered from a comparable hybrid CPC system are suitable for domestic energy use patterns. Of course, maximum temperatures obtained from a hybrid prism system will always be limited by the need to cool the cell to a reasonable operating temperature and by the absorptivity of the cell surface.

\[
X_{\text{max}} = \frac{n \sin(\theta_{\text{max}})}{\sin(\theta_{\text{max}})} \frac{\sin(\theta_{\text{max}})}{\sin(\theta_{\text{max}})} \quad \text{if } \theta_{\text{max}} \leq 90^\circ
\]

\[
X_{\text{max}} = \frac{n \sin(\theta_{\text{max}})}{\sin(\theta_{\text{max}})} \frac{\sin(\theta_{\text{max}})}{\sin(\theta_{\text{max}})} \quad \text{if } \theta_{\text{max}} \geq 90^\circ
\]

where \( \theta_{\text{max}} \) is the angle of acceptance for radiation at the entrance aperture and \( \theta_{\text{max}} \), the angular range of radiation from the exit aperture.

For the prism

\[
\theta_{\text{max}} = h_{\text{max}} = \cos^{-1}\left\{ n \cos(\cos^{-1}(\frac{1}{n}) + 2a) \right\}
\]

\[
\theta'_{\text{max}} = h_{\text{max}} = \cos^{-1}(\frac{1}{n}) + a.
\]

We know from geometry that

\[
X_{\text{max}} = \frac{\sin(h_{\text{max}})}{\sin \alpha} \quad h_{\text{max}} = 90^\circ
\]

or

\[
X_{\text{max}} = -\frac{1}{\sin \alpha} \quad h_{\text{max}} = 90^\circ
\]

It therefore remains only to prove that

\[
\frac{\sin(h_{\text{max}})}{n \sin(h_{\text{max}})} = \sin \alpha.
\]

\[
\text{L.H.S.} = 1 - \frac{\cos(h_{\text{max}})}{2n \sin(h_{\text{max}})}
\]

\[
= 1 - \frac{\cos^{-1}\left(\frac{1}{n}\right) + \alpha - \sin \left(\cos^{-1}\left(\frac{1}{n}\right) + \alpha\right)}{2\sin \left(\cos^{-1}\left(\frac{1}{n}\right) + \alpha\right)}
\]

\[
= 1 + 2\sqrt{(n^2 - 1) \cos \alpha + \sin^2 \alpha - \cos^2 \alpha}
\]

\[
= \sin \alpha.
\]

Hence the simple prism concentrator is an ideal transformer.

APPENDIX A

To satisfy the conditions of an ideal transformer, a simple prism must satisfy the following relationship:

\[
X_{\text{max}} = n \sin(\theta_{\text{max}}/2) \sin(\theta_{\text{max}}) \quad \text{if } \theta_{\text{max}} \leq 90^\circ
\]

\[
X_{\text{max}} = n \sin(\theta_{\text{max}}/2) \sin(\theta_{\text{max}}) \quad \text{if } \theta_{\text{max}} \geq 90^\circ
\]

\[
X_{\text{max}} = n \sin(\theta_{\text{max}}/2) \sin(\theta_{\text{max}}) \quad \theta_{\text{max}} \text{ defined in Appendix A}
\]
TECHNICAL NOTE

New ideal concentrators for distant radiation sources

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(Received 6 October 1978, revision accepted 17 January 1979)

INTRODUCTION

Winston and Hinterberger[1] have shown how to construct ideal two-dimensional and linear concentrators for any convex cylindrical receiver using the general constraint that all rays from the extreme directions of the two-dimensional angle of acceptance are reflected tangent to the receiver surface. Such a concentrator is useful for a distant point source of radiation, which may be approximated by the solar disk. Rabl and Winston[2] have also described ideal concentrators for finite extended sources of radiation. These are constructed using the general constraint that all rays from the point sources of radiation are reflected tangent to the receiver surface. The latter condition is expressed in eqn (2) of this paper and the former in eqn (1).

Both these constraints may be used to develop an additional family of both ideal and nearly ideal concentrators for distant point sources of radiation. The next section gives a general description of concentrators of this type, while the section following describes a specific case with applications in solar thermal collection. This method of mirror profile development owes much to papers by Winston and Hinterberger[1], and Rabl and Winston[2], and attempts to follow their format as closely as possible.

GENERAL PROFILE DESCRIPTION

Figure 1 shows an asymmetrical two-dimensional concentrator which accepts radiation from an acceptance angle \( k = \theta_{\text{max}} \) and distributes it upon the convex receiver shown. The mirror profile for the right side of the concentrator is composed of three sections. The first, an involute, extends from a point \( S_0 \) on the receiver surface to a point \( P_1 \) on the profile. This curved mirror segment, which will be called \( S_0 P_1 \), is intended to reflect all rays crossing \( S_1 P_1 \) such that they arrive at the receiver after striking \( S_0 P_1 \) one or more times. At any point \( S \), on the absorber, \( \vec{l} \) is a unit vector tangent to the absorber perimeter at that point, and the point \( P \), lies on the mirror profile at its intersection with the tangent defined by \( \vec{l} \). Then the distance \( r \), from \( P \), to \( S \), is given by

\[
   r = (R - P) \cdot \vec{l}
\]

where \( R \) and \( P \), are vectors from \( S_0 \) and \( P \), respectively.

