Mixed finite element methods for nonlinear equations: a priori and a posteriori error estimates

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Mixed finite element methods for nonlinear equations: a priori and a posteriori error estimates

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A priori error estimation provides information about the asymptotic behavior of the approximate solution and information on convergence rates of the problem. Contrarily, a posteriori error estimation derives the estimation of the exact error by employing the approximate solution and provides a practical accurate error estimation. Additionally, a posteriori error estimates can be used to steer adaptive schemes, that is to decide the refinement processes, namely local mesh refinement or local order refinement schemes. Adaptive schemes of finite element methods for numerical solutions of partial differential equations are considered standard tools in science and engineering to achieve better accuracy with minimum degrees of freedom.

In this thesis, we focus on a posteriori error estimations of mixed finite element methods for nonlinear time dependent partial differential equations. Mixed finite element methods are methods which are based on mixed formulations of the problem. In a mixed formulation, the derivative of the solution is introduced as a separate dependent variable in a different finite element space than the solution itself. We implement the H^1-Galerkin mixed finite element method (H1MFEM) to approximate the solution and its derivative. Two nonlinear time dependent partial differential equations are considered in this thesis, namely the Benjamin-Bona-Mahony (BBM) equation and Burgers equation. Our a posteriori error estimations are based on implicit schemes of a posteriori error estimations, where the error estimators are locally computed on each element. We propose a posteriori error estimates by using the approximate solution produced by H1MFEM and use the a posteriori error estimates to compute the local error estimators, respectively for the BBM and Burgers equations. Then, we prove that the introduced a posteriori error estimates are accurate and efficient estimations of the exact errors.

The last part of this study is on numerical studies of adaptive mesh refinement schemes for the two equations mentioned above. By implementing the introduced a posteriori error estimates, we propose adaptive mesh refinement schemes of H1MFEM for both equations.

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ABSTRACT

A priori error estimation provides information about the asymptotic behavior of the approximate solution and information on convergence rates of the problem. Contrarily, a posteriori error estimation derives the estimation of the exact error by employing the approximate solution and provides a practical accurate error estimation. Additionally, a posteriori error estimates can be used to steer adaptive schemes, that is to decide the refinement processes, namely local mesh refinement or local order refinement schemes. Adaptive schemes of finite element methods for numerical solutions of partial differential equations are considered standard tools in science and engineering to achieve better accuracy with minimum degrees of freedom.

In this thesis, we focus on a posteriori error estimations of mixed finite element methods for nonlinear time dependent partial differential equations. Mixed finite element methods are methods which are based on mixed formulations of the problem. In a mixed formulation, the derivative of the solution is introduced as a separate dependent variable in a different finite element space than the solution itself. We implement the H^1-Galerkin mixed finite element method (H1MFEM) to approximate the solution and its derivative. Two nonlinear time dependent partial differential equations are considered in this thesis, namely the Benjamin-Bona-Mahony (BBM) equation and Burgers equation. Our a posteriori error estimations are based on implicit schemes of a posteriori error estimations, where the error estimators are locally computed on each element. We propose a posteriori error estimates by using the approximate solution produced by H1MFEM and use the a posteriori error estimates to compute the local error estimators, respectively for the BBM and Burgers equations. Then, we prove that the introduced a posteriori error estimates are accurate and efficient estimations of the exact errors.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Subject of study

Disciplines such as engineering, physics, economics and biology involve many real models which are translated into solving mathematical models, e.g. partial differential equations in space and time. One of the well-known mathematical models is the boundary value problem (BVP). For example, a description of waves in electromagnetic and fluid dynamics is represented by a wave equation with specified boundary conditions, which is often stated as a boundary value problem.

Consistent with the application of the BVP in real problems, the number of numerical methods and analysis for solving the BVP is rapidly growing. In general, there is no closed form for the exact solution u of the BVP. Numerical methods as the finite difference method, the finite element method, the finite volume method, and spline interpolation are used as a tool to compute the approximate solution U_h of the exact solution u. Motivated by this situation, our study is considering the finite element method (FEM) for the boundary value problems.

During the approximation of the BVP, it is normal to question *"How good are the approximate solutions produced by the numerical methods? When should we stop the computation process and which of the approximate solutions should be taken as the best approximation to the real problem?"* In order to answer these questions, one way is to perform another approximation in a specified norm of the exact error e,

$$\|e\|_H = \|u - U_h\|_H \,,$$

i.e. by letting the exact error be approximated such as

$$\|E\|_H \approx \|e\|_H \,.$$

Extra concern should be put on deciding the methods to compute values of E, the error estimator. The computability and cost of computation are factors that are considered in deciding the efficiency of the error estimation.

In this study, we focus on a posteriori error estimation which is a method to compute the error estimator. Details about the a posteriori error estimation of finite element methods can be found in [6, 8] and the references therein. The a posteriori error estimation is based on a situation where we have the approximate solutions U_h which are generated by a FEM, then our aim is to obtain a quantitative estimate for the exact error e measured in a specified norm.

A posteriori error estimates provide useful indications of the accuracy of a calculation and provide a basis for adaptive mesh refinement schemes. We will give the details of a posteriori error estimation in Chapter 3.

1.2 Scope of study

This study focuses on a posteriori error estimation of a mixed finite element method (mixed FEM) for the Benjamin-Bona-Mahony equation and Burgers equation. The mixed FEM is a FEM which is based on a mixed formulation of the problem. In a mixed formulation, the derivative of the solution u is introduced as a separate dependent variable in a different finite element space than the solution itself. In this study, we implement the H^1-Galerkin mixed finite element method which is based on the procedure introduced by Pani [45]. Our scope of study can be categorized into three main parts.

The first part is devoted to a posteriori error estimation of the H^1-Galerkin mixed finite element method for the Benjamin-Bona-Mahony equation. The Benjamin-Bona-Mahony equation is a nonlinear equation which is widely used in modelling physical problems involving long waves. The Benjamin-Bona-Mahony equation is studied by Benjamin et al. as an alternative to the Korteweg-de Vries equation for describing unidirectional long dispersive waves [12].

Secondly, we study a posteriori error estimation of the H^1-Galerkin mixed finite element method for the Burgers equation. The Burgers equation is a well-known equation and named after Johannes Martinus Burgers [18, 19]. The Burgers equation is also known as a nonlinear diffusion equation, and a simplified version of Navier-Stokes equation. We will give details about the H^1-Galerkin mixed finite element method, the Benjamin-Bona-Mahony equation and the Burgers equation in the following chapter (Chapter 2).

The last part of this study is on adaptive schemes for two equations mentioned above. The a posteriori error estimates are known as a fundamental component in the designation of efficient adaptive algorithms for solving partial differential equations. By implementing the a posteriori error estimates introduced in the first two objectives, our third objective is on numerical studies of adaptive schemes for the Benjamin-Bona-Mahony equation and the Burgers equation.

1.3 Structure of thesis

This thesis consists of six chapters. Chapter 1 is the introduction. In Chapter 2, some important function spaces, theorems and results are reviewed. We complete Chapter 2 with an introduction for H^1-Galerkin mixed finite element method, the Benjamin-Bona-Mahony equation and the Burgers equation.

Chapter 3 is devoted to a general framework of a posteriori error estimation. In this chapter, we present some known a posteriori error estimation techniques and the procedure of a posteriori error estimation considered in this study.

In Chapter 4, we present the first contribution of the thesis which is a posteriori error estimation of H^1-Galerkin mixed finite element method for the Benjamin-Bona-Mahony equation. In this chapter, we propose some error estimators to compute the error estimation of the Benjamin-Bona-Mahony equation. We prove that the introduced a posteriori error estimates are accurate and efficient approximations of the exact errors.

We finish this chapter with some numerical experiments.

Our second contribution (Chapter 5) is a posteriori error estimation of H^1-Galerkin mixed finite element method for the Burgers equation. This chapter consists of analysis and numerical studies of a posteriori error estimation of H^1-Galerkin mixed finite element method for the Burgers equation.

Our third contribution (Chapter 6) is numerical studies of adaptive schemes for the Benjamin-Bona-Mahony and Burgers equations. We present the procedure of the adaptive schemes for both equations where the approximate solutions are computed by H^1-Galerkin mixed finite element method and the a posteriori error estimations are proposed in Chapter 4 and Chapter 5. We finish the chapter with numerical experiments.

Chapter 2

Preliminaries

This chapter provides a range of fundamental results which will be used in the remainder of the thesis. We begin by introducing some important function spaces. We then introduce the fundamental results for variational formulation of differential equations. Important theorems and results that will be used in the analysis of finite element methods are introduced in the next section. We finish this chapter with an introduction on finite element discretization, the H^1-Galerkin mixed finite element method, the Benjamin-Bona-Mahony equation and the Burgers equation, respectively in Section 2.5, Section 2.6 and Section 2.7.

2.1 Function spaces

All of the results stated in this section are well-known and can be found in different literatures; see e.g. [16, 53]. Since the equations we study in this thesis are posed in one spatial dimension, we mention only results for this case.

We let $\Omega = (a, b)$ be an open subset in $\mathbb{R}$ and u be a scalar function defined on Ω. For $p \in [1, \infty]$, the Lebesgue space $L^p(\Omega)$ is defined as

$$L^p(\Omega) = \{u : \Omega \to \mathbb{R} \big| \, \|u\|_{L^p(\Omega)} < \infty\}.$$

The $L^p(\Omega)$-norm is defined by

$$\|u\|_{L^p(\Omega)} := \left(\int_{\Omega} |u(x)|^p \, dx \right)^{1/p},$$

for $0 < p < \infty$, and

$$\|u\|_{L^\infty(\Omega)} := \inf\{C \geq 0 : |u(x)| \leq C \quad \text{for almost all} \quad x \in \Omega\}$$

for $p = \infty$. A special role is taken when $p = 2$. The $L^2(\Omega)$ is a Hilbert space with the inner product

$$\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u(x)v(x)\,dx \quad u, v \in L^2(\Omega)$$

and the norm $\|u\|_{L^2(\Omega)}$.

Let $\mathbb{C}^m(\Omega)$ be the space of all functions $\phi : \Omega \longrightarrow \mathbb{R}$ such that $\phi, \phi', \cdots, \phi^{(m)}$ are continuous on Ω. The space $\mathbb{C}_0^m(\Omega)$ denoted the space of all functions $\phi \in \mathbb{C}^m$ such that $\phi(x) = 0$ for all $x \in \Omega_0$ for some bounded subset Ω_0 of Ω.

We recall the definition of derivative in a weak sense. A function $u \in L^p(\Omega)$ is called the weak derivative of order $m = 1, 2, 3, \dots$ of a function $v \in L^p(\Omega)$ if

$$\int_\Omega u(x)\phi(x)\,dx = (-1)^m \int_\Omega v(x)\phi^{(m)}(x)\,dx \quad \forall \phi \in \mathbb{C}_0^m(\Omega).$$

In the following part, we recall the Sobolev spaces and norms to be used in this thesis. The Sobolev space $W_p^k(\Omega)$, $1 \leq p < \infty$ and $k = 1, 2, \dots$ is defined as

$$W_p^k(\Omega) := \{u \in L^p(\Omega) : u', u'', \cdots, u^{(k)} \text{exist in the weak sense}\}$$

and $W_p^k(\Omega)$ norm is defined by

$$\|u\|_{W_p^k(\Omega)} := \left(\sum_{i=0}^k \left\|u^{(i)}\right\|_{L^p(\Omega)}^p\right)^{1/p}.$$

When $p = 2$, we have $W_2^k(\Omega) = H^k(\Omega)$, which is a Hilbert space equipped with the inner product

$$\langle u, v\rangle_{H^k(\Omega)} := \int_\Omega \left(uv + u'v' + \cdots + u^{(k)}v^{(k)}\right)\,dx \quad \forall u, v \in H^k(\Omega),$$

and norm

$$\|u\|_{H^k(\Omega)} := \left(\|u\|_{L^2(\Omega)}^2 + \left\|u'\right\|_{L^2(\Omega)}^2 + \cdots + \left\|u^{(k)}\right\|_{L^2(\Omega)}^2\right)^{1/2} \quad \forall u \in H^k(\Omega).$$

The space $H_0^k(\Omega)$ contains all functions in $H^k(\Omega)$ whose traces are zero at a and b.

In the case $p = \infty$, $W_\infty^k(\Omega)$ norm is defined by

$$\|u\|_{W_\infty^k(\Omega)} := \max_{0 \leq i \leq k} \left\|u^{(i)}\right\|_{L^\infty(\Omega)}.$$

2.2 Notations

In the remaining part of the thesis, we use the following notations for the norm spaces and inner products. For any $p \in [1, \infty]$ and any normed vector space D, by $L^p(D)$ we denote the space $L^p(0, T; D)$ of all functions defined in $[0, T]$, with values in D. We will write $\|\cdot\|_{L^p(L^\infty)}$ and $\|\cdot\|_{L^p(H^1)}$ instead of $\|\cdot\|_{L^p(L^\infty(D))}$ and $\|\cdot\|_{L^p(H^1(D))}$. We will also write $H^0(D) = L^2(D)$. The $H^n(D)$ norm, for $n = 0, 1, \ldots$ is represented by $\|u(t)\|_n$ instead of $\|u(t)\|_{H^n(D)}$. Similarly, we will write $\|u\|_{W^1_\infty(H^n)}$ instead of $\|u\|_{W^1_\infty(0,T;H^n(D))}$.

In general, the inner product in $H^s(X)$ is denoted by $\langle \cdot, \cdot \rangle_{H^s(X)}$, where $s = 0, 1, \ldots$ and X is a subset of $\mathbb{R}$. In particular, when $s = 0$ and $X = \Omega$ we write $\langle \cdot, \cdot \rangle_0$ instead of $\langle \cdot, \cdot \rangle_{H^0(\Omega)}$. When $s = 1$ and $X = \Omega$, we write $\langle \cdot, \cdot \rangle_1$ instead of $\langle \cdot, \cdot \rangle_{H^1(\Omega)}$.

Besides that, when there is no confusions we omit the dependence of the function on t to avoid crowded notations. For example, we write $\langle u, v \rangle_s$ instead of $\langle u(t), v(t) \rangle_{H^s(\Omega)}$. Finally, for $l > 0$, we define the local inner product in $H^s(\Omega_l)$ by

$$\langle u, v \rangle_{s,\Omega_l} = \int_{\Omega_l} u(x)v(x)dx \quad \forall u, v \in H^s(\Omega_l). \tag{2.2.1}$$

2.3 Important theorems and results

The following well-known results will be frequently used. They are recalled here for the reader's convenience.

Theorem 2.3.1 (Imbedding Theorem [16, Theorem 1.4.6]). *Let k be a positive integer and p be a real number in the range $1 \le p < \infty$ such that*

$$k \ge 1 \quad \textit{when} \quad p = 1$$
$$k > 1/p \quad \textit{when} \quad p > 1.$$

Then there is a constant C such that for all $u \in W^k_p(\Omega)$

$$\|u\|_{L^\infty(\Omega)} \le C \|u\|_{W^k_p(\Omega)}.$$

Lemma 2.3.2 (Gronwall's Lemma [11, Theorem 4.2] or [28]). *Let φ, ψ and θ be locally integrable functions defined on $[0, T]$ which satisfy*

$$\theta(t) \ge 0 \quad \textit{and} \quad \varphi(t) \le \psi(t) + \int_0^t \theta(s)\varphi(s)\, ds \qquad \forall t \in [0, T].$$

Then

$$\varphi(t) \leq \psi(t) + \int_0^t \theta(s)\psi(s)\exp\left[\int_s^t \theta(r)dr\right] ds \quad \forall t \in [0,T].$$

If ψ *is a constant, then*

$$\varphi(t) \leq \psi \exp\left[\int_0^t \theta(s)\, ds\right].$$

Lemma 2.3.3 (General Gronwall's Lemma [11, Theorem 4.3] or [13, Section 3])**.** *If* β *is a positive constant and* θ *is a non-decreasing function satisfying* $\theta(s) > 0$ *for* $s > 0$, *then the inequality*

$$\varphi(t) \leq \beta + \int_0^t \theta[\varphi(\tau)]\, d\tau \quad \forall t \in [0,T]$$

implies

$$\varphi(t) \leq \Theta^{-1}(t) \quad \forall t \in [0,T^*]$$

where Θ^{-1} *is the inverse of*

$$\Theta(\sigma) = \int_\beta^\sigma \frac{ds}{\theta(s)}, \quad \sigma \geq 0,$$

and $T^* = \min(T,T_1)$ *with* $[0,T_1]$ *being the range of* Θ.

Proof. Let

$$\psi(t) = \beta + \int_0^t \theta[\varphi(\tau)]\, d\tau.$$

Then from $\varphi(t) \leq \psi(t)$ and the monotonicity of θ we deduce

$$\frac{\psi'(t)}{\theta[\psi(t)]} = \frac{\theta[\varphi(t)]}{\theta[\psi(t)]} \leq 1.$$

This implies

$$\frac{d}{dt}\Theta[\psi(t)] \leq 1.$$

By integrating from 0 to t and noting that $\Theta[\psi(0)] = 0$, we obtain

$$\Theta[\psi(t)] \leq t.$$

Now if $t \in [0,T^*]$ then by applying the inverse function Θ^{-1}, we obtain $\psi(t) \leq \Theta^{-1}(t)$, and thus the required inequality follows from $\varphi(t) \leq \psi(t)$. □

2.4 Finite element discretization

In this section, we introduce the hierarchical basis functions of finite element spaces used in the thesis. We partition the interval $\Omega = (a, b)$ into

$$a = x_1 < x_2 < \cdots < x_{N+1} = b, \tag{2.4.1}$$

and let $h_l := x_{l+1} - x_l$, $l = 1, \ldots, N$, and $h := \max_l h_l$. We define the linear basis functions by using the hat functions ϕ_{l1} on (x_{l-1}, x_{l+1}) for $l = 2, \ldots, N$, i.e.,

$$\phi_{l1}(x) = \begin{cases} \frac{x - x_{l-1}}{h_{l-1}}, & x_{l-1} \le x < x_l, \\ \frac{x_{l+1} - x}{h_l}, & x_l \le x < x_{l+1}, \\ 0, & \text{otherwise.} \end{cases}$$

At the endpoints of Ω we define

$$\phi_{11}(x) = \begin{cases} \frac{x_2 - x}{h_1}, & x_1 \le x < x_2, \\ 0, & \text{otherwise,} \end{cases} \tag{2.4.2}$$

and

$$\phi_{N+1,1}(x) = \begin{cases} \frac{x - x_N}{h_N}, & x_N \le x < x_{N+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.3}$$

For $l = 1, \ldots, N$ and $k = 2, 3, 4, \ldots$, functions ϕ_{lk} are defined as antiderivatives of the Legendre polynomials P_{k-1} of degree $k - 1$ scaled to the subinterval $[x_l, x_{l+1}]$, i.e.,

$$\phi_{lk}(x) = \begin{cases} \frac{\sqrt{2(2k-1)}}{h_l} \int_{x_l}^{x} P_{k-1}(y)\, dy, & x_l \le x < x_{l+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.4}$$

Figure 2.4 shows functions $\phi_{l,k}$ of degree $k = 2, \ldots, 5$ on the reference element $[-1, 1]$.

Let $\mathcal{S}_h$ be the space of piecewise linear functions on Ω i.e.,

$$\mathcal{S}_h := \text{span}\ \{\phi_{11}, \phi_{21} \ldots, \phi_{N+1,1}\},$$

and $\mathring{\mathcal{S}}_h$ its subspace consisting of functions vanishing at a and b, i.e.,

$$\mathring{\mathcal{S}}_h := \text{span}\ \{\phi_{21}, \ldots, \phi_{N,1}\}.$$

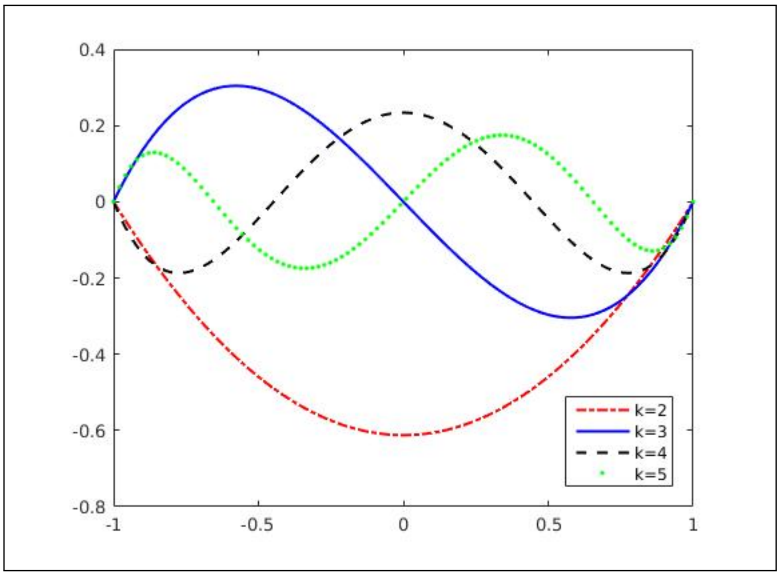

Figure 2.1: Hierarchical shape functions of degrees 2 $(-\cdot)$, 3 (solid), 4 $(--)$ and 5 $(\cdot\cdot)$ on reference element $[-1, 1]$.

The spaces of bubble functions in Ω are defined by

$$\mathcal{S}_h^k := \text{span } \{\phi_{1k}, \dots, \phi_{Nk}\}, \quad k \geq 2,$$

where, for $l = 1, \dots, N$ and $k = 2, 3, 4, \dots$, ϕ_{lk} is defined by (2.4.4).

For $p \in \mathbb{N}$ and $p \geq 2$, let $\mathcal{V}_h^p$ and $\mathring{\mathcal{V}}_h^p$ be finite dimensional subspaces of $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively, defined by

$$\mathcal{V}_h^p := \mathcal{S}_h + \sum_{k=2}^{p} \mathcal{S}_h^k, \quad \text{and} \quad \mathring{\mathcal{V}}_h^p := \mathring{\mathcal{S}}_h + \sum_{k=2}^{p} \mathcal{S}_h^k. \tag{2.4.5}$$

With $\chi_h \in \mathring{\mathcal{V}}_h^p$ and $w_h \in \mathcal{V}_h^p$, we have the following approximation properties

$$\inf_{\chi_h \in \mathring{\mathcal{V}}_h^p} \left\{ \|u - \chi_h\|_0 + h \|\partial_x(u - \chi_h)\|_0 \right\} \leq Ch^{p+1} \|u\|_{p+1} \quad \forall u \in H_0^1(\Omega) \cap H^{p+1}(\Omega) \tag{2.4.6}$$

and

$$\inf_{w_h \in \mathcal{V}_h^p} \left\{ \|v - w_h\|_0 + h \|\partial_x(v - w_h)\|_0 \right\} \leq Ch^{p+1} \|v\|_{p+1} \quad \forall v \in H^{p+1}(\Omega). \tag{2.4.7}$$

2.5 H^1-Galerkin mixed finite element method

The mathematical analysis and applications of mixed FEM have been widely developed since decades ago. For example, a general analysis for this kind of methods is studied by Brezzi [17]. A mixed FEM is a type of FEM which is based on a mixed formulation of the problem. The mixed FEM is originally considered for problems where there are possibilities of having numerical ill-posedness if discretized by using the normal FEM. An example of such problems is computation of stress and strain fields in an almost incompressible elastic body. Besides that, the mixed FEM is also applied for cases where we have to discretize the gradient of the solution. The need to approximate the gradient of the solution is originated from solid mechanics problems which require more accurate approximations of certain derivatives of the displacement [16]. The mathematical elements of classical mixed FEM can be found in the books on mathematical theory of FEM [16, 53].

By using mixed FEM, the original problem is reformulated into a problem of two bilinear forms and two finite element spaces. As an example we consider the following one dimensional parabolic partial differential equation:

$$\partial_t u(x,t) - \partial_{xx} u(x,t) = f(x,t), \quad x \in \Omega = (0,1), \quad t \in (0,T], \quad T < \infty, \tag{2.5.1}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in [0,T], \tag{2.5.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{2.5.3}$$

By using a mixed formulation, the derivative of the solution u is introduced as a second unknown. The second order problem is reformulated into a system of first order equations having the form

$$v(x,t) = \partial_x u(x,t), \tag{2.5.4}$$

$$\partial_t u(x,t) - \partial_x v(x,t) = f(x,t) \tag{2.5.5}$$

with boundary condition (2.5.2) and initial condition (2.5.3). We note that, with $\alpha \in H^1(\Omega)$ and $\beta \in H^0(\Omega)$ are arbitrary test functions, the solution $(u, v) \in H^0(\Omega) \times H^1(\Omega)$ also solves the weak formulation

$$\langle v(t), \alpha \rangle_0 = - \langle u(t), \partial_x \alpha \rangle_0 \quad \forall \alpha \in H^1(\Omega) \tag{2.5.6}$$

$$\langle \partial_t u(t), \beta \rangle_0 - \langle \partial_x v(t), \beta \rangle_0 = \langle f(t), \beta \rangle_0 \quad \forall \beta \in H^0(\Omega). \tag{2.5.7}$$

It should be noted that the boundary condition $u = 0$ (see (2.5.2)) is implicitly contained in the formulation (2.5.6)–(2.5.7). Using integration by parts in (2.5.6), we have

$$\langle v, \alpha \rangle_0 = - \langle u, \partial_x \alpha \rangle_0 = \langle \partial_x u, \alpha \rangle_0 \quad \forall \alpha \in H^1(\Omega),$$

and hence, formally, $v = \partial_x u$ in Ω and $u = 0$ at the endpoints of Ω. Since $v = \partial_x u$ from (2.5.6), noting that $\partial_x v \in H^0(\Omega)$ and taking $\beta = \partial_t u - \partial_x v - f \in H^0(\Omega)$ in (2.5.7), we have (2.5.1).

This way of mixed formulation needs two finite dimensional spaces $\mathcal{W} \subset H^0(\Omega)$ and $\mathcal{V} \subset H^1(\Omega)$ which are required to satisfy inf-sup condition or Ladyzhenskaya-Babuška-Brezzi (LBB) condition to have a stable numerical scheme. Details of the mixed formulation by classical mixed FEM for a general parabolic partial differential equation can be found in [30].

In this study, we implement a mixed FEM called H^1-Galerkin mixed finite element method (H1MFEM) which is based on an approach suggested by Pani et. al for nonlinear parabolic equations and second order hyperbolic equations [45, 46]. The H1MFEM is closely related to least square mixed methods in that the second order partial differential equation is reformulated into a system of first order partial differential equations with a new unknown defined as the flux. Studies on the least square mixed finite element method can be found in [20, 21, 49, 48] and the references therein. By using the H1MFEM, a problem is reformulated into a system of first order partial differential equations, which allows the approximation for u and its gradient v.

As an example, we consider parabolic partial differential equation (2.5.1)–(2.5.3). Using the H1MFEM, equation (2.5.1) is reduced to a system of first order equations by defining a new variable $v = \partial_x u$. As a consequence, we have (2.5.4)–(2.5.5). By multiplying (2.5.4) by $\partial_x \alpha$ and (2.5.5) by $-\partial_x \beta$ where $\alpha \in H_0^1(\Omega)$ and $\beta \in H^1(\Omega)$ we have

$$\langle v, \partial_x \alpha \rangle_0 = \langle \partial_x u, \partial_x \alpha \rangle_0 \quad \forall \alpha \in H_0^1(\Omega) \tag{2.5.8}$$

and

$$\langle \partial_t v, \beta \rangle_0 + \langle \partial_x v, \partial_x \beta \rangle_0 = -\langle f, \partial_x \beta \rangle_0 \quad \forall \beta \in H^1(\Omega). \tag{2.5.9}$$

For the first term in (2.5.9), we have used integration by parts and the Dirichlet boundary conditions $\partial_t u(0,t) = \partial_t u(1,t) = 0$.

The weak formulation by H1MFEM is formulated as: Given $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, find $(u,v) : [0,T] \to H_0^1(\Omega) \times H^1(\Omega)$, satisfying for $t > 0$

$$\langle v(t), \partial_x \alpha \rangle_0 = \langle \partial_x u(t), \partial_x \alpha \rangle_0 \quad \forall \alpha \in H_0^1(\Omega) \tag{2.5.10}$$

$$\langle \partial_t v(t), \beta \rangle_0 + \langle \partial_x v(t), \partial_x \beta \rangle_0 = -\langle f(t), \partial_x \beta \rangle_0 \quad \forall \beta \in H^1(\Omega) \tag{2.5.11}$$

and for $t = 0$,

$$\langle v(0), \beta \rangle_0 = \langle \partial_x u_0, \beta \rangle_0 \quad \forall \beta \in H^1(\Omega). \tag{2.5.12}$$

If $u \in W_\infty^1(0,T; H_0^1(\Omega) \cap H^2(\Omega))$, $v \in W_\infty^1(0,T; H^1(\Omega))$ and (u,v) satisfies (2.5.10)–(2.5.11) then (u,v) satisfies (2.5.4)–(2.5.5). Indeed, by using integration by parts we deduce from (2.5.10) that $\partial_x(v - \partial_x u) = 0 \in W_\infty^1(0,T; H^0(\Omega))$, which implies

$$v(x,t) = \partial_x u(x,t) + g(t) \quad \text{a.e. in} \quad \Omega \times (0,T) \tag{2.5.13}$$

for some function g depending on t. We note that we also have

$$v(x,0) = \partial_x u(x,0) + g(0).$$

By integrating over Ω, noting (2.5.12), we infer $g(0) = 0$. On the other hand, it follows from (2.5.13) and (2.5.11) (with $\beta = 1$) that

$$\int_\Omega \partial_{tx} u + g'(t) = 0, \tag{2.5.14}$$

implying $g'(t) = 0$. Hence $g \equiv 0$, i.e. (u,v) satisfies (2.5.4). This immediately gives (2.5.5).

Some of the attractive features of the H1MFEM are firstly this method does not require the LBB condition. Secondly, finite element spaces of u and v are allowed to be of different polynomial degrees. For example, by using the H1MFEM, the approximate solutions U_h and V_h of the finite element spaces $\mathring{\mathcal{V}}_h^p$ and $\mathcal{V}_h^q$ (see (2.4.5)) are allowed to be of different polynomial degrees, i.e. we can have different values of p and q where $p, q \geq 1$. Thirdly, this procedure required extra regularity of the solution which gives a better order of convergence for v, in $H^0(\Omega)$ norm [45]. For one dimensional cases, the orders of convergence obtained by H1MFEM are

$$\|u - U_h\|_1 \leq Ch^{\min(p,q+1)} \tag{2.5.15}$$

and

$$\|v - V_h\|_1 \leq Ch^{\min(p+1,q)}, \tag{2.5.16}$$

which are comparable with results generated by a classical mixed FEM. Details of the mixed formulation by the H1MFEM for a general parabolic partial differential equation can be found in [45].

In 2007, the H1MFEM is adapted for a priori error estimation of the Burgers equation [47]. Besides that, Tripathy et. al studied on the superconvergence properties of the H1MFEM for second order elliptic equations [56]. Recently, Zhang et. al studied the H1MFEM with the linearised Crank-Nicolson for couple BBM equations [59].

In this thesis, we are interested in a posteriori error estimations of the H1MFEM for the BBM and Burgers equations. Mixed finite element methods allow approximation to the solution of the BBM and Burgers equations and its derivative, by reformulating the BBM and Burgers equations into a system of first order equations. Therefore, instead of dealing with second order nonlinear partial differential equations, the problem is reformulated and the computation is less hard compared to the approximation by using a normal finite element method. Mixed finite element methods give better orders of convergence for the unknown derivative by requiring extra regularity of the unknown. To the best of our knowledge, this is the first time the procedure of a posteriori error estimation in this study (to be explained in Chapter 3) is applied to the BBM and Burgers equations, where the approximate solutions are computed by using the H1MFEM.

2.6 The Benjamin-Bona-Mahony equation

The Benjamin-Bona-Mahony (BBM) equation

$$\partial_t u(x,t) - \partial_{xxt} u(x,t) + u(x,t)\partial_x u(x,t) + \partial_x u(x,t) = 0, \tag{2.6.1}$$

where $\partial_t = \partial/\partial t$, $\partial_{xxt} = \partial^3/\partial x^2 \partial t$ and $\partial_x = \partial/\partial x$ is studied by Benjamin et al., with $u(x,t)$ being considered in a class of real nonperiodic functions defined for $-\infty < x < \infty$ and $t \geq 0$ [12]. The BBM equation is studied in flows of fluid. Examples where the BBM equation is used are acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma and acoustic waves in anharmonic crystal.

The BBM equation is studied as an alternative and improvement of the Korteweg-de Vries (KdV) equation

$$\partial_t u(x,t) + \partial_{xxx} u(x,t) + u(x,t)\partial_x u(x,t) + \partial_x u(x,t) = 0, \tag{2.6.2}$$

particularly for describing unidirectional long dispersive waves. In general, the KdV model in physical science and engineering has difficulty with the dispersion ratio; a ratio of dispersion's effect in a medium, when a wave is travelling within the medium. The dispersion term $\partial_{xxx} u$ in the KdV model has a tendency to emphasise the short-wave components which is unnatural with respect to the original physical problem. The dispersion relation $\partial_{xxt} u$ in the BBM model overcomes this difficulty by giving a bounded dispersion relation [37]. Besides that, modelling with the BBM equation also overcomes the stability problem with high wave number components in the KdV model.

Details on the uniqueness and stability of the BBM model for long waves in nonlinear dispersive systems can be found in [12]. The existence and uniqueness of (2.6.1) and its non-homogeneous form are studied by Benjamin et al. Besides that, the uniqueness, global existence and continuous dependence of solutions on initial and boundary data for model equation (2.6.1) with an additional term $-\partial_{xx} u$ are studied by Bona and Dougalis [14]. Another general case of the BBM equation, namely

$$\partial_t u(x,t) - \partial_{xxt} u(x,t) + \partial_x f(u) = g(x,t) \tag{2.6.3}$$

where $f \in C^1(\mathbb{R})$ and $g \in L^\infty(0,T; L^2(0,1))$, is studied by Medeiros and Miranda [39]. They prove existence, uniqueness and regularity of (2.6.3). The BBM equation is also studied for periodic solutions (periodic with respect to the x variable) [23], [38]. For higher dimensions, a study on the existence, uniqueness and regularity is conducted by Goldstein et. al [27].

Since decades ago, initial boundary value problems for various generalized BBM equations have been studied. For example, in [32], a linearised method which is based on a differential quadrature method is studied as a new method to approximate the BBM equation on a semi-infinite interval. A linearised Crank-Nicolson H1MFEM is studied for coupled BBM equations in [59]. Besides that, a numerical study on the BBM equation with a mixed FEM (differently from the method studied in Chapter 4 of this thesis) can be found in [33]. In this study, we are interested on a posteriori error estimation for the BBM equation, where the approximate solutions are computed by using the H1MFEM.

2.7 The Burgers equation

The Burgers equation

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = \nu \partial_{xx} u(x,t) \tag{2.7.1}$$

is a fundamental one dimensional nonlinear partial differential equation occurring in various areas of mathematical modelling, particularly in mathematical models of turbulence and shock wave theory. Solution $u(x,t)$ can be considered as a quantity of a velocity for space x and time t. The value of ν is a small parameter known as a viscosity coefficient of the fluid motion, which is related to the Reynolds number $R = \frac{1}{\nu}$. The Burgers model has been studied as the simplest form of nonlinear advection term $u\partial_x u$ and dissipation term $\nu\partial_{xx} u$ for simulating the physical phenomena of wave motions.

Since decades ago, the Burgers model became an interest of researchers due to the tendency of a steep gradient (shocks) which almost becomes discontinuous when the viscosity coefficient $\nu = 0$ in (2.7.1) i.e.

$$\partial_t u(x,t) + u(x,t)\partial_x u(x,t) = 0. \tag{2.7.2}$$

Equation (2.7.2) is also known as inviscid Burgers equation.

Because of the nonlinear convection term and the occurrence of the viscosity term, the Burgers equation (2.7.1) shows similar features with Navier-Stokes equation and it is viewed as the simplified version of the Navier-Stokes equation. Due to the complexity in obtaining the analytical solutions, many researchers have used numerical methods as a tool to approximate the solution, e.g. finite element methods and spline interpolation.

In 1950, Hopf and Cole introduced a method to solve (2.7.1), which is known as a Hopf-Cole transformation [29]. By the Hopf-Cole transformation, a new dependent variable $w(x,t)$ is introduced such that

$$u(x,t) = -2\nu \left(\frac{\partial_x w(x,t)}{w(x,t)} \right).$$

Then, the nonlinear Burgers equation (2.7.1) is transformed to a linear heat equation

$$\partial_t w(x,t) = \nu \partial_{xx} w(x,t).$$

Since the heat equation is explicitly solvable in terms of the so-called heat kernel, then the general solution of the Burgers equation can be obtained. There are many numerical studies conducted which are relied on the Hopf-Cole transformation of the Burgers equation, e.g. [43, 44].

There are many studies have been done on the numerical methods for the Burgers equation. Numerical studies of Burgers equation by FEM can be found in [4, 25, 43]. A series of study on application of FEM and spline in approximating the Burgers equation can be found in [5, 31, 44, 60] and the references therein. Some studies on the a posteriori error estimations for the Burgers equation are studied by Patera et.al [41, 50].

Considering the importance of the Burgers equation as a mathematical model of turbulence and shock wave theory and a simplified model to study the Navier-Stokes equation, we are interested to study the error estimation of the Burgers equation. In this study, we focus on approximation of the Burgers equation without the Hopf-Cole transformation. We first implement the H1MFEM to compute the approximate solution of the Burgers equation. Secondly, we design a posteriori error estimation for the Burgers equation, using the approximate solution produced by the H1MFEM.

Chapter 3

A posteriori error estimation: a general framework

In this chapter, we give the general framework of a posteriori error estimation in finite element analysis. We begin this chapter with an introduction about a posteriori error estimation, and properties of a good a posteriori error estimation. Section 3.2 is devoted to discussing some known a posteriori error estimation techniques. We finish this chapter with a discussion on a posteriori error estimation for time dependent partial differential equations in Section 3.3.

3.1 Introduction

Error estimation of finite element solutions has been studied since the late 1970s [6]. A priori error estimation gives information about the asymptotic behavior of the approximation. With a priori error estimates, we obtain information on convergence rates of the problem. This is not enough to give a quantitative error information of the problem.

On the other hand, a posteriori error estimation derives the estimation of the exact error by employing the approximate solution and provides a practical accurate estimation. Additionally, a posteriori error estimates can be used to steer adaptive schemes, that is to decide the refinement processes, namely local mesh refinement or local order refinement schemes. After having computed an approximate solution, it is possible to obtain a posteriori error estimates which give quantitative information about the accuracy of the solution. A posteriori error estimation of finite element solutions for one dimensional problems have been first studied by Babuška et al. in 1981 [8]. Babuška et al. developed the theory for a posteriori error estimates under any L^p-energy norm for $2 \leq p \leq \infty$.

In general, a good a posteriori error estimation should have several properties. Firstly, an error estimator should be accurate in the sense that it is close to the exact error. Secondly, the error estimation should be applicable for a wide range of mesh spacings and different polynomial degrees. Thirdly, the procedure of computing the error estimators should be inexpensive relative to the cost of computing the approximate solutions. This means, the error estimator should be computed *locally* on each element. The localization of the error estimate equations to the element level is a key step in reducing the computation cost in dealing with the global problem. This theme is applied in the derivation of all a posteriori error estimates. Lastly, an error estimation should be possible to be applied in the refinement process in adaptive schemes. Error estimators which are computed *locally* for an element provide an indication about where approximation accuracy is insufficient and where the refinement process should be applied.

3.2 Techniques of a posteriori error estimation

In this section, we mention some of the known a posteriori error estimations. The materials contained in this section are mainly taken from the book by M. Ainsworth and J. T. Oden [3].

In order to discuss the techniques of a posteriori error estimation, we consider a one dimensional model elliptic boundary value problem of finding the solution u of

$$-u''(x) + u(x) = f(x) \quad x \in \Omega = (0, 1), \tag{3.2.1}$$

where $f \in H^0(\Omega)$ and $u = 0$ at the endpoints of Ω. A weak formulation of the problem is to find $u \in H_0^1(\Omega)$ such that

$$\mathscr{B}(u, \alpha) = \langle f, \alpha \rangle_0 \quad \forall \alpha \in H_0^1(\Omega),$$

where $\mathscr{B}(\cdot,\cdot)$ represents the $H^1(\Omega)$-inner product.

