Existence, uniqueness and approximation of solutions to the stochastic Landau-Lifshitz-Gilbert equation on the real line

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Publication Date:
2022

DOI:
https://doi.org/10.26190/unsworks/24013

EXISTENCE, UNIQUENESS AND APPROXIMATION OF SOLUTIONS TO THE STOCHASTIC LANDAU-LIFSHITZ-GILBERT EQUATION ON THE REAL LINE

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A THESIS IN FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

School of Mathematics and Statistics

Faculty of Science

UNSW Sydney

May 9, 2022

Thesis Title and Abstract | Declarations | Inclusion of Publications Statement | Corrected Thesis and Responses

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Acknowledgements

I would like to start by thanking my supervisors Prof. Thanh Tran and Prof. Beniamin Goldys for their continued support, countless long discussions and guidance throughout my years of study. The depth of knowledge paired with their passion for mathematics were a true inspiration for me and integral to the completion of this dissertation.

I am also very thankful to the University of New South Wales for funding me during my candidature and for giving me the opportunity to attend conferences which contributed greatly to my learning experience.

Not to forget my colleagues in the Department of Mathematics and Statistics Haya Aldosari and Debopriya Mukherjee whom I am very grateful for their friendship. They did not hesitate to share their learnings from past experiences which benefitted me considerably especially during the early days.

A huge thank you to my mum and dad for their complete support from start to finish and for standing by my side through the thick and thin. Also, a special thanks to my three and a half years old son, Jude, for creating happy moments midst all the Covid-19 pandemic challenges over the last 18 months.

Last but not least, there are no words to express my gratitude and appreciation to the most loving and supportive husband, Alaa Mikati, for encouraging me every step of this journey and helping me get back up every time I stumbled. THANK YOU.

This research has been supported by an Australian Government Research Training Program (RTP) Scholarship.

Abstract

The Landau-Lifshitz-Gilbert(LLG) equation is a partial differential equation describing the motion of magnetic moments in a ferromagnetic material. In the theory of ferromagnetism, an important problem is to study noise-induced transitions between different equilibrium states. Hence, the LLG equation needs to be modified in order to incorporate random fluctuations into the dynamics of the magnetisation. Including the noise effects in the theory of evolution of magnetic moments requires a proper study of the stochastic version of the LLG equation. The aim of this thesis is to lay foundation of the theory of the stochastic LLG equation for magnetic nanowire of infinite length that is widely used in physics to study the dynamics of the domain walls. The deterministic version of this equation has been intensely studied in recent years due its importance for fabrication of magnetic devices. It is customary to study the nanowire of infinite length. This approach allows for relatively simple mathematical description and at the same time provides a useful approximation of the wires of finite length.

Firstly, we propose a semi-discrete finite difference method to find approximate solutions to the stochastic LLG equation on the real line. Then, we transform the discretised equation into a partial differential equation with random coefficients, without the Itô term, to prove the convergence of approximate solutions. We deduce the existence and uniqueness of a global unique strong solution to the stochastic problem on the whole real line. The main novelty of our approach is that we prove the existence of pathwise solutions, unique for each given in advance trajectory of the noise.

Secondly, in order to solve numerically the stochastic LLG equation on real line, we truncate the infinite line into a bounded interval. We consider the stochastic problem

on a bounded interval $[-L, L]$ with physically relevant homogeneous Neumann boundary conditions and we show that when L tends to infinity, the solution of the problem on a bounded interval converges to the solution of the original stochastic problem on real line. We also provide pathwise error estimates depending on L.

Finally, in order to solve the stochastic LLG equation numerically we propose a fully-discrete finite difference scheme based on the midpoint rule for the stochastic LLG equation on a bounded interval. We perform first numerical experiments which shows that the fully-discrete solutions converge to the solution of the stochastic problem on a bounded interval $[-L, L]$ for vanishing discretisation parameters. Next, we implement a numerical experiment which validates the convergence of the solution on a bounded interval $[-L, L]$ to the solution on real line when L is large enough.

Contents

Chapter 1

Introduction

1.1 Physical background of the Landau-Lifshitz-Gilbert Equation

Landau and Lifshitz proposed in [37] a model for the dynamics of the magnetisation vector $\mathbf{u} \in \mathbb{R}^3$ of a ferromagnetic material occupying a region D in space at temperatures below the Curie temperature. According to this model, at every point $x \in D$ and every time $t \geq 0$ the vector $\mathbf{u}(t,x)$ has constant length and without loss of generality we can assume that $|\mathbf{u}(t,x)| = 1$. To every configuration $\{\mathbf{u}(t,x)\}$ of magnetisation vectors (spins), Landau and Lifshitz associate the energy $\mathcal{E}(\mathbf{u})$ that in general is a rather complicated expression including the exchange energy, stray energy, anisotropy energy and many others. If at time $t = 0$ the energy of the spin configuration $\mathbf{u}_0(x) = \mathbf{u}(0,x)$ does not minimise the energy functional $\mathcal{E}(\mathbf{u}_0)$ then, again according to Landau and Lifschitz, the configuration will evolve following the dynamics given by the equation

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u} \times \frac{\partial \mathcal{E}}{\partial \mathbf{u}},$$

where $\mathbf{u} \times \mathbf{v}$ stands for the vector product in $\mathbb{R}^3$. Let us note here that in physical literature a different notation is used:

$$\mathbf{H}_{eff} = -\frac{\partial \mathcal{E}}{\partial \mathbf{u}},$$

where $\mathbf{H}_{eff}$ is the so-called effective field, see [16]. Gilbert [24] studied the Landau-Lifshitz equation with a small dissipative term stabilising the equation. In fact, Gilbert considered a model that is now widely known as the so-called Landau-Lifshitz-Gilbert(LLG) equation

$$\frac{\partial \mathbf{u}}{\partial t}(t,x) = \mu \mathbf{u}(t,x) \times \mathbf{H}_{eff}(t,x) - \lambda \mathbf{u}(t,x) \times (\mathbf{u}(t,x) \times \mathbf{H}_{eff}(t,x)) \tag{1.1.1}$$

for $t \geq 0$, $x \in D$ where $D \subseteq \mathbb{R}^d (d = 1, 2, 3)$, $\mu \neq 0$ and $\lambda > 0$ are constants. In fact, as $\mathbf{u}(t,x)$ is on the unit sphere, we have $\left\langle \mathbf{u}, \frac{\partial \mathbf{u}}{\partial t} \right\rangle = 0$. Geometrically this means that we have a sphere of radius 1 and $\frac{\partial \mathbf{u}}{\partial t}$ is in the tangent plane. The idea is to choose two orthogonal vectors $\mathbf{u} \times \mathbf{H}_{eff}$ and $\mathbf{u} \times (\mathbf{u} \times \mathbf{H}_{eff})$ which form an orthogonal basis in the tangent plane.

In this project we follow Stoner and Wohlfarth [47] and consider the energy functional consisting of the exchange energy only, in which case

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2} \int_D |\nabla \mathbf{u}(x)|^2 dx \quad \text{and} \quad \mathbf{H}_{eff} = \Delta \mathbf{u}\,.$$

Then, the LLG equation takes the form

$$\frac{\partial \mathbf{u}}{\partial t}(t,x) = \mu \mathbf{u}(t,x) \times \Delta \mathbf{u}(t,x) - \lambda \mathbf{u}(t,x) \times (\mathbf{u}(t,x) \times \Delta \mathbf{u}(t,x)). \tag{1.1.2}$$

Following physical considerations, this equation must be supplanted with the homogeneous Neumann boundary conditions. We note that the other types of energy contribute to the $\mathbf{H}_{eff}$ with lower order differential operators (nonlocal in the case of stray energy). Therefore, evolution of spins driven by the exchange energy only, while relatively simple, is mathematically the most challenging.

When $D \subset \mathbb{R}^3$ is a bounded domain, the existence of weak solutions to (1.1.1) has been studied by Visintin in [50] where all energy contibutions have been taken into account. Alouges and Soyeur [3] and Bertsch et.al. [9] have proved the existence of weak solutions on a bounded three-dimensional domain where the effective field $\mathbf{H}_{eff}$ contains the exchange energy only. The existence of regular solutions in three-dimensional domains is a challenging problem; the strongest results in this direction can be found in [41].

When the domain is unbounded, $D = \mathbb{R}$, Zhou Yulin, Guo Boling and Tan Shaobin [52] have proved global existence and uniqueness of solutions to (1.1.2). Furthermore, Guo Boling and Min-Chun Hong [26] have proved global existence with small initial data and uniqueness of solutions to (1.1.2) on $\mathbb{R}^2$. When $D = \mathbb{R}^3$, Carbou and Fabrie [15] established the local in time existence, the global existence for small initial data and uniqueness of solutions. Fuwa and Tsutsumi [22] have studied the local in time existence, the global in time existence for small initial data and uniqueness of solutions to (1.1.2) by a semi-discrete finite difference method. In fact, they extended the approach considered by P.L. Sulem, C. Sulem and C. Bardos [48], who studied the Heisenberg equation, to the case of LLG equation.

To solve numerically LLG equation (1.1.2), Weinan E and Xiao-Ping Wang [18] proposed a method which preserves the length $|\mathbf{u}| = 1$ of the numerical solution but there is no guarantee that the scheme preserves energy bounds (see [21]). The discretisation of scalar non-linear partial differential equations which preserves energy bounds has been studied by Furihata [39] who verified the efficacity of this scheme. Based on this study, Fuwa, Ishiwata and Tsutsumi proposed in [21] a finite difference scheme and established error estimates for this problem. This scheme satisfies the length preserving property $|\mathbf{u}| = 1$ and preserves the energy bounds. A similar scheme has been considered in [4] and [5] where the authors applied the finite element method to prove weak convergence of numerical solutions to the LLG equation and studied the stability of this method.

1.2 The Stochastic Landau-Lifshitz-Gilbert Equation

An important problem in the theory of ferromagnetism is to describe noise-induced transitions between equilibrium states and the associated random movements of the domain walls. Therefore, we need to include random fluctuations into the effective field and to adjust correspondingly the LLG equation that will describe random dynamics of the magnetisation vector $\mathbf{u}$. The program to analyse noise-induced transitions was first formulated by Néel [43] and further developped in [10] and [33]. In this dissertation, we add noise to the effective field following [6] and [11]. Therefore, the stochastic version of the LLG equation considered is

$$d\mathbf{u} = \big(\mu\mathbf{u} \times \mathbf{H}_{eff} - \lambda\mathbf{u} \times (\mathbf{u} \times \mathbf{H}_{eff})\big)dt + \mu(\mathbf{u} \times \mathbf{g}) \circ dW(t), \qquad (1.2.1)$$

where $\mathbf{g} : D \to \mathbb{R}^3$ is a given bounded function, W is a real-valued Wiener process and $\circ dW(t)$ stands for the Stratonovich differential. We remark that in the stochastic equation (1.2.1) noise is ignored in the second term on the right hand side of the equation because of the smallness of the parameter λ in physical problems (see [23],[34]). We note that the stochastic term should be understood in the Stratonovich sense in order to accommodate for the pathwise sphere constraint $|\mathbf{u}| = 1$ (see [10], [35], [8]).

When $D \subset \mathbb{R}^3$ is a bounded domain, the existence of a weak martingale solution to (1.2.1) has been studied in [11] using Faedo–Galerkin approximations. In [12], the authors considered a similar equation on a bounded interval and proved the existence of a pathwise unique and regular solution. For bounded domains $D \subset \mathbb{R}^2$, A. Hocquet proved the existence of pathwise Struwe solutions in [32]. A convergent finite element discretisation to (1.2.1) on a bounded domain is considered in [25]. The authors first transformed the stochastic equation into a partial differential equation without the Itô term and then proposed a convergent linear scheme to show the existence of weak martingale solutions. Another finite element scheme has been proposed in [6]. We note that this is a non-linear fully-discrete scheme based on the midpoint rule to ensure that numerical solutions satisfy the sphere constraint. They use a different approach that works directly with the Itô equation to prove convergence of numerical solutions to a weak martingale solution for vanishing discretisation parameters. A new convergent time semi-discrete scheme for (1.2.1) is proposed in [2] to prove the existence of a martingale solution. This scheme is only linearly implicit and does not require the resolution of a non-linear problem at each time step. In [7], the authors proposed an efficient non-linear solver which makes the scheme proposed computationally more attractive. We note that other semi-implicit numerical methods for the stochastic LLG equation have been proposed in [42]. In fact, there is a vast literature on numerical methods for stochastic linear and non-linear partial differential equations including [29], [38] and [30].

When the domain is unbounded, $D = \mathbb{R}^d$ for any $d > 0$, an equation similar to (1.2.1) has been studied in [27] using difference method. In this work, a very restrictive assumption is made that noise is constant in $x \in D$. This assumption corresponds to a choice of the function $\mathbf{g}$ in equation (1.2.1) to be constant across the domain D. The authors established the existence of global weak solutions to the equation. When $D = \mathbb{R}$, they show that the Cauchy problem of this equation has a unique global smooth solution.

1.3 Problem at Issue

In recent years, there was an intense study of magnetic nanowires of infinite length, especially of their travelling wave solutions, see for example [13, 14] and references therein. Interest in such problems is driven by the desire to understand the dynamics of the domain walls, the problem of utmost importance for fabrication of magnetic memories. While real nonowires are of finite length, infinite nanowires offer useful approximation. In particular, explicit formula for the travelling wave solutions can be found.

In this dissertation, we initiate systematic mathematical and numerical analysis of the stochastic LLG equation (1.2.1) on the whole real line driven by the noise depending on space and time. More precisely, the initial value problem with the stochastic Landau-Lifshitz equation studied in this thesis takes the form

$$d\mathbf{u} = (\mu\mathbf{u} \times \Delta\mathbf{u} - \lambda\mathbf{u} \times (\mathbf{u} \times \Delta\mathbf{u}))\, dt + \mu(\mathbf{u} \times \mathbf{g}) \circ dW \quad \text{in } \mathbb{R}^+ \times \mathbb{R}, \tag{1.3.1}$$

$$\mathbf{u}(0, x) = \mathbf{u}_0(x) \quad x \in \mathbb{R}, \tag{1.3.2}$$

$$|\mathbf{u}_0(x)| = 1 \quad x \in \mathbb{R}, \tag{1.3.3}$$

where $\mathbf{u} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^3$ is the magnetisation of a ferromagnetic material, $\mu, \lambda > 0$, the given function $\mathbf{g} : \mathbb{R} \to \mathbb{R}^3$ is bounded and W is a Wiener process defined on a certain probability space. We note that this problem is posed on $\mathbb{R}$ with homogeneous Neumann boundary conditions at infinity. Furthermore, the driving noise can be multi-dimensional but for simplicity of presentation we assume that it is one-dimensional.

Equation (1.3.1) belongs to a difficult class of the so-called critical problems. The concept of criticality comes from physics and is not rigorously defined. We will adopt here the understanding of this concept as introduced in [31]. The main idea is to zoom in the solution by change of variables

$$\mathbf{u}^{\delta}(t,x) = u\left(\delta^{b}t, \delta^{c}x\right) ,$$

where $b, c > 0$. If, for $\delta \to 0$, the nonlinear part of the equation vanishes, while the linear part and the noise remain unchanged, then we say that the problem is subcritical. If the magnitude of the nonlinear part remains the same on small space-time scales, then we say that the problem is critical, and finally it is supercritical, if the magnitude of the nonlinear term grows to infinity for $\delta \to 0$. The subcritical case is the easiest one (that does not mean that it is easy) and the critical case requires a more delicate analysis. Differentianting $\mathbf{u}^{\delta}(t,x) = u\left(\delta^{b}t, \delta^{b/2}x\right)$ and plugging in equation (1.3.1), it is easy to check that this equation is critical.

1.3.1 Different Concepts of Solution

In what follows, we define a global strong solution to (1.3.1)- (1.3.3). This solution is the main object of study in this dissertation. In the literature, there are other concepts of solution which are used when the strong solution is difficult to prove.

Definition 1.3.1. Let $T \in (0,\infty)$ be given. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a given probability space endowed with a filtration $(\mathcal{F}_t)$ on which a one-dimensional Wiener process W adapted to $(\mathcal{F}_t)$ is defined. We say that an $(\mathcal{F}_t)$-adapted stochastic process $\mathbf{u} = \{\mathbf{u}(t);\, t \leq T\}$ taking values in L^2_m for every $m > 0$, a strong solution to (3.1.1)-(3.1.3) for the time interval $[0,T]$, if $\mathbf{u}$ satisfies (1)-(4) below:

(1) for every $m > 0$

$$\mathbf{u}(\cdot) \in C([0,T], L^2_m), \quad \mathbb{P}\text{-a.s.}\,,$$

(2) for every $t \in [0,T]$ and a.e. $x \in \mathbb{R}$

$$|\mathbf{u}(t,x)| = 1,$$

(3)

$$\operatorname*{ess\,sup}_{t\in[0,T]} |\nabla \mathbf{u}(t)|^2_{L^2} + \int_0^T |\Delta \mathbf{u}(t)|^2_{L^2}\, dt < \infty, \quad \mathbb{P}\text{-a.s.}\,,$$

(4) for every $t \in [0,T]$, the following equation holds in L^2_m $\mathbb{P}$-a.s.:

$$\begin{aligned}\mathbf{u}(t) = \mathbf{u}_0 + \mu \int_0^t \mathbf{u}(s) \times \Delta \mathbf{u}(s) ds - \lambda \int_0^t \mathbf{u}(s) \times (\mathbf{u}(s) \times \Delta \mathbf{u}(s))\, ds \\ + \mu \int_0^t (\mathbf{u}(s) \times \mathbf{g}) \circ dW(s)\,. \end{aligned} \tag{1.3.4}$$

Moreover, if $\mathbf{u}$ is a strong solution on $[0,T]$ for all $T \geq 0$, we say that $\mathbf{u}$ is a global strong solution.

Next, we define a weak martingale solution to (1.3.1)- (1.3.3).

Definition 1.3.2. Given $T \in (0,\infty)$, we say that a tuple $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}, W, \mathbf{u})$ is a weak martingale solution to (1.3.1), for the time interval $[0,T]$, if there exists:

(a) a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with the filtration satisfying the usual conditions,

(b) a one-dimensional $(\mathcal{F}_t)$-adapted Wiener process $W = (W_t)_{t\in[0,T]}$,

(c) a progressively measurable process $\mathbf{u} : [0,T] \times \Omega \to L^2_m$

such that

(1) $\mathbf{u}(\cdot,\omega) \in C([0,T], H^{-1})$ for $\mathbb{P} - a.s.\ \omega \in \Omega$;

(2) $\mathbb{E}\left[\operatorname{ess\,sup}_{t\in[0,T]} |\nabla \mathbf{u}(t)|^2_{L^2}\right] < \infty$;

(3) $|\mathbf{u}(t,x)| = 1$ for each $t \in [0,T]$, $a.e. x$, and $\mathbb{P} - a.s.$;

(4) for every $t \in [0,T]$, for all $\psi \in \mathbb{C}^\infty_0$, $\mathbb{P} a.s.$:

$$\begin{aligned}\langle \mathbf{u}(t), \psi\rangle_{L^2_m} = \langle \mathbf{u}_0, \psi\rangle_{L^2_m} - \mu \int_0^t \langle \mathbf{u} \times \nabla \mathbf{u}, \nabla \psi\rangle_{L^2_m} ds - \lambda \int_0^t \langle \mathbf{u} \times \nabla \mathbf{u}, \nabla(\mathbf{u} \times \psi)\rangle_{L^2_m} ds \\ + \mu \int_0^t \langle \mathbf{u} \times \mathbf{g}, \psi\rangle_{L^2_m} \circ dW(s)\,.\end{aligned}$$

If (c) holds for any given in advance filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ and any given in advance Wiener process defined on this space then we say that $\mathbf{u}$ is a weak pathwise solution.

We note that a strong solution is a weak martingale solution and a weak pathwise solution as well.

The definitions of operators, probability spaces, filtration, stochastic processes, Stratonovich integral and the weighted space L^2_m related to the stochastic problem above can be found in Chapter 2.

1.4 Contributions

In this dissertation, we prove the existence and uniqueness of pathwise regular solutions to the stochastic LLG equation (1.3.1)-(1.3.3) on real line, when the noise is space and time dependent. We note that during our work on this project a similar result was obtained by E. Gussetti and A. Hocquet in [28] for the stochastic LLG defined on the circle. They prove the existence and uniqueness of pathwise solutions using the rough paths theory. Their argument is much more complicated than ours but allows them to consider a more general class of noises including fractional Brownian Motion. Our proof is considerably simpler than all other proofs available in the literature. We directly prove the existence and uniqueness of pathwise solutions using rather classical arguments and avoid the arguments based on tightness of approximating measures and the Skorokhod theorem. To this end we start from a semi-discrete finite difference method to find approximate solutions to the stochastic problem (1.3.1)-(1.3.3) following the scheme considered in [3]. A modified version of the semi-discrete scheme is considered to show the existence of approximate solutions. Uniqueness of the semi-discrete solutions is also proved. Then, we transform the discretised stochastic equation into a partial differential equation with random coefficients, where the Itô term vanishes, to prove the strong convergence of the semi-discrete solution. We follow the paper by Goldys, Le and Tran [25] where the same idea is used to prove the existence of a weak martingale solution on a bounded three-dimensional domain. The crucial step in our argument is a new result about the continuous dependence of solutions to the stochastic LLG equation on the trajectories of the driving Wiener process. In some sense, it is a counterpart of the rough paths approach taken in [28]. Then, we derive uniform estimates and use the weak convergence of approximating equations in the space $L^2\left((0,T)\times\Omega;L^2_m\right)$, where L^2_m stands for the weighted L^2-space. Finally, we deduce the existence of a global regular pathwise solution to the stochastic problem (1.3.1)-(1.3.3) on the whole real line. Let us note that our approach provides a natural starting point for numerical algorithms. We also note that differently from [28] we need to choose carefully the functional spaces for the proofs of uniform estimates and weak convergence because the initial condition is not an element of $L^2(\mathbb{R})$.

Our second contribution in this dissertation is approximating the stochastic problem (1.3.1)-(1.3.3) on the whole real line. In fact, we truncate the infinite domain $\mathbb{R}$ into a bounded domain $[-L,L]$ for $L>0$, consider the stochastic problem (1.3.1)-(1.3.3) on this bounded interval with physically relevant homogeneous Neumann boundary conditions, and then show that when $L\to\infty$ the solution $\mathbf{u}_L$ of the stochastic problem on a bounded domain converges to the solution $\mathbf{u}$ of the original problem on $\mathbb{R}$. In the course of the proof, we also obtain pathwise estimates on the rate of convergence of solutions on bounded intervals to solution on the whole real line.

Our next contribution is proposing a fully-discrete finite difference scheme for numerical solution of the stochastic problem on a bounded domain $[-L,L]$. Following Fuwa [21], we propose the fully-discrete scheme based on the midpoint rule to guarantee that the numerical solution is always on the unit sphere. First, we carry out numerical experiments to show numerically that the fully-discrete solutions converge to the solution $\mathbf{u}_L$ of the stochastic problem on a bounded interval $[-L,L]$ for vanishing discretisation parameters. Next, we perform an additional numerical experiment which validates the convergence of $\mathbf{u}_L$ to the solution $\mathbf{u}$ of the original problem on $\mathbb{R}$ when $L\to\infty$.

As a summary, the contributions of this dissertation are:

- Firstly, we prove the existence and uniqueness of a global regular pathwise solution to the stochastic LLG equation on $\mathbb{R}$ by first proposing a semi-discrete finite difference scheme and then proving the convergence of the finite difference solutions to a global regular pathwise solution of the problem, Theorem 3.1.3.

- Secondly, we truncate the infinite domain $\mathbb{R}$ into a bounded domain $[-L, L]$ and consider the stochastic problem on this bounded interval with homogeneous Neumann conditions at the boundaries. We show the convergence of the solution on a bounded domain to the solution of the stochastic problem on $\mathbb{R}$, Theorem 4.1.2.
- Thirdly, we propose a fully-discrete finite difference scheme to solve numerically the stochastic problem on a bounded interval. We perform numerical experiments to show convergence of the fully-discrete solutions to the solution of the stochastic problem on $\mathbb{R}$ for vanishing discretisation parameters and a large domain $[-L, L]$.

We note that our results cannot be extended to equations on $\mathbb{R}^d$, for $d \geq 2$. Indeed, even the deterministic LLG equation has no H^2-valued solutions for $d \geq 2$; in fact, solutions have singularities at which the gradient blows up to infinity. Analysis of such problems in the stochastic case remains a very challenging open problem.

The dissertation consists of five chapters. Chapter 1 is the introduction. Chapter 2 reviews some important spaces and elementary formulas. The definitions of stochastic integrals and Stratonovich differentials which are important for the rest of the dissertation are also discussed in this chapter. It also reviews important function spaces on a lattice and some discrete functions results which will be frequently used in the following chapters.

Chapter 3 is devoted to our first contribution. In this chapter, we propose a semi-discrete finite difference scheme to find approximate solutions of (1.3.1)-(1.3.3). We prove that the finite difference solutions converge to a global regular pathwise solution of the stochastic LLG problem on the whole real line. Uniqueness of the global regular solution is also proved.

In Chapter 4, we present our second contribution. For the aim of solving numerically the stochastic problem (1.3.1)-(1.3.3) on the whole real line, we truncate the infinite domain into a bounded domain $[-L, L]$. We consider the problem on a bounded domain with homogeneous Neumann boundary conditions and prove the convergence of the solution $\mathbf{u}_L$ on a bounded domain to the solution $\mathbf{u}$ of the original problem on $\mathbb{R}$ when L tends to infinity.

In Chapter 5, we propose a fully-discrete finite difference scheme to solve numerically the stochastic problem on a bounded domain $[-L, L]$. We perform numerical experiments to show convergence of the fully-discrete finite difference solutions to the solution of the original problem on $\mathbb{R}$ for vanishing discretisation parameters and for L large enough.

Chapter 2

Preliminaries

2.1 Function Spaces

We start with the definition of the operators involved in the stochastic LLG equation. Let U be an open subset of $\mathbb{R}$ and $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$ be an $\mathbb{R}^3$-valued function defined on U. The *gradient* and *Laplace* operators acting on $\boldsymbol{\phi}$ are respectively denoted as

$$\nabla\boldsymbol{\phi} := \left(\frac{\partial\phi_1}{\partial x}, \frac{\partial\phi_2}{\partial x}, \frac{\partial\phi_3}{\partial x}\right),$$
$$\Delta\boldsymbol{\phi} := \left(\frac{\partial^2\phi_1}{\partial x^2}, \frac{\partial^2\phi_2}{\partial x^2}, \frac{\partial^2\phi_3}{\partial x^2}\right),$$

if the derivatives exist in the weak sense.

We denote by $|\cdot|_X$ the norm in a Banach space X. If X is a Hilbert space, we denote by $\langle\cdot,\cdot\rangle_X$ the inner product in X. When subscripts are omitted, $|\cdot|$ and $\langle\cdot,\cdot\rangle$ refer to the Euclidean norm and inner product in $\mathbb{R}^3$ respectively.

We also denote some standard spaces of functions defined on U as follows

- $L^\infty(U) := \{\mathbf{v} : U \to \mathbb{R}^3 | \quad \operatorname{ess\,sup}_{x\in U} |\mathbf{v}(x)| < \infty\}$,
- $L^2(U) := \{\mathbf{v} : U \to \mathbb{R}^3 | \quad \int_U |\mathbf{v}(x)|^2 dx < \infty\}$,
- $H^1(U) := \{\mathbf{v} : U \to \mathbb{R}^3 | \quad \int_U |\mathbf{v}(x)|^2 dx + \int_U |\nabla\mathbf{v}(x)|^2 dx < \infty\}$,
- $H^2(U) := \{\mathbf{v} : U \to \mathbb{R}^3 | \quad \int_U |\mathbf{v}(x)|^2 dx + \int_U |\nabla\mathbf{v}(x)|^2 dx + \int_U |\Delta\mathbf{v}(x)|^2 dx < \infty\}$.

The following well known fact will be important for Chapters 3 and 4.

Lemma 2.1.1. *Let $L > 0$ and $\boldsymbol{\varphi} \in H^1(-L, L)$. Then*

$$|\boldsymbol{\varphi}|_{L^\infty}^2 \leq k|\boldsymbol{\varphi}|_{L^2}|\boldsymbol{\varphi}|_{H^1} \,,$$

where

$$k = 2\sqrt{2}\max\left(1, \frac{1}{L}\right) .$$

Proof. Consider first the case when $\boldsymbol{\varphi} \in H^1(-1,1)$ and $\boldsymbol{\varphi}(-1) = 0$. Then,

$$|\boldsymbol{\varphi}(x)|^2 = \left|\int_{-1}^{x} \frac{d}{dt}\boldsymbol{\varphi}^2(t)dt\right| = \left|\int_{-1}^{x} 2\boldsymbol{\varphi}(t)\cdot\nabla\boldsymbol{\varphi}(t)dt\right| \leq 2|\boldsymbol{\varphi}|_{L^2}|\nabla\boldsymbol{\varphi}|_{L^2}.$$

Hence,

$$|\boldsymbol{\varphi}|_{L^\infty}^2 \leq 2|\boldsymbol{\varphi}|_{L^2}|\nabla\boldsymbol{\varphi}|_{L^2}. \tag{2.1.1}$$

The same result holds if $\boldsymbol{\varphi}(-1) \neq 0$ but $\boldsymbol{\varphi}(1) = 0$.

Next, we consider the case when $\boldsymbol{\varphi} \in H^1(-1,1)$ and $\boldsymbol{\varphi}(\pm 1) \neq 0$. Let

$$m(x) = \begin{cases} x+1, & x \in [-1,0], \\ 1, & x \in [0,1], \end{cases}$$

and

$$m^*(x) = \begin{cases} 1, & x \in [-1,0], \\ 1-x, & x \in [0,1]. \end{cases}$$

Using (2.1.1), we get

$$|m\boldsymbol{\varphi}|_{L^\infty}^2 \leq 2|m\boldsymbol{\varphi}|_{L^2}\left|\nabla(m\boldsymbol{\varphi})\right|_{L^2} \leq 2\sqrt{2}|\boldsymbol{\varphi}|_{L^2}|\boldsymbol{\varphi}|_{H^1}$$

and

$$|m^*\boldsymbol{\varphi}|_{L^\infty}^2 \leq 2\sqrt{2}|\boldsymbol{\varphi}|_{L^2}|\boldsymbol{\varphi}|_{H^1}.$$

Hence, for $x \in [0,1]$ we have

$$|\boldsymbol{\varphi}(x)|^2 = |m\boldsymbol{\varphi}(x)|^2 \le 2\sqrt{2}|\boldsymbol{\varphi}|_{L^2}|\boldsymbol{\varphi}|_{H^1}$$

and for $x \in [-1,0]$

$$|\boldsymbol{\varphi}(x)|^2 = |m^*\boldsymbol{\varphi}(x)|^2 \le 2\sqrt{2}|\boldsymbol{\varphi}|_{L^2}|\boldsymbol{\varphi}|_{H^1}.$$

Consequently,

$$|\boldsymbol{\varphi}|^2_{L^\infty} \le 2\sqrt{2}|\boldsymbol{\varphi}|_{L^2}|\boldsymbol{\varphi}|_{H^1}. \tag{2.1.2}$$

Finally, we consider the case when $\boldsymbol{\varphi} \in H^1(-L,L)$ for some $L > 0$. We define

$$\boldsymbol{\psi}(y) = \boldsymbol{\varphi}(Ly), \quad y \in [-1,1].$$

Then $\boldsymbol{\psi} \in H^1(-1,1)$ and we have

$$\begin{aligned}
|\boldsymbol{\psi}|^2_{L^2} &= \int_{-1}^{1} |\boldsymbol{\psi}(y)|^2 dy = \int_{-1}^{1} |\boldsymbol{\varphi}(Ly)|^2 dy = \int_{-L}^{L} |\boldsymbol{\varphi}(x)|^2 \frac{1}{L} dx = \frac{1}{L}|\boldsymbol{\varphi}|^2_{L^2},\\
|\nabla\boldsymbol{\psi}|^2_{L^2} &= \int_{-1}^{1} |\nabla\boldsymbol{\psi}(y)|^2 dy = \int_{-1}^{1} |\nabla\boldsymbol{\varphi}(Ly)|^2 dy = \int_{-L}^{L} L^2|\nabla\boldsymbol{\varphi}(x)|^2 \frac{1}{L} dx = L|\nabla\boldsymbol{\varphi}|^2_{L^2}.
\end{aligned}$$

Therefore, using (2.1.2) we get

$$\begin{aligned}
|\boldsymbol{\varphi}|^2_{L^\infty} = |\boldsymbol{\psi}|^2_{L^\infty} \le 2\sqrt{2}|\boldsymbol{\psi}|_{L^2}|\boldsymbol{\psi}|_{H^1} &\le 2\sqrt{2}\left(\frac{1}{\sqrt{L}}|\boldsymbol{\varphi}|_{L^2}\right)\left(\frac{1}{L}|\boldsymbol{\varphi}|^2_{L^2} + L|\nabla\boldsymbol{\varphi}|^2_{L^2}\right)^{\frac{1}{2}}\\
&\le 2\sqrt{2}\frac{1}{\sqrt{L}} \max\left(\frac{1}{\sqrt{L}}, \sqrt{L}\right)|\boldsymbol{\varphi}|_{L^2}|\boldsymbol{\varphi}|_{H^1}\\
&\le 2\sqrt{2}\max\left(\frac{1}{L}, 1\right)|\boldsymbol{\varphi}|_{L^2}|\boldsymbol{\varphi}|_{H^1}
\end{aligned}$$

and the lemma follows. □

We define the following weighted spaces

$$L^2_m(U) := \left\{\mathbf{v} : U \to \mathbb{R}^3 | \quad \int_U |\mathbf{v}(x)|^2 \rho_m(x) dx < \infty\right\},$$

$$H^1_m(U) := \left\{ \mathbf{v} : U \to \mathbb{R}^3 | \quad \int_U |\mathbf{v}(x)|^2 \rho_m(x) dx + \int_U |\nabla \mathbf{v}(x)|^2 \rho_m(x) dx < \infty \right\},$$

where $\rho_m(x) = \mathrm{e}^{-\frac{|x|}{m}}$, $m > 0$.

We finish this section by the following compact embedding result.

Lemma 2.1.2. *For every $m > 0$, the embedding*

$$H^1_m(\mathbb{R}) \hookrightarrow L^2_{m/2}(\mathbb{R})$$

where $\rho_{m/2}(x) = \mathrm{e}^{-\frac{2|x|}{m}}$ is compact.

Proof. The lemma is well known but we were not able to locate a theorem in a suitable form and we decided to provide an independent proof. Let

$$I : H^1_m(\mathbb{R}) \hookrightarrow L^2_{m/2}(\mathbb{R})$$

be the embedding operator: $I\varphi = \varphi$. It is easy to see that I is well defined and bounded. For every $n \geq 1$ we define a bounded operator

$$I_n : H^1_m(\mathbb{R}) \hookrightarrow L^2_{m/2}(\mathbb{R})$$

by the formula $I_n\varphi = \varphi I_{[-,n,n]}$. By Theorem 6.12 in [44] the lemma will follow if we can prove that each operator I_n is compact and

$$\lim_{n\to\infty} \|I_n - I\| = 0\,. \tag{2.1.3}$$

Let B be a unit ball in $H^1_m(\mathbb{R})$. Then the set

$$B_n = \{I_n\varphi;\, \varphi \in B\}$$

can be considered as a bounded closed subset of $H^1(-n,n)$. Therefore, by the Rellich-Kondrachov theorem, see Theorem 6.3 in [1] we find that B_n is compact in $L^2\,(-n,n)$, hence in $L^2_{m/2}(\mathbb{R})$ as well. We will show (2.1.3). Denote $H = L^2_{m/2}(\mathbb{R})$. For $\varphi \in H^1_m(\mathbb{R})$ we have

$$\begin{aligned}
|(I_n - I)\,\varphi|_H^2 &= \int_{|x|\geq n} |\varphi|^2 \rho_{m/2}\, dx \\
&= \int_{|x|\geq n} |\varphi|^2 \frac{\rho_{m/2}}{\rho_m} \rho_m\, dx \\
&\leq e^{-n/m} \int_{\mathbb{R}} |\varphi|^2 \rho_m\, dx \\
&\leq e^{-n/m} |\varphi|_{H^1_m(\mathbb{R})}\,.
\end{aligned}$$

Therefore

$$\|I_n - I\| \leq e^{-n/m} \longrightarrow 0\,,$$

and the proof is complete. □

2.2 Elementary Formulas

We recall some cross product elementary properties which will be frequently used in the following chapters. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^3$. Then

$$\langle \mathbf{a}, \mathbf{a}\times\mathbf{b}\rangle = 0, \tag{2.2.1}$$

$$\langle \mathbf{a}, (\mathbf{a}\times\mathbf{b})\times\mathbf{b}\rangle = -|\mathbf{a}\times\mathbf{b}|^2, \tag{2.2.2}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \langle \mathbf{a},\mathbf{c}\rangle\mathbf{b} - \langle \mathbf{a},\mathbf{b}\rangle\mathbf{c}, \tag{2.2.3}$$

$$|\mathbf{a}\times\mathbf{b}|^2 + \langle \mathbf{a},\mathbf{b}\rangle^2 = |\mathbf{a}|^2|\mathbf{b}|^2, \tag{2.2.4}$$

$$\langle \mathbf{a}, \mathbf{b}\times\mathbf{c}\rangle = \langle \mathbf{b}, \mathbf{c}\times\mathbf{a}\rangle = \langle \mathbf{c}, \mathbf{a}\times\mathbf{b}\rangle. \tag{2.2.5}$$

Next, we state the Young inequality. If $a, b \geq 0$ and $p, q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}. \tag{2.2.6}$$

Furthermore, we present the following Gronwall's inequality which can be found in [49].

Lemma 2.2.1. *Let α, β and u be real-valued positive functions defined on an interval $[a,b]$. Assume that β and u are continuous. If α is non-decreasing and u satisfies the integral inequality*

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds, \quad \forall t \in [a,b],$$

then

$$u(t) \leq \alpha(t)e^{\int_a^t \beta(s)ds}, \quad t \in [a,b]. \tag{2.2.7}$$

2.3 Stochastic Analysis

We start by recalling the definition of a measurable space and a probability space.

Definition 2.3.1. Let Ω be a nonempty set and $\mathcal{F}$ be a collection of subsets of Ω. Then, we call $\mathcal{F}$ a σ-algebra on Ω if the following properties hold:

- $\emptyset \in \Omega$,
- If $F \in \mathcal{F}$, then $F^c \in \mathcal{F}$, where F^c is the complement of F in Ω,
- If $I_1, I_2, I_3, ... \in \mathcal{F}$, then $\cup_{i=1}^{\infty} I_i \in \mathcal{F}$.

$(\Omega, \mathcal{F})$ is called a measurable space.

Definition 2.3.2. A probability measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function on $\mathcal{F}$ taking values in $[0,1]$ such that

- $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$,
- If $\{I_i\} \subset \mathcal{F}$, then $\mathbb{P}(\cup_{i=1}^{\infty} I_i) \leq \sum_{i=1}^{\infty} \mathbb{P}(I_i)$,
- If $\{I_i\} \subset \mathcal{F}$ and $I_i \cap I_j = \emptyset$ for $i \neq j$ then $\mathbb{P}(\cup_{i=1}^{\infty} I_i) = \sum_{i=1}^{\infty} \mathbb{P}(I_i)$.

$(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Remark 2.3.3.
- A set $A \subset \mathcal{F}$ is called an event.
- $\mathbb{P}(A)$ is the probability of the event A.
- A property which is true except for an event of probability zero is said to hold almost surely (abbreviated "a.s.").

We introduce next the Borel-Cantelli lemma which can be found in [20] and will be used in Chapter 4. We start by the following definition.

Definition 2.3.4. Let $A_1, ..., A_n, ...$ be events in a probability space. Then, the event

$$\cap_{n=1}^{\infty} \cup_{m=n}^{\infty} A_m = \{\omega \in \Omega \,|\, \omega \text{ belongs to infinitely many of the } A_n\}$$

is called "A_n infinitely often" and abbreviated by "A_n *i.o.*".

Lemma 2.3.5. *If* $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$*, then* $\mathbb{P}(A_n \text{ } i.o.) = 0$.

Now, we recall the definition of random variables and stochastic processes.

Definition 2.3.6. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathbb{B}$ be a real Banach space endowed with the Borel σ-algebra $\mathcal{B}$. A mapping $X : \Omega \to \mathbb{B}$ is called a $\mathbb{B}$-valued random variable if for each $B \in \mathcal{B}$, we have $X^{-1}(B) \in \mathcal{F}$.

Notation: We usually write X and not $X(\omega)$.

Definition 2.3.7. A collection $\{X(t), t \geq 0\}$ of $\mathbb{B}$-valued random variables is called a $\mathbb{B}$-valued stochastic process.

Definition 2.3.8. An $\mathbb{R}$-valued stochastic process W is called a Wiener process or Brownian motion if

- $W(0) = 0$ a.s.
- $W(t) - W(s)$ has normal distribution $\mathcal{N}(0, t-s)$ for all $t \geq s \geq 0$.
- for all $0 < t_1 < t_2 < ... < t_n$, the random variables $W(t_1), W(t_2) - W(t_1), ..., W(t_n) - W(t_{n-1})$ are independent.

Furthermore, we mention the Chebyshev's inequality (see [20]) which will be important in Chapters 3 and 4.

Lemma 2.3.9. *If X is a random variable and $1 \leq p < \infty$, then*

$$\mathbb{P}\left(|X| \geq \lambda\right) \leq \frac{1}{\lambda^p} \mathbb{E}\left[|X|^p\right]$$

for all $\lambda > 0$.

We will recall now the definition of a filtration.

Definition 2.3.10. An increasing family $\{\mathcal{F}_t\}_{t \geq 0}$ of σ-algebras such that $\mathcal{F}_t \subset \mathcal{F}, t \geq 0$ is called a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$.

Definition 2.3.11. Let $\mathbb{B}$ be a separable Banach space and $X = \{X(t), t \geq 0\}$ be a $\mathbb{B}$-valued stochastic process. The process X is called adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if for any $t \geq 0$, the random variable $X(t)$ is $\mathcal{F}_t$-measurable.

Furthermore, we recall the general definition of a stopping time.

Definition 2.3.12. A random variable $\tau : \Omega \to [0, +\infty]$ is called a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ provided $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

We will recall briefly the basic properties of the Itô stochastic integral in the case of processes taking values in Hilbert spaces in the case of one dimensional Wiener process. For a more general and more detailed construction see for example [17]. In order to introduce the Itô integral, we start by defining a step process. Let $\mathbb{H}$ be a real separable Hilbert space.

Definition 2.3.13. Let X be an $(\mathscr{F}_t)$-adapted stochastic process defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$, taking values in $\mathbb{H}$ and such that $\mathbb{E}\left[|X(t)|_{\mathbb{H}}^2\right] < \infty$ for all $t \leq T$. We will say that X is a step process if there exists a partition $P = \{0 = t_0 < t_1 < ... < t_m = T\}$ such that

$$X(t) \equiv X_k \quad \text{for } t_k \leq t < t_{k+1} \quad (k = 0, ..., m-1).$$

Then, the Itô integral for a step process X is defined as follows.

Definition 2.3.14. Let X be a step process and let W be a one-dimensional $(\mathscr{F}_t)$-adapted Wiener process. Then,

$$\int_0^T X dW = \sum_{k=0}^{m-1} X_k \left(W(t_{k+1}) - W(t_k)\right)$$

is the Itô stochastic integral of X on the interval $[0, T]$.

Now, we will define the Itô integral for arbitrary adapted $\mathbb{H}$-valued process such that

$$\mathbb{E}\left[\int_0^T |X(t)|_{\mathbb{H}}^2 \, dt\right] < \infty \, .$$

The class of such processes will be denoted by $\mathscr{M}_T^2(\mathbb{H})$. It is known, see [17] (Proposition 4.22), that for every $X \in \mathscr{M}_T^2(\mathbb{H})$ there exists a sequence of bounded step processes X^n such that

$$\mathbb{E}\left[\int_0^T |X(t) - X^n(t)|_{\mathbb{H}}^2 dt\right] \to 0 \quad \text{as } n \to \infty. \tag{2.3.1}$$

Theorem 2.3.15. *Let $X \in \mathscr{M}_T^2(\mathbb{H})$ and let (X^n) be a sequence of step processes that satisfy* (2.3.1)*. Then,*

$$\mathbb{E}\left[\left|\int_0^T X^n dW - \int_0^T X^m dW\right|_{\mathbb{H}}^2\right] = \mathbb{E}\left[\int_0^T |X^n - X^m|_{\mathbb{H}}^2 \, dt\right] \to 0 \quad \text{as } n, m \to \infty.$$

For the L^2-limit of stochastic integrals $\int_0^T X^n dW$ we will use the notation of the Itô integral:

$$\int_0^T X dW := \lim_{n\to\infty} \int_0^T X^n dW \, .$$

We will recall the definition of Itô processes.

Definition 2.3.16. Let $\mathbb{H}$ be a separable Hilbert space and W be a one-dimensional Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$. An $\mathbb{H}$-valued stochastic process $\{X(t)\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called an Itô process if its paths are a.s. continuous and if it is represented as

$$X(t) = X(0) + \int_0^t y(s)ds + \int_0^t z(s)dW(s) \tag{2.3.2}$$

where $X(0)$ is $\mathcal{F}_0$-measurable, y and z are $(\mathcal{F}_t)$-adapted, y is Bochner integrable and $z \in \mathscr{M}_T^2(\mathbb{H})$ for all $T \geq 0$.

If $\{X(t)\}_{t\geq 0}$ is an Itô process of the form (2.3.2), it can sometimes be written in the following differential form

$$dX(t) = y(t)dt + z(t)dW(t).$$

Throughout this dissertation, we will also use the relation between Stratonovich and Itô differentials. We will recall this connection in the case of specific processes that are important in the dissertation. Let $\mathbf{g} \in L^\infty(\mathbb{R})$. Then, the mapping

$$G : L^2(0,T; L^2_m(\mathbb{R})) \to L^2(0,T; L^2_m(\mathbb{R}))$$

$$\mathbf{u} \mapsto \mathbf{u} \times \mathbf{g}$$

is well defined and differentiable with

$$G'(\mathbf{u})\mathbf{h} = G(\mathbf{h}), \quad \mathbf{u}, \mathbf{h} \in L^2(0,T; L^2_m(\mathbb{R})).$$

In particular, $G'(\mathbf{u})[G(\mathbf{u})] = G^2(\mathbf{u})$. We will use the fact that for every adapted $L^2_m(\mathbb{R})$-valued process $\mathbf{u}$ such that

$$\mathbb{E}\left[\int_0^T |\mathbf{u}(t)|^2_{L^2_m(\mathbb{R})}\, dt\right] < \infty\,,$$

we have

$$\int_0^t (G\mathbf{u}) \circ dW(s) = \frac{1}{2}\int_0^t G'(\mathbf{u})G(\mathbf{u})(s)ds + \int_0^t G(\mathbf{u})(s)dW(s), \qquad \text{a.s.} \tag{2.3.3}$$

Now, we state a lemma which will be frequently used in the following chapters.

Lemma 2.3.17. *Let $\{X(t)\}_{t\geq 0}$ be an Itô process taking values in a Hilbert space $\mathbb{H}$ and satisfying the stochastic differential*

$$dX(t) = y(t)dt + z(t)dW(t)$$

where y and z are such that $\int_0^T |y(s)|^2_{\mathbb{H}} ds < \infty$ and $z \in \mathscr{M}^2_T(\mathbb{H})$ for all $T \geq 0$. Then,

$$d\left(|X(t)|^2_{\mathbb{H}}\right) = \left(2\langle X(t), y(t)\rangle_{\mathbb{H}} + |z(t)|^2_{\mathbb{H}}\right) dt + 2\left\langle X(t), z(t)\right\rangle_{\mathbb{H}} dW(t).$$

We note that the above lemma is a special case of the Itô formula.

Next, we recall a special case of the Burkholder-Davis-Gundy(BDG) inequality that will be used in the following chapters.

Lemma 2.3.18. *For every $1 \leq p < \infty$, there exists a constant $C_p > 0$ such that for every $(\mathcal{F}_t)$-adapted stochastic process ϕ taking values in a separable Hilbert space $\mathbb{H}$, we have*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t \phi(s)dW(s)\right|_{\mathbb{H}}^p\right] \leq C_p\mathbb{E}\left[\left(\int_0^T |\phi(s)|_{\mathbb{H}}^2 ds\right)^{\frac{p}{2}}\right].$$

We finish this section by an important proposition which will be used in Chapter 3. First, we define the spaces of progressively measurable processes $\mathcal{H}_m := L^2(\Omega; L^2(0,T; L^2_m))$ and

$$\mathcal{H}_m^1 := \left\{\phi \in \mathcal{H}_m | \quad \mathbb{E}\left[\int_0^T |\nabla\phi(t)|_{L^2}^2 dt\right] < \infty\right\},$$
$$\mathcal{H}_m^2 := \left\{\phi \in \mathcal{H}_m^1 | \quad \mathbb{E}\left[\int_0^T |\Delta\phi(t)|_{L^2}^2 dt\right] < \infty\right\},$$

with the corresponding norms

$$|\phi|_{\mathcal{H}_m}^2 = \mathbb{E}\left[\int_0^T |\phi(t)|_{L^2_m}^2 \, dt\right],$$

$$|\phi|_{\mathcal{H}_m^1}^2 = |\phi|_{\mathcal{H}_m}^2 + \mathbb{E}\left[\int_0^T |\nabla\phi(t)|_{L^2}^2 \, dt\right],$$

and

$$|\phi|_{\mathcal{H}_m^2}^2 = |\phi|_{\mathcal{H}_m^1}^2 + \mathbb{E}\left[\int_0^T |\Delta\phi(t)|_{L^2}^2 \, dt\right].$$

The operator $\boldsymbol{\nabla} = I_{L^2([0,T]\times\Omega)} \otimes \nabla$ is well defined on $\mathcal{H}_m^1$ and the operator $\boldsymbol{\Delta} = I_{L^2([0,T]\times\Omega)} \otimes \Delta$ is well defined on $\mathcal{H}_m^2$. Next, we present the following proposition.

Proposition 2.3.19. *If a sequence ϕ_n satisfies as $n \to \infty$*

$$\phi_n \to \phi \quad \textit{and} \quad \boldsymbol{\nabla}\phi_n \to \psi$$

weakly in $\mathcal{H}_m$, for $\phi, \psi \in \mathcal{H}_m$, then $\phi \in \mathcal{H}_m^1$ and $\psi = \boldsymbol{\nabla}\phi$. In addition, if as $n \to \infty$

$$\boldsymbol{\Delta}\phi_n \to \xi$$

weakly in $\mathcal{H}_m$, for $\xi \in \mathcal{H}_m$, then $\phi \in \mathcal{H}_m^2$ and $\xi = \boldsymbol{\Delta}\phi$.

Proof. The spaces $\mathcal{H}_m^1$ and $\mathcal{H}_m^2$ with their respective norms are Hilbert spaces that have continuous embeddings into $\mathcal{H}_m$. Therefore, if a sequence ϕ_n satisfies as $n \to \infty$

$$\phi_n \to \phi \quad \text{and} \quad \boldsymbol{\nabla}\phi_n \to \psi$$

in norm in $\mathcal{H}_m$, for $\phi, \psi \in \mathcal{H}_m$, then $\phi \in \mathcal{H}_m^1$ and $\psi = \boldsymbol{\nabla}\phi$. If additionally

$$\boldsymbol{\Delta}\phi_n \to \xi$$

in norm in $\mathcal{H}_m$, for $\xi \in \mathcal{H}_m$, then, $\phi \in \mathcal{H}_m^2$ and $\xi = \boldsymbol{\Delta}\phi$. Since, by the Mazur Theorem, see Theorem 2.5.16 in [40], convex subsets of the Hilbert space $\mathcal{H}_m \times \mathcal{H}_m$ are norm closed if and only if they are weakly closed, the lemma follows. $\square$

2.4 Function Spaces on a Bounded Lattice

In this section, we consider $0 < L < \infty$. Given $h > 0$, let $\mathbb{X}_h$ be a discretization of $[-L, L]$ with the vertices $x_i = ih$, i.e.

$$\mathbb{X}_h = \{ih | i = 0, \pm 1, ..., \pm I\},$$

with $h = \frac{L}{I}$. Given $k, T > 0$, let $\mathbb{T}_k$ be a discretisation of $[0, T]$ with the vertices $t_n = nk$, i.e.

$$\mathbb{T}_k = \{nk | n = 0, 1, ..., N\},$$

with $k = \frac{T}{N}$.

For any $\mathbf{v} : \mathbb{X}_h \to \mathbb{R}^3$, we define the functions $\tau^{\pm}\mathbf{v} : \mathbb{X}_h \to \mathbb{R}^3$ by

$$\tau^{+}\mathbf{v}(x_i) := \mathbf{v}(x_i + h) \quad \text{for all } -I \leq i \leq I - 1,$$
$$\tau^{-}\mathbf{v}(x_i) := \mathbf{v}(x_i - h) \quad \text{for all } -(I - 1) \leq i \leq I.$$

For any $\mathbf{v} : \mathbb{X}_h \to \mathbb{R}^3$, we denote by D^+ and D^- the following finite difference operators

$$
\begin{aligned}
D^+\mathbf{v}(x_i) &:= \frac{\tau^+\mathbf{v}(x_i) - \mathbf{v}(x_i)}{h} \quad \text{for all } -I \le i \le I-1, \\
D^-\mathbf{v}(x_i) &:= \frac{\mathbf{v}(x_i) - \tau^-\mathbf{v}(x_i)}{h} \quad \text{for all } -(I-1) \le i \le I.
\end{aligned}
$$

For any $\mathbf{v} : \mathbb{X}_h \to \mathbb{R}^3$, we also denote by $\tilde{\Delta}$ the following discretised Laplacian

$$
\begin{aligned}
\tilde{\Delta}\mathbf{v}(x_i) &:= D^+D^-\mathbf{v}(x_i) = D^-D^+\mathbf{v}(x_i) \\
&= \frac{\tau^+\mathbf{v}(x_i) - 2\mathbf{v}(x_i) + \tau^-\mathbf{v}(x_i)}{h^2} \quad \text{for all } -(I-1) \le i \le I-1.
\end{aligned}
$$

We recall below the definitions of some L^p-spaces of functions defined on the lattice $\mathbb{X}_h$:

- $$L_h^\infty(\mathbb{X}_h) := \{\mathbf{v} : \mathbb{X}_h \to \mathbb{R}^3\}$$

  with the associated norm $|\mathbf{v}|_{L_h^\infty} := \sup_{x_i \in \mathbb{X}_h} |\mathbf{v}(x_i)|$,
- for $1 \le p < \infty$

  $$L_h^p(\mathbb{X}_h) := \{\mathbf{v} : \mathbb{X}_h \to \mathbb{R}^3\}$$

  with the associated norm $|\mathbf{v}|_{L_h^p} := \left(h \sum_{x_i \in \mathbb{X}_h} |\mathbf{v}(x_i)|^p\right)^{\frac{1}{p}}$.
- In particular, for $p = 2$ we obtain a Hilbert space

  $$L_h^2(\mathbb{X}_h) := \{\mathbf{v} : \mathbb{X}_h \to \mathbb{R}^3\}$$

  with the associated inner product $\langle \mathbf{u}, \mathbf{v} \rangle_{L_h^2} := h \sum_{x_i \in \mathbb{X}_h} \langle \mathbf{u}(x_i), \mathbf{v}(x_i) \rangle$.

Furthermore, we present the following discrete Gronwall's inequality which will be used in Chapter 5.

Lemma 2.4.1. *Let y_n and g_n be non-negative sequences and C a non-negative constant. If*

$$y_n \le C + \sum_{k=0}^{n-1} g_k y_k \quad \textit{for } n \ge 0,$$

then

$$y_n \leq Ce^{\sum_{j=0}^{n-1} g_j} \quad \textit{for } n \geq 0.$$

Proof. The lemma follows immediately from Theorem 5.1 on page 498 of [19]. It is enough to define a discrete measure $\mu(\{k\}) = g_k$. □

2.5 Function Spaces on an Unbounded Lattice

In this section, we consider $U = \mathbb{R}$. Given $h > 0$, let $\mathbb{Z}_h$ be a discretization of $\mathbb{R}$ with the vertices $x_i = ih$, i.e.

$$\mathbb{Z}_h = \{ih | i = 0, \pm 1, ...\}.$$

For any $\mathbf{v} : \mathbb{R} \to \mathbb{R}^n$ $(n = 1, 3)$, we define the functions $\tau^{\pm}\mathbf{v} : \mathbb{R} \to \mathbb{R}^n$ by

$$\tau^{\pm}\mathbf{v}(x) := \mathbf{v}(x \pm h) \quad \text{for all } x \in \mathbb{R}.$$

For any $\mathbf{v} : \mathbb{R} \to \mathbb{R}^n$ $(n = 1, 3)$, we denote by D^+ and D^- the following finite difference operators

$$\begin{aligned} D^+\mathbf{v}(x) &:= \frac{\tau^+\mathbf{v}(x) - \mathbf{v}(x)}{h} \quad \text{for all } x \in \mathbb{R}, \\ D^-\mathbf{v}(x) &:= \frac{\mathbf{v}(x) - \tau^-\mathbf{v}(x)}{h} \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

For any $\mathbf{v} : \mathbb{R} \to \mathbb{R}^n$ $(n = 1, 3)$, we also denote by $\tilde{\Delta}$ the following discretised Laplacian

$$\begin{aligned} \tilde{\Delta}\mathbf{v}(x) &:= D^+D^-\mathbf{v}(x) = D^-D^+\mathbf{v}(x) \\ &= \frac{\tau^+\mathbf{v}(x) - 2\mathbf{v}(x) + \tau^-\mathbf{v}(x)}{h^2} \quad \text{for all } x \in \mathbb{R}. \end{aligned}$$

We note that all the operators $\tau^{\pm}$, $D^{\pm}$ and $\tilde{\Delta}$ depend on h but we omit h for simplicity of notations. We will use the same notations for the restrictions of the operators $\tau^{\pm}$, $D^{\pm}$ and $\tilde{\Delta}$ acting on functions $\mathbf{v} : \mathbb{Z}_h \to \mathbb{R}^3$.