Since \( S_0 P_1 \) is an involute, the length of arc from \( S_0 \) to any point \( S \), in \( S_0 S_1 \), is given by

\[
   S_1 - S_0 = (R - P) \cdot \vec{l} = r
\]

The arc length from \( S_0 \) to \( S_1 \), is therefore

\[
   S_1 - S_0 = (R - P) \cdot \vec{l} = r
\]

The left profile side may be described in a similar manner using primed variables, giving

\[
   S'_1 - S'_0 = (R' - P'_1) \cdot \vec{l}' = r'
\]

In the second section of the mirror profile, between \( P_1 \) and \( P_2 \), a new constraint is introduced. This is

\[
   ds = d(l + r)
\]

where \( l \) is the distance from \( P_1 \) to \( P \). This ensures that all rays which cross the concentrator aperture \( A \) and strike \( P_1 P_2 \) will either strike the receiver directly after reflection or cross \( S_1 P_2 \). If eqn 2 is integrated from \( S_1 \) to \( S_2 \), the result is

\[
   S_2 - S_1 = \int_{P_1}^{P_2} d(l + r) = (l_2 + r_2) - (l_1 + r_1)
\]

for the right profile side, and

\[
   S'_1 - S'_0 = (l'_1 + r'_1) - (l'_1 + r'_1)
\]

for the left profile side.

Between \( P_2 \) and \( P_3 \), the profile uses the Winston-Hinterberger curvature. This curvature has the property that all rays from the direction \( \vec{k} \) incident upon \( P_3 P_2 \) are reflected tangent to the receiver, and is described by

\[
   \frac{dP}{ds} \cdot \vec{k} = \frac{dP}{ds} \cdot \vec{k}
\]

The direction \( \vec{k} \) may be taken as one limiting direction within the angle of acceptance because all rays crossing \( A \) which strike

Fig 1 A schematic of the most general type of asymmetrical concentrator using eqns (2) and (3).
$P_2 P_3$ from directions anticlockwise to $k$ will miss the receiver and be reflected away.

From Fig 1,

$$P = R - ri \quad \text{where} \quad i = \frac{dR}{ds}$$

and

$$\frac{dP}{ds} = 1 - \frac{dr}{ds} = \frac{dP}{ds} k$$

Integrating from $S_2$ to $S_3$,

$$(S_1 - S_2) - (r_5 - r_6) = (P_3 - P_2) k$$

The perimeter of the receiver, $S_p$, may now be determined in terms of the concentrator dimensions, and it is found that

$$S_p = (S_1 - S_2) + (S_2 - S_3) + (S_3 - S_4) + (S_4 - S_5)$$

$$= r_1 + (l_1 + r_2) + (l_1 + r_3) + (r_3 - r_2)$$

$$= (P_3 - P_2) k + r_1$$

$$= l_1 - l_1 + r_1 + r_1 + (P_3 - P_2) k \quad \text{(right side)}$$

$$= l_1 - l_1 + r_3 + r_3 + (P_3 - P_2) k \quad \text{(left side)}$$

Since $l_1 = r_1 + r_3$ and $l_1 = r_1 + r_3$, it follows that

$$S_p = \frac{1}{2} \left[ (l_1 + l_1) + (P_3 - P_2) k + (P_3 - P_2) k \right]$$

(5)

The directions $k$ and $k'$ are taken to make angles of $\phi$ and $\phi'$ respectively with the aperture $A$, as shown in Fig 1. Therefore, from the geometry of the figure,

$$l_1 = A \cos \phi - (P_3 - P_2) k$$

$$l_1' = A \cos \phi' - (P_3 - P_2) k'$$

and, after substitution into eqn (5),

$$S_p = \frac{1}{2} [A(\cos \phi + \cos \phi')]$$

The ratio of entrance aperture width to receiver perimeter is, therefore

$$A/S_p = 2(\cos \phi + \cos \phi')$$

In a previous paper [3], it was shown that, in order to be ideal, a linear non-refractive concentrator must accept all rays within $\theta_{\text{max}}$ and possess an entrance to exit aperture ratio of $A_1/A_2$ such that,

$$A_1/A_2 = \left[ \sin(\theta_{\text{max}}/2) \sin(\phi' + \theta_{\text{max}}/2) \right]^{-1}$$

$$= 2(\cos \phi' - \cos(\phi' + \theta_{\text{max}}))$$

From Fig 1, $\phi = \pi - \phi' - \theta_{\text{max}}$, giving

$$A_1/A_2 = 2(\cos \phi + \cos \phi')$$

Hence, the concentrator is ideal. It should be emphasized that in Fig 1 no radiation is permitted to strike the receiver directly from outside $k$ or $k'$. The construction of the concentrator must fulfill this requirement [4, 5].

Ideal transformers, in which the exit angle for rays at the exit aperture is restricted, and nearly ideal concentrators, in which some rays may strike the receiver from outside $\theta_{\text{max}}$, may both be constructed using the new method with appropriate modifications [2, 3].

### IDEAL AND PRACTICAL EXTREME ASYMMETRICAL CONCENTRATORS

A special case of the new construction method occurs if the Winston-Hinterberger curve is eliminated on one side of the concentrator if $P_2 P_3$ is eliminated, in order for the concentrator to remain ideal. $P_2 P_3$ must be extended to meet the shadow line $k'$ to prevent rays from outside $\theta_{\text{max}}$ striking the absorber directly. Using methods similar to those in the last section, such a concentrator may easily be proved to be ideal. The acceptance angle for the new concentrator will be the angle between $k$ and the aperture $A$. Because one extreme ray direction runs parallel to the aperture, the concentrator may be described as an Extreme Asymmetrical Concentrator (EAC), as described in a previous paper [6].

A practical example of how the new technique may lend to a more desirable concentrator is illustrated in Fig 2. Figure 2(a) shows an ideal EAC after a design by Smith [7]. The properties and advantages of such EAC concentrators for solar energy collection have been discussed by Smith [7], Rabl [8] and Mills [6]. However, previous EAC designs in the literature, such as in Fig 2(a), are difficult to cast. or form in one piece, leading to high manufacturing costs. In addition (and perhaps more seriously) they have suffered from a high average number of reflections and, consequently, higher absorption losses in the mirror surface.

The design in Fig 2(b) uses the new technique to create a nearly ideal concentrator in which the receiver tube is well exposed to directions within the acceptance angle. In addition, the mirror is kept relatively close to the receiver, so that the

![Fig 2](image-url)
latter occupies a relatively larger solid angle from the point of view of rays being reflected from the mirror. Both considerations tend to reduce the average number of reflections. Note that the profile 2(b) is designed for use with a cover tube. Instead of eliminating the mirror profile in the region between the cover tube and receiver tube, the profile has been designed so that the involutes coincide at the cover tube surface (i.e. $S_0 = S_0$). This allows an increase in real aperture area which compensates for the loss in thermal output due to rays passing through the gap between receiver and mirror [9].

