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Author:
Jegaraj, Terence Joseph

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ON SMALL TIME ASYMPTOTICS OF SOLUTIONS OF STOCHASTIC EQUATIONS IN INFINITE DIMENSIONS

Terence Jegaraj

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Abstract

This thesis investigates the small time asymptotics of solutions of stochastic equations in infinite dimensions. In this abstract H denotes a separable Hilbert space, A denotes a linear operator on H generating a strongly continuous semigroup and $(W(t))_{t\geq 0}$ denotes a separable Hilbert space-valued Wiener process.

In chapter 2 we consider the mild solution $(X_x(t))_{t\in[0,1]}$ of a stochastic initial value problem

$$\begin{aligned} dX &= AX\,dt + dW \quad t \in (0,1] \\ X(0) &= x \in H\,, \end{aligned}$$

where the equation has an invariant measure μ. Under some conditions $\mathcal{L}(X_x(t))$ has a density $k(t,x,\cdot)$ with respect to μ and we can find the limit $\lim_{t\to 0} t \ln k(t,x,y)$. For infinite dimensional H this limit only provides the lower bound of a large deviation principle (LDP) for the family of continuous trajectory-valued random variables $\{\, t \in [0,1] \to X_x(\epsilon t) : \epsilon \in (0,1]\}$.

In each of chapters 3, 4 and 5 we find an LDP which describes the small time asymptotics of the continuous trajectories of the solution of a stochastic initial value problem. A crucial role is played by the LDP associated with the Gaussian trajectory-valued random variable of the noise.
Chapter 3 considers the initial value problem

$$\begin{aligned} dX(t) &= (AX(t) + F(t,X(t)))\,dt + G(X(t))\,dW(t) \quad t \in (0,1] \\ X(0) &= x \in H, \end{aligned}$$

where drift function $F(t,\cdot)$ is Lipschitz continuous on H uniformly in $t \in [0,1]$ and diffusion function G is Lipschitz continuous, taking values that are Hilbert-Schmidt operators.
Chapter 4 considers an equation with dissipative drift function F defined on a separable Banach space continuously embedded in H; the solution has continuous trajectories in the

Banach space.

Chapter 5 considers a linear initial value problem with fractional Brownian motion noise.

In chapter 6 we return to equations with Wiener process noise and find a lower bound for $\liminf_{t\to 0} t \ln P\{X(0) \in B, X(t) \in C\}$ for arbitrary $\mathcal{L}(X(0))$ and Borel subsets B and C of H. We also obtain an upper bound for $\limsup_{t\to 0} t \ln P\{X(0) \in B, X(t) \in C\}$ when the equation has an invariant measure μ, $\mathcal{L}(X(0))$ is absolutely continuous with respect to μ and the transition semigroup is holomorphic.

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Contents

Chapter 1

Introduction

In section 1.1 we look at some of the literature on small time asymptotics of diffusion processes which motivates this thesis. Section 1.2 is devoted to a list of notation. In section 1.3 we outline the contents of the chapters. We end this chapter with a review of some relevant theory.

1.1 Previous work motivating this thesis

The theme of this thesis is the small time asymptotics of solutions of stochastic differential equations in a separable real Hilbert space. Let H be a separable Hilbert space. Consider the stochastic initial value problem

$$\left.\begin{array}{rcl} dX(t) & = & (AX(t) + F(t, X(t)))\, dt + G(t, X(t))\, dW(t)\,, \quad t \in (0, 1], \\ X(0) & = & \xi\,, \end{array}\right\} \tag{1.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on H, F is a H-valued function on $[0, 1] \times H$, G is a Hilbert-Schmidt operator-valued function on $[0, 1] \times H$, $(W(t))_{t\geq 0}$ is a separable Hilbert space-valued Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$ with associated filtration $(\mathcal{F}_t)_{t\geq 0}$ and ξ is an H-valued $\mathcal{F}_0$-measurable random variable. Details on the properties of A, F, G, ξ and the image space of $W(t)$ are omitted for the time being. The mild solution of problem (1.1) is defined to be the solution of the equation

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s, X(s))\, ds + \int_0^t S(t-s)G(s, X(s))\, dW(s) \quad P \text{ a.e.} \tag{1.2}$$

for each t in $[0,1]$. Of course the properties of A, F, G, ξ and the solution process itself must allow the Bochner integral and Itô integral on the right hand side of equation (1.2) to exist. Typically one uses tools of analysis such as a fixed point theorem on an appropriate function space to show that the mild solution exists and is unique and has a version with continuous trajectories. Even when we know existence, uniqueness and continuity of the mild solution, it is generally hard to quantify the behaviour of the solution at positive times t because the distributions involved are complicated. Thus it is a consolation that there may be a relatively simple estimate of the limiting behaviour of the solution in time interval $[0,t]$ as t goes to zero.

Varadhan [31] was one of the pioneers in small time asymptotics of diffusion processes. He studied the small time asymptotics of diffusion processes in $\mathbb{R}^n$, for n a natural number. In [31] Varadhan considered the solution $(z_\zeta(t))_{t\in[0,1]}$ of a stochastic initial value problem

$$\begin{aligned} dz(t) &= b(z(t))\,dt + \sigma(z(t))\,dB(t)\,, \quad t \in (0,1], \\ z(0) &= \zeta \in \mathbb{R}^n\,, \end{aligned}$$

where $(B(t))_{t\geq 0}$ is a Brownian motion in $\mathbb{R}^k$ for some natural number k and, among other conditions,

1. the function $b : \mathbb{R}^n \to \mathbb{R}^n$ is Hölder continuous and bounded,
2. σ is a real $n \times k$ matrix-valued function on $\mathbb{R}^n$ such that for some positive real numbers $\alpha_1 < \alpha_2$ we have

$$\alpha_1 \sum_{j=1}^{n} \eta_j^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} (\sigma(x)\,\sigma^*(x))_{ij}\,\eta_j\eta_i \leq \alpha_2 \sum_{j=1}^{n} \eta_j^2 \quad \forall(\eta_1,\ldots,\eta_n) \in \mathbb{R}^n \text{ and } \forall x \in \mathbb{R}^n$$

   and

3. the distribution of $z_\zeta(t)$ has density $y \mapsto p(t,\zeta,y)$ with respect to Lebesgue measure on $\mathbb{R}^n$ for each $t \in (0,1]$.

To simplify notation set $a(x) := \sigma(x)\,\sigma^*(x)$ for all $x \in \mathbb{R}^n$. For x and y in $\mathbb{R}^n$ define

$$\begin{aligned} d(x,y) \;:=\; \inf\Bigg\{ \int_0^1 \sqrt{\langle \dot{u}(\tau),\, a^{-1}(u(\tau))\dot{u}(\tau)\rangle_{\mathbb{R}^n}}\, d\tau \;:\; u : [0,1] \to \mathbb{R}^n \text{ is} \\ \text{absolutely continuous with derivative } \dot{u} \text{ and } u(0) = x \text{ and } u(1) = y \Bigg\}\,; \quad (1.3) \end{aligned}$$

in this equation $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ is the usual inner product in $\mathbb{R}^n$. Varadhan showed that

$$\lim_{t \to 0} t \ln p(t, x, y) = -\frac{1}{2} d^2(x, y)\,, \tag{1.4}$$

where the convergence is uniform in x and y on bounded subsets of $\mathbb{R}^n$.

For each $\epsilon \in (0, 1]$ set $z^\epsilon_\zeta(t) := z_\zeta(\epsilon t)$ for all $t \in [0, 1]$ and let z^ϵ_ζ be the random variable whose values are the continuous trajectories in $\mathbb{R}^n$ of the process $(z^\epsilon_\zeta(t))_{t \in [0,1]}$. Varadhan used the limit in equation (1.4) to obtain a large deviation principle for the family of distributions $\{\mathcal{L}(z^\epsilon_\zeta) : \epsilon \in (0, 1]\}$ on the Banach space $C([0, 1]; \mathbb{R}^n)$ of continuous functions mapping $[0, 1]$ into $\mathbb{R}^n$ with the supremum norm. He showed that for closed subsets C of $C([0, 1]; \mathbb{R}^n)$ we have

$$\lim_{r \to 0} \sup_{\epsilon < r} \epsilon \ln P\{z^\epsilon_\zeta \in C\} \leq - \inf_{u \in C} \mathcal{J}(u)$$

and for open subsets G we have

$$\lim_{r \to 0} \inf_{\epsilon < r} \epsilon \ln P\{z^\epsilon_\zeta \in G\} \geq - \inf_{u \in G} \mathcal{J}(u).$$

In these inequalitites P is the probability measure in the underlying probability space and the rate function is

$$\mathcal{J}(u) := \begin{cases} \frac{1}{2} \int_0^1 \langle \dot{u}(\tau),\, a^{-1}(u(\tau)) \dot{u}(\tau) \rangle_{\mathbb{R}^n}\, d\tau & \text{if } u : [0, 1] \to \mathbb{R}^n \text{ is absolutely continuous} \\ & \text{and } u(0) = \zeta \text{ and } \dot{u} \text{ is square integrable,} \\ & \\ \infty & \text{for all other } u \in C([0, 1]; \mathbb{R}^n). \end{cases} \tag{1.5}$$

More recently, working in infinite dimensional separable Hilbert spaces, Fang and Zhang [13] followed the same line of investigation as Varadhan. The framework of Fang and Zhang is as follows:

1. there are two Hilbert spaces H and H_1 such that the embedding of H into H_1 is Hilbert-Schmidt and

2. there is a linear operator A on H which is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on H and

3. $(W(t))_{t \geq 0}$ is a H_1-valued Wiener process such that the reproducing kernel Hilbert space $(H_\nu, |\cdot|_{H_\nu})$ of $\nu := \mathcal{L}(W(1))$ is continuously embedded in H and the embedding

of H_ν in H_1 is trace class.

Fang and Zhang showed that the short time asymptotics of the continuous solution of the equation

$$\begin{aligned} dY &= AY\,dt + dW\,, \quad t \in (0,1], \\ Y(0) &= x \in H_1 \end{aligned} \tag{1.6}$$

is described by a large deviation principle in trajectory space $C([0,1];H_1)$ with the same rate function as the large deviation principle describing the short time asymptotics of the shifted Wiener process $(x+W(t))_{t\in[0,1]}$. Fang and Zhang implicitly made use of the exponential equivalence concept from large deviations theory.

In the paper [13] Fang and Zhang also considered the situation where there exists an invariant measure μ on H_1 for equation (1.6) and the transition operators on the space of real-valued square integrable functions $L^2(H_1,\mu)$ are symmetric. Under these conditions Fang and Zhang studied the small time limiting behaviour of $P\{Y(0)\in B,\, Y(t)\in C\}$, where P is the probability measure in the underlying probability space, $(Y(t))_{t\in[0,1]}$ is the mild solution of equation (1.6) with initial distribution μ and B and C are Borel subsets of H_1.
Working in a more abstract setting, Hino and Ramirez [17] were able to better Fang's and Zhang's upper bound for $\limsup_{t\to 0} t\ln P\{Y(0)\in B, Y(t)\in C\}$ by obtaining an upper bound for $P\{Y(0)\in B, Y(t)\in C\}$ which holds at all times t in $(0,1]$. While Fang's and Zhang's upper bound was derived using a property specific to symmetric Markov processes, Hino's and Ramirez's approach used the basic theory of Dirichlet forms.

Zhang [33] continued the investigation started in [13]. In [33] the embedding of H into H_1 is still Hilbert-Schmidt but the embedding of H_ν into H_1 need not be trace class. Zhang obtained a large deviation principle in trajectory space $C([0,1];H_1)$ describing the small time asymptotics of the mild solution $(Y_x(t))_{t\in[0,1]}$ of an initial value problem:

$$\begin{aligned} dY(t) &= (AY(t)+F(Y(t)))\,dt \,+\, G(Y(t))\,dW(t)\,, \quad t\in(0,1], \\ Y(0) &= x\in H_1. \end{aligned}$$

Here the diffusion function G takes values in $L_2(H_\nu,H)$, the space of Hilbert-Schmidt operators mapping H_ν into H, and is Lipschitz continuous and bounded and the drift function F is Lipschitz continuous. For each $\epsilon\in(0,1]$ set $Y_x^\epsilon(t) := Y_x(\epsilon t)$ for all t in $[0,1]$ and denote by Y_x^ϵ the corresponding trajectory-valued random variable in $C([0,1];H_1)$.

Zhang proved that the family $\{Y_x^\epsilon : \epsilon \in (0,1]\}$ satisfies a large deviation principle with rate function

$$\begin{aligned}\mathcal{I}(u) \quad := \quad & \frac{1}{2}\inf\left\{\int_0^1 |\phi(s)|_{H_\nu}^2\, ds \,:\, \phi : [0,1] \to H_\nu \text{ is square integrable and}\right.\\ & \left. u(t) = x + \int_0^t G(u(s))\phi(s)\, ds \quad \text{for all } t \in [0,1]\right\}.\end{aligned}$$

Zhang's proof uses the exponential equivalence concept explicitly and the Hilbert-Schmidt embedding of H into H_1 plays an important role. Zhang's result is particularly impressive because the diffusion function G depends on the state; this makes the task of relating the small time behaviour of $(Y_x(t))$ to that of $(W(t))$ considerably harder. Note that if the trajectories of $(Y_x(t))$ lie in $C([0,1];H)$ then Zhang's large deviation principle in $C([0,1];H_1)$ does not automatically imply that a large deviation principle also holds in $C([0,1];H)$.

1.2 Common notation

In each of the following chapters we define notation whose scope is restricted to that chapter. However, there are some notational conventions common to all the chapters; these we list below. In the list $(E, \|\cdot\|)$ and $(E_1, \|\cdot\|_1)$ are separable Banach spaces, $(H, \langle\cdot,\cdot\rangle, |\cdot|)$ and $(H_1, \langle\cdot,\cdot\rangle_1, |\cdot|_1)$ are separable Hilbert spaces, $(M, \mathcal{M}, \mu)$ and $(M_1, \mathcal{M}_1, \mu_1)$ are measure spaces and X is a topological space.