We recall below the definitions of L^p-spaces of functions defined on the lattice $\mathbb{Z}_h$:

- $$L_h^\infty(\mathbb{Z}_h) := \left\{ \mathbf{v} : \mathbb{Z}_h \to \mathbb{R}^3 | \quad \sup_{x_i \in \mathbb{Z}_h} |\mathbf{v}(x_i)| < \infty \right\}$$

  with the associated norm $|\mathbf{v}|_{L_h^\infty} := \sup_{x_i \in \mathbb{Z}_h} |\mathbf{v}(x_i)|$,

- for $1 \leq p < \infty$

  $$L_h^p(\mathbb{Z}_h) := \left\{ \mathbf{v} : \mathbb{Z}_h \to \mathbb{R}^3 | \quad h \sum_{x_i \in \mathbb{Z}_h} |\mathbf{v}(x_i)|^p < \infty \right\}$$

  with the associated norm $|\mathbf{v}|_{L_h^p} := \left(h \sum_{x_i \in \mathbb{Z}_h} |\mathbf{v}(x_i)|^p \right)^{\frac{1}{p}}$.

- In particular, for $p = 2$ we obtain a Hilbert space

  $$L_h^2(\mathbb{Z}_h) := \left\{ \mathbf{v} : \mathbb{Z}_h \to \mathbb{R}^3 | \quad h \sum_{x_i \in \mathbb{Z}_h} |\mathbf{v}(x_i)|^2 < \infty \right\}$$

  with the associated inner product $\langle \mathbf{u}, \mathbf{v} \rangle_{L_h^2} := h \sum_{x_i \in \mathbb{Z}_h} \langle \mathbf{u}(x_i), \mathbf{v}(x_i) \rangle$.

- $$H_h^1(\mathbb{Z}_h) := \left\{ \mathbf{v} : \mathbb{Z}_h \to \mathbb{R}^3 | \quad |\mathbf{v}|_{L_h^2} + |D^+ \mathbf{v}|_{L_h^2} < \infty \right\}$$

  with the associated norm $|\mathbf{v}|_{H_h^1} := \sqrt{|\mathbf{v}|_{L_h^2}^2 + |D^+ \mathbf{v}|_{L_h^2}^2}$,

- for $1 \leq p < \infty$ and $m > 0$

  $$L_{m,h}^p(\mathbb{Z}_h) := \left\{ \mathbf{v} : \mathbb{Z}_h \to \mathbb{R}^3 | \quad h \sum_{x_i \in \mathbb{Z}_h} \mathrm{e}^{-\frac{|x_i|}{m}} |\mathbf{v}(x_i)|^p < \infty \right\}$$

  with the associated norm $|\mathbf{v}|_{L_{m,h}^p} := \left(h \sum_{x_i \in \mathbb{Z}_h} \mathrm{e}^{-\frac{|x_i|}{m}} |\mathbf{v}(x_i)|^p \right)^{\frac{1}{p}}$.

- In particular, for $p = 2$

  $$L_{m,h}^2(\mathbb{Z}_h) := \left\{ \mathbf{v} : \mathbb{Z}_h \to \mathbb{R}^3 | \quad h \sum_{x_i \in \mathbb{Z}_h} \mathrm{e}^{-\frac{|x_i|}{m}} |\mathbf{v}(x_i)|^2 < \infty \right\}$$

  is a Hilbert space with the associated inner product

  $$\langle \mathbf{u}, \mathbf{v} \rangle_{L_{m,h}^2} := h \sum_{x_i \in \mathbb{Z}_h} \mathrm{e}^{-\frac{|x_i|}{m}} \langle \mathbf{u}(x_i), \mathbf{v}(x_i) \rangle.$$

Furthermore, we will state the following result which will be used in Chapter 3.

Lemma 2.5.1. *We define*

$$\zeta_m(x_i) := \mathrm{e}^{-\frac{|x_i|}{2m}} = \sqrt{\rho_m(x_i)}, \qquad x_i \in \mathbb{Z}_h.$$

For sufficiently large m, we have

$$|\zeta_m(x_i)|^2 - |D^-\zeta_m(x_i)|^2 \geq 0 \tag{2.5.1}$$

for every i and $h < 1$.

Proof. By simple calculation, we get for $\alpha := \frac{1}{2m}$

$$|\zeta_m(x_i)|^2 - |D^-\zeta_m(x_i)|^2 = \mathrm{e}^{-2\alpha|x_i|}\left(1 - \frac{\left(\mathrm{e}^{-\alpha(|x_i-h|-|x_i|)} - 1\right)^2}{h^2}\right).$$

For $x_i \geq h$, we have

$$|\zeta_m(x_i)|^2 - |D^-\zeta_m(x_i)|^2 = \mathrm{e}^{-2\alpha x_i}\left(1 - \frac{\left(\mathrm{e}^{\alpha h} - 1\right)^2}{h^2\alpha^2}\alpha^2\right).$$

By taking m sufficiently large, we get

$$|\zeta_m(x_i)|^2 - |D^-\zeta_m(x_i)|^2 = \mathrm{e}^{-2\alpha x_i}(1-\alpha^2) + O(h\alpha)$$

which implies (2.5.1). For $x_i < h$, we have

$$|\zeta_m(x_i)|^2 - |D^-\zeta_m(x_i)|^2 = \mathrm{e}^{2\alpha x_i}\left(1 - \frac{\left(\mathrm{e}^{-\alpha h} - 1\right)^2}{h^2\alpha^2}\alpha^2\right).$$

By taking m sufficiently large, we get

$$|\zeta_m(x_i)|^2 - |D^-\zeta_m(x_i)|^2 = \mathrm{e}^{2\alpha x_i}(1-\alpha^2) + O(h\alpha)$$

which implies (2.5.1). $\square$

Throughout this dissertation, we will frequently use the following result.

Lemma 2.5.2. *For any two functions* $\mathbf{u} : \mathbb{Z}_h \to \mathbb{R}^3$ *and* $v : \mathbb{Z}_h \to \mathbb{R}$ *, we have*

$$|\tau^+\mathbf{u}|_{L^p_h} = |\mathbf{u}|_{L^p_h} = |\tau^-\mathbf{u}|_{L^p_h}, \tag{2.5.2}$$

$$|D^+\mathbf{u}|_{L^p_h} = |D^-\mathbf{u}|_{L^p_h}, \tag{2.5.3}$$

$$\left|(D^+)^2\mathbf{u}\right|_{L^p_h} = |\tilde{\Delta}\mathbf{u}|_{L^p_h}, \tag{2.5.4}$$

$$\left|(\tau^+v)\mathbf{u}\right|_{L^p_h} = \left|v(\tau^-\mathbf{u})\right|_{L^p_h}, \tag{2.5.5}$$

$$\left|(\tau^+v)D^+\mathbf{u}\right|_{L^p_h} = |vD^-\mathbf{u}|_{L^p_h}, \tag{2.5.6}$$

$$|D^+vD^+\mathbf{u}|_{L^p_h} = |D^-vD^-\mathbf{u}|_{L^p_h} \tag{2.5.7}$$

for $1 \leq p \leq \infty$.

Proof. We will prove (2.5.3) and all the other equalities follow in the same manner. For $1 \leq p < \infty$, by simple calculations

$$\begin{aligned}
|D^+\mathbf{u}|^p_{L^p_h} &= \sum_{x_i\in\mathbb{Z}_h} h|D^+\mathbf{u}(x_i)|^p = \sum_{x_i\in\mathbb{Z}_h} h\left|\frac{\tau^+\mathbf{u}(x_i) - \mathbf{u}(x_i)}{h}\right|^p \\
&= \sum_{x_i\in\mathbb{Z}_h} h\left|\frac{\mathbf{u}(x_i) - \tau^-\mathbf{u}(x_i)}{h}\right|^p = \sum_{x_i\in\mathbb{Z}_h} h|D^-\mathbf{u}(x_i)|^p = |D^-\mathbf{u}|^p_{L^p_h}.
\end{aligned}$$

For $p = \infty$,

$$|D^+\mathbf{u}|_{L^\infty_h} = \sup_{x_i\in\mathbb{Z}_h}\left|\frac{\tau^+\mathbf{u}(x_i) - \mathbf{u}(x_i)}{h}\right| = \sup_{x_i\in\mathbb{Z}_h}\left|\frac{\mathbf{u}(x_i) - \tau^-\mathbf{u}(x_i)}{h}\right| = |D^-\mathbf{u}|_{L^\infty_h}.$$

□

Next, we state some lemmas which will be used throughout the rest of this dissertation.

Lemma 2.5.3. *For any two functions* $\mathbf{u}$,$\mathbf{v} : \mathbb{R} \to \mathbb{R}^3$, *we have for every* $x \in \mathbb{R}$

$$\begin{aligned}
D^+\langle\mathbf{u}(x), \mathbf{v}(x)\rangle &= \langle D^+\mathbf{u}(x), \tau^+\mathbf{v}(x)\rangle + \langle\mathbf{u}(x), D^+\mathbf{v}(x)\rangle \\
&= \langle D^+\mathbf{u}(x), \mathbf{v}(x)\rangle + \langle\tau^+\mathbf{u}(x), D^+\mathbf{v}(x)\rangle,
\end{aligned}$$

$$\begin{aligned} D^+(\mathbf{u}\times\mathbf{v})(x) &= D^+\mathbf{u}(x)\times\tau^+\mathbf{v}(x)+\mathbf{u}(x)\times D^+\mathbf{v}(x) \\ &= D^+\mathbf{u}(x)\times\mathbf{v}(x)+\tau^+\mathbf{u}(x)\times D^+\mathbf{v}(x). \end{aligned}$$

The same equalities hold if we replace τ^+ and D^+ by τ^- and D^- respectively.

Proof. Simple calculation reveals the following

$$\begin{aligned} D^+\langle\mathbf{u}(x),\mathbf{v}(x)\rangle &= \frac{1}{h}\Big(\langle\tau^+\mathbf{u}(x),\tau^+\mathbf{v}(x)\rangle-\langle\mathbf{u}(x),\mathbf{v}(x)\rangle\Big) \\ &= \frac{1}{h}\Big(\langle\tau^+\mathbf{u}(x)-\mathbf{u}(x),\tau^+\mathbf{v}(x)\rangle+\langle\mathbf{u}(x),\tau^+\mathbf{v}(x)-\mathbf{v}(x)\rangle\Big) \\ &= \langle D^+\mathbf{u}(x),\tau^+\mathbf{v}(x)\rangle+\langle\mathbf{u}(x),D^+\mathbf{v}(x)\rangle. \end{aligned}$$

The remaining equalities can be obtained in the same manner. □

Lemma 2.5.4. *For any two functions $\mathbf{u},\mathbf{v}:\mathbb{Z}_h\to\mathbb{R}^3$ in L^2_h, we have the discrete integration by parts formula*

$$\langle\mathbf{u},D^+\mathbf{v}\rangle_{L^2_h}=-\langle D^-\mathbf{u},\mathbf{v}\rangle_{L^2_h}.$$

Proof. It is clear that

$$\begin{aligned} \langle\mathbf{u},D^+\mathbf{v}\rangle_{L^2_h} &= \left\langle\mathbf{u},\frac{\tau^+\mathbf{v}-\mathbf{v}}{h}\right\rangle_{L^2_h} \\ &= \left\langle\mathbf{u},\frac{\tau^+\mathbf{v}}{h}\right\rangle_{L^2_h}-\left\langle\mathbf{u},\frac{\mathbf{v}}{h}\right\rangle_{L^2_h} \\ &= \left\langle\frac{\tau^-\mathbf{u}}{h},\mathbf{v}\right\rangle_{L^2_h}-\left\langle\frac{\mathbf{u}}{h},\mathbf{v}\right\rangle_{L^2_h} \\ &= -\left\langle\frac{\mathbf{u}-\tau^-\mathbf{u}}{h},\mathbf{v}\right\rangle_{L^2_h} \\ &= -\langle D^-\mathbf{u},\mathbf{v}\rangle_{L^2_h}. \end{aligned}$$

□

Lemma 2.5.5. *For any function $\mathbf{u}:\mathbb{Z}_h\to\mathbb{R}^3$ such that $|\mathbf{u}|=1$, we have the following relation*

$$2\langle \mathbf{u},\tilde{\Delta}\mathbf{u}\rangle+|D^+\mathbf{u}|^2+|D^-\mathbf{u}|^2=0.$$

Proof. From Lemma 2.5.3, we get

$$\begin{aligned}\tilde{\Delta}(|\mathbf{u}|^2)&=D^+\left(\langle D^-\mathbf{u},\mathbf{u}\rangle+\langle \tau^-\mathbf{u},D^-\mathbf{u}\rangle\right)\\&=\left\langle \tilde{\Delta}\mathbf{u},\mathbf{u}\right\rangle+\langle \tau^+D^-\mathbf{u},D^+\mathbf{u}\rangle+\langle D^+\tau^-\mathbf{u},D^-\mathbf{u}\rangle+\left\langle \mathbf{u},\tilde{\Delta}\mathbf{u}\right\rangle\\&=2\left\langle \mathbf{u},\tilde{\Delta}\mathbf{u}\right\rangle+|D^+\mathbf{u}|^2+|D^-\mathbf{u}|^2.\end{aligned}$$

Having $|\mathbf{u}|=1$, we deduce that $\tilde{\Delta}(|\mathbf{u}|^2)=0$ and the result follows. □

The following lemma can be found in [51] (Inequality (11)).

Lemma 2.5.6. *For every $p\in[2,\infty]$, any integer $n\geq 1$ and any non-negative integer $k<n$, there exists a constant $K=K(p,k,n)$ independent of h such that for any function $\mathbf{u}:\mathbb{Z}_h\to\mathbb{R}^3$*

$$\left|(D^+)^k\mathbf{u}\right|_{L^p_h}\leq K\,|\mathbf{u}|_{L^2_h}^{1-\frac{(k+\frac{1}{2}-\frac{1}{p})}{n}}\left|(D^+)^n\mathbf{u}\right|_{L^2_h}^{\frac{k+\frac{1}{2}-\frac{1}{p}}{n}}.$$

For any $\mathbf{f}:\mathbb{R}\to\mathbb{R}^3$, we will denote by $\mathbf{f}^h:\mathbb{Z}_h\to\mathbb{R}^3$ the restriction of $\mathbf{f}$ to $\mathbb{Z}_h$. We define from $\mathbf{f}^h$ a piecewise constant function defined on the whole real line as follows. For each x, let x_i be such that $x\in[x_i,x_{i+1})$. Then,

$$r_h\mathbf{f}^h(x)=\mathbf{f}^h(x_i).$$

The following lemma will be frequently used throughout this dissertation.

Lemma 2.5.7. *(a) If $\mathbf{f}:\mathbb{R}\to\mathbb{R}^3$ is such that $\mathbf{f}\in H^1(\mathbb{R})$ then $\mathbf{f}^h\in L^2_h(\mathbb{Z}_h)$.*

(b) Let $\mathbf{f}:\mathbb{R}\to\mathbb{R}^3$ have the property

$$\int_{\mathbb{R}}|\nabla\mathbf{f}(x)|^2\mathrm{d}x<\infty\,.$$

Then,

$$\lim_{h\to 0}\int_{\mathbb{R}} \left|r_h\mathbf{f}^h(x)-\mathbf{f}(x)\right|^2 \mathrm{d}x = 0\,.$$

Proof. We start by proving part (a). Simple calculation reveals

$$\begin{aligned}
|\mathbf{f}^h|^2_{L^2_h} &= \sum_{x_i\in\mathbb{Z}_h} h|\mathbf{f}(x_i)|^2\\
&= \sum_{x_i\in\mathbb{Z}_h} h|\mathbf{f}(x_i)|^2 - \int_{\mathbb{R}}|\mathbf{f}(x)|^2dx + \int_{\mathbb{R}}|\mathbf{f}(x)|^2dx\\
&= \sum_{x_i\in\mathbb{Z}_h} \int_{x_i}^{x_{i+1}}(|\mathbf{f}(x_i)|^2-|\mathbf{f}(x)|^2)dx + \int_{\mathbb{R}}|\mathbf{f}(x)|^2dx\\
&= -\sum_{x_i\in\mathbb{Z}_h} \int_{x_i}^{x_{i+1}}\left(\int_{x_i}^{x}\nabla|\mathbf{f}(s)|^2ds\right)dx + \int_{\mathbb{R}}|\mathbf{f}(x)|^2dx\\
&= -\sum_{x_i\in\mathbb{Z}_h} \int_{x_i}^{x_{i+1}}\left(2\int_{x_i}^{x}\langle\mathbf{f}(s),\nabla\mathbf{f}(s)\rangle\, ds\right)dx + \int_{\mathbb{R}}|\mathbf{f}(x)|^2dx\\
&\le 2\sum_{x_i\in\mathbb{Z}_h} \int_{x_i}^{x_{i+1}}\left(\int_{x_i}^{x_{i+1}}|\mathbf{f}(s)||\nabla\mathbf{f}(s)|ds\right)dx + \int_{\mathbb{R}}|\mathbf{f}(x)|^2dx\\
&\le \sum_{x_i\in\mathbb{Z}_h} h\int_{x_i}^{x_{i+1}}|\mathbf{f}(s)|^2ds + \sum_{x_i\in\mathbb{Z}_h} h\int_{x_i}^{x_{i+1}}|\nabla\mathbf{f}(s)|^2ds + \int_{\mathbb{R}}|\mathbf{f}(x)|^2dx\\
&\le h\int_{\mathbb{R}}(|\mathbf{f}(s)|^2+|\nabla\mathbf{f}(s)|^2)ds + \int_{\mathbb{R}}|\mathbf{f}(x)|^2dx\\
&\le h|\mathbf{f}|^2_{H^1} + |\mathbf{f}|^2_{L^2}\\
&< \infty.
\end{aligned}$$

Next, we prove part (b). We have,

$$\begin{aligned}
\int_{\mathbb{R}}\left|r_h\mathbf{f}^h(x)-\mathbf{f}(x)\right|^2\mathrm{d}x &= \sum_{x_i\in\mathbb{Z}_h}\int_{x_i}^{x_{i+1}}\left|\mathbf{f}(x_i)-\mathbf{f}(x)\right|^2dx\\
&= \sum_{x_i\in\mathbb{Z}_h}\int_{x_i}^{x_{i+1}}\left(\int_{x_i}^{x}\nabla\mathbf{f}(s)ds\right)^2dx\\
&\le \sum_{x_i\in\mathbb{Z}_h}\int_{x_i}^{x_{i+1}}\left(\int_{x_i}^{x_{i+1}}|\nabla\mathbf{f}(s)|ds\right)^2dx\\
&\le h^2\sum_{x_i\in\mathbb{Z}_h}\int_{x_i}^{x_{i+1}}|\nabla\mathbf{f}(s)|^2ds
\end{aligned}$$

$$\le h^2 \int_{\mathbb{R}} |\nabla \mathbf{f}(s)|^2 ds.$$

Then, using the assumption on $\mathbf{f}$ we get

$$\lim_{h\to 0} \int_{\mathbb{R}} \left| r_h \mathbf{f}^h(x) - \mathbf{f}(x) \right|^2 \mathrm{d}x = 0.$$

$\square$

Throughout the rest of the thesis, we will use the notation C to refer to a generic constant which may take different values at different occurences.

Now for $\mathbf{v}^h : \mathbb{Z}_h \to \mathbb{R}^3$, we denote by p_h the interpolation operator such that for each x, let x_i be such that $x \in [x_i, x_{i+1})$ and

$$p_h \mathbf{v}^h(x) = \mathbf{v}^h(x_i) + D^+ \mathbf{v}^h(x_i)(x - x_i).$$

We note that $p_h \mathbf{v}^h$ is continuous in $\mathbb{R}$ and that

$$\nabla p_h \mathbf{v}^h = r_h D^+ \mathbf{v}^h. \tag{2.5.8}$$

We have with these operators the following proposition. We note that a similar proposition can be found in [36] (see Lemma 3.1, 3.2 (page 224-226)).

Proposition 2.5.8. *(a) Assume that*

$$|D^+ \mathbf{v}^h|_{L^2_h} \le C$$

where C is independent of h. If one of the interpolants $p_h \mathbf{v}^h, r_h \mathbf{v}^h$ converges strongly in L^2_m when h tends to 0, then the other one also converges strongly to the same limit.

(b) Assume that

$$|\mathbf{v}^h|_{L^2_{m,h}} + |D^+ \mathbf{v}^h|_{L^2_h} \le C$$

where C is independent of h. If one of the interpolants $p_h\mathbf{v}^h, r_h\mathbf{v}^h$ converges weakly in L^2_m when h tends to 0, then the other one also converges weakly to the same limit.

Proof. First we prove (a). For any $x \in [x_i, x_{i+1})$, we have

$$p_h\mathbf{v}^h(x) - r_h\mathbf{v}^h(x) = D^+\mathbf{v}^h(x_i)(x - x_i).$$

Then,

$$\sup_{x_i \le x \le x_{i+1}} |p_h\mathbf{v}^h(x) - r_h\mathbf{v}^h(x)| \le h|D^+\mathbf{v}^h(x_i)|.$$

Therefore,

$$\begin{aligned}
|p_h\mathbf{v}^h - r_h\mathbf{v}^h|^2_{L^2_m} &= \int_{\mathbb{R}} |p_h\mathbf{v}^h(x) - r_h\mathbf{v}^h(x)|^2\rho_m(x)dx \\
&= \sum_{x_i \in \mathbb{Z}_h} \int_{x_i}^{x_{i+1}} |p_h\mathbf{v}^h(x) - r_h\mathbf{v}^h(x)|^2\rho_m(x)dx \\
&\le h^2 \sum_{x_i \in \mathbb{Z}_h} |D^+\mathbf{v}^h(x_i)|^2 \int_{x_i}^{x_{i+1}} \rho_m(x)dx \\
&\le h^2|D^+\mathbf{v}^h|^2_{L^2_h} \\
&\le Ch^2.
\end{aligned}$$

Consequently,

$$\lim_{h\to 0} |p_h\mathbf{v}^h - r_h\mathbf{v}^h|^2_{L^2_m} = 0. \tag{2.5.9}$$

Now, we assume that there exists $\mathbf{v} \in L^2_m$ such that

$$\lim_{h\to 0} |p_h\mathbf{v}^h - \mathbf{v}|_{L^2_m} = 0. \tag{2.5.10}$$

We have

$$|r_h\mathbf{v}^h - \mathbf{v}|_{L^2_m} \le |r_h\mathbf{v}^h - p_h\mathbf{v}^h|_{L^2_m} + |p_h\mathbf{v}^h - \mathbf{v}|_{L^2_m}.$$

Hence, from (2.5.9) and (2.5.10) we get

$$\lim_{h\to 0} |r_h\mathbf{v}^h - \mathbf{v}|_{L^2_m} = 0.$$

Similarly, if there exists $\mathbf{v} \in L^2_m$ such that

$$\lim_{h\to 0} |r_h \mathbf{v}^h - \mathbf{v}|_{L^2_m} = 0,$$

we obtain

$$\lim_{h\to 0} |p_h \mathbf{v}^h - \mathbf{v}|_{L^2_m} = 0.$$

Next, we prove (b). We note that

$$\begin{aligned}
|r_h \mathbf{v}^h|^2_{L^2_m} &= \sum_{x_i\in\mathbb{Z}_h} \int_{x_i}^{x_{i+1}} |r_h \mathbf{v}^h(x)|^2 \rho_m(x) dx \\
&= \sum_{x_i\in\mathbb{Z}_h} |\mathbf{v}^h(x_i)|^2 \int_{x_i}^{x_{i+1}} \rho_m(x) dx \\
&\leq C |\mathbf{v}^h|^2_{L^2_{m,h}} \\
&\leq C.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|p_h \mathbf{v}^h|^2_{L^2_m} &= \sum_{x_i\in\mathbb{Z}_h} \int_{x_i}^{x_{i+1}} |p_h \mathbf{v}^h(x)|^2 \rho_m(x) dx \\
&\leq 2 \sum_{x_i\in\mathbb{Z}_h} \int_{x_i}^{x_{i+1}} |\mathbf{v}^h(x_i)|^2 \rho_m(x) dx + 2 \sum_{x_i\in\mathbb{Z}_h} \int_{x_i}^{x_{i+1}} |D^+ \mathbf{v}^h(x_i)|^2 |x - x_i|^2 \rho_m(x) dx \\
&\leq C \sum_{x_i\in\mathbb{Z}_h} |\mathbf{v}^h(x_i)|^2 \int_{x_i}^{x_{i+1}} \rho_m(x) dx + C \sum_{x_i\in\mathbb{Z}_h} |\mathbf{v}^h(x_{i+1})|^2 \int_{x_i}^{x_{i+1}} \rho_m(x) dx \\
&\leq C |\mathbf{v}^h|^2_{L^2_{m,h}} \\
&\leq C.
\end{aligned}$$

We consider $\boldsymbol{\phi} \in C_c^\infty(\mathbb{R})$. Then,

$$\lim_{h\to 0} |r_h \boldsymbol{\phi}^h - \boldsymbol{\phi}|_{L^2_m} = 0.$$

Thus, we have

$$\lim_{h\to 0} \left\langle r_h \mathbf{v}^h, \boldsymbol{\phi} - r_h \boldsymbol{\phi}^h \right\rangle_{L^2_m} = 0, \tag{2.5.11}$$

$$\lim_{h\to 0} \left\langle p_h \mathbf{v}^h, \boldsymbol{\phi} - r_h \boldsymbol{\phi}^h \right\rangle_{L^2_m} = 0. \tag{2.5.12}$$

Moreover,

$$\begin{aligned}
\langle r_h \mathbf{v}^h - p_h \mathbf{v}^h, r_h \boldsymbol{\phi}^h \rangle_{L^2_m} &= \sum_{x_i \in \mathbb{Z}_h} \int_{x_i}^{x_{i+1}} \left\langle D^+ \mathbf{v}^h(x_i)(x - x_i), \boldsymbol{\phi}^h(x_i) \right\rangle \rho_m(x) dx \\
&\leq h \sum_{x_i \in \mathbb{Z}_h} |D^+ \mathbf{v}^h(x_i)||\boldsymbol{\phi}^h(x_i)| \int_{x_i}^{x_{i+1}} \rho_m(x) dx \\
&\leq h^2 \sum_{x_i \in \mathbb{Z}_h} |D^+ \mathbf{v}^h(x_i)||\boldsymbol{\phi}^h(x_i)| \rho_m(x_i) \\
&\leq \frac{h^2}{2} \sum_{x_i \in \mathbb{Z}_h} |D^+ \mathbf{v}^h(x_i)|^2 \rho_m(x_i) + \frac{h^2}{2} \sum_{x_i \in \mathbb{Z}_h} |\boldsymbol{\phi}^h(x_i)|^2 \rho_m(x_i) \\
&\leq \frac{h}{2} \bigg(|D^+ \mathbf{v}^h|^2_{L^2_{m,h}} + |\boldsymbol{\phi}^h|^2_{L^2_{m,h}} \bigg) \\
&\leq \frac{h}{2} \bigg(|D^+ \mathbf{v}^h|^2_{L^2_h} + |\boldsymbol{\phi}^h|^2_{L^2_{m,h}} \bigg).
\end{aligned}$$

Hence,

$$\left\langle r_h \mathbf{v}^h - p_h \mathbf{v}^h, r_h \boldsymbol{\phi}^h \right\rangle_{L^2_m} \leq Ch,$$

which implies

$$\lim_{h\to 0} \left\langle r_h \mathbf{v}^h - p_h \mathbf{v}^h, r_h \boldsymbol{\phi}^h \right\rangle_{L^2_m} = 0. \tag{2.5.13}$$

Now, we assume that there exists $\mathbf{v} \in L^2_m$ such that

$$\lim_{h\to 0} \left\langle r_h \mathbf{v}^h - \mathbf{v}, \boldsymbol{\phi} \right\rangle_{L^2_m} = 0. \tag{2.5.14}$$

We have

$$\begin{aligned}
\langle p_h \mathbf{v}^h - \mathbf{v}, \boldsymbol{\phi} \rangle_{L^2_m} = &\langle p_h \mathbf{v}^h - r_h \mathbf{v}^h, r_h \boldsymbol{\phi}^h \rangle_{L^2_m} + \langle p_h \mathbf{v}^h, \boldsymbol{\phi} - r_h \boldsymbol{\phi}^h \rangle_{L^2_m} + \langle r_h \mathbf{v}^h, r_h \boldsymbol{\phi}^h - \boldsymbol{\phi} \rangle_{L^2_m} \\
&+ \langle r_h \mathbf{v}^h - \mathbf{v}, \boldsymbol{\phi} \rangle_{L^2_m}.
\end{aligned}$$

Hence, from (2.5.11), (2.5.12), (2.5.13) and (2.5.14) we get

$$\lim_{h\to 0} \left\langle p_h \mathbf{v}^h - \mathbf{v}, \boldsymbol{\phi} \right\rangle_{L^2_m} = 0.$$

Similarly, if there exists $\mathbf{v} \in L^2_m$ such that

$$\lim_{h\to 0} \left\langle p_h \mathbf{v}^h - \mathbf{v}, \boldsymbol{\phi} \right\rangle_{L^2_m} = 0,$$

we obtain

$$\lim_{h\to 0} \left\langle r_h \mathbf{v}^h - \mathbf{v}, \boldsymbol{\phi} \right\rangle_{L^2_m} = 0.$$

□

In what follows, we denote by $\mathcal{H}_{m,h} := L^2(\Omega; L^2(0,T; L^2_{m,h}))$ the space of progressively measurable processes taking values in $L^2_{m,h}$. The operator $\boldsymbol{D}^{\pm} = I_{L^2([0,T]\times\Omega)} \otimes D^{\pm}$ is well defined on $\mathcal{H}_{m,h}$ and the operator $\tilde{\boldsymbol{\Delta}} = I_{L^2([0,T]\times\Omega)} \otimes \tilde{\Delta}$ is well defined on $\mathcal{H}_{m,h}$.

For $m > 0$ and $h > 0$, we define two operators

$$\begin{aligned} I \otimes r_h &: \mathcal{H}_{m,h} \to \mathcal{H}_m, \\ I \otimes p_h &: \mathcal{H}_{m,h} \to \mathcal{H}_m, \end{aligned}$$

which will still be denoted by r_h and p_h for simplicity and will be used in Chapter 3.

Next, we state that we have the following proposition which can be proved similarly to Proposition 2.5.8.

Proposition 2.5.9. *(a) Assume that ϕ^h satisfies*

$$\mathbb{E}\left[\int_0^T |D^+\phi^h(t)|^2_{L^2_h} dt\right] \le C$$

where C is independent of h. If one of the interpolants $p_h\phi^h, r_h\phi^h$ converges strongly in $\mathcal{H}_m$ when h tends to 0, then the other one also converges strongly to the same limit.

(b) Assume that ϕ^h satisfies

$$\mathbb{E}\Big[\int_0^T |\phi^h(t)|^2_{L^2_{m,h}}\,dt\Big] + \mathbb{E}\Big[\int_0^T |D^+\phi^h(t)|^2_{L^2_h}\,dt\Big] \le C$$

where C is independent of h. If one of the interpolants $p_h\phi^h, r_h\phi^h$ converges weakly in $\mathcal{H}_m$ when h tends to 0, then the other one also converges weakly to the same limit.

Chapter 3

A Semi-Discrete Finite Difference Scheme on Real Line

3.1 Introduction

In this chapter, we employ a semi-discrete finite difference method to derive the existence of global strong solutions to the stochastic LLG problem (1.3.1)-(1.3.3). We start to prove the uniqueness of solutions. The proof of existence of solutions is long and complicated and here we will sketch the main steps and their importance. We start with semi-discrete finite difference approximations (3.3.1). We prove the existence and uniqueness of semi-discrete solutions to equation (3.3.1). Then, we transform the discrete stochastic LLG equation into a partial differential equation with random coefficients (3.6.1) (without the Itô term). The resulting equation has time-differentiable solutions. Finally, we prove uniform estimates which allow us to use the method of compactness in order to get the limit when the discretisation parameter goes to zero. We emphasis that we apply method of compactness to the random partial differential equation and therefore we do not have to use method of martingale solution exploited in other papers ([11], [25]). We note that the proof can't be transferred to equations in multidimensional domains because uniform estimates for Laplacian and fourth power norm of gradient can't be obtained in higher dimensions.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P})$ be a given probability space endowed with a filtration satisfying the usual assumptions. Invoking the relation between Stratonovich and Itô differentials given in (2.3.3), problem (1.3.1)-(1.3.3) can be written as follows

$$d\mathbf{u} = \left(\mu\mathbf{u}\times\Delta\mathbf{u} - \lambda\mathbf{u}\times(\mathbf{u}\times\Delta\mathbf{u}) + \frac{\mu^2}{2}(\mathbf{u}\times\mathbf{g})\times\mathbf{g}\right)dt + \mu(\mathbf{u}\times\mathbf{g})dW, \quad (3.1.1)$$

$$\mathbf{u}(0,x) = \mathbf{u}_0(x), \quad (3.1.2)$$

$$|\mathbf{u}_0(x)| = 1. \quad (3.1.3)$$

We assume that $\lambda > 0$ and $\mu \neq 0$. We recall that a global strong solution of the problem (3.1.1)-(3.1.3) is defined in Chapter 1.

Remark 3.1.1. (a) Definition 1.3.1 implies that for almost every $t \leq T$ and almost every $x \in \mathbb{R}$, we have $\nabla\mathbf{u}(t,x)$ well defined. Then, by part (2) of the definition, we have

$$\nabla\big(|\mathbf{u}(t,x)|^2\big) = 0, \quad (t,x)\text{-a.e.},$$

which implies

$$\langle \mathbf{u}(t,x), \nabla\mathbf{u}(t,x)\rangle = 0, \quad (t,x)\text{-a.e.}$$

In addition, we have

$$\langle \mathbf{u}(t,x), \Delta\mathbf{u}(t,x)\rangle = \nabla\Big(\ \langle \mathbf{u}(t,x), \nabla\mathbf{u}(t,x)\rangle\ \Big) - |\nabla\mathbf{u}(t,x)|^2 = -|\nabla\mathbf{u}(t,x)|^2, \quad (t,x)\text{-a.e.}$$

(b) The elementary property (2.2.4) yields

$$|\mathbf{u}(t,x)\times\Delta\mathbf{u}(t,x)|^2 + \langle \mathbf{u}(t,x), \Delta\mathbf{u}(t,x)\rangle^2 = |\mathbf{u}(t,x)|^2|\Delta\mathbf{u}(t,x)|^2.$$

Then, from Definition 1.3.1 and part (a) of this remark, we get

$$|\mathbf{u}(t,x)\times\Delta\mathbf{u}(t,x)|^2 + |\nabla\mathbf{u}(t,x)|^4 = |\Delta\mathbf{u}(t,x)|^2.$$

Integrating with respect to x and t, we deduce from the definition

$$\int_0^T |\nabla\mathbf{u}(t)|_{L^4}^4 \, dt < \infty\,.$$

Lemma 3.1.2. *A process* $\mathbf{u}$ *is a strong solution of problem* (3.1.1)-(3.1.3) *if and only if it satisfies conditions (1)-(3) of Definition 1.3.1 and*

$$\begin{aligned}\mathbf{u}(t) =& \mathbf{u}_0 + \lambda \int_0^t \Delta \mathbf{u}(s)\, ds + \mu \int_0^t \mathbf{u}(s) \times \Delta \mathbf{u}(s)\, ds + \lambda \int_0^t |\nabla \mathbf{u}(s)|^2 \mathbf{u}(s)\, ds \\ &+ \frac{\mu^2}{2} \int_0^t (\mathbf{u}(s) \times \mathbf{g}) \times \mathbf{g} ds + \mu \int_0^t (\mathbf{u}(s) \times \mathbf{g}) dW(s)\,. \end{aligned} \tag{3.1.4}$$

Proof. Using the elementary property (2.2.3) and Remark 3.1.1 (a) we deduce

$$\begin{aligned}\mathbf{u} \times (\mathbf{u} \times \Delta \mathbf{u}) &= \langle \mathbf{u}, \Delta \mathbf{u} \rangle \, \mathbf{u} - \langle \mathbf{u}, \mathbf{u} \rangle \, \Delta \mathbf{u} \\ &= -|\nabla \mathbf{u}|^2 \mathbf{u} - \Delta \mathbf{u}.\end{aligned}$$

Thus (1.3.4) and (3.1.4) are equivalent. □

The main theorem of this chapter is stated as follows.

Theorem 3.1.3. *Assume that* $|\mathbf{u}_0(x)| = 1$ *for every* $x \in \mathbb{R}$, $\nabla \mathbf{u}_0 \in L^2$ *and* $\mathbf{g} \in H^2$. *Then there exists a unique global strong solution* $\mathbf{u}$ *to* (3.1.1)-(3.1.3)*, such that for every* $p \geq 1$

$$\mathbb{E}\left[\operatorname*{ess\,sup}_{t \in [0,T]} |\nabla \mathbf{u}(t)|_{L^2}^{2p} \right] + \mathbb{E}\left[\left(\int_0^T |\Delta \mathbf{u}(t)|_{L^2}^2 dt \right)^p \right] < \infty\,. \tag{3.1.5}$$

Corollary 3.1.4. *Under assumptions of Theorem 3.1.3, for every* $t > 0$ *we have*

$$\lim_{|x| \to \infty} |\nabla \mathbf{u}(t, x)| = 0\,.$$

3.2 Uniqueness of Global Strong Solution

In this section, we prove the uniqueness of solutions to problem (3.1.1)-(3.1.3). We will start with a more general result.

Theorem 3.2.1. *Let* $\mathbf{u_1}$ *and* $\mathbf{u_2}$ *be two solutions of* (3.1.1)-(3.1.3) *on* $[0, T]$ *in the sense of Definition 1.3.1, starting with* $\mathbf{u_1}(0)$ *and* $\mathbf{u_2}(0)$*, such that* $\mathbf{u_1}(0) - \mathbf{u_2}(0) \in L^2$. *Then*

$\mathbf{u_1}(t) - \mathbf{u_2}(t) \in L^2$ $\mathbb{P}$-*a.s. for every* $t \in [0,T]$ *and there exists a random variable* C_T *such that*

$$|\mathbf{u_1}(t) - \mathbf{u_2}(t)|_{L^2} \leq C_T\, |\mathbf{u_1}(0) - \mathbf{u_2}(0)|_{L^2}\,, \quad t \in [0,T]\,.$$

Proof. In order to simplify notations, we will assume in the proof without loss of generality, that $\lambda = \mu = 1$. Setting $\bar{\mathbf{u}} = \mathbf{u_1} - \mathbf{u_2}$ and using Lemma 3.1.2, we obtain

$$\begin{aligned} d\bar{\mathbf{u}} = (&\Delta\bar{\mathbf{u}} + \mathbf{u_1} \times \Delta\bar{\mathbf{u}} + \bar{\mathbf{u}} \times \Delta\mathbf{u_2} + |\nabla\mathbf{u_1}|^2\bar{\mathbf{u}} + \langle\nabla\mathbf{u_1} + \nabla\mathbf{u_2}, \nabla\bar{\mathbf{u}}\rangle\, \mathbf{u_2} \\ &+ \frac{1}{2}(\bar{\mathbf{u}} \times \mathbf{g}) \times \mathbf{g})dt + (\bar{\mathbf{u}} \times \mathbf{g})dW. \end{aligned}$$

Multiplying by $\zeta_m(x) = \sqrt{\rho_m(x)} = \mathrm{e}^{-\frac{|x|}{2m}}$ and then using Lemma 2.3.17, we get

$$\begin{aligned} \frac{1}{2}d|\zeta_m\bar{\mathbf{u}}|^2_{L^2} - \frac{1}{2}|\zeta_m\bar{\mathbf{u}} \times \mathbf{g}|^2_{L^2}dt = \Bigg(&\int_{\mathbb{R}} \langle\zeta_m\bar{\mathbf{u}}, \zeta_m\Delta\bar{\mathbf{u}}\rangle\, dx + \int_{\mathbb{R}}\langle\zeta_m\bar{\mathbf{u}}, \zeta_m\mathbf{u_1} \times \Delta\bar{\mathbf{u}}\rangle dx \\ &+ \int_{\mathbb{R}}\langle\zeta_m\bar{\mathbf{u}}, \zeta_m\bar{\mathbf{u}} \times \Delta\mathbf{u_2}\rangle dx + \int_{\mathbb{R}}\langle\zeta_m\bar{\mathbf{u}}, \zeta_m|\nabla\mathbf{u_1}|^2\bar{\mathbf{u}}\rangle dx \\ &+ \int_{\mathbb{R}} \langle\zeta_m\bar{\mathbf{u}}, \zeta_m\mathbf{u_2}\rangle\, \langle\nabla\mathbf{u_1} + \nabla\mathbf{u_2}, \nabla\bar{\mathbf{u}}\rangle\, dx \\ &+ \frac{1}{2}\int_{\mathbb{R}} \langle\zeta_m\bar{\mathbf{u}}, (\zeta_m\bar{\mathbf{u}} \times \mathbf{g}) \times \mathbf{g}\rangle\, dx\Bigg)dt \\ &+ \int_{\mathbb{R}}\langle\zeta_m\bar{\mathbf{u}}, \zeta_m\bar{\mathbf{u}} \times \mathbf{g}\rangle dxdW. \end{aligned}$$

Using (2.2.1) and (2.2.2), we find that

$$\langle\zeta_m\bar{\mathbf{u}}, \zeta_m\bar{\mathbf{u}} \times \mathbf{g}\rangle = 0$$

and

$$\frac{1}{2}\int_{\mathbb{R}} \langle\zeta_m\bar{\mathbf{u}}, (\zeta_m\bar{\mathbf{u}} \times \mathbf{g}) \times \mathbf{g}\rangle\, dx = -\frac{1}{2}\, |\zeta_m\bar{\mathbf{u}} \times \mathbf{g}|^2_{L^2}\,.$$

Therefore, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\zeta_m\bar{\mathbf{u}}|^2_{L^2} = &\int_{\mathbb{R}}\langle\zeta_m\bar{\mathbf{u}}, \zeta_m\Delta\bar{\mathbf{u}}\rangle dx + \int_{\mathbb{R}}\langle\zeta_m\bar{\mathbf{u}}, \zeta_m\mathbf{u_1} \times \Delta\bar{\mathbf{u}}\rangle dx \\ &+ \int_{\mathbb{R}}\langle\zeta_m\bar{\mathbf{u}}, \zeta_m|\nabla\mathbf{u_1}|^2\bar{\mathbf{u}}\rangle dx \end{aligned}$$

$$+\int_{\mathbb{R}} \langle \zeta_m \bar{\mathbf{u}}, \zeta_m \mathbf{u_2} \rangle \langle \nabla \mathbf{u_1} + \nabla \mathbf{u_2}, \nabla \bar{\mathbf{u}} \rangle \, dx. \tag{3.2.1}$$

We consider the first term on the right hand side,

$$\begin{aligned}
\int_{\mathbb{R}} \langle \zeta_m \bar{\mathbf{u}}, \zeta_m \Delta \bar{\mathbf{u}} \rangle \, dx &= \int_{\mathbb{R}} \left\langle \zeta_m^2 \bar{\mathbf{u}}, \Delta \bar{\mathbf{u}} \right\rangle dx = -\int_{\mathbb{R}} \left\langle \nabla (\zeta_m^2 \bar{\mathbf{u}}), \nabla \bar{\mathbf{u}} \right\rangle dx \\
&= -2 \int_{\mathbb{R}} \langle \zeta_m \bar{\mathbf{u}}, \nabla \zeta_m \nabla \bar{\mathbf{u}} \rangle \, dx - \int_{\mathbb{R}} |\zeta_m \nabla \bar{\mathbf{u}}|^2 dx.
\end{aligned}$$

For the second term on the right hand side, using (2.2.1) and (2.2.5) we have

$$\begin{aligned}
\int_{\mathbb{R}} \langle \zeta_m \bar{\mathbf{u}}, \zeta_m \mathbf{u_1} \times \Delta \bar{\mathbf{u}} \rangle \, dx &= \int_{\mathbb{R}} \langle \Delta \bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \times \zeta_m \mathbf{u_1} \rangle \, dx \\
&= -\int_{\mathbb{R}} \left\langle \nabla \bar{\mathbf{u}}, \nabla (\zeta_m^2 \bar{\mathbf{u}} \times \mathbf{u_1}) \right\rangle dx \\
&= -2 \int_{\mathbb{R}} \langle \nabla \bar{\mathbf{u}}, \zeta_m \nabla \zeta_m \bar{\mathbf{u}} \times \mathbf{u_1} \rangle \, dx - \int_{\mathbb{R}} \left\langle \nabla \bar{\mathbf{u}}, \zeta_m^2 \nabla \bar{\mathbf{u}} \times \mathbf{u_1} \right\rangle dx \\
&\quad - \int_{\mathbb{R}} \left\langle \nabla \bar{\mathbf{u}}, \zeta_m^2 \bar{\mathbf{u}} \times \nabla \mathbf{u_1} \right\rangle dx \\
&= -2 \int_{\mathbb{R}} \langle \mathbf{u_1} \times \nabla \zeta_m \nabla \bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \rangle \, dx - \int_{\mathbb{R}} \langle \nabla \mathbf{u_1} \times \zeta_m \nabla \bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \rangle \, dx.
\end{aligned}$$

Consequently, we get from (3.2.1)

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} |\zeta_m \bar{\mathbf{u}}|_{L^2}^2 + |\zeta_m \nabla \bar{\mathbf{u}}|_{L^2}^2 = &-2 \int_{\mathbb{R}} \langle \nabla \zeta_m \nabla \bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \rangle \, dx \\
&- 2 \int_{\mathbb{R}} \langle \mathbf{u_1} \times \nabla \zeta_m \nabla \bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \rangle \, dx \\
&- \int_{\mathbb{R}} \langle \nabla \mathbf{u_1} \times \zeta_m \nabla \bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \rangle \, dx \\
&+ \int_{\mathbb{R}} \langle \zeta_m \bar{\mathbf{u}}, \zeta_m \mathbf{u_2} \rangle \langle \nabla \mathbf{u_1} + \nabla \mathbf{u_2}, \nabla \bar{\mathbf{u}} \rangle \, dx \\
&+ \int_{\mathbb{R}} \left\langle \zeta_m \bar{\mathbf{u}}, \zeta_m |\nabla \mathbf{u_1}|^2 \bar{\mathbf{u}} \right\rangle dx.
\end{aligned}$$

Integrating with respect to t and using part (2) of Definition 1.3.1 gives

$$\begin{aligned}\frac{1}{2}|\zeta_m\bar{\mathbf{u}}|_{L^2}^2 - \frac{1}{2}|\zeta_m\bar{\mathbf{u}}(0)|_{L^2}^2 + \int_0^t |\zeta_m\nabla\bar{\mathbf{u}}|_{L^2}^2 ds \leq 4\int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}|\nabla\zeta_m\nabla\bar{\mathbf{u}}|_{L^2} ds \\ + \int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}|\nabla\mathbf{u_1}|_{L^\infty}|\zeta_m\nabla\bar{\mathbf{u}}|_{L^2} ds \\ + \int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}\,|\nabla(\mathbf{u_1}+\mathbf{u_2})|_{L^\infty}\,|\zeta_m\nabla\bar{\mathbf{u}}|_{L^2} ds \\ + \int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}^2|\nabla\mathbf{u_1}|_{L^\infty}^2 ds.\end{aligned}$$

Thus we get, using Young's inequality (2.2.6) for $p = q = 2$,

$$\begin{aligned}\frac{1}{2}|\zeta_m\bar{\mathbf{u}}|_{L^2}^2 - \frac{1}{2}|\zeta_m\bar{\mathbf{u}}(0)|_{L^2}^2 + \frac{1}{2}\int_0^t |\zeta_m\nabla\bar{\mathbf{u}}|_{L^2}^2 ds \leq 8\int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}^2 ds + \frac{1}{2}\int_0^t |\nabla\zeta_m\nabla\bar{\mathbf{u}}|_{L^2}^2 ds \\ + \int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}^2|\nabla\mathbf{u_1}|_{L^\infty}^2 ds \\ + \int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}^2\left(|\nabla\mathbf{u_1}|_{L^\infty}^2 + |\nabla\mathbf{u_2}|_{L^\infty}^2\right) ds \\ + \int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}^2|\nabla\mathbf{u_1}|_{L^\infty}^2 ds.\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}&|\zeta_m\bar{\mathbf{u}}|_{L^2}^2 - |\zeta_m\bar{\mathbf{u}}(0)|_{L^2}^2 + \int_0^t |\zeta_m\nabla\bar{\mathbf{u}}|_{L^2}^2 ds - \int_0^t |\nabla\zeta_m\nabla\bar{\mathbf{u}}|_{L^2}^2 ds \\ &\leq C\int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}^2\left(1 + |\nabla\mathbf{u_1}|_{L^\infty}^2 + |\nabla\mathbf{u_2}|_{L^\infty}^2\right) ds.\end{aligned}$$

Using the fact that $|\nabla\zeta_m| = \frac{1}{2m}|\zeta_m|$ a.e. we find that for $m > 1$

$$\int_0^t |\zeta_m\nabla\bar{\mathbf{u}}|_{L^2}^2 ds - \int_0^t |\nabla\zeta_m\nabla\bar{\mathbf{u}}|_{L^2}^2 ds = \int_0^t \left(1 - \frac{1}{4m^2}\right)|\zeta_m\nabla\bar{\mathbf{u}}|_{L^2}^2 ds \geq 0,$$

and therefore

$$|\zeta_m\bar{\mathbf{u}}|_{L^2}^2 \leq |\zeta_m\bar{\mathbf{u}}(0)|_{L^2}^2 + C\int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2}^2\left(1 + |\nabla\mathbf{u_1}|_{L^\infty}^2 + |\nabla\mathbf{u_2}|_{L^\infty}^2\right) ds\,. \tag{3.2.2}$$

Hence, using Lemma 2.1.1 we have $\mathbb{P}$-a.s.

$$
\begin{aligned}
C_T &:= \int_0^T (1 + |\nabla \mathbf{u_1}|^2_{L^\infty} + |\nabla \mathbf{u_2}|^2_{L^\infty}) ds \\
&\leq C \int_0^T (1 + |\nabla \mathbf{u_1}|^2_{H^1} + |\nabla \mathbf{u_2}|^2_{H^1}) ds \\
&\leq C \int_0^T (1 + |\nabla \mathbf{u_1}|^2_{L^2} + |\Delta \mathbf{u_1}|^2_{L^2} + |\nabla \mathbf{u_2}|^2_{L^2} + |\Delta \mathbf{u_2}|^2_{L^2}) ds \\
&\leq CT + CT \operatorname*{ess\,sup}_{t\in[0,T]} |\nabla \mathbf{u_1}|^2_{L^2} + C \int_0^T |\Delta \mathbf{u_1}|^2_{L^2} ds + CT \operatorname*{ess\,sup}_{t\in[0,T]} |\nabla \mathbf{u_2}|^2_{L^2} + C \int_0^T |\Delta \mathbf{u_2}|^2_{L^2} ds \\
&< \infty.
\end{aligned}
$$

Then, using Lemma 2.2.1, we obtain from (3.2.2)

$$
\begin{aligned}
|\zeta_m \bar{\mathbf{u}}|^2_{L^2} &\leq |\zeta_m \bar{\mathbf{u}}(0)|^2_{L^2} e^{C \int_0^t (1+|\nabla \mathbf{u_1}|^2_{L^\infty} + |\nabla \mathbf{u_2}|^2_{L^\infty}) ds} \\
&\leq C_T |\zeta_m \bar{\mathbf{u}}(0)|^2_{L^2}.
\end{aligned} \tag{3.2.3}
$$

Finally, by taking m to ∞, we deduce using the Monotone Convergence Theorem

$$
|\mathbf{u_1}(t) - \mathbf{u_2}(t)|^2_{L^2} \leq C_T \, |\mathbf{u_1}(0) - \mathbf{u_2}(0)|^2_{L^2}, \quad t \in [0,T],
$$

and the result follows. □

Corollary 3.2.2. *Let $\mathbf{u_1}$ and $\mathbf{u_2}$ be two solutions of (3.1.1)-(3.1.3) on $[0,T]$ in the sense of Definition 1.3.1, such that $\mathbf{u_1}(0) = \mathbf{u_2}(0)$. Then $\mathbf{u_1}(t) = \mathbf{u_2}(t)$ in L^2_m $\mathbb{P}$-a.s. for every $t \in [0,T]$.*

Proof. The result follows from the theorem above. More precisely, it follows from (3.2.3). □

3.3 The Semi-Discrete Finite Difference Scheme

In this section, we design a semi-discrete finite difference method to find approximate solutions to (3.1.1)-(3.1.3). More precisely, we prove in the following sections that the finite difference solutions converge to a global strong solution of (3.1.1)-(3.1.3).

We assume that $\mathbf{g}$ and $\mathbf{u}_0$ satisfy the conditions in Theorem 3.1.3. We recall that $\mathbf{g}^h : \mathbb{Z}_h \to \mathbb{R}^3$ is the restriction of $\mathbf{g}$ to $\mathbb{Z}_h$. Our first aim is to prove the existence and uniqueness of solution $\mathbf{u}^h : \mathbb{R}^+ \times \mathbb{Z}_h \to \mathbb{R}^3$ to the following problem

$$d\mathbf{u}^h = \left(\mu\mathbf{u}^h \times \tilde{\Delta}\mathbf{u}^h - \lambda\mathbf{u}^h \times (\mathbf{u}^h \times \tilde{\Delta}\mathbf{u}^h) + \frac{\mu^2}{2}(\mathbf{u}^h \times \mathbf{g}^h) \times \mathbf{g}^h\right) dt + \mu(\mathbf{u}^h \times \mathbf{g}^h)dW, \tag{3.3.1}$$

$$\mathbf{u}^h(0, x_i) = \mathbf{u}_0(x_i), \tag{3.3.2}$$

$$|\mathbf{u}_0(x_i)| = 1. \tag{3.3.3}$$

A global strong solution of the problem (3.3.1)-(3.3.3) is defined as follows.

Definition 3.3.1. Given $T \in (0,\infty)$, we call an $(\mathcal{F}_t)$-adapted stochastic process $\mathbf{u}^h = \{\mathbf{u}^h(t); t \le T\}$ taking values in $L^2_{m,h}$ for every $m > 0$, a strong solution to (3.3.1)-(3.3.3) for the time interval $[0,T]$, if $\mathbf{u}^h$ satisfies (1)-(4) below:

(1) for every $m > 0$

$$\mathbf{u}^h(\cdot) \in C([0,T], L^2_{m,h}), \quad \mathbb{P}\text{-a.s.},$$

(2) for every $t \in [0,T]$ and $x_i \in \mathbb{Z}_h$

$$|\mathbf{u}^h(t, x_i)| = 1,$$

(3)

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|D^+\mathbf{u}^h(t)\right|^2_{L^2_h}\right] < \infty\,,$$

(4) the following equation holds $\mathbb{P}$-a.s. for all $t \in [0,T]$ in $L^2_{m,h}$:

$$\begin{aligned}\mathbf{u}^h(t) = \mathbf{u}_0 &+ \mu\int_0^t \mathbf{u}^h(s) \times \tilde{\Delta}\mathbf{u}^h(s)ds - \lambda\int_0^t \mathbf{u}^h(s) \times \left(\mathbf{u}^h(s) \times \tilde{\Delta}\mathbf{u}^h(s)\right) ds \\ &+ \frac{\mu^2}{2}\int_0^t \left(\mathbf{u}^h(s) \times \mathbf{g}^h\right) \times \mathbf{g}^h ds + \mu\int_0^t \left(\mathbf{u}^h(s) \times \mathbf{g}^h\right) dW(s).\end{aligned} \tag{3.3.4}$$

Moreover, if $\mathbf{u}^h$ is a strong solution on $[0,T]$ for all $T \ge 0$, we say that $\mathbf{u}^h$ is a global strong solution.

We note that in equation (3.3.4) the first three integrals are Bochner integrals in the space $L^2_{m,h}$ and the last one is the Itô integral in $L^2_{m,h}$.

Remark 3.3.2. By definition of the discrete Laplacian $\tilde{\Delta}$, part (3) of Definition 3.3.1 immediately yields

$$\mathbb{E}\Big[\int_0^T \left|\tilde{\Delta}\mathbf{u}^h(t)\right|^2_{L^2_h} dt\Big] < \infty.$$

3.3.1 Uniqueness of Semi-Discrete Solution

In the subsection, we assume the existence of a global strong solution to (3.3.1)-(3.3.3) and we prove its uniqueness.