The right side of Fig. 2(b) uses only involute plus Winston-Hinterberger curvature, while the left side uses only an involute plus a section described by eqn (2). Considerable freedom of symmetry and absorber pipe location may be obtained by using various combinations of these curvatures, however.

The profile Fig. 2(c) is similar to 2(b), but uses an additional plane section of mirror to reduce both cover plate area and reflection losses from the cover. Although this increases mirror area, it would be a cost-effective measure in a concentrator composed of plastic reflective film bonded to an inexpensive cast substrate and protected by a glass cover plate. Acceptance at low angles of incidence is also enhanced. Unlike profile 2(a), profiles 2(b) and 2(c) may be moulded either singly or in nested modules.

Figure 3 shows calculated net flux concentration corrected for absorption losses in a mirror of reflectivity equal to 0.9 as a function of radiation entrance angle. Profile 2(c) gives the best performance for low angles of incidence.

Reflection losses from the cover plate and cover tube are not included. A decrease in reflectivity would depress all the performance curves, but increase the performance advantage of the new designs.

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THE TECHNICAL NOTE

The place of extreme asymmetrical non-focussing concentrators in solar energy utilization

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(Received 14 March 1978, accepted 22 May 1978)

The physical form of an ideal trough concentrator is very closely (but not invariably) related to the orientation of its acceptance angle with respect to its entrance aperture. The “tilt” of the acceptance angle rigidly affects the performance of the concentrator and can be used to quantify a property which will be called the “symmetry” of the device. This paper will show that asymmetrical non-focussing ideal concentrators—in particular, the extreme asymmetrical forms—are often to be preferred to symmetrical concentrators such as the CPC in solar applications.

Figure 1 shows three possible conditions of ideal trough concentrator symmetry, along with the peak concentrations attainable from ideal concentrators of those symmetries, derived in a previous paper [1]. Note that peak concentration, defined as the maximum radiation flux concentration possible for radiation from a distant point source, is highest in the extreme asymmetrical case. We may also define the instantaneous flux concentration achieved to be the area of the effective entrance aperture area normal to a distant point source within the acceptance angle, divided by the exit aperture area. The maximum concentration of the device is, simply, the maximum such instantaneous flux concentration possible. It is clear from Fig. 1(c) that the instantaneous flux concentration varies considerably for such a concentrator when a distant source traverses the acceptance angle. Compared to a definition of maximum concentration based upon the geometrical ratio of actual entrance to exit aperture [2], this definition of concentration better allows for a variation in performance for different concentrator symmetries and locations of a distant point source.

Comparisons of ideal concentrators may now be (and indeed, must be) on a basis comparing time-averaged performance for a given point source motion. Non-isotropic scattered radiation may also be treated in a similar manner. In practical concentrators, the question of how well the output matches demand must also be taken into account. Using these criteria, three situations can be suggested where extreme asymmetrical concentrators offer advantages in solar applications.

1. COMPLETELY STATIONARY E-W TROUGH CONCENTRATOR SYSTEMS

The solid lines in Fig. 2 show the ideal concentration variation at noon and 3:30 p.m. (8 a.m.) for an ideal extreme asymmetrical reflector of acceptance angle $72^\circ$. Practical profiles approaching this performance are OAB in Fig. 3, which we have designed to make best use of the Winston-Hinterberger curve [3], and certain designs by Smith [4] and Rabl [5], which use parabolic curvature Profiles of the type shown in Fig. 3 exhibit a lower average number of reflections for sources in the high concentration region of the angular acceptance, thereby giving a lower “weighted” average number of reflections over the whole acceptance angle. The solid curves in Fig. 2 describe a concentrator which has been oriented so that maximum performance occurs near winter solstice, the maximum effective aperture being seen by the solar disk at that time. However, the dashed lines in the same figure show that another orientation is possible which covers exactly the same solal angle of the sky and exhibits enhanced performance in summer. The dotted lines show the average performance of a system using equal numbers of identical extreme asymmetrical units in both the summer and winter orientations. Note that this is the same performance that would be obtained from an untruncated symmetrical (CPC) concentrator [2, 3] of the same acceptance angle. In other words, a symmetrical ideal concentrator system may be constructed of asymmetrical components.

The concept goes further than this, however, for any system symmetry between the symmetrical and extreme asymmetrical can be synthesized by altering the relative number of component EAC modules in the two orientations.

![Fig 1 Symmetry groups for ideal linear concentrators](image-url)
**Technical Note**

**Fig 2** Yearly concentration variation for profile OAC (Fig 3) arranged in two pure EAC systems with performance based toward opposite solstices, and one symmetrical system composed of EAC components.

**Fig 3** Two stationary (OA, OAB) and one semi-stationary (OAC) nearly ideal mirror profiles which use the Winston–Hinterberger curve [3].

This is illustrated in Fig 4, which shows an intermediate asymmetrical system composed of identical ideal EAC modules in two different orientations, here called the A and B orientations. The same angle of acceptance $\theta_{\text{max}}$ is covered by all EAC components in the system. To be an ideal concentrator, the system must achieve an instantaneous concentration

$$X = \frac{n_2 \sin(\alpha + \beta)}{n_1 \sin(\alpha + \theta_{\text{max}}/2) \sin(\theta_{\text{max}}/2)}$$  \hspace{1cm} (1)

for radiation incident within $\theta_{\text{max}}$ at some arbitrary angle $\beta$ from one of the two limits of the angle of acceptance. Variables $n_1$ and $n_2$ are the indices of refraction at the entrance and exit apertures of the component EACs.