1. *Asterisk superscript* The asterisk superscript * has two different meanings.

   E^* denotes the Banach space of continuous linear functionals on E with norm

   $$\|l\|_{E^*} := \sup\{|l(x)| : x \in E \text{ and } \|x\| = 1\} \quad \forall l \in E^*.$$

   If $l \in E^*$ we write

   $$_{E^*}\langle l, x\rangle_E := l(x) \quad \text{for all } x \in E.$$

   If T is a linear operator mapping a dense subspace of H into H_1 then T^* denotes the adjoint operator, that is,

   $$\langle Tx, y\rangle_{H_1} = \langle x, T^*y\rangle_H$$

   for all x in the domain of T and for all y in the domain of T^*.

2. *Balls* For any point $x \in E$ and positive real r we define

$$\begin{aligned} B_E(x,r) &:= \{y \in E : \|y-x\| < r\} \quad \text{and} \\ \overline{B}_E(x,r) &:= \{y \in E : \|y-x\| \le r\}. \end{aligned}$$

If K is a subset of E and r is a positive real number we define

$$B_E(K,r) := \bigcup_{x \in K} B_E(x,r).$$

3. *Borel σ-algebra* We denote the Borel σ-algebra of X by $\mathcal{B}_X$.

4. *Closure of a set* If S is a subset of X then the closure of S is denoted by $\overline{S}$.

5. *Spaces of continuous functions* If a and b are real numbers and $a < b$ then $(C([a,b];E), \|\cdot\|_{C([a,b];E)})$ is the Banach space of continuous functions mapping $[a,b]$ into E with the supremum norm

$$\|f\|_{C([a,b];E)} := \sup_{t \in [a,b]} \|f(t)\| \quad \text{for all } f \in C([a,b];E).$$

6. *More spaces of continuous functions* If $\mathcal{O}$ is a bounded open subset of $\mathbb{R}^n$, where n is a natural number, then

   $C_c^\infty(\mathcal{O})$ denotes the set of all continuous functions which map $\mathcal{O}$ into $\mathbb{R}$ and have compact support contained in $\mathcal{O}$ and have continuous partial derivatives of all orders;

   $C_0(\overline{\mathcal{O}})$ is the set of all continuous real-valued functions defined on $\overline{\mathcal{O}}$ and vanishing on the boundary of $\mathcal{O}$. This set equipped with the supremum norm is a Banach space.

7. *A continuous linear functional on $C([0,1];E)$* If $t \in [0,1]$ and l is a continuous linear functional on E then we define

$$(\delta_t \otimes l)(u) := l(u(t)) \quad \text{for all } u \in C([0,1];E);$$

   if $x \in H$ we define

$$(\delta_t \otimes x)(u) := \langle u(t), x \rangle \quad \text{for all } u \in C([0,1];H).$$

8. *Domains* If A is a linear operator on H or a bilinear form on H we denote the linear

subspace of H on which A is defined by $D(A)$.

9. *Embedding* If E is continuously embedded in E_1 we write

$$E \hookrightarrow E_1.$$

10. *Identity operator* I_E denotes the identity operator on E.

11. *Indicator function* If B is a set in the σ-algebra $\mathcal{M}$ then

$$1_B(x) := \begin{cases} 1 & \text{if } x \in B, \\ 0 & \text{if } x \in M \backslash B. \end{cases}$$

12. *Kernel* If $T : E \to E_1$ is a bounded linear operator then the kernel of T is denoted by

$$\text{ker}T := \{x \in E : Tx = 0\}.$$

13. *Law or distribution* If Z is a random variable then $\mathcal{L}(Z)$ denotes the distribution of Z.

14. *Bounded linear operators* $(L(E, E_1), \|\cdot\|_{L(E,E_1)})$ denotes the Banach space of bounded linear operators mapping E into E_1 with the operator norm

$$\|T\|_{L(E,E_1)} := \sup\{\|Tx\|_1 : x \in E \text{ and } \|x\| = 1\} \quad \text{for all } T \in L(E, E_1).$$

15. *Linear operators that are Hilbert-Schmidt* $(L_2(H, H_1), \langle\cdot,\cdot\rangle_{L_2(H,H_1)}, \|\cdot\|_{L_2(H,H_1)})$ denotes the Hilbert space of Hilbert-Schmidt operators mapping H into H_1 with inner product

$$\langle T, S\rangle_{L_2(H,H_1)} := \sum_{k=1}^{\infty} \langle Te_k, Se_k\rangle_1\,, \qquad S \text{ and } T \in L_2(H, H_1),$$

where $\{e_k : k \in \mathbb{N}\}$ is any orthonormal basis of H.

16. L^p *spaces* If $p \in [1, \infty)$ then $(L^p(M, \mathcal{M}, \mu;\, H), \|\cdot\|_{L^p(M,\mathcal{M},\mu;\, H)})$ denotes the Banach space of measurable functions $u : (M, \mathcal{M}) \to (H, \mathcal{B}_H)$ such that $\int_M |u(x)|^p\, d\mu(x) < \infty$ and we define

$$\|u\|_{L^p(M,\mathcal{M},\mu;\, H)} := \left(\int_M |u(x)|^p\, d\mu(x)\right)^{\frac{1}{p}} \qquad \text{for all } u \in L^p(M, \mathcal{M}, \mu;\, H);$$

strictly speaking we refer to the space of equivalence classes of functions which are equal μ a.e.. We write $L^p(M, \mathcal{M}, \mu)$ when the image space of the functions is $\mathbb{R}$. When the σ-algebra $\mathcal{M}$ or the measure μ are obvious we sometimes omit them from the symbol as well.

$L^\infty(M, \mathcal{M}, \mu)$ denotes the set of (equivalence classes of μ a.e. equal) measurable functions $u : (M, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}_\mathbb{R})$ such that the essential supremum of $|u|$ with respect to μ is finite.

17. *Orthogonal complement* If U is a subset of H then

$$U^\perp := \{x \in H : \langle x, u \rangle = 0 \quad \text{for all } u \in U\}$$

is the orthogonal complement of U.

18. *Product σ-algebra and product measure* $\mathcal{M} \otimes \mathcal{M}_1$ denotes the product σ-algebra of subsets of the cartesian product $M \times M_1$ and $\mu \times \mu_1$ denotes the product measure on $\mathcal{M} \otimes \mathcal{M}_1$ or some sub σ-algebra.

1.3 Summary of the chapters and our results

We now summarise the substance of the following chapters. In this section H denotes a separable Hilbert space. To simplify notation, we overuse some notation where there is no ambiguity.

In chapter 2 we consider the solution $(X_x(t))_{t\in[0,1]}$ of the initial value problem

$$\begin{aligned} dX &= AX\,dt + dW\,, \quad t \in (0,1], \\ X(0) &= x \in H\,, \end{aligned} \tag{1.7}$$

where the linear operator A on H generates a strongly continuous semigroup and $(W(t))_{t\geq 0}$ is a H-valued Wiener process. We assume that equation (1.7) has an invariant measure μ and that the transition semigroup on $L^2(H,\mu)$ is symmetric and strongly Feller. Then for each x in H and each $t > 0$ the distribution of the random variable $X_x(t)$ is absolutely continuous with respect to invariant measure μ and has a continuous Radon-Nikodym derivative $k(t, x, \cdot)$. We show that under some conditions the small time asymptotics of the Radon-Nikodym derivative $k(t, x, \cdot)$ resembles the asymptotics found by Varadhan for $p(t, \zeta, \cdot)$; this is Proposition 2.2. Our conclusion is that one cannot simply adapt

Varadhan's methods in $\mathbb{R}^n$ to find a large deviation principle in $C([0,1];H)$ for the short time asymptotics of $(X_x(t))$.

In chapter 3 we find a large deviation principle in trajectory space $C([0,1];H)$ which describes the small time asymptotics of the solution $(X_x(t))_{t\in[0,1]}$ of a stochastic initial value problem in H:

$$\begin{aligned} dX(t) &= (AX(t) + F(t, X(t)))\,dt + G(X(t))\,dW(t)\,, \quad t \in (0,1], \\ X(0) &= x \in H. \end{aligned}$$

Here $(W(t))_{t\geq 0}$ is a separable Hilbert space-valued Wiener process and $\nu := \mathcal{L}(W(1))$ has reproducing kernel Hilbert space H_ν, A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators on H, $F : [0,1] \times H \to H$ is Lipschitz continuous in H uniformly in $[0,1]$ and the diffusion function $G : H \to L_2(H_\nu, H)$ is Lipschitz continuous and not necessarily bounded.
There is no need to work in a Hilbert space containing H via a Hilbert-Schmidt embedding, as Zhang [33] did. We follow the method which Peszat [25] originally employed to obtain a large deviation principle describing the small noise asymptotics of solutions of stochastic differential equations. To clarify the difference between our problem and Peszat's problem: in our small time asymptotics problem, for each $\epsilon \in (0,1]$ we consider the process $(X_x^\epsilon(t))_{t\in[0,1]}$ which is the continuous solution of

$$\begin{aligned} dX^\epsilon(t) &= \epsilon(AX^\epsilon(t) + F(\epsilon t, X^\epsilon(t)))\,dt + \epsilon^{\frac{1}{2}} G(X^\epsilon(t))\,dW(t)\,, \quad t \in (0,1], \\ X^\epsilon(0) &= x \in H; \end{aligned}$$

in Peszat's small noise asymptotics problem, for each $\epsilon \in (0,1]$ Peszat considered the process $(Y_x^\epsilon(t))_{t\in[0,1]}$ which is the continuous solution of

$$\begin{aligned} dY^\epsilon(t) &= (AY^\epsilon(t) + F(t, Y^\epsilon(t)))\,dt + \epsilon^{\frac{1}{2}} G(Y^\epsilon(t))\,dW(t)\,, \quad t \in (0,1], \\ Y^\epsilon(0) &= x \in H. \end{aligned}$$

We only need to make small modifications to each step of Peszat's method. The fact that we work just with Hilbert spaces, unlike Peszat who also had a more general Banach space to deal with, makes the assumptions we need less restrictive compared to those Peszat needed; however since the small time asymptotics problem puts the parameter ϵ which goes to zero in front of the drift terms as well as the noise term in the stochastic differential equation, the assumption we make about the strongly continuous semigroup generated by the unbounded linear operator A seems more restrictive. Our main result is the large deviation principle in Corollary 3.4 for the family of $C([0,1];H)$-valued random variables

$$\{(t \in [0,1] \mapsto X_x(\epsilon t)(\cdot)) : \epsilon \in (0,1]\}.$$

In chapter 4 we study the small time asymptotics of the solution of a stochastic equation:

$$\left.\begin{array}{rcl} dX & = & (AX + F(X))\,dt + dW\,, \quad t \in (0,1], \\ X(0) & = & x \in E\,, \end{array}\right\} \tag{1.8}$$

whose dissipative nonlinear drift function $F : E \to E$ is defined in a separable Banach space E continuously embedded in H. This drift function might, for example, arise in a stochastic reaction-diffusion equation, the reaction rate being a decreasing polynomial function with degree greater than one in the concentration and E being a space of continuous functions on the bounded domain where the reaction is taking place. Fantozzi [14] investigated the small noise asymptotics problem for this type of equation but, unlike in chapter 3, we cannot simply modify the methods used in the small noise asymptotics problem to find a solution for our small time asymptotics problem. Instead we use exponential equivalence to show that if a large deviation principle in trajectory space $C([0,1];E)$ describes the small time asymptotics of the Ornstein-Uhlenbeck process which is the solution of equation (1.8) with F identically zero, then a large deviation principle with the same rate function describes the short time asymptotics of the solution of equation (1.8) when F is nonzero. This is Proposition 4.2. Proving a large deviation principle in $C([0,1];E)$ to describe the short time asymptotics of the Ornstein-Uhlenbeck process is not as straightforward as one might hope despite the fact that we are dealing with a family of Gaussian random variables in $C([0,1];E)$ for which the large deviation principle in $C([0,1];H)$ is known. Working in a general separable Banach space E is what complicates matters. To prove the large deviation principle in $C([0,1];E)$ we assume that the Wiener process $(W(t))$ is E-valued. We also impose an additional condition on the strongly continuous semigroup generated by the unbounded linear operator A in order to ensure uniform tightness of a family of Gaussian random variables in E. Our main result is Corollary 4.12.

In chapter 5 we need no new ideas to obtain a large deviation principle in trajectory space $C([0,1];H)$ which describes the small time asymptotics of the solution of a stochastic equation with unbounded linear drift on H and additive fractional Brownian motion noise. Our framework is that of Duncan, Maslowski and Pasik-Duncan [12]. We employ essentially the same method we used in the second half of the previous chapter for the Banach space-valued Ornstein-Uhlenbeck process; our task is now simpler because we work in a Hilbert space and the only difference compared to Wiener process noise is in the technical details. The large deviation principle is in Theorem 5.1.

In chapter 6 we return to equations with Wiener process noise and study the small time asymptotics of the probability of moving from one set to another $P\{X(0) \in B,\, X(t) \in C\}$. If we have a lower bound for $\liminf_{t \to 0} t \ln P\{X_x(t) \in C\}$ for all initial states x in H then we can obtain a lower bound for $\liminf_{t \to 0} t \ln P\{X(0) \in B, X(t) \in C\}$ when $\mathcal{L}(X(0))$ is arbitrary and B is any Borel subset of H. Our lower bound is in Theorem 6.1.
Our upper bound for $P\{X(0) \in B, X(t) \in C\}$ in Theorem 6.3 applies when $(X(t))_{t \in [0,1]}$ is an Ornstein-Uhlenbeck process and $\mathcal{L}(X(0))$ is absolutely continuous with respect to invariant measure μ and the semigroup of transition operators on $L^2(H, \mu)$ is holomorphic. In the case when $\mathcal{L}(X(0)) = \mu$ and the transition operators are symmetric our upper bound agrees with the upper bound of Hino and Ramirez [17]. In fact we obtain Theorem 6.3 by adapting the method Hino and Ramirez used. Compared to the upper bound of $\limsup_{t \to 0} t \ln P\{X(0) \in B, X(t) \in C\}$ when the transition semigroup is symmetric, the upper bound when the transition semigroup is holomorphic is increased by a factor which depends on how nonsymmetric the transition semigroup is.