Lemma 3.3.3. *Let $\mathbf{u}_1^h$ and $\mathbf{u}_2^h$ be two solutions of* (3.3.1)-(3.3.3) *on* $[0,T]$ *in the sense of Definition 3.3.1, such that $\mathbf{u}_1^h(0) = \mathbf{u}_2^h(0)$. Then $\mathbf{u}_1^h(t) = \mathbf{u}_2^h(t)$ in $L^2_{m,h}$ $\mathbb{P}$-a.s. for every $t \in [0,T]$.*

Proof. The proof is similar to the proof of Theorem 3.2.1. In order to simplify notations, we will assume in the proof without loss of generality, that $\lambda = \mu = 1$. Setting $\overline{\mathbf{u}^h} = \mathbf{u}_1^h - \mathbf{u}_2^h$, then using part (2) of Definition 3.3.1, (2.2.3) and Lemma 2.5.5, one has

$$\begin{aligned} d\overline{\mathbf{u}^h} = \Bigg(&\tilde{\Delta}\overline{\mathbf{u}^h} + \mathbf{u}_1^h \times \tilde{\Delta}\overline{\mathbf{u}^h} + \overline{\mathbf{u}^h} \times \tilde{\Delta}\mathbf{u}_2^h + \frac{1}{2}|D^+\mathbf{u}_1^h|^2\overline{\mathbf{u}^h} + \frac{1}{2}\left\langle D^+\mathbf{u}_1^h + D^+\mathbf{u}_2^h, D^+\overline{\mathbf{u}^h}\right\rangle \mathbf{u}_2^h \\ &+ \frac{1}{2}|D^-\mathbf{u}_1^h|^2\overline{\mathbf{u}^h} + \frac{1}{2}\left\langle D^-\mathbf{u}_1^h + D^-\mathbf{u}_2^h, D^-\overline{\mathbf{u}^h}\right\rangle \mathbf{u}_2^h + \frac{1}{2}(\overline{\mathbf{u}^h} \times \mathbf{g}^h) \times \mathbf{g}^h\Bigg)dt \\ &+ (\overline{\mathbf{u}^h} \times \mathbf{g}^h)dW, \end{aligned}$$

$$\overline{\mathbf{u}^h}(0) = 0\,.$$

Multiplying by $\zeta_m(x_i) = \sqrt{\rho_m(x_i)} = \mathrm{e}^{-\frac{|x_i|}{2m}}$, then by using Lemma 2.3.17 and the elementary property (2.2.1), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left|\zeta_m\overline{\mathbf{u}^h}\right|^2_{L^2_h} - \frac{1}{2}\left|\zeta_m\overline{\mathbf{u}^h} \times \mathbf{g}^h\right|^2_{L^2_h} \\ &= \left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\tilde{\Delta}\overline{\mathbf{u}^h}\right\rangle_{L^2_h} + \left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\mathbf{u}_1^h \times \tilde{\Delta}\overline{\mathbf{u}^h}\right\rangle_{L^2_h} \\ &\quad + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\overline{\mathbf{u}^h}|D^+\mathbf{u}_1^h|^2\right\rangle_{L^2_h} + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\mathbf{u}_2^h\left\langle D^+\mathbf{u}_1^h + D^+\mathbf{u}_2^h, D^+\overline{\mathbf{u}^h}\right\rangle\right\rangle_{L^2_h} \end{aligned}$$

$$+\frac{1}{2}\left\langle\zeta_m\overline{\mathbf{u}^h},\zeta_m\overline{\mathbf{u}^h}|D^-\mathbf{u}_1^h|^2\right\rangle_{L_h^2}+\frac{1}{2}\left\langle\zeta_m\overline{\mathbf{u}^h},\zeta_m\mathbf{u}_2^h\left\langle D^-\mathbf{u}_1^h+D^-\mathbf{u}_2^h,D^-\overline{\mathbf{u}^h}\right\rangle\right\rangle_{L_h^2}$$

$$+\frac{1}{2}\left\langle\zeta_m\bar{\mathbf{u}}^h,(\zeta_m\overline{\mathbf{u}^h}\times\mathbf{g}^h)\times\mathbf{g}^h\right\rangle_{L_h^2}. \tag{3.3.5}$$

We consider the first term on the right hand side, using Lemmas 2.5.2, 2.5.3 and 2.5.4 we get

$$\begin{aligned}\left\langle\zeta_m\overline{\mathbf{u}^h},\zeta_m\tilde{\Delta}\overline{\mathbf{u}^h}\right\rangle_{L_h^2}&=\left\langle\zeta_m^2\overline{\mathbf{u}^h},\tilde{\Delta}\overline{\mathbf{u}^h}\right\rangle_{L_h^2}=-\left\langle D^+(\zeta_m^2\overline{\mathbf{u}^h}),D^+\overline{\mathbf{u}^h}\right\rangle_{L_h^2}\\&=-\left\langle D^+(\zeta_m^2)\overline{\mathbf{u}^h},D^+\overline{\mathbf{u}^h}\right\rangle_{L_h^2}-\left\langle\tau^+\zeta_m^2D^+\overline{\mathbf{u}^h},D^+\overline{\mathbf{u}^h}\right\rangle_{L_h^2}\\&=-\left\langle\zeta_m\overline{\mathbf{u}^h},D^+\zeta_mD^+\overline{\mathbf{u}^h}\right\rangle_{L_h^2}-\left\langle\tau^+\zeta_m\overline{\mathbf{u}^h},D^+\zeta_mD^+\overline{\mathbf{u}^h}\right\rangle_{L_h^2}-|\zeta_mD^-\overline{\mathbf{u}^h}|_{L_h^2}^2.\end{aligned}$$

For the second term on the right hand side, using Lemmas 2.5.3, 2.5.4 and properties (2.2.1), (2.2.5) we have

$$\begin{aligned}&\left\langle\zeta_m\overline{\mathbf{u}^h},\zeta_m\mathbf{u}_1^h\times\tilde{\Delta}\overline{\mathbf{u}^h}\right\rangle_{L_h^2}\\&=\left\langle\tilde{\Delta}\overline{\mathbf{u}^h},\zeta_m\overline{\mathbf{u}^h}\times\zeta_m\mathbf{u}_1^h\right\rangle_{L_h^2}\\&=-\left\langle D^+\overline{\mathbf{u}^h},D^+(\zeta_m^2\overline{\mathbf{u}^h}\times\mathbf{u}_1^h)\right\rangle_{L_h^2}\\&=-\left\langle D^+\overline{\mathbf{u}^h},D^+(\zeta_m^2)\overline{\mathbf{u}^h}\times\tau^+\mathbf{u}_1^h\right\rangle_{L_h^2}-\left\langle D^+\overline{\mathbf{u}^h},\tau^+\zeta_m^2D^+\overline{\mathbf{u}^h}\times\tau^+\mathbf{u}_1^h\right\rangle_{L_h^2}\\&\quad-\left\langle D^+\overline{\mathbf{u}^h},\zeta_m^2\overline{\mathbf{u}^h}\times D^+\mathbf{u}_1^h\right\rangle_{L_h^2}\\&=-\left\langle D^+\overline{\mathbf{u}^h},\zeta_mD^+\zeta_m\overline{\mathbf{u}^h}\times\tau^+\mathbf{u}_1^h\right\rangle_{L_h^2}-\left\langle D^+\overline{\mathbf{u}^h},\tau^+\zeta_mD^+\zeta_m\overline{\mathbf{u}^h}\times\tau^+\mathbf{u}_1^h\right\rangle_{L_h^2}\\&\quad-\left\langle D^+\overline{\mathbf{u}^h},\tau^+\zeta_m^2D^+\overline{\mathbf{u}^h}\times\tau^+\mathbf{u}_1^h\right\rangle_{L_h^2}-\left\langle D^+\overline{\mathbf{u}^h},\zeta_m^2\overline{\mathbf{u}^h}\times D^+\mathbf{u}_1^h\right\rangle_{L_h^2}\\&=-\left\langle\tau^+\mathbf{u}_1^h\times D^+\zeta_mD^+\overline{\mathbf{u}^h},\zeta_m\overline{\mathbf{u}^h}\right\rangle_{L_h^2}-\left\langle\tau^+\mathbf{u}_1^h\times D^+\zeta_mD^+\overline{\mathbf{u}^h},\tau^+\zeta_m\overline{\mathbf{u}^h}\right\rangle_{L_h^2}\\&\quad-\left\langle D^+\mathbf{u}_1^h\times\zeta_mD^+\overline{\mathbf{u}^h},\zeta_m\overline{\mathbf{u}^h}\right\rangle_{L_h^2}.\end{aligned}$$

Consequently, we get from (3.3.5)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\zeta_m\overline{\mathbf{u}^h}\right|^2_{L^2_h} - \frac{1}{2}\left|\zeta_m\overline{\mathbf{u}^h}\times\mathbf{g}^h\right|^2_{L^2_h} + \left|\zeta_m D^-\overline{\mathbf{u}^h}\right|^2_{L^2_h} \\
&= -\left\langle \zeta_m\overline{\mathbf{u}^h}, D^+\zeta_m D^+\overline{\mathbf{u}^h}\right\rangle_{L^2_h} - \left\langle \tau^+\zeta_m\overline{\mathbf{u}^h}, D^+\zeta_m D^+\overline{\mathbf{u}^h}\right\rangle_{L^2_h} \\
&\quad - \left\langle \tau^+\mathbf{u}^h_1\times D^+\zeta_m D^+\overline{\mathbf{u}^h}, \zeta_m\overline{\mathbf{u}^h}\right\rangle_{L^2_h} - \left\langle \tau^+\mathbf{u}^h_1\times D^+\zeta_m D^+\overline{\mathbf{u}^h}, \tau^+\zeta_m\overline{\mathbf{u}^h}\right\rangle_{L^2_h} \\
&\quad - \left\langle D^+\mathbf{u}^h_1\times \zeta_m D^+\overline{\mathbf{u}^h}, \zeta_m\overline{\mathbf{u}^h}\right\rangle_{L^2_h} \\
&\quad + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\overline{\mathbf{u}^h}|D^+\mathbf{u}^h_1|^2\right\rangle_{L^2_h} + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\mathbf{u}^h_2\left\langle D^+\mathbf{u}^h_1 + D^+\mathbf{u}^h_2, D^+\overline{\mathbf{u}^h}\right\rangle\right\rangle_{L^2_h} \\
&\quad + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\overline{\mathbf{u}^h}|D^-\mathbf{u}^h_1|^2\right\rangle_{L^2_h} + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\mathbf{u}^h_2\left\langle D^-\mathbf{u}^h_1 + D^-\mathbf{u}^h_2, D^-\overline{\mathbf{u}^h}\right\rangle\right\rangle_{L^2_h} \\
&\quad + \frac{1}{2}\left\langle \zeta_m\bar{\mathbf{u}}^h, (\zeta_m\overline{\mathbf{u}^h}\times\mathbf{g}^h)\times\mathbf{g}^h\right\rangle_{L^2_h}.
\end{aligned}
$$

Then, using (2.2.2)

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left|\zeta_m\overline{\mathbf{u}^h}\right|^2_{L^2_h} + \left|\zeta_m D^-\overline{\mathbf{u}^h}\right|^2_{L^2_h} \\
&= -\left\langle \zeta_m\overline{\mathbf{u}^h}, D^+\zeta_m D^+\overline{\mathbf{u}^h}\right\rangle_{L^2_h} - \left\langle \tau^+\zeta_m\overline{\mathbf{u}^h}, D^+\zeta_m D^+\overline{\mathbf{u}^h}\right\rangle_{L^2_h} \\
&\quad - \left\langle \tau^+\mathbf{u}^h_1\times D^+\zeta_m D^+\overline{\mathbf{u}^h}, \zeta_m\overline{\mathbf{u}^h}\right\rangle_{L^2_h} - \left\langle \tau^+\mathbf{u}^h_1\times D^+\zeta_m D^+\overline{\mathbf{u}^h}, \tau^+\zeta_m\overline{\mathbf{u}^h}\right\rangle_{L^2_h} \\
&\quad - \left\langle D^+\mathbf{u}^h_1\times \zeta_m D^+\overline{\mathbf{u}^h}, \zeta_m\overline{\mathbf{u}^h}\right\rangle_{L^2_h} \\
&\quad + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\overline{\mathbf{u}^h}|D^+\mathbf{u}^h_1|^2\right\rangle_{L^2_h} + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\mathbf{u}^h_2\left\langle D^+\mathbf{u}^h_1 + D^+\mathbf{u}^h_2, D^+\overline{\mathbf{u}^h}\right\rangle\right\rangle_{L^2_h} \\
&\quad + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\overline{\mathbf{u}^h}|D^-\mathbf{u}^h_1|^2\right\rangle_{L^2_h} + \frac{1}{2}\left\langle \zeta_m\overline{\mathbf{u}^h}, \zeta_m\mathbf{u}^h_2\left\langle D^-\mathbf{u}^h_1 + D^-\mathbf{u}^h_2, D^-\overline{\mathbf{u}^h}\right\rangle\right\rangle_{L^2_h}.
\end{aligned}
$$

Integrating with respect to t, using Lemma 2.5.2, part (2) of Definition 3.3.1 and the fact that for $\mathbf{v}:\mathbb{Z}_h\to\mathbb{R}^3$

$$
|(\tau^{\pm}\zeta_m)\mathbf{v}|_{L^2_h} \le C|\zeta_m\mathbf{v}|_{L^2_h},
$$

we get

$$
\frac{1}{2}\left|\zeta_m\overline{\mathbf{u}^h}\right|^2_{L^2_h} + \int_0^t \left|\zeta_m D^-\overline{\mathbf{u}^h}\right|^2_{L^2_h} ds \le C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L^2_h}\left|D^-\zeta_m D^-\overline{\mathbf{u}^h}\right|_{L^2_h} ds
$$

$$
\begin{aligned}
&+ C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2} \left|D^+\mathbf{u}_1^h\right|_{L_h^\infty} \left|\zeta_m D^-\overline{\mathbf{u}^h}\right|_{L_h^2} ds \\
&+ C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2} \left|D^+(\mathbf{u}_1^h+\mathbf{u}_2^h)\right|_{L_h^\infty} \left|\zeta_m D^-\overline{\mathbf{u}^h}\right|_{L_h^2} ds \\
&+ C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 \left|D^+\mathbf{u}_1^h\right|_{L_h^\infty}^2 ds.
\end{aligned}
$$

Thus, using Young's inequality (2.2.6) for $p = q = 2$, we obtain

$$
\begin{aligned}
\frac{1}{2}\left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 + \frac{1}{2}\int_0^t \left|\zeta_m D^-\overline{\mathbf{u}^h}\right|_{L_h^2}^2 ds \le C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 ds &+ \frac{1}{2}\int_0^t \left|D^-\zeta_m D^-\overline{\mathbf{u}^h}\right|_{L_h^2}^2 ds \\
&+ C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 \left|D^+\mathbf{u}_1^h\right|_{L_h^\infty}^2 ds \\
&+ C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 \left(\left|D^+\mathbf{u}_1^h\right|_{L_h^\infty}^2 + \left|D^+\mathbf{u}_2^h\right|_{L_h^\infty}^2\right) ds \\
&+ C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 \left|D^+\mathbf{u}_1^h\right|_{L_h^\infty}^2 ds.
\end{aligned}
$$

Consequently, we have

$$
\begin{aligned}
&\left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 + \int_0^t \left|\zeta_m D^-\overline{\mathbf{u}^h}\right|_{L_h^2}^2 ds - \int_0^t \left|D^-\zeta_m D^-\overline{\mathbf{u}^h}\right|_{L_h^2}^2 ds \\
&\le C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 \left(1 + \left|D^+\mathbf{u}_1^h\right|_{L_h^\infty}^2 + \left|D^+\mathbf{u}_2^h\right|_{L_h^\infty}^2\right) ds.
\end{aligned}
$$

Using Lemma 2.5.1, we have

$$
\int_0^t \left|\zeta_m D^-\overline{\mathbf{u}^h}\right|_{L_h^2}^2 ds - \int_0^t \left|D^-\zeta_m D^-\overline{\mathbf{u}^h}\right|_{L_h^2}^2 ds \ge 0
$$

for m sufficiently large. Therefore

$$
\left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 \le C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 \left(1 + \left|D^+\mathbf{u}_1^h\right|_{L_h^\infty}^2 + \left|D^+\mathbf{u}_2^h\right|_{L_h^\infty}^2\right) ds\,.
$$

Then, using Lemma 2.5.6

$$
\left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 \le C\int_0^t \left|\zeta_m\overline{\mathbf{u}^h}\right|_{L_h^2}^2 \left(1 + \left|D^+\mathbf{u}_1^h\right|_{L_h^2} \left|\tilde{\Delta}\mathbf{u}_1^h\right|_{L_h^2} + \left|D^+\mathbf{u}_2^h\right|_{L_h^2} \left|\tilde{\Delta}\mathbf{u}_2^h\right|_{L_h^2}\right) ds\,.
$$

By Definition 3.3.1 and Remark 3.3.2, we have that

$$1 + \left|D^+\mathbf{u}_1^h\right|_{L_h^2} \left|\tilde{\Delta}\mathbf{u}_1^h\right|_{L_h^2} + \left|D^+\mathbf{u}_2^h\right|_{L_h^2} \left|\tilde{\Delta}\mathbf{u}_2^h\right|_{L_h^2} \in L^1(0,T)$$

$\mathbb{P}$-a.s.. Then, using Gronwall's inequality (2.2.7), we obtain $\left|\zeta_m \overline{\mathbf{u}^h}(t)\right|_{L_h^2} = 0$ for every $t \geq 0$ as $\overline{\mathbf{u}^h} : [0,T] \to L_{m,h}^2$ is continuous and the lemma follows. $\square$

3.3.2 A Modified Version of Semi-Discrete Scheme

Let $R > 0$, we define a cutoff function $\psi \in C^2(\mathbb{R})$ such that

$$\psi(r) = \begin{cases} 1 & \text{if} \quad r \in [0, R] \\ 0 & \text{if} \quad r \in (-\infty, -1] \cup [R+1, \infty). \end{cases}$$

We define the following space

$$E_h := \{\mathbf{v} : \mathbb{Z}_h \to \mathbb{R}^3 | \quad |D^+\mathbf{v}|_{L_h^2}^2 + |\mathbf{v}|_{L_h^\infty}^2 < \infty\},$$

with the norm $|\mathbf{v}|_{E_h} := \sqrt{|D^+\mathbf{v}|_{L_h^2}^2 + |\mathbf{v}|_{L_h^\infty}^2}$. The space E_h endowed with the norm $|\cdot|_{E_h}$ is a Banach space.

We will consider a modified version of problem (3.3.1)-(3.3.3):

$$\begin{aligned} d\mathbf{u}^{h,R} = &\left(\mu\psi\left(\left|\mathbf{u}^{h,R}\right|_{E_h}\right)\mathbf{u}^{h,R} \times \tilde{\Delta}\mathbf{u}^{h,R} - \lambda\psi\left(\left|\mathbf{u}^{h,R}\right|_{E_h}\right)\mathbf{u}^{h,R} \times (\mathbf{u}^{h,R} \times \tilde{\Delta}\mathbf{u}^{h,R}) \right. \\ &\left. + \frac{\mu^2}{2}(\mathbf{u}^{h,R} \times \mathbf{g}^h) \times \mathbf{g}^h\right) dt + \mu(\mathbf{u}^{h,R} \times \mathbf{g}^h)dW, \end{aligned} \tag{3.3.6}$$

$$\mathbf{u}^{h,R}(0, x_i) = \mathbf{u}_0(x_i), \tag{3.3.7}$$

$$|\mathbf{u}_0(x_i)| = 1. \tag{3.3.8}$$

A global strong solution of problem (3.3.6)-(3.3.8) is defined as follows.

Definition 3.3.4. Given $T, R \in (0, \infty)$, we call an $(\mathcal{F}_t)$-adapted stochastic process $\mathbf{u}^{h,R} = \{\mathbf{u}^{h,R}(t); t \leq T\}$ taking values in $L_{m,h}^2$ for every $m > 0$, a strong solution to (3.3.6)-(3.3.8) for the time interval $[0, T]$, if $\mathbf{u}^{h,R}$ satisfies (1)-(3) below:

(1) for every $m > 0$

$$\mathbf{u}^{h,R}(\cdot) \in C([0,T]; L^2_{m,h}), \quad \mathbb{P}\text{-a.s,}$$

(2) for every $t \in [0,T]$ and $x_i \in \mathbb{Z}_h$

$$|\mathbf{u}^{h,R}(t, x_i)| = 1,$$

(3) for every $t \in [0,T]$, the following equation holds in $L^2_{m,h}$ $\mathbb{P}$-a.s.:

$$\begin{aligned}\mathbf{u}^{h,R}(t) &= \mathbf{u}_0 + \mu \int_0^t \psi\left(\left|\mathbf{u}^{h,R}\right|_{E_h}\right) \mathbf{u}^{h,R}(s) \times \tilde{\Delta}\mathbf{u}^{h,R}(s) ds \\ &\quad - \lambda \int_0^t \psi\left(\left|\mathbf{u}^{h,R}\right|_{E_h}\right) \mathbf{u}^{h,R}(s) \times \left(\mathbf{u}^{h,R}(s) \times \tilde{\Delta}\mathbf{u}^{h,R}(s)\right) ds \\ &\quad + \frac{\mu^2}{2} \int_0^t \left(\mathbf{u}^{h,R}(s) \times \mathbf{g}^h\right) \times \mathbf{g}^h ds + \mu \int_0^t \left(\mathbf{u}^{h,R}(s) \times \mathbf{g}^h\right) dW(s). \qquad (3.3.9)\end{aligned}$$

Moreover, if $\mathbf{u}^{h,R}$ is a strong solution on $[0,T]$ for all $T \geq 0$, we say that $\mathbf{u}^{h,R}$ is a global strong solution.

For every $h > 0$ and $\mathbf{v} \in E_h$, we define the maps

$$\begin{aligned}I_h^{R,1}(\mathbf{v}) &:= \psi\left(|\mathbf{v}|_{E_h}\right) \mathbf{v} \times \tilde{\Delta}\mathbf{v}, \\ I_h^{R,2}(\mathbf{v}) &:= \psi\left(|\mathbf{v}|_{E_h}\right) \mathbf{v} \times (\mathbf{v} \times \tilde{\Delta}\mathbf{v}), \\ J_h(\mathbf{v}) &:= (\mathbf{v} \times \mathbf{g}^h) \times \mathbf{g}^h.\end{aligned}$$

The next lemma will be used to prove the unique solvability of the semi-discrete scheme (3.3.6)-(3.3.8).

Lemma 3.3.5. *Assume that $\mathbf{g}^h \in L^2_h$. Then, for every $h > 0$, the following holds*

- *the mappings $I_h^{R,k} : E_h \to E_h$, $k = 1, 2$ are Lipshitz.*
- *the mapping $J_h : E_h \to E_h$ defines a linear bounded operator on E_h.*

Proof. Let us prove that the map $I_h^{R,1}$ is Lipshitz. For $\mathbf{v}_1,\, \mathbf{v}_2 \in E_h$

$$\begin{aligned}
\left| I_h^{R,1}(\mathbf{v}_1) - I_h^{R,1}(\mathbf{v}_2) \right|_{E_h}^2 &= \left| \psi\left(|\mathbf{v}_1|_{E_h}\right) \mathbf{v}_1 \times \tilde{\Delta}\mathbf{v}_1 - \psi\left(|\mathbf{v}_2|_{E_h}\right) \mathbf{v}_2 \times \tilde{\Delta}\mathbf{v}_2 \right|_{E_h}^2 \\
&= \Big| \psi\left(|\mathbf{v}_1|_{E_h}\right) \mathbf{v}_1 \times \frac{1}{h^2}(\tau^+\mathbf{v}_1 - 2\mathbf{v}_1 + \tau^-\mathbf{v}_1) \\
&\quad - \psi\left(|\mathbf{v}_2|_{E_h}\right) \mathbf{v}_2 \times \frac{1}{h^2}(\tau^+\mathbf{v}_2 - 2\mathbf{v}_2 + \tau^-\mathbf{v}_2) \Big|_{E_h}^2 \\
&= \frac{1}{h^4} \Big| \psi\left(|\mathbf{v}_1|_{E_h}\right) \mathbf{v}_1 \times \tau^+\mathbf{v}_1 + \psi\left(|\mathbf{v}_1|_{E_h}\right) \mathbf{v}_1 \times \tau^-\mathbf{v}_1 \\
&\quad - \psi\left(|\mathbf{v}_2|_{E_h}\right) \mathbf{v}_2 \times \tau^+\mathbf{v}_2 - \psi\left(|\mathbf{v}_2|_{E_h}\right) \mathbf{v}_2 \times \tau^-\mathbf{v}_2 \Big|_{E_h}^2 \\
&\le \frac{2}{h^4} \left| \psi\left(|\mathbf{v}_1|_{E_h}\right) \mathbf{v}_1 \times \tau^+\mathbf{v}_1 - \psi\left(|\mathbf{v}_2|_{E_h}\right) \mathbf{v}_2 \times \tau^+\mathbf{v}_2 \right|_{E_h}^2 \\
&\quad + \frac{2}{h^4} \left| \psi\left(|\mathbf{v}_1|_{E_h}\right) \mathbf{v}_1 \times \tau^-\mathbf{v}_1 - \psi\left(|\mathbf{v}_2|_{E_h}\right) \mathbf{v}_2 \times \tau^-\mathbf{v}_2 \right|_{E_h}^2 .
\end{aligned}$$

We estimate now the first term on the right hand side, the last term follows in the same way. Since $\psi \in C^2$, we can assume without loss of generality that $|\psi'(r)| \le 1$. Elementary calculations reveal

1. If $|\mathbf{v}_1|_{E_h} \le R$ and $|\mathbf{v}_2|_{E_h} \le R$, then using Lemmas 2.5.2 and 2.5.3 we get

$$\begin{aligned}
&\frac{2}{h^4} \left| \psi\left(|\mathbf{v}_1|_{E_h}\right) \mathbf{v}_1 \times \tau^+\mathbf{v}_1 - \psi\left(|\mathbf{v}_2|_{E_h}\right) \mathbf{v}_2 \times \tau^+\mathbf{v}_2 \right|_{E_h}^2 \\
&= \frac{2}{h^4} \left| \mathbf{v}_1 \times \tau^+\mathbf{v}_1 - \mathbf{v}_2 \times \tau^+\mathbf{v}_2 \right|_{E_h}^2 \\
&\le \frac{4}{h^4} \left| (\mathbf{v}_1 - \mathbf{v}_2) \times \tau^+\mathbf{v}_1 \right|_{E_h}^2 + \frac{4}{h^4} \left| \mathbf{v}_2 \times (\tau^+\mathbf{v}_1 - \tau^+\mathbf{v}_2) \right|_{E_h}^2 \\
&\le \frac{8}{h^4} \left| D^+(\mathbf{v}_1 - \mathbf{v}_2) \right|_{L_h^2}^2 |\mathbf{v}_1|_{L_h^\infty}^2 + \frac{8}{h^4} |\mathbf{v}_1 - \mathbf{v}_2|_{L_h^\infty}^2 \left| D^+\mathbf{v}_1 \right|_{L_h^2}^2 + \frac{4}{h^4} |\mathbf{v}_1 - \mathbf{v}_2|_{L_h^\infty}^2 |\mathbf{v}_1|_{L_h^\infty}^2 \\
&\quad + \frac{8}{h^4} \left| D^+(\mathbf{v}_1 - \mathbf{v}_2) \right|_{L_h^2}^2 |\mathbf{v}_2|_{L_h^\infty}^2 + \frac{8}{h^4} |\mathbf{v}_1 - \mathbf{v}_2|_{L_h^\infty}^2 \left| D^+\mathbf{v}_2 \right|_{L_h^2}^2 + \frac{4}{h^4} |\mathbf{v}_1 - \mathbf{v}_2|_{L_h^\infty}^2 |\mathbf{v}_2|_{L_h^\infty}^2 \\
&\le \frac{C}{h^4} R^2 \left| D^+(\mathbf{v}_1 - \mathbf{v}_2) \right|_{L_h^2}^2 + \frac{C}{h^4} R^2 |\mathbf{v}_1 - \mathbf{v}_2|_{L_h^\infty}^2 \\
&\le \frac{C}{h^4} R^2 |\mathbf{v}_1 - \mathbf{v}_2|_{E_h}^2 .
\end{aligned}$$

2. If $|\mathbf{v}_1|_{E_h} \le R$ and $R \le |\mathbf{v}_2|_{E_h} \le (R+1)$ then, by using the same argument as in case 1., we have

$$
\begin{aligned}
&\frac{2}{h^4}\left|\psi\left(|\mathbf{v}_1|_{E_h}\right)\mathbf{v}_1\times\tau^+\mathbf{v}_1-\psi\left(|\mathbf{v}_2|_{E_h}\right)\mathbf{v}_2\times\tau^+\mathbf{v}_2\right|^2_{E_h}\\
&=\frac{2}{h^4}\left|(\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right))\mathbf{v}_1\times\tau^+\mathbf{v}_1+\psi\left(|\mathbf{v}_2|_{E_h}\right)\left(\mathbf{v}_1\times\tau^+\mathbf{v}_1-\mathbf{v}_2\times\tau^+\mathbf{v}_2\right)\right|^2_{E_h}\\
&\le\frac{4}{h^4}\left|(\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right))\mathbf{v}_1\times\tau^+\mathbf{v}_1\right|^2_{E_h}\\
&\quad+\frac{4}{h^4}\left|\psi\left(|\mathbf{v}_2|_{E_h}\right)\left(\mathbf{v}_1\times\tau^+\mathbf{v}_1-\mathbf{v}_2\times\tau^+\mathbf{v}_2\right)\right|^2_{E_h}\\
&\le\frac{4}{h^4}\left|\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right)\right|^2\left|\mathbf{v}_1\times\tau^+\mathbf{v}_1\right|^2_{E_h}+\frac{4}{h^4}\left|\mathbf{v}_1\times\tau^+\mathbf{v}_1-\mathbf{v}_2\times\tau^+\mathbf{v}_2\right|^2_{E_h}\\
&\le\frac{4}{h^4}|\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right)|^2|\mathbf{v}_1|^2_{L^\infty_h}|\mathbf{v}_1|^2_{E_h}+\frac{C}{h^4}R^2|\mathbf{v}_1-\mathbf{v}_2|^2_{E_h}\\
&\le\frac{4}{h^4}R^4\sup_r|\psi'(r)|^2\left||\mathbf{v}_1|_{E_h}-|\mathbf{v}_2|_{E_h}\right|^2+\frac{C}{h^4}R^2|\mathbf{v}_1-\mathbf{v}_2|^2_{E_h}\\
&\le\frac{C}{h^4}R^2(R^2+1)|\mathbf{v}_1-\mathbf{v}_2|^2_{E_h}.
\end{aligned}
$$

3. If $|\mathbf{v}_1|_{E_h} \le R$ and $|\mathbf{v}_2|_{E_h} \ge (R+1)$, then $\big($noting that $\psi\left(|\mathbf{v}_2|_{E_h}\right)=0\big)$

$$
\begin{aligned}
&\frac{2}{h^4}\left|\psi\left(|\mathbf{v}_1|_{E_h}\right)\mathbf{v}_1\times\tau^+\mathbf{v}_1-\psi\left(|\mathbf{v}_2|_{E_h}\right)\mathbf{v}_2\times\tau^+\mathbf{v}_2\right|^2_{E_h}\\
&=\frac{2}{h^4}\left|(\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right))\mathbf{v}_1\times\tau^+\mathbf{v}_1\right|^2_{E_h}\\
&\le\frac{2}{h^4}\left|\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right)\right|^2\left|\mathbf{v}_1\times\tau^+\mathbf{v}_1\right|^2_{E_h}\\
&\le\frac{C}{h^4}R^4\sup_r|\psi'(r)|^2\left||\mathbf{v}_1|_{E_h}-|\mathbf{v}_2|_{E_h}\right|^2\\
&\le\frac{C}{h^4}R^4|\mathbf{v}_1-\mathbf{v}_2|^2_{E_h}.
\end{aligned}
$$

4. If $R \le |\mathbf{v}_1|_{E_h} \le (R+1)$ and $R \le |\mathbf{v}_2|_{E_h} \le (R+1)$, then

$$
\begin{aligned}
&\frac{2}{h^4}\left|\psi\left(|\mathbf{v}_1|_{E_h}\right)\mathbf{v}_1\times\tau^+\mathbf{v}_1-\psi\left(|\mathbf{v}_2|_{E_h}\right)\mathbf{v}_2\times\tau^+\mathbf{v}_2\right|^2_{E_h}\\
&\frac{2}{h^4}\left|(\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right))\mathbf{v}_1\times\tau^+\mathbf{v}_1+\psi\left(|\mathbf{v}_2|_{E_h}\right)\left(\mathbf{v}_1\times\tau^+\mathbf{v}_1-\mathbf{v}_2\times\tau^+\mathbf{v}_2\right)\right|^2_{E_h}
\end{aligned}
$$

$$\leq \frac{4}{h^4}\left|(\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right))\mathbf{v}_1\times\tau^+\mathbf{v}_1\right|^2_{E_h}+\frac{4}{h^4}\left|\mathbf{v}_1\times\tau^+\mathbf{v}_1-\mathbf{v}_2\times\tau^+\mathbf{v}_2\right|^2_{E_h}$$
$$\leq \frac{4}{h^4}|\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right)|^2|\mathbf{v}_1|^2_{L^\infty_h}|\mathbf{v}_1|^2_{E_h}+\frac{C}{h^4}(R+1)^2|\mathbf{v}_1-\mathbf{v}_2|^2_{E_h}$$
$$\leq \frac{C}{h^4}((R+1)^4+(R+1)^2)|\mathbf{v}_1-\mathbf{v}_2|^2_{E_h}.$$

5. If $R\leq|\mathbf{v}_1|_{E_h}\leq(R+1)$ and $|\mathbf{v}_2|_{E_h}\geq(R+1)$, then

$$\frac{2}{h^4}\left|\psi\left(|\mathbf{v}_1|_{E_h}\right)\mathbf{v}_1\times\tau^+\mathbf{v}_1-\psi\left(|\mathbf{v}_2|_{E_h}\right)\mathbf{v}_2\times\tau^+\mathbf{v}_2\right|^2_{E_h}$$
$$=\frac{2}{h^4}\left|(\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right))\mathbf{v}_1\times\tau^+\mathbf{v}_1\right|^2_{E_h}$$
$$\leq\frac{2}{h^4}(R+1)^4\left|\psi\left(|\mathbf{v}_1|_{E_h}\right)-\psi\left(|\mathbf{v}_2|_{E_h}\right)\right|^2$$
$$\leq\frac{C}{h^4}(R+1)^4|\mathbf{v}_1-\mathbf{v}_2|^2_{E_h}.$$

6. If $|\mathbf{v}_1|_{E_h}\geq(R+1)$ and $|\mathbf{v}_2|_{E_h}\geq(R+1)$, then

$$\frac{2}{h^4}\left|\psi\left(|\mathbf{v}_1|_{E_h}\right)\mathbf{v}_1\times\tau^+\mathbf{v}_1-\psi\left(|\mathbf{v}_2|_{E_h}\right)\mathbf{v}_2\times\tau^+\mathbf{v}_2\right|^2_{E_h}=0.$$

We deduce that the map $I_h^{R,1}$ is Lipshitz on E_h. The proof for $I_h^{R,2}$ follows in the same manner.

Next, let us prove that

$$|J_h(\mathbf{v})|^2_{E_h}\leq C|\mathbf{v}|^2_{E_h}.$$

In fact, using Lemmas 2.5.2, 2.5.3 and 2.5.6 we obtain

$$\begin{aligned}|J_h(\mathbf{v})|^2_{E_h}&=|(\mathbf{v}\times\mathbf{g}^h)\times\mathbf{g}^h|^2_{E_h}\\&=|(\mathbf{v}\times\mathbf{g}^h)\times\mathbf{g}^h|^2_{L^\infty_h}+\left|D^+\big((\mathbf{v}\times\mathbf{g}^h)\times\mathbf{g}^h\big)\right|^2_{L^2_h}\\&\leq|\mathbf{g}^h|^4_{L^\infty_h}|\mathbf{v}|^2_{L^\infty_h}+C|\mathbf{g}^h|^4_{L^\infty_h}|D^+\mathbf{v}|^2_{L^2_h}+C|\mathbf{g}^h|^2_{L^\infty_h}|D^+\mathbf{g}^h|^2_{L^2_h}|\mathbf{v}|^2_{L^\infty_h}\\&\leq C|\mathbf{g}^h|^2_{L^2_h}|D^+\mathbf{g}^h|^2_{L^2_h}|\mathbf{v}|^2_{L^\infty_h}+C|\mathbf{g}^h|^2_{L^2_h}|D^+\mathbf{g}^h|^2_{L^2_h}|D^+\mathbf{v}|^2_{L^2_h}+C|\mathbf{g}^h|_{L^2_h}|D^+\mathbf{g}^h|^3_{L^2_h}|\mathbf{v}|^2_{L^\infty_h}\\&\leq C|\mathbf{v}|^2_{E_h}.\end{aligned}$$

$\square$

For $T > 0$, let $\mathcal{E}_h(T)$ denote the Banach space of E_h-valued processes $\mathbf{v}$ such that the process $\mathbf{v}(\cdot, x)$ is progressively measurable for every $x \in \mathbb{Z}_h$ and

$$|\mathbf{v}|^2_{\mathcal{E}_h(T)} := \mathbb{E}\left[\sup_{t\in[0,T]} |\mathbf{v}(t)|^2_{E_h}\right] < \infty.$$

For every $\mathbf{v} \in \mathcal{E}_h(T)$, we define the process

$$M(\mathbf{v})(t, x_i) := \int_0^t \mathbf{v}(s, x_i) \times \mathbf{g}^h(x_i)\, dW(s), \quad x_i \in \mathbb{Z}_h, \quad t \geq 0.$$

In the sequel, we will use frequently the following obvious fact. Assume that $\mathbf{v} \in \mathcal{E}_h(T)$. Then, $\mathbb{P}$-a.s.

$$D^+\left(\int_0^t \mathbf{v}(s)\, dW(s)\right) = \int_0^t D^+\mathbf{v}(s)\, dW(s), \quad t \leq T\,. \tag{3.3.10}$$

The lemma that follows will be used to prove the unique solvability of (3.3.6)-(3.3.8).

Lemma 3.3.6. *Let $h > 0$ be fixed. We assume that $\mathbf{g}^h \in L^2_h$. Then $M(\mathbf{v}) \in \mathcal{E}_h(T)$ for every $\mathbf{v} \in \mathcal{E}_h(T)$. Moreover, there exists $C = C(h) > 0$ such that for every $\mathbf{v} \in \mathcal{E}_h(T)$*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |M(\mathbf{v})(t)|^2_{H^1_h}\right] \leq TC|\mathbf{v}|^2_{\mathcal{E}_h(T)}. \tag{3.3.11}$$

Hence the mapping $M : \mathcal{E}_h(T) \to \mathcal{E}_h(T)$ defines a linear bounded operator.

Proof. We first note that, by definition of one-dimensional Itô integrals, the process $M(\mathbf{v})(\cdot, x_i)$ is progressively measurable for every $x_i \in \mathbb{Z}_h$. Next, we note that Lemma 2.5.6 gives

$$|\mathbf{v}|_{L^\infty_h} \leq C|\mathbf{v}|_{H^1_h} \quad \text{for all } \mathbf{v} \in H^1_h,$$

(which implies H^1_h is continuously embedded into E_h). Thus it suffices to prove (3.3.11). Using Lemma 2.3.18 we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|M(\mathbf{v})(t)\right|^2_{L^2_h}\right] = \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \mathbf{v}(s) \times \mathbf{g}^h dW(s)\right|^2_{L^2_h}\right]$$

$$
\begin{aligned}
&\le C\mathbb{E}\bigg[\int_0^T |\mathbf{v}(s)\times \mathbf{g}^h|^2_{L^2_h}\,ds\bigg]\\
&\le CT|\mathbf{g}^h|^2_{L^2_h}\mathbb{E}\bigg[\sup_{t\in[0,T]}|\mathbf{v}(s)|^2_{L^\infty_h}\bigg]\\
&< CT|\mathbf{v}|^2_{\mathcal{E}_h(T)}.
\end{aligned}
$$

Moreover, using (3.3.10) and Lemmas 2.5.2, 2.5.3, 2.5.6 and 2.3.18, we find that

$$
\begin{aligned}
&\mathbb{E}\bigg[\sup_{t\in[0,T]}\big|D^+M(\mathbf{v})(t)\big|^2_{L^2_h}\bigg]\\
&= \mathbb{E}\bigg[\sup_{t\in[0,T]}\bigg|\int_0^t D^+(\mathbf{v}(s)\times \mathbf{g}^h)dW(s)\bigg|^2_{L^2_h}\bigg]\\
&\le C\mathbb{E}\bigg[\int_0^T |D^+(\mathbf{v}(s)\times \mathbf{g}^h)|^2_{L^2_h}\,ds\bigg]\\
&\le 2C\mathbb{E}\bigg[\int_0^T |D^+\mathbf{v}(s)\times \tau^+\mathbf{g}^h|^2_{L^2_h}\,ds\bigg] + 2C\mathbb{E}\bigg[\int_0^T |\mathbf{v}(s)\times D^+\mathbf{g}^h|^2_{L^2_h}\,ds\bigg]\\
&\le 2C|\mathbf{g}^h|^2_{L^\infty_h}\mathbb{E}\bigg[\int_0^T |D^+\mathbf{v}(s)|^2_{L^2_h}\,ds\bigg] + 2C|D^+\mathbf{g}^h|^2_{L^2_h}\mathbb{E}\bigg[\int_0^T |\mathbf{v}(s)|^2_{L^\infty_h}\,ds\bigg]\\
&\le CT(|\mathbf{g}^h|^2_{L^\infty_h} + |D^+\mathbf{g}^h|^2_{L^2_h})\mathbb{E}\bigg[\sup_{t\in[0,T]}|\mathbf{v}(s)|^2_{E_h}\bigg]\\
&\le CT(|\mathbf{g}^h|_{L^2_h}|D^+\mathbf{g}^h|_{L^2_h} + |D^+\mathbf{g}^h|^2_{L^2_h})\mathbb{E}\bigg[\sup_{t\in[0,T]}|\mathbf{v}(s)|^2_{E_h}\bigg]\\
&< CT|\mathbf{v}|^2_{\mathcal{E}_h(T)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\mathbb{E}\bigg[\sup_{t\in[0,T]}\big|M(\mathbf{v})(t)\big|^2_{H^1_h}\bigg] &= \mathbb{E}\bigg[\sup_{t\in[0,T]}\big|M(\mathbf{v})(t)\big|^2_{L^2_h}\bigg] + \mathbb{E}\bigg[\sup_{t\in[0,T]}\big|D^+M(\mathbf{v})(t)\big|^2_{L^2_h}\bigg]\\
&\le CT|\mathbf{v}|^2_{\mathcal{E}_h(T)}.
\end{aligned}
$$

This completes the proof of the lemma. □

Now, we prove the existence of a global unique strong solution to (3.3.6)-(3.3.8).

Lemma 3.3.7. *Let $h > 0$ and $T > 0$ be fixed and assume that $\mathbf{u}^h(0) \in E_h$ and $\mathbf{g}^h \in L^2_h$. Then, for every $R > 0$, there exists a global unique strong solution $\mathbf{u}^{h,R}$ of equation* (3.3.6) *belonging to $\mathcal{E}_h(T)$.*

Proof. In order to simplify notations, we assume in the proof without loss of generality that $\lambda = \mu = 1$. For $\mathbf{v} \in \mathcal{E}_h(T)$, we define the mapping $\mathcal{H}$

$$\begin{aligned}\mathcal{H}(\mathbf{v})(t) := \mathbf{u}_0 &+ \int_0^t \psi\left(|\mathbf{v}(s)|_{E_h}\right)\mathbf{v}(s) \times \tilde{\Delta}\mathbf{v}(s)ds \\ &- \int_0^t \psi\left(|\mathbf{v}(s)|_{E_h}\right)\mathbf{v}(s) \times \left(\mathbf{v}(s) \times \tilde{\Delta}\mathbf{v}(s)\right)\,ds \\ &+ \frac{1}{2}\int_0^t \left(\mathbf{v}(s) \times \mathbf{g}^h\right) \times \mathbf{g}^h\,ds + \int_0^t \mathbf{v}(s) \times \mathbf{g}^h\,dW(s) \\ = \mathbf{u}_0 &+ \int_0^t I_h^{R,1}(\mathbf{v}(s))\,ds - \int_0^t I_h^{R,2}(\mathbf{v}(s))\,ds + \frac{1}{2}\int_0^t J_h(\mathbf{v}(s))\,ds + M(\mathbf{v})(t)\end{aligned}$$

for $t \in [0, T]$. First, we prove that $\mathcal{H} : \mathcal{E}_h(T) \to \mathcal{E}_h(T)$. Let $\mathbf{v} \in \mathcal{E}_h(T)$. Clearly, the process $t \to \mathcal{H}(\mathbf{v})(t, x)$ is progressively measurable for every $x \in \mathbb{Z}_h$. We will prove that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\mathcal{H}(\mathbf{v})(t)|^2_{E_h}\right] < \infty. \tag{3.3.12}$$

By Lemma 3.3.5 we have

$$\left|I_h^{R,k}(\mathbf{v})\right|_{E_h} \leq C_R\left(1 + |\mathbf{v}|_{E_h}\right), \quad k = 1, 2\,,$$

and

$$\left|J_h(\mathbf{v})\right|_{E_h} \leq C|\mathbf{v}|_{E_h}\,.$$

Therefore,

$$|\mathcal{H}(\mathbf{v})(t)|_{E_h} \leq |\mathbf{u}_0|_{E_h} + C\int_0^t \left(1 + |\mathbf{v}(s)|_{E_h}\right)\,ds + |M(\mathbf{v})(t)|_{E_h},$$

hence, invoking Lemma 3.3.6 we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\mathcal{H}(\mathbf{v})(t)|^2_{E_h}\right]\leq C\,|\mathbf{u}_0|^2_{E_h}+CT^2+CT^2\mathbb{E}\left[\sup_{t\in[0,T]}|\mathbf{v}(t)|^2_{E_h}\right]$$

and (3.3.12) follows. We deduce that $\mathcal{H}:\mathcal{E}_h(T)\to\mathcal{E}_h(T)$.

Next, we prove that the mapping $\mathcal{H}$ is a contraction in $\mathcal{E}_h(T)$ for T small enough. More precisely, for a fixed $R>|\mathbf{u}_0|_{E_h}$, we will prove that for T small enough there exists $C\in(0,1)$ such that for $\mathbf{v}_1,\mathbf{v}_2\in\mathcal{E}_h(T)$

$$|\mathcal{H}(\mathbf{v}_1)-\mathcal{H}(\mathbf{v}_2)|^2_{\mathcal{E}_h(T)}\leq C|\mathbf{v}_1-\mathbf{v}_2|^2_{\mathcal{E}_h(T)}.$$

In fact, using Lemmas 3.3.5 and 3.3.6 we have

$$\begin{aligned}
&|\mathcal{H}(\mathbf{v}_1)-\mathcal{H}(\mathbf{v}_2)|^2_{\mathcal{E}_h(T)}\\
&\leq C\left|\int_0^t I_h^{R,1}(\mathbf{v}_1(s))-I_h^{R,1}(\mathbf{v}_2(s))ds\right|^2_{\mathcal{E}_h(T)}+C\left|\int_0^t I_h^{R,2}(\mathbf{v}_1(s))-I_h^{R,2}(\mathbf{v}_2(s))ds\right|^2_{\mathcal{E}_h(T)}\\
&\quad+C\left|\int_0^t J_h(\mathbf{v}_1(s))-J_h(\mathbf{v}_2(s))ds\right|^2_{\mathcal{E}_h(T)}+C\left|M(\mathbf{v}_1)(t)-M(\mathbf{v}_2)(t)\right|^2_{\mathcal{E}_h(T)}\\
&\leq C\mathbb{E}\left[\int_0^T\left|I_h^{R,1}(\mathbf{v}_1(s))-I_h^{R,1}(\mathbf{v}_2(s))\right|^2_{E_h}ds\right]+C\mathbb{E}\left[\int_0^T\left|I_h^{R,2}(\mathbf{v}_1(s))-I_h^{R,2}(\mathbf{v}_2(s))\right|^2_{E_h}ds\right]\\
&\quad+C\mathbb{E}\left[\int_0^T\left|J_h(\mathbf{v}_1(s))-J_h(\mathbf{v}_2(s))\right|^2_{E_h}ds\right]+C\mathbb{E}\left[\sup_{t\in[0,T]}\left|M(\mathbf{v}_1)(t)-M(\mathbf{v}_2)(t)\right|^2_{E_h}\right]\\
&\leq CT|\mathbf{v}_1-\mathbf{v}_2|^2_{\mathcal{E}_h(T)}.
\end{aligned}$$

We deduce that the mapping $\mathcal{H}$ is a contraction in $\mathcal{E}_h(T)$ for T small enough.

From Banach fixed point theorem applied to the mapping $\mathcal{H}:\mathcal{E}_h(T)\to\mathcal{E}_h(T)$, the lemma follows for T small enough and then for any $T>0$ by standard argument. This completes the proof of the lemma. $\square$

3.3.3 A Priori Estimates

In this subsection, we introduce and prove some uniform estimates which will be used to prove the solvability of (3.3.1)-(3.3.3).

Lemma 3.3.8. *Let $\mathbf{u}^{h,R}$ be the solution of equation* (3.3.6)*. Assume that* $|\mathbf{u}^h(0,x_i)| = 1$ *for all* $x_i \in \mathbb{Z}_h$*,* $D^+\mathbf{u}^h(0) \in L^2_h$ *and* $\mathbf{g}^h \in L^2_h$*. Then, for every* $t \in [0,T]$ *and all* $x_i \in \mathbb{Z}_h$*, we have*

$$|\mathbf{u}^{h,R}(t,x_i)| = 1. \tag{3.3.13}$$

Moreover, for $1 \le p < \infty$ *and* $T \in (0,\infty)$*, there exists a constant* C *which does not depend on* R *but may depend on* $|\mathbf{g}|_{H^1}$*,* p*,* $|\nabla \mathbf{u}_0|_{L^2}$ *and* T *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |D^+\mathbf{u}^{h,R}(t)|^{2p}_{L^2_h}\right] \le C. \tag{3.3.14}$$

Proof. In order to simplify notations, we assume in the proof without loss of generality, that $\lambda = \mu = 1$. First, we prove (3.3.13). Using Lemma 2.3.17, (2.2.1) and (2.2.2), we get from equation (3.3.6)

$$\begin{aligned}
&\frac{1}{2}d\big(|\mathbf{u}^{h,R}(t)|^2\big) - \frac{1}{2}\big|\mathbf{u}^{h,R}(t) \times \mathbf{g}^h\big|^2 dt \\
&= \psi\left(|\mathbf{u}^{h,R}(t)|_{E_h}\right)\left\langle \mathbf{u}^{h,R}(t), \mathbf{u}^{h,R}(t) \times \tilde{\Delta}\mathbf{u}^{h,R}(t)\right\rangle dt \\
&\quad - \psi\left(|\mathbf{u}^{h,R}(t)|_{E_h}\right)\left\langle \mathbf{u}^{h}(t), \mathbf{u}^{h,R}(t) \times \big(\mathbf{u}^{h,R}(t) \times \tilde{\Delta}\mathbf{u}^{h,R}(t)\big)\right\rangle dt \\
&\quad + \frac{1}{2}\left\langle \mathbf{u}^{h,R}(t), \big(\mathbf{u}^{h,R}(t) \times \mathbf{g}^h\big) \times \mathbf{g}^h\right\rangle dt + \left\langle \mathbf{u}^{h,R}(t), \mathbf{u}^{h,R}(t) \times \mathbf{g}^h\right\rangle dW(t) \\
&= -\frac{1}{2}\big|\mathbf{u}^{h,R}(t) \times \mathbf{g}^h\big|^2 dt
\end{aligned}$$

for $t \in [0,T]$. It follows that

$$d\big(|\mathbf{u}^{h,R}(t)|^2\big) = 0.$$

Then, by integrating with respect to t, we obtain

$$|\mathbf{u}^{h,R}(t,x_i)| = |\mathbf{u}_0(x_i)| = 1$$

for every $t \in [0,T]$ and all $x_i \in \mathbb{Z}_h$.

Next we prove (3.3.14). Applying D^+ to the discrete equation (3.3.6), we have

$$\begin{aligned}
dD^+\mathbf{u}^{h,R} &= \psi\left(|\mathbf{u}^{h,R}|_{E_h}\right) D^+\left(\mathbf{u}^{h,R}\times\tilde{\Delta}\mathbf{u}^{h,R}\right)dt\\
&\quad-\psi\left(|\mathbf{u}^{h,R}|_{E_h}\right) D^+\left(\mathbf{u}^{h,R}\times(\mathbf{u}^{h,R}\times\tilde{\Delta}\mathbf{u}^{h,R})\right)dt\\
&\quad+\frac{1}{2}D^+\left((\mathbf{u}^{h,R}\times\mathbf{g}^h)\times\mathbf{g}^h\right)dt+D^+\left(\mathbf{u}^{h,R}\times\mathbf{g}^h\right)dW.
\end{aligned}$$

Then, by using Lemma 2.3.17, we get at every $x_i\in\mathbb{Z}_h$

$$\begin{aligned}
d|D^+\mathbf{u}^{h,R}|^2 &= 2\psi\left(|\mathbf{u}^{h,R}|_{E_h}\right)\left\langle D^+\mathbf{u}^{h,R}, D^+(\mathbf{u}^{h,R}\times\tilde{\Delta}\mathbf{u}^{h,R})\right\rangle dt\\
&\quad-2\psi\left(|\mathbf{u}^{h,R}|_{E_h}\right)\left\langle D^+\mathbf{u}^{h,R}, D^+\big(\mathbf{u}^{h,R}\times(\mathbf{u}^{h,R}\times\tilde{\Delta}\mathbf{u}^{h,R})\big)\right\rangle dt\\
&\quad+\left\langle D^+\mathbf{u}^{h,R}, D^+\big((\mathbf{u}^{h,R}\times\mathbf{g}^h)\times\mathbf{g}^h\big)\right\rangle dt+2\left\langle D^+\mathbf{u}^{h,R}, D^+(\mathbf{u}^{h,R}\times\mathbf{g}^h)\right\rangle dW\\
&\quad+|D^+(\mathbf{u}^{h,R}\times\mathbf{g}^h)|^2dt.
\end{aligned}$$

Hence, by taking the summation over $x_i\in\mathbb{Z}_h$, multiplying by h, using (2.2.1), (2.2.2) and Lemma 2.5.4, we obtain

$$\begin{aligned}
d|D^+\mathbf{u}^{h,R}|^2_{L^2_h} &= 2\psi\left(|\mathbf{u}^{h,R}|_{E_h}\right)\left\langle\tilde{\Delta}\mathbf{u}^{h,R}, \mathbf{u}^{h,R}\times(\mathbf{u}^{h,R}\times\tilde{\Delta}\mathbf{u}^{h,R})\right\rangle_{L^2_h}dt\\
&\quad+\left\langle D^+\mathbf{u}^{h,R}, D^+\big((\mathbf{u}^{h,R}\times\mathbf{g}^h)\times\mathbf{g}^h\big)\right\rangle_{L^2_h}dt\\
&\quad+2\left\langle D^+\mathbf{u}^{h,R}, D^+(\mathbf{u}^{h,R}\times\mathbf{g}^h)\right\rangle_{L^2_h}dW+\left|D^+(\mathbf{u}^{h,R}\times\mathbf{g}^h)\right|^2_{L^2_h}dt\\
&= -2\psi\left(|\mathbf{u}^{h,R}|_{E_h}\right)\left|\mathbf{u}^{h,R}\times\tilde{\Delta}\mathbf{u}^{h,R}\right|^2_{L^2_h}dt\\
&\quad+\left\langle D^+\mathbf{u}^{h,R}, D^+\big((\mathbf{u}^{h,R}\times\mathbf{g}^h)\times\mathbf{g}^h\big)\right\rangle_{L^2_h}dt\\
&\quad+2\left\langle D^+\mathbf{u}^{h,R}, D^+(\mathbf{u}^{h,R}\times\mathbf{g}^h)\right\rangle_{L^2_h}dW+\left|D^+(\mathbf{u}^{h,R}\times\mathbf{g}^h)\right|^2_{L^2_h}dt.
\end{aligned}$$

Then for $t\in[0,T]$,

$$\begin{aligned}
&|D^+\mathbf{u}^{h,R}(t)|^2_{L^2_h}-|D^+\mathbf{u}_0|^2_{L^2_h}+2\int_0^t\psi\left(|\mathbf{u}^{h,R}(s)|_{E_h}\right)\left|\mathbf{u}^{h,R}(s)\times\tilde{\Delta}\mathbf{u}^{h,R}(s)\right|^2_{L^2_h}ds\\
&=\int_0^t\left\langle D^+\mathbf{u}^{h,R}(s), D^+\big((\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\times\mathbf{g}^h\big)\right\rangle_{L^2_h}ds\\
&\quad+2\int_0^t\left\langle D^+\mathbf{u}^{h,R}(s), D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\right\rangle_{L^2_h}dW(s)
\end{aligned}$$

$$+\int_0^t \left|D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\right|^2_{L^2_h}\,ds. \tag{3.3.15}$$

Since

$$2\int_0^t \psi\left(|\mathbf{u}^{h,R}(s)|_{E_h}\right)\left|\mathbf{u}^{h,R}(s)\times\tilde{\Delta}\mathbf{u}^{h,R}(s)\right|^2_{L^2_h}\,ds\geq 0,$$

we get

$$\begin{aligned}
&|D^+\mathbf{u}^{h,R}(t)|^2_{L^2_h}\\
&\leq |D^+\mathbf{u}_0|^2_{L^2_h}+\int_0^t \left|D^+\mathbf{u}^{h,R}(s)\right|_{L^2_h}\left|D^+\big((\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\times\mathbf{g}^h\big)\right|_{L^2_h}ds\\
&\quad+2\left|\int_0^t \left\langle D^+\mathbf{u}^{h,R}(s),D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\right\rangle_{L^2_h}dW(s)\right|+\int_0^t \left|D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\right|^2_{L^2_h}ds\\
&\leq |D^+\mathbf{u}_0|^2_{L^2_h}\\
&\quad+C\int_0^t\left(\left|D^+\mathbf{u}^{h,R}(s)\right|^2_{L^2_h}+|D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)|^2_{L^2_h}+\left|D^+\big((\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\times\mathbf{g}^h\big)\right|^2_{L^2_h}\right)ds\\
&\quad+2\left|\int_0^t \left\langle D^+\mathbf{u}^{h,R}(s),D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\right\rangle_{L^2_h}dW(s)\right|.
\end{aligned}\tag{3.3.16}$$

The middle term in the integrand of the first integral on the right hand side can be estimated by using the product rule (Lemma 2.5.3), Lemma 2.5.6 and (3.3.13) as follows

$$\begin{aligned}
\left|D^+(\mathbf{u}^{h,R}\times\mathbf{g}^h)\right|^2_{L^2_h}&\leq 2|D^+\mathbf{u}^{h,R}\times\tau^+\mathbf{g}^h|^2_{L^2_h}+2|\mathbf{u}^{h,R}\times D^+\mathbf{g}^h|^2_{L^2_h}\\
&\leq 2|D^+\mathbf{u}^{h,R}|^2_{L^2_h}|\mathbf{g}^h|^2_{L^\infty_h}+2|D^+\mathbf{g}^h|^2_{L^2_h}\\
&\leq 2K|D^+\mathbf{u}^{h,R}|^2_{L^2_h}|\mathbf{g}^h|_{L^2_h}|D^+\mathbf{g}^h|_{L^2_h}+2|D^+\mathbf{g}^h|^2_{L^2_h}\\
&\leq K|D^+\mathbf{u}^{h,R}|^2_{L^2_h}|\mathbf{g}^h|^2_{L^2_h}+K|D^+\mathbf{u}^{h,R}|^2_{L^2_h}|D^+\mathbf{g}^h|^2_{L^2_h}+2|D^+\mathbf{g}^h|^2_{L^2_h}\\
&\leq C|D^+\mathbf{u}^{h,R}|^2_{L^2_h}+C
\end{aligned}\tag{3.3.17}$$

where the constants depend only on $|\mathbf{g}|_{H^1}$ but not on R. Similarly, we can show that

$$\left|D^+((\mathbf{u}^{h,R}\times\mathbf{g}^h)\times\mathbf{g}^h)\right|^2_{L^2_h}\leq C|D^+\mathbf{u}^{h,R}|^2_{L^2_h}+C.$$

Hence, it follows from (3.3.16) that

$$
\begin{aligned}
|D^+\mathbf{u}^{h,R}(t)|^2_{L^2_h} &\le |D^+\mathbf{u}_0|^2_{L^2_h} + Ct + C\int_0^t |D^+\mathbf{u}^{h,R}(s)|^2_{L^2_h} ds \\
&\quad + C\left|\int_0^t \left\langle D^+\mathbf{u}^{h,R}(s), D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\right\rangle_{L^2_h} dW(s)\right|.
\end{aligned}
$$

By raising both sides to the power p and using the following Jensen's inequality

$$
\left(\sum_{i=1}^n a_i\right)^q \le n^{q-1}\sum_{i=1}^n a_i^q, \quad q\ge 1, \quad n=1,2,3,...., \tag{3.3.18}
$$

we deduce

$$
\begin{aligned}
|D^+\mathbf{u}^{h,R}(t)|^{2p}_{L^2_h} &\le C + C\left(\int_0^t |D^+\mathbf{u}^{h,R}(s)|^2_{L^2_h} ds\right)^p \\
&\quad + C\left|\int_0^t \left\langle D^+\mathbf{u}^{h,R}(s), D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\right\rangle_{L^2_h} dW(s)\right|^p \\
&\le C + C\int_0^t |D^+\mathbf{u}^{h,R}(s)|^{2p}_{L^2_h} ds \\
&\quad + C\left|\int_0^t \left\langle D^+\mathbf{u}^{h,R}(s), D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\right\rangle_{L^2_h} dW(s)\right|^p
\end{aligned}
$$

where in the last step we used Hölder's inequality. The constants depend on $|\mathbf{g}|_{H^1}$, p, $|\nabla\mathbf{u}_0|_{L^2}$ and T. Therefore,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]} |D^+\mathbf{u}^{h,R}(s)|^{2p}_{L^2_h}\right] &\le C + C\int_0^t \mathbb{E}\left[|D^+\mathbf{u}^{h,R}(s)|^{2p}_{L^2_h}\right] ds \\
&\quad + C\mathbb{E}\left[\sup_{s\in[0,t]}\left|\int_0^s \left\langle D^+\mathbf{u}^{h,R}(\tau), D^+(\mathbf{u}^{h,R}(\tau)\times\mathbf{g}^h)\right\rangle_{L^2_h} dW(\tau)\right|^p\right] \\
&\le C + C\int_0^t \mathbb{E}\left[|D^+\mathbf{u}^{h,R}(s)|^{2p}_{L^2_h}\right] ds \\
&\quad + C\mathbb{E}\left[\left(\int_0^t \left|\left\langle D^+\mathbf{u}^{h,R}(s), D^+(\mathbf{u}^{h,R}(s)\times\mathbf{g}^h)\right\rangle_{L^2_h}\right|^2 ds\right)^{\frac{p}{2}}\right]
\end{aligned}
$$

where in the last step we used the BDG inequality (Lemma 2.3.18). It follows from (3.3.17) that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,t]}|D^+\mathbf{u}^{h,R}(s)|^{2p}_{L^2_h}\Big]\\
&\le C + C\int_0^t \mathbb{E}\Big[|D^+\mathbf{u}^{h,R}(s)|^{2p}_{L^2_h}\Big]ds + C\mathbb{E}\Big[\Big(\int_0^t \Big(|D^+\mathbf{u}^{h,R}(s)|^2_{L^2_h} + |D^+\mathbf{u}^{h,R}(s)|^4_{L^2_h}\Big)\,ds\Big)^{\frac{p}{2}}\Big]\\
&\le C + C\int_0^t \mathbb{E}\Big[|D^+\mathbf{u}^{h,R}(s)|^{2p}_{L^2_h}\Big]ds + C\mathbb{E}\Big[\Big(\int_0^t \Big(1 + |D^+\mathbf{u}^{h,R}(s)|^4_{L^2_h}\Big)\,ds\Big)^{\frac{p}{2}}\Big]\\
&\le C + C\int_0^t \mathbb{E}\Big[|D^+\mathbf{u}^{h,R}(s)|^{2p}_{L^2_h}\Big]ds + C(T) + C\mathbb{E}\Big[\Big(\int_0^t |D^+\mathbf{u}^{h,R}(s)|^4_{L^2_h}ds\Big)^{\frac{p}{2}}\Big] \qquad (3.3.19)
\end{aligned}
$$

for $p \ge 2$ where in the last step we used Jensen's inequality (3.3.18). Next, Hölder's inequality implies

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\int_0^t |D^+\mathbf{u}^{h,R}(s)|^4_{L^2_h}ds\Big)^{\frac{p}{2}}\Big] &\le C(T)\mathbb{E}\Big[\int_0^t \big|D^+\mathbf{u}^{h,R}(s)\big|^{2p}_{L^2_h}\,ds\Big]\\
&\le C(T)\int_0^t \mathbb{E}\Big[\big|D^+\mathbf{u}^{h,R}(s)\big|^{2p}_{L^2_h}\Big]ds. \qquad (3.3.20)
\end{aligned}
$$

Hence, inequalities (3.3.19) and (3.3.20) yield

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,t]}\big|D^+\mathbf{u}^{h,R}(s)\big|^{2p}_{L^2_h}\Big] &\le C + C\int_0^t \mathbb{E}\Big[\big|D^+\mathbf{u}^{h,R}(s)\big|^{2p}_{L^2_h}\Big]ds\\
&\le C + C\int_0^t \mathbb{E}\Big[\sup_{\tau\in[0,s]}\big|D^+\mathbf{u}^{h,R}(\tau)\big|^{2p}_{L^2_h}\Big]ds.
\end{aligned}
$$

Then, by using Gronwall's inequality (2.2.7), we obtain for $p \ge 2$

$$
\mathbb{E}\Big[\sup_{s\in[0,t]}\big|D^+\mathbf{u}^{h,R}(s)\big|^{2p}_{L^2_h}\Big] \le C
$$

where the constant depend on $|\mathbf{g}|_{H^1}$, p, $|\nabla\mathbf{u}_0|_{L^2}$ and T but not on R. Finally, the inequality $a \le \frac{1+a^2}{2}$ implies the result for $p \ge 1$ and the proof of (3.3.14) is complete. □

Remark 3.3.9. By definition of the discrete Laplacian $\tilde{\Delta}$ Lemma 3.3.8 immediately yields

$$
\mathbb{E}\Big[\Big(\int_0^T \big|\tilde{\Delta}\mathbf{u}^{h,R}(t)\big|^2_{L^2_h}\,dt\Big)^p\Big] \le C
$$

where C is independent of R but depends on h, T, p, $\mathbf{g}^h$ and $\mathbf{u}^h(0)$.

