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Trotter-Kato product formula and an approximation formula for a propagator in symmetric operator ideals

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Abstract

The Trotter-Kato product formula is a mathematical clarification of path integration in quantum theory [62]. It gives a precise meaning to Feynman's path integral representation of the solutions to Schrödinger equations with time-dependent potentials. In this thesis, we consider the Trotter-Kato product formula in arbitrary symmetrically F-normed ideal closed with respect to the logarithmic submajorization.

An abstract non-autonomous evolution equation is widely used in various fields of mathematics and quantum mechanics. For example, Schrödinger equation and linear partial differential equations of parabolic or hyperbolic type [53, 70]. The second problem we consider is the existence of the propagator for such an equation and its approximation formula in an arbitrary symmetric Banach ideal. The approximation formula in the autonomous case corresponds to the Trotter product formula.

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Contents

Chapter 1

Introduction

One of the main objects of this thesis is the exponential formula

$$e^{A+B} = e^A \cdot e^B, \tag{1.1}$$

for some suitable operators A and B. Note that (1.1) is true when A and B are complex numbers or commuting bounded operators. However, (1.1) does not hold in general for noncommuting bounded or unbounded operators. A well known formula due to Lie [55, Theorem VIII.29] states that for a pair of $m \times m$ matrices A and B, one has

$$\lim_{n\to\infty} \left(e^{-tA/n}e^{-tB/n}\right)^n = e^{-t(A+B)}, \quad t \geq 0, \tag{1.2}$$

where the convergence holds in the uniform norm, and uniformly in t on compact intervals of $[0,\infty)$. Note that the same assertion also holds for pair of bounded operators A, $B \in \mathcal{L}(H)$ on a Hilbert space H (see, Section 4.1).

An analogue of (1.2) in more general setting was first obtained by Trotter [69]. Let A and B be non-negative self-adjoint operators on a Hilbert space H and $A+B$ be essentially self-adjoint on $\mathrm{dom}(A) \cap \mathrm{dom}(B)$. Then, the following Trotter product formula is known

$$\operatorname*{s-lim}_{n\to\infty} \left(e^{-tA/n}e^{-tB/n}\right)^n = e^{-t(A+B)}, \quad t \geq 0, \tag{1.3}$$

where the convergence holds in the strong operator topology, and uniformly in t on compact intervals of $[0,\infty)$. The Trotter product formula is widely applicable in quantum field theory. For example, it gives a precise meaning for Feynman's path integral [62, 50].

Later, Kato [31] extended (1.3) to a more general case of the form-sum $C := A\dot{+}B$ of two non-negative self-adjoint operators A and B (more details in Section 2.2.3). Namely, he proved the following

$$\operatorname*{s-lim}_{n\to\infty}(e^{-tA/n}e^{-tB/n})^n = e^{-tC}P, \quad t > 0, \tag{1.4}$$

where the convergence holds uniformly in t on compact intervals of $(0, \infty)$, and P denotes the orthogonal projection from H onto $\operatorname{dom}(C) = \overline{\operatorname{dom}(A^{1/2}) \cap \operatorname{dom}(B^{1/2})}$. Furthermore, in [30, 31], Kato generalized (1.4) for a class of Kato functions (see Definition 4.2.1)

$$\operatorname*{s-lim}_{n\to\infty}(f(tA/n)g(tB/n))^n = e^{-tC}P, \quad t > 0, \tag{1.5}$$

where the convergence holds uniformly in t on compact intervals of $(0, \infty)$. The simple examples of Kato functions are the following: $f(t) = e^{-t}$, $t \geq 0$ and $f(t) = (1 + t)^{-1}$, $t \geq 0$. In the literature, the latter type formula (1.5) is known as Trotter-Kato product formula.

A particular point of interest has been to strengthen the convergence in (1.5). One such result was first obtained by Rogava [57, 56]. He proved the convergence in the operator norm and gave an estimate on error bound (see Section 4.2).

Further, in a series of papers by Neidhardt, Zagrebnov, Ichinose, Tamura and etc. (see, for example, [21, 37, 48, 39, 67, 24, 20]), the Trotter-Kato product formula in the operator norm topology was considered. They obtained the optimal error bounds of such formulas for certain subclasses of Kato functions for both algebraic sum $A + B$ and form-sum $A\dot{+}B$ of two non-negative self-adjoint operators A, B under suitable conditions on them. For a more detailed review of these results, we refer the reader to Chapter 4.

The first attempt to prove the Trotter product formula in the trace norm topology was made by Zagrebnov in [74]. He proved the Trotter product formula in trace norm topology for some classes of Gibbs semigroups. Later, in [38, 49], Neidhardt and Zagrebnov proved that the Trotter-Kato product formula also holds in trace norm provided at least one of the operators A and B generates a self-adjoint Gibbs semigroup.

First results concerning some symmetric (quasi-)normed ideals were obtained by Hiai in [15], where the Trotter product formula was proved in Schatten ideals $\mathcal{L}_p(H)$ for any $0 < p < \infty$ under the assumption that $e^{-A} \in \mathcal{L}_p(H)$. Moreover, Hiai conjectured that the Trotter-Kato product formula holds in an arbitrary fully symmetric ideal of bounded operators (see [15, Problem 3.16]). In [40] (see also [75, Chapter 6] and [77]), Neidhardt and Zagrebnov answered the conjecture in the affirmative by showing that the Trotter-Kato product formula holds in any fully symmetric ideal $\mathcal{I}(H)$ of bounded operators provided the so-called Kac operator (the transfer matrix) $e^{-tB/2}e^{-tA}e^{-tB/2}$ belongs to this ideal for some t. Namely, they showed that

$$\begin{aligned}\lim_{n\to\infty}\left(e^{-tA/n}e^{-tB/n}\right)^n &= e^{-tC}P,\\ \lim_{r\to\infty}\left(e^{-tB/2r}e^{-tA/r}e^{-tB/2r}\right)^r &= e^{-tC}P,\end{aligned} \tag{1.6}$$

where the limits are taken with respect to the norm of a fully symmetric ideal, and P denotes the orthogonal projection from H onto $\mathrm{dom}(C) = \overline{\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})}$. Moreover, they proved similar formulas for the whole class of Kato functions f and g.

In this thesis, we present a result given in [2, Theorem 3.6], where we further extend the results of Hiai [15], Neidhardt and Zagrebnov [40] to an arbitrary symmetrically F-normed ideal closed with respect to the logartihmic submajorization. Namely, let A and B be non-negative self-adjoint operators on a separable Hilbert space H and $\mathcal{I}(H)$ be a symmetrically F-normed ideal closed with respect to the logarithmic submajorization. Then, we prove the Trotter-Kato product formula (1.6) with respect to the F-norm of $\mathcal{I}(H)$, for a suitable class of Kato function f and g (cf. Theorem 5.3.1).

Note that the class of all symmetrically F-normed ideals closed with respect to the logarithmic submajorization contains all symmetric quasi-Banach ideals. Hence, the latter result is also true in an arbitrary symmetric quasi-Banach ideal (see, Corollary 5.3.2). We also consider error bounds of such formulas and show some examples similar to [40, Theorem 5.1] (see Proposition 5.4.3 and Corollary 5.4.4).

In the second part of the thesis, we further investigate an abstract non-autonomous Cauchy problem for the evolution equation on a Hilbert space H. Let A and $B(t)$, $t \geq 0$ be non-negative self-adjoint operators on H. We consider the following non-autonomous evolution equation

$$\begin{cases} \frac{du(t)}{dt} = -(A + B(t))u(t), \\ u(s) = \xi \in H, \end{cases} \qquad 0 \leq s \leq t \leq 1. \tag{1.7}$$

The main problem solving (1.7) is to find a so-called propagator (solution operator)

$\{U(t,s)\}_{0\leq s\leq t\leq 1}$, such that $u(t) = U(t,s)\xi$ is a solution of (1.7) in a certain sense for an appropriate set of initial data ξ. Note that if $B(t) = 0$, $t \geq 0$, then the propagator can be written via strongly continuous semigroup, i.e., $U(t,s) = e^{-(t-s)A}$. However, when $B(t)$ is not trivial, the situation becomes more complicated, and there is no such simple expression as in the autonomous case.

In the non-autonomous case, the problem of finding the propagator of the evolution equation (1.7) was first considered by Phillips [53], where the family $\{B(t)\}_{t\in[0,1]}$ was assumed to consist of bounded operators. Later, Kato studied a more general setting of unbounded perturbations in [28, 29]. Since then, there has been a series of papers concerning the evolution equation in different settings, especially for unbounded perturbations [12, 72, 73, 22, 71, 70, 76].

One common point of all the works mentioned earlier was to prove the existence of a propagator and that some families of well-behaved approximants tend to converge to the original propagator in some given topology. For instance, an example of a family of approximants can be given as follows

$$U_n(t,s) := \prod_{k=n}^{1} e^{-\frac{(t-s)A}{n}} \cdot e^{-\frac{t-s}{n}B\left(s+\frac{k(t-s)}{n}\right)}, \quad 0 \leq s \leq t \leq 1. \tag{1.8}$$

Note that if $B(t) = B$, $t \geq 0$ is independent of a time variable t, then the given approximation problem corresponds to the Trotter product formula (see, for example, [21, 22, 45, 70]).

The above approximation problem in the operator norm topology was successfully considered in a series of papers [22, 47, 46, 44]. The approximation formula of a propagator in the operator norm can be used as a lifting tool for other stronger topologies. Hence, in Section 4.5, we present a detailed review of the results in the operator norm topology.

In this thesis, following the above method, we extend the results of Zagrebnov [76], where the same problem was considered in the trace norm topology. Namely, we first prove the existence of a propagator $\{U(t,s)\}_{0\leq s\leq t\leq 1}$ in an arbitrary symmetric Banach ideal $\mathcal{I}(H)$ of bounded operators under the assumptions given in Section 6.1 (cf. Theorem 6.1.1 or [3, Theorem 1.2]).

Furthermore, we prove that the approximating families as (1.8) converge to the original propagator $\{U(t,s)\}_{0\leq s\leq t\leq 1}$ in the norm of symmetric Banach ideal $\mathcal{I}(H)$ (cf. Theorem

6.1.2 or [3, Theorem 1.3]).

Structure of the thesis

The thesis is organised in the following way. In Chapter 2, we recall the necessary preliminary material employed in the following chapters. Firstly, we recall the notion of an F-norm and F-normed space, which contains all quasi-normed spaces. We further recall some necessary material from the theory of bounded and unbounded operators. Primarily, we review the non-negative self-adjoint operators, the densely defined non-negative closed forms, and the connection between them to construct the form-sum of two non-negative self-adjoint operators.

In Chapter 3, we present one of the main objects of this thesis, the symmetric operator ideals. As an essential tool, we recall the notion of a logarithmic submajorization and closedness of the above ideals with respect to the logarithmic submajorization. In Section 3.3, we further present different results regarding the (F, respectively, quasi)-norm of a symmetric operator ideal.

Chapter 4 consists of a detailed overview of current results on the Trotter-Kato product formula and an approximation formula for a propagator in the operator norm topology. Firstly, we discuss the product formulas for elementary cases of finite matrices and bounded operators. In Sections 4.2 and 4.3, we recall various existing results regarding the Trotter-Kato product formula in the operator norm topology for the algebraic sum and the form-sum of two unbounded operators, respectively. Section 4.4 presents a criterion for convergence of the Trotter-Kato product formula in the operator norm. Finally, Section 4.5 consists of the basics of the theory of evolution equation and different results on the existence of a propagator of the non-autonomous evolution equation. We also present an approximation formula for a propagator in the operator norm topology and various estimates of error bounds for this approximation formula.

In Chapter 5, we present the Trotter-Kato product formula in symmetrically F-normed ideal closed with respect to the logarithmic submajorization. First, we prove the equivalence between formulas for various families generated by the Kato functions. We further present a lifting method, which helps obtain the product formula in symmetric operator ideal from the same formula in the operator norm topology. Section 5.3 consists of the main results of this chapter. Finally, we discuss the error bounds of the Trotter-Kato product formula in symmetrically F-normed ideals closed with respect to the logarithmic submajorization.

Chapter 6 consists of results regarding the existence of a propagator of an abstract non-autonomous Cauchy problem for evolution equation and its approximation formula in symmetric Banach ideals. The main results of this chapter are proved in Sections 6.3 and 6.4.

List of publications during the candidature

[2] M. Akhymbek and G. Levitina. Trotter–Kato product formula in symmetric F-normed ideals. *Studia Math.*, 266(2):167–191, 2022. ISSN: 0039-3223. DOI: 10.4064/sm210708-4-11. URL: https://doi.org/10.4064/sm210708-4-11

[3] M. Akhymbek and D. Zanin. Approximation formula for a propagator in symmetrically normed ideals, 2022. Submitted

Chapter 2

Preliminaries

In this chapter, we present all notations and preliminary material used throughout this thesis. In Section 2.1, we recall basic notions and notations regarding the bounded and unbounded operators. Section 2.2 presents the notion of a densely defined non-negative closed form on a Hilbert space H. Moreover, we recall the connection between densely defined non-negative closed forms and non-negative self-adjoint operators on a Hilbert space H. Using this connection, we further define a form-sum of two non-negative self-adjoint operators on a Hilbert space H.

2.1 Basic definitions and notations

Let $\mathbb{R}$ and $\mathbb{C}$ be the set of all real and complex numbers, respectively. We denote the integer and remainder parts of a real number $r \in \mathbb{R}$ as usual, by $[r]$ and $\{r\}$, respectively. In the following we use notation Const to denote a positive constant which may vary from line to line.

We now recall the definition of F-norms.

Definition 2.1.1. *Let Ω be a linear space over the field $\mathbb{C}$. A function $\|\cdot\|$ from Ω to $[0, \infty)$ is called an* F*-norm, if for any $x, y \in \Omega$ the following conditions hold:*

(i) $\|x\| \geq 0$, *and* $\|x\| = 0 \Leftrightarrow x = 0$;

(ii) $\|\alpha x\| \leq \|x\|$ *for any* $\alpha \in \mathbb{C}$ *such that* $|\alpha| \leq 1$;

(iii) $\lim_{\alpha \to 0} \|\alpha x\| = 0$;

(iv) $\|x + y\| \leq \|x\| + \|y\|$.

The couple $(\Omega, \|\cdot\|)$ is called an F-normed space. Note that if $\|\cdot\|$ is an F-norm on Ω, then it induces a metrizable topology [27, Chapter 1] and conversely if Ω is metrizable then it can be equipped with an equivalent F-norm (see e.g. [32, Section 15.11], [27, Chapter 1]).

Remark 2.1.2. *To avoid the confusion we do not use the notion of* F*-space for complete* F*-normed spaces, since the notion of* F*-space often refers to complete metrizable locally convex spaces.*

Another important notion is the notion of a quasi-norm.

Definition 2.1.3. *Let Ω be a linear space over the field $\mathbb{C}$. A function $\|\cdot\|$ from Ω to $[0,\infty)$ is called an quasi-norm, if for any $x, y \in \Omega$ the following conditions hold:*

(i) $\|x\| \geq 0$, *and* $\|x\| = 0 \Leftrightarrow x = 0$;

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ *for any* $\alpha \in \mathbb{C}$;

(iii) $\|x + y\| \leq C\,(\|x\| + \|y\|)$, $C \geq 1$.

The least constant C satisfying property (iii) is called the modulus of concavity of the quasi-norm. In this case, the pair $(\Omega, \|\cdot\|)$ is called a quasi-normed space. Note that any quasi-normed space is locally bounded and metrizable [32, Section 15.10]. Hence, it can be equipped with an equivalent F-norm. Therefore, any quasi-normed space is an F-normed space [27, Chapter 1, Section 3].

One can quickly note that if $C = 1$, in (iii), then Definition 2.1.3 coincides with a usual definition of a norm.

Next, we recall some preliminary material regarding the bounded operators. Let H be a separable Hilbert space and $\mathcal{L}(H)$ be the C^*-algebra of all bounded linear operators on H equipped with the uniform norm $\|\cdot\|_\infty$. Let us remind various topologies in $\mathcal{L}(H)$ that are used throughout this thesis. More information about these topologies can be found in [25, Section 5.1].

Definition 2.1.4. *Let $\{X_n\}_{n\geq 1}$ be a sequence of bounded operators on a Hilbert space H. Then,*

(i) the sequence X_n is said to converge to X in the operator norm topology, if

$$\|X_n - X\|_\infty \to 0, \quad n \to \infty.$$

(ii) the sequence X_n is said to converge to X in the strong operator topology and denoted as $\text{s-lim}_{n\to\infty} X_n = X$*, if*

$$\|X_n\xi - X\xi\| \to 0, \quad n \to \infty$$

for any $\xi \in H$.

(iii) the sequence X_n is said to converge to X in the weak operator topology and denoted as $\text{w-lim}_{n\to\infty} X_n = X$*, if*

$$\langle (X_n - X)\xi, \eta\rangle \to 0, \quad n \to \infty$$

for any $\xi, \eta \in H$.

Since the unbounded non-negative self-adjoint operators on a Hilbert space H are the main objects of this thesis, we now recall the basics regarding the unbounded operators on a Hilbert space H. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Let X be a densely-defined linear operator on H with domain $\text{dom}(X)$. Set

$$\text{dom}(X^*) = \{\eta \in H : \exists u \in H : \langle X\xi, \eta\rangle = \langle \xi, u\rangle, \quad \text{for any} \quad \xi \in \text{dom}(X)\}.$$

Since $\text{dom}(X)$ is dense in H, the vector $u \in H$ satisfying $\langle X\xi, \eta\rangle = \langle \xi, u\rangle$ for all $\xi \in \text{dom}(X)$ is uniquely determined by η. Therefore, setting $X^*\eta = u$, we obtain a well-defined mapping X^* on H. It is easily seen that X^* is linear and it is called the adjoint operator of X.

Let X be a densely defined linear operator on H. Then X is *self adjoint* if $X = X^*$ which means $\text{dom}(X) = \text{dom}(X^*)$ and $X\xi = X^*\xi$, $\xi \in \text{dom}(X)$. We say that X is essentially self-adjoint if its closure $\overline{X}$ is self-adjoint. We also say that X is non-negative and write $X \geq 0$ if $\langle X\xi, \xi\rangle \geq 0$ for all $\xi \in \text{dom}(X)$. For further details in this topic, we refer the reader to [60].

Let $X \in \mathcal{L}(H)$ be a bounded operator on a Hilbert space H. It is known that one can define the exponential function of X as the following uniform norm convergent series

$$e^X := \sum_{j=0}^{\infty} \frac{X^j}{j!}. \tag{2.1}$$

However, in general, the series (2.1) fails to be convergent for unbounded operators.

In this case, Hille-Yosida's theory gives a precise meaning to the exponential function of an unbounded operator in terms of a strongly continuous semigroup. Let us recall the definition of a strongly continuous semigroup.

Definition 2.1.5. *A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators on a Hilbert space H is called a strongly continuous semigroup if*

$$T(t+s) = T(t)T(s), \quad T(0) = I, \quad t, s \geq 0,$$

and $T(t)$ is continuous with respect to the strong operator topology.

To understand the strongly continuous semigroup as an exponential function, we first need to recall the definition of the generator of a strongly continuous semigroup.

Definition 2.1.6. *[10, Definition II.1.2] The generator $X : \mathrm{dom}(X) \subset H \to H$ of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on a Hilbert space H is the operator*

$$X\xi := \lim_{h\downarrow 0} \frac{\xi - T(h)\xi}{h}$$

defined for every ξ in its domain

$$\mathrm{dom}(X) := \{\xi \in H : \lim_{h\downarrow 0} \frac{\xi - T(h)\xi}{h} \quad \textit{exists}\}.$$

The important result here is the following from [10, Theorem II.1.4]

Proposition 2.1.7. *The generator of a strongly continuous semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Since, by Proposition 2.1.7, the strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ is determined uniquely by its generator X, in the following we denote $T(t) := e^{-tX}$, $t \geq 0$ and the exponential function is understood as a strongly continuous semigroup. For more details regarding the semigroup theory, we refer the reader to [10] and [52].

2.2 Connection between sesquilinear forms and self-adjoint operators

In this section, we recall a notion of a densely defined non-negative closed form and its connection with a non-negative self-adjoint operator on a Hilbert space H (for more details see [60, Chapter 10]). Using these notions we recall an extended version of a sum, so-called a form-sum of two non-negative self-adjoint operators on a Hilbert space H, which is used in Chapters 4 and 5.

2.2.1 Closed sesquilinear forms

Definition 2.2.1. *[60, Definition 10.1] A mapping $t[\cdot,\cdot] : \mathrm{dom}(t) \times \mathrm{dom}(t) \to \mathbb{C}$ is called a sesquilinear form (or, shortly, form) on a linear subspace $\mathrm{dom}(t)$ of a Hilbert space H if it is linear on a first variable and conjugate linear on a second variable. In this case, $\mathrm{dom}(t)$ is called a domain of a form $t[\cdot,\cdot]$.*

The sum of two forms $s[\cdot,\cdot]$ on $\mathrm{dom}(s)$ and $t[\cdot,\cdot]$ on $\mathrm{dom}(t)$, and a scalar multiplication by $\alpha \in \mathbb{C}$ of a form are defined as follows

$$(s+t)[\xi,\eta] := s[\xi,\eta] + t[\xi,\eta], \quad \xi,\eta \in \mathrm{dom}(s+t) := \mathrm{dom}(s) \cap \mathrm{dom}(t)$$
$$(\alpha s)[\xi,\eta] := \alpha \cdot s[\xi,\eta], \quad \xi,\eta \in \mathrm{dom}(\alpha s) := \mathrm{dom}(s).$$

Definition 2.2.2. *A form $t[\cdot,\cdot]$ on a linear subspace $\mathrm{dom}(t)$ is called symmetric, if*

$$t[\xi,\eta] = \overline{t[\eta,\xi]}, \quad \textit{for all} \quad x,y \in \mathrm{dom}(t).$$

A symmetric form is called non-negative if

$$t[\xi,\xi] \geq 0, \quad \textit{for all} \quad \xi \in \mathrm{dom}(t).$$

Let $t[\cdot,\cdot]$ be a non-negative form on $\mathrm{dom}(t)$. Then, one can define the inner product $\langle\cdot,\cdot\rangle_t$ on $\mathrm{dom}(t)$ as

$$\langle\xi,\eta\rangle_t := t[\xi,\eta] + \langle\xi,\eta\rangle, \quad \xi,\eta \in \mathrm{dom}(t).$$

The properties of an inner product $\langle\cdot,\cdot\rangle_t$ trivially follows from the properties of an inner product $\langle\cdot,\cdot\rangle$ on a Hilbert space H.

Therefore, one can define a norm $\|\cdot\|_t$ on $\mathrm{dom}(t)$ associated with the inner product $\langle\cdot,\cdot\rangle_t$ as usual

$$\|\xi\|_t = \left(t[\xi,\xi] + \|\xi\|^2\right)^{1/2}, \quad \xi \in \mathrm{dom}(t),$$

where $\|\cdot\|$ is a norm associated with an inner product of a Hilbert space H.

Definition 2.2.3. *[60, Definition 10.2] A non-negative symmetric form $t[\cdot,\cdot]$ on $\mathrm{dom}(t) \subset H$ is said to be closed if $(\mathrm{dom}(t), \|\cdot\|_t)$ is complete.*

The closed forms have the following important property. Later, it is used to define a form-sum of certain unbounded operators on a Hilbert space H.

Proposition 2.2.4. *[60, Corollary 10.2] Let $\{t_j\}_{1\le j\le n}$ be a finite number of closed forms on $\mathrm{dom}(t_j)$, $1 \le j \le n$, respectively. Then, $t_1 + t_2 + ... + t_n$ is also closed on $\cap_{j=1}^n \mathrm{dom}(t_j)$.*

2.2.2 A form associated with a self-adjoint operator

In this subsection, we assume that X is a non-negative self-adjoint operator with a domain $\mathrm{dom}(X) \subseteq H$. Let $E_X(\cdot)$ be a spectral measure of X. One can associate a densely-defined non-negative symmetric form t_X to X as follows

$$\mathrm{dom}(t_X) := \mathrm{dom}(X^{1/2}) = \left\{\xi \in H, \quad \int_{\mathbb{R}} |\lambda| d\langle E_X(\lambda)\xi, \xi\rangle < \infty\right\},$$
$$t_X[\xi,\eta] := \int_{\mathbb{R}} \lambda d\langle E_X(\lambda)\xi, \eta\rangle, \quad \xi,\eta \in \mathrm{dom}(t_X).$$

Then, t_X is called a form associated with X and $\mathrm{dom}[X] := \mathrm{dom}(X^{1/2})$ is called a form-domain of X.

The next proposition shows another description of the form t_X associated with a non-negative self-adjoint operator X, avoiding the direct use of spectral representation. For the proof of it and more properties of the forms associated with self-adjoint operators, we refer the reader to [60, Proposition 10.4].

Proposition 2.2.5. *Let X be a non-negative self-adjoint operator with domain $\mathrm{dom}(X) \subset H$, then*

$$t_X[\xi,\eta] = \langle X^{1/2}\xi, X^{1/2}\eta\rangle \quad \textit{for all} \quad \xi,\eta \in \mathrm{dom}(X^{1/2}).$$

Conversely, if a form $t[\cdot,\cdot]$ with dense domain $\mathrm{dom}(t)$ is given, then one can associate an operator X_t to it as follows: the domain $\mathrm{dom}(X_t)$ contains all vectors $\xi \in \mathrm{dom}(t)$ for which there exists a vector $u_\xi \in H$ such that $t[\xi,\eta] = \langle u_\xi, \eta\rangle$ for any $\eta \in \mathrm{dom}(t)$. In this case, $X_t\xi = u_\xi$, $\xi \in \mathrm{dom}(X_t)$. Note that since $\mathrm{dom}(t)$ is dense in H, the vector u_ξ is unique for a given $\xi \in \mathrm{dom}(t)$. Therefore, X_t is well-defined and linear. Now, we present the main result of this subsection. The proof can be found in [60, Proposition 10.7 and Corollary 10.8].