All of the propositions, theorems and corollaries we have referred to in this section contain results that appear to be new.

1.4 Some background theory

In this section we remind the reader of some theory which will be used in the following chapters. No proofs are given; the books by Da Prato and Zabczyk [10] and [11] provide a comprehensive development of the theory. Throughout this section let $(E, \|\cdot\|)$ be a separable Banach space and let $(H, \langle\cdot,\cdot\rangle, |\cdot|)$ and $(H_1, \langle\cdot,\cdot\rangle_1, |\cdot|_1)$ be separable Hilbert spaces and let $(\Omega, \mathcal{F}, P)$ be a probability space.

1.4.1 Gaussian measures on a separable Banach space

Let ν be a probability measure on the measurable space $(E, \mathcal{B}_E)$. The measure ν is symmetric Gaussian if and only if each continuous linear functional $l \in E^*$, considered as a random variable on $(E, \mathcal{B}_E, \nu)$, has symmetric Gaussian distribution νl^{-1} on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. A fundamental property of symmetric Gaussian measures on $(E, \mathcal{B}_E)$ is stated in Fernique's theorem.

Theorem 1.1 (Fernique's theorem. See [10, Theorem 2.6] for a proof) *If ν is a symmetric Gaussian measure on $(E, \mathcal{B}_E)$ and $\lambda > 0$ and $r > 0$ satisfy*

$$\ln\left(\frac{1-\nu(\overline{B}_E(0,r))}{\nu(\overline{B}_E(0,r))}\right) + 32\lambda r^2 \leq -1$$

then

$$\int_E \mathrm{e}^{\lambda\|x\|^2}\,d\nu(x) \leq \mathrm{e}^{16\lambda r^2} + \frac{\mathrm{e}^2}{\mathrm{e}^2-1}.$$

If ν is symmetric Gaussian then its covariance operator Q is the positive definite and symmetric bounded linear operator from E^* into E such that

$$\int_E {}_{E^*}\langle l_1, x\rangle_E\, {}_{E^*}\langle l_2, x\rangle_E\,d\nu(x) = {}_{E^*}\langle l_2, Ql_1\rangle_E \quad \forall l_1,\, l_2 \in E^*;$$

that this definition makes sense follows from Fernique's theorem. Hence the characteristic function of symmetric Gaussian ν is

$$\hat{\nu}(l) := \int_E \mathrm{e}^{i\,{}_{E^*}\langle l,x\rangle_E}\,d\nu(x) = \mathrm{e}^{-\frac{1}{2}\,{}_{E^*}\langle l,Ql\rangle_E} \quad \forall l \in E^*.$$

Since a probability measure on $(E, \mathcal{B}_E)$ is uniquely determined by its characteristic function, a symmetric Gaussian measure on $(E, \mathcal{B}_E)$ is uniquely determined by its covariance operator. The convolution of the point mass at $x \in E$ and a symmetric Gaussian measure on $(E, \mathcal{B}_E)$ is a Gaussian measure with mean x and with covariance operator of the symmetric Gaussian measure.

If ν is symmetric Gaussian there is a unique Hilbert space H_ν such that the embedding $i : H_\nu \to E$ is continuous and

$$\int_E {}_{E^*}\langle l, x\rangle_E^2\,d\nu(x) = \|l \circ i\|_{H_\nu^*}^2 \quad \text{for all } l \in E^*.$$

H_ν is called the reproducing kernel Hilbert space of ν. Further details on reproducing kernel Hilbert spaces can be found in [10, Section 2.2.2].

Now let ν be a symmetric Gaussian measure on $(H_1, \mathcal{B}_{H_1})$. In the Hilbert space setting we modify the definition of the covariance operator: the covariance operator of ν is defined to be the bounded linear operator Q on H_1 such that

$$\int_{H_1} \langle x, u\rangle_1 \langle x, v\rangle_1\,d\nu(x) = \langle Qu, v\rangle_1 \quad \forall u,\, v \in H_1.$$

The family of covariance operators of symmetric Gaussian measures on $(H_1, \mathcal{B}_{H_1})$ is precisely the family of positive definite, symmetric, trace class operators on H_1. Thus $Q^{\frac{1}{2}}$ is a well defined positive definite, symmetric, Hilbert-Schmidt operator on H_1 and one can show using [10, Proposition B.1] that $H_\nu = Q^{\frac{1}{2}}(H_1)$ and the norm $|\cdot|_{H_\nu}$ in H_ν is given by $|u|_{H_\nu} = |Q^{-\frac{1}{2}}u|_1$, where $Q^{-\frac{1}{2}}(u)$ is taken as the element of $Q^{-\frac{1}{2}}\{u\}$ which belongs to the orthogonal complement of the kernel of $Q^{\frac{1}{2}}$.

1.4.2 The stochastic integral with respect to a Wiener process

Let ν be a symmetric Gaussian measure on $(H_1, \mathcal{B}_{H_1})$ with covariance operator Q and let $(W(t) : (\Omega, \mathcal{F}, P) \to (H_1, \mathcal{B}_{H_1}))_{t \geq 0}$ be a Q-Wiener process; this means that

1. $W(0) = 0$ P a.e.,
2. $\mathcal{L}(W(t) - W(s))$ is symmetric Gaussian with covariance operator $(t-s)Q$ whenever $0 \leq s < t$,
3. $W(t_1) - W(t_0), \ldots, W(t_n) - W(t_{n-1})$ are independent whenever $n \geq 2$ and $0 \leq t_0 < t_1 < \cdots < t_n$
4. and the trajectories $t \mapsto W(t)(\omega)$, $\omega \in \Omega$, are continuous H_1-valued functions.

Associated with $(W(t))_{t\geq 0}$ is a filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $W(t)$ is $\mathcal{F}_t$-measurable for each $t \geq 0$ and $W(t) - W(s)$ is independent of $\mathcal{F}_s$ whenever $0 \leq s < t$. Set $\mathcal{Z} := \{B \in \mathcal{F} : P(B) = 0\}$ and for each $t \geq 0$ set $\mathcal{G}_t := \sigma(W(r) : r \in [0,t])$, that is, the σ-algebra generated by the random variables $W(r)$ for all $r \in [0,t]$. In this thesis we may take $\mathcal{F}_t = \sigma(\mathcal{Z} \cup \mathcal{G}_t)$ for each $t \geq 0$. If we require the filtration $(\mathcal{F}_t)_{t\geq 0}$ to be right continuous we may take $\mathcal{F}_t := \cap_{n=1}^{\infty} \sigma(\mathcal{Z} \cup \mathcal{G}_{t+\frac{1}{n}})$ for each $t \geq 0$. We specify the filtration only when it is important for the analysis.

Fix a positive real number T. The $(\mathcal{F}_t)$-predictable σ-algebra of subsets of $[0,T] \times \Omega$, $\mathcal{P}_T$, is generated by sets of the form $\{0\} \times B$, where $B \in \mathcal{F}_0$ and $(a,b] \times B$, where $0 \leq a < b \leq T$ and $B \in \mathcal{F}_a$.

Let H_ν be the reproducing kernel Hilbert space of ν and denote the embedding of H_ν into H_1 by i. An elementary process is a finite linear combination of terms of the form

$$1_{(a,b] \times B} S i$$

where $0 \leq a < b \leq T$ and $B \in \mathcal{F}_a$ and S is a bounded linear operator mapping H_1 into H. Note that Si is a Hilbert-Schmidt operator mapping H_ν into H. One can show that the elementary processes form a dense subspace of $L^2([0,T]\times\Omega,\mathcal{P}_T,\lambda\times P;L_2(H_\nu,H))$, where λ is Lebesgue measure on $\mathcal{B}_{[0,T]}$ and $\lambda\times P$ is the product measure of λ and P restricted to $\mathcal{P}_T$. Any elementary process can be written in the form

$$\Phi(s,\omega)=\sum_{k=0}^{n-1}1_{(t_k,t_{k+1}]}(s)\Phi_k(\omega)\circ i, \tag{1.9}$$

where n is a natural number and $0=t_0<t_1<\cdots<t_n=T$ and for each k from 0 to $n-1$, Φ_k is a $\mathcal{F}_{t_k}$-measurable simple function in $L(H_1,H)$. For the elementary process Φ in equation (1.9) and each $t\in[0,T]$ we define the Ito integral

$$\int_0^t \Phi(s)\,dW(s):=\sum_{k=0}^{n-1}\Phi_k(W(t_{k+1}\wedge t)-W(t_k\wedge t)). \tag{1.10}$$

The process $(\int_0^t\Phi(s)\,dW(s))_{t\in[0,T]}$ is a continuous square integrable martingale in H and we have

$$E\left[\left|\int_0^T\Phi(s)\,dW(s)\right|^2\right]=E\int_0^T\|\Phi(s)\|^2_{L_2(H_\nu,H)}\,ds.$$

Equation (1.10) is a linear isometry from a dense subspace of $L^2([0,T]\times\Omega,\mathcal{P}_T,\lambda\times P;L_2(H_\nu,H))$ into the space of continuous square integrable martingales and we define the stochastic integral of an arbitrary process in $L^2([0,T]\times\Omega,\mathcal{P}_T,\lambda\times P;L_2(H_\nu,H))$ by extending the domain of the isometry.

If $\Phi:([0,T]\times\Omega,\mathcal{P}_T)\to(L_2(H_\nu,H),\mathcal{B}_{L_2(H_\nu,H)})$ is measurable and satisfies

$$P\left\{\int_0^T\|\Phi(s)\|^2_{L_2(H_\nu,H)}\,ds<\infty\right\}=1$$

then the stochastic integral of Φ with respect to W can be defined by the localization procedure. For details see the localization lemma [10, Lemma 4.9] and the paragraph following that lemma.

1.4.3 Stochastic convolution and Ornstein-Uhlenbeck process

The definitions of the previous subsection hold in this subsection. Let $A:D(A)\subset H\to H$ be the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on H. If $\Phi:([0,T]\times\Omega,\mathcal{P}_T)\to(L_2(H_\nu,H),\mathcal{B}_{L_2(H_\nu,H)})$ is a measurable function then the stochastic convolution process is the process of Itô integrals

$$\int_0^t S(t-s)\Phi(s)\,dW(s) \quad \text{for all } t \in [0,T],$$

provided the Itô integrals exist. Stochastic convolution processes are of interest to us because one appears in the definition of the mild solution of a stochastic initial value problem.

Let

$$F : ([0,T] \times H, \mathcal{B}_{[0,T]} \otimes \mathcal{B}_H) \to (H, \mathcal{B}_H)$$

and

$$G : ([0,T] \times H, \mathcal{B}_{[0,T]} \otimes \mathcal{B}_H) \to (L_2(H_\nu, H), \mathcal{B}_{L_2(H_\nu,H)})$$

be measurable functions and let ξ be a $\mathcal{F}_0$-measurable H-valued random variable. By definition, the mild solution $(X_\xi(t))_{t\in[0,T]}$ of the initial value problem

$$\begin{aligned} dX(t) &= (AX(t) + F(t, X(t)))\,dt + G(t, X(t))\,dW(t)\,, \quad t \in (0,T], \\ X(0) &= \xi \end{aligned}$$

satisfies the integral equation

$$X(t) = S(t)\xi + \int_0^t S(t-s)F(s, X(s))\,ds + \int_0^t S(t-s)G(s, X(s))\,dW(s) \quad P \text{ a.e.}$$

for each $t \in [0,T]$. Thus the limiting behaviour of stochastic convolution processes is an important consideration when studying the small time behaviour of mild solutions.

We can simplify the analysis by focusing on the initial value problem where $(W(t))_{t\geq 0}$ is a H-valued Wiener process, F is identically zero and G is the embedding i of H_ν into H:

$$\begin{aligned} dX &= AX\,dt + dW\,, \quad t \in (0,T], \\ X(0) &= \xi. \end{aligned} \tag{1.11}$$

Notice that, by convention, the operator i is omitted from equation (1.11). The mild solution of this initial value problem is an Ornstein-Uhlenbeck process:

$$X_\xi(t) := S(t)\xi + \int_0^t S(t-s)i\,dW(s)\,, \quad t \in [0,T].$$

From the definition of the Itô integral, $\int_0^t S(t-s)i\,dW(s)$ has symmetric Gaussian distribution with covariance operator

$$Q_t x := \int_0^t S(s)QS(s)^* x\, ds\,, \quad x \in H.$$

We say that a probability measure μ is an invariant measure of a stochastic differential equation if the mild solution with initial distribution μ also has distribution μ at all times $t > 0$. If $\int_0^\infty \|S(s)Q^{\frac{1}{2}}\|^2_{L_2(H,H)}\, ds < \infty$ then

$$Q_\infty x := \int_0^\infty S(s)QS(s)^* x\, ds\ , \quad x \in H,$$

is the covariance operator of a symmetric Gaussian invariant measure μ for equation (1.11); thus if $\mathcal{L}(\xi) = \mu$ then $\mathcal{L}(S(t)\xi + \int_0^t S(t-s)i\, dW(s)) = \mu$ for all $t \geq 0$. When the invariant measure μ exists, for each $t \in [0, T]$ the operator

$$(R_t\, \phi)(x) := E\left[\phi(S(t)x + \textstyle\int_0^t S(t-s)i\, dW(s))\right]\ , \quad x \in H,$$

on the bounded, Borel measurable, real-valued functions ϕ on H extends to a bounded linear operator on $L^2(H, \mu)$ with operator norm equal to 1. These operators on $L^2(H, \mu)$ form the strongly continuous semigroup of transition operators. If, in addition, $S(t)(H) \subset Q_t^{\frac{1}{2}}(H)$ for all $t > 0$ then the semigroup of transition operators is said to be strongly Feller. More details on Ornstein-Uhlenbeck processes and invariant measures can be found in [10, chapters 5 and 11] and [11, chapter 10].

1.4.4 Large deviation principle

Basic large deviations theory provides useful tools for finding the short time asymptotics of solutions of stochastic differential equations. We outline some ideas here in the context of the separable Banach space E.