3.3.4 Existence of Semi-Discrete Solutions

In this subsection, we prove the existence of solutions to (3.3.1)-(3.3.3). For fixed $h, T > 0$ and for every $R > 0$, we define the stopping time τ^R as follows

$$\tau^R := \inf\left\{0 \le t \le T | \quad |\mathbf{u}^{h,R}(t)|_{E_h} \ge R\right\}.$$

We note that if $|\mathbf{u}^{h,R}(t)|_{E_h} < R$ for every $0 \le t \le T$, then $\tau^R = T$.

In fact, we aim to prove the following lemma.

Lemma 3.3.10. *Let $h > 0$ and $T > 0$ be fixed and assume that $|\mathbf{u}^h(0,x)| = 1$ for all $x \in \mathbb{Z}_h$, $D^+\mathbf{u}^h(0) \in L_h^2$ and $\mathbf{g}^h \in L_h^2$. Then, there exists a global unique strong solution $\mathbf{u}^h$ of equation* (3.3.1) *belonging to $\mathcal{E}_h(T)$.*

Proof. We consider $\mathbf{u}^{h,R_1}$ and $\mathbf{u}^{h,R_2}$ two elements of the sequence $(\mathbf{u}^{h,R})_R$ such that $R_1 < R_2$. We denote $\sigma^R := \min(\tau^{R_1}, \tau^{R_2})$. For all $t \le \sigma^R$, we have that

$$\psi\left(\left|\mathbf{u}^{h,R_1}\right|_{E_h}\right) = \psi\left(\left|\mathbf{u}^{h,R_2}\right|_{E_h}\right) = 1,$$

hence $\mathbf{u}^{h,R_1}$ and $\mathbf{u}^{h,R_2}$ verify respectively

$$\begin{aligned}
d\mathbf{u}^{h,R_1} &= \left(\mu\mathbf{u}^{h,R_1} \times \tilde{\Delta}\mathbf{u}^{h,R_1} - \lambda\mathbf{u}^{h,R_1} \times \left(\mathbf{u}^{h,R_1} \times \tilde{\Delta}\mathbf{u}^{h,R_1}\right) + \frac{\mu^2}{2}\left(\mathbf{u}^{h,R_1} \times \mathbf{g}^h\right) \times \mathbf{g}^h\right)dt \\
&\quad + \mu\left(\mathbf{u}^{h,R_1} \times \mathbf{g}^h\right) dW, \\
d\mathbf{u}^{h,R_2} &= \left(\mu\mathbf{u}^{h,R_2} \times \tilde{\Delta}\mathbf{u}^{h,R_2} - \lambda\mathbf{u}^{h,R_2} \times \left(\mathbf{u}^{h,R_2} \times \tilde{\Delta}\mathbf{u}^{h,R_2}\right) + \frac{\mu^2}{2}\left(\mathbf{u}^{h,R_2} \times \mathbf{g}^h\right) \times \mathbf{g}^h\right)dt \\
&\quad + \mu\left(\mathbf{u}^{h,R_2} \times \mathbf{g}^h\right) dW.
\end{aligned}$$

Using Lemma 3.3.3, we deduce that $\mathbf{u}^{h,R_1}(t) = \mathbf{u}^{h,R_2}(t)$ for all $t \le \sigma^R$. Consequently, since $R_1 < R_2$ we get

$$\tau^{R_1} \le \tau^{R_2}, \tag{3.3.21}$$

which means that $\sigma^R = \tau^{R_1}$. Indeed, assume that $\sigma^R = \tau^{R_2} < \tau^{R_1}$. By continuity, for t less than σ^R and sufficiently close to σ^R,

$$R_1 < \left|\mathbf{u}^{h,R_1}(t)\right|_{E_h} < \left|\mathbf{u}^{h,R_1}(\tau^{R_2})\right|_{E_h} = \left|\mathbf{u}^{h,R_2}(\tau^{R_2})\right|_{E_h} = R_2.$$

This contradicts the definition of τ^{R_1}.

Since $\{\tau^R\}$ is non-decreasing and bounded above by T, the stopping time

$$\tau^\infty := \lim_{R\to\infty} \tau^R$$

is well defined. Using Lemma 2.3.9 and (3.3.14), we have

$$\begin{aligned}
\mathbb{P}(\tau^\infty < T) &= \mathbb{P}\left(\lim_{R\to\infty} \tau^R < T\right) = \lim_{R\to\infty} \mathbb{P}\left(\tau^R < T\right) = \lim_{R\to\infty} \mathbb{P}\left(\sup_{0\le t\le T} |\mathbf{u}^{h,R}(t)|^2_{E_h} > R^2\right) \\
&= \lim_{R\to\infty} \mathbb{P}\left(\sup_{0\le t\le T} |D^+\mathbf{u}^{h,R}(t)|^2_{L^2_h} > R^2 - 1\right) \\
&\le \lim_{R\to\infty} \frac{1}{R^2-1}\mathbb{E}\left[\sup_{0\le t\le T} |D^+\mathbf{u}^{h,R}(t)|^2_{L^2_h}\right] \\
&\le \lim_{R\to\infty} \frac{C}{R^2-1} \\
&= 0
\end{aligned}$$

with C independent of R. Hence, $\tau^\infty = T$ $\mathbb{P}$-a.s.

We define a stochastic process $\mathbf{u}^h = \{\mathbf{u}^h(t)\}$ such that $\mathbf{u}^h(t) = \mathbf{u}^{h,R}(t)$ for all $t \le \tau^R$. Since $\lim_{R\to\infty} \tau^R = T$, the process $\mathbf{u}^h = \{\mathbf{u}^h(t); t \le T\}$ is well defined. Moreover, by the definition of τ^R and $\mathbf{u}^{h,R}$, the process $\mathbf{u}^h$ satisfies the following equation for every $t \in [0, T]$

$$\begin{aligned}
d\mathbf{u}^h(t) = &\left(\mu\mathbf{u}^h(t) \times \tilde{\Delta}\mathbf{u}^h(t) - \lambda\mathbf{u}^h(t) \times \left(\mathbf{u}^h(t) \times \tilde{\Delta}\mathbf{u}^h(t)\right) + \frac{\mu^2}{2}\left(\mathbf{u}^h(t) \times \mathbf{g}^h\right) \times \mathbf{g}^h\right)dt \\
&+ \mu\left(\mathbf{u}^h(t) \times \mathbf{g}^h\right) dW
\end{aligned}$$

and the lemma follows. $\square$

3.4 A Priori Estimates

In this section, we introduce and prove some uniform estimates which will be used to prove the main result of this chapter, Theorem 3.1.3.

Lemma 3.4.1. *Assume that* $|\mathbf{u}^h(0, x_i)| = 1$ *for all* $x_i \in \mathbb{Z}_h$*,* $D^+\mathbf{u}^h(0) \in L^2_h$ *and* $\mathbf{g}^h \in L^2_h$*. Then, for every* $t \in [0, T]$ *and all* $x_i \in \mathbb{Z}_h$*, we have*

$$|\mathbf{u}^h(t, x_i)| = 1. \tag{3.4.1}$$

Moreover, for $1 \leq p < \infty$ *and* $T \in (0, \infty)$*, there exists a constant* C *which does not depend on* h *but may depend on* $|\mathbf{g}|_{H^1}$*,* p*,* $|\nabla \mathbf{u}_0|_{L^2}$ *and* T *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left|D^+\mathbf{u}^h(t)\right|^{2p}_{L^2_h}\right] \leq C, \tag{3.4.2}$$

$$\mathbb{E}\left[\left(\int_0^T \left|\tilde{\Delta}\mathbf{u}^h(t)\right|^2_{L^2_h} dt\right)^p\right] \leq C. \tag{3.4.3}$$

Proof. The proof for (3.4.1) and (3.4.2) follows in the same manner as proving Lemma 3.3.8 by considering $\psi\left(\left|\mathbf{u}^{h,R}\right|_{E_h}\right) = 1$ and replacing $\mathbf{u}^{h,R}$ by $\mathbf{u}^h$. In fact, following the same reasoning, we can see that the constant in (3.3.14) is independent of h. Next, we prove (3.4.3). By using (3.4.1) and the elementary property (2.2.4), we have

$$|\tilde{\Delta}\mathbf{u}^h|^2 = |\mathbf{u}^h \times \tilde{\Delta}\mathbf{u}^h|^2 + \left\langle \mathbf{u}^h, \tilde{\Delta}\mathbf{u}^h\right\rangle^2.$$

Then, from Lemma 2.5.5 we deduce

$$|\tilde{\Delta}\mathbf{u}^h|^2 \leq |\mathbf{u}^h \times \tilde{\Delta}\mathbf{u}^h|^2 + \frac{1}{2}|D^+\mathbf{u}^h|^4 + \frac{1}{2}|D^-\mathbf{u}^h|^4.$$

Applying summation over $x \in \mathbb{Z}_h$, multiplying by h and using Lemma 2.5.2 we get

$$|\tilde{\Delta}\mathbf{u}^h|^2_{L^2_h} \leq |\mathbf{u}^h \times \tilde{\Delta}\mathbf{u}^h|^2_{L^2_h} + |D^+\mathbf{u}^h|^4_{L^4_h}.$$

Integrating with respect to $t \in [0,T]$, raising to the power p and applying expectation

$$\begin{aligned}\mathbb{E}\bigg[\bigg(\int_0^T \Big|\tilde{\Delta}\mathbf{u}^h(t)\Big|_{L_h^2}^2 dt\bigg)^p\bigg] &\le \mathbb{E}\bigg[\bigg(\int_0^T \Big|\mathbf{u}^h(t)\times\tilde{\Delta}\mathbf{u}^h(t)\Big|_{L_h^2}^2 dt\bigg)^p\bigg] \\ &\quad + \mathbb{E}\bigg[\bigg(\int_0^T \big|D^+\mathbf{u}^h(t)\big|_{L_h^4}^4 dt\bigg)^p\bigg] \\ &\le B_1 + B_2. \qquad (3.4.4)\end{aligned}$$

In order to estimate B_1, we proceed from (3.3.15) by considering $\psi\left(\left|\mathbf{u}^{h,R}\right|_{E_h}\right) = 1$ and replacing $\mathbf{u}^{h,R}$ by $\mathbf{u}^h$ and write down the same inequalities but with $\int_0^t |\mathbf{u}^h(s)\times\tilde{\Delta}\mathbf{u}^h(s)|_{L_h^2}^2 ds$ instead of $|D^+\mathbf{u}^h(t)|_{L_h^2}^2$ on the left hand side. In fact, we have for any $t\in[0,T]$

$$\mathbb{E}\bigg[\bigg(\int_0^t \Big|\mathbf{u}^h(s)\times\tilde{\Delta}\mathbf{u}^h(s)\Big|_{L_h^2}^2 ds\bigg)^p\bigg] \le C + C\int_0^t \mathbb{E}\bigg[\sup_{\tau\in[0,s]}\Big|D^+\mathbf{u}^h(\tau)\Big|_{L_h^2}^{2p}\bigg]ds.$$

From (3.4.2), we deduce for any $t\in[0,T]$

$$\mathbb{E}\bigg[\bigg(\int_0^t \Big|\mathbf{u}^h(s)\times\tilde{\Delta}\mathbf{u}^h(s)\Big|_{L_h^2}^2 ds\bigg)^p\bigg] \le C \qquad (3.4.5)$$

where the constant C depends on $|\mathbf{g}|_{H^1}$, p, $|\nabla\mathbf{u}_0|_{L^2}$ and T but not on h. Now, we estimate B_2. We know that

$$|D^+\mathbf{u}^h|_{L_h^4}^4 \le |D^+\mathbf{u}^h|_{L_h^\infty}^2 |D^+\mathbf{u}^h|_{L_h^2}^2. \qquad (3.4.6)$$

From Lemma 2.5.6, for the discrete function $D^+\mathbf{u}^h$, we have

$$|D^+\mathbf{u}^h|_{L_h^\infty} \le K\left|D^+\mathbf{u}^h\right|_{L_h^2}^{\frac{1}{2}}\left|(D^+)^2\mathbf{u}^h\right|_{L_h^2}^{\frac{1}{2}}.$$

From Lemma 2.5.2, we notice that $|(D^+)^2\mathbf{u}^h|_{L_h^2} = |\tilde{\Delta}\mathbf{u}^h|_{L_h^2}$. Then, from (3.4.6)

$$\begin{aligned}|D^+\mathbf{u}^h|_{L_h^4}^4 &\le K^2\left|D^+\mathbf{u}^h\right|_{L_h^2}^3 |\tilde{\Delta}\mathbf{u}^h|_{L_h^2} \\ &\le C_\varepsilon\left|D^+\mathbf{u}^h\right|_{L_h^2}^6 + \varepsilon|\tilde{\Delta}\mathbf{u}^h|_{L_h^2}^2.\end{aligned}$$

Integrating with respect to time, raising to the power p and using the inequality $(a+b)^p \leq 2^{p-1}(a^p+b^p)$, for $a,b \geq 0$ and $p \geq 1$, we get

$$\begin{aligned}\left(\int_0^T \left|D^+\mathbf{u}^h(t)\right|_{L_h^4}^4 dt\right)^p &\leq \left(C_\varepsilon \int_0^T \left|D^+\mathbf{u}^h(t)\right|_{L_h^2}^6 dt + \varepsilon \int_0^T \left|\tilde{\Delta}\mathbf{u}^h(t)\right|_{L_h^2}^2 dt\right)^p \\ &\leq C_\varepsilon T^p \sup_{t\in[0,T]} \left|D^+\mathbf{u}^h(t)\right|_{L_h^2}^{6p} + \varepsilon \left(\int_0^T \left|\tilde{\Delta}\mathbf{u}^h(t)\right|_{L_h^2}^2 dt\right)^p.\end{aligned}$$

Hence,

$$\mathbb{E}\left[\left(\int_0^T \left|D^+\mathbf{u}^h(t)\right|_{L_h^4}^4 dt\right)^p\right] \leq C_\varepsilon T^p \mathbb{E}\left[\sup_{t\in[0,T]} \left|D^+\mathbf{u}^h(t)\right|_{L_h^2}^{6p}\right] + \varepsilon \mathbb{E}\left[\left(\int_0^T \left|\tilde{\Delta}\mathbf{u}^h(t)\right|_{L_h^2}^2 dt\right)^p\right].$$

Consequently, using (3.4.2) and (3.4.5), we deduce from (3.4.4)

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T \left|\tilde{\Delta}\mathbf{u}^h(t)\right|_{L_h^2}^2 dt\right)^p\right] &\leq C_\varepsilon \mathbb{E}\left[\left(\int_0^T \left|\mathbf{u}^h(t)\times\tilde{\Delta}\mathbf{u}^h(t)\right|_{L_h^2}^2 dt\right)^p\right] \\ &\quad + C_\varepsilon T^p \mathbb{E}\left[\sup_{t\in[0,T]} \left|D^+\mathbf{u}^h(t)\right|_{L_h^2}^{6p}\right] \\ &\leq C \end{aligned} \tag{3.4.7}$$

where the constant C may depend on $|\mathbf{g}|_{H^1}$, p, $|\nabla \mathbf{u}_0|_{L^2}$ and T but not on h. Then, the proof of (3.4.3) is complete. □

3.5 Some Technical Results

In this section, we prove some properties of a transformation which will be used in the next section to define a new variable $\mathbf{m}^h$.

In the following, we define $\mathbf{G}: L_m^2 \to L_m^2$ by

$$\mathbf{G}\boldsymbol{\phi} := \boldsymbol{\phi}\times\mathbf{g} \quad \forall \boldsymbol{\phi} \in L_m^2$$

and since $\mathbf{G}: L_m^2 \to L_m^2$ is bounded, we can define the operator $e^{t\mathbf{G}}: L_m^2 \to L_m^2$ by

$$e^{t\mathbf{G}} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{G}^k, \quad t \geq 0.$$

We recall that $\mathbf{g}^h$ is the restriction of the given function $\mathbf{g}$ to $\mathbb{Z}_h$. We note that the first lemma and parts of the second lemma are proved in [25] but we reproduce the proof for completeness.

Lemma 3.5.1. *Assume that $\mathbf{g}^h \in L_h^\infty$. We define $\mathbf{G}_h : L^2_{m,h} \to L^2_{m,h}$ by*

$$\mathbf{G}_h \boldsymbol{\phi} := \boldsymbol{\phi} \times \mathbf{g}^h \quad \forall \boldsymbol{\phi} \in L^2_{m,h}.$$

Then, $\mathbf{G}_h$ is well defined and for any $\boldsymbol{\phi}, \boldsymbol{\psi} \in L^2_{m,h}$,

$$\mathbf{G}_h^* = -\mathbf{G}_h, \tag{3.5.1}$$

$$\boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi} = \left\langle \boldsymbol{\phi}, \mathbf{g}^h \right\rangle \boldsymbol{\psi} - \left\langle \boldsymbol{\phi}, \boldsymbol{\psi} \right\rangle \mathbf{g}^h, \tag{3.5.2}$$

$$\boldsymbol{\phi} \times \mathbf{G}_h^2 \boldsymbol{\psi} = \left\langle \boldsymbol{\phi}, \mathbf{g}^h \right\rangle \mathbf{G}_h \boldsymbol{\psi} - \mathbf{G}_h \boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi}, \tag{3.5.3}$$

$$\mathbf{G}_h \boldsymbol{\phi} \times \mathbf{G}_h^2 \boldsymbol{\psi} = -\mathbf{G}_h^2 \boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi}, \tag{3.5.4}$$

$$\mathbf{G}_h^{2n+1} \boldsymbol{\phi} = (-1)^n \mathbf{G}_h \boldsymbol{\phi} \quad n \geq 0, \tag{3.5.5}$$

$$\mathbf{G}_h^{2n+2} \boldsymbol{\phi} = (-1)^n \mathbf{G}_h^2 \boldsymbol{\phi} \quad n \geq 0. \tag{3.5.6}$$

Assume further that $|\mathbf{g}^h| = 1$, we have

$$\mathbf{G}_h \boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi} = \left\langle \mathbf{g}^h, \boldsymbol{\phi} \times \boldsymbol{\psi} \right\rangle \mathbf{g}^h = \mathbf{G}_h^2 \boldsymbol{\phi} \times \mathbf{G}_h^2 \boldsymbol{\psi}. \tag{3.5.7}$$

Proof. We start by proving (3.5.1). By using (2.2.5), we have for $\boldsymbol{\phi}, \boldsymbol{\psi} \in L^2_{m,h}$

$$\left\langle \mathbf{G}_h \boldsymbol{\phi}, \boldsymbol{\psi} \right\rangle = \left\langle \boldsymbol{\phi} \times \mathbf{g}^h, \boldsymbol{\psi} \right\rangle = \left\langle \mathbf{g}^h \times \boldsymbol{\psi}, \boldsymbol{\phi} \right\rangle = -\left\langle \boldsymbol{\phi}, \boldsymbol{\psi} \times \mathbf{g}^h \right\rangle = -\left\langle \boldsymbol{\phi}, \mathbf{G}_h \boldsymbol{\psi} \right\rangle.$$

Next, we prove (3.5.2). Using (2.2.3), we get

$$\boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi} = \boldsymbol{\phi} \times \left(\boldsymbol{\psi} \times \mathbf{g}^h \right) = \left\langle \boldsymbol{\phi}, \mathbf{g}^h \right\rangle \boldsymbol{\psi} - \left\langle \boldsymbol{\phi}, \boldsymbol{\psi} \right\rangle \mathbf{g}^h.$$

Now, we prove (3.5.3). Using (3.5.2), we obtain

$$\boldsymbol{\phi} \times \mathbf{G}_h^2 \boldsymbol{\psi} = \left\langle \boldsymbol{\phi}, \mathbf{g}^h \right\rangle \mathbf{G}_h \boldsymbol{\psi} - \left\langle \boldsymbol{\phi}, \mathbf{G}_h \boldsymbol{\psi} \right\rangle \mathbf{g}^h. \tag{3.5.8}$$

We know that, using (2.2.1) and (2.2.3),

$$\mathbf{G}_h \boldsymbol{\psi} \times \mathbf{G}_h \boldsymbol{\phi} = \mathbf{G}_h \boldsymbol{\psi} \times \left(\boldsymbol{\phi} \times \mathbf{g}^h \right) = \left\langle \mathbf{G}_h \boldsymbol{\psi}, \mathbf{g}^h \right\rangle \boldsymbol{\phi} - \left\langle \mathbf{G}_h \boldsymbol{\psi}, \boldsymbol{\phi} \right\rangle \mathbf{g}^h = - \left\langle \mathbf{G}_h \boldsymbol{\psi}, \boldsymbol{\phi} \right\rangle \mathbf{g}^h. \tag{3.5.9}$$

Then, we get from (3.5.8)

$$\boldsymbol{\phi} \times \mathbf{G}_h^2 \boldsymbol{\psi} = \left\langle \boldsymbol{\phi}, \mathbf{g}^h \right\rangle \mathbf{G}_h \boldsymbol{\psi} - \mathbf{G}_h \boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi}.$$

We continue by proving (3.5.4). From (3.5.3), we have using (2.2.1)

$$\mathbf{G}_h \boldsymbol{\phi} \times \mathbf{G}_h^2 \boldsymbol{\psi} = \left\langle \mathbf{G}_h \boldsymbol{\phi}, \mathbf{g}^h \right\rangle \mathbf{G}_h \boldsymbol{\psi} - \mathbf{G}_h^2 \boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi} = -\mathbf{G}_h^2 \boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi}.$$

Let us prove (3.5.5) by induction. it is clear that (3.5.5) is true for $n = 0$. We assume that it holds for n and we prove that it holds for $n + 1$. In fact,

$$\mathbf{G}_h^{2n+3} \boldsymbol{\phi} = \mathbf{G}_h^2 \mathbf{G}_h^{2n+1} \boldsymbol{\phi} = (-1)^n \mathbf{G}_h^3 \boldsymbol{\phi} = (-1)^{n+1} \mathbf{G}_h \boldsymbol{\phi}.$$

Similarly, we can prove (3.5.6). Finally, we prove (3.5.7). From (3.5.9) and using (2.2.5), we have

$$\mathbf{G}_h \boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi} = \left\langle \mathbf{G}_h \boldsymbol{\psi}, \boldsymbol{\phi} \right\rangle \mathbf{g}^h = \left\langle \boldsymbol{\psi} \times \mathbf{g}^h, \boldsymbol{\phi} \right\rangle \mathbf{g}^h = \left\langle \mathbf{g}^h, \boldsymbol{\phi} \times \boldsymbol{\psi} \right\rangle \mathbf{g}^h. \tag{3.5.10}$$

Using successfully (2.2.3), (2.2.1) and (2.2.5), we get

$$\begin{aligned} \mathbf{G}_h^2 \boldsymbol{\phi} \times \mathbf{G}_h^2 \boldsymbol{\psi} &= \mathbf{G}_h^2 \boldsymbol{\phi} \times \left(\mathbf{G}_h \boldsymbol{\psi} \times \mathbf{g}^h \right) = \left\langle \mathbf{G}_h^2 \boldsymbol{\phi}, \mathbf{g}^h \right\rangle \mathbf{G}_h \boldsymbol{\psi} - \left\langle \mathbf{G}_h^2 \boldsymbol{\phi}, \mathbf{G}_h \boldsymbol{\psi} \right\rangle \mathbf{g}^h \\ &= - \left\langle \mathbf{G}_h \boldsymbol{\phi} \times \mathbf{g}^h, \mathbf{G}_h \boldsymbol{\psi} \right\rangle \mathbf{g}^h = \left\langle \mathbf{G}_h \boldsymbol{\phi} \times \mathbf{G}_h \boldsymbol{\psi}, \mathbf{g}^h \right\rangle \mathbf{g}^h. \end{aligned}$$

Then from (3.5.10), using the assumption $|\mathbf{g}^h| = 1$, we obtain

$$\mathbf{G}_h^2\boldsymbol{\phi} \times \mathbf{G}_h^2\boldsymbol{\psi} = \left\langle \mathbf{g}^h, \boldsymbol{\phi} \times \boldsymbol{\psi} \right\rangle |\mathbf{g}^h|^2 \mathbf{g}^h = \left\langle \mathbf{g}^h, \boldsymbol{\phi} \times \boldsymbol{\psi} \right\rangle \mathbf{g}^h.$$

Thus, the proof of (3.5.7) is completed. □

Since $\mathbf{G}_h : L^2_{m,h} \to L^2_{m,h}$ is bounded, we can define the operator $e^{t\mathbf{G}_h} : L^2_{m,h} \to L^2_{m,h}$ by the formula

$$e^{t\mathbf{G}_h} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbf{G}_h^k, \quad t \geq 0 \,.$$

Lemma 3.5.2. *For any $t \in \mathbb{R}$ and $\boldsymbol{\phi}, \boldsymbol{\psi} \in L^2_{m,h}$,*

$$e^{t\mathbf{G}_h}\boldsymbol{\phi} = \boldsymbol{\phi} + \sin(t)\mathbf{G}_h\boldsymbol{\phi} + (1 - \cos(t))\, \mathbf{G}_h^2\boldsymbol{\phi}, \tag{3.5.11}$$

$$e^{-t\mathbf{G}_h} e^{t\mathbf{G}_h}(\boldsymbol{\phi}) = \boldsymbol{\phi}, \tag{3.5.12}$$

$$e^{t\mathbf{G}_h}\mathbf{G}_h\boldsymbol{\phi} = \mathbf{G}_h e^{t\mathbf{G}_h}\boldsymbol{\phi}, \tag{3.5.13}$$

$$e^{t\mathbf{G}_h}(\boldsymbol{\phi} \times \boldsymbol{\psi}) = e^{t\mathbf{G}_h}\boldsymbol{\phi} \times e^{t\mathbf{G}_h}\boldsymbol{\psi}. \tag{3.5.14}$$

Proof. First, we prove (3.5.11). We get using Taylor's expansion, (3.5.5) and (3.5.6)

$$\begin{aligned}
e^{t\mathbf{G}_h}\boldsymbol{\phi} &= \sum_{n=0}^{\infty} \frac{t^n}{n!} \mathbf{G}_h^n \boldsymbol{\phi} \\
&= \boldsymbol{\phi} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} \mathbf{G}_h^{2k+1}\boldsymbol{\phi} + \sum_{k=0}^{\infty} \frac{t^{2k+2}}{(2k+2)!} \mathbf{G}_h^{2k+2}\boldsymbol{\phi} \\
&= \boldsymbol{\phi} + \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} (-1)^k \mathbf{G}_h\boldsymbol{\phi} + \sum_{k=0}^{\infty} \frac{t^{2k+2}}{(2k+2)!} (-1)^k \mathbf{G}_h^2\boldsymbol{\phi} \\
&= \boldsymbol{\phi} + \sin(t)\mathbf{G}_h\boldsymbol{\phi} + (1 - \cos(t))\mathbf{G}_h^2\boldsymbol{\phi}.
\end{aligned}$$

Next, we prove (3.5.12). We have from (3.5.11) and (3.5.6)

$$\begin{aligned}
e^{-t\mathbf{G}_h} e^{t\mathbf{G}_h}(\boldsymbol{\phi}) &= e^{-t\mathbf{G}_h} \left(\boldsymbol{\phi} + \sin(t)\mathbf{G}_h\boldsymbol{\phi} + (1 - \cos(t))\, \mathbf{G}_h^2\boldsymbol{\phi}\right) \\
&= \boldsymbol{\phi} - \sin(t)\mathbf{G}_h\boldsymbol{\phi} + (1 - \cos(t))\, \mathbf{G}_h^2\boldsymbol{\phi} + \sin(t)\mathbf{G}_h\boldsymbol{\phi} - \sin^2(t)\mathbf{G}_h^2\boldsymbol{\phi} \\
&\quad + \sin(t)\,(1 - \cos(t))\, \mathbf{G}_h^3\boldsymbol{\phi} + (1 - \cos(t))\, \mathbf{G}_h^2\boldsymbol{\phi} - \sin(t)\,(1 - \cos(t))\, \mathbf{G}_h^3\boldsymbol{\phi}
\end{aligned}$$

$$
\begin{aligned}
&+ (1-\cos(t))^2\, \mathbf{G}_h^4 \boldsymbol{\phi} \\
&= \boldsymbol{\phi} + 2\,(1-\cos(t))\, \mathbf{G}_h^2 \boldsymbol{\phi} - \sin^2(t) \mathbf{G}_h^2 \boldsymbol{\phi} - (1-\cos(t))^2\, \mathbf{G}_h^2 \boldsymbol{\phi} \\
&= \boldsymbol{\phi}.
\end{aligned}
$$

We continue by proving (3.5.13). Using (3.5.11), we get

$$
\begin{aligned}
e^{t\mathbf{G}_h} \mathbf{G}_h \boldsymbol{\phi} &= \mathbf{G}_h \boldsymbol{\phi} + \sin(t) \mathbf{G}_h^2 \boldsymbol{\phi} + (1-\cos(t))\, \mathbf{G}_h^3 \boldsymbol{\phi} \\
&= \mathbf{G}_h \left(\boldsymbol{\phi} + \sin(t) \mathbf{G}_h \boldsymbol{\phi} + (1-\cos(t))\, \mathbf{G}_h^2 \boldsymbol{\phi} \right) = \mathbf{G}_h e^{t\mathbf{G}_h} \boldsymbol{\phi}.
\end{aligned}
$$

Finally, we prove (3.5.14). if $\mathbf{g}^h(x_i) = 0$, then using (3.5.11) we get that (3.5.14) holds. Now, if $\mathbf{g}^h(x_i) \neq 0$, we consider

$$
\begin{aligned}
\mathbf{g}_1^h(x_i) &= \frac{\mathbf{g}^h(x_i)}{|\mathbf{g}^h(x_i)|}, \\
\mathbf{G}_{h,1} \boldsymbol{\phi} &:= \boldsymbol{\phi} \times \mathbf{g}_1^h, \\
s &= |\mathbf{g}^h(x_i)| t.
\end{aligned}
$$

We start by proving

$$
e^{s\mathbf{G}_{h,1}} (\boldsymbol{\phi} \times \boldsymbol{\psi}) = e^{s\mathbf{G}_{h,1}} \boldsymbol{\phi} \times e^{s\mathbf{G}_{h,1}} \boldsymbol{\psi}. \tag{3.5.15}
$$

Using (3.5.11), we have

$$
\begin{aligned}
e^{s\mathbf{G}_{h,1}} \boldsymbol{\phi} \times e^{s\mathbf{G}_{h,1}} \boldsymbol{\psi} = \;& \boldsymbol{\phi} \times \boldsymbol{\psi} + \sin(s)\, (\boldsymbol{\phi} \times \mathbf{G}_{h,1} \boldsymbol{\psi} + \mathbf{G}_{h,1} \boldsymbol{\phi} \times \boldsymbol{\psi}) \\
&+ (1-\cos(s)) \left(\boldsymbol{\phi} \times \mathbf{G}_{h,1}^2 \boldsymbol{\psi} + \mathbf{G}_{h,1}^2 \boldsymbol{\phi} \times \boldsymbol{\psi} \right) \\
&+ \sin(s)\, (1-\cos(s)) \left(\mathbf{G}_{h,1} \boldsymbol{\phi} \times \mathbf{G}_{h,1}^2 \boldsymbol{\psi} + \mathbf{G}_{h,1}^2 \boldsymbol{\phi} \times \mathbf{G}_{h,1} \boldsymbol{\psi} \right) \\
&+ \sin^2(s) \mathbf{G}_{h,1} \boldsymbol{\phi} \times \mathbf{G}_{h,1} \boldsymbol{\psi} + (1-\cos(s))^2\, \mathbf{G}_{h,1}^2 \boldsymbol{\phi} \times \mathbf{G}_{h,1}^2 \boldsymbol{\psi} \\
:= \;& \boldsymbol{\phi} \times \boldsymbol{\psi} + N_1 + \ldots + N_5.
\end{aligned}
$$

Using (2.2.3), we have that $N_1 = \sin(s)\mathbf{G}_{h,1}(\boldsymbol{\phi} \times \boldsymbol{\psi})$. From (3.5.4), we get that $N_3 = 0$. Finally, using (3.5.3) and (3.5.7) we have

$$\begin{aligned}
&N_2 + N_4 + N_5 \\
&= (1-\cos(s))\left(\left\langle \boldsymbol{\phi}, \mathbf{g}_1^h\right\rangle \mathbf{G}_{h,1}\boldsymbol{\psi} - \mathbf{G}_{h,1}\boldsymbol{\phi} \times \mathbf{G}_{h,1}\boldsymbol{\psi} - \left\langle \boldsymbol{\psi}, \mathbf{g}_1^h\right\rangle \mathbf{G}_{h,1}\boldsymbol{\phi} + \mathbf{G}_{h,1}\boldsymbol{\psi} \times \mathbf{G}_{h,1}\boldsymbol{\phi}\right) \\
&\quad + \sin^2(s)\mathbf{G}_{h,1}\boldsymbol{\phi} \times \mathbf{G}_{h,1}\boldsymbol{\psi} + (1-\cos(s))^2\,\mathbf{G}_{h,1}\boldsymbol{\phi} \times \mathbf{G}_{h,1}\boldsymbol{\psi} \\
&= (1-\cos(s))\left(\left\langle \boldsymbol{\phi}, \mathbf{g}_1^h\right\rangle \mathbf{G}_{h,1}\boldsymbol{\psi} - 2\mathbf{G}_{h,1}\boldsymbol{\phi} \times \mathbf{G}_{h,1}\boldsymbol{\psi} - \left\langle \boldsymbol{\psi}, \mathbf{g}_1^h\right\rangle \mathbf{G}_{h,1}\boldsymbol{\phi}\right) \\
&\quad + 2\mathbf{G}_{h,1}\boldsymbol{\phi} \times \mathbf{G}_{h,1}\boldsymbol{\psi} - 2\cos(s)\mathbf{G}_{h,1}\boldsymbol{\phi} \times \mathbf{G}_{h,1}\boldsymbol{\psi} \\
&= (1-\cos(s))\left(\left\langle \boldsymbol{\phi}, \mathbf{g}_1^h\right\rangle \mathbf{G}_{h,1}\boldsymbol{\psi} - \left\langle \boldsymbol{\psi}, \mathbf{g}_1^h\right\rangle \mathbf{G}_{h,1}\boldsymbol{\phi}\right) \\
&= -(1-\cos(s))\,\mathbf{G}_{h,1}\left(\left\langle \mathbf{g}_1^h, \boldsymbol{\psi}\right\rangle \boldsymbol{\phi} - \left\langle \mathbf{g}_1^h, \boldsymbol{\phi}\right\rangle \boldsymbol{\psi}\right) \\
&= -(1-\cos(s))\,\mathbf{G}_{h,1}\left(\mathbf{g}_1^h \times \left(\boldsymbol{\phi} \times \boldsymbol{\psi}\right)\right) \\
&= (1-\cos(s))\,\mathbf{G}_{h,1}^2(\boldsymbol{\phi} \times \boldsymbol{\psi}).
\end{aligned}$$

Therefore,

$$e^{s\mathbf{G}_{h,1}}\boldsymbol{\phi} \times e^{s\mathbf{G}_{h,1}}\boldsymbol{\psi} = \boldsymbol{\phi} \times \boldsymbol{\psi} + \sin(s)\mathbf{G}_{h,1}(\boldsymbol{\phi} \times \boldsymbol{\psi}) + (1-\cos(s))\,\mathbf{G}_{h,1}^2(\boldsymbol{\phi} \times \boldsymbol{\psi}).$$

Using (3.5.11), the proof of (3.5.15) is complete. We know that

$$\begin{aligned}
e^{s\mathbf{G}_{h,1}}(\boldsymbol{\phi} \times \boldsymbol{\psi}) &= \sum_{k=0}^{\infty} \frac{s^k}{k!}\mathbf{G}_{h,1}^k(\boldsymbol{\phi} \times \boldsymbol{\psi}) = \sum_{k=0}^{\infty} \frac{|\mathbf{g}^h(x_i)|^k t^k}{k!}\mathbf{G}_h^k(\boldsymbol{\phi} \times \boldsymbol{\psi})\frac{1}{|\mathbf{g}^h(x_i)|^k} \\
&= \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathbf{G}_h^k(\boldsymbol{\phi} \times \boldsymbol{\psi}) = e^{t\mathbf{G}_h}(\boldsymbol{\phi} \times \boldsymbol{\psi}).
\end{aligned}$$

Similarly, we have

$$e^{s\mathbf{G}_{h,1}}\boldsymbol{\phi} \times e^{s\mathbf{G}_{h,1}}\boldsymbol{\psi} = e^{t\mathbf{G}_h}\boldsymbol{\phi} \times e^{t\mathbf{G}_h}\boldsymbol{\psi}.$$

We deduce that

$$e^{t\mathbf{G}_h}(\boldsymbol{\phi} \times \boldsymbol{\psi}) = e^{t\mathbf{G}_h}\boldsymbol{\phi} \times e^{t\mathbf{G}_h}\boldsymbol{\psi}$$

which completes the proof of the lemma. □

Lemma 3.5.3. *For any $t \in \mathbb{R}$ and $\boldsymbol{\phi} \in L^2_{m,h}$, we have*

$$|e^{t\mathbf{G}_h}\boldsymbol{\phi}| = |\boldsymbol{\phi}|. \tag{3.5.16}$$

Proof. By using (3.5.11) and (3.5.1), we have

$$\begin{aligned}
|e^{t\mathbf{G}_h}\boldsymbol{\phi}|^2 &= \left\langle \boldsymbol{\phi} + \sin(t)\mathbf{G}_h\boldsymbol{\phi} + (1-\cos(t))\,\mathbf{G}_h^2\boldsymbol{\phi}, \boldsymbol{\phi} + \sin(t)\mathbf{G}_h\boldsymbol{\phi} + (1-\cos(t))\,\mathbf{G}_h^2\boldsymbol{\phi}\right\rangle \\
&= |\boldsymbol{\phi}|^2 + 2\,(1-\cos(t))\left\langle \boldsymbol{\phi}, \mathbf{G}_h^2\boldsymbol{\phi}\right\rangle - \sin^2(t)\left\langle \boldsymbol{\phi}, \mathbf{G}_h^2\boldsymbol{\phi}\right\rangle + (1-\cos(t))^2\left\langle \boldsymbol{\phi}, \mathbf{G}_h^4\boldsymbol{\phi}\right\rangle.
\end{aligned}$$

From (3.5.6), we get

$$\begin{aligned}
|e^{t\mathbf{G}_h}\boldsymbol{\phi}|^2 &= |\boldsymbol{\phi}|^2 + 2\,(1-\cos(t))\left\langle \boldsymbol{\phi}, \mathbf{G}_h^2\boldsymbol{\phi}\right\rangle - \left(1-\cos^2(t)\right)\left\langle \boldsymbol{\phi}, \mathbf{G}_h^2\boldsymbol{\phi}\right\rangle \\
&\quad - \left(1 - 2\cos(t) + \cos^2(t)\right)\left\langle \boldsymbol{\phi}, \mathbf{G}_h^2\boldsymbol{\phi}\right\rangle \\
&= |\boldsymbol{\phi}|^2.
\end{aligned}$$

□

We note that Lemmas 3.5.1, 3.5.2 and 3.5.3 holds for $\mathbf{G} : L^2_m \to L^2_m$ as well.

In the proof of existence of strong solutions, we will use the following results for the operators $\mathbf{G}_h$ and $e^{t\mathbf{G}_h}$.

Lemma 3.5.4. *For any $\boldsymbol{\phi} \in L^2_{m,h}$, we have*

$$-\tilde{\Delta}\mathbf{G}_h\boldsymbol{\phi} + \mathbf{G}_h\tilde{\Delta}\boldsymbol{\phi} = -\mathbf{C}\boldsymbol{\phi}$$

with

$$\mathbf{C}\boldsymbol{\phi} = \boldsymbol{\phi} \times \tilde{\Delta}\mathbf{g}^h + D^+\boldsymbol{\phi} \times D^+\mathbf{g}^h + D^-\boldsymbol{\phi} \times D^-\mathbf{g}^h.$$

Proof. Simple calculation reveals, using Lemma 2.5.3

$$\begin{aligned}
-\tilde{\Delta}\mathbf{G}_h\boldsymbol{\phi} + \mathbf{G}_h\tilde{\Delta}\boldsymbol{\phi} &= -D^-D^+(\boldsymbol{\phi}\times\mathbf{g}^h) + \tilde{\Delta}\boldsymbol{\phi}\times\mathbf{g}^h \\
&= -D^-(D^+\boldsymbol{\phi}\times\tau^+\mathbf{g}^h + \boldsymbol{\phi}\times D^+\mathbf{g}^h) + \tilde{\Delta}\boldsymbol{\phi}\times\mathbf{g}^h \\
&= -(\tilde{\Delta}\boldsymbol{\phi}\times\mathbf{g}^h + D^+\boldsymbol{\phi}\times D^+\mathbf{g}^h + D^-\boldsymbol{\phi}\times D^-\mathbf{g}^h + \boldsymbol{\phi}\times\tilde{\Delta}\mathbf{g}^h) + \tilde{\Delta}\boldsymbol{\phi}\times\mathbf{g}^h \\
&= -\mathbf{C}\boldsymbol{\phi}.
\end{aligned}$$

□

Lemma 3.5.5. *For any $t\in\mathbb{R}$ and $\boldsymbol{\phi}\in L^2_{m,h}$, we have*

$$\tilde{\mathbf{C}}(t, e^{-t\mathbf{G}_h}\boldsymbol{\phi}) = -\tilde{\Delta}e^{-t\mathbf{G}_h}\boldsymbol{\phi} + e^{-t\mathbf{G}_h}\tilde{\Delta}\boldsymbol{\phi}$$

where

$$\begin{aligned}
\tilde{\mathbf{C}}(t,\boldsymbol{\phi}) &= e^{-t\mathbf{G}_h}\left(\sin(t)\mathbf{C} + (1-\cos(t))\left(\mathbf{G}_h\mathbf{C}+\mathbf{C}\mathbf{G}_h\right)\right)\boldsymbol{\phi} \\
&= e^{-t\mathbf{G}_h}\mathbf{C}_0(t,\boldsymbol{\phi}).
\end{aligned}$$

The operator $\mathbf{C}$ is defined in Lemma 3.5.4.

Proof. We consider $\tilde{\boldsymbol{\phi}} = e^{-t\mathbf{G}_h}\boldsymbol{\phi}$ and then by using the definition of $\tilde{\mathbf{C}}$, we get

$$\begin{aligned}
\tilde{\mathbf{C}}(t, e^{-t\mathbf{G}_h}\boldsymbol{\phi}) &= \tilde{\mathbf{C}}(t,\tilde{\boldsymbol{\phi}}) \\
&= e^{-t\mathbf{G}_h}\left(\sin(t)\mathbf{C} + (1-\cos(t))\left(\mathbf{G}_h\mathbf{C}+\mathbf{C}\mathbf{G}_h\right)\right)\tilde{\boldsymbol{\phi}}.
\end{aligned}$$

Using Lemma 3.5.4, we obtain

$$\begin{aligned}
\tilde{\mathbf{C}}(t, e^{-t\mathbf{G}_h}\boldsymbol{\phi}) &= \sin(t)e^{-t\mathbf{G}_h}\tilde{\Delta}\mathbf{G}_h\tilde{\boldsymbol{\phi}} - \sin(t)e^{-t\mathbf{G}_h}\mathbf{G}_h\tilde{\Delta}\tilde{\boldsymbol{\phi}} \\
&\quad + (1-\cos(t))e^{-t\mathbf{G}_h}\mathbf{G}_h\tilde{\Delta}\mathbf{G}_h\tilde{\boldsymbol{\phi}} - (1-\cos(t))e^{-t\mathbf{G}_h}\mathbf{G}_h^2\tilde{\Delta}\tilde{\boldsymbol{\phi}} \\
&\quad + (1-\cos(t))e^{-t\mathbf{G}_h}\tilde{\Delta}\mathbf{G}_h^2\tilde{\boldsymbol{\phi}} - (1-\cos(t))e^{-t\mathbf{G}_h}\mathbf{G}_h\tilde{\Delta}\mathbf{G}_h\tilde{\boldsymbol{\phi}} \\
&= \sin(t)e^{-t\mathbf{G}_h}\tilde{\Delta}\mathbf{G}_h\tilde{\boldsymbol{\phi}} - \sin(t)e^{-t\mathbf{G}_h}\mathbf{G}_h\tilde{\Delta}\tilde{\boldsymbol{\phi}} \\
&\quad + (1-\cos(t))e^{-t\mathbf{G}_h}\tilde{\Delta}\mathbf{G}_h^2\tilde{\boldsymbol{\phi}} - (1-\cos(t))e^{-t\mathbf{G}_h}\mathbf{G}_h^2\tilde{\Delta}\tilde{\boldsymbol{\phi}}.
\end{aligned}$$

By simple calculation, we get

$$\begin{aligned}\tilde{\mathbf{C}}(t, e^{-t\mathbf{G}_h}\boldsymbol{\phi}) &= e^{-t\mathbf{G}_h}\tilde{\Delta}\left(\mathbf{I} + \sin(t)\mathbf{G}_h + (1-\cos(t))\mathbf{G}_h^2\right)\tilde{\boldsymbol{\phi}} \\ &\quad - e^{-t\mathbf{G}_h}\left(\mathbf{I} + \sin(t)\mathbf{G}_h + (1-\cos(t))\mathbf{G}_h^2\right)\tilde{\Delta}\tilde{\boldsymbol{\phi}}.\end{aligned}$$

Then, by using (3.5.11) and (3.5.12)

$$\begin{aligned}\tilde{\mathbf{C}}(t, e^{-t\mathbf{G}_h}\boldsymbol{\phi}) &= e^{-t\mathbf{G}_h}\tilde{\Delta}(e^{t\mathbf{G}_h}\tilde{\boldsymbol{\phi}}) - e^{-t\mathbf{G}_h}(e^{t\mathbf{G}_h}\tilde{\Delta}\tilde{\boldsymbol{\phi}}) \\ &= e^{-t\mathbf{G}_h}\tilde{\Delta}(e^{t\mathbf{G}_h}e^{-t\mathbf{G}_h}\boldsymbol{\phi}) - e^{-t\mathbf{G}_h}(e^{t\mathbf{G}_h}\tilde{\Delta}e^{-t\mathbf{G}_h}\boldsymbol{\phi}) \\ &= e^{-t\mathbf{G}_h}\tilde{\Delta}\boldsymbol{\phi} - \tilde{\Delta}e^{-t\mathbf{G}_h}\boldsymbol{\phi}.\end{aligned}$$

□

3.6 Equivalence of Approximate Solutions

In this section, we benefit from the operator $\mathbf{G}_h$ defined in the above section to define a new process $\mathbf{m}^h$ from $\mathbf{u}^h$. Let

$$\mathbf{m}^h(t) := e^{-W(t)\mathbf{G}_h}\mathbf{u}^h(t) \quad \forall t \in [0, T].$$

We note that with this new variable, the differential $dW(t)$ vanishes in the equation satisfied by $\mathbf{m}^h$. In the following, we introduce the equation satisfied by $\mathbf{m}^h$ so that $\mathbf{u}^h$ is a solution to (3.3.1).

Lemma 3.6.1. *If $\mathbf{m}^h(t)$ satisfies*

$$\frac{d\mathbf{m}^h}{dt} = \mu\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h - \lambda\mathbf{m}^h \times (\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h) + F(W, \mathbf{m}^h), \quad \mathbb{P}a.s. \tag{3.6.1}$$

where

$$F(W(t), \mathbf{m}^h(t)) = \mu\mathbf{m}^h \times \tilde{\mathbf{C}}\left(W(t), \mathbf{m}^h(t)\right) - \lambda\mathbf{m}^h \times \left(\mathbf{m}^h \times \tilde{\mathbf{C}}\left(W(t), \mathbf{m}^h(t)\right)\right).$$

Then, $\mathbf{u}^h(t) = e^{W(t)\mathbf{G}_h}\mathbf{m}^h(t)$ *satisfies* (3.3.1) $\mathbb{P}a.s.$ *and* $\mathbf{m}^h(t)$ *is the unique solution of equation* (3.6.1)*.*

Proof. In order to simplify notations, we will assume in the proof without loss of generality, that $\lambda = \mu = 1$. Using Itô formula for $\mathbf{u}^h = e^{W(t)\mathbf{G}_h}\mathbf{m}^h$ (see [17]), we get

$$\mathbf{u}^h(t) = \mathbf{u}^h(0) + \int_0^t \mathbf{G}_h e^{W(s)\mathbf{G}_h}\mathbf{m}^h(s)dW(s) + \int_0^t \left(e^{W(s)\mathbf{G}_h}\frac{d\mathbf{m}^h}{dt} + \frac{1}{2}\mathbf{G}_h^2 e^{W(s)\mathbf{G}_h}\mathbf{m}^h(s)\right) ds.$$

Then,

$$\begin{aligned}
\mathbf{u}^h(t) = \mathbf{u}^h(0) &+ \int_0^t \mathbf{G}_h e^{W(s)\mathbf{G}_h}\mathbf{m}^h(s)dW(s) + \int_0^t e^{W(s)\mathbf{G}_h}\left(\mathbf{m}^h(s) \times \tilde{\Delta}\mathbf{m}^h(s)\right) ds \\
&- \int_0^t e^{W(s)\mathbf{G}_h}\left(\mathbf{m}^h(s) \times \left(\mathbf{m}^h(s) \times \tilde{\Delta}\mathbf{m}^h(s)\right)\right) ds + \int_0^t e^{W(s)\mathbf{G}_h} F\left(W(s), \mathbf{m}^h(s)\right) ds \\
&+ \frac{1}{2}\int_0^t \mathbf{G}_h^2 e^{W(s)\mathbf{G}_h}\mathbf{m}^h(s)ds.
\end{aligned}$$

From the definition of F, we obtain

$$\mathbf{u}^h(t) = \mathbf{u}^h(0) + \frac{1}{2}\int_0^t \mathbf{G}_h^2\mathbf{u}^h(s)ds + \int_0^t \mathbf{G}_h\mathbf{u}^h(s)dW(s) + \int_0^t (T_1 + T_2 + T_3 + T_4)\, ds \quad (3.6.2)$$

with

$$\begin{aligned}
T_1 &= e^{W(s)\mathbf{G}_h}\left(\mathbf{m}^h(s) \times \tilde{\Delta}\mathbf{m}^h(s)\right) \\
T_2 &= e^{W(s)\mathbf{G}_h}\left(\mathbf{m}^h(s) \times \tilde{\mathbf{C}}\left(W(s), \mathbf{m}^h(s)\right)\right) \\
T_3 &= -e^{W(s)\mathbf{G}_h}\left(\mathbf{m}^h(s) \times \left(\mathbf{m}^h(s) \times \tilde{\Delta}\mathbf{m}^h(s)\right)\right) \\
T_4 &= -e^{W(s)\mathbf{G}_h}\left(\mathbf{m}^h(s) \times \left(\mathbf{m}^h(s) \times \tilde{\mathbf{C}}\left(W(s), \mathbf{m}^h(s)\right)\right)\right),
\end{aligned}$$

where $\tilde{\mathbf{C}}$ is defined in Lemma 3.5.5. By using (3.5.12), (3.5.14) and the definition $\mathbf{m}^h(t) = e^{-W(t)\mathbf{G}_h}\mathbf{u}^h(t)$, we obtain

$$\begin{aligned}
T_2 &= e^{W(s)\mathbf{G}_h}\left(e^{-W(s)\mathbf{G}_h}\mathbf{u}^h(s) \times \tilde{\mathbf{C}}\left(W(s), e^{-W(s)\mathbf{G}_h}\mathbf{u}^h(s)\right)\right) \\
&= \mathbf{u}^h(s) \times e^{W(s)\mathbf{G}_h}\tilde{\mathbf{C}}\left(W(s), e^{-W(s)\mathbf{G}_h}\mathbf{u}^h(s)\right).
\end{aligned}$$

From Lemma 3.5.5, we get

$$T_2 = \mathbf{u}^h(s) \times e^{W(s)\mathbf{G}_h}\left(-\tilde{\Delta} e^{-W(s)\mathbf{G}_h}\mathbf{u}^h(s) + e^{-W(s)\mathbf{G}_h}\tilde{\Delta}\mathbf{u}^h(s)\right).$$

Using (3.5.12)

$$\begin{aligned} T_2 &= -\mathbf{u}^h(s) \times e^{W(s)\mathbf{G}_h}\left(\tilde{\Delta} e^{-W(s)\mathbf{G}_h}\mathbf{u}^h(s)\right) + \mathbf{u}^h(s) \times \tilde{\Delta}\mathbf{u}^h(s) \\ &= -T_1 + \mathbf{u}^h(s) \times \tilde{\Delta}\mathbf{u}^h(s). \end{aligned}$$

Consequently,

$$T_1 + T_2 = \mathbf{u}^h(s) \times \tilde{\Delta}\mathbf{u}^h(s).$$

Similarly, we have

$$T_3 + T_4 = -\mathbf{u}^h(s) \times \left(\mathbf{u}^h(s) \times \tilde{\Delta}\mathbf{u}^h(s)\right).$$

Finally, we obtain from (3.6.2)

$$\begin{aligned} \mathbf{u}^h(t) = \mathbf{u}^h(0) &+ \int_0^t \mathbf{u}^h(s) \times \tilde{\Delta}\mathbf{u}^h(s)ds - \int_0^t \mathbf{u}^h(s) \times \left(\mathbf{u}^h(s) \times \tilde{\Delta}\mathbf{u}^h(s)\right) ds \\ &+ \frac{1}{2}\int_0^t \mathbf{G}_h^2\mathbf{u}^h(s)ds + \int_0^t \mathbf{G}_h\mathbf{u}^h(s)dW(s). \end{aligned}$$

In addtition, if we consider two solutions $\mathbf{m}_1^h$ and $\mathbf{m}_2^h$ to (3.6.1), then since $e^{-W(t)\mathbf{G}_h}$ is an isometry there exist two solutions $\mathbf{u}_1^h$ and $\mathbf{u}_2^h$ to (3.3.1). In fact, Lemma 3.3.3 implies the uniqueness of $\mathbf{m}^h$ which completes the proof. □

3.7 Uniform Estimates

In this section, we introduce and prove some uniform estimates for $\mathbf{m}^h$ which will be used to prove the main result of this chapter, Theorem 3.1.3.