Let $U_h \in \mathring{\mathcal{V}}_h^p$ be a finite element approximation of u such that

$$\mathscr{B}(U_h, \alpha_h) = \langle f, \alpha_h \rangle_0 \quad \forall \alpha_h \in \mathring{\mathcal{V}}_h^p.$$

By noting the weak formulation and the finite element approximation, the residual equation of the exact error $e = u - U_h$ satisfies

$$\mathscr{B}(e, \alpha) = \langle f, \alpha \rangle_0 - \mathscr{B}(U_h, \alpha) \quad \forall \alpha \in H_0^1(\Omega). \tag{3.2.2}$$

The first two techniques are explicit and implicit schemes of a posteriori error estimation. The third technique of a posteriori error estimation is the recovery methods. And finally, the forth technique is a posteriori error estimation which is based on hierarchical bases.

Explicit schemes

Explicit schemes involve a direct computation using available data in which the error estimators are computed directly from the finite element approximations. As an example, the residual equation (3.2.2) of the exact error is decomposed into local contributions from each element l for $l = 1, \ldots, N$, namely

$$\mathscr{B}(e, \alpha) = \sum_{l=1}^{N} \Big\{ \langle f, \alpha \rangle_{0,\Omega_l} - \mathscr{B}_l(U_h, \alpha) \Big\} \quad \forall \alpha \in H_0^1(\Omega),$$

where $\mathscr{B}_l(\cdot,\cdot)$ represents elementwise of the bilinear form, on Ω_l. Let U_h' and U_h'' be the piecewise first order and second order derivatives of U_h. By using integration by parts on $\langle U_h', \alpha' \rangle_{0,\Omega_l}$ of $\mathscr{B}_l(U_h, \alpha)$ and rearranging the terms, we have

$$\mathscr{B}(e, \alpha) = \sum_{l=1}^{N} \langle R, \alpha \rangle_{0,\Omega_l} \quad \forall \alpha \in H_0^1(\Omega)$$

where $R = f + U_h'' - U_h$ is the element residual. Recalling the Galerkin orthogonality condition

$$\mathscr{B}(e, \alpha_h) = 0 \quad \forall \alpha_h \in \mathring{\mathcal{V}}_h^p$$

and introducing the interpolation operator $\Pi_h : H_0^1 \to \mathring{\mathcal{V}}_h^p$, we have

$$\mathscr{B}(e, \alpha) = \sum_{l=1}^{N} \langle R, \alpha - \Pi_h \alpha \rangle_{0,\Omega_l} \quad \forall \alpha \in H_0^1(\Omega).$$

Then, by using the Cauchy Schwarz inequality elementwise and

$$\|\alpha - \Pi_h \alpha\|_{H^0(\Omega_l)} \leq C h_l \|\alpha\|_{H^1(\Omega_l)}$$

with h_l is the diameter of Ω_l, the error estimate is computed such that

$$\|e\|_{H^1(\Omega)} \leq \left\{ \sum_{l=1}^{N} C h_l^2 \|R\|_{H^0(\Omega_l)}^2 \right\}^{1/2} . \tag{3.2.3}$$

Explicit estimators are derived originally by Babuška and Rheinboldt for problem posed in one space dimension [7]. Babuška and Rheinboldt show the existence of unique mesh distribution for a two point boundary value problem by using the explicit a posteriori error estimates. Explicit schemes generally require less computational effort but involving compromises in robustness and utility as a way to have an accurate and quantitative error estimation [3].

Implicit schemes

On the other hand, implicit schemes involve the solution of an algebraic system of equations. The interest in implicit schemes results from the fact that in explicit schemes the whole information for the total error is obtained only from the computed solution, when it might be possible to obtain more accurate information on the error by solving additional auxiliary problems. It is known as implicit error estimator because the approximation of the error must be solved over each element to determine the error estimator. The global problem is replaced by sequence of uncoupled local boundary value problems which are posed either over a single element (element residual method) or over a small patch of elements (subdomain residual method). The error estimator is obtained by evaluating the norms of the solutions of the local problems and summing the local approximations over the elements.

In general, the subdomain residual method involves a decomposition process of the global error equation into a number of local residual problems on small element patches with homogeneous Dirichlet boundary conditions. Consider this method in our model problem for a mesh of N nodes. Let $\psi_i \in \mathring{\mathcal{V}}_h^p$ be the shape function corresponding to node i. These functions are characterized by the conditions

$$\psi_i(x_j) = \delta_{ij},$$

where x_j is any of the N nodes and

$$\sum_{i=1}^{N} \psi_i(x) = 1 \quad x \in \bar{\Omega}. \tag{3.2.4}$$

We introduce on each patch of elements, $\tilde{\Omega}_i$ as the support of node function ψ_i containing x_i. Using (3.2.4), equation (3.2.2) is replaced by a sequence of independent equations posed on small subdomains such that

$$\begin{aligned} \mathscr{B}(e, \alpha) &= \left\langle f, \alpha \sum_{i=1}^{N} \psi_i \right\rangle_0 - \mathscr{B}(U_h, \alpha \sum_{i=1}^{N} \psi_i) \\ &= \sum_{i=1}^{N} \langle f, \alpha\psi_i \rangle_0 - \sum_{i=1}^{N} \mathscr{B}(U_h, \alpha\psi_i) \quad \forall \alpha \in H_0^1(\Omega). \end{aligned} \tag{3.2.5}$$

The function $\alpha\psi_i$ is supported on the set $\tilde{\Omega}_i$ and vanishes on the boundary. Therefore, it follows that $\alpha\psi_i$ belongs to the space $H_0^1(\tilde{\Omega}_i)$. The local bilinear and linear forms associated with this space are respectively given by

$$\mathscr{B}_i(u, \alpha) = \int_{\tilde{\Omega}_i} (u\alpha + u'\alpha')\, dx$$

and

$$\langle f, \alpha \rangle_{0,i} = \int_{\tilde{\Omega}_i} f\alpha\, dx.$$

The subdomain residual problem consists of finding $e_i \in H_0^1(\tilde{\Omega}_i)$ such that

$$\mathscr{B}_i(e_i, \alpha) = \langle f, \alpha \rangle_{0,i} - \mathscr{B}_i(U_h, \alpha) \quad \forall \alpha \in H_0^1(\tilde{\Omega}_i). \tag{3.2.6}$$

By referring to (3.2.5), the error estimator η_i associated with the subdomain $\tilde{\Omega}_i$ is taken to be

$$\eta_i = \|e_i\|_{H^1(\tilde{\Omega}_i)}$$

and the global error estimator η is computed by summing the contributions from the subdomains

$$\eta = \left\{ \sum_{i=1}^{N} \eta_i^2 \right\}^{1/2}.$$

There are two disadvantages of this method. Firstly, the local patch problems are rather expensive to solve accurately. Secondly, each element is treated several times according to the number of patches associated with it. Solving local problems posed over individual elements (element residual method) is an alternative to overcome these disadvantages.

In the element residual method, a local error e_l is defined on a single element l, i.e.

$$\mathscr{B}_l(e_l, \alpha) = \langle f, \alpha \rangle_{0,\Omega_l} - \mathscr{B}_l(U_h, \alpha) \quad \forall \alpha \in H_0^1(\Omega_l), \tag{3.2.7}$$

recalling that Ω_l is element l^{th} whereas the subdomain $\tilde{\Omega}_i$ is a union of Ω_l's. It is noted that all of the local errors e_l have disjoint supports and thus, are orthogonal in $H_0^1(\Omega)$. Then the local problem means to find a function $e_l \in H_0^1(\Omega_l)$ that satisfies (3.2.7). With the solution e_l of the single element known, the error estimator is computed by using

$$\|e\|^2_{H^1(\Omega)} = \sum_{l=1}^{N} \|e_l\|^2_{H^1(\Omega)} \, .$$

In this study, we implement an implicit scheme of a posteriori error estimation, where the error estimators are locally computed on the elements. Details of the implicit scheme a posteriori error estimation consider in this study will be explained in Section 3.3.

Recovery methods

The main idea of this technique is to smoothen the gradients of the finite element solution. Then, the error estimator is computed by comparing the unsmoothed and the smoothed gradients of the approximation. Let the approximation to the gradient of the exact solution be denoted by $G[U_h]$, then the a posteriori error estimator is taken to be

$$\eta^2 = \int_\Omega |G[U_h] - U_h'|^2 dx.$$

This technique uses the fact that the gradient of the finite element solution is generally discontinuous across the interelement boundaries. The main ingredient is on the derivation of the recovery operator G. Since the finite element approximation U_h is itself piecewise linear, meaning that the same numerical procedures already present in the finite element code may be reused to store and handle the post-processed gradient G. As an example, the post-processed gradient is measured by interpolating the gradient of the finite element approximation at the centroids of the elements that sharing the node. Then the estimator associated with element l is then defined to be

$$\eta_l = \left\| G(U_h) - U_h' \right\|_{H^0(\Omega_l)},$$

and the global estimator is computed as the summation of η_l for $l = 1, \ldots, N$. In some cases, there is a superconvergence phenomenon, where the unsmoothed approximation U_h' is closer to the smoothed gradient $G[U_h]$ than to the exact gradient u', i.e. for a linear interpolation case, we have

$$\left\| G(U_h) - U_h' \right\|_{H^0(\Omega)} \leq Ch^2$$

and

$$\left\| u' - U_h' \right\|_{H^0(\Omega)} \leq Ch.$$

Hierarchical bases error estimates

A study on general theory of hierarchical a posteriori error estimations can be found in [10]. Details of an analysis of hierarchical bases error estimation technique can be found in [3, 10] and the references therein.

The hierarchical bases error estimate is based on the idea of obtaining the computable error estimates by solving the problem of interest using two discretization schemes of different accuracy. Then, the estimate of the exact error is measured from the difference in the approximations. There are two ways in enriching the space, for example by augmenting the original space with higher order basis functions or by using uniform refinement on the mesh used to construct the original space.

Suppose that the finite element approximation $U_h \in \mathring{\mathcal{V}}_h^p$ is known. Let the finite dimensional subspace $\mathring{\mathcal{W}}_h^p \subset H_0^1(\Omega)$ be an enrichment of the original finite element subspace $\mathring{\mathcal{V}}_h^p$. It is assumed that these spaces satisfy

$$\mathring{\mathcal{V}}_h^p \cap \mathring{\mathcal{W}}_h^p = \{0\}.$$

An improved approximation of the exact solution u is obtained from the space

$$\mathring{\mathcal{V}}_h^{p*} = \mathring{\mathcal{V}}_h^p \oplus \mathring{\mathcal{W}}_h^p$$

by solving

$$\mathscr{B}(U_h^*, \alpha_h^*) = \langle f, \alpha_h^* \rangle_0 \quad \forall \alpha_h^* \in \mathring{\mathcal{V}}_h^{p*}. \tag{3.2.8}$$

As mentioned before, the enhanced space $\mathring{\mathcal{V}}_h^{p*}$ can be constructed by augmenting the original space $\mathring{\mathcal{V}}_h^p$ with higher order basis functions or by using a uniform refinement of the mesh to construct the space $\mathring{\mathcal{V}}_h^p$. Let U_h^* be the solution of (3.2.8), then

$$\|e\|_{H^1(\Omega)} = \|u - U_h\|_{H^1(\Omega)} \approx \|U_h^* - U_h\|_{H^1(\Omega)} = \|e^*\|_{H^1(\Omega)}.$$

The difference between two approximations $U_h^* - U_h$ will provide a computable estimate for the error.

3.3 A posteriori error estimation for time dependent partial differential equations

The solution of time dependent partial differential equations is involved in many engineering applications. From a numerical point of view, in addition to the usual space discretization, a time discretization has to be applied for time dependent problems. Moreover, the relationship between space discretization and time discretization should be considered to ensure the accuracy of the approximation. In this study, we focus on nonlinear time dependent partial differential equations, which are the BBM and Burgers equations.

A posteriori error estimation for time dependent problems considered in this study is based on the procedure developed by Adjerid et al. for one dimensional parabolic systems [2]. In their study, a finite element method of lines using hierarchical piecewise polynomial bases of degree $p \geq 1$ is considered to approximate the solution. Then, the a posteriori error estimates of the spatial discretization error is calculated by solving *local* parabolic or *local* elliptic finite element problems with piecewise polynomial functions of degree $p + 1$. Details on the analysis and numerical studies of a posteriori error estimation with finite element methods of lines for one dimensional and two dimensional linear parabolic systems can be found in [2, 1] and the references therein. This procedure of a posteriori error estimates is then studied for one dimensional nonlinear parabolic systems [40, 54]. In 2005, Tran et al. studied this procedure of a posteriori error estimation with finite element method of lines for a Sobolev equation [55]. The interest points of this idea include the simplicity of the derivation, the ease of implementation and the provision of a basis for a local mesh refinement or local order refinement schemes, h and p refinement, respectively.

As an example, we consider a one dimensional parabolic system

$$\partial_t u(x,t) + g(u) = \partial_{xx} u(x,t) \quad x \in \Omega = (0,1), \quad t > 0 \tag{3.3.1}$$

with

$$u(x,0) = u_0(x), \quad x \in \Omega, \quad u(0,t) = u(1,t) = 0 \quad t > 0, \tag{3.3.2}$$

where g is a smooth functions satisfying a global Lipschitz condition. A weak formulation of this problem is to find $u \in H_0^1(\Omega)$ such that

$$\langle \partial_t u(t), \alpha \rangle_0 + \langle g(u), \alpha \rangle_0 + \langle \partial_x u(t), \partial_x \alpha \rangle_0 = 0 \quad \forall \alpha \in H_0^1(\Omega).$$

By using the FEM, the approximate solution $U_h \in \mathring{\mathcal{V}}_h^p$ of $u \in H_0^1(\Omega)$ is computed by solving

$$\langle \partial_t U_h(t), \alpha_h \rangle_0 + \langle g(U_h), \alpha_h \rangle_0 + \langle \partial_x U_h(t), \partial_x \alpha_h \rangle_0 = 0 \quad \forall \alpha_h \in \mathring{\mathcal{V}}_h^p.$$

Denoting the exact error by $e(x,t) = u(x,t) - U_h(x,t)$, we infer that e satisfies the following error representation

$$\langle \partial_t e(t), \alpha_h \rangle_0 + \langle \partial_x e(t), \partial_x \alpha_h \rangle_0 = - \langle g(U_h + e), \alpha_h \rangle_0 - \langle \partial_t U_h(t), \alpha_h \rangle_0 - \langle \partial_x U_h(t), \partial_x \alpha_h \rangle_0$$

for any $\alpha_h \in \mathring{\mathcal{V}}_h^p$.

Due to superconvergence property at mesh points, we are able to approximate the exact error e by E having the form

$$E(x,t) = \sum_{l=1}^{N} E_l(t) \phi_{l,p+1}(x),$$

where $\phi_{l,p+1}$ is defined by (2.4.4). For a given partition (2.4.1), we associate with every subinterval Ω_l an error estimator E_l, $l = 1, \dots, N$. The *local* error estimator E_l is computable in terms of U_h on Ω_l, by solving *local* parabolic or *local* elliptic equations using piecewise polynomials of degree $p+1$. Noting (2.2.1), E is computed by solving a *local* parabolic equation

$$\begin{aligned}\langle \partial_t E(t), \phi_{l,p+1} \rangle_{0,\Omega_l} + \langle \partial_x E(t), \partial_x \phi_{l,p+1} \rangle_{0,\Omega_l} = &- \langle g(U+E), \phi_{l,p+1} \rangle_{0,\Omega_l} - \langle \partial_t U(t), \phi_{l,p+1} \rangle_{0,\Omega_l} \\ &- \langle \partial_x U(t), \partial_x \phi_{l,p+1} \rangle_{0,\Omega_l}\end{aligned}$$

or a *local* elliptic equation

$$\begin{aligned}\langle \partial_x E(t), \partial_x \phi_{l,p+1} \rangle_{0,\Omega_l} = &- \langle g(U+E), \phi_{l,p+1} \rangle_{0,\Omega_l} - \langle \partial_t U(t), \phi_{l,p+1} \rangle_{0,\Omega_l} \\ &- \langle \partial_x U(t), \partial_x \phi_{l,p+1} \rangle_{0,\Omega_l} .\end{aligned}$$

Since these approximations are locally computed, we are able to identify elements for mesh refinement in an adaptive scheme. Then, the quality of error estimate E is judged by the global effectivity index, Θ which is defined by

$$\Theta(t) = \frac{\|E(t)\|_1}{\|e(t)\|_1}.$$

In 2014, Tripathy et al. implement a residual approach of a posteriori error estimation for two dimensionals parabolic problems with the approximate solution is computed by the H1MFEM. References on the residual approach of a posteriori error estimation can be found in [57] and the references therein. By using the residual approach of a posteriori error estimation of the H1MFEM for parabolic problem (3.3.1)–(3.3.2), the equivalence of residual equations and exact errors are analysed by using the standard energy argument. Then, error indicators with respect to both time and space are compared with the residual equations.

In this study, we use a mixed formulation of finite element methods of lines (H1MFEM) for the approximations of the solution and its derivative. Then, we propose a posteriori error estimations of the H1MFEM by using the procedure developed by Adjerid et al. To the best of our knowledge, this is the first time this way of a posteriori error estimation is considered for nonlinear equations such as the BBM and Burgers equations, where the approximate solution is computed by the H1MFEM. Details of the H1MFEM, the BBM and Burgers equations are stated in Section 2.5–Section 2.7.

Chapter 4

Benjamin-Bona-Mahony equation: a mixed finite element method

In this chapter, we focus on a priori and a posteriori error estimations of H1MFEM for the BBM equation. In Section 4.1, we elaborate on a formulation of weak solutions and a finite element scheme for the BBM equation. Section 4.2 and Section 4.3 are respectively devoted to the analysis of a priori and a posteriori error estimations. The chapter ends with implementation and numerical experiments in Section 4.4 and Section 4.5, respectively.

4.1 Formulation of weak solutions and finite element scheme

We consider the following BBM equation

$$\partial_t u(x,t) - \partial_{xxt} u(x,t) + u(x,t)\partial_x u(x,t) + \partial_x u(x,t) = 0 \quad x \in \Omega, \quad t \in (0,T], \tag{4.1.1}$$

with Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in [0,T], \tag{4.1.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{4.1.3}$$

where $\partial_t := \partial/\partial t$, $\partial_x := \partial/\partial x$, $\partial_{xxt} := \partial^3/\partial x^2 \partial t$, T is a positive constant and $\Omega := (0,1)$.

In this chapter, C denotes a generic constant which may take different values at different occurrences.

Following [45, 47], we use the H1MFEM which reduces equation (4.1.1) to a system of first order equations by defining a new variable $v = \partial_x u$. As a consequence, (4.1.1) is reformulated as

$$\partial_x u(x,t) = v(x,t), \tag{4.1.4}$$

$$\partial_t u(x,t) - \partial_{tx} v(x,t) + u(x,t)v(x,t) + v(x,t) = 0 \tag{4.1.5}$$

and for $t = 0$

$$v(x,0) = v_0(x) = \partial_x u_0(x). \tag{4.1.6}$$

By multiplying (4.1.4) by $\partial_x \chi$ where $\chi \in H_0^1(\Omega)$, we obtain

$$\langle \partial_x u(t), \partial_x \chi \rangle_0 = \langle v(t), \partial_x \chi \rangle_0 \quad \forall \chi \in H_0^1(\Omega).$$

On the other hand, by multiplying (4.1.5) by $-\partial_x w$ where $w \in H^1(\Omega)$ we have

$$\langle \partial_t v(t), w \rangle_1 = \langle u(t)v(t), \partial_x w \rangle_0 + \langle v(t), \partial_x w \rangle_0 \quad \forall w \in H^1(\Omega),$$

where we have used integration by parts and the Dirichlet boundary conditions $\partial_t u(0,t) = \partial_t u(1,t) = 0$ for the first term.

A weak formulation of the problem reads: Given $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, find $(u,v) : [0,T] \to H_0^1(\Omega) \times H^1(\Omega)$ such that for $t > 0$

$$\langle \partial_x u(t), \partial_x \chi \rangle_0 = \langle v(t), \partial_x \chi \rangle_0 \quad \forall \chi \in H_0^1(\Omega), \tag{4.1.7}$$

$$\langle \partial_t v(t), w \rangle_1 = \langle u(t)v(t), \partial_x w \rangle_0 + \langle v(t), \partial_x w \rangle_0 \quad \forall w \in H^1(\Omega), \tag{4.1.8}$$

and that for $t = 0$

$$\langle v(0), w \rangle_1 = \langle v_0, w \rangle_1 \quad \forall w \in H^1(\Omega), \tag{4.1.9}$$

where $v_0 = \partial_x u_0$.

Lemma 4.1.1. *If* $u \in W_\infty^1(0,T; H_0^1(\Omega) \cap H^2(\Omega))$ *and* $v \in W_\infty^1(0,T; H^1(\Omega))$ *and* (u,v) *are solutions to* (4.1.7)–(4.1.8)*, then we have* (u,v) *satisfies* (4.1.4)–(4.1.5).

Proof. To show this fact, firstly by using integration by parts to (4.1.7), we have $\partial_x(v - \partial_x u) = 0$ in $W^1_\infty(0,T;H^0(\Omega))$, which implies

$$v(x,t) = \partial_x u(x,t) + f(t) \tag{4.1.10}$$

for all t and a.e. in Ω, and in particular

$$v(x,0) = \partial_x u(x,0) + f(0) \tag{4.1.11}$$

for some function f depending on t. Secondly, by integrating (4.1.11) over Ω, we obtain

$$\int_\Omega v(x,0) = \int_\Omega \partial_x u(x,0) + f(0).$$

Noting $v(0) = v_0$ by (4.1.9) and $v_0 = \partial_x u_0$ by (4.1.6), we infer that $f(0) = 0$. By differentiating with respect to t and integrating over Ω, it follows from (4.1.10) that

$$\int_\Omega \partial_t v(x,t) = \int_\Omega \partial_{tx} u(x,t) + f'(t).$$

This together with (4.1.8) with $w = 1$ yields $f'(t) = 0$. Hence, we have (u,v) satisfies (4.1.4). A similar argument using integration by parts gives (4.1.5). $\square$

Let p and q be two positive integers. A semidiscrete approximation to (4.1.7)–(4.1.9) reads: Find $(U_h, V_h) : [0,T] \to \mathring{\mathcal{V}}^p_h \times \mathcal{V}^q_h$ such that

$$\langle \partial_x U_h(t), \partial_x \chi_h \rangle_0 = \langle V_h(t), \partial_x \chi_h \rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}^p_h, \quad t \in (0,T], \tag{4.1.12}$$

$$\langle \partial_t V_h(t), w_h \rangle_1 = \langle U_h(t) V_h(t), \partial_x w_h \rangle_0 + \langle V_h(t), \partial_x w_h \rangle_0 \quad \forall w_h \in \mathcal{V}^q_h, \quad t \in (0,T], \tag{4.1.13}$$

and at $t = 0$

$$\langle V_h(0), w_h \rangle_1 = \langle v_0, w_h \rangle_1 \quad \forall w_h \in \mathcal{V}^q_h, \tag{4.1.14}$$

where the finite dimensional subspaces $\mathring{\mathcal{V}}^p_h$ and $\mathcal{V}^q_h$ are defined by (2.4.5).

4.2 A priori error estimation

In this section we present an analysis for a priori error estimates for the approximation of the solution of (4.1.7)–(4.1.9) by that of (4.1.12)–(4.1.14). We first show the boundedness of the families $\{U_h\}$ and $\{V_h\}$.

Lemma 4.2.1. *If* (u, v) *and* (U_h, V_h) *are solutions to* (4.1.7)–(4.1.6) *and* (4.1.12)–(4.1.14) *respectively, then the following inequality holds*

$$\|U_h\|_{L^\infty(L^\infty)} + \|V_h\|_{L^\infty(L^\infty)} \leq C(T).$$

Proof. Substituting $\chi_h = U_h$ into (4.1.12) and noting Hölder's inequality, we deduce

$$\|\partial_x U_h(t)\|_0 \leq \|V_h(t)\|_0 \,.$$

By using the Poincaré inequality and the Sobolev imbedding theorem, we obtain

$$\|U_h(t)\|_{L^\infty(\Omega)} \leq C \|V_h(t)\|_0 \,. \tag{4.2.1}$$

Letting $w_h = V_h$ in (4.1.13), we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \|V_h(t)\|_1^2 &\leq \|U_h(t)\|_{L^\infty(\Omega)} \|V_h(t)\|_0 \|\partial_x V_h(t)\|_0 + \|V_h(t)\|_0 \|\partial_x V_h(t)\|_0 \\ &\leq \Big(C \|V_h(t)\|_0^2 + \|V_h(t)\|_0 \Big) \|\partial_x V_h(t)\|_0 \,, \end{aligned}$$

where we use (4.2.1) in the last inequality. Integrating from 0 to t, noting from (4.1.14) that

$$\|V_h(0)\|_1 \leq \|v_0\|_1 \,,$$

and using the inequality

$$2ab \leq \epsilon a^2 + \frac{b^2}{\epsilon} \quad \forall a, b \geq 0, \quad \epsilon > 0, \tag{4.2.2}$$

we have

$$\begin{aligned} \|V_h(t)\|_1^2 &\leq \|V_h(0)\|_1^2 + C \int_0^t \Big(\|V_h(s)\|_0^2 + \|V_h(s)\|_0 \Big) \|\partial_x V_h(s)\|_0 \, ds \\ &\leq C \left(1 + \int_0^t \Big(\|V_h(s)\|_0^4 + \|V_h(s)\|_0^2 \Big) \, ds \right) + C \int_0^t \|\partial_x V_h(s)\|_0^2 \, ds \\ &\leq C \left(1 + \int_0^t \Big(\|V_h(s)\|_1^4 + \|V_h(s)\|_1^2 \Big) \, ds \right). \end{aligned}$$

By using Lemma 2.3.3 (with $\varphi(t) = \|V_h(t)\|_1^2$, $a = C$, and $\theta(s) = C(s^2 + s)$ for $s \geq 0$) we deduce from the above inequality

$$\|V_h(t)\|_1^2 \leq C \quad \forall t \in [0, T^*],$$

where T^* is defined in Lemma 2.3.3. Since

$$\int_a^\sigma \frac{ds}{\theta(s)} \to \infty \quad \text{as} \quad \sigma \to 0,$$

we have $T^* = T$. Therefore, we deduce that

$$\|V_h(t)\|_1^2 \leq C(T) \quad \forall t \in [0, T].$$

This together with (4.2.1) implies

$$\|U_h\|_{L^\infty(L^\infty)} + \|V_h\|_{L^\infty(L^\infty)} \leq C(T),$$

thus completing the proof of the lemma. $\square$

It is usual in the error analysis for parabolic equations to consider elliptic projections; see [53, 58]. We define $\bar{U}_h \in W^1_\infty(0, T; \mathring{\mathcal{V}}^p_h)$ and $\bar{V}_h \in W^1_\infty(0, T; \mathcal{V}^q_h)$ satisfying, for $t \in [0, T]$,

$$\left\langle \partial_x u(t) - \partial_x \bar{U}_{h,p}(t), \partial_x \chi_h \right\rangle_0 = 0 \quad \forall \chi_h \in \mathring{\mathcal{V}}^p_h. \tag{4.2.3}$$

$$\left\langle v(t) - \bar{V}_h(t), w_h \right\rangle_1 = 0 \quad \forall w_h \in \mathcal{V}^q_h. \tag{4.2.4}$$

Let the errors in the approximation of (4.1.7)–(4.1.6) by (4.1.12)–(4.1.14) be denoted by e_h and f_h, i.e.,

$$e_h(x, t) := u(x, t) - U_h(x, t) \tag{4.2.5}$$

and

$$f_h(x, t) := v(x, t) - V_h(x, t). \tag{4.2.6}$$

By defining the following notations

$$\eta(x, t) = u(x, t) - \bar{U}_h(x, t) \tag{4.2.7}$$

$$\zeta(x, t) = \bar{U}_h(x, t) - U_h(x, t), \tag{4.2.8}$$

$$\rho(x, t) = v(x, t) - \bar{V}_h(x, t), \tag{4.2.9}$$

$$\xi(x, t) = \bar{V}_h(x, t) - V_h(x, t), \tag{4.2.10}$$

we then rewrite e_h and f_h respectively as

$$e_h(x, t) = \eta(x, t) + \zeta(x, t) \quad \text{and} \quad f_h(x, t) = \rho(x, t) + \xi(x, t). \tag{4.2.11}$$

Therefore, to estimate e_h and f_h, we estimate each of the terms η, ζ, ρ, and ξ. The next lemma shows the approximation properties of the elliptic projections defined in (4.2.3)–(4.2.4).

Lemma 4.2.2. *Assume that* $u \in W^1_\infty(0,T;H^1_0(\Omega)\cap H^{p+1}(\Omega))$. *Assume further that* $v \in W^1_\infty(0,T;H^{q+1}(\Omega))$. *Then the functions* η *and* ρ *defined by* (4.2.7) *and* (4.2.9) *satisfy for* $j = 0, 1$

$$\|\eta(t)\|_j \le Ch^{p+1-j}\,\|u(t)\|_{p+1}\,, \tag{4.2.12}$$

$$\|\partial_t\eta(t)\|_j \le Ch^{p+1-j}\,\|\partial_t u(t)\|_{p+1}\,, \tag{4.2.13}$$

$$\|\rho(t)\|_j \le Ch^{q+1-j}\,\|v(t)\|_{q+1}\,, \tag{4.2.14}$$

$$\|\partial_t\rho(t)\|_j \le Ch^{q+1-j}\left(\|v(t)\|_{q+1} + \|\partial_t v(t)\|_{q+1}\right). \tag{4.2.15}$$

Proof. We start with $j = 1$ for (4.2.12). Noting (4.2.7) and using (4.2.3) we have for any $\bar\chi_h \in \mathring{\mathcal{V}}^p_h$

$$\begin{aligned}\|\partial_x\eta(t)\|_0^2 &= \left\langle \partial_x u - \partial_x\bar U_h, \partial_x u - \partial_x\bar U_h\right\rangle_0\\ &= \left\langle \partial_x u - \partial_x\bar U_h, \partial_x u - \partial_x\bar\chi_h\right\rangle_0 + \left\langle \partial_x u - \partial_x\bar U_h, \partial_x\bar\chi_h - \partial_x\bar U_h\right\rangle_0\\ &= \left\langle \partial_x u - \partial_x\bar U_h, \partial_x u - \partial_x\bar\chi_h\right\rangle_0 \le \left\|\partial_x u(t) - \partial_x\bar U_h(t)\right\|_0 \|\partial_x u(t) - \partial_x\bar\chi_h\|_0\,.\end{aligned}$$

Hence by using the Poincaré inequality and (2.4.6) we have

$$\|\eta(t)\|_1 \le C\,\|\partial_x\eta(t)\|_0 \le C \inf_{\bar\chi_h\in\mathring{\mathcal{V}}^p_h} \|\partial_x u(t) - \partial_x\bar\chi_h\|_0 \le Ch^p\,\|u(t)\|_{p+1}\,. \tag{4.2.16}$$

Similarly we prove (4.2.14) with $j = 1$, noting that the H^1-inner product is used in (4.2.4).

The results for $j = 0$ can be obtained by using the Aubin-Nitsche trick [53, 58]. We prove only (4.2.12) for $j = 0$. Let $\varphi \in H^0(\Omega)$ and consider the problem of finding $\psi \in H^2(\Omega)\cap H^1_0(\Omega)$ such that

$$-\partial_{xx}\psi = \varphi \quad \text{in} \quad \Omega \tag{4.2.17}$$

with $\psi(0) = \psi(1) = 0$. Recalling that the solution ψ of (4.2.17) is smoother by two derivatives in $H^0(\Omega)$ than the right hand side φ, thus we have

$$\|\psi\|_2 \le C\,\|\partial_{xx}\psi\|_0 = C\,\|\varphi\|_0\,. \tag{4.2.18}$$

For any $\psi_h \in \mathring{\mathcal{V}}_h^p$ we have

$$\left\langle u - \bar{U}_h, \varphi \right\rangle_0 = -\left\langle u - \bar{U}_h, \partial_{xx}\psi \right\rangle_0 = \left\langle \partial_x u - \partial_x \bar{U}_h, \partial_x \psi \right\rangle_0 = \left\langle \partial_x u - \partial_x \bar{U}_h, \partial_x \psi - \partial_x \psi_h \right\rangle_0$$
$$\leq \left\| \partial_x u - \partial_x \bar{U}_h \right\|_0 \left\| \partial_x \psi - \partial_x \psi_h \right\|_0 .$$

From (2.4.6), we have

$$\left\| \partial_x u - \partial_x \bar{U}_h \right\|_0 \leq \inf_{\chi_h \in \mathring{\mathcal{V}}_h^p} \left\| \partial_x u - \partial_x \chi_h \right\|_0 \leq C h^p \left\| u \right\|_{p+1} .$$

This together with

$$\left\| \partial_x \psi - \partial_x \psi_h \right\|_0 \leq h \left\| \psi \right\|_2$$

yield

$$\left\langle u - \bar{U}_h, \varphi \right\rangle_0 \leq C h^{p+1} \left\| u \right\|_{p+1} \left\| \psi \right\|_2 .$$

Then, (4.2.18) gives

$$\left\langle u - \bar{U}_h, \varphi \right\rangle_0 \leq C h^{p+1} \left\| u \right\|_{p+1} \left\| \varphi \right\|_0 .$$

We obtain (4.2.12) by choosing $\varphi = u - \bar{U}_h$.

On the other hand, by differentiating (4.2.3) and (4.2.4) with respect to t, it can be seen that $\partial_t \bar{U}_h$ and $\partial_t \bar{V}_h$ are respectively the elliptic projections of $\partial_t u$ and $\partial_t v$, namely

$$\left\langle \partial_{tx} u(t) - \partial_{tx} \bar{U}_h(t), \partial_x \chi_h \right\rangle_0 = 0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^p \tag{4.2.19}$$

and

$$\left\langle \partial_t v(t) - \partial_t \bar{V}_h(t), w_h \right\rangle_1 = 0 \quad \forall w_h \in \mathcal{V}_h^q. \tag{4.2.20}$$

Therefore, (4.2.13) and (4.2.15) can be proved in a similar way to (4.2.12) and (4.2.14).

□

It is well-known that the elliptic projection approximates the Galerkin solution better than the exact solution; see e.g. [53]–[55], [58]. The next lemma shows this property, namely the superconvergence property of ξ and ζ.

Lemma 4.2.3. *Assume that $u \in W^1_\infty(0,T; H^1_0(\Omega) \cap H^{p+1}(\Omega))$ and $v \in W^1_\infty(0,T; H^{q+1}(\Omega))$. Then, the following estimates hold*

$$\left\| \zeta \right\|_{W^1_\infty(H^1)} \leq C h^\gamma \left(\left\| u \right\|_{L^\infty(H^{p+1})} + \left\| v \right\|_{L^\infty(H^{q+1})} \right) \tag{4.2.21}$$

and

$$\|\xi\|_{W^1_\infty(H^1)} \le Ch^\gamma \left(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})}\right), \tag{4.2.22}$$

where $\gamma = \min(p+1, q+1)$.

Proof. Firstly, by subtracting (4.1.12) from (4.1.7) and using (4.2.7)–(4.2.10) we have

$$\langle \partial_x \eta, \partial_x \chi_h \rangle_0 + \langle \partial_x \zeta, \partial_x \chi_h \rangle_0 = \langle \rho, \partial_x \chi_h \rangle_0 + \langle \xi, \partial_x \chi_h \rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}^p_h. \tag{4.2.23}$$

By noting (4.2.3) the first term in (4.2.23) vanishes, and thus we obtain

$$\langle \partial_x \zeta, \partial_x \chi_h \rangle_0 = \langle \rho, \partial_x \chi_h \rangle_0 + \langle \xi, \partial_x \chi_h \rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}^p_h. \tag{4.2.24}$$

Substituting $\chi_h = \zeta \in \mathring{\mathcal{V}}^p_h$ in (4.2.24) and using the Hölder inequality give

$$\|\partial_x \zeta(t)\|_0^2 \le \|\rho(t)\|_0 \|\partial_x \zeta(t)\|_0 + \|\xi(t)\|_0 \|\partial_x \zeta(t)\|_0 .$$

This and the Poincaré inequality yield

$$\|\zeta(t)\|_1 \le C\Big(\|\rho(t)\|_0 + \|\xi(t)\|_0 \Big). \tag{4.2.25}$$

Therefore, (4.2.21) is proved if we prove (4.2.22), noting the bound for $\|\rho(t)\|_0$ given in Lemma 4.2.2.

By subtracting (4.1.13) from (4.1.8) and noting $v - V_h = f_h = \rho + \xi$, for any $w_h \in \mathcal{V}^q_h$ we have

$$\langle \partial_t \rho, w_h \rangle_1 + \langle \partial_t \xi, w_h \rangle_1 = \langle uv, \partial_x w_h \rangle_0 + \langle \rho, \partial_x w_h \rangle_0 + \langle \xi, \partial_x w_h \rangle_0 - \langle U_h V_h, \partial_x w_h \rangle_0 \tag{4.2.26}$$

By referring to (4.2.20), we note that $\langle \partial_t \rho, w_h \rangle_1 = 0$. Rewriting

$$uv - U_h V_h = u(f_h + V_h) - U_h V_h = u f_h + V_h(u - U_h) = u f_h + V_h e_h = u(\rho + \xi) + V_h e_h$$

we deduce from (4.2.26)

$$\begin{aligned} \langle \partial_t \xi, w_h \rangle_1 &= \langle u(\rho + \xi), \partial_x w_h \rangle_0 + \langle V_h e_h, \partial_x w_h \rangle_0 + \langle \rho, \partial_x w_h \rangle_0 + \langle \xi, \partial_x w_h \rangle_0 \\ &= \langle (u+1)(\rho + \xi), \partial_x w_h \rangle_0 + \langle V_h e_h, \partial_x w_h \rangle_0 \quad \forall w_h \in \mathcal{V}^q_h. \end{aligned} \tag{4.2.27}$$

Letting $w_h = \xi \in \mathcal{V}_h^q$ and using

$$\langle \partial_t \xi, \xi \rangle_1 = \frac{1}{2} \frac{d}{dt} \|\xi(t)\|_1^2$$

we have from (4.2.27)

$$\frac{1}{2} \frac{d}{dt} \|\xi(t)\|_1^2 = \langle (u+1)(\rho+\xi), \partial_x \xi \rangle_0 + \langle V_h e_h, \partial_x \xi \rangle_0 .$$

By integrating from 0 to t, and using the fact that $\xi(0) = 0$, we have

$$\begin{aligned}
\|\xi(t)\|_1^2 &\leq 2 \int_0^t \Big| \langle (u(s)+1)(\rho(s)+\xi(s)), \partial_x \xi(s) \rangle_0 \Big| \, ds + 2 \int_0^t \Big| \langle V_h(s) e_h(s), \partial_x \xi(s) \rangle_0 \Big| \, ds \\
&=: \mathscr{T}_1 + \mathscr{T}_2.
\end{aligned} \tag{4.2.28}$$

By using the Cauchy-Schwarz inequality and (4.2.11), we have

$$\begin{aligned}
\mathscr{T}_1 &\leq 2 \Big(\|u\|_{L^\infty(L^\infty)} + 1 \Big) \int_0^t \Big(\|\rho(s)\|_0 \|\partial_x \xi(s)\|_0 + \|\xi(s)\|_0 \|\partial_x \xi(s)\|_0 \Big) \, ds \\
&\leq C \int_0^t \Big(\|\rho(s)\|_0 \|\partial_x \xi(s)\|_0 + \|\xi(s)\|_0 \|\partial_x \xi(s)\|_0 \Big) \, ds \\
&\leq C \|\rho\|_{L^2(H^0)}^2 + C \int_0^t \|\xi(s)\|_1^2 \, ds.
\end{aligned}$$

Noting the bound of $\|\rho(t)\|_0$ in Lemma 4.2.1, we have

$$\mathscr{T}_1 \leq C h^{2(q+1)} + C \int_0^t \|\xi(s)\|_1^2 \, ds. \tag{4.2.29}$$

Similarly, by referring to Lemma 4.2.1 and using (4.2.25) we have

$$\begin{aligned}
\mathscr{T}_2 &\leq 2 \|V_h\|_{L^\infty(L^\infty)} \int_0^t \|e_h(s)\|_0 \|\partial_x \xi(s)\|_0 \, ds \\
&\leq C \int_0^t \Big(\|\eta(s)\|_0 + \|\zeta(s)\|_0 \Big) \|\partial_x \xi(s)\|_0 \, ds \\
&\leq C \int_0^t \Big(\|\eta(s)\|_0 + \|\rho(s)\|_0 + \|\xi(s)\|_0 \Big) \|\partial_x \xi(s)\|_0 \, ds \\
&\leq C \Big(\|\eta\|_{L^2(H^0)}^2 + \|\rho\|_{L^2(H^0)}^2 \Big) + C \int_0^t \|\xi(s)\|_1^2 \, ds \\
&\leq C h^{2\gamma} \Big(\|u\|_{L^\infty(H^{p+1})}^2 + \|v\|_{L^\infty(H^{q+1})}^2 \Big) + C \int_0^t \|\xi(s)\|_1^2 \, ds.
\end{aligned} \tag{4.2.30}$$

Thus, (4.2.28)–(4.2.30) yield

$$\|\xi(t)\|_1^2 \leq C h^{2\gamma} \Big(\|u\|_{L^\infty(H^{p+1})}^2 + \|v\|_{L^\infty(H^{q+1})}^2 \Big) + C \int_0^t \|\xi(s)\|_1^2 \, ds.$$

The desired estimate (4.2.22) now follows from Lemma 2.3.2 (Gronwall's Lemma). $\square$

In the following lemma, we prove the bounds for $\|\partial_t \xi(t)\|_1$ and $\|\partial_t \zeta(t)\|_1$.

