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INTERACTION OF PHOTONS WITH ELECTRONS IN DIELECTRIC MEDIA

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A possible modification of some of the properties of the electromagnetic wave by a medium in which it propagates, are studied. Assuming the validity of the Maxwell equations, a new way of deriving the formula for the black body radiation in a medium is presented. The mechanism for the mass generation is investigated. It is shown that a medium may be replaced, for some purposes, by a curved space. Furthermore, if the non-linear interaction terms between the electromagnetic wave and the curvature of space-time are included, it is found that the resulting photons acquire mass, proportional to the curvature scalar. This is further supported by the conclusions following from the analysis of the gauge theories. Some of the consequences of this non-linear interaction are then examined.
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I. INTRODUCTION

Theory of light has been attracting attention of many for a long time. Here, it suffices to begin with the work of Newton, who believed that light consists of a stream of massive corpuscles travelling with very high velocities. This result was accepted in his time as it successfully explained the then available experiments.

There was also another proposal, put forward by Huygens, who claimed that the light has wave-like properties. However, in the absence of decisive experiments that theory did not come to the fore for a long time. Towards the end of the XVIII century, new experimental data became available, which could not, any longer, be explained within the framework of the Newton theory. Consequently, the wave theory of light was then accepted as being the only correct theory describing the propagation of light. This situation remained, more or less unchanged, till the end of the last century. Then, once again, the wealth of experimental evidence precipitated a contention regarding the nature of light.

In those confusing times, Planck was forced to introduce, in his work on the black body radiation, the discontinuous aspect of the radiation field, in order to resolve the discrepancy between theory and experiment. As a result, the particle nature of radiation was reintroduced. Later, the name "photon" was given to these particles. When investigating the photoelectric effect, this concept was successfully used by Einstein. However, as distinct from the earlier times, the particle theory, now, did not become the dominant one, but was considered to be complementary to the wave theory. Put otherwise,
depending on the question posed, the radiation phenomenon could be accounted for by either the particle or the wave theories.

At present, the complementarity of these different aspects of radiation is well established. For instance, the classical wave theory can explain neither the photoelectric phenomenon nor the Compton effect, because these belong to the particle domain. On the other hand, the occurrence of interference, diffraction, and polarization manifest the wave nature of radiation and, therefore, cannot be explained by the photon theory. Thus, in order to really comprehend radiation phenomena, this duality must be inevitably accepted.

In spite of differences, there is a close connection between the wave and particle pictures of radiation. Classically, the notion of frequency and wavelength are inherent in the former, whilst energy and momentum belong to the latter. However, it is now customary to associate with the electromagnetic wave of angular frequency $\omega$, a particle with energy $E$ given by

$$E = \frac{\hbar}{2\pi} \omega$$

where $2\pi\hbar = \hbar$ is the Planck constant. Although this relation was proposed by Planck already at the beginning of this century and is now well accepted, it meaning as to the physical connection between energy and frequency remains rather obscure /1/. Nevertheless, (1) in conjunction with one of the most fundamental principles of the special theory of relativity, viz., the principle of the equivalence of mass $m$ and any kind of energy, $E = mc^2$, suggest that an effective mass $m_0/c^2$ can be attached to the photon. Now, according to that theory, the energy of a particle is given by

$$E^2 = p^2c^2 + m_0^2 c^4$$

(2)
where $p$ stands for the momentum, $m_0$ refers to the rest mass of a particle, and $c$ is the vacuum speed of light. As the rest mass vanishes in the case of a photon, its momentum has the value $\frac{\hbar \nu}{c}$.

In addition, there is another, equivalent expression for the photon's momentum. Thus, on substituting the relation between the frequency $\nu$ and the corresponding wavelength $\lambda$, which in vacuo takes the form

$$\nu \lambda = c,$$

(3)

together with (1) and (2), one obtains

$$p = \frac{\hbar}{c} k,$$

(4)

where $k = 2\pi \lambda^{-1}$ is the wave vector. Consequently, (1) and (4) express the relation between the particle and the wave aspects of radiation. Notice that the ratio between the two is given in terms of the quantum of action. This, in essence, reflects the wave-particle parallelism which is now usually adopted when dealing with radiation effects.

The relations (1) and (4) remain formally correct also for particles with non-vanishing rest mass, as was first recognized by de Broglie /2/. From the similarity between the phase of a plane wave and the Hamiltonian action of a particle, he deduced that the Fermat principle, valid in the domain of geometrical optics, is analogous to the Maupertuis theorem in the analytical mechanics. Since the Fermat principle has no meaning in the realm of the wave optics, he concluded that the traditional mechanics is also an approximation with the same range of validity. Thus, de Broglie applied the wave concept of radiation to matter and was able to show that (1) and (4) are, indeed, relativistically invariant also for massive particles.
Despite these similarities, massless photons and massive particles are not the same. Thus, in order for a particle to be observed, it must interact. Consequently, its energy and momentum will be changed in accordance with the conservation laws. Then, high energy material particles are like high energy photons, because they change energy by changing their effective mass. This is the domain of the applicability of the geometrical optics, which, by definition, is valid in the region of comparatively small wavelengths. On the other hand, the differences between photons and massive particles are very pronounced at the other end of the energy spectrum. The energy of material particles approaches, in that case, the rest energy, whilst the photon energy approaches zero. Therefore, in this limiting case, the theory of radiation is effectively replaced by the static field theory. As a result, our conception of matter is represented, classically, by the particle nature, whereas that of radiation assumes the wave character.

Bearing this in mind, the present work attempts to clarify some aspects of interaction of electromagnetic radiation with a general medium. As the essential difference between real particles and photons is the value of their mass, it may prove instructive to study this concept in some detail. In the absence of the generally accepted theory, different possibilities will be investigated.

This work is divided into two parts: one, in which the assumption of the linearity is invoked in the problem at hand, whilst in the other this assumption is dropped and some of the consequences are then examined. It should be emphasized, at this point, that the proper treatment of interaction phenomenon should take into account the mutual effect of the radiation and a medium. However, the scope
of this work is more limited, as it examines, to a large extent, only the effect of a medium on the radiation field.

With this in mind, we begin, in Chapter II, by studying the effect of the medium on the electromagnetic field moving within it. The main new contribution of that Chapter, already to be found in literature /3/, deals with the generalization of the Brownian motion to include a medium with the dispersion relation of the form encountered in the theory of a collisionless plasma. The philosophy of the approach parallels that of Einstein's original article /4/. The results are, evidently, different. It may be said, in view of that analysis, that the black body radiation inside a medium differs from the one in vacuo. Although this result has been known /5/ for some time, the method of its derivation is new. In the process, it was confirmed that, in the region where $\hbar \nu >> kT$, i.e., in the domain of the validity of the geometrical optics, the exchanged momentum between the radiation field and medium is proportional to the refractive index of the latter. The form of this momentum was derived by Minkowski /6/ who substituted the Maxwell equations to the conservation relations. As these field equations are linear, it is implied that, within the specified restrictions, the conclusion of Chapter II may be looked upon as following from the linearity assumption.

Generally, it is agreed that the classical electrodynamics is only an approximate theory, valid when the accelerations of electrons are small. In an attempt to make it more accurate, one encounters the main difficulty in the problem of the electron structure. Thus, assumption of a point charge leads to an infinite self-energy, whilst the spatial distribution of charge is not compatible with the theory of relativity. In the treatment of this problem, Dirac /7/ proposed a radically different theory for the motion of electrons. As is well
known, the potentials $A^\mu_\mu$ in the field theory are not unique. Normally, one restricts the arbitrariness in $A^\mu_\mu$ by the condition

$$\partial^\mu A^\mu = 0.$$  (5)

Dirac argues that a more powerful electrodynamic theory can be built, whilst abandoning the above condition. In the process, the gauge invariance is destroyed. However, as the existence of the gauge transformations signifies that there are more variables in the theory than necessary, from physical point of view, the lack of gauge invariance is not considered to be a serious drawback. On the contrary, these superfluous variables in the theory without charges, are then used to introduce the charges, rather than bringing in new variables to describe them, as is done in the classical theory.

Realising that the classical electrodynamics is an approximate theory, we shall concentrate on some of the ways put forward as a possible generalization. These are presented in Chapter III. We shall discuss the modifications of Mie /8/ in his attempt to find the field equations inside an electron. Further, an interesting theory of Born /9/ and Born and Infeld /10/ is reviewed. It is a non-linear theory which, for reasons to be mentioned, did not become completely successful. Another theory, which is relevant to the present investigation, belongs to the class of scalar-tensor theories and has been developed by Brans and Dicke /11/. We shall conclude that Chapter III with the results of a very interesting paper by Rainich /12/ where a possible way of geometrizing the electromagnetic field is presented. That Chapter covers a wide range of different approaches in investigation of topics closely related to the present study - the concept of mass and a possible modification of the Maxwell Theory.

As the concept of mass is one of the issues examined here, it may
be instructive to study it within the framework of the general relativity. It will be seen, in Chapter IV, that the effect of a medium on the electromagnetic wave closely resembles the effect of the curved space on the same wave. This may suggest, then, that, for some purposes, a medium may be replaced by a curved space. Consequently, if the interaction between the electromagnetic field and that (curved) space, however small, is taken into account, then it seems plausible to say that photons have acquired a non-vanishing mass. Put otherwise, allowing an existence of a non-linear interaction is seen as a possible way for mass generation.

A somewhat similar conclusion, regarding the mass of the photon, was obtained by Kibble /13/. From the analogy between the refraction of a beam of radiation by an electron cloud and of a beam of electrons by the radiation field, his study reveals that the mass-squared shift is proportional to the particle density divided by the particle energy.

Let us look now at the same problems from a completely different standpoint, viz., using the formalism of the gauge theories. It will be shown that due to the spontaneous symmetry breaking, via the Higgs mechanism, photons, under certain conditions, may be looked upon as being massive. In addition, the question of mass generation using the dynamical symmetry generation is also considered. These, as well as the related issues are the subject of Chapter V.

In view of the results obtained, we shall reexamine the question of the electromagnetic field energy-momentum tensor in a medium. We emphasize here that the question of the total energy-momentum tensor is not the object of the present analysis. Rather, the medium-induced modifications to the radiation field are examined. It is then concluded that the previous results /3, 14/ remain, in essence,
unchanged. However, in view of the extensive analysis, they may appear in a clearer perspective. The necessary details are to be found in Chapter VI. It should be noticed that the experimental evidence is in complete agreement with the theoretical conclusions of that Chapter— the fact which should not be easily discarded.

The concluding Chapter summarizes the important points of this study and, by bringing together different fields of physics, evaluates the new results.
As part of our investigation of the interaction between the electromagnetic field and matter, concentrate now on the thermodynamics of the problem. First of all, it is necessary to study the constitutive laws, which characterize the particular medium in question. Recalling the way they were introduced to physics, these constitutive relations are not universal laws, although most materials can be treated as belonging to one of the mutually very different groups. We restrict our attention to a system which, from the point of view of thermodynamics, behaves as a simple fluid. Such a system is readily found, e.g., one component plasma. The constitutive relations can be deduced from thermodynamics and describe the effect that the fluid has on the electromagnetic field. In addition, they reflect the influence of the electromagnetic field on the motion of the fluid, which obeys the conservation laws involving the energy-momentum tensor.

Conventionally, the constitutive relations are defined by the ratios of the electric \( E \) and magnetic \( H \) fields to the electric displacement \( D \) and magnetic induction \( B \), respectively. In general, when directional effects become important, they form permittivity and permeability tensors. As our study is restricted to simple media, these quantities are considered to be constants. Evidently, in reality they depend on many parameters, in particular, frequency, temperature, density, field strengths. Although they are defined as the ratios of quantities occurring in the macroscopic Maxwell equations (which do not take into account the detailed structure of matter), their theoretical description is accomplished by the microscopic Maxwell equations (which, by restricting their region of applicability, consider the charges in vacuo, thus completely disregarding the ponderable
The microscopic theory is more fundamental and is supposed to hold everywhere. The macroscopic equations are, by their nature, statistic averages of the microscopic fields and charges. The connection between the two theories has been established by Lorentz /15/.

In many cases, the wavelength $\lambda$ of the electromagnetic radiation is much greater than the interparticle distance, hence, it is possible to analyse the wave propagation in a medium from the point of view of the macroscopic theory. As a result, the discreteness of the constituents is ignored and only the smeared-out, average fields are considered. This brings us directly to the ambiguity involved in the definition of the dielectric permittivity. This question has attracted a lot of attention and there exists diversity of opinions /16, 17, 18/.

The difficulty involves the local electric field $E_{\text{loc}}$. It is assumed, on one hand, that $E_{\text{loc}}$ is equal to the external field $E$, and, on the other hand, that the contribution of the Lorentz polarization term, $4\pi P/3$, must also be taken into account. However, due to the complexity of the problem, the resolution is not a trivial matter. A detailed analysis /17/ reveals that either conclusion is viable and the choice depends on the type of a medium in question. Thus, inside matter, where electrons are neutralized by a positive continuum, the local field $E_{\text{loc}} = E$. Furthermore, a medium consisting of a set of bound electrons experiences polarization, therefore, the Lorentz formula $E_{\text{loc}} = E + 4\pi P/3$ should be used in the derivation of dielectric constant. The index of refraction $n$ is connected with this constant via the relation

$$n^2 = \varepsilon \mu$$

valid in the high frequency domain.
In order to appreciate the restrictions of the following calculations, it was considered useful to draw attention to a rather unclear differentiation in the dielectric permittivity and, in turn, the index of refraction. As a consequence, the following discussion is valid, strictly speaking, for optical frequencies. In this case, the dispersion relation for many different media approaches that of a collisionless plasma, as the frequency of the electromagnetic wave is, often, much larger than the characteristic frequency of a given medium.

Next, let us turn to the important topic of the black body radiation. It deals, as is well known, with the equilibrium spectral distribution between the thermal radiation and matter in an evacuated cavity. The presence of matter is necessary to satisfy the equilibrium conditions and its amount is normally minute. When, however, the cavity is filled with matter, it affects this spectral distribution and it is quite plausible to suppose that the density of radiation in a cavity now differs from the one in an evacuated cavity. Thus, if $\varepsilon(\omega,T)$ is the mean energy of harmonic oscillators of frequency $\omega$ at the temperature $T$, then

$$
\varepsilon(\omega,T) = n_\omega \left( e^{\frac{\hbar \omega}{kT}} - 1 \right)^{-1}
$$

with $k$ being the Boltzmann constant, so that the energy density $u(\omega)$ of the radiation can be expressed by

$$
u(\omega)d\omega = \varepsilon(\omega,T) g(\omega)d\omega ,
$$

where $g(\omega)d\omega$ refers to the density of radiation states in the frequency interval $\omega$ to $\omega + d\omega$. This density of states is usually defined /19/ in terms of the wave vector $k$, as

$$
g(\xi)dk = \pi^{-2} k^2 dk .
$$
In order to arrive at the density of states, in terms of frequencies, recall that the wave vector in a medium $k$ is related to that in vacuo $k_0$ by

$$k = n k_0$$

where $n$ is the refractive index of a medium. When (3) is rewritten in terms of the frequency, we have

$$g(\omega) d\omega = \pi^{-2} k^2(\omega) \frac{dk(\omega)}{d\omega} d\omega .$$

In accordance with the above mentioned proposition to consider radiation interacting with a simple fluid, e.g., a collisionless electron plasma, the dispersion relation for the electromagnetic waves in such a region is

$$k^2 = \frac{\omega^2}{c^2} n^2$$

with $\omega_p^2 = \frac{4\pi e^2 N_e}{m}$ denoting the collective oscillations of electrons. It is now clear, from (6), that $k$ is a function of frequency $\omega$. Therefore, in order to evaluate (5), calculate first the derivative as

$$\frac{dk(\omega)}{d\omega} = \frac{n}{c} + \frac{\omega}{c} \frac{\omega_p^2}{n\omega^2} ,$$

using (6) and (7). Substitution of (8) into (5), utilizing (6), leads to

$$g(\omega) d\omega = \pi^{-2} c^{-3} \omega^2 n^2 \left[ n + \frac{\omega_p^2}{\omega^2} \frac{1}{n} \right] d\omega$$

or, in view of (7),

$$g(\omega) d\omega = n \frac{\omega^2}{\pi^2 c^3} d\omega .$$
The same expression can be found in literature \( /20/ \) and is correct, providing one takes into account the \( dn/d\omega \) contribution. As is often done, however, this dispersion term is neglected, thus resulting in a non-dispersive formula
\[
g(\omega)d\omega = n^3 \omega^2 d\omega/\pi^2 c^3.
\]

Now, if (9), together with (1), is inserted into (2), we obtain the following modified Planck law
\[
u_M = n \nu_0(\omega), \quad (10)
\]
where \( \nu_0(\omega) \) corresponds to the vacuum value of the energy density. This result was also obtained, by different route, by Bekefi \( /5/ \) and by Dawson \( /21/ \). Furthermore, Case and Chiu \( /22/ \) used the fluctuation-dissipation theorem to arrive at the same conclusion. A modified form of the Planck law in anisotropic media was described by Cole \( /23/ \), which in the isotropic limit reduces to (10).

As it is important to establish the correctness of the radiation energy density, we dwelt on that subject for a while. We are now in a position to present the generalization of the Brownian motion to include dielectric media. In the process, it will be confirmed that the exchanged momentum between the electromagnetic wave and a medium, in the \( \hbar \omega \gg kT \) limit, is proportional to the refractive index.

The following investigation parallels the work of Einstein \( /4/ \) developed in his studies of emission and absorption processes. An attempt was made by Skobel'tsyn \( /24/ \) to apply the method to optically dense media. His work, however, appears to be inconclusive. In the present treatment, the original (Einstein) method will be adapted to include simple media. The extension to other phases of matter is trivial, if the knowledge of the radiation energy density is assumed.
It should be pointed out, in order to avoid any possible misunderstanding, that the Einstein method is based on a general hypothesis regarding interaction of radiation with matter. Thus, the objects interacting with radiation (the Einstein molecules) have no special intrinsic properties and their role could be equivalently taken over by the Planck oscillators. Here, these objects will be referred to as elementary systems.

During any absorption (or emission) process, a momentum is transferred to an elementary system. The velocity distribution of such systems are entirely due to their interactions with the radiation field. Thus, this distribution must be the same as the one, these systems attain through their mutual collisions alone. Hence, it is necessary that the average kinetic energy of such a system in the radiation field be equal to $\frac{1}{2} kT$. Suppose now that we have an ensemble of these elementary systems, each of which can be in one of the two possible quantum states $Z^m$ and $Z^n$ with the corresponding energies $\epsilon_m$ and $\epsilon_n$. Next, let the transition from state $Z^m$ to state $Z^n$ be allowed. That transition will result if the system in state $Z^m$ absorbs a quantum of energy of magnitude $\epsilon_n - \epsilon_m$. Similarly, the transition from $Z^n$ to $Z^m$ will result when the system emits that quantum of energy. Now, there are two different processes an elementary system can undergo, in order to make a transition. First, there is a process independent of the external radiation field, viz., the so called spontaneous emission. The probability for such a process to occur in time $dt$ is

$$dP = A_{nm} \ dt,$$

(11)

where $A_{nm}$ is the Einstein "A" coefficient. Second, a transition which depends on the stimulation by the external field. Processes
of this type are known as Induced radiation processes. Thus, the probability \( dP \) that a system under the influence of the radiation density \( u(v) \) makes the transition from \( Z_m \) to \( Z_n \) by absorbing the energy quantum is \( dP = B_{mn} u(v) \, dt \). Similarly, the probability for the transition from \( Z_n \) to \( Z_m \) to occur is \( dP = B_{nm} u(v) dt \). Again, \( B_{mn} \) and \( B_{nm} \) are the Einstein "B" coefficients. Thus, in those two cases, the transition probability is the function of the radiation energy density \( u(v) \) surrounding our elementary system.

In other words, as the system is in a plasma, \( u(v) \) refers to the radiation density inside the latter. This density is given by (10) and its use is imperative. We shall prove below that this choice does not disturb the thermodynamic equilibrium conditions.

Note that the approach used by Skobel'tsyn /24/ is quite different. Originally, he assumed that the radiation energy density in a dielectric is the same as the one in vacuo, but, in the same article, he was forced to make an ad hoc assumption that density ought to be \( u(v)/n \) for \( n > 1 \).

In view of the above discussion, therefore, the probabilities for upward and downward transitions can now be written, respectively, as

\[
dP = n B_{mn} u_o(v) \, dt; \quad dP = n B_{nm} u_o(v) \, dt. \tag{12}
\]

Hence, the probability for the radiation induced processes in a collisionless plasma is decreased relative to vacuo. This is so, because, as the plasma density increases, the refractive index decreases and, consequently, the amount of radiation density is diminished, in accordance with (10).
We are now in a position to study the effect of the radiations field on the Brownian motion of test particles embedded in plasma. In other words, if a particle emits or absorbs an energy quantum $h\nu$, then it experiences a recoil. If the wave is spherical, then there is no recoil. The question now arises, as to how is the magnitude of the exchanged momentum modified by the presence of plasma. To simplify the calculations and, at the same time, to make the physics of the problem more transparent, it will be assumed that the particle of mass $M$ is moving with velocity $v$ in the $x$-direction and that the conditions are such that the higher powers of $v/c$ can be neglected. In a radiation field, the linear momentum $Mv$ of this particle changes, in time $\tau$, due to the effect of the radiation force $f = Rv$, $R$ being a coefficient to be evaluated below. In addition, during the same time, the fluctuations of the action of radiation on the particle, cause a momentum transfer $\Delta$ to the latter. The magnitude and the direction of this momentum transfer are constantly changing, as a result of the fluctuations of the radiation pressure. As the radiation field must not affect the overall velocity distribution in a thermal equilibrium, the average momentum of the particle, at both ends of the time interval $\tau$, must be $<Mv>$. Therefore, we must have

$$<(Mv - R\nu\tau + \Delta)^2 > = <(Mv)^2 > . \quad (13)$$

Furthermore, in order not to disturb this equilibrium, we shall assume the validity of the equipartition law, $Mv^2/2 = kT/2$. If, in addition, terms containing $V^2$ are dropped as being small and $\nu\Delta$ set equal to zero, since $v$ and $\Delta$ are uncorrelated, then the above equation reduces to
\[ \frac{\Delta^2}{\tau} = 2RkT \]  \hspace{1cm} (14)

This is the fundamental equation required by the theory of heat and a comparison with \( h \) affirms that the presence of a plasma does not affect it.

Now, \( \frac{\Delta^2}{\tau} \) and \( R \) are calculated and then inserted in (14), to see if that relation is satisfied. However, in order to calculate these quantities, using the method devised by Einstein, it is necessary to know the radiation energy density and the exchanged momentum. The former is given by (10) and, as far as the momentum is concerned, we assume it to be proportional to the refractive index of the surrounding medium. Then, if the equality (14) holds, our momentum assumption will be justified.

In order to determine the propagation of light in moving media, it is only required to apply the Lorentz transformation to the corresponding laws for stationary body - a prescription given by e.g., Pauli /25/. As the radiation laws are defined only for the inertial frame moving with the medium, in calculating constant \( R \), the above prescription is followed. To distinguish the quantities referring to the rest frame, they will be marked with a dash. Thus, radiation per unit volume, in the frequency range \( dv \), within a solid angle \( d\Omega \), transforms according to

\[ u'(v',\phi') = u(v) \frac{dv}{dv'} \frac{d\Omega}{d\Omega'} (1 - 2\beta \cos \phi), \]  \hspace{1cm} (15)

where \( \beta = v/c \) and \( u'(v',\phi') \) depends on the direction defined by the angle \( \phi' \) it makes with the \( x'-axis \), and the angle \( \psi' \) between its projection in the \( y'z' \)-plane and the \( y'-axis \). To facilitate the
transformation of the argument of $u'(v', \phi')$, the transformation formulae for the Doppler effect and the aberration are needed. These, however, are well known in the theory of relativity and, in the approximation applicable here, take the forms

\[ v' = v(1 - n \beta \cos \phi') \] (16)

and

\[ \cos \phi' = \cos \phi - n \beta + n \beta \cos^2 \phi', \] (17)

respectively. Further, no relevant information is lost if we put

\[ \psi' = \psi. \] (18)

The inverse transformation to that in (16) is

\[ v = v'(1 + n \beta \cos \phi') \] (19)

which, on substitution in $u(v)$ and subsequent expansion, gives

\[ u(v) = u(v') + v' n \beta \cos \phi' \frac{\partial u(v')}{\partial v'} \] (20)

A solid angle $d\Omega$ is (in the present approximation) connected with $d\Omega'$ by

\[ \frac{d\Omega}{d\Omega'} = 1 - 2 n \beta \cos \phi'. \] (21)

Thus, the substitution of (19), (20), and (21) in (15) results in

\[ u'(v', \phi') = \left[u(v') + v' n \beta \cos \phi' \frac{\partial u(v')}{\partial v'}\right] (1 - 3 n \beta \cos \phi'). \] (22)

Making use of the last relation and the assumption about the elementary processes, discussed earlier, allows the calculation of the average momentum transferred, per unit time, to the test particle. Let, then, the number of elementary transitions from level $m$ to level $n$ be
\[ \frac{1}{S} \sum_{m} w_{m} e^{-\varepsilon_{m}/kT} B_{mn} \int_{\Omega_{n}} u'(\nu',\phi')d\Omega'/4\pi, \quad (23) \]

where
\[ S = \sum_{m} w_{m} e^{-\varepsilon_{m}/kT} + \sum_{n} w_{n} e^{-\varepsilon_{n}/kT}. \]

and \( w_{m}, w_{n} \) are the statistical weights corresponding, respectively, to levels \( m \) and \( n \). An analogous form holds for the transition from level \( n \) to level \( m \).

As assumed earlier, the momentum of the exchanged photon in plasma is given by the Minkowski relation. Therefore, in each elementary process, the momentum \( nhv \cos \phi'/c \) is transferred to a particle. Combining the upward and downward transitions, the total momentum transferred to the particle is then expressed, using (23), as

\[ nhv(Sc)^{-1} \sum_{m} \int_{\Omega_{n}} u'(\nu',\phi') \cos \phi' d\Omega'/4\pi, \quad (24) \]

Substituting (22) into (24) and putting \( Q = \sum_{m} w_{m} B_{mn} e^{-\varepsilon_{m}/kT} \), the total average momentum transferred, per unit time, to a particle through the induced processes can be written as

\[ -\frac{n^2 hv}{c^2} \beta \left[ u(\nu) - \frac{v}{3} \frac{3u(\nu)}{3\nu} \right] Q(1 - e^{-hv/kT}). \quad (25) \]

For the sake of clarity, the dashes over symbols have been dropped here.

As the spontaneous emission does not have any preferred direction, the average momentum transfer, due to processes of this type, vanishes. Consequently, the coefficient \( R \) can be written, finally, as

\[ R = \frac{n^2 hv}{c^2} \beta \left[ u(\nu) - \frac{v}{3} \frac{3u(\nu)}{3\nu} \right] Q(1 - e^{-hv/kT}). \quad (26) \]
Prior to utilizing this result, let us inquire about the effect of the presence of a plasma on the momentum fluctuation $\Delta$. Denote by $\delta$ the momentum transferred to the particle during a single process and assume that its sign and magnitude vary in time in such a way that the average value of $\delta$ vanishes. Now, if $\delta_i$ refers to the transfer of momentum through the $i$-th independent process, then the resultant transfer is $\Delta = \sum \delta_i$. Because the average value of each $\delta_i$ vanishes, it is necessary to write (assuming $\overline{\delta_i^2} = \overline{\delta^2}$)

$$\overline{\Delta^2} = N \overline{\delta^2},$$  \hspace{1cm} (27)

where $N$ is the number of all elementary transitions in time $\tau$. If the momentum corresponding to each elementary process $\delta = nh\nu \cos \phi/c$, is substituted into (27), it then yields

$$\overline{\Delta^2} = \frac{N}{3} \left( \frac{nh\nu}{c} \right)^2,$$  \hspace{1cm} (28)

As the number of induced processes from level $m$ to level $n$, in time $\tau$, in equilibrium, is $N/2$, it is, then, possible to write

$$N = \frac{2}{\tau} Q u(\nu),$$

which, on combining with (28), gives

$$\frac{\overline{\Delta^2}}{\tau} = \frac{2}{3} \left( \frac{nh\nu}{c} \right)^2 Q u(\nu).$$  \hspace{1cm} (29)

As previously emphasized, the expressions (26) and (29) are determined in terms of the radiation energy density $u(\nu)$ which, evidently, differs from its vacuum value $u_0(\nu)$. To show the validity of the claim that the momentum of the exchanged photon inside plasma is proportional to the refractive index of the latter, we must show that the thermodynamic equilibrium is not disturbed when
the radiation momentum is transferred to a particle. This, in turn, will be so if, after calculating $R$ and $\delta \gamma$ in terms of the proposed energy and momentum densities of radiation, their subsequent insertion in (14) yields an identity. Therefore, let us express $R$ in terms of vacuum energy density $u_0(v)$ and substitute the modified Planck law (10) into (26). The step by step evaluation proceeds as follows:

$$\frac{3u(v)}{3v} = \frac{3}{3v} n(v) u_o(v) = u_0(v) \frac{3n(v)}{3v} + n(v) \frac{3u_o(v)}{3v}, \quad (30)$$

where

$$u_0(v) = \frac{8\pi \hbar \nu^3}{c^3} (e^{\hbar \nu / kT} - 1)^{-1}. \quad (31)$$

Then, substitute (31) in (30) and calculate the individual terms

$$u_0(v) \frac{3n(v)}{3v} = \frac{8\pi \hbar \nu^3}{c^3} e^{\hbar \nu / kT} - 1 \frac{1 - n^2(v)}{n(v)}$$

and

$$n(v) \frac{3u_o(v)}{3v} = \frac{8\pi \hbar \nu^3}{c^3} e^{\hbar \nu / kT} - 1 \left[ 3n - \frac{n e^{\hbar \nu / kT}}{3(e^{\hbar \nu / kT} - 1)} \frac{\nu}{kT} \right].$$

Finally,

$$\left[ \frac{u(v)}{3} - \frac{v}{3} \frac{3u(v)}{3v} \right] = \frac{8\pi \hbar \nu^3}{c^3} e^{\hbar \nu / kT} - 1 \left[ n - \frac{1 - n^2}{3n} - n + \frac{n e^{\hbar \nu / kT}}{3(e^{\hbar \nu / kT} - 1)} \frac{\nu}{kT} \right].$$

Substituting for the square bracket in (26), the above expression leads to

$$R = \frac{n \hbar \nu}{c^3} \frac{u_o(v)}{3} \left\{ n^2 \frac{\hbar \nu}{kT} + n \frac{e^{\hbar \nu / kT}}{e^{\hbar \nu / kT} - 1} + n^2 - 1 \right\} Q \left( 1 - e^{-\hbar \nu / kT} \right). \quad (32)$$

In order that the Brownian particle is not internally affected by the radiation field, $\hbar \nu = \epsilon_n - \epsilon_m$ must be much larger than $kT$. 
Effectively, restricting our considerations to the region of validity of the Wien Law, the above reduces to

\[ R = \frac{n^3}{c^2} \frac{h\nu}{3} \frac{u_0(\nu)}{\kappa T} Q. \quad (33) \]

In addition, when (10) is inserted in (29), it results in

\[ \frac{\Delta^2}{\tau} = \frac{2}{3} n^3 \left( \frac{h\nu}{c} \right)^2 u_0(\nu) Q. \quad (34) \]

Finally, if (33) and (34) are substituted into (14), the identity is, indeed, obtained. Therefore, it can be concluded that the Minkowski form of the momentum density, used in the preceding analysis, appears to be satisfactory as it fulfills the condition for the thermal equilibrium, at least in the region where \( h\nu >> \kappa T \), when surrounded by a collisionless plasma.
Large part of the present investigation is concerned with the concept of mass and a possible modification of the classical electrodynamics. Therefore, it seems appropriate to review some of the earlier studies, which bear some relation to the present work. In doing so, the conceptual background will be discussed, which should prove useful for our purposes.

As already pointed out, one of the aims of this work is the study of the feasibility of generalization of the Maxwell equations when the electromagnetic wave propagates in a non-empty space. As will be seen in what follows, this is closely connected with the mass generation problem.

The first theory to be mentioned here, was proposed by Mie /8/. As is well accepted, the Maxwell and Lorentz theories do not hold inside the electron. Therefore, Mie has attempted to generalise the electrodynamics in such a way that the repulsive Coulomb forces are balanced inside an electron, whilst the field in the region outside the electron is unaffected by the proposed modifications. It turns out, according to Mie, that it is possible to generate the mass from the field. Thus, the first set of the Maxwell equations

$$\frac{\partial}{\partial t} F_{\alpha\beta} + \frac{\partial}{\partial x_{\gamma}} F_{\gamma\alpha} + \frac{\partial}{\partial y_{\alpha}} F_{\alpha\beta} = 0 \quad (1)$$

with $F_{\alpha\beta}$ describing the electromagnetic field tensor, together with the continuity condition

$$\frac{\partial}{\partial t} j^\alpha = 0 \quad (2)$$

with $j^\alpha$ denoting the four-current density, is retained in that theory. It follows from (2) that there must exist an asymmetric
tensor $H^\mu\nu$ such that

$$j^\mu = \partial_\nu H^\mu\nu.$$  

(3)

In the Lorentz theory it is assumed that $H^\mu\nu = F^\mu\nu$.

According to Mie, however, $H^\mu\nu$ has a real significance and its components are universal functions of $F^{\alpha\beta}$ and $\phi^\alpha$,

$$H^\mu\nu = \alpha_{\mu\nu}(F^\gamma\delta, \phi^\gamma).$$  

(4)

In addition, the four-current density is then given by

$$j^\alpha = \beta_\alpha (F^\gamma\delta, \phi^\gamma).$$  

(5)

In view of these assumptions, the field equations acquire a new physical content. Thus, relations (4) differ from those of conventional electrodynamics, as $H^\mu\nu$ now depends explicitly on $\phi^\gamma$.

That is, not only the potential difference, but also the potential itself acquires a real existence. By virtue of the above assumption, (4) and (5), Mie introduced in the theory ten universal functions. However, by invoking the energy principle, these unknown can be reduced to a single one, resulting in a noticeable simplification.

This unknown is often called the world function $L(F^\mu\nu, \phi^\mu)$. If this invariant function exists, then it is true to say that

$$H^\mu\nu = \frac{\partial L}{\partial F_{\mu\nu}}$$

and, consequently, we can write

$$\partial L = H^\mu\nu \delta F_{\mu\nu} - 2j^\mu \delta \phi_\mu.$$  

(6)

One of the difficulties encountered at this stage is the determination of $L$. Quite generally, the only invariants that can be formed from $F^\mu\nu$ and $\phi^\mu$ are: the square of the vector $\phi_\alpha - \phi_\alpha \phi^\mu$; the square of the tensor $F_{\alpha\beta}^\mu - \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta}$; the square of the vector
and the square of the tensor of the fourth order with components $\Sigma^{\alpha\beta}_{\gamma\delta} F_{\alpha\beta} \phi^{\gamma\delta}$; it is then true to assert that $L$ must be expressible in terms of these four invariants. The problem of matter in the theory of Mie can, thus, be replaced by the determination of the appropriate expression for $L$. As is well known, if $L$ is set equal to the second of the above invariants, the Mie theory reduces to the electron theory in the space containing no charges. Evidently, $L$ must be different from this invariant inside the electron.

Turn the attention now to the determination of the energy-momentum tensor $S_{\alpha\beta}$. Using the notation of the general theory of relativity, together with the variation of $g_{\alpha\beta}$, Weyl simplified the calculations of Mie, who was using, obviously, the special theory of relativity. Weyl showed that $\phi$ and $F_{\alpha\beta}$ remain ordinary tensors, whilst $j^\alpha$ and $H^{\alpha\beta}$ should be replaced by tensor densities $\gamma^\alpha$ and $H^{\alpha\beta}$, respectively. Consequently, we have to write

$$\gamma^\alpha = 0$$ (8)

and

$$\gamma^\alpha = \partial_\beta H^{\alpha\beta}. \quad (9)$$

The action function is now written as

$$\delta \int L \ dx = 0 \quad (10)$$

and it is used to determine the energy tensor $S_{\alpha\beta}$. Since $L$ is now independent of the derivatives of the $g_{\alpha\beta}$, we can write

$$\delta L = L_{\alpha\beta} \delta g^{\alpha\beta}$$

and, in turn,

$$S_{\alpha\beta} = \frac{\partial L}{\partial g^{\gamma\delta}} g^{\gamma\delta} - \frac{1}{2} L \delta_{\alpha}^{\beta}, \quad (11)$$
where the assumption of a constant electromagnetic field is implied.

Now, if the variation of the field quantities with respect to the infinitesimal coordinate transformations is substituted into the general expression for $\delta L$, the latter must vanish, resulting in

$$\frac{\partial L}{\partial \phi^\alpha} g^{\beta \gamma} = H^{\beta \gamma} F_{\alpha \beta} - j^\beta \phi_\alpha.$$  \hspace{1cm} (12)

A subsequent substitution into (11) leads to the expression for the energy tensor, already obtained by Mie using a different method,

$$S_{\alpha}^\beta = H^{\beta \gamma} F_{\alpha \beta} + j^\beta \phi_\alpha - \frac{1}{2} L \delta_\alpha^\beta.$$  \hspace{1cm} (13)

Now, knowing the energy tensor $S_{\alpha}^\beta$, it is possible to determine the rest mass of the particle from

$$m_0 = - \int S_{\alpha}^\alpha \, d\tau,$$  \hspace{1cm} (14)

where $d\tau$ is the spatial volume. In addition, using the mixed components of $S_{\alpha}^\beta$, the Lorentz force can also be determined within the framework of the Mie theory.