Proposition 2.2.6. *(i) If $t[\cdot,\cdot]$ is a densely defined non-negative closed form, then the operator X_t is non-negative self-adjoint and $t[\cdot,\cdot] \equiv t_{X_t}$ (for a form associated with X).*

(ii) The mapping $X \to t_X$ is a bijection between the set of non-negative self-adjoint operators on a Hilbert space H and the set of densely defined non-negative closed forms on H.

2.2.3 The form-sum of two non-negative self-adjoint operators

Let $(X, \mathrm{dom}(X))$ and $(Y, \mathrm{dom}(Y))$ be self-adjoint operators on a Hilbert space H. In this subsection, we define a notion of a form-sum of X and Y, which extends the notion of the usual algebraic sum. It is essential to do so since the algebraic sum of two self-adjoint operators is not always self-adjoint on the common domain $\mathrm{dom}(X)\cap\mathrm{dom}(Y)$. For example, let Y be a self-adjoint operator defined as $Y = -X$ on $\mathrm{dom}(X)$. In this case, the algebraic sum $X+Y$ is a zero operator defined on $\mathrm{dom}(X)$. Therefore, if $\mathrm{dom}(X) \neq H$, the algebraic sum $X+Y$ is not self-adjoint. For further deep investigation of a problem of a sum of two unbounded self-adjoint operators, we refer the reader to [6, 13].

In order to define the form-sum, assume that X and Y are non-negative self-adjoint operators with domains $\mathrm{dom}(X)$ and $\mathrm{dom}(Y)$, respectively. By Proposition 2.2.6, there exist unique densely defined non-negative closed forms $t_X[\cdot,\cdot]$ and $t_Y[\cdot,\cdot]$ with domains $\mathrm{dom}[X] = \mathrm{dom}(X^{1/2})$ and $\mathrm{dom}[Y] = \mathrm{dom}(Y^{1/2})$, respectively. Moreover, by Proposition 2.2.5, they are defined as follows

$$t_X[\xi,\eta] = \langle X^{1/2}\xi, X^{1/2}\eta\rangle, \quad \xi,\eta \in \mathrm{dom}(X^{1/2}),$$
$$t_Y[\mu,\nu] = \langle Y^{1/2}\mu, Y^{1/2}\nu\rangle, \quad \mu,\nu \in \mathrm{dom}(Y^{1/2}).$$

By Proposition 2.2.4, the sum of t_X and t_Y is a closed form on $\mathrm{dom}(X^{1/2}) \cap \mathrm{dom}(Y^{1/2})$.

Therefore, by Proposition 2.2.6, there exists an operator Z associated with a closed form $t_X + t_Y$, which is self-adjoint on $\overline{\mathrm{dom}(X^{1/2}) \cap \mathrm{dom}(Y^{1/2})}$. The operator Z is called a form-sum of the operators X and Y and denoted as $Z := X \dot{+} Y$.

Note that $\mathrm{dom}(X) \subset \mathrm{dom}(X^{1/2}) := \mathrm{dom}[X]$, hence, for an unbounded operator X, the form-domain $\mathrm{dom}[X]$ is strictly larger than the domain $\mathrm{dom}(X)$. Therefore, in particular, $\mathrm{dom}(X) \cap \mathrm{dom}(Y) \subset \mathrm{dom}(X^{1/2}) \cap \mathrm{dom}(Y^{1/2})$. Hence, assuming that the form-sum is densely defined is weaker than the requirement that the algebraic sum is densely defined. Furthermore, the form-sum can exist and be self-adjoint on $\overline{\mathrm{dom}(X^{1/2}) \cap \mathrm{dom}(Y^{1/2})}$ even if the algebraic sum is not self-adjoint (see, for example, [6, p. 54-55]).

Chapter 3

Symmetric operator ideals

This chapter contains necessary definitions of one of the main objects of this thesis, the symmetric operator ideals, and their various properties. In Section 3.1, we recall the basic definitions regarding the two-sided ideals of compact operators and introduce the class of symmetrically F-normed ideals. Section 3.2 presents the essential tool, so-called logarithmic submajorization, and we introduce the class of ideals closed with respect to the logarithmic submajorization. Furthermore, in Section 3.3, we present various results in these ideals, which are necessary for later chapters.

3.1 Symmetrically F-normed ideals

Let H be a separable Hilbert space. Recall that a proper non-trivial subspace $\mathcal{I}(H) \subset \mathcal{L}(H)$ is called a two-sided ideal (or, shortly, an ideal) of $\mathcal{L}(H)$ if $XY, YX \in \mathcal{I}(H)$ whenever $X \in \mathcal{I}(H)$ and $Y \in \mathcal{L}(H)$. Denote by $\mathcal{K}(H) \subset \mathcal{L}(H)$ and $\mathcal{L}_\infty(H) \subset \mathcal{L}(H)$ the ideal of all finite rank and compact operators on H, respectively. The Calkin's theorem [5] states that for any ideal $\mathcal{I}(H)$ of $\mathcal{L}(H)$, one has

$$\mathcal{K}(H) \subseteq \mathcal{I}(H) \subseteq \mathcal{L}_\infty(H).$$

For any compact operator $X \in \mathcal{L}_\infty(H)$, we denote by $\{s_j(X)\}_{j\geq 0}$ the singular values of the operator X, i.e. the eigenvalues $\{\lambda_j(|X|)\}_{j\geq 0}$ of the operator $|X| = (X^*X)^{1/2}$ arranged in decreasing order, and counting multiplicities. For the sequences of singular values $\{s_j(\cdot)\}_{j\geq 0}$ of any two compact operators $X, Y \in \mathcal{L}_\infty(H)$ and a bounded operator $B \in \mathcal{L}(H)$, we have the following classical inequalities (see, e.g., [14, Sections 2.1 and 2.4, respectively]):

$$\begin{aligned} s_j(BX) &\le \|B\|_\infty s_j(X), \quad j \ge 0, \\ \prod_{j=0}^m s_j(XY) &\le \prod_{j=0}^m s_j(X) \prod_{j=0}^m s_j(Y), \quad m \ge 0. \end{aligned} \tag{3.1}$$

Note that there is a remarkable correspondence between sequence spaces generated by singular values and two-sided ideals of compact operators due to Calkin [5]. Let $\{e_j\}_{j\ge0}$ be an orthonormal basis in the Hilbert space H, then, for any bounded sequence $z = \{z_j\}_{j\ge0}$, we can define the diagonal operator $\mathrm{diag}(z) = \sum_{j\ge0} z_j\langle\cdot, e_j\rangle e_j$ on H. Let c_0 be the space of all sequences $x = \{x_j\}_{j\ge0}$ which tends to zero, i.e., $x_j \to 0$ as $j \to \infty$. A subspace J of c_0 is called a *(F-, respectively, quasi-)normed Calkin sequence space* if it is equipped with (F-, respectively, quasi-)norm $\|\cdot\|_J$ such that for any bounded sequence $y = \{y_j\}_{j\ge0}$, $y_j^* \le x_j^*$, $j \ge 0$ and $x \in J$ imply that $y \in J$ and $\|y\|_J \le \|x\|_J$, where $z^* = \{z_j^*\}_{j\ge0}$ is the decreasing rearrangement of $\{z_j\}_{j\ge0}$. One can write the above-mentioned correspondence in the following way: If J is a Calkin sequence space, then the associated two-sided ideal $\mathcal{J}(H)$ is defined as

$$\mathcal{J}(H) := \{X \in \mathcal{L}_\infty(H) : \quad s(X) = \{s_j(X)\}_{j\ge0} \in J\}. \tag{3.2}$$

Conversely, if $\mathcal{J}(H)$ is a two-sided ideal of compact operators, then the associated sequence space is defined as

$$J := \{z \in c_0 : \quad \mathrm{diag}(z) \in \mathcal{J}(H)\}. \tag{3.3}$$

The following result is known as a Calkin correspondence (see, [5] or [33, Theorem 1.2.3]).

Proposition 3.1.1. *The correspondence $J \leftrightarrow \mathcal{J}(H)$ is a bijection between Calkin sequence spaces and two-sided ideals of compact operators.*

We now present a definition of a symmetrically F-normed ideal.

Definition 3.1.2. *Let $\mathcal{I}(H)$ be a two-sided ideal in $\mathcal{L}(H)$ equipped with an* F*-norm $\|\cdot\|_{\mathcal{I}}$. We say that $\mathcal{I}(H)$ is a symmetrically* F*-normed ideal if $X \in \mathcal{I}(H)$, $Y \in \mathcal{L}(H)$ and $s_j(Y) \le s_j(X)$ for any $j \in \mathbb{N}\cup\{0\}$ imply that $Y \in \mathcal{I}(H)$ and $\|Y\|_{\mathcal{I}} \le \|X\|_{\mathcal{I}}$. In the particular case when $\|\cdot\|_{\mathcal{I}}$ is a (quasi-)norm, we say that $\mathcal{I}(H)$ is symmetrically* *(quasi)-normed ideal.*

Note that if symmetrically (quasi)-normed ideal $\mathcal{I}(H)$ is equipped with a complete (quasi)-norm $\|\cdot\|_{\mathcal{I}}$, then it is called a symmetric (quasi)-Banach ideal.

Throughout the thesis, we assume for convenience that $\|U\|_{\mathcal{I}} = \|U\|_\infty$ for any rank-one operator from $\mathcal{L}(H)$.

We now state the Calkin correspondence in terms of symmetrically (quasi-)normed ideals (see [34, Section 3.1] or [66], for more details).

Proposition 3.1.3. *[34, Theorem 3.1.1]*

(i) Let J be a Calkin sequence space equipped with a (quasi-)norm $\|\cdot\|_J$. Then, an expression

$$X \mapsto \|s(X)\|_J, \quad X \in \mathcal{J}(H),$$

is a (quasi-)norm on $\mathcal{J}(H)$, and $\mathcal{J}(H)$ is a symmetrically (quasi-)normed ideal equipped with the given (quasi-)norm (cf. (3.2)).

(ii) Let $(\mathcal{J}(H), \|\cdot\|_{\mathcal{J}})$ be a symmetrically (quasi-)normed ideal. Then, an expression

$$z \mapsto \|\mathrm{diag}(z)\|_{\mathcal{J}}, \quad z \in J,$$

is a (quasi-)norm on J, and J is a (quasi-)normed Calkin sequence space (cf. (3.3)).

(iii) The correspondence $(J, \|\cdot\|_J) \leftrightarrow (\mathcal{J}(H), \|\cdot\|_{\mathcal{J}})$ is one-to-one, and it preserves completeness.

Note that a similar result (as in Proposition 3.1.3) is also true regarding symmetrically F-normed ideals and F-normed Calkin sequence spaces. For more details, we refer the reader to [18].

By *the separable part of a symmetrically normed ideal* $\mathcal{I}(H)$, equipped with a norm $\|\cdot\|_{\mathcal{I}}$, we understand the closure of finite rank operators in $\mathcal{L}(H)$ with respect to $\|\cdot\|_{\mathcal{I}}$. A symmetrically normed ideal $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ is called *separable* if it is coincides with its separable part i.e., the class of finite rank operators in $\mathcal{L}(H)$ is dense in $\mathcal{I}(H)$ with respect to $\|\cdot\|_{\mathcal{I}}$.

Let us present some examples of symmetrically F-normed ideals.

Example 3.1.4. *(i) [14, Section III.7] The basic example is the class of Schatten-von-Neumann ideals*

$$\mathcal{L}_p := \{X \in \mathcal{L}_\infty(H), \quad \sum_{k\geq 0} s_k(X)^p < \infty\}, \quad 0 < p < +\infty,$$

equipped with a standard Schatten (quasi-)norm

$$\|X\|_p := \left(\sum_{k\geq 0} s_k(X)^p\right)^{1/p}, \quad X \in \mathcal{L}_p(H).$$

The pair $(\mathcal{L}_p(H), \|\cdot\|_p)$ is a symmetric Banach ideal for $1 \leq p < \infty$ and symmetric quasi-Banach ideal for $0 < p < 1$ [15, Section 2.2]. Note that $(\mathcal{L}_p(H), \|\cdot\|_p)$, $p \geq 1$ is separable [14, Theorem 7.1].

(ii) [33, Example 1.2.6] Another important example is a class of weak-l_p ideals $\mathcal{L}_{p,\infty}(H)$ for $0 < p < \infty$, defined as

$$\mathcal{L}_{p,\infty}(H) := \{X \in \mathcal{L}_\infty(H), \quad \sup_{k\geq 0}(k+1)^{1/p} s_k(X) < \infty\}$$

equipped with

$$\|X\|_{p,\infty} := \sup_{k\geq 0}(k+1)^{1/p} s_k(X), \quad X \in \mathcal{L}_{p,\infty}(H).$$

The pair $(\mathcal{L}_{p,\infty}(H), \|\cdot\|_{p,\infty})$ is a symmetric quasi-Banach ideal for any $0 < p < \infty$. If $p > 1$, there exists an equivalent Calderón norm and $\mathcal{L}_{p,\infty}(H)$ becomes a symmetric Banach ideal equipped with a Calderón norm [63, Section 1.7]. However, it is known that the ideal $\mathcal{L}_{p,\infty}(H)$ fails to be normable for $0 < p \leq 1$ (see, for example, [54, pp. 210] and [19, pp. 259-260]).

(iii) [9, Section 4] A simple example of a complete symmetrically F*-normed ideal is a trace class ideal $\mathcal{L}_1(H)$ equipped with a functional*

$$\|X\|_{\log} := \sum_{k=0}^{\infty} \log(1 + s_k(X)), \quad X \in \mathcal{L}_1(H).$$

The properties (i)-(iii) of F*-norm (see Definition 2.1.1) follow from the fact that*

$s_k(\alpha X) = |\alpha| s_k(X)$ for any $k \geq 0$, $\alpha \in \mathbb{C}$ and $X \in \mathcal{L}_\infty(H)$. Since $f(x) = \log(1+x)$, $x \geq 0$ is a concave function, the triangle inequality follows from [59, Theorem 1]. Now assume that $s_k(Y) \leq s_k(X)$, $k \geq 0$ for $X \in \mathcal{L}_1(H)$ and $Y \in \mathcal{L}_\infty(H)$. Hence, it easily follows that $Y \in \mathcal{L}_1(H)$. Moreover, since $f(x) = \log(1+x)$, $x \geq 0$ is an increasing function, we have $\|Y\|_{\log} \leq \|X\|_{\log}$. Hence, $(\mathcal{L}_1(H), \|\cdot\|_{\log})$ is a symmetrically F*-normed ideal. Furthermore, since*

$$\frac{x}{1+x} \leq \log(1+x) \leq x, \quad x \geq 0,$$

it follows that $\mathcal{L}_1(H)$ coincides as a set with

$$\mathcal{L}_{\log}(H) := \{X \in \mathcal{L}_\infty(H) : \quad \|X\|_{\log} < \infty\}.$$

Therefore, the completeness follows from the fact that $(\mathcal{L}_{\log}(H), \|\cdot\|_{\log})$ is complete (see [9, Section 4]).

(iv) *Now, we present an example of a symmetrically* F*-normed ideal from [58, Section 4], which cannot be equipped with an equivalent quasi-norm. Let $\{t_k\}_{k\geq 1}$ be a sequence of positive numbers such that t_1 is an arbitrary positive number and t_n is a positive solution of the equation $t_n^2 + t_n = t_{n-1}^2$ for each $n \geq 2$. Define a function $M(t)$ on $[0,\infty)$ in the following way:*

$$M(t) = \begin{cases} \frac{t^2(t^2+t)}{t_{2k}^2}, & t_{2k+1} \leq t < t_{2k}, \quad k \geq 1; \\ t_{2k+1}^2, & t_{2k+2} \leq t < t_{2k+1}, \quad k \geq 0; \\ t^2, & t \geq t_1; \end{cases}$$
$$M(0) = 0.$$

Then Orlicz sequence space l^M, defined as

$$l^M = \{x = \{x_k\}_{k\geq 0} \in l_\infty, \quad \sum_{k=0}^{\infty} M(x_k) < \infty\}$$

equipped with

$$\|x\|_{l^M} = \inf\{\varepsilon : \quad \varepsilon > 0, \quad \sum_{k=0}^{\infty} M\left(\frac{x_k}{\varepsilon}\right) < \varepsilon\},$$

is a complete F*-normed Calkin sequence space (for details, see [58, Section 4], [35, Theorem 2] and [33, Section 2.4]). Moreover, as shown in [58, Section 4], the function* M *does not satisfy the condition (b) of [58, Theorem 1]. Hence, by [58, Theorem 1], the space* l^M *is not locally bounded, that is* l^M *cannot be equipped with an equivalent quasi-norm (see e.g. [27, Chapter I.3] or [32, Chapter 3, Section 15.10]). Passing to operator ideals, [18, Theorem 3.8] guarantees that the corresponding Orlicz ideal*

$$\mathcal{L}^M := \{X \in \mathcal{L}_\infty(H), \quad s(X) := \{s_k(X)\}_{k\geq 0} \in l^M\}$$

equipped with

$$\|X\|_{\mathcal{L}^M} = \inf\{\varepsilon : \quad \varepsilon > 0, \quad \sum_{k=0}^{\infty} M\left(\frac{s_k(X)}{\varepsilon}\right) < \varepsilon\}, \quad X \in \mathcal{L}^M,$$

is a complete symmetrically F*-normed ideal, which cannot be equipped with a quasi-norm (since otherwise the corresponding sequence space* l^M *would be quasi-normed too, [18, Theorem 3.9]).*

We now present some properties of F-norm $\|\cdot\|_{\mathcal{I}}$.

Proposition 3.1.5. *Let* $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ *be a symmetrically* F*-normed ideal. Then,*

(i) $\|XY\|_{\mathcal{I}} \leq \|\|X\|_\infty Y\|_{\mathcal{I}} \leq \lceil\|X\|_\infty\rceil \|Y\|_{\mathcal{I}}$ *for* $X \in \mathcal{L}(H)$ *and* $Y \in \mathcal{I}(H)$*, where* $\lceil\cdot\rceil$ *denotes the ceiling function;*

(ii) $\|\cdot\|_{\mathcal{I}}$ *is unitarily invariant, i.e., for any unitary operator* $U \in \mathcal{L}(H)$ *and any* $X \in \mathcal{I}(H)$*, one has*

$$\|UX\|_{\mathcal{I}} = \|XU\|_{\mathcal{I}} = \|X\|_{\mathcal{I}};$$

(iii) $\|X^*\|_{\mathcal{I}} = \|X\|_{\mathcal{I}}$ *for any* $X \in \mathcal{I}(H)$.

Proof. (i). Since $s_j(XY) \leq \|X\|_\infty s_j(Y) = s_j(\|X\|_\infty Y), j \geq 0$, the first inequality follows from the definition of a symmetrically F-normed ideal and the second follows from the repeated use of triangle inequality.

(ii). Let $U, V \in \mathcal{L}(H)$ be the unitary operators. Then,

$$s_j(UXV) \leq \|U\|_\infty s_j(X)\|V\|_\infty = s_j(X), \quad j \geq 0,$$

which implies that $\|UXV\|_{\mathcal{I}} \leq \|X\|_{\mathcal{I}}$. On the other hand,

$$s_j(X) = s_j(U^*UXVV^*) \leq \|U^*\|_\infty s_j(UXV)\|V^*\|_\infty = s_j(UXV), \quad j \geq 0,$$

which implies that $\|X\|_{\mathcal{I}} \leq \|UXV\|_{\mathcal{I}}$. Hence, assuming that $V = I$ is the identity operator and combining the latter two inequalities, we prove the assertion.

(iii) Let $X = U|X|$ be the polar decomposition of the operator X. Then, by (i), one has

$$\|X\|_{\mathcal{I}} \leq \|U\|_\infty \||X|\|_{\mathcal{I}} = \||X|\|_{\mathcal{I}}.$$

Since $U^*X = |X|$, we also have

$$\||X|\|_{\mathcal{I}} \leq \|U^*\|_\infty \|X\|_{\mathcal{I}} = \|X\|_{\mathcal{I}},$$

which together with the previous inequality implies that $\|X\|_{\mathcal{I}} = \||X|\|_{\mathcal{I}}$.

Using the same argument for the representation $X^* = |X|U^*$, one can also obtain that $\|X^*\|_{\mathcal{I}} = \||X|\|_{\mathcal{I}}$, which together with $\|X\|_{\mathcal{I}} = \||X|\|_{\mathcal{I}}$ ends the proof. □

3.2 Logarithmic submajorization

First, we define a notion of logarithmic submajorization for compact operators in $\mathcal{L}_\infty(H)$. For any $X, Y \in \mathcal{L}_\infty(H)$ we say that Y is logarithmically submajorised by X (denoted by $Y \prec\prec_{\log} X$) if

$$\prod_{j=0}^{k} s_j(Y) \leq \prod_{j=0}^{k} s_j(X), \quad \forall k \in \mathbb{N} \cup \{0\}.$$

The important class of all symmetrically F-normed ideals is the following symmetrically F-normed ideals closed with respect to the logarithmic submajorization.

Definition 3.2.1. *A symmetrically* F*-normed ideal $\mathcal{I}(H)$ is said to be closed with respect to the logarithmic submajorisation if*

(i) $X \in \mathcal{I}(H)$, $Y \in \mathcal{L}(H)$ and $Y \prec\prec_{\log} X$ imply that $Y \in \mathcal{I}(H)$;

(ii) there is a constant $C_{\mathcal{I}} > 0$ such that $\|Y\|_{\mathcal{I}} \leq C_{\mathcal{I}} \|X\|_{\mathcal{I}}$ for any $X, Y \in \mathcal{I}(H)$ with $Y \prec\prec_{\log} X$.

We note that our definition of the closedness with respect to the logarithmic submajorization is stronger than that in [65, Definition 6] and includes inequality for the F-norm (ii) of Definition 3.2.1. However, if $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ is symmetric (quasi)-Banach ideal, then the latter definition is superfluous since all symmetric (quasi)-Banach ideals are closed with respect to the logarithmic submajorization. We show the latter fact in the following proposition.

Proposition 3.2.2. *Any symmetric quasi-Banach ideal $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ is closed with respect to the logarithmic submajorization.*

Proof. By [26, Proposition 3.2] any symmetric quasi-Banach ideal is geometrically stable, i.e., if $X \in \mathcal{I}(H)$, then

$$\begin{aligned}&\overline{\operatorname{diag}}\,(s_j(X))\\ &:= \operatorname{diag}(s_0(X), (s_0(X)s_1(X))^{1/2}, ..., (s_0(X)s_1(X)...s_{j-1}(X))^{1/j}, ...) \in \mathcal{I}(H),\end{aligned}$$

and

$$\|\overline{\operatorname{diag}}(s_j(X))\|_{\mathcal{I}} \leq \operatorname{Const} \cdot \|X\|_{\mathcal{I}} \tag{3.4}$$

for some constant $\operatorname{Const} > 0$ depending only on the modulus of concavity of the quasi-norm. Furthermore, by [65, Lemma 35], any geometrically stable ideal satisfies the condition *(i)* of Definition 3.2.1. Since $s_j(Y) \leq \left(\prod_{k=0}^{j-1} s_k(Y)\right)^{1/j}$, $j \geq 0$, the symmetricity of the quasi-norm $\|\cdot\|_{\mathcal{I}}$ and (3.4) imply that

$$\|Y\|_{\mathcal{I}} \leq \left\|\overline{\operatorname{diag}}(s_j(Y))\right\|_{\mathcal{I}} \leq \left\|\overline{\operatorname{diag}}(s_j(X))\right\|_{\mathcal{I}} \leq \operatorname{Const} \cdot \|X\|_{\mathcal{I}}$$

provided that $Y \prec\prec_{\log} X$. Thus, the condition *(ii)* of Definition 3.2.1 is satisfied too. □

However, Proposition 3.2.2 is no longer true for a general symmetrically F-normed ideal (even in the case when the ideal $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ is complete). To demonstrate this, we present the following example.

Example 3.2.3. *Consider the ideal of trace-class operators*

$$\mathcal{L}_1(H) = \{X \in \mathcal{L}_\infty(H) : \quad \sum_{j=0}^{\infty} s_j(X) < +\infty\},$$

and the functional $\|\cdot\|_{\log\log}$ defined as

$$\|X\|_{\log\log} = \sum_{j=0}^{\infty} \log(1+\log(1+s_j(X))), \quad \forall X \in \mathcal{L}(H).$$

Note that for any positive real number $x > 0$, one has

$$\log(1+x) \leq x.$$

Hence,

$$\|X\|_{\log\log} = \sum_{j=0}^{\infty} \log(1+\log(1+s_j(X))) \leq \sum_{j=0}^{\infty} \log(1+s_j(X)) \leq \sum_{j=0}^{\infty} s_j(X),$$

which means that $\|\cdot\|_{\log\log}$ is finite on $\mathcal{L}_1(H)$. Furthermore, an argument similar to Example 3.1.4(iii) shows that $(\mathcal{L}_1(H), \|\cdot\|_{\log\log})$ is a complete symmetrically F*-normed ideal. From the fact that any quasi-Banach ideal is closed with respect to the logarithmic submajorization (see Proposition 3.2.2) we know that $\mathcal{L}_1(H)$ satisfies the condition (i) of the Definition 3.2.1, i.e. $Y \prec\prec_{\log} X$, $X \in \mathcal{L}_1(H)$ implies $Y \in \mathcal{L}_1(H)$. For any finite $n \in \mathbb{N}$ consider the finite-rank operators*

$$Y_n = \operatorname{diag}(\underbrace{e^n, e^n, ..., e^n}_{n\text{-times}}, 0, ...), \quad X_n = \operatorname{diag}(e^{n^2}, \underbrace{1, ..., 1}_{(n\text{-}1)\text{-times}}, 0, ...).$$

It is easy to see that $Y_n \prec\prec_{\log} X_n$, $n \in \mathbb{N}$, and

$$\frac{\|Y_n\|_{\log\log}}{\|X_n\|_{\log\log}} = \frac{n \cdot \log(1+\log(1+e^n))}{\log(1+\log(1+e^{n^2})) + (n-1)\log(1+\log(2))} \to \infty,\ n \to \infty.$$

Therefore, there is no constant $C_{\mathcal{I}} > 0$ such that $\|Y_n\|_{\log\log} \leq C_{\mathcal{I}} \|X_n\|_{\log\log}$, $\forall n \in \mathbb{N}$.