Lemma 3.7.1. *Assume that $|\mathbf{u}^h(0, x_i)| = 1$ for all $x_i \in \mathbb{Z}_h$, $D^+\mathbf{u}^h(0) \in L_h^2$ and $\mathbf{g}^h \in L_h^2$. For every $t \in [0, T]$ and all $x_i \in \mathbb{Z}_h$, we have*

$$\left|\mathbf{m}^h(t, x_i)\right| = 1. \tag{3.7.1}$$

Moreover, for $T \in (0,\infty)$ there exists a deterministic constant C which does not depend on h but which may depend on $|\mathbf{g}|_{H^2}$, $|\nabla \mathbf{u}_0|_{L^2}$ and T such that $\mathbb{P}$-a.s.

$$\sup_{t\in[0,T]} \left|D^+\mathbf{m}^h(t)\right|^2_{L^2_h} \le C, \tag{3.7.2}$$

$$\int_0^T \left|\tilde{\Delta}\mathbf{m}^h(t)\right|^2_{L^2_h} dt \le C. \tag{3.7.3}$$

Proof. In order to simplify notations, we assume in the proof without loss of generality, that $\lambda = \mu = 1$. We first prove (3.7.1). From (3.4.1), the definition $\mathbf{m}^h(t) = e^{-W(t)\mathbf{G}_h}\mathbf{u}^h(t)$ and using (3.5.12), we obtain

$$\left|e^{W(t)\mathbf{G}_h}\mathbf{m}^h(t)\right| = 1.$$

Then, using (3.5.16) we get

$$\left|\mathbf{m}^h(t,x_i)\right| = 1,$$

for every $t \in [0,T]$ and all $x_i \in \mathbb{Z}_h$ and (3.7.1) follows.

Next we prove (3.7.2). Applying D^+ to the discrete equation (3.6.1), we have

$$\begin{aligned}\frac{d}{dt}D^+\mathbf{m}^h &= D^+\left(\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h\right) - D^+\left(\mathbf{m}^h \times \left(\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h\right)\right)\\ &\quad + D^+\left(\mathbf{m}^h \times \tilde{\mathbf{C}}\left(W(t),\mathbf{m}^h\right)\right) - D^+\left(\mathbf{m}^h \times \left(\mathbf{m}^h \times \tilde{\mathbf{C}}\left(W(t),\mathbf{m}^h\right)\right)\right).\end{aligned}$$

Multiplying by $D^+\mathbf{m}^h$

$$\begin{aligned}\frac{d}{dt}\left|D^+\mathbf{m}^h\right|^2 &= \left\langle D^+\left(\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h\right), D^+\mathbf{m}^h\right\rangle - \left\langle D^+\left(\mathbf{m}^h \times \left(\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h\right)\right), D^+\mathbf{m}^h\right\rangle\\ &\quad + \left\langle D^+\left(\mathbf{m}^h \times \tilde{\mathbf{C}}\left(W(t),\mathbf{m}^h\right)\right), D^+\mathbf{m}^h\right\rangle\\ &\quad - \left\langle D^+\left(\mathbf{m}^h \times \left(\mathbf{m}^h \times \tilde{\mathbf{C}}\left(W(t),\mathbf{m}^h\right)\right)\right), D^+\mathbf{m}^h\right\rangle.\end{aligned}$$

Hence, by taking the summation over $x_i \in \mathbb{Z}_h$, multiplying by h, using (2.2.1), (2.2.2) and Lemma 2.5.4, we obtain

$$\frac{d}{dt}\left|D^+\mathbf{m}^h\right|^2_{L^2_h} = -\left|\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h\right|^2_{L^2_h} + S_1 + S_2 \tag{3.7.4}$$

with

$$S_1 = \left\langle D^+ \left(\mathbf{m}^h \times \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right) \right), D^+ \mathbf{m}^h \right\rangle_{L_h^2}$$
$$S_2 = - \left\langle D^+ \left(\mathbf{m}^h \times \left(\mathbf{m}^h \times \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right) \right) \right), D^+ \mathbf{m}^h \right\rangle_{L_h^2}.$$

Using Lemma 2.5.4 and the elementary property (2.2.5)

$$\begin{aligned}
S_1 &= - \left\langle \mathbf{m}^h \times \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right), \tilde{\Delta} \mathbf{m}^h \right\rangle_{L_h^2} \\
&= - \left\langle \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right), \tilde{\Delta} \mathbf{m}^h \times \mathbf{m}^h \right\rangle_{L_h^2} \\
&= \left\langle \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right), \mathbf{m}^h \times \tilde{\Delta} \mathbf{m}^h \right\rangle_{L_h^2} \\
&\le \left| \tilde{\mathbf{C}}(W(t), \mathbf{m}^h) \right|_{L_h^2} \left| \mathbf{m}^h \times \tilde{\Delta} \mathbf{m}^h \right|_{L_h^2} \\
&\le C_\varepsilon \left| \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right) \right|_{L_h^2}^2 + \varepsilon \left| \mathbf{m}^h \times \tilde{\Delta} \mathbf{m}^h \right|_{L_h^2}^2.
\end{aligned}$$

Similarly, for S_2 using (3.7.1) we get

$$\begin{aligned}
S_2 &= \left\langle \mathbf{m}^h \times \left(\mathbf{m}^h \times \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right) \right), \tilde{\Delta} \mathbf{m}^h \right\rangle_{L_h^2} \\
&= \left\langle \mathbf{m}^h \times \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right), \tilde{\Delta} \mathbf{m}^h \times \mathbf{m}^h \right\rangle_{L_h^2} \\
&= - \left\langle \mathbf{m}^h \times \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right), \mathbf{m}^h \times \tilde{\Delta} \mathbf{m}^h \right\rangle_{L_h^2} \\
&\le \left| \mathbf{m}^h \times \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right) \right|_{L_h^2} \left| \mathbf{m}^h \times \tilde{\Delta} \mathbf{m}^h \right|_{L_h^2} \\
&\le \left| \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right) \right|_{L_h^2} \left| \mathbf{m}^h \times \tilde{\Delta} \mathbf{m}^h \right|_{L_h^2} \\
&\le C_\varepsilon \left| \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right) \right|_{L_h^2}^2 + \varepsilon \left| \mathbf{m}^h \times \tilde{\Delta} \mathbf{m}^h \right|_{L_h^2}^2.
\end{aligned}$$

Then, from (3.7.4)

$$\frac{d}{dt} \left| D^+ \mathbf{m}^h \right|_{L_h^2}^2 + C_\varepsilon \left| \mathbf{m}^h \times \tilde{\Delta} \mathbf{m}^h \right|_{L_h^2}^2 \le C_\varepsilon \left| \tilde{\mathbf{C}} \left(W(t), \mathbf{m}^h \right) \right|_{L_h^2}^2.$$

Integrating with respect to t and applying $\sup_{s\in[0,t]}$ for $t \in [0,T]$

$$\sup_{s\in[0,t]} \left|D^+\mathbf{m}^h(s)\right|^2_{L^2_h} + C_\varepsilon \int_0^t \left|\mathbf{m}^h(s) \times \tilde{\Delta}\mathbf{m}^h(s)\right|^2_{L^2_h} ds \le \left|D^+\mathbf{m}^h_0\right|^2_{L^2_h} + C_\varepsilon \int_0^t \left|\tilde{\mathbf{C}}\left(W(s), \mathbf{m}^h(s)\right)\right|^2_{L^2_h} ds. \tag{3.7.5}$$

We proceed with the last term on the right hand side. From the definition of $\tilde{\mathbf{C}}$ and property (3.5.16), we obtain

$$\begin{aligned}
&\int_0^t \left|\tilde{\mathbf{C}}\left(W(s), \mathbf{m}^h(s)\right)\right|^2_{L^2_h} ds \\
&= \int_0^t \left|e^{-W(s)\mathbf{G}_h}\left(\sin W(s)\mathbf{C} + (1-\cos W(s))(\mathbf{C}\mathbf{G}_h + \mathbf{G}_h\mathbf{C})\right)\mathbf{m}^h(s)\right|^2_{L^2_h} ds \\
&= \int_0^t \left|\left(\sin W(s)\mathbf{C} + (1-\cos W(s))(\mathbf{C}\mathbf{G}_h + \mathbf{G}_h\mathbf{C})\right)\mathbf{m}^h(s)\right|^2_{L^2_h} ds.
\end{aligned}$$

From the definition of $\mathbf{C}$ in Lemma 3.5.4

$$\begin{aligned}
&\int_0^t \left|\tilde{\mathbf{C}}\left(W(s), \mathbf{m}^h(s)\right)\right|^2_{L^2_h} ds \\
&\le C\int_0^t \left|\sin W(s)\left(\mathbf{m}^h \times \tilde{\Delta}\mathbf{g}^h + D^+\mathbf{m}^h \times D^+\mathbf{g}^h + D^-\mathbf{m}^h \times D^-\mathbf{g}^h\right)\right|^2_{L^2_h} ds \\
&\quad + C\int_0^t \left|(1-\cos W(s))\left(\mathbf{G}_h\mathbf{m}^h \times \tilde{\Delta}\mathbf{g}^h + D^+\mathbf{G}_h\mathbf{m}^h \times D^+\mathbf{g}^h + D^-\mathbf{G}_h\mathbf{m}^h \times D^-\mathbf{g}^h\right)\right|^2_{L^2_h} ds \\
&\quad + C\int_0^t \left|(1-\cos W(s))\left(\mathbf{G}_h\left(\mathbf{m}^h \times \tilde{\Delta}\mathbf{g}^h + D^+\mathbf{m}^h \times D^+\mathbf{g}^h + D^-\mathbf{m}^h \times D^-\mathbf{g}^h\right)\right)\right|^2_{L^2_h} ds.
\end{aligned}$$

Consequently, using (3.7.1), the definition of the operator $\mathbf{G}_h$ and Lemma 2.5.2

$$\begin{aligned}
&\int_0^t \left|\tilde{\mathbf{C}}\left(W(s), \mathbf{m}^h(s)\right)\right|^2_{L^2_h} ds \\
&\le CT\left(|\tilde{\Delta}\mathbf{g}^h|^2_{L^2_h} + |\mathbf{g}^h|^2_{L^2_h} + |\mathbf{g}^h|^2_{L^\infty_h}|\tilde{\Delta}\mathbf{g}^h|^2_{L^2_h} + |D^+\mathbf{g}^h|^2_{L^\infty_h}|D^+\mathbf{g}^h|^2_{L^2_h}\right) \\
&\quad + C\left(|\mathbf{g}^h|^2_{L^\infty_h}|D^+\mathbf{g}^h|^2_{L^\infty_h} + |D^+\mathbf{g}^h|^2_{L^\infty_h}\right)\int_0^t \left|D^+\mathbf{m}^h(s)\right|^2_{L^2_h} ds.
\end{aligned}$$

From Lemma 2.5.6, we have

$$\begin{aligned}
&\int_0^t \left|\tilde{\mathbf{C}}\left(W(s), \mathbf{m}^h(s)\right)\right|^2_{L^2_h} ds \\
&\leq CT\left(|\tilde{\Delta}\mathbf{g}^h|^2_{L^2_h} + |\mathbf{g}^h|^2_{L^2_h} + K^2|\mathbf{g}^h|_{L^2_h}|D^+\mathbf{g}^h|_{L^2_h}|\tilde{\Delta}\mathbf{g}^h|^2_{L^2_h} + K^2|D^+\mathbf{g}^h|^3_{L^2_h}|\tilde{\Delta}\mathbf{g}^h|_{L^2_h}\right) \\
&\quad + C\left(K^4|\mathbf{g}^h|_{L^2_h}|D^+\mathbf{g}^h|^2_{L^2_h}|\tilde{\Delta}\mathbf{g}^h|_{L^2_h} + K^2|D^+\mathbf{g}^h|_{L^2_h}|\tilde{\Delta}\mathbf{g}^h|_{L^2_h}\right)\int_0^t \left|D^+\mathbf{m}^h(s)\right|^2_{L^2_h} ds. \quad (3.7.6)
\end{aligned}$$

Then, from (3.7.5) we obtain

$$\begin{aligned}
&\sup_{s\in[0,t]} \left|D^+\mathbf{m}^h(s)\right|^2_{L^2_h} + C_\varepsilon \int_0^t \left|\mathbf{m}^h(s) \times \tilde{\Delta}\mathbf{m}^h(s)\right|^2_{L^2_h} ds \\
&\leq \left|D^+\mathbf{m}^h(0)\right|^2_{L^2_h} + C_\varepsilon\left(T, |\mathbf{g}|_{H^2}\right) + C_\varepsilon\left(|\mathbf{g}|_{H^2}\right)\int_0^t \sup_{r\in[0,s]}\left|D^+\mathbf{m}^h(r)\right|^2_{L^2_h} ds. \quad (3.7.7)
\end{aligned}$$

Since

$$C_\varepsilon \int_0^t \left|\mathbf{m}^h(s) \times \tilde{\Delta}\mathbf{m}^h(s)\right|^2_{L^2_h} ds \geq 0,$$

we get by using Gronwall's inequality (2.2.7)

$$\sup_{s\in[0,t]} \left|D^+\mathbf{m}^h(s)\right|^2_{L^2_h} \leq \left(\left|D^+\mathbf{m}^h(0)\right|^2_{L^2_h} + C_\varepsilon\left(T, |\mathbf{g}|_{H^2}\right)\right) e^{C_\varepsilon\left(|\mathbf{g}|_{H^2}\right)T}$$

which completes the proof of (3.7.2).

Finally, we prove (3.7.3). By using (3.7.1) and the elementary property (2.2.4), we have

$$\left|\tilde{\Delta}\mathbf{m}^h\right|^2 = \left|\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h\right|^2 + \left\langle \mathbf{m}^h, \tilde{\Delta}\mathbf{m}^h\right\rangle^2.$$

Then, from Lemma 2.5.5 we deduce

$$\left|\tilde{\Delta}\mathbf{m}^h\right|^2 \leq \left|\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h\right|^2 + \frac{1}{2}\left|D^+\mathbf{m}^h\right|^4 + \frac{1}{2}\left|D^-\mathbf{m}^h\right|^4.$$

Applying summation over $x \in \mathbb{Z}_h$, multiplying by h, using Lemma 2.5.2 and then integrating with respect to t we get

$$\int_0^T \left|\tilde{\Delta}\mathbf{m}^h\right|_{L_h^2}^2 dt \le \int_0^T \left|\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h\right|_{L_h^2}^2 dt + \int_0^T \left|D^+\mathbf{m}^h\right|_{L_h^4}^4 dt. \tag{3.7.8}$$

From (3.7.7) and using (3.7.2), we obtain

$$\int_0^T \left|\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h\right|_{L_h^2}^2 dt \le C\left(|\nabla \mathbf{u}_0|_{L^2}, |\mathbf{g}|_{H^2}, T\right).$$

Now for the second term on the right hand side of (3.7.8), using Lemma 2.5.6 for the discrete function $D^+\mathbf{m}^h$ and Lemma 2.5.2, we have

$$\begin{aligned}
\left|D^+\mathbf{m}^h\right|_{L_h^4}^4 &\le \left|D^+\mathbf{m}^h\right|_{L_h^\infty}^2 \left|D^+\mathbf{m}^h\right|_{L_h^2}^2 \\
&\le K^2 \left|D^+\mathbf{m}^h\right|_{L_h^2}^3 \left|\tilde{\Delta}\mathbf{m}^h\right|_{L_h^2} \\
&\le C_\varepsilon \left|D^+\mathbf{m}^h\right|_{L_h^2}^6 + \varepsilon \left|\tilde{\Delta}\mathbf{m}^h\right|_{L_h^2}^2 .
\end{aligned}$$

Consequently, using (3.7.2) we get from (3.7.8)

$$\int_0^T \left|\tilde{\Delta}\mathbf{m}^h\right|_{L_h^2}^2 dt \le C_\varepsilon\left(|\nabla \mathbf{u}_0|_{L^2}, |\mathbf{g}|_{H^2}, T\right)$$

which completes the proof of (3.7.3). □

Lemma 3.7.2. *Assume that* $|\mathbf{u}^h(0, x_i)| = 1$ *for all* $x_i \in \mathbb{Z}_h$, $D^+\mathbf{u}^h(0) \in L_h^2$ *and* $\mathbf{g}^h \in L_h^2$. *Let* $W_1(\cdot), W_2(\cdot)$ *be two trajectories of the Wiener process and let* $\mathbf{m}_1^h$, $\mathbf{m}_2^h$ *be the corresponding solutions of equation* (3.6.1). *Then, there exists a constant* C *which does not depend on* h *but which may depend on* $|\mathbf{g}|_{H^2}$, $|\nabla \mathbf{u}_0|_{L^2}$ *and* T *such that*

$$\sup_{t\in[0,T]} \left|\mathbf{m}_1^h(t) - \mathbf{m}_2^h(t)\right|_{L_h^2} \le C \sup_{t\in[0,T]} |W_1(t) - W_2(t)| .$$

Proof. In order to simplify notations, we assume in the proof without loss of generality, that $\lambda = \mu = 1$. Using (2.2.3) and (3.7.1), we have

$$\begin{aligned}\frac{d\mathbf{m}^h}{dt} &= \mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h - \left\langle \mathbf{m}^h, \tilde{\Delta}\mathbf{m}^h \right\rangle \mathbf{m}^h + \tilde{\Delta}\mathbf{m}^h \\ &\quad + \mathbf{m}^h \times \tilde{\mathbf{C}}\left(W(t), \mathbf{m}^h(t)\right) - \left\langle \mathbf{m}^h, \tilde{\mathbf{C}}\left(W(t), \mathbf{m}^h(t)\right)\right\rangle \mathbf{m}^h + \tilde{\mathbf{C}}\left(W(t), \mathbf{m}^h(t)\right).\end{aligned}$$

From Lemma 2.5.5, we get

$$\begin{aligned}\frac{d\mathbf{m}^h}{dt} &= \mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h + \frac{1}{2}\left|D^+\mathbf{m}^h\right|^2 \mathbf{m}^h + \frac{1}{2}\left|D^-\mathbf{m}^h\right|^2 \mathbf{m}^h + \tilde{\Delta}\mathbf{m}^h \\ &\quad + \mathbf{m}^h \times \tilde{\mathbf{C}}\left(W(t), \mathbf{m}^h(t)\right) - \left\langle \mathbf{m}^h, \tilde{\mathbf{C}}\left(W(t), \mathbf{m}^h(t)\right)\right\rangle \mathbf{m}^h + \tilde{\mathbf{C}}\left(W(t), \mathbf{m}^h(t)\right).\end{aligned}$$

We denote $\overline{\mathbf{m}^h} := \mathbf{m}_1^h - \mathbf{m}_2^h$. Substracting equations for $\mathbf{m}_1^h$ and $\mathbf{m}_2^h$ we obtain

$$\begin{aligned}\frac{d\overline{\mathbf{m}^h}}{dt} &= \tilde{\Delta}\overline{\mathbf{m}^h} + \overline{\mathbf{m}^h} \times \tilde{\Delta}\mathbf{m}_1^h + \mathbf{m}_2^h \times \tilde{\Delta}\overline{\mathbf{m}^h} \\ &\quad + \frac{1}{2}\left|D^+\mathbf{m}_1^h\right|^2 \overline{\mathbf{m}^h} + \frac{1}{2}\left\langle D^+\overline{\mathbf{m}^h}, D^+\mathbf{m}_1^h + D^+\mathbf{m}_2^h\right\rangle \mathbf{m}_2^h \\ &\quad + \frac{1}{2}\left|D^-\mathbf{m}_1^h\right|^2 \overline{\mathbf{m}^h} + \frac{1}{2}\left\langle D^-\overline{\mathbf{m}^h}, D^-\mathbf{m}_1^h + D^-\mathbf{m}_2^h\right\rangle \mathbf{m}_2^h \\ &\quad + \overline{\mathbf{m}^h} \times \tilde{\mathbf{C}}\left(W_1, \mathbf{m}_1^h\right) + \mathbf{m}_2^h \times \left(\tilde{\mathbf{C}}\left(W_1, \mathbf{m}_1^h\right) - \tilde{\mathbf{C}}\left(W_2, \mathbf{m}_1^h\right)\right) \\ &\quad + \mathbf{m}_2^h \times \tilde{\mathbf{C}}\left(W_2, \overline{\mathbf{m}^h}\right) \\ &\quad - \left\langle \mathbf{m}_1^h, \tilde{\mathbf{C}}\left(W_1, \mathbf{m}_1^h\right)\right\rangle \overline{\mathbf{m}^h} - \left\langle \overline{\mathbf{m}^h}, \tilde{\mathbf{C}}\left(W_1, \mathbf{m}_1^h\right)\right\rangle \mathbf{m}_2^h \\ &\quad - \left\langle \mathbf{m}_2^h, \tilde{\mathbf{C}}\left(W_1, \mathbf{m}_1^h\right) - \tilde{\mathbf{C}}\left(W_2, \mathbf{m}_1^h\right)\right\rangle \mathbf{m}_2^h - \left\langle \mathbf{m}_2^h, \tilde{\mathbf{C}}\left(W_2, \overline{\mathbf{m}^h}\right)\right\rangle \mathbf{m}_2^h \\ &\quad + \left(\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}\left(W_2, \mathbf{m}_1^h\right)\right) + \tilde{\mathbf{C}}\left(W_2, \overline{\mathbf{m}^h}\right).\end{aligned}$$

Multiplying by $\overline{\mathbf{m}^h}$ and using (2.2.1)

$$\begin{aligned}\frac{1}{2}\frac{d|\overline{\mathbf{m}^h}|^2}{dt} &= \left\langle \overline{\mathbf{m}^h}, \tilde{\Delta}\overline{\mathbf{m}^h}\right\rangle + \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h \times \tilde{\Delta}\overline{\mathbf{m}^h}\right\rangle \\ &\quad + \frac{1}{2}\left|\overline{\mathbf{m}^h}\right|^2 \left|D^+\mathbf{m}_1^h\right|^2 + \frac{1}{2}\left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h\right\rangle \left\langle D^+\overline{\mathbf{m}^h}, D^+\mathbf{m}_1^h + D^+\mathbf{m}_2^h\right\rangle \\ &\quad + \frac{1}{2}\left|\overline{\mathbf{m}^h}\right|^2 \left|D^-\mathbf{m}_1^h\right|^2 + \frac{1}{2}\left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h\right\rangle \left\langle D^-\overline{\mathbf{m}^h}, D^-\mathbf{m}_1^h + D^-\mathbf{m}_2^h\right\rangle \\ &\quad + \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h \times \left(\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right)\right\rangle\end{aligned}$$

$$
\begin{aligned}
&+\left\langle\overline{\mathbf{m}^h},\mathbf{m}_2^h\times\tilde{\mathbf{C}}\left(W_2,\overline{\mathbf{m}^h}\right)\right\rangle\\
&-\left|\overline{\mathbf{m}^h}\right|^2\left\langle\mathbf{m}_1^h,\tilde{\mathbf{C}}(W_1,\mathbf{m}_1^h)\right\rangle-\left\langle\overline{\mathbf{m}^h},\mathbf{m}_2^h\right\rangle\left\langle\overline{\mathbf{m}^h},\tilde{\mathbf{C}}(W_1,\mathbf{m}_1^h)\right\rangle\\
&-\left\langle\overline{\mathbf{m}^h},\mathbf{m}_2^h\right\rangle\left\langle\mathbf{m}_2^h,\tilde{\mathbf{C}}(W_1,\mathbf{m}_1^h)-\tilde{\mathbf{C}}(W_2,\mathbf{m}_1^h)\right\rangle\\
&-\left\langle\overline{\mathbf{m}^h},\mathbf{m}_2^h\right\rangle\left\langle\mathbf{m}_2^h,\tilde{\mathbf{C}}\left(W_2,\overline{\mathbf{m}^h}\right)\right\rangle\\
&+\left\langle\overline{\mathbf{m}^h},\left(\tilde{\mathbf{C}}(W_1,\mathbf{m}_1^h)-\tilde{\mathbf{C}}(W_2,\mathbf{m}_1^h)\right)\right\rangle+\left\langle\overline{\mathbf{m}^h},\tilde{\mathbf{C}}\left(W_2,\overline{\mathbf{m}^h}\right)\right\rangle. \qquad (3.7.9)
\end{aligned}
$$

Multiplying by h, taking summation over $x_i \in \mathbb{Z}_h$ and using Lemma 2.5.4 we get for the first term on the right hand side

$$
\left\langle\overline{\mathbf{m}^h},\tilde{\Delta}\overline{\mathbf{m}^h}\right\rangle_{L_h^2}=-\left\langle D^+\overline{\mathbf{m}^h},D^+\overline{\mathbf{m}^h}\right\rangle_{L_h^2}=-\left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2.
$$

For the second term on the right hand side of (3.7.9), using Lemmas 2.5.3, 2.5.4 and the elementary property (2.2.1) we have

$$
\begin{aligned}
\left\langle\overline{\mathbf{m}^h},\mathbf{m}_2^h\times\tilde{\Delta}\overline{\mathbf{m}^h}\right\rangle_{L_h^2}&=\left\langle\overline{\mathbf{m}^h},D^-\left(\mathbf{m}_2^h\times D^+\overline{\mathbf{m}^h}\right)\right\rangle_{L_h^2}-\left\langle\overline{\mathbf{m}^h},D^-\mathbf{m}_2^h\times D^-\overline{\mathbf{m}^h}\right\rangle_{L_h^2}\\
&=-\left\langle D^+\overline{\mathbf{m}^h},\mathbf{m}_2^h\times D^+\overline{\mathbf{m}^h}\right\rangle_{L_h^2}-\left\langle\overline{\mathbf{m}^h},D^-\mathbf{m}_2^h\times D^-\overline{\mathbf{m}^h}\right\rangle_{L_h^2}\\
&=-\left\langle\overline{\mathbf{m}^h},D^-\mathbf{m}_2^h\times D^-\overline{\mathbf{m}^h}\right\rangle_{L_h^2}.
\end{aligned}
$$

Then, from (3.7.9)

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2+\left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2=&-\left\langle\overline{\mathbf{m}^h},D^-\mathbf{m}_2^h\times D^-\overline{\mathbf{m}^h}\right\rangle_{L_h^2}\\
&+\frac{1}{2}\sum h\left|\overline{\mathbf{m}^h}\right|^2\left|D^+\mathbf{m}_1^h\right|^2\\
&+\frac{1}{2}\sum h\left\langle\overline{\mathbf{m}^h},\mathbf{m}_2^h\right\rangle\left\langle D^+\overline{\mathbf{m}^h},D^+\mathbf{m}_1^h+D^+\mathbf{m}_2^h\right\rangle\\
&+\frac{1}{2}\sum h\left|\overline{\mathbf{m}^h}\right|^2\left|D^-\mathbf{m}_1^h\right|^2\\
&+\frac{1}{2}\sum h\left\langle\overline{\mathbf{m}^h},\mathbf{m}_2^h\right\rangle\left\langle D^-\overline{\mathbf{m}^h},D^-\mathbf{m}_1^h+D^-\mathbf{m}_2^h\right\rangle\\
&+\left\langle\overline{\mathbf{m}^h},\mathbf{m}_2^h\times\left(\tilde{\mathbf{C}}(W_1,\mathbf{m}_1^h)-\tilde{\mathbf{C}}(W_2,\mathbf{m}_1^h)\right)\right\rangle_{L_h^2}\\
&+\left\langle\overline{\mathbf{m}^h},\mathbf{m}_2^h\times\tilde{\mathbf{C}}\left(W_2,\overline{\mathbf{m}^h}\right)\right\rangle_{L_h^2}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum h \left|\overline{\mathbf{m}^h}\right|^2 \left\langle \mathbf{m}_1^h, \tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h)\right\rangle \\
&-\sum h \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h\right\rangle \left\langle \overline{\mathbf{m}^h}, \tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h)\right\rangle \\
&-\sum h \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h\right\rangle \left\langle \mathbf{m}_2^h, \tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right\rangle \\
&-\sum h \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h\right\rangle \left\langle \mathbf{m}_2^h, \tilde{\mathbf{C}}\left(W_2, \overline{\mathbf{m}^h}\right)\right\rangle \\
&+\left\langle \overline{\mathbf{m}^h}, \left(\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right)\right\rangle_{L_h^2} \\
&+\left\langle \overline{\mathbf{m}^h}, \tilde{\mathbf{C}}\left(W_2, \overline{\mathbf{m}^h}\right)\right\rangle_{L_h^2}.
\end{aligned}
$$

We integrate with respect to t to obtain

$$
\frac{1}{2}\left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 + \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 = \sum_{i=1}^{13} T_i. \tag{3.7.10}
$$

We will estimate each term T_i separately. For T_1, using Lemma 2.5.6 and Young's inequality (2.2.6) for $p = 4$ and $q = \frac{4}{3}$ we have

$$
\begin{aligned}
T_1 &= -\int_0^t \left\langle \overline{\mathbf{m}^h}, D^-\mathbf{m}_2^h \times D^-\overline{\mathbf{m}^h}\right\rangle_{L_h^2} ds \\
&\leq \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|D^+\mathbf{m}_2^h\right|_{L_h^2} \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2} \\
&\leq K \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^{\frac{1}{2}} \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^{\frac{3}{2}} \left|D^+\mathbf{m}_2^h\right|_{L_h^2} ds \\
&\leq C_\varepsilon \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 \left|D^+\mathbf{m}_2^h\right|_{L_h^2}^4 ds + \varepsilon \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds.
\end{aligned}
$$

For T_2, using Lemma 2.5.6

$$
\begin{aligned}
T_2 = \frac{1}{2}\int_0^t \sum h \left|\overline{\mathbf{m}^h}\right|^2 \left|D^+\mathbf{m}_1^h\right|^2 ds &\leq C \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty}^2 \left|D^+\mathbf{m}_1^h\right|_{L_h^2}^2 ds \\
&\leq C \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2} \left|D^+\mathbf{m}_1^h\right|_{L_h^2}^2 ds \\
&\leq C_\varepsilon \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 \left|D^+\mathbf{m}_1^h\right|_{L_h^2}^4 ds + \varepsilon \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds.
\end{aligned}
$$

For T_3, using (3.7.1), Lemma 2.5.6 and Young's inequality (2.2.6) for $p = 4$ and $q = \frac{4}{3}$

$$
\begin{aligned}
T_3 &= \frac{1}{2}\int_0^t \sum h \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h \right\rangle \left\langle D^+\overline{\mathbf{m}^h}, D^+\mathbf{m}_1^h + D^+\mathbf{m}_2^h \right\rangle ds \\
&\le C \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2} \left|D^+\mathbf{m}_1^h + D^+\mathbf{m}_2^h\right|_{L_h^2} ds \\
&\le C \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^{\frac{1}{2}} \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^{\frac{3}{2}} \left(\left|D^+\mathbf{m}_1^h\right|_{L_h^2} + \left|D^+\mathbf{m}_2^h\right|_{L_h^2}\right) ds \\
&\le C_\varepsilon \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 \left(\left|D^+\mathbf{m}_1^h\right|_{L_h^2}^4 + \left|D^+\mathbf{m}_2^h\right|_{L_h^2}^4\right) ds + \varepsilon \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds.
\end{aligned}
$$

For T_4 and T_5, using the same reasoning we get

$$
T_4 = \frac{1}{2}\int_0^t \sum h \left|\overline{\mathbf{m}^h}\right|^2 \left|D^-\mathbf{m}_1^h\right|^2 ds \le C_\varepsilon \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 \left|D^+\mathbf{m}_1^h\right|_{L_h^2}^4 ds + \varepsilon \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds,
$$

$$
\begin{aligned}
T_5 &= \frac{1}{2}\int_0^t \sum h \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h \right\rangle \left\langle D^-\overline{\mathbf{m}^h}, D^-\mathbf{m}_1^h + D^-\mathbf{m}_2^h \right\rangle ds \\
&\le C_\varepsilon \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 \left(\left|D^+\mathbf{m}_1^h\right|_{L_h^2}^4 + \left|D^+\mathbf{m}_2^h\right|_{L_h^2}^4\right) ds + \varepsilon \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds.
\end{aligned}
$$

We continue by estimating T_6, using (3.7.1) we obtain

$$
\begin{aligned}
T_6 &= \int_0^t \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h \times \left(\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right)\right\rangle_{L_h^2} ds \\
&\le \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right|_{L_h^2} ds \\
&\le C \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds + C \int_0^t \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right|_{L_h^2}^2 ds.
\end{aligned}
$$

For T_7, using (3.7.1) we have

$$
T_7 = \int_0^t \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h \times \tilde{\mathbf{C}}(W_2, \overline{\mathbf{m}^h})\right\rangle_{L_h^2} ds \le \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\tilde{\mathbf{C}}(W_2, \overline{\mathbf{m}^h})\right|_{L_h^2} ds.
$$

For term T_8, using (3.7.1)

$$
\begin{aligned}
T_8 &= -\int_0^t \sum h \left|\overline{\mathbf{m}^h}\right|^2 \left\langle \mathbf{m}_1^h, \tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h)\right\rangle ds \\
&\le \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h)\right|_{L_h^2} ds.
\end{aligned}
$$

Using the same resoning, we have for T_9

$$T_9 = -\int_0^t \sum h \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h \right\rangle \left\langle \overline{\mathbf{m}^h}, \tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) \right\rangle ds \leq \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h)\right|_{L_h^2} ds.$$

Now, for T_{10} using (3.7.1)

$$\begin{aligned}
T_{10} &= -\int_0^t \sum h \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h \right\rangle \left\langle \mathbf{m}_2^h, \tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h) \right\rangle ds \\
&\leq \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right|_{L_h^2} ds \\
&\leq C \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds + C \int_0^t \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right|_{L_h^2}^2 ds.
\end{aligned}$$

For T_{11},

$$T_{11} = -\int_0^t \sum h \left\langle \overline{\mathbf{m}^h}, \mathbf{m}_2^h \right\rangle \left\langle \mathbf{m}_2^h, \tilde{\mathbf{C}}(W_2, \overline{\mathbf{m}^h}) \right\rangle ds \leq \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\tilde{\mathbf{C}}(W_2, \overline{\mathbf{m}^h})\right|_{L_h^2} ds.$$

For T_{12},

$$\begin{aligned}
T_{12} &= \int_0^t \left\langle \overline{\mathbf{m}^h}, \left(\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right) \right\rangle_{L_h^2} ds \\
&\leq \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right|_{L_h^2} ds \\
&\leq C \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds + C \int_0^t \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right|_{L_h^2}^2 ds.
\end{aligned}$$

Finally, for T_{13} we get

$$T_{13} = \int_0^t \left\langle \overline{\mathbf{m}^h}, \tilde{\mathbf{C}}(W_2, \overline{\mathbf{m}^h}) \right\rangle_{L_h^2} ds \leq \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\tilde{\mathbf{C}}(W_2, \overline{\mathbf{m}^h})\right|_{L_h^2} ds.$$

Combining all the above and applying $\sup_{s\in[0,t]}$ for $t \in [0, T]$, we obtain from (3.7.10)

$$\begin{aligned}
\sup_{s\in[0,t]} \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 + C_\varepsilon \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds &\leq C_\varepsilon \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 \left(\left|D^+\mathbf{m}_1^h\right|_{L_h^2}^4 + \left|D^+\mathbf{m}_2^h\right|_{L_h^2}^4 + 1\right) ds \\
&\quad + C(S_1 + S_2 + S_3) \qquad (3.7.11)
\end{aligned}$$

with

$$
\begin{aligned}
S_1 &= \int_0^t \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right|_{L_h^2}^2 ds \\
S_2 &= \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\tilde{\mathbf{C}}\left(W_2, \overline{\mathbf{m}^h}\right)\right|_{L_h^2} ds \\
S_3 &= \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h)\right|_{L_h^2} ds.
\end{aligned}
$$

We proceed by estimating $S_i (i = 1, 2, 3)$. For S_1, we have

$$
\begin{aligned}
S_1 &= \int_0^t \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h) - \tilde{\mathbf{C}}(W_2, \mathbf{m}_1^h)\right|_{L_h^2}^2 ds \\
&= \int_0^t \left|e^{-W_1 \mathbf{G}_h} \mathbf{C}_0(W_1, \mathbf{m}_1^h) - e^{-W_2 \mathbf{G}_h} \mathbf{C}_0(W_2, \mathbf{m}_1^h)\right|_{L_h^2}^2 ds \\
&= \int_0^t \left|\left(e^{-W_1 \mathbf{G}_h} - e^{-W_2 \mathbf{G}_h}\right) \mathbf{C}_0(W_2, \mathbf{m}_1^h) + e^{-W_1 \mathbf{G}_h} \left(\mathbf{C}_0(W_1, \mathbf{m}_1^h) - \mathbf{C}_0(W_2, \mathbf{m}_1^h)\right)\right|_{L_h^2}^2 ds \\
&\leq C(S_4 + S_5)
\end{aligned}
$$

with $\mathbf{C}_0$ defined in Lemma 3.5.5 and

$$
\begin{aligned}
S_4 &= \int_0^t \left|\left(e^{-W_1 \mathbf{G}_h} - e^{-W_2 \mathbf{G}_h}\right) \mathbf{C}_0(W_2, \mathbf{m}_1^h)\right|_{L_h^2}^2 ds \\
S_5 &= \int_0^t \left|e^{-W_1 \mathbf{G}_h} \left(\mathbf{C}_0(W_1, \mathbf{m}_1^h) - \mathbf{C}_0(W_2, \mathbf{m}_1^h)\right)\right|_{L_h^2}^2 ds.
\end{aligned}
$$

We have from (3.5.11) and the definition of the operator $\mathbf{G}_h$

$$
\begin{aligned}
S_4 &= \int_0^t \left|(\sin W_2 - \sin W_1) \mathbf{G}_h \left(\mathbf{C}_0(W_2, \mathbf{m}_1^h)\right) + (\cos W_2 - \cos W_1) \mathbf{G}_h^2 \left(\mathbf{C}_0(W_2, \mathbf{m}_1^h)\right)\right|_{L_h^2}^2 ds \\
&\leq C \int_0^t \left|(\sin W_2 - \sin W_1) \mathbf{G}_h \left(\mathbf{C}_0(W_2, \mathbf{m}_1^h)\right)\right|_{L_h^2}^2 ds \\
&\quad + C \int_0^t \left|(\cos W_2 - \cos W_1) \mathbf{G}_h^2 \left(\mathbf{C}_0(W_2, \mathbf{m}_1^h)\right)\right|_{L_h^2}^2 ds \\
&\leq C \sup_{s \in [0,t]} |W_2 - W_1|^2 (|\mathbf{g}^h|_{L_h^\infty}^2 + |\mathbf{g}^h|_{L_h^\infty}^4) \int_0^t \left|\mathbf{C}_0(W_2, \mathbf{m}_1^h)\right|_{L_h^2}^2 ds.
\end{aligned}
$$

We estimate $\int_0^t |\mathbf{C}_0(W_2, \mathbf{m}_1^h)|^2_{L^2_h} ds$ in the same resoning as (3.7.6) is done, then by using (3.7.2) and Lemma 2.5.6 we get

$$
\begin{aligned}
S_4 &\le C \sup_{s\in[0,t]} |W_2 - W_1|^2 (K^2 |\mathbf{g}^h|_{L^2_h} |D^+\mathbf{g}^h|_{L^2_h} + K^4 |\mathbf{g}^h|^2_{L^2_h} |D^+\mathbf{g}^h|^2_{L^2_h}) C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \\
&\le C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \sup_{s\in[0,t]} |W_2 - W_1|^2.
\end{aligned}
$$

Now, using (3.5.16) we obtain for S_5

$$
\begin{aligned}
S_5 &= \int_0^t \left| e^{-W_1 \mathbf{G}_h} \left(\mathbf{C}_0(W_1, \mathbf{m}_1^h) - \mathbf{C}_0(W_2, \mathbf{m}_1^h) \right) \right|^2_{L^2_h} ds \\
&= \int_0^t \left| \mathbf{C}_0(W_1, \mathbf{m}_1^h) - \mathbf{C}_0(W_2, \mathbf{m}_1^h) \right|^2_{L^2_h} ds \\
&= \int_0^t \left| (\sin W_1 - \sin W_2) \mathbf{C}\mathbf{m}_1^h + (\cos W_2 - \cos W_1)(\mathbf{C}\mathbf{G}_h + \mathbf{G}_h\mathbf{C})\mathbf{m}_1^h \right|^2_{L^2_h} ds \\
&\le C \int_0^t \left| (\sin W_1 - \sin W_2) \mathbf{C}\mathbf{m}_1^h \right|^2_{L^2_h} ds \\
&\quad + C \int_0^t \left| (\cos W_2 - \cos W_1)(\mathbf{C}\mathbf{G}_h + \mathbf{G}_h\mathbf{C})\mathbf{m}_1^h \right|^2_{L^2_h} ds \\
&\le C \sup_{s\in[0,t]} |W_1 - W_2|^2 \int_0^t \left| \mathbf{C}\mathbf{m}_1^h \right|^2_{L^2_h} ds + C \sup_{s\in[0,t]} |W_2 - W_1|^2 \int_0^t \left| (\mathbf{C}\mathbf{G}_h + \mathbf{G}_h\mathbf{C})\mathbf{m}_1^h \right|^2_{L^2_h} ds \\
&\le C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \sup_{s\in[0,t]} |W_1 - W_2|^2
\end{aligned}
$$

where the last inequality is obtained in the same reasoning as (3.7.6) is done. Combining the above, we get

$$
S_1 \le C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \sup_{s\in[0,t]} |W_1 - W_2|^2.
$$

Using (3.5.16) and Lemma 2.5.2, we continue by estimating S_2

$$
\begin{aligned}
S_2 &= \int_0^t \left| \overline{\mathbf{m}^h} \right|_{L^2_h} \left| \tilde{\mathbf{C}} \left(W_2, \overline{\mathbf{m}^h} \right) \right|_{L^2_h} ds \\
&= \int_0^t \left| \overline{\mathbf{m}^h} \right|_{L^2_h} \left| e^{-W_2 \mathbf{G}_h} \mathbf{C}_0 \left(W_2, \overline{\mathbf{m}^h} \right) \right|_{L^2_h} ds \\
&= \int_0^t \left| \overline{\mathbf{m}^h} \right|_{L^2_h} \left| \left(\sin W_2 \mathbf{C} + (1 - \cos W_2)(\mathbf{C}\mathbf{G}_h + \mathbf{G}_h\mathbf{C}) \right) \overline{\mathbf{m}^h} \right|_{L^2_h} ds
\end{aligned}
$$

$$
\begin{aligned}
&\le C\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\mathbf{C}\overline{\mathbf{m}^h}\right|_{L_h^2} ds + C\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|(\mathbf{C}\mathbf{G}_h + \mathbf{G}_h\mathbf{C})\overline{\mathbf{m}^h}\right|_{L_h^2} ds \\
&\le C\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h} \times \tilde{\Delta}\mathbf{g}^h + D^+\overline{\mathbf{m}^h} \times D^+\mathbf{g}^h + D^-\overline{\mathbf{m}^h} \times D^-\mathbf{g}^h\right|_{L_h^2} ds \\
&\quad + C\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\mathbf{G}_h\overline{\mathbf{m}^h} \times \tilde{\Delta}\mathbf{g}^h + D^+\mathbf{G}_h\overline{\mathbf{m}^h} \times D^+\mathbf{g}^h + D^-\mathbf{G}_h\overline{\mathbf{m}^h} \times D^-\mathbf{g}^h\right|_{L_h^2} ds \\
&\quad + C\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\mathbf{G}_h\left(\overline{\mathbf{m}^h} \times \tilde{\Delta}\mathbf{g}^h + D^+\overline{\mathbf{m}^h} \times D^+\mathbf{g}^h + D^-\overline{\mathbf{m}^h} \times D^-\mathbf{g}^h\right)\right|_{L_h^2} ds \\
&\le C\Bigg(|\tilde{\Delta}\mathbf{g}^h|_{L_h^2} + |\mathbf{g}^h|_{L_h^\infty}|\tilde{\Delta}\mathbf{g}^h|_{L_h^2} + |D^+\mathbf{g}^h|_{L_h^\infty}|D^+\mathbf{g}^h|_{L_h^2}\Bigg)\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} ds \\
&\quad + C\Bigg(|D^+\mathbf{g}^h|_{L_h^\infty} + |\mathbf{g}^h|_{L_h^\infty}|D^+\mathbf{g}^h|_{L_h^\infty}\Bigg)\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2} ds.
\end{aligned}
$$

Then, using Lemma 2.5.6 and Young's inequality (2.2.6) for $p = 4$ and $q = \frac{4}{3}$ we get

$$
\begin{aligned}
S_2 &\le C_\varepsilon\left(|\mathbf{g}|_{H^2}\right)\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^{\frac{3}{2}} \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^{\frac{1}{2}} ds + C_\varepsilon\left(|\mathbf{g}|_{H^2}\right)\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds + \varepsilon\int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds \\
&\le C_\varepsilon\left(|\mathbf{g}|_{H^2}\right)\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds + \varepsilon\int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds.
\end{aligned}
$$

For S_3, using (3.5.16) and (3.7.1) we have

$$
\begin{aligned}
S_3 &= \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|\tilde{\mathbf{C}}(W_1, \mathbf{m}_1^h)\right|_{L_h^2} ds \\
&= \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|\mathbf{C}_0(W_1, \mathbf{m}_1^h)\right|_{L_h^2} ds \\
&= \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|(\sin W_1\mathbf{C} + (1-\cos W_1)(\mathbf{C}\mathbf{G}_h + \mathbf{G}_h\mathbf{C}))\,\mathbf{m}_1^h\right|_{L_h^2} ds \\
&\le C\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|\mathbf{m}_1^h \times \tilde{\Delta}\mathbf{g}^h + D^+\mathbf{m}_1^h \times D^+\mathbf{g}^h + D^-\mathbf{m}_1^h \times D^-\mathbf{g}^h\right|_{L_h^2} ds \\
&\quad + C\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|\mathbf{G}_h\mathbf{m}_1^h \times \tilde{\Delta}\mathbf{g}^h + D^+\mathbf{G}_h\mathbf{m}_1^h \times D^+\mathbf{g}^h + D^-\mathbf{G}_h\mathbf{m}_1^h \times D^-\mathbf{g}^h\right|_{L_h^2} ds \\
&\quad + C\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|\mathbf{G}_h\left(\mathbf{m}_1^h \times \tilde{\Delta}\mathbf{g}^h + D^+\mathbf{m}_1^h \times D^+\mathbf{g}^h + D^-\mathbf{m}_1^h \times D^-\mathbf{g}^h\right)\right|_{L_h^2} ds \\
&\le C\Bigg(|\tilde{\Delta}\mathbf{g}^h|_{L_h^2} + |\mathbf{g}^h|_{L_h^\infty}|\tilde{\Delta}\mathbf{g}^h|_{L_h^2} + |D^+\mathbf{g}^h|_{L_h^2}|D^+\mathbf{g}^h|_{L_h^\infty}\Bigg)\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} ds \\
&\quad + C\Bigg(|D^+\mathbf{g}^h|_{L_h^\infty} + |\mathbf{g}^h|_{L_h^\infty}|D^+\mathbf{g}^h|_{L_h^\infty}\Bigg)\int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} \left|D^+\mathbf{m}_1^h\right|_{L_h^2} ds.
\end{aligned}
$$

Then, using (3.7.2), Lemma 2.5.6 and Young's inequality (2.2.6) for $p = 4$ and $q = \frac{4}{3}$ we deduce

$$\begin{aligned} S_3 &\leq C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2} \left|\overline{\mathbf{m}^h}\right|_{L_h^\infty} ds \\ &\leq C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^{\frac{3}{2}} \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^{\frac{1}{2}} ds \\ &\leq C_\varepsilon(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds + \varepsilon \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds. \end{aligned}$$

Finally, combining all the estimates for $S_i (i = 1, 2, 3)$ and using (3.7.2), we get from (3.7.11)

$$\begin{aligned} &\sup_{s\in[0,t]} \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 + C_\varepsilon \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds \\ &\leq C_\varepsilon(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \int_0^t \left|\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds + C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \sup_{s\in[0,t]} |W_1 - W_2|^2 \\ &\leq C_\varepsilon(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \int_0^t \sup_{r\in[0,s]} \left|\overline{\mathbf{m}^h}(r)\right|_{L_h^2}^2 ds + C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) \sup_{s\in[0,t]} |W_1 - W_2|^2. \end{aligned}$$

From the fact that

$$C_\varepsilon \int_0^t \left|D^+\overline{\mathbf{m}^h}\right|_{L_h^2}^2 ds \geq 0,$$

we obtain using Gronwall's inequality (2.2.7)

$$\sup_{s\in[0,t]} \left|\overline{\mathbf{m}^h}(s)\right|_{L_h^2}^2 \leq C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2}) e^{C_\varepsilon(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2})} \sup_{s\in[0,t]} |W_1 - W_2|^2$$

and the proof is now complete. □

3.8 Proof of the Main Theorem

In this section, we will prove the main theorem of this chapter, Theorem 3.1.3. We recall that the operators r_h and p_h are defined in Chapter 2. In what follows, we will repeatedly use the simple fact that for any functions $\mathbf{v}^h$, $\mathbf{w}^h : \mathbb{Z}_h \to R^3$

$$r_h \left(\mathbf{v}^h \times \mathbf{w}^h\right) = r_h \mathbf{v}^h \times r_h \mathbf{w}^h.$$

We start by proving the following lemmas.

Lemma 3.8.1. *Assume that $|\mathbf{u}_0| = 1$, $\nabla \mathbf{u}_0 \in L^2$ and $\mathbf{g} \in H^2$. Then, for $T \in (0, \infty)$, there exists a constant C which does not depend on h but may depend on $|\mathbf{g}|_{H^2}$, $|\nabla \mathbf{u}_0|_{L^2}$ and T such that $\mathbb{P}$-a.s.*

$$\int_0^T \left|p_h \mathbf{m}^h(t)\right|_{L^2_m}^2 dt \leq C, \tag{3.8.1}$$
$$\int_0^T \left|\nabla p_h \mathbf{m}^h(t)\right|_{L^2}^2 dt \leq C, \tag{3.8.2}$$
$$\int_0^T \left|\frac{d}{dt} p_h \mathbf{m}^h(t)\right|_{L^2}^2 dt \leq C. \tag{3.8.3}$$

Proof. In order to simplify notations, we assume in the proof without loss of generality, that $\lambda = \mu = 1$. First we prove (3.8.1). Using (3.7.1), we have

$$\begin{aligned}
\int_0^T \left|p_h \mathbf{m}^h(t)\right|_{L^2_m}^2 dt &= \int_0^T \sum_{x_i \in \mathbb{Z}_h} \int_{x_i}^{x_{i+1}} \left|\mathbf{m}^h(t, x_i) + D^+ \mathbf{m}^h(t, x_i)(x - x_i)\right|^2 \rho_m(x) dx dt \\
&\leq C \int_0^T \sum_{x_i \in \mathbb{Z}_h} \int_{x_i}^{x_{i+1}} \left|\mathbf{m}^h(t, x_i)\right|^2 \rho_m(x) dx dt \\
&\quad + C \int_0^T \sum_{x_i \in \mathbb{Z}_h} \int_{x_i}^{x_{i+1}} \left|\mathbf{m}^h(t, x_{i+1}) - \mathbf{m}^h(t, x_i)\right|^2 \rho_m(x) dx dt \\
&\leq C \int_0^T \int_{\mathbb{R}} \rho_m(x) dx dt \\
&\leq C(T).
\end{aligned}$$

Next, we prove (3.8.2). Using property (2.5.8), we get

$$\begin{aligned}
\int_0^T \left|\nabla p_h \mathbf{m}^h(t)\right|_{L^2}^2 dt = \int_0^T \left|r_h D^+ \mathbf{m}^h(t)\right|_{L^2}^2 dt &= \int_0^T \left|D^+ \mathbf{m}^h(t)\right|_{L^2_h}^2 dt \\
&\leq T \sup_{t \in [0,T]} \left|D^+ \mathbf{m}^h(t)\right|_{L^2_h}^2 \\
&\leq C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2})
\end{aligned}$$

where the last step follows from (3.7.2).

Finally, we prove (3.8.3). Using (3.7.1), we have

$$\begin{aligned}
&\int_0^T \left|\frac{d}{dt} p_h \mathbf{m}^h(t)\right|_{L^2}^2 dt \\
&= \int_0^T \sum_{x_i \in \mathbb{Z}_h} \int_{x_i}^{x_{i+1}} \left|\frac{d}{dt}\left(\mathbf{m}^h(t,x_i) + D^+\mathbf{m}^h(t,x_i)(x-x_i)\right)\right|^2 dx dt \\
&\leq C \int_0^T \left|\frac{d}{dt}\mathbf{m}^h(t)\right|_{L_h^2}^2 dt \\
&\leq C \int_0^T \Big|\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h - \mathbf{m}^h \times (\mathbf{m}^h \times \tilde{\Delta}\mathbf{m}^h) \\
&\quad + \mathbf{m}^h \times \tilde{\mathbf{C}}\left(W, \mathbf{m}^h\right) - \mathbf{m}^h \times \left(\mathbf{m}^h \times \tilde{\mathbf{C}}\left(W, \mathbf{m}^h\right)\right)\Big|_{L_h^2}^2 dt \\
&\leq C \int_0^T \left|\tilde{\Delta}\mathbf{m}^h\right|_{L_h^2}^2 dt + C \int_0^T \left|\tilde{\mathbf{C}}\left(W, \mathbf{m}^h\right)\right|_{L_h^2}^2 dt \\
&\leq C(|\nabla \mathbf{u}_0|_{L^2}, T, |\mathbf{g}|_{H^2})
\end{aligned}$$

where the last step follows from (3.7.6), (3.7.2) and (3.7.3). The proof is now complete.

□

Lemma 3.8.2. *There exists a sequence* $(h_n)_{n\geq 0}$ *and* $\mathbf{m} \in L^2(0,T;L_m^2)$ *such that as* $h_n \to 0$,

$$\int_0^T \left|r_{h_n}\mathbf{m}^{h_n}(t,\omega) - \mathbf{m}(t,\omega)\right|_{L_m^2}^2 dt \to 0$$

for every ω.