**Fig 4** An intermediate asymmetrical system based upon EAC components.

The instantaneous concentration of a component EAC in the A orientation can be shown [1] to be equal

$$X_A = \frac{n_2 \sin(\theta_{\text{max}} - \beta)}{n_1 \sin^2(\theta_{\text{max}}/2)}$$  \hspace{1cm} (2)

and for the B orientation to be equal to

$$X_B = \frac{n_2 \sin \beta}{n_1 \sin^2(\theta_{\text{max}}/2)}$$  \hspace{1cm} (3)

For the whole system, the average instantaneous concentration is, therefore,

$$X_S = \frac{n_2 \left[N_A \sin(\theta_{\text{max}} - \beta) + N_B \sin \beta \right]}{n_1 (N_A + N_B) \sin^2(\theta_{\text{max}}/2)}$$  \hspace{1cm} (4)

where $N_A$ is the number of EAC units in the A orientation and $N_B$ is the number of EAC units in the B orientation. $X_S$ can be
shown to be equal to $X_a$ in Fig 1 if use is made of the geometric identity

$$\cos \alpha = \left| N_B - N_A \cos \theta_{\text{max}} \right| \left( N_A^2 + N_B^2 - 2N_A N_B \cos \theta_{\text{max}} \right)^{-1}$$  \hspace{1cm} (5)

Of course, there is no necessity that a V type configuration be adopted for a practical system and it is highly likely that some separation between the $A$ and $B$ modules (which could themselves be in several differentiated sections) would be adopted in practice.

The advantages of yearly output modulation using symmetry variation are immediately apparent. For the first time we can "phase match" yearly output cycles to approximate yearly demand cycles in a completely non-moving system. This is an entirely different procedure from biasing, for example, a flat plate toward the particular solstice where higher demand exists, because total yearly output is significantly reduced by such a procedure. Clear sky yearly output is not, however, reduced by symmetry alterations. Of course if maximum yearly output is biased toward a time of year with increased cloud, yearly output will suffer. But other situations exist, such as air conditioning applications, where yearly output would be increased over that of a symmetrical system by biasing toward the sunny summer months. Whatever the locale, symmetry variation provides the means whereby the yearly performance of a stationary collector system can be optimized to allow for weather conditions and demand. Long term alterations in demand after original installation can be accommodated by adding a few more EAC modules in the appropriate orientation, or switching modules from one orientation to another.

Two general groups of extreme asymmetrical concentrators can be used in this way. The prism-type ideal $\theta - \theta_e$ extreme asymmetrical transformer, described elsewhere[6] in this issue, combines manufacturing ease with great simplicity in a concentrator for photovoltaics. It is always ideal and cannot be 'truncated.' For thermal work, the mirror type EAC described in this paper and elsewhere[4,7] can be designed for a wide range of acceptance angles and ideality is not reduced by truncation[7].

The acceptance angle merely increases. All the advantages offocusing concentrators are retained, such as high scattered radiation acceptance and relaxed mirror shaping requirements. Individual extreme asymmetrical panels are much shallower than equivalent CPC panels, but flow through thermal systems using two orientations must be more carefully managed.

2. OCCASIONALLY TILTING SYSTEMS

Higher concentrations than those attainable in a completely stationary system can be achieved by using a concentrator of smaller acceptance angle which is tilted occasionally throughout the year. Figure 4 shows the effect of tilting profile OAC in Fig 3 six times a year preferentially in winter. A less "serrated" output form could be achieved by tracking more frequently, while a higher output in summer could be achieved by tilting at that time of the year. It may be seen that, to a limited extent, performance is a function of tilting frequency.

There are two consequences of this. The first is that, if we tilt frequently enough we can outperform a symmetrical concentrator of similar acceptance angle and mirror area by keeping the solar disk in the higher concentration region of the acceptance angle. This amounts to a substitution of labour for mirror costs, an important consideration in both developing countries and in industrial situations where tilt adjustments could be combined with routine maintenance at little increase in cost. Indeed, in most cases, more frequent adjustments than once a month would seldom be necessary.

The second consequence of an ability to change output by altering adjustment frequency is an operational flexibility unrivalled in any other solar collector. Provided there is enough capacity in the system, unexpected variations in demand, dirty mirrors, aging components and weather conditions can be compensated for to a limited extent by changing the adjustment frequency. Tilting the concentrator less frequently in periods of low demand would save labour and limit thermal ageing processes such as outgassing and fatigue. In practice, one would simply adjust the system when its performance dropped too low, rather than according to a fixed schedule.

Running temperatures over 100°C should be able to be

![Image 5](image-url) Performance of profile OAC during the year with six tilt adjustments carried out in winter.
obtained with such mirror concentrators using inexpensive un-
evacuated thermal absorber tubes with a glass tube cover, so that
developing nations may be able to make use of these systems
Occasionally tilting prism concentrators are under investigation
for photovoltaic applications.

3. FREQUENTLY TILTED SYSTEMS

If a non-focusing extreme asymmetrical trough system is
 tilted frequently enough, an optimum orientation can be mainte-
tained such that the solar disk will always occupy the highest
concentration region of the acceptance angle at noon, when the
sun apparently moves most slowly. An EAC will outperform its
symmetrical counterpart on a daily time-averaged basis under
such conditions, as has been shown in previous papers [2, 7].

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Practical concentrators attaining maximal concentration

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Received September 13, 1979

A new class of radiation concentrators is described that achieve maximal concentration of radiation from a uniform source. Unlike ideal concentrators, which accept all radiation within a given acceptance angle and none outside, the new maximal concentration collectors may reject some radiation from within the nominal acceptance angle. However, the new concentrators offer small mirror or refractor area, high practical concentration levels (unlike ideal designs, which must be truncated), and an adaptable concentration response versus radiation incident angle. The new concentrators are exceptionally well suited to solar-energy applications and should also prove useful for radiation detection or distribution.