Lemma 4.2.4. *Assume that* $\partial_t u \in L^\infty(0,T; H_0^1(\Omega) \cap H^{p+1}(\Omega))$ *and* $\partial_t v \in L^\infty(0,T; H^{q+1}(\Omega))$. *Then, the following estimates hold*

$$\|\partial_t \xi\|_{L^\infty(H^1)} \leq Ch^\gamma \left(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} \right) \tag{4.2.31}$$

and

$$\|\partial_t \zeta\|_{L^\infty(H^1)} \leq Ch^\gamma \left(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^\infty(H^{q+1})} \right) \tag{4.2.32}$$

where $\gamma = \min(p+1, q+1)$.

Proof. Substituting $w_h = \partial_t \xi \in \mathcal{V}_h^q$ in (4.2.27) yields

$$\|\partial_t \xi(t)\|_1^2 = \langle (u+1)(\rho + \xi), \partial_{tx} \xi \rangle_0 + \langle V_h e_h, \partial_{tx} \xi \rangle_0$$

By using Lemma 4.2.1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \|\partial_t \xi(t)\|_1^2 &\leq \left(1 + \|u\|_{L^\infty(L^\infty)}\right) \left(\|\rho(t)\|_0 + \|\xi(t)\|_0 \right) \|\partial_{tx} \xi(t)\|_0 \\ &\quad + \|V_h\|_{L^\infty(L^\infty)} \|e_h(t)\|_0 \|\partial_{tx} \xi(t)\|_0 \\ &\leq C\Big(\|\rho(t)\|_0 + \|\xi(t)\|_0 + \|e_h(t)\|_0 \Big) \|\partial_t \xi(t)\|_1 . \end{aligned}$$

Noting $e_h = \eta + \zeta$, we have

$$\|\partial_t \xi(t)\|_1 \leq C\Big(\|\rho(t)\|_0 + \|\xi(t)\|_0 + \|\eta(t)\|_0 + \|\zeta(t)\|_0 \Big).$$

Hence, inequality (4.2.31) is obtained by referring to Lemma 4.2.2 and Lemma 4.2.3.

On the other hand, in order to have (4.2.32), firstly we differentiate (4.2.24) with respect to t to have

$$\langle \partial_{tx} \zeta, \partial_x \chi_h \rangle_0 = \langle \partial_t \rho, \partial_x \chi_h \rangle_0 + \langle \partial_t \xi, \partial_x \chi_h \rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^p.$$

Letting $\chi_h = \partial_t \zeta \in \mathring{\mathcal{V}}_h^p$ and using the Cauchy-Schwarz inequality we have

$$\|\partial_{tx} \zeta(t)\|_0^2 \leq \|\partial_t \rho(t)\|_0 \|\partial_{tx} \zeta(t)\|_0 + \|\partial_t \xi(t)\|_0 \|\partial_{tx} \zeta(t)\|_0 .$$

Using the Poincaré inequality yields

$$\|\partial_t \zeta(t)\|_1 \leq C\Big(\|\partial_t \rho(t)\|_0 + \|\partial_t \xi(t)\|_0 \Big). \tag{4.2.33}$$

Therefore, we complete the proof by noting (4.2.15) and (4.2.31). $\square$

By combining the results in Lemma 4.2.2–Lemma 4.2.4, we are now ready to state the results for a priori error estimates of (4.1.7)–(4.1.8) by (4.1.12)–(4.1.13).

Theorem 4.2.5. *Assume that* $u \in W^1_\infty(0,T;H^1_0(\Omega) \cap H^{p+1}(\Omega))$ *and* $v \in W^1_\infty(0,T;H^{q+1}(\Omega))$. *Assume further that* V_h *and* $\bar{V}_h$ *satisfy* $V_h - \bar{V}_h = \partial_t V_h - \partial_t \bar{V}_h = 0$ *at* $t = 0$. *Then, for* $j = 0, 1$ *there exists a positive constant* $C > 0$ *independent of* h *such that*

$$\begin{aligned}
\|e_h(t)\|_j &\le C h^{\min(p+1-j,q+1)} \Big(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} \Big), \\
\|f_h(t)\|_j &\le C h^{\min(p+1,q+1-j)} \Big(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} \Big), \\
\|\partial_t e_h(t)\|_j &\le C h^{\min(p+1-j,q+1)} \Big(\|u\|_{L^\infty(H^{p+1})} + \|\partial_t u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} \\
&\qquad + \|\partial_t v\|_{L^\infty(H^{q+1})} \Big), \\
\|\partial_t f_h(t)\|_j &\le C h^{\min(p+1,q+1-j)} \Big(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^\infty(H^{q+1})} \Big).
\end{aligned}$$

4.3 A posteriori error estimation

In this section, we design strategies to compute a posteriori error estimates from the finite element solutions U_h and V_h computed by using (4.1.12)–(4.1.13).

We approximate the exact errors e_h and f_h, see (4.2.5)–(4.2.6), by $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$, respectively. As in [54] and [55], the approximate errors E and F should maintain some properties of the exact errors. Moreover, our strategies involve the *local* computation of E and F on each subinterval to facilitate parallel computation.

From (4.1.12)–(4.1.13) and (4.1.7)–(4.1.8), we infer that the exact errors satisfy for all $\chi_h \in \mathring{\mathcal{V}}_h^p$

$$\langle \partial_x e_h(t), \partial_x \chi_h \rangle_0 = \langle f_h(t), \partial_x \chi_h \rangle_0 \tag{4.3.1}$$

and for all $w_h \in \mathcal{V}_h^q$

$$\begin{aligned}
&\langle \partial_t f_h(t), w_h \rangle_1 - \langle e_h(t) f_h(t), \partial_x w_h \rangle_0 - \langle U_h(t) f_h(t), \partial_x w_h \rangle_0 - \langle V_h(t) e_h(t), \partial_x w_h \rangle_0 \\
&\quad - \langle f_h(t), \partial_x w_h \rangle_0 = \langle U_h(t) V_h(t), \partial_x w_h \rangle_0 + \langle V_h(t), \partial_x w_h \rangle_0 - \langle \partial_t V_h(t), w_h \rangle_1 .
\end{aligned} \tag{4.3.2}$$

At $t = 0$, noting (4.1.9) and (4.1.14), we have

$$\langle f_h(0), w_h \rangle_1 = 0 \quad \forall w_h \in \mathcal{V}_h^q. \tag{4.3.3}$$

Due to (4.1.13), the right hand side of (4.3.2) vanishes. However, for the purpose of developing a posteriori error estimates, we keep these terms in the equation as an indication of how the a posteriori error estimation should be.

Recalling $\langle\cdot,\cdot\rangle_{0,\Omega_l}$ defined by (2.2.1) on the subinterval Ω_l, $l = 1, \ldots, N$, we propose to compute E and F locally on each subinterval Ω_l, by one of the following two methods.

Method (i): ***Nonlinear parabolic error estimate***

Let $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$ be defined on Ω_l by

$$\langle \partial_x E(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} = \langle F(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} \quad \forall \hat{\chi}_h \in \mathcal{S}_h^{p+1}, \tag{4.3.4}$$

and

$$\begin{aligned} &\langle \partial_t F(t), \hat{w}_h \rangle_{1,\Omega_l} - \langle E(t)F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle U_h(t)F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle V_h(t)E(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} \\ &- \langle F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} = \langle U_h(t)V_h(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} \\ &- \langle \partial_t V_h(t), \hat{w}_h \rangle_{1,\Omega_l} \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}, \end{aligned} \tag{4.3.5}$$

when $t \in (0, T]$ and

$$\langle F(0), \hat{w}_h \rangle_{1,\Omega_l} = \langle v_0, \hat{w}_h \rangle_{1,\Omega_l} - \langle V_h(0), \hat{w}_h \rangle_{1,\Omega_l} \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1} \tag{4.3.6}$$

when $t = 0$.

Method (ii): ***Linear parabolic error estimate***

Let $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$ be defined on Ω_l by

$$\langle \partial_x E(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} = \langle F(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} \tag{4.3.7}$$

for all $\hat{\chi}_h \in \mathcal{S}_h^{p+1}$ and

$$\begin{aligned} &\langle \partial_t F(t), \hat{w}_h \rangle_{1,\Omega_l} - \langle U_h(t)F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle V_h(t)E(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} \\ &= \langle U_h(t)V_h(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle \partial_t V_h(t), \hat{w}_h \rangle_{1,\Omega_l} \end{aligned} \tag{4.3.8}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$ and when $t \in (0, T]$. The initial condition at $t = 0$ is defined by

$$\langle F(0), \hat{w}_h \rangle_{1,\Omega_l} = \langle v_0, \hat{w}_h \rangle_{1,\Omega_l} - \langle V_h(0), \hat{w}_h \rangle_{1,\Omega_l} \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}. \tag{4.3.9}$$

In the computation of a posteriori error estimator F, an additional saving on the computation cost can be obtained by neglecting the term $\langle E(t)F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l}$, therefore reducing the nonlinear equation (4.3.5) of Method (i) by the linear equation (4.3.8) of Method (ii).

Since $\operatorname{supp} \phi_{l,p+1} = \bar{\Omega}_l$, if $\psi \in H^0(\Omega)$ then the statement

$$\langle \psi, \hat{\chi}_h \rangle_0 = 0 \quad \forall \hat{\chi}_h \in \mathcal{S}_h^{p+1}$$

is equivalent to

$$\langle \psi, \hat{\chi}_h \rangle_{0,\Omega_l} = 0 \quad \forall \hat{\chi}_h \in \mathcal{S}_h^{p+1}, \quad l = 1, \ldots, N.$$

Hence, in fact (4.3.4)–(4.3.9) hold for the inner product on the whole domain Ω.

In order to emphasize on the polynomial degrees p and q, as in the remaining part we will need different projection onto spaces of different polynomial degree, we rewrite $\bar{U}_{h,p} := \bar{U}_h \in W^1_\infty(0,T; \mathring{\mathcal{V}}_h^p)$ and $\bar{V}_{h,q} := \bar{V}_h \in W^1_\infty(0,T; \mathcal{V}_h^q)$, where $\bar{U}_h$ and $\bar{V}_h$ are defined by (4.2.3)–(4.2.4). For example, by $\bar{U}_{h,p+1}$ we mean the projection defined by (4.2.3) onto the space $\mathring{\mathcal{V}}_h^{p+1}$ instead of $\mathring{\mathcal{V}}_h^p$.

Let $\bar{V}_{h,q+1} \in \mathcal{V}_h^{q+1}$ be defined by

$$\left\langle v - \bar{V}_{h,q+1}, w_h \right\rangle_1 = 0 \quad \forall w_h \in \mathcal{V}_h^{q+1}. \tag{4.3.10}$$

By noting that $\mathcal{V}_h^{q+1} = \mathcal{V}_h^q \oplus \mathcal{S}_h^{q+1}$, we can write $\bar{V}_{h,q+1}$ as

$$\bar{V}_{h,q+1} = \tilde{V}_{h,q} + \tilde{f}_h \quad \text{where} \quad \tilde{V}_{h,q} \in \mathcal{V}_h^q \quad \text{and} \quad \tilde{f}_h \in \mathcal{S}_h^{q+1}. \tag{4.3.11}$$

Letting

$$\hat{e}(t) := \|e_h(t)\|_1 + \|f_h(t)\|_1$$

where e_h and f_h are defined by (4.2.5)–(4.2.6) and letting

$$\hat{E}(t) := \|E(t)\|_1 + \|F(t)\|_1 \,,$$

we define the effectivity index by

$$\Theta(t) = \frac{\hat{E}(t)}{\hat{e}(t)}.$$

We now state the main result of this section.

Conjecture 4.3.1. *For any $V \in H^{q+1}(\Omega)$, let $\tilde{V}_{h,q}, \bar{V}_{h,q} \in \mathcal{V}_h^q$ be defined respectively by* (4.3.11) *and* (4.2.4). *Then we conjecture that:*

$$\left\|\tilde{V}_{h,q} - \bar{V}_{h,q}\right\|_1 \leq Ch^{q+1}, \tag{4.3.12}$$

where C is independent of h.

We computed $\left\|\tilde{V}_{h,q} - \bar{V}_{h,q}\right\|_1$ for the following functions V, with $\Omega = [0, 1]$.

1. Example 1: $V(x) = x^2 \exp(2x + 1) - x \exp(3)$;
2. Example 2: $V(x) = \sin(\pi x)$;
3. Example 3: $V(x) = \log(1 + x) - x \log(2)$;
4. Example 4: $V(x) = x^{5/2}$;
5. Example 5: $V(x) = x^{7/3}$.

The numerical convergence orders κ_V presented in Table 4.1–Table 4.5 justify our conjecture. In fact, the results of Example 1, Example 2 and Example 3, where the functions V are smooth show a higher convergence order for $q \geq 2$, namely $O(h^{q+2})$ rather than $O(h^{q+1})$.

Theorem 4.3.2. *Let $\tilde{V}_{h,q}$ and $\bar{V}_{h,q}$ be defined respectively by* (4.3.11) *and* (4.2.4). *Assume that the Conjecture 4.3.1 holds and*

$$\hat{e}(t) \geq Ch^{\min(p,q)}. \tag{4.3.13}$$

Then the approximate errors $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$ defined by Method (i) and Method (ii) satisfy for almost all $t \in [0, T]$

$$\lim_{h \to 0} \Theta(t) = 1.$$

We now provide the proof of Theorem 4.3.2, which is based on the following lemmas. For the analysis, we first define $\bar{e}_h \in \mathcal{S}_h^{p+1}$ and $\bar{f}_h \in \mathcal{S}_h^{q+1}$ such that for $l = 1, \ldots, N$

$$\langle \partial_x \bar{e}_h(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} = \left\langle \partial_x u(t) - \partial_x \bar{U}_{h,p}(t), \partial_x \hat{\chi}_h \right\rangle_{0,\Omega_l} \quad \forall \hat{\chi}_h \in \mathcal{S}_h^{p+1} \tag{4.3.14}$$

q	h	$\left\Vert \tilde{V}_{h,q} - \bar{V}_{h,q} \right\Vert_1$	κ_V
1	1/4	4.2572E-01	
	1/8	1.0773E-01	1.983
	1/16	2.7017E-02	1.996
	1/32	6.7594E-03	1.998
	1/64	1.6902E-03	1.999
	1/128	4.2257E-04	2.000
2	1/4	1.8450E-03	
	1/8	1.1865E-04	3.958
	1/16	7.4697E-06	3.989
	1/32	4.6771E-07	3.997
	1/64	2.9245E-08	3.999
	1/128	1.8275E-09	4.000
3	1/4	3.5090E-05	
	1/8	1.1253E-06	4.963
	1/16	3.5397E-08	4.991
	1/32	1.1080E-09	4.998
	1/64	3.8795E-11	4.836

Table 4.1: Experiments justifying (4.3.12) by Example 1.

q	h	$\left\Vert \tilde{V}_{h,q} - \bar{V}_{h,q} \right\Vert_1$	κ_V
1	1/4	3.3110E-02	
	1/8	8.2216E-03	2.009
	1/16	2.0521E-03	2.002
	1/32	5.1281E-04	2.001
	1/64	1.2819E-04	2.000
	1/128	3.2047E-05	2.000
2	1/4	1.1657E-04	
	1/8	7.3449E-06	3.988
	1/16	4.5997E-07	3.997
	1/32	2.8762E-08	3.999
	1/64	1.7979E-09	3.999
	1/128	1.1243E-10	3.999
3	1/4	2.2898E-06	
	1/8	7.2066E-08	4.989
	1/16	2.2561E-09	4.997
	1/32	7.0535E-11	4.999
	1/64	2.9263E-12	4.591

Table 4.2: Experiments justifying (4.3.12) by Example 2.

q	h	$\left\Vert \tilde{V}_{h,q} - \bar{V}_{h,q} \right\Vert_1$	κ_V
1	1/4	2.6102E-03	
	1/8	6.5485E-04	1.995
	1/16	1.6386E-04	1.998
	1/32	4.0974E-05	1.999
	1/64	1.0244E-05	1.999
	1/128	2.5611E-06	2.000
2	1/4	4.5141E-06	
	1/8	2.8871E-07	3.967
	1/16	1.8155E-08	3.991
	1/32	1.1364E-09	3.998
	1/64	7.1053E-11	3.999
	1/128	4.4416E-12	3.999
3	1/4	7.1648E-08	
	1/8	2.3339E-09	4.940
	1/16	7.3752E-11	4.984
	1/32	2.3113E-12	4.996
	1/64	1.4046E-13	4.041

Table 4.3: Experiments justifying (4.3.12) by Example 3.

q	h	$\left\|\tilde{V}_{h,q} - \bar{V}_{h,q}\right\|_1$	κ_V
1	1/4	1.3135e-02	
	1/8	3.2771e-03	2.003
	1/16	8.1861e-04	2.001
	1/32	2.0459e-04	2.001
	1/64	5.1141e-05	2.000
	1/128	1.2785e-05	2.000
2	1/4	1.8952e-05	
	1/8	1.2979e-06	3.868
	1/16	8.7633e-08	3.889
	1/32	5.8552e-09	3.904
	1/64	3.8811e-10	3.915
	1/128	2.5601e-11	3.922
3	1/4	4.9335e-07	
	1/8	3.0902e-08	3.996
	1/16	1.9326e-09	3.999
	1/32	1.2081e-10	4.999
	1/64	7.5972e-12	3.991

Table 4.4: Experiments justifying (4.3.12) by Example 4.

q	h	$\left\Vert \tilde{V}_{h,q} - \bar{V}_{h,q} \right\Vert_1$	κ_V
1	1/4	1.2242e-02	
	1/8	3.0525e-03	2.004
	1/16	7.6225e-04	2.002
	1/32	1.9047e-04	2.001
	1/64	4.7609e-05	2.000
	1/128	1.1901e-05	2.000
2	1/4	1.4244e-05	
	1/8	1.0518e-06	3.759
	1/16	7.6583e-08	3.780
	1/32	5.5227e-09	3.794
	1/64	3.9556e-10	3.803
	1/128	2.8213e-11	3.809
3	1/4	5.4899e-07	
	1/8	3.8583e-08	3.831
	1/16	2.7082e-09	3.833
	1/32	1.9002e-10	3.833
	1/64	1.3360e-11	3.830

Table 4.5: Experiments justifying (4.3.12) by Example 5.

and

$$\left\langle \bar{f}_h(t), \hat{w}_h \right\rangle_{1,\Omega_l} = \left\langle v(t) - \bar{V}_{h,q}(t), \hat{w}_h \right\rangle_{1,\Omega_l} \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}. \tag{4.3.15}$$

Letting

$$\bar{\eta} = u - \left(\bar{U}_{h,p} + \bar{e}_h\right), \tag{4.3.16}$$

$$\bar{\zeta} = \bar{e}_h - E, \tag{4.3.17}$$

$$\bar{\rho} = v - \left(\bar{V}_{h,q} + \bar{f}_h\right), \tag{4.3.18}$$

and

$$\bar{\xi} = \bar{f}_h - F, \tag{4.3.19}$$

we rewrite

$$e_h - E = \bar{\eta} + \bar{\zeta} + \zeta \tag{4.3.20}$$

and

$$f_h - F = \bar{\rho} + \bar{\xi} + \xi. \tag{4.3.21}$$

In the following part, our aim is to estimate each of the terms $\bar{\eta}$, $\bar{\zeta}$, ζ, $\bar{\rho}$, $\bar{\xi}$ and ξ of (4.3.20)–(4.3.21). The estimates of $\|\zeta(t)\|_1$ and $\|\xi(t)\|_1$ are presented in Section 4.2.

We first focus on estimating $\|\bar{\eta}(t)\|_1$.

Lemma 4.3.3. *Assume that $u \in W^1_\infty(0,T; H^1_0(\Omega) \cap H^{p+2}(\Omega))$ then there holds*

$$\|\bar{\eta}\|_{W^1_\infty(H^1)} \leq C h^{p+1} \|u\|_{L^\infty(H^{p+2})}. \tag{4.3.22}$$

Proof. Due to orthogonality of the Legendre polynomials, we have

$$\langle \partial_x \bar{e}_h(t), \partial_x \chi_h \rangle_0 = 0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^p. \tag{4.3.23}$$

Therefore, by noting $\mathring{\mathcal{V}}_h^{p+1} = \mathring{\mathcal{V}}_h^p \oplus \mathcal{S}_h^{p+1}$, using (4.3.14), (4.2.3) and (4.3.23), we have

$$\langle \bar{\eta}(t), \partial_x \chi_h \rangle_0 = \left\langle \partial_x u(t) - \partial_x \bar{U}_{h,p}(t) - \partial_x \bar{e}_h(t), \partial_x \chi_h \right\rangle_0 = 0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^{p+1}.$$

By noting that

$$\left\langle \partial_x u(t) - \partial_x \bar{U}_{h,p+1}(t), \partial_x \chi_h \right\rangle_0 = 0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^{p+1}$$

thus, we have $\bar{U}_{h,p} + \bar{e}_h$ is the elliptic projection $\bar{U}_{h,p+1}$ of u into $\mathring{\mathcal{V}}_h^{p+1}$. Therefore, we have for any $\bar{\chi}_h \in \mathring{\mathcal{V}}_h^{p+1}$

$$\begin{aligned}
\|\partial_x \bar{\eta}(t)\|_0^2 &= \left\langle \partial_x u - \partial_x \bar{U}_h - \partial_x \bar{e}_h, \partial_x u - \partial_x \bar{U}_h - \partial_x \bar{e}_h \right\rangle_0 \\
&= \left\langle \partial_x u - \partial_x \bar{U}_h - \partial_x \bar{e}_h, \partial_x u - \partial_x \bar{\chi}_h \right\rangle_0 \\
&\quad + \left\langle \partial_x u - \partial_x \bar{U}_h - \partial_x \bar{e}_h, \partial_x \bar{\chi}_h - \partial_x \bar{U}_h - \partial_x \bar{e}_h \right\rangle_0 \\
&= \left\langle \partial_x u - \partial_x \bar{U}_h - \partial_x \bar{e}_h, \partial_x u - \partial_x \bar{\chi}_h \right\rangle_0 \\
&\le \left\| \partial_x u(t) - \partial_x \bar{U}_h(t) - \partial_x \bar{e}_h(t) \right\|_0 \|\partial_x u(t) - \partial_x \bar{\chi}_h\|_0 .
\end{aligned}$$

Hence, by using the Poincaré inequality we have

$$\|\bar{\eta}(t)\|_1 \le C \|\partial_x \bar{\eta}(t)\|_0 \le C \inf_{\bar{\chi}_h \in \mathring{\mathcal{V}}_h^{p+1}} \|\partial_x u(t) - \partial_x \bar{\chi}_h\|_0 \le C h^{p+1} \|u(t)\|_{p+2} ,$$

thus proving the lemma. □

We now focus on estimating $\|\bar{\rho}(t)\|_1$, $\left\|\bar{\zeta}(t)\right\|_1$ and $\left\|\bar{\xi}(t)\right\|_1$.

Lemma 4.3.4. *Assume that $v \in W^1_\infty(0,T;H^{q+2}(\Omega))$. Let $\tilde{V}_{h,q}$ and $\bar{V}_{h,q}$ be defined respectively by* (4.3.11) *and* (4.2.4)*, and assume that the Conjecture 4.3.1 holds. Then there holds*

$$\|\bar{\rho}\|_{W^1_\infty(H^1)} \le C h^{q+1} \|v\|_{L^\infty(H^{q+2})} .$$

Proof. By noting (4.3.10) and using standard finite element arguments we have

$$\left\| v(t) - \bar{V}_{h,q+1}(t) \right\|_1 \le C h^{q+1} \|v(t)\|_{q+2} .$$

By using the triangle inequality we obtain

$$\begin{aligned}
\|\bar{\rho}(t)\|_1 &= \left\| v(t) - \bar{V}_{h,q}(t) - \bar{f}_h(t) \right\|_1 \\
&\le \left\| v(t) - \bar{V}_{h,q+1}(t) \right\|_1 + \left\| \bar{V}_{h,q+1}(t) - \bar{V}_{h,q}(t) - \bar{f}_h(t) \right\|_1 \\
&\le C h^{q+1} + \left\| \bar{V}_{h,q+1}(t) - \bar{V}_{h,q}(t) - \bar{f}_h(t) \right\|_1 . \qquad (4.3.24)
\end{aligned}$$

The lemma will be proved if we can prove $\left\| \bar{V}_{h,q+1}(t) - \bar{V}_{h,q}(t) - \bar{f}_h(t) \right\|_1 \le C h^{q+1}$. Noting that $\bar{V}_{h,q+1} - \bar{V}_{h,q} - \bar{f}_h \in \mathcal{V}_h^{q+1}$, due to (4.3.10) we have

$$\left\langle \bar{V}_{h,q+1}, \bar{V}_{h,q+1} - \bar{V}_{h,q} - \bar{f}_h \right\rangle_1 = \left\langle v, \bar{V}_{h,q+1} - \bar{V}_{h,q} - \bar{f}_h \right\rangle_1$$

Therefore, (4.3.11) together with (4.3.15) give

$$
\begin{aligned}
&\left\|\bar{V}_{h,q+1}(t)-\bar{V}_{h,q}(t)-\bar{f}_h(t)\right\|_1^2=\left\langle v-\bar{V}_{h,q}-\bar{f}_h,\bar{V}_{h,q+1}-\bar{V}_{h,q}-\bar{f}_h\right\rangle_1\\
&=\left\langle v-\bar{V}_{h,q}-\bar{f}_h,\tilde{V}_{h,q}-\bar{V}_{h,q}\right\rangle_1+\left\langle v-\bar{V}_{h,q}-\bar{f}_h,\tilde{f}_h-\bar{f}_h\right\rangle_1\\
&=\left\langle \bar{V}_{h,q+1}-\bar{V}_{h,q}-\bar{f}_h,\tilde{V}_{h,q}-\bar{V}_{h,q}\right\rangle_1\\
&\le\left\|\bar{V}_{h,q+1}(t)-\bar{V}_{h,q}(t)-\bar{f}_h(t)\right\|_1\left\|\tilde{V}_{h,q}(t)-\bar{V}_{h,q}(t)\right\|_1.
\end{aligned}
$$

This implies

$$
\left\|\bar{V}_{h,q+1}(t)-\bar{V}_{h,q}(t)-\bar{f}_h(t)\right\|_1\le\left\|\tilde{V}_{h,q}(t)-\bar{V}_{h,q}(t)\right\|_1.
$$

If $v\in H_0^1(\Omega)$ and if $\bar{V}_{h,q}$ and $\bar{V}_{h,q+1}$ are defined with the H_0^1-inner product then it follows from (4.3.23) that $\tilde{V}_{h,q}=\bar{V}_{h,q}$. In the present case with the H^1-inner product our numerical experiments show that $\left\|\tilde{V}_{h,q}(t)-\bar{V}_{h,q}(t)\right\|_1\le Ch^{q+1}$. Since we are unable to prove this result, we resort to posing a conjecture (see Conjecture 4.3.1). With this conjecture, we obtain the required result. □

In the following lemma, we estimate $\bar{\zeta}$ and $\bar{\xi}$ (see (4.3.19) and (4.3.17)) in the $H^1(\Omega)$-norm.

Lemma 4.3.5. *Assume that $u\in W^1_\infty(0,T;H_0^1(\Omega)\cap H^{p+2}(\Omega))$ and $v\in W^1_\infty(0,T;H^{q+2}(\Omega))$. Then the following estimates hold:*

$$
\|\bar{\zeta}\|_{W^1_\infty(H^1)}\le Ch^\gamma\left(\|u\|_{L^\infty(H^{p+2})}+\|v\|_{L^\infty(H^{q+2})}\right)\tag{4.3.25}
$$

and

$$
\|\bar{\xi}\|_{W^1_\infty(H^1)}\le Ch^\gamma\left(\|u\|_{L^\infty(H^{p+2})}+\|v\|_{L^\infty(H^{q+2})}\right)\tag{4.3.26}
$$

where $\gamma=\min(p+1,q+1)$.

Proof. We present the proof only for the case that E and F are defined by (4.3.7) and (4.3.8). It follows from (4.1.7) and (4.3.7) that

$$
\langle\partial_x u,\partial_x\hat{\chi}_h\rangle_0-\langle\partial_x E,\partial_x\hat{\chi}_h\rangle_0=\langle v,\partial_x\hat{\chi}_h\rangle_0-\langle F,\partial_x\hat{\chi}_h\rangle_0-\langle V_h,\partial_x\hat{\chi}_h\rangle_0\tag{4.3.27}
$$

for any $\hat{\chi}_h\in\mathcal{S}_h^{p+1}$. It follows from (4.3.20) that

$$
u-E=U_h+e_h-E=U_h+\bar{\eta}+\bar{\zeta}+\zeta.
$$

Similarly, it follows from (4.3.21) that

$$v - F - V_h = f_h - F = \bar\rho + \bar\xi + \xi.$$

Hence, (4.3.27) can be rewritten as

$$\begin{aligned}&\langle \partial_x U_h, \partial_x \hat\chi_h\rangle_0 + \langle \partial_x \bar\eta, \partial_x \hat\chi_h\rangle_0 + \left\langle \partial_x \bar\zeta, \partial_x \hat\chi_h\right\rangle_0 + \langle \partial_x \zeta, \partial_x \hat\chi_h\rangle_0\\ &= \langle \bar\rho, \partial_x \hat\chi_h\rangle_0 + \left\langle \bar\xi, \partial_x \hat\chi_h\right\rangle_0 + \langle \xi, \partial_x \hat\chi_h\rangle_0 \quad \forall \hat\chi_h \in \mathcal{S}_h^{p+1}. \qquad (4.3.28)\end{aligned}$$

Due to orthogonality of the Legendre polynomials we have $\langle \partial_x U_h, \partial_x \hat\chi_h\rangle_0 = 0$ and due to (4.3.14) we have $\langle \partial_x \bar\eta, \partial_x \hat\chi_h\rangle_0 = 0$. Therefore, (4.3.28) gives

$$\left\langle \partial_x \bar\zeta, \partial_x \hat\chi_h\right\rangle_0 = \langle \bar\rho, \partial_x \hat\chi_h\rangle_0 + \left\langle \bar\xi, \partial_x \hat\chi_h\right\rangle_0 + \langle \xi, \partial_x \hat\chi_h\rangle_0 - \langle \partial_x \zeta, \partial_x \hat\chi_h\rangle_0 \quad \forall \hat\chi_h \in \mathcal{S}_h^{p+1}.$$

Substituting $\hat\chi_h = \bar\zeta \in \mathcal{S}_h^{p+1}$ and using Hölder's inequality give

$$\left\|\partial_x \bar\zeta(t)\right\|_0^2 \le \Big(\|\bar\rho(t)\|_0 + \left\|\bar\xi(t)\right\|_0 + \|\xi(t)\|_0 + \|\partial_x \zeta(t)\|_0 \Big) \left\|\partial_x \bar\zeta(t)\right\|_0.$$

By dividing both sides by $\left\|\partial_x \bar\zeta(t)\right\|_0$ and using the Poincaré inequality, we deduce

$$\left\|\bar\zeta(t)\right\|_1 \le \|\bar\rho(t)\|_0 + \left\|\bar\xi(t)\right\|_0 + \|\xi(t)\|_0 + \|\zeta(t)\|_1. \qquad (4.3.29)$$

Referring to Lemma 4.3.4 for $\|\bar\rho(t)\|_0$ and Lemma 4.2.3 for $\|\xi(t)\|_0$ and $\|\zeta(t)\|_1$, we obtain

$$\left\|\bar\zeta\right\|_{L^\infty(H^1)} \le C h^\gamma \Big(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+2})} \Big) + \left\|\bar\xi\right\|_{L^\infty(H^0)}. \qquad (4.3.30)$$

Therefore, (4.3.25) is proved if we prove (4.3.26).

Due to (4.3.19) and (4.3.18), we have

$$\bar\xi = \bar f_h - F = v - \bar V_{h,q} - \bar\rho - F = \rho - \bar\rho - F.$$

Therefore, for any $\hat w_h \in \mathcal{S}_h^{q+1}$

$$\left\langle \partial_t \bar\xi, \hat w_h\right\rangle_1 = \langle \partial_t \rho, \hat w_h\rangle_1 - \langle \partial_t \bar\rho, \hat w_h\rangle_1 - \langle \partial_t F, \hat w_h\rangle_1. \qquad (4.3.31)$$

By differentiating (4.3.15) with respect to t, the second term on the right hand side of (4.3.31) vanishes:

$$\langle \partial_t \bar\rho, \hat w_h\rangle_1 = \left\langle \partial_t v - \partial_t \bar V_{h,q} - \partial_t \bar f_h, \hat w_h\right\rangle_1 = 0 \quad \forall \hat w_h \in \mathcal{S}_h^{q+1}. \qquad (4.3.32)$$

From (4.3.8),

$$\langle \partial_t F, \hat{w}_h \rangle_1 = \langle U_h F, \partial_x \hat{w}_h \rangle_0 + \langle V_h E, \partial_x \hat{w}_h \rangle_0 + \langle F, \partial_x \hat{w}_h \rangle_0 + \langle U_h V_h, \partial_x \hat{w}_h \rangle_0 + \langle V_h, \partial_x \hat{w}_h \rangle_0 - \langle \partial_t V_h, \hat{w}_h \rangle_1 \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}. \tag{4.3.33}$$

Combining (4.3.31)–(4.3.33), we have

$$\left\langle \partial_t \bar{\xi}, \hat{w}_h \right\rangle_1 = \langle \partial_t \rho, \hat{w}_h \rangle_1 - \langle U_h F, \partial_x \hat{w}_h \rangle_0 - \langle V_h E, \partial_x \hat{w}_h \rangle_0 - \langle F, \partial_x \hat{w}_h \rangle_0 - \langle U_h V_h, \partial_x \hat{w}_h \rangle_0 - \langle V_h, \partial_x \hat{w}_h \rangle_0 + \langle \partial_t V_h, \hat{w}_h \rangle_1 \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}. \tag{4.3.34}$$

For the last term on the right hand side of (4.3.34) we use (4.1.8) to have

$$\langle \partial_t V_h, \hat{w}_h \rangle_1 = \langle \partial_t v, \hat{w}_h \rangle_1 - \langle \partial_t f_h, \hat{w}_h \rangle_1 = \langle uv, \partial_x \hat{w}_h \rangle_0 + \langle v, \partial_x \hat{w}_h \rangle_0 - \langle \partial_t f_h, \hat{w}_h \rangle_1 \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}. \tag{4.3.35}$$

Therefore, (4.3.35) and (4.3.34) give

$$\left\langle \partial_t \bar{\xi}, \hat{w}_h \right\rangle_1 = \langle \partial_t \rho, \hat{w}_h \rangle_1 - \langle U_h F, \partial_x \hat{w}_h \rangle_0 - \langle V_h E, \partial_x \hat{w}_h \rangle_0 - \langle F, \partial_x \hat{w}_h \rangle_0 - \langle U_h V_h, \partial_x \hat{w}_h \rangle_0 - \langle V_h, \partial_x \hat{w}_h \rangle_0 + \langle uv, \partial_x \hat{w}_h \rangle_0 + \langle v, \partial_x \hat{w}_h \rangle_0 - \langle \partial_t f_h, \hat{w}_h \rangle_1$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$. Noting that $\rho - f_h = -\xi$, $F = \rho - \bar{\rho} - \bar{\xi}$, $E = \eta - \bar{\eta} - \bar{\zeta}$ and $v - V_h - F = \bar{\rho} + \bar{\xi} + \xi$, we have

$$\left\langle \partial_t \bar{\xi}, \hat{w}_h \right\rangle_1 = - \langle \partial_t \xi, \hat{w}_h \rangle_1 - \left\langle U_h(\rho - \bar{\rho} - \bar{\xi}), \partial_x \hat{w}_h \right\rangle_0 - \left\langle V_h(\eta - \bar{\eta} - \bar{\zeta}), \partial_x \hat{w}_h \right\rangle_0 + \left\langle \bar{\rho} + \bar{\xi} + \xi, \partial_x \hat{w}_h \right\rangle_0 - \langle U_h V_h, \partial_x \hat{w}_h \rangle_0 + \langle uv, \partial_x \hat{w}_h \rangle_0 \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}.$$

Rewriting

$$uv - U_h V_h = v(e_h + U_h) - U_h V_h = v\, e_h + U_h(v - V_h) = v\, e_h + U_h f_h,$$

we obtain

$$\left\langle \partial_t \bar{\xi}, \hat{w}_h \right\rangle_1 = - \langle \partial_t \xi, \hat{w}_h \rangle_1 - \left\langle U_h(\rho - \bar{\rho} - \bar{\xi}), \partial_x \hat{w}_h \right\rangle_0 - \left\langle V_h(\eta - \bar{\eta} - \bar{\zeta}), \partial_x \hat{w}_h \right\rangle_0 + \left\langle \bar{\rho} + \bar{\xi} + \xi, \partial_x \hat{w}_h \right\rangle_0 + \langle v\, e_h, \partial_x \hat{w}_h \rangle_0 + \langle U_h f_h, \partial_x \hat{w}_h \rangle_0 . \tag{4.3.36}$$

Substituting $\hat{w}_h = \bar{\xi} \in \mathcal{S}_h^{q+1}$, integrating (4.3.36) from 0 to t and noting $\bar{\xi}(0) = 0$ give

$$\begin{aligned}
\left\|\bar{\xi}(t)\right\|_1^2 = &- \int_0^t \left\langle \partial_t \xi(s), \bar{\xi}(s) \right\rangle_1 \, ds - \int_0^t \left\langle U_h(s)\big(\rho(s) - \bar{\rho}(s) - \bar{\xi}(s)\big), \partial_x \bar{\xi}(s) \right\rangle_0 \, ds \\
&- \int_0^t \left\langle V_h(s)\big(\eta(s) - \bar{\eta}(s) - \bar{\zeta}(s)\big), \partial_x \bar{\xi}(s) \right\rangle_0 \, ds \\
&+ \int_0^t \left\langle \bar{\rho}(s) + \bar{\xi}(s) + \xi(s), \partial_x \bar{\xi}(s) \right\rangle_0 \, ds + \int_0^t \left\langle v(s) e_h(s), \partial_x \bar{\xi}(s) \right\rangle_0 \, ds \\
&+ \int_0^t \left\langle U_h(s) f_h(s), \partial_x \bar{\xi}(s) \right\rangle_0 \, ds =: \mathscr{T}_1 + \cdots + \mathscr{T}_6. \qquad (4.3.37)
\end{aligned}$$

Recalling the Hölder inequality and Lemma 4.2.1, we estimate

$$\begin{aligned}
|\mathscr{T}_2| &\le \|U_h\|_{L^\infty(L^\infty)} \int_0^t \left(\|\rho(s)\|_0 + \|\bar{\rho}(s)\|_0 + \left\|\bar{\xi}(s)\right\|_0\right) \left\|\partial_x \bar{\xi}(s)\right\|_0 \, ds \\
&\le C \left(\|\rho\|_{L^2(H^0)}^2 + \|\bar{\rho}\|_{L^2(H^0)}^2\right) + \int_0^t \left\|\bar{\xi}(s)\right\|_1^2 \, ds \\
&\le C h^{2(q+1)} \|v\|_{L^\infty(H^{q+2})} + \int_0^t \left\|\bar{\xi}(s)\right\|_1^2 \, ds.
\end{aligned}$$

Similarly, together with (4.3.22) and (4.3.29), we estimate

$$\begin{aligned}
|\mathscr{T}_3| &\le \|V_h\|_{L^\infty(L^\infty)} \int_0^t \left(\|\eta(s)\|_0 + \|\bar{\eta}(s)\|_0 + \left\|\bar{\zeta}(s)\right\|_0 \right) \left\|\partial_x \bar{\xi}(s)\right\|_0 \, ds \\
&\le C \left(\|\eta\|_{L^2(H^0)}^2 + \|\bar{\eta}\|_{L^2(H^0)}^2 + \left\|\bar{\zeta}\right\|_{L^2(H^0)}^2\right) + \int_0^t \left\|\bar{\xi}(s)\right\|_1^2 \, ds \\
&\le C h^{2(p+1)} \|u\|_{L^\infty(H^{p+2})} + C \left(\|\bar{\rho}\|_{L^2(H^0)}^2 + \left\|\bar{\xi}\right\|_{L^2(H^0)}^2 + \|\xi\|_{L^2(H^0)}^2 + \|\zeta\|_{L^2(H^0)}^2\right) \\
&\quad + \int_0^t \left\|\bar{\xi}(s)\right\|_1^2 \, ds \\
&\le C h^{2\gamma} \left(\|u\|_{L^\infty(H^{p+2})} + \|v\|_{L^\infty(H^{q+2})}\right) + \int_0^t \left\|\bar{\xi}(s)\right\|_1^2 \, ds
\end{aligned}$$

where $\gamma = \min(p+1, q+1)$. By referring to Theorem 4.2.5, we estimate

$$\begin{aligned}
|\mathscr{T}_5| + |\mathscr{T}_6| &\le \|v\|_{L^\infty(L^\infty)} \int_0^t \|e_h(s)\|_0 \left\|\bar{\xi}(s)\right\|_1 \, ds + \|U_h\|_{L^\infty(L^\infty)} \int_0^t \|f_h(s)\|_0 \left\|\bar{\xi}(s)\right\|_1 \, ds \\
&\le C \|e_h\|_{L^2(H^0)}^2 + C \|f_h\|_{L^2(H^0)}^2 + C \int_0^t \left\|\bar{\xi}(s)\right\|_1^2 \, ds \\
&\le C h^{2\gamma} \left(\|u\|_{L^\infty(H^{p+1})}^2 + \|v\|_{L^\infty(H^{q+1})}^2\right) + C \int_0^t \left\|\bar{\xi}(s)\right\|_1^2 \, ds.
\end{aligned}$$

The remaining terms can be estimated in a simpler way, noting that (4.2.31) is used for $\|\partial_t \xi(t)\|_0$. Hence, (4.3.37) yields

$$\left\|\bar{\xi}(t)\right\|_1^2 \le C h^{2\gamma} \left(\|u\|_{L^\infty(H^{p+2})}^2 + \|v\|_{L^\infty(H^{q+2})}^2\right) + C \int_0^t \left\|\bar{\xi}(s)\right\|_1^2 \, ds.$$

The desired estimate (4.3.26) followed by using Lemma 2.3.2. □

Proof of Theorem 4.3.2

We are now ready to prove the main result stated in the Theorem 4.3.2.