We further recall previously established results without proofs regarding the logarithmic submajorization. The proofs can be found in their respective references.

Lemma 3.2.4. *[40, Lemma 2.5] Let X, X_0, Y, Y_0 be non-negative self-adjoint compact operators such that $X^r \leq X_0$ and $Y^r \leq Y_0$ for some $r \geq 1$, then*

$$\left(Y^{1/2} X Y^{1/2}\right)^r \prec\prec_{\log} Y_0^{1/2} X_0 Y_0^{1/2}.$$

Lemma 3.2.5. *[16, Theorem 2.3] Let X and Y be non-negative self-adjoint operators on a separable Hilbert space H and $Z = X \dot{+} Y$ be their from sum. Then,*

$$e^{-tZ}P \prec\prec_{\log} \left(e^{-tY/2r}e^{-tX/r}e^{-tY/2r}\right)^r, \quad t \geq 0,$$

where P is the orthogonal projection onto $\mathrm{dom}(Z)$.

3.3 Some results in symmetric operator ideals

We now present some properties of symmetric Banach ideals, which are helpful in the following chapters.

Lemma 3.3.1. *Let $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ be a symmetric Banach ideal. Let A be a positive self-adjoint operator such that $e^{-tA} \in \mathcal{I}(H)$ for all $t > 0$ and let $\{B_j\}_{j=1}^n \subset \mathcal{L}(H)$. For every strictly positive sequence $\{t_j\}_{j=1}^n$, we have*

$$\left\|\prod_{j=1}^n B_j e^{-t_j A}\right\|_{\mathcal{I}} \leq \mathrm{Const} \cdot \prod_{j=1}^n \|B_j\|_\infty \left\|e^{-(t_1+t_2+\cdots+t_n)A}\right\|_{\mathcal{I}},$$

where $\mathrm{Const} > 0$ is some constant which depends only on $\|\cdot\|_{\mathcal{I}}$.

Proof. Without loss of generality assume that $\|B_j\|_\infty = 1$ for every $1 \leq j \leq n$. Hence, by (3.1), we have

$$\begin{aligned} \prod_{k=0}^m s_k\left(\prod_{j=1}^n B_j e^{-t_j A}\right) &\leq \prod_{j=1}^n \left(\prod_{k=0}^m s_k\left(B_j e^{-t_j A}\right)\right) \\ &\leq \prod_{j=1}^n \left(\prod_{k=0}^m s_k\left(e^{-t_j A}\right)\right) = \prod_{k=0}^m \left(\prod_{j=1}^n s_k\left(e^{-t_j A}\right)\right). \end{aligned} \tag{3.5}$$

Observe that for any positive number $t_j > 0$, one has

$$s_k(e^{-t_j A}) = s_k(e^{-A})^{t_j}, \quad k \geq 0.$$

Therefore,

$$\prod_{j=1}^n s_k\left(e^{-t_j A}\right) = s_k\left(e^{-(t_1+t_2+\cdots+t_n)A}\right), \quad k \geq 0.$$

Substituting the latter equality in (3.5) yields

$$\prod_{k=0}^{m} s_k(\prod_{j=1}^{n} B_j e^{-t_j A}) \leq \prod_{k=0}^{m} s_k\left(e^{-(t_1+t_2+\cdots+t_n)A}\right), \quad m \geq 0.$$

In other words,

$$\prod_{j=1}^{n} B_j e^{-t_j A} \prec\prec_{\log} e^{-(t_1+t_2+\cdots+t_n)A}.$$

Since every symmetric Banach ideal is closed with respect to the logarithmic submajorization, it follows from Proposition 3.2.2 that

$$\left\|\prod_{j=1}^{n} B_j e^{-t_j A}\right\|_{\mathcal{I}} \leq \text{Const} \cdot \left\|e^{-\sum_{j=1}^{n} t_j A}\right\|_{\mathcal{I}}$$

for some positive constant Const which only depends on norm $\|\cdot\|_{\mathcal{I}}$. $\square$

For a bounded operator $X \in \mathcal{L}(H)$, *the left support* and *the right support* mean the smallest projections $P \in \mathcal{L}(H)$ and $Q \in \mathcal{L}(H)$ such that $PX = X$ and $XQ = X$, respectively. We denote them by $\mathrm{supp}_l(X) := P$ and $\mathrm{supp_r(X)} := Q$, respectively.

We now present two convergence results for the symmetric Banach ideal, which are helpful in Chapter 6.

Lemma 3.3.2. *Let $\mathcal{I}(H)$ be a symmetric Banach ideal and let X belongs to the separable part of $\mathcal{I}(H)$. Let the functions $f : [a, b] \to \mathcal{L}(H)$ and $g : [c, d] \to \mathcal{L}(H)$ be continuous in the strong operator topology. Then, the mapping*

$$(t, s) \to f(t)Xg(s), \quad t \in [a, b], \quad s \in [c, d],$$

is jointly continuous with respect to $\|\cdot\|_{\mathcal{I}}$. The same assertion holds for not necessarily closed intervals.

Proof. For every $\xi \in H$, the mapping $t \to f(t)\xi$, $t \in [a, b]$, is a continuous H-valued function. In particular, it is H-bounded. By the Uniform Boundedness Principle, $f : [a, b] \to \mathcal{L}(H)$ is bounded in the uniform norm. So is the mapping $g : [c, d] \to \mathcal{L}(H)$. We can assume without loss of generality that

$$\|f(t)\|_\infty \leq 1, \quad \|g(s)\|_\infty \leq 1, \quad t \in [a, b], \quad s \in [c, d].$$

Without loss of generality, $\|X\|_{\mathcal{I}} \leq 1$.

Fix $\epsilon \in (0,1)$. Since X belongs to the separable part of the ideal we can choose finite rank operator $Y \in \mathcal{L}(H)$ such that $\|X-Y\|_{\mathcal{I}} < \epsilon$. Note that $\|Y\|_{\mathcal{I}} < 2$. The mapping $t \to f(t)\cdot \mathrm{supp}_l(Y)$, $t \in [a,b]$, is continuous in the uniform norm. The mapping $s \to \mathrm{supp}_r(Y)\cdot g(s)$, $s \in [c,d]$, is continuous in the uniform norm. Choose $\delta > 0$ such that

$$\|f(t_1)\cdot \mathrm{supp}_l(Y) - f(t_2)\cdot \mathrm{supp}_l(Y)\|_\infty < \epsilon,$$

whenever $|t_1 - t_2| < \delta$ and

$$\|\mathrm{supp}_r(Y)\cdot g(s_1) - \mathrm{supp}_r(Y)\cdot g(s_2)\|_\infty < \epsilon,$$

whenever $|s_1 - s_2| < \delta$.

We write

$$f(t_1)Xg(s_1) - f(t_2)Xg(s_2) = f(t_1)(X-Y)g(s_1) - f(t_2)(X-Y)g(s_2) +$$

$$+(f(t_1)\cdot\mathrm{supp}_l(Y) - f(t_2)\cdot\mathrm{supp}_l(Y))\cdot Y\cdot g(s_1) + f(t_2)\cdot Y\cdot(\mathrm{supp}_r(Y)\cdot g(s_1) - \mathrm{supp}_r(Y)\cdot g(s_2)).$$

By triangle inequality and the symmetricity of $\|\cdot\|_{\mathcal{I}}$, we have

$$\|f(t_1)Xg(s_1) - f(t_2)Xg(s_2)\|_{\mathcal{I}} \leq \|f(t_1)\|_\infty\|X-Y\|_{\mathcal{I}}\|g(s_1)\|_\infty + \|f(t_2)\|_\infty\|(X-Y)g(s_2)+$$

$$+\|f(t_1)\cdot\mathrm{supp}_l(Y) - f(t_2)\cdot\mathrm{supp}_l(Y)\|_\infty\|Y\|_{\mathcal{I}}\|g(s_1)\|_\infty +$$

$$+\|f(t_2)\|_\infty\|Y\|_{\mathcal{I}}\|\mathrm{supp}_r(Y)\cdot g(s_1) - \mathrm{supp}_r(Y)\cdot g(s_2)\|_\infty <$$

$$< 1\cdot\epsilon\cdot 1 + 1\cdot\epsilon\cdot 1 + \epsilon\cdot 2\cdot 1 + 1\cdot 2\cdot\epsilon = 6\epsilon$$

whenever $|t_1 - t_2| < \delta$ and $|s_1 - s_2| < \delta$. Since ϵ is arbitrary, it follows that our mapping is jointly continuous with respect to $\|\cdot\|_{\mathcal{I}}$. This proves the assertion in the case of a closed intervals.

Now, let us consider not necessarily closed intervals. For example, let the first interval be $(a,b]$ and let the second interval be $[c,d)$. Consider our mapping on the set $[a+\frac{1}{n},b]\times[c,d-\frac{1}{n}]$. There, it is continuous with respect to $\|\cdot\|_{\mathcal{I}}$. Hence, it is continuous with respect to $\|\cdot\|_{\mathcal{I}}$

on the set

$$\bigcup_{n\geq 1}[a+\frac{1}{n},b]\times[c,d-\frac{1}{n}]=(a,b]\times[c,d).$$

This proves the assertion for not necessarily closed intervals. $\square$

Lemma 3.3.3. *Let $\mathcal{I}(H)$ be a symmetrically normed ideal and let X belong to the separable part of $\mathcal{I}(H)$. Let $f:[a,b]\to\mathcal{L}(H)$ be continuously differentiable in strong operator topology. The mapping*

$$t\to f(t)X,\quad t\in(a,b],$$

is continuously differentiable with respect to $\|\cdot\|_{\mathcal{I}}$. Its derivative is $t\to f'(t)X$, $t\in[a,b]$. The same assertion holds for not necessarily closed intervals.

Proof. The mapping f' is continuous in the strong operator topology and is, therefore, bounded in the uniform norm (see the proof of Lemma 3.3.2). Since f is continuously differentiable in the strong operator topology, it follows that $g(\cdot)=\langle f(\cdot)\xi,\eta\rangle$ is continuously differentiable and its derivative is $g'(\cdot)=\langle f'(\cdot)\xi,\eta\rangle$ for any $\xi,\eta\in H$. Hence, the fundamental theorem of calculus implies that

$$\langle (g(t_1)-g(t_2))\,\xi,\eta\rangle=\int_{t_1}^{t_2}\langle f'(t)\xi,\eta\rangle dt,\quad \forall\xi,\eta\in H.$$

Therefore,

$$f(t_1)-f(t_2)=\int_{t_1}^{t_2}f'(t)dt,$$

where the integral is understood in the weak operator topology (see [8, Section II.3] or [61, Chapter 2]). It implies that

$$\|f(t_1)-f(t_2)\|_\infty\leq\int_{t_1}^{t_2}\left\|f'(t)\right\|_\infty dt\leq\sup_{t\in[a,b]}\left\|f'(t)\right\|_\infty|t_1-t_2|,$$

which means that f is Lipschitz in the uniform norm. Without loss of generality,

$$\|f'(t)\|_\infty\leq 1,\quad \|f(t_1)-f(t_2)\|_\infty\leq|t_1-t_2|,\quad t,t_1,t_2\in[a,b].$$

Without loss of generality, $\|X\|_{\mathcal{I}}\leq 1$.

Fix $\epsilon\in(0,1)$. Since X belongs to the separable part of the ideal we can choose finite rank operator $Y\in\mathcal{L}(H)$ such that $\|X-Y\|_{\mathcal{I}}<\epsilon$. Note that $\|Y\|_{\mathcal{I}}<2$. The mapping

$t \to f(t) \cdot \mathrm{supp}_l(Y)$, $t \in [a, b]$, is continuously differentiable in the uniform norm. Choose $\delta > 0$ such that

$$\left\| \frac{f(t_2) \cdot \mathrm{supp}_l(Y) - f(t_1) \cdot \mathrm{supp}_l(Y)}{t_2 - t_1} - f'(t_1) \cdot \mathrm{supp}_l(Y) \right\|_\infty < \epsilon,$$

whenever $|t_1 - t_2| < \delta$.

We write

$$\frac{f(t_2)X - f(t_1)X}{t_2 - t_1} - f'(t_1)X = \Big(\frac{f(t_2) - f(t_1)}{t_2 - t_1} - f'(t_1) \Big) \cdot (X - Y) +$$

$$+ \Big(\frac{f(t_2) \cdot \mathrm{supp}_l(Y) - f(t_1) \cdot \mathrm{supp}_l(Y)}{t_2 - t_1} - f'(t_1) \cdot \mathrm{supp}_l(Y) \Big) \cdot Y.$$

By the triangle inequality and the symmetricity of $\| \cdot \|_{\mathcal{I}}$, we have

$$\left\| \frac{f(t_2)X - f(t_1)X}{t_2 - t_1} - f'(t_1)X \right\|_{\mathcal{I}} \leq \left\| \frac{f(t_2) - f(t_1)}{t_2 - t_1} - f'(t_1) \right\|_\infty \|X - Y\|_{\mathcal{I}} +$$

$$+ \left\| \frac{f(t_2) \cdot \mathrm{supp}_l(Y) - f(t_1) \cdot \mathrm{supp}_l(Y)}{t_2 - t_1} - f'(t_1) \cdot \mathrm{supp}_l(Y) \right\|_\infty \|Y\|_{\mathcal{I}} <$$

$$< 2 \cdot \epsilon + \epsilon \cdot 2 = 4\epsilon$$

whenever $|t_1 - t_2| < \delta$. This proves the differentiability of our mapping with respect to $\| \cdot \|_{\mathcal{I}}$. That the derivative is continuous with respect to $\| \cdot \|_{\mathcal{I}}$ follows from Lemma 3.3.2. This proves the assertion in the case of a closed interval.

Now, let us consider not necessarily closed interval. For example, let the interval be $(a, b]$. Consider our mapping on the interval $[a + \frac{1}{n}, b]$. There, it is continuously differentiable with respect to $\| \cdot \|_{\mathcal{I}}$. Hence, it is continuously differentiable with respect to $\| \cdot \|_{\mathcal{I}}$ on the set

$$\bigcup_{n \geq 1} [a + \frac{1}{n}, b] = (a, b].$$

This proves the assertion for not necessarily closed intervals. □

Finally, we present an extension of [40, Lemma 2.6] to the general class of symmetrically F-normed ideals.

Lemma 3.3.4. *Let $X \in \mathcal{I}(H)$, $Y \in \mathcal{L}_\infty(H)$ and $Z \in \mathcal{L}(H)$ be self-adjoint operators, where $\mathcal{I}(H)$ is a symmetrically* F*-normed ideal in $\mathcal{L}(H)$. If $\{Z(t)\}_{t \geq 0}$ is a family of self-adjoint* *bounded operators such that* s-$\lim_{t\to+0} Z(t) = Z$*, then*

$$\lim_{r\to\infty} \sup_{t\in[0,T]} \|(Z(t/r) - Z)YX\|_{\mathcal{I}} = \lim_{r\to\infty} \sup_{t\in[0,T]} \|XY(Z(t/r) - Z)\|_{\mathcal{I}} = 0$$

for any $T \in (0, \infty)$*, where* $r \in \mathbb{R}$.

Proof. Fix $T \in (0,\infty)$ and arbitrary $\varepsilon > 0$. It is sufficient to prove

$$\lim_{r\to\infty} \sup_{t\in[0,T]} \|(Z(t/r) - Z)YX\|_{\mathcal{I}} = 0,$$

since the second equality can be showed by taking adjoints. Since Y is a compact operator, it can be represented as $Y = Y_1 + Y_2$, where Y_1 is a finite-rank operator and Y_2 satisfies $\|Y_2\|_\infty < \delta$ for a given $\delta > 0$. Then, by triangle inequality, we have

$$\begin{aligned}\sup_{t\in[0,T]} \|(Z(t/r) - Z)YX\|_{\mathcal{I}} \leq \sup_{t\in[0,T]} \|(Z(t/r) - Z)Y_1X\|_{\mathcal{I}} \\ + \sup_{t\in[0,T]} \|(Z(t/r) - Z)Y_2X\|_{\mathcal{I}}.\end{aligned} \tag{3.6}$$

We consider the two terms in (3.6) separately. Writing $Y_1 = \sum_{k=1}^m \langle\cdot,\xi_k\rangle\eta_k$ for some $\{\xi_k\}_{k=1}^m, \{\eta_k\}_{k=1}^m \subset H$ and $m \in \mathbb{N}$, and using triangle inequality repeatedly, we have

$$\begin{aligned}\sup_{t\in[0,T]} \|(Z(t/r) - Z)Y_1X\|_{\mathcal{I}} &\leq \sum_{k=1}^m \sup_{t\in[0,T]} \|(Z(t/r) - Z)\left(\langle\cdot,\xi_k\rangle\eta_k\right)X\|_{\mathcal{I}} \\ &= \sum_{k=1}^m \sup_{t\in[0,T]} \|(Z(t/r) - Z)\left(\langle\cdot,\xi_k\rangle\eta_k\right)X\|_\infty \\ &\leq \|X\|_\infty \sum_{k=1}^m \sup_{t\in[0,T]} \|(Z(t/r) - Z)\eta_k\|.\end{aligned}$$

Since s-$\lim_{t\to+\infty} Z(t) = Z$, there exist $r_k \in \mathbb{R}$, $k = 1, ..., m$, such that

$$\sup_{t\in[0,T]} \|(Z(t/r_k) - Z)\eta_k\| < \frac{\varepsilon}{2m\|X\|_\infty}.$$

Setting $R_1 = \max_{1\leq k\leq m} r_k$, for any $r \geq R_1$, we have

$$\sup_{t\in[0,T]} \|\left(Z(t/r) - Z\right)Y_1X\|_{\mathcal{I}} < \frac{\varepsilon}{2}. \tag{3.7}$$

Now we consider the second term of (3.6). Since s-$\lim_{t\to+0} Z(t) = Z$, it follows that s-$\lim_{r\to\infty} Z(t/r) = Z$ uniformly in $t \in [0,T]$. Therefore, there exists a constant $C > 0$ and a large enough number $R_2 \in \mathbb{R}_+$ such that $\sup_{t\in[0,T]} \|Z(t/r) - Z\|_\infty \le C$ for any $r \ge R_2$. Then, for a given $\varepsilon > 0$, using axiom (iii) of the F-norm, we can choose $\delta > 0$ such that $\|C\delta X\|_{\mathcal{I}} < \varepsilon$. By the symmetricity of the F-norm, (i) and the choice of the operator Y_2, we have

$$\begin{aligned} \sup_{t\in[0,T]} \|(Z(t/r) - Z)Y_2X\|_{\mathcal{I}} &\le \sup_{t\in[0,T]} \| \|(Z(t/r) - Z)\|_\infty \|Y_2\|_\infty X\|_{\mathcal{I}} \\ &\le \|C\delta X\|_{\mathcal{I}} < \frac{\varepsilon}{2} \end{aligned} \tag{3.8}$$

for any $r \ge R_2$. Therefore, since $\varepsilon > 0$ is arbitrary, combining (3.7) and (3.8) with $r \ge \max\{R_1, R_2\}$, we conclude the proof. $\square$

Chapter 4

Trotter-Kato product formula in the operator norm topology

Note that the results of the Trotter-Kato product formula and the approximation formula for a propagator in the operator norm topology are crucial to considering these formulas in symmetric operator ideals. Hence, in this chapter, we first overview existing results regarding the Trotter-Kato product formula and approximation formula for a propagator in the operator norm topology. Section 4.1 discusses the elementary cases of finite matrices and bounded operators. Section 4.2 presents the results on the Trotter-Kato product formula in the case of the algebraic sum of two (possibly unbounded) operators on a Hilbert space H. In Section 4.3, we recall similar results in the case of the form-sum of two (possibly unbounded) operators on a Hilbert space H. The criterion of the Trotter-Kato product formula in the operator norm topology is given in Section 4.4. Section 4.5 first recalls the basics of the abstract Cauchy problem for the evolution equation. Moreover, in Subsection 4.5.3, we present some existence results for a propagator of the non-autonomous evolution equation. In Subsection 4.5.4, we overview various approximation formulas for a propagator in the operator norm topology.

4.1 Product formulas for bounded operators

It was Sophus Lie who first considered the exponential formulas for a case of finite matrices (see, for example, [55, Theorem VIII.29]). We present this result in the following lemma.

Proposition 4.1.1. *Let A and B be finite $k \times k$ matrices. Then, one has*

$$\lim_{n\to\infty} \left(e^{-tA/n}e^{-tB/n}\right)^n = e^{-t(A+B)}$$

in the operator norm topology and uniformly in $t \in [0,T]$, $0 < T < \infty$. Moreover, there exists a constant $\mathrm{Const} = \mathrm{Const}\,(\|A\|_\infty, \|B\|_\infty, t) > 0$ *such that*

$$\left\| \left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)} \right\|_\infty \leq \frac{\mathrm{Const}}{n}, \quad n \geq 1.$$

Proof. Firstly, note that for any matrices X, Y with dimensions $k \times k$, one has the following telescopic representation

$$X^n - Y^n = \sum_{j=0}^{n-1} X^j (X - Y) Y^{n-j-1}. \tag{4.1}$$

Hence, taking $X = e^{-tA/n}e^{-tB/n}$ and $Y = e^{-t(A+B)/n}$, (4.1) implies

$$\begin{aligned}\left(e^{-tA/n}e^{-tB/n}\right)^n &- \left(e^{-t(A+B)/n}\right)^n \\ &= \sum_{j=0}^{n-1} \left(e^{-tA/n}e^{-tB/n}\right)^j \left(e^{-tA/n}e^{-tB/n} - e^{-t(A+B)/n}\right) e^{-t(A+B)(n-j-1)/n}.\end{aligned}$$

Thus,

$$\begin{aligned}&\left\| \left(e^{-tA/n}e^{-tB/n}\right)^n - \left(e^{-t(A+B)/n}\right)^n \right\|_\infty \\ &\leq \sum_{j=0}^{n-1} \left\| \left(e^{-tA/n}e^{-tB/n}\right) \right\|_\infty^j \left\| e^{-tA/n}e^{-tB/n} - e^{-t(A+B)/n} \right\|_\infty \left\| e^{-t(A+B)/n} \right\|_\infty^{n-j-1} \\ &\qquad n \cdot e^{-t((\|A\|_\infty + \|B\|_\infty))} \left\| e^{-tA/n}e^{-tB/n} - e^{-t(A+B)/n} \right\|_\infty .\end{aligned}$$

Note that

$$\left\| e^{-tA/n}e^{-tB/n} - e^{-t(A+B)/n} \right\|_\infty = \left\| \sum_{k=0}^{\infty} \frac{1}{k!} \frac{(-tA)^k}{n^k} \sum_{j=0}^{\infty} \frac{1}{j!} \frac{(-tB)^j}{n^j} - \sum_{i=0}^{\infty} \frac{1}{i!} \frac{(-t(A+B))^i}{n^i} \right\|_\infty$$

$$\begin{aligned}&\leq \left\| \frac{t^2}{2n^2}(AB - BA) + \frac{t^3}{6n^3}\left(ABA + BAB + BA^2 + B^2A - 2AB^2 - 2A^2B\right) \right\|_\infty + O\left(\frac{1}{n^4}\right) \\ &\qquad \leq \frac{t^2}{n^2} \|A\|_\infty \|B\|_\infty + \frac{2t^3}{3n^3}\left(\|A\|_\infty^2 \|B\|_\infty + \|A\|_\infty \|B\|_\infty^2\right) + O\left(\frac{1}{n^4}\right)\end{aligned}$$

$$\leq \frac{\text{Const}}{n^2}, \quad n \to \infty,$$

for some positive constant depending on $\|A\|_\infty$ and $\|B\|_\infty$. Hence, from the latter inequalities it follows that

$$\begin{aligned}\sup_{t\in[0,T]} \left\| \left(e^{-tA/n} e^{-tB/n}\right)^n - \left(e^{-t(A+B)/n}\right)^n \right\|_\infty &\leq \frac{\text{Const}}{n} \sup_{t\in[0,T]} e^{-t(\|A\|_\infty + \|B\|_\infty)} \\ &\leq \frac{\text{Const}}{n},\end{aligned}$$

which completes the proof. □

Moreover, one can also prove a similar product formula for a symmetrized family as below. Note that the convergence rate is better in this case.