Suppose the function $\mathcal{I} : E \to [0, \infty]$ is lower semicontinuous. The family of probability measures $\{\mu_\epsilon : \epsilon \in (0, 1]\}$ on $(E, \mathcal{B}_E)$ is said to satisfy a large deviation principle with rate function $\mathcal{I}$ if for each closed set $F \subset E$ we have the upper bound

$$\lim_{r\to 0} \sup_{\epsilon\in(0,r]} \epsilon \ln \mu_\epsilon(F) \leq - \inf_{x\in F} \mathcal{I}(x) \tag{1.12}$$

and for each open set $G \subset E$ we have the lower bound

$$\lim_{r \to 0} \inf_{\epsilon \in (0,r]} \epsilon \ln \mu_\epsilon(G) \geq - \inf_{x \in G} \mathcal{I}(x). \tag{1.13}$$

If $\{\mathcal{I} \leq r\}$ is a compact subset of E for all $r \in [0, \infty)$ then $\mathcal{I}$ is called a good rate function. We will only consider rate functions that are good. If $\mathcal{I}$ is a good rate function then the upper bound condition in (1.12) is equivalent to the Freidlin-Wentzell condition:
given $r \in (0, \infty)$ and $\delta \in (0, \infty)$ and $\gamma \in (0, \infty)$ there exists $\epsilon_0 \in (0, 1]$ such that

$$\mu_\epsilon(B_E(\{\mathcal{I} \leq r\}, \delta)) \geq 1 - \mathrm{e}^{\frac{-r+\gamma}{\epsilon}} \quad \text{for all } \epsilon \in (0, \epsilon_0].$$

The lower bound condition in (1.13) is equivalent to the Freidlin-Wentzell condition:
given $x \in E$ and $\delta \in (0, \infty)$ and $\gamma \in (0, \infty)$ there exists $\epsilon_0 \in (0, 1]$ such that

$$\mu_\epsilon(B_E(x, \delta)) \geq \mathrm{e}^{\frac{-\mathcal{I}(x)-\gamma}{\epsilon}} \quad \text{for all } \epsilon \in (0, \epsilon_0].$$

Of special interest to us is the following large deviation principle. Let μ be a symmetric Gaussian measure on $(E, \mathcal{B}_E)$ and let $(H_\mu, |\cdot|_{H_\mu})$ be its reproducing kernel Hilbert space.

Theorem 1.2 *The family of symmetric Gaussian measures*

$$\{\mu_\epsilon(B) := \mu(\epsilon^{-\frac{1}{2}} B) \quad \forall B \in \mathcal{B}_E : \epsilon \in (0, 1]\}$$

satisfies a large deviation principle with rate function

$$\mathcal{I}(x) := \begin{cases} \frac{1}{2}|x|^2_{H_\mu} & , x \in H_\mu \\ \infty & , x \in E \backslash H_\mu. \end{cases}$$

A proof of this theorem is given in [10, Section 12.1.2]. For two applications see the end of section 3.4 and Corollary 5.3.

Suppose we have two families of random variables in E: $\{\xi_\epsilon : \epsilon \in (0, 1]\}$ and $\{\eta_\epsilon : \epsilon \in (0, 1]\}$, defined on $(\Omega, \mathcal{F}, P)$. The families are said to be exponentially equivalent if for each $\delta > 0$

$$\lim_{\epsilon \to 0} \epsilon \ln P\{\|\xi_\epsilon - \eta_\epsilon\| \geq \delta\} = -\infty.$$

Theorem 1.3 *If* $\{\xi_\epsilon : \epsilon \in (0, 1]\}$ *and* $\{\eta_\epsilon : \epsilon \in (0, 1]\}$ *are exponentially equivalent then* $\{\mathcal{L}(\xi_\epsilon) : \epsilon \in (0, 1]\}$ *satisfies a large deviation principle if and only if* $\{\mathcal{L}(\eta_\epsilon) : \epsilon \in (0, 1]\}$ *satisfies a large deviation principle and these two large deviation principles have the same rate function.*

This basic result is particularly useful to us when combined with Theorem 1.2. For a proof of Theorem 1.3 see [18, Lemma 27.13].

Chapter 2

Small time asymptotics via densities

2.1 Introduction

We remarked in chapter 1 that Varadhan [31] investigated the small time asymptotics of an $\mathbb{R}^n$-valued diffusion process $(z_\zeta(t))_{t\in[0,1]}$ with initial point $\zeta \in \mathbb{R}^n$. He used the limiting behaviour of the probability density $p(t,\zeta,\cdot)$ of $z_\zeta(t)$:

$$\lim_{t\to 0} t \ln\, p(t,\zeta,y) = -\frac{1}{2}d^2(\zeta,y) \tag{2.1}$$

uniformly for ζ and y in any bounded subset of $\mathbb{R}^n$; the function d is defined in equation (1.3).

In the setting of an infinite dimensional separable Hilbert space H, let $(X_x(t))_{t\in[0,1]}$ be the mild solution of the stochastic initial value problem

$$\left.\begin{array}{rcll} dX &=& AX\,dt + dW & t\in(0,1] \\ X(0) &=& x\in H; & \end{array}\right\} \tag{2.2}$$

we define A and W in section 2.2. Only in special situations is the distribution of $X_x(t)$ absolutely continuous with respect to a natural reference measure on H at all times $t \in (0,1]$. In this chapter we consider one such special situation, namely when an invariant measure μ exists and the transition semigroup is strongly Feller and symmetric on $L^2(H,\mu)$. We shall see that even when we can obtain the small time limiting behaviour of the probability density of $X_x(t)$, it may not lead to the large deviation principle for the small time asymptotics in trajectory space $C([0,1];H)$. The density $k(t,x,\cdot)$ in equation (2.5) is valid under assumptions (H1) and (H2) and we have the small time limit in equation (2.11) when assumption (H3) also holds. The form of the limit in equation (2.11) is not very different from that in equation (2.1). However equation (2.11) only enables us to obtain the lower bound of the large deviation principle in $C([0,1];H)$.

2.2 Small time limiting behaviour of densities

Let $(H,\langle\cdot,\cdot\rangle,|\cdot|)$ be a separable infinite dimensional Hilbert space. Let $A: D(A)\subset H\to H$ be the infinitesimal generator of the strongly continuous semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on H. We use the symbol $\mathcal{N}(m,C)$ to denote a Gaussian measure on $(H,\mathcal{B}_H)$ with mean m and covariance operator C. Let $(W(t):(\Omega,\mathcal{F},P)\to H)_{t\geq 0}$ be an H-valued Wiener process and let the distribution of $W(1)$ be $\nu=\mathcal{N}(0,Q)$, where $\ker Q=\{0\}$. The reproducing kernel Hilbert space of ν is denoted by $(H_\nu = Q^{\frac{1}{2}}(H),\ |\cdot|_{H_\nu}=|Q^{-\frac{1}{2}}\cdot|)$ and the embedding of H_ν into H is denoted by

$$i: H_\nu \hookrightarrow H.$$

Suppose that

$$Q_\infty x := \int_0^\infty S(t)QS^*(t)x\,dt\ , \quad x\in H,$$

defines a trace class operator and set $\mu:=\mathcal{N}(0,Q_\infty)$. Then for each $t>0$ the operator

$$Q_t x := \int_0^t S(s)QS^*(s)x\,ds\ , \quad x\in H,$$

is trace class and $\ker Q_t=\{0\}$. The mild solution of the initial value problem (2.2) at time $t\in(0,1]$,

$$X_x(t) := S(t)x + \int_0^t S(t-s)i\,dW(s), \tag{2.3}$$

has distribution $\mathcal{N}(S(t)x,Q_t)$. Define the strongly continuous semigroup $(R_t)_{t\in[0,1]}$ on $L^2(H,\mu)$ by

$$(R_t\phi)(x) := \int_H \phi(y)\,d\mathcal{N}(S(t)x,Q_t)(y) \quad \text{for } \mu \text{ a.e. } x\in H$$

and for all $\phi\in L^2(H,\mu)$. We assume that

(H1) R_t is strongly Feller, that is, $S(t)(H)\subset Q_t^{\frac{1}{2}}(H)$ for each positive time t.

Chojnowska-Michalik and Goldys have shown in [6, Proposition 2] that

$$S_0(t) := Q_\infty^{-\frac{1}{2}} S(t) Q_\infty^{\frac{1}{2}}\ , \quad t \geq 0,$$

defines a strongly continuous semigroup of contractions on H. Some consequences of assumption (H1) are that for each $t > 0$

1. $Q_\infty^{\frac{1}{2}}(H) = Q_t^{\frac{1}{2}}(H)$, which is equivalent to $\|S_0(t)\|_{L(H,H)} < 1$ and
2. $S_0(t)$ is Hilbert-Schmidt.

As shown in [11, Lemma 10.3.3], it follows that for each $t > 0$ and each $x \in H$ the Gaussian measure $\mathcal{N}(S(t)x, Q_t)$ is absolutely continuous with respect to μ and its Radon-Nikodym derivative $\frac{d\mathcal{N}(S(t)x,Q_t)}{d\mu}$ is

$$\begin{aligned} \frac{d\mathcal{N}(S(t)x, Q_t)}{d\mu}(y) \quad = \quad & (\det(I_H - \Theta_t))^{-\frac{1}{2}} \exp \Big[-\frac{1}{2} \langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t)x,\, Q_\infty^{-\frac{1}{2}} S(t)x \rangle \\ & + \langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t)x,\, Q_\infty^{-\frac{1}{2}} y \rangle \\ & - \frac{1}{2} \langle \Theta_t (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} y,\, Q_\infty^{-\frac{1}{2}} y \rangle \Big] \qquad (2.4) \end{aligned}$$

for μ a.e. $y \in H$, where $\Theta_t := S_0(t) S_0^*(t)$. We remark that the second and third terms appearing in the argument of the exponential function in equation (2.4) are defined for only μ a.e. y, in terms of limits (see for example [11, Proposition 1.2.10]). An equation similar to (2.4) holds under a weaker condition than (H1) [6, Theorem 2] but, for simplicity, we work with (H1).
We make another assumption:
(H2) R_t is symmetric for each $t > 0$.

Chojnowska-Michalik and Goldys [7, Lemma 2.2] have shown that symmetry of R_t is equivalent to symmetry of $S_0(t)$ and this allows us to prove there is a continuous version of the Radon-Nikodym derivative in equation (2.4).

Proposition 2.1 *The symmetry of R_t implies there is a continuous version of the Radon-Nikodym derivative $\frac{d\mathcal{N}(S(t)x,Q_t)}{d\mu}$, which we denote by $k(t, x, \cdot)$:*

$$\begin{aligned} k(t, x, y) \quad := \quad & (\det(I_H - S_0(2t)))^{-\frac{1}{2}} \times \\ & \exp \Big[-\frac{1}{2} |Q_t^{-\frac{1}{2}} S(t)x|^2 + \langle Q_t^{-\frac{1}{2}} S(t/2)x,\, Q_t^{-\frac{1}{2}} S(t/2)y \rangle - \frac{1}{2} |Q_t^{-\frac{1}{2}} S(t)y|^2 \Big] \qquad (2.5) \end{aligned}$$

for all $y \in H$.

Proof. We have $\Theta_t = S_0(t)S_0^*(t) = S_0(2t)$ thus

$$(\det(I_H - \Theta_t))^{-\frac{1}{2}} = (\det(I_H - S_0(2t)))^{-\frac{1}{2}}. \tag{2.6}$$

The operators

$$J(t) := Q_\infty^{-\frac{1}{2}} Q_t^{\frac{1}{2}}\ , \quad t > 0,$$

are bounded linear bijections and we have

$$J^{-1}(t) = Q_t^{-\frac{1}{2}} Q_\infty^{\frac{1}{2}}\ , \quad t > 0.$$

The identity $Q_\infty = Q_t + S(t)Q_\infty S^*(t)$ yields

$$J(t)J^*(t) = I_H - S_0(t)S_0^*(t) = I_H - \Theta_t \quad \text{for } t > 0,$$

thus

$$(I_H - \Theta_t)^{-1} = (J^{-1}(t))^* J^{-1}(t) \quad \text{for } t > 0. \tag{2.7}$$

From equation (2.7) we have

$$\begin{aligned}\langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t)x,\, Q_\infty^{-\frac{1}{2}} S(t)x\rangle &= \langle J^{-1}(t) Q_\infty^{-\frac{1}{2}} S(t)x,\, J^{-1}(t) Q_\infty^{-\frac{1}{2}} S(t)x\rangle \\ &= |Q_t^{-\frac{1}{2}} S(t)x|^2. \end{aligned} \tag{2.8}$$

The other two terms in the argument of exp in equation (2.4) are defined in terms of limits. Let (f_k) be an orthonormal basis of H made up of eigenvectors of Q_∞. For each natural number n define

$$P_n x := \sum_{k=1}^{n} \langle x, f_k\rangle f_k \quad \text{for all } x \in H.$$

In the following expressions (n_k) denotes some strictly increasing sequence of natural numbers. We have

$$
\begin{aligned}
\langle \Theta_t (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} y,\, Q_\infty^{-\frac{1}{2}} y\rangle &= \lim_{k\to\infty} \langle \Theta_t (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} P_{n_k} y,\, Q_\infty^{-\frac{1}{2}} P_{n_k} y\rangle \ , \quad \mu \text{ a.e. } y\in H,\\
&= \lim_{k\to\infty} \langle (I_H - \Theta_t)^{-1} \Theta_t^{\frac{1}{2}} Q_\infty^{-\frac{1}{2}} P_{n_k} y,\, \Theta_t^{\frac{1}{2}} Q_\infty^{-\frac{1}{2}} P_{n_k} y\rangle\\
&= \lim_{k\to\infty} \langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t) P_{n_k} y,\, Q_\infty^{-\frac{1}{2}} S(t) P_{n_k} y\rangle\\
&= \langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t) y,\, Q_\infty^{-\frac{1}{2}} S(t) y\rangle\\
&= |Q_t^{-\frac{1}{2}} S(t) y|^2.
\end{aligned}
\tag{2.9}
$$

We have

$$
\begin{aligned}
\langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t) x,\, Q_\infty^{-\frac{1}{2}} y\rangle &= \lim_{k\to\infty} \langle Q_\infty^{-\frac{1}{2}} P_{n_k} (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t) x, y\rangle \ , \quad \mu \text{ a.e. } y\in H,\\
&= \lim_{k\to\infty} \langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t) x,\, Q_\infty^{-\frac{1}{2}} P_{n_k} y\rangle\\
&= \lim_{k\to\infty} \langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t/2) x,\, S_0(t/2) Q_\infty^{-\frac{1}{2}} P_{n_k} y\rangle\\
&= \lim_{k\to\infty} \langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t/2) x,\, Q_\infty^{-\frac{1}{2}} S(t/2) P_{n_k} y\rangle\\
&= \langle (I_H - \Theta_t)^{-1} Q_\infty^{-\frac{1}{2}} S(t/2) x,\, Q_\infty^{-\frac{1}{2}} S(t/2) y\rangle\\
&= \langle Q_t^{-\frac{1}{2}} S(t/2) x,\, Q_t^{-\frac{1}{2}} S(t/2) y\rangle.
\end{aligned}
\tag{2.10}
$$

Substituting the expressions from equations (2.6), (2.8), (2.9) and (2.10) into the right hand side of equation (2.4), we get the formula for $k(t,x,y)$ shown in equation (2.5). This completes the proof.