Proof. Using Lemmas 2.1.2 and 3.8.1, we get that for every fixed ω there exists $h_n(\omega)$ and $\mathbf{m}^0 \in L^2(0,T;L_{m'}^2)$ such that as $h_n \to 0$

$$\int_0^T \left|p_{h_n}\mathbf{m}^{h_n}(t,\omega) - \mathbf{m}^0(t,\omega)\right|_{L_{m'}^2}^2 dt \to 0$$

where $m' = \frac{m}{2}$. In what follows, m' will be denoted by m. Using Proposition 2.5.8, we obtain that for every fixed ω and the corresponding trajectory of the Wiener process

$W(\omega)$, there exists $h_n(\omega)$ and $\mathbf{m}^0 \in L^2(0,T;L^2_m)$ such that as $h_n \to 0$

$$\int_0^T \left| r_{h_n}\mathbf{m}^{h_n}(t,\omega) - \mathbf{m}^0(t,\omega)\right|^2_{L^2_m} dt \to 0.$$

Since $C_0([0,T])$ is a separable Banach space and $\mathbb{P}(W(\cdot) \in B) = 1$ for every ball B in $C_0([0,T])$, see for example the seminal work [46], we can find a countable set $\{\omega_k, k \geq 1\}$ such that the corresponding trajectories $t \to W(t,\omega_k)$ form a dense set in $C_0([0,T])$. Then, for every fixed $\{\omega_k\}$ and the corresponding trajectory of the Wiener process $W(\omega_k)$, there exists $h_n^k := h_n(\omega_k)$ and $\mathbf{m}^0 \in L^2(0,T;L^2_m)$ such that as $h_n^k \to 0$

$$\int_0^T \left| r_{h_n^k}\mathbf{m}^{h_n^k}(t,\omega_k) - \mathbf{m}^0(t,\omega_k)\right|^2_{L^2_m} dt \to 0$$

for every $k \geq 1$. From the diagonal procedure, we can find a subsequence h_n independent of k such that for every $k \geq 1$

$$\lim_{n\to\infty} h_n(\omega_k) = 0$$

and

$$\int_0^T \left| r_{h_n}\mathbf{m}^{h_n}(t,\omega_k) - \mathbf{m}^0(t,\omega_k)\right|^2_{L^2_m} dt \to 0. \tag{3.8.4}$$

Next, we fix ω and a sequence ω_k such that when $k \to \infty$,

$$\sup_{t\in[0,T]} |W(t,\omega_k) - W(t,\omega)| \to 0.$$

Then, for $n, l \geq 0$

$$\begin{aligned}
&\int_0^T \left| r_{h_n}\mathbf{m}^{h_n}(t,\omega) - r_{h_l}\mathbf{m}^{h_l}(t,\omega)\right|^2_{L^2_m} dt \\
&\leq \int_0^T \left| r_{h_n}\mathbf{m}^{h_n}(t,\omega) - r_{h_n}\mathbf{m}^{h_n}(t,\omega_k)\right|^2_{L^2_m} dt + \int_0^T \left| r_{h_n}\mathbf{m}^{h_n}(t,\omega_k) - r_{h_l}\mathbf{m}^{h_l}(t,\omega_k)\right|^2_{L^2_m} dt \\
&\quad + \int_0^T \left| r_{h_l}\mathbf{m}^{h_l}(t,\omega_k) - r_{h_l}\mathbf{m}^{h_l}(t,\omega)\right|^2_{L^2_m} dt.
\end{aligned} \tag{3.8.5}$$

Using Lemma 3.7.2, we have that for $\varepsilon > 0$ there exists $f_0 > 0$ such that for $n, l \geq f_0$

$$\begin{aligned}\int_0^T \left|r_{h_n}\mathbf{m}^{h_n}(t,\omega) - r_{h_n}\mathbf{m}^{h_n}(t,\omega_k)\right|_{L^2_m}^2 dt &\leq T \sup_{t\in[0,T]} \left|r_{h_n}\mathbf{m}^{h_n}(t,\omega) - r_{h_n}\mathbf{m}^{h_n}(t,\omega_k)\right|_{L^2_m}^2 \\ &\leq CT \sup_{t\in[0,T]} |W(t,\omega) - W(t,\omega_k)|^2 \\ &\leq \frac{\varepsilon}{2},\end{aligned}$$

and

$$\begin{aligned}\int_0^T \left|r_{h_l}\mathbf{m}^{h_l}(t,\omega) - r_{h_l}\mathbf{m}^{h_l}(t,\omega_k)\right|_{L^2_m}^2 dt &\leq T \sup_{t\in[0,T]} \left|r_{h_l}\mathbf{m}^{h_l}(t,\omega) - r_{h_l}\mathbf{m}^{h_l}(t,\omega_k)\right|_{L^2_m}^2 \\ &\leq CT \sup_{t\in[0,T]} |W(t,\omega) - W(t,\omega_k)|^2 \\ &\leq \frac{\varepsilon}{2}.\end{aligned}$$

Therefore, from (3.8.5) and using (3.8.4), we obtain

$$\begin{aligned}&\limsup_{h_n,h_l\to 0} \int_0^T \left|r_{h_n}\mathbf{m}^{h_n}(t,\omega) - r_{h_l}\mathbf{m}^{h_l}(t,\omega)\right|_{L^2_m}^2 dt \\ &\leq \varepsilon + \limsup_{h_n,h_l\to 0} \int_0^T \left|r_{h_n}\mathbf{m}^{h_n}(t,\omega_k) - r_{h_l}\mathbf{m}^{h_l}(t,\omega_k)\right|_{L^2_m}^2 dt \\ &\leq \varepsilon.\end{aligned}$$

Since ε is arbitrary, we get

$$\lim_{h_n,h_l\to 0} \int_0^T \left|r_{h_n}\mathbf{m}^{h_n}(t,\omega) - r_{h_l}\mathbf{m}^{h_l}(t,\omega)\right|_{L^2_m}^2 dt = 0.$$

Consequently, $\{r_{h_n}\mathbf{m}^{h_n}\}$ is a Cauchy sequence in a complete space $L^2(0,T;L^2_m)$. Hence, there exists $\mathbf{m} \in L^2(0,T;L^2_m)$ such that

$$\lim_{h_n\to 0} \int_0^T \left|r_{h_n}\mathbf{m}^{h_n}(t,\omega) - \mathbf{m}(t,\omega)\right|_{L^2_m}^2 dt = 0$$

for every ω and the result follows. $\square$

Lemma 3.8.3. *There exists a sequence $(h_n)_{n\geq 0}$ and $\mathbf{u} \in \mathcal{H}_m$ such that as $h_n \to 0$,*

$$\mathbb{E}\bigg[\int_0^T \big|r_{h_n}\mathbf{u}^{h_n}(t) - \mathbf{u}(t)\big|^2_{L^2_m}\,dt\bigg] \to 0$$

where $\mathbf{u}^{h_n}(t) = e^{W(t)\mathbf{G}_{h_n}}\mathbf{m}^{h_n}(t)$ and $\mathbf{u}(t) = e^{W(t)\mathbf{G}}\mathbf{m}(t)$. We note that $\mathbf{u}^{h_n}(t)$ satisfies (3.3.1)*. In addition,*

$$|\mathbf{u}(\omega,t,x)| = 1 \quad d\mathbb{P}\,dt\,dx\,a.e.$$

Proof. We have that, using Lemma 3.5.3 applied on $\mathbf{G}: L^2_m \to L^2_m$

$$\begin{aligned}
\mathbb{E}\bigg[\int_0^T \big|r_{h_n}\mathbf{u}^{h_n}(t) - \mathbf{u}(t)\big|^2_{L^2_m}\,dt\bigg] &= \mathbb{E}\bigg[\int_0^T \big|r_{h_n}e^{W(t)\mathbf{G}_{h_n}}\mathbf{m}^{h_n}(t) - e^{W(t)\mathbf{G}}\mathbf{m}(t)\big|^2_{L^2_m}\,dt\bigg] \\
&\leq C\mathbb{E}\bigg[\int_0^T \big|r_{h_n}e^{W(t)\mathbf{G}_{h_n}}\mathbf{m}^{h_n}(t) - e^{W(t)\mathbf{G}}r_{h_n}\mathbf{m}^{h_n}(t)\big|^2_{L^2_m}\,dt\bigg] \\
&\quad + C\mathbb{E}\bigg[\int_0^T \big|e^{W(t)\mathbf{G}}r_{h_n}\mathbf{m}^{h_n}(t) - e^{W(t)\mathbf{G}}\mathbf{m}(t)\big|^2_{L^2_m}\,dt\bigg] \\
&\leq C\mathbb{E}\bigg[\int_0^T \big|r_{h_n}e^{W(t)\mathbf{G}_{h_n}}\mathbf{m}^{h_n}(t) - e^{W(t)\mathbf{G}}r_{h_n}\mathbf{m}^{h_n}(t)\big|^2_{L^2_m}\,dt\bigg] \\
&\quad + C\mathbb{E}\bigg[\int_0^T \big|r_{h_n}\mathbf{m}^{h_n}(t) - \mathbf{m}(t)\big|^2_{L^2_m}\,dt\bigg] \\
&\leq A_1 + A_2. \qquad (3.8.6)
\end{aligned}$$

For A_1, by using (3.5.11) we get

$$\begin{aligned}
A_1 &= \mathbb{E}\bigg[\int_0^T \big|\sin W(t)\big(r_{h_n}\mathbf{G}_{h_n}\mathbf{m}^{h_n}(t) - \mathbf{G}r_{h_n}\mathbf{m}^{h_n}(t)\big) \\
&\quad + (1-\cos W(t))\big(r_{h_n}\mathbf{G}^2_{h_n}\mathbf{m}^{h_n}(t) - \mathbf{G}^2 r_{h_n}\mathbf{m}^{h_n}(t)\big)\big|^2_{L^2_m}\,dt\bigg] \\
&\leq C\mathbb{E}\bigg[\int_0^T \big|r_{h_n}\mathbf{G}_{h_n}\mathbf{m}^{h_n}(t) - \mathbf{G}r_{h_n}\mathbf{m}^{h_n}(t)\big|^2_{L^2_m}\,dt\bigg] \\
&\quad + C\mathbb{E}\bigg[\int_0^T \big|r_{h_n}\mathbf{G}^2_{h_n}\mathbf{m}^{h_n}(t)) - \mathbf{G}^2 r_{h_n}\mathbf{m}^{h_n}(t)\big|^2_{L^2_m}\,dt\bigg] \\
&\leq C\mathbb{E}\bigg[\int_0^T \big|r_{h_n}\mathbf{m}^{h_n}(t) \times (r_{h_n}\mathbf{g}^{h_n} - \mathbf{g})\big|^2_{L^2_m}\,dt\bigg] \\
&\quad + C\mathbb{E}\bigg[\int_0^T \big|\big(r_{h_n}\mathbf{m}^{h_n}(t) \times r_{h_n}\mathbf{g}^{h_n}\big) \times (r_{h_n}\mathbf{g}^{h_n} - \mathbf{g})
\end{aligned}$$

$$
\begin{aligned}
&+ \big(r_{h_n}\mathbf{m}^{h_n}(t) \times (r_{h_n}\mathbf{g}^{h_n} - \mathbf{g})\big) \times \mathbf{g}\big|_{L^2_m}^2\, dt\bigg] \\
&\le C\mathbb{E}\bigg[\int_0^T \big|r_{h_n}\mathbf{m}^{h_n}(t) \times (r_{h_n}\mathbf{g}^{h_n} - \mathbf{g})\big|_{L^2_m}^2\, dt\bigg] \\
&\quad + C\mathbb{E}\bigg[\int_0^T \big|\big(r_{h_n}\mathbf{m}^{h_n}(t) \times r_{h_n}\mathbf{g}^{h_n}\big) \times (r_{h_n}\mathbf{g}^{h_n} - \mathbf{g})\big|_{L^2_m}^2\, dt\bigg] \\
&\quad + C\mathbb{E}\bigg[\int_0^T \big|\big(r_{h_n}\mathbf{m}^{h_n}(t) \times (r_{h_n}\mathbf{g}^{h_n} - \mathbf{g})\big) \times \mathbf{g}\big|_{L^2_m}^2\, dt\bigg].
\end{aligned}
$$

Then, using (3.7.1) we obtain

$$
A_1 \le CT|r_{h_n}\mathbf{g}^{h_n} - \mathbf{g}|_{L^2}^2 + CT|r_{h_n}\mathbf{g}^{h_n}|_{L^\infty}^2 |r_{h_n}\mathbf{g}^{h_n} - \mathbf{g}|_{L^2}^2 + CT|\mathbf{g}|_{L^\infty}^2 |r_{h_n}\mathbf{g}^{h_n} - \mathbf{g}|_{L^2}^2.
$$

From the assumption $\mathbf{g} \in H^1$, we get using Lemma 2.5.7

$$
\lim_{h_n \to 0} A_1 = 0.
$$

On the other hand, we have from Lemma 3.8.2

$$
\lim_{h_n \to 0} A_2 = 0.
$$

Then, (3.8.6) implies

$$
\lim_{h_n \to 0} \mathbb{E}\bigg[\int_0^T \big|r_{h_n}\mathbf{u}^{h_n}(t) - \mathbf{u}(t)\big|_{L^2_m}^2\, dt\bigg] = 0. \tag{3.8.7}
$$

Furthermore, from Lemma 3.6.1 we get that $\mathbf{u}^{h_n}$ satisfies (3.3.1). Next, using (3.8.7), we can find a sequence $(r_{h_n}\mathbf{u}^{h_n})$ such that as $h_n \to 0$,

$$
\begin{gathered}
r_{h_n}\mathbf{u}^{h_n} \to \mathbf{u} \quad \text{in } \mathcal{H}_m \\
r_{h_n}\mathbf{u}^{h_n}(\omega, t, x) \to \mathbf{u}(\omega, t, x) \quad d\mathbb{P}\, dt\, dx\, a.e.
\end{gathered}
$$

Since $|r_{h_n}\mathbf{u}^{h_n}(\omega, t, x)| = |\mathbf{u}(\omega, t, x)|\, d\mathbb{P}\, dt\, dx\, a.e.$ and from (3.4.1) the lemma follows. $\square$

Lemma 3.8.4. *There exists a sequence* $(h_n)_{n\geq 0}$ *and* $\mathbf{u} \in \mathcal{H}_m$ *such that as* $h_n \to 0$*,*

$$r_{h_n}\mathbf{u}^{h_n} \to \mathbf{u} \quad \textit{strongly in } \mathcal{H}_m \textit{ and } d\mathbb{P}\, dt\, dx\textit{-a.e.,} \tag{3.8.8}$$

$$r_{h_n}\tilde{\boldsymbol{\Delta}}\mathbf{u}^{h_n} \to \boldsymbol{\Delta}\mathbf{u} \quad \textit{weakly in } \mathcal{H}_m. \tag{3.8.9}$$

Proof. The first convergence (3.8.8) follows from Lemma 3.8.3 and the existence of an a.e. converging subsequence follows. Next we prove (3.8.9). From (3.8.8) and Proposition 2.5.9, we get that as $h_n \to 0$

$$p_{h_n}\mathbf{u}^{h_n} \to \mathbf{u} \quad \text{strongly in } \mathcal{H}_m.$$

From Lemma 3.4.1, we also find that there exists $\mathbf{v} \in \mathcal{H}_m$ such that as $h_n \to 0$

$$r_{h_n}\boldsymbol{D}^{+}\mathbf{u}^{h_n} = \boldsymbol{\nabla} p_{h_n}\mathbf{u}^{h_n} \to \mathbf{v} \quad \text{weakly in } \mathcal{H}_m. \tag{3.8.10}$$

Therefore, from Proposition 2.3.19

$$\mathbf{u} \in \mathcal{H}_m^1 \quad \text{and} \qquad \mathbf{v} = \boldsymbol{\nabla}\mathbf{u}.$$

Next, we have from Lemma 3.4.1 that there exists $\mathbf{w} \in \mathcal{H}_m$ such that as $h_n \to 0$

$$r_{h_n}\tilde{\boldsymbol{\Delta}}\mathbf{u}^{h_n} = r_{h_n}(\boldsymbol{D}^{+}\boldsymbol{D}^{-}\mathbf{u}^{h_n}) = \boldsymbol{\nabla}(p_{h_n}\boldsymbol{D}^{-}\mathbf{u}^{h_n}) \to \mathbf{w} \quad \text{weakly in } \mathcal{H}_m.$$

It follows from (3.8.10) that as $h_n \to 0$

$$r_{h_n}\boldsymbol{D}^{-}\mathbf{u}^{h_n} \to \mathbf{v} \quad \text{weakly in } \mathcal{H}_m,$$

then using Proposition 2.5.9, we have as $h_n \to 0$

$$p_{h_n}\boldsymbol{D}^{-}\mathbf{u}^{h_n} \to \mathbf{v} = \boldsymbol{\nabla}\mathbf{u} \quad \text{weakly in } \mathcal{H}_m.$$

Therefore, from Proposition 2.3.19

$$\mathbf{u} \in \mathcal{H}^2_m \quad \text{and} \quad \mathbf{w} = \mathbf{\Delta u}$$

then the lemma follows. $\square$

Lemma 3.8.5. *There exists a sequence $(h_n)_{n\geq 0}$ such that we have as $h_n \to 0$,*

$$\int_0^t r_{h_n}\mathbf{u}^{h_n}(s) \times r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}(s)ds \to \int_0^t \mathbf{u}(s) \times \Delta\mathbf{u}(s)ds, \tag{3.8.11}$$

$$\int_0^t r_{h_n}\mathbf{u}^{h_n}(s) \times \left(r_{h_n}\mathbf{u}^{h_n}(s) \times r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}(s)\right) ds \to \int_0^t \mathbf{u}(s) \times (\mathbf{u}(s) \times \Delta\mathbf{u}(s))\, ds \tag{3.8.12}$$

weakly in $\mathcal{H}_m$.

Proof. First, we prove (3.8.11). We have for $\phi \in \mathcal{H}_m$, using (2.2.5)

$$\begin{aligned}
\left\langle r_{h_n}\mathbf{u}^{h_n} \times r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}, \phi\right\rangle_{\mathcal{H}_m} &= \left\langle (r_{h_n}\mathbf{u}^{h_n} - \mathbf{u}) \times r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}, \phi\right\rangle_{\mathcal{H}_m} + \left\langle \mathbf{u} \times r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}, \phi\right\rangle_{\mathcal{H}_m} \\
&= \left\langle \phi \times (r_{h_n}\mathbf{u}^{h_n} - \mathbf{u}), r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}\right\rangle_{\mathcal{H}_m} + \left\langle \phi \times \mathbf{u}, r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}\right\rangle_{\mathcal{H}_m} \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, we have

$$\begin{aligned}
I_1 &\leq \mathbb{E}\left[\int_0^T \left|\phi \times (r_{h_n}\mathbf{u}^{h_n} - \mathbf{u})\right|_{L^2_m} \left|r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}\right|_{L^2_m} dt\right] \\
&\leq \mathbb{E}\left[\left(\int_0^T \left|\phi \times (r_{h_n}\mathbf{u}^{h_n} - \mathbf{u})\right|^2_{L^2_m} dt\right)^{\frac{1}{2}} \left(\int_0^T \left|r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}\right|^2_{L^2_m} dt\right)^{\frac{1}{2}}\right] \\
&\leq \mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\phi|^2 |r_{h_n}\mathbf{u}^{h_n} - \mathbf{u}|^2 \rho_m dxdt\right]^{\frac{1}{2}} \mathbb{E}\left[\int_0^T \left|r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}\right|^2_{L^2_m} dt\right]^{\frac{1}{2}}.
\end{aligned}$$

Using Lemma 3.4.1, we get that

$$I_1 \leq C\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} |\phi|^2 |r_{h_n}\mathbf{u}^{h_n} - \mathbf{u}|^2 \rho_m dxdt\right]^{\frac{1}{2}}.$$

By Lemma 3.8.3, we can find a sequence $(r_{h_n}\mathbf{u}^{h_n})$ such that as $h_n \to 0$

$$r_{h_n}\mathbf{u}^{h_n} \to \mathbf{u} \quad \text{in } \mathcal{H}_m$$
$$r_{h_n}\mathbf{u}^{h_n}(\omega,t,x) \to \mathbf{u}(\omega,t,x) \quad d\mathbb{P}\,dt\,dx\,a.e.$$

Since $|r_{h_n}\mathbf{u}^{h_n}(\omega,t,x)| = |\mathbf{u}(\omega,t,x)|\,d\mathbb{P}\,dt\,dx\,a.e.$, the dominated convergence theorem yields

$$I_1 \to 0.$$

For I_2, as $\phi \times \mathbf{u} \in \mathcal{H}_m$ and using (3.8.9) we obtain as $h_n \to 0$

$$I_2 \to \langle \phi \times \mathbf{u}, \Delta\mathbf{u}\rangle_{\mathcal{H}_m}.$$

We deduce using (2.2.5) that as $h_n \to 0$

$$\left\langle r_{h_n}\mathbf{u}^{h_n} \times r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}, \phi\right\rangle_{\mathcal{H}_m} \to \langle \mathbf{u} \times \Delta\mathbf{u}, \phi\rangle_{\mathcal{H}_m} \tag{3.8.13}$$

which implies (3.8.11). Next, we prove (3.8.12). Let $Y_n := r_{h_n}\mathbf{u}^{h_n} \times r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}$ and $Y := \mathbf{u} \times \Delta\mathbf{u}$. We have for $\phi \in \mathcal{H}_m$, using (2.2.5)

$$\begin{aligned}
\left\langle r_{h_n}\mathbf{u}^{h_n} \times Y_n, \phi\right\rangle_{\mathcal{H}_m} &= \left\langle (r_{h_n}\mathbf{u}^{h_n} - \mathbf{u}) \times Y_n, \phi\right\rangle_{\mathcal{H}_m} + \langle \mathbf{u} \times Y_n, \phi\rangle_{\mathcal{H}_m} \\
&= \left\langle \phi \times (r_{h_n}\mathbf{u}^{h_n} - \mathbf{u}), Y_n\right\rangle_{\mathcal{H}_m} + \langle \phi \times \mathbf{u}, Y_n\rangle_{\mathcal{H}_m} \\
&= J_1 + J_2.
\end{aligned}$$

For J_1, following the same resoning as I_1 and using Lemma 3.4.1, we get

$$J_1 \to 0.$$

For J_2, as $\phi \times \mathbf{u} \in \mathcal{H}_m$, using (3.8.13) and (2.2.5) we obtain as $h_n \to 0$

$$J_2 \to \langle \mathbf{u} \times (\mathbf{u} \times \Delta\mathbf{u}), \phi\rangle_{\mathcal{H}_m}$$

which implies (3.8.12). $\square$

Lemma 3.8.6. *Assume $\mathbf{g} \in H^1$. We have as $h_n \to 0$,*

$$\int_0^t (r_{h_n}\mathbf{u}^{h_n} \times r_{h_n}\mathbf{g}^{h_n}) \times r_{h_n}\mathbf{g}^{h_n} ds \to \int_0^t (\mathbf{u}(s) \times \mathbf{g}) \times \mathbf{g}\, ds \tag{3.8.14}$$

$$\int_0^t r_{h_n}\mathbf{u}^{h_n}(s) \times r_{h_n}\mathbf{g}^{h_n} dW(s) \to \int_0^t \mathbf{u}(s) \times \mathbf{g}\, dW(s) \tag{3.8.15}$$

strongly in $\mathcal{H}_m$.

Proof. First we prove (3.8.14). We have

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \Big| \int_0^t \big((r_{h_n}\mathbf{u}^{h_n}(s) \times r_{h_n}\mathbf{g}^{h_n}) \times r_{h_n}\mathbf{g}^{h_n} - (\mathbf{u}(s) \times \mathbf{g}) \times \mathbf{g}\big)\, ds \Big|^2_{L^2_m}\Big] \\
&\le T\mathbb{E}\Big[\int_0^T \big|\big(r_{h_n}\mathbf{u}^{h_n}(s) \times r_{h_n}\mathbf{g}^{h_n}\big) \times r_{h_n}\mathbf{g}^{h_n} - (\mathbf{u}(s) \times \mathbf{g}) \times \mathbf{g}\big|^2_{L^2_m}\, ds\Big] \\
&\le CT\mathbb{E}\Big[\int_0^T \big|\big(r_{h_n}\mathbf{u}^{h_n}(s) \times (r_{h_n}\mathbf{g}^{h_n} - \mathbf{g})\big) \times r_{h_n}\mathbf{g}^{h_n}\big|^2_{L^2_m}\, ds\Big] \\
&\quad + CT\mathbb{E}\Big[\int_0^T \big|\big(r_{h_n}\mathbf{u}^{h_n}(s) \times \mathbf{g}\big) \times (r_{h_n}\mathbf{g}^{h_n} - \mathbf{g})\big|^2_{L^2_m}\, ds\Big] \\
&\quad + CT\mathbb{E}\Big[\int_0^T \big|\big(\big(r_{h_n}\mathbf{u}^{h_n}(s) - \mathbf{u}(s)\big) \times \mathbf{g}\big) \times \mathbf{g}\big|^2_{L^2_m}\, ds\Big].
\end{aligned}$$

Then, using Lemma 3.4.1

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^T \Big| \int_0^t \big(\big(r_{h_n}\mathbf{u}^{h_n}(s) \times r_{h_n}\mathbf{g}^{h_n}\big) \times r_{h_n}\mathbf{g}^{h_n} - (\mathbf{u}(s) \times \mathbf{g}) \times \mathbf{g}\big)\, ds \Big|^2_{L^2_m}\Big] \\
&\le CT^2 |r_{h_n}\mathbf{g}^{h_n} - \mathbf{g}|^2_{L^2_m} |r_{h_n}\mathbf{g}^{h_n}|^2_{L^\infty} + CT^2 |r_{h_n}\mathbf{g}^{h_n} - \mathbf{g}|^2_{L^2_m} |\mathbf{g}|^2_{L^\infty} \\
&\quad + CT\mathbb{E}\Big[\int_0^T \int_{\mathbb{R}} |r_{h_n}\mathbf{u}^{h_n} - \mathbf{u}|^2 |\mathbf{g}|^4 \rho_m dx ds\Big].
\end{aligned}$$

Hence, by Lemma 2.5.7 and from Lemma 3.8.4 the dominated convergence theorem yields as $h_n \to 0$,

$$\mathbb{E}\Big[\int_0^T \Big| \int_0^t \big(\big(r_{h_n}\mathbf{u}^{h_n}(s) \times r_{h_n}\mathbf{g}^{h_n}\big) \times r_{h_n}\mathbf{g}^{h_n} - (\mathbf{u}(s) \times \mathbf{g}) \times \mathbf{g}\big)\, ds \Big|^2_{L^2_m}\Big] \to 0.$$

Next, we prove (3.8.15). Using Lemma 3.4.1, we have

$$
\begin{aligned}
&\mathbb{E}\Big[\int_0^T\Big|\int_0^t\big(r_{h_n}\mathbf{u}^{h_n}(s)\times r_{h_n}\mathbf{g}^{h_n}-\mathbf{u}(s)\times\mathbf{g}\big)\,dW(s)\Big|^2_{L^2_m}\Big]\\
&\le CT\mathbb{E}\Big[\int_0^T\big|r_{h_n}\mathbf{u}^{h_n}(s)\times r_{h_n}\mathbf{g}^{h_n}-\mathbf{u}(s)\times\mathbf{g}\big|^2_{L^2_m}\,ds\Big]\\
&\le CT\mathbb{E}\Big[\int_0^T\big|r_{h_n}\mathbf{u}^{h_n}(s)\times(r_{h_n}\mathbf{g}^{h_n}-\mathbf{g})+\big(r_{h_n}\mathbf{u}^{h_n}(s)-\mathbf{u}(s)\big)\times\mathbf{g}\big|^2_{L^2_m}\,ds\Big]\\
&\le CT\mathbb{E}\Big[\int_0^T\big|r_{h_n}\mathbf{u}^{h_n}(s)\times(r_{h_n}\mathbf{g}^{h_n}-\mathbf{g})\big|^2_{L^2_m}\,ds\Big]\\
&\quad+CT\mathbb{E}\Big[\int_0^T\big|\big(r_{h_n}\mathbf{u}^{h_n}(s)-\mathbf{u}(s)\big)\times\mathbf{g}\big|^2_{L^2_m}\,ds\Big]\\
&\le CT^2|r_{h_n}\mathbf{g}^{h_n}-\mathbf{g}|^2_{L^2_m}+CT\mathbb{E}\Big[\int_0^T\int_{\mathbb{R}}|r_{h_n}\mathbf{u}^{h_n}-\mathbf{u}|^2|\mathbf{g}|^2\rho_m dxds\Big].
\end{aligned}
$$

Then, using Lemma 2.5.7 and from Lemma 3.8.4 the dominated convergence theorem yields that as $h_n\to 0$,

$$
\mathbb{E}\Big[\int_0^T\Big|\int_0^t\big(r_{h_n}\mathbf{u}^{h_n}(s)\times r_{h_n}\mathbf{g}^{h_n}-\mathbf{u}(s)\times\mathbf{g}\big)\,dW(s)\Big|^2_{L^2_m}\Big]\to 0.
$$

□

Now, we are ready to prove the main result of this chapter, Theorem 3.1.3.

Proof of Theorem 3.1.3: We have for $\phi\in\mathcal{H}_m$

$$
\begin{aligned}
\big\langle r_{h_n}\mathbf{u}^{h_n},\phi\big\rangle_{\mathcal{H}_m}&=\langle\mathbf{u}_0,\phi\rangle_{\mathcal{H}_m}+\mu\Big\langle\int_0^t r_{h_n}\mathbf{u}^{h_n}\times r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}ds,\phi\Big\rangle_{\mathcal{H}_m}\\
&\quad-\lambda\Big\langle\int_0^t r_{h_n}\mathbf{u}^{h_n}\times\Big(r_{h_n}\mathbf{u}^{h_n}\times r_{h_n}\tilde{\Delta}\mathbf{u}^{h_n}\Big)\,ds,\phi\Big\rangle_{\mathcal{H}_m}\\
&\quad+\frac{\mu^2}{2}\Big\langle\int_0^t(r_{h_n}\mathbf{u}^{h_n}\times r_{h_n}\mathbf{g}^{h_n})\times r_{h_n}\mathbf{g}^{h_n}ds,\phi\Big\rangle_{\mathcal{H}_m}\\
&\quad+\mu\Big\langle\int_0^t(r_{h_n}\mathbf{u}^{h_n}\times r_{h_n}\mathbf{g}^{h_n})dW(s),\phi\Big\rangle_{\mathcal{H}_m}.
\end{aligned}
$$

Taking the limit when $h_n \to 0$, we obtain from Lemmas 3.8.3, 3.8.5 and 3.8.6 that for every $\phi \in \mathcal{H}_m$

$$\begin{aligned}\langle \mathbf{u}, \phi\rangle_{\mathcal{H}_m} = \langle \mathbf{u}_0, \phi\rangle_{\mathcal{H}_m} &+ \mu \left\langle \int_0^t \mathbf{u} \times \Delta \mathbf{u} \, ds, \phi \right\rangle_{\mathcal{H}_m} - \lambda \left\langle \int_0^t \mathbf{u} \times (\mathbf{u} \times \Delta \mathbf{u}) \, ds, \phi \right\rangle_{\mathcal{H}_m} \\ &+ \frac{\mu^2}{2} \left\langle \int_0^t (\mathbf{u} \times \mathbf{g}) \times \mathbf{g} \, ds, \phi \right\rangle_{\mathcal{H}_m} + \mu \left\langle \int_0^t (\mathbf{u} \times \mathbf{g}) \, dW(s), \phi \right\rangle_{\mathcal{H}_m} .\end{aligned}$$

Equivalently, we have $d\mathbb{P}\, dt\, dx - a.e.$

$$\begin{aligned}\mathbf{u}(t) =& \mathbf{u}_0 + \mu \int_0^t \mathbf{u}(s) \times \Delta \mathbf{u}(s) \, ds - \lambda \int_0^t \mathbf{u}(s) \times (\mathbf{u}(s) \times \Delta \mathbf{u}(s)) \, ds \\ &+ \frac{\mu^2}{2} \int_0^t (\mathbf{u}(s) \times \mathbf{g}) \times \mathbf{g} \, ds + \mu \int_0^t (\mathbf{u}(s) \times \mathbf{g}) \, dW(s).\end{aligned} \tag{3.8.16}$$

By (3.8.11), we have

$$\mathbb{E}\left[\int_0^T |\mathbf{u} \times \Delta \mathbf{u}|^2_{L^2_m} \, dt \right] < \infty$$

then the process

$$t \to \int_0^t \mathbf{u} \times \Delta \mathbf{u} \, ds$$

taking values in L^2_m is continuous $\mathbb{P}$-a.s. Using (3.8.12) and (3.8.14) we show by similar arguments, that the L^2_m-valued processes

$$t \to \int_0^t \mathbf{u} \times (\mathbf{u} \times \Delta \mathbf{u}) \, ds \quad \text{and} \quad t \to \int_0^t (\mathbf{u} \times \mathbf{g}) \times \mathbf{g} \, ds$$

are continuous $\mathbb{P}$-a.s. The L^2_m-valued process

$$M_t = \int_0^t (\mathbf{u} \times \mathbf{g}) \, dW(s)$$

is by (3.8.15) a square-integrable martingale hence has a continuous modification by Theorem 4.27 in [17]. Therefore, there exists an L^2_m-continuous version of the process $\mathbf{u}^h$ and the property (1) of Definition 1.3.1 holds. Next, property (2) of Definition 1.3.1 follows from Lemma 3.4.1 and (3.8.8). Furthermore, from Lemma 3.4.1 we can find that there exists $\mathbf{v} \in L^2(\Omega; L^\infty(0,T;L^2))$ such that as $h_n \to 0$

$$r_{h_n} D^+ \mathbf{u}^{h_n} \to \mathbf{v} \quad \text{weak}^\star \text{ in } L^2(\Omega; L^\infty(0,T;L^2)), \tag{3.8.17}$$

hence (for a subsequence still denoted as h_n)

$$r_{h_n} D^+ \mathbf{u}^{h_n} \to \mathbf{v} \quad \text{weakly in } \mathcal{H}_m.$$

Therefore, there exists $\mathbf{v} \in \mathcal{H}_m$ such that as $h_n \to 0$

$$r_{h_n} D^+ \mathbf{u}^{h_n} \to \mathbf{v} \quad \text{weakly in } \mathcal{H}_m.$$

From the proof of Lemma 3.8.4, we have that $\mathbf{v} = \nabla \mathbf{u}$. Then, (3.8.17) implies

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\nabla \mathbf{u}(t)|_{L^2}^2 \right] < \infty.$$

Similarly, using Lemma 3.4.1 we can find that there exists $\mathbf{w} \in L^2(\Omega; L^2(0,T;L^2))$ such that as $h_n \to 0$

$$r_{h_n} \tilde{\Delta} \mathbf{u}^{h_n} \to \mathbf{w} \quad \text{weak}^\star \ \text{ in } L^2(\Omega; L^2(0,T;L^2)). \tag{3.8.18}$$

Hence, there exists $\mathbf{w} \in \mathcal{H}_m$ such that as $h_n \to 0$

$$r_{h_n} \tilde{\Delta} \mathbf{u}^{h_n} \to \mathbf{w} \quad \text{weakly in } \mathcal{H}_m.$$

From Lemma 3.8.4, we have that $\mathbf{w} = \Delta \mathbf{u}$. Then, (3.8.18) implies

$$\mathbb{E}\left[\int_0^T |\Delta \mathbf{u}(t)|_{L^2}^2 \, dt \right] < \infty$$

and property (3) of Definition 1.3.1 is satisfied. Finally, property (4) follows from (3.8.16) and the fact that each term of the equation is in L_m^2.

Remark 3.8.7. In fact, inspection of the proof of the theorem shows that the process $\mathbf{u} - \mathbf{u}_0$ has a continuous modification in $L^2(\mathbb{R})$.

Chapter 4

Reduction of the Whole Real Line to a Bounded Interval

4.1 Introduction

Throughout the rest of this dissertation, we aim to solve numerically the one-dimensional stochastic problem (3.1.1)-(3.1.3). We recall that this problem is posed on $\mathbb{R}$ with homogeneous Neumann boundary conditions at infinity. In order to solve (3.1.1)-(3.1.3) numerically, a truncation of the domain $\mathbb{R}$ to some bounded domain is necessary. This also requires boundary conditions for $\mathbf{u}$. In this chapter, we truncate the infinite domain to a bounded domain $[-L, L]$ and we employ physically relevant Neumann boundary conditions for $\mathbf{u}$. Then, we prove that when the domain $[-L, L]$ is large enough the solution $\mathbf{u}_L$ of the problem on this bounded domain approximates the solution $\mathbf{u}$ of the original problem on $\mathbb{R}$.

We truncate the domain $\mathbb{R}$ into a bounded domain $[-L, L]$ and impose homogeneous Neumann boundary conditions. In this case, the stochastic LLG equation takes the form

$$\begin{aligned} d\mathbf{u}_L = \Big(\mu\mathbf{u}_L \times \Delta\mathbf{u}_L - \lambda\mathbf{u}_L \times \big(\mathbf{u}_L \times \Delta\mathbf{u}_L\big) + \frac{\mu^2}{2}\big(\mathbf{u}_L \times \mathbf{g}\big) \times \mathbf{g}\Big)\, dt \\ + \mu\big(\mathbf{u}_L \times \mathbf{g}\big)dW \qquad \text{on } (0,T) \times (-L, L), \end{aligned} \tag{4.1.1}$$

$$\mathbf{u}_L(0, x) = \mathbf{u}_0^L(x), \quad x \in [-L, L], \tag{4.1.2}$$

$$|\mathbf{u}_0^L(x)| = 1, \quad x \in [-L, L], \tag{4.1.3}$$

$$\frac{\partial \mathbf{u}_L}{\partial x}(t, \pm L) = 0, \quad t \in (0, T], \tag{4.1.4}$$

where $\mathbf{u}_L : [0,T] \times [-L,L] \to \mathbb{R}^3$.

The existence and uniqueness of solutions to this equation and their regularity was proved in [12]. The folowing result summarises Theorems 3.1, 4.1 and 5.2 in [12].

Theorem 4.1.1. *Let $\mathbf{u}_0^L$ denote the function $\mathbf{u}_0$ restricted to the interval $[-L,L]$. Assume that $|\mathbf{u}_0(x)| = 1$ for $x \in \mathbb{R}$, $\nabla \mathbf{u}_0 \in L^2(\mathbb{R})$ and $\mathbf{g} \in W^{1,\infty}(\mathbb{R})$. Then equation (4.1.1) has a unique strong solution $\mathbf{u}_L$, such that*

$$|\mathbf{u}_L(t,x)| = 1 \tag{4.1.5}$$

for all $t \in [0,T]$ and $x \in [-L,L]$. Moreover, for every $p \geq 1$

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\nabla \mathbf{u}_L(t)|^{2p}_{L^2(-L,L)}\right] + \mathbb{E}\left[\left(\int_0^T |\Delta \mathbf{u}_L(t)|^2_{L^2(-L,L)}\, dt\right)^p\right] < \infty\,. \tag{4.1.6}$$

We note that (3.1.5) where the whole real line is considered confer estimate (4.1.6) obtained in [12].

The main theorem of this chapter is stated as follows.

Theorem 4.1.2. *Let $\mathbf{u} : [0,T] \times \mathbb{R} \to \mathbb{R}^3$ be the unique global strong solution to (3.1.1)-(3.1.3). Assume that $|\mathbf{u}_0(x)| = 1$ for all $x \in \mathbb{R}$, $\nabla \mathbf{u}_0 \in L^2(\mathbb{R})$ and $\mathbf{g} \in W^{1,\infty}(\mathbb{R}) \cap H^2(\mathbb{R})$. Then, for every sequence $(L_n)_{n\geq 0}$ such that*

$$\sum_{n=1}^{\infty} \frac{1}{L_n^p} < \infty \tag{4.1.7}$$

for $1 \leq p < \infty$, we have for every $m > 1$

$$\lim_{n\to\infty} \left(\sup_{t\in[0,T]} \int_{-L}^{L} |\mathbf{u}(t,x) - \mathbf{u}_{L_n}(t,x)|^2\, \rho_m(x)\, dx\right) = 0 \quad \mathbb{P}\text{-a.s.}.$$

for every $L > 0$. Moreover, for a fixed $\alpha > 0$ small enough and every $m > 1$, there exists a random variable C independent of n and such that

$$\sup_{t\in[0,T]} \int_{-L}^{L} |\mathbf{u}(t,x) - \mathbf{u}_{L_n}(t,x)|^2 \rho_m(x) dx \leq C e^{-\frac{\alpha L_n}{2m}}.$$

4.2 A Priori Estimates

In this section, we will improve a priori estimates in (4.1.6). They will be used in the next section to prove Theorem 4.1.2. In the following, we denote $D_l := (-l, l)$ for any $l > 0$.

Lemma 4.2.1. *Let* $\mathbf{u}_0$ *and* $\mathbf{g}$ *satisfy the assumptions of Theorem 4.1.2. Suppose that* $L \geq 1$ *and let* $\mathbf{u}_L$ *be the corresponding solution to equation* (4.1.1). *Then, for every* $T > 0$ *and* $p \geq 1$

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\nabla \mathbf{u}_L|^{2p}_{L^2(D_L)}\Big] \leq C, \tag{4.2.1}$$

$$\mathbb{E}\Big[\Big(\int_0^T |\Delta \mathbf{u}_L|^2_{L^2(D_L)} ds\Big)^p\Big] \leq C, \tag{4.2.2}$$

with C *depending only on* $|\nabla \mathbf{u}_0|_{L^2}, |\mathbf{g}|_{H^1}$, T *and* p *but not on* L.

Proof. We assume in the proof, without loss of generality, that $\lambda = \mu = 1$. First, we prove (4.2.1). Applying ∇ to (4.1.1), we get

$$\begin{aligned} d\nabla \mathbf{u}_L &= \nabla(\mathbf{u}_L \times \Delta \mathbf{u}_L)dt - \nabla\left(\mathbf{u}_L \times (\mathbf{u}_L \times \Delta \mathbf{u}_L)\right)dt + \frac{1}{2}\nabla\left((\mathbf{u}_L \times \mathbf{g}) \times \mathbf{g}\right)dt \\ &\quad + \nabla\big(\mathbf{u}_L \times \mathbf{g}\big)dW. \end{aligned}$$

Then, by using Lemma 2.3.17, we obtain for every $x \in D_L$

$$\begin{aligned} d\big|\nabla \mathbf{u}_L\big|^2 &= 2\left\langle \nabla \mathbf{u}_L, \nabla\big(\mathbf{u}_L \times \Delta \mathbf{u}_L\big)\right\rangle dt - 2\left\langle \nabla \mathbf{u}_L, \nabla\big(\mathbf{u}_L \times \big(\mathbf{u}_L \times \Delta \mathbf{u}_L\big)\big)\right\rangle dt \\ &\quad + \left\langle \nabla \mathbf{u}_L, \nabla\big(\big(\mathbf{u}_L \times \mathbf{g}\big) \times \mathbf{g}\big)\right\rangle dt + 2\left\langle \nabla \mathbf{u}_L, \nabla\big(\mathbf{u}_L \times \mathbf{g}\big)\right\rangle dW + \big|\nabla\big(\mathbf{u}_L \times \mathbf{g}\big)\big|^2 dt. \end{aligned}$$

Integrating with respect to $x \in D_L$, we obtain

$$\begin{aligned} d\left(\big|\nabla \mathbf{u}_L\big|^2_{L^2(D_L)}\right) &= \left(2\int_{-L}^{L} \left\langle \nabla \mathbf{u}_L, \nabla\big(\mathbf{u}_L \times \Delta \mathbf{u}_L\big)\right\rangle dx\right) dt \\ &\quad - \left(2\int_{-L}^{L} \left\langle \nabla \mathbf{u}_L, \nabla\big(\mathbf{u}_L \times \big(\mathbf{u}_L \times \Delta \mathbf{u}_L\big)\big)\right\rangle dx\right) dt \end{aligned}$$

$$
+\left(\int_{-L}^{L}\big\langle \nabla\mathbf{u}_L, \nabla\big(\big(\mathbf{u}_L\times\mathbf{g}\big)\times\mathbf{g}\big)\big\rangle\,dx\right)dt
$$
$$
+\left(2\int_{-L}^{L}\big\langle \nabla\mathbf{u}_L, \nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big\rangle\,dx\right)dW+\big|\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big|^2_{L^2(D_L)}dt.
$$

Using integration by parts and the elementary properties (2.2.1) and (2.2.2), we get

$$
\begin{aligned}
d\left(\big|\nabla\mathbf{u}_L\big|^2_{L^2(D_L)}\right) &= \left(2\int_{-L}^{L}\big\langle \Delta\mathbf{u}_L, \mathbf{u}_L\times\big(\mathbf{u}_L\times\Delta\mathbf{u}_L\big)\big\rangle\,dx\right)dt\\
&\quad+\left(\int_{-L}^{L}\big\langle \nabla\mathbf{u}_L, \nabla\big(\big(\mathbf{u}_L\times\mathbf{g}\big)\times\mathbf{g}\big)\big\rangle\,dx\right)dt\\
&\quad+\left(2\int_{-L}^{L}\big\langle \nabla\mathbf{u}_L, \nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big\rangle\,dx\right)dW+\big|\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big|^2_{L^2(D_L)}dt\\
&= -2\left|\mathbf{u}_L\times\Delta\mathbf{u}_L\right|^2_{L^2(D_L)}dt+\left(\int_{-L}^{L}\big\langle \nabla\mathbf{u}_L, \nabla\big(\big(\mathbf{u}_L\times\mathbf{g}\big)\times\mathbf{g}\big)\big\rangle\,dx\right)dt\\
&\quad+\left(2\int_{-L}^{L}\big\langle \nabla\mathbf{u}_L, \nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big\rangle\,dx\right)dW+\big|\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big|^2_{L^2(D_L)}dt.
\end{aligned}
$$

We obtain for all $t\in[0,T]$,

$$
\begin{aligned}
&\big|\nabla\mathbf{u}_L(t)\big|^2_{L^2(D_L)}-\big|\nabla\mathbf{u}_0\big|^2_{L^2(D_L)}+2\int_0^t\big|\mathbf{u}_L\times\Delta\mathbf{u}_L\big|^2_{L^2(D_L)}ds\\
&=\int_0^t\int_{-L}^{L}\big\langle \nabla\mathbf{u}_L, \nabla\big(\big(\mathbf{u}_L\times\mathbf{g}\big)\times\mathbf{g}\big)\big\rangle\,dxds+2\int_0^t\int_{-L}^{L}\big\langle \nabla\mathbf{u}_L, \nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big\rangle\,dxdW(s)\\
&\quad+\int_0^t\big|\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big|^2_{L^2(D_L)}ds.
\end{aligned}\tag{4.2.3}
$$

Since

$$
2\int_0^t\big|\mathbf{u}_L\times\Delta\mathbf{u}_L\big|^2_{L^2(D_L)}ds\ge 0,
$$

we get

$$
\begin{aligned}
&\big|\nabla\mathbf{u}_L(t)\big|^2_{L^2(D_L)}\\
&\le\big|\nabla\mathbf{u}_0\big|^2_{L^2(D_L)}+\int_0^t|\nabla\mathbf{u}_L|_{L^2(D_L)}|\nabla\big(\big(\mathbf{u}_L\times\mathbf{g}\big)\times\mathbf{g}\big)|_{L^2(D_L)}ds\\
&\quad+2\left|\int_0^t\int_{-L}^{L}\big\langle \nabla\mathbf{u}_L, \nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big\rangle\,dxdW(s)\right|+\int_0^t\big|\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big|^2_{L^2(D_L)}ds\\
&\le\big|\nabla\mathbf{u}_0\big|^2_{L^2(D_L)}+C\int_0^t\left(|\nabla\mathbf{u}_L|^2_{L^2(D_L)}+\big|\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big|^2_{L^2(D_L)}+|\nabla\big(\big(\mathbf{u}_L\times\mathbf{g}\big)\times\mathbf{g}\big)|^2_{L^2(D_L)}\right)ds
\end{aligned}
$$

$$+2\left|\int_0^t\int_{-L}^{L}\left\langle\nabla\mathbf{u}_L,\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\right\rangle dx dW(s)\right|. \tag{4.2.4}$$

The middle term in the integrand of the first integral on the right hand side can be estimated by using (4.1.5) as follows

$$\begin{aligned}
\big|\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\big|^2_{L^2(D_L)} &\le 2\big|\nabla\mathbf{u}_L\times\mathbf{g}\big|^2_{L^2(D_L)}+2\big|\mathbf{u}_L\times\nabla\mathbf{g}\big|^2_{L^2(D_L)}\\
&\le 2\big|\nabla\mathbf{u}_L\big|^2_{L^2(D_L)}|\mathbf{g}|^2_{L^\infty(D_L)}+2\big|\nabla\mathbf{g}\big|^2_{L^2(D_L)}\\
&\le C\big|\nabla\mathbf{u}_L\big|^2_{L^2(D_L)}+C
\end{aligned} \tag{4.2.5}$$

where the constants depend only on $|\mathbf{g}|_{H^1}$. Similarly, we can show that

$$\big|\nabla\big(\big(\mathbf{u}_L\times\mathbf{g}\big)\times\mathbf{g}\big)\big|^2_{L^2(D_L)}\le C\big|\nabla\mathbf{u}_L\big|^2_{L^2(D_L)}+C.$$

Hence, it follows from (4.2.4) that

$$\begin{aligned}
\big|\nabla\mathbf{u}_L(t)\big|^2_{L^2(D_L)} \le \big|\nabla\mathbf{u}_0\big|^2_{L^2(D_L)}+Ct+C\int_0^t|\nabla\mathbf{u}_L|^2_{L^2(D_L)}ds\\
+C\left|\int_0^t\int_{-L}^{L}\left\langle\nabla\mathbf{u}_L,\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\right\rangle dx dW(s)\right|.
\end{aligned}$$

By raising both sides to the power p and using Jensen's inequality (3.3.18), we deduce

$$\begin{aligned}
\big|\nabla\mathbf{u}_L(t)\big|^{2p}_{L^2(D_L)} &\le C+C\left(\int_0^t|\nabla\mathbf{u}_L|^2_{L^2(D_L)}ds\right)^p\\
&\quad+C\left|\int_0^t\int_{-L}^{L}\left\langle\nabla\mathbf{u}_L,\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\right\rangle dx dW(s)\right|^p\\
&\le C+C\int_0^t|\nabla\mathbf{u}_L|^{2p}_{L^2(D_L)}ds\\
&\quad+C\left|\int_0^t\int_{-L}^{L}\left\langle\nabla\mathbf{u}_L,\nabla\big(\mathbf{u}_L\times\mathbf{g}\big)\right\rangle dx dW(s)\right|^p
\end{aligned}$$

where in the last step we used Hölder's inequality. The constants depend on $|\nabla\mathbf{u}_0|_{L^2}$, $|\mathbf{g}|_{H^1}$, T and p. Therefore,

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,t]}\big|\nabla\mathbf{u}_L(s)\big|^{2p}_{L^2(D_L)}\Big] &\le C + C\int_0^t \mathbb{E}\Big[|\nabla\mathbf{u}_L(s)|^{2p}_{L^2(D_L)}\Big]ds \\
&\quad + C\mathbb{E}\Big[\sup_{s\in[0,t]}\Big|\int_0^s\int_{-L}^{L}\big\langle\nabla\mathbf{u}_L(\tau),\nabla\big(\mathbf{u}_L(\tau)\times\mathbf{g}\big)\big\rangle\,dxdW(\tau)\Big|^p\Big] \\
&\le C + C\int_0^t \mathbb{E}\Big[|\nabla\mathbf{u}_L(s)|^{2p}_{L^2(D_L)}\Big]ds \\
&\quad + C\mathbb{E}\Big[\Big(\int_0^t\Big|\int_{-L}^{L}\big\langle\nabla\mathbf{u}_L(s),\nabla\big(\mathbf{u}_L(s)\times\mathbf{g}\big)\big\rangle\,dx\Big|^2ds\Big)^{\frac{p}{2}}\Big]
\end{aligned}
$$

where in the last step we used the BDG inequality (Lemma 2.3.18). It follows from (4.2.5) that

$$
\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[0,t]}\big|\nabla\mathbf{u}_L(s)\big|^{2p}_{L^2(D_L)}\Big] \\
&\le C + C\int_0^t \mathbb{E}\Big[|\nabla\mathbf{u}_L(s)|^{2p}_{L^2(D_L)}\Big]ds + C\mathbb{E}\Big[\Big(\int_0^t\Big(|\nabla\mathbf{u}_L(s)|^2_{L^2(D_L)} + |\nabla\mathbf{u}_L(s)|^4_{L^2(D_L)}\Big)ds\Big)^{\frac{p}{2}}\Big] \\
&\le C + C\int_0^t \mathbb{E}\Big[|\nabla\mathbf{u}_L(s)|^{2p}_{L^2(D_L)}\Big]ds + C\mathbb{E}\Big[\Big(\int_0^t\Big(1 + |\nabla\mathbf{u}_L(s)|^4_{L^2(D_L)}\Big)ds\Big)^{\frac{p}{2}}\Big] \\
&\le C + C\int_0^t \mathbb{E}\Big[|\nabla\mathbf{u}_L(s)|^{2p}_{L^2(D_L)}\Big]ds + C(T) + C\mathbb{E}\Big[\Big(\int_0^t|\nabla\mathbf{u}_L(s)|^4_{L^2(D_L)}ds\Big)^{\frac{p}{2}}\Big] \qquad (4.2.6)
\end{aligned}
$$

for $p \ge 2$ where in the last step we used Jensen's inequality (3.3.18). Next, Hölder's inequality implies

$$
\begin{aligned}
E\Big[\Big(\int_0^t|\nabla\mathbf{u}_L(s)|^4_{L^2(D_L)}ds\Big)^{\frac{p}{2}}\Big] &\le C(T)\mathbb{E}\Big[\int_0^t|\nabla\mathbf{u}_L(s)|^{2p}_{L^2(D_L)}ds\Big] \\
&\le C(T)\int_0^t\mathbb{E}\Big[|\nabla\mathbf{u}_L(s)|^{2p}_{L^2(D_L)}\Big]ds. \qquad (4.2.7)
\end{aligned}
$$

Hence, inequalities (4.2.6) and (4.2.7) yield

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{s\in[0,t]}\big|\nabla\mathbf{u}_L(s)\big|^{2p}_{L^2(D_L)}\Big] &\le C + C\int_0^t \mathbb{E}\Big[|\nabla\mathbf{u}_L(s)|^{2p}_{L^2(D_L)}\Big]ds \\
&\le C + C\int_0^t \mathbb{E}\Big[\sup_{\tau\in[0,s]}|\nabla\mathbf{u}_L(\tau)|^{2p}_{L^2(D_L)}\Big]ds.
\end{aligned}
$$

Then, by using Gronwall's inequality (2.2.7), we obtain for $p \geq 2$

$$\mathbb{E}\left[\sup_{s\in[0,t]} \left|\nabla \mathbf{u}_L(s)\right|^{2p}_{L^2(D_L)}\right] \leq C$$

where the constant depends on $|\nabla \mathbf{u}_0|_{L^2}, |\mathbf{g}|_{H^1}$, T and p but not on L. Finally, the inequality $a \leq \frac{1+a^2}{2}$ implies the result for $p \geq 1$ and the proof of (4.2.1) is complete.