Proposition 4.1.2. *Let A and B be finite $k \times k$. Then, one has*

$$\lim_{n\to\infty} \left(e^{-tA/2n} e^{-tB/n} e^{-tA/2n}\right)^n = e^{-t(A+B)}$$

in the operator norm topology and uniformly in $t \in [0,T]$, $0 < T < \infty$. Moreover, there exists a constant $\text{Const} = \text{Const}(\|A\|_\infty, \|B\|_\infty, t) > 0$ *such that*

$$\left\| \left(e^{-tA/2n} e^{-tB/n} e^{-tA/2n}\right)^n - e^{-t(A+B)} \right\|_\infty \leq \frac{\text{Const}}{n^2}, \quad n \geq 1.$$

Proof. Using the similar arguments as in Proposition 4.1.1, one has

$$\begin{aligned}&\left\| \left(e^{-tA/2n} e^{-tB/n} e^{-tA/2n}\right)^n - \left(e^{-t(A+B)/n}\right)^n \right\|_\infty \\ &\qquad \leq n \cdot e^{-t(\|A\|_\infty + \|B\|_\infty)} \left\| e^{-tA/2n} e^{-tB/n} e^{-tA/2n} - e^{-t(A+B)/n} \right\|_\infty .\end{aligned}$$

Note that

$$\begin{aligned}&\left\| e^{-tA/2n} e^{-tB/n} e^{-tA/2n} - e^{-t(A+B)/n} \right\|_\infty \\ &= \left\| \sum_{k=0}^\infty \frac{1}{k!} \frac{(-tA)^k}{(2n)^k} \sum_{j=0}^\infty \frac{1}{j!} \frac{(-tB)^j}{n^j} \sum_{i=0}^\infty \frac{1}{i!} \frac{(-tA)^i}{(2n)^i} - \sum_{m=0}^\infty \frac{1}{m!} \frac{(-t(A+B))^m}{n^m} \right\|_\infty \\ &= \left\| \frac{t^3}{24n^3} \left(A^2B + BA^2 + 4BAB - 2ABA - 2AB^2 - 2B^2A\right) + O\left(\frac{1}{n^4}\right) \right\|_\infty \\ &\leq \frac{t^3}{6n^3} \left(\|A\|_\infty^2 \|B\|_\infty + 2\|A\|_\infty \|B\|_\infty^2\right) + O\left(\frac{1}{n^4}\right) \leq \frac{\text{Const}}{n^3}, \quad n \to \infty,\end{aligned}$$

for some positive constant depending on $\|A\|_\infty$ and $\|B\|_\infty$. Hence, we finally have that

$$\begin{aligned}\sup_{t\in[0,T]}\left\|\left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n-\left(e^{-t(A+B)/n}\right)^n\right\|_\infty &\leq \frac{\text{Const}}{n^2}\sup_{t\in[0,T]} e^{-t(\|A\|_\infty+\|B\|_\infty)}\\ &\leq \frac{\text{Const}}{n^2}.\end{aligned}$$

□

Note that a further extension for a couple of bounded operators $A, B \in \mathcal{L}(H)$ is straightforward and it is shown in the following proposition. A proof mainly follows the same line of reasoning as in the proof of Propositions 4.1.1 and 4.1.2, hence, is omitted.

Proposition 4.1.3. *Let A and B be bounded operators on a separable Hilbert space H. Then, the following convergences in the operator norm*

$$\lim_{n\to\infty}\left(e^{-tA/n}e^{-tB/n}\right)^n = e^{-t(A+B)},$$
$$\lim_{n\to\infty}\left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n = e^{-t(A+B)}$$

holds uniformly in $t \in [0,T]$, $0 < T < \infty$. Moreover, there exists a constant $\text{Const} = \text{Const}(\|A\|_\infty, \|B\|_\infty, t) > 0$ *such that*

$$\left\|\left(e^{-tA/n}e^{-tB/n}\right)^n - e^{-t(A+B)}\right\|_\infty \leq \frac{\text{Const}}{n}, \quad n \geq 1,$$

$$\left\|\left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n - e^{-t(A+B)}\right\|_\infty \leq \frac{\text{Const}}{n^2}, \quad n \geq 1.$$

4.2 Product formulas with the algebraic sum of unbounded operators

Let A and B be possibly unbounded operators on a Hilbert space H. Note that the obvious domain $\text{dom}(A) \cap \text{dom}(B)$ might be a trivial set. Hence, the algebraic sum of A and B is not defined in general. However, one can impose the assumption that the algebraic sum $A + B$ is self-adjoint on $\text{dom}(A) \cap \text{dom}(B)$. In this setting, it is possible to obtain the Trotter-Kato product formula in the operator norm topology with various rates of convergences under additional suitable conditions on domains $\text{dom}(A), \text{dom}(B)$ and Kato functions f, g.

The first result regarding the convergence of the Trotter product formula in the operator norm topology was obtained by Rogava [57] (see also [56]). He considered an error bound of the Trotter-product formula for a non-symmetric product case of exponential functions. Namely, let A and B be non-negative self-adjoint operators on a Hilbert space H such that the algebraic sum $C := A + B$ is self-adjoint on $\mathrm{dom}(C) = \mathrm{dom}(A) \subset \mathrm{dom}(B)$. Then, he proved the following

$$\left\| \left(e^{-tB/n}e^{-tA/n}\right)^n - e^{-tC}\right\|_{\infty} = O\left(\frac{\ln n}{\sqrt{n}}\right), \quad n \to \infty, \tag{4.2}$$

where the convergence holds uniformly in t on compact interval $[0, T] \subset [0, \infty)$. The detailed proof of this result can be found in [56, Theorem 1].

Later, Ichinose and Tamura [21] obtained a stronger estimate for an error bound of the Trotter product formula in symmetric product case. They considered two non-negative self-adjoint operators A and B such that $\mathrm{dom}(A^\alpha) \subset \mathrm{dom}(B)$ for some $\alpha \in [0, 1)$. Then, one has that the algebraic sum $C := A + B$ is non-negative self-adjoint on $\mathrm{dom}(C) = \mathrm{dom}(A)$. Moreover, they proved the following convergence result

$$\left\| \left(e^{-tA/2n}e^{-tB/n}e^{-tA/2n}\right)^n - e^{-tC}\right\|_{\infty} = O\left(\frac{1}{n^{2/(3+\alpha)}}\right), \quad n \to \infty, \tag{4.3}$$

where the convergence holds uniformly in t on compact interval $[0, T] \subset [0, \infty)$.

Before proceeding further, let us recall the notion of a Kato function.

Definition 4.2.1. *[30] A Borel measurable function $f(\cdot)$ defined on $[0, \infty)$ is called a Kato function if it satisfies the following*

$$0 \le f(x) \le 1, \quad f(0) = 1, \quad f'(0) = -1.$$

For example, the exponential function $f(x) = e^{-x}$, $x \ge 0$, is a simple example of a Kato function.

In [37], Neidhardt and Zagrebnov further extended the above results of Rogava, and Ichinose and Tamura, for a certain subclass of Kato functions under weaker conditions than in [21]. It was assumed that A and B are non-negative self-adjoint operators such that $A \ge I$ and $B \ge I$. It was also assumed that $\mathrm{dom}(A) \subset \mathrm{dom}(B)$ and B is a relatively

A-bounded with bound a less than 1, i.e.,

$$\|B\xi\| \le a\|A\xi\|, \quad \xi \in \mathrm{dom}(A).$$

Let f and g be Kato functions such that

$$\begin{aligned}
C_{1/2\gamma} &:= \sup_{x>0} \frac{xf(x)^{1/2}}{1-f(x)} < \infty,\\
C_1 &:= \sup_{x>0} \frac{1-f(x)}{x} < \infty,\\
C_2 &:= \sup_{x>0} \left| \left(f(x) - \frac{1}{1+x} \right) \frac{1}{x^2} \right| < \infty,\\
S_1 &:= \sup_{x>0} \frac{1-g(x)}{x} < \infty,\\
S_2 &:= \sup_{x>0} \left| \left(g(x) - \frac{1}{1+x} \right) \frac{1}{x^2} \right| < \infty.
\end{aligned} \tag{4.4}$$

Moreover, assume that

$$0 < aC_0S_1 < 1.$$

Then, one has the following convergences for both symmetric and non-symmetric product cases

$$\begin{aligned}
&\left\| \left(f(tA/n)^{1/2} g(tB/n) f(tA/n)^{1/2} \right)^n - e^{-tC} \right\|_\infty = O\left(\frac{\ln n}{n} \right), \quad n \to \infty,\\
&\left\| \left(f(tA/n) g(tB/n) \right)^n - e^{-tC} \right\|_\infty = O\left(\frac{\ln n}{n} \right), \quad n \to \infty,
\end{aligned} \tag{4.5}$$

where the convergences hold uniformly in $t \ge 0$.

One can easily see that an estimate (4.5) is, indeed, stronger than both (4.2) and (4.3). Moreover, the conditions on operators A and B in this case are weaker than in [21] (for more details, see [37, Section 3]).

Note that the simple examples of functions which satisfy (4.4) are the following

$$f(x) = g(x) = e^{-x}, \quad x \ge 0$$

and

$$f(x) = g(x) = \left(1 + \frac{x}{k}\right)^{-k}, \quad x \ge 0, \quad k \ge 2.$$

4.2.1 An optimal error bound

One can easily notice that all previous results in this section were obtained under the assumption that the algebraic sum of the pair of given operators is self-adjoint on common domain $\mathrm{dom}(A) \cap \mathrm{dom}(B)$. This subsection presents the optimal estimates for error bounds under the analogues assumption. Further details can be found in [23] and [24].

Let us first recall the subclass $\widehat{\mathcal{K}}_\gamma$, $\gamma \in (1, 2]$ of Kato functions.

Definition 4.2.2. *[75, Definition 5.24] A Kato function f is said to belong to the class $\widehat{\mathcal{K}}_\gamma$ for $\gamma \in (1, 2]$, if*

(i) for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) < 1$ such that

$$f(x) \leq 1 - \delta(\varepsilon), \quad x \geq \varepsilon;$$

(ii)

$$[f]_\gamma := \sup_{x>0} \frac{|f(x) - 1 + x|}{x^\gamma} < \infty.$$

The standard examples of functions from class $\widehat{\mathcal{K}}_\gamma$ are

$$f(x) = e^{-x}, \quad f(x) = \left(1 + \frac{x}{k}\right)^{-k}, \quad k > 0, \quad x \geq 0.$$

Let A and B be non-negative self-adjoint operators on a separable Hilbert space H such that the algebraic sum $C = A + B$ is self-adjoint on $\mathrm{dom}(A) \cap \mathrm{dom}(B)$. Let f, g be Kato functions from class $\widehat{\mathcal{K}}_\gamma$ for some given $3/2 \leq \gamma \leq 2$. Then, in [23, Theorem 1], Ichinose and Tamura proved that

$$\begin{aligned} &\left\|(f(tA/2n)g(tB/n)f(tA/2n))^n - e^{-tC}\right\|_\infty = O\left(n^{-1/2}\right), \quad n \to \infty, \\ &\left\|(f(tA/n)g(tB/n))^n - e^{-tC}\right\|_\infty = O\left(n^{-1/2}\right), \quad n \to \infty, \end{aligned} \tag{4.6}$$

uniformly in t on compact interval $[0, T] \subset [0, \infty)$. Moreover, they showed that if C is a strictly positive operator, then the convergences in (4.6) are uniform on $[T, \infty)$, $T > 0$.

In the subsequent paper [24], the given approach was further studied and the authors proved the following

Theorem 4.2.3. *([24], Theorem 1 and Corollary 1]) Let A and B be non-negative self-adjoint operators on a separable Hilbert H such that their algebraic sum $C = A + B$ is self-adjoint on* $\mathrm{dom}(A) \cap \mathrm{dom}(B)$. *Let f, g be Kato functions from the class $\widehat{\mathcal{K}}_2$. Then, one has*

$$\left\|(g(tB/2n)f(tA/n)g(tB/2n))^n - e^{-tC}\right\|_\infty = O\left(\frac{1}{n}\right), \quad n \to \infty,$$

$$\left\|(f(tA/2n)g(tB/n)f(tA/2n))^n - e^{-tC}\right\|_\infty = O\left(\frac{1}{n}\right), \quad n \to \infty,$$

$$\left\|(f(tA/n)g(tB/n))^n - e^{-tC}\right\|_\infty = O\left(\frac{1}{n}\right), \quad n \to \infty,$$

$$\left\|(g(tB/n)f(tA/n))^n - e^{-tC}\right\|_\infty = O\left(\frac{1}{n}\right), \quad n \to \infty,$$

in t on compact interval $[0, T] \subset [0, \infty)$. Moreover, if C is a strictly positive operator, then the convergence above is uniform on $[T, \infty)$, $T > 0$.

Note that the given estimate $O\left(\frac{1}{n}\right)$ for an error bound is optimal, although, only in case of $k = 2$. Hence, it improves and extends all the given results in this section for the algebraic sum of two non-negative self-adjoint operators.

In the following example we show that the error bound in Theorem 4.2.3 is, indeed, optimal. For more details, we refer the reader to [67].

Example 4.2.4. *Let $\left(H_k := \left(\mathbb{R}^2, \langle\cdot,\cdot\rangle_k\right)\right)_{k\geq 1}$ be a countable family of Hilbert spaces. One can define a direct sum $H = \oplus_{k=1}^{\infty} H_k$ with the inner product $\langle \xi, \eta\rangle_k = \sum_{k\geq 1}\langle \xi_k, \eta_k\rangle_k$ for $\xi = (\xi_k)_{k\geq 1}$ and $\eta = (\eta_k)_{k\geq 1}$ in H. For each $k \geq 1$, define the following bounded non-negative self-adjoint operators on H_k*

$$A_k = k(S + E), \quad B_k = k\left(S\cos\theta_k + T\sin\theta_k + I\right),$$

where

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and the parameter $\theta_k \in (0, \pi/2]$ is chosen to satisfy

$$\cos\theta_k = 1 - \varepsilon_k, \quad \varepsilon = \frac{1}{2k^2}.$$

Define two unbounded non-negative self-adjoint operators as follows

$$A = \oplus \left(A_k\right)_{k\geq 1}, \quad \mathrm{dom}(A) = \left\{ \xi = (\xi_k)_{k\geq 1} \in H, \quad \sum_{k\geq 1} \|A_k \xi_k\|_k^2 < \infty \right\},$$

$$B = \oplus \left(B_k\right)_{k\geq 1}, \quad \mathrm{dom}(B) = \left\{ \eta = (\eta_k)_{k\geq 1} \in H, \quad \sum_{k\geq 1} \|B_k \eta_k\|_k^2 < \infty \right\},$$

where $\|\cdot\|_k$ *is a norm with respect to the inner product* $\langle\cdot,\cdot\rangle_k$*. One can also define the algebraic sum* $A+B = (A_k + B_k)_{k\geq 1}$ *which is symmetric and non-negative on domain* $\mathrm{dom}(A) \cap \mathrm{dom}(B)$*.*

By [24, Proposition 1], it follows that $\mathrm{dom}(A) = \mathrm{dom}(B)$ *and the algebraic sum* $A+B$ *is self-adjoint on* $\mathrm{dom}(A) \cap \mathrm{dom}(B) = \mathrm{dom}(A) = \mathrm{dom}(B)$*. Moreover, by [24, Proposition 2], there exists a positive bounded continuous function* $L(t)$ *for* $t > 0$ *independent of* $n \geq 1$ *such that*

$$\left\| \left(e^{-tB/2n} e^{-tA/n} e^{-tB/2n}\right)^n - e^{-t(A+B)} \right\|_\infty \geq \frac{L(t)}{n}, \tag{4.7}$$

for every $t > 0$ *and* $n \geq 1$*. Therefore, (4.7) shows that an error bound which is shown in Theorem 4.2.3 is, indeed, optimal.*

4.3 Product formulas with the form-sum of unbounded operators

In contrast to previous results, we consider the Trotter-Kato product formula for a more general type of sum of two unbounded operators, the so-called form-sum (see Section 2.2.3 for more details). Note that if the algebraic sum of two unbounded operators is self-adjoint on their common domain, then the form-sum is also self-adjoint; moreover, it coincides with the algebraic sum.

In [48], Neidhardt and Zagrebnov further extended their result by considering two self-adjoint operators A and B such that

$$A \geq I, \quad B \geq 0,$$
$$\mathrm{dom}(A^\alpha) \subset \mathrm{dom}(B^\alpha), \quad \text{for some} \quad \frac{1}{2} < \alpha < 1,$$

and

$$\|B^\alpha \xi\| \leq a\|A^\alpha \xi\|, \quad \xi \in \mathrm{dom}(A^\alpha), \quad 0 < a < 1.$$

Let f and g be Kato functions such that

$$\begin{aligned}
C_{1/2\gamma} &:= \sup_{x>0} \frac{x f(x)^{1/2\alpha}}{1-f(x)} < \infty, \\
C_1 &:= \sup_{x>0} \frac{1-f(x)}{x} < \infty, \\
C_2 &:= \sup_{x>0} \left| \left(f(x) - \frac{1}{1+x} \right) \frac{1}{x^2} \right| < \infty,
\end{aligned}$$

and

$$\begin{aligned}
S_1 &:= \sup_{x>0} \frac{1-g(x)}{x} < \infty, \\
S_2 &:= \sup_{x>0} \left| \left(g(x) - \frac{1}{1+x} \right) \frac{1}{x^2} \right| < \infty.
\end{aligned}$$

Moreover, assume that

$$0 < a^{1/\alpha} C_0 S_1 < 1.$$

Then, for the self-adjoint form-sum $C := A \dot{+} B$ on $\overline{\mathrm{dom}(A^{1/2}) \cap \mathrm{dom}(B^{1/2})} = H$, they proved the following

$$\begin{aligned}
&\left\| \left(f(tA/n)^{1/2} g(tB/n) f(tA/n)^{1/2} \right)^n - e^{-tC} \right\|_\infty = O\left(\frac{1}{n^{\alpha-1/2}} \right), \quad n \to \infty, \\
&\left\| (f(tA/n) g(tB/n))^n - e^{-tC} \right\|_\infty = O\left(\frac{1}{n^{\alpha-1/2}} \right), \quad n \to \infty,
\end{aligned}$$

where the convergence is uniform in $t \in (0, \infty)$. Moreover, they obtained stronger estimates

$$\begin{aligned}
&\left\| \left(f(tA/n)^{1/2} g(tB/n) f(tA/n)^{1/2} \right)^n - e^{-tC} \right\|_\infty = O\left(\frac{\ln n}{n^{2\alpha-1}} \right), \quad n \to \infty, \\
&\left\| (f(tA/n) g(tB/n))^n - e^{-tC} \right\|_\infty = O\left(\frac{\ln n}{n^{2\alpha-1}} \right), \quad n \to \infty,
\end{aligned}$$

where the convergence is uniform in $[\varepsilon, \infty)$, $\varepsilon > 0$. Additionally, if

$$\mathrm{dom}(C^\alpha) \subset \mathrm{dom}(A^\alpha), \quad \frac{1}{2} < \alpha < 1,$$

then

$$\left\| \left(f(tA/n)^{1/2} g(tB/n) f(tA/n)^{1/2} \right)^n - e^{-tC} \right\|_\infty = O\left(\frac{1}{n^{2\alpha-1}} \right), \quad n \to \infty,$$

$$\left\| (f(tA/n) g(tB/n))^n - e^{-tC} \right\|_\infty = O\left(\frac{1}{n^{2\alpha-1}} \right), \quad n \to \infty,$$

uniformly in $t \in [0, \infty)$. Note that the latter estimate of an error bound $O\left(n^{1-2\alpha}\right)$ turned out to be optimal under the given fractional conditions on operators A and B for any $\alpha \in (\frac{1}{2}, 1)$ (see [67]). Moreover, in [67], a counter-example was given. It shows that under the given fractional conditions on operators A and B with $0 < \alpha \leq \frac{1}{2}$, the convergence of the Trotter-Kato product formula in the operator norm is not possible in general. Lastly, note that a case with $\alpha = 1$ is considered in an earlier paper [37].

4.3.1 An optimal error bound

Now we present the results from [20], which further extended the results of [48] and obtained an optimal error bound assuming weaker conditions than in [48]. Authors in [20] use the methods from [23] and [24] with some modifications.

Define

$$m_f(x) := \sup_{y \geq x} f(y), \quad x > 0. \tag{4.8}$$

In [20, Theorem 1.1], authors proved the following

Theorem 4.3.1. *Let A and B be non-negative self-adjoint operators with the form-sum $C = A \dot{+} B$. Let $\beta \in (0, 1]$ be given and f and g be Kato functions such that $[f]_{2\beta} < \infty$ and $[g]_{2\beta} < \infty$ (see Definition 4.2.2) with $m_f(x) < 1$ for $x > 0$ (cf. (4.8)). If $\mathrm{dom}(C^\alpha) \subseteq \mathrm{dom}(A^\alpha) \cap \mathrm{dom}(B^\alpha)$ for some $\alpha \in (1/2, 1)$ and $\mathrm{dom}(A^{1/2}) \subseteq \mathrm{dom}(B^{1/2})$, then for any compact interval $[0, T] \subset (0, \infty)$ there exists a constant* Const *which only depends on α and $T > 0$ such that*

$$\left\| (f(tA/2n) g(tB/n) f(tA/2n))^n - e^{-tC} \right\|_\infty \leq \mathrm{Const} \cdot \frac{1}{n^{2\alpha-1}},$$

for $t \in [0, T]$ and $n \geq 1$.

Note that Theorem 4.3.1 improves the results of [48, Theorem 5.5 and Corollary 5.6], where an additional condition involving the domains of fractional powers A^α and B^α,

$\alpha \in (1/2, 1]$ was assumed. Nevertheless, the given error bound in Theorem 4.3.1 is still optimal, as shown in [67]. Although putting $\alpha = 1$, one can get the error bound shown in Theorem 4.2.3 of the previous section, the methods of [20] do not allow including the case $\alpha = 1$. Note also that in the case of $\alpha = 1$ (see, Theorem 4.2.3), the additional condition $\mathrm{dom}(A^{1/2}) \subset \mathrm{dom}(B^{1/2})$ is not needed.

4.4 Criteria for a convergence in the operator norm

In [39], Neidhardt and Zagrebnov further investigated the problem of the convergence of the Trotter-Kato product formula in the operator norm topology. They obtained some necessary and sufficient conditions using the generalization of Chernoff's theorem in the operator norm topology (see [39, Theorem 2.2] and [6, Theorem 1.1]).

We now mention some results from [39], which are helpful in the next chapter. Let $f(\cdot)$ be a Kato function. Define

$$0 \leq \varphi_0(x) := \inf_{0<s\leq x} s^{-1}\left(\frac{1}{f(s)} - 1\right), \quad 0^{-1} := +\infty$$

and

$$f_0(x) := \begin{cases} 1, & x = 0 \\ (1 + x\varphi_0(x))^{-1}, & x > 0. \end{cases}$$

Note that $f_0(\cdot)$ is also a Kato function.

They proved the following convergence result.

Theorem 4.4.1. *[39, Theorem 3.2] Let A and B be non-negative self-adjoint operators on a separable Hilbert space H with form-sum C. Let f and g be Kato functions. If $f_0(t_0A) \in \mathcal{L}_\infty$ is a compact operator for some $t_0 > 0$, then the Trotter-Kato product formula converges in the operator norm, uniformly on compact intervals of $(0, \infty)$.*

Note that if $f(x) = g(x) = e^{-x}$, $x \geq 0$, then Theorem 4.4.1 implies that the Trotter product formula (for exponential functions) converges in the operator norm uniformly on compact intervals of $(0, \infty)$ if $(I + t_0A)^{-1} \in \mathcal{L}_\infty(H)$ is a compact operator for some $t_0 > 0$.

We now recall the notion of a regular Kato function (see [40] or [2, Definition 2.10]).

Definition 4.4.2. *A Kato function $f(\cdot)$ is called regular if $0 \leq \sup_{s\in[x,+\infty)} f(s) < 1$ for*

$x > 0$ and

$$\lim_{x\to+\infty} \frac{\sup_{0\le s\le x} sf(s)}{x} = 0.$$

The authors of [39] further obtained the criterion of the convergence of the Trotter-Kato product formula in the operator norm topology using Theorem 4.4.1.

Theorem 4.4.3. *[39, Theorem 3.7] Let A be a non-negative self-adjoint operator. The Trotter-Kato product formula converges in the operator norm uniformly on compact intervals of $(0,\infty)$ for any regular Kato function $f(\cdot)$, any Kato function $g(\cdot)$ and any non-negative self-adjoint operator B if and only if $(I+A)^{-1} \in \mathcal{L}_\infty(H)$ is a compact operator.*

4.5 Abstract Cauchy problem for evolution equation

In this section, we present some preliminary material regarding the abstract Cauchy problem for evolution equation. Moreover, we overview existing results on existence of a propagator of such equation and its various approximation results in the operator norm.

4.5.1 Autonomous evolution equation

Let H be a Hilbert space and $C : \mathrm{dom}(C) \subseteq H \to H$ be a linear operator. An autonomous Cauchy problem for an evolution equation is given as follows

$$\begin{cases} \frac{du(t)}{dt} = Cu(t), \quad t \ge 0, \\ u(0) = \xi \in H, \end{cases} \tag{4.9}$$

where t is a time variable and $u(\cdot)$ is Hilbert space H valued function. Firstly, we recall the notion of a solution and well-posedness of (4.9).

Definition 4.5.1. *([36, Definition 1.1])*

1. *A function $u : [0,+\infty) \to H$ is called a (classical) solution of (4.9) if $u(\cdot)$ is continuously differentiable such that $u(t) \in \mathrm{dom}(C)$ for each $t \in [0,+\infty)$ and (4.9) holds.*

2. *A continuous function $u : [0,+\infty) \to H$ is called a mild solution of (4.9) if $\int_0^t u(\tau)d\tau \in \mathrm{dom}(C)$ for all $t \ge 0$ and $u(t) = C\int_0^t u(\tau)d\tau + \xi$.*

3. *The abstract Cauchy problem (4.9) is called well-posed if*

(i) the subspace $\mathrm{dom}(A)$ *is dense on* H *and for every* $\xi \in H$ *there exists a classical solution* $u(\cdot;\xi)$ *of* (4.9)*;*

(ii) this classical solution is unique;

(iii) for a sequence $\mathrm{dom}(C) \ni \xi_n \to 0$ *it follows that* $u(t;\xi_n) \to 0$ *uniformly on* t *in compact intervals* $[0,T]$*,* $0 < T < +\infty$.

Cauchy problem for an evolution equation in autonomous case can be fully described by Hille-Yosida's semigroup theory. Namely, the following result is true.

Proposition 4.5.2. *([36, Theorem 1.2]) An autonomous Cauchy problem for an evolution equation* (4.9) *is well-posed if and only if* $(C, \mathrm{dom}(C))$ *is the generator of a strongly continuous semigroup* $\{T(t)\}_{t\geq 0}$*. In this case, for any* $\xi \in \mathrm{dom}(C)$*, the classical solution of* (4.9) *is given by* $t \mapsto T(t)\xi = u(t)$*. Moreover, for any* $\xi \in H$*, the function* $t \mapsto T(t)\xi$ *is a mild solution of* (4.9)*.*

Note that there are other definitions of well-posedness of (4.9) which do not obtain any strongly continuous semigroup on a given Hilbert space. Since, we are not interested in these different concepts of well-posedness, we refer the reader to [4, 7, 51].