When x and y belong to $Q^{\frac{1}{2}}(H)$ we can write $k(t,x,y)$ in terms of only t and the eigenvalues of A_0, the infinitesimal generator of $(S_0(t))_{t\geq 0}$; then it is straightforward to find $\lim_{t\to 0} t \ln k(t,x,y)$. The results obtained in this way can be of interest only if $\mu(Q^{\frac{1}{2}}(H)) = 1$. We now introduce a further assumption to ensure $\mu(Q^{\frac{1}{2}}(H)) = 1$. Chojnowska-Michalik and Goldys [7, Theorems 2.7 and 2.9] showed that the symmetry of R_t implies that

$$S_Q(t) := Q^{-\frac{1}{2}} S(t) Q^{\frac{1}{2}} \ , \quad t\geq 0,$$

defines a strongly continuous semigroup of symmetric contractions on H and there is an isometric isomorphism $U : H \to H$ such that

$$S_Q(t) = U S_0(t) U^{-1} \quad \text{for all } t\geq 0.$$

Hence, like $S_0(t)$, $S_Q(t)$ is a Hilbert-Schmidt strict contraction for each $t > 0$ and the infinitesimal generators A_Q of $(S_Q(t))$ and A_0 of $(S_0(t))$ are related by

$$D(A_Q) = U(D(A_0)) \text{ and } A_Q x = U A_0 U^{-1} x \text{ for } x \in D(A_Q).$$

Since $(S_Q(t))$ is a compact, symmetric semigroup of contractions, A_Q is self-adjoint and its spectrum consists of real eigenvalues

$$0 > -\alpha_1 \geq -\alpha_2 \geq -\alpha_3 \geq \cdots$$

where $-\alpha_j \to -\infty$ as $j \to \infty$ (see [19, Theorem 13 in chapter 34] and [24, Theorems 2.3 and 2.4 in chapter 2]). We have $-\alpha_1 < 0$ because $\mathrm{e}^{-\alpha_1 t} = \|S_Q(t)\|_{L(H,H)} < 1$ for each $t > 0$. By [24, Theorem 3.3 in chapter 2], A_Q^{-1} is compact as well as symmetric and hence there is an orthonormal basis (g_k) of H composed of eigenvectors of A_Q:

$$A_Q g_k = -\alpha_k g_k \quad \text{for all } k \in \mathbb{N}.$$

We assume that
(H3) A_Q^{-1} is trace class, that is, $\sum_{k=1}^{\infty} \frac{1}{\alpha_k} < \infty$.
Chojnowska-Michalik and Goldys [7, Theorem 5.1] showed that $\mu(Q^{\frac{1}{2}}(H)) = 1$ if and only if

$$\int_0^\infty \|S_Q(t)\|_{L_2(H,H)}^2 \, dt < \infty,$$

where $\|\cdot\|_{L_2(H,H)}$ denotes the Hilbert-Schmidt norm. We have

$$\begin{aligned}
\int_0^\infty \|S_Q(t)\|_{L_2(H,H)}^2 \, dt &= \int_0^\infty \sum_{k=1}^{\infty} |S_Q(t) g_k|^2 \, dt \\
&= \sum_{k=1}^{\infty} \int_0^\infty |S_Q(t) g_k|^2 \, dt \\
&= \sum_{k=1}^{\infty} \int_0^\infty \mathrm{e}^{-2\alpha_k t} \, dt \\
&= \sum_{k=1}^{\infty} \frac{1}{2\alpha_k}.
\end{aligned}$$

Thus assumption (H3) is equivalent to the assumption that $\mu(Q^{\frac{1}{2}}(H)) = 1$.

Proposition 2.2 *Under assumption (H3) we have for all x and y in $Q^{\frac{1}{2}}(H)$*

$$\lim_{t\to 0} t \ln k(t,x,y) = -\frac{1}{2}|Q^{-\frac{1}{2}}(x-y)|^2 \tag{2.11}$$

and convergence is uniform for $Q^{-\frac{1}{2}}x$ and $Q^{-\frac{1}{2}}y$ in any compact subset of H.

Remark In the example following the proof we show that equation (2.11) does not necessarily hold if $x-y$ is in $Q^{\frac{1}{2}}(H)$ but x and y are in $H\backslash Q^{\frac{1}{2}}(H)$.

Proof.

Assumption (H3) is sufficient (but not necessary) to ensure that

$$\lim_{t\to 0} t \ln \det(I_H - S_0(2t)) = 0.$$

We have

$$\begin{aligned} t\ln\det(I_H - S_0(2t)) &= t\ln\prod_{j=1}^{\infty}(1-\mathrm{e}^{-2\alpha_j t}) \\ &= \sum_{j=1}^{\infty} t\ln(1-\mathrm{e}^{-2\alpha_j t}) \ , \quad t>0. \end{aligned}$$

We can write for $t>0$

$$t\ln(1-\mathrm{e}^{-2\alpha_j t}) = \frac{t}{1/\ln(1-\mathrm{e}^{-2\alpha_j t})}$$

and by L'Hôpital's rule

$$\begin{aligned} \lim_{t\to 0} t\ln(1-\mathrm{e}^{-2\alpha_j t}) &= \lim_{t\to 0}\frac{-(\ln(1-\mathrm{e}^{-2\alpha_j t}))^2(1-\mathrm{e}^{-2\alpha_j t})}{2\alpha_j \mathrm{e}^{-2\alpha_j t}} \\ &= 0 \quad \text{for each } j\in\mathbb{N}. \end{aligned} \tag{2.12}$$

Since the function $x\in(0,\infty)\mapsto x\ln(1-\mathrm{e}^{-x})$ is bounded we have

$$\begin{aligned} t\ln\det(I_H - S_0(2t)) &= \sum_{j=1}^{\infty}\frac{2\alpha_j t\ln(1-\mathrm{e}^{-2\alpha_j t})}{2\alpha_j} \\ &\to 0 \quad \text{as } t\to 0 \end{aligned} \tag{2.13}$$

by equation (2.12) and Lebesgue's dominated convergence theorem.

It remains to find the limit of t times the argument of the exponential function in equation (2.5).

Let $t > 0$. We have

$$\begin{aligned} Q_t x &= \int_0^t S(2r) Q x \, dr \\ &= Q^{\frac{1}{2}} \int_0^t A_Q S_Q(2r) A_Q^{-1} Q^{\frac{1}{2}} x \, dr \\ &= \frac{1}{2} Q^{\frac{1}{2}} \int_0^t \frac{d}{dr} (S_Q(2r) A_Q^{-1} Q^{\frac{1}{2}} x) \, dr \\ &= \frac{1}{2} Q^{\frac{1}{2}} (S_Q(2t) A_Q^{-1} Q^{\frac{1}{2}} x - A_Q^{-1} Q^{\frac{1}{2}} x) \\ &= \frac{1}{2} Q^{\frac{1}{2}} (I_H - S_Q(2t))(-A_Q)^{-1} Q^{\frac{1}{2}} x \ , \quad x \in H. \end{aligned}$$

Substituting $x = Q^{-\frac{1}{2}} y$ into this equation, where $y \in Q^{\frac{1}{2}}(H)$, we have

$$Q^{-\frac{1}{2}} Q_t Q^{-\frac{1}{2}} y = \frac{1}{2} (I_H - S_Q(2t))(-A_Q)^{-1} y \quad \text{for } y \in Q^{\frac{1}{2}}(H). \tag{2.14}$$

By [7, Proposition 2.10]

$$Q_t^{\frac{1}{2}}(H) = Q^{\frac{1}{2}}(D(\sqrt{-A_Q})) \quad \text{for } t > 0, \tag{2.15}$$

therefore $Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}}$ is a bounded linear operator with range $D(\sqrt{-A_Q})$. Since $Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}}$ is one to one and has a dense range, its adjoint $(Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}})^*$ has the same properties. From equation (2.14) we have

$$Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}} (Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}})^* = \frac{1}{2} (I_H - S_Q(2t))(-A_Q)^{-1}; \tag{2.16}$$

notice that, since $\|S_Q(2t)\|_{L(H,H)} < 1$, $(I_H - S_Q(2t))$ is invertible and the range of the operator in equation (2.16) is $D(A_Q)$. Taking inverses on both sides of equation (2.16) we have

$$((Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}})^{-1})^* Q_t^{-\frac{1}{2}} Q^{\frac{1}{2}} x = -2(I_H - S_Q(2t))^{-1} A_Q x \ , \quad x \in D(A_Q). \tag{2.17}$$

Let $r > 0$. Then since A_Q is self-adjoint,

$$S_Q(r)(H) \subset D(A_Q).$$

Hence for $u, v \in H$ equation (2.17) yields

$$\begin{aligned}
-2\langle (I_H - S_Q(2t))^{-1} A_Q S_Q(r)u,\, S_Q(r)v\rangle &= \langle (((Q^{-\frac{1}{2}} Q_t^{\frac{1}{2}})^{-1})^* Q_t^{-\frac{1}{2}} Q^{\frac{1}{2}} S_Q(r)u,\, S_Q(r)v\rangle \\
&= \langle Q_t^{-\frac{1}{2}} Q^{\frac{1}{2}} S_Q(r)u,\, Q_t^{-\frac{1}{2}} Q^{\frac{1}{2}} S_Q(r)v\rangle \\
&= \langle Q_t^{-\frac{1}{2}} S(r) Q^{\frac{1}{2}} u,\, Q_t^{-\frac{1}{2}} S(r) Q^{\frac{1}{2}} v\rangle. \qquad (2.18)
\end{aligned}$$

The expression on the right hand side of equation (2.18) appears in equation (2.5) when x and y are both in $Q^{\frac{1}{2}}(H)$. The expression on the left hand side of equation (2.18) can be written in terms of the eigenvalues $(-\alpha_j)$ of A_Q.
Recall that (g_k) is an orthonormal basis of H such that $A_Q g_k = -\alpha_k g_k$ for each $k \in \mathbb{N}$. Setting $u_k := \langle u, g_k\rangle$ and $v_k := \langle v, g_k\rangle$ for $k \in \mathbb{N}$, we have from equation (2.18):

$$\begin{aligned}
t\langle Q_t^{-\frac{1}{2}} S(t/2) Q^{\frac{1}{2}} u,\, Q_t^{-\frac{1}{2}} S(t/2) Q^{\frac{1}{2}} v\rangle &= -2t\langle (I_H - S_Q(2t))^{-1} A_Q S_Q(t/2)u,\, S_Q(t/2)v\rangle \\
&= -2t \sum_{k=1}^{\infty} (1 - \mathrm{e}^{-2\alpha_k t})^{-1} (-\alpha_k) \mathrm{e}^{-\alpha_k t} u_k v_k \\
&= \sum_{k=1}^{\infty} \frac{2\alpha_k t}{\mathrm{e}^{\alpha_k t} - \mathrm{e}^{-\alpha_k t}} u_k v_k \qquad (2.19) \\
&\to \sum_{k=1}^{\infty} u_k v_k \;=\; \langle u, v\rangle \quad \text{as } t \to 0, \qquad (2.20)
\end{aligned}$$

and the convergence is uniform for u and v in any compact subset of H. The uniform convergence on compact sets is because for any compact set $K \subset H$ we have $\sup\{\sum_{j=n}^{\infty} \langle u, g_j\rangle^2 : u \in K\} \to 0$ as n goes to infinity.
Similarly we have

$$\begin{aligned}
t|Q_t^{-\frac{1}{2}} S(t) Q^{\frac{1}{2}} u|^2 &= \sum_{k=1}^{\infty} \frac{2\alpha_k t}{\mathrm{e}^{2\alpha_k t} - 1} u_k^2 \\
&\to \sum_{k=1}^{\infty} u_k^2 \;=\; |u|^2 \quad \text{as } t \to 0, \qquad (2.21)
\end{aligned}$$

and the convergence is uniform for u in any compact subset of H.

Finally, using equations (2.13), (2.20) and (2.21), we have for x and y in $Q^{\frac{1}{2}}(H)$:

$$
\begin{aligned}
\lim_{t\to 0} t \ln k(t,x,y) &= \lim_{t\to 0} -\frac{1}{2}(t|Q_t^{-\frac{1}{2}}S(t)x|^2 - 2t\langle Q_t^{-\frac{1}{2}}S(t/2)x,\, Q_t^{-\frac{1}{2}}S(t/2)y\rangle + t|Q_t^{-\frac{1}{2}}S(t)y|^2) \\
&= -\frac{1}{2}(|Q^{-\frac{1}{2}}x|^2 - 2\langle Q^{-\frac{1}{2}}x, Q^{-\frac{1}{2}}y\rangle + |Q^{-\frac{1}{2}}y|^2) \\
&= -\frac{1}{2}|Q^{-\frac{1}{2}}x - Q^{-\frac{1}{2}}y|^2 \ ,
\end{aligned}
$$

and the convergence is uniform for $Q^{-\frac{1}{2}}x$ and $Q^{-\frac{1}{2}}y$ in any compact subset of H. This completes the proof.