Let us now discuss some of the reasons why this theory has not succeeded. As it has not been possible to establish the form of the world function $L$ which corresponds to the reality, the theory allows the existence of elementary particles with arbitrary value of their total charge. It may be possible, in principle, to choose other expression for $L$, which could well correlate with experiment, however, this situation is unsatisfactory, as we have no method of determining the world function. Another difficulty, already pointed out by Mie, seems to follow from his very assumption. As in the Mie electrodynamics the absolute value of the potential plays an important role, it follows that if $\phi$ is a solution (the electrostatic potential) of the field equations, $\phi + \text{constant}$ does not satisfy them.
To put it in other words, a massive particle cannot exist in a static external potential field. This is, perhaps, the main reason leading to a downfall of the Mie theory.

Another interesting proposal for the nonlinear theory of the electromagnetic field was put forward by Born /9/ and Born and Infeld /10/. Recall that the field equations are determined by the principle of the least action. The Lagrangian of the Maxwell theory \( L = \frac{1}{2} (\mathbf{H}^2 - \mathbf{E}^2) \) leads to infinities which violate the principle of finiteness. In order to restore it, the assumption of the upper limit of the field strength is invoked, resulting in the Born modification of the action function

\[
L = \beta^2 \left[ 1 + \frac{1}{\beta^2} (H^2 - E^2) \right]^{1/2} - 1
\]  

(15)

where \( \beta \) has the dimensions of a field strength and may be called the absolute field. In order to take the guess out of this expression, they /9,10/ start from the principle of general covariance, i.e., requiring that all laws of nature are covariant under all space-time transformations. Thus, it is necessary to find a transformation group which is larger than the Lorentz group of the special theory of relativity. They proceed to show that the Lagrangian (15) belongs, indeed, to the group of general relativity. Consequently, the relation between the Born theory and that of Maxwell is analogous to that between the mechanics of Einstein and Newton. Thus, having found the Lagrangian density, the field equations as well as the conservation laws can be determined. From the action principle, the field equations are

\[
\begin{align*}
\beta \sqrt{-g} \, F^{\mu \nu} & = 0 \\
\beta \sqrt{-g} \, H^{\mu \nu} & = 0 
\end{align*}
\]  

(16)
where $F^\mu\nu$ is the dual tensor and the relation between $H^{\mu\nu}$ and $F^{\mu\nu}$ is similar to that of the dielectric displacement and magnetic induction to the field strengths in the Maxwell theory. In order to gain some insight into this theory, consider the electrostatic field. It is then found that the solution for the potential is

$$\phi(r) = \frac{e}{r_0} f\left(\frac{r}{r_0}\right),$$  \hspace{1cm} (17)

where

$$f(x) = \int_x^\infty \frac{dy}{\sqrt{1 + y^2}}; \quad r_0 = \frac{\sqrt{e}}{\beta}.$$  \hspace{1cm} (18)

Thus, (17) represents a modification to the Coulomb law, which, of course, regains its established form if $r >> r_0$. The advantage of the potential (17) is its finiteness as $x \to 0$. It is found that the potential has the maximum at this point which is given by

$$\phi(0) = 1.85 \frac{e}{r_0}.$$  \hspace{1cm} (19)

Using the conservation laws, the field mass is determined. Its value is found to be

$$m = 1.24 \frac{e^2}{c^2 r_0},$$  \hspace{1cm} (20)

which is finite. In order to complete the theory, the value of the critical field $\beta$ is calculated. In view of (20), this value is

$$\beta = 9.18 \times 10^{-12} \text{ e.s.u.}.$$  \hspace{1cm} (21)

This large magnitude of the critical field justifies the use of the Maxwell equations in most cases. However, for the wavelengths of the order of $r_0$, the modifications are mandatory.

The philosophical superiority of the theory which assumes only one physical entity, viz., the electromagnetic field, resulting in a mass being a secondary (derived) concept, is unquestioned. Yet, the reason why this theory has not been generally accepted can be
According to the Mach principle, as is well known, the only meaningful motion is that relative to the rest of the matter in the Universe. There are two types of fields which transmit the long-range forces, viz., the gravitational and electromagnetic fields. In line with the Mach principle, Brans and Dicke \cite{11} built a theory assuming the existence of still another kind of long-range forces, produced by scalar fields. Their theory is a generalization of the general theory of relativity. The resulting gravitational effects are in part geometrical and in part due to the postulated scalar interaction. The inertial masses of the elementary particles are not the fundamental constants, but arise from their interaction with the surrounding field. Consequently, the particle's mass can be determined by measuring the gravitational acceleration $Gm/r^2$.

Put otherwise, the gravitational constant $G$ is related to the scalar field $\phi$, which is, in turn, coupled to the mass density in the Universe.

$$\delta \left[ 16\pi G L + R \right] \sqrt{-g} \, d^4x = 0, \quad (22)$$

where $L$ is the Lagrangian density and $R$ is the scalar of curvature. The proposed generalization of the above takes the form

$$\delta \left[ 16\pi L + \phi R - \omega (\partial_\alpha \phi \partial^\alpha \phi/\phi) \right] \sqrt{-g} \, d^4x = 0, \quad (23)$$

where $\omega$ is a dimensionless constant and it is assumed that $G^{-1}$ is proportional to $\phi$ so that the Lagrangian density of this scalar field is included in (23). Now, in order to obtain the wave equations for $\phi$, we vary $\phi$ and $\partial_\alpha \phi$ in (23). This equation can be expressed as
\[ \Box \phi = \frac{8\pi}{3 + 2\omega} T^\alpha_\alpha, \] (24)

where $\Box$ is the generally covariant d'Alembertian. Thus, the source term for the generation of $\phi$ waves is expressed by the trace of the energy-momentum tensor $T^\alpha_\alpha$ of matter alone, what agrees with the content of the Mach principle.

Further, in order to find the field equations for the metric field, the metric tensor and the corresponding first derivatives are varied this time, leading to

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi}{\phi} T_{\mu\nu} \frac{\omega}{\phi^2} \left( \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \phi \partial_\beta \phi \right) - \frac{1}{\phi} \left( \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \Box \phi \right). \tag{25}
\]

In this equation the terms on the RHS have the following meaning: the first term denotes the source, as in the relativity theory; the second term refers to the energy-momentum tensor of the scalar field; the last term arises from the existence of the second derivatives of the metric tensor. To see the point of contact with the Einstein theory, consider the limit $\omega \to \infty$. The relations (24) and (25) then reduce to the well known results

\[ \Box \phi = 0 \]
\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu}. \tag{26} \]

The theory of Brans and Dicke is not widely accepted. Although it can explain the deflection of light, the gravitational redshift, and the perihelion rotation of the orbit of Mercury, the computed results differ from those of the general relativity. Yet, as there is not a sufficient experimental accuracy to determine the
correct theory, it is too early to choose between the two. It is also very difficult to establish the numerical value of the coupling parameter $\omega$.

It seems pertinent, at this point, to mention an interesting conclusion of the Brans-Dicke theory: It predicts the time-dependence of the gravitational constant. Although a rather unconventional result, it seems to be based on sound physical reasoning. It follows from the argument that in the Mach picture there is no direct way in which the mass of two particles at different space-time points can be compared. However, there is no ambiguity if the mass ratios are involved. Thus, using the characteristic mass of gravitation, $(\hbar c/G)^{\frac{1}{2}}$, the mass ratio of a particle $m(G/c)^{\frac{1}{2}}$ should be uniquely defined and, further, provides the means to compare the masses at different space-time points. Now, if $\hbar$ and $c$ are assumed to be constants and, in addition, if the number $(27)$ varies, then it may be asserted that either $m$ or $G$, or both, are space-time dependent. In principle, there is no difference between the theories involving constant mass of a particle or the constant value of $G$, although they may lead, formally, to a different structure of the theory. The metric tensors, corresponding to either alternative, are then also different /11/ and their relation is expressed by a conformal transformation.

Finally, we shall recapitulate here a very interesting proposition, put forward by Rainich /12/ and, later, further developed by Misner and Wheeler /28/ in their study of the space theory matter. Thus, there exist two basic tensors commonly used in physics: the
electromagnetic field tensor and the Riemann tensor. The former combines, in the domain of the special theory of relativity, the electric and magnetic field vectors, whilst the latter determines the curvature of the space which, in turn, accounts for the gravitational phenomena. Clearly, these tensors have quite different roles. Thus, from the curvature of space, described by the Riemann tensor, one can deduce the gravitational properties — we say that the gravitation can be geometrized. On the other hand, the electromagnetic tensor is commonly understood to be superimposed on this space and is treated independently of the space. There were several attempts to include the electromagnetic tensor in the geometry of space, of which the most successful appears to be the model proposed by Weyl /26/. In essence, the Riemann tensor was replaced by a more general curvature tensor, describing a different space geometry. It was then conjectured that one part of this tensor belongs to the gravitational domain, whilst the other one describes the realm of electromagnetism.

Not wanting to change the curvature tensor, Rainich /12/ turned his attention to the energy-momentum tensor which connects the electromagnetic tensor with the Riemann tensor. As a consequence of that investigation, it became clear that under certain (Rainich) assumptions the electromagnetic field is completely determined by the space curvature. In a nutshell, Rainich starts by comparing the electromagnetic and Riemann tensors at a given point, without taking into account the appropriate tensor fields. Put otherwise, the algebraic properties of the tensors of the second rank are analyzed first. As is well known from the special theory of relativity, we have three types of vectors: time-like, null, and
space-like, depending on the value of their scalar product. Furthermore, there are three kinds of planes: those containing two-, one-, and none- of the null vectors. It is then argued that only the first and the last case have any connection with reality, as the two corresponding invariants of the tensor are not simultaneously zero.

Now, when the four, mutually perpendicular unit vectors $i, j, k, l$, with the square of each of the length $+1$, are introduced, the electromagnetic tensor can be written in the form

$$f(x) = \lambda \left[ j(x) - j(i x) \right] + \mu \left[ k(x) - k(i x) \right], \quad (28)$$

where $f(x)$ denotes $F_{\alpha \beta} x^\alpha$, and $\mu$ and $\lambda$ are real and imaginary numbers, respectively.

Next, according to the general theory of relativity, the energy-momentum tensor at a given point can be obtained from the Ricci tensor $R_{\alpha \beta}$. This tensor is given by $R_{\alpha \beta} = \frac{1}{2} g_{\alpha \beta} R$, where $R$ is the scalar of curvature. If there is no matter in the region in question, then the just mentioned tensor must be equal to the electromagnetic energy-momentum tensor, leading to the equation

$$R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R = F_{\alpha \sigma} F_{\beta \sigma} - \frac{1}{2} g_{\alpha \beta} F_{\sigma \tau} F^{\sigma \tau}. \quad (29)$$

This is a clear indication that the electromagnetic energy tensor can be expressed in terms of the geometry of the surrounding space. This relationship is further studied by Rainich. He then finds that if (28) is to satisfy (29), the constants $\lambda$ and $\mu$ satisfy

$$\lambda = \omega \sqrt{2} \sin \phi \quad \mu = \omega \sqrt{2} \cos \phi, \quad (30)$$

where

$$2\omega^2 = \mu^2 - \lambda^2.$$
It now emerges that the electromagnetic tensor is not determined by the Riemann tensor uniquely, as the scalar $\phi$ remains arbitrary. Put otherwise, in view of (30), there is an infinite number of electromagnetic tensors satisfying the given curvature tensor. Therefore, in order to complete the proposed theory, it is necessary to eliminate this arbitrariness. Recall that so far only the algebraic properties of tensors were considered. Now, by studying the differential properties of the energy-momentum tensor, it is possible /12/ to reduce this arbitrariness to one constant of integration. In order to remove even this ambiguity, the analogy and difference between the theory of electromagnetic field and the theory of analytic functions of complex variables are analyzed. It is then found that, assuming the Rainich conditions are satisfied, the electromagnetic field is completely determined by the curvature of the space-time. The Maxwell equations thus reduce to geometric expressions connecting the Ricci tensor and its rate of change.

Developing the idea of Rainich further, Misner and Wheeler /28/ allowed for multiple connected topologies and showed their compatibility with the Riemann space. The charge is then interpreted in terms of source free Maxwell field in vacuo which is, however, trapped in the "worm holes". In addition, the gravitational mass originates entirely in the energy of the electromagnetic field. Therefore, mass, as well as charge, are certain representations of the concept of curved space. This theory is not without difficulties, as it can explain only masses larger than $10^{42}$ kg - and these magnitudes are, simply, unknown. Elementary particles and real masses are excluded, on the assumption as belonging to the realm of micro-physics.
IV GENERALLY COVARIANT APPROACH

In essence, we study the interaction of electromagnetic waves with matter. Commonly, this phenomenon is dealt with in a linear approximation and in many cases that is sufficient to explain the results of experiments. However, from the theoretical point of view, the linear theory is only an approximation which, strictly speaking, should be replaced by a more general, nonlinear theory. This idea has already been touched upon by the present author /29/ and is further developed here. It stemmed from an inquiry into the feasibility of the assumptions /3/ used to describe the relative merit of the Abraham and Minkowski tensors. The premise of that work, viz., that an electromagnetic wave propagating through a medium is influenced by the latter and that this effect is manifested via the generation of massive photons, is further analyzed.

This brings us to the most familiar, yet, the least understood concept in physics. I am referring to the notion of mass. Our comprehension of this issue is minute, although, recently, some information has been obtained from elementary particle theory. An excellent monograph on the subject of mass, from a general standpoint, has been written by Jammer /30/, who not only covers the historical development of this concept but also explains the difficulties encountered, at the time of writing. Therefore, we shall not review the related literature, as it would not be possible to add anything of substance to Jammer's book, but concentrate, instead, on the more recent developments, with the aim of bringing a new insight to the old issue.
The idea to be elucidated bears some resemblance to the Mach principle. It will be recalled that the latter suggests that, roughly speaking, the mass of a body is due to the influence of all stars in the Universe on that body. Expressed in mathematical terms, this dependence takes the form

\[ m(x) = \xi Z(x), \]  

(1)

where \( m(x) \) is the mass of the particle, \( \xi \) - a coupling constant and \( Z(x) \) is the mass generating field. The central idea is, therefore, not to consider the mass of a particle as an intrinsic, fixed quantity but, rather, to think of it as a variable quantity depending on the field in which it moves.

The Mach principle is not universally accepted and opinions about its validity differ \(^31\)/. There are many who dismiss this idea altogether on the basis that it is not testable. On the other hand, as the general theory of relativity is built on this concept, and some of the predictions of that theory were experimentally confirmed, it is difficult to see that the connection between the two could be coincidental. In any case, we shall not go into this debate here, but simply assume the validity (in the absence of any proof to the contrary) of the relation (1).

To take the effect of all stars in the Universe, Einstein introduced the concept of curved space into physics. Let us start, then, with modifications of electrodynamics in such a space. Using the generally covariant expressions, we can write the Maxwell equations in an arbitrary system of coordinates as \(^32\) 

\[ \frac{1}{\sqrt{|g|}} \partial_\beta \sqrt{|g|} F^{\alpha \beta} = j^\alpha, \]  

(2)
where $F^{\alpha\beta}$ is the electromagnetic field tensor and $j^\alpha$ is the four-current density. Clearly, in every local inertial frame the above reduce to the standard Maxwell equations in vacuo. On account of simplifications, introduce a system of coordinates (also known as time-orthogonal system) in which

$$\varepsilon_{\lambda_4} = 0 .$$  \hspace{1cm} (3)

In addition, introduce the antisymmetric tensors $H^{\alpha\beta}$ and $B^{\alpha\beta}$ by

$$F^{\alpha\beta} = \frac{H^{\alpha\beta}}{\sqrt{-\varepsilon_{\lambda_4}}} = B^{\alpha\beta} \hspace{1cm} (4)$$

and the vectors $D^\alpha$ and $E^\alpha$ by

$$F_{\alpha\lambda_4} = \frac{-D^\alpha}{\sqrt{-\varepsilon_{\lambda_4}}} = -\frac{E^\alpha}{\varepsilon_{\lambda_4}}.$$  \hspace{1cm} (5)

The corresponding covariant expressions are

$$F_{\alpha\beta} = \varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} F^{\gamma\delta} = \varepsilon_{\alpha\gamma} \varepsilon_{\beta\delta} B^{\gamma\delta} = B_{\alpha\beta} = \frac{H_{\alpha\beta}}{\sqrt{-\varepsilon_{\lambda_4}}} \hspace{1cm} (5a)$$

$$F_{\alpha\lambda_4} = \varepsilon_{\alpha\gamma} \varepsilon_{\lambda_4\delta} F^{\gamma\delta} = \varepsilon_{\alpha\beta} \sqrt{-\varepsilon_{\lambda_4}} B^{\beta} = \sqrt{-\varepsilon_{\lambda_4}} D_{\alpha} = E_{\alpha} .$$  \hspace{1cm} (5b)

This allows us to write the Maxwell equations (2) in the form

$$\partial_{\gamma} B_{\alpha\beta} + \partial_{\alpha} B_{\beta\gamma} + \partial_{\beta} B_{\gamma\alpha} = 0 \hspace{1cm} (6)$$

$$\frac{1}{\sqrt{-g}} \partial_{\beta} (\sqrt{-g} H^{\alpha\beta}) - \frac{1}{\sqrt{-g}} \partial_{\alpha} (\sqrt{-g} D^{\alpha}) = \rho u^\alpha .$$

In order to rewrite these in a more familiar form, introduce the dual vectors $B$ and $H$ corresponding to the tensors $B_{\alpha\beta}$ and $H^{\alpha\beta}$ by the
standard prescription, resulting in

\[ \begin{align*}
B^1 &= \frac{1}{\sqrt{-g}} B_{23} \\
B^2 &= \frac{1}{\sqrt{-g}} B_{31} \\
B^3 &= \frac{1}{\sqrt{-g}} B_{12}
\end{align*} \]

\[ H_1 = \sqrt{-g} H^{23} \quad H_2 = \sqrt{-g} H^{31} \quad H_3 = \sqrt{-g} H^{12}. \]

Finally, on substituting (7) into (6), we obtain the Maxwell equations in the form

\[
\begin{align*}
\nabla \times E &= -\frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} B) \\
\nabla \times H &= \frac{1}{\sqrt{-g}} \partial_t (\sqrt{-g} D) = \rho u \\
\n\nabla \cdot B &= 0 \\
\n\nabla \cdot D &= \rho .
\end{align*}
\]

(8)

The connection between the electromagnetic waves in a medium and the same waves in vacuo in a curved space now emerges. Thus, if \( g \) in the above equations does not depend on time, and if we put

\[ \varepsilon = \mu = (\varepsilon_{\mu\nu} - 1), \quad (9) \]

then the field, responsible for the space curvature, behaves like a medium with the dielectric permittivity and magnetic susceptibility given by (9). In the following, it will be assumed that inverse of this statement is also valid. In other words, in studying the electromagnetic wave propagating in a dielectric medium, one may, equivalently, replace that medium by vacuo, providing a non-vanishing space-time curvature is simultaneously introduced.

In support of the above idea, investigate the effect of the intense electromagnetic field on the geometry of space-time. Thus, if the Minkowski metric is denoted by \( \eta_{\mu\nu} \), the metric produced by the intense electromagnetic field can then be written as

\[
g_{\mu\nu} (x, t) = \eta_{\mu\nu} + h_{\mu\nu} (x, t). \quad (10)
\]
The perturbation $h_{\mu \nu}(\mathbf{r},t)$ can be obtained from the linearized Einstein equations /32/

$$h_{\mu \nu}(\mathbf{r},t) = -16\pi G(T_{\mu \nu}(\mathbf{r},t) - \frac{1}{2} \eta_{\mu \nu} T_{\kappa \kappa}(\mathbf{r},t)), \quad (11)$$

where $G$ denotes the gravitational constant and $T_{\mu \nu}$ is the energy-momentum tensor defined by

$$T_{\mu \nu} = \frac{1}{2\kappa} (\mathbf{F} \cdot \mathbf{F} - \eta_{\mu \nu} F^{\alpha \beta} F_{\alpha \beta}). \quad (12)$$

Now, to calculate the perturbation $h_{\mu \nu}(\mathbf{r},t)$ consider a simple example. Thus, an intense radiation beam normally incident on a mirror is reflected and travels in the same line with its direction reversed. For the argument's sake, this process may be imagined as a head-on collision of a particle with vanishing mass $m$ (representing an electromagnetic wave) and a very massive particle of mass $M$ (describing the mirror). Furthermore, if the initial four-momentum $p_{m}^{\mu}$ of a particle with mass $m$ has the components

$$p_{m}^{\mu} = (E_{m}, 0, p \sin \theta, p \cos \theta), \quad (13a)$$

then the final momentum is given by

$$p_{mf}^{\mu} = (E_{m}, 0, -p \sin \theta, -p \cos \theta). \quad (13b)$$

Similarly, the initial and final four-momenta corresponding to the other particle are, respectively,

$$p_{M}^{\mu} = (E_{M}, 0, -p \sin \theta, -p \cos \theta)$$

and

$$p_{Mf}^{\mu} = (E_{M}, 0, p \sin \theta, p \cos \theta)$$

In addition, if the involved particles are assumed to be point-like, then the appropriate energy-momentum tensor is given by /33/
\[ T^{\nu\tau}(r,t) = \frac{p^\mu_{mi}}{E_{mi}} \ p^\nu_{mi} \ \delta^3(r - v_{mi} \ t) \theta(-t) + \frac{p^\mu_{mf}}{E_{mf}} \ p^\nu_{mf} \ \delta^3(r - v_{mf} \ t) \theta(t) + \]

\[ + \frac{p^\mu_{mi}}{E_{mi}} \ p^\nu_{mi} \ \delta^3(r - v_{mi} \ t) \theta(-t) + \frac{p^\mu_{mf}}{E_{mf}} \ p^\nu_{mf} \ \delta^3(r - v_{mf} \ t) \theta(t), \]

(15)

where, e.g., \( E_{mf} \) describes the final energy of a particle with mass \( M_f \), \( v \) denotes the appropriate velocity, and \( \theta(t) \) is the step function.

Next, as is well known /33/, the solution of (11) is given by

\[ h_{\mu\nu}(r,t) = \frac{4G}{\pi} \int S_{\mu\nu} \frac{(r',t - |r - r'|)}{|r - r'|} d^3r'. \]

(16)

where

\[ S_{\mu\nu}(r,t) = T_{\mu\nu}(r,t) - \frac{1}{\mu} \int S_{\mu\nu}(k,\omega) e^{-i\omega(t-\tau)} d\omega + c.c., \]

(17)

At distances much greater than the source dimensions, the above can be approximated by

\[ h_{\mu\nu}(r,t) = \frac{4G}{\pi} \int_0^\infty S_{\mu\nu}(k,\omega) e^{-i\omega(t-\tau)} d\omega + c.c., \]

(18)

where \( S_{\mu\nu}(k,\omega) \) is the Fourier transform of \( S_{\mu\nu}(r,t) \) given by

\[ S_{\mu\nu}(k,\omega) = \frac{1}{2\pi^3} \int_0^\infty S_{\mu\nu}(k,\omega) e^{-i\omega t} d^3k. \]

(19)

Utilizing (15), (17), and (18) and performing the required operations, we find that the metric perturbation is given by

\[ h_{\mu\nu}(r,t) = \frac{4G}{\pi} \left\{ \frac{p^\mu_{mi} p^\nu_{mi}}{E_{mi}} \frac{\delta^3(r - v_{mi} \ t) \theta(-t) + \frac{p^\mu_{mf} p^\nu_{mf}}{E_{mf}} \ \delta^3(r - v_{mf} \ t) \theta(t)}{1} + \right. \]

\[ + \frac{p^\mu_{mi} p^\nu_{mi}}{E_{mi}} \ \delta^3(r - v_{mi} \ t) \theta(-t) + \frac{p^\mu_{mf} p^\nu_{mf}}{E_{mf}} \ \delta^3(r - v_{mf} \ t) \theta(t), \]

(15)

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At distances much greater than the source dimensions, the above can be approximated by

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(18)

where \( S_{\mu\nu}(k,\omega) \) is the Fourier transform of \( S_{\mu\nu}(r,t) \) given by

\[ S_{\mu\nu}(k,\omega) = \frac{1}{2\pi^3} \int_0^\infty S_{\mu\nu}(k,\omega) e^{-i\omega t} d^3k. \]

(19)

Utilizing (15), (17), and (18) and performing the required operations, we find that the metric perturbation is given by
Far away from the source, all components of $h_{\mu\nu}(r,t)$ except $h_{11}$, $h_{12}$, and $h_{22}$ ($= - h_{11}$) can be transformed away /33/ and (20) reduces to

$$h_{11}(r,t) = \frac{2G}{r} \left[ \left( \frac{m^2}{E_m - p \cos \theta} + \frac{M^2}{E_m + p \cos \theta} \right) \delta(r-t) + \left( \frac{M^2}{E_m + p \cos \theta} - \frac{m^2}{E_m - p \cos \theta} \right) \delta(t-r) \right],$$

(21)

$$h_{12}(r,t = 0)$$

We can, therefore, conclude that, although using a rather crude example, an intense radiation field introduces a non-zero curvature to the space-time metric. Certainly, a more realistic example could be chosen, but the additional complexity of the calculation did not warrant it, as we were only interested in finding out if the metric perturbation is finite. Consequently, the following working hypothesis is proposed: as the intense electromagnetic wave generates a curvature of the space-time, it must have an active gravitational energy. Let us investigate if such an assertion can be incorporated in the standard theoretical framework.

In order to arrive at the Maxwell equations in a curved space, the Principle of general covariance is invoked and applied to the corresponding equations in the flat space. However, terms containing the curvature tensor are normally neglected. Let us, therefore, study what will happen if these terms are included in the theory. Thus, consider a space-time manifold, described by the metric tensor $g_{\mu\nu}$. In addition, allow an electromagnetic wave, characterised by the four-potential $A_\mu$, to travel through and, also,
interact with that tensor field. Furthermore, for the sake of simplicity, assume that the electromagnetic wave is weak and does not perturb the metric. The Lagrangian for this system can then be written as

$$L = -\frac{1}{16\pi} \sqrt{-g} F_{\mu\nu} F^{\mu\nu} + \frac{1}{\kappa} \sqrt{-g} g^{\mu\nu} R_{\mu\nu} + \frac{1}{\kappa} \sqrt{-g} \chi A_{\mu} A^{\mu} g^{\mu\nu} R_{\mu\nu},$$

where the first term refers to the free electromagnetic field, described by the electromagnetic tensor $F_{\mu\nu}$, the second term is the Lagrangian of the curved space with $R_{\mu\nu}$ being the Ricci tensor, and the last term specifies the coupling, with $\chi$ being the proportionality constant.

We now enquire as to the form of the field equations which are determined by the above Lagrangian density $L$. For this purpose, a variational calculus is employed. In order to use the Euler-Lagrange equations of motion

$$\frac{\partial}{\partial g_{\alpha\beta}} \frac{\delta L}{\delta g_{\alpha\beta}} - \frac{\delta L}{\delta g_{\mu\nu}} = 0,$$  

(23)

the variation of (22) is needed. Thus, if $g_{\mu\nu}$ are subject to the variation

$$g_{\mu\nu} \rightarrow g_{\mu\nu} + \delta g_{\mu\nu},$$

then the relevant transformation formulae are required. These, however, can be calculated /25, 33/. We shall show here explicitly the calculation of $\delta F_{\mu\nu}$; other transformation formulae will only be stated. Thus,

$$\delta F_{\mu\nu} = F_{\alpha\beta} \delta(g^{\alpha\mu} g^{\beta\nu}) = F_{\alpha\beta} \delta g^{\alpha\mu} g^{\beta\nu} + F_{\alpha\beta} g^{\alpha\mu} \delta g^{\beta\nu}.$$  

(24)
From the equality $\delta (g_{\mu \nu} g^{\nu \rho}) = 0$, it follows that

$$\delta g^{\mu \nu} = -g^{\nu \gamma} g^{\epsilon \beta} \delta g_{\gamma \epsilon}.$$  \hfill (25)

Substitution of (25) into (24) leads to

$$\delta F^{\mu \nu} = -F^{\mu \epsilon} g^{\nu \gamma} \delta g_{\gamma \epsilon} - F^{\gamma \nu} g^{\mu \epsilon} \delta g_{\gamma \epsilon}.$$

The following are also required

$$\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu \nu} \delta g_{\mu \nu}$$ \hfill (27)

and

$$\delta R = 0.$$ \hfill (28)

Consequently, from the knowledge of the results involved in the variational calculus, we obtain, after some algebra, the equations of motion in the form

$$G_{\mu \nu} + \chi A^2 G_{\mu \nu} + \chi R A_\mu A_\nu + \chi R^2 \frac{\partial^2}{\partial x^\mu \partial x^\nu} (\chi A^2) = -\kappa T_{\mu \nu}. \hfill (29)$$

The meaning of the symbols in the above equation is as follows:

$G_{\mu \nu}$ - the Einstein tensor, semicolon denotes the covariant derivative, and $A^2 = A_\mu A^\mu$. Taking the trace of that equation result in

$$R = -3 \chi \frac{\partial}{\partial x} A^2.$$ \hfill (30)

On the other hand, if the four-potential $A^\mu$ are varied in the Lagrangian density, then the modified Maxwell equations are obtained

$$F^{\mu \nu} ;_\nu + \chi \frac{\partial}{\partial x} A^\mu = 0.$$

Substituting (30) into (31) leads to

$$F^{\mu \nu} ;_\nu - \frac{3 \chi}{\kappa} (\Box A^2) A^\mu = 0,$$

and the resulting nonlinearity, due to the coupling of the electromagnetic radiation with the curvature of space-time, is clearly exhibited.
Let us now reflect on the intermediate results. First of all, it has been shown that an electromagnetic wave in a curved space can also be treated as if it were in a medium. As a consequence, it was assumed that the inverse statement is also valid, viz., that a medium through which the electromagnetic wave travels may be replaced by a curved space. In addition, it was shown that if a medium is replaced by a high intensity radiation, the Minkowski metric is perturbed, thus, again, leading to a deviation from a flat space. There, it appears that, for our purposes, the exact nature of the medium was not required as long as it was assumed that the relevant region was curved. Subsequently, some questions involving the radiation-medium interaction may be replaced by the appropriate studies including the interaction with the curvature of the space-time. Obviously, this kind of problem is rather complex as it introduces nonlinear terms in the field equations.

A possible way of deriving the field equations is through the calculus of variations, as already indicated. Starting with the Lagrangian density of the form

\[ L = \frac{1}{k \sqrt{g}} R, \tag{33} \]

one is led to the Einstein field equations

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}, \tag{34} \]

These are nonlinear partial differential equations. As the gravitational fields carry energy and momentum, the effect of gravitation on itself is represented by the nonlinear terms. This may be contrasted with the Maxwell equations, which are linear.

Equations (34) represent ten apparently independent relations. However, in view of the Bianchi identities
the situation is different. There are, in fact only six independent equations, so that four out of ten unknowns are undetermined. This is the result of a general coordinate transformation involving four arbitrary functions \( x^\alpha(x) \). There is an analogous ambiguity regarding the four-potential in the Maxwell theory.

To elucidate the meaning of the Einstein equations, write the metric tensor as

\[
\gamma_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.
\]

The Ricci tensor, linear in \( h_{\mu\nu} \), is then written as /

\[
R^{(1)}_{\mu\kappa} = \frac{1}{8\pi G} \left( \gamma_{\mu\kappa} - \frac{1}{2} \gamma_{\mu\lambda} R^{\lambda}_{\kappa \lambda} - \frac{1}{2} \gamma_{\mu\kappa} R^{(1)}_{\lambda} + \frac{1}{2} \eta_{\mu\kappa} \gamma^{(1)}_{\lambda} \right). 
\]

Then, if \( t_{\mu\kappa} \) is defined as

\[
t_{\mu\kappa} = \frac{1}{8\pi G} (R_{\mu\kappa} - \frac{1}{2} \gamma_{\mu\lambda} R^{\lambda}_{\kappa \lambda} + \frac{1}{2} \gamma_{\mu\kappa} \gamma^{(1)}_{\lambda} + \frac{1}{2} \eta_{\mu\kappa} \gamma^{(1)}_{\lambda} ),
\]

the exact Einstein equations can be written as

\[
R^{(1)}_{\mu\kappa} - \frac{1}{2} \gamma_{\mu\kappa} R^{(1)}_{\lambda} = - 8\pi G (T_{\mu\kappa} + t_{\mu\kappa}) .
\]

The interesting feature of the theory now emerges: as \( t_{\mu\kappa} \) may be interpreted as the energy-momentum pseudo-tensor of the gravitational field, the RHS of (39) - the source term - depends on the values of the components of the metrical tensor \( \gamma_{\mu\nu} \).

Turn the attention now to the electrodynamics of massive spin one particles. Again, starting with the Lagrangian formalism, one arrives at the equations of motion. Thus, if the Lagrangian density is of the form

\[
L = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8\pi} A_\alpha A^\alpha ,
\]

also known as the Proca Lagrangian, then the equations of motion have
the form

\[ \partial_{\mu} \text{F}^{\mu\nu} + \mu^2 A^\nu = 0, \quad (41) \]

where the reciprocal length is associated with the rest mass of a
particle by \( \mu = \frac{m_o c}{\hbar} \). As is well known, the Proca equations
describe a wave aspect of massive spin one field. The formulation
of such a field, in terms of its particle properties, is available
and was first worked out by Duffin /34/ and Kemmer /35/. Guided by
the similarity with the Dirac equation of electron, they arrived at

\[ \beta_{\mu} \beta_{\nu} \psi + \mu \psi = 0 \tag{42} \]

together with the commutation rules for \( \beta \)'s

\[ \beta_{\mu} \beta_{\nu} \beta_{\rho} + \beta_{\rho} \beta_{\nu} \beta_{\mu} = \beta_{\mu} \delta_{\nu\rho} + \beta_{\rho} \delta_{\nu\mu} \tag{43} \]

These are, however, nothing else but the 10 x 10 matrices. In analogy
to the Dirac theory, we therefore define a ten component spinor in
terms of the field strengths \( E \) and \( H \) and, also, include the term
proportional to the four-potential. Hence, if the spinor has the form

\[ \psi = (E^k, H^k, -\mu A^k, -\mu V)^T \quad k = 1, 2, 3 \]

then we can write (42) in the form

\[
\begin{pmatrix}
0 & 0 & -\partial_t & -\partial_k \\
0 & 0 & \text{curl} & 0 \\
-\partial_t & \text{curl} & 0 & 0 \\
-\partial_k & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\partial_{\nu} \\
\beta_{\nu} \\
\beta_{\rho} \\
\delta_{\nu\mu} \\
\end{pmatrix}
\begin{pmatrix}
E^k \\
H^k \\
-\mu A^k \\
-\mu V \\
\end{pmatrix}
\]

where \( I \) is 10 x 10 unit matrix and the following substitutions were
used

\[
\partial_t = \begin{pmatrix}
\partial_t & 0 & 0 \\
0 & \partial_t & 0 \\
0 & 0 & \partial_t \\
\end{pmatrix} ; \quad \text{curl} = \begin{pmatrix}
\partial_3 & 0 & -\partial_1 \\
0 & \partial_2 & 0 \\
-\partial_2 & \partial_1 & 0 \\
\end{pmatrix}
\]
When $\mu$ is set equal to zero, then (42) reduces to the free Maxwell equations. There is, however, a fundamental difference between the massless and massive electrodynamics, as the latter is not gauge invariant. As is well known, there are two types of gauge invariance in electrodynamics: global gauge invariance, expressed as

$$\psi(x) \rightarrow \psi(x) e^{ie\lambda}$$

with $\lambda$ being a constant, and local gauge invariance

$$\psi(x) \rightarrow \psi(x) e^{ie\lambda(x)}$$

where $e$ denotes the charge of a particle and $\lambda$ is now a function of space-time. As we shall return to the gauge theories later, it suffices to say now that the physical meaning underlying (45) and (46) is rather different. Thus, the global invariance is closely related to the charge conservation and, consequently, its violation undermines the very basics of the theory. On the other hand, a consistent theory is available even if the local gauge invariance is broken.

Let us now investigate the similarities between the Maxwell equations in a curved space (allowing for their interaction with the curvature) and the Proca equations in the Minkowski (flat) space. Clearly, in both cases we are dealing with ten algebraic equations. However, whilst the Proca equations obey the Lorentz invariance, the gravitational field equations do not; they are invariant under a broader group of all point transformations. In addition, due to the occurrence of mass term in the Proca equations, these are not gauge-invariant, whilst the gauge invariance holds for the Maxwell equations, provided the coupling with the curvature tensor is excluded. The reason for this exclusion should already be apparent. As seen above, the Maxwell equations, when coupled with the surrounding curved space-time have the form
On the other hand, the Proca equations, which are defined in the flat space, have the form

\[ \nabla_{\mu} F_{\mu} + \frac{\kappa}{\kappa} R A^{\mu} = 0 . \] (31)

The similarity between these two equations is rather striking. Therefore, it is plausible to identify the mass of the neutral vector meson \( \mu \) as being proportional to the scalar of curvature \( R \). In other words, equation (31) was obtained by considering the coupling of the electromagnetic wave with the curved space, assuming the validity of the Maxwell theory. On the contrary, the Proca equations are derived in the Minkowski space, by adding the mass term explicitly to the Maxwell Lagrangian. Hence, the following identification seems possible

\[ \mu^2 = \frac{\kappa}{\kappa} R. \] (47)

Assuming the validity of (47), it follows that a mass of a neutral meson vanishes if \( R = 0 \), i.e., if the space surrounding that meson is flat. Now, as is well known /36/, the Proca equations are the unique generalizations of the Maxwell equations. As a consequence, we may be able to assert that in a non-empty space, where the curvature is not vanishing, a photon acquires a mass, as a result of its interaction with the surroundings.