Next, we prove (4.2.2). By using the elementary property (2.2.4) and from (4.1.5), we have

$$|\Delta \mathbf{u}_L|^2 = |\mathbf{u}_L \times \Delta \mathbf{u}_L|^2 + \langle \mathbf{u}_L, \Delta \mathbf{u}_L\rangle^2 .$$

Then, by using $\langle \mathbf{u}_L, \Delta \mathbf{u}_L\rangle = -\left|\nabla \mathbf{u}_L\right|^2$, we deduce

$$|\Delta \mathbf{u}_L|^2 = |\mathbf{u}_L \times \Delta \mathbf{u}_L|^2 + \left|\nabla \mathbf{u}_L\right|^4.$$

Integrating with respect to $x \in D_L$, we get

$$\left|\Delta \mathbf{u}_L\right|^2_{L^2(D_L)} = \left|\mathbf{u}_L \times \Delta \mathbf{u}_L\right|^2_{L^2(D_L)} + \left|\nabla \mathbf{u}_L\right|^4_{L^4(D_L)}.$$

Integrating with respect to $t \in [0, T]$, raising to the power p and applying expectation we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T \left|\Delta \mathbf{u}_L\right|^2_{L^2(D_L)} ds\right)^p\right] \\
&\leq C\mathbb{E}\left[\left(\int_0^T \left|\mathbf{u}_L \times \Delta \mathbf{u}_L\right|^2_{L^2(D_L)} ds\right)^p\right] + C\mathbb{E}\left[\left(\int_0^T \left|\nabla \mathbf{u}_L\right|^4_{L^4(D_L)} ds\right)^p\right] \\
&\leq S_1 + S_2 \qquad (4.2.8)
\end{aligned}$$

where the constants depend only on p. In order to estimate S_1, we proceed from (4.2.3) and write down the same inequalities but with $\int_0^t |\mathbf{u}_L \times \Delta \mathbf{u}_L|^2_{L^2(D_L)} ds$ instead of $|\nabla \mathbf{u}_L(t)|^2_{L^2(D_L)}$ on the left hand side. In fact, we have

$$\mathbb{E}\left[\left(\int_0^t \left|\mathbf{u}_L \times \Delta \mathbf{u}_L\right|^2_{L^2(D_L)} ds\right)^p\right] \leq C + C\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]} \left|\nabla \mathbf{u}_L(r)\right|^{2p}_{L^2(D_L)}\right] ds.$$

From (4.2.1), we deduce

$$\mathbb{E}\Big[\Big(\int_0^t \big|\mathbf{u}_L \times \Delta \mathbf{u}_L\big|^2_{L^2(D_L)} ds\Big)^p\Big] \leq C \tag{4.2.9}$$

where the constant C depends only on $|\nabla \mathbf{u}_0|_{L^2}$, $|\mathbf{g}|_{H^1}$, T and p but not on L. Now, we estimate S_2. We know that

$$\big|\nabla \mathbf{u}_L\big|^4_{L^4(D_L)} \leq \big|\nabla \mathbf{u}_L\big|^2_{L^\infty(D_L)} \big|\nabla \mathbf{u}_L\big|^2_{L^2(D_L)}. \tag{4.2.10}$$

Since $L \geq 1$, Lemma 2.1.1 yields

$$\begin{aligned}
\big|\nabla \mathbf{u}_L\big|^2_{L^\infty(D_L)} &\leq 2\sqrt{2}\big|\nabla \mathbf{u}_L\big|_{L^2(D_L)} \left(\big|\nabla \mathbf{u}_L\big|^2_{L^2(D_L)} + \big|\Delta \mathbf{u}_L\big|^2_{L^2(D_L)}\right)^{1/2} \\
&\leq 2\sqrt{2}\big|\nabla \mathbf{u}_L\big|_{L^2(D_L)} \left(\big|\nabla \mathbf{u}_L\big|_{L^2(D_L)} + \big|\Delta \mathbf{u}_L\big|_{L^2(D_L)}\right).
\end{aligned}$$

Then, from (4.2.10) we get

$$\begin{aligned}
\big|\nabla \mathbf{u}_L\big|^4_{L^4(D_L)} &\leq 2\sqrt{2}\big|\nabla \mathbf{u}_L\big|^4_{L^2(D_L)} + 2\sqrt{2}\big|\nabla \mathbf{u}_L\big|^3_{L^2(D_L)}\big|\Delta \mathbf{u}_L\big|_{L^2(D_L)} \\
&\leq 2\sqrt{2}\big|\nabla \mathbf{u}_L\big|^4_{L^2(D_L)} + C_\varepsilon\big|\nabla \mathbf{u}_L\big|^6_{L^2(D_L)} + \varepsilon\big|\Delta \mathbf{u}_L\big|^2_{L^2(D_L)}.
\end{aligned}$$

Integrating with respect to time, raising to the power p and using Jensen's inequality (3.3.18), we get for $\varepsilon \in (0,1)$

$$\begin{aligned}
\Big(\int_0^T \big|\nabla \mathbf{u}_L\big|^4_{L^4(D_L)} ds\Big)^p &\leq \Big(2\sqrt{2}\int_0^T \big|\nabla \mathbf{u}_L\big|^4_{L^2(D_L)} ds + C_\varepsilon \int_0^T \big|\nabla \mathbf{u}_L\big|^6_{L^2(D_L)} ds \\
&\quad + \varepsilon \int_0^T \big|\Delta \mathbf{u}_L\big|^2_{L^2(D_L)} ds\Big)^p \\
&\leq C \sup_{t\in[0,T]} \big|\nabla \mathbf{u}_L(t)\big|^{4p}_{L^2(D_L)} + C_\varepsilon \sup_{t\in[0,T]} \big|\nabla \mathbf{u}_L(t)\big|^{6p}_{L^2(D_L)} \\
&\quad + \varepsilon\Big(\int_0^T \big|\Delta \mathbf{u}_L\big|^2_{L^2(D_L)} ds\Big)^p.
\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{E}\Big[\Big(\int_0^T |\nabla \mathbf{u}_L|^4_{L^4(D_L)} ds\Big)^p\Big] &\le C\mathbb{E}\Big[\sup_{t\in[0,T]} |\nabla \mathbf{u}_L(t)|^{4p}_{L^2(D_L)}\Big] + C_\varepsilon \mathbb{E}\Big[\sup_{t\in[0,T]} |\nabla \mathbf{u}_L(t)|^{6p}_{L^2(D_L)}\Big] \\ &\quad + \varepsilon \mathbb{E}\Big[\Big(\int_0^T |\Delta \mathbf{u}_L|^2_{L^2(D_L)} ds\Big)^p\Big] \\ &\le C + \varepsilon \mathbb{E}\Big[\Big(\int_0^T |\Delta \mathbf{u}_L|^2_{L^2(D_L)} ds\Big)^p\Big] \qquad (4.2.11)\end{aligned}$$

where in the last step we used (4.2.1). Then, inequalities (4.2.8), (4.2.9) and (4.2.11) yield (4.2.2) where the constant C depends only on $|\nabla \mathbf{u}_0|_{L^2}$, $|\mathbf{g}|_{H^1}$, T and p but not on L. This completes the proof of the lemma. □

4.3 Proof of the Main Theorem

We are ready now to prove the main result of this chapter, Theorem 4.1.2. We assume in the proof without loss of generality, that $\lambda = \mu = 1$. We consider a sequence $(L_n)_{n\ge 0}$ verifying assumption (4.1.7) and denote $L_n^+ := L_n + 1$. We consider $\phi_n : \mathbb{R} \to [0,1]$ such that $\phi_n \in C_c^\infty(\mathbb{R})$ and

$$\phi_n(x) = \begin{cases} 1 & x \in D_{L_n}, \\ 0 & x \ge (L_n+1) \text{ or } x \le -(L_n+1). \end{cases}$$

We multiply (3.1.1) by ϕ_n, use the elementary property (2.2.3) and property (2) in Definition 1.3.1 to get

$$d(\mathbf{u}\phi_n) = \Big(\phi_n \mathbf{u} \times \Delta \mathbf{u} - \phi_n \langle \mathbf{u}, \Delta \mathbf{u}\rangle \mathbf{u} + \phi_n \Delta \mathbf{u} + \frac{1}{2}\phi_n (\mathbf{u}\times \mathbf{g})\times \mathbf{g}\Big) dt + \phi_n\big(\mathbf{u}\times \mathbf{g}\big) dW.$$

Furthermore, by using (2.2.3) and (4.1.5), we obtain from equation (4.1.1) on $D_{L_n^+}$

$$\begin{aligned} d\mathbf{u}_{L_n^+} &= \Big(\mathbf{u}_{L_n^+} \times \Delta \mathbf{u}_{L_n^+} - \big\langle \mathbf{u}_{L_n^+}, \Delta \mathbf{u}_{L_n^+}\big\rangle \mathbf{u}_{L_n^+} + \Delta \mathbf{u}_{L_n^+} + \frac{1}{2}\big(\mathbf{u}_{L_n^+} \times \mathbf{g}\big)\times \mathbf{g}\Big) dt \\ &\quad + \big(\mathbf{u}_{L_n^+}\times \mathbf{g}\big) dW. \end{aligned}$$

In the following, we denote

$$\bar{\mathbf{u}} := \mathbf{u}\phi_n - \mathbf{u}_{L_n^+}.$$

Substracting the two equations above and using $\langle \mathbf{u}, \Delta\mathbf{u}\rangle = -\big|\nabla\mathbf{u}\big|^2$, we get

$$\begin{aligned}
d\bar{\mathbf{u}} = \Big(&\bar{\mathbf{u}} \times \Delta\mathbf{u} + \mathbf{u}_{L_n^+} \times \Delta\bar{\mathbf{u}} + \mathbf{u}_{L_n^+} \times \Delta(\mathbf{u} - \mathbf{u}\phi_n) + \big|\nabla\mathbf{u}\big|^2\bar{\mathbf{u}} \\
&+ \big\langle \nabla\bar{\mathbf{u}}, \nabla\mathbf{u} + \nabla\mathbf{u}_{L_n^+}\big\rangle\, \mathbf{u}_{L_n^+} + \big\langle \nabla(\mathbf{u} - \mathbf{u}\phi_n), \nabla\mathbf{u} + \nabla\mathbf{u}_{L_n^+}\big\rangle\, \mathbf{u}_{L_n^+} \\
&+ (\phi_n - 1)\Delta\mathbf{u} + \Delta(\mathbf{u} - \mathbf{u}\phi_n) + \Delta\bar{\mathbf{u}} + \frac{1}{2}((\bar{\mathbf{u}} \times \mathbf{g}) \times \mathbf{g})\Big)dt \\
&+ (\bar{\mathbf{u}} \times \mathbf{g})dW, \qquad \forall x \in D_{L_n^+}.
\end{aligned}$$

Multiplying both sides by $\zeta_m(x) = e^{-\frac{|x|}{2m}}$, using Lemma 2.3.17 and the elementary property (2.2.1) we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\zeta_m\bar{\mathbf{u}}|^2 =&\frac{1}{2}|\zeta_m\bar{\mathbf{u}} \times \mathbf{g}|^2 \\
&+ \langle \zeta_m\bar{\mathbf{u}}, \zeta_m\Delta\bar{\mathbf{u}}\rangle + \langle \zeta_m\bar{\mathbf{u}}, (\phi_n - 1)\zeta_m\Delta\mathbf{u}\rangle + \langle \zeta_m\bar{\mathbf{u}}, \zeta_m\Delta(\mathbf{u} - \mathbf{u}\phi_n)\rangle \\
&+ \big\langle \zeta_m\bar{\mathbf{u}}, \zeta_m\mathbf{u}_{L_n^+} \times \Delta\bar{\mathbf{u}}\big\rangle + \big\langle \zeta_m\bar{\mathbf{u}}, \zeta_m\mathbf{u}_{L_n^+} \times \Delta(\mathbf{u} - \mathbf{u}\phi_n)\big\rangle \\
&+ \langle \zeta_m\bar{\mathbf{u}}, \zeta_m\bar{\mathbf{u}}\rangle\, \big|\nabla\mathbf{u}\big|^2 + \big\langle \zeta_m\bar{\mathbf{u}}, \big\langle \nabla\bar{\mathbf{u}}, \nabla\mathbf{u} + \nabla\mathbf{u}_{L_n^+}\big\rangle\, \zeta_m\mathbf{u}_{L_n^+}\big\rangle \\
&+ \big\langle \zeta_m\bar{\mathbf{u}}, \big\langle \nabla(\mathbf{u} - \mathbf{u}\phi_n), \nabla\mathbf{u} + \nabla\mathbf{u}_{L_n^+}\big\rangle\, \zeta_m\mathbf{u}_{L_n^+}\big\rangle + \frac{1}{2}\,\langle \zeta_m\bar{\mathbf{u}}, (\zeta_m\bar{\mathbf{u}} \times \mathbf{g}) \times \mathbf{g}\rangle\,.
\end{aligned} \tag{4.3.1}$$

For the second term on the right hand side of (4.3.1), using the fact that $\nabla\bar{\mathbf{u}}(\pm L_n^+) = 0$, we obtain

$$\begin{aligned}
\int_{-L_n^+}^{L_n^+} \langle \zeta_m\bar{\mathbf{u}}, \zeta_m\Delta\bar{\mathbf{u}}\rangle\, dx &= \int_{-L_n^+}^{L_n^+} \big\langle \zeta_m^2\bar{\mathbf{u}}, \Delta\bar{\mathbf{u}}\big\rangle\, dx = -\int_{-L_n^+}^{L_n^+} \big\langle \nabla(\zeta_m^2\bar{\mathbf{u}}), \nabla\bar{\mathbf{u}}\big\rangle\, dx \\
&= -2\int_{-L_n^+}^{L_n^+} \langle \zeta_m\bar{\mathbf{u}}, \nabla\zeta_m\nabla\bar{\mathbf{u}}\rangle\, dx - \int_{-L_n^+}^{L_n^+} |\zeta_m\nabla\bar{\mathbf{u}}|^2\, dx.
\end{aligned} \tag{4.3.2}$$

For the fifth term on the right hand side of (4.3.1), using elementary properties (2.2.1), (2.2.5) and the fact that $\nabla\bar{\mathbf{u}}(\pm L_n^+) = 0$ we get

$$\begin{aligned}
&\int_{-L_n^+}^{L_n^+} \left\langle \zeta_m \bar{\mathbf{u}}, \zeta_m \mathbf{u}_{L_n^+} \times \Delta\bar{\mathbf{u}} \right\rangle dx \\
&= \int_{-L_n^+}^{L_n^+} \left\langle \Delta\bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \times \zeta_m \mathbf{u}_{L_n^+} \right\rangle dx \\
&= -\int_{-L_n^+}^{L_n^+} \left\langle \nabla\bar{\mathbf{u}}, \nabla(\zeta_m^2 \bar{\mathbf{u}} \times \mathbf{u}_{L_n^+}) \right\rangle dx \\
&= -2\int_{-L_n^+}^{L_n^+} \left\langle \nabla\bar{\mathbf{u}}, \zeta_m \nabla\zeta_m \bar{\mathbf{u}} \times \mathbf{u}_{L_n^+} \right\rangle dx - \int_{-L_n^+}^{L_n^+} \left\langle \nabla\bar{\mathbf{u}}, \zeta_m^2 \nabla\bar{\mathbf{u}} \times \mathbf{u}_{L_n^+} \right\rangle dx \\
&\quad - \int_{-L_n^+}^{L_n^+} \left\langle \nabla\bar{\mathbf{u}}, \zeta_m^2 \bar{\mathbf{u}} \times \nabla\mathbf{u}_{L_n^+} \right\rangle dx \\
&= -2\int_{-L_n^+}^{L_n^+} \left\langle \mathbf{u}_{L_n^+} \times \nabla\zeta_m \nabla\bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \right\rangle dx - \int_{-L_n^+}^{L_n^+} \left\langle \nabla\mathbf{u}_{L_n^+} \times \zeta_m \nabla\bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \right\rangle dx. \qquad (4.3.3)
\end{aligned}$$

Integrating both sides of (4.3.1) with respect to $x \in D_{L_n^+}$, using (4.3.2) and (4.3.3) we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}|\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} + \left|\zeta_m \nabla\bar{\mathbf{u}}\right|^2_{L^2(D_{L_n^+})} \\
&= \frac{1}{2}\int_{-L_n^+}^{L_n^+} |\zeta_m \bar{\mathbf{u}} \times \mathbf{g}|^2 dx - 2\int_{-L_n^+}^{L_n^+} \langle \zeta_m \bar{\mathbf{u}}, \nabla\zeta_m \nabla\bar{\mathbf{u}} \rangle\, dx - 2\int_{-L_n^+}^{L_n^+} \left\langle \mathbf{u}_{L_n^+} \times \nabla\zeta_m \nabla\bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \right\rangle dx \\
&\quad - \int_{-L_n^+}^{L_n^+} \left\langle \nabla\mathbf{u}_{L_n^+} \times \zeta_m \nabla\bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \right\rangle dx + \int_{-L_n^+}^{L_n^+} \langle \zeta_m \bar{\mathbf{u}}, (\phi_n - 1)\zeta_m \Delta\mathbf{u} \rangle\, dx \\
&\quad + \int_{-L_n^+}^{L_n^+} \langle \zeta_m \bar{\mathbf{u}}, \zeta_m \Delta(\mathbf{u} - \mathbf{u}\phi_n) \rangle\, dx + \int_{-L_n^+}^{L_n^+} \left\langle \zeta_m \bar{\mathbf{u}}, \zeta_m \mathbf{u}_{L_n^+} \times \Delta(\mathbf{u} - \mathbf{u}\phi_n) \right\rangle dx \\
&\quad + \int_{-L_n^+}^{L_n^+} \langle \zeta_m \bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}} \rangle \left|\nabla\mathbf{u}\right|^2 dx + \int_{-L_n^+}^{L_n^+} \left\langle \zeta_m \bar{\mathbf{u}}, \left\langle \nabla\bar{\mathbf{u}}, \nabla\mathbf{u} + \nabla\mathbf{u}_{L_n^+} \right\rangle \zeta_m \mathbf{u}_{L_n^+} \right\rangle dx \\
&\quad + \int_{-L_n^+}^{L_n^+} \left\langle \zeta_m \bar{\mathbf{u}}, \left\langle \nabla(\mathbf{u} - \mathbf{u}\phi_n), \nabla\mathbf{u} + \nabla\mathbf{u}_{L_n^+} \right\rangle \zeta_m \mathbf{u}_{L_n^+} \right\rangle dx + \frac{1}{2}\int_{-L_n^+}^{L_n^+} \langle \zeta_m \bar{\mathbf{u}}, (\zeta_m \bar{\mathbf{u}} \times \mathbf{g}) \times \mathbf{g} \rangle\, dx.
\end{aligned}$$

(4.3.4)

We integrate with respect to t to obtain

$$\frac{1}{2}|\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} + \int_0^t \left|\zeta_m \nabla\bar{\mathbf{u}}\right|^2_{L^2(D_{L_n^+})} ds = \frac{1}{2}|\zeta_m \bar{\mathbf{u}}(0)|^2_{L^2(D_{L_n^+})} + \sum_{k=1}^{11} N_k\,.$$

We note first that, using the elementary property (2.2.2), we have $N_1 + N_{11} = 0$. We will estimate each term $N_k (k = 2, ..., 10)$ separately. For N_2,

$$\begin{aligned}
N_2(t) &= -2\int_0^t \int_{-L_n^+}^{L_n^+} \langle \zeta_m \bar{\mathbf{u}}, \nabla\zeta_m \nabla\bar{\mathbf{u}}\rangle \, dxds \\
&\le 2\int_0^t |\zeta_m \bar{\mathbf{u}}|_{L^2(D_{L_n^+})} |\nabla\zeta_m \nabla\bar{\mathbf{u}}|_{L^2(D_{L_n^+})} ds \\
&\le C\int_0^t |\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} ds + \frac{1}{4}\int_0^t |\nabla\zeta_m \nabla\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} ds.
\end{aligned}$$

For N_3, using (4.1.5) we obtain

$$\begin{aligned}
N_3(t) &= -2\int_0^t \int_{-L_n^+}^{L_n^+} \left\langle \mathbf{u}_{L_n^+} \times \nabla\zeta_m \nabla\bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}}\right\rangle \, dxds \\
&\le 2\int_0^t |\zeta_m \bar{\mathbf{u}}|_{L^2(D_{L_n^+})} |\nabla\zeta_m \nabla\bar{\mathbf{u}}|_{L^2(D_{L_n^+})} ds \\
&\le C\int_0^t |\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} ds + \frac{1}{4}\int_0^t |\nabla\zeta_m \nabla\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} ds.
\end{aligned}$$

For N_4, using Lemma 2.1.1 we get

$$\begin{aligned}
N_4(t) &= -\int_0^t \int_{-L_n^+}^{L_n^+} \left\langle \nabla\mathbf{u}_{L_n^+} \times \zeta_m \nabla\bar{\mathbf{u}}, \zeta_m \bar{\mathbf{u}}\right\rangle \, dxds \\
&\le \int_0^t |\zeta_m \bar{\mathbf{u}}|_{L^\infty(D_{L_n^+})} \left|\zeta_m \nabla\bar{\mathbf{u}}\right|_{L^2(D_{L_n^+})} \left|\nabla\mathbf{u}_{L_n^+}\right|_{L^2(D_{L_n^+})} ds \\
&\le C\int_0^t |\zeta_m \bar{\mathbf{u}}|^{\frac{1}{2}}_{L^2(D_{L_n^+})} \left(|\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} + |\nabla\zeta_m \bar{\mathbf{u}} + \zeta_m \nabla\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}\right)^{\frac{1}{4}} \\
&\quad \left|\zeta_m \nabla\bar{\mathbf{u}}\right|_{L^2(D_{L_n^+})} \left|\nabla\mathbf{u}_{L_n^+}\right|_{L^2(D_{L_n^+})} ds \\
&\le C\int_0^t |\zeta_m \bar{\mathbf{u}}|^{\frac{1}{2}}_{L^2(D_{L_n^+})} \left(|\zeta_m \bar{\mathbf{u}}|^{\frac{1}{2}}_{L^2(D_{L_n^+})} + |\nabla\zeta_m \bar{\mathbf{u}}|^{\frac{1}{2}}_{L^2(D_{L_n^+})} + |\zeta_m \nabla\bar{\mathbf{u}}|^{\frac{1}{2}}_{L^2(D_{L_n^+})}\right) \\
&\quad \left|\zeta_m \nabla\bar{\mathbf{u}}\right|_{L^2(D_{L_n^+})} \left|\nabla\mathbf{u}_{L_n^+}\right|_{L^2(D_{L_n^+})} ds.
\end{aligned}$$

Let us recall that $|\nabla\zeta_m| = \frac{1}{2m}|\zeta_m|$. Then, using Young's inequality (2.2.6) for $p = 4$ and $q = \frac{4}{3}$ we obtain

$$N_4 \le C_m \int_0^t |\zeta_m \bar{\mathbf{u}}|_{L^2(D_{L_n^+})} \left|\zeta_m \nabla\bar{\mathbf{u}}\right|_{L^2(D_{L_n^+})} \left|\nabla\mathbf{u}_{L_n^+}\right|_{L^2(D_{L_n^+})} ds$$

$$
\begin{aligned}
&+C\int_0^t |\zeta_m\bar{\mathbf{u}}|^{\frac{1}{2}}_{L^2(D_{L_n^+})}|\zeta_m\nabla\bar{\mathbf{u}}|^{\frac{3}{2}}_{L^2(D_{L_n^+})}\big|\nabla\mathbf{u}_{L_n^+}\big|_{L^2(D_{L_n^+})}ds\\
&\le C_m\int_0^t |\zeta_m\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}\left(\big|\nabla\mathbf{u}_{L_n^+}\big|^2_{L^2(D_{L_n^+})}+\big|\nabla\mathbf{u}_{L_n^+}\big|^4_{L^2(D_{L_n^+})}\right)ds\\
&+\frac{1}{8}\int_0^t |\zeta_m\nabla\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}ds.
\end{aligned}
$$

For N_5,

$$
\begin{aligned}
N_5(t) &= \int_0^t\int_{-L_n^+}^{L_n^+}\langle\zeta_m\bar{\mathbf{u}},(\phi_n-1)\zeta_m\Delta\mathbf{u}\rangle\,dxds\\
&\le \int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2(D_{L_n^+})}\,|(\phi_n-1)\zeta_m\Delta\mathbf{u}|_{L^2(D_{L_n^+})}\,ds\\
&\le \frac{1}{2}\int_0^t\int_{-L_n^+}^{L_n^+}|\zeta_m\bar{\mathbf{u}}|^2dxds+\frac{1}{2}\int_0^t\int_{-L_n^+}^{L_n^+}(\phi_n-1)^2\zeta_m^2\big|\Delta\mathbf{u}\big|^2dxds\\
&\le C\int_0^t|\zeta_m\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}ds+C\int_0^t\int_{-L_n^+}^{-L_n}(\phi_n-1)^2\zeta_m^2\big|\Delta\mathbf{u}\big|^2dxds\\
&\quad +C\int_0^t\int_{L_n}^{L_n^+}(\phi_n-1)^2\zeta_m^2\big|\Delta\mathbf{u}\big|^2dxds.
\end{aligned}
$$

For N_6,

$$
\begin{aligned}
N_6(t) &= \int_0^t\int_{-L_n^+}^{L_n^+}\langle\zeta_m\bar{\mathbf{u}},\zeta_m\Delta(\mathbf{u}-\mathbf{u}\phi_n)\rangle\,dxds\\
&\le \int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2(D_{L_n^+})}|\zeta_m\Delta(\mathbf{u}-\mathbf{u}\phi_n)|_{L^2(D_{L_n^+})}ds\\
&\le C\int_0^t|\zeta_m\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}ds+C\int_0^t\int_{-L_n^+}^{-L_n}\zeta_m^2\big|\Delta\mathbf{u}-\Delta\mathbf{u}\phi_n-2\nabla\mathbf{u}\nabla\phi_n-\mathbf{u}\Delta\phi_n\big|^2dxds\\
&\quad +C\int_0^t\int_{L_n}^{L_n^+}\zeta_m^2\big|\Delta\mathbf{u}-\Delta\mathbf{u}\phi_n-2\nabla\mathbf{u}\nabla\phi_n-\mathbf{u}\Delta\phi_n\big|^2dxds.
\end{aligned}
$$

Now we estimate N_7, using (4.1.5) we get

$$
\begin{aligned}
N_7(t) &= \int_0^t\int_{-L_n^+}^{L_n^+}\big\langle\zeta_m\bar{\mathbf{u}},\zeta_m\mathbf{u}_{L_n^+}\times\Delta(\mathbf{u}-\mathbf{u}\phi_n)\big\rangle\,dxds\\
&\le \int_0^t |\zeta_m\bar{\mathbf{u}}|_{L^2(D_{L_n^+})}|\zeta_m\Delta(\mathbf{u}-\mathbf{u}\phi_n)|_{L^2(D_{L_n^+})}ds\\
&\le C\int_0^t|\zeta_m\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}ds+C\int_0^t\int_{-L_n^+}^{-L_n}\zeta_m^2\big|\Delta\mathbf{u}-\Delta\mathbf{u}\phi_n-2\nabla\mathbf{u}\nabla\phi_n-\mathbf{u}\Delta\phi_n\big|^2dxds
\end{aligned}
$$

$$+ C\int_0^t\int_{L_n}^{L_n^+}\zeta_m^2\big|\Delta\mathbf{u}-\Delta\mathbf{u}\phi_n-2\nabla\mathbf{u}\nabla\phi_n-\mathbf{u}\Delta\phi_n\big|^2dxds.$$

For N_8, using Lemma 2.1.1 and the same resoning as N_4, we get

$$\begin{aligned}
&\int_0^t\int_{-L_n^+}^{L_n^+}\langle\zeta_m\bar{\mathbf{u}},\zeta_m\bar{\mathbf{u}}\rangle\left|\nabla\mathbf{u}\right|^2dxds\\
&\le\int_0^t|\zeta_m\bar{\mathbf{u}}|^2_{L^\infty(D_{L_n^+})}\big|\nabla\mathbf{u}\big|^2_{L^2(D_{L_n^+})}ds\\
&\le 2\sqrt{2}\int_0^t|\zeta_m\bar{\mathbf{u}}|_{L^2(D_{L_n^+})}\left(|\zeta_m\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}+\big|\nabla\zeta_m\bar{\mathbf{u}}+\zeta_m\nabla\bar{\mathbf{u}}\big|^2_{L^2(D_{L_n^+})}\right)^{\frac{1}{2}}\big|\nabla\mathbf{u}\big|^2_{L^2(D_{L_n^+})}ds\\
&\le 2\sqrt{2}\int_0^t|\zeta_m\bar{\mathbf{u}}|_{L^2(D_{L_n^+})}\left(|\zeta_m\bar{\mathbf{u}}|_{L^2(D_{L_n^+})}+\big|\nabla\zeta_m\bar{\mathbf{u}}\big|_{L^2(D_{L_n^+})}+\big|\zeta_m\nabla\bar{\mathbf{u}}\big|_{L^2(D_{L_n^+})}\right)\big|\nabla\mathbf{u}\big|^2_{L^2(D_{L_n^+})}ds\\
&\le C_m\int_0^t|\zeta_m\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}\big|\nabla\mathbf{u}\big|^2_{L^2(D_{L_n^+})}ds+C\int_0^t|\zeta_m\bar{\mathbf{u}}|_{L^2(D_{L_n^+})}\big|\zeta_m\nabla\bar{\mathbf{u}}\big|_{L^2(D_{L_n^+})}\big|\nabla\mathbf{u}\big|^2_{L^2(D_{L_n^+})}ds\\
&\le C_m\int_0^t|\zeta_m\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}\left(\big|\nabla\mathbf{u}\big|^2_{L^2(D_{L_n^+})}+\big|\nabla\mathbf{u}\big|^4_{L^2(D_{L_n^+})}\right)ds+\frac{1}{8}\int_0^t|\zeta_m\nabla\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}ds.
\end{aligned}$$

For N_9, using (4.1.5) we get following the same reasoning as N_4

$$\begin{aligned}
&\int_0^t\int_{-L_n^+}^{L_n^+}\left\langle\zeta_m\bar{\mathbf{u}},\left\langle\nabla\bar{\mathbf{u}},\nabla\mathbf{u}+\nabla\mathbf{u}_{L_n^+}\right\rangle\zeta_m\mathbf{u}_{L_n^+}\right\rangle dxds\\
&\le\int_0^t|\zeta_m\bar{\mathbf{u}}|_{L^\infty(D_{L_n^+})}\big|\zeta_m\nabla\bar{\mathbf{u}}\big|_{L^2(D_{L_n^+})}\big|\nabla\mathbf{u}+\nabla\mathbf{u}_{L_n^+}\big|_{L^2(D_{L_n^+})}ds\\
&\le C_m\int_0^t|\zeta_m\bar{\mathbf{u}}|_{L^2(D_{L_n^+})}\big|\zeta_m\nabla\bar{\mathbf{u}}\big|_{L^2(D_{L_n^+})}\big|\nabla\mathbf{u}+\nabla\mathbf{u}_{L_n^+}\big|_{L^2(D_{L_n^+})}ds\\
&\quad+C\int_0^t|\zeta_m\bar{\mathbf{u}}|^{\frac{1}{2}}_{L^2(D_{L_n^+})}\big|\zeta_m\nabla\bar{\mathbf{u}}\big|^{\frac{3}{2}}_{L^2(D_{L_n^+})}\big|\nabla\mathbf{u}+\nabla\mathbf{u}_{L_n^+}\big|_{L^2(D_{L_n^+})}ds\\
&\le C_m\int_0^t|\zeta_m\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}\left(\big|\nabla\mathbf{u}\big|^2_{L^2(D_{L_n^+})}+\big|\nabla\mathbf{u}\big|^4_{L^2(D_{L_n^+})}+\big|\nabla\mathbf{u}_{L_n^+}\big|^2_{L^2(D_{L_n^+})}+\big|\nabla\mathbf{u}_{L_n^+}\big|^4_{L^2(D_{L_n^+})}\right)ds\\
&\quad+\frac{1}{4}\int_0^t|\zeta_m\nabla\bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}ds.
\end{aligned}$$

Finally we estimate N_{10}, using (4.1.5) we obtain

$$\begin{aligned}
N_{10}(t)&=\int_0^t\int_{-L_n^+}^{L_n^+}\left\langle\zeta_m\bar{\mathbf{u}},\left\langle\nabla(\mathbf{u}-\mathbf{u}\phi_n),\nabla\mathbf{u}+\nabla\mathbf{u}_{L_n^+}\right\rangle\zeta_m\mathbf{u}_{L_n^+}\right\rangle dxds\\
&\le\int_0^t|\zeta_m\bar{\mathbf{u}}|_{L^2(D_{L_n^+})}\left|\left\langle\zeta_m\nabla(\mathbf{u}-\mathbf{u}\phi_n),\nabla\mathbf{u}+\nabla\mathbf{u}_{L_n^+}\right\rangle\right|_{L^2(D_{L_n^+})}ds
\end{aligned}$$

$$\leq \frac{1}{2}\int_0^t \int_{-L_n^+}^{L_n^+} |\zeta_m \bar{\mathbf{u}}|^2 dxds + \frac{1}{2}\int_0^t \int_{-L_n^+}^{L_n^+} \zeta_m^2 \big|\nabla(\mathbf{u}-\mathbf{u}\phi_n)\big|^2 \big|\nabla\mathbf{u}+\nabla\mathbf{u}_{L_n^+}\big|^2 dxds$$
$$\leq C\int_0^t |\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} ds$$
$$+ C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\nabla\mathbf{u}-\nabla\mathbf{u}\phi_n - \mathbf{u}\nabla\phi_n\big|^2 \big|\nabla\mathbf{u}+\nabla\mathbf{u}_{L_n^+}\big|^2 dxds$$
$$+ C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\nabla\mathbf{u}-\nabla\mathbf{u}\phi_n - \mathbf{u}\nabla\phi_n\big|^2 \big|\nabla\mathbf{u}+\nabla\mathbf{u}_{L_n^+}\big|^2 dxds.$$

Then, combining all the above and applying $\sup_{s\in[0,t]}$ for any $t \in [0,T]$ we obtain

$$\sup_{s\in[0,t]} |\zeta_m \bar{\mathbf{u}}(s)|^2_{L^2(D_{L_n^+})} + \int_0^t \big|\zeta_m \nabla\bar{\mathbf{u}}\big|^2_{L^2(D_{L_n^+})} ds - \int_0^t \big|\nabla\zeta_m \nabla\bar{\mathbf{u}}\big|^2_{L^2(D_{L_n^+})} ds$$
$$\leq |\zeta_m \bar{\mathbf{u}}(0)|^2_{L^2(D_{L_n^+})}$$
$$+ C\int_0^t \int_{-L_n^+}^{-L_n} (\phi_n - 1)^2 \zeta_m^2 \big|\Delta\mathbf{u}\big|^2 dxds + C\int_0^t \int_{L_n}^{L_n^+} (\phi_n - 1)^2 \zeta_m^2 \big|\Delta\mathbf{u}\big|^2 dxds$$
$$+ C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\Delta\mathbf{u} - \Delta\mathbf{u}\phi_n - 2\nabla\mathbf{u}\nabla\phi_n - \mathbf{u}\Delta\phi_n\big|^2 dxds$$
$$+ C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\Delta\mathbf{u} - \Delta\mathbf{u}\phi_n - 2\nabla\mathbf{u}\nabla\phi_n - \mathbf{u}\Delta\phi_n\big|^2 dxds$$
$$+ C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\nabla\mathbf{u}-\nabla\mathbf{u}\phi_n - \mathbf{u}\nabla\phi_n\big|^2 \left(\big|\nabla\mathbf{u}\big|^2 + \big|\nabla\mathbf{u}_{L_n^+}\big|^2\right) dxds$$
$$+ C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\nabla\mathbf{u}-\nabla\mathbf{u}\phi_n - \mathbf{u}\nabla\phi_n\big|^2 \left(\big|\nabla\mathbf{u}\big|^2 + \big|\nabla\mathbf{u}_{L_n^+}\big|^2\right) dxds$$
$$+ C\int_0^t |\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}$$
$$\left(1 + \big|\nabla\mathbf{u}\big|^2_{L^2(D_{L_n^+})} + \big|\nabla\mathbf{u}\big|^4_{L^2(D_{L_n^+})} + \big|\nabla\mathbf{u}_{L_n^+}\big|^2_{L^2(D_{L_n^+})} + \big|\nabla\mathbf{u}_{L_n^+}\big|^4_{L^2(D_{L_n^+})}\right) ds. \quad (4.3.5)$$

Since, for $m > 1$,

$$\int_0^t \big|\zeta_m \nabla\bar{\mathbf{u}}\big|^2_{L^2(D_{L_n^+})} ds - \int_0^t \big|\nabla\zeta_m \nabla\bar{\mathbf{u}}\big|^2_{L^2(D_{L_n^+})} ds = \int_0^t \left(1 - \frac{1}{4m^2}\right) \big|\zeta_m \nabla\bar{\mathbf{u}}\big|^2_{L^2(D_{L_n^+})} ds \geq 0,$$

(4.3.5) and Gronwall's inequality (2.2.7) yield

$$\sup_{s\in[0,t]} |\zeta_m \bar{\mathbf{u}}(s)|^2_{L^2(D_{L_n^+})}$$

$$\leq \Bigg(|\zeta_m \bar{\mathbf{u}}(0)|^2_{L^2(D_{L_n^+})} + C\int_0^t \int_{-L_n^+}^{-L_n} (\phi_n - 1)^2 \zeta_m^2 \big|\Delta \mathbf{u}\big|^2 dxds$$
$$+ C\int_0^t \int_{L_n}^{L_n^+} (\phi_n - 1)^2 \zeta_m^2 \big|\Delta \mathbf{u}\big|^2 dxds$$
$$+ C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\Delta \mathbf{u} - \Delta \mathbf{u}\phi_n - 2\nabla \mathbf{u}\nabla\phi_n - \mathbf{u}\Delta\phi_n\big|^2 dxds$$
$$+ C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\Delta \mathbf{u} - \Delta \mathbf{u}\phi_n - 2\nabla \mathbf{u}\nabla\phi_n - \mathbf{u}\Delta\phi_n\big|^2 dxds$$
$$+ C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\nabla \mathbf{u} - \nabla \mathbf{u}\phi_n - \mathbf{u}\nabla\phi_n\big|^2 \left(\big|\nabla \mathbf{u}\big|^2 + \big|\nabla \mathbf{u}_{L_n^+}\big|^2\right) dxds$$
$$+ C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\nabla \mathbf{u} - \nabla \mathbf{u}\phi_n - \mathbf{u}\nabla\phi_n\big|^2 \left(\big|\nabla \mathbf{u}\big|^2 + \big|\nabla \mathbf{u}_{L_n^+}\big|^2\right) dxds\Bigg) e^{\mu_{L_n}(t)}$$

with $\mu_{L_n}(t) = C\int_0^t \left(1 + \big|\nabla \mathbf{u}\big|^2_{L^2(D_{L_n^+})} + \big|\nabla \mathbf{u}\big|^4_{L^2(D_{L_n^+})} + \big|\nabla \mathbf{u}_{L_n^+}\big|^2_{L^2(D_{L_n^+})} + \big|\nabla \mathbf{u}_{L_n^+}\big|^4_{L^2(D_{L_n^+})}\right) ds.$
Consequently, we have

$$\begin{aligned}
\sup_{s\in[0,t]} |\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} \leq \Bigg(& C\int_{-L_n^+}^{-L_n} |\zeta_m|^2 dx + C\int_{L_n}^{L_n^+} |\zeta_m|^2 dx \\
& + C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\Delta \mathbf{u}\big|^2 dxds + C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\Delta \mathbf{u}\big|^2 dxds \\
& + C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\nabla \mathbf{u}\nabla\phi_n\big|^2 dxds + C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\Delta\phi_n\big|^2 dxds \\
& + C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\nabla \mathbf{u}\nabla\phi_n\big|^2 dxds + C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\Delta\phi_n\big|^2 dxds \\
& + C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\nabla \mathbf{u}\big|^4 dxds + C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\nabla \mathbf{u}_{L_n^+}\big|^4 dxds \\
& + C\int_0^t \int_{-L_n^+}^{-L_n} \zeta_m^2 \big|\nabla\phi_n\big|^2 \big|\nabla \mathbf{u}_{L_n^+}\big|^2 dxds \\
& + C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\nabla \mathbf{u}\big|^4 dxds + C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\nabla \mathbf{u}_{L_n^+}\big|^4 dxds \\
& + C\int_0^t \int_{L_n}^{L_n^+} \zeta_m^2 \big|\nabla\phi_n\big|^2 \big|\nabla \mathbf{u}_{L_n^+}\big|^2 dxds\Bigg) e^{\mu_{L_n}(t)}. \qquad (4.3.6)
\end{aligned}$$

Then, as $e^{-\frac{|x|}{m}} \leq e^{-\frac{L_n}{m}}$ for $-L_n^+ \leq x \leq -L_n$ and $L_n \leq x \leq L_n^+$, we obtain

$$\sup_{s\in[0,t]} |\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} \leq e^{-\frac{L_n}{m}} X_{L_n} e^{\mu_{L_n}(t)} \qquad (4.3.7)$$

with

$$
\begin{aligned}
X_{L_n} = C &+ C\int_0^t \int_{-L_n^+}^{-L_n} \left|\Delta \mathbf{u}\right|^2 dxds + C\int_0^t \int_{L_n}^{L_n^+} \left|\Delta \mathbf{u}\right|^2 dxds \\
&+ C\int_0^t \int_{-L_n^+}^{-L_n} \left|\nabla \mathbf{u}\nabla\phi_n\right|^2 dxds + C\int_0^t \int_{-L_n^+}^{-L_n} \left|\Delta \phi_n\right|^2 dxds \\
&+ C\int_0^t \int_{L_n}^{L_n^+} \left|\nabla \mathbf{u}\nabla\phi_n\right|^2 dxds + C\int_0^t \int_{L_n}^{L_n^+} \left|\Delta \phi_n\right|^2 dxds \\
&+ C\int_0^t \int_{-L_n^+}^{-L_n} \left|\nabla \mathbf{u}\right|^4 dxds + C\int_0^t \int_{-L_n^+}^{-L_n} \left|\nabla \mathbf{u}_{L_n^+}\right|^4 dxds \\
&+ C\int_0^t \int_{-L_n^+}^{-L_n} \left|\nabla \phi_n\right|^2 \left|\nabla \mathbf{u}_{L_n^+}\right|^2 dxds \\
&+ C\int_0^t \int_{L_n}^{L_n^+} \left|\nabla \mathbf{u}\right|^4 dxds + C\int_0^t \int_{L_n}^{L_n^+} \left|\nabla \mathbf{u}_{L_n^+}\right|^4 dxds \\
&+ C\int_0^t \int_{L_n}^{L_n^+} \left|\nabla \phi_n\right|^2 \left|\nabla \mathbf{u}_{L_n^+}\right|^2 dxds.
\end{aligned}
$$

Next, for $\alpha \in (0,1)$ and $\beta = 1 - \frac{\alpha}{2}$, we denote $A_n := e^{-\frac{\beta L_n}{2m}} X_{L_n}$ and $B_n := e^{-\frac{\beta L_n}{2m}} e^{\mu_{L_n}(t)}$. Then, from (4.3.7) we get

$$
\sup_{s\in[0,t]} |\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} \leq e^{-\frac{\alpha L_n}{2m}} A_n B_n. \tag{4.3.8}
$$

We have, for $\varepsilon > 0$

$$
\sum_n \mathbb{P}\left(A_n > \varepsilon\right) = \sum_n \mathbb{P}(X_{L_n} > \varepsilon e^{\frac{\beta L_n}{2m}}).
$$

Using Lemma 2.3.9, we get for $1 \leq p < \infty$

$$
\sum_n \mathbb{P}(A_n > \varepsilon) \leq \sum_n \frac{1}{\varepsilon^p} e^{-\frac{\beta p L_n}{2m}} \mathbb{E}\left[X_{L_n}^p\right].
$$

From Theorem 3.1.3 and Lemma 4.2.1 we obtain

$$\begin{aligned}\sum_n \mathbb{P}(A_n > \varepsilon) &\le C\frac{1}{\varepsilon^p}\sum_n e^{-\frac{p\beta L_n}{2m}} \\ &\le C\frac{1}{\varepsilon^p}\sum_n \left(\frac{1}{1+\frac{\beta L_n}{2m}}\right)^p \\ &\le C\left(\frac{2m}{\beta\varepsilon}\right)^p \sum_n \frac{1}{L_n^p}.\end{aligned}$$

Then, from condition (4.1.7)

$$\sum_n \mathbb{P}(A_n > \varepsilon) < \infty.$$

Consequently, Lemma 2.3.5 gives

$$\mathbb{P}\left(\{A_n > \varepsilon\}\ i.o.\right) = 0, \quad \text{for every } \varepsilon > 0$$

which means that

$$\lim_{n\to\infty} A_n = 0, \quad \mathbb{P}\text{-a.s.} \tag{4.3.9}$$

On the other hand, using Lemma 2.3.9, Theorem 3.1.3 and Lemma 4.2.1 we have for any $\varepsilon > 0$ and $1 \le p < \infty$

$$\begin{aligned}\sum_n \mathbb{P}\left(\frac{\mu_{L_n}}{L_n} > \varepsilon\right) &\le \sum_n \frac{1}{\varepsilon^p L_n^p}\mathbb{E}\left[\mu_{L_n}^p\right] \\ &\le \frac{C}{\varepsilon^p}\sum_{n=1}^{\infty}\frac{1}{L_n^p} < \infty\,,\end{aligned}$$

hence, by Lemma 2.3.5 again we obtain

$$\lim_{n\to\infty}\frac{\mu_{L_n}}{L_n} = 0, \quad \mathbb{P}\text{-a.s.}.$$

We deduce that

$$\sup_{n\ge 1} B_n = \sup_{n\ge 1}\left[e^{-L_n(\frac{\beta}{2m}-\frac{1}{L_n}\mu_{L_n})}\right] < \infty, \quad \mathbb{P}\text{-a.s.}. \tag{4.3.10}$$

Finally, we deduce from (4.3.8), (4.3.9) and (4.3.10) that there exists a finite random variable C, such that

$$\sup_{s\in[0,t]} |\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})} \le Ce^{-\frac{\alpha L_n}{2m}}.$$

In particular,

$$\lim_{n\to\infty}\left(\sup_{s\in[0,t]} |\zeta_m \bar{\mathbf{u}}|^2_{L^2(D_{L_n^+})}\right) = 0 \quad \mathbb{P}\text{-a.s.}.$$

Then, we obtain

$$\sup_{s\in[0,t]} \int_{-L_n^+}^{L_n^+} |\zeta_m(\mathbf{u}\phi_n - \mathbf{u}_{L_n^+})|^2 dx \le Ce^{-\frac{\alpha L_n}{2m}}$$

and

$$\lim_{n\to\infty}\left(\sup_{s\in[0,t]} \int_{-L_n^+}^{L_n^+} |\zeta_m(\mathbf{u}\phi_n - \mathbf{u}_{L_n^+})|^2 dx\right) = 0 \quad \mathbb{P}\text{-a.s.}.$$

Therefore, we get

$$\sup_{s\in[0,t]} \int_{-L_n}^{L_n} |\mathbf{u} - \mathbf{u}_{L_n^+}|^2 \rho_m(x) dx \le Ce^{-\frac{\alpha L_n}{2m}}$$

and

$$\lim_{n\to\infty}\left(\sup_{s\in[0,t]} \int_{-L_n}^{L_n} |\mathbf{u} - \mathbf{u}_{L_n^+}|^2 \rho_m(x) dx\right) = 0 \quad \mathbb{P}\text{-a.s.}.$$

Consequently, we conclude that

$$\lim_{n\to\infty}\left(\sup_{s\in[0,t]} \int_{-L}^{L} |\mathbf{u}(s,x) - \mathbf{u}_{L_n}(s,x)|^2 \rho_m(x) dx\right) = 0 \quad \mathbb{P}\text{-a.s.}$$

for any $L > 0$. Moreover, for a fixed $\alpha > 0$ small enough, there exists a random variable C independent of n and such that

$$\sup_{s\in[0,t]} \int_{-L}^{L} |\mathbf{u}(s,x) - \mathbf{u}_{L_n}(s,x)|^2 \rho_m(x) dx \le Ce^{-\frac{\alpha L_n}{2m}},$$

completing the proof of Theorem 4.1.2.

Corollary 4.3.1. *Assume that for a certain $1 \le p < \infty$,*

$$\sum_{n=1}^{\infty} \frac{1}{L_n^p} < \infty.$$

Then, for every $L > 0$ and $r \geq 2$

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |\mathbf{u}(t) - \mathbf{u}_{L_n}(t)|^r_{L^2_m(D_L)}\right] = 0\,.$$

Proof. For every $r \geq 2$, we have

$$\begin{aligned}
|\mathbf{u}(t) - \mathbf{u}_{L_n}(t)|^r_{L^2_m(D_L)} &= \left(\int_{-L}^{L} |\mathbf{u}(t,x) - \mathbf{u}_{L_n}(t,x)|^2 \rho_m(x)dx\right)^{\frac{r}{2}} \\
&\leq 2^r \left(\int_{-L}^{L} \rho_m(x)dx\right)^{\frac{r}{2}} \\
&\leq 2^r \left|2m - 2me^{-\frac{L}{m}}\right|^{\frac{r}{2}} \\
&\leq 2^{\frac{3r}{2}} m^{\frac{r}{2}}.
\end{aligned}$$

Hence, the corollary follows from Theorem 4.1.2 and the Dominated Convergence Theorem. $\square$

Chapter 5

A Fully-Discrete Finite Difference Scheme on a Bounded Domain

5.1 Introduction

In this chapter, we solve numerically the reduced problem (4.1.1)-(4.1.4). In fact, we design a fully-discrete finite difference scheme to find approximate solutions and we carry out numerical experiments supporting the conjecture that the finite difference solutions converge to the solution of the reduced problem (4.1.1)-(4.1.4) for vanishing discretisation parameters. In addition, we perform a numerical experiment which validate the theoretical result in the previous chapter.

In the following, we recall that $\mathbb{T}_k$ and $\mathbb{X}_h$ are defined in Chapter 2 (section 2.4). For any $\mathbf{v} : \mathbb{T}_k \times \mathbb{X}_h \to \mathbb{R}^3$, we denote

$$
\begin{aligned}
\mathbf{v}_i^n &:= \mathbf{v}(t_n, x_i), \quad -I \le i \le I, \quad 0 \le n \le N,\\
\mathbf{v}_i^{n+1/2} &:= \frac{\mathbf{v}_i^{n+1} + \mathbf{v}_i^n}{2}, \quad -I \le i \le I, \quad 0 \le n \le N-1,\\
\mathbf{v}^n &:= \mathbf{v}(t_n, \cdot), \quad 0 \le n \le N,\\
\mathbf{v}^{n+1/2} &:= \frac{\mathbf{v}^{n+1} + \mathbf{v}^n}{2}, \quad 0 \le n \le N-1.
\end{aligned}
$$

Let $\mathbf{U}^{k,h} : \mathbb{T}_k \times \mathbb{X}_h \to \mathbb{R}^3$ and $\mathbf{g}^h : \mathbb{X}_h \to \mathbb{R}^3$ such that

$$
\begin{aligned}
\mathbf{g}^h(x_i) &= \mathbf{g}(x_i), \qquad -(I-1) \le i \le I-1,\\
\mathbf{g}^h(x_{-I}) &= \mathbf{g}^h(x_{-I+1}),\\
\mathbf{g}^h(x_I) &= \mathbf{g}^h(x_{I-1}).
\end{aligned}
$$

Note that in this case we have

$$D^-\mathbf{g}^h(x_I) = D^+\mathbf{g}^h(x_{-I}) = 0.$$

Let $\mathbf{u}_0^h : \mathbb{X}_h \to \mathbb{R}^3$ be defined by

$$\mathbf{u}_0^h(x_i) = \mathbf{u}^h(0, x_i), \quad -(I-1) \le i \le I-1, \tag{5.1.1}$$

$$\mathbf{u}_0^h(x_{-I}) = \mathbf{u}_0^h(x_{-I+1}), \tag{5.1.2}$$

$$\mathbf{u}_0^h(x_I) = \mathbf{u}_0^h(x_{I-1}). \tag{5.1.3}$$

In what follows, we denote

$$\mathbf{U}_i^n := \left(\mathbf{U}^{k,h}\right)_i^n, \quad -I \le i \le I, \quad 0 \le n \le N,$$
$$W^n := W(t_n), \quad 0 \le n \le N.$$

We propose the following fully-discrete finite difference scheme for problem (4.1.1)-(4.1.4):

$$\begin{aligned}\mathbf{U}_i^{n+1} - \mathbf{U}_i^n = {} & \mu k \mathbf{U}_i^{n+1/2} \times \tilde{\Delta}\mathbf{U}_i^{n+1} - \lambda k \mathbf{U}_i^{n+1/2} \times (\mathbf{U}_i^{n+1/2} \times \tilde{\Delta}\mathbf{U}_i^{n+1}) \\ & + \mu\big(\mathbf{U}_i^{n+1/2} \times \mathbf{g}^h\big)(W^{n+1} - W^n), \quad -(I-1) \le i \le (I-1), 0 \le n \le N-1,\end{aligned} \tag{5.1.4}$$

$$\mathbf{U}_i^0 = \mathbf{u}_0^h(ih), \qquad -I \le i \le I, \tag{5.1.5}$$

$$D^-\mathbf{U}_I^n = D^+\mathbf{U}_{-I}^n = 0, \qquad 0 \le n \le N, \tag{5.1.6}$$

$$|\mathbf{u}_0^h(ih)| = 1, \qquad -I \le i \le I. \tag{5.1.7}$$

We note that we consider the average $\mathbf{U}_i^{n+1/2}$ in the fully-discrete equation (5.1.4) to guarantee that the sphere restriction $|\mathbf{U}_i^n| = 1$ is preserved.

In the deterministic case, this type of scheme was studied by Fuwa, Ishiwata and Tsutsumi in [21] . In our case, where the stochastic term appears, modifications in the scheme are needed and they are similar to the modifications made for the finite element method considered in [6].

5.2 The Fully-Discrete Finite Difference Scheme

In this section, we prove the unique solvability of the fully-discrete scheme (5.1.4)-(5.1.7). We note that (5.1.4) is equivalent to

$$\mathbf{U}_i^{n+1} = \mathbf{u}_0^h + \mu k \sum_{m=0}^{n} \mathbf{U}_i^{m+1/2} \times \tilde{\Delta}\mathbf{U}_i^{m+1} - \lambda k \sum_{m=0}^{n} \mathbf{U}_i^{m+1/2} \times (\mathbf{U}_i^{m+1/2} \times \tilde{\Delta}\mathbf{U}_i^{m+1})$$
$$+ \mu \sum_{m=0}^{n} \big(\mathbf{U}_i^{m+1/2} \times \mathbf{g}^h\big)(W^{m+1} - W^m).$$

We define the following space

$$E_{h,k} = \left\{ \mathbf{v} : \mathbb{T}_k \times \mathbb{X}_h \to \mathbb{R}^3 \,|\ \ D^- \mathbf{v}_I^n = D^+ \mathbf{v}_{-I}^n = 0, \quad \forall\, 0 \le n \le N \right\},$$

with the norm

$$|\mathbf{v}|_{E_{h,k}} := \sup_{0 \le n \le N} \sup_{-I \le i \le I} |\mathbf{v}(t_n, x_i)|.$$

The space $E_{h,k}$ endowed with the norm $|\cdot|_{E_{h,k}}$ is a Banach space.

For every $h, k > 0$, we define the maps $I_{h,k}^s : E_{h,k} \to E_{h,k}$, $s = 1, 2$ and $J_{h,k} : E_{h,k} \to E_{h,k}$ by

$$\big(I_{h,k}^1(\mathbf{v})\big)_i^n := \mu k \sum_{m=0}^{n} \mathbf{v}_i^{m+1/2} \times \tilde{\Delta}\mathbf{v}_i^{m+1},$$
$$\big(I_{h,k}^2(\mathbf{v})\big)_i^n := \lambda k \sum_{m=0}^{n} \mathbf{v}_i^{m+1/2} \times (\mathbf{v}_i^{m+1/2} \times \tilde{\Delta}\mathbf{v}_i^{m+1}),$$
$$\left(J_{h,k}(\mathbf{v})\right)_i^n := \mu \sum_{m=0}^{n} \left(\mathbf{v}_i^{m+1/2} \times \mathbf{g}^h\right) \left(W^{m+1} - W^m\right),$$

for all $i = \{-I, ..., I\}$, $n = \{0, ...N\}$ and for all $\mathbf{v} \in E_{h,k}$. For completness of the definition of these maps, for every $\mathbf{v} \in E_{h,k}$, we extend the function $\mathbf{v}$ as follows

$$\mathbf{v}_{I+1}^n = \mathbf{v}_I^n \quad , \quad \mathbf{v}_{-I-1}^n = \mathbf{v}_{-I}^n, \quad 0 \le n \le N,$$

$$\mathbf{v}_i^{N+1} = \mathbf{v}_i^N, \quad -I-1 \le i \le I+1,$$

and we extend W as follows

$$W^{N+1} = W^N.$$

It is clear that $I_{h,k}^1(\mathbf{v})$ and $I_{h,k}^2(\mathbf{v})$ belongs to $E_{h,k}$ if $\mathbf{v}$ belongs to $E_{h,k}$. The same property also holds for $J_{h,k}$ due to the fact that $D^-\mathbf{g}^h(x_I) = D^+\mathbf{g}^h(x_{-I}) = 0$.

The next lemma will be used to prove the unique solvability of the fully-discrete scheme (5.1.4)-(5.1.7).

Lemma 5.2.1. *Assume $\mathbf{g} \in L^\infty(\mathbb{R})$. For every $h, k \in [0,1]$, the following holds*

- *the mappings $I_{h,k}^s : E_{h,k} \to E_{h,k}$, $s = 1, 2$ are locally Lipshitz.*
- *the mapping $J_{h,k} : E_{h,k} \to E_{h,k}$ is Lipshitz in the following sense: there exists a random variable $C := C(T) \ge 0$ such that $\mathbb{E}[C^p] < \infty$ for every $p > 0$ and*

$$|J_{h,k}(\mathbf{v}) - J_{h,k}(\mathbf{w})|_{E_{h,k}} \le C\,|\mathbf{v}-\mathbf{w}|_{E_{h,k}}, \quad \mathbf{v}, \mathbf{w} \in E_{h,k}.$$

Proof. Consider $\mathbf{v}, \mathbf{w} \in B_R$, a ball centered at 0 having radius R in $E_{h,k}$. Let us prove that the map $I_{h,k}^1$ is locally Lipshitz. We have

$$\begin{aligned}
\left|I_{h,k}^1(\mathbf{v}) - I_{h,k}^1(\mathbf{w})\right|_{E_{h,k}} &= \sup_{0\le n\le N}\ \sup_{-I\le i\le I}\left|\mu k \sum_{m=0}^{n}\left(\mathbf{v}_i^{m+1/2}\times\tilde{\Delta}\mathbf{v}_i^{m+1} - \mathbf{w}_i^{m+1/2}\times\tilde{\Delta}\mathbf{w}_i^{m+1}\right)\right| \\
&\le \mu k \sup_{0\le n\le N}\ \sup_{-I\le i\le I}\sum_{m=0}^{n}\left|\mathbf{v}_i^{m+1/2}\times\tilde{\Delta}\mathbf{v}_i^{m+1} - \mathbf{w}_i^{m+1/2}\times\tilde{\Delta}\mathbf{w}_i^{m+1}\right| \\
&\le 2\mu T \sup_{0\le n\le N}\ \sup_{-I\le i\le I}\left|\mathbf{v}_i^{n+1/2}\times\tilde{\Delta}\mathbf{v}_i^{n+1} - \mathbf{w}_i^{n+1/2}\times\tilde{\Delta}\mathbf{w}_i^{n+1}\right|.
\end{aligned} \tag{5.2.1}$$

We know that

$$\begin{aligned}
&\left|\mathbf{v}_i^{n+1/2}\times\tilde{\Delta}\mathbf{v}_i^{n+1} - \mathbf{w}_i^{n+1/2}\times\tilde{\Delta}\mathbf{w}_i^{n+1}\right| \\
&= \frac{1}{h^2}\left|\mathbf{v}_i^{n+1/2}\times\left(\mathbf{v}_{i+1}^{n+1} - 2\mathbf{v}_i^{n+1} + \mathbf{v}_{i-1}^{n+1}\right) - \mathbf{w}_i^{n+1/2}\times\left(\mathbf{w}_{i+1}^{n+1} - 2\mathbf{w}_i^{n+1} + \mathbf{w}_{i-1}^{n+1}\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{h^2}\bigg|\frac{1}{2}\mathbf{v}_i^{n+1}\times\mathbf{v}_{i+1}^{n+1} + \frac{1}{2}\mathbf{v}_i^{n}\times\mathbf{v}_{i+1}^{n+1} - \mathbf{v}_i^{n}\times\mathbf{v}_{i}^{n+1} + \frac{1}{2}\mathbf{v}_i^{n+1}\times\mathbf{v}_{i-1}^{n+1} + \frac{1}{2}\mathbf{v}_i^{n}\times\mathbf{v}_{i-1}^{n+1} \\
&\qquad - \frac{1}{2}\mathbf{w}_i^{n+1}\times\mathbf{w}_{i+1}^{n+1} - \frac{1}{2}\mathbf{w}_i^{n}\times\mathbf{w}_{i+1}^{n+1} + \mathbf{w}_i^{n}\times\mathbf{w}_{i}^{n+1} - \frac{1}{2}\mathbf{w}_i^{n+1}\times\mathbf{w}_{i-1}^{n+1} - \frac{1}{2}\mathbf{w}_i^{n}\times\mathbf{w}_{i-1}^{n+1}\bigg| \\
&= \frac{1}{h^2}\bigg|\frac{1}{2}\left(\mathbf{v}_i^{n+1}-\mathbf{w}_i^{n+1}\right)\times\mathbf{v}_{i+1}^{n+1} + \frac{1}{2}\mathbf{w}_i^{n+1}\times\left(\mathbf{v}_{i+1}^{n+1}-\mathbf{w}_{i+1}^{n+1}\right) \\
&\qquad + \frac{1}{2}\left(\mathbf{v}_i^{n}-\mathbf{w}_i^{n}\right)\times\mathbf{v}_{i+1}^{n+1} + \frac{1}{2}\mathbf{w}_i^{n}\times\left(\mathbf{v}_{i+1}^{n+1}-\mathbf{w}_{i+1}^{n+1}\right) \\
&\qquad - \left(\mathbf{v}_i^{n}-\mathbf{w}_i^{n}\right)\times\mathbf{v}_{i}^{n+1} - \mathbf{w}_i^{n}\times\left(\mathbf{v}_{i}^{n+1}-\mathbf{w}_{i}^{n+1}\right) \\
&\qquad + \frac{1}{2}\left(\mathbf{v}_i^{n+1}-\mathbf{w}_i^{n+1}\right)\times\mathbf{v}_{i-1}^{n+1} + \frac{1}{2}\mathbf{w}_i^{n+1}\times\left(\mathbf{v}_{i-1}^{n+1}-\mathbf{w}_{i-1}^{n+1}\right) \\
&\qquad + \frac{1}{2}\left(\mathbf{v}_i^{n}-\mathbf{w}_i^{n}\right)\times\mathbf{v}_{i-1}^{n+1} + \frac{1}{2}\mathbf{w}_i^{n}\times\left(\mathbf{v}_{i-1}^{n+1}-\mathbf{w}_{i-1}^{n+1}\right)\bigg| \\
&\le \frac{1}{2h^2}\left|\mathbf{v}_i^{n+1}-\mathbf{w}_i^{n+1}\right|\left|\mathbf{v}_{i+1}^{n+1}\right| + \frac{1}{2h^2}\left|\mathbf{w}_i^{n+1}\right|\left|\mathbf{v}_{i+1}^{n+1}-\mathbf{w}_{i+1}^{n+1}\right| \\
&\quad + \frac{1}{2h^2}\left|\mathbf{v}_i^{n}-\mathbf{w}_i^{n}\right|\left|\mathbf{v}_{i+1}^{n+1}\right| + \frac{1}{2h^2}\left|\mathbf{w}_i^{n}\right|\left|\mathbf{v}_{i+1}^{n+1}-\mathbf{w}_{i+1}^{n+1}\right| \\
&\quad + \frac{1}{h^2}\left|\mathbf{v}_i^{n}-\mathbf{w}_i^{n}\right|\left|\mathbf{v}_{i}^{n+1}\right| + \frac{1}{h^2}\left|\mathbf{w}_i^{n}\right|\left|\mathbf{v}_{i}^{n+1}-\mathbf{w}_{i}^{n+1}\right| \\
&\quad + \frac{1}{2h^2}\left|\mathbf{v}_i^{n+1}-\mathbf{w}_i^{n+1}\right|\left|\mathbf{v}_{i-1}^{n+1}\right| + \frac{1}{2h^2}\left|\mathbf{w}_i^{n+1}\right|\left|\mathbf{v}_{i-1}^{n+1}-\mathbf{w}_{i-1}^{n+1}\right| \\
&\quad + \frac{1}{2h^2}\left|\mathbf{v}_i^{n}-\mathbf{w}_i^{n}\right|\left|\mathbf{v}_{i-1}^{n+1}\right| + \frac{1}{2h^2}\left|\mathbf{w}_i^{n}\right|\left|\mathbf{v}_{i-1}^{n+1}-\mathbf{w}_{i-1}^{n+1}\right|.
\end{aligned}
$$

Then, from (5.2.1) we deduce

$$
\left|I_{h,k}^1(\mathbf{v}) - I_{h,k}^1(\mathbf{w})\right|_{E_{h,k}} \le \frac{C\mu TR}{h^2}\left|\mathbf{v}-\mathbf{w}\right|_{E_{h,k}}. \tag{5.2.2}
$$

Similarly, we can prove that

$$
\left|I_{h,k}^2(\mathbf{v}) - I_{h,k}^2(\mathbf{w})\right|_{E_{h,k}} \le \frac{C\lambda TR^2}{h^2}\left|\mathbf{v}-\mathbf{w}\right|_{E_{h,k}}. \tag{5.2.3}
$$

Finally, let us prove that $J_{h,k}$ is Lipshitz. We have

$$
\begin{aligned}
\left|J_{h,k}(\mathbf{v}) - J_{h,k}(\mathbf{w})\right|_{E_{h,k}} &= \sup_{0\le n\le N}\ \sup_{-I\le i\le I}\left|\mu\sum_{m=0}^{n}\left(\left(\mathbf{v}_i^{m+1/2}-\mathbf{w}_i^{m+1/2}\right)\times\mathbf{g}^h\right)\left(W^{m+1}-W^m\right)\right| \\
&\le \mu\sup_{0\le n\le N}\ \sup_{-I\le i\le I}\sum_{m=0}^{n}\left|\left(\mathbf{v}_i^{m+1/2}-\mathbf{w}_i^{m+1/2}\right)\times\mathbf{g}^h\right|\ \left|W^{m+1}-W^m\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mu(N+1)|\mathbf{g}|_{L^\infty(\mathbb{R})} \sup_{0\leq n\leq N} \sup_{-I\leq i\leq I} \left|\mathbf{v}_i^{n+1/2} - \mathbf{w}_i^{n+1/2}\right| \sup_{0\leq n\leq N} (2|W^n|) \\
&\leq \frac{1}{2}\mu(N+1)|\mathbf{g}|_{L^\infty(\mathbb{R})} \sup_{0\leq n\leq N} \sup_{-I\leq i\leq I} \left|\mathbf{v}_i^{n+1} - \mathbf{w}_i^{n+1}\right| \sup_{0\leq n\leq N} (2|W^n|) \\
&\quad + \frac{1}{2}\mu(N+1)|\mathbf{g}|_{L^\infty(\mathbb{R})} \sup_{0\leq n\leq N} \sup_{-I\leq i\leq I} |\mathbf{v}_i^{n} - \mathbf{w}_i^{n}| \sup_{0\leq n\leq N} (2|W^n|) \\
&\leq \mu(N+1)|\mathbf{g}|_{L^\infty(\mathbb{R})} \sup_{0\leq n\leq N} \sup_{-I\leq i\leq I} |\mathbf{v}_i^{n} - \mathbf{w}_i^{n}| \sup_{0\leq n\leq N} (2|W^n|)\,.
\end{aligned} \tag{5.2.4}
$$

We will use the Law of Iterated Logarithm for Brownian motion (see Theorem 1.9 on p. 56 of [45]),

$$
\limsup_{t\to 0} \frac{|W(t)|}{\sqrt{2t\log\log(\frac{1}{t})}} = 1.
$$

Equivalently, for t sufficiently small, there exists a random variable C such that

$$
|W(t)| \leq C\sqrt{2t\log\log\big(\frac{1}{t}\big)}
$$

$\mathbb{P}$-a.s. Then, from (5.2.4) we get

$$
|J_{h,k}(\mathbf{v}) - J_{h,k}(\mathbf{w})|_{E_{h,k}} \leq C\mu(N+1)|\mathbf{g}|_{L^\infty(\mathbb{R})}\sqrt{T\log\log\big(\frac{1}{T}\big)}\,|\mathbf{v}-\mathbf{w}|_{E_{h,k}} \tag{5.2.5}
$$

$\mathbb{P}$-a.s. □

In the next lemma, we prove the existence of a global solution to (5.1.4)-(5.1.7).