Proof. By referring to (4.3.20)–(4.3.21) and using the triangle inequality, we deduce

$$
\begin{aligned}
|\Theta(t)-1| = \left|\frac{\hat{E}(t)}{\hat{e}(t)}-1\right| &= \frac{\left|\left(\|E(t)\|_1+\|F(t)\|_1\right)-\left(\|e_h(t)\|_1+\|f_h(t)\|_1\right)\right|}{\left|\hat{e}(t)\right|} \\
&\leq \frac{\left|\,\|E(t)\|_1-\|e_h(t)\|_1\right|+\left|\,\|F(t)\|_1-\|f_h(t)\|_1\right|}{\left|\hat{e}(t)\right|} \\
&\leq \frac{\|\bar{\eta}(t)\|_1+\left\|\bar{\zeta}(t)\right\|_1+\|\zeta(t)\|_1+\|\bar{\rho}(t)\|_1+\left\|\bar{\xi}(t)\right\|_1+\|\xi(t)\|_1}{\left|\hat{e}(t)\right|}.
\end{aligned}
$$

By using (4.3.22), Lemma 4.3.5, Lemma 4.2.3, Lemma 4.3.4 and inequality (4.3.13), we infer

$$
|\Theta(t)-1| \leq Ch,
$$

thus proving the theorem. □

4.4 Implementation issues

In this section, we show the computations of (U_h, V_h) and (E, F) respectively by using (4.1.12)–(4.1.14) and Method (ii) (see page 40).

With $\phi_{l,k}$, $l = 1, \ldots, N$ and $k = 1, 2, \ldots$, defined by (2.4.2)– (2.4.4), the approximate solution (U_h, V_h) can be represented as

$$
\begin{aligned}
U_h(x,t) &= \sum_{l=2}^{N} U_{l,1}(t)\phi_{l,1}(x) + \sum_{l=1}^{N}\sum_{k=2}^{p} U_{l,k}(t)\phi_{l,k}(x), \\
U_h(x,t) &= \sum_{l=1}^{N+1} V_{l,1}(t)\phi_{l,1}(x) + \sum_{l=1}^{N}\sum_{k=2}^{q} V_{l,k}(t)\phi_{l,k}(x).
\end{aligned}
$$

Let the L^2-inner products

$$
\alpha_{k,k'}^{l,l'} = \left\langle \phi_{l,k}, \phi_{l',k'} \right\rangle_0, \tag{4.4.1}
$$

$$
\bar{\alpha}_{k,k'}^{l,l'} = \left\langle \partial_x\phi_{l,k}, \partial_x\phi_{l',k'} \right\rangle_0, \tag{4.4.2}
$$

and

$$\beta_{k,k'}^{l,l'} = \left\langle \phi_{l,k}, \partial_x \phi_{l',k'} \right\rangle_0 . \tag{4.4.3}$$

For each $l = 1, \ldots, N$, and $r, r' = 2, 3, \ldots$, we define a 2×2 matrix $\boldsymbol{M}_{1,1}^l$, a $2 \times (r-1)$ matrix $\boldsymbol{M}_{1,r}^l$, and an $(r-1) \times (r'-1)$ matrix $\boldsymbol{M}_{r,r'}^l$ with entries, respectively,

$$\begin{aligned}
(\boldsymbol{M}_{1,1}^l)_{ij} &= \alpha_{1,1}^{l+j-1,l+i-1}, && i,j = 1,2 \\
(\boldsymbol{M}_{1,r}^l)_{ij} &= \alpha_{j,1}^{l,l+i-1}, && i = 1,2, \quad j = 2,\ldots,r, \\
(\boldsymbol{M}_{r,r'}^l)_{ij} &= \alpha_{j,i}^{l,l}, && i = 2,\ldots,r,\ j = 2,\ldots,r'.
\end{aligned}$$

Similarly, we define $\boldsymbol{S}_{1,1}^l$, $\boldsymbol{S}_{1,r}^l$, $\boldsymbol{S}_{r,r'}^l$ by using $\bar{\alpha}_{r,r}^{l,l'}$, and $\boldsymbol{B}_{1,1}^l$, $\boldsymbol{B}_{1,r}^l$, $\boldsymbol{B}_{r,r'}^l$ by using $\beta_{r,r}^{l,l'}$, instead of $\alpha_{r,r}^{l,l'}$. By using these notations, we then define matrices

$$\boldsymbol{M}_r^l = \begin{pmatrix} \boldsymbol{M}_{1,1}^l & \boldsymbol{M}_{1,r}^l \\ \left(\boldsymbol{M}_{1,r}^l\right)^\top & \boldsymbol{M}_{rr}^l \end{pmatrix}, \tag{4.4.4}$$

$$\boldsymbol{S}_r^l = \begin{pmatrix} \boldsymbol{S}_{1,1}^l & \boldsymbol{S}_{1,r}^l \\ \left(\boldsymbol{S}_{1,r}^l\right)^\top & \boldsymbol{S}_{rr}^l \end{pmatrix} \tag{4.4.5}$$

and

$$\boldsymbol{B}_{r,r'}^l = \begin{pmatrix} \boldsymbol{B}_{1,1}^l & \boldsymbol{B}_{1,r'}^l \\ \left(\boldsymbol{B}_{1,r}^l\right)^\top & \boldsymbol{B}_{r,r'}^l \end{pmatrix}, \tag{4.4.6}$$

where $\top$ denotes the transpose matrix. We note that the matrices $\boldsymbol{M}_r^l$ and $\boldsymbol{S}_r^l$ have size $(r+1) \times (r+1)$, whereas the matrix $\boldsymbol{B}_{r,r'}^l$ has size $(r+1) \times (r'+1)$.

By using $\boldsymbol{M}_r^l$, $\boldsymbol{S}_r^l$ and $\boldsymbol{B}_{r,r'}^l$, we assemble the global matrices $\boldsymbol{M}_r$, $\boldsymbol{S}_r$ and $\boldsymbol{B}_{r,r'}$, respectively. We note that the sizes of the global matrices $\boldsymbol{M}_r$ and $\boldsymbol{S}_r$ are $(Nr+1) \times (Nr+1)$ and of $\boldsymbol{B}_{r,r'}$ is $(Nr+1) \times (Nr'+1)$.

We also define, for each $l = 1, \ldots, N$ and $p, q = 2, 3, \ldots$,

$$\boldsymbol{U}^l = [U_{l,1}, U_{l+1,1}, U_{l,2} \ldots, U_{l,p}]^\top$$

and

$$\boldsymbol{V}^l = [V_{l,1}, V_{l+1,1}, V_{l,2} \ldots, V_{l,q}]^\top$$

where $U_{1,1}$ and $U_{N+1,1}$ are zeros. The vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ are of size $(Np+1) \times 1$ and $(Nq+1) \times 1$, respectively, and are assembled from the vectors $\boldsymbol{U}^l$ and $\boldsymbol{V}^l$.

With the above matrices defined, the matrix representation of (4.1.12)–(4.1.13) is of the form

$$\boldsymbol{S}_p \boldsymbol{U}(t) = \boldsymbol{B}_{p,q} \boldsymbol{V}(t), \tag{4.4.7}$$

$$(\boldsymbol{M}_q + \boldsymbol{S}_q)\, \partial_t \boldsymbol{V}(t) = \boldsymbol{H}(\boldsymbol{U}(t), \boldsymbol{V}(t)). \tag{4.4.8}$$

Here, the vector $\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{V})$ is an $(Nq+1) \times 1$ vector defined by

$$\boldsymbol{H}(\boldsymbol{U}, \boldsymbol{V}) = [\boldsymbol{H}^{(0)}, \boldsymbol{H}^{(1)}, \dots, \boldsymbol{H}^{(N)}]^\top$$

where

$$\boldsymbol{H}^{(0)} = \left[\langle (\boldsymbol{U}+1)\boldsymbol{V}, \phi_{1,1} \rangle_0, \langle (\boldsymbol{U}+1)\boldsymbol{V}, \phi_{2,1} \rangle_0, \dots, \langle (\boldsymbol{U}+1)\boldsymbol{V}, \phi_{N+1,1} \rangle_0\right]^\top$$

and

$$\boldsymbol{H}^{(l)} = \left[\langle (\boldsymbol{U}+1)\boldsymbol{V}, \phi_{l,2} \rangle_0, \langle (\boldsymbol{U}+1)\boldsymbol{V}, \phi_{l,3} \rangle_0, \dots, \langle (\boldsymbol{U}+1)\boldsymbol{V}, \phi_{l,q} \rangle_0\right]^\top$$

for $l = 1, \dots, N$. We use the Matlab ODE solver to solve (4.4.7)–(4.4.8). Therefore, the right hand side of (4.4.8) is computed by first solving (4.4.7) for a given $\boldsymbol{V}(t)$.

In the following part, we discuss the computation of (E, F) which are computed locally on each Ω_l, for $l = 1, \dots, N$, from the approximate solutions (U_h, V_h), and have the forms

$$E(x,t) = \sum_{l=1}^{N} E_l(t)\phi_{l,p+1}(x) \quad \text{and} \quad F(x,t) = \sum_{l=1}^{N} F_l(t)\phi_{l,q+1}(x).$$

Equations (4.3.8) and (4.3.7) are rewritten as

$$\begin{aligned} &\langle \partial_t F(t), \phi_{l,q+1} \rangle_{1,\Omega_l} - \langle U_h(t)F(t), \partial_x \phi_{l,q+1} \rangle_{0,\Omega_l} - \langle V_h(t)E(t), \partial_x \phi_{l,q+1} \rangle_{0,\Omega_l} \\ &\quad - \langle F(t), \partial_x \phi_{l,q+1} \rangle_{0,\Omega_l} = \langle U_h(t)V_h(t), \partial_x \phi_{l,q+1} \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \phi_{l,q+1} \rangle_{0,\Omega_l} \\ &\quad - \langle \partial_t V_h(t), \phi_{l,q+1} \rangle_{1,\Omega_l} \end{aligned} \tag{4.4.9}$$

and

$$\langle \partial_x E(t), \partial_x \phi_{l,p+1} \rangle_{0,\Omega_l} = \langle F(t), \partial_x \phi_{l,p+1} \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \phi_{l,p+1} \rangle_{0,\Omega_l}. \tag{4.4.10}$$

Recalling (4.4.1)–(4.4.3), we have

$$\begin{aligned} \langle \partial_t V_h(t), \phi_{l,q+1} \rangle_{1,\Omega_l} &= \partial_t V_{l+1,1}(t) \left(\alpha_{1,q+1}^{l+1,l} + \bar{\alpha}_{1,q+1}^{l+1,l}\right) + \sum_{k'=1}^{q} \partial_t V_{l,k'}(t) \left(\alpha_{k',q+1}^{l,l} + \bar{\alpha}_{k',q+1}^{l,l}\right) \\ &:= T_1, \end{aligned}$$

$$\langle V_h(t), \partial_x \phi_{l,q+1} \rangle_{0,\Omega_l} = V_{l+1,1}(t)\beta^{l+1,l}_{1,q+1} + \sum_{k'=1}^{q} V_{l,k'}(t)\beta^{l,l}_{k',q+1} := T_2 \tag{4.4.11}$$

and

$$\langle V_h(t), \partial_x \phi_{l,p+1} \rangle_{0,\Omega_l} = V_{l+1,1}(t)\beta^{l+1,l}_{1,p+1} + \sum_{k'=1}^{p} V_{l,k'}(t)\beta^{l,l}_{k',p+1} := T_3. \tag{4.4.12}$$

We note that

$$\alpha^{l,l}_{p+1,p+1} = \frac{h_l}{(2p+3)(2p-1)}, \tag{4.4.13}$$

$$\bar{\alpha}^{l,l}_{p+1,p+1} = \frac{2}{h_l} \tag{4.4.14}$$

and

$$\beta^{l,l}_{p+1,q+1} = \begin{cases} \frac{1}{\sqrt{(2q+3)(2q+1)}}, & p = q+1, \\ \frac{-1}{\sqrt{(2q+1)(2q-1)}}, & p = q-1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4.15}$$

By defining

$$\bar{\beta}^{l}_{\bar{k},k',q} = \left\langle \phi_{l,\bar{k}}\phi_{l,k'}, \partial_x \phi_{l,q+1} \right\rangle_{0,\Omega_l},$$

$$\tilde{\beta}^{l}_{\bar{k},k',q} = \left\langle \phi_{l+1,\bar{k}}\phi_{l,k'}, \partial_x \phi_{l,q+1} \right\rangle_{0,\Omega_l}$$

and

$$\hat{\beta}^{l}_{\bar{k},k',q} = \left\langle \phi_{l+1,\bar{k}}\phi_{l+1,k'}, \partial_x \phi_{l,q+1} \right\rangle_{0,\Omega_l},$$

we have

$$\begin{aligned} \langle U_h(t)F(t), \partial_x \phi_{l,q+1} \rangle_{0,\Omega_l} &= F_l(t) \left[U_{l+1,1}(t)\tilde{\beta}^{l}_{1,q+1,q} + \sum_{k=1}^{p} U_{l,k}(t)\bar{\beta}^{l}_{k,q+1,q} \right] \\ &:= T_4 F_l(t), \end{aligned} \tag{4.4.16}$$

$$\begin{aligned} \langle V_h(t)E_l(t), \partial_x \phi_{l,q+1} \rangle_{0,\Omega_l} &= E_l \left[V_{l+1,1}(t)\tilde{\beta}^{l}_{1,p+1,q} + \sum_{k'=1}^{q} V_{l,k'}(t)\bar{\beta}^{l}_{k',p+1,q} \right] \\ &:= T_5 E_l(t), \end{aligned} \tag{4.4.17}$$

and

$$
\begin{aligned}
\langle U_h(t)V_h(t), \partial_x\phi_{l,q+1}\rangle_{0,\Omega_l} &= U_{l+1,1}(t)\left[V_{l+1,1}(t)\hat{\beta}^l_{1,1,q} + \sum_{k'=1}^{q} V_{l,k'}(t)\bar{\beta}^l_{1,k',q+1}\right] \\
&\quad + \sum_{k=1}^{p} U_{l,k}(t)\left[V_{l+1,1}(t)\hat{\beta}^l_{k,1,q} + \sum_{k'=1}^{q} V_{l,k'}(t)\bar{\beta}^l_{k,k',q}\right] \\
&:= T_6. \qquad (4.4.18)
\end{aligned}
$$

Note that, the values of $\bar{\beta}^l_{k,k',q}$, $\tilde{\beta}^l_{k,k',q}$ and $\hat{\beta}^l_{k,k',q}$ can be computed by using Matlab or Maple.

By using the above notations, (4.4.9)–(4.4.10) can be rewritten as

$$
\left(\frac{h_l}{(2q+3)(2q-1)} + \frac{2}{h_l}\right)\partial_t F_l(t) - \left(T_4 + \beta^{l,l}_{q+1,q+1}\right)F_l(t) - T_5 E_l(t) = T_6 + T_2 - T_1
$$

and

$$
\frac{2}{h_l}E_l(t) = \beta^{l,l}_{p+1,q+1}F_l(t) + T_3.
$$

Then, by using the Backward Euler Formulation, we compute $F_l(t_j)$ recursively by

$$
F_l(t_j) = \frac{T_6 + T_2 - T_1 + \frac{h_l}{2}T_5T_3 + dF_l(t_{j-1})}{d - T_4 - \beta^{l,l}_{q+1,q+1} - \frac{h_l}{2}\beta^{l,l}_{p+1,q+1}T_5} \qquad (4.4.19)
$$

where

$$
d = \left(\frac{h_l}{(2q+3)(2q-1)} + \frac{2}{h_l}\right)\left(\frac{1}{t_j - t_{j-1}}\right)
$$

and $t_j = j\triangle t$ for $j = 1, 2, 3, \ldots$. The time step $\triangle t$ is chosen to be not less than h.

4.5 Numerical experiment

In this section, we present the numerical results when we solve the following BBM equation

$$
\begin{aligned}
\partial_t u(x,t) - \frac{\mu}{d^2}\partial_{xxt}u(x,t) + \frac{1}{d}u(x,t)\partial_x u(x,t) &= g(x,t), \quad x \in (0,1),\ t \in (0,1], \\
u(0,t) = u(1,t) &= 0, \qquad\quad t \in [0,1],
\end{aligned}
$$

whose exact solution (u, v) is

$$
u(x,t) = 3\,\mathrm{sech}^2\big(k(dx - a)\big) - 3(1-x)\,\mathrm{sech}^2\big(-k\,a\big) - 3x\,\mathrm{sech}^2\big(k(d-a)\big)
$$

and

$$v(x,t) = -6kd\,\mathrm{sech}^2\big(k(dx-a)\big)\tanh\big(k(dx-a)\big) - 3\mathrm{sech}^2\big(k(d-a)\big) + 3\mathrm{sech}^2\big(-k\,a\big),$$

where the constants are $a = x_0 + \nu t$, $k = 0.35$, $\mu = 1/4k^2$, $\nu = 0.5$, $x_0 = 20$, and $d = 2x_0$. The initial value v_0 is given by

$$v_0(x) = -6kd\,\mathrm{sech}^2\big(k(dx-x_0)\big)\tanh\big(k(dx-x_0)\big) - 3\mathrm{sech}^2\big(k(d-x_0)\big) + 3\mathrm{sech}^2\big(-k\,x_0\big).$$

In the numerical experiment, we compute the approximate solutions (U_h, V_h) by solving (4.1.12)–(4.1.13). After that, the errors e_h and f_h are computed and the order of convergence given by Theorem 4.2.5 is checked. Finally, we compute the error estimates E and F by using the linear parabolic error estimate (see Method (ii) in page 40).

In Table 4.6, the relative exact errors $\frac{\|e_h(t)\|_1}{\|u(t)\|_1}$ and $\frac{\|f_h(t)\|_1}{\|v(t)\|_1}$ for N elements and $t = 0.8$ are presented. We compute the relative errors for $p = q+1$, with $p = 2, 3, 4$.

In Table 4.7, we present the error estimations $\hat{E}(t)$ and exact errors $\hat{e}(t)$ at $t = 0.8$. For these numerical results, we choose $\triangle t = 0.4$. Figure 4.5 shows the computed effectivity indices for the numbers in Table 4.7. The results show that our a posteriori error estimations are efficient.

dof_u	dof_v	p	q	N	$\|e_h(t)\|_1 \,/\, \|u(t)\|_1$	κ_u	$\|f_h(t)\|_1 \,/\, \|v(t)\|_1$	κ_v
39	21			20	1.6102E-01		4.8905E-01	
79	41			40	3.5722E-02	2.172	2.3826E-01	1.037
159	81			80	8.8314E-03	2.016	1.1937E-01	0.997
319	161	2	1	160	2.2015E-03	2.004	5.9711E-02	0.999
639	321			320	5.4999E-04	2.001	2.9859E-02	1.000
1279	641			640	1.3747E-04	2.000	1.4930E-02	1.000
2559	1281			1280	3.4349E-05	2.001	7.4651E-03	1.000
59	41			20	1.7780E-02		1.1054E-01	
119	81			40	2.8360E-03	2.648	3.2042E-02	1.787
239	161	3	2	80	3.6641E-04	2.952	8.1051E-03	1.983
479	321			160	4.6186E-05	2.988	2.0321E-03	1.996
959	641			320	5.7854E-06	2.997	5.0839E-04	1.999
79	61			20	3.5710E-03		2.8850E-02	
159	121			40	2.1257E-04	4.070	3.4321E-03	2.703
239	181	4	3	60	4.2714E-05	3.958	1.0320E-03	2.964
319	241			80	1.3589E-05	3.981	4.3735E-04	2.984
399	301			100	5.5803E-06	3.989	2.2440E-04	2.990

Table 4.6: The orders of convergence κ_u and κ_v at $t = 0.8$.

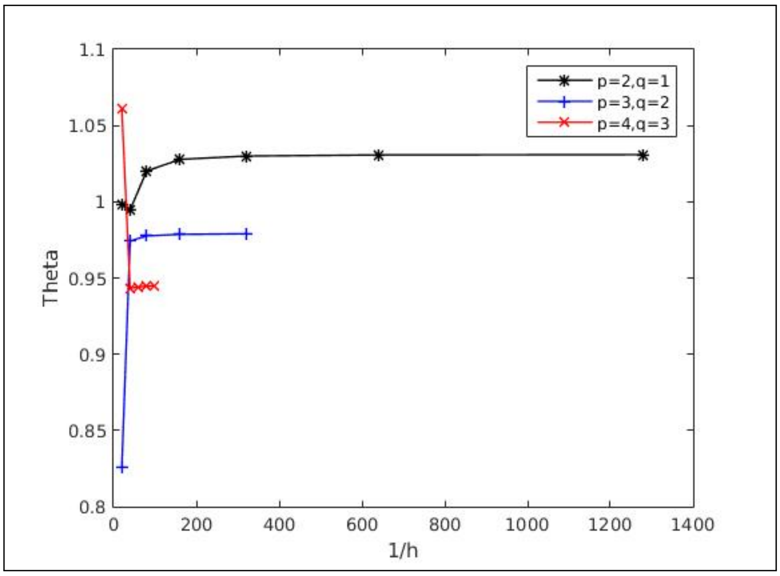

Figure 4.1: Effectivity indices Θ by Method (ii) at $t = 0.8$ with different values of h.

p	q	h	$\hat{E}(t)$	$\hat{e}(t)$
2	1	1/20	1.3590E+02	1.3616E+02
		1/40	6.5493E+01	6.5840E+01
		1/80	3.3539E+01	3.2879E+01
		1/160	1.6879E+01	1.6422E+01
		1/320	8.4519E+00	8.2053E+00
		1/640	4.2271E+00	4.1012E+00
		1/1280	2.1136E+00	2.0502E+00
3	2	1/20	2.5227E+01	3.0560E+01
		1/40	8.6034E+00	8.8314E+00
		1/80	2.1798E+00	2.2298E+00
		1/160	5.4657E-01	5.5853E-01
		1/320	1.3673E-01	1.3967E-01
4	3	1/20	8.4505E+00	7.9635E+00
		1/40	8.9120E-01	9.4488E-01
		1/60	2.6800E-01	2.8387E-01
		1/80	1.1359E-01	1.2025E-01
		1/100	5.8283E-02	6.1681E-02

Table 4.7: Values of $\hat{E}$ by Method (ii) at $t = 0.8$.

Chapter 5

Burgers equation: a mixed finite element method

In this chapter, we focus on a priori and a posteriori error estimations of H1MFEM for the Burgers equation. In Section 5.1, we elaborate on a formulation of weak solutions and finite element scheme for the Burgers equation. Section 5.2 provides an analysis of a priori error estimation. Section 5.3 focuses on an analysis of a posteriori error estimation. The chapter ends with implementation and numerical experiments in Section 5.4 and Section 5.5.

Throughout this chapter, C denoted a generic constant which may take different values at different occurrences.

5.1 Formulation of weak solutions and finite element scheme

Let $\Omega := (0,1)$ and T and ν (viscosity coefficient) be positive constants. We consider the following Burgers equation

$$\partial_t u(x,t) - \nu \partial_{xx} u(x,t) + u(x,t) \partial_x u(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T], \tag{5.1.1}$$

with Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \in [0,T], \tag{5.1.2}$$

and initial condition

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{5.1.3}$$

where $\partial_t := \partial/\partial t$, $\partial_x := \partial/\partial x$, $\partial_{xx} := \partial^2/\partial x^2$.

By H1MFEM, equation (5.1.1) is reduced to a system of first order equations using a new variable define as $v = \partial_x u$ [45, 47]. Thus, (5.1.1) is reformulated as

$$\partial_x u(x,t) = v(x,t), \tag{5.1.4}$$

$$\partial_t u(x,t) - \nu \partial_x v(x,t) + u(x,t)v(x,t) = 0 \tag{5.1.5}$$

and for $t = 0$

$$v(x,0) = v_0(x) = \partial_x u_0(x). \tag{5.1.6}$$

Multiplying (5.1.4) by $\partial_x \chi$ and (5.1.5) by $-\partial_x w$, where $\chi \in H_0^1(\Omega)$ and $w \in H^1(\Omega)$, we have

$$\langle \partial_x u(t), \partial_x \chi \rangle_0 = \langle v(t), \partial_x \chi \rangle_0 \quad \forall \chi \in H_0^1(\Omega)$$

and

$$\langle \partial_t v(t), w \rangle_0 + \nu \langle \partial_x v(t), \partial_x w \rangle_0 = \langle u(t)v(t), \partial_x w \rangle_0 \quad \forall w \in H^1(\Omega),$$

noting that $\partial_t u(0,t) = \partial_t u(1,t) = 0$.

A weak formulation of the problem reads: Given $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, find $(u,v) : [0,T] \to H_0^1(\Omega) \times H^1(\Omega)$ such that for $t > 0$

$$\langle \partial_x u(t), \partial_x \chi \rangle_0 = \langle v(t), \partial_x \chi \rangle_0 \quad \forall \chi \in H_0^1(\Omega), \tag{5.1.7}$$

$$\langle \partial_t v(t), w \rangle_0 + \nu \langle \partial_x v(t), \partial_x w \rangle_0 = \langle u(t)v(t), \partial_x w \rangle_0 \quad \forall w \in H^1(\Omega), \tag{5.1.8}$$

and

$$\langle v(0), w \rangle_0 = \langle \partial_x u_0, w \rangle_0 \quad \forall w \in H^1(\Omega) \tag{5.1.9}$$

for $t = 0$.

Lemma 5.1.1. *If $u \in W_\infty^1(0,T; H_0^1(\Omega) \cap H^2(\Omega))$, $v \in W_\infty^1(0,T; H^1(\Omega))$ and (u,v) satisfies (5.1.7)–(5.1.8), then we have (u,v) are solutions to (5.1.4)–(5.1.5).*

Proof. Indeed, by using integration by parts we deduce from (5.1.7) that

$$\partial_x \left(v(x,t) - \partial_x u(x,t)\right) = 0$$

in $W^1_\infty(0,T;H^0(\Omega))$ which implies that

$$v(x,t) = \partial_x u(x,t) + g(t) \tag{5.1.10}$$

for all t and a.e. in Ω, and for some function g depending on t. Particularly, we have

$$v(x,0) = \partial_x u(x,0) + g(0). \tag{5.1.11}$$

By integrating (5.1.11) over Ω, noting (5.1.9) and (5.1.6), we infer that $g(0) = 0$. On the other hand, it follows from differentiating (5.1.10) with respect to t and (5.1.8) with $w = 1$, that

$$\int_\Omega \partial_{tx} u + g('t) = 0,$$

implying that $g'(t) = 0$. Therefore, we have $g \equiv 0$, that is (u,v) satisfies (5.1.4). A similar argument using integration by parts gives (5.1.5). □

Let p and q be two positive integers and $\mathring{\mathcal{V}}_h^p$ and $\mathcal{V}_h^q$ are the finite dimensional subspaces of $H_0^1(\Omega)$ and $H^1(\Omega)$ (see (2.4.5)). A semidiscrete approximation to (5.1.7)–(5.1.9) reads: Find $(U_h, V_h) : [0,T] \to \mathring{\mathcal{V}}_h^p \times \mathcal{V}_h^q$ such that for $t > 0$

$$\langle \partial_x U_h(t), \partial_x \chi_h \rangle_0 = \langle V_h(t), \partial_x \chi_h \rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^p, \tag{5.1.12}$$

$$\langle \partial_t V_h(t), w_h \rangle_0 + \nu \langle \partial_x V_h(t), \partial_x w_h \rangle_0 = \langle U_h(t) V_h(t), \partial_x w_h \rangle_0 \quad \forall w_h \in \mathcal{V}_h^q, \tag{5.1.13}$$

and for $t = 0$

$$\langle V_h(0), w_h \rangle_0 = \langle \partial_x u_0, w_h \rangle_0 \quad \forall w_h \in \mathcal{V}_h^q. \tag{5.1.14}$$

5.2 A priori error estimation

In this section we carry out a rigorous analysis for a priori error estimates for the approximation of the solution of (5.1.12)–(5.1.14) by that of (5.1.7)–(5.1.9), filling the gap in [47]. We first show the boundedness of the sequences $\{U_h\}$ and $\{V_h\}$.

Lemma 5.2.1. *If* (u, v) *and* (U_h, V_h) *are solutions to* (5.1.7)–(5.1.9) *and* (5.1.12)–(5.1.14), *respectively, and if*

$$\|\partial_t V_h(0)\|_0 \le C, \tag{5.2.1}$$

then the following inequality holds

$$\|U_h\|_{L^\infty(L^\infty)} + \|\partial_t U_h\|_{L^\infty(L^\infty)} + \|V_h\|_{L^\infty(L^\infty)} + \|\partial_t V_h\|_{L^2(L^\infty)} \le C(T).$$

Proof. Substituting $\chi_h = U_h$ and $w_h = V_h$ respectively in (5.1.12) and (5.1.13), and using the Hölder inequality, we have

$$\|\partial_x U_h(t)\|_0 \le \|V_h(t)\|_0 \tag{5.2.2}$$

and

$$\frac{1}{2}\frac{d}{dt}\|V_h(t)\|_0^2 + \nu\,\|\partial_x V_h(t)\|_0^2 \le \|U_h(t)\|_{L^\infty(\Omega)}\,\|V(t)\|_0\,\|\partial_x V_h(t)\|_0 . \tag{5.2.3}$$

The Sobolev imbedding theorem, Poincaré inequality, and (5.2.2) imply

$$\|U_h(t)\|_{L^\infty(\Omega)} \le C\,\|V_h(t)\|_0 . \tag{5.2.4}$$

Hence, by integrating (5.2.3) from 0 to t, using (4.2.2) and noting from (5.1.14) that

$$\|V_h(0)\|_0 \le \|\partial_x u_0\|_0 ,$$

we obtain

$$\begin{aligned}\|V_h(t)\|_0^2 + 2\nu\int_0^t \|\partial_x V_h(\tau)\|_0^2\,d\tau &\le \|V_h(0)\|_0^2 + C\int_0^t \|V_h(\tau)\|_0^2\,\|\partial_x V_h(\tau)\|_0\,d\tau \\ &\le C\left(1 + \int_0^t \|V_h(\tau)\|_0^4\,ds + \epsilon\int_0^t \|\partial_x V_h(\tau)\|_0^2\,d\tau\right).\end{aligned}$$

By choosing $\epsilon > 0$ sufficiently small so that $2\nu - C\epsilon > 0$ we deduce

$$\|V_h(t)\|_0^2 + \int_0^t \|\partial_x V_h(s)\|_0^2\,ds \le C\left(1 + \int_0^t \|V_h(s)\|_0^4\,ds\right). \tag{5.2.5}$$

Lemma 2.3.3 (with $\varphi(t) = \|V_h(t)\|_0^2$, $a = C$ and $\theta(s) = Cs^2$ for $s \ge 0$) gives

$$\|V_h(t)\|_0^2 \le C \quad \forall t \in [0, T^*],$$

where T^* is defined in Lemma 2.3.3. Since

$$\int_a^\sigma \frac{ds}{\theta(s)} \to \infty \quad \text{as} \quad \sigma \to 0,$$

we have $T^* = T$. Therefore, noting (5.2.5) we deduce

$$\|V_h(t)\|_0^2 + \int_0^t \|\partial_x V_h(s)\|_0^2 \, ds \le C(T) \quad \forall t \in [0, T].$$

This together with (5.2.4) implies

$$\|U_h\|_{L^\infty(L^\infty)} + \|V_h\|_{L^\infty(H^0)} + \|V_h\|_{L^2(H^1)} \le C(T). \tag{5.2.6}$$

We next show a bound for $\|\partial_t V_h(t)\|_0$. By differentiating (5.1.12) and (5.1.13) with respect to t, we have the following results for $t > 0$

$$\langle \partial_{tx} U_h(t), \partial_x \chi_h \rangle_0 = \langle \partial_t V_h(t), \partial_x \chi_h \rangle_0 \tag{5.2.7}$$

for any $\chi_h \in \mathring{\mathcal{V}}_h^p$ and

$$\begin{aligned} \langle \partial_{tt} V_h(t), w_h \rangle_0 + \nu \, \langle \partial_{tx} V_h(t), \partial_x w_h \rangle_0 &= \langle \partial_t U_h(t) \, V_h(t), \partial_x w_h \rangle_0 \\ &\quad + \langle U_h \partial_t V_h(t), \partial_x w_h \rangle_0 \end{aligned} \tag{5.2.8}$$

for any $w_h \in \mathcal{V}_h^q$. Substituting $\chi_h = \partial_t U_h$ into (5.2.7), $w_h = \partial_t V_h$ into (5.2.8), and using the same argument as above we obtain

$$\|\partial_t U_h(t)\|_{L^\infty(\Omega)} \le C \, \|\partial_t V_h(t)\|_0 \tag{5.2.9}$$

and

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\partial_t V_h(t)\|_0^2 + \nu \, \|\partial_{tx} V_h(t)\|_0^2 &\le \|\partial_t U_h(t)\|_{L^\infty(\Omega)} \|V_h(t)\|_0 \, \|\partial_{tx} V_h(t)\|_0 \\ &\quad + \|U_h(t)\|_{L^\infty(\Omega)} \|\partial_t V_h(t)\|_0 \, \|\partial_{tx} V_h(t)\|_0 \\ &\le C(T) \, \|\partial_t V_h(t)\|_0 \, \|\partial_{tx} V_h(t)\|_0 \,, \end{aligned}$$

where in the last step we have used (5.2.6) and (5.2.9). By integrating the above inequality from 0 to t, using (5.2.1), inequality (4.2.2) and invoking Lemma 2.3.2 (Gronwall's Lemma), we deduce

$$\|\partial_t V_h(t)\|_0^2 + \int_0^t \|\partial_{tx} V_h(s)\|_0^2 \, ds \le C(T).$$

By noting (5.2.9), we have

$$\|\partial_t U_h\|_{L^\infty(L^\infty)} + \|\partial_t V_h\|_{L^\infty(H^0)} + \|\partial_t V_h\|_{L^2(H^1)} \le C(T). \tag{5.2.10}$$

By using [35, Lemma 1.2] or [36, Theorem 3.1] we deduce from

$$\|V_h\|_{L^2(H^1)} + \|\partial_t V_h\|_{L^2(H^1)} \le C(T),$$

see (5.2.6) and (5.2.10), that (after a possible modification on a set of measure zero)

$$\|V_h\|_{L^\infty(H^1)} \le C(T),$$

which implies by the Sobolev imbedding theorem

$$\|V_h\|_{L^\infty(L^\infty)} \le C(T).$$

The final estimate $\|\partial_t V_h\|_{L^2(L^\infty)} \le C(T)$ also follows from (5.2.10) and the Sobolev imbedding theorem, completing the proof of the lemma. □

As is usual in the error analysis for parabolic equations, it is necessary to consider elliptic projections. We define $\bar{U}_h \in W^1_\infty(0,T;\mathring{\mathcal{V}}^p_h)$ and $\bar{V}_h \in W^1_\infty(0,T;\mathcal{V}^q_h)$ satisfying, for $t \in [0,T]$,

$$\left\langle \partial_x u(t) - \partial_x \bar{U}_h(t), \partial_x \chi_h \right\rangle_0 = 0 \quad \forall \chi_h \in \mathring{\mathcal{V}}^p_h, \tag{5.2.11}$$

$$\mathscr{A}(u(t); v(t) - \bar{V}_h(t), w_h) = 0 \quad \forall w_h \in \mathcal{V}^q_h, \tag{5.2.12}$$

where the bounded bilinear form $\mathscr{A}(u;\cdot,\cdot)$ on $H^1(\Omega)$ is defined by (see [45])

$$\mathscr{A}(u; \upsilon_1, \upsilon_2) = \nu \left\langle \partial_x \upsilon_1, \partial_x \upsilon_2 \right\rangle_0 - \left\langle u\upsilon_1, \partial_x \upsilon_2 \right\rangle_0 + \lambda \left\langle \upsilon_1, \upsilon_2 \right\rangle_0 \quad \forall \upsilon_1, \upsilon_2 \in H^1(\Omega). \tag{5.2.13}$$

Here, the constant $\lambda > 0$ is chosen appropriately to ensure that $\mathscr{A}(u;\cdot,\cdot)$ is $H^1(\Omega)$-coercive. We note that

$$\begin{aligned}
\mathscr{A}(u; v, v) &\ge \nu \|\partial_x v\|_0^2 - \|u\|_{L^\infty(\Omega)} \|v\|_0 \|\partial_x v\|_0 + \lambda \|v\|_0^2 \\
&\ge \nu \|\partial_x v\|_0^2 - C(\frac{1}{2\varepsilon} \|v\|_0^2 + \frac{\varepsilon}{2} \|\partial_x v\|_0^2) + \lambda \|v\|_0^2 \\
&\ge (\nu - \frac{C\varepsilon}{2}) \|\partial_x v\|_0^2 + (\lambda - \frac{C}{2\varepsilon}) \|v\|_0^2 .
\end{aligned}$$

By choosing $\epsilon > 0$ sufficiently small and $\lambda > 0$ sufficiently large so that

$$\alpha := \min\left(\nu - \frac{C\varepsilon}{2}, \lambda - \frac{C}{2\varepsilon}\right) > 0, \tag{5.2.14}$$

we deduce that $\mathscr{A}(u;\cdot,\cdot)$ is coercive and satisfies

$$\mathscr{A}(u;v,v) \geq \alpha \left\|v\right\|_1^2. \tag{5.2.15}$$

Let the errors in the approximation of (5.1.7)–(5.1.9) by (5.1.12)–(5.1.14) be denoted by e_h and f_h, i.e.,

$$e_h(x,t) := u(x,t) - U_h(x,t) \quad \text{and} \quad f_h(x,t) := v(x,t) - V_h(x,t). \tag{5.2.16}$$

Recalling the symbols η, ζ, ρ and ξ defined by (4.2.7)–(4.2.10) as

$$\eta = u - \bar{U}_h, \quad \zeta = \bar{U}_h - U_h, \quad \rho = v - \bar{V}_h, \quad \xi = \bar{V}_h - V_h,$$

we rewrite e_h and f_h as:

$$e_h(x,t) = \eta(x,t) + \zeta(x,t) \quad \text{and} \quad f_h(x,t) = \rho(x,t) + \xi(x,t). \tag{5.2.17}$$

Therefore, to estimate e_h and f_h, we estimate each of the terms η, ζ, ρ, and ξ. In the following lemma, we show the approximation properties of the elliptic projections defined in (5.2.11) and (5.2.12).