4.5.2 Non-autonomous evolution equation

In this subsection, we further consider (4.9) where the operator on the right hand side depends on time variable t. Let $C(t), t \geq 0$ be a family of linear operators on H with domains $\mathrm{dom}(C(t)), t \geq 0$. The non-autonomous Cauchy problem for an evolution equation is given as follows

$$\begin{cases} \frac{du(t)}{dt} = C(t)u(t), \quad t \geq 0, \\ u(t) = \xi_s \in H, \end{cases} \tag{4.10}$$

where u_s is an initial value. In this case, the solvability of (4.10) is not straightforward as in autonomous case and it heavily depends on the initial time s and initial value ξ_s. We first recall a definition of a classical solution and well-posedness of (4.10) is this setting.

Definition 4.5.3. *([36, Definitions 2.1 and 2.2])*

1. *For a given* $s \in [0, +\infty)$ *and* $\xi_s \in H$*, a continuous function* $u : [0, +\infty) \to H$ *is called a classical solution of* (4.10) *if* $u(\cdot)$ *is continuously differentiable such that* $u(t) \in \mathrm{dom}(C(t))$ *for all* $s \geq t$ *and* (4.10) *holds.*

2. *The abstract Cauchy problem* (4.10) *is called well-posed with regularity subpaces* $H_s, s \in [0, +\infty)$ *if*

   (i) *For each* $s \in [0, +\infty)$, *the subspace*

$$H_s := \{\eta \in H : \text{there exists a solution for } (4.10)\} \subset \operatorname{dom}(C(t))$$

   is dense in H;

   (ii) *For each* $s \in [0, +\infty)$ *and* $\eta \in H_s$ *the solution* $u(\cdot; s, \eta)$ *is unique;*

   (iii) *if* $s_n \to s \in [0, +\infty)$ *and* $H_{s_n} \ni \eta_n \to \eta \in H_s$, *then one has*

$$\|\widehat{u}(t; s_n, \eta_n) - \widehat{u}(t; s, \eta)\| \to 0$$

   uniformly for t *in compact intervals of* $[0, +\infty)$, *where*

$$\widehat{u}(t; s, \eta) := \begin{cases} u(t; s, \eta), & t \geq s, \\ \eta, & t < s. \end{cases}$$

In contrast to the autonomous case, there is no explicit necessary and sufficient conditions for a well-posedness of (4.10). However, in the series of papers there were obtained best available sufficient conditions, for which we refer the reader to the next subsection.

In this case, one need a notion of a propagator (or solution operator) of (4.10).

Definition 4.5.4. *([36, Deinition 3.1]) A family* $\{U(t, s)\}_{0 \leq s \leq t}$ *of bounded linear operators on a Hilbert space* H *is called a (strongly continuous) family of propagators (shortly, propagator) if*

(i) $U(t, r)U(r, s) = U(t, s)$ *and* $U(s, s) = I$ *for* $0 \leq s \leq r \leq t$;

(ii) *the mapping* $\{(t, s) \in [0, +\infty) \times [0, +\infty) : t \geq s\} \ni (t, s) \mapsto U(t, s)$ *is strongly continuous.*

One can easily see that if $\{T(t)\}_{t \geq 0}$ is a strongly continuous semigroup on H, then it gives rise to a propagator $U(t, s) = T(t - s), t \geq s$ with special property that all operators $U(t, s)$ commute with each other.

Definition 4.5.5. *The family of propagators* $\{U(t,s)\}_{0\le s\le t}$ *is said to be solving* (4.10) *on subspaces* H_s *if there are dense subspaces* H_s *such that* $U(t,s)H_s \subset H_t \subset \mathrm{dom}(C(t))$ *for* $t \ge s$ *and a function* $u(\cdot; s, \xi_s) : t \mapsto U(t,s)\xi_s$ *is a classical solution of* (4.10) *for* $s \in [0, +\infty)$ *and* $\xi_s \in H_s$.

Using the notion of a propagator solving non-autonomous Cauchy problem (4.10), one can give the following equivalency of well-posedness.

Proposition 4.5.6. *A non-autonomous Cauchy problem* (4.10) *is well-posed with regularity subspaces* H_s *if and only if there is a family of propagators* $\{U(t,s)\}_{0\le s\le t}$ *solving* (4.10) *on subspaces* H_s.

Hence, in order to solve the non-autonomous evolution equation (4.10), one can consider the problem of finding the family of propagators solving (4.10). Therefore, in the next subsection, we recall some important results presenting sufficient conditions of existence of a propagator.

4.5.3 Results about the existence of a propagator

Let H be a complex Hilbert space and consider the following non-autonomous evolution equation

$$\frac{du(t)}{dt} = -A(t)u(t), \quad 0 \le t \le T, \tag{4.11}$$

where the unknown $u(t)$ is a Hilbert space H valued function and $A(t)$ is a (possibly unbounded) linear operator on H. In [68], the non-autonomous evolution equation (4.11) was considered under the following conditions

(i) The operators $-A(t)$ are the generators of some semigroups of bounded operators for each $t \in [0, T]$.

(ii) The domain $D = \mathrm{dom}(A(t))$, $0 \le t \le T$ is independent of time t. Note that an operator $B(t,s) := (I + A(t))(I + A(s))^{-1}$ is bounded for fixed $t, s \in [0, T]$ since $\mathrm{Ran}((I + A(s))^{-1}) = \mathrm{dom}(A(s)) = D = \mathrm{dom}(A(t))$. Assume that $B(t,s)$ is uniformly bounded for $0 \le t, s \le T$, i.e.,

$$\sup_{0\le t,s\le T} \|B(t,s)\|_\infty < +\infty.$$

Moreover, $B(t,s)$ is Lipschitz continuous in t for every s in the operator norm, i.e., for $0 \le s \le T$

$$\|B(t,s) - B(\tau,s)\|_\infty \le \text{Const} \cdot |t-\tau|, \quad 0 \le t, \tau \le T.$$

(iii) $B(t,s)$ is strongly continuously differentiable in t for every s.

(iv) For each $0 \le s \le T$ and $t > 0$, $(d/dt)e^{-tA(s)}$ is a bounded operator and there exist positive constants Const and t_0 such that

$$\left\| \frac{d}{dt} e^{-tA(s)} \right\|_\infty = \left\| A(s) e^{-tA(s)} \right\|_\infty \le \frac{\text{Const}}{t},$$

for any s and $t \le t_0$.

Note that even though the assumptions *(i)* – *(iii)* are enough for the existence of an unique propagator family $U_0(t,s)$, by [28, Theorem 1 and Theorem 2], the additional more restrictive condition *(iv)* makes it easy to deduce various properties of the propagator family $U_0(t,s)$.

In [68, Theorem 1.1], the following important existence result of a classical solution of (4.11) was obtained.

Proposition 4.5.7. *Let the assumptions (i)-(iv) be satisfied. Then, there exists a propagator $U_0(t,s)$ for $0 \le s \le t \le T$, which is strongly jointly continuous in t and s and strongly differentiable in t for each fixed $s < t \le T$. The propagator $U_0(t,s)$ is an unique solution of (4.11), i.e., $\frac{\partial}{\partial t}U_0(t,s)$ and $A(t)U_0(t,s)$ are bounded operators for $s < t$ and*

$$\frac{\partial}{\partial t} U_0(t,s) = -A(t)U_0(t,s), \quad U(s,s) = I.$$

Moreover, there exists a positive constant Const *such that*

$$\left\| \frac{\partial}{\partial t} U_0(t,s) \right\|_\infty = \|A(t)U_0(t,s)\|_\infty \le \text{Const} \cdot (t-s)^{-1},$$

for $0 \le s < t \le T$.

Note that a propagator in Proposition 4.5.7 is identical to the one constructed by Kato [28]. Note also that if non-negative self-adjoint operators $A(t) = A$, $0 \le t \le T$ are independent of time variable t with domain $\text{dom}(A(t)) = \text{dom}(A)$. Then, the assumptions (i)-(iv) are satisfied automatically and as a result of Proposition 4.5.7, there exists a solution in the following form

$$U_0(t,s) = e^{-(t-s)A}, \quad 0 \le s \le t \le T.$$

Now consider the following perturbed non-autonomous evolution equation

$$\frac{du(t)}{dt} = -\left(A(t) + B(t)\right) u(t), \quad 0 \le t \le T. \tag{4.12}$$

Additional to the assumptions (i)-(iv) we consider the following assumptions

(i') The operators $A(t)$, $0 \le t \le T$ have a common domain D. Moreover, the operators $B(t)$, $0 \le t \le T$ are closed such that $D = \text{dom}(A(t)) \subset \text{dom}(B(t))$, $0 \le t \le T$.

(ii') The operator $B(t)A(s)^{-1}$ is continuous in $0 \le t \le T$ for each s in the operator norm.

(iii') There exists constants Const > 0 and $\rho \le 1$ such that

$$\left\| B(t)e^{-\tau A(s)} \right\|_\infty \le \frac{\text{Const}}{\tau^{1-\rho}},$$
$$\left\| (B(t) - B(t'))e^{-\tau A(s)} \right\|_\infty \le \frac{\text{Const}}{|t-t'|^{1-\rho}}$$

for $0 \le t, t', s \le T$ and $\tau > 0$.

The solution of (4.12) can formally be constructed as a following Dyson-Phillips series

$$U(t,s) = \sum_{k=0}^{\infty} U_k(t,s),$$

where $U_0(t,s)$ is a solution of (4.11) and

$$U_k(t,s) = -\int_s^t U_0(t,\sigma)B(\sigma)U_{k-1}(\tau,s)d\sigma, \quad k \ge 1.$$

In fact, in [68], Tanabe showed the following

Proposition 4.5.8. *Let the assumptions (i)-(iv) and (i')-(iii') be satisfied. Then, the solution $U(t,s)$, $0 \le s \le t \le T$ defined as above Dyson-Phillips series is an unique solution of (4.12) which is strongly jointly continuous in t and s for $0 \le s \le t \le T$ and strongly differentiable in t for each fixed $s < t \le T$. The operators $\frac{\partial}{\partial t}U(t,s)$ and $(A(t)+B(t))U(t,s)$*

are bounded operators and

$$\frac{\partial}{\partial t}U(t,s) = -(A(t)+B(t))U(t,s), \quad U(s,s) = I.$$

Moreover, there exists a constant Const > 0 *such that*

$$\begin{aligned}&\left\|\frac{\partial}{\partial t}U(t,s)\right\|_\infty \le \frac{\text{Const}}{t-s}, \quad \|A(t)U(t,s)\|_\infty \le \frac{\text{Const}}{t-s},\\ &\|B(t)U(t,s)\|_\infty \le \frac{\text{Const}}{t-s}.\end{aligned}$$

Note that if $A(t)+B(t)$ has the form $A+B(t)$ and satisfy the Assumptions **(A1)**-**(A4)** (see Section 6.1), then the Assumptions (i)-(iv) and (i')-(iii') are satisfied automatically and the above Proposition 4.5.8 implies that there exists a unique propagator constructed as above Dyson-Phillips series, where $U_0(t,s) = e^{-(t-s)A}$. The given propagator is a solution of (4.12) with $A+B(t)$ on the right hand side.

In [64, Theorem 1], Sobolevskiĭ proved a similar existence result of a propagator in Hilbert space H under the different conditions. We use this result in Chapter 6 to prove the existence of a propagator $U(t,s)$ in symmetric normed ideals.

Theorem 4.5.9. *Let $A(t)$, $t \in [0,T]$ be a family of linear operators on H with time independent domain $D := \mathrm{dom}(A(t))$, $t \in [0,T]$. Let the operator $A(\cdot)A^{-1}(\tau)$ be a Hölder continuous function in $[0,T]$ for any $\tau \in [0,T]$. Assume that for any λ with $\mathrm{Re}\lambda \ge 0$, the operator $A(t)+\lambda I$ is invertible and*

$$\left\|(A(t)+\lambda I)^{-1}\right\|_\infty \le \frac{\text{Const}}{|\lambda|+1}.$$

Then, there exists an operator valued function $U(t,s)$ which is jointly continuous in t and s for $0 \le s \le t \le T$ in the strong operator topology. Furthermore, it is differentiable in t for $t > s$ in the strong operator topology and

$$\frac{\partial U(t,s)}{\partial t} = -A(t)U(t,s).$$

Moreover, $U(t,s)$ satisfy the following property

$$U(t,s) = U(t,\tau)U(\tau,s), \quad U(s,s) = I, \quad 0 \le s \le \tau \le t \le T.$$

The expression $u(t) = U(t,0)\xi_0$ defines a continuous and continuously differentiable in $t \in [0,T]$ solution of (4.11) with initial value condition $u(0) = u_0$. For any $\xi_0 \in H$, the same expression defines a continuous in $t \in [0,T]$ and differentiable in $t \in (0,T]$ solution of (4.11) with initial value condition $u(0) = u_0$.

For more detailed investigation of the operator $U(t,s)$ and its various properties such as differentiability in the second variable and the inequalities involving $U(t,s)$ and fractional powers of $A(t)$, we refer the reader to [64].

4.5.4 Approximation formula for a propagator

In this subsection we recall some existing results on the approximation formula for a propagator of an abstract Cauchy problem for non-autonomous evolution equation

$$\begin{cases} \frac{du(t)}{dt} = -(A(t)+B(t))u(t), \\ u(s) = \xi_s \in H, \end{cases} \qquad 0 \le s \le t \le T. \tag{4.13}$$

Note that some results presented in this subsection were initially considered in Banach spaces. However, for convenience, we consider them in Hilbert spaces.

The first result in this direction was obtained by Faris [12], where he proved the approximation formula in the strong operator topology. Namely, he proved the following

Theorem 4.5.10. *Let $-A(t)$ and $-B(t)$ are the infinitesimal generators of the strongly continuous semigroups $\{e^{-\tau A(t)}\}_{\tau\ge 0}$ and $\{e^{-\tau B(t)}\}_{\tau \ge 0}$ for each $t \in [0,T]$. Let also $A(t) + B(t)$ be a closed operator with a time independent dense domain D in H for each $t \in [0,T]$. Let $\{U(t,s)\}_{0\le s\le t\le T}$ be a propagator consisting contractions and solving (4.13) on D (see, Definition 4.5.5). Then, for $t \in [0,T]$, one has*

$$U(t,0) = \lim_{n\to\infty} \prod_{k=n-1}^{0} \exp\left(-A\left(\frac{kt}{n}\right)\frac{t}{n}\right)\cdot \exp\left(-B\left(\frac{kt}{n}\right)\frac{t}{n}\right),$$

where the convergence holds in the strong operator topology.

Further improved results in the strong operator topology were obtained in [71] and [70]. Authors supposed more detailed assumptions on operators $A(t)$ and $B(t)$ to prove the existence of a propagator $\{U(t,s)\}_{0\le s\le t\le T}$ which is continuous and continuously differentiable in both t and s, and satisties (4.13) in the operator norm. Moreover, they proved the following approximation formula (similar to the one in Theorem 4.5.10) in the strong operator topology

$$U(t,s) = \lim_{n\to\infty} \prod_{k=n-1}^{0} \exp\left(-\frac{t-s}{n} A\left(s + \frac{k(t-s)}{n}\right)\right) \cdot \exp\left(-\frac{t-s}{n} B\left(s + \frac{k(t-s)}{n}\right)\right).$$

The first approximation formula in the operator norm topology was proved in [22]. Ichinose and Tamura proved the approximation formula in terms of error bounds. They assumed the operators $A(t) = A$ be independent of time variable and proved the following

Theorem 4.5.11. *Let A and $B(t)$, $t \in [0,T]$ be strictly positive self-adjoint operators on a Hilbert space H. Let also that $\mathrm{dom}(A^\alpha) \subset \mathrm{dom}(B(t))$, $t \in [0,T]$ for some $\alpha \in [0,1)$ independent of t. Assume that $B(t)A^{-\alpha} : H \to H$ is uniformly bounded and an operator valued function $[0,T] \ni t \mapsto A^{-\alpha}B(t)A^{-\alpha}$ is Lipschitz continuous. Then, one has the following formula for a propagator*

$$\left\| U(t,0) - \prod_{k=n-1}^{0} \exp\left(-\frac{tA}{2n}\right) \cdot \exp\left(-\frac{t}{n} B\left(\frac{kt}{n}\right)\right) \cdot \exp\left(-\frac{tA}{2n}\right) \right\|_\infty = O\left(\frac{\log n}{n}\right),$$

as $n \to \infty$, where the convergence holds uniformly in $t \in [0,T]$.

Note that the above assumption of strictly positivity is not essential and one might assume that the operators involved are semi-bounded uniformly in t. Furthermore, one can use the same approximating family as in Theorem 4.5.10 and prove the following convergence (see, [22, Section 7]):

$$\left\| U(t,0) - \prod_{k=n-1}^{0} \exp\left(-\frac{tA}{n}\right) \cdot \exp\left(-\frac{t}{n} B\left(\frac{kt}{n}\right)\right) \right\|_\infty = O\left(\frac{\log n}{n}\right), \quad n \to \infty,$$

Later, there were series of papers on the direction of approximation formula for a propagator in the operator norm topology. The common part of those papers are using the "relatively" new method introduced by Howland and Evans [17, 11], and Neidhardt [41, 42, 43]. This method does not use any approximants. However, its main point is to reformulate non-autonomous evolution equation as autonomous one on the Banach space $L^p([0,T], H)$. Note that these results used modified versions of the previous assumptions.

For example, in [47, Theorem 5.6], Neidhardt, Stephan and Zagrebnov proved the following in case of $A(t) = A$, $t \in [0,T]$ being independent of the time variable t

Theorem 4.5.12. *Let the operators $A \geq I$ and $B(t), t \in [0,T]$ be positive self-adjoint operators on a separable Hilbert space H. Assume that there exists $\alpha \in (\frac{1}{2}, 1)$ such that $D(A^\alpha) \subseteq D(B(t))$ for a.e. $t \in [0,1]$ and the function $B(\cdot)A^{-\alpha} : [0,1] \to \mathcal{L}(H)$ is strongly measurable and essentially bounded in the operator norm. Moreover, assume that the mapping $[0,T] \ni t \mapsto A^{-\alpha}B(t)A^{-\alpha} \in \mathcal{L}(H)$ is Lipschitz continuous. Then, one has*

$$\left\| U(t,s) - \prod_{k=n-1}^{1} \exp\left(-\frac{(t-s)A}{n}\right) \cdot \exp\left(-\frac{t-s}{n} B\left(s + \frac{k(t-s)}{n}\right)\right) \right\|_\infty = O\left(\frac{1}{n^{1-\alpha}}\right),$$

as $n \to \infty$.

In [44, Theorem 7.11], the same authors further extended the result as follows

Theorem 4.5.13. *Let the operators $A \geq I$ and $B(t)$, $t \in [0,T]$ be positive self-adjoint operators on a separable Hilbert space H. Assume that there exists $\alpha \in (0,1)$ such that $D(A^\alpha) \subseteq D(B(t))$ for a.e. $t \in [0,1]$ and the function $B(\cdot)A^{-\alpha} : [0,1] \to \mathcal{L}(H)$ is strongly measurable and essentially bounded in the operator norm. Moreover, assume that the mapping $[0,T] \ni t \mapsto A^{-1}B(t)A^{-\alpha} \in \mathcal{L}(H)$ is Hölder continuous with Hölder exponent $\beta \in (\alpha, 1)$. Then, one has*

$$\left\| U(t,s) - \prod_{k=n-1}^{1} \exp\left(-\frac{(t-s)A}{n}\right) \cdot \exp\left(-\frac{t-s}{n} B\left(s + \frac{k(t-s)}{n}\right)\right) \right\|_\infty = O\left(\frac{1}{n^{\beta-\alpha}}\right),$$

as $n \to \infty$.

Another work of these authors further improved the Hölder continuity case with a better estimate of approximation formula (see, [46, Theorem 3.14]). Namely, they proved the following

Theorem 4.5.14. *Let the operators $A \geq I$ and $B(t)$, $t \in [0,T]$ be positive self-adjoint operators on a separable Hilbert space H. Assume that there exists $\alpha \in (0,1)$ such that $D(A^\alpha) \subseteq D(B(t))$ for a.e. $t \in [0,1]$ and the function $B(\cdot)A^{-\alpha} : [0,1] \to \mathcal{L}(H)$ is strongly measurable and essentially bounded in the operator norm. Moreover, assume that the mapping $[0,T] \ni t \mapsto A^{-\alpha}B(t)A^{-\alpha} \in \mathcal{L}(H)$ be Hölder continuous with Hölder exponent $\beta \in (2\alpha - 1, 1)$. Then, one has*

$$\left\| U(t,s) - \prod_{k=n-1}^{1} \exp\left(-\frac{(t-s)A}{n}\right) \cdot \exp\left(-\frac{t-s}{n} B\left(s + \frac{k(t-s)}{n}\right)\right) \right\|_\infty = O\left(\frac{1}{n^{\beta}}\right),$$

as $n \to \infty$.

We further refer the reader to [1, Section 7] and [71, 70] for a detailed discussion of various assumptions under which the evolution equation (4.13) and the approximation formula of its propagator were considered.

Chapter 5

Trotter-Kato product formula in symmetric operator ideals

In this chapter, we extend the convergence of the Trotter-Kato product formula for arbitrary symmetrically F-normed ideal closed with respect to the logarithmic submajorization. The latter class of ideals contains all symmetric (quasi-)Banach ideals of compact operators. Hence, the results of this chapter extend the results of Hiai [15], Neidhardt and Zagrebnov [40].

This chapter is based on the results of [2] and is organised as follows. In Section 5.1, we first introduce the notion of convergence in this case. Furthermore, we show the equivalence of the Trotter-Kato product formula for various families generated by pair of Kato functions. In Section 5.2, we present the lifting method which helps to obtain the convergence in symmetrically F-normed ideals closed with respect to the logarithmic submajorization via the similar convergence in the operator norm topology. Section 5.3 consists of the main results of this chapter, the Trotter-Kato product formula in symmetrically F-normed ideals closed with respect to the logarithmic submajorization. Moreover, we present a criterion of the latter convergence for a suitable class of Kato functions. In Section 5.4, we describe the error bound of the Trotter-Kato product formula in symmetrically F-normed ideal closed with respect to the logarithmic submajorization and present some examples where it can be computed directly.

5.1 An equivalence of the Trotter-Kato product formula for various families

In this section, we first introduce the notion of convergence of the Trotter-Kato product formula in the norm of a symmetrically F-normed ideal. Moreover, we show an equivalence result regarding the Trotter-Kato product formula for various families generated by Kato functions.

Throughout this chapter, we assume that A and B are non-negative self-adjoint operators on a separable Hilbert space H and $C = A\dot{+}B$ is the form-sum of A and B (see Subsection 2.2.3). Moreover, we assume that $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ is an arbitrary symmetrically F-normed ideal closed with respect to the logarithmic submajorization (see Definitions 3.1.2 and 3.2.1).

We now introduce a notion of a convergence of the Trotter-Kato product formula for various families generated by Kato functions.

Definition 5.1.1. *Let $f(\cdot)$, $g(\cdot)$ be Kato functions (see Definition 4.2.1). Then we say that*

(i) the Trotter-Kato product formula for the family $\{f(tA)g(tB)\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ if for any bounded interval $[\tau_0, \tau] \subset (t_0, \infty)$ there is a natural number $n_0 \geq 1$ such that

$$e^{-tC} \in \mathcal{I}(H_0), \quad (f(tA/n)g(tB/n))^n \in \mathcal{I}(H),$$

for any $t \in [\tau_0, \tau]$ and $n \geq n_0$, and the convergence

$$\lim_{n\to\infty} \sup_{t\in[\tau_0,\tau]} \left\|(f(tA/n)g(tB/n))^n - e^{-tC}P_0\right\|_{\mathcal{I}} = 0$$

holds.

(ii) the Trotter-Kato product formula for the symmetrized family $\{g(tB)^{1/2} f(tA)g(tB)^{1/2}\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ if for any bounded interval $[\tau_0, \tau] \subset (t_0, \infty)$ there is a real number $r_0 \geq 1$ such that

$$e^{-tC} \in \mathcal{I}(H_0), \quad \left(g(tB/r)^{1/2}f(tA/r)g(tB/r)^{1/2}\right)^r \in \mathcal{I}(H),$$

for any $t \in [\tau_0, \tau]$ and $r \geq r_0$, and the convergence

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \left\| \left(g(tB/r)^{1/2} f(tA/r) g(tB/r)^{1/2} \right)^r - e^{-tC} P_0 \right\|_{\mathcal{I}} = 0$$

holds.

In a similar way, one can define the notion of convergence for the families $\{g(tB)f(tA)\}_{t\geq 0}$ and $\{f(tA)^{1/2}g(tB)f(tA)^{1/2}\}_{t\geq 0}$ by exchanging f with g and A with B. Moreover, if the convergence holds for all these families, then we say that the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ for all families generated by $f(\cdot)$ and $g(\cdot)$.

In the case when $\|\cdot\|_{\mathcal{I}}$ is the operator norm $\|\cdot\|_\infty$ we say that the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in the operator norm.

For the convenience, we introduce the following notations:

$$F(t) = g(tB)^{1/2} f(tA) g(tB)^{1/2}, \quad t \geq 0,$$
$$G(t) = f(tA)^{1/2} g(tB) f(tA)^{1/2}, \quad t \geq 0.$$

First we want to demonstrate that it is sufficient to consider the convergence of the Trotter-Kato product formula just for one of the families $\{F(t)\}_{t\geq 0}$, $\{G(t)\}_{t\geq 0}$, $\{f(tA)g(tB)\}_{t\geq 0}$ and $\{g(tB)f(tA)\}_{t\geq 0}$. We prove a proposition similar to [40, Proposition 3.1] that establishes equivalence of the convergences of the Trotter-Kato product formula for different families considered in Definition 5.1.1.