If a uniform, stationary radiation source completely fills the acceptance angle of a concentrator, and if the brightness, \(B\), seen from any direction anywhere on the exit aperture is equal to the brightness of the source, the concentrator may be said to have achieved maximal concentration. A special class of maximally performing concentrators are the "ideal" concentrators' described in recent years, which accept all radiation crossing the entrance aperture within the acceptance angle. However, maximal concentration may also be achieved by a class of two-stage concentrators to be described in this Letter even though they reflect some radiation within the nominal acceptance angle. The concentrators would appear to be more practical than ideal concentrators in many cases.

A maximally performing two-dimensional concentrator will satisfy the equation

\[
B \int_{\theta_1}^{\theta_2} f(\theta) A_1 \cos \theta \, d\theta = B \int_{-\pi/2}^{\pi/2} A_2 \cos \theta' \, d\theta' = 2BA_2, \tag{1}
\]

where \(\theta\) is the angle of incidence to the entrance aperture \(A_1\), \(\theta'\) is the angle of incidence to the exit aperture \(A_2\), \(\theta_1\) and \(\theta_2\) are the limits of the acceptance angle, \(\theta_{\text{max}}\), \(\theta_1 = \theta_2 = \theta_{\text{max}}\), and \(f(\theta)\) is the fraction of \(A_1\) that accepts radiation from the direction \(\theta\) and is a geometrical attribute of the concentrator.

Rearranging Eq. (1), one obtains the ratio of aperture as

\[
A_1 \quad A_2 = 2 \int_{\theta_1}^{\theta_2} f(\theta) \cos \theta \, d\theta. \tag{2}
\]

It can be seen that the smaller the average value of \(f(\theta)\), the larger is \(A_2\) to compensate. This reduces the efficiency of radiation collection. However, requirement for the extra mirror may be quite insignificant compared with the mirror saving gained by using such concentrators in place of ideal concentrators.

To continue, if the uniform source is now replaced by a distant point source of radiation that traverses \(\theta_{\text{max}}\) with a constant angular velocity from \(\theta_1\) to \(\theta_2\), the time-averaged instantaneous flux concentration at \(A_2\) may be expressed as

\[
X_{\text{avg}} = \frac{1}{t} \int A_1 f(\theta(t)) \cos \theta(t) \, dt = \frac{A_1}{A_2 \theta_{\text{max}}} \int_{\theta_1}^{\theta_2} f(\theta) \cos \theta \, d\theta, \tag{3}
\]

since \(\theta\) is a linear function of \(t\).

When Eqs. (2) and (3) are combined, the average concentration that results is

\[
X_{\text{avg}} = \frac{2}{\theta_{\text{max}}}, \tag{4}
\]

which is independent of \(f(\theta)\), \(\theta_1\), and \(\theta_2\). [Equation (4) becomes \(X_{\text{avg}} = \pi/\Omega\) in three dimensions, where \(\Omega\) is the solid acceptance angle.] If the exit aperture has a limited exit angle \(\theta_{\text{max}}' = \theta_2' - \theta_1'\), and \(A_1\) and \(A_2\) are situated in media of refractive indices \(n_1\) and \(n_2\), respectively, Eq. (4) may be generalized to

\[
X_{\text{avg}} = \frac{n_2}{n_1} \frac{\sin \theta_2' - \sin \theta_1'}{\theta_{\text{max}}'}, \tag{5}
\]

where \(\alpha' = 90 - \theta_1'\) or \(\alpha' = 90 - \theta_2'\).

The significance of Eq. (4) is that, in principle, a maximally performing concentrator may be designed.
Two-stage high-performance concentrators have been suggested previously but strictly as systems in which a secondary ideal concentrator upgrades the performance of a conventional imaging condenser. Wide-acceptance-angle, two-stage concentrators were considered by Tabor as far back as 1958, but the required maximally performing second stage did not exist at that time.

Compared with CPC and other ideal designs, the mirror material required in the new concentrators is extremely low. For example, for an acceptance angle of \( \theta_{\text{max}} = 12^\circ \), our designs require a mirror length of approximately 1.5 times \( A \), whereas a comparable ideal concentrator requires about 10 times \( A \). Tilting concentrators of the new type have recently been outlined in both asymmetrical and symmetrical forms by Mills, and maximal-concentration systems applied to solar energy will soon be described in detail by the authors. Applications in other areas, such as radiation detection, appear likely.

Support for this research was provided under the National Energy Research Development and Demonstration Program administered by the Commonwealth Department of National Development.

References

10. Ref. 5, p 85.
10. Ref 5, p 85.
allowing $\theta_{\text{max}} = 180^\circ$, and the ideal transformer may now be called an ideal concentrator. If the form of the concentrator is symmetrical, 

$$\int_{a = -\theta_{\text{max}}}^{a = \theta_{\text{max}}} \frac{dr dp}{d_1 d_2} = 2n d_2 l,$$

and

$$\phi = 0.$$

This gives

$$X_{\text{max}} = \frac{d_1}{d_2} = \frac{n_2}{n_1} \frac{\sin(\theta_{\text{max}}/2)}{\sin(\theta_{\text{max}}/2)},$$

agreement with the Winston result\(^6\) for compound parabolic (CPC) concentrators.

In the other limit, in which the ideal concentrator is so asymmetrical that one reflecting wall vanishes completely,

$$\int_{a = -\theta_{\text{max}}}^{a = \theta_{\text{max}}} \frac{dr dp}{d_1 d_2} = 2n d_2$$

and

$$\phi = 0,$$

so that

$$X_{\text{max}} = \frac{n_2}{n_1} \frac{\sin(\theta_{\text{max}}/2)}{\sin^2(\theta_{\text{max}}/2)},$$

This agrees with our results derived geometrically for the extreme semiparabolic (SP) concentrators in a previous paper.\(^6\) Figure 2 shows such a concentrator

Equations (8) and (9) may now be used to determine the maximum possible concentration for an ideal concentrator in terms of the parameter $\alpha$. Note that values of $\alpha$ such that $180 - \theta_{\text{max}} < \alpha < 0$ are not meaningful because they correspond to cases where incoming radiation within $\theta_{\text{max}}$ cannot enter the aperture. Figure 3 shows that peak concentration reaches a maximum for the case corresponding to Eq (9) with $\alpha = 0$ or $\alpha = 180^\circ - \theta_{\text{max}}$. Hence the highest possible peak concentration for an ideal concentrator of $\theta_{\text{max}}$ is that given by Eq (13), the SP case. The lowest peak concentration is seen to correspond to the symmetrical (CPC) case, described by Eq (1).