Before discussing the gauge theory and the symmetry breaking in the massive field theories, let us consider some analogies between the Maxwell and Einstein theories. A theory of general relativity is based on a powerful principle, which refers to general covariance. Concisely, it demands that equations preserve their form under the point coordinate
transformations. In addition, when the metric tensor $g_{\mu\nu}$ equals to the Minkowski tensor $\eta_{\mu\nu}$, then these equations adopt the form which is accepted in the special theory of relativity. However, it was pointed out \cite{37} that this principle, by itself, does not have any physical significance. The meaning of the principle reflects only the validity of an equation in a gravitational field, if such an equation is valid in the absence of those fields. The presence of the gravitational field is then described by the additional terms involving $g_{\mu\nu}$ and the affine connection $\Gamma^\alpha_{\mu\nu}$. On the contrary, the Lorentz invariance demands that no additional terms (in particular the velocity of the reference frame) enter into the transformed equation. Therefore, one may say that general covariance does not imply the Lorentz invariance. In other words, the principle of general covariance is not an invariance principle, but, as it is restricted to the gravitational field, it is also known as the dynamic symmetry \cite{38}. This brings us to the point of contact with the dynamic symmetry in electrodynamics, viz., the local gauge invariance.

According to that symmetry, the fields $\psi(x)$ and the electromagnetic potentials $A_{\mu}$ transform as

$$\begin{align*}
\psi(x) &\longrightarrow \psi(x) e^{i\epsilon(x)} \\
A_{\mu}(x) &\longrightarrow A_{\mu}(x) + \partial_{\mu}\lambda(x).
\end{align*}$$

(48)

Now, as the $\partial_{\mu}\psi$, in general, does not obey (48), an equation of the form

$$(\partial_{\alpha} \partial^\alpha + m^2) \psi(x) = 0$$

(49)

is not gauge-invariant. In order for (49) to be gauge-invariant, it must be derived only from $\psi(x)$ and the respective gauge-covariant
derivatives $D_\alpha \psi(x)$ which transform like the fields,

$$D_\alpha \psi(x) \rightarrow D_\alpha \psi(x) e^{ie\lambda(x)}.$$  \hspace{1cm} (50)

The above equation is then written in a gauge-invariant form as

$$(D_\alpha D^\alpha + m^2) \psi(x) = 0.$$  \hspace{1cm} (51)

In analogy, the Lorentz invariant equation would be invariant under the point coordinate transformations, provided only tensors and their covariant derivatives, which include the affine connection and transform as tensors, are used. Evidently, there is a close correspondence between the general covariance of the theory of gravitation and the local gauge invariance in the Maxwell theory. This section aimed to provide some insight into this relation.

So far the possibility of replacing the medium by a curved space was studied. It appears that, if the nonlinear interaction is allowed in the theory, the photons may be described by the Proca equations. Essentially, the argument is dependent on the choice of the metric: if the Riemann metric is adopted, the Maxwell equations are modified by the inclusion of some additional nonlinear terms, whilst if the Minkowski metric is preferred, the photons appear to behave as if they possessed a non-vanishing mass. A similar conclusion, for particles with spin 0 and $\frac{1}{2}$, was recently obtained /39/ using the supersymmetry technique. They found that a masslessness of a particle in the de Sitter space or a massiveness of the same particle in the Minkowski space are, in fact, just two different gauge choices of one and the same theory.
V. GAUGE THEORIES AND THE SPONTANEOUS SYMMETRY BREAKING

Earlier, we have studied a possibility of attaching a non-vanishing mass to the photon, as a result of the nonlinear interaction between the electromagnetic wave and the curvature of space-time. In the process, we have discussed some of the similarities and differences between the notions of the Lorentz invariance and the General Covariance. As it was shown, the medium, in which that wave propagates, may be replaced by a curved space.

However, in order to confirm the correctness of this supposition, an analysis dealing with the gauge theory is required. The reason being, as already mentioned earlier, is that, classically, the Maxwell equations in a medium are invariant under the gauge transformations, whilst the Proca equations, due to the appearance of the mass term, violate that invariance principle. Before invoking the concept of the spontaneous symmetry breaking, as a mechanism for the mass generation, let us reflect upon the gauge theories.

It is a common practice to adopt the Lagrangian formalism in field theories. Of prime importance, in such a description, is the action integral \( A \), defined as /40/

\[
A = \int L(\psi_i(x), \partial_\alpha \psi_i(x)) \, d\tau ,
\]

where \( d\tau \) is the four-dimensional volume element and the Lagrangian density is the function of the fields and their gradients. The field equations then follow from the Euler-Lagrange equations of motion

\[
\partial^\nu \frac{\partial L}{\partial (\partial^\mu \psi)} - \frac{\partial L}{\partial \psi} = 0 .
\]
Next, it is well established that the symmetry principles in physics are closely linked with the conservation laws. Every symmetry principle is built on an assumption that a particular quantity is not measurable. For instance, the lack of the absolute position in space, implies translational invariance. In fact, one can construct transformations on the fields, which leave $L$ invariant, for every conserved quantity. So it follows that the energy, momentum, and angular momentum are conserved if $L$ is the Lorentz invariant.

Now we are interested in transformations which do not affect the space-time point - the so-called internal symmetries. The simplest transformations of this kind are defined by

$$\psi(x) \rightarrow \psi(x) e^{i\lambda} ; \partial_a \psi(x) \rightarrow \partial_a \psi(x) e^{i\lambda},$$  \hspace{1cm} (3)

where $\lambda$ is a phase, independent of the position $x$. A transformation of that kind is called a global gauge transformation. Physically, the invariance means that the phase of the field cannot be determined and is purely arbitrary. Hence, this phase cannot be fixed within the region of the experiment and any change in $\lambda$ must be the same at all times. The group of transformation (3) is also referred to as the Abelian U(1) group - the group of unitary transformations in one dimension.

There is another kind of transformations, with larger symmetry, known as the local gauge transformations, which are defined by

$$\psi(x) \rightarrow \psi(x) e^{i\lambda(x)},$$ \hspace{1cm} (4)

where the phase $\lambda(x)$ is now dependent on $x$. However, complications arise, as the gradient of the field now does not transform as the field itself. This implies that e.g., the Lagrangian density of a free
electron field is not invariant under \( (4) \). As already indicated, the invariance of the Lagrangian under \( (4) \) can be restored if a new, the so called covariant, derivative is introduced. It has the property that it transforms as the field itself, and is defined by

\[
D_{\alpha} \equiv \partial_{\alpha} + i e A_{\alpha},
\]

(5)

where \( A_{\alpha} \) is the gauge field which transforms according to

\[
A_{\alpha}(x) \rightarrow A_{\alpha}(x) - \frac{1}{e} \partial_{\alpha} \lambda(x).
\]

(6)

This is well known from electrodynamics and it represents the non-uniqueness of the potentials, as the gradient of an arbitrary scalar can be added to the vector potential. In addition, the definition (5) is often referred to as the minimal coupling.

The interpretation of gauge invariance was extended beyond that of the phase transformation by Yang and Mills /41/, to describe an internal symmetry. Thus, a phase transformation was generalized to a rotation in the internal space. By studying the similarities between the gauge theories and the general theory of relativity, it was shown /42/ that the gauge invariance may be expressed using elements of geometry. It seems pertinent to ponder over this issue for a while.

Thus, in the general relativity we study the motion of particles in a curved space-time manifolds. To facilitate this investigation, a concept of a free-falling frame of reference is often used. Such a frame is well defined only in a limited region of space-time. Because the gravitational field is inhomogeneous, it is impossible to extend a free-falling frame of reference far from a given world point. Hence, in the vicinity of such a point the particles appear unaccelerated, despite the presence of gravitation field. As a consequence, a particle can then be described at each point in space-
time, with respect to the free-falling frame, as if the gravitational field was absent. The effect of the gravity appears when the motion of a particle is described from different non-inertial frames and one inquires about the relation between the two. On the other hand, the gauge theory deals with the internal properties of a particle, like charge (global invariance) or isotopic spin (invariance under SU(2) group). Thus, the abstract internal symmetry space is quite different from the ordinary physical space-time. Consequently, in view of /42/, one may identify the internal symmetry space with the local free-falling frame of reference, which is used to describe the motion of a particle through the external field. An observer, located inside such an internal space, would notice a rotation of the phase of the wave function, describing a particle. The angle of rotation would then depend on the strengths of the interaction with the external gauge field as well as on the distance the particle travelled through that field. Therefore, in analogy with the gravitational field, this external gauge field determines the modifications of the coordinates in the internal space, as the particle moves from point to point. Hence, the external gauge field specifies the behaviour of the internal coordinates at different points in the space-time.

There is a close connection between the gauge invariance and the Lorentz invariance. The mutual interdependence is clearly seen in the following example. Consider two observers moving in different inertial frames. For a given (the same) problem in electrodynamics they both choose the Coulomb gauge. As is well known /43/, such a gauge is not Lorentz invariant. Nevertheless, the electromagnetic
fields, as measured in those inertial frames, are connected by the Lorentz transformation. This implies, therefore, that the Lorentz transformation not only changes the space-time coordinates in the inertial frames, but also affects the rotation in the internal space. Put otherwise, the Lorentz transformation also implies a gauge transformation. To illustrate this important point further, consider the following example. In quantum field theory \( ^4\) the transformation law for the four-potential is given by

\[
A'_\alpha = L^\beta_\alpha A^\beta - \partial^\alpha \lambda ,
\]

where \( L^\beta_\alpha \) is the Lorentz matrix. Evidently, the four-potential is not a real four-vector as, under the Lorentz transformation, an extra term is required. That term is interpreted as being due to the rotation in the internal space. The effect of this term on the Lorentz invariance is now obvious. Thus, using the Lorentz gauge implies

\[
\partial^\alpha A^\alpha = 0 ,
\]

However, (7) will only be satisfied if

\[
\partial^\alpha \partial^\alpha \lambda = 0 ,
\]

which is precisely the constraint on \( \lambda \) in the Lorentz gauge. The meaning of (8) is clear now: in order for \( \partial^\alpha A^\alpha \) to be treated as the Lorentz invariant, the rotation in the internal space must be eliminated. The reason for the non-invariance of the Coulomb gauge is now self-evident.

Unlike electric charge conservation, there are other symmetries in Nature that are not exact. It is well established by now that if the physical vacuum is not invariant under the symmetry group, an
exactly symmetric Lagrangian will yield a spontaneously broken symmetry. To illustrate this point, consider a Lagrangian density describing the dynamics of a scalar field $\phi(x)$ interacting with an electromagnetic field

$$L = -\frac{1}{4} F_{\mu\nu}^2 + |D_\mu \phi|^2 - \mu^2 |\phi|^2 - \lambda |\phi|^4 . \quad (9)$$

This Lagrangian, in view of its construction, is invariant under the local gauge transformation $U(1)$. The first term describes the energy contained in the electromagnetic field, the second term is a minimal coupling term, and the last two terms are interpreted as referring to the potential energy. Now, the behaviour of the potential energy $V(\phi)$ depends on quantities $\mu$ and $\lambda$. Thus, $V(\phi)$ has a minimum only if $\lambda > 0$. In order to determine the value of this minimum, we need to know the sign of $\mu^2$. Thus, if $\mu^2$ is non-negative, the minimum of $V(\phi)$ is at $\phi = 0$, the well known classical solution. However, if $\mu^2$ is negative, the minimum of $V(\phi)$ is at $\phi \neq 0$ and there are many solutions which yield the required minimum. For a moment, assume that $\mu^2$ is negative and choose one of the possible solutions in the form

$$\phi(x) = \frac{1}{\sqrt{2}} (a + \beta(x)) e^{i\epsilon(x)/\alpha}. \quad (10)$$

Substituting (10) into (9) yields

$$L_1 = -\frac{1}{4} B_{\mu\nu}^2 + \frac{a^2}{2} \frac{e^2}{\mu^2} B_{\mu}^2 + \frac{1}{2} \left( \frac{\alpha^2}{\mu^2} \beta(x)^2 - \lambda a^2 \beta^2(x) \right) - \frac{1}{4} \lambda \beta^4(x) + \frac{1}{2} B_{\mu} e^2 (2a \beta(x) + \beta^2(x)). \quad (11)$$

In the process we have used

$$A_\alpha(x) = B_\alpha(x) + \frac{1}{\alpha e^2} \beta \epsilon(x).$$
Evidently, the Lagrangian density $\mathcal{L}$ is not invariant under the gauge transformations. It contains only massive particles, thus, displaying the particle aspect of the theory. To put it succinctly, the spontaneous symmetry breaking is responsible for the mass of the gauge particles. The above described mechanism is known as the Higgs mechanism $/45/$.

A common occurrence, in the many-body systems, is the existence of collective modes of excitation. For instance, photons in a superfluid helium, the spin waves in the ferromagnet. The description of these collective phenomena is facilitated by the introduction of a degenerate ground state. Thus, in the BCS model of superconductivity $/46/$, the ground state is characterized by a nonvanishing isospin density which, in turn, determines the energy gap $/47/$. Hence, the ground state is degenerate and the symmetry is broken. Put otherwise, in this theory, the photon acquires a mass, manifested by e.g., the Meissner effect, through coupling to the Cooper pairs. An analogous situation arises in the theory of a ferromagnet. At temperatures $T > T_c$, the ground state of the ferromagnet is invariant under the rotation, however, this invariance breaks down for $T > T_c$, as the spins point in the preferred direction. Again, as the ground state does not possess the full symmetry of the interaction describing the system, the spontaneous symmetry breaking occurs.

In the light of the above discussion, consider an interaction of electromagnetic wave with a medium. As most of the characteristics of a medium are described by considering only valence electrons, we restrict our analysis by replacing a medium by an electron distribution. Alternatively, consider a medium irradiated by an intense radiation
and investigate the interaction between such a system and a test radiation beam. If the medium in question is not in thermal equilibrium, the corresponding electron density may be written in the form

\[ n(r,t) = n_o + a(r,t) \]  

where \( n_o \) is the density in the thermal equilibrium and \( a(r,t) \) is the effect of the excitation of the medium in question. If a medium is irradiated by high intensity source, the density will fluctuate around a point \( n_o + a_o \), where \( a_o \) may be of the order of \( n_o \). As in the absence of irradiation \( a(r,t) \) vanishes, the ground state is determined by the condition

\[ a(r,t) = a_o. \]  

Let us now introduce the notion of the order parameter by considering the following example /48/. In the mean field theory of the ferromagnet, the direct interaction between the spins is viewed as a two-step process: 1. different spins generate the magnetization; 2. this magnetization then determines the behaviour of the individual spins. The magnetization, which is a macroscopic quantity, can be referred to as an order parameter, as, apart from determining the behaviour of each micro-system, it also gives the degree of order - in analogy to that concept in the phase transition theory.

Let us now construct the Lagrangian density for laser-medium system, using the just mentioned order parameters, in analogy to the Ginzburg-Landau theory of superconductivity /49/. A possible form in the absence of the electromagnetic field is

\[ L = \frac{i}{2}(\psi^* i \partial_t \psi + \psi i \partial_t \psi^*) - \frac{1}{2m} \left[ i \nabla \psi^* \right] \left[ -i \nabla \psi \right] - V \psi^* \psi. \]
Next, introduce the interaction with the medium through the minimal coupling. The Lagrangian for the complete system then becomes

$$L = \frac{1}{8\pi} (E^2 - H^2) + \frac{1}{2} \left[ \psi^* (i \partial_t - eA) \psi + \psi (-i \partial_t - eA) \psi^* \right] - \frac{1}{2m} \left[ (i \nabla - eA) \psi^* \right] \left[ (-i \nabla - eA) \psi \right] - V \psi^* \psi,$$

(16)

where $\psi$ is the order parameter. To see the effect of the symmetry breaking, transform the order parameter, in analogy to the Higgs picture, as

$$\psi = \left[ \alpha_0 + \psi_2 (r, t) \right] e^{i e \delta (r, t)}$$

(17)

Substitution of (17) into (16) leads to

$$L = \frac{1}{8\pi} (E^2 - H^2) - \frac{1}{2m} \left[ (\nabla \psi_2)^2 + e^2 B^2 (\psi_2^2 + 2 \alpha_0 \psi_2) + e^2 \alpha_0^2 \phi^2 \right] - (\alpha_0 + \psi_2)^2 V - (\alpha_0 + \psi_2)^2 e \phi,$$

(18)

where the following relations have been used

$$B = A + \nabla \delta \quad \phi = \phi - \partial_t \delta.$$ 

(19)

Although the Lagrangians (18) and (16) are formally different, it must be emphasized that they both describe the same physical system. One is derived from the other by a transformation, which in no way can change the physical content. Although the latter form is not gauge invariant, it readily lends itself to the important interpretation: it describes the interaction of the massive vector field $B_\mu$ with the massive scalar field $\psi_2 (r, t)$. In other words, the electromagnetic field, as a result of its interaction with the medium, far from the thermal equilibrium, has acquired a real mass. Despite the formalism of the elementary particle physics being involved, the
philosophy of this result is quite novel. Thus, in particle physics the symmetry is broken even in the absence of the electromagnetic field, and is due purely to the auxiliary Higgs field. In the present case, however, the symmetry is broken as a direct consequence of the interaction between the electromagnetic fields. So far, we have seen how a photon can acquire a nonvanishing mass, using the formalism of the Higgs model.

Let us, now, diverge for a while, whilst considering the interaction of radiation with a medium. As is well known, the electrodynamics is based on the Maxwell equations and the Lorentz equations, describing the motions of electrons. Although not widely appreciated, this is not an exact theory. The application of the theory to various interaction processes often makes use of the perturbation approximation. Commonly, only the first non-vanishing order in the electric charge (coupling parameter) is taken into account, as it is, generally, assumed that the magnitude of the higher orders in the expansion is progressively smaller. Occasionally, however, one includes these higher orders in the calculation and these are then termed the radiative corrections. These are quantum effects which are closely linked with problems of self-charge and self-mass. Although these corrections are small in comparison with the first order, they are of fundamental importance from the theoretical point of view. It is still a major unsolved difficulty of the theory that the infinities, resulting from self-interaction, cannot be satisfactorily accounted for. In order to avoid this difficulty, the following argument is used: as the observable quantities, the total mass and the total charge, are finite, the renormalization procedure is employed. In essence, this method involves separation and subsequent subtraction of these divergent
quantities, leading to the finite measured values. A further question arises when the additional terms in the expansion are included. Although it is generally assumed that the series converges, it is rather improbable as the number of the Feynman diagrams corresponding to each higher order term increases rapidly with the power of $e^2$. The reasons for these unsatisfactory features of the theory are not known. It may be possible that the expansion in powers of $e^2$ is not allowed. Or, there may be another possibility, involving a radical modification on the more basic level. However, whatever the changes in the improved theory may be, they must, in the zeroth order approximation, include its present form, which in many cases leads to highly accurate results.

We are now in a position to investigate another mechanism for mass generation, which is often called dynamic symmetry breaking. This phenomenon was already investigated /50/ and, as it may improve our understanding of the concept of mass, the pertinent details of that model will now be recalled.

The essential difference between the Higgs model and the one being analyzed now is the driving mechanism of the instability. Thus, in the former case its role is assumed by the negative term in the Lagrangian density, arising from the postulated scalar field; in the latter case, the radiative corrections are responsible for the symmetry breaking.

Using the method of Jona-Lasinio /51/, the effective potential is introduced. Then, the minima of this potential determine the exact vacuum states of the theory. Thus, the knowledge of the effective potential will yield the necessary information about the symmetry breaking. With Jona-Lasinio, consider the modifications to the
dynamics of the scalar field \( \phi(x) \), by inclusion of its interaction with an external source \( J(x) \)

\[
L(\phi, \partial_x \phi) \rightarrow L(\phi, \partial_x \phi) + \phi(x) J(x)
\]  

(20)

The effective action \( S(\phi_c) \) is then defined by

\[
S(\phi_c) = W(J) - \int J(x) \phi_c(x) \, d^4x,
\]

(21)

where \( \phi_c \), the classical field, is given by

\[
\phi_c(x) = \frac{\delta W}{\delta J(x)} = \frac{\langle 0^+ | \phi(x) | 0^- \rangle}{\langle 0^+ | 0^- \rangle} |_J
\]

(22)

and \( W(J) \), the generating functional, is expressed in terms of the transition amplitude as

\[
e^{iW(J)} = \langle 0^+ | 0^- \rangle_J
\]

(23)

It follows from (21) that we can write

\[
\frac{\delta S}{\delta \phi_c(x)} = - J(x)
\]

(24)

Now, the effective action can be expanded in powers of momentum as

\[
S = \int d^4x \left[ - \frac{1}{2} \left( \partial^2 \phi_c \right)^2 + V(\phi_c) + \ldots \right],
\]

(25)

where \( V(\phi_c) \) is the effective potential. According to the perturbation theory, one can express the mass and coupling constant in terms of functions in (25) as

\[
u^2 = \left. \frac{d^2V}{d \phi_c^2} \right|_0
\]

(26)

\[
\lambda = \left. \frac{d^4V}{d \phi_c^4} \right|_0
\]

(27)
Now, assuming that the Lagrangian density has an internal symmetry and the source $j(x)$ is absent, the appearance of the spontaneous symmetry breaking signifies that the quantum field $\phi$ has a non-zero vacuum expectation value. It follows, from (22) and (24), that

$$\frac{\delta S}{\delta \phi_c} = 0 \quad \phi_c \neq 0.$$  \tag{26}

Further, if we require that the vacuum expectation value be translationally invariant, then, in view of (25), the above simplifies to

$$\frac{dV}{d\phi_c} = 0.$$  \tag{29}

If the new quantum field, having a vanishing vacuum expectation value, is defined by

$$\phi' = \phi - <\phi>,$$  \tag{30}

then the classical field obeys

$$\phi'_c = \phi_c - <\phi>,$$  \tag{31}

where $<\phi>$ is the value of $\phi_c$ at which the potential is minimum. Therefore, in order to use (26) and (27), in calculations of mass and coupling constant, they must be evaluated at $<\phi>$.

Next, consider a massless charged meson field interacting with the electromagnetic field. The corresponding Lagrangian density is

$$L = \frac{1}{2} F_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu \phi_1 - e A_\mu \phi_2)^2 + \frac{1}{2} (\partial_\mu \phi_2 + e A_\mu \phi_1)^2 - \frac{\lambda}{4!} (\phi_1^2 + \phi_2^2)^2,$$  \tag{32}

where $\phi_1$ and $\phi_2$ are real fields. In order to eliminate the contribution of certain classes of the Feynman graphs to the effective Lagrangian, the Landau gauge is used. After some calculation /50/, the effective potential is found to be in the form
where $M$ is an arbitrary renormalization mass, which we chose to be the location of the minimum $<\phi>$. As there is no a priori reason to the contrary, we restrict our analysis to the case where the magnitude of $\lambda$ is of the order of $e^4$. This restriction is shown to be valid /50/. Consequently, (33) is simplified to yield

$$
V = \frac{\lambda}{4!} \phi^4 + \frac{3e^4}{64\pi^2} \phi^4 \left( \ln \frac{\phi^2}{<\phi>^2} - \frac{25}{6} \right)
$$

(34)

We are now in a position to determine $\lambda$ in terms of $e$. Since $<\phi>$ is the minimum of $V$, i.e. $V(<\phi>) = 0$, it is straightforward to obtain

$$
\lambda = \frac{33}{8\pi^2} e^4.
$$

(35)

The effective potential can now be written, in view of (35), as

$$
V = \frac{3e^4}{64\pi^2} \phi^4 \left( \ln \frac{\phi^2}{<\phi>^2} - \frac{1}{2} \right).
$$

(36)

This, in essence, is the dynamical model for the mass generation. From now on, there is an overlap with the Higgs model. Therefore, following the standard procedure, we find that the mass of the scalar meson is given by

$$
\frac{m^2}{M^2} = V'' (<\phi>) = \frac{3e^4}{8\pi^2} <\phi>^2
$$

(37)

and that of the photon, which now becomes massive, by

$$
m^2 = e^2 <\phi>^2.
$$

It is essential to note that the mass of scalar and vector mesons is expressed in terms of the higher order corrections in $e$. Therefore, within the restrictions imposed throughout this calculation, it is
valid to say that the radiative corrections are responsible for the
degeneracy of the vacuum state and, consequently, are instrumental
in mass generation. Put otherwise, if only the first order
corrections in the interaction process are dealt with, it is correct
to assume that photons are massless. However, in the circumstances
where it becomes necessary to include the higher order radiative
effects, it seems plausible to attach a non-vanishing mass to the
photons.
VI. SOME ASPECTS OF INTERACTION AND EXPERIMENTAL EVIDENCE

Reader, who has carefully followed the preceding pages, may feel that the somewhat different issues, which were studied, present a rather disjoint picture. Two different approaches were adopted to verify whether the results were consistent. It must be clear that the concept of mass is rather complex and its complete understanding is still quite far. For instance, the ambiguity involved in determining the form of the Lagrangian density is not satisfactory, if the proposed theory is to be successful. Nevertheless, within the imposed restrictions, the general relativity as well as the gauge theory approaches lead us, independently, to the conclusion that, under certain conditions, the originally massless photons behave as the neutral vector mesons. Let us now apply this suggestion to some specific problems and, then, compare the theoretical predictions with the experimental results.

In the meantime, as was argued above, the Maxwell equations inside a medium may be replaced by the corresponding equations in a curved space. Or, on the other hand, if the Minkowski (flat) space-time is preferred, than these equations should be generalized and substituted by the Proca equations /52/. Assume, for a moment, that this supposition is valid and investigate what would be the conclusions derived from this premise. Thus, as the mass of the photon is now finite, there would be three states of polarization. Thence, apart from two transverse polarizations, there also exists a longitudinal one. Furthermore, the faster the longitudinal photon moves, the weaker is the associated electric field /36/, thus, implying only a rather weak interaction with a medium. In addition, the electric field diminishes exponentially
with distance and the corresponding flux lines fade away, even in
vacuo. Similarly, magnetic field is also affected and the associated
lines are compressed around equator. In addition, the velocity of
energy propagation is now frequency dependent, as can be seen from
the following argument. A free electromagnetic wave, corresponding to
a massive photon, may be described by
\[ A = A_0 e^{i(\omega t - k\cdot x)} \]  
with
\[ (\omega/c)^2 - k^2 = \mu^2, \]  
where \( \mu \) is the rest mass of the photon. Then, from its definition,
the group speed is \( v_g = d\omega/dk \), and, on using (2), we obtain
\[ v_g = \frac{c}{\omega} \left( \omega^2 - \mu^2 \right)^{\frac{1}{2}}, \]  
in accord with the experimental results /53, 36/.

Consider now a static field, i.e., a case when \( \omega = 0 \). It then
follows from (2) that \( k = \pm i \mu \). Subsequent substitution into the
wave equation (1) implies that the spherical static field decays
exponentially, i.e., \( A = A_0 e^{-\mu r} (A_0 \propto r^{-1}) \). On selecting \( k = i \mu \),
we have a behaviour similar to that displayed by the Yukawa model of
nuclear binding. Thus, one of the consequences of adopting a non-
vanishing mass for photons is a deviation from the Coulomb law.

As the photon is now assumed to have a non-vanishing mass, a
mass term has to be added to the Lagrangian density describing the
Maxwell field. The thus obtained Lagrangian density, in the absence
of charges, becomes
\[ L_P = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} + \frac{\mu^2}{8} A_\alpha A^\alpha \]  
and is termed the Proca Lagrangian density. The reciprocal length
\( \mu \) in (4) is associated with the rest mass of the photon by \( \mu = m_0 c/\hbar \).
The Proca equations of motion are then written as

$$\partial_{\mu} F^{\mu\nu} + \mu^2 A^{\nu} = 0 .$$

(5)

Thus, (5) together with the definition of the electromagnetic field tensor, $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$, represent the covariant Proca equations for massive vector field. It should also be noted that these are the only possible linear generalizations of the Maxwell equations. In addition, as a result of the inclusion of the coupling constant $\mu$, the potentials now acquire real physical characteristics. That is, in contradistinction to the potentials in the Maxwell theory, they now become observable.

As already indicated, the massive photon has three degrees of polarization. It appears that the result of Hora /54/, dealing with an exact Maxwellian beam in vacuo or in homogeneous medium, may be related to the Proca description. Studying the intense irradiation, that work concentrates on the non-linear effects, whilst relying on the quivering motion of electrons in laser field. It is then found that an unexpected longitudinal component appears, as a result of the exact description of high intensity beam. It seems necessary to include this component in the description of the non-linear force emission of electrons in radial direction from the laser beam. In addition, it is also required to explain the internal diffraction of the beam.

Next, it seems appropriate to mention here the Schwartz-Hora effect /55/, as the resemblance with the present work is quite intriguing. The effect arises when an electron beam traverses a solid slab which is simultaneously irradiated by a crossing laser beam. An analogous
situation with the electron beam being replaced by the electromagnetic wave was considered above. The electron diffraction pattern, then, is modulated by the colour of laser photons. As the existence of plasmons in solids is well established /43/, Hora /56/ used this result to extend it to the electron wave modulation by plasmons. In that work, the differences between the classical bunching and the quantum mechanical modulation were studied. It was shown that a superposition of two electromagnetic waves differing by $\Delta \omega$, leads to a beating wavelength

$$\lambda = \frac{\omega}{\Delta \omega} \lambda_0.$$

Further, in investigating the similarity between the light waves and the matter waves, it was found that in the latter case one must include an additional beating wavelength

$$\Lambda = 8 \lambda_d \left( \frac{E}{\hbar \omega} \right)^2,$$

where $\lambda_d$ is the de Broglie wavelength. It was pointed out that this difference is related to the second order term in the momentum expansion. The result was then used to describe the side shift at the total reflection - the Goos-Haenchen effect. It was then concluded that matter waves allowed the rigorous mathematical description of the wave bundle /57/, whilst the use of light waves represented only an approximation.

One of the topics which for a long time has occupied the mind of many and is still not completely understood, as the number of papers appearing recently can prove, is the ambiguity regarding the correct form of the energy-momentum tensor in a medium. Although this problem has been studied by the present author /3, 14/, the background of this work is different and it seems appropriate to include it here. An extensive literature survey, which was covered in that work, will not be repeated now. Here, we shall concentrate on the effect of the medium
on the electromagnetic wave propagating in it, drawing on our knowledge from the previous sections. In other words, to emphasize the point which may be easily misunderstood, we are interested, exclusively, in the properties of the electromagnetic field, modified by the presence of the non-empty surroundings. Commonly, one is interested in the total energy-momentum tensor, which includes the field as well as the mechanical contributions. It can then be shown that, from mathematical standpoint, i.e., by formally altering the form of the matter tensor, one arrives at the conclusion that all these descriptions are equivalent /58, 59/. This may be so, but only formally. In other words, the physical content of electrodynamics of Abraham, Minkowski, and others differ, as argued by Brevik /60/ and Novak /14/. One of the reasons in support of that latter claim is an excellent correlation with the experimental findings, as will be detailed shortly.

Consider then the energy-momentum tensor of the free massive field. As is well known, it can be formed from the Lagrange density. Thus, if the Hamiltonian density is defined by

$$H_P = \frac{\partial L}{\partial \left( \partial_t A_\alpha \right)} \partial_\alpha A_t - L_P,$$  \hspace{1cm} (6)

then the covariant formulation of (6) leads, in view of the Lorentz transformation properties of $H$, to the canonical energy-momentum density tensor

$$T^\mu_\nu = \frac{\partial L}{\partial (\partial_\mu A^\kappa)} \partial^\nu A^\kappa - \eta^\mu_\nu L_P.$$ \hspace{1cm} (7)

Substituting (4) into (7) and making use of the field equations (5), the components of the above tensor become
In order to eliminate the unwanted divergencies in (8), three-dimensional space integration results in the following expressions for the energy and momentum of the Proca field:

\[
\begin{align*}
T^{\mu\nu} & = \frac{1}{8\pi} \left[ (E^2 + H^2) + \mu^2 (V^2 + A^2) \right] + \frac{1}{4\pi} B \cdot (\text{BE}) \\
T^{0i} & = \frac{1}{8\pi} \left[ (E^2 + H^2) + \mu^2 A_i^2 V^2 + V \cdot (A_i E) \right] \\
T^{io} & = \frac{1}{4\pi} \left[ (E^2 + H^2) + \mu^2 A_i^2 V^2 + (V + VH)_i - \delta_t(\text{VE}) \right].
\end{align*}
\] (8)

The shortcomings of the canonical tensor are well known /19, 3/. In an attempt to meet them, Belinfante /61/ proposed a symmetrical form for the tensor density. Recently, Hehl /62/ gave the more general symmetrization procedure, of which the Belinfante method is a special case. The primary concern in defining the symmetrical energy-momentum tensor \( T^{\mu\nu} \) is to modify the canonical tensor \( T^{\mu\nu} \) in such a way that the total distribution of energy and momentum densities in the field be given by

\[
\varepsilon^{\mu\nu} = T^{\mu\nu} + T^{\mu\nu}_S, \tag{10}
\]

where the divergenceless tensor \( T^{\mu\nu}_S \) must be chosen in such a way that the conservation laws involving \( T^{\mu\nu} \) still hold. It is then found that

\[
T^{\mu\nu}_S = \frac{1}{4\pi} \delta^{\mu\nu} \quad A^\nu. \tag{11}
\]

Since

\[
\int T^{\lambda\nu} d^3x = \int T^{\lambda\nu}_S d^3x, \tag{12}
\]

\( T^{\mu\nu}_S \) gives no contribution to the total energy and momentum.
Consequently, $T_{\mu}^{\nu}$ may be interpreted as the spin energy momentum tensor.

Next, the total angular momentum density tensor $J_{\lambda}^{\mu\nu}$ is generally written as

$$J_{\lambda}^{\mu\nu} = M_{\lambda}^{\mu\nu} + N_{\lambda}^{\mu\nu},$$

i.e., as a sum of an orbital $M_{\lambda}^{\mu\nu}$ and spin $N_{\lambda}^{\mu\nu}$ angular momentum density tensors. However, this last subdivision may be shown /63/ to be unique only when the rest frame of the particle can be defined.

And a meson, having a non-vanishing rest mass, satisfies this condition.

The total angular momentum density tensor, which is defined as

$$J_{\lambda}^{\mu\nu} = x_{\lambda}^{\mu} \theta_{\lambda}^{\mu\nu} - x_{\mu}^{\lambda} \theta_{\lambda}^{\mu\nu}$$

is, in view of $\theta_{\mu\nu}$ being a symmetric tensor, conserved, as its divergence, $\delta_{\lambda} J_{\lambda}^{\mu\nu}$, vanishes. It is, therefore, evident that the contribution by $T_{\lambda}^{\mu\nu}$ ensures the conservation of $J_{\lambda}^{\mu\nu}$.

Taking into the account the expressions for $T_{\mu}^{\nu}$, (8), and that for $T_{\lambda}^{\mu\nu}$, (11), the tensor $\theta_{\mu\nu}$ can be expressed in the component form as

$$\theta_{00} = \frac{1}{8\pi} \left[ E^2 + H^2 + \mu^2 (v^2 + \lambda^2) \right]$$

$$\theta_{i0} = \theta_{0i} = \frac{1}{4\pi} \left[ (E \times H)_i + \mu^2 \lambda_i \right].$$

The influence of the mass term on the energy-momentum tensor is now quite explicit. As $\mu$ is assumed to be, in most cases, small, the deviation of the above results from those corresponding to the Maxwell field will be small. Nevertheless, these results are of utmost importance from the theoretical point of view.
Next, using the Proca equations, the linear momentum derived from the canonical tensor $T^{\mu\nu}$ becomes

$$\mathcal{E} = \frac{1}{4\pi} \int Y (E \cdot A) \, d^3x. \quad (16)$$

The corresponding expression, obtained from the symmetric tensor $\sigma^{\mu\nu}$, takes the form

$$\mathcal{E} = \frac{1}{4\pi} \int \{ A(\mathbf{v} \cdot E) + (E \times H) \} \, d^3x. \quad (17)$$

The integrated total angular momentum can be written, therefore, as

$$J = \frac{1}{4\pi} \int \mathbf{v} \times \left[ A(\mathbf{v} \cdot E) + (E \times H) \right] \, d^3x. \quad (18)$$

Using standard vector relations, the above integrand yields

$$\mathbf{j} = \left[ v (E \cdot A) - (E \cdot v) A + A(\mathbf{v} \cdot E) \right]. \quad (19)$$

Now, in view of (16), the above is interpreted as the moment of the canonical momentum and the spin momentum densities. Consequently, substitution of (19) into (18), followed by an integration, results in

$$J = \frac{1}{4\pi} \int E \cdot (\mathbf{x} \times \mathbf{v}) A \, d^3x + \frac{1}{4\pi} \int (E \times A) \, d^3x. \quad (20)$$

Thus, the modification of $J$ due to the symmetric tensor $\sigma^{\mu\nu}$ is now clear.

In accord with our supposition that the Maxwell equations in a medium may be replaced by the free Proca equations, let us apply now the reasoning required in the massive electrodynamics to the actual, proposed forms of the energy-momentum tensors. There are many different forms of this tensor, which were put forward by various workers. Their advantages as well as disadvantages were already
discussed /3, 60/. Here, to emphasize, we are interested in the mutual relation between the two of them, viz., the tensors of Minkowski and Abraham. This choice is, to certain extent, governed by the availability of the experimental data.