Lemma 5.2.2. *Assume $\mathbf{g} \in L^\infty(\mathbb{R})$. Let $R > 1$ be fixed and let B_R be a ball centered at 0 having radius R in $E_{h,k}$. Then, for every $h, k \in [0,1]$ and $T > 0$, there exists a unique solution $(\mathbf{U}_i^n), 0 \leq n \leq N, -I \leq i \leq I$ to (5.1.4)-(5.1.7) in B_R $\mathbb{P}$-a.s..*

Proof. In order to simplify notations, we assume in the proof without loss of generality, that $\lambda = \mu = 1$. We consider the mapping $\mathcal{H} : E_{h,k} \to E_{h,k}$ defined by

$$
\mathcal{H}(\mathbf{v}) = \mathbf{u}_0^h + I_{h,k}^1(\mathbf{v}) - I_{h,k}^2(\mathbf{v}) + J_{h,k}(\mathbf{v}), \quad \forall\, \mathbf{v} \in E_{h,k}.
$$

It is clear that $\mathcal{H}: E_{h,k} \to E_{h,k}$ due to (5.1.2)-(5.1.3) .

First, we prove that $\mathcal{H}: B_R \to B_R$. For $\mathbf{v} \in B_R$, we deduce from (5.2.2), (5.2.3) and (5.2.5) that

$$\begin{aligned}
|\mathcal{H}(\mathbf{v})|_{E_{h,k}} &\leq |\mathcal{H}(\mathbf{v}) - \mathcal{H}(0)|_{E_{h,k}} + |\mathcal{H}(0)|_{E_{h,k}} \\
&\leq |\mathbf{u}_0^h|_{E_{h,k}} + |I^1_{h,k}(\mathbf{v})|_{E_{h,k}} + |I^2_{h,k}(\mathbf{v})|_{E_{h,k}} + |J_{h,k}(\mathbf{v})|_{E_{h,k}} \\
&\leq 1 + \left|I^1_{h,k}(\mathbf{v}) - I^1_{h,k}(0)\right|_{E_{h,k}} + \left|I^2_{h,k}(\mathbf{v}) - I^2_{h,k}(\mathbf{v})\right|_{E_{h,k}} + |J_{h,k}(\mathbf{v}) - J_{h,k}(0)|_{E_{h,k}} \\
&\leq 1 + \left(\frac{CTR(R+1)}{h^2} + C(N+1)|\mathbf{g}|_{L^\infty(\mathbb{R})}\sqrt{T\log\log\left(\frac{1}{T}\right)}\right)|\mathbf{v}|_{E_{h,k}} \\
&\leq 1 + \left(\frac{CTR(R+1)}{h^2} + C(N+1)|\mathbf{g}|_{L^\infty(\mathbb{R})}\sqrt{T\log\log\left(\frac{1}{T}\right)}\right)R.
\end{aligned}$$

Hence, for T sufficiently small (recalling $R > 1$)

$$|\mathcal{H}(\mathbf{v})|_{E_{h,k}} \leq R.$$

Next, we prove that $\mathcal{H}$ is a contraction in B_R for T sufficiently small. For any $\mathbf{v}, \mathbf{w} \in B_R$, we have

$$\begin{aligned}
&|\mathcal{H}(\mathbf{v}) - \mathcal{H}(\mathbf{w})|_{E_{h,k}} \\
&\leq \left|I^1_{h,k}(\mathbf{v}) - I^1_{h,k}(\mathbf{w})\right|_{E_{h,k}} + \left|I^2_{h,k}(\mathbf{v}) - I^2_{h,k}(\mathbf{w})\right|_{E_{h,k}} + |J_{h,k}(\mathbf{v}) - J_{h,k}(\mathbf{w})|_{E_{h,k}} \\
&\leq \left(\frac{CR(R+1)}{h^2}T + C(N+1)|\mathbf{g}|_{L^\infty(\mathbb{R})}\sqrt{T\log\log\left(\frac{1}{T}\right)}\right)|\mathbf{v} - \mathbf{w}|_{E_{h,k}} \\
&\leq c\,|\mathbf{v} - \mathbf{w}|_{E_{h,k}}
\end{aligned}$$

with $c \in (0,1)$ if T is sufficiently small. We deduce that the mapping $\mathcal{H}$ is a contraction in B_R for T sufficiently small. From Banach fixed point theorem, we obtain that for every $h, k \in [0,1]$, there exists a unique solution $(\mathbf{U}_i^n), 0 \leq n \leq N, -I \leq i \leq I$ to (5.1.4)-(5.1.7) in B_R $\mathbb{P}$-a.s., for T sufficiently small.

Next, multiplying (5.1.4) by $\left(\mathbf{U}_i^{n+1}+\mathbf{U}_i^n\right)$ and by using the elementary property (2.2.1), we obtain

$$\left\langle \mathbf{U}_i^{n+1}-\mathbf{U}_i^n, \mathbf{U}_i^{n+1}+\mathbf{U}_i^n \right\rangle = 0, \qquad \forall i \in \{-(I-1), ..., I-1\},\ \forall n \in \{0, ..., N-1\},$$

with $t_N = T$ where T is sufficiently small, which means that

$$|\mathbf{U}_i^{n+1}| = |\mathbf{U}_i^n|, \qquad \forall i \in \{-(I-1), ..., I-1\},\ \forall n \in \{0, ..., N-1\}.$$

Consequently,

$$|\mathbf{U}_i^n| = 1, \qquad \forall i \in \{-I, ..., I\},\ \forall n \in \{1, ..., N\},$$

for every $t_n \in [0,T]$ with T sufficiently small and all $x_i \in [-L, L]$. Then, since $|\mathbf{U}^{k,h}(T, ih)| = 1$ for every $-I \le i \le I$, we can repeat the same calculation as above with initial condition $\mathbf{U}^{k,h}(T)$ where T is sufficiently small. Then, we obtain that for each $h, k \in [0,1]$ and $T > 0$, there exists a unique solution $\mathbf{U}_i^n$ in B_R $\mathbb{P}$-a.s. and the lemma follows. $\square$

5.3 A Priori Estimates

In this section, we introduce and prove some uniform estimates which are important to prove order of convergence of the fully-discrete solutions to the solution of the reduced problem (4.1.1)-(4.1.4). We note that the proof of convergence order is subject to further study. In fact, the analysis using Taylor's expansion requires some estimates of higher order derivatives of the solution. These estimates should be uniform with respect to the discretisation parameters and this couldn't be achieved at this stage.

Lemma 5.3.1. *Suppose that* $|\mathbf{u}_0| = 1$*,* $\nabla \mathbf{u}_0 \in L^2(\mathbb{R})$ *and* $\mathbf{g} \in H^2(\mathbb{R})$*. If* $\{\mathbf{U}_i^n\}_{-I\le i\le I, 0\le n\le N}$ *is the solution to* (5.1.4)-(5.1.7) *then*

(1) $|\mathbf{U}_i^n| = 1$ *for all* $-I \le i \le I$ *and all* $1 \le n \le N$*,*

(2)

$$\mathbb{E}\left[\sup_{1\le n\le N} \left|D^+\mathbf{U}^n\right|_{L_h^2}^2\right] + \lambda\,\mathbb{E}\left[k\sum_{n=0}^{N-1}\left|\mathbf{U}^{n+1/2}\times\tilde{\Delta}\mathbf{U}^{n+1}\right|_{L_h^2}^2\right] \le C,$$

where C is a constant which does not depend on h and k but may depend on $|\nabla \mathbf{u}_0|_{L^2}$, $|\mathbf{g}|_{H^2}$ and T.

Proof. In order to simplify notations, we assume in the proof without loss of generality, that $\lambda = \mu = 1$. We note that in this proof, $\langle \cdot, \cdot \rangle_{L^2_h}$ and $|\cdot|_{L^2_h}$ are the inner product and norm in $L^2_h(\mathbb{X}_h)$. The corresponding inner product and norm in $L^2_h(\mathbb{Z}_h)$ will be clearly written with $\mathbb{Z}_h$.

First, part (1) is proved above in Lemma 5.2.2.

Next, we prove part (2). We multiply (5.1.4) by $-\tilde{\Delta}\mathbf{U}_i^{n+1}$,

$$\begin{aligned} -\left\langle \mathbf{U}_i^{n+1} - \mathbf{U}_i^n, \tilde{\Delta}\mathbf{U}_i^{n+1} \right\rangle &= -k \left\langle \mathbf{U}_i^{n+1/2} \times \tilde{\Delta}\mathbf{U}_i^{n+1}, \tilde{\Delta}\mathbf{U}_i^{n+1} \right\rangle \\ &\quad + k \left\langle \mathbf{U}_i^{n+1/2} \times (\mathbf{U}_i^{n+1/2} \times \tilde{\Delta}\mathbf{U}_i^{n+1}), \tilde{\Delta}\mathbf{U}_i^{n+1} \right\rangle \\ &\quad - \left\langle \mathbf{U}_i^{n+1/2} \times \mathbf{g}^h, \tilde{\Delta}\mathbf{U}_i^{n+1} \right\rangle (W^{n+1} - W^n). \end{aligned} \tag{5.3.1}$$

We extend the grid $\mathbb{X}_h$ to $\mathbb{Z}_h$ and, for all $n = 0, ..., N$, extend $\mathbf{U}^n$ to $\mathbb{Z}_h$ by

$$\mathbf{U}_i^n = \mathbf{U}_I^n \quad \textit{and} \quad \mathbf{U}_{-i}^n = \mathbf{U}_{-I}^n \quad \text{for all } i > I.$$

This yields

$$D^+\mathbf{U}_i^n = D^-\mathbf{U}_i^n = \tilde{\Delta}\mathbf{U}_i^n = 0 \quad \text{for all } |i| \geq I. \tag{5.3.2}$$

By multiplying both sides of (5.3.1) by h, summing over i from $-I$ to I and using (2.2.1), we obtain

$$\begin{aligned} -\left\langle \mathbf{U}^{n+1} - \mathbf{U}^n, \tilde{\Delta}\mathbf{U}^{n+1} \right\rangle_{L^2_h} &= k \left\langle \mathbf{U}^{n+1/2} \times (\mathbf{U}^{n+1/2} \times \tilde{\Delta}\mathbf{U}^{n+1}), \tilde{\Delta}\mathbf{U}^{n+1} \right\rangle_{L^2_h} \\ &\quad - \left\langle \mathbf{U}^{n+1/2} \times \mathbf{g}^h, \tilde{\Delta}\mathbf{U}^{n+1} \right\rangle_{L^2_h} (W^{n+1} - W^n). \end{aligned} \tag{5.3.3}$$

For the term on the left hand side, using (5.3.2), Lemma 2.5.4 and the equality $2\langle a, a-b\rangle = |a|^2 - |b|^2 + |a-b|^2$ for $a, b \in \mathbb{R}^3$, we derive

$$\begin{aligned}
&-\left\langle \mathbf{U}^{n+1} - \mathbf{U}^n, \tilde{\Delta}\mathbf{U}^{n+1} \right\rangle_{L^2_h} \\
&= -\left\langle \mathbf{U}^{n+1} - \mathbf{U}^n, \tilde{\Delta}\mathbf{U}^{n+1} \right\rangle_{L^2_h(\mathbb{Z}_h)} \\
&= \left\langle D^+\mathbf{U}^{n+1} - D^+\mathbf{U}^n, D^+\mathbf{U}^{n+1} \right\rangle_{L^2_h(\mathbb{Z}_h)} \\
&= \left\langle D^+\mathbf{U}^{n+1} - D^+\mathbf{U}^n, D^+\mathbf{U}^{n+1} \right\rangle_{L^2_h} \\
&= \frac{1}{2}\sum_{i=-I}^{I} h\left(|D^+\mathbf{U}_i^{n+1}|^2 - |D^+\mathbf{U}_i^n|^2 + |D^+\mathbf{U}_i^{n+1} - D^+\mathbf{U}_i^n|^2\right) \\
&= \frac{1}{2}\left(|D^+\mathbf{U}^{n+1}|^2_{L^2_h} - |D^+\mathbf{U}^n|^2_{L^2_h} + |D^+\mathbf{U}^{n+1} - D^+\mathbf{U}^n|^2_{L^2_h}\right).
\end{aligned}$$

For the first term on the right hand side of (5.3.3), we have using (2.2.2)

$$k\left\langle \mathbf{U}^{n+1/2} \times \left(\mathbf{U}^{n+1/2} \times \tilde{\Delta}\mathbf{U}^{n+1}\right), \tilde{\Delta}\mathbf{U}^{n+1} \right\rangle_{L^2_h} = -k\left|\mathbf{U}^{n+1/2} \times \tilde{\Delta}\mathbf{U}^{n+1}\right|^2_{L^2_h}.$$

Combining and taking summation over m from 0 to n, we obtain from (5.3.3)

$$\begin{aligned}
&\frac{1}{2}\left(|D^+\mathbf{U}^{n+1}|^2_{L^2_h} - |D^+\mathbf{U}^0|^2_{L^2_h} + \sum_{m=0}^{n}|D^+\mathbf{U}^{m+1} - D^+\mathbf{U}^m|^2_{L^2_h}\right) + k\sum_{m=0}^{n}\left|\mathbf{U}^{m+1/2} \times \tilde{\Delta}\mathbf{U}^{m+1}\right|^2_{L^2_h} \\
&= -\sum_{m=0}^{n}\left\langle \mathbf{U}^{m+1/2} \times \mathbf{g}^h, \tilde{\Delta}\mathbf{U}^{m+1} \right\rangle_{L^2_h}(W^{m+1} - W^m). \qquad (5.3.4)
\end{aligned}$$

We proceed with the term on the right hand side. Using (5.3.2), Lemmas 2.5.4, 2.5.3 and the elementary property (2.2.1) we get

$$\begin{aligned}
&-\sum_{m=0}^{n}\left\langle \mathbf{U}^{m+1/2} \times \mathbf{g}^h, \tilde{\Delta}\mathbf{U}^{m+1} \right\rangle_{L^2_h}(W^{m+1} - W^m) \\
&= -\sum_{m=0}^{n}\left\langle \mathbf{U}^{m+1/2} \times \mathbf{g}^h, \tilde{\Delta}\mathbf{U}^{m+1} \right\rangle_{L^2_h(\mathbb{Z}_h)}(W^{m+1} - W^m) \\
&= \sum_{m=0}^{n}\left\langle D^+\left(\mathbf{U}^{m+1/2} \times \mathbf{g}^h\right), D^+\mathbf{U}^{m+1} \right\rangle_{L^2_h(\mathbb{Z}_h)}(W^{m+1} - W^m)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{m=0}^{n} \left\langle D^+ \left(\mathbf{U}^{m+1/2} \times \mathbf{g}^h\right), D^+\mathbf{U}^{m+1} \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&= \sum_{m=0}^{n} \left\langle D^+\mathbf{U}^{m+1/2} \times \tau^+\mathbf{g}^h, D^+\mathbf{U}^{m+1} \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&\quad + \sum_{m=0}^{n} \left\langle \mathbf{U}^{m+1/2} \times D^+\mathbf{g}^h, D^+\mathbf{U}^{m+1} \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&= \frac{1}{2} \sum_{m=0}^{n} \left\langle D^+\mathbf{U}^{m} \times \tau^+\mathbf{g}^h, D^+\mathbf{U}^{m+1} \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&\quad + \sum_{m=0}^{n} \left\langle \mathbf{U}^{m+1/2} \times D^+\mathbf{g}^h, D^+\mathbf{U}^{m+1} \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&= M_1 + M_2.
\end{aligned} \tag{5.3.5}
$$

Now, we estimate M_1. Using the elementary property (2.2.1), we obtain

$$
\begin{aligned}
M_1 &= \frac{1}{2} \sum_{m=0}^{n} \left\langle D^+\mathbf{U}^{m} \times \tau^+\mathbf{g}^h, D^+\mathbf{U}^{m+1} \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&= \frac{1}{2} \sum_{m=0}^{n} \left\langle D^+\mathbf{U}^{m} \times \tau^+\mathbf{g}^h, D^+\mathbf{U}^{m+1} - D^+\mathbf{U}^{m} \right\rangle_{L_h^2} (W^{m+1} - W^m).
\end{aligned}
$$

Then,

$$
|M_1| \le \frac{1}{8} \sum_{m=0}^{n} \left| D^+\mathbf{U}^{m+1} - D^+\mathbf{U}^{m} \right|_{L_h^2}^2 + \frac{1}{2} |\mathbf{g}^h|_{L_h^\infty}^2 \sum_{m=0}^{n} \left| D^+\mathbf{U}^{m} \right|_{L_h^2}^2 (W^{m+1} - W^m)^2. \tag{5.3.6}
$$

Furthermore, for the term M_2 we get

$$
\begin{aligned}
M_2 &= \frac{1}{2} \sum_{m=0}^{n} \left\langle (\mathbf{U}^{m+1} + \mathbf{U}^{m}) \times D^+\mathbf{g}^h, D^+\mathbf{U}^{m+1} \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&= \frac{1}{2} \sum_{m=0}^{n} \left\langle (\mathbf{U}^{m+1} - \mathbf{U}^{m}) \times D^+\mathbf{g}^h, D^+\mathbf{U}^{m+1} \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&\quad + \sum_{m=0}^{n} \left\langle \mathbf{U}^{m} \times D^+\mathbf{g}^h, D^+\mathbf{U}^{m+1} \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&= \frac{1}{2} \sum_{m=0}^{n} \left\langle (\mathbf{U}^{m+1} - \mathbf{U}^{m}) \times D^+\mathbf{g}^h, D^+(\mathbf{U}^{m+1} - \mathbf{U}^{m}) \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&\quad + \frac{1}{2} \sum_{m=0}^{n} \left\langle (\mathbf{U}^{m+1} - \mathbf{U}^{m}) \times D^+\mathbf{g}^h, D^+\mathbf{U}^{m} \right\rangle_{L_h^2} (W^{m+1} - W^m)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{m=0}^{n} \left\langle \mathbf{U}^m \times D^+\mathbf{g}^h, D^+(\mathbf{U}^{m+1} - \mathbf{U}^m) \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&+ \sum_{m=0}^{n} \left\langle \mathbf{U}^m \times D^+\mathbf{g}^h, D^+\mathbf{U}^m \right\rangle_{L_h^2} (W^{m+1} - W^m).
\end{aligned}
$$

Then, using part (1) of the lemma

$$
\begin{aligned}
|M_2| &\le \frac{1}{16} \sum_{m=0}^{n} \left| D^+\mathbf{U}^{m+1} - D^+\mathbf{U}^m \right|_{L_h^2}^2 + \sum_{m=0}^{n} (W^{m+1} - W^m)^2 |D^+\mathbf{g}^h|_{L_h^2}^2 |\mathbf{U}^{m+1} - \mathbf{U}^m|_{L_h^\infty}^2 \\
&\quad + \frac{1}{8} \sum_{m=0}^{n} |\mathbf{U}^{m+1} - \mathbf{U}^m|_{L_h^2}^2 + \frac{1}{2} \sum_{m=0}^{n} (W^{m+1} - W^m)^2 |D^+\mathbf{g}^h|_{L_h^\infty}^2 |D^+\mathbf{U}^m|_{L_h^2}^2 \\
&\quad + \frac{1}{16} \sum_{m=0}^{n} \left| D^+\mathbf{U}^{m+1} - D^+\mathbf{U}^m \right|_{L_h^2}^2 + 4 \sum_{m=0}^{n} (W^{m+1} - W^m)^2 |D^+\mathbf{g}^h|_{L_h^2}^2 \\
&\quad + \left| \sum_{m=0}^{n} \left\langle \mathbf{U}^m \times D^+\mathbf{g}^h, D^+\mathbf{U}^m \right\rangle_{L_h^2} (W^{m+1} - W^m) \right| \\
&\le \frac{1}{8} \sum_{m=0}^{n} \left| D^+\mathbf{U}^{m+1} - D^+\mathbf{U}^m \right|_{L_h^2}^2 + \frac{1}{8} \sum_{m=0}^{n} |\mathbf{U}^{m+1} - \mathbf{U}^m|_{L_h^2}^2 \\
&\quad + \left| \sum_{m=0}^{n} \left\langle \mathbf{U}^m \times D^+\mathbf{g}^h, D^+\mathbf{U}^m \right\rangle_{L_h^2} (W^{m+1} - W^m) \right| \\
&\quad + C \sum_{m=0}^{n} (W^{m+1} - W^m)^2 \left(|D^+\mathbf{g}^h|_{L_h^2}^2 + |D^+\mathbf{g}^h|_{L_h^\infty}^2 |D^+\mathbf{U}^m|_{L_h^2}^2 \right). \qquad (5.3.7)
\end{aligned}
$$

We need to control $|\mathbf{U}^{m+1} - \mathbf{U}^m|_{L_h^2}$. We multiply (5.1.4) by $\mathbf{U}^{m+1} - \mathbf{U}^m$ to obtain

$$
\begin{aligned}
\left| \mathbf{U}_i^{m+1} - \mathbf{U}_i^m \right|^2 &= k \left\langle \mathbf{U}_i^{m+1/2} \times \tilde{\Delta}\mathbf{U}_i^{m+1}, \mathbf{U}_i^{m+1} - \mathbf{U}_i^m \right\rangle \\
&\quad - k \left\langle \mathbf{U}_i^{m+1/2} \times \left(\mathbf{U}_i^{m+1/2} \times \tilde{\Delta}\mathbf{U}_i^{m+1} \right), \mathbf{U}_i^{m+1} - \mathbf{U}_i^m \right\rangle \\
&\quad + \left\langle \mathbf{U}_i^{m+1/2} \times \mathbf{g}^h, \mathbf{U}_i^{m+1} - \mathbf{U}_i^m \right\rangle (W^{m+1} - W^m).
\end{aligned}
$$

Multiplying by h and taking summation over i from $-I$ to I, we obtain

$$
\begin{aligned}
\left| \mathbf{U}^{m+1} - \mathbf{U}^m \right|_{L_h^2}^2 &= k \left\langle \mathbf{U}^{m+1/2} \times \tilde{\Delta}\mathbf{U}^{m+1}, \mathbf{U}^{m+1} - \mathbf{U}^m \right\rangle_{L_h^2} \\
&\quad - k \left\langle \mathbf{U}^{m+1/2} \times \left(\mathbf{U}^{m+1/2} \times \tilde{\Delta}\mathbf{U}^{m+1} \right), \mathbf{U}^{m+1} - \mathbf{U}^m \right\rangle_{L_h^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \left\langle \mathbf{U}^{m+1/2} \times \mathbf{g}^h, \mathbf{U}^{m+1} - \mathbf{U}^m \right\rangle_{L_h^2} (W^{m+1} - W^m) \\
&\le \frac{1}{6} \left| \mathbf{U}^{m+1} - \mathbf{U}^m \right|_{L_h^2}^2 + \frac{3}{2} k^2 \left| \mathbf{U}^{m+1/2} \times \tilde{\Delta} \mathbf{U}^{m+1} \right|_{L_h^2}^2 \\
&\quad + \frac{1}{6} \left| \mathbf{U}^{m+1} - \mathbf{U}^m \right|_{L_h^2}^2 + \frac{3}{2} k^2 \left| \mathbf{U}^{m+1/2} \right|_{L_h^\infty}^2 \left| \mathbf{U}^{m+1/2} \times \tilde{\Delta} \mathbf{U}^{m+1} \right|_{L_h^2}^2 \\
&\quad + \frac{1}{6} \left| \mathbf{U}^{m+1} - \mathbf{U}^m \right|_{L_h^2}^2 + \frac{3}{2} \left| \mathbf{U}^{m+1/2} \times \mathbf{g}^h \right|_{L_h^2}^2 (W^{m+1} - W^m)^2.
\end{aligned}
$$

Then, we get using part (1) of the lemma

$$
\left| \mathbf{U}^{m+1} - \mathbf{U}^m \right|_{L_h^2}^2 \le C k^2 \left| \mathbf{U}^{m+1/2} \times \tilde{\Delta} \mathbf{U}^{m+1} \right|_{L_h^2}^2 + C \left| \mathbf{U}^{m+1/2} \times \mathbf{g}^h \right|_{L_h^2}^2 (W^{m+1} - W^m)^2.
$$

This inequality, (5.3.7) and part (1) of the lemma yield

$$
\begin{aligned}
|M_2| &\le \frac{1}{8} \sum_{m=0}^{n} \left| D^+ \mathbf{U}^{m+1} - D^+ \mathbf{U}^m \right|_{L_h^2}^2 + C k^2 \sum_{m=0}^{n} \left| \mathbf{U}^{m+1/2} \times \tilde{\Delta} \mathbf{U}^{m+1} \right|_{L_h^2}^2 \\
&\quad + \left| \sum_{m=0}^{n} \left\langle \mathbf{U}^m \times D^+ \mathbf{g}^h, D^+ \mathbf{U}^m \right\rangle_{L_h^2} (W^{m+1} - W^m) \right| \\
&\quad + C \sum_{m=0}^{n} (W^{m+1} - W^m)^2 \left(\left| \mathbf{g}^h \right|_{L_h^2}^2 + |D^+ \mathbf{g}^h|_{L_h^2}^2 + |D^+ \mathbf{g}^h|_{L_h^\infty}^2 |D^+ \mathbf{U}^m|_{L_h^2}^2 \right). \qquad (5.3.8)
\end{aligned}
$$

It follows from (5.3.4), (5.3.5), (5.3.6) and (5.3.8),

$$
\begin{aligned}
&\frac{1}{2} |D^+ \mathbf{U}^{n+1}|_{L_h^2}^2 - \frac{1}{2} |D^+ \mathbf{U}^0|_{L_h^2}^2 + \frac{1}{4} \sum_{m=0}^{n} \left| D^+ \mathbf{U}^{m+1} - D^+ \mathbf{U}^m \right|_{L_h^2}^2 \\
&\quad + (1 - Ck) k \sum_{m=0}^{n} \left| \mathbf{U}^{m+1/2} \times \tilde{\Delta} \mathbf{U}^{m+1} \right|_{L_h^2}^2 \\
&\le C \left(\left| \mathbf{g}^h \right|_{L_h^2}^2 + |D^+ \mathbf{g}^h|_{L_h^2}^2 \right) \sum_{m=0}^{n} (W^{m+1} - W^m)^2 \\
&\quad + C \left(|\mathbf{g}^h|_{L_h^\infty}^2 + |D^+ \mathbf{g}^h|_{L_h^\infty}^2 \right) \sum_{m=0}^{n} (W^{m+1} - W^m)^2 \left| D^+ \mathbf{U}^m \right|_{L_h^2}^2 \\
&\quad + \left| \sum_{m=0}^{n} \left\langle \mathbf{U}^m \times D^+ \mathbf{g}^h, D^+ \mathbf{U}^m \right\rangle_{L_h^2} (W^{m+1} - W^m) \right|.
\end{aligned}
$$

Applying $\sup_{0\le n\le N-1}$ and expectation, we get

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\Big[\sup_{0\le n\le N-1}|D^+\mathbf{U}^{n+1}|^2_{L^2_h}\Big]+\frac{1}{4}\mathbb{E}\Big[\sum_{n=0}^{N-1}\left|D^+(\mathbf{U}^{n+1}-\mathbf{U}^n)\right|^2_{L^2_h}\Big]\\
&\quad+(1-Ck)\mathbb{E}\Big[\sum_{n=0}^{N-1}k\left|\mathbf{U}^{n+1/2}\times\tilde{\Delta}\mathbf{U}^{n+1}\right|^2_{L^2_h}\Big]\\
&\le C|D^+\mathbf{U}^0|^2_{L^2_h}+C\left(|\mathbf{g}^h|^2_{L^2_h}+|D^+\mathbf{g}^h|^2_{L^2_h}\right)\mathbb{E}\Big[\sum_{n=0}^{N-1}(W^{n+1}-W^n)^2\Big]\\
&\quad+C\left(|D^+\mathbf{g}^h|^2_{L^\infty_h}+|\mathbf{g}^h|^2_{L^\infty_h}\right)\mathbb{E}\Big[\sum_{n=0}^{N-1}(W^{n+1}-W^n)^2\left|D^+\mathbf{U}^n\right|^2_{L^2_h}\Big]\\
&\quad+\mathbb{E}\Big[\sup_{0\le n\le N-1}\left|\sum_{m=0}^{n}\left\langle\mathbf{U}^m\times D^+\mathbf{g}^h,D^+\mathbf{U}^m\right\rangle_{L^2_h}(W^{m+1}-W^m)\right|\Big].
\end{aligned}
\tag{5.3.9}
$$

We proceed with the last term. We define from $\mathbf{U}^n, 0\le n\le N$, a piecewise constant function defined on $[0,T]$ as follows. For each t, let t_n be such that $t\in[t_n,t_{n+1})$. Then,

$$
r_k\mathbf{U}(t,\cdot)=\mathbf{U}(t_n,\cdot)=\mathbf{U}^n.
$$

Consequently, we have

$$
\left\langle\mathbf{U}^m\times D^+\mathbf{g}^h,D^+\mathbf{U}^m\right\rangle_{L^2_h}=\left\langle r_k\mathbf{U}(t)\times D^+\mathbf{g}^h,r_kD^+\mathbf{U}(t)\right\rangle_{L^2_h}
$$

for $t\in[t_m,t_{m+1})$. Then,

$$
\begin{aligned}
&\sum_{m=0}^{n}\left\langle\mathbf{U}^m\times D^+\mathbf{g}^h,D^+\mathbf{U}^m\right\rangle_{L^2_h}(W^{m+1}-W^m)\\
&=\sum_{m=0}^{n}\int_{t_m}^{t_{m+1}}\left\langle\mathbf{U}^m\times D^+\mathbf{g}^h,D^+\mathbf{U}^m\right\rangle_{L^2_h}dW(t)\\
&=\int_0^{t_{n+1}}\left\langle r_k\mathbf{U}(t)\times D^+\mathbf{g}^h,r_kD^+\mathbf{U}(t)\right\rangle_{L^2_h}dW(t).
\end{aligned}
$$

Using Lemma 2.3.18 and part (1) of this lemma, we get

$$
\mathbb{E}\Big[\sup_{0\le n\le N-1}\left|\sum_{m=0}^{n}\left\langle\mathbf{U}^m\times D^+\mathbf{g}^h,D^+\mathbf{U}^m\right\rangle_{L^2_h}(W^{m+1}-W^m)\right|\Big]
$$

$$
\begin{aligned}
&= \mathbb{E}\Bigg[\sup_{0\le n\le N-1}\left|\int_0^{t_{n+1}} \left\langle r_k\mathbf{U}(t)\times D^+\mathbf{g}^h, r_kD^+\mathbf{U}(t)\right\rangle_{L_h^2} dW(t)\right|\Bigg] \\
&\le C\mathbb{E}\Bigg[\left(\int_0^T \left|\left\langle r_k\mathbf{U}(t)\times D^+\mathbf{g}^h, r_kD^+\mathbf{U}(t)\right\rangle_{L_h^2}\right|^2 dt\right)^{\frac{1}{2}}\Bigg] \\
&\le C\mathbb{E}\Bigg[1+\int_0^T \left|\left\langle r_k\mathbf{U}(t)\times D^+\mathbf{g}^h, r_kD^+\mathbf{U}(t)\right\rangle_{L_h^2}\right|^2 dt\Bigg] \\
&\le C + C\mathbb{E}\left[\int_0^T \left|D^+\mathbf{g}^h\right|_{L_h^2}^2 \left|r_kD^+\mathbf{U}(t)\right|_{L_h^2}^2 dt\right] \\
&\le C + C\left|D^+\mathbf{g}^h\right|_{L_h^2}^2 \mathbb{E}\left[\int_0^T \left|r_kD^+\mathbf{U}(t)\right|_{L_h^2}^2 dt\right] \\
&\le C + C\left|D^+\mathbf{g}^h\right|_{L_h^2}^2 \mathbb{E}\left[\sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}} \left|D^+\mathbf{U}^n\right|_{L_h^2}^2 dt\right] \\
&\le C + C\left|D^+\mathbf{g}^h\right|_{L_h^2}^2 \mathbb{E}\left[\sum_{n=0}^{N-1} k\left|D^+\mathbf{U}^n\right|_{L_h^2}^2\right] \\
&\le C + C\left|D^+\mathbf{g}^h\right|_{L_h^2}^2 k\sum_{n=0}^{N-1}\mathbb{E}\left[\sup_{0\le r\le n}|D^+\mathbf{U}^r|_{L_h^2}^2\right].
\end{aligned}
$$

We deduce from (5.3.9) that, for k sufficiently small so that $(1-Ck)>0$,

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\Bigg[\sup_{0\le n\le N-1}|D^+\mathbf{U}^{n+1}|_{L_h^2}^2\Bigg] + \frac{1}{4}\mathbb{E}\Bigg[\sum_{n=0}^{N-1}\left|D^+(\mathbf{U}^{n+1}-\mathbf{U}^n)\right|_{L_h^2}^2\Bigg] \\
&\quad + (1-Ck)\mathbb{E}\Bigg[\sum_{n=0}^{N-1} k\left|\mathbf{U}^{n+1/2}\times\tilde{\Delta}\mathbf{U}^{n+1}\right|_{L_h^2}^2\Bigg] \\
&\le C + C|D^+\mathbf{U}^0|_{L_h^2}^2 + C\left(|\mathbf{g}^h|_{L_h^2}^2 + |D^+\mathbf{g}^h|_{L_h^2}^2\right)\mathbb{E}\Bigg[\sum_{n=0}^{N-1}(W^{n+1}-W^n)^2\Bigg] \\
&\quad + C\left(|D^+\mathbf{g}^h|_{L_h^\infty}^2 + |\mathbf{g}^h|_{L_h^\infty}^2\right)\mathbb{E}\Bigg[\sum_{n=0}^{N-1}(W^{n+1}-W^n)^2\left|D^+\mathbf{U}^n\right|_{L_h^2}^2\Bigg] \\
&\quad + C\left|D^+\mathbf{g}^h\right|_{L_h^2}^2 k\sum_{n=0}^{N-1}\mathbb{E}\left[\sup_{0\le r\le n}|D^+\mathbf{U}^r|_{L_h^2}^2\right]. \qquad (5.3.10)
\end{aligned}
$$

We know that

$$
\begin{aligned}
\mathbb{E}\Bigg[(W^{n+1}-W^n)^2\left|D^+\mathbf{U}^n\right|_{L_h^2}^2\Bigg] &= \mathbb{E}\Bigg[\mathbb{E}\Bigg[(W^{n+1}-W^n)^2\left|D^+\mathbf{U}^n\right|_{L_h^2}^2\Bigg|\mathcal{F}_{t_n}\Bigg]\Bigg] \\
&= \mathbb{E}\Bigg[\left|D^+\mathbf{U}^n\right|_{L_h^2}^2\mathbb{E}\left[(W^{n+1}-W^n)^2\middle|\mathcal{F}_{t_n}\right]\Bigg]
\end{aligned}
$$

$$= k\mathbb{E}\left[\left|D^+\mathbf{U}^n\right|^2_{L^2_h} \right],$$

and for k sufficiently small so that $(1 - Ck) > 0$,

$$(1 - Ck)\mathbb{E}\left[\sum_{n=0}^{N-1} k \left| \mathbf{U}^{n+1/2} \times \tilde{\Delta}\mathbf{U}^{n+1} \right|^2_{L^2_h} \right] \geq 0,$$

consequently, we deduce from (5.3.10)

$$\mathbb{E}\left[\sup_{1\leq n\leq N} |D^+\mathbf{U}^n|^2_{L^2_h} \right] \leq C + C|D^+\mathbf{U}^0|^2_{L^2_h} + C\left(|\mathbf{g}^h|^2_{L^2_h} + |D^+\mathbf{g}^h|^2_{L^2_h} \right) T$$
$$+ C\left(|D^+\mathbf{g}^h|^2_{L^2_h} + |D^+\mathbf{g}^h|^2_{L^\infty_h} + |\mathbf{g}^h|^2_{L^\infty_h} \right) k \sum_{n=0}^{N-1} \mathbb{E}\left[\sup_{0\leq r\leq n} |D^+\mathbf{U}^r|^2_{L^2_h} \right].$$

Finally, using Lemma 2.4.1 we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{1\leq n\leq N} |D^+\mathbf{U}^n|^2_{L^2_h} \right] \\
&\leq \left(C + C|D^+\mathbf{U}^0|^2_{L^2_h} + CT\left(|\mathbf{g}^h|^2_{L^2_h} + |D^+\mathbf{g}^h|^2_{L^2_h} \right) \right) \mathrm{e}^{CT\left(|D^+\mathbf{g}^h|^2_{L^2_h} + |D^+\mathbf{g}^h|^2_{L^\infty_h} + |\mathbf{g}^h|^2_{L^\infty_h} \right)} \\
&= C\left(|\nabla \mathbf{u}_0|_{L^2}, |\mathbf{g}|_{H^2}, T \right)
\end{aligned}$$

where in the last step we used Theorem 2 on page 6 of [51].

We note that from (5.3.10), for k sufficiently small so that $(1 - Ck) > 0$, we deduce

$$\mathbb{E}\left[\sum_{n=0}^{N-1} k \left| \mathbf{U}^{n+1/2} \times \tilde{\Delta}\mathbf{U}^{n+1} \right|^2_{L^2_h} \right] \leq C\left(|\nabla \mathbf{u}_0|_{L^2}, |\mathbf{g}|_{H^2}, T \right)$$

and the lemma follows. □

5.4 Numerical Experiments

In this section, we carry out numerical experiments to solve the one-dimensional stochastic problem (3.1.1)-(3.1.3).

In the first two experiments, we show convergence of the solution of (5.1.4)-(5.1.7) to the solution of (4.1.1)-(4.1.4) when h and k tends to zero. We solve an example of the stochastic LLG equation (4.1.1) on the domain $D = [-1, 1]$. We consider the initial condition $\mathbf{u}_0 = \big(\sin(e^{-|x|}), \cos(e^{-|x|}), 0\big)$ and the function $\mathbf{g}$ is given by $\mathbf{g} = (e^{-|x|}, 0, 0)$. We set the values for the parameters in (5.1.4) as $\lambda = \frac{1}{12}$ and $\mu = \frac{1}{8}$.

In the following experiments, we consider $T = 1$ and we generate for each time step k a discrete Brownian path by:

$$W_k(t_{n+1}) - W_k(t_n) \sim \mathcal{N}(0, k) \qquad \text{for all } n = 0, ..., N-1.$$

We approximate any expected value by the average of A discrete Brownian paths. In our experiments, we choose $A = 50$.

We compute $\mathbf{U}_i^n$ by using the following algorithm:

1. Set $n = 0$. Choose $\mathbf{U}_i^0 = \mathbf{u}_0^h(ih)$.
2. Find $\mathbf{U}_i^n$ satisfying the stochastic equation (5.1.4).
3. Set $n = n + 1$, and return to step 2 if $n \leq N - 1$. Stop if $n = N$.

We note that the procedure of calculating $\mathbf{U}^{n+1}$ from $\mathbf{U}^n$ is implicit and nonlinear. We employ a fixed point algorithm in the experiments and choose the tolerance to be 10^{-6}.

Experiment 1: To observe convergence of the numerical method when h tends to zero, we solve with $N = 8000$ and $h = \dfrac{1}{I}$ where $I = 5, 10, 20, 40, 80$. Since the exact solution of problem (3.1.1)-(3.1.3) is unknown, we compute a reference solution $\mathcal{U}(t, x)$ by solving this problem when $L = 1$ but with a finer mesh size by considering $I = 160$ and $N = 16000$. For each value of h, the domain D is uniformly partitioned into intervals of size h. We note that

$$\mathcal{E}_{k,h}^2 := \mathbb{E}\Bigg[\sup_{0\leq n\leq N} |\mathbf{U}^n - \mathcal{U}^n|^2_{L^2_h(-1,1)}\Bigg].$$

We plot in Figure 5.1 the error $\mathcal{E}_{k,h}^2$ for different values of h. The figure shows a clear convergence of the method when h tends to zero. In Figure 5.2, we plot the logarithm of the error $\mathcal{E}_{k,h}^2$ with respect to the logarithm of different values of h. The figure shows that the rate of convergence with respect to h is 2.

Experiment 2: We observe convergence of the numerical method when k tends to zero. In fact, we solve with $h = \dfrac{1}{I}$ where $I = 20$ and $N = 500, 1000, 2000, 4000,\ 8000$. We choose h that big because we assume that we have the condition on the parameters h and k considered in the deterministic case (see [21]) since we still do not have the study for the stochastic case. We note that the reference solution $\mathcal{U}(t,x)$ is the one considered in the first experiment. For each value of k, the domain $[0,T]$ is uniformly partitioned into intervals of size k. We plot in Figure 5.3 the error $\mathcal{E}^2_{k,h}$ for different values of k. The figure shows convergence of the numerical method when k tends to zero. We note that the value of the error for the smallest k is 0.0001329. In Figure 5.4, we plot the logarithm of the error $\mathcal{E}^2_{k,h}$ with respect to the logarithm of different values of k and the figure shows that the rate of convergence with respect to k is the same as the rate for h and is equal to 2. In fact, our numerical results are consistent with the analytical results proved in [21] for the deterministic case. We note that the stochastic case is still under study.

In the last two experiments, we show convergence of the solution of (4.1.1)-(4.1.4) on a bounded domain $[-L, L]$ to the solution of (3.1.1)-(3.1.3) on the whole real line when L is large enough. The function $\mathbf{g}$ is considered as above and the initial condition is $\mathbf{u}_0 = (\sin(x), \cos(x), 0)$. We note that $\mathbf{u}_0$ should satisfy the conditions in Lemma 4.2.1 on $\mathbb{R}$ but in the experiments it is enough that it satisfies the conditions on a bounded domain. We set the values for the parameters λ and μ as previously. We note that we choose the tolerance to be 10^{-12}. In the following experiments, we compute a reference solution $\mathcal{U}(t,x)$ by solving this problem when $L = 100$ but with a finer mesh size by considering $k = 0.0005$ and $h = 0.025$.

Experiment 3: To observe convergence when $L \to \infty$, we fix k and h to be equal to 0.001 and 0.05 respectively. We note that

$$E^2_{k,h} := \mathbb{E}\left[\sup_{0\le n\le N} |\mathbf{U}^n - \mathcal{U}^n|^2_{L^2_h(-20,20)}\right].$$

We solve with $L = 5, 10, 20, 40, 80$ and compute the error in the interval $[-20, 20]$. It is clear that when L increases, the errors decrease. However, we observe that the difference

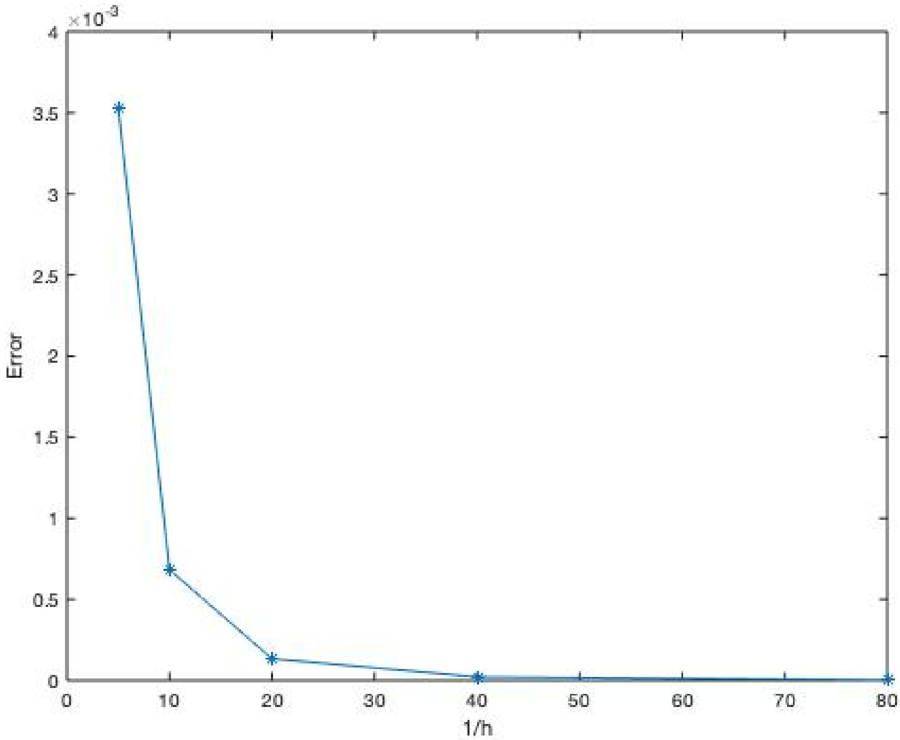

Figure 5.1: Plot of error $\mathcal{E}^2_{k,h}$.

in errors between $L = 40$ and $L = 80$ is not clear. Indeed, the error for $L = 80$ does not reduce much because of large h. This is confirmed when we compute the error with $L = 80$ and $h = 0.025$. We present in Table 5.1, the error $E^2_{k,h}$ for different values of L. We note that, when necessary, we extend the solutions by their values at the endpoints. The error values show a clear convergence of the method.

Experiment 4: To observe more data, we fix again k and h to be equal to 0.001 and 0.05 respectively. We solve with $L = 73, 73.5, 74, 74.5, ..., 79, 79.5$. We note that

$$\mathbf{E}^2_{k,h} := \mathbb{E}\left[\sup_{0\le n\le N} |\mathbf{U}^n - \mathcal{U}^n|^2_{L^2_h(-76,76)}\right].$$

We present in Table 5.2, the error $\mathbf{E}^2_{k,h}$ for different values of L. The table shows convergence of the method.

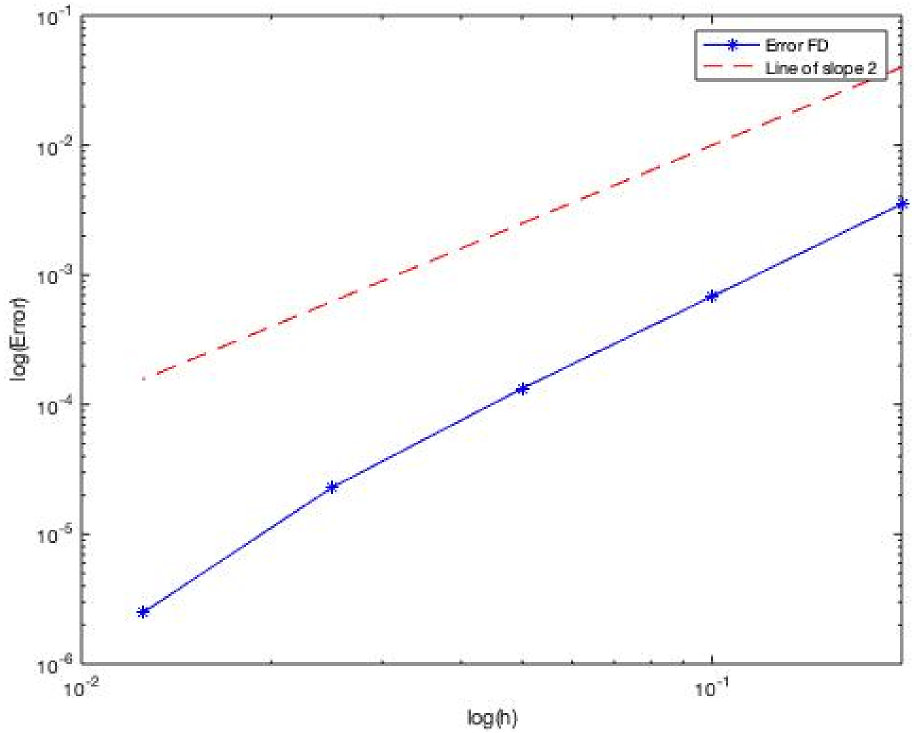

Figure 5.2: Plot of $\log(\mathcal{E}^2_{k,h})$.

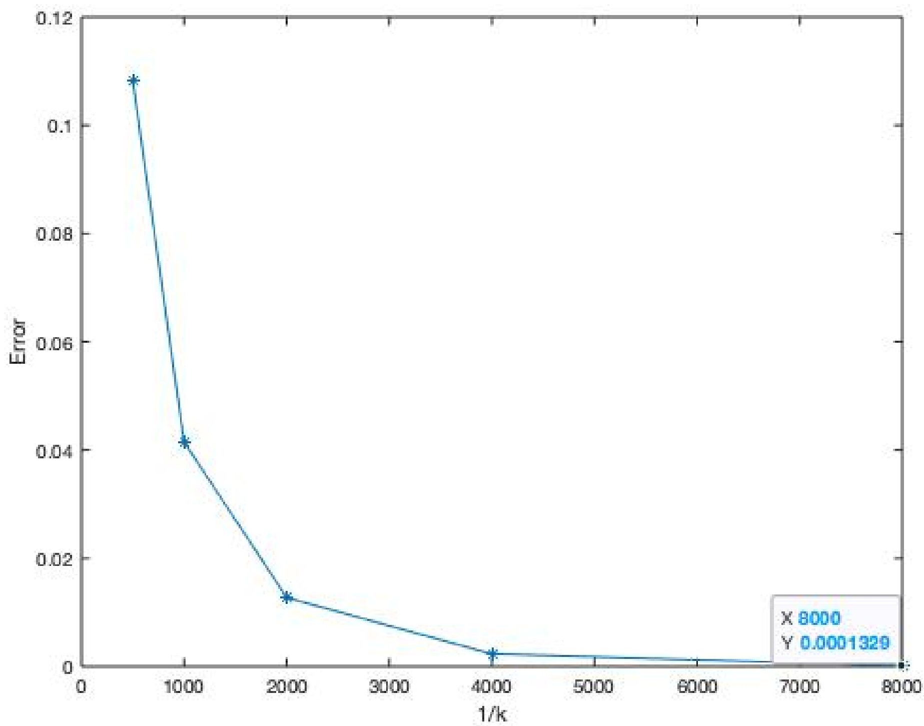

Figure 5.3: Plot of error $\mathcal{E}^2_{k,h}$.

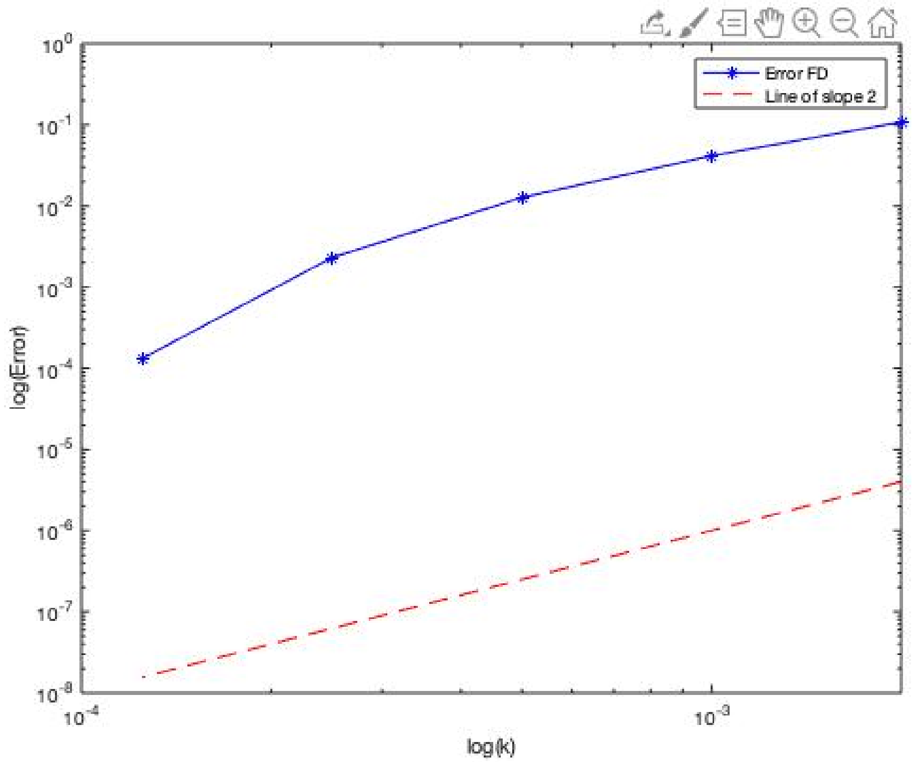

Figure 5.4: Plot of $\log(\mathcal{E}^2_{k,h})$.

Table 5.1: Error $E^2_{k,h}$

L	Error
5	$6.0699e^{+01}$
10	$4.5071e^{+01}$
20	$1.0522e^{-01}$
40	$3.9200e^{-09}$
80 (h=0.05)	$3.9199e^{-09}$
80 (h=0.025)	$4.3412e^{-10}$

Table 5.2: Error $\mathbf{E}_{k,h}^2$

L	Error
74	7.2135
74.5	4.0969
75	1.8761
75.5	0.6051
76	0.1052
76.5	0.0097
77	0.0005
77.5	$1.2952e^{-05}$
78	$1.5951e^{-07}$
78.5	$2.6251e^{-09}$
79	$2.6243e^{-09}$
79.5	$2.6241e^{-09}$

References

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