Concentrators and transformers using receivers which are not flat will require different reflector shapes to achieve ide-
If the time average is taken over the whole acceptance angle, this becomes

$$X_{\text{ave}} = \frac{X_{\text{max}}}{\theta_{\text{max}}} \left( \sin \theta_2 - \sin \theta_1 \right)$$  \hspace{1cm} (15)$$

For uniformly diffuse radiation, the input is time independent, and the resultant concentration is also given by Eq (15).

This result, when applied to ideal concentrators, is of interest. For ideal concentrators obeying Eq (8),

$$X_{\text{max}} = \frac{n_1}{(n_1 \sin \alpha + \theta_{\text{max}}/2)} \sin(\theta_{\text{max}}/2)$$

$$\theta_1 = 90 - \alpha,$$

$$\theta_2 = 90 - (\alpha + \theta_{\text{max}}),$$

giving an average concentration of

$$X_{\text{ave}} = \frac{n_1}{n_2} \frac{\sin(90 - \alpha - \theta_{\text{max}}) - \sin(90 - \alpha)}{\sin(\alpha + \theta_{\text{max}}/2) \sin(\theta_{\text{max}}/2)}$$

$$= \frac{2n_2}{n_1} \theta_{\text{max}},$$  \hspace{1cm} (16)$$

a result which is independent of concentrator symmetry. For concentrators obeying Eq (9), $X_{\text{max}}$ in Eq (15) must be changed to $X_{\text{max}}/\sin \alpha$ or $X_{\text{max}}/\sin \alpha + \theta_{\text{max}}$, whichever is appropriate, to take account of the fact that $X_{\text{max}}$ does not occur when incident radiation is perpendicular to the concentrator aperture. When this is done the result of Eq (16) is again obtained illustrating that the average concentration of an ideal concentrator for a constant instant velocity (or uniformly diffuse) source is independent of concentrator symmetry. This is also true for ideal transformers but the actual value of $X_{\text{ave}}$ will be less than that given by Eq (16).

However, for a source which displays apparent accelerating motion across the acceptance angle, such as the sun, the symmetry of the transformer or concentrator assumes considerable importance. If the motion of the accelerating source is represented by $t(t)$,

$$X_{\text{ave}} = X_{\text{max}} \left( \frac{t_2 - t_1}{n_1} \int_{t_1}^{t_2} \cos(t) \, dt \right)$$  \hspace{1cm} (17)$$

which is not purely dependent upon $\theta_2$ and $\theta_1$.

**TIME-AVERAGED CONCENTRATION**

For an infinitely distant point source of radiation moving across the acceptance angle with a constant angular velocity $\omega$,

$$X_{\text{ave}} = X_{\text{max}} \left( \frac{t_2 - t_1}{n_1} \int_{t_1}^{t_2} \cos\omega t \, dt \right)$$

$$= \frac{X_{\text{max}}}{\omega t_2 - \omega t_1} \left( \sin\omega t_2 - \sin\omega t_1 \right)$$

$$= \frac{X_{\text{max}}}{\theta_2 - \theta_1} \left( \sin\theta_2 - \sin\theta_1 \right)$$  \hspace{1cm} (14)$$

**Fig. 3**  Peak concentration as a function of the symmetry parameter angle $\alpha$ for an ideal concentrator of $\theta_{\text{max}} = 60^\circ$.

**Fig. 4**  Concentrator versus hour of day for both symmetrical and extreme asymmetrical concentrators of $\theta_{\text{max}} = 25^\circ$ at solstice and equinox. Two general orientations are possible for the asymmetrical concentrator at solstice. The orientation which dips to a minimum at noon would go to zero at noon if the whole acceptance angle were used but gives a reasonable output if only half the acceptance angle is used, as shown.
and CPC solar concentrators of identical acceptance angle at solstice is present in Fig 4. There are two possible orientations available at solstice for the asymmetrical SP unit, and one of these gives superior performance to the comparable symmetrical CPC. It is always possible to find one orientation for which an asymmetrical transformer will give superior time-averaged performance over a symmetrical transformer of similar \( \theta_{\text{max}} \) and \( \theta_{\text{min}} \), so long as the average motion of the source is biased to one side of the angle of acceptance. This is also true for diffuse radiation if the latter is not uniform over the angle of acceptance.

If the source of radiation makes no apparent movement across the acceptance angle (as is the case in Fig 4 for an E-W linear solar concentrator at equinox), the asymmetrical transformer exceeds the performance of its symmetrical counterpart by the ratio of their peak concentrations.

Asymmetrical concentrators and transformers have possible uses in areas where CPC concentrators are proving useful, such as solar energy work, radiation detection, astronomy, and retinal theory. Deconcentrators working in reverse from a highly diffuse brilliant source, may also prove useful. For example, radiant bar heaters or fluorescent tubes may now be backed by an appropriate asymmetrical or symmetrical reflector to give a desired distribution of radiation across a room.

ACKNOWLEDGMENT

The authors wish to acknowledge the very helpful assistance of Mr. J. R. Shepanak in the preparation of this paper.

APPENDIX II

THE COMPUTED CLEAR SKY PERFORMANCE OF PRISM SOLAR CONCENTRATORS
INTRODUCTION

Computer programs for calculation of clear sky performance for both stationary and tilting prisms are now available. We have based the computer calculations on two important solar cell dimensions offered by Solarex, 2 cm and 6 cm. The prisms have been taken to be 10% oversize; that is a 2 cm prism has a 2.2 cm face to allow for fittings etc.