Thus, Minkowski, applying the macroscopic Maxwell equations to the energy and momentum conservation laws, arrived /6/ at the energy-momentum tensor of the form

$$ S_{\mu \nu}^M = \frac{1}{4\pi} \left[ \eta^{\mu \alpha} F_{\alpha \lambda} H_{\lambda \nu} + \frac{1}{2} \eta^{\mu \nu} F_{\alpha \beta} H_{\alpha \beta} \right], \quad (21) $$

where $H_{\mu \nu}$ is obtained from $F_{\mu \nu}$ by substituting $E \rightarrow D$ and $H \rightarrow B$.

In an attempt to overcome the deficiencies /3, 25/ of the Minkowski tensor, all arising from the asymmetric character of that tensor, Abraham /64/ proposed a different form

$$ S_{\alpha \beta}^{\mu \nu} = \frac{1}{4\pi} \left[ \eta^{\mu \alpha} F_{\alpha \lambda} H_{\lambda \nu} + \frac{1}{2} \eta^{\mu \nu} F_{\alpha \beta} H_{\alpha \beta} + (\epsilon \mu - 1) \eta^{\mu \alpha} \omega_{\alpha} \nu \right], \quad (22) $$

where $\epsilon$ and $\mu$ are, respectively, a dielectric permittivity and magnetic permeability and $v^\lambda$ denotes the four-velocity of the medium.

Furthermore, the last term in (22) can be written as

$$ (\epsilon \mu - 1) F_{\alpha \beta} \nu^\kappa (v^\mu H_{\beta \kappa} + v^\kappa H_{\beta \mu}) \nu^\nu \equiv S_{sp}^{\mu \nu} \quad (23) $$

with $F_{\alpha \beta} = F_{\alpha \beta}^{\mu \nu}$. Evidently, the Minkowski tensor differs from the Abraham one by the expression (23), which, in the rest frame of the medium, can be written in the component form as

$$ S_{sp}^{00} = 0 \quad \quad S_{sp}^{01} = 0 $$

$$ S_{sp}^{10} = \frac{\epsilon \mu - 1}{4 \pi c} \quad \quad E \times H. \quad (24) $$

Clearly, the only non-vanishing component of $S_{sp}^{\mu \nu}$ is the space-time one. Let us now inquire about the role of this tensor in the
energy-momentum tensor problem. It is immediately obvious that for frequencies at which \( \epsilon_\omega \) may be replaced by unity, the space-time components of \( S_{\mu\nu}^{sp} \) also vanish. The magnitude of frequencies at which this statement applies, depends on the medium in question. For instance, in a collisionless plasma, the frequency domain is given by \( \omega > \omega_p \) (\( \omega_p \) - plasma frequency); in a solid, frequencies larger than those of the electronic vibrations, i.e., in the optical frequency domain, are required. This is, however, the region where WKB approximation applies, i.e., a domain of the geometrical optics. In this case the electromagnetic wave may be replaced by particles with no internal structure.

These considerations, together with the Belinfante symmetrization procedure, suggest that the tensor \( S_{sp}^{\mu\nu} \), defined as

\[
S_{sp}^{\mu\nu} = S_A^{\mu\nu} - S_M^{\mu\nu}
\]

may be identified as the spin tensor. In the (optical) frequency limit, \( S_{sp}^{\mu\nu} \) may be neglected and, as the effect of the medium on the electromagnetic wave is minute, the energy-momentum tensor of the latter is given by the Minkowski expression. On the other hand, at the opposite end of the frequency spectrum, the electromagnetic wave is considerably modified by travelling in a non-empty space. This influence is reflected in the Abraham form for the energy-momentum tensor. Furthermore, it is accepted that the energy and spin localizations are related by the angular momentum conservation. In addition, they (mass and spin) are the unitary representations of the Poincare group. From the arguments detailed in the previous section, it follows that mass is not an intrinsic concept, but arises as a result of the interactions. In virtue of the just presented discussion, it seems plausible to extend this conclusion also to the
concept of spin. Thus, high frequency electromagnetic wave does not "see" the medium — a conclusion represented by $\epsilon W$ approaching unity. In this case, then, the electromagnetic wave is described by spinless photons, as is well accepted. In this limit, the spin tensor $S_{\text{sp}}^{\mu \nu}$ vanishes. If, on the other hand, the frequency of the electromagnetic wave is of the order of the characteristic frequency of a medium in question, then the interaction term cannot be neglected. The spin tensor $S_{\text{sp}}^{\mu \nu}$ is, in this case, non-vanishing, what amounts to saying that photons display spin. This is connected with polarization currents. It is to be noted that they give no contribution to the total energy and momentum fluxes, since

$$\int s_{A}^{\mu \nu} d^3x = \int s_{M}^{\mu \nu} d^3x. \quad (26)$$

In the limit of low frequency, the corresponding force is also finite and, in the rest frame, can be written as

$$f_{\text{sp}} = \frac{\epsilon W - 1}{4 \pi} \mathbf{a}_t (\mathbf{E} \times \mathbf{H}). \quad (27)$$

In virtue of the above arguments, this force will almost vanish at high frequencies, whilst it will increase towards the low frequency end of the spectrum.

Let us now corroborate this result by turning our attention to the experimental evidence. Although the available data are not in abundance, nevertheless, they well correlate with the above predictions. It should also be pointed out that there is a general agreement as to the correctness of the experimental findings. Briefly, the experiments of Jones and Richards /65/, Jones and Leslie /66/, and Ashkin and Dziedzic /67/, to mention just a few, were all performed using the optical frequencies. The obtain results are in accord with the Minkowski form for the energy-momentum tensor. On the other hand, the experiments
of Hakim and Higham /68/ and Walker et al. /69, 70/ were all conducted at the low frequencies. The conclusion they reached agrees with the Abraham result.

It is pertinent to mention one other point. The Minkowski momentum is often referred to as a pseudo-momentum /71, 72/ of the medium. This does not disagree with our result. As was already emphasized, in this work we study the effect of a general medium on the electromagnetic wave propagating in it and assume that the conditions are such that the effect of the wave on the medium can be neglected. Then, as was shown in the investigation of the black body radiation /3/, in the domain of the geometrical optics, the exchanged momentum between the electromagnetic wave and the medium is given by the Minkowski expression. In this frequency range, this is the total momentum of the photon, which is carried away by the medium, as a pseudo-momentum.

A possible way to support the predictions about the momentum of the electromagnetic wave in a medium would be to devise an experiment covering a large frequency range, preferably with a high intensity source. At the frequency extremes we would then expect the results as detailed above, whilst for intermediate frequencies one would obtain the average value of those at both ends of the range. In other words, the momentum $g$ of the electromagnetic wave of intermediate frequencies propagating in a medium would be given by

$$g = \frac{g_A + g_B}{2} \quad (28)$$

This result can, indeed, be recovered from the work of Peierls /72/, if the correction term of the second order is dropped. Klina and Petrzilka /73/ found the same expression for the wave in a collisionless plasma and Hora /74/ further substantiated it by showing that the Fresnel formulae can be derived by using that momentum.
It remains now only to synthesize the intermediate results of the individual sections, in order to obtain a clearer picture of some aspects of interaction phenomena.

Accepting, implicitly, the linearity of the field equations, the new derivation of the black body radiation inside a medium was presented. At the same time, it was found that the magnitude of the momentum exchanged between the radiation field and a medium is, in the region of validity of the Wien law, proportional to the refractive index of the latter.

There is a sufficient evidence that the classical electrodynamics holds only as long as the wavelengths are long compared to the "radius" of the electron. When the field contains shorter wavelengths, this theory breaks down. It is, therefore, desirable to modify the theory in such a way, as to widen its region of applicability. Some interesting, and very different proposals of others were discussed.

The concept of mass, which was interpreted as arising from the higher order interaction, played a dominant role in this work. It was seen that, first of all, a medium may be replaced by a curved space. If the non-linear interaction between the electromagnetic wave and the curvature of that space is taken into account, one is naturally led to the non-linear photons, whose mass is proportional to the scalar of curvature.

In support of this result, the framework of elementary particle physics was invoked. The detailed analysis of a simple experiment then confirmed that a photon, in a medium, may, under certain conditions, acquire a non-vanishing rest mass. The underlying philosophy is similar to that occurring in the theory of super-
conductivity, where photon is considered to be massive. This mass generation, in either case, is a result of the symmetry breaking and a subsequent loss of the gauge invariance.

In the light of this result, it is suggested that, for most media, at optical frequencies, the energy-momentum tensor of the electromagnetic field inside that medium is given by Minkowski. However, when conditions are such that the photon may be considered massive, then the Abraham result seems more appropriate.

In conclusion, when conditions are such that the Maxwell theory may be considered to be exact, then the corresponding photons are massless. However, if this is not the case, e.g., in the presence of the strong fields, which substantially modify the character of the electromagnetic wave, then this theory must be altered to include the non-linear interaction terms, thus, giving rise to the non-vanishing mass of the photons.
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Interaction of Photons with Electrons in Dielectric Media

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Abstract

The question of the field energy-momentum tensor in a medium is not new and many workers, in the past, attempted to find the answer to it. Nevertheless, there was no general agreement about the form of such a tensor, thus resulting in a confusion involving the very fundamentals of physics. Although the present work uses well established theories, the underlying philosophy is completely novel. Investigations are mainly centered around the collisionless plasma as, in that state, development of the argument is most transparent.

On the basis of a simple theoretical reasoning, it is, first of all, postulated that the momentum of the photon in a plasma is given by the Minkowski expression. This hypothesis becomes more viable as it directly leads to the accepted form for the field energy density. Furthermore, utilizing the concepts of stimulated and spontaneous emission and absorption, it is confirmed that, in the equilibrium (between radiation and plasma), the transferred momentum has the value given by the Minkowski theory. However, as will be seen, this result is valid only in the region of high frequencies. Put otherwise, when conditions are such that the laws of geometrical optics apply, the momentum of the photon is, to a good approximation, given by the Minkowski theory.

In order to arrive at a more general result, valid in the domain of wave optics, a similarity between the dispersion relation (describing the wave propagation through a medium) and the relativistic energy-momentum of a particle moving through vacuum, is explored. Accepting the equivalence of these two relations implies that some of the properties of radiation in a medium, can also be described by a massive particle travelling through empty space. In other words, the task of analysing the behaviour of electromagnetic waves in a medium, can be replaced by the analysis of the free neutral vector meson field. Adoption of this equivalence results in the field energy-momentum tensor being symmetrical.

A thorough study of the field theory then provides the basis for interpreting the Abraham tensor as the sum of the "orbital" and "spin" tensors. The former is directly connected with the energy transport, whereas the latter one is not. Realising that the magnitude of the spin term increases towards the lower end of the frequency spectrum provides the resolution of the Abraham-Minkowski dilemma: the energy-momentum tensor, corresponding to the electromagnetic wave in a medium, is that given by Abraham, while the one of Minkowski is only an approximation valid at high frequencies.

Although this conclusion stems from studies involving collisionless isotropic plasma, it is in excellent agreement with the experimental data.

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I. Introduction

The idea that light has purely undulatory character was generally accepted in the last century. The corresponding theory was able to explain empirical facts quite well and, thus, superseded the older, corpuscular theory of Newton. As the new experimental data became available, the theory was altered, but this change was only gradual. However, towards the end of that century, the understanding of the physical phenomena was, once again, thrown into confusion. There was an accumulation of the experimental evidence which could not be accounted for by the existing theory.

Thus, Planck, in his work on the black body radiation, was forced to introduce the discontinuous aspect of radiation, in order to resolve this discrepancy between theory and experiment. This meant that the corpuscular nature of radiation was reintroduced. Einstein used and further expanded this concept, when investigating the photo-electric effect. The name “photon” was introduced to denote these corpuscles. Nevertheless, he did not mean that the corpuscular (photon) theory became (as was the case in the time of Newton) the dominant one. As distinct from the exclusiveness of the previous theories, the radiation was now to be understood as having two distinct, complementary aspects: undulatory and corpuscular. In other words, depending on the question posed, the radiation-induced effects could be accounted for by either the wave or the particle theory. The complementarity of these different characters of radiation is now well established. For instance, the classical wave theory can explain neither the photoelectric phenomenon nor the Compton effect, because these apparently belong to the particle domain. On the other hand, the occurrence of interference, diffraction, and polarization manifest the wave nature of radiation and, therefore, cannot be explained by the photon theory. Thus, in order to really comprehend the radiation phenomenon, this duality must be inevitably accepted.

In spite of differences, there is a close connection between the undulatory and corpuscular pictures. Classically, the notion of frequency and wavelength are inherent in the former, while energy and momentum belong to the latter. However, it is now customary to associate with the electromagnetic wave of angular frequency \(\omega\), a particle with energy \(E\) given by

\[
E = h\omega,
\]

(1.1)

where \(2\pi\hbar = h\) is the Planck constant. Although (1.1) was proposed by Planck already at the beginning of this century and is now well accepted, its meaning as to a physical connection between the energy and the frequency remains rather obscure [1]. Nevertheless, (1.1) in conjunction with one of the most fundamental principles of the special theory of relativity, viz. the principle of the equivalence of mass \(m\) and any kind of energy \(E\), \(E = mc^2\), suggest that an effective mass \(m_0/c^2\) can be attached to the photon. Now, according to the special theory of relativity, the energy \(E\) of a particle is given by

\[
E^2 = p^2c^2 + m_0^2c^4,
\]

(1.2)

where \(p\) stands for the momentum and \(m_0\) refers to the rest mass of a particle. Of course, the rest mass vanishes in the case of a photon, hence its momentum has the value \(h\omega/c\).

In addition, there is another, equivalent expression for the photon's momentum. Thus, the substitution of the relation between the frequency \(\nu\) and the corresponding wavelength \(\lambda\), which in vacuo takes the form

\[
\nu\lambda = c,
\]

(1.3)

together with (1.1) into (1.2) results in \(p = h/\lambda\), or

\[
p = h\nu,
\]

(1.4)

where \(k = 2\pi/\lambda\) is the wave vector. Hence, (1.1) and (1.4) emphasize the relation between the corpuscular and the wave aspects of radiation. It is noticed that the ratio between the two is given in terms of the quantum of action. This, in essence, reflects the wave-corpuscular parallelism, which Einstein applied to particles with zero rest mass — photons.

The relations (1.1) and (1.4) remain, however, formally correct also for corpuscles with non-vanishing rest mass, as was first recognized by de Broglie [3]. He noted the similarity between the phase of a plane wave, \(\varphi (r, t) = A_0 \exp (i(Et - p \cdot r)/h)\), and the Hamiltonian action of the particle. Thus, he deduced that the Fermat principle, valid in the domain of geometrical optics, is analogous to the Maupertuis theorem in the analytical mechanics. Since the Fermat theorem has no meaning in the realm of the wave optics, he concluded that the traditional mechanics is also an approximation with the same range of validity. Therefore, de Broglie applied the wave concept of radiation to matter and showed that (1.1) and (1.4) are, indeed, relativistically invariant also for massive particles.

Subsequently, utilizing the definitions for the phase \(\varphi\) and the group \(v_g\) speeds, given by the wave theory as

\[
v_p = \omega/k, \quad v_g = d\omega/dk,
\]

(1.5)

he arrived at the well known group speed theorem [4]. In view of its derivation, it is valid in the geometrical optics approximation and expresses the relation between the speed \(v\) of a particle and the phase speed \(v_p\) of the associated wave — the so called matter wave

\[
v_p = c/\nu.
\]

(1.6)

It is then easily shown [4] that the group speed of the \(\varphi\) (matter) waves connected with the free particle is equal to the speed of the latter. Hence, a photon travelling in vacuum appears as a particle whose speed (and also group speed) is equal to the phase speed of the corresponding wave, as expected.

Despite these similarities, massless photons and massive particles are not the same. Thus, in order for a particle to be observed, it must interact. Consequently, its energy and momentum will be changed in accordance with the conservation laws. Then, high energy material particles are like high energy photons, because they change energy by changing their effective mass. This is the domain of the applicability of the geometrical optics, which, by definition [5], is valid in the region of small wavelengths. On the other hand,
In this case, the rest energy, whereas the photon energy (and frequency) approaches zero. The interaction of an electromagnetic field with a dielectric medium is characterized by the differences between photons and particles are very pronounced at the other end of the energy spectrum, i.e., in the low energy limit. The energy of material particles approaches zero. Hence, in this limit, the energy of radiation is represented by the electromagnetic field. Thus, the energy of radiation is represented by the electromagnetic field, while that of radiation assumes the wave character.

To complicate matters even further, other forms for the electromagnetic energy-momentum tensor have been put forward as well. Thus, Einstein and Laub [21] examined the problem and arrived at a completely different result from those of Minkowski and Abraham. Another proposal is due to Beck [22], who arrived at a conclusion similar to that of Marx and Gyrogi [23].

From the in-depth analysis of the properties of a field in a medium, Dr. Groot and Suttorp [24] obtained two different expressions for the energy-momentum tensor. The first one was derived by the statistical averaging of the relevant microscopic quantities, while the second one was found by subtracting the total energy-momentum in the absence of the field from the one when the field was present.

It would appear that this confused situation should be easy to resolve experimentally. That is not so, however. The few experiments actually performed do not correlate. Thus, the data collected by Jones and Richards [26], and in the more accurate, recent version of the same experiment by Jones and Leslie [27], lead to the conclusion in agreement with the Minkowski form of the momentum. This finding is also supported by the experiment of Ashkin and Dejord [27]. Nevertheless, the empirical data from Walker et al. [28] and Walker and Walker [29] suggest that the photon momentum in a medium is given by the Abraham value. In addition, results obtained in the static limit by Hawkins [30] confirm the Abraham theory.

On the basis of the brief description of the theoretical and experimental results, it is seen that the confusion is rather profound. The obvious difficulty is part in the complexity of the question of the nature of the photon momentum in a medium. Furthermore, there is a rather large variation in the conditions under which experiments were performed. For instance, in the Jones and Richards experiment, visible light ($\lambda \approx 10^5$ Hz) was used and liquid under investigation had the refractive index of the order of unity. On the other hand, Walker et al. employed very low frequency ($< 100$ Hz) and a ceramic sample had very high dielectric constant ($\approx 3200$). It is, therefore, not surprising that such a wide range of the involved variables produces confusing results.

In order to comprehend this, rather peculiar, behaviour, it is necessary, first of all, to consider the simplest case. As is well known, a medium can be in any of the four states: solid, liquid, gas, and plasma. Each of these states is characterized by the different set of variables. Therefore, in order to simplify the structure and to emphasize the physical nature of this fundamental problem, we shall concentrate our attention on the case of the plasma state. It will be assumed that there is no external magnetic field present and that collisions among electrons are rare and, hence, can be neglected. In other words, our main effort will be directed toward an isotropic, collisionless plasma with the dispersion relation of the form

$$\epsilon(\omega) = 1 - (\omega_p^2/\omega^2),$$

where $\epsilon(\omega)$ is the frequency dependent dielectric permeability, $\omega_p^2 = 4\pi Ne^2/m$ is the plasma frequency, $\omega$ denotes the frequency of the incident wave, and $\omega_p^2$ is the total number density, charge, and mass of the electrons, respectively. The reason for this selection is due to the weak nature of the interaction among plasma particles. This, consequently, considerably simplifies the procedure. Once this relatively simple case has been well understood, the extension to other states is then more likely to lead to the correct conclusion. Furthermore, most of the results obtained through the investigation of a tenuous plasma will also apply for the case of dielectrics and metals when the incident frequency is approximately in (or above) the ultraviolet range, since (1.7) may be shown [8] to be valid at high frequencies for matter in any state.

As may be deduced by now, the central task of the present work involves the analysis of the interaction of an electromagnetic field with a dielectric medium. Hence, to facilitate...
the inquiry, the following scheme is proposed: The field theory, which provides the necessary background information for this type of problem, is briefly outlined in
the next chapter. The four-dimensional formalism of the special theory of relativity is used throughout this chapter. Starting from the covariant form of the Maxwell equations, the energy momentum tensor of the total system is derived employing the concept of the Lagrangian density. It then found that part of this tensor refers to the electromagnetic field, while the other part describes the background medium. This "splitting-up" procedure of the total tensor is, however, not unique, and this is responsible for the above ambiguity in selecting the form of the field energy-momentum tensor. A comprehensive review and an analysis of the various proposed forms of that tensor are presented in Chapter III. As already indicated, they lead to different conclusions which are then also examined. In addition, the correlation between the theory and existing empirical facts is discussed.

As mentioned above, the present analysis is centered on the fourth state of matter, i.e., plasma. Various kinds of electromagnetic waves that such a state can support. It is, therefore, instructive to include some details about the wave propagation and also to recall the essential features of the supporting medium. The problem of defining dielectric permittivity of such a system is a rather complex one and it has attracted much attention [34, 35]. These questions are treated in Chapter IV.

Elementary considerations, which often help in understanding the dual nature of radiation are examined in Chapter V. It will be found that, within the WKB approximation, this simple theory is in agreement with the accepted results. The de Broglie group velocity theorem is utilized in order to arrive at some conclusions of that chapter. Further, it is of interest to inquire into the momentum transfer from the radiation field to a particle moving in a plasma. The method used by Einstein [34] in analysis of the black-body radiation is, first of all, adopted to include the medium with the refractive index differing from unity, along the lines attempted by Skobeltsyn [36]. It is then applied in derivation of the momentum transfer, thus directly leading to the accepted value [35, 37, 38] of the radiation energy density inside plasma. Let us emphasize again that these results are applicable to the case of an isotropic, collisionless plasma. A possible extension to the collisional plasma is also suggested, in analogy to the treatment of a gaseous body radiation is, first of all, adapted to include the media with the refractive index differing from unity.

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where the antisymmetric tensor, is equal to
1 for any even permutation of indices,
-1 for any odd permutation,
0 if two indices are equal.

The homogeneous field equations in terms of the dual field tensor assume the form
\[ \partial_\mu F^\mu = 0. \]  
(2.8)

Mathematically speaking, (2.6) and (2.8) are equivalent. However, as in an ordinary space the electric field vector \( E \) is polar and that due to the magnetic field \( H \) is axial, it seems more correct [2] to represent the electromagnetic field by \( F^\mu \).

The inhomogeneous pair of the Maxwell equations can also be written in the relativistically invariant form. Thus, in terms of \( F^\mu \) and the four-current \( j^\mu \), these take on the form
\[ \partial_\mu j^\mu = \frac{4\pi}{c} \rho. \]  
(2.9)

The space components of the four-current, \( j = e \sigma \), form a vector in three dimensions, the so-called current density vector. The time component is the product of the charge density \( \rho \) and the vacuum speed of light \( c \). The relation between these components of the four-current is expressed through the continuity equation
\[ \partial_\mu j^\mu = 0. \]  
(2.10)

In a continuous medium we think of the charges as smeared over space-time manifold. Four-current then takes this effect into account. Of course, if there are no charges present, the right-hand side of (2.9) vanishes. In the macroscopic formulation it is, however, necessary, because of polarization and magnetization effects, to introduce the additional electric field tensor \( \mathbf{H}^\mu \). This tensor is obtained from \( F^\mu \) by the transformation \( \mathbf{E} \rightarrow \mathbf{D} \) and \( \mathbf{H} \rightarrow \mathbf{B} \). The covariant form of the macroscopic Maxwell equations is then
\[ \partial_\mu j^\mu = \frac{4\pi}{c} \rho. \]  
(2.11)

The definitions of the four-current \( j^\mu \), the four-potential \( A^\mu \), and the field tensors \( F^\mu \) and \( H^\mu \), in conjunction with the Maxwell equations conclusively establish the covariance of the field equations [20].

Turn attention now to the covariant formulation of the conservation laws. It will be assumed here that the system under investigation consists of an electromagnetic field and that there also are some charged particles moving in it. Hence, we shall examine the interaction between these two subsystems, resorting to the Lagrangian formalism. Of prime importance, in the description, is the action integral \( A \), defined as [5]
\[ A = \int \mathcal{L} \, dt, \]  
(2.12)

where the four-dimensional volume element is
\[ dx = c \, dt \, dx \, dy \, dz. \]  
(2.13)

These equations can also be expressed through the dual field tensor \( \mathcal{J}^\mu \), which is defined as
\[ \mathcal{J}^\mu = \frac{1}{2} \varepsilon^{\mu
u\rho\sigma} F_{\nu\rho}, \]  
(2.7)

and \( \varepsilon^{\mu
u\rho\sigma} \), the antisymmetric tensor, is equal to
1 for any even permutation of indices,
-1 for any odd permutation,
0 if two indices are equal.

The first term in (2.14), \( A^\mu \), refers to the action of the field alone, i.e., to system in the absence of any charges. If one knows this term, the field equations can be easily [2] obtained. The Lagrangian density corresponding to this term must be Lorentz invariant, in order that the field equations remain linear. Furthermore, as this function must be Lorentz invariant, it cannot involve coordinates explicitly. In addition, it cannot be expressed in terms of the four-potentials, as these are not uniquely determined. In view of these restrictions, the only Lorentz invariant quadratic forms are \( F_{\mu\nu} F^{\mu\nu} \) and \( F_{\mu
u} F_{\mu
u} \). However, the latter scalar changes sign under the inversion. Therefore, only the former scalar, \( \mathcal{J}^\mu \mathcal{J}_{\mu} \), constitutes the free field Lagrangian. Hence, the action \( A^\mu \) corresponding to the free field, takes the form
\[ A_f = -\frac{1}{16\pi} \int \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \, dx, \]  
(2.15)

which, in the three-dimensional representation, becomes
\[ A_f = \frac{1}{8\pi} \int (E^2 - H^2) \, dV \, dt. \]  
(2.15a)

In order to find the Lagrangian density for the free field, it is only necessary to compare (2.12) with (2.15a). One then obtains
\[ \mathcal{L}_f = \frac{1}{16\pi} (E^2 - H^2). \]  
(2.16)

The second term in (2.14), \( A_{\sigma} \), represents that part of the action which depends exclusively on the properties of the particles, i.e., the system in the absence of the field. If there are several particles present, then the general form for the action is [5]
\[ A_{\sigma} = -\sum \frac{q}{c} \int A^\mu \, dx^\mu. \]  
(2.17)

In other words, \( A_{\sigma} \) is equal to the sum of the actions referring separately to each particle. The last term in (2.14), \( A_{\mu\nu} \), describes the interaction between the components of our system, viz., between the field and matter. Therefore, it must contain information regarding both subsystems. Hence, it is found convenient to represent matter by the electromagnetic charge \( q \), while the field is determined by the four-potential, \( A^\mu \). The action term then takes the form
\[ A_{\mu\nu} = -\int \frac{q}{c} \sum A^\mu \, dx^\mu. \]  
(2.18)

If discrete charges are replaced by the continuous charge density \( \rho \), then (2.18) assumes the form
\[ A_{\mu\nu} = -\int \frac{\rho}{c^2} \sum A^\mu \, dx^\mu. \]  
(2.19)

Combining (2.15), (2.17), and (2.19), the action for the whole system (with a continuous charge distribution) then becomes
\[ A = A_f + A_{\sigma} + A_{\mu\nu}. \]  
(2.14)
With this in mind, the electromagnetic Lagrangian density becomes

$$\mathcal{L} = \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} j_\mu A^\mu.$$  \hspace{1cm} (2.21)

Having found the Lagrangian density, the field equations can be readily obtained. Thus, utilizing the Euler-Lagrange equations of motion

$$\nabla^\alpha \left( \frac{\partial \mathcal{L}}{\partial (\nabla_\alpha A^\mu)} \right) - \frac{\partial \mathcal{L}}{\partial A^\mu} = 0,$$  \hspace{1cm} (2.22)

together with (2.21), one arrives at the field equations. To elucidate the meaning of (2.22), the required operations are first performed. After some calculation, we obtain

$$\frac{1}{4\pi} \nabla^\alpha F_{\alpha\beta} + \frac{1}{c} j_\beta = 0.$$  \hspace{1cm} (2.23)

As is easily recognized, these are nothing else but the inhomogeneous Maxwell equations (2.9). To obtain the conservation of the current density, we proceed to take the divergence of (2.23), thus yielding

$$\frac{1}{4\pi} \nabla^\alpha j_\alpha = 0.$$  \hspace{1cm} (2.24)

However, it is recalled that the product of a symmetric operator $\nabla^\alpha j_\alpha$ and the antisymmetric tensor $F_{\alpha\beta}$ vanishes, resulting in

$$\nabla^\alpha j_\alpha = 0,$$  \hspace{1cm} (2.25)

and the Hamiltonian density $\mathcal{H}$ then becomes

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial (\dot{q}_i)} q_i - \mathcal{L},$$  \hspace{1cm} (2.26)

where $\dot{q}_i$ refers to the time derivative of the generalized coordinates $q_i$. Similarly, the field canonical momentum is obtained from (2.25) by the appropriate replacement for $\dot{q}_i$. Thus, it is then defined by

$$P_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i},$$  \hspace{1cm} (2.27)

what means that the field momentum is canonically conjugate to $A_\nu$. In analogy, the expression for the field Hamiltonian density can be obtained. The result is then known as the canonical energy-momentum tensor [48]

$$T^{\alpha\nu} = \frac{\partial \mathcal{L}}{\partial (\nabla_\alpha A_\nu)} \dot{A}_\nu - \dot{\mathcal{L}}.$$  \hspace{1cm} (2.28)

In general, this tensor is not symmetric. Furthermore, the component corresponding to the energy density is not always positive [48]. These facts have caused some difficulties. To make this tensor more transparent, consider first the free electromagnetic field. The second term on the right hand side of (2.21) then vanishes (as there are no charges present). The free field Lagrangian, $\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}$, is then substituted in (2.28) to give

$$T^{\alpha\nu} = \frac{1}{4\pi} g_{\alpha\nu} F_{\mu\rho} \dot{E}^\mu \dot{A}^\rho + \frac{1}{16\pi} g_{\alpha\nu} F_{\mu\rho} F^{\mu\rho}.$$  \hspace{1cm} (2.29)

This expression becomes clearer when it is written in the three-dimensional form. Thus, substituting (2.5), the matrix form for the field tensor $F_{\mu\nu}$, together with the free field Lagrangian (2.16) into (2.24) gives us the following components for the electromagnetic energy-momentum tensor

$$T^{00} = \frac{1}{8\pi} (E^2 + H^2) + \frac{1}{4\pi} V \cdot (\nabla E),$$  \hspace{1cm} (2.30)

$$T^{0i} = \frac{1}{4\pi} (E \times H)_i + \frac{1}{4\pi} V \cdot (A_i E),$$

$$T^{ij} = \frac{1}{4\pi} (E \times H)_i + \frac{1}{4\pi} [V \times (\nabla E) - \partial_0 (\nabla E)].$$

It is seen that all expressions differ from the commonly accepted ones for the energy and momentum densities by the addition of the last term. This difficulty, however, is usually obviated. It is recalled that the components of the energy-momentum tensor refer to the densities. However, in order to obtain energy and momentum of the field, they must be integrated over all space. As it is assumed that the fields are localized, the contributions from the added terms in (2.30) vanish when the integration is performed, thus leading to the standard expression for the field energy and momentum.

Knowledge of the energy-momentum tensor enables one to write down the conservation laws. Thus, the Poynting theorem, which expresses the conservation of energy and momentum, can be written [2] in the covariant form as

$$\delta_\alpha T^{\alpha\nu} = 0.$$  \hspace{1cm} (2.31)

In analogy, the vanishing of the four-divergence of the angular momentum density implies that the orbital angular momentum of the field is conserved. That is, if the angular momentum tensor is

$$M_{\mu\nu} = T^{\alpha\nu} x^\alpha - T^{\alpha\nu} x^\alpha,$$  \hspace{1cm} (2.32)

then

$$\delta_\mu M^{\mu\nu} = 0,$$  \hspace{1cm} (2.33)

expresses the appropriate conservation law. However, in order for (2.33) to hold, the direct calculation requires that $T^{\alpha\nu}$ must be symmetric. As was already noticed, and is also clearly exhibited by comparing $T^{\alpha\nu}$ and $T^{\alpha\nu}$ in (2.30), the tensor $T^{\alpha\nu}$ does not satisfy this symmetry requirement. Hence, the angular momentum, with $M_{\mu\nu}$ as defined by (2.32), is not conserved. In addition, according to the definition of $T^{\alpha\nu}$, (2.29), it depends explicitly on the potentials. However, it is known that the potentials are not uniquely determined. Consequently, the tensor $T^{\alpha\nu}$ is not gauge invariant [50]. Moreover, direct calculation shows that its trace, $T^{\alpha\nu}$, does not vanish.

As the just considered free field is free (in view of (2.16)), it is difficult to accept the tensor $T^{\alpha\nu}$ as physically significant. Therefore, it was necessary to find another energy-momentum...
tensor, which would remove the above mentioned deficiencies. Here lies the origin of the symmetric energy-momentum tensor $\Theta^\mu$. The prescription for such a tensor was first given by Belinfante $[46, 47]$. Thus, to symmetrize $T^\mu_\nu$, the term $\partial^4 A^\mu$ in (2.29) is replaced, utilizing (2.4), by $\partial^4 A^\mu - \partial^4 A^\nu$. Then, the tensor $T^\mu_\nu$ assumes the form

$$T^\mu_\nu = \frac{1}{4\pi} \left[ \frac{g^{\mu\rho} F_{\alpha\beta} F^{\rho\alpha}}{4} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right] - \frac{1}{4\pi} g^{\mu\nu} F_{\alpha\beta} \partial^\alpha A^\beta. \quad (2.34)$$

The last term in above can now be rewritten as

$$T^\mu_\nu = \frac{1}{4\pi} \partial^\gamma (F^\mu_{\gamma\nu} - F^\nu_{\gamma\mu}). \quad (2.35)$$

It is assumed that the field is free. That is, as there are no charges present, the Maxwell equations have the form (2.3), $\partial_\mu F^\mu = 0$. Thus, a term $A^\gamma \partial^\gamma F^{\mu\nu}$ can be added to (2.35). Consequently, we can write

$$T^\mu_\nu = \frac{1}{4\pi} \partial^\gamma (F^\mu_{\gamma\nu} - F^\nu_{\gamma\mu}). \quad (2.36)$$

The symmetric energy-momentum tensor $\Theta^\mu_\nu$ is now defined as

$$\Theta^\mu_\nu = T^\mu_\nu - T^\nu_\mu. \quad (2.37)$$

or, written in terms of the field tensors it becomes

$$\Theta^\mu_\nu = \frac{1}{4\pi} \left[ g^{\mu\alpha} F_{\alpha\beta} F^{\rho\beta} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right].$$

It is clear that the potentials are now not explicitly involved in the definition, therefore, the tensor $\Theta^\mu_\nu$ is gauge invariant $[5]$. If the field energy tensor (2.5) is substituted in (2.37), then $\Theta^\mu_\nu$ can be written in the component form as

$$\Theta^{\gamma\delta} = \frac{1}{8\pi} (E^2 + H^2) \quad (2.38)$$

$$\Theta^{ij} = \frac{1}{4\pi} \left[ E_i E_j + H_i H_j - \frac{1}{2} \delta_{ij} (E^2 + H^2) \right].$$

It is immediately obvious that the expression for the energy and momentum densities given by the time-time and space-space components of $\Theta^{\mu\nu}$, respectively, are in accordance with the accepted formulae. Furthermore, the space-space components represent the Maxwell stress tensor taken with the negative sign.

The four-divergence of $T^\mu_\nu$ vanishes as it is the product of symmetrical and anti-symmetrical tensors. Therefore, in view of its derivation, the tensor $\Theta^\mu_\nu$ also has vanishing divergence. Expressed mathematically,

$$\partial^\gamma \Theta_{\gamma\nu} = 0. \quad (2.39)$$

Of course, (2.39) is nothing else but the covariant form of the conservation of energy and momentum. In other words, it is the four-dimensional equivalent of the Poynting theorem. It is to be noted that the comparison of (2.29) with (2.37) yields the equality of the total energy and momentum integrals over space-like volumes $[48]$. Pauli has, furthermore, emphasized that for the unique localization in space-time of the energy and momentum, the gravitational theory is necessary. The introduction of the gravitational field gives, then, an unambiguous meaning to the energy-momentum tensor. The total angular momentum density of the field is now defined, using $\Theta^\mu_\nu$, in terms of the tensor $M^{\mu\nu}$, as

$$M^\mu_\nu = \Theta^\nu_\rho \omega^\rho - \Theta^\mu_\rho \omega^\rho. \quad (2.40)$$

In view of the symmetry of $\Theta^\mu_\nu$, the total angular momentum is conserved, which is mathematically expressed by

$$\partial_\mu M^\mu_{\nu} = 0. \quad (2.41)$$

The total angular momentum is again obtained by the integration of the corresponding density over space-like volumes.

So far, the considerations were mainly directed towards a free, i.e., non-interacting electromagnetic field. If the charges are included, interaction occurs. Consequently, the Lagrangian for the Maxwell field is also changed and is now given by (2.21). This, in turn, implies that the energy-momentum tensor $\Theta^\mu_\nu$ is no longer divergenceless. Then, instead of (2.39), we now write

$$\partial_\gamma \Theta_{\gamma\nu} = -\frac{1}{c} j^\nu. \quad (2.42)$$

The right hand side of (2.42) describes the interaction between the electromagnetic field, here represented by $F^\mu_\nu$, and the four-current $j^\nu$, describing the source. This coupling term is known as the Lorentz force density. The integral of this force density over the space volume then gives the time rate of change of the total four-momentum of all particles. On the other hand, the space integral of $\partial_\mu \Theta^\mu_\nu$ describes the time rate of change of the field four-momentum. Therefore, the conservation of the momentum of the whole system (field and particles) is written as

$$\int (\partial_\mu \Theta^\mu_\nu + j_\nu) dV = 0, \quad (2.43)$$

where the Lorentz force $j_\nu = 1/c F^\mu_\nu j_\mu$. Thus, the sum of the total momentum and energy of the system is conserved.