Lemma 5.2.2. *Assume that $u \in W^1_\infty(0,T;H^1_0(\Omega) \cap H^{p+1}(\Omega))$. Assume further that $v \in W^1_\infty(0,T;H^{q+1}(\Omega))$. Then the functions η and ρ defined by* (4.2.7) *and* (4.2.9) *satisfy for $j = 0,1$*

$$\left\|\eta(t)\right\|_j \leq Ch^{p+1-j} \left\|u(t)\right\|_{p+1}, \tag{5.2.18}$$

$$\left\|\partial_t\eta(t)\right\|_j \leq Ch^{p+1-j} \left\|\partial_t u(t)\right\|_{p+1}, \tag{5.2.19}$$

$$\left\|\rho(t)\right\|_j \leq Ch^{q+1-j} \left\|v(t)\right\|_{q+1}, \tag{5.2.20}$$

$$\left\|\partial_t\rho(t)\right\|_j \leq Ch^{q+1-j} \left(\left\|v(t)\right\|_{q+1} + \left\|\partial_t v(t)\right\|_{q+1}\right). \tag{5.2.21}$$

Moreover, if $\partial_{tt}v \in L^\infty(0,T;H^{q+1}(\Omega))$ and $\partial_{tt}\bar{V}_h$ belongs to $L^\infty(0,T;H^1(\Omega))$, then

$$\left\|\partial_{tt}\rho(t)\right\|_j \leq Ch^{q+1} \left(\left\|v(t)\right\|_{q+1} + \left\|\partial_t v(t)\right\|_{q+1} + \left\|\partial_{tt} v(t)\right\|_{q+1}\right), \quad j = 0,1. \tag{5.2.22}$$

Proof. By differentiating (5.2.11) with respect to t, it can be seen that $\partial_t \bar{U}_h$ is the elliptic projection of $\partial_t u$ such that

$$\langle \partial_{tx} U_h(t), \partial_x \chi_h \rangle_0 = \langle \partial_t V_h(t), \partial_x \chi_h \rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^p.$$

Hence, (5.2.18) and (5.2.19) with $j = 1$ follow from standard argument. The results for $j = 0$ can be obtained by using the Aubin–Nitsche trick, which is also called the duality argument; see e.g. [53, 58].

Similarly, (5.2.20) can be proved. The proof of (5.2.21) is slightly different since $\partial_t \bar{V}_h$ is not the elliptic projection of $\partial_t v$ if one differentiates (5.2.12) due to the fact that u also depends on t. More precisely, we have from (5.2.12)

$$\mathscr{A}(u(t); \partial_t \rho(t), w_h) = \langle (\partial_t u(t)) \rho(t), \partial_x w_h \rangle_0 \quad \forall w_h \in \mathcal{V}_h^q. \tag{5.2.23}$$

We follow the technique used in the proof of [58, Lemma 3.2]. Let $\bar{V}_h^* \in \mathcal{V}_h^q$ be the elliptic projection of $\partial_t v$ defined by

$$\mathscr{A}(u(t); \partial_t v(t) - \bar{V}_h^*(t), w_h) = 0 \quad \forall w_h \in \mathcal{V}_h^q. \tag{5.2.24}$$

Then

$$\left\| \partial_t v(t) - \bar{V}_h^*(t) \right\|_1 \le C h^q \left\| \partial_t v(t) \right\|_{q+1}. \tag{5.2.25}$$

By using (5.2.15) and (5.2.24) we obtain

$$\begin{aligned} \left\| \bar{V}_h^*(t) - \partial_t \bar{V}_h(t) \right\|_1^2 &\le C \mathscr{A}(u; \bar{V}_h^* - \partial_t \bar{V}_h, \bar{V}_h^* - \partial_t \bar{V}_h) \\ &= C \mathscr{A}(u; (\bar{V}_h^* - \partial_t v) + (\partial_t v - \partial_t \bar{V}_h), \bar{V}_h^* - \partial_t \bar{V}_h) \\ &= C \mathscr{A}(u; \partial_t \rho, \bar{V}_h^* - \partial_t \bar{V}_h). \end{aligned}$$

This and (5.2.23) yield

$$\begin{aligned} \left\| \bar{V}_h^*(t) - \partial_t \bar{V}_h(t) \right\|_1^2 &= C \left\langle (\partial_t u) \rho, \partial_x (\bar{V}_h^* - \partial_t \bar{V}_h) \right\rangle_0 \\ &\le C \left\| \partial_t u(t) \right\|_{L^\infty(\Omega)} \left\| \rho(t) \right\|_0 \left\| (\bar{V}_h^* - \partial_t \bar{V}_h)(t) \right\|_1, \end{aligned}$$

so that due to (5.2.20)

$$\begin{aligned} \left\| \bar{V}_h^*(t) - \partial_t \bar{V}_h(t) \right\|_1 &\le C \left\| \partial_t u(t) \right\|_{L^\infty(\Omega)} \left\| \rho(t) \right\|_0 \le C h^{q+1} \left\| \partial_t u(t) \right\|_{L^\infty(\Omega)} \left\| v(t) \right\|_{q+1} \\ &\le C h^{q+1} \left\| v(t) \right\|_{q+1} \end{aligned}$$

for h sufficiently small. This inequality together with (5.2.25) and the triangle inequality gives (5.2.21) in the case $j = 1$. The result for $j = 0$ can be obtained by using Aubin–Nitsche's trick again.

Finally, by differentiating (5.2.23) with respect to t we obtain

$$\mathscr{A}(u; \partial_{tt}\rho(t), w_h) = \langle 2\partial_t u(t)\ \partial_t \rho(t) + \partial_{tt} u(t)\rho(t), \partial_x w_h\rangle_0 \quad \forall w_h \in \mathcal{V}_h^q.$$

Inequality (5.2.22) follows by using the same argument as above, thus completing the proof. $\square$

The next lemma shows the superconvergence property of ξ and ζ.

Lemma 5.2.3. *Assume that $u \in W^1_\infty(0,T; H^1_0(\Omega) \cap H^{p+1}(\Omega))$ and $v \in W^1_\infty(0,T; H^{q+1}(\Omega))$. Then, the following estimates hold*

$$\|\zeta\|_{W^1_\infty(H^1)} \le Ch^\gamma \Big(\|u\|_{L^\infty(H^{p+1})} + \|\partial_t u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^\infty(H^{q+1})} \Big) \tag{5.2.26}$$

and

$$\|\xi\|_{L^\infty(H^1)} + \|\partial_t \xi\|_{L^\infty(H^0)} + \|\partial_t \xi\|_{L^2(H^1)} \le Ch^\gamma \Big(\|u\|_{L^\infty(H^{p+1})} + \|\partial_t u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^\infty(H^{q+1})} \Big), \tag{5.2.27}$$

where $\gamma = \min(p+1, q+1)$. The positive constant C depends on T, α and λ defined in (5.2.13) *and* (5.2.14).

Proof. First we note that by subtracting (5.1.12) from (5.1.7) we have

$$\langle \partial_x \eta, \partial_x \chi_h\rangle_0 + \langle \partial_x \zeta, \partial_x \chi_h\rangle_0 = \langle \rho, \partial_x \chi_h\rangle_0 + \langle \xi, \partial_x \chi_h\rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^p.$$

By using (5.2.11) we have

$$\langle \partial_x \zeta, \partial_x \chi_h\rangle_0 = \langle \rho, \partial_x \chi_h\rangle_0 + \langle \xi, \partial_x \chi_h\rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^p. \tag{5.2.28}$$

Substituting $\chi_h = \zeta \in \mathring{\mathcal{V}}_h^p$ in (5.2.28) and using the Cauchy Schwarz-inequality, we deduce

$$\|\partial_x \zeta(t)\|_0 \le \|\rho(t)\|_0 + \|\xi(t)\|_0 .$$

Since $\zeta(t) \in H_0^1(\Omega)$ this implies

$$\|\zeta(t)\|_1 \leq C\Big(\|\rho(t)\|_0 + \|\xi(t)\|_0 \Big). \tag{5.2.29}$$

Moreover, by differentiating (5.2.28) with respect to t we have

$$\langle \partial_{tx}\zeta, \partial_x \chi_h \rangle_0 = \langle \partial_t \rho, \partial_x \chi_h \rangle_0 + \langle \partial_t \xi, \partial_x \chi_h \rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^p. \tag{5.2.30}$$

By using the same argument we obtain

$$\|\partial_t\zeta(t)\|_1 \leq C\Big(\|\partial_t\rho(t)\|_0 + \|\partial_t\xi(t)\|_0 \Big). \tag{5.2.31}$$

Consequently, (5.2.26) is proved if we prove (5.2.27), noting that the bounds for $\|\rho(t)\|_0$ and $\|\partial_t\rho(t)\|_0$ are given by Lemma 5.2.2.

In the following part, we prove (5.2.27), which is divided into three steps.

Step 1: We first prove

$$\|\xi\|_{L^\infty(H^0)} + \|\xi\|_{L^2(H^1)} \leq Ch^\gamma \left(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^\infty(H^{q+1})} \right) \tag{5.2.32}$$

and prove (5.2.26).

By subtracting (5.1.13) from (5.1.8) and recalling (5.2.16), we obtain

$$\langle \partial_t f_h, w_h \rangle_0 + \nu \langle \partial_x f_h, \partial_x w_h \rangle_0 = \langle uv - U_h V_h, \partial_x w_h \rangle_0 \quad \forall w_h \in \mathcal{V}_h^q.$$

Since

$$uv - U_h V_h = u(f_h + V_h) - U_h V_h = u f_h + V_h e_h,$$

we have

$$\langle \partial_t f_h, w_h \rangle_0 + \nu \langle \partial_x f_h, \partial_x w_h \rangle_0 = \langle u f_h, \partial_x w_h \rangle_0 + \langle V_h e_h, \partial_x w_h \rangle_0 \quad \forall w_h \in \mathcal{V}_h^q. \tag{5.2.33}$$

On the other hand, it follows successively from (5.2.17) and (5.2.12)

$$\mathscr{A}(u; \xi, w_h) = \mathscr{A}(u; f_h + \rho, w_h) = \mathscr{A}(u; f_h, w_h),$$

and the definition (5.2.13) of $\mathscr{A}(u; \cdot, \cdot)$ that

$$\mathscr{A}(u; \xi, w_h) = \nu \langle \partial_x f_h, \partial_x w_h \rangle_0 - \langle u f_h, \partial_x w_h \rangle_0 + \lambda \langle f_h, w_h \rangle_0 . \tag{5.2.34}$$

Rewriting

$$\langle \partial_t f_h, w_h \rangle_0 = \langle \partial_t \rho, w_h \rangle_0 + \langle \partial_t \xi, w_h \rangle_0$$

and adding (5.2.33) and (5.2.34) give

$$\langle \partial_t \xi, w_h \rangle_0 + \mathscr{A}(u; \xi, w_h) = - \langle \partial_t \rho, w_h \rangle_0 + \langle V_h e_h, \partial_x w_h \rangle_0 + \lambda \langle f_h, w_h \rangle_0 \tag{5.2.35}$$

for any $w_h \in \mathcal{V}_h^q$. Letting $w_h = \xi \in \mathcal{V}_h^q$ we have

$$\frac{1}{2} \frac{d}{dt} \|\xi(t)\|_0^2 + \mathscr{A}(u; \xi, \xi) = - \langle \partial_t \rho, \xi \rangle_0 + \langle V_h e_h, \partial_x \xi \rangle_0 + \lambda \langle f_h, \xi \rangle_0 . \tag{5.2.36}$$

By integrating (5.2.36) from 0 to t, using the coercivity of $\mathscr{A}(u; \cdot, \cdot)$ and the fact that V_h and $\bar{V}_h$ can be chosen such that $\xi(0) = 0$, we obtain

$$\begin{aligned} \|\xi(t)\|_0^2 + \int_0^t \|\xi(s)\|_1^2 \, ds \le C \int_0^t \Big| \langle \partial_t \rho(s), \xi(s) \rangle_0 \Big| \, ds + C \int_0^t \Big| \langle V_h(s) e_h(s), \partial_x \xi(s) \rangle_0 \Big| \, ds \\ + C \int_0^t \Big| \langle f_h(s), \xi(s) \rangle_0 \Big| \, ds. \end{aligned} \tag{5.2.37}$$

For the middle term on the right-hand side, Lemma 5.2.1, (5.2.29), and Hölder's inequality give, for any $\epsilon > 0$,

$$\begin{aligned} \int_0^t \Big| \langle V_h(s) e_h(s), \partial_x \xi(s) \rangle_0 \Big| \, ds &\le \|V_h\|_{L^\infty(L^\infty)} \int_0^t \Big(\|\eta(s)\|_0 + \|\zeta(s)\|_0 \Big) \|\partial_x \xi(s)\|_0 \, ds \\ &\le C \int_0^t \Big(\|\eta(s)\|_0 + \|\rho(s)\|_0 + \|\xi(s)\|_0 \Big) \|\partial_x \xi(s)\|_0 \, ds \\ &\le C \Big(\|\eta\|_{L^2(H^0)}^2 + \|\rho\|_{L^2(H^0)}^2 \Big) + C\epsilon \int_0^t \|\xi(s)\|_1^2 \, ds. \end{aligned}$$

The first and last terms on the right-hand side of (5.2.37) can be estimated as follows.

$$\int_0^t \Big| \langle \partial_t \rho(s), \xi(s) \rangle_0 \Big| \, ds \le \int_0^t \|\partial_t \rho(s)\|_0 \|\xi(s)\|_0 \, ds \le C \|\partial_t \rho\|_{L^2(H^0)}^2 + C\epsilon \int_0^t \|\xi(s)\|_1^2 \, ds$$

and

$$\int_0^t \Big| \langle f_h(s), \xi(s) \rangle_0 \Big| \, ds \le \int_0^t \|f_h(s)\|_0 \|\xi(s)\|_0 \, ds \le C \|\rho\|_{L^2(H^0)}^2 + C\epsilon \int_0^t \|\xi(s)\|_1^2 \, ds.$$

Thus, (5.2.37) gives

$$\begin{aligned} \|\xi(t)\|_0^2 + \int_0^t \|\xi(s)\|_1^2 \, ds \le C \Big(\|\eta\|_{L^\infty(H^0)}^2 + \|\rho\|_{L^\infty(H^0)}^2 + \|\partial_t \rho\|_{L^\infty(H^0)}^2 \Big) \\ + C\epsilon \int_0^t \|\xi(s)\|_1^2 \, ds. \end{aligned}$$

By choosing $\epsilon > 0$ sufficiently small such that $1 - C\epsilon > 0$ we deduce

$$\|\xi(t)\|_0^2 + \int_0^t \|\xi(s)\|_1^2 \, ds \le C \left(\|\eta\|^2_{L^\infty(H^0)} + \|\rho\|^2_{L^\infty(H^0)} + \|\partial_t \rho\|^2_{L^\infty(H^0)} \right).$$

Inequality (5.2.32) now follows from Lemma 5.2.2.

Step 2: We next prove

$$\|\xi\|_{L^\infty(H^1)} + \|\partial_t \xi\|_{L^2(H^0)} \le Ch^\gamma \Big(\|u\|_{L^\infty(H^{p+1})} + \|\partial_t u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{p+1})} + \|\partial_t v\|_{L^\infty(H^{q+1})} \Big). \tag{5.2.38}$$

By choosing $w_h = \partial_t \xi$ in (5.2.35) we have

$$\|\partial_t \xi(t)\|_0^2 + \mathscr{A}(u; \xi, \partial_t \xi) = - \langle \partial_t \rho, \partial_t \xi \rangle_0 + \langle V_h e_h, \partial_{tx} \xi \rangle_0 + \lambda \langle f_h, \partial_t \xi \rangle_0 \quad \forall w_h \in \mathcal{V}_h^q. \tag{5.2.39}$$

Since

$$\langle u\xi, \partial_{tx} \xi \rangle_0 = \frac{d}{dt} \langle u\xi, \partial_x \xi \rangle_0 - \langle \xi \partial_t u, \partial_x \xi \rangle_0 - \langle u \partial_t \xi, \partial_x \xi \rangle_0 ,$$

we have

$$\begin{aligned} \mathscr{A}(u; \xi, \partial_t \xi) &= \nu \langle \partial_x \xi, \partial_{tx} \xi \rangle - \langle u\xi, \partial_{tx} \xi \rangle + \lambda \langle \xi, \partial_t \xi \rangle \\ &= \frac{\nu}{2} \frac{d}{dt} \|\partial_x \xi(t)\|_0^2 - \frac{d}{dt} \langle u\xi, \partial_x \xi \rangle_0 + \langle \xi \partial_t u, \partial_x \xi \rangle_0 + \langle u \partial_t \xi, \partial_x \xi \rangle_0 + \frac{\lambda}{2} \frac{d}{dt} \|\xi(t)\|_0^2 . \end{aligned}$$

Moreover, it is easy to see that

$$\langle V_h e_h, \partial_{tx} \xi \rangle_0 = \frac{d}{dt} \langle V_h e_h, \partial_x \xi \rangle_0 - \langle (\partial_t V_h) e_h, \partial_x \xi \rangle_0 - \langle V_h \partial_t e_h, \partial_x \xi \rangle_0 .$$

Hence, it follows from (5.2.39) after rearranging the terms that

$$\begin{aligned} &\frac{\nu}{2} \frac{d}{dt} \|\partial_x \xi(t)\|_0^2 + \frac{\lambda}{2} \frac{d}{dt} \|\xi(t)\|_0^2 + \|\partial_t \xi(t)\|_0^2 \\ &= \frac{d}{dt} \langle u\xi, \partial_x \xi \rangle_0 + \frac{d}{dt} \langle V_h e_h, \partial_x \xi \rangle_0 - \langle \partial_t \rho, \partial_t \xi \rangle_0 + \lambda \langle f_h, \partial_t \xi \rangle_0 - \langle \xi \, \partial_t u, \partial_x \xi \rangle_0 \\ &\quad - \langle u \, \partial_t \xi, \partial_x \xi \rangle_0 - \langle (\partial_t V_h) e_h, \partial_x \xi \rangle_0 - \langle V_h \partial_t e_h, \partial_x \xi \rangle_0 . \end{aligned}$$

By integrating from 0 to t and using $\xi = \partial_x \xi = 0$ at $t = 0$, we deduce

$$\begin{aligned}
&\frac{\nu}{2} \|\partial_x \xi(t)\|_0^2 + \frac{\lambda}{2} \|\xi(t)\|_0^2 + \int_0^t \|\partial_t \xi(s)\|_0^2 \, ds \\
&= \langle u(t)\xi(t), \partial_x \xi(t)\rangle_0 + \langle V_h(t)e_h(t), \partial_x \xi(t)\rangle_0 - \int_0^t \langle \partial_t \rho(s), \partial_t \xi(s)\rangle_0 \, ds \\
&\quad + \lambda \int_0^t \langle f_h(s), \partial_t \xi(s)\rangle_0 \, ds - \int_0^t \langle \xi(s)\partial_t u(s), \partial_x \xi(s)\rangle_0 \, ds - \int_0^t \langle u(s)\partial_t \xi(s), \partial_x \xi(s)\rangle_0 \, ds \\
&\quad - \int_0^t \langle (\partial_t V_h(s)) e_h(s), \partial_x \xi\rangle_0 - \int_0^t \langle V_h(s)\partial_t e_h(s), \partial_x \xi\rangle_0 =: \mathscr{T}_1 + \cdots + \mathscr{T}_8. \qquad (5.2.40)
\end{aligned}$$

Recalling (4.2.2) and using the Hölder inequality, we estimate $\mathscr{T}_1$, $\mathscr{T}_5$ and $\mathscr{T}_6$ as

$$\begin{aligned}
|\mathscr{T}_1| &\le \|u\|_{L^\infty(L^\infty)} \|\xi(t)\|_0 \|\partial_x \xi(t)\|_0 \le C \|\xi\|_{L^\infty(H^0)}^2 + C\epsilon \|\partial_x \xi(t)\|_0^2, \\
|\mathscr{T}_5| &\le \|\partial_t u\|_{L^\infty(L^\infty)} \left(\int_0^t \|\xi(s)\|_0 \|\partial_x \xi(s)\|_0 \, ds \right) \le C \left(\|\xi\|_{L^\infty(H^0)}^2 + \|\xi\|_{L^2(H^1)}^2 \right), \\
|\mathscr{T}_6| &\le \|u\|_{L^\infty(L^\infty)} \int_0^t \|\partial_t \xi(s)\|_0 \|\partial_x \xi(s)\|_0 \, ds \le C\epsilon \int_0^t \|\partial_t \xi(s)\|_0^2 \, ds + C \|\xi\|_{L^2(H^1)}^2 .
\end{aligned}$$

Similarly, by using Lemma 5.2.1 and (5.2.29) we estimate $\mathscr{T}_2$, $\mathscr{T}_7$ and $\mathscr{T}_8$ as

$$\begin{aligned}
|\mathscr{T}_2| &\le \|V_h\|_{L^\infty(L^\infty)} \|e_h(t)\|_{L^2(\Omega)} \|\partial_x \xi(t)\|_{L^2(\Omega)} \le C \left(\|\eta(t)\|_0^2 + \|\zeta(t)\|_0^2 \right) + C\epsilon \|\partial_x \xi(t)\|_0^2 \\
&\le C \left(\|\eta\|_{L^\infty(H^0)}^2 + \|\rho\|_{L^\infty(H^0)}^2 + \|\xi\|_{L^\infty(H^0)}^2 \right) + C\epsilon \|\partial_x \xi(t)\|_0^2, \\
|\mathscr{T}_7| &\le \int_0^t \|\partial_t V_h(s)\|_{L^\infty(\Omega)} \|e_h(s)\|_0 \|\partial_x \xi(s)\|_0 \, ds \\
&\le \|e_h\|_{L^\infty(H^0)} \|\partial_t V_h\|_{L^2(L^\infty)} \|\partial_x \xi\|_{L^2(H^0)} \\
&\le C \left(\|e_h\|_{L^\infty(H^0)}^2 + \|\partial_x \xi\|_{L^2(H^0)}^2 \right) \le C \left(\|\eta\|_{L^\infty(H^0)}^2 + \|\zeta\|_{L^\infty(H^0)}^2 + \|\partial_x \xi\|_{L^2(H^0)}^2 \right) \\
&\le C \left(\|\eta\|_{L^\infty(H^0)}^2 + \|\rho\|_{L^\infty(H^0)}^2 + \|\xi\|_{L^\infty(H^0)}^2 + \|\xi\|_{L^2(H^1)}^2 \right),
\end{aligned}$$

and

$$\begin{aligned}
|\mathscr{T}_8| &\le \|V_h\|_{L^\infty(L^\infty)} \int_0^t \|\partial_t e_h(s)\|_0 \|\partial_x \xi(s)\|_0 \, ds \\
&\le C\epsilon \int_0^t \|\partial_t e_h(s)\|_0^2 \, ds + C \|\partial_x \xi\|_{L^2(H^0)}^2 \\
&\le C \int_0^t \|\partial_t \eta(s)\|_0^2 \, ds + C\epsilon \int_0^t \|\partial_t \zeta(s)\|_0^2 \, ds + C \|\xi\|_{L^2(H^1)}^2 \\
&\le C \left(\|\partial_t \eta\|_{L^2(H^0)}^2 + \|\partial_t \rho\|_{L^2(H^0)}^2 + \|\xi\|_{L^2(H^1)}^2 \right) + C\epsilon \int_0^t \|\partial_t \xi(s)\|_0^2 \, ds,
\end{aligned}$$

where in the last inequality we used (5.2.31). Finally,

$$|\mathscr{T}_3| \le C\, \|\partial_t \rho\|^2_{L^2(H^0)} + C\epsilon \int_0^t \|\partial_t \xi(s)\|_0^2 \, ds,$$

and

$$|\mathscr{T}_4| \le C\, \|\rho\|^2_{L^\infty(H^0)} + \|\xi\|^2_{L^\infty(H^0)} + C\epsilon \int_0^t \|\partial_t \xi(s)\|_0^2 \, ds.$$

Hence, (5.2.40) becomes

$$\begin{aligned}
&\frac{\nu}{2} \|\partial_x \xi(t)\|_0^2 + \frac{\lambda}{2} \|\xi(t)\|_0^2 + \int_0^t \|\partial_t \xi(s)\|_0^2 \, ds \\
&\quad \le C \Big(\|\xi\|^2_{L^\infty(H^0)} + \|\xi\|^2_{L^2(H^1)} + \|\eta\|^2_{L^\infty(H^0)} + \|\rho\|^2_{L^\infty(H^0)} + \|\partial_t \eta\|^2_{L^2(H^0)} + \|\partial_t \rho\|^2_{L^2(H^0)} \Big) \\
&\qquad + C\epsilon\, \|\partial_x \xi(t)\|_0^2 + C\epsilon \int_0^t \|\partial_t \xi(s)\|_0^2 \, ds.
\end{aligned}$$

By choosing $\epsilon > 0$ sufficiently small so that

$$\frac{\nu}{2} - C\epsilon > 0 \quad \text{and} \quad 1 - C\epsilon > 0,$$

we deduce

$$\begin{aligned}
\|\partial_x \xi(t)\|_0^2 + \int_0^t \|\partial_t \xi(s)\|_0^2 \, ds \le C \Big(&\|\xi\|^2_{L^\infty(H^0)} + \|\xi\|^2_{L^2(H^1)} + \|\eta\|^2_{L^\infty(H^0)} + \|\rho\|^2_{L^\infty(H^0)} \\
&+ \|\partial_t \eta\|^2_{L^2(H^0)} + \|\partial_t \rho\|^2_{L^2(H^0)} \Big).
\end{aligned}$$

Thus, (5.2.38) follows from (5.2.32) and Lemma 5.2.2.

Step 3: Finally we prove

$$\begin{aligned}
\|\partial_t \xi\|_{L^\infty(H^0)} + \|\partial_t \xi\|_{L^2(H^1)} \le C h^\gamma \Big(&\|u\|_{L^\infty(H^{p+1})} + \|\partial_t u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} \\
&+ \|\partial_t v\|_{L^\infty(H^{q+1})} \Big).
\end{aligned} \tag{5.2.41}$$

By differentiating (5.2.35) with respect to t, noting

$$\mathscr{A}(u; \partial_t \xi, w_h) = \langle \xi \partial_t u, \partial_x w_h \rangle_0$$

and rearranging the terms, we have

$$\begin{aligned}
\langle \partial_{tt} \xi, w_h \rangle_0 + \mathscr{A}(u; \partial_t \xi, w_h) = \langle \xi \partial_t u, \partial_x w_h \rangle_0 - \langle \partial_{tt} \rho, \partial_x w_h \rangle_0 + \langle (\partial_t V_h) e_h, \partial_x w_h \rangle_0 \\
+ \langle V_h \partial_t e_h, \partial_x w_h \rangle_0 + \lambda \langle \partial_t f_h, w_h \rangle_0 \quad \forall w_h \in \mathcal{V}_h^q.
\end{aligned}$$

Setting $w_h = \partial_t \xi$, integrating from 0 to t, and using the coercivity of $\mathscr{A}(u;\cdot,\cdot)$ and $\partial_t \xi = 0$ at $t=0$, we obtain

$$\begin{aligned}
&\|\partial_t \xi(t)\|_0^2 + \int_0^t \|\partial_t \xi(s)\|_1^2 \\
&\qquad \le C \left(\int_0^t \left| \langle \xi(s)\partial_t u(s), \partial_{tx}\xi(s)\rangle_0 \right| ds + \int_0^t \left| \langle \partial_{tt}\rho(s), \partial_{tx}\xi(s)\rangle_0 \right| ds \right. \\
&\qquad\quad + \int_0^t \left| \langle (\partial_t V_h(s)) e_h(s), \partial_{tx}\xi(s)\rangle_0 \right| ds + \int_0^t \left| \langle V_h(s)\partial_t e_h(s), \partial_{tx}\xi(s)\rangle_0 \right| ds \\
&\qquad\quad \left. + \int_0^t \left| \langle \partial_t f_h(s), \partial_t \xi(s)\rangle_0 \right| ds \right) =: \mathscr{T}_1 + \cdots + \mathscr{T}_5.
\end{aligned}$$

The right-hand side can be estimated in the same manner as in the proof of Step 2, such that

$$\begin{aligned}
|\mathscr{T}_1| &\le \|\partial_t u\|_{L^\infty(L^\infty)} \int_0^t \|\xi(s)\|_0 \|\partial_{tx}\xi(s)\|_0 \, ds \le C\|\xi\|_{L^\infty(H^0)} + C\epsilon \|\partial_t \xi(t)\|_1^2, \\
|\mathscr{T}_2| &\le C\|\partial_{tt}\rho\|_{L^2(H^0)}^2 + C\epsilon \int_0^t \|\partial_t \xi(s)\|_1^2 \, ds, \\
|\mathscr{T}_5| &\le C\|\partial_t \rho\|_{L^\infty(H^0)}^2 + \|\partial_t \xi\|_{L^\infty(H^0)}^2 + C\epsilon \int_0^t \|\partial_t \xi(s)\|_1^2 \, ds,
\end{aligned}$$

and finally we estimate $\mathscr{T}_3$ and $\mathscr{T}_4$ as

$$\begin{aligned}
|\mathscr{T}_3| &\le \int_0^t \|\partial_t V_h(s)\|_{L^\infty(\Omega)} \|e_h(s)\|_0 \|\partial_{tx}\xi(s)\|_0 \, ds \\
&\le \|e_h\|_{L^\infty(H^0)} \|\partial_t V_h\|_{L^2(L^\infty)} \|\partial_{tx}\xi\|_{L^2(H^0)} \le C\left(\|e_h\|_{L^\infty(H^0)}^2 + \|\partial_{tx}\xi\|_{L^2(H^0)}^2 \right) \\
&\le C\left(\|\eta\|_{L^\infty(H^0)}^2 + \|\zeta\|_{L^\infty(H^0)}^2 + \|\partial_{tx}\xi\|_{L^2(H^0)}^2 \right) \\
&\le C\left(\|\eta\|_{L^\infty(H^0)}^2 + \|\rho\|_{L^\infty(H^0)}^2 + \|\xi\|_{L^\infty(H^0)}^2 + \|\partial_t \xi\|_{L^2(H^1)}^2 \right),
\end{aligned}$$

and

$$\begin{aligned}
|\mathscr{T}_4| &\le \|V_h\|_{L^\infty(L^\infty)} \int_0^t \|\partial_t e_h(s)\|_0 \|\partial_{tx}\xi(s)\|_0 \, ds \le C\epsilon \int_0^t \|\partial_t e_h(s)\|_0^2 \, ds + C\|\partial_{tx}\xi\|_{L^2(H^0)}^2 \\
&\le C\int_0^t \|\partial_t \eta(s)\|_0^2 \, ds + C\epsilon \int_0^t \|\partial_t \zeta(s)\|_0^2 \, ds + C\|\partial_t \xi\|_{L^2(H^1)}^2 \\
&\le C\left(\|\partial_t \eta\|_{L^2(H^0)}^2 + \|\partial_t \rho\|_{L^2(H^0)}^2 + \|\partial_t \xi\|_{L^2(H^0)}^2 \right) + C\epsilon \int_0^t \|\partial_t \xi(s)\|_1^2 \, ds,
\end{aligned}$$

where in the last inequality we used (5.2.31). Thus, we have

$$\begin{aligned}\|\partial_t \xi(t)\|_0^2 + (1 - C\epsilon) \int_0^t \|\partial_t \xi(s)\|_1^2 \leq C \Big(&\|\xi\|^2_{L^\infty(H^0)} + \|\partial_{tt}\rho\|^2_{L^2(H^0)} + \|\eta\|^2_{L^\infty(H^0)} \\ &+ \|\rho\|^2_{L^\infty(H^0)} + \|\partial_t \eta\|^2_{L^2(H^0)} + \|\partial_t \rho\|^2_{L^2(H^0)} \\ &+ \|\partial_t \xi\|^2_{L^2(H^0)} + \|\partial_t \rho\|^2_{L^\infty(H^0)} \Big) .\end{aligned}$$

Choosing $1 - C\epsilon > 0$, noting (5.2.32), (5.2.38) and Lemma 5.2.2, thus complete the proof of the lemma. □

Combining the results in Lemmas 5.2.2 and 5.2.3 we are now ready to state the a priori error estimates in the approximation of (5.1.7)–(5.1.8) by (5.1.12)–(5.1.13).

Theorem 5.2.4. *Assume that $u \in W^1_\infty(0, T; H^1_0(\Omega) \cap H^{p+1}(\Omega))$, and $v \in W^1_\infty(0, T; H^{q+1}(\Omega))$. Assume further that V and $\bar{V}_h$ satisfy $V - \bar{V}_h = \partial_t V - \partial_t \bar{V}_h = 0$ at $t = 0$. Then, there exist positive constants $C > 0$ independent of h such that*

$$\begin{aligned}\|e_h(t)\|_j &\leq C h^{\min(p+1-j,q+1)} \Big(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^\infty(H^{q+1})} \Big), \\ \|\partial_t e_h(t)\|_0 + \|\partial_t f_h(t)\|_0 &\leq C h^{\min(p+1,q+1)} \Big(\|u\|_{L^\infty(H^{p+1})} + \|\partial_t u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} \\ &\quad + \|\partial_t v\|_{L^\infty(H^{q+1})} \Big), \\ \|\partial_t e_h(t)\|_1 &\leq C h^{\min(p,q+1)} \Big(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^\infty(H^{q+1})} \Big), \\ \|f_h(t)\|_j &\leq C h^{\min(p+1,q+1-j)} \Big(\|u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} + \|\partial_t v\|_{L^\infty(H^{q+1})} \Big)\end{aligned}$$

and

$$\begin{aligned}\int_0^t \|\partial_t f_h(s)\|_1 \, ds \leq C h^{\min(p+1,q)} \Big(&\|u\|_{L^\infty(H^{p+1})} + \|\partial_t u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+1})} \\ &+ \|\partial_t v\|_{L^\infty(H^{q+1})} \Big)\end{aligned}$$

for $j = 0, 1$.

5.3 A posteriori error estimation

In this section, we design strategies to compute a posteriori error estimates from the finite element solutions U_h and V_h computed by using (5.1.12)–(5.1.13).

We approximate the true errors e_h and f_h, see (5.2.16), by $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$, respectively. From (5.1.12)–(5.1.13) and (5.1.7)–(5.1.8), we infer that the true errors satisfy

$$\langle \partial_x e_h(t), \partial_x \chi_h \rangle_0 = \langle f_h(t), \partial_x \chi_h \rangle_0 \quad \forall \chi_h \in \mathring{\mathcal{V}}_h^p, \tag{5.3.1}$$

and

$$\begin{aligned} &\langle \partial_t f_h(t), w_h \rangle_0 + \nu \langle \partial_x f_h(t), \partial_x w_h \rangle_0 - \langle e_h(t) f_h(t), \partial_x w_h \rangle_0 - \langle U_h(t) f_h(t), \partial_x w_h \rangle_0 \\ &\quad - \langle V_h(t) e_h(t), \partial_x w_h \rangle_0 = -\nu \langle \partial_x V_h(t), \partial_x w_h \rangle_0 + \langle U_h(t) V_h(t), \partial_x w_h \rangle_0 \\ &\quad - \langle \partial_t V_h(t), w_h \rangle_0 \quad \forall w_h \in \mathcal{V}_h^q. \end{aligned} \tag{5.3.2}$$

At $t = 0$, noting (5.1.9) and (5.1.14), we have

$$\langle f_h(0), w_h \rangle_0 = 0 \quad \forall w_h \in \mathcal{V}_h^q. \tag{5.3.3}$$

Due to (5.1.13), the right hand side of (5.3.2) vanishes. However, for the purpose of developing a posteriori error estimates, we keep these terms in the equation as indication of how the a posteriori error estimation should be.

By using the local inner product (2.2.1), we propose to approximate the exact errors e_h and f_h, respectively, by $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$, where E and F are computed locally on Ω_l, $l = 1, \dots, N$, by one of the following four methods.

Method (i): ***Nonlinear parabolic error estimate***

Let $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$ be defined on Ω_l by

$$\langle \partial_x E(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} = \langle F(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \hat{\chi} \rangle_{0,\Omega_l} \tag{5.3.4}$$

for any $\hat{\chi}_h \in \mathcal{S}_h^{p+1}$ and

$$\begin{aligned} &\langle \partial_t F(t), \hat{w}_h \rangle_{0,\Omega_l} + \nu \langle \partial_x F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle E(t) F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle U_h(t) F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} \\ &\quad - \langle V_h(t) E(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} = \langle U_h(t) V_h(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle \partial_t V_h(t), \hat{w}_h \rangle_{0,\Omega_l} \end{aligned} \tag{5.3.5}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$ at $t \in (0, T]$. An initial condition at $t = 0$ is defined by

$$\langle F(0), \hat{w}_h \rangle_{0,\Omega_l} = \langle \partial_x u_0, \hat{w}_h \rangle_{0,\Omega_l} - \langle V_h(0), \hat{w}_h \rangle_{0,\Omega_l} \tag{5.3.6}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$.

Method (ii): ***Nonlinear elliptic error estimate***

Let $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$ be defined on Ω_l by

$$\langle \partial_x E(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} = \langle F(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \hat{\chi} \rangle_{0,\Omega_l} \tag{5.3.7}$$

for any $\hat{\chi}_h \in \mathcal{S}_h^{p+1}$ and

$$\begin{aligned} \nu \langle \partial_x F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle E(t)F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle U_h(t)F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} \\ - \langle V_h(t)E(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} = \langle U_h(t)V_h(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle \partial_t V_h(t), \hat{w}_h \rangle_{0,\Omega_l} \end{aligned} \tag{5.3.8}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$ at $t \in (0,T]$. An initial condition at $t=0$ is defined by

$$\langle F(0), \hat{w}_h \rangle_{0,\Omega_l} = \langle \partial_x u_0, \hat{w}_h \rangle_{0,\Omega_l} - \langle V_h(0), \hat{w}_h \rangle_{0,\Omega_l} \tag{5.3.9}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$.