Proposition 5.1.2. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and $\mathcal{I}(H)$ be any symmetrically* F*-normed ideal closed with respect to the logarithmic submajorization. Let also $f(\cdot)$ and $g(\cdot)$ be Kato-functions. Then, the following assertions are equivalent:*

- *(i) The Trotter-Kato product formula for the family $\{F(t)\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$;*
- *(ii) The Trotter-Kato product formula for the family $\{G(t)\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$;*
- *(iii) The Trotter-Kato product formula for the family $\{f(tA)g(tB)\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$;*

(iv) The Trotter-Kato product formula for the family $\{g(tB)f(tA)\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$.

Proof. First we show that (i) implies (ii). Fix a bounded interval $[\tau_0, \tau] \subset (t_0, +\infty)$. Choose an interval $[a, b] \subset (t_0, \infty)$ such that $[\tau_0, \tau] \subset (a, b)$. Assume a real number $r \in \mathbb{R}_+$ is given, and it can be decomposed as $r = [r] + \{r\}$ for integer and fractional parts. For any $t > 0$ we can write

$$G(t/r)^r = G(t/r)^{\{r\}} f(tA/r)^{1/2} g(tB/r)^{1/2} F(t/r)^{[r]-1} g(tB/r)^{1/2} f(tA/r)^{1/2}. \tag{5.1}$$

Note that $F(t/r)^{[r]-1} = F(\theta/([r]-1))^{[r]-1}$, where $\theta = t([r]-1)/r$. Since the assumption of the proposition is true for the family $\{F(t)\}_{t\geq 0}$, for the interval $[a, b]$ we can find a number $R_1 \in \mathbb{N}$ such that $F(t/r)^{[r]-1} = F(\theta/([r]-1))^{[r]-1} \in \mathcal{I}(H)$ for any $\theta \in [a, b]$ and $[r] \geq R_1$. Note that condition $\theta \in [a, b]$ is equivalent to $t \in [ra/([r]-1), rb/([r]-1)]$. However, since $[\tau_0, \tau] \subset (a, b)$ and $\frac{r}{[r]-1} \to 1$ as $r \to \infty$, it follows that there exists a large enough number $R_2 \in \mathbb{R}_+$ such that

$$[\tau_0, \tau] \subset [ra/([r]-1), b] \subseteq [ra/([r]-1), rb/([r]-1)]$$

for any $r \geq R_2$. Therefore, if $r \geq \max\{R_1, R_2\}$, then we have that $F(t/r)^{[r]-1} \in \mathcal{I}(H)$ for any $t \in [\tau_0, \tau]$ and $r \geq R_{\max}$. Thus, (5.1) implies that $G(t/r)^r \in \mathcal{I}(H)$ for any $t \in [\tau_0, \tau]$ and $r \geq \max\{R_1, R_2\}$. Similarly, since $e^{-\theta C} \in \mathcal{I}(H_0)$ for any $\theta \in [a, b]$, it follows that $e^{-tC} \in \mathcal{I}(H_0)$ for any $t \in [\tau_0, \tau]$.

We have left to show the convergence from Definition 5.1.1(ii). Note that, for any $t \geq 0$ and $r \in \mathbb{R}_+$ we can write

$$\begin{aligned}
&G(t/r)^r - e^{-tC}P_0 \\
&= G(t/r)^{\{r\}} f(tA/r)^{1/2} g(tB/r)^{1/2} \left[F(t/r)^{[r]-1} - e^{-tC}P_0\right] g(tB/r)^{1/2} f(tA/r)^{1/2} \\
&+ G(t/r)^{\{r\}} f(tA/r)^{1/2} g(tB/r)^{1/2} e^{-tC}P_0 \left[g(tB/r)^{1/2} - I\right] f(tA/r)^{1/2} \\
&+ G(t/r)^{\{r\}} f(tA/r)^{1/2} g(tB/r)^{1/2} e^{-tC}P_0 \left[f(tA/r)^{1/2} - I\right] \\
&+ G(t/r)^{\{r\}} f(tA/r)^{1/2} \left[g(tB/r)^{1/2} - I\right] e^{-tC}P_0 \\
&+ G(t/r)^{\{r\}} \left[f(tA/r)^{1/2} - I\right] e^{-tC}P_0 + \left[G(t/r)^{\{r\}} - I\right] e^{-tC}P_0.
\end{aligned}$$

Therefore, using the triangle inequality and (i), noting that $\left\|G(t/r)^{\{r\}}\right\|_\infty \le 1$, $\left\|f(tA/r)^{1/2}\right\|_\infty \le 1$ and $\left\|g(tB/r)^{1/2}\right\|_\infty \le 1$ for any $t>0$ and $r \in \mathbb{R}_+$, we have

$$
\begin{aligned}
&\sup_{t\in[\tau_0,\tau]} \left\|G(t/r)^r - e^{-tC}P_0\right\|_{\mathcal{I}} \le \sup_{t\in[\tau_0,\tau]} \left\|F(t/r)^{[r]-1} - e^{-t([r]-1)C/r}P_0\right\|_{\mathcal{I}} \\
&+ \sup_{t\in[\tau_0,\tau]} \left\|e^{-t([r]-1)C/r}P_0 - e^{-tC}P_0\right\|_{\mathcal{I}} + \sup_{t\in[\tau_0,\tau]} \left\|e^{-tC}P_0\left(I - g(tB/r)^{1/2}\right)\right\|_{\mathcal{I}} \\
&+ \sup_{t\in[\tau_0,\tau]} \left\|e^{-tC}P_0\left(I - f(tA/r)^{1/2}\right)\right\|_{\mathcal{I}} + \sup_{t\in[\tau_0,\tau]} \left\|\left(I - g(tB/r)^{1/2}\right)e^{-tC}P_0\right\|_{\mathcal{I}} \\
&+ \sup_{t\in[\tau_0,\tau]} \left\|\left(I - f(tA/r)^{1/2}\right)e^{-tC}P_0\right\|_{\mathcal{I}} + \sup_{t\in[\tau_0,\tau]} \left\|\left(I - G(t/r)^{\{r\}}\right)e^{-tC}P_0\right\|_{\mathcal{I}}.
\end{aligned}
\tag{5.2}
$$

Since $[\tau_0,\tau] \subset (a,b)$ there exists small enough number $\delta > 0$ such that $\tau_0 - \delta > a$ and $e^{-tC}P_0 = e^{-(\tau_0-\delta)C}e^{-(t-\tau_0+\delta)C}P_0$, where $e^{-(\tau_0-\delta)C} \in \mathcal{I}(H_0)$ and $e^{-(t-\tau_0+\delta)C}P_0 \in \mathcal{L}_\infty(H)$. Moreover, the spectral theorem implies that $\text{s-lim}_{t\to+0} f(tA)^{1/2} = I$, $\text{s-lim}_{t\to+0} g(tB)^{1/2} = I$, and, additionally, $\text{s-lim}_{t\to+0} G(t) = I$. Therefore, by Lemma 3.3.4, we have

$$
\begin{aligned}
&\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \left\|e^{-tC}P_0\left(I - g(tB/r)^{1/2}\right)\right\|_{\mathcal{I}} \\
&\qquad = \lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \left\|\left(I - g(tB/r)^{1/2}\right)e^{-tC}P_0\right\|_{\mathcal{I}} = 0
\end{aligned}
\tag{5.3}
$$

and

$$
\begin{aligned}
&\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \left\|e^{-tC}P_0\left(I - f(tA/r)^{1/2}\right)\right\|_{\mathcal{I}} \\
&\qquad = \lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \left\|\left(I - f(tA/r)^{1/2}\right)e^{-tC}P_0\right\|_{\mathcal{I}} = 0
\end{aligned}
\tag{5.4}
$$

and

$$
\begin{aligned}
&\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \left\|\left(I - G(t/r)^{\{r\}}\right)e^{-tC}P_0\right\|_{\mathcal{I}} \\
&\qquad \le \lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \left\|\left(I - G(t/r)\right)e^{-tC}P_0\right\|_{\mathcal{I}} = 0,
\end{aligned}
\tag{5.5}
$$

where the last inequality follows from the symmetricity of the F-norm.

Therefore, using (5.3), (5.4), (5.5) and (5.2), we obtain that

$$\begin{aligned}
&\lim_{r\to\infty}\sup_{t\in[\tau_0,\tau]}\left\|G(t/r)^r - e^{-tC}P_0\right\|_{\mathcal{I}} \\
&\qquad\le \lim_{r\to\infty}\sup_{t\in[\tau_0,\tau]}\left\|F(t/r)^{[r]-1} - e^{-t([r]-1)C/r}P_0\right\|_{\mathcal{I}} \qquad (5.6)\\
&\qquad\qquad + \lim_{r\to\infty}\sup_{t\in[\tau_0,\tau]}\left\|e^{-t([r]-1)C/r}P_0 - e^{-tC}P_0\right\|_{\mathcal{I}}.
\end{aligned}$$

We estimate two terms of (5.6) separately. Since the Trotter-Kato product formula converges for the family $\{F(t)\}_{t\geq 0}$ and the interval $[a,b]$, we have

$$\lim_{r\to\infty}\sup_{\theta\in[a,b]}\left\|F(\theta/([r]-1))^{[r]-1} - e^{-\theta C}P_0\right\|_{\mathcal{I}} = 0,$$

which is equivalent to say that

$$\lim_{r\to\infty}\sup_{t([r]-1)/r\in[a,b]}\left\|F(t/r)^{[r]-1} - e^{-t([r]-1)C/r}P_0\right\|_{\mathcal{I}} = 0,$$

and, hence,

$$\lim_{r\to\infty}\sup_{t\in[\tau_0,\tau]}\left\|F(t/r)^{[r]-1} - e^{-t([r]-1)C/r}P_0\right\|_{\mathcal{I}} = 0, \qquad (5.7)$$

since $[\tau_0,\tau]\subset[ra/([r]-1), rb/([r]-1)]$ for $r\geq\max\{R_1,R_2\}$. Hence, the first term of (5.6) is equal to zero. For the second term, since $[\tau_0,\tau]\subset(a,b)$ and $\frac{[r]-1}{r}\to 1$ as $r\to\infty$, we can find $\varepsilon>0$ and $R_3\in\mathbb{R}_+$ such that $t([r]-1)/r-\varepsilon\in[a,b]$ for any $t\in[\tau_0,\tau]$ and $r\geq R_3$. Hence, $e^{-t([r]-1)C/r}P_0 = e^{-(t([r]-1)/r-\varepsilon)C}P_0e^{-\varepsilon C}P_0$, where $e^{-(t([r]-1)/r-\varepsilon)C}P_0\in\mathcal{I}(H)$ and $e^{-\varepsilon C}P_0\in\mathcal{L}_\infty(H)$. Therefore, by Lemma 3.3.4 for the second term of (5.6) we have

$$\lim_{r\to\infty}\sup_{t\in[\tau_0,\tau]}\left\|e^{-t([r]-1)C/r}P_0 - e^{-tC}P_0\right\|_{\mathcal{I}}$$

$$= \lim_{r\to\infty}\sup_{t\in[\tau_0,\tau]}\left\|e^{-(t([r]-1)/r-\varepsilon)C}P_0e^{-\varepsilon C}P_0\left(I - e^{-(t-t([r]-1)/r))C}P_0\right)\right\|_{\mathcal{I}} = 0$$

which together with (5.7) applied to (5.6) proves the convergence. Therefore, for arbitrarily given interval $[\tau_0,\tau]$ we find a number $R_{\max}:=\max\{R_1,R_2,R_3\}$ such that the convergence of the Trotter-Kato product formula locally uniformly away from $t_0>0$ in $\mathcal{I}(H)$ holds for the family $\{G(t)\}_{t\geq 0}$.

Note that we have the equalities

$$(f(tA/n)g(tB/n))^n = f(tA/n)^{1/2}G(t/n)^{n-1}f(tA/n)^{1/2}g(tB/n)$$

$$(g(tB/n)f(tA/n))^n = g(tB/n)\left(f(tA/n)g(tB/n)\right)^{n-1} f(tA/n)$$

and

$$F(t/r)^r = F(t/r)^{\{r\}}g(tB/r)^{1/2}f(tA/r)\left(g(tB/r)f(tA/r)\right)^{[r]-1} g(tB/r)^{1/2},$$

where $r = [r] + \{r\}$. Therefore, the proof of implications (ii)⇒(iii), (iii)⇒(iv) and (iv)⇒(i) is similar to the proof above, hence, is omitted. □

5.2 Lifting results

In this section, we present a lifting result similar to [40, Proposition 3.2] which implies the convergence of the Trotter-Kato product formula in symmetrically F-normed ideals from operator-norm convergence. Since the convergence of the Trotter-Kato product formula in symmetrically F-normed ideals for different families hold if convergence holds for one of these families (as established in Proposition 5.1.2), it is sufficient to show the convergence for the family $\{F(t)\}_{t\geq 0}$.

Proposition 5.2.1. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H. Let also $\mathcal{I}(H)$ be symmetrically* F*-normed ideal closed with respect to the logarithmic submajorization and $f(\cdot)$, $g(\cdot)$ be Kato functions. Let some real number $t_0 > 0$ be given. Assume that*

(i) the Trotter-Kato product formula for $\{F(t)\}_{t\geq 0}$ converges locally uniformly away from zero in the operator norm;

(ii) for any bounded interval $[\tau_0, \tau] \subset (t_0, +\infty)$ there exists a number $r_0 \geq 1$ such that $F(t/r)^r \in \mathcal{I}(H)$ for any $t \in [\tau_0, \tau]$ and $r \geq r_0$ and

$$M([\tau_0, \tau]) := \sup_{r\geq r_0} \sup_{t\in[\tau_0,\tau]} \|F(t/r)^r\|_{\mathcal{I}} < +\infty;$$

(iii) $e^{-tC} \in \mathcal{I}(H_0)$ for $t > t_0$.

Then the Trotter-Kato product formula for the family $\{F(t)\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$.

Proof. Fix a bounded interval $[\tau_0, \tau] \subset (t_0, \infty)$ and arbitrary $\varepsilon > 0$. Then there exists a number $\alpha \in (0,1)$ such that $\tau_0' := \alpha\tau_0 > t_0$. It is clear that for any $t \in [\tau_0, \tau]$ we have that $\alpha t \in [\tau_0', \tau]$. Denoting by $E_C(\cdot)$ the spectral measure of C, we can write

$$F(t/r)^r - e^{-tC}P_0 = (F(t/r)^{(1-\alpha)r} - e^{-(1-\alpha)tC}P_0)F(t/r)^{\alpha r}$$

$$+e^{-(1-\alpha)tC}E_C([0,N))P_0(F(t/r)^{\alpha r} - e^{-\alpha tC}P_0)$$

$$+e^{-(1-\alpha)tC}E_C([N,\infty))P_0(F(t/r)^{\alpha r} - e^{-\alpha tC}P_0).$$

Using the triangle inequality, we infer

$$\begin{aligned}
&\sup_{t\in[\tau_0,\tau]} \left\|F(t/r)^r - e^{-tC}P_0\right\|_{\mathcal{I}} \\
&\quad\leq \sup_{t\in[\tau_0,\tau]} \left\|(F(t/r)^{(1-\alpha)r} - e^{(1-\alpha)tC}P_0)F(t/r)^{\alpha r}\right\|_{\mathcal{I}} \\
&\quad+ \sup_{t\in[\tau_0,\tau]} \left\|e^{-(1-\alpha)tC}E_C([0,N))P_0(F(t/r)^{\alpha r} - e^{-\alpha tC}P_0)\right\|_{\mathcal{I}} \\
&\quad+ \sup_{t\in[\tau_0,\tau]} \left\|e^{-(1-\alpha)tC}E_C([N,\infty))P_0(F(t/r)^{\alpha r} - e^{-\alpha tC}P_0)\right\|_{\mathcal{I}}.
\end{aligned} \tag{5.8}$$

We estimate each term on the right hand side separately. For any $t \geq 0$ we have $F(t/r)^{\alpha r} = F(\alpha t/\alpha r)^{\alpha r}$. Hence, by (ii), for the interval $[\tau_0', \tau]$, there exists a number $r_0 \in \mathbb{R}_+$ such that $F(\alpha t/\alpha r) \in \mathcal{I}(H)$ for any $\alpha t \in [\tau_0', \tau]$, $r \geq r_0$ and

$$M([\tau_0', \tau]) := \sup_{\alpha t\in[\tau_0',\tau]} \sup_{r\geq r_0} \|F(\alpha t/\alpha r)^{\alpha r}\|_{\mathcal{I}} < +\infty.$$

Since $\alpha t \in [\tau_0', \tau]$ for any $t \in [\tau_0, \tau]$, we have that $F(t/r)^{\alpha r} \in \mathcal{I}(H)$ for any $t \in [\tau_0, \tau]$, $r \geq r_0$ and

$$\|F(t/r)^{\alpha r}\|_{\mathcal{I}} \leq M([\tau_0', \tau]). \tag{5.9}$$

We now estimate the third term of (5.8). Since $CE_C([N,\infty)) \geq NE_C([N,\infty))$ we have

$$\sup_{t\in[\tau_0,\tau]} \left\|e^{-(1-\alpha)tC}E_C([N,\infty))P_0\right\|_{\infty} \leq e^{-(1-\alpha)\tau_0 N}$$

for $N \geq 1$. Therefore, by (i) and triangle inequality, we have

$$\sup_{t\in[\tau_0,\tau]} \left\| e^{-(1-\alpha)tC} E_C([N,\infty)) P_0 (F(t/r)^{\alpha r} - e^{-\alpha tC} P_0) \right\|_{\mathcal{I}}$$

$$\leq \sup_{t\in[\tau_0,\tau]} \left\| \sup_{t\in[\tau_0,\tau]} \left\| e^{-(1-\alpha)tC} E_C([N,\infty)) P_0 \right\|_\infty F(t/r)^{\alpha r} \right\|_{\mathcal{I}}$$

$$+ \sup_{t\in[\tau_0,\tau]} \left\| \sup_{t\in[\tau_0,\tau]} \left\| e^{-(1-\alpha)tC} E_C([N,\infty)) P_0 \right\|_\infty e^{-\alpha tC} P_0 \right\|_{\mathcal{I}}$$

$$\leq \sup_{t\in[\tau_0,\tau]} \left\| e^{-(1-\alpha)\tau_0 N} F(t/r)^{\alpha r} \right\|_{\mathcal{I}} + \sup_{t\in[\tau_0,\tau]} \left\| e^{-(1-\alpha)\tau_0 N} e^{-\alpha\tau_0 C} P_0 \right\|_{\mathcal{I}}$$

for $N \geq 1$. Note that the sequence of real numbers $\{e^{-(1-\alpha)\tau_0 N}\}_{N\geq 1}$ converges to zero as N goes to infinity. By (5.9) and the fact that $\|e^{-\alpha\tau_0 C} P_0\|_{\mathcal{I}} < +\infty$, Definition 2.1.1(iii) implies that for given $\varepsilon > 0$, there exists large enough natural number $N_{\max}$ such that

$$\sup_{t\in[\tau_0,\tau]} \left\| e^{-(1-\alpha)\tau_0 N} F(t/r)^{\alpha r} \right\|_{\mathcal{I}} + \sup_{t\in[\tau_0,\tau]} \left\| e^{-(1-\alpha)\tau_0 N} e^{-\alpha\tau_0 C} P_0 \right\|_{\mathcal{I}} < \varepsilon/3 \tag{5.10}$$

for any $r \geq r_0$ and $n \geq N_{\max}$. We fix this number $N_{\max}$.

To estimate the second term of (5.8), we firstly note that $s_j(e^{-t\beta C}) = s_j(e^{-tC})^\beta, j \geq 1$ for any $\beta > 0$, implies that e^{-tC} is compact for any $t > 0$. Therefore, the spectrum of C is discrete with the only accumulation point at infinity. Hence, the projection $E_C([0,N))$ is finite-rank operator for each $N = 1, 2, ...$, and, in particular, $E_C([0,N)) \in \mathcal{I}(H)$ for any $N \geq 1$. Since $F(t/r)^{\alpha r} \to e^{-\alpha tC}$ in the strong operator topology [30, 31], Lemma 3.3.4 (with $X = Y = E_C([0,N))$ and $Z(t/r) = F(t/r)^{\alpha r}$, $Z = e^{-\alpha tC}$) implies that

$$\lim_{r\to\infty} \sup_{t\in[\tau_0,\tau]} \|E_C([0,N)) P_0 (F(t/r)^{\alpha r} - e^{-\alpha tC})\|_{\mathcal{I}} = 0.$$

Therefore, by (i), we can find large enough $r_{\max} \geq r_0$ such that

$$\sup_{t\in[\tau_0,\tau]} \left\| e^{-(1-\alpha)tC} E_C([0,N_{\max})) P_0 (F(t/r)^{\alpha r} - e^{-\alpha tC} P_0) \right\|_{\mathcal{I}} < \varepsilon/3, \tag{5.11}$$

for any $r \geq r_{\max}$.

Lastly, since the Trotter-Kato product formula converges locally uniformly away from zero in the operator-norm, we have

$$\lim_{r\to+\infty}\sup_{t\in[\tau_0,\tau]}\left\|F(t/r)^{(1-\alpha)r}-e^{(1-\alpha)tC}P_0\right\|=0.$$

Therefore, by (5.9), the symmetricity of the F-norm and Definition 2.1.1(iii) imply that there exists a real number $R\geq r_{\max}$ such that

$$\begin{aligned}\sup_{t\in[\tau,\tau_0]}&\left\|(F(t/r)^{(1-\alpha)r}-e^{(1-\alpha)tC}P_0)F(t/r)^{\alpha r}\right\|_{\mathcal{I}}\\&\leq\sup_{t\in[\tau,\tau_0]}\left\|\sup_{t\in[\tau,\tau_0]}\left\|F(t/r)^{(1-\alpha)r}-e^{(1-\alpha)tC}P_0\right\|_{\infty}F(t/r)^{\alpha r}\right\|_{\mathcal{I}}<\varepsilon/3\end{aligned}\tag{5.12}$$

for any $r\geq R$.

Finally, combining the estimates (5.10), (5.11), (5.12) and representation (5.8), for any bounded interval $[\tau_0,\tau]$ and $\varepsilon>0$, we can find a real number $R\in\mathbb{R}_+$ such that

$$\sup_{t\in[\tau_0,\tau]}\left\|F(t/r)^{r}-e^{-tC}P_0\right\|_{\mathcal{I}}<\varepsilon$$

for $r\geq R$. This proves that the Trotter-Kato product formula for the family $\{F(t)\}_{t\geq0}$ converges locally uniformly away from $t_0>0$ in $\mathcal{I}(H)$. $\square$

In the following Lemmas 5.2.3 and 5.2.4, we verify that conditions (ii) and (iii) of the Proposition 5.2.1 are satisfied. Let us first recall the notion of a dominated Kato function (see [40] or [2, Definition 2.11]).

Definition 5.2.2. *Let $f^D(\cdot):[0,\infty)\to[0,\infty)$ be a Borel measurable function. A Kato function $f(\cdot)$ is said to be a dominated by $f^D(\cdot)$ if for any $x\geq0$ and $0<q\leq1$ one has*

$$f(qx)^{1/q}\leq f^D(x).$$

Lemma 5.2.3. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and let $f^D(\cdot):[0,\infty)\to[0,\infty)$ and $g^D(\cdot):[0,\infty)\to[0,\infty)$ be bounded Borel measurable functions such that $F^D(t_0)=g^D(t_0B)^{1/2}f^D(t_0A)g^D(t_0B)^{1/2}\in\mathcal{I}(H)$ for some $t_0>0$, where $\mathcal{I}(H)$ is a symmetrically* F*-normed ideal closed with respect to the logarithmic submajorization. If Kato functions $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, then*

$F(t/r)^r \in \mathcal{I}(H)$ and

$$\|F(t/r)^r\|_{\mathcal{I}} \le C_{\mathcal{I}} \cdot \left\|F^D(t_0)\right\|_{\mathcal{I}}$$

for $t_0 \le t \le rt_0$, $r \ge 1$ and constant $C_{\mathcal{I}} > 0$ from Definition 3.2.1.

Proof. Let $X = f(tA/r)$, $Y = g(tB/r)$ and $X_0 = f^D(t_0 A)$, $Y_0 = g^D(t_0 B)$. We have

$$X^r = f(tA/r)^r \le f(tA/r)^{rt_0/t} \le f^D(t_0 A) = X_0$$

$$Y^r = g(tB/r)^r \le g(tB/r)^{rt_0/t} \le g^D(t_0 B) = Y_0$$

for $t_0 \le t \le rt_0$ and $r \ge 1$. Therefore, by Lemma 3.2.4 we have

$$F(t/r)^r = (Y^{1/2} X Y^{1/2})^r \prec\prec_{\log} Y_0^{1/2} X_0 Y_0^{1/2} = F^D(t_0),$$

for $t_0 \le t \le rt_0$ and $r \ge 1$. Since $\mathcal{I}(H)$ is closed with respect to the logarithmic submajorization, it follows that $F(t/r)^r \in \mathcal{I}(H)$ and

$$\|F(t/r)^r\|_{\mathcal{I}} \le C_{\mathcal{I}} \cdot \left\|F^D(t_0)\right\|_{\mathcal{I}}.$$

□

Next we show that under the assumption of Lemma 5.2.3 one has $e^{-tC} \in \mathcal{I}(H_0)$ for $t > t_0$.

Lemma 5.2.4. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and let $f^D(\cdot) : [0,\infty) \to [0,\infty)$ and $g^D(\cdot) : [0,\infty) \to [0,\infty)$ be bounded Borel measurable functions such that $F^D(t_0) = g^D(t_0 B)^{1/2} f^D(t_0 A) g^D(t_0 B)^{1/2} \in \mathcal{I}(H)$ for some $t_0 > 0$, where $\mathcal{I}(H)$ is a symmetrically* F*-normed ideal closed with respect to the logarithmic submajorization. If Kato functions $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, then $e^{-tC} \in \mathcal{I}(H_0)$ and*

$$\left\|e^{-tC}\right\|_{\mathcal{I}} \le C_{\mathcal{I}} \cdot \left\|F^D(t_0)\right\|_{\mathcal{I}} \tag{5.13}$$

for $t \ge t_0$ and constant $C_{\mathcal{I}} > 0$ from Definition 3.2.1.