The Poynting theorem can also be interpreted as follows: the time-time component of the Lorentz force density multiplied by $c$ is $j_\nu E$. The space volume integration then describes the conversion of the electromagnetic energy into the mechanical or heat energy. Hence, it must be balanced by the corresponding decrease of the field energy. Physically, this is equal to work done by the field per unit time. Thus, it represents the rate of increase of the mechanical energy. In analogy, the considerations of a similar nature apply to the linear momentum.
The derivation of the field energy-momentum tensor inside a macroscopic medium presents a complex problem. We saw above that it was possible to divide the tensor corresponding to the total system, consisting of field and particles, into two parts, each representing the separate component of the system. However, when the macroscopic medium is involved, this separation cannot be established uniquely. The main reason for the implied ambiguity lies in the impossibility to differentiate as to what is electromagnetic and what is not. The following chapter examines the different proposals put forward to avoid this dilemma.

III. The Energy-Momentum Tensor in a Medium

1. Origin of the confusion

The considerations of the preceding chapter apply, essentially, to the electromagnetic field in vacuo interacting with the charged particles. Results arising from such analysis are well established. These do not hold, however, when the field energy-momentum tensor inside a macroscopic body is discussed. Judging on the basis of a number of proposed forms for such a tensor, it is clear that neither is the question trivial nor is the involved subtlety sufficiently appreciated. It may appear surprising that a problem as fundamental as this has still not been satisfactorily resolved, although the main work on the subject was completed decades ago. Numerous articles appearing in the literature testify to the complexity of the problem. The question of the electromagnetic energy-momentum tensor was almost completely solved by Helmholtz, (cf. [57]), decades before the controversy originated. Using the principle of virtual work in conjunction with the macroscopic field equations, Helmholtz found the force which acts on a medium in the static field.

Amongst the early workers investigating this subject was Minkowski. By a straightforward application of the macroscopic Maxwell equations to the energy and momentum conservation laws, he arrived at the energy-momentum tensor of the form:

\[
S_{\mu\nu} = \frac{1}{4\pi} \left[ \varepsilon_{\nu} F_{\mu\alpha} H^{\alpha} + \frac{1}{4} \varepsilon^\sigma \varepsilon_{\sigma} F_{\mu\alpha} H^{\alpha} \right].
\]  

(3.1)

Substitution of the components of the field strength tensors \( F_{\mu\nu} \) and \( H^{\mu} \) into (3.1) results in the space-space part of this tensor being expressed in terms of the field variables, thus becoming

\[
S_{\mu\nu}^{\text{sp}} = \frac{1}{4\pi} \left[ E_i D_k + H_k B_i - \frac{1}{2} (E \cdot D + H \cdot B) \right].
\]  

(3.2)

Recall that \( F_{\mu\nu} = \varepsilon_{\mu\nu} \varepsilon_{\rho\nu} \), and that \( H^{\mu} \) is obtained from \( F^{\mu\nu} \) by substituting \( E \rightarrow D \) and \( H \rightarrow B \). The influence of the medium is taken into account by introducing concepts of the tensors \( \varepsilon_{\mu\nu} \) and \( \mu_{\mu} \) of dielectric permittivity and magnetic permeability, respectively. Hence, we write \( D_i = \varepsilon_{\mu\nu} E_k \) and \( B_i = \mu_{\mu} H_k \).

It is generally accepted [2] that the time-time, \( S_{\mu\nu}^{tt} \), component of the tensor (3.2) refers to the field energy density

\[
S_{\mu\nu}^{tt} = \frac{1}{8\pi} (E \cdot D + H \cdot B).
\]  

(3.3)

while the space-time, \( S_{\mu\nu}^{sp} \), component represents the energy flux

\[
S_{\mu\nu}^{sp} = \frac{1}{4\pi} (E \times H)_i = \frac{S_i}{c},
\]  

(3.4)

where \( S = c/4\pi (E \times H) \) is the well known Poynting vector.

Now, on calculating \( S_{\mu\nu}^{sp} \), the space-time component of the (3.1) tensor, we have

\[
S_{\mu\nu}^{sp} = \frac{1}{4\pi} (D \times B)_i = \frac{\varepsilon_\mu}{4\pi} (E \times H)_i = \frac{\varepsilon_\mu}{c} S_i.
\]  

(3.5)

In view of the relationship

\[
g_{\mu\nu} = \frac{\varepsilon_\mu}{c^2} S = S \frac{c^2}{c^2}
\]  

(3.5a)

we obtain, therefore, the expression

\[
g_{\mu\nu} = \varepsilon_\mu \gamma,
\]

associating a momentum density \( \gamma \) with each energy flux \( S \).

Similarly, a contradiction arises also in connection with the principle of equivalence of energy and mass, \( E = mc^2 \). This is not surprising, as the latter principle is just an integrated form of the Planck principle of inertia [2]. The asymmetry of tensor \( S_{\mu\nu} \) also must give rise to uncompensated mechanical torques which, clearly, are entirely unphysical [2].

The above deficiencies in the Minkowski tensor, all arising from the asymmetric character of that tensor, are rather disturbing and, in an attempt to overcome them, a different form for the energy-momentum tensor (it will be denoted here by \( S_{\mu\nu}^{\text{new}} \)) was proposed by Abraham [52].

\[
S_{\mu\nu}^{\text{new}} = \frac{1}{4\pi} \left[ g^{\mu\nu} F_{\alpha\beta} H^{\alpha\beta} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} H^{\alpha\beta} + (\gamma - 1) g^{tt} \Omega_{\gamma}, \right.
\]

(3.6)

which, evidently, differs from \( S_{\mu\nu} \) by the addition of the last term, in which

\[
g^{tt} \Omega_\gamma = F_{\gamma\alpha} (\gamma H^{\alpha\beta} + \nu H^{\beta\gamma}),
\]

with \( \nu \) being the four-velocity of the medium. The components of this new tensor are

\[
S_{\mu\nu}^{\text{new}} = \frac{1}{8\pi} \left[ (E_i D_k + E_k D_i) + (H_i B_k + H_k B_i) - \delta^i \Omega_\gamma (E \cdot D + H \cdot B) \right]
\]

(3.6a)

\[
S_{\mu\nu}^{\text{new}} = \frac{1}{8\pi} (E \times H)_i,
\]

(3.6b)
The space-time components form a quantity which, again, should be identifiable with the momentum density \( \rho \) of the field, while satisfying

\[
g_A = \frac{S}{c^2},
\]

in accordance with the Planck principle.

Using the prescription (2.42), for the formation of the force density in the field, we then have, on comparing the resulting expressions for the Minkowski and Abraham tensors, the relation

\[
f_A = f_M + \frac{n^2 - 1}{c^2} \frac{\partial S}{\partial t},
\]

while

\[
f_A^0 = f_M^0
\]

is the corresponding time component. It is clear, from inspection of (3.8), that the Abraham force density differs from the Minkowski one by the very small additive term. The macroscopic four-force density \( f_{4}^{\mu} \) is to be regarded as a statistical average of its microscopic equivalents. Therefore, the relation \( f_{\mu} = S_{\mu}^{\nu} \) implies that the macroscopic energy-momentum tensor, in analogy with the four-force, is equal to the averaged microscopic version, because the symmetry of the tensor is preserved under the required operation. It appears that it was this reasoning that initially inclined Pauli [2] towards favouring the symmetric tensor.

Einstein and Laut [21] arrived at an expression for the energy-momentum tensor which was completely different from the two previously discussed. Starting from simple examples, they first calculated the force exerted by electric, or magnetic, fields on a material volume. They then obtained the force due to the electric dipoles. In other words, they assumed that the interaction between matter and an electromagnetic field was mediated through some hypothetical elementary electric and magnetic particles. The observed force density then had to consist of two contributions, viz., a part depending on the velocity of those particles, and a part which did not. Essentially, this separation was to be understood as a splitting of the total observed force into a surface force \((1 - \mu^{-1})(j \times H_{ext})/\mu c\) and a volume force \((j \times H_{int})/\mu c\). Consequently, the corresponding energy-momentum tensor, was written as

\[
S_{\mu}^{\nu} = \frac{1}{4\pi} \left[ E_{\mu} D_{\nu} + H_{\mu} B_{\nu} - \frac{1}{c^2} \partial_\mu \partial_\nu (E^2 + H^2) \right].
\]

(3.10)

It is to be noted that this tensor was not completely determined, as its time-time component was not specified.

Another proposal regarding the energy-momentum tensor was put forward by Beck [22]. In order to reconcile the velocity of the electric field with the (3.16), the Planck principle of inertia (satisfied by the Abraham tensor), on the other, he introduced what he called the radiation tensor \( S_\mu^{\nu\rho} \) of the form

\[
S_\mu^{\nu\rho} = S_\mu^{\nu\rho} + U_\mu^{\nu}\nu
\]

where \( U_\mu^{\nu\rho} \) represents the matter tensor. This radiation tensor may be put in a covariant form

\[
S_\mu^{\nu\rho} = \frac{1}{4\pi} \varepsilon_\mu^{\nu\rho} S_\mu^{\nu\rho} + \frac{1}{4\pi} \varepsilon_\mu^{\nu\rho} S_\mu^{\nu\rho}
\]

(3.11)

It turned out, however, that, although the tensor \( S_\mu^{\nu\rho} \) was symmetric, some conclusions drawn from it by that author were incorrect [10].

The question was further examined by Marx and György [23]. On considering the field and the dielectric as forming a closed system, they also found (similarly to Beck) that the field energy-momentum tensor, as given by Abraham, had to be supplemented by a matter tensor. The covariant form of the Marx and György tensor was

\[
S_\mu^{\nu\rho} = \frac{1}{c^2} \left\{ \frac{1}{4\pi} \left[ g^{\mu\rho} F_\mu H^\nu + \frac{1}{4} g^{\mu\rho} F_\mu H^{\nu\rho} + (\mu \nu - 1) U_\mu^{\nu\rho} \right] \right\},
\]

(3.12)

where

\[
U_\mu^{\nu\rho} = \frac{1}{c^2} \left( \frac{1}{4\pi} \varepsilon_\mu^{\nu\rho} \left( F_\mu H^\nu + \frac{1}{4} F_\mu H^{\nu\rho} \right) \right)
\]

In the rest frame of the medium, the components of the radiation tensor (3.12) are

\[
S_\mu^{\nu\rho} = \frac{1}{n^2} S_\mu^{\nu\rho}, \quad S_\mu^{\nu\rho} = S_\mu^{\nu\rho} = S_\mu^{\nu\rho}
\]

(3.13)

Although the tensors of Beck (3.11) and of Marx and György (3.12) appeared to be quite different, it was shown by Szönyi [33] that, at least in the case of a pure radiation field, they were, in fact, identical. Nevertheless, while Beck [22] advanced arguments in support of the Minkowski tensor being taken together with additional terms due to matter, Marx and György [23] accepted the Abraham version when further supplemented by a contribution again arising from presence of matter.

In an extensive analysis of the question of the energy-momentum tensor, de Groot and Suttorp [24] adopted an approach which was purely microscopic, i.e., one which was related to the behaviour of point particles. According to their thesis, this was the only rigorous way to arrive at the solution of the problem. Thus, they began with a microscopic force law for a constituent particle. The total electromagnetic field acting on such a particle is, of course, equal to the external field, to which the system is subjected, modified by contributions of the neighbouring particles. Put otherwise, the field may be seen as made up of two parts. First, there is the external field together with the addition of the field due to the surrounding particles — this field varies only very little over the atomic distances. Second, there is the intra-atomic field, due to the constituent particles of the atom. Obviously, that field varies quite drastically over the atomic dimensions.

Utilizing this concept, de Groot and Suttorp defined an atomic field energy-momentum tensor \( t^{\nu\rho} \). Then, from the behaviour of a particle in that field, the atomic material energy-momentum tensor \( t^{\nu\rho} \) was obtained. They then assumed that the atomic conservation laws had the form

\[
\partial_\nu t^{\nu\rho} = 0, \quad t^{\nu\rho} = t^{\nu\rho} + t^{\nu\rho}
\]

(3.14)

The tensor \( t^{\nu\rho} \) thus obtained was, naturally, symmetric, hence implying the law of conservation of angular momentum. If then \( m^{\nu\rho} \) is the (density) angular momentum tensor, we have

\[
\partial_\nu m^{\nu\rho} = 0, \quad m^{\nu\rho} = m^{\nu\rho} + m^{\nu\rho}
\]

(3.15)

In order to change from densities to observable quantities, a space integration at constant time is applied, thus, in turn, leading to a conserved quantity.

On performing a statistical averaging on such microscopic systems, de Groot and Suttorp [24] obtained a macroscopic conservation law of the form

\[
\partial_\nu S_\mu^{\nu\rho} = 0, \quad S_\mu^{\nu\rho} = (t^{\nu\rho})
\]

(3.16)
The total macroscopic energy-momentum tensor \( S_0^{\mu\nu} \) is symmetric and, in analogy with its microscopic counterpart \( \epsilon^{\mu\nu} \), consists of the sum of field and material tensors. Although the tensor \( S_0^{\mu\nu} \) is an average of \( \epsilon^{\mu\nu} \), this does not apply to the tensors making up the \( S_0^{\mu\nu} \). In other words, the macroscopic field (matter) tensor is not the mean of the microscopic field (matter) equivalents. Thus, according to the above workers, the averaging procedure applies only to the total energy-momentum tensor.

The symmetry of \( S_0^{\mu\nu} \), fulfilled in isotropic media, breaks down when the system is anisotropic. This is clearly seen when the tensor is written in the three-dimensional formalism

\[
S_0^{\mu\nu} = \frac{1}{4\pi} \left[ E_\mu D_\nu + H_\mu B_\nu - \delta^{\mu\nu} \left( \frac{1}{2} E^2 + \frac{1}{2} B^2 - M \cdot B \right) \right]
\]

where \( M = B - H \) is the magnetization vector.

In polar substances, where \( M = 0 \), the tensor \( S_0^{\mu\nu} \) coincides with the Einstein and Laubf tensor \( S_0^{\mu\nu} \), as seen on comparing (3.17) with (3.10).

De Groot and Suttorp [24] proposed also an alternative energy-momentum tensor. That tensor \( (S_0^{\mu\nu}) \) was defined as the difference between the total energy-momentum tensor in the presence of the electromagnetic field and the corresponding tensor, when such a field was absent. The last of these was, of course, identical with the material tensor. Assuming linear constitutive relations, the new tensor had the following components in the rest frame

\[
S_0^{\mu\nu} = \frac{1}{4\pi} \left[ E_\mu D_\nu + H_\mu B_\nu - \frac{1}{2} \delta^{\mu\nu} \left( \frac{\partial \varepsilon}{\partial T} + B^2 \frac{\partial \mu}{\partial T} \right) \right]
\]


\[
S_0^{\mu\nu} = \frac{1}{5\pi} \left[ E \cdot D + H \cdot B + E^2 \frac{\partial \varepsilon}{\partial T} + B^2 \frac{\partial \mu}{\partial T} \right],
\]

(3.18)

where \( I \) stands for the specific volume, \( T \) for the temperature, and the electric \( x \) and magnetic susceptibilities \( \chi \) are defined as \( P = 2\varepsilon A \) and \( M = \chi B \). If it is assumed that the derivatives of the material constants occurring in (3.18) vanish and the medium is isotropic, then it is immediately seen, on comparing (3.18) with (3.6a), that \( S_0^{\mu\nu} \) is identical with \( S_0^{\mu\nu} \). All of the above described energy-momentum tensors occur frequently in literature. In the section below, the just outlined forms of these tensors will be examined and the arguments put forward to support them will be discussed.

2. Theoretical considerations

We shall now examine the applicability of the various tensors to the description of different physical phenomena. Also, as conclusions reached in numerous papers, published in many different journals, are quite contradictory, it will be instructive to investigate, at this stage, the different approaches employed in an attempt to resolve the present dilemma.

Proponents of a symmetrical field energy-momentum tensor often refer to a simple thought experiment devised by Balazs [9]. Although stemming from a rather naive argument, it is, nevertheless, based on sound physical principles of classical physics and it leads to correct results.

Consider, then, two identical and parallel channels with no external forces acting on them. Let each of these channels contain an identical medium. A beam of radiation can be then chosen in such a way that it passes through the given medium in the first channel, but by-passes it in the second. (The selected paths are assumed, for simplicity, to be parallel.) Now, it is obvious that the speed of energy propagation in the medium is smaller than that in the empty space. Hence, the time a photon takes to travel a certain distance is different in the two channels. But otherwise, a photon (and so the mass associated with its energy) in the first channel requires a longer time to cover that length than does a photon in the second channel. Suppose, the medium does not move while a wave is travelling through it. Then, the displacement of the centre of mass of each channel should be different, in spite of the fact that no external forces are present. This, however, contradicts the velocity invariance. Therefore, the medium, during the passage of an electromagnetic wave through it, must have been displaced by the same amount. It means, in turn, that the medium has a certain momentum and, due to the conservation of the momentum of the total system, value of that momentum in the medium is different from that in vacuo. Put otherwise, as the displacement velocity of the centre of mass of the system is the same in both cases, its position of the first channel is shifted further from the initial one. This, therefore, implies that the medium itself must also move in the same direction. Hence, it may be said, the momentum is transferred to the medium in the direction of motion. This, however, would not be possible had the electromagnetic field momentum in a medium been that given by the Minkowski expression (3.5). (It has been tacitly assumed here that the refractive index of the medium is greater than unity.)

The above may be expressed quantitatively, as follows. If \( x_1 \) is the \( x \)-coordinate of the centre of mass of the system, \( W \) — the mass of the medium, \( w \) — the relativistic mass equivalent of the energy carried by the wave, then the velocity of the centre of mass of the wave-medium system

\[
\begin{align*}
\langle \delta x_1 \rangle &= (W \delta x_1 + w \delta x_2)/(W + w), \\
\langle \delta x_2 \rangle &= \omega c/(W + w),
\end{align*}
\]

(3.20)

where \( X_1 \) and \( X_2 \) are the coordinates of the centre of mass of the material medium and the wave-field, respectively. If, further, \( (\delta x_1) \) is the value of the velocity of centre of mass before the interaction, then

\[
\begin{align*}
\langle \delta x_1 \rangle &= \langle \delta x_1 \rangle_L = (\omega c/w) - p, \\
\langle \delta x_2 \rangle &= \omega c/(W + w),
\end{align*}
\]

(3.19)

From the momentum balance, it then follows that

\[
W \delta x_1 = (\omega c/w) - p,
\]

(3.21)

where \( p \) denotes the momentum associated with the electromagnetic field inside the medium. Let \( (\delta x_1) \) be the velocity of the centre of mass, while the wave-medium interaction is taking place, i.e., while the wave travels through the medium. Then, combining (3.19) with (3.21) leads to

\[
\langle \delta x_1 \rangle = (\omega c/w) - (\omega c/w) = (\omega c/w).
\]

(3.22)
which is the Abraham expression for the field momentum inside the medium.

The above considerations can be summed up in the following way. The total momentum of a system of radiation plus matter can be uniquely split into two parts [9], with one part referring to the change in the field momentum and the other describing the increase in the mechanical momentum. It can be then concluded that only the symmetric field energy-momentum tensor simultaneously satisfies the momentum conservation and the centre of mass theorems.

Essentially the same thought experiment was also used by other workers. Thus, e.g., Burt and Peters [11] employed this argument for illustrative purposes. They pointed out, however, that the controversy was not adequately resolved by the above reasoning, as the assumption of a rigid medium was involved—the concept incompatible with the special theory of relativity.

A similar analysis was also put forward by Jones [14]. Utilizing rather simple concepts, he attempted to find out how the momentum of radiation inside a medium was distributed between its electromagnetic and mechanical parts which then led to the conclusion that the total momentum associated with the radiation field should be that given by the Minkowski expression, \( p = mp/c \). If, then, \( n_p \) refers to the group refractive index, the fraction \( n_p(n_p - 1)/(n^2 - 1) \) (in the conventional phase refractive index) is, according to him, to be associated with the mechanical momentum present in the body. It should be noticed, that in order to arrive at this conclusion, Jones introduces an extra (group) refractive index, \( n_p \), thus, complicating the issue. Besides, in the experiment of Jones and Leslie [26], one of the conclusions was that the refractive index should be given by the ratio of the phase, rather than group, speeds. The above analysis by Jones seems, therefore, rather unsatisfactory.

A more profound approach is that of Haus [15]. In order to arrive at the expression for the electromagnetic momentum density in a linearly polarizable medium, he separates the total system into a part describing the motion of a mechanical continuum and another part corresponding to electromagnetic and thermodynamic properties. Then, the requirement for the mechanical momentum conservation of the continuum takes the form

\[
p = \frac{hv}{c n},
\]

(3.24)

while that for the other part gives

\[
\partial_k (\varepsilon \varepsilon_k + \phi \cdot \sigma_k) = -i \varepsilon_0.
\]

Here, \( \rho \) is the mass density, \( v \) — the velocity, \( \sigma_k \) — the stress tensor, and \( q_k \) — the momentum density of the remaining, non-mechanical components. Although the separation of the system into kinetic and electromagnetic parts is, essentially, embedded in work of Balazs [9], the conclusion reached by Haus is slightly modified by some additive terms. He argues that, as the radiation enters a dielectric, an interaction between field and matter takes place. Thus, the effect of matter upon the field is reflected in the electromagnetic constitutive relations. The force with which the field acts on the medium was calculated by Penfield and Haus [16]. It will be sufficient to mention here that the volume integration of that force results in

\[
\int \mathbf{F} \cdot dV = -\varepsilon_0 \int (\mathbf{E} \times \mathbf{H}) \cdot dV/c^2.
\]

(3.25)

From this, it directly follows that the electromagnetic momentum density in a non-dispersive linear medium is \((\mathbf{E} \times \mathbf{H})/c^2\).

The momentum exchanged between the vacuum and the field, and the medium, is of course, equal to the difference between the field momentum before and after the interaction has taken place. Haus then argued that the momentum density possessed by the electromagnetic wave packet travelling in a medium need not be necessarily given by \((\mathbf{E} \times \mathbf{H})/c^2\) once the said exchange is taken into account.

Furthermore, it is shown that in the case of normal incidence at a sharp boundary between the vacuum and the medium, there are no singularities in the force density \(\mathbf{F}\).

Consequently, there are no surface forces. This implies that all of the momentum is then imparted to the bulk of the medium. Hence, he concludes that all of the momentum is contained in the incident, reflected, and refracted wave, as none is left on the surface. These considerations apply to a medium with negligible electro- and magneto-stiction.

Thus, finally, it appears as if neither \(g_0\), given by (3.7), nor \(g_1\), from (3.5), could represent the wave momentum. According to Haus, the correct value for the momentum is given by (3.7) supplemented by terms describing a mechanical momentum, attributable to a medium. In addition, the energy flux density must also be increased by a contribution from the medium.

The conclusions reached by Haus, essentially, agree with results of Shockley [15]. That author, using an ingenious model, succeeds in interpreting the connection between the macroscopic laws and the structure of the medium. The electromagnetic theory is, first of all, expanded to include the magnetic equivalents of electric charges thus, naturally, leading to the magnetic analogue of the Lorentz force [64]. Consequently, the hidden linear momentum is introduced. Its application to the relativistic energy-momentum tensor is discussed [65] and, in conjunction with the centre of mass theorem, points to the conclusion that, in stationary matter, the electromagnetic momentum density is correctly given by the Abraham expression. This is supported by the experiment of James [55] whose results definitely reject the Minkowski form.

The physical reality of the magnetic dual of the Lorentz force was previously discussed by Costa de Beauregard [56] in his attempt at analyzing the Gojillo [57] experiment. Although the theoretically deduced effect was qualitatively similar to that observed in the experiment, its magnitude was much smaller. In Costa de Beauregard’s thought experiment, a ferromagnetic tubular test body and an axial wire with a constant density of charge are considered. Calculations of the force acting on the test body [58], assuming conservation of linear momentum, lead to the Minkowski value of the momentum density. A more complete interpretation of the driving force is given in [60].

When investigating the so called Weyssenhoff behaviour of a spinning particle, which is, essentially, a classical analogue of the Zitterbewegung of the Dirac electron, Costa de Beauregard [56] concludes that the non-linearity of the velocity and momentum of such particles requires an asymmetric energy-momentum tensor. A similar result was also obtained by Papapetrou [67]. Hence, it is easily found [60] that the four-momentum of photons in a medium should then be space-like, thus, presumably, resulting in a tachyonic type of behaviour.

Another argument in support of the Abraham tensor is supplied by Györgyi [62], who claims that the crucial point in the dispute is the Lorentz force acting on polarization currents. He, then, shows that the Abraham theory gives a correct description of, e.g., the Cherenkov radiation.
where the primed quantities refer to the moving inertial frame and a's represent the transformation matrices, i.e., $\alpha_{\xi \eta} = \delta_{\xi \eta}$, which can be written as

$$\alpha_{\xi \eta} = \begin{pmatrix} \gamma & -i \beta \gamma \nu \\ -i \beta \gamma \nu & \gamma \end{pmatrix}, \quad \gamma = \left(1 - \frac{\nu^2}{c^2}\right)^{-1/2}. \tag{3.27}$$

According to (3.6a), the Abraham tensor in the rest frame takes the form \[10\]

$$S_{\xi \eta} = \begin{pmatrix} u & iu/n \\ in/n & -u \end{pmatrix}, \tag{3.28}$$

where $u$ is the energy density. Similarly, in view of (3.3), (3.4), and (3.5), the Minkowski tensor in the rest frame is written as

$$S_{\xi \eta} = \begin{pmatrix} u & iu/n \\ in/n & -u \end{pmatrix}. \tag{3.29}$$

Transforming $S_{\xi \eta}$ and $S_{\eta \eta}$, according to (3.26) and using (3.27) leads to

$$S_{\xi \eta} = \frac{u_0}{n} \gamma \begin{pmatrix} n(1 + \beta^2) - 2\beta & i(1 - 2\beta n + n\beta) \\ i(1 - 2\beta n + n\beta) & -n(1 + \beta^2) - 2\beta \end{pmatrix} \tag{3.29a}$$

and

$$S_{\eta \eta} = \frac{u_0}{n} \gamma \begin{pmatrix} (n - \beta)(1 - n\beta) & i(n - \beta)^2 \\ i(n - \beta)^2 & -(n - \beta)(1 - n\beta) \end{pmatrix}. \tag{3.29b}$$

These two matrices then describe the corresponding two forms for the energy-momentum tensor in a moving inertial frame. These expressions will now be compared in model situation describing the limiting behaviour of matter, viz., the cases of a rigid body and of a rarefied medium.

In the first of these (the case of rigid body) the total mechanical tensor is represented by the stress tensor, whereas in the latter case it is represented by the product of the momentum density and the velocity of the medium. Thus, in the former case, the total mechanical energy-momentum tensor can be written \[10\] as

$$U^{\xi \eta} = \begin{pmatrix} -(n^2 - 1) u/n^2 & 0 \\ 0 & -\phi \end{pmatrix}, \tag{3.30}$$

where $\phi$ is the mass density in the rest frame. Of course, in the absence of the field, all components of $U^{\xi \eta}$ other than $U^{00}$ vanish. Thus, that part of the mechanical tensor which depends on the electromagnetic field is

$$\begin{pmatrix} -(n^2 - 1) u/n^2 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.31}$$

The component $U^{01} = -(n^2 - 1) u/n^2$ is often referred to as the radiation reaction force \[10\]. Transforming (3.31) into the moving frame, using (3.29a) and (3.27), gives the following expression for the tensor, $P_{\xi \eta}^1$, which represents the response of the medium as observed from the moving frame

$$P_{\xi \eta}^1 = \frac{\gamma^2 p_0}{n^2} \begin{pmatrix} 1 & i \beta \\ -i \beta & -\beta^2 \end{pmatrix}, \tag{3.32}$$

where $p_0 = -(n^2 - 1) u/n$. Now, the sum of $P_{\xi \eta}^1$ and $S_{\xi \eta}$, as given by (3.32) and (3.28), respectively, is the total tensor for the system

$$\begin{pmatrix} \nu \phi(n - \beta)(1 - n\beta) & i(n - \beta)^2 \\ i(n - \beta)^2 & -(n - \beta)(1 - n\beta) \end{pmatrix}. \tag{3.33}$$

Let us next consider, together with Skobel'styn [10], the second limiting case, viz., that of a "dust-like" medium. In that case, all stresses vanish and the corresponding tensor is then a product of the momentum density and the velocity of the medium. This assumes that the mass of each of the constituent particles is approaching infinity, while their corresponding number density is rather small. Consequently, in this approximation, the refractive index may be taken to be independent of the intensity of radiation (as the particle mass is very large). In addition, the radiation is not reflected at the incidence. When an electromagnetic beam enters the medium, the surface of the latter moves towards the source with a mechanical momentum \[10\]

$$p_n = (n - 1) \nu \phi/c. \tag{3.34}$$

It should be remarked that the same expression is obtained from both the Minkowski and the Abraham tensors.

Applying now the conservation equations (3.19), (3.21), and (3.23), the momentum $p_1$ of the material component of the system becomes

$$p_1 = (n - 1) \nu \phi/c_n. \tag{3.35}$$

This momentum, however, is equal to the difference between the medium and surface momenta. Thus, addition of (3.34) and (3.35) leads to the expression

$$p_n = (n^2 - 1) \nu \phi/c. \tag{3.36}$$

for the momentum of the medium.

Furthermore, if the particle number density in the absence of the field is $\nu_0$, then, at some later time $t$, it becomes

$$\phi(t) = \nu_0 + l_0. \tag{3.37}$$
The mechanical energy-momentum tensor may then be obtained in the form [60]

\[ U^{ij} = \begin{pmatrix} \beta \phi \partial_\phi e^2 \\ \frac{n^2 - 1}{n} \phi \partial_\phi - n^2 (n - 1) \phi \partial_\phi \end{pmatrix} \]  

(3.39)

On subtracting the value of \( U^{ij} \) in the absence of the field from (3.39), we obtain an expression for that part of the tensor which is related to the field

\[ P_x^{ij} = \begin{pmatrix} \frac{n^2 - 1}{n} \phi \partial_\phi - (n^2 - 1) \phi \partial_\phi \end{pmatrix} \]  

(3.40)

where \( \beta \phi e = (n^2 - 1) \phi \partial_\phi \). In the limit of infinitesimally small number density, the above tensor reduces to

\[ P_x^{ij} = \frac{n^2 - 1}{n} \phi \partial_\phi \left( \begin{array}{c} 0 \\ -n \end{array} \right) \]  

(3.41)

A subsequent transformation to the moving inertial frame yields

\[ P_x^{ij} = \frac{n^2 - 1}{n} \phi \partial_\phi \left( \begin{array}{c} 0 \\ -n \end{array} \right) \]  

The energy-momentum tensor corresponding to the whole system is then constructed, in analogy to above, resulting in

\[ T_x^{ij} = \phi \partial_\phi \left( \begin{array}{c} \frac{1}{1 + \beta^2} \left( 1 - n \beta \right) \\ \frac{1}{1 + \beta^2} \left( 1 + n \beta \right) \end{array} \right) \]  

(3.43)

Together with SKOBELET'SYN [10, 55], we can now compare the two expressions, (3.33) and (3.43), for the total energy-momentum tensor applicable in each limiting case. It is interesting to note that those forms differ from each other only by the factor \( 1/n^2 \).

We can now see the reason for considering two limiting cases of media, in which the interaction with field is to be investigated. For, clearly, the splitting of the total tensor of the system into contributions corresponding to the field and the matter, is not simple [17, 18, 65, 66]. Skobele'tsyn, therefore, in an attempt to overcome this difficulty, considers two extreme model situations, viz., those of rigid and “dust-like” media. This makes it possible to divide up the total energy-momentum tensor uniquely into field and matter parts, provided that the field energy-momentum tensor satisfies

\[ \varepsilon \partial_\phi S^{ij} = -J^i, \]  

(3.44)

(in accord with the discussion in Chapter II). Skobele'tsyn, further, claims [70] that the momentum density \( S^{ij} \) is obtained from the invariance of the velocity of centre of mass together with the momentum conservation law, while to determine the component \( S^{ij} \), one uses (3.44), (in the form \( \varepsilon \partial_\phi S^{ij} + 0 \)). As in the left-hand side of the last equation expresses the work done by the field, that equation, then describes the continuity of the energy flux (proportional to the Poynting vector). Consequently, \( S^{ij} = S^{ij} \), thus implying the symmetry of \( S^{ij} \). From the requirement that the energy-momentum parts of the total tensor be symmetric, it follows that also the mechanical tensor must be symmetric. Hence, Skobele'tsyn concludes that the correct expression for the field is that given by Abraham.

It should be noted that, initially, von Laue also [67] accepted the reasoning of Abraham. However, in a later article [68], he developed an argument which (so he believed) rejected this tensor and replaced it by the Minkowski one. Calculations of Schewe [68] even further supported this idea. The following considerations led to this change.

It is well known that, in general, when plane electromagnetic waves propagate through a medium, their phase speed is different from their group speed. Further, if the absorption in the medium is low, the latter speed may be identified with the speed of energy propagation. Von Laue [69] required the group velocity to satisfy the velocity superposition theorem due to Einstein, and proceeded to show that only the Minkowski tensor fulfilled this condition. Moller [70] developed a more general version of the same argument. Thus, if \( S^{ij} \) is the electromagnetic energy-momentum tensor, then the velocity of the energy propagation is defined by \( v_{ij} = c S^{ij}/S^{tt} \). Moller finds the condition for this velocity to transform like a particle velocity to be

\[ \frac{\partial v}{\partial x} = 0, \]  

(3.45)

where \( R^\mu = S^\mu - 1/c^2 S^{\mu\nu} c_{\nu} \) and \( e_{\nu} \) represents the four-velocity. He then shows that only the Minkowski tensor satisfies (3.45) and, thus, concludes that this tensor must give an adequate description of electromagnetic phenomena in matter.

Skobele'tsyn [70], nevertheless, doubts the validity of this criterion.

Next, using the thermodynamics of irreversible processes, Tanu and Meixner [69] analyzed the influence of electromagnetic waves on a simple fluid. The total energy-momentum tensor obtained by them consisted of three contributions: a constant field term, an oscillating field term, and a constant matter term. Noting that, in the absence of the field, the velocity of the energy transport does not satisfy the addition of velocities theorem, they split the total energy flow in the separate parts and then examine the von Laue and Moller transformation criterion for these parts. They concluded that Moller's treatment was incomplete and suggested that the Abraham and Minkowski tensors lead to the same result, provided the matter contribution is properly accounted for.

De Groot and Suttorp [24], however, claimed that what Tanu and Meixner [69] did, in fact, disproved the very validity of the von Laue criterion. Recently, this last statement was strongly criticized by Brevik [8].

Let us concentrate now on static fields inside a medium, hoping to obtain at least some correct results. We adopt a variational approach of Brevik and assume that the field energy is known. A similar method as also used by Landau and Lifschitz [8].

The free energy of an electrostatic field in a medium is defined, in the usual way [8], by

\[ \mathcal{F} = \frac{1}{8\pi} \int E \cdot D \, dV. \]  

(3.16)

Now, the rate of change of the field free energy may be interpreted as equal to mechanical work \( \int J \cdot e \, dV \) (\( e \) — velocity, \( J \) — force density). One thus obtains the expression for the force density

\[ J^i = \varepsilon E - \frac{1}{8\pi} E_i E^k \partial_k \phi + \frac{1}{8\pi} \partial_i (E D_k - E_k D), \]  

(3.47)
The divergence of the stress tensor is equal to the force density, i.e.,
\[ \nabla \cdot \sigma = -\frac{\partial S}{\partial t}, \]
so (3.47) is compatible with a tensor of the form
\[ S^{\mu\nu} = \frac{1}{2\varepsilon_0} \left[ E^{\mu} D^\nu + E^{\nu} D^\mu - \frac{1}{2} \partial_\mu E^\nu \right]. \]

Comparing it with (3.6a) shows (3.48) to be the same as the expression for the Abraham tensor of the static limit. This result follows directly from the variational principle. Now, the crucial point in Brevik’s argument consists in the proper accounting for the moment even when the external force is absent. This can happen when there is a distribution, given the torque, there is an additional amount of work done. In this model, the torque density \( \tau = D \times E \) represents an extra contribution to the total mechanical work. Putting the rate of change of the free energy equal to that work done per unit time, leads to the Minkowski result for the force density
\[ f = \nabla \cdot E - \frac{1}{4\varepsilon_0} E \cdot \nabla \varepsilon, \]
while the stress tensor takes on the form
\[ S^{\mu\nu} = \frac{1}{4\varepsilon_0} \left[ E^{\mu} D^\nu - \frac{1}{2} \partial_\mu E^\nu \right]. \]

which is that given by Minkowski for the static field (cf. (3.2)). This, of course, does not prove the validity of the Minkowski force, \( f = \nabla \cdot \sigma \), as its existence, at least in the static case considered here, depends upon the assumption of the body torques. Nonetheless, according to Brevik [18], this supposition is quite natural.

Yet another distinction between the two tensors may be drawn. Thus, if the balance equation is exact in terms of the rotational angle (instead of velocity), then it can be satisfied only by the Minkowski tensor. However, when the quantity, which vanishes after integration, has been added to it, the equality is fulfilled for the Abraham tensor as well. This means that the Abraham tensor already includes the effect of torques, while, in the Minkowski case it is represented by the asymmetry of the tensor. This is, essentially, one of the main conclusions of Brevik’s [18] paper.