Method (iii): ***Linear parabolic error estimate***

Let $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$ be defined on Ω_l by

$$\langle \partial_x E(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} = \langle F(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \hat{\chi} \rangle_{0,\Omega_l} \tag{5.3.10}$$

for any $\hat{\chi}_h \in \mathcal{S}_h^{p+1}$ and

$$\begin{aligned} \langle \partial_t F(t), \hat{w}_h \rangle_{0,\Omega_l} + \nu \langle \partial_x F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle U_h(t)F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} \\ - \langle V_h(t)E(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} = \langle U_h(t)V_h(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle \partial_t V_h(t), \hat{w}_h \rangle_{0,\Omega_l} \end{aligned} \tag{5.3.11}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$ at $t \in (0,T]$. An initial condition at $t=0$ is defined by

$$\langle F(0), \hat{w}_h \rangle_{0,\Omega_l} = \langle \partial_x u_0, \hat{w}_h \rangle_{0,\Omega_l} - \langle V_h(0), \hat{w}_h \rangle_{0,\Omega_l} \tag{5.3.12}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$.

Method (iv): ***Linear elliptic error estimate***

Let $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$ be defined on Ω_l by

$$\langle \partial_x E(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} = \langle F(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \hat{\chi} \rangle_{0,\Omega_l}$$

for any $\hat{\chi}_h \in \mathcal{S}_h^{p+1}$ and

$$\begin{aligned}&\nu \langle \partial_x F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle U_h(t)F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle V_h(t)E(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} \\ &\quad = \langle U_h(t)V_h(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l} - \langle \partial_t V_h(t), \hat{w}_h \rangle_{0,\Omega_l} \end{aligned} \tag{5.3.13}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$ at $t \in (0, T]$. An initial condition at $t = 0$ is defined by

$$\langle F(0), \hat{w}_h \rangle_{0,\Omega_l} = \langle \partial_x u_0, \hat{w}_h \rangle_{0,\Omega_l} - \langle V_h(0), \hat{w}_h \rangle_{0,\Omega_l} \tag{5.3.14}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$.

We note that, in the computation of a posteriori error estimator F, an additional saving on the computation cost is obtained by neglecting the time rate of change in (5.3.5), thus reducing the nonlinear parabolic equation of Method (i) by the nonlinear elliptic equation (5.3.8) of Method (ii). Lastly, for Method (iii) and Method (iv), we neglect the term $\langle E(t)F(t), \partial_x \hat{w}_h \rangle_{0,\Omega_l}$ respectively in (5.3.5) of Method (i) and in (5.3.8) of Method (ii) to have linear parabolic equation (5.3.11) and linear elliptic equation (5.3.13).

Since $\operatorname{supp} \phi_{l,k} = \bar{\Omega}_l$, if $\psi \in H^0(\Omega)$ then the statement

$$\langle \psi, \hat{\chi}_h \rangle_0 = 0 \quad \forall \hat{\chi}_h \in \mathcal{S}_h^{p+1}$$

is equivalent to

$$\langle \psi, \hat{\chi}_h \rangle_{0,\Omega_l} = 0 \quad \forall \hat{\chi}_h \in \mathcal{S}_h^{p+1}, \quad l = 1, \ldots, N.$$

Hence, in fact (5.3.4)–(5.3.14) hold for the inner product on the whole domain Ω.

In order to emphasize on the polynomial degrees p and q, as in the remaining part we will need different projection onto spaces of different polynomial degree, we rewrite $\bar{U}_{h,p} := \bar{U}_h \in W_\infty^1(0, T; \mathring{\mathcal{V}}_h^p)$ and $\bar{V}_{h,q} := \bar{V}_h \in W_\infty^1(0, T; \mathcal{V}_h^q)$, where $\bar{U}_h$ and $\bar{V}_h$ are defined by (5.2.11)–(5.2.12).

Letting $\hat{e}$ such that

$$\hat{e}(t) := \|e_h(t)\|_1 + \|f_h(t)\|_1$$

where $\|e_h(t)\|_1$ and $\|f_h(t)\|_1$ are defined in Theorem 5.2.4 and letting $\hat{E}$ such that

$$\hat{E}(t) := \|E(t)\|_1 + \|F(t)\|_1\,,$$

we then define the effectivity index Θ as follows

$$\Theta(t) = \frac{\hat{E}(t)}{\hat{e}(t)}.$$

We now state the main result of this section.

Theorem 5.3.1. *Let $\bar{V}_{h,q}$ be defined by (5.2.12) and $\check{V}_{h,q} \in \mathcal{V}_h^q$ be defined by*

$$\left\langle v - \check{V}_{h,q}, w_h \right\rangle_1 = 0 \quad \forall w_h \in \mathcal{V}_h^q. \tag{5.3.15}$$

Assume that the Conjecture 4.3.1 holds and assume that

$$\hat{e}(t) \geq Ch^{\min(p,q)}. \tag{5.3.16}$$

Then the approximate errors $E \in \mathcal{S}_h^{p+1}$ and $F \in \mathcal{S}_h^{q+1}$ defined by Method (i)–Method (iv) satisfy for almost all $t \in [0, T]$

$$\lim_{h \to 0} \Theta(t) = 1.$$

We now provide the proof of Theorem 5.3.1, which is based on the following lemmas. For the analysis, we define $\bar{e} \in \mathcal{S}_h^{p+1}$ and $\bar{f} \in \mathcal{S}_h^{q+1}$ such that for $l = 1, \ldots, N$

$$\langle \partial_x \bar{e}(t), \partial_x \hat{\chi}_h \rangle_{0,\Omega_l} = \left\langle \partial_x u(t) - \partial_x \bar{U}_{h,p}(t), \partial_x \hat{\chi}_h \right\rangle_{0,\Omega_l} \quad \forall \hat{\chi}_h \in \mathcal{S}_h^{p+1} \tag{5.3.17}$$

and

$$\mathscr{A}_l(u(t); \bar{f}_h(t), \hat{w}_h) = \mathscr{A}_l(u(t); v(t) - \bar{V}_{h,q}(t), \hat{w}_h) \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}, \tag{5.3.18}$$

where $\mathscr{A}_l(u; \cdot, \cdot)$ is defined as $\mathscr{A}(u; \cdot, \cdot)$ but on Ω_l.

Similar to the previous chapter, letting

$$\bar{\eta} = u - \left(\bar{U}_{h,p} + \bar{e}\right), \quad \bar{\zeta} = \bar{e} - E, \quad \bar{\rho} = v - \left(\bar{V}_{h,q} + \bar{f}\right), \quad \bar{\xi} = \bar{f} - F, \tag{5.3.19}$$

we rewrite:

$$e_h - E = \bar{\eta} + \bar{\zeta} + \zeta \quad \text{and} \quad f_h - F = \bar{\rho} + \bar{\xi} + \xi. \tag{5.3.20}$$

In the remaining part, we focus on estimating $\bar{\zeta}$, $\bar{\rho}$ and $\bar{\xi}$ of (5.3.20). We note that the estimates of $\|\zeta(t)\|_1$ and $\|\xi(t)\|_1$ are presented in Lemma 5.2.3. We also note that from Lemma 4.3.3 we have

$$\|\bar{\eta}(t)\|_1 \leq Ch^{p+1} \|u(t)\|_{p+2}. \tag{5.3.21}$$

We now focus the estimation of $\|\bar{\rho}(t)\|_1$. In general, we have

$$\mathscr{A}(u;\bar{f},w_h)\neq 0 \quad \forall w_h\in\mathcal{V}_h^q,$$

where $\bar{V}_{h,q}+\bar{f}$ is not the elliptic projection $\bar{V}_{h,q+1}$ of v into $\mathcal{V}_h^{q+1}$; see (5.2.12). However, $\bar{V}_{h,q}+\bar{f}$ approximates v with same order of convergence as $\bar{V}_{h,q+1}$. In order to show this we first attempt to obtain two superconvergence results (Lemma 5.3.2 and Lemma 5.3.3).

Lemma 5.3.2. *Assume that $v\in W^1_\infty(0,T;H^{q+2}(\Omega))$. Let $\bar{V}_{h,q}$ be defined by* (5.2.12) *and $\check{V}_{h,q}\in\mathcal{V}_h^q$ be defined by* (5.3.15)*. Then*

$$\left\|\bar{V}_{h,q}(t)-\check{V}_{h,q}(t)\right\|_1\leq Ch^{q+1}\left\|v(t)\right\|_{q+1}, \tag{5.3.22}$$

and

$$\left\|\partial_t\bar{V}_{h,q}(t)-\partial_t\check{V}_{h,q}(t)\right\|_1\leq Ch^{q+1}\left(\left\|v(t)\right\|_{q+1}+\left\|\partial_t v(t)\right\|_{q+1}\right). \tag{5.3.23}$$

Proof. Firstly, we prove (5.3.22). From (5.3.15) we have,

$$\left\|\bar{V}_{h,q}(t)-\check{V}_{h,q}(t)\right\|_1^2=\left\langle(\bar{V}_{h,q}-v)+(v-\check{V}_{h,q}),\bar{V}_{h,q}-\check{V}_{h,q}\right\rangle_1=\left\langle\bar{V}_{h,q}-v,\bar{V}_{h,q}-\check{V}_{h,q}\right\rangle_1.$$

Noting that

$$\begin{aligned}\left\langle\partial_x\bar{V}_{h,q}-\partial_x v,\partial_x\bar{V}_{h,q}-\partial_x\check{V}_{h,q}\right\rangle_0&=\frac{1}{\nu}\mathscr{A}(u;\bar{V}_{h,q}-v,\bar{V}_{h,q}-\check{V}_{h,q})\\&\quad+\frac{1}{\nu}\left\langle u(\bar{V}_{h,q}-v),\partial_x\bar{V}_{h,q}-\partial_x\check{V}_{h,q}\right\rangle_0\\&\quad-\frac{\lambda}{\nu}\left\langle\bar{V}_{h,q}-v,\bar{V}_{h,q}-\check{V}_{h,q}\right\rangle_0\end{aligned}$$

and $\mathscr{A}(u;\bar{V}_{h,q}-v,\bar{V}_{h,q}-\check{V}_{h,q})=0$ due to (5.2.12), we deduce

$$\begin{aligned}\left\|\bar{V}_{h,q}(t)-\check{V}_{h,q}(t)\right\|_1^2&=\left\langle\bar{V}_{h,q}-v,\bar{V}_{h,q}-\check{V}_{h,q}\right\rangle_0+\frac{1}{\nu}\left\langle u(\bar{V}_{h,q}-v),\partial_x\bar{V}_{h,q}-\partial_x\check{V}_{h,q}\right\rangle_0\\&\quad-\frac{\lambda}{\nu}\left\langle\bar{V}_{h,q}-v,\bar{V}_{h,q}-\check{V}_{h,q}\right\rangle_0\\&\leq\left\|\bar{V}_{h,q}(t)-v(t)\right\|_0\left\|\bar{V}_{h,q}(t)-\check{V}_{h,q}(t)\right\|_0\\&\quad+\frac{C}{\nu}\left\|\bar{V}_{h,q}(t)-v(t)\right\|_0\left\|\partial_x\bar{V}_{h,q}(t)-\partial_x\check{V}_{h,q}(t)\right\|_0\\&\quad+\frac{\lambda}{\nu}\left\|\bar{V}_{h,q}(t)-v(t)\right\|_0\left\|\bar{V}_{h,q}(t)-\check{V}_{h,q}(t)\right\|_0\\&\leq C\left\|\bar{V}_{h,q}(t)-v(t)\right\|_0\left\|\bar{V}_{h,q}(t)-\check{V}_{h,q}(t)\right\|_1.\end{aligned}$$

This implies

$$\left\|\bar{V}_{h,q}(t) - \check{V}_{h,q}(t)\right\|_1 \le C \left\|\bar{V}_{h,q}(t) - v(t)\right\|_0$$

and the desired estimate (5.3.22) followed from (5.2.20).

It remains to prove (5.3.23). By differentiating (5.3.15) with respect to t, we have

$$\left\langle \partial_t v - \partial_t \check{V}_{h,q}, w_h \right\rangle_1 = 0 \quad \forall w_h \in \mathcal{V}_h^q.$$

This together with (5.2.23) gives

$$\begin{aligned}
&\left\|\partial_t \bar{V}_{h,q}(t) - \partial_t \check{V}_{h,q}(t)\right\|_1^2 = \left\langle \partial_t \bar{V}_{h,q} - \partial_t v, \partial_t \bar{V}_{h,q} - \partial_t \check{V}_{h,q}\right\rangle_1 \\
&= \left\langle \partial_t \bar{V}_{h,q} - \partial_t v, \partial_t \bar{V}_{h,q} - \partial_t \check{V}_{h,q}\right\rangle_0 + \frac{1}{\nu}\mathscr{A}(u; \partial_t \bar{V}_{h,q} - \partial_t v, \partial_t \bar{V}_{h,q} - \partial_t \check{V}_{h,q}) \\
&\quad + \frac{1}{\nu}\left\langle u(\partial_t \bar{V}_{h,q} - \partial_t v), \partial_{tx} \bar{V}_{h,q} - \partial_{tx} \check{V}_{h,q}\right\rangle_0 - \frac{\lambda}{\nu}\left\langle \partial_t \bar{V}_{h,q} - \partial_t v, \partial_t \bar{V}_{h,q} - \partial_t \check{V}_{h,q}\right\rangle_0 \\
&= \left\langle \partial_t \bar{V}_{h,q} - \partial_t v, \partial_t \bar{V}_{h,q} - \partial_t \check{V}_{h,q}\right\rangle_0 + \frac{1}{\nu}\left\langle (\partial_t u)(\bar{V}_{h,q} - v), \partial_{tx} \bar{V}_{h,q} - \partial_{tx} \check{V}_{h,q}\right\rangle_0 \\
&\quad + \frac{1}{\nu}\left\langle u(\partial_t \bar{V}_{h,q} - \partial_t v), \partial_{tx} \bar{V}_{h,q} - \partial_{tx} \check{V}_{h,q}\right\rangle_0 - \frac{\lambda}{\nu}\left\langle \partial_t \bar{V}_{h,q} - \partial_t v, \partial_t \bar{V}_{h,q} - \partial_t \check{V}_{h,q}\right\rangle_0 \\
&\le C\Big(\left\|\partial_t \bar{V}_{h,q}(t) - \partial_t v(t)\right\|_0 + \left\|\bar{V}_{h,q}(t) - v(t)\right\|_0 \Big) \left\|\partial_t \bar{V}_{h,q}(t) - \partial_t \check{V}_{h,q}(t)\right\|_1,
\end{aligned}$$

implying

$$\left\|\partial_t \bar{V}_{h,q}(t) - \partial_t \check{V}_{h,q}(t)\right\|_1 \le C\Big(\left\|\partial_t \bar{V}_{h,q}(t) - \partial_t v(t)\right\|_0 + \left\|\bar{V}_{h,q}(t) - v(t)\right\|_0 \Big).$$

Thus, by noting that $v - \bar{V}_h = \rho$, (5.3.23) follows from (5.2.21) and (5.2.20). □

Lemma 5.3.3. *Let $\check{f} \in \mathcal{S}_h^{q+1}$ be defined by*

$$\left\langle \check{f}, \hat{w}_h \right\rangle_1 = \left\langle v - \check{V}_{h,q}, \hat{w}_h \right\rangle_1 \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}, \tag{5.3.24}$$

where $\check{V}_{h,q}$ is defined by (5.3.15). *Assume that $v \in W^1_\infty(0,T; H^{q+2}(\Omega))$ and the Conjecture 4.3.1 is satisfied. Then there holds*

$$\left\|\partial_t^i v(t) - \partial_t^i \check{V}_{h,q}(t) - \partial_t^i \check{f}(t)\right\|_1 \le C h^{q+1}\left(\|v(t)\|_{q+2} + \left\|\partial_t^i v(t)\right\|_{q+2}\right), \quad i = 0, 1,$$

where $\partial_t^i = \partial^i / \partial t^i$, $i = 0, 1$.

Proof. Let $\check{V}_{h,q+1} \in \mathcal{V}_h^{q+1}$ be defined by the same equation (5.3.15) with polynomial degree $q+1$ instead of q. Then, by using standard finite element arguments we have

$$\left\|\partial_t^i v(t) - \partial_t^i \check{V}_{h,q+1}(t)\right\|_1 \le Ch^{q+1}\left(\|v(t)\|_{q+2} + \left\|\partial_t^i v(t)\right\|_{q+2}\right).$$

Therefore, by using the triangle inequality we obtain

$$\begin{aligned}
&\left\|\partial_t^i v(t) - \partial_t^i \check{V}_{h,q}(t) - \partial_t^i \check{f}(t)\right\|_1 \\
&\le \left\|\partial_t^i v(t) - \partial_t^i \check{V}_{h,q+1}(t)\right\|_1 + \left\|\partial_t^i \check{V}_{h,q+1}(t) - \partial_t^i \check{V}_{h,q}(t) - \check{f}(t)\right\|_1 \\
&\le Ch^{q+1} + \left\|\partial_t^i \check{V}_{h,q+1}(t) - \partial_t^i \check{V}_{h,q}(t) - \partial_t^i \check{f}(t)\right\|_1. \qquad (5.3.25)
\end{aligned}$$

The lemma will be proved if we can prove $\left\|\partial_t^i \check{V}_{h,q+1}(t) - \partial_t^i \check{V}_{h,q}(t) - \partial_t^i \check{f}(t)\right\|_1 \le Ch^{q+1}$. By noting that $\mathcal{V}_h^{q+1} = \mathcal{V}_h^q \oplus \mathcal{S}_h^{q+1}$, we can write $\check{V}_{h,q+1}$ as

$$\check{V}_{h,q+1} = \tilde{V}_{h,q} + \tilde{f} \quad \text{where} \quad \tilde{V}_{h,q} \in \mathcal{V}_h^q \quad \text{and} \quad \tilde{f} \in \mathcal{S}_h^{q+1}. \qquad (5.3.26)$$

We also note that

$$\left\langle \partial_t \check{f}, \hat{w}_h \right\rangle_1 = \left\langle \partial_t v - \partial_t \check{V}_{h,q}, \hat{w}_h \right\rangle_1 \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}.$$

This together with (5.3.24) and the definition of $\check{V}_{h,q+1}$ gives

$$\begin{aligned}
&\left\|\partial_t^{(i)} \check{V}_{h,q+1}(t) - \partial_t^{(i)} \check{V}_{h,q}(t) - \partial_t^i \check{f}(t)\right\|_1^2 \\
&= \left\langle \partial_t^{(i)} v - \partial_t^{(i)} \check{V}_{h,q} - \partial_t^{(i)} \check{f}), \partial_t^i \check{V}_{h,q+1} - \partial_t^i \check{V}_{h,q} - \partial_t^i \check{f} \right\rangle_1 \\
&= \left\langle \partial_t^{(i)} v - \partial_t^{(i)} \check{V}_{h,q} - \partial_t^i \check{f}, \partial_t^i \tilde{V}_{h,q} - \partial_t^i \check{V}_{h,q} \right\rangle_1 + \left\langle \partial_t^{(i)} v - \partial_t^{(i)} \check{V}_{h,q} - \partial_t^i \check{f}, \partial_t^i \tilde{f} - \partial_t^i \check{f} \right\rangle_1 \\
&= \left\langle \partial_t^{(i)} \check{V}_{h,q+1} - \partial_t^{(i)} \check{V}_{h,q} - \partial_t^i \check{f}, \partial_t^i \tilde{V}_{h,q} - \partial_t^i \check{V}_{h,q} \right\rangle_1 \\
&\le \left\|\partial_t^{(i)} \check{V}_{h,q+1}(t) - \partial_t^{(i)} \check{V}_{h,q}(t) - \partial_t^i \check{f}(t)\right\|_1 \left\|\partial_t^{(i)} \tilde{V}_{h,q}(t) - \partial_t^{(i)} \check{V}_{h,q}(t)\right\|_1.
\end{aligned}$$

This implies

$$\left\|\partial_t^{(i)} \check{V}_{h,q+1}(t) - \partial_t^{(i)} \check{V}_{h,q}(t) - \partial_t^i \check{f}(t)\right\|_1 \le \left\|\partial_t^{(i)} \tilde{V}_{h,q}(t) - \partial_t^{(i)} \check{V}_{h,q}(t)\right\|_1.$$

With Conjecture 4.3.1, we obtain the required result. □

We are now able to estimate $\|\bar{\rho}(t)\|_1$ and $\|\partial_t \bar{\rho}(t)\|_1$.

Lemma 5.3.4. *If* $v \in W^1_\infty(0,T;H^{q+2}(\Omega))$*, then*

$$\|\bar\rho(t)\|_1 \le Ch^{q+1}\|v(t)\|_{q+2}, \tag{5.3.27}$$

$$\|\partial_t\bar\rho(t)\|_1 \le Ch^{q+1}\big(\|v(t)\|_{q+2} + \|\partial_t v(t)\|_{q+2}\big). \tag{5.3.28}$$

Proof. By using the coercivity of the bilinear form $\mathscr{A}(u;\cdot,\cdot)$, noting $\mathscr{A}(u;\bar\rho,\bar f) = 0$ and Hölder's inequality, we have

$$\begin{aligned}\|\bar\rho(t)\|_1^2 &\le C\,\mathscr{A}(u;\bar\rho, v - \bar V_{h,q} - \bar f) = C\,\mathscr{A}(u;\bar\rho, v - \bar V_{h,q}) = C\,\mathscr{A}(u;\bar\rho, v - \bar V_{h,q} - \check f)\\ &\le C\,\|\bar\rho(t)\|_1\left\|v(t) - \bar V_{h,q}(t) - \check f(t)\right\|_1,\end{aligned}$$

implying

$$\|\bar\rho(t)\|_1 \le C\left\|v(t) - \bar V_{h,q}(t) - \check f(t)\right\|_1 \le C\left\|\check V_{h,q}(t) - \bar V_{h,q}(t)\right\|_1 + C\left\|v(t) - \check V_{h,q}(t) - \check f(t)\right\|_1.$$

The desired estimate (5.3.27) now follows from (5.3.22) and Lemma 5.3.3 (at $j = 0$).

It remains to prove (5.3.28). The same argument as in the proof of Lemma 5.2.2 is used here. First we note that $\partial_t\bar f$ is not the projection of $\partial_t\rho$ by the bilinear form $\mathscr{A}(u;\cdot,\cdot)$. In fact, we have

$$\mathscr{A}(u;\partial_t\bar\rho,\hat w_h) = \mathscr{A}(u;\partial_t\rho - \partial_t\bar f,\hat w_h) = \langle(\partial_t u)\bar\rho,\partial_x\hat w_h\rangle_0 \quad \forall \hat w_h \in \mathcal{S}_h^{q+1}. \tag{5.3.29}$$

Let $f^* \in \mathcal{S}_h^{q+1}$ be defined by

$$\mathscr{A}(u;\partial_t\rho - f^*,\hat w_h) = 0 \quad \forall \hat w_h \in \mathcal{S}_h^{q+1}. \tag{5.3.30}$$

By using the triangle inequality, we have

$$\|\partial_t\bar\rho(t)\|_1 \le \|\partial_t\rho(t) - f^*(t)\|_1 + \left\|f^*(t) - \partial_t\bar f(t)\right\|_1 =: \mathscr{T}_1 + \mathscr{T}_2. \tag{5.3.31}$$

Noting (5.3.30) we have

$$\begin{aligned}\mathscr{T}_1^2 &\le C\mathscr{A}(u;\partial_t\rho - f^*,\partial_t\rho - f^*) = C\mathscr{A}(u;\partial_t\rho - f^*,\partial_t\rho) = C\mathscr{A}(u;\partial_t\rho - f^*,\partial_t\rho - \partial_t\check f)\\ &\le C\,\|\partial_t\rho(t) - f^*(t)\|_1\left\|\partial_t\rho(t) - \partial_t\check f(t)\right\|_1,\end{aligned}$$

implying

$$\begin{aligned}\mathscr{T}_1 &\le C\left\|\partial_t\rho(t) - \partial_t\check f(t)\right\|_1 = C\left\|\partial_t v(t) - \partial_t\bar V_{h,q}(t) - \partial_t\check f(t)\right\|_1\\ &\le C\left\|\partial_t\check V_{h,q}(t) - \partial_t\bar V_{h,q}(t)\right\|_1 + C\left\|\partial_t v(t) - \partial_t\check V_{h,q}(t) - \partial_t\check f(t)\right\|_1\\ &\le Ch^{q+1}\Big(\|v(t)\|_{q+2} + \|\partial_t v(t)\|_{q+2}\Big)\end{aligned} \tag{5.3.32}$$

due to (5.3.23) and Lemma 5.3.3 (at $j = 1$). To estimate $\mathscr{T}_2$, by referring to (5.3.30) and using (5.3.29), we have

$$\begin{aligned}\mathscr{T}_2^2 &\le C\,\mathscr{A}(u; f^* - \partial_t\bar{f}, f^* - \partial_t\bar{f}) = C\,\mathscr{A}(u; \partial_t\bar{\rho}, f^* - \partial_t\bar{f}) = C\left\langle(\partial_t u)\bar{\rho}, \partial_x(f^* - \partial_t\bar{f})\right\rangle_0\\ &\le C\,\|\bar{\rho}(t)\|_0\left\|f^*(t) - \partial_t\bar{f}(t)\right\|_1,\end{aligned}$$

implying

$$\mathscr{T}_2 \le C\,\|\bar{\rho}(t)\|_0 \le Ch^{q+1}\,\|v(t)\|_{q+2}\,. \tag{5.3.33}$$

Inequality (5.3.28) now follows from (5.3.32) and (5.3.33). □

The following lemmas give the estimate of $\bar{\xi}$ and $\bar{\zeta}$ in the $H^1(\Omega)$-norm.

Lemma 5.3.5. *Assume that $u \in W^1_\infty(0,T; H^1_0(\Omega)\cap H^{p+2}(\Omega))$, and $v \in W^1_\infty(0,T; H^{q+2}(\Omega))$. Then the following estimates hold:*

$$\begin{aligned}\left\|\bar{\zeta}\right\|_{W^1_\infty(H^1)} \le Ch^\gamma\Big(&\|u\|_{L^\infty(H^{p+2})} + \|\partial_t u\|_{L^\infty(H^{p+2})} + \|v\|_{L^\infty(H^{q+2})}\\ &+ \|\partial_t v\|_{L^\infty(H^{p+2})}\Big)\end{aligned} \tag{5.3.34}$$

and

$$\begin{aligned}\left\|\bar{\xi}\right\|_{L^\infty(H^0)} + \left\|\bar{\xi}\right\|_{L^2(H^1)} \le Ch^\gamma\Big(&\|u\|_{L^\infty(H^{p+2})} + \|\partial_t u\|_{L^\infty(H^{p+2})} + \|v\|_{L^\infty(H^{q+2})}\\ &+ \|\partial_t v\|_{L^\infty(H^{q+2})}\Big),\end{aligned} \tag{5.3.35}$$

where $C = C(\lambda, \nu, T)$ and $\gamma = \min(p+1, q+1)$.

Proof. We present the proof only for the case that E and F are defined by (5.3.10) and (5.3.11). It follows from (5.3.20) and (5.3.17) that for any $\hat{\chi}_h \in \mathcal{S}_h^{q+1}$ we have

$$\left\langle\partial_x\bar{\zeta}, \partial_x\hat{\chi}_h\right\rangle_0 = \langle\partial_x e_h, \partial_x\hat{\chi}_h\rangle_0 - \langle\partial_x E, \partial_x\hat{\chi}_h\rangle_0 - \langle\partial_x\zeta, \partial_x\hat{\chi}_h\rangle_0 \quad \forall\hat{\chi}_h \in \mathcal{S}_h^{p+1}.$$

By orthogonality of the Legendre polynomials we have $\langle\partial_x U_h, \partial_x\hat{\chi}_h\rangle_0 = 0$. Therefore, rewriting $e_h = u - U_h$, using (5.1.7) and (5.3.10) we obtain

$$\left\langle\partial_x\bar{\zeta}, \partial_x\hat{\chi}_h\right\rangle_0 = \langle v, \partial_x\hat{\chi}_h\rangle_0 - \langle F, \partial_x\hat{\chi}_h\rangle_0 - \langle V_h, \partial_x\hat{\chi}_h\rangle_0 - \langle\partial_x\zeta, \partial_x\hat{\chi}_h\rangle_0 \quad \forall\hat{\chi}_h \in \mathcal{S}_h^{p+1}.$$

Recalling that $v - V_h - F = \bar{\rho} + \bar{\xi} + \xi$ we have

$$\left\langle \partial_x \bar{\zeta}, \partial_x \hat{\chi}_h \right\rangle_0 = \left\langle \bar{\rho}, \partial_x \hat{\chi}_h \right\rangle_0 + \left\langle \bar{\xi}, \partial_x \hat{\chi}_h \right\rangle_0 + \left\langle \xi, \partial_x \hat{\chi}_h \right\rangle_0 - \left\langle \partial_x \zeta, \partial_x \hat{\chi}_h \right\rangle_0 \quad \forall \hat{\chi}_h \in \mathcal{S}_h^{p+1}.$$

Substituting $\hat{\chi}_h = \bar{\zeta} \in \mathcal{S}_h^{p+1}$, using Hölder's inequality and simplifying we deduce

$$\left\| \partial_x \bar{\zeta}(t) \right\|_0 \leq \left\| \bar{\rho}(t) \right\|_0 + \left\| \bar{\xi}(t) \right\|_0 + \left\| \xi(t) \right\|_0 + \left\| \partial_x \zeta(t) \right\|_0.$$

The Poincaré inequality yields

$$\left\| \bar{\zeta}(t) \right\|_1 \leq C\Big(\left\| \bar{\rho}(t) \right\|_0 + \left\| \bar{\xi}(t) \right\|_0 + \left\| \xi(t) \right\|_0 + \left\| \zeta(t) \right\|_1 \Big). \tag{5.3.36}$$

By noting Lemma 5.3.4 and Lemma 5.2.3 we have

$$\begin{aligned} \left\| \bar{\zeta} \right\|_{L^\infty(H^1)} &\leq C h^\gamma \Big(\|u\|_{L^\infty(H^{p+1})} + \|\partial_t u\|_{L^\infty(H^{p+1})} + \|v\|_{L^\infty(H^{q+2})} + \|\partial_t v\|_{L^2(H^{q+1})} \Big) \\ &\quad + \left\| \bar{\xi} \right\|_{L^\infty(H^0)}. \end{aligned}$$

Therefore, (5.3.34) is proved if we prove (5.3.35).

Noting the definition of $\bar{\xi}$ in (5.3.19) and using (5.3.11), for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$ we have

$$\begin{aligned} \left\langle \partial_t \bar{\xi}, \hat{w}_h \right\rangle_0 &= \left\langle \partial_t \bar{f}, \hat{w}_h \right\rangle_0 - \left\langle \partial_t F, \hat{w}_h \right\rangle_0 \\ &= \left\langle \partial_t \rho, \hat{w}_h \right\rangle_0 - \left\langle \partial_t \bar{\rho}, \hat{w}_h \right\rangle_0 + \nu \left\langle \partial_x F, \partial_x \hat{w}_h \right\rangle_0 - \left\langle U_h F, \partial_x \hat{w}_h \right\rangle_0 - \left\langle V_h E, \partial_x \hat{w}_h \right\rangle_0 \\ &\quad - \left\langle U_h V_h, \partial_x \hat{w}_h \right\rangle_0 + \left\langle \partial_t V_h, \hat{w}_h \right\rangle_0. \end{aligned}$$

Recalling that $\partial_t V_h = \partial_t v - \partial_t f_h$ and using (5.1.8) we have

$$\begin{aligned} \left\langle \partial_t \bar{\xi}, \hat{w}_h \right\rangle_0 &= \left\langle \partial_t \rho, \hat{w}_h \right\rangle_0 - \left\langle \partial_t \bar{\rho}, \hat{w}_h \right\rangle_0 + \nu \left\langle \partial_x F, \partial_x \hat{w}_h \right\rangle_0 - \left\langle U_h F, \partial_x \hat{w}_h \right\rangle_0 - \left\langle V_h E, \partial_x \hat{w}_h \right\rangle_0 \\ &\quad - \left\langle U_h V_h, \partial_x \hat{w}_h \right\rangle_0 - \nu \left\langle \partial_x v, \partial_x \hat{w}_h \right\rangle_0 + \left\langle uv, \partial_x \hat{w}_h \right\rangle_0 - \left\langle \partial_t f_h, \hat{w}_h \right\rangle_0 \end{aligned}$$

for any $\hat{w}_h \in \mathcal{S}_h^{q+1}$. By rearranging the terms, rewriting $E = \eta - \bar{\eta} - \bar{\zeta}$ and $F = \rho - \bar{\rho} - \bar{\xi}$ we deduce

$$\begin{aligned} \left\langle \partial_t \bar{\xi}, \hat{w}_h \right\rangle_0 &= \left\langle \partial_t \rho, \hat{w}_h \right\rangle_0 - \left\langle \partial_t \bar{\rho}, \hat{w}_h \right\rangle_0 - \left\langle U_h F, \partial_x \hat{w}_h \right\rangle_0 - \left\langle V_h E, \partial_x \hat{w}_h \right\rangle_0 - \left\langle U_h V_h, \partial_x \hat{w}_h \right\rangle_0 \\ &\quad + \left\langle uv, \partial_x \hat{w}_h \right\rangle_0 - \left\langle \partial_t f_h, \hat{w}_h \right\rangle_0 - \nu \left\langle \partial_x v - \partial_x F, \partial_x \hat{w}_h \right\rangle_0 \\ &= \left\langle \partial_t \rho, \hat{w}_h \right\rangle_0 - \left\langle \partial_t \bar{\rho}, \hat{w}_h \right\rangle_0 - \left\langle U_h(\rho - \bar{\rho} - \bar{\xi}), \partial_x \hat{w}_h \right\rangle_0 - \left\langle V_h(\eta - \bar{\eta} - \bar{\zeta}), \partial_x \hat{w}_h \right\rangle_0 \\ &\quad + \left\langle uv - U_h V_h, \partial_x \hat{w}_h \right\rangle_0 - \left\langle \partial_t f_h, \hat{w}_h \right\rangle_0 - \nu \left\langle \partial_x v - \partial_x F, \partial_x \hat{w}_h \right\rangle_0. \end{aligned} \tag{5.3.37}$$

Since $\langle \partial_x V_h, \partial_x \hat{w}_h \rangle_0 = 0$ due to the orthogonality of the Legendre polynomials there holds by using (5.3.20), the definition of the bilinear form $\mathscr{A}(u;\cdot,\cdot)$ and (5.3.18)

$$\begin{aligned}
\nu \langle \partial_x v - \partial_x F, \partial_x \hat{w}_h \rangle_0 &= \nu \langle \partial_x (v - V_h - F), \partial_x \hat{w}_h \rangle_0 = \nu \left\langle \partial_x \xi + \partial_x (\bar{\rho} + \bar{\xi}), \partial_x \hat{w}_h \right\rangle_0 \\
&= \nu \langle \partial_x \xi, \partial_x \hat{w}_h \rangle_0 + \mathscr{A}(u; \bar{\rho} + \bar{\xi}, \hat{w}_h) + \left\langle u(\bar{\rho} + \bar{\xi}), \partial_x \hat{w}_h \right\rangle_0 \\
&\quad - \lambda \left\langle \bar{\rho} + \bar{\xi}, \hat{w}_h \right\rangle_0 \\
&= \nu \langle \partial_x \xi, \partial_x \hat{w}_h \rangle_0 + \mathscr{A}(u; \bar{\xi}, \hat{w}_h) + \left\langle u(\bar{\rho} + \bar{\xi}), \partial_x \hat{w}_h \right\rangle_0 \\
&\quad - \lambda \left\langle \bar{\rho} + \bar{\xi}, \hat{w}_h \right\rangle_0 .
\end{aligned} \tag{5.3.38}$$

Noting that $uv - U_h V_h = u(f_h + V_h) - U_h V_h = e_h v + f_h U_h$, it follows from (5.3.37) and (5.3.38) that

$$\begin{aligned}
\left\langle \partial_t \bar{\xi}, \hat{w}_h \right\rangle_0 + \mathscr{A}(u; \bar{\xi}, \hat{w}_h) &= \langle \partial_t \rho, \hat{w}_h \rangle_0 - \langle \partial_t \bar{\rho}, \hat{w}_h \rangle_0 - \left\langle U_h(\rho - \bar{\rho} - \bar{\xi}), \partial_x \hat{w}_h \right\rangle_0 \\
&\quad - \left\langle V_h(\eta - \bar{\eta} - \bar{\zeta}), \partial_x \hat{w}_h \right\rangle_0 + \langle e_h v, \partial_x \hat{w}_h \rangle_0 + \langle f_h U_h, \partial_x \hat{w}_h \rangle_0 \\
&\quad - \langle \partial_t f_h, \hat{w}_h \rangle_0 - \nu \langle \partial_x \xi, \partial_x \hat{w}_h \rangle_0 - \left\langle u(\bar{\rho} + \bar{\xi}), \partial_x \hat{w}_h \right\rangle_0 \\
&\quad + \lambda \left\langle \bar{\rho} + \bar{\xi}, \hat{w}_h \right\rangle_0 \quad \forall \hat{w}_h \in \mathcal{S}_h^{q+1}.
\end{aligned} \tag{5.3.39}$$

Substituting $\hat{w}_h = \bar{\xi} \in \mathcal{S}_h^{q+1}$ and using Hölder's inequality we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \left\|\bar{\xi}(t)\right\|_0^2 + \alpha \left\|\bar{\xi}(t)\right\|_1^2 &\le C\Big(\|\partial_t \rho(t)\|_0 + \|\partial_t \bar{\rho}(t)\|_0 + \|\rho(t)\|_0 + \|\bar{\rho}(t)\|_0 + \|\eta(t)\|_0 \\
&\quad + \|\bar{\eta}(t)\|_0 + \left\|\bar{\zeta}(t)\right\|_0 + \|e_h(t)\|_0 + \|f_h(t)\|_0 + \|\partial_t f_h(t)\|_0 \\
&\quad + \|\xi(t)\|_1 \Big) \left\|\bar{\xi}(t)\right\|_1 \\
&\le C\Big(\|\partial_t \rho(t)\|_0^2 + \|\partial_t \bar{\rho}(t)\|_0^2 + \|\rho(t)\|_0^2 + \|\bar{\rho}(t)\|_0^2 + \|\eta(t)\|_0^2 \\
&\quad + \|\bar{\eta}(t)\|_0^2 + \|\xi(t)\|_0^2 + \|\zeta(t)\|_1^2 + \|e_h(t)\|_0^2 + \|f_h(t)\|_0^2 \\
&\quad + \|\partial_t f_h(t)\|_0^2 + \|\xi(t)\|_1^2 \Big) + C\epsilon \left\|\bar{\xi}(t)\right\|_1 ,
\end{aligned} \tag{5.3.40}$$

where in the last inequality we used (5.3.36) for the term $\left\|\bar{\zeta}(t)\right\|_1$. Thus, rearranging the terms in (5.3.40), choosing $\epsilon > 0$ sufficiently small such that $\alpha - C\epsilon > 0$ and integrating from 0 to t we have

$$\left\|\bar{\xi}(t)\right\|_0^2+\int_0^t\left\|\bar{\xi}(s)\right\|_1^2\,ds\le C\int_0^t\Big(\|\partial_t\rho(s)\|_0^2+\|\partial_t\bar{\rho}(s)\|_0^2+\|\rho(s)\|_0^2+\|\bar{\rho}(s)\|_0^2+\|\eta(s)\|_0^2$$
$$+\|\bar{\eta}(s)\|_0^2+\|\xi(s)\|_0^2+\|\zeta(s)\|_1^2+\|e_h(s)\|_0^2+\|f_h(s)\|_0^2$$
$$+\|\partial_t f_h(s)\|_0^2+\|\xi(s)\|_1^2\Big)\,ds.$$

Hence, the desired estimate (5.3.35) followed by noting Lemma 5.2.2, Theorem 5.2.4, Lemma 5.2.3, (5.2.32) and Lemma 5.3.4. □

In the remaining part of this section, we estimate $\left\|\bar{\xi}(t)\right\|_1$.