Proof. Note that, for $x \ge 0$, we have

$$e^{-x} = \lim_{r\to\infty} f(x/r)^r \le f^D(x),$$

$$e^{-x} = \lim_{r\to\infty} g(x/r)^r \le g^D(x).$$

Hence, the assumption of the lemma implies that $f_0(x) = g_0(x) = e^{-x}$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively. Lemma 5.2.3 guarantees that $\left(e^{-tB/2r}e^{-tA/r}e^{-tB/2r}\right)^r \in \mathcal{I}(H)$ for $t_0 \leq t \leq rt_0$. By Lemma 3.2.5, we have

$$e^{-tC}P_0 \prec\!\prec_{\log} \left(e^{-tB/2r}e^{-tA/r}e^{-tB/2r}\right)^r \in \mathcal{I}(H), \quad r \geq 1, \tag{5.14}$$

for $t_0 \leq t \leq rt_0$. Therefore, by the assumption on ideal $\mathcal{I}(H)$ and (5.14), we obtain that $e^{-tC}P_0 \in \mathcal{I}(H)$ and (5.13) holds. □

5.3 Trotter-Kato product formula in symmetrically F-normed ideals

Now, using the lifting method from the previous section and the convergence of the Trotter-Kato product formula in the operator norm from Chapter 4, we present the main result of the present chapter.

Theorem 5.3.1. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and $\mathcal{I}(H)$ be a symmetrically* F*-normed ideal closed with respect to the logarithmic submajorization. Let $f^D(\cdot) : [0,\infty) \to [0,\infty)$ be a Borel measurable functions such that $f^D(t_0A) \in \mathcal{I}(H)$ for some $t_0 > 0$. If $g(\cdot)$ is any Kato function and $f(\cdot)$ is any regular Kato function dominated by $f^D(\cdot)$, then the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ for all families generated by $f(\cdot)$ and $g(\cdot)$.*

Proof. Similar argument as in proof of Lemma 5.2.4 implies that e^{-x} is dominated by $f^D(x), x \geq 0$. Thus, one has $e^{-t_0A} \in \mathcal{I}(H)$ provided that $f^D(t_0A) \in \mathcal{I}(H)$. In particular, $(I + t_0A)^{-1} \in \mathcal{L}_\infty(H)$. Then, since $f(\cdot)$ is regular, Theorem 4.4.3 implies that the Trotter-Kato product formula converges locally uniformly away from zero in the operator norm for the family $\{F(t)\}_{t\geq 0}$.

Setting $g^D(x) \equiv 1$, $x \geq 0$, it easily follows that $F^D(t_0) \in \mathcal{I}(H)$. Notice that any Kato function $g(\cdot)$ is dominated by $g^D(\cdot) \equiv 1$. Hence, by Lemma 5.2.3 and Lemma 5.2.4, we have that for any bounded interval $[\tau_0, \tau] \subset (t_0, \infty)$ there exists $r_0 \geq 1$ such that $e^{-tC} \in \mathcal{I}(H_0)$ and $F(t/r)^r \in \mathcal{I}(H)$ for any $t \in [\tau_0, \tau]$ and $r \geq r_0$ such that

$$\sup_{t\in[\tau_0,\tau]} \sup_{r\geq r_0} \|F(t/r)^r\|_{\mathcal{I}} \leq \|F^D(t_0)\|_{\mathcal{I}} < +\infty.$$

Thus, all assumptions of Proposition 5.2.1 are satisfied, and the Trotter-Kato product formula for the family $\{F(t)\}_{t\geq 0}$ converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$. Finally, by Proposition 5.1.2, we have the convergence of the Trotter-Kato product formula in symmetrically F-normed ideal $\mathcal{I}(H)$ for all other families generated by $f(\cdot)$ and $g(\cdot)$. $\square$

As a corollary of Theorem 5.3.1, we have analogues result for symmetric quasi-Banach ideals.

Corollary 5.3.2. *Let $\mathcal{I}(H)$ be a symmetric quasi-Banach ideal and let the operators A, B and the functions $f(\cdot)$, $g(\cdot)$, $f^D(\cdot)$ be as in Theorem 5.3.1. If $f^D(t_0 A) \in \mathcal{I}(H)$ for some $t_0 > 0$, then the Trotter-Kato product formula converges locally uniformly away from $t_0 > 0$ in $\mathcal{I}(H)$ for all families generated by $f(\cdot)$ and $g(\cdot)$.*

We present a concrete example of a symmetric quasi-Banach ideal and the noncommuting operators A and B for Corollary 5.3.2, which is not covered by [15] and [40].

Example 5.3.3. *Let $I(H) = \mathcal{L}_{1,\infty}(H)$ be a weak-l_1 ideal from the part (ii) of Example 3.1.4, equipped with its natural quasi-norm*

$$\|X\|_{1,\infty} = \sup_{k\geq 0}(k+1)s_k(X)$$

Let $D : W^{1,2}(\mathbb{T}) \to L^2(\mathbb{T})$ be a self-adjoint operator given by the formula $Df = \frac{1}{i}\frac{df}{dt}$, $f \in W^{1,2}(\mathbb{T})$, where $\mathbb{T}$ is the unit circle. It is known that $(1+|D|)^{-1} \in \mathcal{L}_{1,\infty}(H)$. Indeed, for $k \geq 0$

$$s_0\left((1+|D|)^{-1}\right) = 1, \quad s_{2k}\left((1+|D|)^{-1}\right) = s_{2k+1}\left((1+|D|)^{-1}\right) = \frac{1}{k+1}.$$

We set $A = \log(1+|D|)$. Hence, we have that $e^{-A} = (1+|D|)^{-1} \in \mathcal{L}_{1,\infty}(H)$. Moreover, $(1+|D|)^{-1}$ generates the principal ideal $\mathcal{L}_{1,\infty}(H)$ and, therefore, e^{-A} cannot belong to any ideal which is strictly smaller than $\mathcal{L}_{1,\infty}(H)$.

We claim that an operator B with non-trivial continuous spectrum does not commute with A. Indeed, otherwise it commutes with spectral projections of A. Let $p_n = \chi_{\{n\}}(|D|)$ be the spectral projection of A. It follows that

$$B = \sum_{n\in\mathbb{Z}_+} Bp_n = \sum_{n\in\mathbb{Z}_+} p_n B p_n,$$

where the sums are taken in the strong operator topology. Each of the operators $p_n B p_n$ *is of finite rank; hence its spectrum consists of eigenvalues. Since the operators* $\{p_n B p_n\}_{n\in\mathbb{Z}_+}$ *are pairwise orthogonal, it follows that the spectrum of* B *consists of eigenvalues as well, a contradiction with the assumption. In particular, if we take* $B = M_h$, $h \in L_\infty(\mathbb{T})$, *then the operators* A *and* B *do not commute.*

Let the function $f^D(\cdot)$ *from Corollary 5.3.2 be an exponential function* $e^{-x}, x \geq 0$. *Hence, it follows that* $f^D(A) = e^{-A} = (1+|D|)^{-1} \in \mathcal{L}_{1,\infty}(H)$ *for* $t_0 = 1$. *Therefore, by Corollary 5.3.2, one has that the Trotter-Kato product formula converges locally uniformly away from* $t_0 = 1$ *in* $\mathcal{L}_{1,\infty}(H)$ *for all families generated by the exponential functions.*

Now we want to specify a particular subclass of Kato functions to present a necessary and sufficient condition of the convergence of the Trotter-Kato product formula in $\mathcal{I}(H)$ similar to [40, Theorem 4.10].

Definition 5.3.4. *A Kato function* $f(\cdot)$ *is said to be a self-dominated if for any* $x \geq 0$ *and* $0 < q \leq 1$, *one has* $f(qx)^{1/q} \leq f(x)$.

Theorem 5.3.5. *Let* A *and* B *be non-negative self-adjoint operators on a separable Hilbert space* H *and let* $\mathcal{I}(H)$ *be a symmetrically* F*-normed ideal closed with respect to the logarithmic submajorization. Let also* $f(\cdot)$ *and* $g(\cdot)$ *be self-dominated Kato functions which additionally satisfy*

$$\sup_{x>0} \frac{xf(x)}{1-f(x)} < \infty, \quad \sup_{x>0} \frac{xg(x)}{1-g(x)} < \infty.$$

Then, there exists $t_0 > 0$ *such that the Trotter-Kato product formula converges locally uniformly away from* $t_0 > 0$ *in* $\mathcal{I}(H)$ *for all families generated by* $f(\cdot)$ *and* $g(\cdot)$ *if and only if there exists* $s_0 > 0$ *and* $p \in \mathbb{Z}_+$ *such that* $F(s_0)^p \in \mathcal{I}(H)$.

Proof. The necessity follows from Definition 5.1.1 of the convergence of the Trotter-Kato product formula in $\mathcal{I}(H)$.

Assume that there exists $s_0 > 0$ and $p \in \mathbb{Z}_+$ such that $F(s_0)^p \in \mathcal{I}(H) \subset \mathcal{L}_\infty(H)$. It follows that $F(s_0)$ is a compact operator and a similar argument to [40, Theorem 4.6] implies that $(I+A)^{-1}(I+B)^{-1} \in \mathcal{L}_\infty(H)$. Hence, by [39, Theorem 5.3], the Trotter-Kato product formula converges locally uniformly away from zero in the operator norm. Finally, the sufficiency follows from Proposition 5.2.1 together with Lemma 5.2.3 and 5.2.4, taking $f^D(x) \equiv f(x)$ and $g^D(x) \equiv g(x)$ for $x \geq 0$ and $t_0 = ps_0$. $\square$

5.4 Error bound of the Trotter-Kato product formula in symmetric operator ideals

In this section we determine the error bounds of the Trotter-Kato product formulas in symmetrically F-normed ideals closed with respect to the logarithmic submajorization and give some examples where they can be computed directly. As before, we assume that A and B are non-negative self-adjoint operators on a separable Hilbert space H with the form-sum $C = A\dot{+}B$ and $\mathcal{I}(H)$ is a symmetrically F-normed ideal closed with respect to the logarithmic submajorization. Let $f(\cdot)$ and $g(\cdot)$ be Kato functions and use the same notation as before

$$F(t) := g(tB)^{1/2} f(tA) g(tB)^{1/2}, \quad t \geq 0,$$
$$G(t) := f(tA)^{1/2} g(tB) f(tA)^{1/2}, \quad t \geq 0.$$

We first define the notion of an error bound in the same manner as in [40]. Let $\varepsilon(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ (or $\varepsilon(\cdot) : \mathbb{N} \to \mathbb{R}_+$) be a function such that $\lim_{x\to\infty} \varepsilon(x) = 0$.

Definition 5.4.1. *A function $\varepsilon_{\mathcal{I}}(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ is called an error bound of the Trotter-Kato product formula for the family $\{F(t)\}_{t\geq 0}$ away from $t_0 > 0$ in $\mathcal{I}(H)$ if for any bounded interval $[a, b] \subset (t_0, \infty)$ there exists $1 \leq r_0 \in \mathbb{R}_+$ such that*

$$F(t/r)^r - e^{-tC} P_0 \in \mathcal{I}(H)$$

and

$$\|F(t/r)^r - e^{-tC} P_0\|_{\mathcal{I}} \leq Const \cdot \varepsilon(r)$$

for any $t \in [a, b]$, $r \geq r_0$ and some constant $Const > 0$.

In a similar way, one can define the notion of an error bound for the families $\{G(t)\}_{t\geq 0}$, $\{f(tA)g(tB)\}_{t\geq 0}$ and $\{g(tB)f(tA)\}_{t\geq 0}$, assuming that $r \in \mathbb{N}$ in the last two cases. In case when $\|\cdot\|_{\mathcal{I}}$ is the operator norm $\|\cdot\|_\infty$, we simply write $\varepsilon_\infty(\cdot)$ instead of $\varepsilon_{\mathcal{I}}(\cdot)$, and call it an error bound of the Trotter-Kato product formula away from $t_0 > 0$ in the operator norm.

First we want to prove an auxiliary lemma.

Lemma 5.4.2. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and $\mathcal{I}(H)$ be a symmetrically* F*-normed ideal. Let also $f^D(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, $g^D(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ be Borel measurable functions such that $F^D(t_0) := g^D(t_0 B)^{1/2} f^D(t_0 A) g^D(t_0 B)^{1/2} \in \mathcal{I}(H)$ for some $t_0 > 0$. Then $G^D(t_0) := f^D(t_0 A)^{1/2} g^D(t_0 B) f^D(t_0 A)^{1/2} \in \mathcal{I}(H)$ and*

$$s_j\left(G^D(t_0)\right) \leq s_j\left(F^D(t_0)\right), \quad j \geq 1. \tag{5.15}$$

Proof. Let

$$E(t) = f^D(tA)^{1/2} g^D(tB)^{1/2}, \quad t \geq 0.$$

Note that $|E(t)| = F^D(t)^{1/2}$ and $|E(t)^*| = G^D(t)^{1/2}$. Therefore, denoting by $U(t)$ the partial isometry in the polar decomposition

$$E(t) = U(t) F^D(t)^{1/2}, \quad t \geq 0$$

we conclude that

$$G^D(t_0) = E(t_0)E(t_0)^* = U(t_0) F^D(t_0) U(t_0)^* \in \mathcal{I}(H)$$

which also implies (5.15). □

Now we present the following result which is helpful to compute the error bounds of the Trotter-Kato product formula in symmetrically F-normed ideals closed with respect to the logarithmic submajorization. The proof is similar to [40, Theorem 5.1].

Proposition 5.4.3. *Let A and B be non-negative self-adjoint operators on a separable Hilbert space H and $\mathcal{I}(H)$ be a symmetrically* F*-normed ideal closed with respect to the logarithmic submajorization. Let also $f^D(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ and $g^D(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ be Borel measurable functions such that $F^D(t_0) = g^D(t_0 B)^{1/2} f^D(t_0 A) g(t_0 B)^{1/2} \in \mathcal{I}(H)$ for some $t_0 > 0$. Assume that the Kato-functions $f(\cdot)$ and $g(\cdot)$ are dominated by $f^D(\cdot)$ and $g^D(\cdot)$, respectively. Then:*

(i) For families $\{F(t)\}_{t\geq 0}$ and $\{G(t)\}_{t\geq 0}$, we have

$$\|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \leq 2C_{\mathcal{I}} \left\| \|F(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty F_D(t_0) \right\|_{\mathcal{I}}$$
$$\|G(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \leq 2C_{\mathcal{I}} \left\| \|G(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty F_D(t_0) \right\|_{\mathcal{I}}$$

for $2t_0 \le t \le rt_0$ *and* $r \ge 2$.

(ii) *For families* $\{f(tA)g(tB)\}_{t\ge 0}$ *and* $\{g(tB)f(tA)\}_{t\ge 0}$, *we have*

$$\begin{aligned}
&\| (f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}} \\
&\qquad \le C_{\mathcal{I}} \left\| \| (f(tA/n)g(tB/n))^k - e^{-ktC/n}P_0\|_\infty F^D(t_0)\right\|_{\mathcal{I}} \\
&\qquad + C_{\mathcal{I}} \left\| \| (f(tA/n)g(tB/n))^m - e^{-mtC/n}P_0\|_\infty F^D(t_0)\right\|_{\mathcal{I}}, \\
&\| (g(tB/n)f(tA/n))^n - e^{-tC}P_0\|_{\mathcal{I}} \\
&\qquad \le C_{\mathcal{I}} \left\| \| (g(tB/n)f(tA/n))^k - e^{-ktC/n}P_0\|_\infty F^D(t_0)\right\|_{\mathcal{I}} \\
&\qquad + C_{\mathcal{I}} \left\| \| (g(tB/n)f(tA/n))^m - e^{-mtC/n}P_0\|_\infty F^D(t_0)\right\|_{\mathcal{I}}
\end{aligned}$$

for $t_0 \le (m-1)t/n \le (m-1)t_0$, $m \ge 2$ *and* $kt/n \ge t_0$, *where* $k := \left[\frac{n}{2}\right]$ *and* $m := \left[\frac{n+1}{2}\right]$ *with* $n \ge 3$.

Proof. (i). For $r \in \mathbb{R}_+$, we write

$$\begin{aligned}
&F(t/r)^r - e^{-tC}P_0 \\
&\quad = \left(F(t/r)^{r/2} - e^{-tC/2}P_0\right) F(t/r)^{r/2} + e^{-tC/2}P_0 \left(F(t/r)^{r/2} - e^{-tC/2}P_0\right).
\end{aligned} \tag{5.16}$$

By (i), we have

$$\begin{aligned}
\left\|F(t/r)^r - e^{-tC}P_0\right\|_{\mathcal{I}} \le &\left\| \left\|F(t/r)^{r/2} - e^{-tC/2}P_0\right\|_\infty F(t/r)^{r/2}\right\|_{\mathcal{I}} \\
&+ \left\| \left\|F(t/r)^{r/2} - e^{-tC/2}P_0\right\|_\infty e^{-tC/2}P_0\right\|_{\mathcal{I}}.
\end{aligned}$$

Since $F^D(t_0) \in \mathcal{I}(H)$, Lemmas 5.2.3 and 5.2.4 imply

$$F(t/r)^{r/2} \prec\prec_{\log} F^D(t_0)$$

for $2t_0 \le t \le rt_0$, $r \ge 2$ and

$$e^{-tC/2}P_0 \prec\prec_{\log} F^D(t_0)$$

for $t \ge 2t_0$. Hence, since the ideal $\mathcal{I}(H)$ is closed with respect to the logarithmic submajorization, it follows that

$$\|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \le 2C_{\mathcal{I}} \left\| \|F(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty F^D(t_0)\right\|_{\mathcal{I}}$$

for $2t_0 \le t \le rt_0$

For the family $\{G(t)\}_{t\ge 0}$ we have a similar estimate via decomposition similar to (5.16) with an additional reference to Lemma 5.4.2.

(ii). We prove the result for the family $\{f(tA)g(tB)\}_{t\ge 0}$. The argument for the family $\{g(tB)f(tA)\}_{t\ge 0}$ is similar, and therefore, is omitted. Let $n \in \mathbb{N}$ and write $n = k + m$ with

$$k := \left[\frac{n}{2}\right], \quad m := \left[\frac{n+1}{2}\right], \quad n \ge 3.$$

We have

$$\begin{aligned}
&(f(tA/n)g(tB/n))^n - e^{-tC}P_0 \\
&\quad = ((f(tA/n)g(tB/n))^k - e^{-ktC/n}P_0)(f(tA/n)g(tB/n))^m \\
&\qquad\qquad + e^{-ktC/n}P_0((f(tA/n)g(tB/n))^m - e^{-mtC/n}P_0).
\end{aligned}$$

Therefore, triangle inequality and (i) imply

$$\begin{aligned}
&\| (f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}} \\
&\quad \le \left\| \| (f(tA/n)g(tB/n))^k - e^{-ktC/n}P_0\|_\infty (f(tA/n)g(tB/n))^m \right\|_{\mathcal{I}} \\
&\qquad + \left\| e^{-ktC/n}P_0 \| (f(tA/n)g(tB/n))^m - e^{-mtC/n}P_0\|_\infty \right\|_{\mathcal{I}}.
\end{aligned} \tag{5.17}$$

We consider two terms of (5.17) separately. Note that

$$(f(tA/n)g(tB/n))^m = f(tA/n)g(tB/n)^{1/2}F(t/n)^{m-1}g(tB/n)^{1/2}.$$

By Lemma 5.2.3, $F(t/n)^{m-1} \in \mathcal{I}(H)$ when $t_0 \le (m-1)t/n \le (m-1)t_0$ and $m-1 \ge 1$, Therefore, $(f(tA/n)g(tB/n))^m \in \mathcal{I}(H)$ and

$$(f(tA/n)g(tB/n))^m \prec\prec_{\log} F(t/n)^{m-1} \prec\prec_{\log} F^D(t_0) \tag{5.18}$$

for $t_0 \le (m-1)t/n \le (m-1)t_0$ and $m-1 \ge 1$. Lemma 5.2.4 implies that $e^{-ktC/n}P_0 \in \mathcal{I}(H)$ and

$$e^{-ktC/n}P_0 \prec\prec_{\log} F^D(t_0) \tag{5.19}$$

for $kt/n \ge t_0$. Hence, by (5.18), (5.19) and (5.17), together with the fact that $\mathcal{I}(H)$ is closed with respect to the logarithmic submajorization, we have

$$\begin{aligned}\|\,(f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}} \\ &\leq C_{\mathcal{I}}\left\|\|\,(f(tA/n)g(tB/n))^k - e^{-ktC/n}P_0\|_\infty F^D(t_0)\right\|_{\mathcal{I}} \\ &+ C_{\mathcal{I}}\left\|\|\,(f(tA/n)g(tB/n))^m - e^{-mtC/n}P_0\|_\infty F^D(t_0)\right\|_{\mathcal{I}}\end{aligned}$$

for $t_0 \leq (m-1)t/n \leq (m-1)t_0$, $m \geq 2$ and $kt/n \geq t_0$.

□

Now we present some examples similar to [40, Theorem 5.1], where the error bounds can be computed directly. Recall that the notion of an error bound in the operator norm is defined as in Definition 5.4.1.

Example 5.4.4. *(i) Let $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ be a symmetrically* F*-normed ideal closed with respect to the logarithmic submajorization and let the functions $f(\cdot)$, $g(\cdot)$, $f^D(\cdot)$ and $g^D(\cdot)$ be as in Proposition 5.4.3 such that $F^D(t_0) \in \mathcal{I}(H)$ for some $t_0 > 0$, and*

$$\left\|\alpha F^D(t_0)\right\|_{\mathcal{I}} = O(\alpha^q), \quad \alpha \to 0. \tag{5.20}$$

Let $\varepsilon(r)$, $r \geq 0$ be an error bound of the Trotter-Kato product formula in the operator norm away from $t_0 > 0$. Then, by Proposition 5.4.3(i), we have

$$\|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \leq 2C_{\mathcal{I}}\left\|\|F(t/r)^{r/2} - e^{-tC/2}P_0\|_\infty F_D(t_0)\right\|_{\mathcal{I}}.$$

Hence, the symmetricity of the F*-norm and (5.20) imply that*

$$\|F(t/r)^r - e^{-tC}P_0\|_{\mathcal{I}} \leq Const \cdot \|\varepsilon(r/2)F_D(t_0)\|_{\mathcal{I}} \leq Const \cdot \varepsilon(r/2)^q$$

for some constant $Const > 0$. Hence, the function $\varepsilon_{\mathcal{I}}(r) = \varepsilon(r/2)^q, r \geq 0$ is an error bound of the Trotter-Kato product formula for the family $\{F(t)\}_{t\geq 0}$ away from $2t_0$ in $\mathcal{I}(H)$. Similarly, by the second inequality of Proposition 5.4.3(i), we have an analogous error bound for a family $\{G(t)\}_{t\geq 0}$ away from $2t_0$ in $\mathcal{I}(H)$. For the family $\{f(tA)g(tB)\}_{t\geq 0}$, using Proposition 5.4.3 (ii) and arguments above, we have

$$\|\,(f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}}$$

$$\leq Const \cdot \left(\left\| \varepsilon\left(\left[\frac{n}{2}\right]\right) F^D(t_0) \right\|_{\mathcal{I}} + \left\| \varepsilon\left(\left[\frac{n+1}{2}\right]\right) F^D(t_0) \right\|_{\mathcal{I}} \right)$$

which together with (5.20) *implies*

$$\|\,(f(tA/n)g(tB/n))^n - e^{-tC}P_0\|_{\mathcal{I}}$$

$$\leq Const \cdot \left(\varepsilon\left(\left[\frac{n}{2}\right]\right)^q + \varepsilon\left(\left[\frac{n+1}{2}\right]\right)^q \right)$$

for some constant $Const > 0$. *Therefore,* $\varepsilon_{\mathcal{I}}(n) = \varepsilon(\left[\frac{n}{2}\right])^q + \varepsilon(\left[\frac{n+1}{2}\right])^q$, $r \geq 0$ *is an error bound of the Trotter-Kato product formula for the family* $\{f(tA)g(tB)\}_{t\geq 0}$ *away from* $2t_0$ *in* $\mathcal{I}(H)$. *In a similar way, by the second inequality of Proposition 5.4.3(ii), we have analogues error bound for the family* $\{g(tB)f(tA)\}_{t\geq 0}$.