Let us now examine the behaviour of the other proposed tensor forms in the time-independent field. Comparison of the Einstein-Laub tensor (3.10) with the first of Groot-Suttorp expression (3.17) shows that, in this (static) limit, both tensors are equal. Hence, it is quite sufficient to consider only one proposal, say, the former one. Thus, the calculation of the force density from the usual Maxwell equations then leads to the expression for the so-called Kelvin force density
\[ f = \nabla \cdot E + (P \cdot V) E. \]

This differs from the corresponding Abraham (3.47) and Minkowski (3.49) results, but is equal to that predicted by Livens [70].

It is necessary, therefore, first to ascertain which expression is correct. A relatively simple possibility for doing this differentiation occurs in the case of a non-polar liquid dielectric. In that case, it is necessary to include the effect of electrostriction [34]. Then, on applying the above variational procedure, we are led to the Helmholtz force
\[ f = \nabla \cdot E + \frac{1}{8\pi} \varepsilon \varepsilon_0 \frac{d}{d\omega} \left( \varepsilon_0 \varepsilon_\omega \right), \]
where the last term refers to the dependence of permittivity on the mass density \( \varepsilon_\omega \). Using the Clausius-Mossotti relation, \( (\varepsilon - 1)/(\varepsilon + 2) = \text{const} \varepsilon_\omega \), we then calculate the excess pressure inside a dielectric as
\[ \Delta P = \frac{1}{8\pi} (\varepsilon - 1)(\varepsilon + 2) E^2. \]

On the other hand, the excess pressure connected with (3.51) is
\[ \Delta P = \frac{1}{8\pi} (\varepsilon - 1) E^2. \]

The experiment to test the above mentioned expression for the excess pressure in the presence of the field has, in fact, been performed by Haki and Higman [30], and will be discussed in the last section of this chapter. It will be sufficient to say here, that it has firmly established the validity of (3.53).

In view of that last finding, it can be asserted, that the Einstein-Laub (and also de Groot-Suttorp) tensors yield results which are (at least for static fields) incompatible with experiment.

Consider now a plane wave travelling in an isotropic and homogeneous medium and follow the arguments of Landau and Lifshitz [8] about the construction of the energy-momentum tensor from the macroscopic considerations. Those authors assumed that the total energy density is the sum of the electrostatic and magnetostatic energy densities, and that the same also applies to the stress tensor. If the frequency of the incident wave is much lower than the characteristic frequency of electronic vibrations, then, for not too strong fields, the relations between the field intensities and corresponding inductions remain linear. Hence, the above described construction of the energy-momentum tensor is valid.

It is found from experiments that the electromagnetic wave travelling through an insulator does not produce, in the first approximation, any heat, as it undergoes an elastic scattering. This, therefore, means that the time component of the four-force vanishes. Consequently, the continuity equation is
\[ \nabla \cdot V = 0 \]
and, from the expression for the energy density, \( \varepsilon = (E \cdot D + H \cdot B)/8\pi \), one obtains the energy flux as
\[ S = \varepsilon (E \cdot D + H \cdot B)/4\pi. \]

Extending the validity of the Planck law of inertia, \( \varepsilon = c^2 \eta \), the expression for the momentum density becomes
\[ \rho = (E \cdot D + H \cdot B)/c. \]

Hence, the energy density, energy flux, and momentum density constitute the Abraham tensor, as seen from (3.6a).

Further, if the mechanical part of the energy-momentum tensor is \( \eta \), then we can write
\[ \eta_{\mu\nu} = f_{\mu\nu}. \]
At last in principle, it is mathematically to equate the Abraham and Minkowski results. For instance, the Abraham force is the real force which may be of no use, and of no matter present in a volume. Reaction to this force results in a mechanical stress. On the other hand, the meaning of the Minkowski force is rather unclear.

Another paper dealing with the radiation forces in dielectric media, written by Gordon [12], demonstrates that the true momentum density of the electromagnetic field in a non-dispersive medium is, indeed given by the Abraham expression. Gordon adopts a microscopic model and begins with a simple, weakly polarizable medium, such as a gas of heavy atoms. He, further, assumes that the corresponding permittivity only slightly exceeds unity and, therefore, the electromagnetic properties of the medium are small described by its polarizability $\alpha$. Now, in the rarefied gas, the ponderomotive force is simply the Lorentz force, which can be written as [12]

$$ f = \mathbf{\alpha} \left( \mathbf{E} \times \mathbf{H} \right). $$

(3.56)

For a plane wave in a medium, the first term vanishes and, thus, the force density due to $N$ atoms is $f = i f$. The time integration of this force gives then the mechanical momentum travelling with the pulse as

$$ g_{\text{mech}} = \frac{n - 1}{4\pi c} \left( \mathbf{E} \times \mathbf{H} \right). $$

(3.57)

From the laws of conservation of energy and momentum, Gordon then obtains

$$ n g = \mathbf{E} + g_{\text{mech}}, $$

thus interpreting $g = \mathbf{E} \times \mathbf{H}/4\pi c$ as the electromagnetic momentum density. Hence, the possible separation of the total momentum into field and matter parts leads, in this model, to a complete solution. Although he proceeds with generalization of the above expressions to include fluids and solids, it does not appear that his model will lend itself to an easy generalization, as it basically applies only to media with low polarizations.

A different resolution of the present dilemma is advocated by Ginzburg [19]. Although he considers that no objection can be made against the Abraham tensor, which is more fundamental, and correct in the general case, he, nevertheless, attempts to explain why the quantum mechanical analysis of the problem leads to the accepted conservation laws. Thus, first of all, he derives the laws, using the Hamiltonian method, and proceeds then to compare them with the results of microscopic electrodynamics. From the point of view of the latter, the conservation laws

$$ \dot{E} S_{\text{el}} = f^t; \quad f^t = f^i + f^s; \quad f^s = \frac{i}{c} \mathbf{E}, $$

(3.58)

and

$$ \dot{E} S_{\text{mag}} = f^i; \quad f^i = \frac{i}{c} \mathbf{E}, $$

(3.59)

where $f^s$ is the Lorentz force density, are identical [19]. They differ only by the separation of the four-force in two different ways. Mathematically speaking, this conclusion is correct. However, from the empirical point of view, this resolution is not satisfactory, as the Abraham force is, in principle, measurable, and Ginzburg is aware of this but, nevertheless, prefers, when convenient, to use the Minkowski tensor. In other words, he emphasizes that, when proper account is taken of the background medium, both tensors will, in most situations, yield the same result.
where $E$ refers to the incident wave. It is to be noted that, as this force is due to dipole-dipole interactions, it is found to be proportional to $(n - 1)^2$. On the contrary, both the Abraham and the Minkowski expressions give only the first order dependence.

The derivation [72] of the surface force (3.64) does not take into account the existence of lateral forces, due to the transverse gradient of the external field and which push the medium into the beam. As a consequence, there are two different momentum fluxes present: that of the momentum of the electromagnetic wave, propagating with the group velocity, and that of the momentum of the pressure wave, travelling with the speed of sound. Above authors [72] explicitly consider the former case only.

In the calculations of the radiation pressure on matter, it is generally assumed that the properties of the latter do not change during the interaction. Naturally, this is not the case in reality. Thus, Askar'yan [73] analyses a medium with a variable polarizability $\alpha$. Such a variation is, in general, associated with changes in the parameters (e.g., shape, volume) characterizing the medium. The resulting change in the dipole moment may greatly affect the field energy and, hence, the corresponding radiation momentum. It is then found that the force connected with the change in polarizability is proportional to $\partial \alpha / \partial \omega$. Consequently, depending on the sign of $\partial \alpha / \partial \omega$, the deformation forces may either enhance or diminish effects due to the ordinary radiation pressure.

A purely macroscopic approach is employed by Robinson [77]. Although he realizes the complexity of problem at hand, he claims that the solution to it already exists and that it has been discussed by Penfield and Haus (cf. [77]). This is done by extending the principle of virtual work, used originally by Helmholtz. Robinson, for reasons stated in his article, describes that approach in a "digested" form. The macroscopic properties of matter, in his view, are described by dielectric permittivity and magnetic permeability. He emphasizes, however, the importance of the electro- and magneto-strictive terms, arguing that the effect of the electronic structure of matter on the electromagnetic phenomena cannot be completely described by the constitutive relations.

Robinson arrives at an identity when he replaces $g$ and $j$, in $E \cdot g + j \times B$, by $V \cdot D$ and $V \times H - \partial D / \partial t$, respectively. In regions where polarization $P$ and magnetization $M$ vector densities vanish, the obtained equation is then interpreted as a momentum balance equation.

Simultaneous considerations were already discussed in the present work. Robinson generalizes that approach and, although not arriving at the complete solution, confirms that the part $\mathbf{E} \times \mathbf{H}$ of the total momentum of a field in an isotropic medium does not refer to moving matter.

Consider a nearly monochromatic pulse impinging on a dielectric slab characterized by constants $\varepsilon$ and $\mu$. It is assumed that the wave energy is conserved, i.e., the kinetic energy associated with the transferred momentum is much smaller than the rest energy of the medium. Wave incident from vacuum is partially reflected and partially transmitted at the boundary. There is also another term in the momentum balance equation, viz., one describing the momentum imparted to the surface. By writing down the momentum conservation law at incidence, it is found [77] that the sum of all surface momenta gives the total momentum transferred to the slab. It is, in fact, equal to the difference between the initial and the final momenta of the waves, i.e., those just before an entry and just after an exit.

Assuming that the total momentum of a wave pulse in a medium is $\hbar k / c$, and that the transferred momentum is $\hbar \varepsilon k / c$, where the coefficients $\varepsilon$ and $\mu$ are related to $\varepsilon$ and $\mu$, it is found that [77], in view of the center of mass theorem, $\varepsilon = \mu = 1$. Thus, there remains a part $(\varepsilon - \mu) \hbar k / c = \hbar \varepsilon / c$ of the total propagating momentum which is not associated with matter and may, therefore, be regarded as corresponding to the field. However, the expression for $\varepsilon$ is not known and this makes it impossible to attach a definite value to the surface momentum. If it is assumed that this value vanishes at normal incidence, then $\varepsilon = \mu = 1/2(\varepsilon + \mu)$ and the total momentum travelling through a medium is

$$g = \frac{h\mu}{2\mu} (\varepsilon + \mu).$$

In his earlier treatment [73], Peierls considered normal incidence and only those media in which the difference between $\mu$ and unity could be neglected, to that the refractive index was given by $n = \sqrt{\mu}$. Consequently, for media where $n - 1$ is small, the results (3.62) of Peierls and (3.65) of Robinson are identical.

As already indicated, the Helmholtz method of virtual work was generalized by Penfield and Haus (cf. [77]). They included moving, dispersive media, by treating the mechanical aspects of the system from a relativistic standpoint. They claimed their so obtained solution to be complete. The essence of their argument is as follows.

The radiation field together with the surrounding medium forms a system which may be divided into two parts, according to the way in which the energy and momentum are stored. In one part, this is related to the motion of the medium, while in the other, it is connected to mechanical or electromagnetic stresses. Clearly, these subsystems are not closed and, hence, there must be some mutual interaction. The conservation laws have, therefore, the forms

$$\dot{e}_i T^{ij} + \dot{\varepsilon} g^{ij} = \rho, \quad \dot{e}_i S^{ij} + \dot{\varepsilon} \mu = 0,$$

where $T^{ij}$ is the momentum flux tensor, $g^{ij}$ the momentum density, and $\rho$ refers to the power transfer density. The tensor $T^{ij}$ includes, however, not only the electromagnetic field energy (e.g., electro- and magneto-strictive variables). This, in turn, should be understood as pointing to the feasibility of splitting up the total momentum into its field and material contributions. This follows from the fact that the rate of change of the mechanical momentum density must now include not only the spatial derivatives of $\hbar{T}^{ij}$ but also $\hbar \varepsilon / c$. The most natural interpretation for the latter term is, then, to identify it with the rate of change of the field momentum density.
A variational approach using the Hamiltonian principle was recently presented by Ikura [20]. Although it is well known that the Lagrangian density cannot be defined uniquely, its form is suggested intuitively and, then, justified a posteriori. Robinson agrees that, in some case, the Minkowski momentum density may enable one to find the total momentum transferred to a medium, however, he advocates [17] the Abraham formula as the rigorously correct one.

An explicit calculation leads to the following form for the material part of the energy-momentum conservation law

\[
\partial_t S^m_{\mu\nu} = F^m_{\mu\nu} q_{\mu} - \frac{1}{2} \partial_\mu F^{\nu\rho} \tau_{\rho} - \frac{1}{2} \partial_\nu F^{\rho\mu} \tau_{\rho},
\]

(3.70)

where \( S^m_{\mu\nu} \) is interpreted as the field energy-momentum tensor.

To find the conservation of the total energy and momentum, Ikura [20] defines the tensor for the total system, although not uniquely, as the sum of the tensors corresponding to the subsystems and, then, combining (3.70) and (3.71), obtains \( \partial_\mu S_{\mu\nu} = 0 \).

It is to be noted that the definitions of \( S_{\mu\nu} \) and \( S^m_{\mu\nu} \) are not properly justified on the physical basis. As already indicated, the total tensor may be split into two parts in several different ways, if the formal requirements are met in the conservation laws for the subsystems. Ikura, then, attempts to find a physical criterion which would make this tensor splitting into material and field parts unambiguous. He postulates, therefore, that: 1. the part of the tensor which has a form \( \eta \) of pure material tensor represents the material component is the total tensor; 2. the part identical with the field tensor in vacuo is associated with the field contribution to \( S_{\mu\nu} \); 3. tensors with nonvanishing space-space components, which are not related to the field tensor. He then finds that both the Abraham and Minkowski tensors satisfy these requirements. Thus, the values of the electromagnetic momentum flux density are the same in both cases (if the medium is stationary), although the momentum densities are different in the two cases. Because the radiation pressure is determined by the flux density, rather than by the momentum density, the result is the same, regardless of the choice of the (Abraham or Minkowski) tensor.

Mikura, nevertheless, favours the Minkowski expression which, in view of results of experiments [35, 37] should represent the true radiation momentum density. The radiation pressure is then given by the product of this momentum density and the velocity of light in the medium, i.e., \( \eta X H/4\pi \), in disagreement with the generally accepted result \( (E \times H)/4\pi \). Moreover, Mikura claims that there is no need for the mechanical pressure, \( D \times B = F \times \eta A \), as, in the steady state experiments, the time average effect of the electro- and magneto-strictive forces is balanced by the elastic hydrostatic force. That pressure could be, however, important in transient phenomena.

Mikura's discussion of the acoustic radiation pressure, caused by the density waves, shows it to be smaller than the electromagnetic radiation pressure by a factor \( \eta X H/4\pi \) (\( c_s \) — speed of sound in a medium). As this factor is rather small, the mechanical pressure arising from the density waves is negligible in comparison with the true electromagnetic pressure.

Thus, according to Mikura [20], although there is no unique way for splitting the total tensor \( S_{\mu\nu} \), when the momentum flux density, instead of the momentum density, is taken into account in calculations, the radiation pressure in matter can be uniquely determined.

Noting the fact that the group speed of light waves is reduced in matter and asserting the equivalence of the mass and energy, together with the conservation of the total energy and momentum, Thorner [74] was led to postulate the existence of transverse forces on matter. Similarly to many workers before him, he was supporting the Abraham hypothesis.

While analyzing the radiation pressure on a multilayer medium, Anand [75] generalized results obtained previously by Haus [16].

Arguments based on the adiabatic invariance and the energy quantization are used by Johnson [76] in the analysis of the radiation force. From the operational standpoint, he distinguishes among the impulsive-, the system-, and the energy-carrying momentum. Further, he differentiates between bare and clothed particles and shows that the theory applies not only to photons but also to phonons [77]. In fact, the theory encompassed

\[
S^m_{\mu\nu} = \frac{1}{4\pi} \left( F_{\mu\nu} F_{\rho\sigma} - \frac{1}{4} F_{\rho\sigma} F^{\rho\sigma} \right) + F^m_{\mu\nu} P^m_{\rho\sigma} + F^m_{\rho\sigma} M^m_{\mu\nu},
\]

(3.72)
both zero and non-zero rest mass particles. Thus, he argues that the Abraham momentum refers to a bare particle, whereas that of Minkowski describes the clothed particle. As the essential point of clothing and localization is often confused when interpreting observations, it is, clearly, necessary to distinguish carefully between these concepts. It would appear [76, 77] that, as clothed particles are the only ones observed, the corresponding theory must lead to the Minkowski result.

The classical relativistic field theory served Dewar [78] as a vehicle to analyze the splitting of the total energy-momentum tensor into its components and also to find the force density acting on a medium in any state. In analogy with the canonical and physical momentum for a particle, Dewar distinguishes between the canonical and physical energy-momentum tensors.

He, first of all, obtains the conservation equations for the whole system using the Noether theorem. This, as is known [79], relates the invariance of equations of motion to the symmetry transformations on the system. It employs two symmetry postulates. The first of these requires the Lagrangian density $\mathcal{L}$ to be invariant under space-time translations. The second postulate requires $\mathcal{L}$ to be invariant under rotation. It is then shown that the angular momentum is conserved and the energy-momentum tensor is symmetrized by the method due to Belinfante [47].

The thus obtained symmetrical energy-momentum tensor $\Theta^\mu\nu$ is then split up in two alternative ways. The first of these, Dewar [78] calls the physical split-up. It has the advantage that the background energy-momentum tensor is not modified by the presence of fields, as all interactions are embedded in the term describing the force density on the medium. Hence, the following equations apply to the subsystems

\[ \varepsilon_\mu \Theta^\mu_{\nu} = f_1; \quad \varepsilon_\mu \Theta^\mu_{\nu} = f_2. \]  

(3.73)

Here, $f_1$ is the force density acting on the background medium, also known as the ponderomotive force density, while $f_2$ is the physical force density acting on the $k$-th subsystem.

The second way of splitting up the tensor $\Theta^\mu\nu$ involves identifying the canonical energy-momentum tensor with the background medium. Dewar [78] calls this method the modified canonical split-up. Although the tensors corresponding to the constituents of the system are unsymmetrical, the total tensor is symmetric.

Finally, after a long and elaborate investigation, Dewar [78] reached the conclusion that it is futile to talk about the pure electromagnetic momentum, when taking into account the influence of the background medium, as those subsystems are coupled. Consequently, if the canonical energy-momentum tensor is used to describe the background medium, then the Minkowski proposal is correct. On the other hand, when the symmetrical tensor is associated with the medium, then the electro- and magneto-strictive terms must be added to the Abraham tensor, in order to yield correct results.

Not only the electromagnetic momentum in a medium but also the closely related acoustic momentum are rather controversial subjects [76, 77]. Thus, the meaning of the acoustical momentum was discussed by Post [80] utilizing results of Eckart [81]. The question of the energy-momentum tensor was also investigated by Jones [66] who extended the problem to the electromagnetic waves and considered the important concept of the wave spin.

Using a method applicable to the plasma state [82], Klima and Petriška [83] proceed to explain the Gordon [12] results. However, while investigating the pressure of the radiation field on a fluid, they do not assume the validity of either the Abraham or the Minkowski expressions, yet arrive at the results consistent with Abraham. Their calculations are, unfortunately, not general, applying only to very dilute gases in which the effects of the order are vanishingly small.

It may be possible, in order to throw some light on the energy-momentum tensor controversy, to regard the asymmetric canonical tensor and the Minkowski tensor as describing the same quantity. Indeed,Novoraztvsky [cf. (23)] has shown that, in the free field, both tensors differ only by the divergence term, so that we may write

\[ S_\mu^{\nu} = T^\mu_{\nu} + \frac{1}{4\pi} \varepsilon_\mu (\mathcal{A}^\nu)^\mu. \]  

(3.74)

However, this does not hold in the case of an open system. Therefore, it is not possible [23], in general, to identify the canonical energy-momentum tensor with a physical tensor supposedly symmetrical. This proposition hinges upon the fact that the equation of conservation of angular momentum applied to a spinless (classical) system leads to the requirement that the energy-momentum tensor be symmetric [84]. An interesting way to approach the task of finding the energy-momentum tensor was suggested by Boll [49]. His investigation is based on the following principles:

1. the field energy flux is proportional to the corresponding momentum; 2. the spin of the radiation field contributes to the total momentum, but not to the energy flux. These requirements were questioned by Heil [53] as they, apparently, did not obey the relativistic invariance. Nevertheless, Boll [50] successfully defended his original arguments. Thus, the total energy-momentum tensor consists of the linear and the spin parts. The latter then corresponds, in the macroscopic physics, to the phenomenon of magnetization. In other words, the spin-angular momentum, as well as the magnetizing current in macrophysics, are not linked with the energy transport.

### 3. Experimental results

In the preceding pages the arguments put forward by various workers were discussed purely from the theoretical point of view. It appears that the present situation is unclear, although more workers tend to favour the Abraham tensor. However, the final word in the controversy must be that of the experimentalist. At present, empirical data are rather scarce. Nevertheless, even these do not provide an unambiguous answer to the question of what is the radiation pressure in a medium. Let us now review these experiments.

The most frequently referred to results, relevant to the problem of the radiation momentum in a medium, were obtained by Jones and Richards [25]. The experiment consisted, essentially, of the following set up: a mirror, submerged in a refractive liquid, was attached to a torsion balance. Thus, when it was illuminated, incident radiation produced a pressure which, in turn, resulted in a torque on the mirror suspension. Consequently, this torque was measured. Six liquids with different refractive indices were used to investigate the pressure variation as a function of this index. A tungsten lamp served as a radiation source. In order to eliminate the convection currents and the resulting forces (which occur in an unheated liquid), Jones and Richards resorted to the use of the radiation chopper. After a careful analysis, these workers claimed that, to within the root mean square error of 1.2%, the radiation pressure is proportional to the refractive index of the liquid. Their conclusion was, therefore, in accord with the Minkowski prediction.

Quite recently, that experiment has been repeated, with an improved accuracy, by Jones and Leslie [26]. In the earlier experiment, due to only about 80% reflectivity of the mirror, the absorbed light was transformed into heat, thus generating convection currents. The differentiation between the forces corresponding to this effect and the
radiation forces was, however, possibly due to the unequal time of response in both cases. Now, in order to improve the accuracy, a highly reflecting mirror was used. As a consequence, the mirror heating was reduced by a factor of 400. Furthermore, the tungsten lamp of the Jones-Leslie experiment was replaced, as the source of illumination, by a He-Ne laser. This served to minimize the uncertainty due to the width of the spectral distribution of the source. The data thus obtained supported the earlier conclusion (as to the validity of the Minkowski expression) with an accuracy of approximately 0.05%.

In addition, it was found that the conventional, i.e., the phase- rather than the group-refractive index. Moreover, the Jones-Leslie experiment showed no dependence of the radiation pressure on the plane of polarization for angles of incidence up to 20°. This was contrary to the Peierls prediction (3.63).

At the beginning of this century Poynting [67] suggested that the reflection always produces a normal pressure on a medium, whereas the refraction causes a normal (outward) pull. This was verified experimentally by Barlow [58] when investigating the torque produced by an obliquely incident light beam. The variation of the light torque with the angle between the plane of polarization and plane of incidence, as well as the torque dependence of the wave length were, amongst other things, determined by Bethe [68]. According to the theory, the torque density \( D \times E \times \Delta \) causes, owing to the high frequency used, a great difficulty in observation. As the wavelength increases, so does the torque. Thus, Carver [69] used the centimetre electromagnetic waves and obtained a much stronger effect, in accordance with Bethe [68] findings.

An experiment aimed at determining the transfer of momentum from the radiation field to material particles was performed by Rawson and May [91], with additional data collected by May, Rawson, and Hara [92]. It was found that the propulsion of dust particles situated in the He-Ne laser beam is due to radiometric forces, due to temperature gradients. Such forces are, usually, orders of magnitude larger than those due to the radiation pressure. It was concluded that, at beam fluxes of 150 W/cm, photophoresis can explain the driving forces and stabilizing torques acting on dust particles.

A similar experiment was performed by Ashkin [52], who investigated the behaviour of micron-size latex spheres in a continuous He-Ne laser beam. In the plane of polarization, the temperature gradients caused by light, particles were suspended in a transparent liquid. Furthermore, the power densities were 10^9 times higher than those reported by Rawson and May [91] and May et al. [92]. It was found, when the spheres were submerged in a liquid with refractive index smaller than that of the spheres, that, in addition to being accelerated along the direction of the beam, they were also drawn toward the center of the laser beam.

Whenever the index of refraction of the spheres was lower than that of surrounding medium (such as the case of air bubbles in the mixture of glycerol and water), the spheres were again accelerated along the direction of the beam but were forced away from the beam axis. A similar behaviour was postulated to apply also to atoms and molecules in a laser field.

In another experiment by Ashkin and Dziedzic [27], the effect of a 4 kW pulsed laser beam on a free surface of a medium was studied. It was found that, after about 300 ns, a positive surface lens developed, thus indicating an outward force around the area of the beam entry into a medium. This also confirmed that, during that time interval, the force acting on a surface must have exceeded the surface tension. Following an additional 700 ns, the lens disappeared, presumably under the delayed influence of surface tension.

Further, when the surface was irradiated from within the liquid, it experienced outward forces. It was concluded, therefore, that both the entrance and exit surfaces experienced forces in the direction away from the medium, thus supporting the Minkowski prediction. The theoretical explanation of this, as well as of the Jones-Richards experiment, was given by Jones and Young [94] and later, including the hydrodynamic time response of the liquid surface, by Lai and Young [94]. Also, Kats and Kontorovich [56] calculated the influence of laser beam on a liquid surface and, whilst including the non-linear effects, obtained results in agreement with the experiment.

All the above-mentioned experiments, measuring the radiation force, employed high (optical) frequencies. They led to the conclusion [25, 26, 27] that correct values were obtained when accepting the Minkowski tensor. Let us now turn our attention toward the other extreme — the region of very low frequencies and, in the limit, that of a static field.

The electrostatic field interacts with a liquid dielectric and, in turn, causes a compression of the latter. Consequently, the refractive index of the medium increases and can be measured by Schlieren photography. By studying the images thus obtained, the effect of the experiment on a phenomenon, this was, essentially, the method used by Hakin and Higham [90] in their investigation of the force acting on a volume element inside a dielectric. To within an accuracy of 5%, their findings agreed with (9.32). In other words, it was the Abraham tensor that correlated well with the experimental results in the static limit.

This conclusion was also supported by results of James [55]. Using a frequency of about 3 kHz interacting with a ferromagnetic medium, James confirmed that the Minkowski tensor was not applicable at low frequencies.

Despite another relevant experiment must be mentioned here. It has not been (to my knowledge) referred to in the literature. That experiment was performed by Walker et al. [28], using the time varying electric field and a magnetic field. The sample of BiTho, a non-magnetic ceramic with a dielectric constant \( \varepsilon = 3320 \pm 15 \), was submitted by Bethe of the poles of a magnet as a torsional pendulum. The varying electric field of frequency of about 0.3 Hz was then applied to the sample. Whenever that field was exactly in phase with the natural oscillation of the pendulum, the mechanical force arising from the interaction of the polarization current with the magnetic field would be expected to enhance the oscillations. In the case of phases being opposite, oscillations were damped. If the polarization current was in phase with the varying electric field, the conduction current force cancelled itself over the complete cycle.

The force density predicted from the Abraham tensor is \( \eta \frac{1}{2} E \times H \times J \), which is extremely small, except at low frequencies. In the case of a constant magnetic field and a non-magnetic sample, it can be expressed as \( E \times H \times J \). This force, which does not arise in the Minkowski theory, was, in fact, confirmed by experiment [28]. Also, when this experiment was modified [92] so as to include the case of time varying magnetic field, the results obtained were again in agreement with those of the Abraham theory.

In this chapter we have reviewed the main arguments leading to different electromagnetic energy-momentum tensors. Although several distinct forms were proposed, the main discussion centered around the Abraham and Minkowski tensors. Theoretical arguments, as well as the experimental data, were examined and it was found that they did not always correlate. Most authors, nowadays, tend so favour the Abraham tensor, as it is considered to be the more fundamental of the two. Furthermore, in view of the suggestion by Bethe [56] and realizing that the magnetization (or spin) effects are, essentially, low frequency phenomena, one can say that experiments are, in general, in agreement with the Abraham theory.

In most of the following, the energy-momentum tensor in a plasma medium will be...
IV. Electromagnetic Waves in a Plasma

The discussion in the preceding chapter covered the gaseous, liquid, and solid states of matter, for which the value of the refractive index, at relevant frequencies, was larger varied. Therefore, in an attempt to shed some light on the correct solution, our arguments will revolve around the plasma state. As the dynamics of a real plasma is not present investigation, our main efforts will be concerned with an approximation, namely, to set out the general properties of the fourth state of matter (in that approximation).

The concept of a plasma refers to a conglomeration of charged and neutral particles, there being approximately equal number densities of positive and negative charges. Often, at least one of these charges may be regarded as highly mobile. However, in order to reproduce typical plasma characteristics, it is necessary for the average distances between the charged particles to be very much smaller than the actual physical charges fixed and the concentration of the conduction electrons reading values of the order of $10^9$ cm$^{-3}$, can often display plasma characteristics. Similarly, one can conceive mainly towards tenuous, gaseous plasmas.

From the macroscopic point of view, that state is characterized by several concepts.

Thus, if subscripts $e$ and $i$ refer to electrons and ions, respectively, and $n_e$, $n_i$, in turn, correspond to the average number density, charge, and mass of these charge carriers, then it follows that

$$\bar{n}_e + \bar{n}_i = 0,$$

provided that the plasma is taken to be electrically neutral. Moreover, the average thermal speeds of the plasma constituents are zero, i.e., $v_e = v_i = 0$. Now, in analogy to the kinetic pressure can be expressed by

$$p = \bar{n}_e T, \quad p_i = \bar{n}_i T.$$

(4.1)

Here, $\alpha$ is the Boltzmann constant and $T$ denotes the temperature. In the case of thermodynamic equilibrium, there applies the equality $T_e = T_i$. However, the temperatures $T_e$ and $T_i$ become distinct when external fields are applied to the plasma. If the external magnetic field $B_0$ is transverse to the particle motion, then the charged particles execute a circular motion. The corresponding (so-called cyclotron) angular frequencies for electrons and ions are then, respectively,

$$\omega_e = q_e B_0/m_e, \quad \omega_i = q_i B_0/m_i.$$

(4.2)

If, on the other hand, the velocity of the particle is parallel to the field, it cannot be affected by that field. In general, therefore, the particles travel along a helical trajectory.

Another characteristic of plasma state is its tendency to preserve charge neutrality. Hence, microscopic charge imbalances, due to thermal motion, are opposed by the induced electric field. Thus, electrons oscillate in the direction of the electric field vector, which, in turn, is parallel to the direction of propagation. This kind of plasma electron oscillations is known as the longitudinal oscillation. Parameters associated with this behaviour are called the electron ($\omega_e$) and the ion ($\omega_i$) plasma frequencies and these are defined by

$$\omega_e^2 = 4\pi n_e q_e^2/m_e, \quad \omega_i^2 = 4\pi n_i q_i^2/m_i.$$  

(4.3)

The plasma state is capable of supporting a number of thermally and externally induced oscillations. For instance, low frequency electromagnetic waves in the presence of an external magnetic field, also known as the hydromagnetic waves, can propagate in a plasma with speeds often many times smaller than that of light in vacuo. Further, not only the transverse but also longitudinal waves occur in plasmas, as will be seen below.

In most cases, the wavelength of the electromagnetic radiation is much greater than the interparticle distance in plasma. Therefore, it is possible to analyze the wave propagation in a plasma from the point of view of the phenomenological electrodynamics. In so far as the concepts of charge density $J$ and current density $j$ are involved, the plasma is considered as an equivalent to a continuum. In other words, the discreteness of the plasma constituents is ignored and, hence, the Maxwell equations are used to describe smeared-out, average fields and not the microscopic, interparticle fields. If these macroscopic, electric and magnetic field vectors are $E$ and $H$, with $D$ and $B$ denoting the corresponding inductions, then the Maxwell field equations yield

$$V \times E = -\frac{1}{c} \varepsilon \dot{H}, \quad V \times H = \frac{4\pi}{c} j + \frac{1}{c} \varepsilon \dot{D},$$

$$V \cdot H = 0, \quad V \cdot D = 4\pi n.$$

(4.4)

Note that, as the magnetic permeability of a tenuous plasma is practically equal to unity, the magnetic induction $B$ was replaced, in (4.4), by the magnetic field $H$. Now, to make use of (4.4), we need the actual $E$-dependence of the electric induction $D$ and current density $j$. In absence of any external magnetic field and for an isotropic plasma, we have

$$D = \varepsilon E, \quad j = \sigma E,$$

(4.5)

where $\varepsilon$ and $\sigma$ are, respectively, the dielectric permittivity and the conductivity of the plasma. The presence of an external magnetic field disturbs the plasma isotropy, so that (4.5) is modified to

$$D_{kl} = \varepsilon_{kl} E_l, \quad j_{kl} = \sigma_{kl} E_l,$$

(4.5a)

where $\varepsilon_{kl}$ and $\sigma_{kl}$ are now tensor quantities corresponding to $\varepsilon$ and $\sigma$. If the magnetic field is assumed to be time independent, relation (4.5a) can be regarded as linear. Fortunately, this approximation is reasonably valid in a majority of cases. However, in very strong fields, (4.5a) becomes non-linear, so that, as a result, interactions appear among the various kinds of waves.

It is convenient to combine the $\varepsilon_{kl}$ and $\sigma_{kl}$ tensors into a single, complex permittivity tensor

$$\varepsilon_{kl} = \varepsilon_{kl} - 4\pi n_0 \sigma_{kl},$$

(4.6)

which, in general, depends on all coordinates.

For the sake of simplicity, consider now a homogeneous plasma, i.e., a plasma in a region where $\varepsilon_{kl}$ is constant in space. Further, assume that a plane monochromatic wave propagates in the positive $z$-direction and at normal incidence to the plasma boundary. While deriving the explicit form of the dispersion relation, it will be seen that, in the present
Let us first eliminate the magnetic field strength from (4.4). Assuming that all variables are simple harmonic functions of time (i.e., proportional to $e^{i \omega t}$), we get

$$ V \times V \times E = \frac{4 \pi i}{c^2} \omega j + \frac{\omega}{c^2} D. \tag{4.7} $$

But, the current density $j$ can be written as

$$ j = \vec{n} \vec{d}_f v^e + \vec{n} \vec{d}_f \vec{v}^i. \tag{4.8} $$

Hence, on substituting the plane monochromatic wave in (4.7) and utilizing (4.5) and (4.8), we obtain

$$ (k^2 \omega^2/\omega^2 - 1) E_x = -4 \pi i j_x / \omega \tag{4.9} $$

$$ (k^2 \omega^2/\omega^2 - 1) E_y = -4 \pi i j_y / \omega \tag{4.9} $$

$$ E_z = -\alpha \pi j_z / \omega. \tag{4.9} $$

where $\alpha$ is $Y - 1$.

Thus, it is clear from (4.9) that $E_x$, is always lagging with respect to $v_x$, whilst $E_y$ and $E_z$ can lag or advance relative to the respective current components, depending on whether $(k^2 \omega^2/\omega^2 - 1)$ is negative or positive.

Next, the conservation equations are determined. It will be assumed that the perturbation of the plasma variables caused by the passage of the wave, is small and can be neglected.

If the symbols $n_e$ and $n_i$ denote the fluctuations of the number density of electrons and ions, respectively, then the corresponding conservation equations are written as

$$ \dot{\vec{n}}_e + V \cdot \vec{n}_e = 0; \quad \dot{\vec{n}}_i + V \cdot \vec{n}_i = 0. \tag{4.10} $$

The momentum conservation equations are then, on neglecting the effect of the magnetic field, for the wave on particles,

$$ \dot{n}_e \vec{v}_e = \vec{n} \vec{d}_f (E + \vec{r}_x \times \vec{B}_x) - \vec{V}_p e + \vec{P}_e \tag{4.11} $$

$$ \dot{n}_i \vec{v}_i = \vec{n} \vec{d}_f (E + \vec{r}_i \times \vec{B}_i) - \vec{V}_p i + \vec{P}_i. \tag{4.11} $$

We assumed here that the phase speed of the wave is much greater than the thermal speed; for otherwise, these momentum equations would be incomplete. The terms $\vec{V}_p$, with subscripts $e$, $n$, $i$, and their permutations, refer to momentum transfers in collisions involving electrons, neutral particles, and ions. It is assumed [96] that they can be expressed as

$$ \vec{V}_p e = -\vec{n}_e \vec{m}_e \vec{v}_e; \quad \vec{V}_p i = -\vec{n}_i \vec{m}_i \vec{v}_i; \quad \vec{V}_p i = -\vec{n}_i \vec{m}_i \vec{v}_i - \vec{V}_p e \tag{4.12} $$

The symbol $\vec{r}$, with appropriate subscripts, refers to the collision frequency between the corresponding plasma constituents. Although this frequency is assumed here to be a constant, in general it also has a tensor character. Further, following [96], the pressure gradient is written in terms of the average thermal speeds $V_{e}$ [97], where $V_e^2 = 3kT_e/m_e$, as

$$ \vec{V}_p e = n_e V_e \vec{V}_p e, $$

with subscript $\gamma$ referring to either electrons or ions. Asserting [96] that the particle conservation may be written as

$$ n_e = kT_e / m_e, $$

which, after substitution in the above, yields

$$ \vec{V}_p e = -i \vec{k} \vec{n}_e \vec{v}_e V_e \vec{v}_e / \omega. \tag{4.13} $$

With the assumption that the magnetic field can be split into two components, $\vec{B}_1$, normal and the other, $\vec{B}_2$, parallel to the direction of propagation, the momentum conservation equations can be written out more explicitly. Thus, the substitution of (4.13) and (4.12) into (4.11) leads to conservation equations in the component form

$$ \frac{\partial n_e}{\partial \tau} + \nabla \cdot (n_e \vec{v}_e) = \alpha \nabla \cdot (\vec{V}_p e) + \Omega_{e} \nabla \cdot (\vec{V}_p e) \tag{4.14} $$

$$ \frac{\partial n_i}{\partial \tau} + \nabla \cdot (n_i \vec{v}_i) = \alpha \nabla \cdot (\vec{V}_p i) + \Omega_{i} \nabla \cdot (\vec{V}_p i) \tag{4.14} $$

$$ \frac{\partial \vec{v}_e}{\partial \tau} + \nabla \cdot (\vec{v}_e \vec{V}_p e) = \frac{\vec{k} \vec{v}_e \vec{V}_p e}{\omega} \tag{4.14} $$

When the above equations describe electron (ion) momentum conservation, then the negative (positive) sign before the electron-ion collision term is used. In conjunction with (4.9), they form a set of nine homogeneous linear equations with nine unknowns. These unknowns are components of vectors $\vec{E}$, $\vec{v}_e$, and $\vec{v}_i$. Hence, in order that such a set has a solution, the determinant formed from the coefficients of the individual terms must be equal to zero. If this is the case, there results an equation of the fourth order in $k$.