Lemma 5.3.6. *Let the assumption in Lemma 5.3.5 be satisfied. Then for any $t>0$, there holds*

$$\left\|\bar{\xi}\right\|_{L^\infty(H^1)}+\left\|\partial_t\bar{\xi}\right\|_{L^2(H^0)}\le Ch^\gamma\Big(\|u\|_{L^\infty(H^{p+2})}+\|\partial_t u\|_{L^\infty(H^{p+2})}+\|v\|_{L^\infty(H^{q+2})}$$
$$+\|\partial_t v\|_{L^\infty(H^{q+2})}\Big),$$

where $C=C(\lambda,\nu,T)$ and $\gamma=\min(p+1,q+1)$.

Proof. Substituting $\hat{w}_h=\partial_t\bar{\xi}\in\mathcal{S}_h^{q+1}$ in (5.3.39) gives

$$\left\|\partial_t\bar{\xi}(t)\right\|_0^2+\mathscr{A}(u;\bar{\xi},\partial_t\bar{\xi})=\left\langle\partial_t\rho,\partial_t\bar{\xi}\right\rangle_0-\left\langle\partial_t\bar{\rho},\partial_t\bar{\xi}\right\rangle_0-\left\langle U_h(\rho-\bar{\rho}-\bar{\xi}),\partial_{tx}\bar{\xi}\right\rangle_0$$
$$-\left\langle V_h(\eta-\bar{\eta}-\bar{\zeta}),\partial_{tx}\bar{\xi}\right\rangle_0+\left\langle e_h v,\partial_{tx}\bar{\xi}\right\rangle_0+\left\langle f_h U_h,\partial_{tx}\bar{\xi}\right\rangle_0$$
$$-\left\langle\partial_t f_h,\partial_t\bar{\xi}\right\rangle_0-\nu\left\langle\partial_x\xi,\partial_{tx}\bar{\xi}\right\rangle_0-\left\langle u(\bar{\rho}+\bar{\xi}),\partial_{tx}\bar{\xi}\right\rangle_0$$
$$+\lambda\left\langle\bar{\rho}+\bar{\xi},\partial_t\bar{\xi}\right\rangle_0.$$

Integrating from 0 to t, noting

$$\mathscr{A}(u;\bar{\xi},\partial_t\bar{\xi})=\frac{\nu}{2}\frac{d}{dt}\left\|\partial_x\bar{\xi}(t)\right\|_0^2+\frac{\lambda}{2}\frac{d}{dt}\left\|\bar{\xi}(t)\right\|_0^2-\frac{d}{dt}\left\langle u\bar{\xi},\partial_x\bar{\xi}\right\rangle_0+\left\langle\bar{\xi}\partial_t u,\partial_x\bar{\xi}\right\rangle_0$$
$$+\left\langle u\partial_t\bar{\xi},\partial_x\bar{\xi}\right\rangle_0,$$

and rearranging the terms give

$$\frac{\nu}{2}\left\|\partial_x\bar{\xi}(t)\right\|_0^2+\frac{\lambda}{2}\left\|\bar{\xi}(t)\right\|_0^2+\int_0^t\left\|\partial_t\bar{\xi}(s)\right\|_0^2\,ds=:\mathscr{T}_1+\mathscr{T}_2+\cdots+\mathscr{T}_{13}\qquad(5.3.41)$$

where

$$\mathscr{T}_1 = \left\langle u(t)\bar{\xi}(t), \partial_x\bar{\xi}(t)\right\rangle, \quad \mathscr{T}_2 = -\int_0^t \left\langle \bar{\xi}(s)\partial_t u(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds,$$
$$\mathscr{T}_3 = -\int_0^t \left\langle u\partial_t\bar{\xi}(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds, \quad \mathscr{T}_4 = \int_0^t \left\langle \partial_t\rho(s), \partial_t\bar{\xi}(s)\right\rangle_0 ds,$$
$$\mathscr{T}_5 = -\int_0^t \left\langle \partial_t\bar{\rho}(s), \partial_t\bar{\xi}(s)\right\rangle_0 ds, \quad \mathscr{T}_6 = -\int_0^t \left\langle U_h(s)(\rho(s) - \bar{\rho}(s) - \bar{\xi}(s)), \partial_{tx}\bar{\xi}(s)\right\rangle_0 ds,$$
$$\mathscr{T}_7 = -\int_0^t \left\langle V_h(s)(\eta(s) - \bar{\eta}(s) - \bar{\zeta}(s)), \partial_{tx}\bar{\xi}(s)\right\rangle_0 ds, \quad \mathscr{T}_8 = \int_0^t \left\langle e_h(s)v(s), \partial_{tx}\bar{\xi}(s)\right\rangle_0 ds,$$
$$\mathscr{T}_9 = \int_0^t \left\langle f_h(s)U_h(s), \partial_{tx}\bar{\xi}(s)\right\rangle_0 ds, \quad \mathscr{T}_{10} = -\int_0^t \left\langle \partial_t f_h(s), \partial_t\bar{\xi}(s)\right\rangle_0 ds,$$
$$\mathscr{T}_{11} = -\int_0^t \nu\left\langle \partial_x\xi(s), \partial_{tx}\bar{\xi}(s)\right\rangle_0 ds, \quad \mathscr{T}_{12} = -\int_0^t \left\langle u(s)(\bar{\rho}(s) + \bar{\xi}(s)), \partial_{tx}\bar{\xi}(s)\right\rangle_0 ds$$

and

$$\mathscr{T}_{13} = \lambda\int_0^t \left\langle \bar{\rho}(s) + \bar{\xi}(s), \partial_t\bar{\xi}(s)\right\rangle_0 ds.$$

It suffices to estimate those terms on the right hand side containing $\partial_{tx}\bar{\xi}$. These terms can be estimated by

$$\begin{aligned}
\mathscr{T}_6 &= \left\langle U_h(t)\left(\rho(t) - \bar{\rho}(t) - \bar{\xi}(t)\right), \partial_x\bar{\xi}(t)\right\rangle_0 - \int_0^t \left\langle (\partial_t U_h(s))\left(\rho - \bar{\rho} - \bar{\xi}\right)(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds \\
&\quad - \int_0^t \left\langle U_h(s)\partial_t\left(\rho - \bar{\rho} - \bar{\xi}\right)(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds, \\
\mathscr{T}_7 &= \left\langle V_h(t)\left(\eta(t) - \bar{\eta}(t) - \bar{\zeta}(t)\right), \partial_x\bar{\xi}(t)\right\rangle_0 - \int_0^t \left\langle (\partial_t V_h(s))\left(\eta - \bar{\eta} - \bar{\zeta}\right)(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds \\
&\quad - \int_0^t \left\langle V_h(s)\partial_t\left(\eta - \bar{\eta} - \bar{\zeta}\right)(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds, \\
\mathscr{T}_8 &= \left\langle e_h(t)v(t), \partial_x\bar{\xi}(t)\right\rangle_0 - \int_0^t \left\langle \partial_t e_h(s)v(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds - \int_0^t \left\langle e_h(s)\partial_t v(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds, \\
\mathscr{T}_9 &= \left\langle f_h(t)U_h(t), \partial_x\bar{\xi}(t)\right\rangle_0 - \int_0^t \left\langle \partial_t f_h(s)U_h(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds \\
&\quad - \int_0^t \left\langle f_h(s)\partial_t U_h(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds, \\
\mathscr{T}_{11} &= \nu\left\langle \partial_x\xi(t), \partial_x\bar{\xi}(t)\right\rangle_0 - \nu\int_0^t \left\langle \partial_{tx}\xi(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds, \\
\mathscr{T}_{12} &= \left\langle u(t)\left(\bar{\rho}(t) + \bar{\xi}(t)\right), \partial_x\bar{\xi}(t)\right\rangle_0 - \int_0^t \left\langle (\partial_t u(s))\left(\bar{\rho} + \bar{\xi}\right)(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds \\
&\quad - \int_0^t \left\langle u(s)\partial_t\left(\bar{\rho} + \bar{\xi}\right)(s), \partial_x\bar{\xi}(s)\right\rangle_0 ds.
\end{aligned}$$

By using Hölder's inequality on these terms, we have

$$|\mathscr{T}_6| + \cdots + |\mathscr{T}_9| + |\mathscr{T}_{11}| + |\mathscr{T}_{12}| \le Ch^{2\gamma}\Big(\|u\|^2_{L^\infty(H^{p+2})} + \|\partial_t u\|^2_{L^\infty(H^{p+2})} + \|v\|^2_{L^\infty(H^{q+2})} + \|\partial_t v\|^2_{L^\infty(H^{q+2})}\Big) + C\epsilon \left\|\partial_x \bar{\xi}(t)\right\|_0^2.$$

The remaining terms are estimated in a simpler. Thus, (5.3.41) yields

$$\begin{aligned}
&(\frac{\nu}{2} - C\epsilon)\left\|\partial_x\bar{\xi}\right\|^2_{L^\infty(H^0)} + \frac{\lambda}{2}\left\|\bar{\xi}\right\|^2_{L^\infty(H^0)} + (1-C\epsilon)\left\|\partial_t\bar{\xi}\right\|^2_{L^2(H^0)} \\
&\le Ch^{2\gamma}\Big(\|u\|^2_{L^\infty(H^{p+2})} + \|\partial_t u\|^2_{L^\infty(H^{p+2})} + \|v\|^2_{L^\infty(H^{q+2})} + \|\partial_t v\|^2_{L^\infty(H^{q+2})}\Big) \\
&\quad + C\Big(\|\partial_t\rho\|^2_{L^2(H^0)} + \|\partial_t\bar{\rho}\|^2_{L^2(H^0)} + \|\partial_t f_h\|^2_{L^2(H^0)} + \|\bar{\rho}\|^2_{L^2(H^0)} + \left\|\bar{\xi}\right\|^2_{L^2(H^0)}\Big).
\end{aligned}$$

Letting $\epsilon > 0$ sufficiently small such that

$$\frac{\nu}{2} - C\epsilon > 0 \quad \text{and} \quad 1 - C\epsilon > 0,$$

noting Lemma 5.2.2, Lemma 5.3.4, Theorem 5.2.4, and Lemma 5.3.5 thus the proof is completed. □

Proof of Theorem 5.3.1

We are now ready to prove the main result stated in the Theorem 5.3.1.

Proof. By referring to (5.3.20) and using the triangle inequality, we deduce

$$\begin{aligned}
|\Theta(t) - 1| &= \left|\frac{\hat{E}(t)}{\hat{e}(t)} - 1\right| = \frac{\left|\big(\,\|E(t)\|_1 + \|F(t)\|_1\,\big) - \big(\,\|e_h(t)\|_1 + \|f_h(t)\|_1\,\big)\right|}{\left|\hat{e}(t)\right|} \\
&\le \frac{\left|\,\|E(t)\|_1 - \|e_h(t)\|_1\,\right| + \left|\,\|F(t)\|_1 - \|f_h(t)\|_1\,\right|}{\left|\hat{e}(t)\right|} \\
&\le \frac{\|\bar{\eta}(t)\|_1 + \left\|\bar{\zeta}(t)\right\|_1 + \|\zeta(t)\|_1 + \|\bar{\rho}(t)\|_1 + \left\|\bar{\xi}(t)\right\|_1 + \|\xi(t)\|_1}{\left|\hat{e}(t)\right|}.
\end{aligned}$$

By using (5.3.21), Lemma 5.3.4, Lemma 5.3.6, Lemma 5.3.5, and Lemma 5.2.3 we infer

$$|\Theta(t) - 1| \le Ch,$$

thus proving the theorem. □

5.4 Implementation issues

In this section, we show the computation of (U_h, V_h) by using (5.1.12)–(5.1.14) and (E, F) by using the linear parabolic (Method (iii)) and linear elliptic (Method (iv)) error estimates introduced in Section 5.3. The results in this section have been reported in [52].

With $\phi_{l,k}$, $l = 1, \dots, N$ and $k = 1, 2, \dots,$ defined by (2.4.2)– (2.4.4), the approximate solutions (U_h, V_h) can be represented as

$$U_h(x,t) = \sum_{l=2}^{N} U_{l,1}(t)\phi_{l,1}(x) + \sum_{l=1}^{N}\sum_{k=2}^{p} U_{l,k}(t)\phi_{l,k}(x),$$
$$V_h(x,t) = \sum_{l=1}^{N+1} V_{l,1}(t)\phi_{l,1}(x) + \sum_{l=1}^{N}\sum_{k=2}^{q} V_{l,k}(t)\phi_{l,k}(x).$$

Recalling the L^2- inner products $\alpha_{k,k'}^{l,l'}$, $\bar{\alpha}_{k,k'}^{l,l'}$ and $\beta_{k,k'}^{l,l'}$ which are defined by (4.4.1)–(4.4.3), matrices $\boldsymbol{M}_r^l$, $\boldsymbol{S}_r^l$, $\boldsymbol{B}_{r,r'}^l$, and vectors $\boldsymbol{U}^l$ and $\boldsymbol{V}^l$ introduced in Section 4.4, then the matrix representation of (5.1.12)–(5.1.13) is of the form

$$\boldsymbol{S}_p \boldsymbol{U}(t) = \boldsymbol{B}_{p,q} \boldsymbol{V}(t), \tag{5.4.1}$$

$$\boldsymbol{M}_q \partial_t \boldsymbol{V}(t) + \nu \boldsymbol{S}_q \boldsymbol{V}(t) = \boldsymbol{G}\left(\boldsymbol{U}(t), \boldsymbol{V}(t)\right). \tag{5.4.2}$$

Here, the vector $\boldsymbol{G}(\boldsymbol{U}, \boldsymbol{V})$ is an $(Nq + 1) \times 1$ vector defined by

$$\boldsymbol{G}(\boldsymbol{U}, \boldsymbol{V}) = [\boldsymbol{G}^{(0)}, \boldsymbol{G}^{(1)}, \dots, \boldsymbol{G}^{(N)}]^\top,$$

where

$$\boldsymbol{G}^{(0)} = \left[\langle \boldsymbol{U}\boldsymbol{V}, \phi_{1,1}\rangle_0, \langle \boldsymbol{U}\boldsymbol{V}, \phi_{2,1}\rangle_0, \dots, \langle \boldsymbol{U}\boldsymbol{V}, \phi_{N+1,1}\rangle_0\right]^\top$$

and

$$\boldsymbol{G}^{(l)} = \left[\langle \boldsymbol{U}\boldsymbol{V}, \phi_{l,2}\rangle_0, \langle \boldsymbol{U}\boldsymbol{V}, \phi_{l,3}\rangle_0, \dots, \langle \boldsymbol{U}\boldsymbol{V}, \phi_{l,q}\rangle_0\right]^\top$$

for $l = 1, \dots, N$. We use the Matlab ODE solver to solve (5.4.1)–(5.4.2). Therefore the right hand side of (5.4.2) is computed by first solving (5.4.1) for a given $\boldsymbol{V}(t)$.

In this section, we discuss the computation of (E, F) which have the forms

$$E(x,t) = \sum_{l=1}^{N} E_l(t)\phi_{l,p+1}(x) \quad \text{and} \quad F(x,t) = \sum_{l=1}^{N} F_l(t)\phi_{l,q+1}(x).$$

Equations (5.3.11) and (5.3.10) are rewritten as

$$\begin{aligned}&\langle \partial_t F(t), \phi_{l,q+1}\rangle_{0,\Omega_l} + \nu \langle \partial_x F(t), \partial_x \phi_{l,q+1}\rangle_{0,\Omega_l} - \langle U_h(t)F(t), \partial_x \phi_{l,q+1}\rangle_{0,\Omega_l}\\ &\quad - \langle V_h(t)E(t), \partial_x \phi_{l,q+1}\rangle_{0,\Omega_l} = \langle U_h(t)V_h(t), \partial_x \phi_{l,q+1}\rangle_{0,\Omega_l} - \langle \partial_t V_h(t), \phi_{l,q+1}\rangle_{0,\Omega_l}\end{aligned} \tag{5.4.3}$$

and

$$\langle \partial_x E(t), \partial_x \phi_{l,p+1}\rangle_{0,\Omega_l} = \langle F(t), \partial_x \phi_{l,p+1}\rangle_{0,\Omega_l} + \langle V_h(t), \partial_x \phi_{l,p+1}\rangle_{0,\Omega_l}. \tag{5.4.4}$$

By using the notations (4.4.11)–(4.4.18) introduced in Section 4.4 and letting

$$\langle \partial_t V_h(t), \phi_{l,q+1}\rangle_{0,\Omega_l} = \partial_t V_{l+1,1}(t)\alpha^{l+1,l}_{1,q+1} + \sum_{k'=1}^{q} \partial_t V_{l,k'}(t)\alpha^{l,l}_{k',q+1} := T_7,$$

equation (5.4.3) is rewritten as

$$\frac{h_l}{(2q+3)(2q-1)}\partial_t F_l(t) + \left(\frac{2\nu}{h_l} - T_4\right) F_l(t) - T_5 E_l(t) = T_6 - T_7.$$

Moreover, (5.4.4) can be rewritten as

$$\frac{2}{h_l} E_l(t) = \beta^{l,l}_{p+1,q+1} F_l(t) + T_3.$$

Then, by using the Backward Euler Formulation, we compute $F_l(t_j)$ recursively by

$$\left(m + \frac{2\nu}{h_l} - T_4 - \frac{h_l}{2}\beta^{l,l}_{p+1,q+1} T_5\right) F_l(t_j) = T_6 - T_7 + mF_l(t_{j-1})$$

where

$$m = \frac{h_l}{(2q+3)(2q-1)(t_j - t_{j-1})}$$

and $t_j = j\triangle t$ for $j = 1, 2, 3, \ldots$. The time step $\triangle t$ is chosen to be not less than h.

By using Method (iv), a similar way is used to computed the error estimator E. However, error estimator F is computed differently. By using (5.3.13) and the above notations, we compute $F_l(t_j)$ of Method (iv) for $l = 1, \ldots, N$ as follows.

$$\left(\frac{2\nu}{h_l} - T_4 - \frac{h_l}{2}\beta^{l,l}_{p+1,q+1} T_5\right) F_l(t_j) = T_6 - T_7.$$

5.5 Numerical experiment

In this section, we present numerical results obtained when solving (5.1.1) - (5.1.3) whose exact solution (u, v) is

$$u(x,t) = \frac{2\nu\pi a \sin(\pi x)}{2 + a\cos(\pi x)}$$

and

$$v(x,t) = \frac{2\nu\pi^2 a \cos(\pi x)}{2 + a\cos(\pi x)} + \frac{2\nu(\pi a)^2 \sin(\pi x)^2}{(2 + a\cos(\pi x))^2},$$

where $a = \exp(-\pi^2\nu t)$ and initial value u_0 is given by

$$u_0(x) = \frac{2\nu\pi \sin(\pi x)}{2 + \cos(\pi x)}.$$

In the following, we choose $p = q + 1$. We obtain satisfactory numerical results for a range of $\nu = [0.01, 1]$. We compute for large and small values of ν, for example for $\nu = 1.0, 0.5, 0.05$ and $\nu = 0.01$.

In the numerical experiments, we compute the approximate solution (U_h, V_h) by solving (5.1.12)–(5.1.14). After that, we compute the error e_h and f_h to check on the order of convergence given by Theorem 5.2.4. Finally, we compute the error estimation E and F by using the linear parabolic (Method (iii)) and linear elliptic (Method (iv)) a posteriori error estimates introduced in Section 5.3.

In Table 5.1, Table 5.3, Table 5.5 and Table 5.7, we present the exact errors $\|e_h(t)\|_1$ and $\|f_h(t)\|_1$ at $t = 0.8$ for $\nu = 1.0, 0.5, 0.05$ and 0.01 respectively. As predicted by Theorem 5.2.4, the convergence rates are $\|e_h(t)\|_1 = O(h^p)$ and $\|f_h(t)\|_1 = O(h^{p-1})$.

In Table 5.2, Table 5.4, Table 5.6, and Table 5.8 we present the computed exact error $\hat{e}$ and a posteriori error estimate $\hat{E}$ at $t = 0.8$ for $\nu = 1.0, 0.5, 0.05$ and 0.01 respectively. We note that for Method (iii) we choose $\triangle t = 0.4$.

Figure 5.1, Figure 5.3, Figure 5.5 and Figure 5.7 represent the effectivity indices Θ at $t = 0.8$ respectively for $\nu = 1.0, 0.5, 0.05$ and 0.01, where Method (iii) is implemented to compute the error estimate (E, F). On the other hand, Figure 5.2, Figure 5.4, Figure 5.6 and Figure 5.8 represent the effectivity indices computed by using Method (iv).

dof_u	dof_v	p	q	N	$\|e_h(t)\|_1$	κ_u	$\|f_h(t)\|_1$	κ_v
39	21			20	2.2651E-04		2.7298E-03	
79	41			40	5.6836E-05	1.995	1.3407E-03	1.026
159	81	2	1	80	1.4221E-05	1.999	6.6726E-04	1.007
199	101			100	9.1023E-06	1.999	5.3351E-04	1.002
319	161			160	3.5554E-06	2.000	3.3324E-04	1.001
59	41			20	4.2389E-07		5.4027E-05	
119	81			40	5.2246E-08	3.020	1.3510E-05	1.999
179	121	3	2	60	1.5442E-08	3.006	6.0046E-06	2.000
239	161			80	6.5658E-09	2.973	3.3777E-06	2.000
299	201			100	3.4812E-09	2.844	2.1617E-06	2.000
19	16			5	9.6823E-07		4.5721E-05	
39	31	4	3	10	6.0520E-08	3.999	5.7343E-06	2.995
59	46			15	1.2031E-08	3.984	1.7001E-06	2.998
79	61			20	3.9962E-09	3.831	7.1740E-07	2.999

Table 5.1: The orders of convergence κ_u and κ_v at $\nu = 1.0$ and $t = 0.8$.

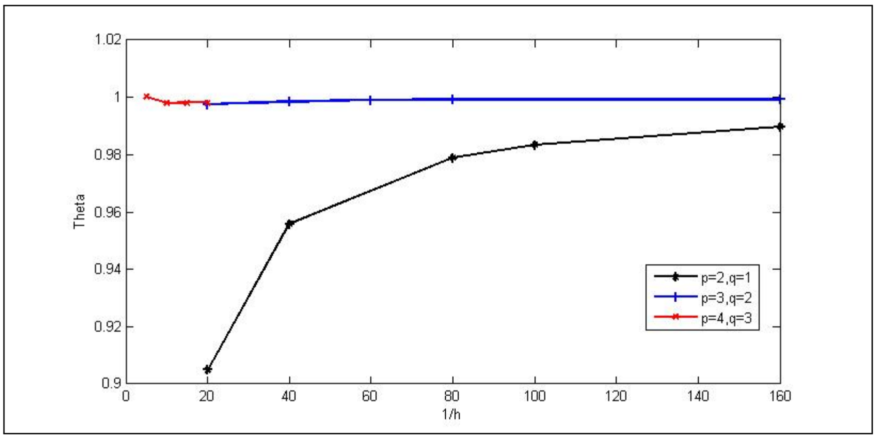

Figure 5.1: Effectivity indices Θ by Method (iii) (refer to Page 80), at $\nu = 1$ and $t = 0.8$ with different values of h.

p	q	h	$\hat{e}(t)$	Method (iii) $\hat{E}(t)$	Method (iv) $\hat{E}(t)$
2	1	1/20	2.9563E-03	2.6751E-03	2.6543E-03
		1/40	1.3975E-03	1.3359E-03	1.3333E-03
		1/80	6.8148E-04	6.6719E-04	6.6687E-04
		1/100	5.4261E-04	5.3361E-04	5.3345E-04
		1/160	3.3680E-04	3.3337E-04	3.3333E-04
3	2	1/20	5.4450E-05	5.4318E-05	5.4218E-05
		1/40	1.3562E-05	1.3542E-05	1.3536E-05
		1/60	6.0201E-06	6.0139E-06	6.0126E-06
		1/80	3.3842E-06	3.3815E-06	3.3811E-06
		1/160	2.1652E-06	2.1636E-06	2.1635E-06
4	3	1/5	4.6689E-05	4.6701E-05	4.6056E-05
		1/10	5.7948E-06	5.7829E-06	5.7629E-06
		1/15	1.7121E-06	1.7089E-06	1.7063E-06
		1/20	7.2139E-07	7.2004E-07	7.1941E-07

Table 5.2: Values of $\hat{E}$ by Method (iii) and Method (iv) at $\nu = 1.0$ and $t = 0.8$.

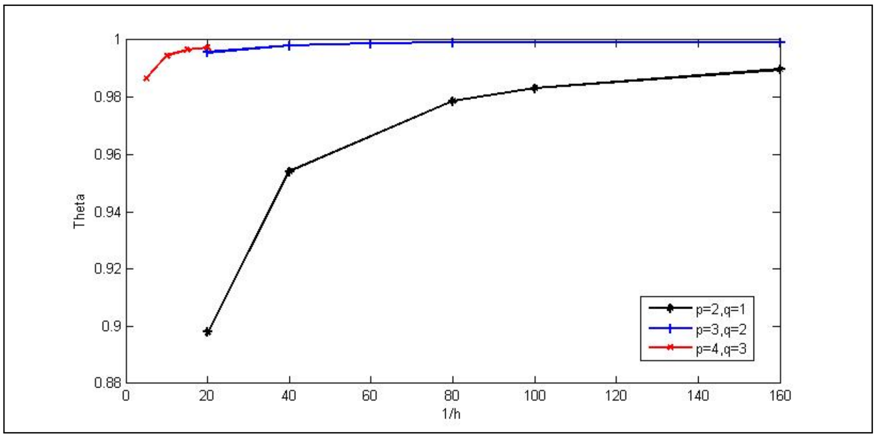

Figure 5.2: Effectivity indices Θ by Method (iv) (refer to Page 80), at $\nu = 1$ and $t = 0.8$ with different values of h.

dof_u	dof_v	p	q	N	$\|e_h(t)\|_1$	κ_u	$\|f_h(t)\|_1$	κ_v
39	21			20	1.0569E-03		2.5964E-02	
79	41			40	2.6461E-04	1.998	1.2950E-02	1.004
159	81	2	1	80	6.6178E-05	1.999	6.4709E-03	1.001
319	161			160	1.6546E-05	2.000	3.2349E-03	1.000
639	321			320	4.1362E-06	2.000	1.6174E-03	1.000
1279	641			640	1.0337E-06	2.000	8.0869E-04	1.000
59	41			20	4.1212E-06		5.3291E-04	
119	81			40	5.1435E-07	3.002	1.3327E-04	1.999
179	121	3	2	60	1.5235E-07	3.000	5.9232E-05	2.000
239	161			80	6.4264E-08	3.000	3.3319E-05	2.000
479	321			160	8.0457E-09	2.998	8.3298E-06	2.000
39	31			10	6.3273E-07		5.9950E-05	
59	46			15	1.2501E-07	3.999	1.7779E-05	2.998
79	61	4	3	20	3.9573E-08	3.998	7.5030E-06	2.999
99	76			25	1.6230E-08	3.994	3.8421E-06	2.999
119	91			30	7.8736E-09	3.967	2.2236E-06	3.000

Table 5.3: The orders of convergence κ_u and κ_v at $\nu = 0.5$ and $t = 0.8$.

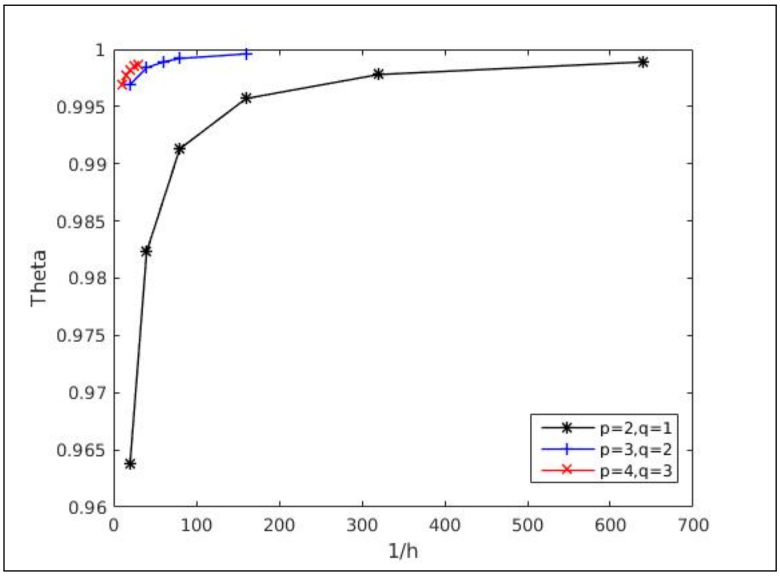

Figure 5.3: Effectivity indices Θ by Method (iii) (refer to Page 80), at $\nu = 0.5$ and $t = 0.8$ with different values of h.

p	q	h	$\hat{e}(t)$	Method (iii)	Method (iv)
				$\hat{E}(t)$	$\hat{E}(t)$
2	1	1/20	2.7021E-02	2.6042E-02	2.5928E-02
		1/40	1.3214E-02	1.2980E-02	1.2966E-02
		1/80	6.5371E-03	6.4799E-03	6.4782E-03
		1/160	3.2515E-03	3.2374E-03	3.2372E-03
		1/320	1.6215E-03	1.6180E-03	1.6180E-03
		1/640	8.0973E-04	8.0886E-04	8.0885E-04
3	2	1/20	5.3704E-04	5.3538E-04	5.3478E-04
		1/40	1.3378E-04	1.3356E-04	1.3352E-04
		1/60	5.9384E-05	5.9318E-05	5.9311E-05
		1/80	3.3383E-05	3.3355E-05	3.3352E-05
		1/160	8.3378E-06	8.3342E-06	8.3341E-06
4	3	1/10	6.0582E-05	6.0393E-05	6.0233E-05
		1/15	1.7904E-05	1.7862E-05	1.7841E-05
		1/20	7.5425E-06	7.5286E-06	7.5237E-06
		1/25	3.8583E-06	3.8524E-06	3.8508E-06
		1/30	2.2315E-06	2.2285E-06	2.2279E-06

Table 5.4: Values of $\hat{E}$ by Method (iii) and Method (iv) at $\nu = 0.5$ and $t = 0.8$.

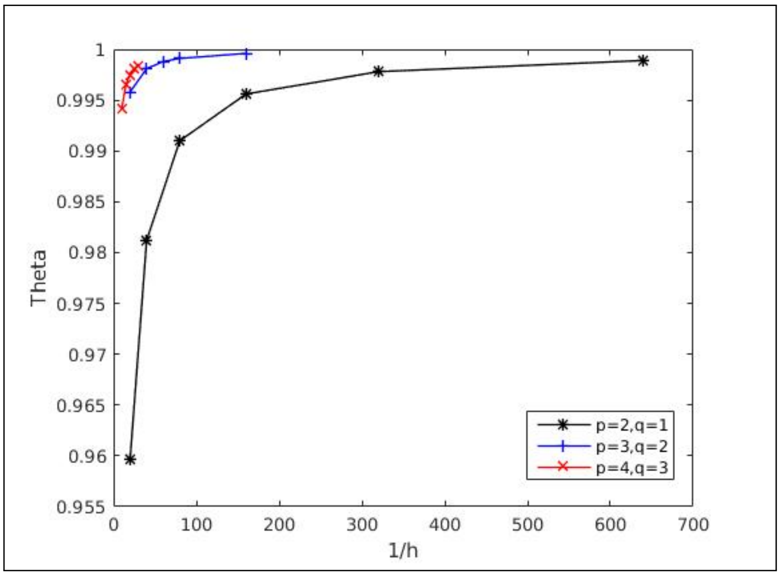

Figure 5.4: Effectivity indices Θ by Method (iv) (refer to Page 80), at $\nu = 0.5$ and $t = 0.8$ with different values of h.

dof_u	dof_v	p	q	N	$\|e_h(t)\|_1$	κ_u	$\|f_h(t)\|_1$	κ_v
39	21			20	1.1338E-03		6.6472E-02	
79	41			40	2.8487E-04	1.993	3.3245E-02	0.999
159	81	2	1	80	7.1305E-05	1.998	1.6624E-02	1.000
319	161			160	1.7831E-05	2.000	8.3120E-03	1.000
639	321			320	4.4577E-06	2.000	4.1560E-03	1.000
1279	641			640	1.1140E-06	2.000	2.0780E-03	1.000
59	41			20	2.2153E-05		2.8620E-03	
119	81			40	2.7675E-06	3.001	7.1684E-04	1.997
179	121	3	2	60	8.1990E-07	3.000	3.1871E-04	1.999
239	161			80	3.4587E-07	3.000	1.7929E-04	2.000
479	321			160	4.3230E-08	3.000	4.4829E-05	2.000
79	61			20	5.4335E-07		1.0302E-04	
159	121			40	3.4027E-08	3.997	1.2908E-05	2.997
239	181	4	3	60	6.7575E-09	3.988	3.8262E-06	2.999
319	241			80	2.2446E-09	3.831	1.6144E-06	3.000
399	301			100	1.0171E-09	3.547	8.2665E-07	3.000

Table 5.5: The orders of convergence κ_u and κ_v at $\nu = 0.05$ and $t = 0.8$.

p	q	h	$\hat{e}(t)$	Method (iii) $\hat{E}(t)$	Method (iv) $\hat{E}(t)$
2	1	1/20	6.7606E-02	6.6942E-02	6.6543E-02
		1/40	3.3530E-02	3.3357E-02	3.3308E-02
		1/80	1.6695E-02	1.6651E-02	1.6645E-02
		1/160	8.3298E-03	8.3187E-03	8.3180E-03
		1/320	4.1604E-03	4.1577E-03	4.1576E-03
		1/640	2.0791E-03	2.0784E-03	2.0784E-03
3	2	1/20	2.8841E-03	2.8725E-03	2.8671E-03
		1/40	7.1961E-04	7.1826E-04	7.1792E-04
		1/60	3.1953E-04	3.1914E-04	3.1907E-04
		1/80	1.7964E-04	1.7948E-04	1.7946E-04
		1/160	4.4872E-05	4.4852E-05	4.4851E-05
4	3	1/20	1.0356E-04	1.0330E-04	1.0319E-04
		1/40	1.2942E-05	1.2927E-05	1.2923E-05
		1/60	3.8329E-06	3.8300E-06	3.8295E-06
		1/80	1.6167E-06	1.6156E-06	1.6155E-06
		1/100	8.2767E-07	8.2714E-07	8.2710E-07

Table 5.6: Values of $\hat{E}$ by Method (iii) and Method (iv) at $\nu = 0.05$ and $t = 0.8$.

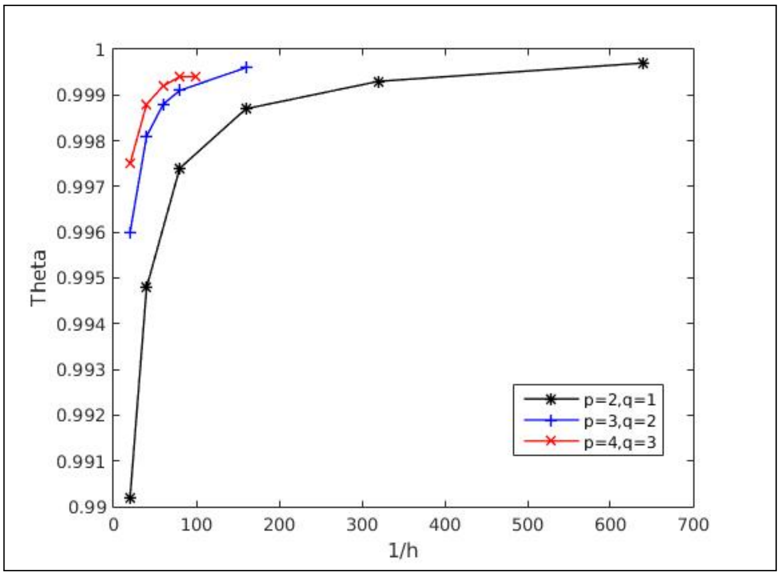

Figure 5.5: Effectivity indices Θ by Method (iii) (refer to Page 80), at $\nu = 0.05$ and $t = 0.8$ with different values of h.

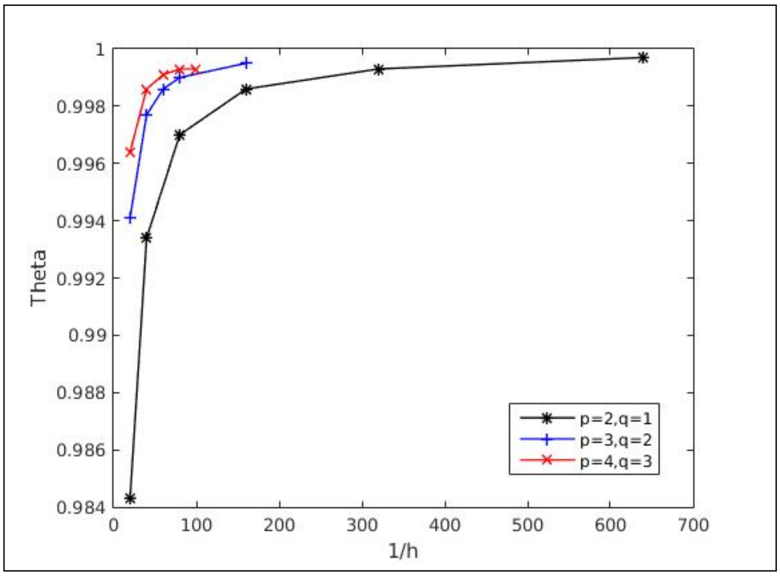

Figure 5.6: Effectivity indices Θ by Method (iv) (refer to Page 80), at $\nu = 0.05$ and $t = 0.8$ with different values of h.

dof_u	dof_v	p	q	N	$\|e_h(t)\|_1$	κ_u	$\|f_h(t)\|_1$	κ_v
39	21	2	1	20	2.7155E-04		3.0153E-02	
79	41			40	6.7352E-05	2.011	1.5085E-02	0.999
159	81			80	1.6804E-05	2.003	7.5435E-03	1.000
199	101			100	1.0752E-05	2.001	6.0349E-03	1.000
319	161			160	4.1988E-06	2.000	3.7719E-03	1.000
59	41	3	2	20	1.2647E-05		1.6474E-03	
119	81			40	1.5915E-06	2.990	4.1308E-04	1.996
179	121			60	4.7214E-07	2.997	1.8369E-04	1.999
239	161			80	1.9927E-07	2.999	1.0335E-04	1.999
299	201			100	1.0205E-07	3.000	6.6148E-05	2.000
39	31	4	3	10	6.2309E-06		5.8969E-04	
59	46			15	1.2437E-06	3.974	1.7675E-04	2.972
79	61			20	3.9485E-07	3.988	7.4857E-05	2.986
99	76			25	1.6198E-07	3.993	3.8396E-05	2.992
119	91			30	7.8178E-08	3.996	2.2242E-05	2.995

Table 5.7: The orders of convergence κ_u and κ_v at $\nu = 0.01$ and $t = 0.8$.

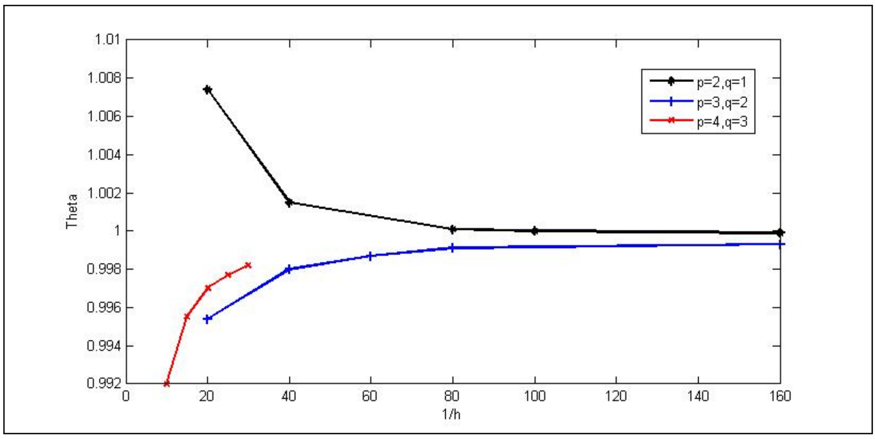

Figure 5.7: Effectivity indices Θ by Method (iii) (refer to Page 80), at $\nu = 0.01$ and $t = 0.8$ with different values of h.

p	q	h	$\hat{e}(t)$	Method (iii) $\hat{E}(t)$	Method (iv) $\hat{E}(t)$
2	1	1/20	3.0424E-02	3.0648E-02	3.0455E-02
		1/40	1.5152E-02	1.5175E-02	1.5148E-02
		1/80	7.5603E-03	7.5610E-03	7.5575E-03
		1/100	6.0457E-03	6.0454E-03	6.0436E-03
		1/160	3.7761E-03	3.7756E-03	3.7752E-03
3	2	1/20	1.6600E-03	1.6524E-03	1.6487E-03
		1/40	4.1467E-04	4.1382E-04	4.1359E-04
		1/60	1.8416E-04	1.8393E-04	1.8388E-04
		1/80	1.0355E-04	1.0345E-04	1.0343E-04
		1/100	6.6250E-05	6.6200E-05	6.6194E-05
4	3	1/10	5.9592E-04	5.9118E-04	5.8819E-04
		1/15	1.7800E-04	1.7720E-04	1.7680E-04
		1/20	7.5252E-05	7.5024E-05	7.4927E-05
		1/25	3.8558E-05	3.8470E-05	3.8438E-05
		1/30	2.2320E-05	2.2279E-05	2.2266E-05

Table 5.8: Values of $\hat{E}$ by Method (iii) and Method (iv) at $\nu = 0.01$ and $t = 0.8$.