The results of the computer run are compared to the performance of a standard encapsulated cell. It is assumed that the solar cell standard is encapsulated in perfectly transparent material of similar refractive index to that of the prism medium. The standard is corrected for reflection losses.

Losses accounted for in the prism are absorption in the medium, absorption losses in the mirror, reflection losses from the prism top surface, and losses due to atmospheric absorption and air mass. The program is corrected for latitude and earth-sun distance. The latitude used in the calculations was 34°S.

The calculations do not take into account the effects of cloud. It is hoped to determine this during our experimental programme.
TILTING PRISM CONCENTRATORS

In these graphs, a tilting prism constructed predominantly of mineral oil (n = 1.47) is carefully tilted so as to optimize yearly performance to about 95% of the performance it would have if tilted daily. The computer tilts the prism when the performance drops to 10% below what could be achieved on that day. This gives about an average of 95% of optimal performance.

In actual fact, the tilt intervals can be varied quite substantially with only marginal effect upon yearly performance, and it is thought that tilt adjustments in practice will simply be carried out when additional input is required. Note that tilt intervals of 1-2 months are possible at the solstices. This coincides well with staff vacation periods, etc. In a large array, tilting sections of the array at different times (but about the same frequency) would smooth out the yearly performance curve.

There are two possible orientations, one in which the prism is roughly vertical, and one roughly horizontal. Although the horizontal orientation gives higher overall yearly performance, the vertical might be chosen for practical reasons, such as a yearly distribution of output better suited to demand, or a smaller susceptibility to sagging in its cradle. However, the horizontal orientation could be expected to experience lower wind loadings.
Total daily clear sky output corrected for all losses, air mass and earth-sun distance. 

\[ n = 1.47 \] 

Air gap mirror reflection = 90% 

Tilt criterion 10% below maximum possible output.
GRAPH 2

**7° TILTING PRISM CONCENTRATOR**

Total daily clear sky output, corrected for all losses, air mass and earth-sun distance. 
n=1.47 air gap mirror reflection = 90% 
Tilt criterion 10% below maximum possible output.

Conventional flat array

Summer solstice 100 Winter solstice 200 Summer solstice

Day of year →
Total daily clear sky output corrected for all losses, air mass and earth-sun distance.

- \( n = 1.47 \) air gap mirror reflection = 90%
- Tilt criterion 10% below maximum possible output.
Total daily clear sky output corrected for all losses, air mass and earth-sun distance. 

n = 1.47 air gap mirror reflection = 90% 
Tilt criterion 10% below maximum possible output.
GRAPHS 5 and 6

HOURLY PERFORMANCE FOR A 7° TILTING PRISM CONCENTRATOR (2cm cell width).
TWO ORIENTATIONS

In these graphs, the relative performance at different times of day is shown for conditions as the previous major graph. Since the performance is symmetrical around noon, performance at, for example, 4.00 p.m. will be the same as that at 8.00 a.m.

Other such hourly outputs are available upon request.
7° TILTING PRISM CONCENTRATOR
"HORIZONTAL ORIENTATION"

Tilt times

Hourly clear sky output, corrected for all losses, air mass and earth-sun distance.

- $h = 1.47$ air gap
- mirror reflection = 90%
- cell size = 2 cm
- Tilt criterion = 10% below maximum possible output.
Hourly clear sky output, corrected for all losses, air mass and earth-sun distance.

- $n = 1.47$
- Air gap mirror reflection: 90%
- Cell size: 2 cm
- Tilt criterion: 10% below maximum possible output.
In this graph the performance of tilting prisms is shown as a function of prism apex angle. This is then compared to that of a flat array. The lower performance of the 6 cm cell is entirely due to increased absorption in the prism medium. However, if the 6 cm cell is significantly cheaper to fabricate, then the 6 cm prism concentrator may be a superior economic solution.

Because diffuse radiation performance and prism efficiency is improved as apex angle increases, the tendency will likely be to choose a prism of 7° or greater. An increased prism efficiency results in less material being used per unit area of land covered.
Yearly performance of small angle prisms vs prism apex angle
Tilt criterion: 90% of best performance clear sky only
Encapsulated flat array
The number of tilt adjustments required per year to achieve 95% of possible performance is shown on graph 8 as a function of apex angle. This is only intended as a rough guide, because the number of tilts will vary according to usage, in practice. Although this is shown for a 2 cm cell, the tilt requirements for other cell sizes will be similar.
Tilt adjustment vs apex angle

$\text{n}=1.47, \; \text{r}=90^\circ$

clear sky only, 2cm solar cell

Apex angle (degrees)
At this early stage in development, we may advance a simplistic argument in order to select an optimal prism apex angle.

Let the yearly concentration advantage over a standard encapsulated cell be $N$. If the number of cells used in a prism is $A$, the required number of cells in a flat array of similar yearly output would be $A.N$.

Let the solar cell price be $x$ per square meter, and let the prism material cost be $y$ per square meter (this could include fabrication costs).

Prism area required is $\frac{\sin(h_{1 \text{max}})}{\sin \alpha} \cdot A$.

Cost of prism + cells is $\frac{\sin(h_{1 \text{max}})}{\sin \alpha} \cdot A.x + A.y$.

Cost of equivalent standard is $A.N.y$

The ratio of prism module cost over standard module cost is

$$R = \frac{\frac{\sin(h_{1 \text{max}})}{\sin \alpha} \cdot A.x + A.y}{\frac{\sin(h_{1 \text{max}})}{\sin \alpha} \cdot A.N.y} = \frac{\sin(h_{1 \text{max}}) \cdot (y/x) + 1}{\sin \alpha}$$

Graphs 9 and 10 show the results of such calculations for cell widths of 2 cm and 6 cm and for various values of $y/x$.

Note that such parameters as land cost, labour and maintenance costs, and the effects of cloud have not been included. Costs associated with electrical equipment outside the collector module can be assumed to be approximately the same in both the concentrator and flat array cases.