We now consider an example where assumptions (H1), (H2) and (H3) hold.
Example. Let $l \in (0,\infty)$ and let $H = L^2((0,l))$ with the usual inner product $\langle u,v\rangle := \int_0^l u(t)v(t)\,dt$ for all u and $v \in H$. Define the operator $(A, D(A))$ on H by

$$
\begin{aligned}
Au &:= u'' \quad \text{for all } u \in D(A) \ \text{ where} \\
D(A) &:= \big\{u \in L^2((0,l)) : u \text{ and } u' \text{ are absolutely continuous and} \\
&\qquad\qquad u'' \in L^2((0,l)) \text{ and } \lim_{t\to 0} u(t) = \lim_{t\to l} u(t) = 0 \Big\} .
\end{aligned}
$$

As shown in [32, Proposition 1 of section 3.1], $(A, D(A))$ is a self-adjoint operator on H and generates the strongly continuous semigroup $(S(t))_{t\geq 0}$ of symmetric bounded linear operators on H:

$$
S(t)u := \sum_{m=1}^{\infty} \mathrm{e}^{-\alpha_m t}\langle u, e_m\rangle e_m \ , \quad u \in H \ , \quad t \geq 0 \ , \tag{2.22}
$$

where

$$
\left\{ e_m(y) := \sqrt{\frac{2}{l}}\sin\left(\frac{m\pi y}{l}\right) \ , \quad y \in (0,l) \ , \quad m \in \mathbb{N} \right\}
$$

is an orthonormal basis of H and

$$
\alpha_m := \frac{\pi^2 m^2}{l^2} \ \text{ for all } m \in \mathbb{N}.
$$

Moreover we have

$$
Ae_m = -\alpha_m e_m \ \text{ for all } m \in \mathbb{N}.
$$

As shown in [32, Theorem 2 of section 3.1], we have

$$\begin{aligned} D(A) &= \{u \in H : \sum_{m=1}^{\infty} \alpha_m^2 \langle u, e_m\rangle^2 < \infty\} \quad \text{and} \\ Au &= \sum_{m=1}^{\infty} -\alpha_m \langle u, e_m\rangle e_m \quad \text{for all } u \in D(A). \end{aligned}$$

From these equations we see that $A : D(A) \to H$ is one to one and onto.
Define

$$Qu := \sum_{m=1}^{\infty} q_m \langle u, e_m\rangle e_m \ , \quad u \in H \ , \tag{2.23}$$

where $q_m > 0$ for all $m \in \mathbb{N}$ and $\sum_{m=1}^{\infty} q_m < \infty$.
It is straightforward to show that

$$Q_t x := \int_0^t S(r)QS(r)x\, dr \ , \quad x \in H \ , \tag{2.24}$$

defines a positive definite, symmetric, trace class operator on H for all $t \in (0, \infty]$, in particular the measure $\mu := \mathcal{N}(0, Q_\infty)$ exists.
Since Q commutes with $S(t)$ for all positive t, [7, Theorem 2.4] tells us that the transition semigroup $(R_t)_{t\geq 0}$ on $L^2(H, \mu)$ consists of symmetric operators; hence assumption (H2) is satisfied.
We have $S_Q(t) := Q^{-\frac{1}{2}} S(t) Q^{\frac{1}{2}} = S(t)$ for all $t \geq 0$ and, by inspection, A^{-1} is trace class; hence assumption (H3) is satisfied.
By [10, Proposition B.1], for each positive t we have $S(t)(H) \subset Q_t^{\frac{1}{2}}(H)$ if and only if there is a positive real number c_t such that

$$|S(t)x| \leq c_t |Q_t^{\frac{1}{2}} x| \quad \text{for all } x \in H. \tag{2.25}$$

Using equations (2.22) and (2.23) in equations (2.24) and (2.25), one arrives at the conclusion that assumption (H1) is satisfied if and only if

$$\sup_{k\in\mathbb{N}} \frac{\alpha_k \mathrm{e}^{-2\alpha_k t}}{q_k} < \infty \quad \text{for each } t > 0. \tag{2.26}$$

For example, inequality (2.26) is satisfied when $q_k := (\alpha_k)^{-r}$ for all $k \in \mathbb{N}$ and $r > \frac{1}{2}$.
We now assume that inequality (2.26) is satisfied. Let x and y be vectors in H and set $x_k := \langle x, e_k\rangle$ and $y_k := \langle y, e_k\rangle$ for all k in $\mathbb{N}$. We shall show that equation (2.11) does not

necessarily hold for x and y in $H\backslash Q^{\frac{1}{2}}(H)$ such that $x-y$ is in $Q^{\frac{1}{2}}(H)$. We have

$$\begin{aligned}
&\lim_{t\to 0} t\ln k(t,x,y)\\
&= \lim_{t\to 0}\left[-\frac{t}{2}|Q_t^{-\frac{1}{2}}S(t)(x-y)|^2 + t\langle Q_t^{-\frac{1}{2}}S(t/2)x, Q_t^{-\frac{1}{2}}S(t/2)y\rangle - t\langle Q_t^{-\frac{1}{2}}S(t)x, Q_t^{-\frac{1}{2}}S(t)y\rangle\right]
\end{aligned}$$

in the sense that if the limit on one side of the equation exists then so does the limit on the other side and they are equal. By equation (2.21), if $x-y$ belongs to $Q^{\frac{1}{2}}(H)$ then $-\frac{t}{2}|Q_t^{-\frac{1}{2}}S(t)(x-y)|^2$ converges to $-\frac{1}{2}|Q^{-\frac{1}{2}}(x-y)|^2$ as t goes to zero.
Proceeding as in equation (2.19) we have

$$t\langle Q_t^{-\frac{1}{2}}S(t/2)x, Q_t^{-\frac{1}{2}}S(t/2)y\rangle - t\langle Q_t^{-\frac{1}{2}}S(t)x, Q_t^{-\frac{1}{2}}S(t)y\rangle = \sum_{k=1}^{\infty}\frac{2\alpha_k t\mathrm{e}^{-\alpha_k t}}{1+\mathrm{e}^{-\alpha_k t}}q_k^{-1}x_k y_k. \quad (2.27)$$

The expression on the right hand side of equation (2.27) converges to zero as t goes to zero if $\sum_{k=1}^{\infty} q_k^{-1}|x_k y_k| < \infty$. On the other hand, if $\alpha_k = k^2$ (so that $l=\pi$) and $q_k = \frac{1}{k^2}$ and $x_k = y_k = \frac{1}{k}$ for each k in $\mathbb{N}$ then

$$\begin{aligned}
t\langle Q_t^{-\frac{1}{2}}S(t/2)x, Q_t^{-\frac{1}{2}}S(t/2)y\rangle - t\langle Q_t^{-\frac{1}{2}}S(t)x, Q_t^{-\frac{1}{2}}S(t)y\rangle &\geq \sum_{k=1}^{\infty} k^2 t\mathrm{e}^{-k^2 t}\\
&\geq \int_0^{\infty} r^2 t\mathrm{e}^{-r^2 t}\,dr - 2\mathrm{e}^{-1}\\
&= \frac{1}{4}\sqrt{\frac{\pi}{t}} - 2\mathrm{e}^{-1} \to \infty \quad \text{as } t\to 0;
\end{aligned}$$

hence in this case we have $x=y$ and $\lim_{t\to 0} t\ln k(t,x,y) = \infty$.

2.3 From limits for densities to short time asymptotics in trajectory space

In this section we assume that $(X_x(t))_{t\in[0,1]}$ is a continuous version of the mild solution defined in equation (2.3). Equation (2.11) enables us to prove the Freidlin-Wentzell formulation of the lower bound of the large deviation principle in $C([0,1];H)$ for small time asymptotics of $(X_x(t))_{t\in[0,1]}$. The proof is similar to the proof of Varadhan's finite dimensional result [31, Lemma 3.4]. The rate function is

$$\mathcal{I}(u) := \begin{cases} \frac{1}{2}\int_0^1 |\phi(s)|_{H_\nu}^2\,ds & \text{if } \phi\in L^2([0,1];H_\nu) \text{ and } u(t) = x+\int_0^t \phi(s)\,ds \quad \forall t\in[0,1],\\ \infty & \text{otherwise.}\end{cases}$$

We remark that equation (2.11) does not yield the upper bound of the large deviation principle so easily. One obstacle is that the limit in equation (2.11) is not uniform on arbitrary bounded subsets of H. The large deviation principle for short time asymptotics of an Ornstein-Uhlenbeck process can be obtained without relying on limits of densities. A fruitful approach is by making clever use of the large deviation principle associated with the Gaussian distribution of the continuous trajectories of $(W(t))_{t\in[0,1]}$; this approach is implemented for solutions of some stochastic equations with nonlinear terms in chapters 3 and 4. Nevertheless we end this chapter by proving the lower bound using equation (2.11) and Peszat's exponential tail estimate for stochastic convolutions. The main use of Peszat's exponential tail estimate in this thesis is in chapter 3, where it is stated in Theorem 3.10. For simplicity suppose that $x \in Q_\infty^{\frac{1}{2}}(H)$. Let $\phi \in L^2([0,1]; H_\nu)$. Define

$$f(t) := x + \int_0^t \phi(s)\, ds \quad \text{for all } t \in [0,1].$$

Proposition 2.3 *Given $\delta > 0$ and $\gamma > 0$ there exists $\epsilon_0 > 0$ such that*

$$P\{\sup_{t\in[0,1]} |X_x(\epsilon t) - f(t)| < \delta\} \geq \mathrm{e}^{\frac{-\frac{1}{2}\int_0^1 |\phi(s)|^2_{H_\nu}\, ds - \gamma}{\epsilon}} \quad \textit{for all } \epsilon < \epsilon_0.$$

Proof. Since $Q_\infty^{\frac{1}{2}}(H)$ is a dense subset of H and of H_ν we can take

$$\psi := \sum_{k=1}^{2^N} 1_{(\frac{k-1}{2^N}, \frac{k}{2^N}]} a_k, \tag{2.28}$$

where N is a natural number and $a_1, \dots, a_{2^N}$ are vectors in $Q_\infty^{\frac{1}{2}}(H)$, such that the continuous function

$$g(t) := x + \int_0^t \psi(s)\, ds\ , \quad t \in [0,1],$$

and ψ and N satisfy

1. $\sup_{t\in[0,1]} |g(t) - f(t)| < \frac{\delta}{6}$ and
2. $-\frac{1}{2}\int_0^1 |\psi(s)|^2_{H_\nu}\, ds > -\frac{1}{2}\int_0^1 |\phi(s)|^2_{H_\nu}\, ds - \frac{\gamma}{2}$ and also
3. $|f(t) - f(s)| < \frac{\delta}{3}$ for all t and s in $[0,1]$ such that $|t-s| \leq \frac{1}{2^N}$.

Let R be a natural number such that

$$B_H(0, \tfrac{\delta}{6}) \supset \overline{B}_{H_\nu}(0, \tfrac{1}{R}).$$

For each $t \in [0,1]$, for any natural number $r \geq R$ we can choose a compact subset of H_ν

$$K_{t,r} \subset B_{H_\nu}(g(t), \tfrac{1}{r})$$

such that $\mu(K_{t,r}) > 0$; this is because $\mu(Q^{\frac{1}{2}}(H)) = 1$ and $g(t)$ belongs to $Q_\infty^{\frac{1}{2}}(H)$, which is a dense subset of H_ν. Set

$$B := \bigcup_{t \in [0,1]} \overline{B}_{H_\nu}(g(t), \tfrac{1}{R}),$$

which is a compact subset of H. We may assume that the natural number N introduced in equation (2.28) is so large that we have

$$\sup\{|(S(s) - I_H)z| : s \in [0, \frac{1}{2^N}] \text{ and } z \in B\} < \frac{\delta}{6}. \tag{2.29}$$

For brevity set for each $t \in [0,1]$

$$B_t := \overline{B}_{H_\nu}(g(t), \tfrac{1}{R}),$$

which is contained in $B_H(f(t), \frac{\delta}{3})$. Also for brevity set

$$t_{n,k} := \frac{k}{2^n}$$

for each natural number $n \geq N$ and $k \in \{0, 1, \ldots, 2^n\}$.