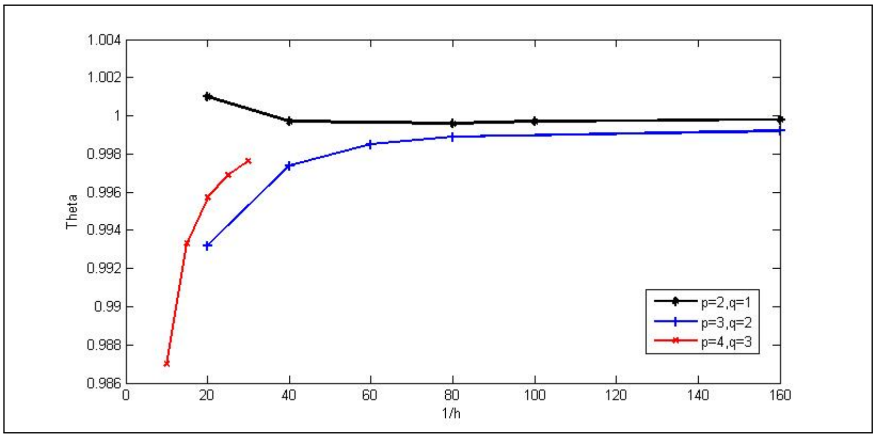

Figure 5.8: Effectivity indices Θ by Method (iv) (refer to Page 80), at $\nu = 0.01$ and $t = 0.8$ with different values of h.

Chapter 6

Adaptive schemes: numerical studies

In this chapter, we focus on the numerical studies of adaptive schemes for the BBM and Burgers equations. The chapter begins with an introduction on adaptive schemes for finite element methods. Then, we explain the adaptive procedures implement in this study in Section 6.2. The chapter ends with numerical experiments of adaptive schemes for the BBM and Burgers equations.

6.1 Introduction

Adaptive schemes of finite element methods for numerical solutions of partial differential equations are considered as a standard tool in science and engineering to achieve better accuracy with minimum degrees of freedom. The adaptive scheme for one dimensional boundary value problems is studied in 1984 by Babuška et. al [9]. Details on the theory of adaptive finite element methods can be found in [42]. Studies about adaptive schemes with mixed finite element methods have been carried out in [15, 22, 24, 26, 51]; see also the references therein.

When solving a BVP by any approximation technique, a scheme is required to guide us on the accuracy of our approximate solutions. By using adaptive schemes of finite element methods, we try to automatically refine a mesh to achieve approximate solutions having a specified accuracy in an optimal way. In general, the computation typically begins with a trial set of approximate solutions generated on a coarse initial mesh. The error estimate of these solutions is appraised. If the error estimate fails to satisfy the prescribed accuracy, refinements are made with a goal to obtain the desired approximate solutions with minimal effort.

In this study, we use (h-type) adaptive mesh refinement schemes for the BBM and Burgers equations. We refer to Chapter 4 and Chapter 5 for a posteriori error estimations of H1MFEM for the BBM and Burgers equations, respectively.

The aim of the adaptive scheme is to generate better approximations of the exact solutions. The basic idea of the adaptive schemes is, given the approximate solution, to create a refined partition (mesh) by subdividing elements where the error estimators indicate that the errors are large. Then, on this refined partition, the next approximate solution is computed. The process is repeated until the desired accuracy of approximate solution is obtained. The final outcomes of the adaptive scheme are the refined mesh and accurate approximate solutions. A principal tool for this adaptive scheme is the availability of local (elementwise) error estimates, namely local a posteriori error estimators.

An adaptive scheme of a finite element method consists of successive loops of the following form:

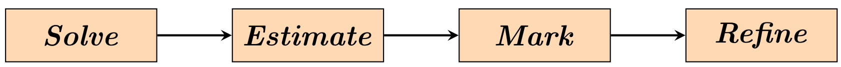

Figure 6.1: Four steps of an adaptive scheme.

The step ***Solve*** involves the computation of the approximate solution. Given the number of elements, the H1MFEM is applied to compute the approximate solution.

The a posteriori error estimation is an essential part of the second step, namely ***Estimate***. By using the approximate solution generated in step ***Solve***, we compute a posteriori error estimators locally on each element. These a posteriori error estimators will be used in the ***Mark*** step. The a posteriori error estimates are then used to appraise the accuracy of the approximate solution and to control the adaptive enrichment through the refinement process.

In the third step, ***Mark***, we perform an evaluation process on a posteriori error estimators of step ***Solve***. Given a tolerance δ, we check the global a posteriori error estimator. If the global a posteriori error estimator is larger than the tolerance δ, then we appraise Φ largest local a posteriori error estimators, $\Phi = 1, 2, 3, \ldots$, where Φ is decided by the user. The elements of these respective local a posteriori error estimators are identified for a mesh refinement process in step ***Refine***.

The adaptive strategy in ***Refine*** is based on halving the Φ intervals where the local a posteriori error estimators are the largest. After the refinement process, a new mesh is generated. By using this finer mesh, the process is repeated and is terminated when the global error estimator is smaller than the assigned tolerance δ.

6.2 Adaptive procedure

In this section, we are now more specific on the adaptive procedure used in this study. We consider examples for the BBM and Burgers equations. For the BBM equation, we refer to the numerical example in Section 4.5. For the Burgers equation, the numerical example in Section 5.5 with $\nu = 0.5$ is considered.

Figure 6.2 shows the general adaptive process applied to the following numerical results. The inputs are the number of elements N for initial mesh τ_0, the polynomial degrees p and q respectively for U_h and V_h and the number of elements Φ of the refinement process. The polynomial degrees are fixed throughout this adaptive scheme.

For the readers convenience, we recall the equations which are used in both numerical examples. Firstly, under the step ***Solve***, we compute the values of (U_h, V_h) by using the Galerkin equations of the problem. Recalling the notations defined in Section 4.4, the approximate solution (U_h, V_h) of the BBM equation is computed by solving

$$\boldsymbol{S}_p \boldsymbol{U}(t) = \boldsymbol{B}_{p,q} \boldsymbol{V}(t),$$

and

$$\Big(\boldsymbol{M}_q + \frac{\mu}{d^2}\boldsymbol{S}_q\Big)\, \partial_t \boldsymbol{V}(t) = \boldsymbol{H}\big(\boldsymbol{U}(t), \boldsymbol{V}(t)\big).$$

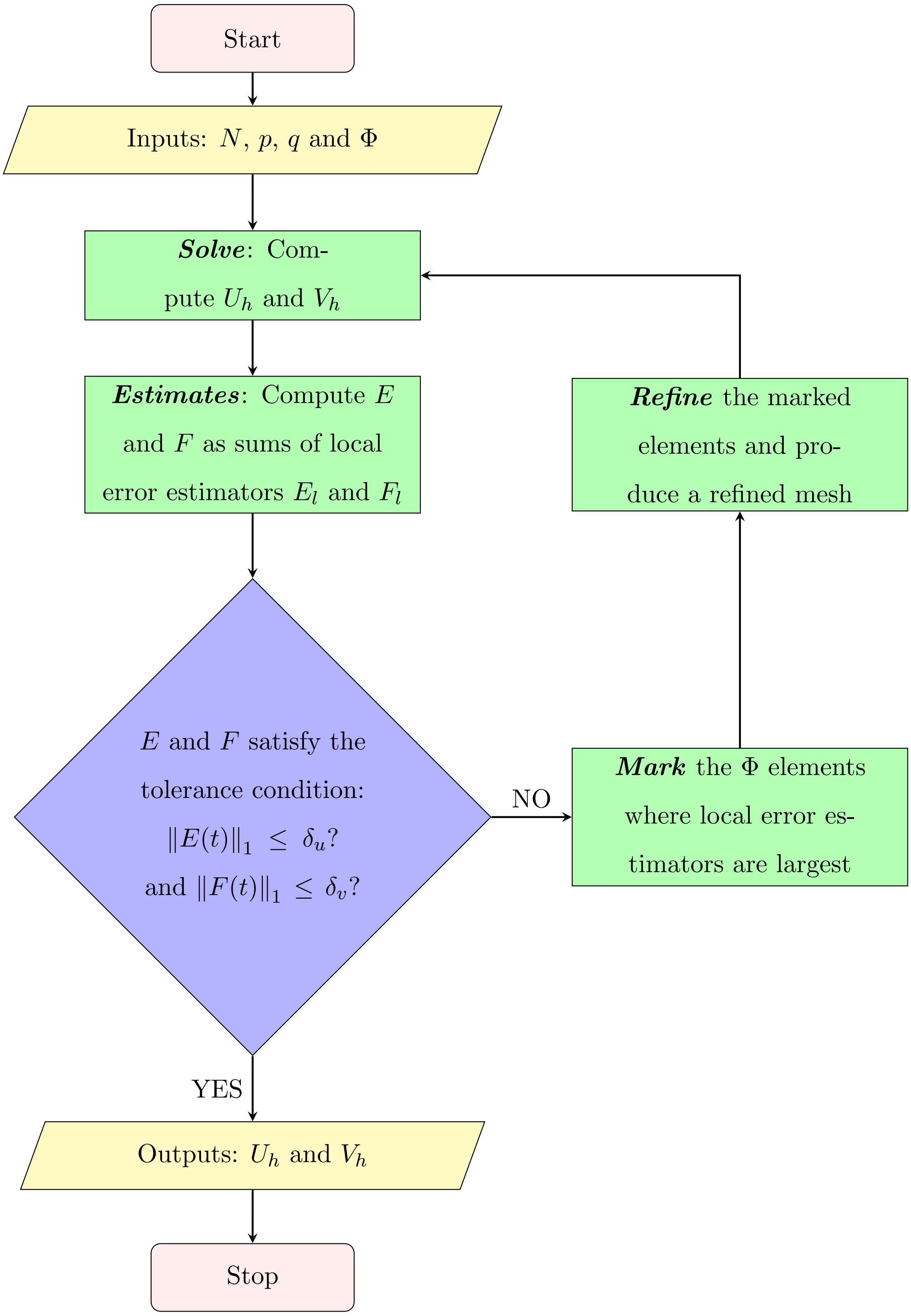

Figure 6.2: Adaptive process applied for the BBM and Burgers equations.

Here, the vector $\boldsymbol{H}(\boldsymbol{U},\boldsymbol{V})$ is an $(Nq+1)\times 1$ vector defined by

$$\boldsymbol{H}(\boldsymbol{U},\boldsymbol{V}) = \big[\boldsymbol{H}^{(0)},\boldsymbol{H}^{(1)},\ldots,\boldsymbol{H}^{(N)}\big]^\top$$

where

$$\boldsymbol{H}^{(0)} = \Big[\Big\langle \frac{1}{d}\boldsymbol{U}\boldsymbol{V}+\boldsymbol{g},\phi_{1,1}\Big\rangle_0, \Big\langle \frac{1}{d}\boldsymbol{U}\boldsymbol{V}+\boldsymbol{g},\phi_{2,1}\Big\rangle_0,\ldots,\Big\langle \frac{1}{d}\boldsymbol{U}\boldsymbol{V}+\boldsymbol{g},\phi_{N+1,1}\Big\rangle_0\Big]^\top$$

and

$$\boldsymbol{H}^{(l)} = \Big[\Big\langle \frac{1}{d}\boldsymbol{U}\boldsymbol{V}+\boldsymbol{g},\phi_{l,2}\Big\rangle_0, \Big\langle \frac{1}{d}\boldsymbol{U}\boldsymbol{V}+\boldsymbol{g},\phi_{l,3}\Big\rangle_0,\ldots,\Big\langle \frac{1}{d}\boldsymbol{U}\boldsymbol{V}+\boldsymbol{g},\phi_{l,q}\Big\rangle_0\Big]^\top$$

for $l=1,\ldots,N$. Similarly, for the Burgers equation, (U_h,V_h) is computed by solving

$$\boldsymbol{S}_p\boldsymbol{U}(t) = \boldsymbol{B}_{p,q}\boldsymbol{V}(t)$$

and

$$\boldsymbol{M}_q\partial_t\boldsymbol{V}(t)+\frac{1}{2}\boldsymbol{S}_q\boldsymbol{V}(t) = \boldsymbol{G}\big(\boldsymbol{U}(t),\boldsymbol{V}(t)\big).$$

Here, the vector $\boldsymbol{G}(\boldsymbol{U},\boldsymbol{V})$ is an $(Nq+1)\times 1$ vector defined by

$$\boldsymbol{G}(\boldsymbol{U},\boldsymbol{V}) = [\boldsymbol{G}^{(0)},\boldsymbol{G}^{(1)},\ldots,\boldsymbol{G}^{(N)}]^\top,$$

where

$$\boldsymbol{G}^{(0)} = [\langle\boldsymbol{U}\boldsymbol{V},\phi_{1,1}\rangle_0, \langle\boldsymbol{U}\boldsymbol{V},\phi_{2,1}\rangle_0,\ldots,\langle\boldsymbol{U}\boldsymbol{V},\phi_{N+1,1}\rangle_0]^\top$$

and

$$\boldsymbol{G}^{(l)} = [\langle\boldsymbol{U}\boldsymbol{V},\phi_{l,2}\rangle_0, \langle\boldsymbol{U}\boldsymbol{V},\phi_{l,3}\rangle_0,\ldots,\langle\boldsymbol{U}\boldsymbol{V},\phi_{l,q}\rangle_0]^\top.$$

Secondly, under the step ***Estimate***, we compute the local a posteriori error estimators (E_l,F_l) for $l=1,\ldots,N$. Noting the notations (4.4.11)–(4.4.18), letting

$$\langle\partial_{tx}V_h(t),\partial_x\phi_{l,q+1}\rangle_{1,\Omega_l} = \partial_t V_{l+1,1}(t)\bar{\alpha}^{l+1,l}_{1,q+1} + \sum_{k'=1}^{q}\partial_t V_{l,k'}(t)\bar{\alpha}^{l,l}_{k',q+1} := T_9$$

and

$$\langle g(t),\partial_x\phi_{l,q+1}\rangle_{0,\Omega_l} = g_{l+1,1}(t)\beta^{l+1,l}_{1,q+1} + \sum_{k'=1}^{q} g_{l,k'}(t)\beta^{l,l}_{k',q+1} := T_{10},$$

the a posteriori error estimators of the BBM equation are computed by using

$$\frac{2}{h_l}E_l(t) = \beta^{l,l}_{p+1,q+1}F_l(t)+T_3 \tag{6.2.1}$$

and

$$\left(\frac{h_l}{(2q+3)(2q-1)}+\frac{2\mu}{d^2h_l}\right)\partial_t F_l(t)-\frac{1}{d}T_4F_l(t)-T_5E_l(t)=T_{10}+T_6-T_7-\frac{\mu}{d^2}T_9. \quad (6.2.2)$$

Similarly, for the Burgers equation, (E_l, F_l) is computed by using

$$\frac{2}{h_l}E_l(t)=\beta^{l,l}_{p+1,q+1}F_l(t)+T_3$$

and

$$\frac{h_l}{(2q+3)(2q-1)}\partial_t F_l(t)+\left(\frac{2\nu}{h_l}-T_4\right)F_l(t)-T_5E_l(t)=T_6-T_7. \quad (6.2.3)$$

After that, the global a posteriori error estimator (E, F) which is the summation of the local a posteriori error estimators (E_l, F_l), for $l = 1, \ldots, N$ is appraised with the assigned tolerances δ_u and δ_v by checking

$$\|E(t)\|_1 \le \delta_u \quad (6.2.4)$$

and

$$\|F(t)\|_1 \le \delta_v. \quad (6.2.5)$$

The process is terminated if (6.2.4) and (6.2.5) are satisfied. Otherwise, the process is continue with ***Mark*** and ***Refine***.

In step ***Mark***, Φ largest local a posteriori error estimators, $\Phi = 1, 2, 3, \ldots$, are identified and appraised for marking process. In the following numerical experiments, we perform the marking process only on the local a posteriori error estimators E_l. For example, if (6.2.4) and (6.2.5) are not satisfied, we appraise Φ largest E_l in the marking process.

For the refinement process in the step ***Refine***, we introduce the Adaptive Method (AM(Φ)). We test the problems in two different ways, namely AM(1) and AM(3). The first method is where we have $\Phi = 1$, that is we refine only the largest local a posteriori error estimator E_l of the marked element. On the other hand, by using AM(3), we have $\Phi = 3$, i.e., three largest local a posteriori error estimators E_l of the marked elements are refined. Figure 6.3 and Figure 6.4 illustrate examples of three refinement processes (R_1–R_3) of both methods, which are applied to an initial mesh τ_0, at $N = 10$. The $x^*_{i,j}$

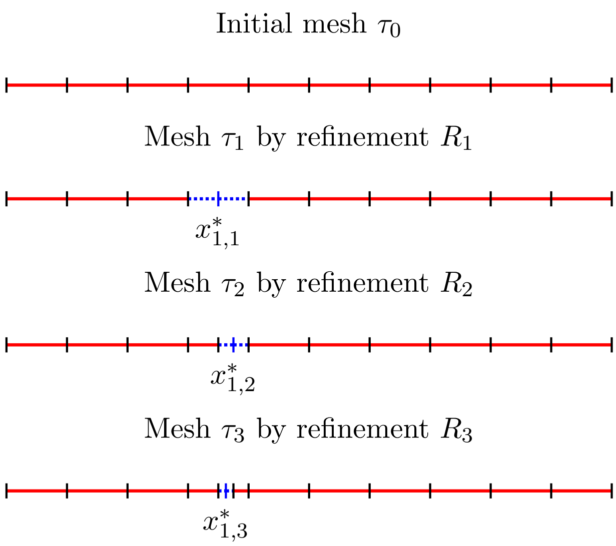

Figure 6.3: Refinement processes by Adaptive Method AM(1).

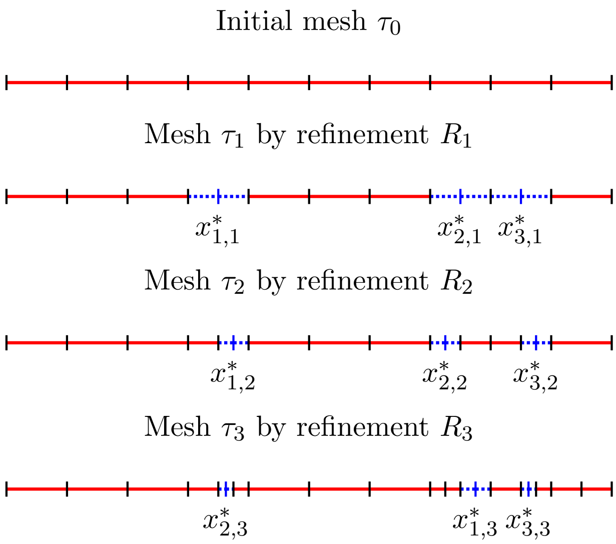

Figure 6.4: Refinement processes by Adaptive Method AM(3).

values, for $i = 1, 2, \ldots$ represent the new points introduced during the refinement process R_j, $j = 1, 2, 3, \ldots$.

Then, the new mesh generated from these two steps (***Mark*** and ***Refine***) is used to generate the new approximate solution in the step ***Solve***. The process is repeated and terminated when the desired accuracy of the approximate solution is obtained.

6.3 Numerical experiments

The numerical results are computed by using three methods, namely AM(1), AM(3) and Non-adaptive Method (NAM). Methods AM(1) and AM(3) are respectively the adaptive schemes introduced in Section 6.2 (see Figure 6.3 and Figure 6.4). By using AM(1), we consider an element ($\Phi = 1$) for the refinement process, while using AM(3), we consider three elements, ($\Phi = 3$).

On the other hand, NAM represents cases with uniform meshes. In the Step ***Mark*** and Step ***Refine*** of NAM, all elements are refined during each of the refinement processes. For example, by using NAM with an initial mesh τ_0, at $N = 10$, we will have 10 new points $\{x^*_{1,1}, x^*_{2,1}, x^*_{3,1}, \ldots, x^*_{10,1}\}$ during the first refinement process, R_1. Similarly with AM(1) and AM(3), the process is terminated if (6.2.4) and (6.2.5) are satisfied.

We compare the numerical results by AM(1) and AM(3) with the numerical results by NAM. In the following numerical experiments, the coarse mesh τ_0 of these methods is started at $N = 10$.

6.3.1 Adaptive scheme for the Benjamin-Bona-Mahony equation

We perform the numerical experiments with different values of tolerances δ_u and δ_v. Figure 6.5 and Figure 6.6 show the graphs of the exact solution (u, v) considered in these numerical experiments.

Example 1 is computed with tolerances $\delta_u = 5 \times 10^{2-p}$ and $\delta_v = 5 \times 10^{1-q}$. Table 6.1 shows the numerical results for $p = 2$ and $q = 1$. The number of elements, N represents the total elements generated by each method after the number of refinements. The number of refinements also represents the total number of problems required to solve, in order to get the approximate solution at the assigned tolerances.

Based on the numerical results in Table 6.1, AM(1) and AM(3) produce comparable error values with NAM. Even though NAM provides a slightly better error values, the CPU time required is 22841s and number of elements N is 640. Meanwhile, only 2089s of CPU time and $N = 156$ by AM(1), and 296s of CPU time and $N = 157$ by AM(3) are required for the error values. Note that, if we use a uniform refinement when

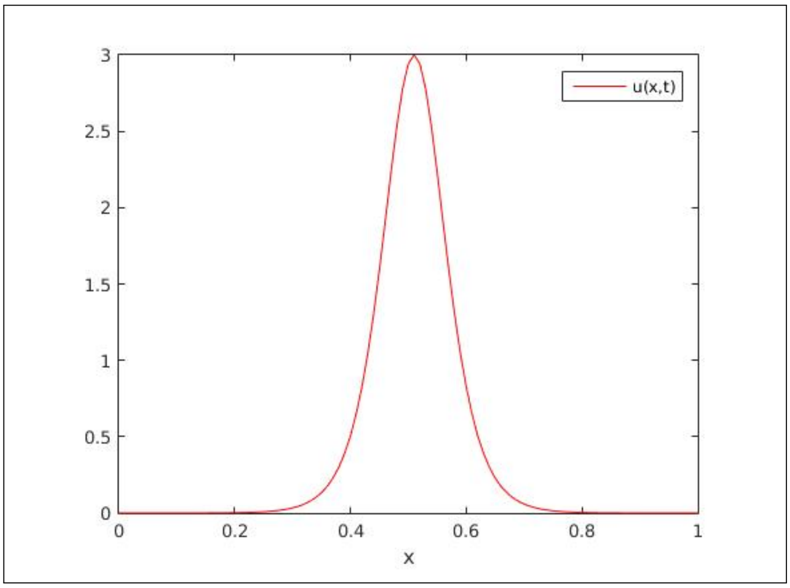

Figure 6.5: The BBM equation: exact solution $u(x,t)$.

	AM(1)	AM(3)	NAM
N	156	157	640
Degrees of freedom u	311	313	1279
Degrees of freedom v	157	158	641
Number of refinements	146	49	6
CPU time (sec)	2089	296	22841
$\Vert e_h(t)\Vert_1 \,/\, \Vert u(t)\Vert_1$	9.5091E-04	7.9001E-04	1.3747E-04
$\Vert f_h(t)\Vert_1 \,/\, \Vert v(t)\Vert_1$	2.2375E-02	2.1463E-02	1.4930E-02

Table 6.1: Example 1 for the BBM equation at $t = 0.8$.

$N = 157$, we have larger error values, which are $\Vert e_h(t)\Vert_1 / \Vert u(t)\Vert_1 = 2.2866\text{E-}03$ and $\Vert f_h(t)\Vert_1 / \Vert v(t)\Vert_1 = 6.0852\text{E-}02$. In this case, the CPU time is smaller (59s) but the accuracy is lower. Figure 6.7 and Figure 6.8 respectively show the final refined meshes at $N = 156$ by AM(1) and $N = 157$ by AM(3).

Example 2 is computed with tolerances $\delta_u = 10^{3-p}$ and $\delta_v = 10^{2-q}$. Table 6.2 shows the numerical results of the three methods for $p = 3$ and $q = 2$. The error values by NAM is slightly better than the error values by AM(1) and AM(3). However, 7566s of

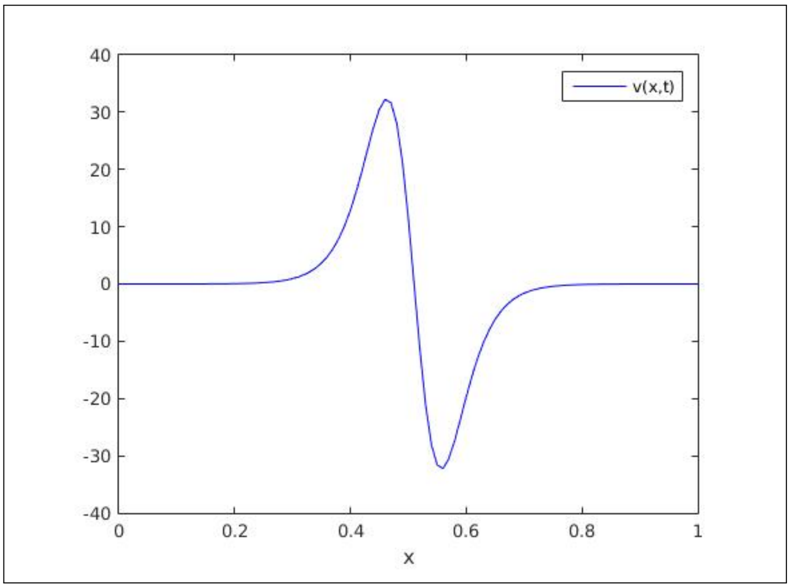

Figure 6.6: The BBM equation: exact solution $v(x,t)$.

CPU time and higher degrees of freedom for u and v are required to obtain the results. Besides that, if a uniform refinement is used at $N = 46$, smaller CPU time (25s) is required and the error values are larger, which are $\|e_h(t)\|_1 \,/\, \|u(t)\|_1 = 1.8845\text{E-}03$ and $\|f_h(t)\|_1 \,/\, \|v(t)\|_1 = 2.4319\text{E-}02$. The final refined meshes of Example 2 by AM(1) and AM(3) are respectively presented in Figure 6.9 and Figure 6.10.

6.3.2 Adaptive scheme for the Burgers equation

Similarly, we perform numerical experiments for the Burgers equation. Figure 6.11 and Figure 6.12 show the graphs of the exact solution (u, v) considered in these numerical experiments.

We use tolerances $\delta_u = 10^{-p-1}$ and $\delta_v = 10^{-q-2}$. Table 6.3 shows the numerical results by the three methods for $p = 2$ and $q = 1$. Based on the numerical results, the error values by NAM are obtained at $N = 320$ and are comparable with the error values by AM(1) and AM(3), which are respectively obtained at $N = 186$ and $N = 187$. If a uniform refinement at $N = 187$ is computed, we have $\|e_h(t)\|_1 = 1.2113\text{E-}05$ and $\|f_h(t)\|_1 = 2.7678\text{E-}03$. The CPU time for this result is 692s. Even though the CPU time is shorter, the error values are at lower accuracy than the error values by AM(1)

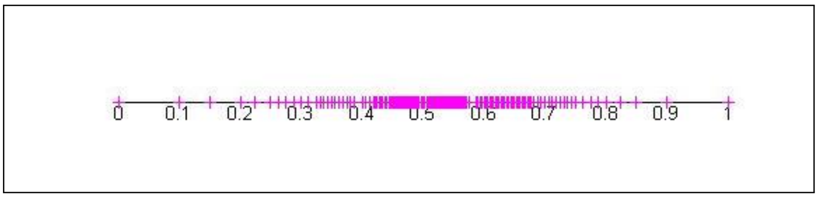

Figure 6.7: Example 1 for BBM by Adaptive Method AM(1).

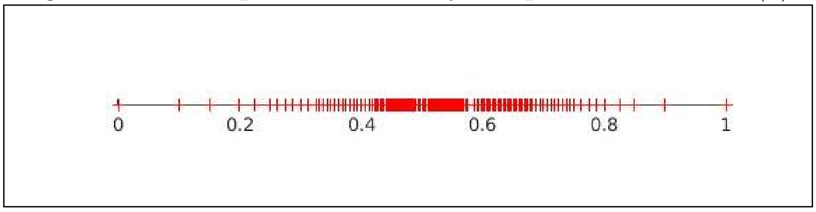

Figure 6.8: Example 1 for BBM by Adaptive Method AM(3).

	AM(1)	AM(3)	NAM
N	44	46	160
Degrees of freedom u	131	137	479
Degrees of freedom v	89	93	321
Number of refinements	34	12	4
CPU time (sec)	229	114	7566
$\|e_h(t)\|_1 \,/\, \|u(t)\|_1$	8.2403E-04	2.8256E-04	4.6186E-05
$\|f_h(t)\|_1 \,/\, \|v(t)\|_1$	6.4244E-03	4.7040E-03	2.0321E-03

Table 6.2: Example 2 for the BBM equation at $t = 0.8$.

and AM(3). Figure 6.13 and Figure 6.13 respectively represent the final refined meshes by AM(1) and AM(3) for this example.

In the following examples, we use $\delta_u = 10^{-2p-2}$ and $\delta_v = 10^{-2q}$. Table 6.4 shows the numerical results for Example 2, where $p = 3$ and $q = 2$. Figure 6.15 and Figure 6.16 show the final refined meshes of AM(1) and AM(3) for this example. Note that, if a uniform refinement at $N = 97$ is computed, the CPU time is 2093s and the error values are $\|e_h(t)\|_1 = 3.6051\text{E-}08$ and $\|f_h(t)\|_1 = 2.2663\text{E-}05$.

In conclusion, from the numerical results of the adaptive schemes for the BBM and Burgers equations, we can see that desired accuracy of the approximate solution can

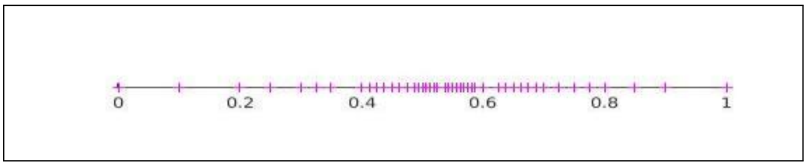

Figure 6.9: Example 2 for BBM by Adaptive Method AM(1).

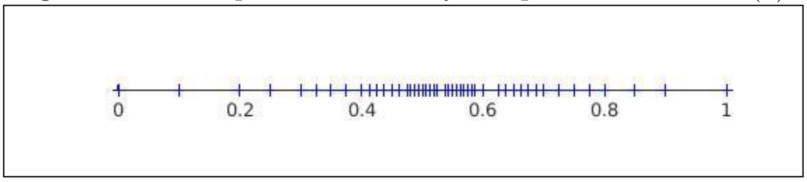

Figure 6.10: Example 2 for BBM by Adaptive Method AM(3).

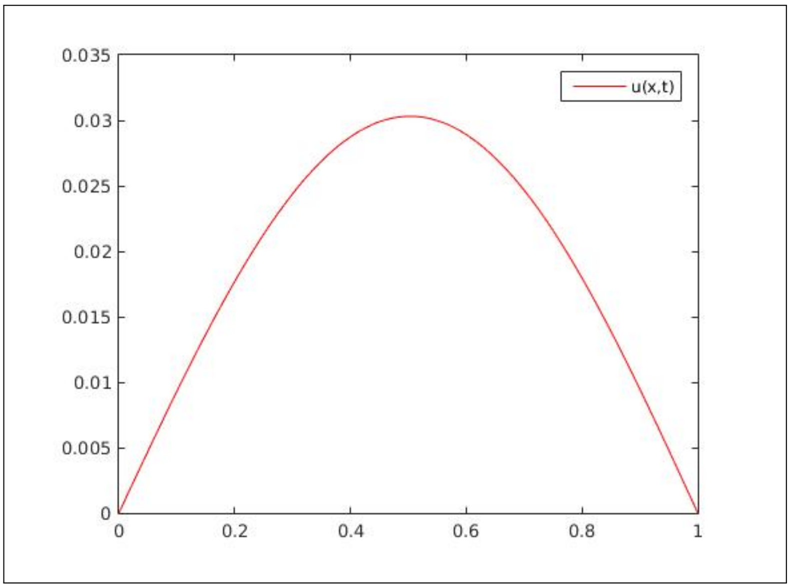

Figure 6.11: The Burgers equation: exact solution $u(x,t)$.

be obtained by using the adaptive schemes introduced in this study. Besides that, a comparable accuracy of approximate solution with lower degrees of freedom is obtained by using these adaptive schemes.

There are several issues that can be considered for the improvement of the adaptive schemes. In the step ***Mark***, we performed the marking process on the local a posteriori error estimators E_l only. This is due to the equation we used to compute the local a posteriori error estimators E_l (see (6.2.1)), which involves information about F_l. Different

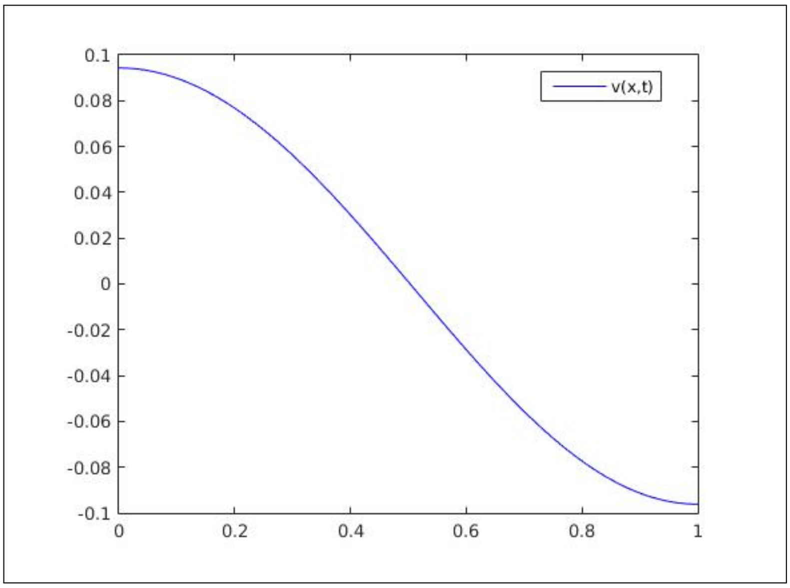

Figure 6.12: The Burgers equation: exact solution $v(x,t)$.

marking processes can also be considered. For example, we can perform the marking process on the local a posteriori error estimators F_l (see (6.2.2) and (6.2.3)) or on both local a posteriori error estimators E_l and F_l.

Lastly, in step ***Refine*** of these adaptive schemes, we refine a marked element by halving it into two new elements. We consider two methods for the refinement process, namely AM(1) and AM(3). Different values of Φ may give better results. Optimal choice of Φ may be a subject of further study.

	AM(1)	AM(3)	NAM
N	186	187	320
Degrees of freedom u	371	373	639
Degrees of freedom v	187	188	321
Number of refinements	176	59	5
CPU time (sec)	12780	5575	22592
$\|e_h(t)\|_1$	8.1367E-06	8.0651E-06	4.1362E-06
$\|f_h(t)\|_1$	2.6714E-03	2.6573E-03	1.6174E-03

Table 6.3: Example 1 for the Burgers equation at $t = 0.8$.

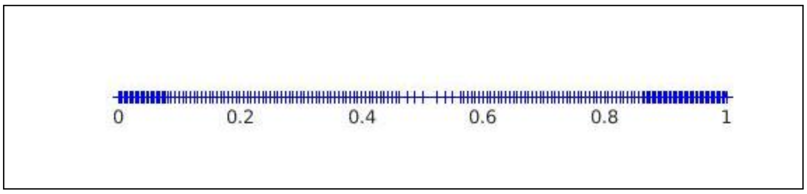

Figure 6.13: Example 1 for Burgers by Adaptive Method AM(1).

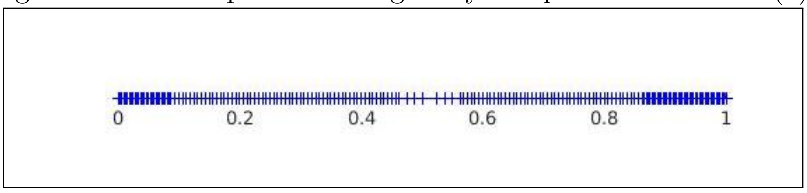

Figure 6.14: Example 1 for Burgers by Adaptive Method AM(3).

	AM(1)	AM(3)	NAM
N	95	97	160
Degrees of freedom u	284	290	479
Degrees of freedom v	191	195	321
Number of refinements	85	29	4
CPU time (sec)	31284	10847	55027
$\|e_h(t)\|_1$	1.8250E-08	1.7577E-08	8.0457E-09
$\|f_h(t)\|_1$	9.5534E-06	9.2243E-06	8.3298E-06

Table 6.4: Example 2 for the Burgers equation at $t = 0.8$.

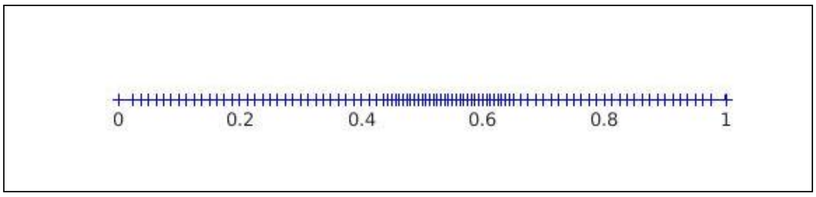

Figure 6.15: Example 2 for Burgers by Adaptive Method AM(1).

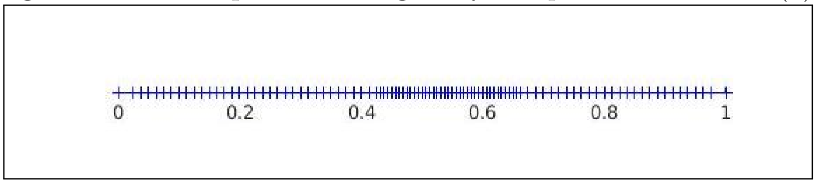

Figure 6.16: Example 2 for Burgers by Adaptive Method AM(3).

Chapter 7

Conclusion

In this thesis, we carried out the study of a priori and a posteriori error estimations of mixed finite element methods for nonlinear equations. We focused on a priori and a posteriori error estimations of H1MFEM for two one-spatial dimensional nonlinear partial differential equations, namely the Benjamin-Bona-Mahony (BBM) and Burgers equations (Chapter 4 and Chapter 5). We proved that the proposed a posteriori error estimates are efficient and the numerical results are consistent with our theoretical results.

In Chapter 6, we conducted numerical studies of adaptive schemes for the BBM and Burgers equations. We presented the procedure and numerical results of the adaptive schemes for both equations, where the approximate solutions are computed by H1MFEM and the a posteriori error estimations are proposed in Chapter 4 and Chapter 5 of this thesis.

It is noted that the one dimensional time dependent incompressible Navier Stokes equations which are studied using a sequential regularization method in [34] are almost similar to the BBM equation considered in this study. Therefore, a posteriori error estimation of H1MFEM for higher-spatial dimension of nonlinear partial differential equations such as the Navier Stokes equations or other nonlinear partial differential equations may be a subject of further study.

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