It, therefore, means that the system has four solutions which describe four, generally different, modes of propagation: the ordinary, extraordinary, electron, and ion modes [96]. Moreover, there are two conjugate solutions for each of $k_2$, which means that the (identical) wave propagates in the opposite directions. When setting up this determinant, the collision frequency is, for simplicity, put equal to zero. Then, the first column refers to the variable $E_x$, second column refers to $v_{ex}$, third column refers to $v_{ix}$, and so on, for the other two components. In addition, the first equation of (4.9) and (4.14), referring to vectors and ions, occupy the first, second, and third rows, respectively. The $y$- and $z$-components follow in the same order — up to row 9. This determinant is written out in [96]. Using the rules applicable to determinants, a more manageable form describing the general dispersion relation is obtained

$$ 1 - \frac{k^2 v^2}{\omega^2} \frac{\Omega_p^2}{\omega^2} = 0. \tag{4.15} $$

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In the dispersion relation (4.15), symbols \( \omega_p^2(Q_2) \) and \( \omega_p^2(Q_2) \) refer to the electron (ion) cyclotron frequency resulting when the magnetic field is split into its transverse and longitudinal components. The relation (4.15), although appearing to be rather complicated, readily lends itself to simplification.

Consider first a plasma in absence of any external magnetic field, \( H_e = 0 \). In this case, both electron and ion cyclotron frequencies vanish. The dispersion relation then reduces to

\[
A_T^2 A_L = 0,
\]

where \( A_T, A_L \) are 3x3 subdeterminants of (4.15), i.e.,

\[
\begin{vmatrix}
A_T & 0 & 0 \\
0 & A_T & 0 \\
0 & 0 & A_L
\end{vmatrix} = 0.
\]

In view of its derivation, (4.16) expresses the relation between \( \omega \) and \( k \), i.e., the dispersion relation, in the absence of the external field. Thus, to find a non-trivial solution of (4.16), we have either \( A_T = 0 \) or \( A_L = 0 \). Consider the former case first. It consists of a product of two identical determinants of order three. Due to the construction of (4.16), the first determinant, \( A_T^{(3)} \), corresponds to the \( x \)-component, whilst the second one, \( A_T^{(2)} \), describes the \( y \)-component. To emphasize again, the direction of propagation is along the \( z \)-axis. If the first determinant is now set equal to zero, all components other than \( E_x, v_x, v_r \) vanish. This, therefore, means that the thus obtained result informs us about the \( x \)-component of the field and of the particle velocity. Put otherwise, if this determinant vanishes, we say that the wave is polarized in the \( x \)-direction.

Similarly, when the second determinant, \( A_T^{(2)} \), is set equal to zero, the obtained solution for \( E_y, v_y, v_r \) immediately implies that the wave is polarized in the \( y \)-direction.

It is clear, therefore, that the equation \( A_T^2 = 0 \) determines the transverse characteristics of the electromagnetic wave.

Concentrate now on the second equation implied in (4.16), i.e., \( A_L = 0 \). In that case, the only non-zero roots are \( E_y, v_y, v_r \). This then means that the wave is polarized in the direction of propagation and, hence, it is a longitudinal wave. Because the equation corresponding to the developed determinant is quadratic in \( k^2 \), there are two roots, thus implying two types of longitudinal wave.

The above conclusions will be mainly used in the following chapters of the present work. However, to help us form on overall picture of the wave propagation in plasma and to deepen our insight into the restrictive assumptions used, other types of waves are introduced in the next few paragraphs.

The presence of a magnetic field parallel to the direction of the wave propagation, does not affect the longitudinal waves, as a glance at (4.15) will reveal. Both the longitudinal electron and ion cyclotron frequencies have now non-zero values, ensuring a coupling between the above mentioned waves which then leads to two distinct modes of propagation. The plasma behaves like an anisotropic, doubly refracting medium and the two waves travel with different velocities. This lends itself to a simple physical interpretation: the plane electromagnetic wave can be considered to consist of two (right- and left-) circularly polarized waves. The properties of those separate waves are determined from their interactions with plasma particles. The electric field of an electromagnetic wave alters the orbiting trajectories of electrons and ions (induced by the static magnetic field) in a way dependent on the relation between the wave and the cyclotron frequencies.

Thus, the left circularly polarized wave rotates in the same direction as would the ions under the influence of the static external magnetic field. The other polarization agrees with the direction of motion of an electron in the magnetic field. Hence, the electromagnetic wave, upon entering the plasma, encounters different particle behaviour, thus leading to an analogy with a doubly refracting crystal.

To generalize this investigation still further, the external magnetic field will be next assumed to be in an arbitrary direction. The resulting dispersion relation becomes, accordingly, very involved. In this case, the transverse waves are coupled to the longitudinal ones, leading to a rather complicated behaviour [96]. This interaction, in turn, results in four distinct modes of propagation and one can no longer talk either about a pure transverse or a pure longitudinal wave. Thus, only mixed modes exist and their discussion is not immediately necessary for our present task.

It follows from the preceding discussion that, in the case where the transverse magnetic field is zero, there are two transverse and two longitudinal modes. Utilizing that part of the determinant which corresponds to the transverse waves implies that these modes of propagation do not depend on the thermal motion of particles.

Consider now a plane monochromatic wave normally incident on a plasma. A simple manipulation of the Maxwell equations then leads to

\[
\Delta E + (\omega^2 \varepsilon(\omega) E_0/c^2) = 0.
\]

As the electromagnetic wave is transverse, the above equation applies to both \( E_x \) and \( E_y \) components. The third component can be found from

\[
\varepsilon(\omega) E_z = 0.
\]

If \( \varepsilon(\omega) \neq 0 \), then \( E_z = 0 \) satisfies (4.18) and the obtained answer is associated with the pure transverse waves. On the other hand, if \( \varepsilon(\omega) = 0 \) and \( E_z \neq 0 \), one refers to a longitudinal wave in an isotropic, homogeneous plasma. The root of \( \varepsilon(\omega) = 0 \) leads, generally, to a complex frequency, meaning that the wave is evanescent. The associated damping is in most cases only slight, since the imaginary part of \( \varepsilon(\omega) \) is small relative to the real part. This approximation is valid for a simple plasma, so that \( \varepsilon(\omega) = 0 \) may then be replaced by

\[
\varepsilon(\omega) = 0.
\]

The solution of (4.19) has a real root, the so called plasma frequency, mentioned above. If the plasma oscillations are simple harmonic, with the phase factor \( \exp \left[ \varepsilon(\omega - k \cdot v) \right] \), the frequency \( \omega \) and the wave vector \( k \) are unrelated. This, clearly, is so, as (4.19) determines only the frequency. Therefore, the group speed of these waves is zero, thus implying no transport energy. This presumed behaviour (which does not agree with the experimental data) results from our neglect of the spatial dispersion. The ratio of the Debye length and the wavelength in the medium determines this effect and its small magnitude permits, often, to omit it.

In the approximation used here, the longitudinal waves are unconnected with the transverse waves [96]. Hence, two types of waves can be regarded as independent, except in a region where the wave frequency has a value close to that of plasma frequency. The separation of the field into transverse and longitudinal modes cannot, however, be effected in an anisotropic plasma.

As implied above, the permittivity is a rather important concept, as it establishes the link between the electric induction and the electric field. It has been, generally, assumed [37], that either the effective field \( \mathbf{E_{eff}} \) is equal to the external field \( \mathbf{E} \), or it is modified by the presence of the Lorentz polarization term, \( 4z^3/3 \). The differentiation, however, between those two possibilities proves to be not a trivial matter. The question has attracted a lot of attention and there exists diversity of opinions [31, 32, 33, 99, 101].
The overall charge of plasma vanishes. It means that the charge of the free electrons must somehow be neutralized. Two possible ways in which this may happen have been put forward: by a positive continuum and by the discrete positive ions. Difficulties arise in attempting to determine the conditions under which these two approaches may be equivalent.

A detailed analysis [27] reveals that \( E_{\text{eff}} = E \) inside a medium in which electrons are neutralized by a positive continuum. On the other hand, a medium consisting of a set of bound atoms obeys the Lorentz formula

\[
E_{\text{eff}} = E + \frac{4\pi}{3} P.
\]

It, therefore, follows that, when considering a fully ionized plasma, the correction term \( 4\pi P/3 \) may be neglected [32, 33].

The main contribution, in determining the value of dielectric permittivity, arises from the motion of charges. The effect of neutral particles is important only when the ionization is very small. In the case under consideration, it may be, therefore, neglected. In order to obtain accurate expression for \( \varepsilon \) and \( \sigma \), one must start with the Boltzmann equation. However, as most of the present work applies to the case of cold plasma, an approximate derivation of the permittivity will be sufficient.

The high conductivity of plasma is well known. It signifies that, although macroscopically neutral, a plasma consists of moving free charged particles. The total current density \( \mathbf{j}_i \), due to this motion, is then written as

\[
\mathbf{j}_i = q \sum_{k=1}^n \mathbf{r}_k - \mathbf{r}_e^{(i)},
\]

where \( q \) is the electronic charge, \( \mathbf{r}_k \) and \( \mathbf{r}_e^{(i)} \) are the time derivatives of the electron and ion position vectors, respectively.

Now, the equation of motion of the electron in a collisionless, isotropic plasma is

\[
m \ddot{\mathbf{r}}_e = qE e^{i\omega t} = qE,
\]

where \( m \) denotes the mean macroscopic field, in agreement with the above discussion. The solution of (4.21) is

\[
\mathbf{r}_e^{(i)} = -(qE/M) \omega^2 + \mathbf{r}_e^{(0)}(t),
\]

where \( \mathbf{r}_e^{(0)}(t) \) is the position vector in the absence of the field. When (4.21) describes the ion motion, then the corresponding solution is

\[
\mathbf{r}_i = -(qE/M) \omega^2 + \mathbf{r}_i^{(0)}(t)
\]

and \( M \) stands for the ion mass.

Now, by definition, the total current density is

\[
\mathbf{j}_i = j + i\omega \mathbf{P} = \left( \sigma + \frac{\varepsilon - 1}{4\pi} \right) E.
\]

Substitution of (4.22) and (4.23) into (4.24) shows then that the conduction vanishes and the polarization vector can be written as

\[
P = -\frac{q^2 E}{\omega^2} \left( \frac{n_e}{m} + \frac{n_i}{M} \right).
\]

As there is no polarization in the absence of the field, then \( \sum n_k = n_i^{(0)} = 0 \) and, hence, the permittivity can be expressed as

\[
\varepsilon = 1 - \frac{4\pi \sigma}{\omega^2} \left( \frac{n_e}{m} + \frac{n_i}{M} \right),
\]

where \( n_e \) and \( n_i \) refer to electron and ion densities, respectively. On account of their mass, ions usually have only small influence of the permittivity and are, therefore, often neglected. Thus, the approximate expression for the permittivity becomes

\[
\varepsilon = 1 - \frac{\omega_p^2}{\omega^2},
\]

where \( \omega_p^2 = 4\pi n_e^0 m \) defines the plasma frequency.

As already indicated the conductivity is zero under the above assumptions. This is so, because here we refer to a collisionless plasma. Hence, electrons do not transfer energy but merely oscillate in the field. These conditions change when collisions are included.

Then, a term proportional to the frictional force, \( \sigma \mathbf{r} \), is added to the right hand side of (4.21). This force is caused by the momentum transfer occurring during collisions. On repeating, essentially, the same arguments as above, we obtain the result

\[
\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 (\omega - i\tau_{\text{eff}})},
\]

where \( \tau_{\text{eff}} \) is the effective collision frequency.

The complex permittivity implies the existence of a non-vanishing conductivity and, hence, the absorption of energy. It must be stressed, however, that (4.26) is only an approximation, and the exact result, obtained from the Boltzmann theory, cannot be reduced [33] to it. Nevertheless, the relation (4.26) is quite adequate for our purpose.

The question now arises as to the validity of applying the classical theory of mechanics to the motion of electrons and ions. It is found [33], that the classical theory is valid if \( \omega_0 \ll \omega_c \), where \( 2\pi \hbar \) is the Planck constant. Another condition for the applicability of the classical theory is that the plasma be regarded only in a non-degenerate case.

When analyzing the propagation of the electromagnetic waves in plasma (or, generally, in a medium) the concepts of the refractive and absorption indices become more prominent. Let us elaborate this point and find out the influence of these indices on some variables pertinent to propagation. Thus, the electromagnetic wave equation in a medium with a constant permittivity tensor can be written [8] as

\[
\Delta \mathbf{E} + (\omega^2 / c^2) \varepsilon(\omega) \mathbf{E} = 0.
\]

If a plane electromagnetic wave, \( E = E_0 \exp(i(\omega t - k \cdot r)) \), is substituted into (4.27), one obtains the dispersion relation

\[
k^2 = (\omega^2 / c^2) \varepsilon(\omega).
\]

To obtain the relation between \( \mathbf{E} \) and \( \mathbf{H} \), the plane wave expression is inserted into the two Maxwell equations (4.4) which involve the curl operator. This yields

\[
\omega \varepsilon(\omega) \mathbf{E} = -\mathbf{c} \times \mathbf{H} \quad \text{and} \quad \omega \mathbf{H} = \mathbf{c} \times \mathbf{E}.
\]

In order to find the condition for transverse waves, (4.29) is (scalarly) multiplied by \( k \), giving \( k \cdot \mathbf{E} = 0 \) and \( k \cdot \mathbf{H} = 0 \). It is usually assumed that \( \varepsilon(\omega) \neq 0 \), otherwise, the longitudinal waves may appear, as indicated above. When the wave vector is complex,
whilst the frequency is real, then the planes of equal phase differ from the planes of equal amplitude, resulting in inhomogeneous waves. On the other hand, when the waves are homogeneous, those two planes coincide. The vector may then be written as \[ k = (\omega/c) (n - i\alpha) (k_l/c), \] (4.30)
where \( n \) and \( \alpha \) are the indices of refraction and absorption, respectively. Replacing the \( k \)-vector in the plane wave expression by (4.30) leads to
\[ E = E_0 e^{-i\omega t} e^{ik_l x/c}, \] (4.31)
Thus, the simple harmonic wave is modulated by the factor \( e^{-i\omega x/c} \), which is connected with the absorption process. As (4.31) implies, the amplitude of the wave in a medium changes by a factor of \( e \) over the distance \( \lambda_0/2\pi n (\lambda_0 \text{ the vacuum wavelength}) \). Now, utilizing (4.31), the relation between the wavelength \( \lambda \) in a medium to that in vacuum becomes
\[ \lambda = \frac{\lambda_0}{n}. \] (4.32)

In the absence of absorption, when only the scattering is present, the refractive index \( n \) is equal to \( \sqrt{\varepsilon} \). This no longer holds when there is also some absorption. Thus, comparing (4.30) with (4.6) leads to
\[ \varepsilon = n^2 - \alpha^2 - 2\alpha x + \frac{2\pi \alpha x}{\varepsilon} \] (4.33)
and
\[ \alpha = \left[ -\frac{\varepsilon}{2} + \left( \frac{\varepsilon}{2} + \frac{2\pi \alpha x}{\varepsilon} \right)^{1/2} \right]^{1/2}. \] (4.34)

Applying expressions (4.34) to a collisionless plasma with parameters \( \varepsilon = 1 - (c_p^2/\omega^2) \) and \( \sigma = 0 \) yields
\[ n = \left( 1 - \frac{\omega_p^2}{\omega^2} \right)^{1/2}, \quad \alpha = 0 \quad \text{if} \quad \omega > \omega_p \]
\[ n = 0, \quad \alpha = \left( \frac{\omega_p^2}{\omega^2} \right)^{1/2} \quad \text{if} \quad \omega < \omega_p \]
\[ n = 0, \quad \alpha = 0 \quad \text{if} \quad \omega = \omega_p. \] (4.35)

Thus, the characteristics of wave propagation differ dramatically, depending on whether the angular frequency \( \omega \) of the wave is larger or smaller than the plasma frequency. Hence, when \( \omega > \omega_p \), the wave travels without attenuation, whilst in the case \( \omega < \omega_p \), the wave cannot propagate in a medium and is completely reflected when \( \omega \equiv \omega_p \). As a consequence, the transverse electromagnetic waves cannot exist. However, under these conditions, the longitudinal waves may appear.

The wave vector \( k \) whose direction indicates the direction of propagation, is related to the wavelength by \( k = 2\pi/\lambda \). Therefore, taking into account (4.32), the wave vector in a medium is related to that in vacuo by
\[ k = \frac{nk_0}{n}. \] (4.36)

There is a close connection between (4.36) and the concept of the phase and group velocities. Thus, the phase velocity is defined as describing the propagation of phase. In other words, it refers to the phase differences between oscillations observed at two different points. Hence, when the frequency \( \omega \) and the wave vector \( k \) are given, the phase speed is expressed as
\[ v_p = \frac{\omega}{k}. \] (4.37)

This speed is primarily [101] used in the treatment of standing waves and in the interference and diffraction phenomena, as phase differences are, then, of particular importance.

The signal, or modulation of the wave, on the other hand, travels (when absorption is small) with the group velocity. The group speed, \( v_g \), is, in a dispersive medium, different from \( v_p \) and is defined by
\[ v_g = \frac{d\omega}{dk}. \] (4.38)

Hence, in vacuum, which is, of course, non-dispersive to electromagnetic waves, \( v_p = v_g = c \) and the signal propagates without distortion. To emphasize again, the phase velocity has nothing to do with propagation [101], but gives us, essentially, the distribution of phase in space. On the other hand, the group velocity refers, in the present approximation, to the energy transport. However, when absorption is high, the signal is strongly frequency dependent, becomes distorted, and the concept of the group velocity then loses its meaning [77].

Let us now apply these concepts to an isotropic, homogeneous plasma. Consider an electromagnetic wave of frequency \( \omega > \omega_p \). Then, assuming absence of absorption, it follows from (4.30) that the wave vector is real. Its subsequent substitution in (4.37) and (4.38) then determines the phase and group speeds of waves, respectively. Thus
\[ v_p = \frac{\omega}{k}, \quad v_g = \frac{d\omega}{dk}, \] (4.39)

which means, as \( n < 1 \) in plasma, that \( v_p > c \) and \( v_g < c \). However, as the phase velocity is not associated with the energy transport, it does not contradict the theory of relativity. At frequencies much larger than the plasma frequency, both speeds approach the vacuum value of light, as is obvious from (4.35).

As already indicated, when \( \omega < \omega_p \), the value of the wave vector is imaginary, the wave exponentially decays, penetrates to a certain thickness, known as the skin depth, and is then totally reflected.

The above described behavior can be explained in relatively simple terms. At very low frequencies, electrons in a plasma have enough time to move in such a way as to screen the plasma interior from the oscillating external electric field. In other words, it follows from the Maxwell equations that the real current density, caused by the motion of charged particles, is greater than the displacement current density. At very high frequencies, on the other hand, electrons cannot move fast enough to follow the oscillating field, and, thus, are unable to screen the plasma interior. In other words, the real current...
be further developed there. In addition, the eigenvalue $S_b$ of the appropriate number operator $S_k = \hbar k^2 / 2m$. It is to be noted that the number of photons with the wave vector $k$ excited in the cavity is determined by the energy $\hbar \omega$.

Another point of view. The former concept arises in the quantization of the field. Thus, $S_k$ refers to a "quantum photon" and a "classical photon", in order to discriminate between another point of view. The latter. Thus, the initially pure, electromagnetic wave becomes now modulated by the radiation, when interfering with matter, assumes some of the characteristics of the latter. This represents, of course, the average of the two momenta: that of the photon and that of the absorbing medium.

The complementary character of the corpuscular and undulatory aspects of electromagnetic radiation is, nowadays, readily accepted. Only the two aspects taken together can reveal the complete nature of that radiation. Very often, in treating radiation phenomena, concepts of the photon and the wave packet (meant to represent the two complementary kinds of radiation) are used interchangeably. This is, of course, quite permissible in the empty space. It is believed, however, that such a (random) alternation between these two notions cannot be validly applied when the radiation propagates in a material medium. Thus, SKobel'tsyn [35] refers to a "quantum photon" and a "classical photon", in order to discriminate between the inherent differences of the two aspects. It is a common mistake to confuse the corpuscular and undulatory characters of radiation, when it is inside a medium. It should be, therefore, emphasized that these two notions represent two distinct (but complementary) aspects of the interaction phenomenon.

Let us now clarify this dissimilarity between the photon and the wave packet from yet another point of view. The former concept arises in the quantization of the field. Thus, if $\hat{a}_k^+$ and $\hat{a}_k$ are the operators which, respectively, create and destroy a quantum of energy $\hbar \omega$, and wave vector $k$, then these quanta are known as the photons [102]. The number of photons with the wave vector $k$ excited in the cavity is determined by the eigenvalue $S_k$ of the appropriate number operator $S_k = \hat{a}_k^+ \hat{a}_k$. It is to be noted that the notion of the energy exchange is of prime importance in this definition. In addition, the value of $S_k$ will vary depending on the surrounding medium.

V. Dual Nature of Radiation

The corresponding phase and group speeds are then, respectively,

$$v_p = c \left(1 - \frac{\omega_p^2}{\omega^2 + \kappa^2 \epsilon_m} \right)^{1/2}$$

$$v_g = c \left[1 - \frac{\omega_p^2}{\omega^2 + \kappa^2 \epsilon_m} \right]^{1/2} \left[\omega^2 + \kappa^2 \epsilon_m \right]^{1/2}$$

Comparing (4.41) with (4.39) confirms that, in the presence of absorption, the phase speed is decreased, whereas the group speed increases.

Considerations in this chapter should serve to provide sufficient information to form a background for analysis in following chapters. Some of the features presented here will be further developed there.

The wave packet, on the other hand, is defined as being the result of a superposition of nearly monochromatic waves and is seen as occupying some finite region in space [50]. It is described mathematically by

$$A(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i(kx - \omega t)} \, dk$$

where it must be remembered that $\omega$ is a function of $k$. The harmonic term contains information about the velocity of propagation of phase and it, therefore, belongs to the undulatory domain.

Next, the momentum density of an elementary radiation field is given by Abraham as

$$g_\lambda = \frac{\hbar \nu}{c}$$

and by Minkowski as

$$g_\mu = \frac{\hbar \nu}{c}$$

In a plasma, where $\kappa < 1$, it holds that

$$g_\mu < g_\lambda < g_\kappa$$

where $g_\lambda = \hbar \nu/c$. One is, therefore, led to suggest that, in a plasma, $g_\lambda$ refers to a "bare" and $g_\mu$ a "dressed" particle, since a "bare" particle influenced by (i.e., interacting with) a medium is referred to as a "dressed" one. As a particle has, in general, a medium-dependent effective mass, only that latter kind is of experimental significance. Hence, it is proposed that the "bare" particle be associated with the concept of a photon, whereas the "dressed" one be linked with that of a wave packet. Putting it in another way, the radiation, when interacting with matter, assumes some of the characteristics of the latter. Thus, the initially pure, electromagnetic wave becomes now modulated by the surrounding medium. However, when a photon is absorbed by a constituent of matter, the wave packet description combines the properties of the material constituent with those of the absorbed photon.

The above discussion distinguishes between the complementary meanings of the photon and the wave packet for radiation in a medium. It is, hence, conjectured that the momentum density $g_\mu$ applies to a photon and $g_\lambda$ to a wave packet, whenever an electromagnetic field inside a medium with the refractive index less than unity, is considered.

The above proposition is in agreement with the work of Horà [39, 103] and Klema and Petržílka [82], who found the total radiation momentum in a plasma medium to be given by

$$g = \frac{1}{2} (g_\lambda + g_\mu)$$

This represents, of course, the average of the two momenta: that of the photon and that of the wave packet. In addition, if the last term in the Peierls expression (3.62), which arises from the presence of dipoles, is neglected, then the total momentum is again given by (5.5). Although this interpretation is somewhat arbitrary, it, nevertheless, leads to satisfactory conclusions.

Consider now radiation propagating in a medium and assume the conditions to be such that laws of geometrical optics apply. This means that the wavelength is much smaller than the typical dimensions involved in the problem. In this case, therefore, waves are treated as corpuscles travelling along the ray trajectories, completely lacking in any...
The role of phase and group speeds in this asymptotic picture becomes clearer from the following considerations, due to D'Alembert wave equation

$$
\Delta \psi + k^2 \psi = 0; \quad k = \frac{\omega}{v}
$$

(5.6)
degenerates in the limit of $\lambda \to 0$. Let us assume a solution of (5.6) to have the form

$$
\psi = a e^{iS}
$$

(5.7)

where $S$ is the eikonal. In such a case, (5.6) will be satisfied by

$$
(FS)^2 = \frac{\omega^2}{c^2} n^2.
$$

(5.8)

This, in turn, means that the surfaces $S = \text{const.}$ describe surfaces of constant phase and, consequently, determine the phase velocity. On the other hand, the Hamilton-Jacobi eikonal

$$
(FS)^2 = 2m(E - V(x))
$$

(5.9)
determines the group velocity which is necessary in order to find the time a particle spends in an optical region. This is connected with the work of Lewis [105], who divided the transport equations into two groups: those describing intensity and those referring to the phase change.

A different approach is adopted here. By extending the Fresnel theorem of stationary phase, de Broglie [106] arrived at the relation between the speed $v$ of a massive particle and the phase velocity $v_p$ of the corresponding wave

$$
v v_p = c^2.
$$

(5.10)

Furthermore, he found that the group speed of the wave, associated with a particle, is equal to the particle speed. Relation (5.10), known as the Broglie group velocity theorem, may be interpreted as pointing to some form of connection in the particle and phase velocities. Of course, this theorem also applies to particles with vanishing rest masses. Then, it expresses the constancy of the product of the corpuscular and undulatory speeds. However, as the energy transport decreases in a medium, the separation into these two aspects is more obvious.

In view of (5.10), it should be remarked that

$$
\mathbf{g}_u \cdot \mathbf{g}_u = g_0^2,
$$

(5.11)

which is a relation connecting the Minkowski, Abraham, and vacuum values of the momentum densities. Therefore, (5.11) can be looked upon as describing the link between the corpuscular and undulatory aspects of the momentum. Although a direct physical significance cannot be attached to the undulatory momentum, this need be no cause for concern, as this analysis applies only in the WKB (i.e., effectively geometrical optics) approximation. It will be seen below that the Minkowski value of the momentum (representing photons) does, indeed, satisfy the condition of equilibrium between radiation and a plasma (or, generally, a medium with $n < 1$), whereas, as expected, the Abraham value does not.

The conservation of momentum and center of mass theorems were already discussed in Chapter III. They were applied there to the analysis of the photon momentum inside an optically dense ($n > 1$) medium. Although that approach is rather simplistic, it will be used here to cover also the domain of optically thin ($n < 1$) media. The qualitative analysis is, essentially, the same as before and, hence, it is not presented here. Nevertheless, it is to be noted that only the Minkowski result simultaneously satisfies both the conservation of momentum and the velocity invariance theorems.

There is some connection between the above approach and that discussed by Fock [107] who draws analogy, from the purely formal standpoint, between classical particles and optical waves. Furthermore, Horak [109] and Carter and Horak [110] applied this similarity to investigate the Goos-Hänchen effect for matter waves. In addition, arguments presented by Boyer [76, 77] attempt to reformulate classical mechanics so as to encompass particles of vanishing as well as non-vanishing rest masses. An application of matter waves to investigate the propagation of light in fibres was presented by Arnold [108]. To illustrate the similarity between those two types of waves, he referred to the quantum mechanical Amaronov-Bohm effect [111, 112], which has been experimentally verified [113]. The classical explanation of this effect was given by Boyer [114]. Basically, the Amaronov-Bohm effect predicts the physical significance of the electromagnetic potentials, and also suggests that their role, in quantum mechanics, is similar to that of the refractive index in classical physics. To verify this proposal, a simple experimental set up is used to observe diffractive effects between matter and light waves.

Thus, consider a row of metallic channels which are at gradually increasing potentials. A plane matter wave (such as that corresponding to electrons) entering those channels at one end, will suffer a deflection upon exit, despite the absence of any classical forces. Similarly, a light wave passing through an optical phase array (a row of channels with increasing optical densities) will be also deflected. Moreover, if the dispersion of the light waves inside this optical system is the same as that for the matter waves in vacuum, then the resultant deflection of the wave will be equal.

This correspondence between matter and optical waves will be further elaborated in the next chapter. Now, let us return to the question of the electromagnetic momentum in a plasma and investigate the important topic of the black body radiation. That last phenomenon describes, as is well known, the equilibrium spectral distribution between the thermal radiation and matter in an evacuated cavity. The presence of matter is necessary to satisfy the equilibrium conditions and its amount is minute. When, however, the cavity is filled with a plasma, it affects this spectral distribution and it is quite natural to suppose that the density of radiation inside plasma differs from the one in vacuum. Thus, if $\epsilon(\nu, T)$ is the mean energy of oscillators of frequency $\nu$ at the temperature $T$, then

$$
\epsilon(\nu, T) = h\nu(\nu/h) - 1)^{-1}
$$

(5.12)

and the energy density $u(\nu)$ of the radiation can be expressed by

$$
u(\nu, T) d\nu = \epsilon(\nu, T) g(\nu) d\nu,
$$

(5.13)

where $g(\nu) d\nu$ refers to the density of radiation states in the frequency interval $\nu$ to $\nu + d\nu$. This density of states is usually defined, in terms of the wave vector $k$, as

$$
\langle k \rangle d\nu = \frac{\pi}{h^2} k d\nu,
$$

(5.14)

In order to arrive at the density of states, in terms of frequencies, (5.14) is transformed, utilizing (4.36), into

$$
\langle k \rangle d\nu = \frac{8\pi}{c^2} u(\nu) d\nu.
$$

(5.15)
Now, if (5.13), together with (5.12), is inserted into (5.13), we obtain the following modified Planck law for an isotropic, collisionless plasma

\[ u(\nu) = n_0(\nu), \quad (5.16) \]

where \( n_0(\nu) \) corresponds to the vacuum value of the energy density. The result (5.16) was obtained in the literature by a completely different method by Pekeris [30] and Dawson [37]. Case and Chu [117] used the fluctuation-dissipation theorem to arrive at the same conclusion. Although Oster [116] also obtained (5.16), his radiative transfer theory was criticized by Croxen [118] as violating the Second Law of thermodynamics. In addition, the transport equation [119] differs from that due to Martin [119]. A modified form of the Planck law in anisotropic media was described by Cole [38], which, in the isotropic limit, reduces to (5.16).

Neglecting the longitudinal waves, Inman [120] developed the thermodynamics for electromagnetic waves in plasma. He found that, as the plasma frequency increases, the number- free energy-, entropy- and energy-densities are all decreasing from their vacuum values. The correctness of the formula for the radiation pressure given by Inman was, however, disputed by More [121].

We are now in a position to develop further the method used by Einstein [34] in his investigation of emission and absorption processes. An attempt was made by Skobel'tzyn [122] to apply this method to optically dense media. His work, however, appears to be inconclusive. In the present treatment, the original (Einstein) method will be adapted to the plasma domain.

During any absorption (or emission) process, a momentum is transferred to an elementary system in the radiation field. The velocity distributions of such systems are merely due to their interactions with the radiation field. Thus, this distribution must be the same as the one these systems attain through their mutual collisions alone. Hence, it is necessary that the average kinetic energy of a system in the radiation field be equal to (1/2)\( P \). Suppose now that there are two possible quantum states \( z_1 \) and \( z_2 \) with the corresponding energies \( e_1 \) and \( e_2 \). Next, let the transition from state \( z_2 \) to state \( z_1 \) be allowed for the system. That transition will then result in the system in state \( z_1 \) absorbing a quantum of energy of magnitude \( e_2 - e_1 \). Similarly, the transition from \( z_1 \) to \( z_2 \) will result when the system emits that quantum of energy. Now, there are two different processes a system can undergo to make a transition. First, there is a process independent of the external radiation field, viz., the so-called spontaneous emission. The probability for such a process to occur in time \( dt \) is

\[ dP = A_{nm} dt, \quad (5.17) \]

where \( A_{nm} \) is the Einstein “A” coefficient. Second, a transition which depends on the stimulation by the external field. Processes of this type are known as “Induced” radiation processes. Thus, the probability \( dP \) of a system under the influence of the radiation density \( \nu(\nu) \) makes the transition from \( z_1 \) to \( z_2 \) by absorbing the energy quantum is \( dP = B_{nm} \nu(\nu) dt \). Similarly, the probability for the transition from \( z_2 \) to \( z_1 \) to occur is \( dP = B_{mn} \nu(\nu) dt \). Again, \( B_{nm} \) and \( B_{mn} \) are the Einstein “B” coefficients. Thus, in these two cases, the transition probability is the function of the radiation density \( \nu(\nu) \) surrounding our elementary system. In other words, when the system is in a plasma, \( \nu(\nu) \) refers to the radiation density inside the latter. This density is given by (5.16) and its use is imperative. (That this choice does not disturb the thermodynamic equilibrium conditions will be shown below.) Never the less, Skobel'tzyn [32] originally assumed that the radiation energy density in a dielectric is the same as the one in vacuum but, in the same article, he later made an ad hoc assumption that that density ought to be \( \nu(\nu)/n \), for \( n > 1 \).

In view of the above discussion, the probabilities for upward and downward transitions can now be written, respectively, as

\[ dP = nB_{nm} \nu(\nu) dt; \quad dP = nB_{mn} \nu(\nu) dt. \quad (5.18) \]

Hence, the probability for the radiation induced processes in a collisionless plasma is decreased relative to vacuum. This is so, because, as the plasma density increases, the refractive index decreases and, thus, the amount of radiation energy density present is diminished, in accordance with the expression (5.16).

Investigate now the motion of a test particle under the influence of the radiation field. To simplify the calculations and to make, at the same time, the physics of the problem more transparent, it will be assumed that the particle of mass \( M \) is moving with velocity \( \nu \) in the \( x \)-direction and that the conditions are such that the higher powers of \( r_j \) can be neglected. In a radiation field, the linear momentum \( M\nu \) of this particle changes, in time \( T \), due to the effect of the radiation force \( -\nu \), \( \nu \) being constant to be evaluated below. Also, during the same time, the fluctuations of the action of radiation on the particle cause a momentum transfer \( \lambda \) to the latter. The magnitude and the direction of this momentum transfer are constantly changing. As the radiation field must not affect the overall velocity distribution, the average momentum of the particle, at both ends of the time interval \( T \), must be \( \lambda T \). Therefore, we must have

\[ \frac{(M\nu - \nu \lambda)}{T} = (1/2)\nu T. \quad (5.19) \]

We shall work in the approximation in which all terms containing \( 1/2 \) can be neglected. Further, in order not to disturb the thermodynamic equilibrium, we shall assume the relation \( (1/2)\nu T = (1/2)\nu T \) to be valid. The above equation then reduces to

\[ \frac{(M\nu - \nu \lambda)}{T} = 2\nu T. \quad (5.20) \]

Thus, this is the fundamental equation required by the theory of heat and a comparison with (5.19) affirms that the presence of a plasma does not affect it.

Now, if \( \nu \) and \( T \) are calculated and then inserted in (5.20), it must lead to an identity, as has already been pointed out by Einstein [34]. However, in order to calculate these quantities using his method, in the conditions of thermal equilibrium, it is necessary to know the radiation energy density and the photon momentum. This, in turn, requires the knowledge of the field energy momentum tensor. \( \nu \), therefore, follows that the condition (5.20) can be used to ascertain the correct form of this tensor.