Let $\epsilon \in (0,1]$. Let $n \geq N$. Let $r \geq R$. Set $y_0 := x$. We have

$$\begin{aligned}
&P\{X_x(\epsilon t_{n,1}) \in B_{t_{n,1}}, \ldots, X_x(\epsilon t_{n,2^n}) \in B_{t_{n,2^n}}\} \\
&\quad \geq \; P\{X_x(\epsilon t_{n,1}) \in K_{t_{n,1},r}, \ldots, X_x(\epsilon t_{n,2^n}) \in K_{t_{n,2^n},r}\} \\
&\quad = \; \int_{K_{t_{n,1},r}} \cdots \int_{K_{t_{n,2^n},r}} \prod_{j=1}^{2^n} k(\epsilon(t_{n,j} - t_{n,j-1}), y_{j-1}, y_j) \, d\mu(y_{2^n}) \ldots d\mu(y_1) \\
&\quad = \; \prod_{j=1}^{2^n} \mu(K_{t_{n,j},r}) \times \\
&\qquad\quad \int_{K_{t_{n,1},r} \times \cdots \times K_{t_{n,2^n},r}} \prod_{j=1}^{2^n} k(\epsilon(t_{n,j} - t_{n,j-1}), y_{j-1}, y_j) \, d(\mu_{t_{n,1},r} \times \cdots \times \mu_{t_{n,2^n},r})(y_1, \ldots, y_{2^n}),
\end{aligned}$$

where $\mu_{t_{n,j},r} := \frac{1}{\mu(K_{t_{n,j},r})}\mu$ on the Borel σ-algebra of $K_{t_{n,j},r}$ for each $j \in 1, \ldots, 2^n$ and

$\mu_{t_{n,1},r}\times\cdots\times\mu_{t_{n,2^n},r}$ is the product measure on the cartesian product $K_{t_{n,1},r}\times\cdots\times K_{t_{n,2^n},r}$. We apply Jensen's inequality to obtain

$$\epsilon\ln P\{X_x(\epsilon t_{n,1})\in B_{t_{n,1}},\ldots,X_x(\epsilon t_{n,2^n})\in B_{t_{n,2^n}}\}\geq$$
$$\epsilon\ln\prod_{j=1}^{2^n}\mu(K_{t_{n,j},r})+$$
$$\int_{K_{t_{n,1},r}\times\cdots\times K_{t_{n,2^n},r}}\textstyle\sum_{j=1}^{2^n}\frac{\epsilon(t_{n,j}-t_{n,j-1})\ln k(\epsilon(t_{n,j}-t_{n,j-1}),y_{j-1},y_j)}{t_{n,j}-t_{n,j-1}}\;d(\mu_{t_{n,1},r}\times\cdots\times\mu_{t_{n,2^n},r})(y_1,\ldots,y_{2^n}).$$

Because $\cup_{j=1}^{2^n}K_{t_{n,j},r}$ is compact in H_ν, equation (2.11) applies to the integrand: given any $\theta>0$ we have for all sufficiently small positive ϵ

$$\sum_{j=1}^{2^n}\frac{\epsilon(t_{n,j}-t_{n,j-1})\ln k(\epsilon(t_{n,j}-t_{n,j-1}),y_{j-1},y_j)}{t_{n,j}-t_{n,j-1}}\geq\sum_{j=1}^{2^n}\frac{-\frac{1}{2}|y_j-y_{j-1}|^2_{H_\nu}}{t_{n,j}-t_{n,j-1}}-\theta$$

uniformly for all $(y_1,\ldots,y_{2^n})\in K_{t_{n,1},r}\times\cdots\times K_{t_{n,2^n},r}$. Thus

$$\begin{aligned}&\liminf_{s\to 0}\inf_{\epsilon<s}\epsilon\ln P\{X_x(\epsilon t_{n,1})\in B_{t_{n,1}},\ldots,X_x(\epsilon t_{n,2^n})\in B_{t_{n,2^n}}\}\\ &\quad\geq\int_{K_{t_{n,1},r}\times\cdots\times K_{t_{n,2^n},r}}\sum_{j=1}^{2^n}\frac{-\frac{1}{2}|y_j-y_{j-1}|^2_{H_\nu}}{t_{n,j}-t_{n,j-1}}\;d(\mu_{t_{n,1},r}\times\cdots\times\mu_{t_{n,2^n},r})(y_1,\ldots,y_{2^n}).\end{aligned}$$

Letting r go to infinity makes $K_{t_{n,j},r}$ shrink in H_ν towards $g(t_{n,j})$ for each $j\in\{1,\ldots,2^n\}$. Thus for all $n\geq N$ we have

$$\begin{aligned}&\liminf_{s\to 0}\inf_{\epsilon<s}\epsilon\ln P\{X_x(\epsilon t_{n,1})\in B_{t_{n,1}},\ldots,X_x(\epsilon t_{n,2^n})\in B_{t_{n,2^n}}\}\\ &\quad\geq\ -\frac{1}{2}\sum_{j=1}^{2^n}\frac{|g(t_{n,j})-g(t_{n,j-1})|^2_{H_\nu}}{t_{n,j}-t_{n,j-1}}\\ &\quad=\ -\frac{1}{2}\int_0^1|\psi(t)|^2_{H_\nu}\,dt\\ &\quad>\ -\frac{1}{2}\int_0^1|\phi(s)|^2_{H_\nu}\,ds-\frac{\gamma}{2}.\end{aligned}\tag{2.30}$$

Let $\epsilon \in (0,1]$ and let $n \geq N$. We have

$$\begin{aligned}
& P\{\sup_{t\in[0,1]} |X_x(\epsilon t) - f(t)| < \delta\} \\
\geq\ & P\{X_x(\epsilon t_{n,1}) \in B_{t_{n,1}}, \ldots, X_x(\epsilon t_{n,2^n}) \in B_{t_{n,2^n}}, \sup_{t\in[0,1]} |X_x(\epsilon t) - f(t)| < \delta\} \\
=\ & P\{X_x(\epsilon t_{n,1}) \in B_{t_{n,1}}, \ldots, X_x(\epsilon t_{n,2^n}) \in B_{t_{n,2^n}}\} \\
& \quad - P\{X_x(\epsilon t_{n,1}) \in B_{t_{n,1}}, \ldots, X_x(\epsilon t_{n,2^n}) \in B_{t_{n,2^n}}, \sup_{t\in[0,1]} |X_x(\epsilon t) - f(t)| \geq \delta\}. \quad (2.31)
\end{aligned}$$

Equation (2.30) gives us a lower bound for the first term on the right hand side of equation (2.31) when ϵ is small. We now want to show that when we choose $n \geq N$ sufficiently large the second term on the right hand side is small compared to the first term. We have

$$\begin{aligned}
& P\{X_x(\epsilon t_{n,1}) \in B_{t_{n,1}}, \ldots, X_x(\epsilon t_{n,2^n}) \in B_{t_{n,2^n}}, \sup_{t\in[0,1]} |X_x(\epsilon t) - f(t)| \geq \delta\} \\
& \quad \leq \sum_{j=1}^{2^n} P\{X_x(\epsilon t_{n,j-1}) \in B_{t_{n,j-1}}, \sup_{t\in[t_{n,j-1},t_{n,j}]} |X_x(\epsilon t) - f(t)| \geq \delta\}. \quad (2.32)
\end{aligned}$$

We can bound the jth summand on the right hand side of inequality (2.32). We have

$$\begin{aligned}
\sup_{t\in[t_{n,j-1},t_{n,j}]} |X_x(\epsilon t) - f(t)| \quad \leq \quad & \sup_{t\in[t_{n,j-1},t_{n,j}]} |X_x(\epsilon t) - X_x(\epsilon t_{n,j-1})| + |X_x(\epsilon t_{n,j-1}) - f(t_{n,j-1})| \\
& + \sup_{t\in[t_{n,j-1},t_{n,j}]} |f(t_{n,j-1}) - f(t)|,
\end{aligned}$$

hence

$$\begin{aligned}
& P\{X_x(\epsilon t_{n,j-1}) \in B_{t_{n,j-1}}, \sup_{t\in[t_{n,j-1},t_{n,j}]} |X_x(\epsilon t) - f(t)| \geq \delta\} \\
& \quad \leq P\{X_x(\epsilon t_{n,j-1}) \in B, \sup_{t\in[t_{n,j-1},t_{n,j}]} |X_x(\epsilon t) - X_x(\epsilon t_{n,j-1})| > \frac{\delta}{3}\} \\
& \quad \leq P\{\sup_{t\in[t_{n,j-1},t_{n,j}]} |\int_{\epsilon t_{n,j-1}}^{\epsilon t} S(\epsilon t - s) i\, dW(s)| > \frac{\delta}{6}\}\ , \quad \text{by (2.29)}, \\
& \quad = P\{\sup_{t\in[t_{n,j-1},t_{n,j}]} |\epsilon^{\frac{1}{2}} \int_{t_{n,j-1}}^{t} S(\epsilon(t-s)) i\, dV(s)| > \frac{\delta}{6}\}, \quad (2.33)
\end{aligned}$$

where $V(s) := \epsilon^{-\frac{1}{2}} W(\epsilon s)$ for all $s \in [0,1]$ is an H-valued Wiener process. Peszat's exponential tail estimate yields

$$P\{\sup_{t\in[t_{n,j-1},t_{n,j}]} |\int_{t_{n,j-1}}^{t} S(\epsilon(t-s))i\,dV(s)| > \frac{\delta}{6\epsilon^{\frac{1}{2}}}\} \leq C\exp\left(\frac{-\delta^2}{36\epsilon\kappa^2\eta}\right), \tag{2.34}$$

where for chosen $\alpha_0 \in (0, \frac{1}{2})$ and $p_0 > 1$ such that $(\alpha_0 - 1)p_0 > -1$ we define

$$\kappa := \frac{\sup_{t\in[0,1]} \|S(t)\|_{L(H,H)}}{((\alpha_0-1)p_0+1)^{\frac{1}{p_0}}}$$

and

$$\eta := \frac{\sup_{t\in[0,1]} \|S(t)\|^2_{L(H,H)} \|i\|^2_{L_2(H_\nu,H)}}{1-2\alpha_0} 2^{-n(1-2\alpha_0)}$$

and

$$C := 4 + \exp(4n_0!)^{\frac{1}{n_0}} \quad \text{and} \quad n_0 := \frac{p_0}{2p_0-2} + 1.$$

Thanks to the factor $2^{-n(1-2\alpha_0)}$ in the definition of η we can choose $n \geq N$ sufficiently large to ensure that

$$2^n C\exp\left(\frac{-\delta^2}{36\epsilon\kappa^2\eta}\right) < \frac{1}{2}\exp\left(\frac{-\frac{1}{2}\int_0^1 |\phi(s)|^2_{H_\nu}\,ds - \frac{\gamma}{2}}{\epsilon}\right) \qquad \forall \epsilon \in (0,1].$$

From inequality (2.30) there exists $s > 0$ such that for all $\epsilon \in (0, s)$ we have

$$P\{X_x(\epsilon t_{n,1}) \in B_{t_{n,1}}, \ldots, X_x(\epsilon t_{n,2^n}) \in B_{t_{n,2^n}}\} > \exp\left(\frac{-\frac{1}{2}\int_0^1 |\phi(s)|^2_{H_\nu}\,ds - \frac{\gamma}{2}}{\epsilon}\right).$$

Using this inequality and inequalities (2.32), (2.33) and (2.34) in the right hand side of (2.31) yields

$$\begin{aligned} &P\{\sup_{t\in[0,1]} |X_x(\epsilon t) - f(t)| < \delta\} \\ &\quad\geq \exp\left(\frac{-\frac{1}{2}\int_0^1 |\phi(s)|^2_{H_\nu}\,ds - \frac{\gamma}{2}}{\epsilon}\right) - 2^n C\exp\left(\frac{-\delta^2}{36\epsilon\kappa^2\eta}\right) \qquad \forall \epsilon \in (0,s) \\ &\quad\geq \exp\left(\frac{-\frac{1}{2}\int_0^1 |\phi(s)|^2_{H_\nu}\,ds - \gamma}{\epsilon}\right) \qquad \forall \epsilon \in (0, s \wedge \frac{\gamma}{2\ln 2}). \end{aligned}$$

This completes the proof.

Chapter 3

Small time asymptotics of the solution when there is a Lipschitz continuous drift and diffusion function

3.1 Introduction

In this chapter we follow Peszat's paper [25] closely to obtain a large deviation principle describing the small time asymptotics of the mild solution of a stochastic differential equation with Lipschitz continuous drift and Lipschitz continuous diffusion function, in a Hilbert space H. Peszat found a large deviation principle describing the small noise asymptotics of mild solutions of stochastic differential equations and his methods require little modification to yield our large deviation principle describing small time asymptotics.

Zhang [33] used exponential equivalence arguments to get a large deviation principle describing the small time asymptotics of the mild solution of a stochastic equation with Lipschitz continuous and bounded diffusion function. To deal with the stochastic convolution term he assumed that the Hilbert space H, in which the unbounded linear drift A and the strongly continuous semigroup $(\mathrm{e}^{At} = S(t))_{t\geq 0}$ it generates are defined, is compactly embedded in another Hilbert space H_1 and $(S(t))_{t\geq 0}$ extends to a strongly continuous semigroup on H_1. His result is a large deviation principle for distributions on the space of continuous H_1-valued trajectories, rather than the space of continuous H-valued trajectories. Using Peszat's methods, we avoid the need to introduce another Hilbert space corresponding to Zhang's H_1 and our result holds if the diffusion function is not bounded. Our main result is the large deviation principle contained in Corollary 3.4.

We now define spaces and functions which we use throughout this chapter. Let $(H, \langle\cdot,\cdot\rangle, |\cdot|)$ and $(U, \langle\cdot,\cdot\rangle_U, |\cdot|_U)$ be separable Hilbert spaces. Let $A : D(A) \subset H \to H$ be the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on H. Set

$$M := \sup_{t\in[0,1]} \|S(t)\|_{L(H,H)}.$$

Let

$$F : ([0,1] \times H, \mathcal{B}_{[0,1]} \otimes \mathcal{B}_H) \to (H, \mathcal{B}_H)$$

be a measurable function and let functions F and

$$G : H \to L_2(U, H)$$

satisfy

$$\begin{aligned}
|F(t,x) - F(t,y)| &\leq \Lambda|x-y| \quad \forall t \in [0,1] \text{ and } \forall x, y \in H \text{ and} && (3.1)\\
|F(t,x)| &\leq \Lambda(1+|x|) \quad \forall t \in [0,1] \text{ and } \forall x \in H \text{ and} && (3.2)\\
\|G(x) - G(y)\|_{L_2(U,H)} &\leq \Lambda|x-y| \quad \forall x, y \in H \text{ and} && (3.3)\\
\|G(x)\|_{L_2(U,H)} &\leq \Lambda(1+|x|) \quad \forall x \in H, && (3.4)
\end{aligned}$$

where Λ is a positive real constant.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(\mathcal{F}_t)_{t\geq 0}$ be a right continuous filtration of sub σ-algebras of $\mathcal{F}$ such that all sets in $\mathcal{F}$ of P measure zero are in $\mathcal{F}_0$. Let (g_k) be an orthonormal basis of U and let $((\beta_k(t))_{t\geq 0})$ be an independent sequence of real valued $(\mathcal{F}_t)$-Brownian motions. A cylindrical Wiener process on U is defined by the series

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t)\, g_k,$$

which does not converge in U but converges in an arbitrary Hilbert space U_1 containing U and such that the embedding

$$J : U \hookrightarrow U_1$$

is Hilbert-Schmidt. Whatever our choice of U_1, the distribution of $W(1)$ in U_1 has reproducing kernel Hilbert space U. We now fix U_1 by taking a decreasing sequence of positive real numbers (λ_k) such that $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$ and defining U_1 to be the completion of U with the inner product

$$\langle u, v \rangle_{U_1} := \sum_{k=1}^{\infty} \lambda_k^2 \langle u, g_k \rangle_U \langle v, g_k \rangle_U \quad \text{for all } u, v \in U. \tag{3.5}$$

We abuse notation and denote the inner product on U_1 still by $\langle \cdot, \cdot \rangle_{U_1}$ and the norm on U_1 is denoted by $|\cdot|_{U_1}$.