(ii) *Let* $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ *be a symmetrically quasi-normed ideal and* $\varepsilon(r)$, $r \in \mathbb{R}_+$, *be an error bound of the Trotter-Kato product formula in the operator norm away from* $t_0 > 0$. *Then, under the assumptions of Proposition 5.4.3, the arguments similar to (i) and the homogeneity of the quasi-norm imply that* $\varepsilon_{\mathcal{I}}(r) = \varepsilon(r/2)$, $r \in \mathbb{R}_+$ *and* $\varepsilon_{\mathcal{I}}(n) = \varepsilon(\left[\frac{n}{2}\right]) + \varepsilon(\left[\frac{n+1}{2}\right])$, $n \in \mathbb{N}$ *are the error bounds locally away from* $2t_0$ *in* $\mathcal{I}(H)$ *for families* $\{F(t)\}_{t\geq 0}$, $\{G(t)\}_{t\geq 0}$ *and* $\{f(tA)g(tB)\}_{t\geq 0}$, $\{g(tB)f(tA)\}_{t\geq 0}$, *respectively.*

(iii) *Let* $\mathcal{L}_1(H)$ *be a trace class ideal equipped with a functional* $\|X\|_{\log} = \sum_{k\geq 1}\log(1 + s_k(X))$, $X \in \mathcal{L}_\infty(H)$. *By [9], it follows that* $(\mathcal{L}_1(H), \|\cdot\|_{\log})$ *is a complete symmetrically* F*-normed ideal. For* $X \in \mathcal{L}_1(H)$ *and a real number* $\alpha \in [0,1]$, *we have the following inequality*

$$\|\alpha X\|_{\log} \leq \|\alpha X\|_1 \leq \alpha(1 + \|X\|_\infty)\|X\|_{\log}, \tag{5.21}$$

where $\|\cdot\|_1$ *is trace class norm. Let* $\varepsilon(r)$, $r \in \mathbb{R}_+$, *be an error bound of the Trotter-Kato product formula in the operator norm away from* $t_0 > 0$. *Therefore, under the assumptions of Proposition 5.4.3,* (5.21) *and the arguments similar to (i) imply that the error bounds are* $\varepsilon_{\mathcal{I}}(r) = \varepsilon(r/2)$, $r \in \mathbb{R}_+$ *and* $\varepsilon_{\mathcal{I}}(n) = \varepsilon(\left[\frac{n}{2}\right]) + \varepsilon(\left[\frac{n+1}{2}\right])$, $n \in \mathbb{N}$ *for families* $\{F(t)\}_{t\geq 0}$, $\{G(t)\}_{t\geq 0}$ *and* $\{f(tA)g(tB)\}_{t\geq 0}$, $\{g(tB)f(tA)\}_{t\geq 0}$, *respectively.*

Chapter 6

Approximation formula for a propagator in symmetric Banach ideals

In this chapter, we consider abstract non-autonomous Cauchy problem for an evolution equation and an approximation formula of its propagator. Firstly, in Section 6.1, we state the assumptions under which the evolution equation is considered and present main results of this chapter. Section 6.2 consists basic properties of a propagator. In Sections 6.3 and 6.4, we prove the main results of this chapter, Theorem 6.1.1 and Theorem 6.1.2, respectively. This chapter is based on the results of [3].

6.1 Assumptions and Main results

Let H be a separable Hilbert space and A and $B(t)$, $t \geq 0$ be the non-negative self-adjoint operators on H. The main object of this chapter is the following abstract non-autonomous evolution equation on a compact interval $[0, 1]$

$$\begin{cases} \frac{du(t)}{dt} = -(A + B(t))u(t), \\ u(s) = u_s \in H, \end{cases} \qquad 0 \leq s \leq t \leq 1. \tag{6.1}$$

We consider the non-autonomous Cauchy problem (6.1) with the following assumptions:

(A1) An operator $A \geq I$ is self-adjoint and a family $\{B(t)\}_{0 \leq t \leq 1}$ consists of positive self-adjoint operators on a separable Hilbert space H.

(A2) There is a number $\alpha \in (0,1)$ such that $D(A^\alpha) \subseteq D(B(t))$ for a.e. $t \in [0,1]$ and the function $B(\cdot)A^{-\alpha} : [0,1] \to \mathcal{L}(H)$ is strongly measurable and essentially bounded in the operator norm, i.e.,

$$\operatorname*{ess\,sup}_{t\in[0,1]} \left\| B(t)A^{-\alpha} \right\|_\infty < \infty.$$

(A3) The mapping $[0,1] \ni t \mapsto B(t)A^{-1} \in \mathcal{L}(H)$ is Hölder continuous in the operator norm.

(A4) Let $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ be an arbitrary symmetric Banach ideal (see, Definition 3.1.2). Then, the operator A is the generator of a strongly continuous semigroup $\{e^{-tA}\}_{t\geq 0}$ such that $\{e^{-tA}\}_{t>0} \subset \mathcal{I}(H)$.

Following the classical methods of solving (6.1), we prove the following existence result for a propagator.

Theorem 6.1.1. *Let $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ be an arbitrary symmetric Banach ideal and let the assumptions **(A1)**-**(A4)** be satisfied. We have*

$$\|U(t,s)\|_{\mathcal{I}} \leq \mathrm{Const}\|e^{-\frac{t-s}{2}A}\|_{\mathcal{I}}, \quad s \leq t.$$

Furthermore, the propagator $U(\cdot,\cdot)$ is continuous (in the norm of $\mathcal{I}(H)$) in both variables and is continuously differentiable (in the norm of $\mathcal{I}(H)$) in the first variable.

We introduce the following approximants for the propagator

$$U_n(t,s) := \prod_{k=n}^{1} e^{-\frac{t-s}{n}A} e^{-\frac{t-s}{n}B(s+\frac{k}{n}(t-s))},$$

$$U_n'(t,s) := \prod_{k=n}^{1} e^{-\frac{t-s}{n}B(s+\frac{k}{n}(t-s))} e^{-\frac{t-s}{n}A}, \tag{6.2}$$

$$U_n''(t,s) := \prod_{k=n}^{1} e^{-\frac{t-s}{2n}A} e^{-\frac{t-s}{n}B(s+\frac{k}{n}(t-s))} e^{-\frac{t-s}{2n}A}. \tag{6.3}$$

Note that when $B(t) = B$ does not depend on time, then the given approximation problem corresponds to the Trotter product formula in symmetric Banach ideals (see, [40] or [2]).

Throughout this paper, we are using the notation

$$\epsilon_\infty(n) = \operatorname*{ess\,sup}_{0\le s\le t\le 1} \|U_n(t,s) - U(t,s)\|_\infty, \quad n \in \mathbb{N}. \tag{6.4}$$

Various estimates on $\epsilon_\infty(n)$ are available in Subsection 4.5.4.

The first lifting result from the operator norm topology to other various topologies was first considered in [76]. Zagrebnov considered the same problem (6.1) under slightly different assumptions (in **(A4)**, $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ was assumed to be a trace class ideal $(\mathcal{L}_1, \|\cdot\|_1)$) and proved the following approximation formula in the trace norm

$$\operatorname*{ess\,sup}_{0\le s<t\le 1} \|U_n(t,s) - U(t,s)\|_1 \le \mathrm{Const}\cdot \|e^{-\frac{t-s}{16}A}\|_1 \epsilon_\infty(n), \quad n\in\mathbb{N},$$

where Const depends only on α, β.

In this thesis, we further extend the last result to the general class of all symmetric Banach ideals in $\mathcal{L}(H)$ and prove the following approximation formula.

Theorem 6.1.2. *Let $(\mathcal{I}(H), \|\cdot\|_{\mathcal{I}})$ be an arbitrary symmetric Banach ideal and let the assumptions **(A1)**-**(A4)** be satisfied. We have*

$$\epsilon_{\mathcal{I}}(n) \le \mathrm{Const}\cdot \left\|e^{-\frac{(t-s)A}{4}}\right\|_{\mathcal{I}} \cdot \left(\epsilon_\infty\left(\left[\frac{n}{2}\right]\right) + \epsilon_\infty\left(\left[\frac{n+1}{2}\right]\right)\right), \quad 2\le n\in\mathbb{N},$$

where Const > 0 *depends only on the ideal $\mathcal{I}(H)$ and on the constant α and where*

$$\epsilon_{\mathcal{I}}(n) = \operatorname*{ess\,sup}_{0\le s<t\le 1} \|U_n(t,s) - U(t,s)\|_{\mathcal{I}}.$$

6.2 Existence and basic properties of the propagator

Note that Assumptions **(A1)** and **(A2)** imply that the operators $C(t) := A + B(t)$ for $t \in [0,1]$ have a common domain $\mathrm{dom}(C(t)) = \mathrm{dom}(A)$ and $C(t)$, $t\in[0,1]$ are self-adjoint in this common domain $\mathrm{dom}(A)$.

Proposition 6.2.1. *Let Assumptions **(A1)**-**(A3)** be satisfied. Then, there exists a propagator $\{U(t,s)\}_{0\le s\le t\le 1}$ solving (6.1) on dense subspace $H_0 = \mathrm{dom}(A)$.*

Proof. We need to verify the assumptions in Theorem 4.5.9 in our setting.

Firstly,

$$\|B(t)A^{-\alpha}\|_\infty \leq c, \quad t \in [0,1].$$

Let $\lambda > 0$ be such that

$$c \cdot \lambda^{\alpha-1} \cdot \sup_{u>0} \frac{u^\alpha}{u+1} = \frac{1}{2}.$$

We have

$$\|B(t)(A+\lambda)^{-1}\|_\infty \leq \|B(t)A^{-\alpha}\|_\infty \cdot \|A^\alpha(A+\lambda)^{-1}\|_\infty \leq c \cdot \sup_{u>0} \frac{u^\alpha}{u+\lambda} = \frac{1}{2}.$$

Thus,

$$\|A \cdot (A+B(t))^{-1}\|_\infty \leq \|A \cdot (A+B(t)+\lambda)^{-1}\|_\infty \cdot \|(A+B(t)+\lambda) \cdot (A+B(t))^{-1}\|_\infty$$

$$\leq (1+\lambda) \cdot \|A \cdot (A+B(t)+\lambda)^{-1}\|_\infty$$

$$\leq (1+\lambda)\|A(A+\lambda)^{-1}\|_\infty \cdot \|(A+\lambda) \cdot (A+B(t)+\lambda)^{-1}\|_\infty$$

$$\leq (1+\lambda) \cdot \|(1+(A+\lambda)^{-1}B(t))^{-1}\|_\infty \leq 2(1+\lambda).$$

In particular, we have

$$\|(A+B(s)-A-B(t)) \cdot (A+B(t))^{-1}\|_\infty \leq \|(B(s)-B(t))A^{-1}\|_\infty \cdot \|A(A+B(t))^{-1}\|_\infty$$

$$\leq 2(1+\lambda)\|B(s)A^{-1} - B(t)A^{-1}\|_\infty \leq 2(1+\lambda) \cdot c'|t-s|^\beta, \quad t,s \in [0,1].$$

This verifies the assumption of Hölder continuity in Theorem 4.5.9. The assumption of bounded invertability in Theorem 4.5.9 is immediate in our setting. □

The following integral equation for the propagator is known (see, for example, [10, Section VI.9.c] or [52, Section 5.6])

$$U(t,s) = e^{-(t-s)A} - \int_s^t e^{-(t-\tau)A} B(\tau) U(\tau,s) d\tau.$$

Here, the integral is understood in the strong operator topology.

The following assertion is established in [76].

Proposition 6.2.2. *The propagator $\{U(t,s)\}_{0\leq s\leq t\leq 1}$ can be constructed as an uniformly* *operator norm convergent Dyson-Phillips series*

$$U(t,s)=\sum_{n=0}^{\infty}S_n(t,s),\quad S_0(t,s)=e^{-(t-s)A},$$
$$S_n(t,s)=-\int_s^t e^{-(t-\tau)A}B(\tau)S_{n-1}(\tau,s)d\tau,\quad n\geq 1,$$

where each $S_n(t,s), n\geq 1$ is bounded, strongly continuous and strongly differentiable for $0\leq s<t\leq 1$, and satisfies the following operator norm estimate for $n\geq 1$:

$$\|S_n(t,s)\|_\infty \leq (\mathrm{Const})^n\frac{\Gamma(1-\alpha)^n}{n(1-\alpha)\Gamma(n(1-\alpha))}(t-s)^{n(1-\alpha)}.$$

Proof. It follows from the assumptions **(A1)** and **(A2)** that

$$\|e^{-(t-\tau)A}B(\tau)\|_\infty \leq \frac{\mathrm{Const}}{(t-\tau)^\alpha},\quad 0\leq\tau<t\leq 1. \tag{6.5}$$

Hence,

$$\|S_n(t,s)\|_\infty \leq \mathrm{Const}\int_s^t (t-\tau)^{-\alpha}\|S_{n-1}(\tau,s)\|_\infty d\tau.$$

We conclude by induction that

$$\|S_n(t,s)\|_\infty \leq (\mathrm{Const})^n\cdot I_n(t,s),$$

where, for $n\geq 1$

$$I_n(t,s):=\int_s^t\frac{d\tau_n}{(t-\tau_n)^\alpha}\int_s^{\tau_n}\frac{d\tau_{n-1}}{(\tau_n-\tau_{n-1})^\alpha}\int_s^{\tau_{n-1}}\frac{d\tau_{n-2}}{(\tau_{n-1}-\tau_{n-2})^\alpha}\cdots\int_s^{\tau_2}\frac{d\tau_1}{(\tau_2-\tau_1)^\alpha(\tau_1-s)^\alpha}.$$

It is easy to see that

$$I_n(t,s)=\frac{\Gamma(1-\alpha)^n}{n(1-\alpha)\Gamma(n(1-\alpha))}(t-s)^{n(1-\alpha)},\quad n\geq 1.$$

□

6.3 Proof of Theorem 6.1.1

First of all, using Assumption **(A4)** we need to prove that the propagator defined as in Subsection 6.2 belongs to $\mathcal{I}(H)$ and is continuous in $\|\cdot\|_{\mathcal{I}}$ for $0\leq s<t\leq 1$. To prove this

fact we now present an extension of the lemma given in [76, Lemma 2.2] to the case of symmetric Banach ideals, which is helpful in further discussion.

Lemma 6.3.1. *Let Assumptions (A1)-(A4) be satisfied. For the propagator* $\{U(t,s)\}_{0\le s\le t\le 1}$ *we have the following estimate*

$$\|U(t,s)\|_{\mathcal{I}} \le \mathrm{Const}\|e^{-\frac{t-s}{2}A}\|_{\mathcal{I}}, \quad s\le t.$$

Proof. Recall that the propagator is defined as following Dyson-Phillips series

$$\begin{aligned} U(t,s) &= \sum_{n=0}^{\infty} S_n(t,s),\\ S_n(t,s) &= (-1)^n \int_s^t d\tau_n \int_s^{\tau_n} d\tau_{n-1} ... \int_s^{\tau_2} d\tau_1 \\ &\qquad e^{-(t-\tau_n)A}B(\tau_n)e^{-(\tau_n-\tau_{n-1})A}\cdots e^{-(\tau_2-\tau_1)A}B(\tau_1)e^{-(\tau_1-s)A}. \end{aligned}$$

Denote by $V(x,y)$ the following

$$V(x,y) := B(x)e^{-(x-y)A/2}, \quad 0\le y<x\le 1.$$

Hence, by (6.5), it follows that $V(x,y)$ is bounded for any $0\le x<y\le 1$ and

$$\|V(x,y)\|_\infty = \left\|B(x)e^{-(x-y)A/2}\right\|_\infty \le \|B(x)A^{-\alpha}\|_\infty \left\|A^\alpha e^{-(x-y)A/2}\right\|_\infty \le \frac{\mathrm{Const}}{(x-y)^\alpha}.$$

We write (using the convention $\tau_0 = s$)

$$e^{-(t-\tau_n)A}B(\tau_n)e^{-(\tau_n-\tau_{n-1})A}\cdots e^{-(\tau_2-\tau_1)A}B(\tau_1)e^{-(\tau_1-s)A} =$$

$$= e^{-(t-\tau_n)A}\prod_{j=n}^{1} V(\tau_j,\tau_{j-1})e^{-(\tau_j-\tau_{j-1})A/2}.$$

By Lemma 3.3.1, we have

$$\left\|e^{-(t-\tau_n)A}B(\tau_n)e^{-(\tau_n-\tau_{n-1})A}\cdots e^{-(\tau_2-\tau_1)A}B(\tau_1)e^{-(\tau_1-s)A}\right\|_{\mathcal{I}} \le$$

$$\le \mathrm{Const}\cdot\|e^{-(t-s)A/2}\|_{\mathcal{I}}\prod_{j=n}^{1}\|V(\tau_j,\tau_{j-1})\|_\infty \le$$

$$\leq \text{Const} \cdot \|e^{-(t-s)A/2}\|_{\mathcal{I}} \prod_{j=n}^{1} (\tau_j - \tau_{j-1})^{-\alpha}.$$

Therefore,

$$\|S_n(t,s)\|_{\mathcal{I}} \leq \text{Const}\|e^{-(t-s)A/2}\|_{\mathcal{I}} \cdot \int_s^t d\tau_n \int_s^{\tau_n} d\tau_{n-1} \cdots \int_s^{\tau_2} d\tau_1 \prod_{j=n}^{1} (\tau_j - \tau_{j-1})^{-\alpha}.$$

Computing the integral, we arrive at

$$\|S_n(t,s)\|_{\mathcal{I}} \leq \text{Const} \left\| e^{-(t-s)A/2} \right\|_{\mathcal{I}} \cdot \frac{\Gamma(1-\alpha)^n}{n(1-\alpha)\Gamma(n(1-\alpha))} (t-s)^{n(1-\alpha)}, \quad n \geq 1. \tag{6.6}$$

Hence, the above Dyson-Phillips series converges in $\|\cdot\|_{\mathcal{I}}$ for $t > s$. Since symmetric Banach ideal is norm-complete, it follows that the propagator belongs to $\mathcal{I}(H)$ and, moreover by (6.6), one has

$$\|U(t,s)\|_{\mathcal{I}} \leq \text{Const} \cdot \|e^{-(t-s)A/2}\|_{\mathcal{I}}.$$

$\square$

Lemma 6.3.2. *Let Assumptions **(A1)**-**(A4)** be satisfied. The propagator is continuous with respect to $\|\cdot\|_{\mathcal{I}}$ for $t > s$.*

Proof. Let $0 < u_1 < u_2 < 1$. We claim that the propagator is continuous with respect to $\|\cdot\|_{\mathcal{I}}$ on the domain $(u_2, 1] \times [1, u_1)$.

Indeed, we have

$$U(t,s) = U(t,u_2) \cdot U(u_2,u_1) \cdot U(u_1,s), \quad t \in (u_2,1], \quad s \in [0,u_1).$$

By Theorem 4.5.9, the mappings $f : t \to U(t,u_2)$, $t \in (u_2,1]$, and $g : s \to U(u_1,s)$, $s \in [0,u_1)$ are continuous in the strong operator topology. By Lemma 6.3.1, we have

$$U(u_1,u_2) = U(u_1, \frac{u_1+u_2}{2}) \cdot U(\frac{u_1+u_2}{2}, u_2) \in \mathcal{I}(H) \cdot \mathcal{I}(H).$$

In particular, $U(u_1,u_2)$ belongs to the separable part of the ideal $\mathcal{I}(H)$. The claim follows now from Lemma 3.3.2.

By the above claim, the propagator is norm-continuous on the set

$$\bigcup_{0<u_1<u_2<1} (u_2, 1] \times [1, u_1) = \{(t, s) : \ 0 \leq s < t \leq 1\}.$$

$\square$

Lemma 6.3.3. *Let Assumptions (A1)-(A4) be satisfied. The propagator is continuously differentiable in the first variable with respect to $\|\cdot\|_{\mathcal{I}}$ for $t > s$. Its derivative is $-(A + B(t))U(t, s)$.*

Proof. Let $s < u < 1$. We claim that the propagator is continuously differentiable in the first variable with respect to $\|\cdot\|_{\mathcal{I}}$ on the interval $(u, 1]$.

Indeed, we have

$$U(t, s) = U(t, u) \cdot U(u, s).$$

By Theorem 4.5.9, the mapping $f : t \to U(t, u)$, $t \in (u, 1]$, is continuous in the strong operator topology. By Lemma 6.3.1, we have

$$U(u, s) = U\left(u, \frac{u+s}{2}\right) \cdot U\left(\frac{u+s}{2}, s\right) \in \mathcal{I}(H) \cdot \mathcal{I}(H).$$

In particular, $U(u, s)$ belongs to the separable part of the ideal $\mathcal{I}(H)$. The claim follows now from Lemma 3.3.3.

By the above claim, the propagator is norm-continuously differentiable in the first variable on the set

$$\bigcup_{s<u<1} (u, 1] = (s, 1].$$

$\square$

Proof of Theorem 6.1.1. The norm estimate for the propagator is established in Lemma 6.3.1. The norm-continuity of the propagator in both variables is established in Lemma 6.3.2. The norm-differentiability of the propagator is established in Lemma 6.3.3. $\square$

6.4 Proof of Theorem 6.1.2

Now we prove the approximation formula for a propagator $\{U(t, s)\}_{0 \leq s < t \leq 1}$.

Lemma 6.4.1. *We have*

$$U_n(t,s) = U_{\left[\frac{n+1}{2}\right]}\left(t, \frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}\right) \cdot U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}, s\right).$$

Proof. Denote for brevity

$$W_{n,k}(t,s) = e^{-\frac{t-s}{n}A} e^{-\frac{t-s}{n}B\left(s+\frac{k}{n}(t-s)\right)}, \quad 1 \le k \le n.$$

We write

$$U_n(t,s) = \prod_{k=n}^{1} W_{n,k}(t,s) = \prod_{k=n}^{\left[\frac{n}{2}\right]+1} W_{n,k}(t,s) \cdot \prod_{k=\left[\frac{n}{2}\right]}^{1} W_{n,k}(t,s).$$

Note that

$$W_{\left[\frac{n+1}{2}\right],k}\left(t, \frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}\right) = W_{n,k+\left[\frac{n}{2}\right]}(t,s), \quad 1 \le k \le \left[\frac{n+1}{2}\right],$$

$$W_{\left[\frac{n}{2}\right],k}\left(\frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}, s\right) = W_{n,k}(t,s), \quad 1 \le k \le \left[\frac{n}{2}\right].$$

Therefore,

$$\prod_{k=n}^{\left[\frac{n}{2}\right]+1} W_{n,k}(t,s) = \prod_{k=\left[\frac{n+1}{2}\right]}^{1} W_{\left[\frac{n+1}{2}\right],k}\left(t, \frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}\right) = U_{\left[\frac{n+1}{2}\right]}\left(t, \frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}\right),$$

$$\prod_{k=\left[\frac{n}{2}\right]}^{1} W_{n,k}(t,s) = \prod_{k=\left[\frac{n}{2}\right]}^{1} W_{\left[\frac{n}{2}\right],k}\left(\frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}, s\right) = U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}, s\right).$$

Combining these equalities, we complete the proof. □

Proof of Theorem 6.1.2. By Lemma 6.4.1 and by the properties of the propagator, we have

$$U_n(t,s) - U(t,s) = U_{\left[\frac{n+1}{2}\right]}\left(t, \frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}\right) \cdot U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}, s\right) -$$

$$-U\left(t, \frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}\right) \cdot U\left(\frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}, s\right) =$$

$$= \left(U_{\left[\frac{n+1}{2}\right]}\left(t, \frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}\right) - U\left(t, \frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}\right)\right) \cdot U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right] + s\left[\frac{n+1}{2}\right]}{n}, s\right) +$$

$$+U\left(t, \frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}\right) \cdot \left(U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}, s\right) - U\left(\frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}, s\right)\right).$$

Hence, by the triangle inequality and the symmetricity of the norm $\|\cdot\|_{\mathcal{I}}$, we have

$$\begin{aligned}
\|U_n(t,s) - U(t,s)\|_{\mathcal{I}} \leq & \left\| U_{\left[\frac{n+1}{2}\right]}\left(t, \frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}\right) - U\left(t, \frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}\right)\right\|_{\infty} \\
& \times \left\| U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}, s\right)\right\|_{\mathcal{I}} \\
+ \left\| U\left(t, \frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}\right)\right\|_{\mathcal{I}} & \left\| U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}, s\right) - U\left(\frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}, s\right)\right\|_{\infty}.
\end{aligned}$$

By (6.4), we have

$$\left\| U_{\left[\frac{n+1}{2}\right]}\left(t, \frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}\right) - U\left(t, \frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}\right)\right\|_{\infty} \leq \epsilon_{\infty}\left(\left[\frac{n+1}{2}\right]\right),$$

$$\left\| U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}, s\right) - U\left(\frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}, s\right)\right\|_{\infty} \leq \epsilon_{\infty}\left(\left[\frac{n}{2}\right]\right).$$

Therefore,

$$\begin{aligned}
\|U_n(t,s) - U(t,s)\|_{\mathcal{I}} \leq & \, \epsilon_{\infty}\left(\left[\frac{n+1}{2}\right]\right) \left\| U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}, s\right)\right\|_{\mathcal{I}} \\
& + \epsilon_{\infty}\left(\left[\frac{n}{2}\right]\right) \left\| U\left(t, \frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}\right)\right\|_{\mathcal{I}}.
\end{aligned} \tag{6.7}$$

Since $B(\cdot)$ is positive, it follows from Lemma 3.3.1 that

$$\|U_m(t,s)\|_{\mathcal{I}} = \left\| \prod_{k=m}^{1} e^{-\frac{t-s}{m}A} e^{-\frac{t-s}{m}B\left(s+k\frac{t-s}{m}\right)}\right\|_{\mathcal{I}} \leq \text{Const}\cdot \left\| e^{-m\cdot\frac{t-s}{m}A}\right\|_{\mathcal{I}} = \text{Const}\cdot\left\| e^{-(t-s)A}\right\|_{\mathcal{I}}.$$

Therefore,

$$\left\| U_{\left[\frac{n}{2}\right]}\left(\frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}, s\right)\right\|_{\mathcal{I}} \leq \text{Const}\cdot\left\| e^{-\frac{(t-s)\left[\frac{n}{2}\right]}{n}}\right\|_{\mathcal{I}} \leq \text{Const}\cdot\left\| e^{-\frac{t-s}{4}A}\right\|_{\mathcal{I}}.$$

Also, by Lemma 6.3.1, we have

$$\left\| U\left(t, \frac{t\left[\frac{n}{2}\right]+s\left[\frac{n+1}{2}\right]}{n}\right)\right\|_{\mathcal{I}} \leq \text{Const}\cdot\left\| e^{-\frac{\left[\frac{n+1}{2}\right](t-s)A}{2n}}\right\|_{\mathcal{I}} \leq \text{Const}\cdot\left\| e^{-\frac{t-s}{4}A}\right\|_{\mathcal{I}}.$$

Substituting those estimates into (6.7), we complete the proof. ☐

Note that Lemma 6.4.1 is also valid for other families of approximants introduced in (6.2) and (6.3). Furthermore, the same line of reasoning as in proof of Theorem 6.1.2 also holds for those families, hence, we omit them.

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