As the radiation laws are defined only for those inertial frames in which our particle is at rest, in calculating constant \( \nu \), the transformation formulae [1] of the special theory of relativity are used. Further, the quantities referring to the rest frame of the particle will be marked with a dash. Thus, radiation per unit volume in the frequency range \( dv \), within a solid angle \( d\Omega \), transforms according to

\[ u' = \nu - \psi \nu \frac{d\Omega}{d\Omega'} (1 - 2\nu \beta \cos \phi), \quad (5.21) \]

where \( \beta = \nu_0 / c \) and \( u' \) and \( u' \) depend on the direction defined by the angle \( \phi \) it makes with the \( x' \)-axis, and the angle \( \theta' \) between its projection in the \( y'z' \)-plane and the \( y' \)-axis.
To facilitate the transformation of the argument of \( u(v', \phi') \), the formulae for the Doppler effect and the aberration are needed. These, however, are well known in the theory of relativity and, in the approximation applicable here, take the forms

\[
\nu' = \nu(1 - n \beta \cos \phi) \quad (5.22)
\]

and

\[
\cos \phi' = \cos \phi - n \beta + n \beta \cos^2 \phi, \quad (5.23)
\]

respectively. Further, no relevant information is lost if we put

\[
\psi' = \psi. \quad (5.24)
\]

The inverse transformation to that in (5.22) is

\[
\nu = \nu'(1 + n \beta \cos \phi'), \quad (5.25)
\]

which, on substitution in \( u(\nu) \) and subsequent expansion, gives

\[
\frac{u(\nu) = u(\nu') + \nu' n \beta \cos \phi' \frac{\partial u(\nu')}{\partial \nu'}}{5.26}
\]

The solid angle \( d\Omega \) is (in our approximation) connected with \( d\Omega' \) by

\[
d\Omega' d\Omega = 1 - 2 n \beta \cos \phi'. \quad (5.27)
\]

Thus, the substitution of (5.25), (5.26), and (5.27) in (5.21) results in

\[
u'(\nu', \phi') = \left[ u(\nu') + \nu' n \beta \cos \phi' \frac{\partial u(\nu')}{\partial \nu'} \right] (1 - 3 n \beta \cos \phi'), \quad (5.28)
\]

Making use of the last relation and the assumption about the elementary processes, discussed earlier, allows calculation of the average momentum transferred, per unit time, to the particle. Let, then, the number of elementary transitions from level \( m \) to level \( n \) be

\[
\frac{1}{S} \int w_m e^{-\nu m} \nu d\nu \int u(\nu', \phi') d\Omega' / 4\pi, \quad (5.29)
\]

where

\[
S = w_m e^{-\nu m} \nu + w_n e^{-\nu n} \nu
\]

and \( w_m, w_n \) are the statistical weights corresponding, respectively, to levels \( m \) and \( n \). An analogous form holds for the transition from level \( n \) to level \( m \).

As suggested earlier, the photon momentum in plasma is given by the Minkowski relation. Therefore, in each elementary process, the momentum \( nh \nu \cos \phi'/c \) is transferred to a particle. Combining the upward and downward transitions, the total momentum transferred to the particle is then expressed, using (5.29), as

\[
nh \nu \cos \phi'/c \frac{n}{S} w_n B_n \nu e^{-\nu} e^{-\nu m} T \int u(\nu', \phi') d\Omega' / 4\pi. \quad (5.30)
\]

Substituting (5.28) into (5.30) and putting \( Q = (1/S) \int w_m B_m e^{-\nu m} \nu d\nu \) the total averaged momentum transferred, per unit time, to a particle through the induced processes is

\[
-\frac{n^2 h \nu}{c} \beta \left[ \nu(\nu) - \frac{\nu}{3} \frac{\partial u(\nu)}{\partial \nu} \right] Q(1 - e^{-\nu m} T), \quad (5.31)
\]

The dashes over symbols have been dropped here, for the sake of clarity.

As the spontaneous emissions do not have any preferred directions, average momentum transfers, due to these processes, vanish.

Thus, finally, the coefficient \( R \) can be written as

\[
R = \frac{n^2 h \nu}{c^2} \left[ \frac{\nu}{3} \frac{\partial u(\nu)}{\partial \nu} \right] Q(1 - e^{-\nu m} T). \quad (5.32)
\]

Prior to utilizing this result, let us inquire about the effect of the presence of a plasma on the momentum \( \Delta \). Denote by \( \delta \) the momentum transferred to the particle during some process and assume that its sign and magnitude vary in time in such a way that the average value of \( \delta \) vanishes. Now, if \( \delta i \) refers to the transfer of momentum through the \( i \)-th independent process, then the resultant transfer is \( \Delta = \sum \delta i \). Because the average value of each \( \delta i \) vanishes, it is necessary to write (assuming \( \delta i^2 = \overline{\delta^2} \))

\[
\overline{\Delta^2} = N \overline{\delta^2}, \quad (5.33)
\]

where \( N \) is the number of all elementary transitions in time \( \tau \). If the momentum corresponding to each elementary process, \( \delta = nh \nu \cos \phi/c \), is substituted into (5.33), it then yields

\[
\overline{\Delta^2} = \frac{N}{3} \left( \frac{nh \nu}{c} \right)^2 Q(\nu). \quad (5.34)
\]

As the number of induced processes from level \( m \) to level \( n \), in time \( \tau \), in equilibrium, is \( N/2 \), it is, then, possible to write

\[
\overline{\Delta^2} = 2 Q(\nu) \tau, \quad \overline{\Delta^2} = 2 \frac{\nu}{3} \left( \frac{nh \nu}{c} \right)^2 Q(\nu). \quad (5.35)
\]

As already previously emphasized, the expressions (5.32) and (5.35) are determined in terms of the radiation energy density \( u(\nu) \) which, of course, differs from its vacuum value \( u(\nu) \). An attempt will now be made to show the validity of the claim that the photon momentum inside plasma is correctly described by the Minkowski relation. For that purpose, we must show that the thermodynamic equilibrium is not disturbed when the radiation momentum is transferred to a particle. This, in turn, will be so if, after calculating \( R \) and \( \overline{\Delta^2} \) in terms of the proposed energy and momentum densities of radiation, their consequent insertion in (5.20) yields an identity.

Thus, substitution of (5.16) in (5.32) results in

\[
R = \frac{n^2 h \nu u(\nu)}{c^2} \left( \frac{\nu}{x T} \frac{\nu}{x T} + \nu^2 - 1 \right) Q(1 - e^{-\nu m} T). \quad (5.36)
\]

Now, if conditions are such that \( \nu m /x T \gg 1 \), then (5.36) reduces to

\[
R = \frac{n^2 h \nu u(\nu)}{c^2} \left( \frac{\nu}{x T} \right) Q. \quad (5.36a)
\]

Furthermore, when (5.16) is inserted in (5.35), the result is

\[
\overline{\Delta^2} = \frac{2}{3} \nu \left( \frac{nh \nu}{c} \right)^2 u(\nu) Q. \quad (5.37)
\]

As already previously emphasized, the expressions (5.32) and (5.35) are determined in terms of the radiation energy density \( u(\nu) \) which, of course, differs from its vacuum value \( u(\nu) \). An attempt will now be made to show the validity of the claim that the photon momentum inside plasma is correctly described by the Minkowski relation. For that purpose, we must show that the thermodynamic equilibrium is not disturbed when the radiation momentum is transferred to a particle. This, in turn, will be so if, after calculating \( R \) and \( \overline{\Delta^2} \) in terms of the proposed energy and momentum densities of radiation, their consequent insertion in (5.20) yields an identity.
VI. The Proca Equations

Results obtained in the previous chapter should be understood as the asymptotic solutions to the wave optics viewed from the classical standpoint. It appears, in this light, that the only expression for the momentum exchange satisfying (5.20), the momentum exchange between field and a particle, is the Minkowski one. It is to be noted, however, that no generally applicable conclusion about the energy-momentum tensor can be reached through utilizing the method of the preceding chapter, as that formulation applies only in the case of geometrical optics. Furthermore, it is not relativistically covariant.

In order to arrive at a reliable result, some other, more rigorous, method must be used. Thus, the procedure to be presented here, resort to the formalism of the covariant field theory, which encompasses the realm of wave phenomena, as well as the corpuscular approach, just presented.

Quite generally, a medium, in which the phase speed is frequency dependent, is known as a dispersive one. In such a case, the relation between the angular frequency \( \omega \) of the wave and the corresponding magnitude of its complex counterpart. This is, essentially, the approach used by Hora [39] whilst treating the interaction of laser radiation with an inhomogeneous plasma. In this chapter, results, valid in what is, essentially, a WKB approximation, were presented. In the following pages, a more general approach will be adopted to facilitate our inquiry into the form of the electromagnetic energy-momentum tensor in a plasma.

\[
\omega^2 = k^2 c^2 + \alpha_p \omega_p^2,
\]

(6.1)

where \( \alpha_p \) is the plasma frequency, whilst \( \omega_p \) describes the characteristic excitation frequency of the medium. In a plasma, \( \omega_p \) vanishes. The expression (6.1) is only approximate, as it is derived under the assumption that the dispersion is determined solely by electrons. In the special theory of relativity, the relation between the energy \( E \) and the momentum \( p \) of a particle, having rest mass \( m_0 \), is given by

\[
E^2 = p^2 c^2 + m_0^2 c^4.
\]

(6.2)

A quick comparison of (6.1) with (6.2) reveals a striking similarity between both relations. Thus, it was suggested [40, 41] that the propagation of a wave in a medium may be treated analogously to the motion of a massive particle in vacuo. Put otherwise, as the wave propagates through a collisionless plasma [43], collisional plasma [122], or a non-dispersive fluid [83], it interacts with its surroundings and, thus, takes on some of the properties of the medium. Thus, for instance, the effect of a collisionless plasma on the field is contained in the \( \alpha_p \) term in (6.1), since, in that case, \( \alpha = 1 \). Hence, a photon, which, before entering a plasma region, had a zero rest mass, acquires, whilst travelling inside it, an effective rest mass of \( \hbar \omega_p^2 / c^2 \). This is readily obtained from (6.1) and (6.2).

Although it is normally assumed that photons are massless particles, this requirement is not necessary from the point of view of the special theory of relativity. What is really important, is that \( c \), the speed of light in vacuo, remains constant in moving inertial frames. This, however, would point to a particle with a vanishing rest mass. On the other hand, the speed of a massive photon must be less than \( c \) and ought to depend on its energy, \( \hbar \), and, hence, its frequency. This would imply that electromagnetic waves of different frequencies would propagate with different speeds. In fact, it has been suggested that high frequency waves should propagate at higher speeds than the low frequency ones. As the waves of progressively lower frequencies are considered, the limit is eventually reached when the propagation speed drops to zero. This happens when \( \hbar \omega = m c^2 \), i.e., in view of the above discussion, when the frequency of the incident wave is equal to the plasma frequency. Experimental results [129] obtained from pulsars agrees with the just presented interpretation.

Thus, it seems plausible to replace an electromagnetic wave in a medium by an equivalent massive photon in vacuo. It will be recalled that the principal difficulty in establishing the correct form of the energy-momentum tensor in a macroscopic body lies in the impossibility of splitting that tensor uniquely into the field and the matter parts. That separation becomes possible here, if we are willing to accept the above interpretation.

The consequences of replacing propagation of radiation in a medium by a photon with a nonvanishing rest mass are rather intriguing. Thus, it follows from the Maxwell theory that the photon can be polarized in two directions. However, as the photon has now an effective mass, the generalization of the Maxwell equations leads to the theory of Proca equations. Consequently, there should occur now all three states of polarization which are predicted by the latter theory. Thence, apart from two transverse polarizations, there also exists a longitudinal one. Furthermore, the longitudinal photon moves, the weaker is the associated electric field [124], thus, implying only a rather weak interaction with a medium. In addition, the electric field diminishes exponentially with distance [124] and the corresponding flux lines fade away, even in vacuo. Similarly, magnetic field is also affected and the associated lines are compressed around around.

Using techniques of classical electrodynamics [30], we proceed now to analyze this problem more rigorously. Although there will be some overlap with Chapter II, as far as the presentation is concerned, the conclusions we reach here will be, in general, quite different.

It is necessary, at this stage, to discuss the problem of the photon spin. Thus, according to quantum mechanics, the total angular momentum of the radiation field is equal [125] to the sum of the spin and the orbital angular momenta. The spin of the photon is, generally, accepted to be unity and to lie in the direction of propagation. This was verified by observation [89]. We shall, therefore, assume here that the spin of the massive photon is also unity. Hence, in view of the above discussion, the Maxwell field in a medium is, essentially, replaced by a neutral vector meson field in vacuo. Then, it is applied to the question of the energy-momentum tensor. Thus, a free electromagnetic wave corresponding to a massive photon may be described by

\[
A = A_\phi e^{-i(\mathbf{k} \cdot \mathbf{x} - \omega t)}.
\]

(6.3)
with
\[ \left( \frac{\omega}{c} \right)^2 - k^2 = \mu^2, \]  
(6.4)
where \( \mu \) is the photon's rest mass, in units of inverse length. From its definition, \( v_p = \omega/k \),
the phase speed is
\[ v_p = c \left(1 + \frac{\mu^2}{k^2} \right)^{1/2}, \]  
(6.5)
whilst, for the group speed, \( v_g = d\omega/dk \), we have the expression
\[ v_g = \frac{c}{\omega} (\omega^2 - \mu^2 k^2)^{1/2}. \]  
(6.6)

A glance at (6.6) shows that also for such a massive field, the dependence of the energy propagation on frequency is in accordance with experimental findings [123, 124].

Let us consider, for a moment, the case of a static field, i.e., \( \omega = 0 \). It then follows from (6.4) that \( k = \pm \mu \). Consequently, substitution into the wave equation (6.3) implies that the spherical static field decays exponentially, i.e., \( A = A_0 e^{-\mu r} (A_0 \propto 1/r) \) as already indicated. On selecting \( k = +\mu \), we have a behavior similar to that displayed by the Yukawa model of nuclear binding. Thus, as one of the consequences of the photon being massive, we must have a deviation from the Coulomb law. Hence, in view of our supposition, the field in a collisionless, isotropic plasma should decay exponentially with distance. The same conclusion was also obtained by Cole [40], in his study of wave propagation in an inhomogeneous medium. He replaced \( \omega \) by \( \omega_0 \) and \( k \) by \( -\delta \), in the dispersion relation and arrived at the Klein-Gordon equation
\[ \phi = \frac{\tau}{r} e^{-\omega_0 \tau r}. \]  
(6.8)

This Yukawa type behavior applies to the case of an isotropic plasma, in view of the form of the dispersion relation (6.1). On introducing, in addition, a magnetic field, the solution of the Klein-Gordon equation, in the static limit, takes the form of the Coulomb potential, thus, effectively, removing the shielding effect of \( e/\omega_0 \).

There is another interesting point reported by Cole [40]. He noticed that the parameter \( e/\omega_0 \) also occurs in the solid state literature, where it is referred to as the London penetration depth. However, as we saw above, it also distinguishes a plasma without a magnetic field from one with one in the field. In other words, it distinguishes between superconducting and conducting states.

As the photon now is assumed to have a nonvanishing effective rest mass, a mass term has to be added to the Lagrangian density \( \mathcal{L} \). The thus obtained Lagrangian becomes [59]
\[ \mathcal{L}_p = \frac{1}{16\pi} F^{\mu\nu} F_{\mu\nu} - \frac{1}{c} J^\mu A^\mu + \frac{\mu^2}{8\pi} A^\mu A_\mu, \]  
(6.9)
and is termed the Proca Lagrangian. The reciprocal length in (6.9) is associated with the rest mass of the photon by \( \mu = m c/\hbar \). The Proca equations of motion are then written as (cf. 2.3)
\[ \partial_{\mu} F^{\mu\nu} + \mu^2 A^\nu = \frac{4\pi}{c} j^\nu, \]  
(6.10)
where \( F^{\mu\nu} \), the electromagnetic field tensor, is given by (2.4). Thus, (6.10), together with (2.4), gives us the covariant Proca equations for a massive vector field, and is should be noted that it is the only possible linear generalization of the Maxwell equations [126].

It is to be noted that, as a result of the inclusion of the coupling constant \( \mu \), the potentials now acquire real physical characteristics. In other words, in contradistinction to the potentials in the Maxwell theory, they now become observable [127].

If we, further, assume the validity of the Lorentz gauge, \( \dot{\varepsilon}_\mu A^\mu = 0 \), then (6.10) may be rewritten as
\[ \Box A^\mu + \mu^2 A^\mu = \frac{4\pi}{c} j^\mu, \]  
(6.11)
where the four-dimensional Laplacian operator (also known as the d'Alembertian) is defined by \( \Box \equiv \partial_\mu \partial^\mu \). When time independence of \( A^\mu \) is assumed, so that the field may be regarded as static, the solution of (6.11) leads to the Yukawa expression for the potential, as already discussed.

In three dimensional notation, the modified Maxwell equations are [126]
\[ V \times E = -\frac{1}{c} \dot{\varepsilon}_\mu H, \quad V \times H = \frac{1}{c} \dot{\varepsilon}_\mu E + \frac{4\pi}{c} j - \mu^2 A, \]  
(6.12)
\[ V \cdot E = 4\pi \rho - \mu^2 V, \quad V \cdot H = 0 \]
\[ \frac{1}{c} \dot{\varepsilon}_\mu V + V \cdot A = 0, \quad V \times A = \mu^2 H, \quad V \cdot \frac{1}{c} \dot{\varepsilon}_\mu A = -\mu^2 E. \]

These comprise the set of fifteen Proca equations and may be viewed as forming two physically distinct groups [127]. Thus, four top equations represent, apart from the coupling constant \( \mu \), the Maxwell field through the six vector \( (E, H) \); the bottom three equations, on the other hand, describe a four-gradient of a scalar potential. A closer inspection reveals that these groups may be seen as corresponding to the transverse and longitudinal waves, respectively.

It is generally accepted that, in the Maxwell model, the vectors \( E, H, \) and \( k \) are all mutually orthogonal. Furthermore, the scalar potential may be made to vanish there, so that the vector potential \( A \) is parallel to \( E \). Assuming that the same should apply also in the Proca theory, for fields corresponding to transverse polarizations, we obtain from the Lorentz transformation relations [2] that [127]
\[ E = -\frac{\omega}{\mu} A \quad \frac{\partial H}{\partial t} = kE \]  
(6.13)
Similarly, in the case of longitudinal waves, the vectors \( E, A \), and \( k \) are mutually parallel and, further, \( H = 0 \). Therefore, the following relations hold
\[ E = -\frac{\mu}{\omega} A \quad \frac{\partial E}{\partial t} = kA. \]  
(6.14)

Now, utilizing the dispersion relation, \( \omega^2 - k^2 c^2 = \mu^2 c^2 \), it becomes clear that the four-vector \( (\omega_0, k_0) \) must be time-like. Hence, it is possible to transform it to the rest frame of the massive photon. (In the case of a massless photon this transformation, of course, fails, as the four-vector is then light-like.) Hence, in the heavy photon's rest frame \( k = 0 \). Consequently, (6.13) yields \( H = 0 \) and (6.14) gives \( V = 0 \). Because, in both cases, the vectors \( E \) and \( A \) are parallel, the resulting \( E \)-field is just the expression of mutual transmutability between transverse and longitudinal waves. This, in turn, is a direct consequence of the fact that the state of polarization is not the Lorentz invariant [127].
where the equality holds for low conductivity. Due to the factor $(\omega/\epsilon \mu)^2$, the transparency of the stellar medium (which is in plasma state) to the longitudinal photons is nearly perfect, thus, explaining the lack of their observation. It is further pointed out by Wigner (cf. [24]) that the longitudinal waves cannot be responsible for cooling of the stars, as the rate of their production must be minute. He then concluded that the thermodynamics of a system in question would be barely affected by the presence of the longitudinal photons.

It is important to emphasize, at this point, that although the above mentioned workers assumed the photon mass to be non-vanishing in vacuum, the underlying philosophy of the present work is diametrically opposite. We assume the standard Maxwell field in vacuum, implying a zero rest mass of a photon in empty space. Only the study of fields inside a cavity with perfectly reflecting walls. Applying the Fresnel formulae [30] gives (27) a transverse wave which is nearly completely reflected but contains, after reflection, a small fraction of a longitudinal wave. A small amount of energy is, at the same time, carried away by the transmitted longitudinal wave. The effect of a longitudinal wave on the matter in its path is very small (22), so that virtually no transfer of momentum to the obstacle occurs. Because the electric field is so small (as its energy is proportional to $\mu_0/\epsilon_0$), the interaction occurring between the longitudinal wave and matter can be, for most practical purposes, neglected. In other words, the approach of the longitudinal photon to equilibrium is very slow. Thus, for example, assuming the rest mass of a photon $m_0 = 10^{-8} \text{g}$ and an angular frequency $\omega$ in the optical region in a physical cavity, it can be estimated (27, 22) that the time necessary for the transverse photons to transform into longitudinal ones is of the order of $10^3$ years. This transformation effect is, indeed, minute.

Furthermore, in order to avoid the contradiction with the black body radiation law, GOLDHAIER and NETO [132] suggested that the longitudinal photons may have a rest mass of several GeV so that the interaction length is so small that the effect cannot be, essentially, observed. Their reasoning follows from the rule known in particle physics which states that the range of a propagated interaction is inversely proportional to the mass of the particle responsible for its transmission. They claim that a redshift results from the slowing down of emitted photons by the radiation field surrounding the source (27). Putting it differently, they suggest that the interaction of a transverse background photon with a (transverse) emitted photon may be mediated by a longitudinal photon — thus assuming that photons should have non-vanishing rest masses. This explanation was queried by WOODWARD and YOCHEL [133] as not fully meeting the requirements of the "tired light" hypothesis. Another anomalous red shift effect was observed by GOLDSTEIN and NETO [134], when the satellite Pioneer 6 reached its perihelion behind the Sun. MOSER et al. [135] have analysed this shift, which is symmetrical with respect to the center of the Sun, and concluded that the effect was due to the interaction of the photons emitted by Pioneer 6 with the radiation emanating from the Sun. These anomalous shifts $\delta\nu$ and the associated energy losses, could be interpreted, so it was suggested, as being caused by the non-elastic photon-photon scattering.

In the just discussed explanations of the non-velocity red shifts, it was assumed that the mass of a photon, although minute, was still finite. However, the effect of the interstellar medium was not taken into account. Thus, a modified explanation can be put forward. As the collisionless plasma is a good approximation (28), in explaining the wave phenomena in the interstellar space, it can be assumed that a photon (originally massless) acquires a non-vanishing mass when propagating through such a medium. And only then it interacts in the way proposed above. This last reasoning is in agreement with the supposition of this work.

Turn the attention now to the energy-momentum tensor for the free massive field. Recalling that the free field is one with no charges or currents present, we see that (6.11) becomes simply

$$\Box A^\mu + \mu A^\mu = 0,$$  
(6.16)
or, written in terms of the electromagnetic field tensor, we have

$$\epsilon_{\mu \nu} F^{\mu \nu} = 0.$$

As a consequence of the mutual dependence of $A^\mu$ and $F^{\mu \nu}$ which stems from (6.17) and (2.4), the $\mu \neq 0$ theory cannot be gauge invariant [48]. This needs an explanation, since gauge invariance is an important concept. The vector potential, introduced as an auxi-
The shortcomings of the canonical tensor were discussed in Chapter II. In an attempt to meet them, the total angular momentum tensor \( J^{\mu
u} \) can be written as

\[
\frac{\partial J_{\mu}^{\nu}}{\partial (\partial \mathbf{A})} = J_{\mu}^{\nu} + \nabla \mathbf{A}.
\]

is then termed the gauge transformation. Hence, in order that a field quantity has a physical meaning, it must be invariant under gauge transformations (6.18). In this connection, we may recall the idea of Aharonov and Bohm [111, 112] to consider the vector potentials \( A^\mu \) (rather than the field tensor \( F^\mu\nu \)) as the true physical field quantities. It can be shown [25] that the theoretical requirement of the gauge invariance implies the vanishing photon mass. Nevertheless, Coester [129] was able to construct a theory of a neutral vector meson which, whilst not being gauge invariant, remained in accord with observations.

Now, as is well known, the energy-momentum tensor can be formed from the Lagrange density. Thus, if the Hamiltonian density is defined by

\[
\mathcal{H}_p = \frac{\partial \mathcal{H}_p}{\partial (\partial \mathbf{A})},
\]

the covariant formulation of (6.19) leads (in view of the Lorentz transformation properties of \( \mathcal{H}_p \)) to the canonical energy-momentum density tensor (2.28). Substitution of the Proca Lagrangian density (6.9) into (2.28) and utilizing the corresponding field equations (6.12), yields the following expressions for components of this tensor

\[
T^{00} = \frac{1}{8\pi} \left[(E^2 + H^2) + \mu^2(V^2 + A^2)\right] + \frac{1}{4\pi} V \cdot (VE)
\]

\[
T^{0i} = \frac{1}{4\pi} \left[(E \times H)_i + \mu^2 A_i V + V \cdot (A_i E)\right],
\]

\[
T^{ij} = \frac{1}{4\pi} \left[(E \times H)_i + \mu^2 A_i V + (V + \nabla)H_i - \nabla_i (VE_i)\right].
\]

Generally speaking, the energy density corresponding to (6.9) is not positive definite [48]. Components \( T^{00} \) and \( T^{0i} \) in (6.20), corresponding to the energy and momentum densities, respectively, are clearly seen to differ from the accepted ones. Nevertheless (cf. Chapter II), by a three dimensional space integration, the following expressions for the energy and momentum, associated with the Proca field, are obtained

\[
\int T^{00} d^3x = \frac{1}{8\pi} \int \left[(E^2 + H^2) + \mu^2(V^2 + A^2)\right] d^3x
\]

\[
\int T^{0i} d^3x = \frac{1}{4\pi} \int \left[(E \times H)_i + \mu^2 A_i V\right] d^3x.
\]

The shortcomings of the canonical tensor were discussed in Chapter II. In an attempt to meet them, Belinfante [46] proposed a symmetrical form for the tensor density. Thus, the total angular momentum density tensor \( J^{\mu\nu} \) can be written as

\[
J_{\mu}^{\nu} = \mathbf{M}^{\mu\nu} + \mathbf{N}^{\mu\nu},
\]

a sum of an orbital \( M^{\mu\nu} \) and spin \( N^{\mu\nu} \) angular momentum density tensors. However, this last subdivision may be shown to be unique only when the rest frame of the particle can be defined. Thus, a meson, having a non-vanishing rest mass, satisfies this condition. Therefore, it lends itself for the analysis of the energy-momentum tensor of the electromagnetic field in a plasma.

Our primary concern in defining the symmetrical energy-momentum tensor \( \Theta^{\mu\nu} \) is to modify \( T^{\mu\nu} \) in such a way that the total distribution of energy and momentum densities in the field, described by \( \Theta^{\mu\nu} \), be given by

\[
\Theta^{\mu\nu} = T^{\mu\nu} + T_{s}^{\mu\nu}.
\]

A divergenceless tensor \( T_{s}^{\mu\nu} \) may be then interpreted as the spin energy-momentum tensor, and must be chosen in such a way that conservation laws involving \( T^{\mu\nu} \) still hold. It is then found [48] that

\[
T_{s}^{\mu\nu} = \frac{1}{4\pi} \partial_i F_{\mu\nu} A^i.
\]

It is to be noted that \( T_{s}^{\mu\nu} \) gives no contribution to the total energy and momentum [46], since

\[
\int \Theta^{\mu\nu} d^3x = \int T^{\mu\nu} d^3x.
\]

The total angular momentum density tensor

\[
J^{\mu\nu} = \mathbf{M}^{\mu\nu} - \mathbf{N}^{\mu\nu},
\]

is now conserved, as its divergence, \( \partial_i J^{\mu\nu} \), vanishes. Thus, the contribution of \( T_{s}^{\mu\nu} \) ensures the conservation of \( J^{\mu\nu} \).

To elucidate the meaning of (6.23), we express it in a component form, making use of the Proca equations (6.12), as

\[
\Theta^{00} = \frac{1}{8\pi} \left[E^2 + \frac{H^2}{\mu^2} + \frac{V^2(V^2 + A^2)}{\mu^2}\right]
\]

\[
\Theta^{0i} = \Theta^{00} = \frac{1}{4\pi} \left[(E \times H)_i + \mu^2 A_i V\right].
\]

Similar expressions for the massive vector field were obtained by Schrödinger [129]. At least, for a relatively loose coupling between the tensor and the potential fields, there will be no mixed terms in (6.27). As \( \mu \) is assumed to be small, the above expressions very closely resemble the ordinary Maxwell field.

There is one other point worth mentioning. Using the field equations (6.12), the linear momentum derived from the canonical tensor \( T^{\mu\nu} \) becomes

\[
g = \frac{1}{4\pi} \int \mathbf{V} \cdot \mathbf{A} d^3x.
\]

The corresponding expression, obtained from the symmetric tensor \( \Theta^{\mu\nu} \), takes the form

\[
g = \frac{1}{4\pi} \int \left[(\mathbf{A} \cdot \mathbf{V}) + (\mathbf{E} \times \mathbf{H})\right] d^3x.
\]

The integrated total angular momentum can be written, therefore, as

\[
\mathbf{J} = \frac{1}{4\pi} \int \mathbf{r} \times \left[(\mathbf{A} \cdot \mathbf{V}) + (\mathbf{E} \times \mathbf{H})\right] d^3x.
\]
Using standard vector relations \[ \mathbf{E} \times \mathbf{H} = \mathbf{B} \], the integrand of (6.30) assumes the form
\[ \mathbf{r} \times \left( \mathbf{E} \cdot \mathbf{A} \right) - \left( \mathbf{E} \cdot \mathbf{r} \right) \mathbf{A} + \mathbf{A} \left( \mathbf{E} \times \mathbf{r} \right). \] (6.31)

Now, in view of (6.28), the above is interpreted as the moment of the canonical momentum density and the spin momentum density. Consequently, substitution of (6.31) into (6.30) followed by an integration, results in
\[ \mathbf{J} = \frac{1}{4 \pi} \int \mathbf{E} \cdot \left( \mathbf{r} \times \mathbf{A} \right) d^3x + \frac{1}{4 \pi} \int \left( \mathbf{E} \times \mathbf{A} \right) d^3x, \] (6.32)
in accord with the relevant expression of Bopp [49].

This point, essentially, concludes the derivation of the field energy-momentum tensor for the neutral vector meson in vacuum. Thus, it is found that the total tensor is symmetric and consists of an "orbital" part, which determines the energy and momentum, and a "spin" part, which does not contribute to either, but is important in calculations of the total angular momentum.

A remark about the energy-momentum tensor (6.27) is in order here. The time component of the vector field \( \mathbf{v} \) vanishes in the case of a transverse field, as was noted in comments following (6.12). Therefore, its contribution to \( \mathbf{E} \cdot \mathbf{v} \) in (6.27) is nil. Moreover, the space-time component of \( \mathbf{B} \) is now \( \mathbf{E} \times \mathbf{H} \) and the wave is, therefore, nearly, pure transverse wave.

Hence, in accordance with the thesis of this work, the above conclusion must also apply to an electromagnetic field in an isotropic, collisionless plasma. Put otherwise, in order that a tensor have a physical meaning, it must be symmetric.

Next, it is well accepted [18, 23, 73] that the Minkowski tensor \( S^{\mu \nu} \) readily adjusts itself to the canonical formalism. Recall, further, that that tensor (similarly to the canonical tensor) is not symmetric. Thus, in analogy to the above discussion, the tensor \( S^{\mu \nu} = S_{\mu \nu} - S_{\nu \mu} \) ought to be interpreted as the spin energy-momentum tensor. Expressed covariantly, it becomes
\[ S^{\mu \nu} = (\epsilon \mu - 1) g^{\nu \sigma} \mathbf{q}_\sigma \] (6.33)
and, in the three dimensional form in the rest frame
\[ S^{0 0} = 0, \quad S^{0 i} = 0, \quad S^{ij} = \frac{\epsilon \mu - 1}{4 \pi} \mathbf{E} \times \mathbf{H}. \] (6.34)

It follows, from (6.34) that the only non-vanishing component of \( S^{\mu \nu} \) is the space-time one. Let us inquire, then, about its effect on the total tensor which, in view of the definition of \( S^{\mu \nu} \), is identical with the Abraham form \( S_{\mu \nu} \). Recalling the classical meson theory, presented above, we see that all high frequency phenomena can be approximately described by the Maxwell theory. Put otherwise, the characteristic differences between two theories become pronounced only in the low frequency limit.

A somewhat similar conclusion can be reached when investigating the propagation of the electromagnetic radiation in a medium. In the process, this medium becomes, generally speaking, magnetic. On a microscopic level, this effect corresponds to the spin of the field quanta. Furthermore, it is known [8] that the magnetic effects are much smaller than dielectric effects and that they belong, essentially, to a low frequency domain. Let us apply now this consideration to the force corresponding to (6.34). Thus, in the rest frame, the corresponding expression takes the form
\[ f^\mu = \frac{\epsilon \mu - 1}{4 \pi} \mathbf{E} \times \mathbf{H}. \] (6.35)

Now, recall that, in the low frequency domain, the value of the dielectric permittivity is considerably different from that of the vacuum. On the other hand, it is well accepted that the value of the permittivity, at high frequencies, asymptotically approaches unity, as a glance at (4.25) will reveal. Therefore, this behaviour also affects the magnitude of \( f^\mu \). Thus, it may be deduced that the magnitude of \( f^\mu \) will almost vanish at high frequencies, whilst it will increases for lower frequencies.

Although this work concentrated mostly on the case of a collisionless, isotropic plasma, nevertheless, the above interpretation is in excellent agreement with the experimental results obtained by using liquid and solid dielectrics. Thus, the conclusion reached here predicts a vanishing spin energy-momentum tensor (6.33) in the region of high frequencies. Therefore, in view of the above discussion, the field energy-momentum tensor is quite correctly given by the Minkowski theory, when the frequency is high. This conclusion is well supported by experiments [25, 26, 27]. At the other end of the frequency spectrum, (6.33) does not vanish and, thus, the field energy-momentum tensor is then a sum of \( S^{\mu \nu} \) and \( S^{\mu \nu} \) directly leading to the form put forward by Abraham. The experimental evidence supplied by [25, 29, 30] well supports this conclusion.

This concludes our discussion of the Proca theory of neutral vector mesons and of its applications to the problem of the electromagnetic energy-momentum tensor in a medium. In fact, we have, essentially, completed the task attempted in this work. It only remains for us to summarize and synthesize our findings, placing them in proper perspective. This will be done in the final chapter, where also some extensions for future work will be suggested.

VII. Conclusion

In the preceding chapters, we dealt with a few aspects of the interaction phenomena between the electromagnetic wave and a medium. Our main effort was devoted to deducing the correct expression for the field energy-momentum tensor in a medium. As discussed in Chapter III, many different suggestions had been previously put forward and it was shown that the emerging picture reflected a rather confused situation with no consensus of opinions as to which form of that tensor was correct. This was shown to be partly due to some very different interpretations being given to certain physical quantities.

In view of the complexity of the problem, it was considered necessary to make a few approximations, in order to elucidate the physics of the question at hand. Thus, the main effort was directed to the case of an isotropic, collisionless plasma, i.e., a collisionless plasma in the absence of an external magnetic field. Although this resulted in a rather artificial problem, it helped to clarify the distinction between the suggested tensors. Then, as a further step towards generalization, collisions could be included, as the above method is readily adaptable.

According to the present writer's knowledge, all relevant works, hitherto, deal with the interaction of the electromagnetic field with a medium whose refractive index is greater than unity. It was, therefore, considered essential to concentrate instead on interactions with media whose refractive index is smaller than unity. Although the most obvious example of such a medium is a plasma, we know that similar considerations also apply to a gas, liquid, or a solid, provided that the frequency of the incident radiation is very much larger than the characteristic frequency of a medium.
It proved illustrative to consider the radiation from both points of view — those of corpuscles and waves. The differentiation between these two aspects turned out to be more profound when radiation was inside a medium. Within the boundaries of geometrical optics approximations, it was found (in Chapter V) that the momentum of a photon in a plasma is given by the Minkowski expression. In the process, the method of Einstein, to investigate the interaction of radiation with matter, was expanded and used, in turn, to support the Minkowski result. It should be emphasized, however, that this conclusion is valid only in the region of high frequencies. There is some experimental evidence, viz., that of Jones and Richards [25], Jones and Leslie [29], and Ashkin and Dziedzic [27] which tends to support this finding.

To investigate the behavior of the electromagnetic field in a medium, the similarity of the relativistic form of the energy-momentum relation for a massive particle to the dispersion equation for the wave motion was invoked. Although the preliminary steps were introduced earlier [40, 43], the resulting neutral vector meson theory had not been previously applied to the problem of the form of the energy-momentum tensor. It should be emphasized that, although the theory itself is not new, the underlying philosophy is certainly entirely novel. Furthermore, it should be stressed that, throughout this work, the mass of the photon in vacuo is assumed to be zero. However, when the radiation field propagates in a medium, it may be represented by photons having non-zero rest mass. The coupling between the radiation field and a medium is responsible for the existence of the massive photons-like behavior.

Therefore, the meson theory, together with the symmetrization procedure of Bell-Penrose [42], was utilized to determine the form of the energy-momentum tensor. The work of Bopp [49] was instrumental in drawing an analogy between the tensors $\Theta_{ij}$ and $\Theta^*_{ij}$ of classical electrodynamics, on one hand, and those suggested by Minkowski and Abraham, on the other. It was then found that the Abraham form of the tensor is, generally, the correct one. The Minkowski tensor was definitely ruled out in the low frequency limit. This agrees with results of the work of Walker et al. [28], and Walker and Walker [29]. Further, the results obtained by Hakin and Higham [30], in the static limit, definitely supported the Abraham conclusion. (It is instructive to notice that in experiments [28, 29], the usual frequencies were in the range $5–100$ Hz and, thus, obviously, did not satisfy the WKB-condition). Although Shockley and James [42] did not actually conclude that the value of the momentum was that given by the Abraham expression, they, nevertheless, definitely excluded the Minkowski result, while using the frequency of $5$ kHz.

The formal equivalence of the meson theory in vacuo with the Maxwell theory in a medium was derived while utilizing the concept of a collisionless plasma. The first step towards generalization would be to apply the above method to the case of a collisional plasma. No real difficulty is envisaged and, it appears that, the extension would be straightforward. Further, although there is qualitative agreement with the above method to apply to any state of matter. Moreover, the interaction of a high density field, e.g., that of laser, with a medium would certainly warrant further investigation.

To summarize, the correct form for the field-energy momentum tensor has been given by Abraham. The Minkowski form for that tensor, on the other hand, appears to represent a fair approximation in the region of high frequencies, where the spin contributions may be neglected.

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