Although the curves in graph 10 are lower, the 6 cm cell price may be sufficiently lower than the 2 cm to give an overall cost advantage to the 6 cm cell module.
Tilting prism cost advantage (2 cm cell)
Collector module only
at present, $x/y = 20-50$
$x = \text{solar cell cost} / \text{m}^2$, $y = \text{prism material cost} / \text{m}^2$
Tilting prism cost advantage (6 cm cell)
Collector module only
at present, \( x/y = 20 \)
\( x = \) solar cell cost/m\(^2\), \( y = \) prism material/m\(^2\)

\( x/y = 50 \)
\( x/y = 20 \)
\( x/y = 10 \)
\( x/y = 5 \)
\( x'/y = 2 \)
These graphs show the yearly performance of a completely fixed 16° prism compared to that of a standard encapsulated array of similar photovoltaic area. Differences in curve shape between the "horizontal" and "vertical" orientations are caused by a combination of the apparent solar path and the acceptance angle shape.

The dotted line in Graph 11 shows the characteristic which would be obtained from an array composed of 2N "vertically" and N "horizontally" oriented prism concentrators of 16° apex angle. In fact, by varying the relative number of prisms in the two orientations, an array can be made to give almost any output between the two extremes. This allows a stationary array to compensate for demand or weather conditions in order to deliver energy just when it is required. The result is a much reduced battery storage requirement for an unattended photovoltaic installation. Clear sky yearly total output is not sacrificed by the use of this technique.
16° STATIONARY PRISM CONCENTRATOR

"Horizontal" orientation

"Vertical" orientation

System of 2N "Vertical" and N "Horizontal" modules.

Flat array output

Total daily clear sky output corrected for losses, air mass and earth-sun distance. 
\( n = 1.47 \) air gap mirror reflection = 90% cell size 2 cm
17° STATIONARY PRISM CONCENTRATOR

Total daily clear sky output corrected for losses, air mass and earth-sun distance.

n = 1.47  air gap mirror reflection = 90%

cell size 2 cm
Total daily clear sky output corrected for losses, air mass and earth-sun distance.

n = 1.47  air gap mirror reflection = 90%
cell size 2 cm
Graph 14

19° STATIONARY PRISM CONCENTRATOR

"Horizontal" orientation

"Vertical" orientation

Flat array output

Relative corrected clear sky output

Total daily clear sky output corrected for losses, air mass and earth-sun distance.

n = 1.47 air gap, mirror reflection = 90%

Cell size 2 cm

Day of year

Summer solstice
Winter solstice
Summer solstice
Total daily clear sky output corrected for losses, air mass and earth-sun distance.

\( n = 1.47 \) air gap mirror reflection = 90% 

cell size 2 cm
GRAPHS 16 and 17

These graphs show the hourly variation in output for the concentrator in Graph 11 for both possible orientations.
16° Stationary prism concentrator
"vertical orientation"

hourly clear sky output corrected for losses
n=1.47 air gap mirror reflection=90°
cell size 2 cm
16° Stationary prism concentrator

"horizontal orientation"

Hourly clear sky output corrected for losses

$\eta = 1.47$ air gap mirror reflection = 90%
cell size 2 cm
These graphs show the yearly output of wide angle prisms as a function of apex angle. It appears that, for fixed, unattended operation, a 16° prism in either the "horizontal" or "vertical" orientation gives a good performance.

If the prism is allowed to tilt occasionally, a better performance may be achieved, as is shown in the same graph.
Yearly performance of wide angle prisms vs prism apex angle for 2 cm solar cell width clear sky only.
Yearly performance of wide angle prisms vs prism apex angle for 6 cm solar cell width clear sky only.
VARIATIONS IN PERFORMANCE DUE TO THE TILT SETTING CHOSEN

The tilt selected by the computer for the stationary prisms previously described was optimized for best daily output at one solstice. This, however, may not be the optimal tilt for maximum output throughout the year.

In graph 20, a 13° apex angle prism is set at 3 different tilts. The tilt setting of $\beta = 160.5^\circ$ is that chosen by the computer to optimize output at summer solstice. The setting of $\beta = 165^\circ$ allows a performance increase of about 7% but summer solstice performance deteriorates slightly. Tilting it further, to $\beta = 170^\circ$, causes yearly performance to decline again to the level of the automatic setting, and summer solstice performance is seriously affected although winter performance is improved. It can be seen, therefore, that even with a single prism the concentration gradient throughout the year can be modified within certain limits.

Such tactics are of particular use when the prism acceptance angle is on the small side for stationary operation. The prism of 13° described is the highest performing stationary prism yet found when set at $\beta = 165^\circ$, and delivers a concentration advantage over a flat array of 2.7. This figure, corrected for losses as it is, compares well with an ideal figure (loss-free) of about 3 which is based upon thermodynamic limitations of concentration for large acceptance angles.

In practice, the researchers believe that an apex angle of 16° or more will usually be chosen for economic reasons.

Graph 21 shows a 15° prism for a 6.6 cm cell oriented to peak near equinox in the vertical orientation and near solstice in the horizontal orientation.
13° Stationary prism concentrator
yearly output for three different tilts
horizontal setting, cell size 2 cm
n = 1.47 mirror reflection = 90°
Stationary prism concentrator

Total daily clear sky output corrected for all losses, air mass and earth-sun distance. Apex angle = $15^\circ$, $n = 1.47$, air gap mirror reflectivity = 90%, cell size = 6 cm.
Graph 22 shows the acceptance solid angle of a 7° prism concentrator (heavy solid line) and the exit solid angle for the same concentrator (dashed line). These are projected on a two-dimensional polar diagram.

The solid exit angle has a distribution of radiation which is biased heavily toward the normal to the solar cell surface. For most solar cells, this is desirable for efficient operation.

The acceptance angle has a curvature which is a consequence of Snell's Law being obeyed by skew rays. The top curvature of this acceptance happens to fit the sun's path at equinox quite well, so that daylight hours of operation are comparatively high for such a small solid acceptance angle.
Entrance and exit solid angles for 7° prism