Our aim is to find a large deviation principle describing the small time asymptotics of the mild solution of the initial value problem

$$\left.\begin{array}{rcl} dX(t) & = & (AX(t) + F(t, X(t)))\, dt + G(X(t))\, dW(t), \\ X(0) & = & x \in H. \end{array}\right\} \tag{3.6}$$

The mild solution of (3.6) is the $(\mathcal{F}_t)$-predictable process $(X_x(t))_{t \in [0,1]}$ such that

$$P\{\int_0^1 |X_x(t)|^2\, dt < \infty\} = 1 \tag{3.7}$$

and

$$X_x(t) = S(t)x + \int_0^t S(t-s)F(s, X_x(s))\, ds + \int_0^t S(t-s)G(X_x(s))\, dW(s) \quad P \text{ a.e.} \tag{3.8}$$

for each $t \in [0, 1]$.

The existence, uniqueness and continuity result underlying this work is Theorem 3.18 in the appendix of this chapter. Notice that Theorem 3.18 applies to more general nonlinear drift functions than F; this fact will be useful later when a change of probability measure introduces an auxiliary problem whose nonlinear drift term is of the type in Theorem 3.18.

Specifically, we will find a large deviation principle for the family of distributions in trajectory space $C([0, 1]; H)$:

$$\mu_x^\epsilon := \mathcal{L}(\, \omega \in \Omega \mapsto (t \in [0, 1] \mapsto X_x(\epsilon t)(\omega))\,) \quad : \epsilon \in (0, 1]. \tag{3.9}$$

From equation (3.8), for each $\epsilon \in (0, 1]$ and $t \in [0, 1]$ we have P a.e.

$$\begin{aligned} X_x(\epsilon t) &= S(\epsilon t)x + \int_0^{\epsilon t} S(\epsilon t - s)F(s, X_x(s))\, ds + \int_0^{\epsilon t} S(\epsilon t - s)G(X_x(s))\, dW(s) \\ &= S(\epsilon t)x + \epsilon \int_0^t S(\epsilon(t-u))F(\epsilon u, X_x(\epsilon u))\, du + \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-u))G(X_x(\epsilon u))\, dV^\epsilon(u), \end{aligned} \tag{3.10}$$

where

$$V^\epsilon(t) := \epsilon^{-\frac{1}{2}} W(\epsilon t) \quad \forall t \geq 0$$

is a U_1-valued $(\mathcal{F}_{\epsilon t})$-Wiener process and $\mathcal{L}(V^\epsilon(1)) = \mathcal{L}(W(1))$. By Proposition 3.19, for each $\epsilon \in (0,1]$ the continuous $(\mathcal{F}_t)$-predictable process $(X_x^\epsilon(t))_{t\in[0,1]}$ satisfying the equation

$$X_x^\epsilon(t) = S(\epsilon t)x + \epsilon \int_0^t S(\epsilon(t-u))F(\epsilon u, X_x^\epsilon(u))\,du + \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-u))G(X_x^\epsilon(u))\,dW(u) \tag{3.11}$$

P a.e. for each $t \in [0,1]$ also has the distribution μ_x^ϵ in trajectory space. Thus for each $\epsilon \in (0,1]$ we consider the process $(X_x^\epsilon(t))_{t\in[0,1]}$, which is the mild solution of the problem

$$\left.\begin{array}{rcl} dX^\epsilon(t) & = & (\epsilon AX^\epsilon(t) + \epsilon F(\epsilon t, X^\epsilon(t)))\,dt + \epsilon^{\frac{1}{2}} G(X^\epsilon(t))\,dW(t) \\ X^\epsilon(0) & = & x \end{array}\right\}$$

and we define the corresponding trajectory-valued random variable $X_x^\epsilon : \Omega \to C([0,1];H)$ by

$$X_x^\epsilon(\omega) := (t \in [0,1] \mapsto X_x^\epsilon(t)(\omega)) \quad \forall \omega \in \Omega. \tag{3.12}$$

For each $\phi \in L^2([0,1];U)$ and $x \in H$ we denote by z_x^ϕ the function in $C([0,1];H)$ such that

$$z_x^\phi(t) = x + \int_0^t G(z_x^\phi(s))\phi(s)\,ds \quad \forall t \in [0,1].$$

For each $x \in H$ we define the prospective rate function $\mathcal{I}_x : C([0,1];H) \to [0,\infty]$ by

$$\mathcal{I}_x(u) := \frac{1}{2} \inf \left\{ \int_0^1 |\psi(s)|_U^2\,ds : \psi \in L^2([0,1];U) \text{ and } u = z_x^\psi \right\} \tag{3.13}$$

for all $u \in C([0,1];H)$. We will prove the following theorem in Section 3.2; it verifies that for each $x \in H$ the function $\mathcal{I}_x$ is well defined and a good rate function.

Theorem 3.1 *1. Given $\phi \in L^2([0,1];U)$ and $x \in H$, z_x^ϕ is well defined; that is, there is a unique function $u \in C([0,1];H)$ such that*

$$u(t) = x + \int_0^t G(u(s))\phi(s)\,ds \quad \forall t \in [0,1].$$

2. For fixed $u \in C([0,1];H)$ the linear operator

$$\psi \in L^2([0,1];U) \mapsto \left(t \mapsto \int_0^t G(u(s))\psi(s)\,ds \right) \in C([0,1];H)$$

is compact.

3. *Let $B \subset L^2([0,1];U)$ be weakly sequentially compact and let $K \subset H$ be compact. Then the set*

$$\mathcal{C} := \{u \in C([0,1];H) : u = z_x^\phi \text{ for some } \phi \in B \text{ and some } x \in K\}$$

is compact. In particular $\{\mathcal{I}_x \leq r\}$ is compact for any $x \in H$ and any $r \in (0,\infty)$ because the closed ball $\{\phi \in L^2([0,1];U) : \|\phi\|_{L^2([0,1];U)} \leq \sqrt{2r}\}$ is weakly sequentially compact.

For each natural number n let $\Pi_n : U \to U$ be the orthogonal projection of U onto the span of $\{g_1, \ldots, g_n\}$:

$$\Pi_n x := \sum_{j=1}^{n} \langle x, g_j\rangle_U \, g_j \quad \forall x \in U.$$

In our proof of the upper bound of the large deviation principle we use the fact that Π_n can be written in terms of the bounded linear operator from U_1 into U:

$$\Pi_n^1 u := \sum_{k=1}^{n} \lambda_k^{-2} \langle u, Jg_k\rangle_{U_1} \, g_k \quad \forall u \in U_1.$$

We have $\Pi_n^1 Jx = \Pi_n x$ for all $x \in U$, which follows from the definition of U_1.
We can now state two additional assumptions (A1) and (A2) on G and $(S(t))_{t\geq 0}$, respectively, which will only be used in the proof of the upper bound of the large deviation principle.

(**A1**) For each $r \in (0,\infty)$

$$\sup_{h \in B_H(0,r)} \|G(h)(I_U - \Pi_n)\|_{L_2(U,H)} \to 0 \quad \text{as } n \to \infty.$$

(**A2**) For each $a \in (0,1]$ the family of functions in $L(H,H)$ with the norm topology:

$$\{t \in [a,1] \mapsto S(\epsilon t) \in L(H,H) \ : \epsilon \in (0,1]\}$$

is uniformly equicontinuous.

Assumption (A1) is true when G is of the form $G(x) = G_1(x)B \ \forall x \in H$, where B is a constant operator in $L_2(U,U)$ and $G_1 : H \to L(U,H)$ is Lipschitz continuous.
Assumption (A2) is true when $(S(t))_{t\geq 0}$ is an analytic semigroup. Then there is a positive real constant c such that $\|AS(t)\|_{L(H,H)} \leq \frac{c}{t}$ for all $t \in (0,1]$ (see [24, Theorem 5.2 in

chapter 2]) and consequently $\|S(t)-S(r)\|_{L(H,H)} \leq c\ln(\frac{t}{r})$ for all $t,r \in (0,1]$. We remark that in the small noise asymptotics paper [25] Peszat only needed to assume that $(S(t))_{t\geq 0}$ is continuous on $(0,1]$ in the norm topology.

Our two main theorems are the following.

Theorem 3.2 *Let K be a compact subset of H and let $\phi \in L^2([0,1];U)$. Let $\delta > 0$ and $\gamma > 0$. There exists $\epsilon_0 > 0$ such that for all $x \in K$ and for all $\epsilon \in (0,\epsilon_0]$*

$$P\left\{\sup_{t\in[0,1]} |X_x^\epsilon(t) - z_x^\phi(t)| < \delta\right\} \geq \exp\left(\frac{-\frac{1}{2}\int_0^1 |\phi(s)|_U^2\, ds - \gamma}{\epsilon}\right).$$

Theorem 3.3 *Assume that (A1) and (A2) hold. Let K be a compact subset of H. Let $r > 0$ and $\delta > 0$ and $\gamma > 0$. There exists $\epsilon_0 > 0$ such that for all $x \in K$ and for all $\epsilon \in (0,\epsilon_0]$*

$$P\{X_x^\epsilon \notin B_{C([0,1];H)}(\{\mathcal{I}_x \leq r\},\, \delta)\} \leq \exp\left(\frac{-r+\gamma}{\epsilon}\right).$$

The following result follows immediately from these theorems.

Corollary 3.4 *Assume that (A1) and (A2) hold. Let $x \in H$. The family of distributions $\{\mu_x^\epsilon : \epsilon \in (0,1]\}$ defined in equation (3.9) satisfies a large deviation principle with rate function $\mathcal{I}_x$.*

Proof. When $K = \{x\}$ Theorem 3.2 implies the Freidlin-Wentzell formulation of the lower bound of the large deviation principle of $\{\mathcal{L}(X_x^\epsilon) = \mu_x^\epsilon : \epsilon \in (0,1]\}$ with rate function $\mathcal{I}_x$ and Theorem 3.3 is the corresponding upper bound.

We will show in Section 3.3 that if Theorems 3.2 and 3.3 hold for bounded diffusion functions $G : H \to L_2(U,H)$ then the theorems also hold when the function G is not bounded. Section 3.4 presents some important inequalities from Peszat's paper [25], which are used to prove Theorems 3.2 and 3.3 in the case of bounded G in Sections 3.5 and 3.6.

3.2 The rate function

In this section we prove Theorem 3.1.

Proof of Theorem 3.1(1). Let $\phi \in L^2([0,1];U)$ and let $x \in H$. Take $N \in \mathbb{N}$ such that

$$\frac{1}{\sqrt{N}}\Lambda\|\phi\|_{L^2([0,1];U)} < 1.$$

Let $y \in H$, let $n \in \{0, 1, \ldots, N-1\}$ and let ϕ_n be the element of $L^2([\frac{n}{N}, \frac{n+1}{N}]; U)$ defined by

$$\phi_n(t) := \phi(t) \quad \forall t \in [\tfrac{n}{N}, \tfrac{n+1}{N}].$$

Define the map $F_{n,y} : C([\frac{n}{N}, \frac{n+1}{N}]; H) \to C([\frac{n}{N}, \frac{n+1}{N}]; H)$ by

$$(F_{n,y}(u))(t) := y + \int_{\frac{n}{N}}^{t} G(u(s))\phi_n(s)\, ds \quad \forall t \in [\tfrac{n}{N}, \tfrac{n+1}{N}].$$

For arbitrary $u, v \in C([\frac{n}{N}, \frac{n+1}{N}]; H)$ and $t \in [\frac{n}{N}, \frac{n+1}{N}]$ we have

$$\begin{aligned} |F_{n,y}(u)(t) - F_{n,y}(v)(t)| &= |\int_{\frac{n}{N}}^{t} (G(u(s)) - G(v(s)))\phi_n(s)\, ds| \\ &\le \Lambda \int_{\frac{n}{N}}^{t} |u(s) - v(s)||\phi_n(s)|_U\, ds \\ &\le \Lambda \frac{1}{\sqrt{N}} \|\phi\|_{L^2([0,1];U)} \sup_{s \in [\frac{n}{N}, \frac{n+1}{N}]} |u(s) - v(s)|. \end{aligned}$$

Thus $F_{n,y}$ is a contraction on the Banach space $C([\frac{n}{N}, \frac{n+1}{N}]; H)$ with the sup norm. Let u_0 be the fixed point of $F_{0,x}$. For $1 \le n \le N-1$ let u_n be the fixed point of $F_{n,u_{n-1}(\frac{n}{N})}$. Define

$$u(t) := u_n(t) \quad \text{for each } t \in [\tfrac{n}{N}, \tfrac{n+1}{N}] \text{ and each } n \in \{0, 1, \ldots, N-1\}.$$

Then, by inspection, $u \in C([0,1]; H)$ and one can show by induction on n that

$$u(t) = x + \int_0^t G(u(s))\phi(s)\, ds \quad \forall t \in [0, \tfrac{n+1}{N}] \text{ and } \forall n \in \{0, 1, \ldots, N-1\}.$$

If also $v \in C([0,1]; H)$ and

$$v(t) = x + \int_0^t G(v(s))\phi(s)\, ds \quad \forall t \in [0,1]$$

then $u(t) = v(t)$ for all $t \in [0, \frac{1}{N}]$ since $F_{0,x}$ has a unique fixed point and one can show by induction on n that $u(t) = v(t)$ for all $t \in [0, \frac{n+1}{N}]$ and for all $n \in \{0, 1, \ldots, N-1\}$.

Proof of Theorem 3.1(2). This proof follows the lines of the proof of [10, Proposition 8.4]. Let $u \in C([0,1]; H)$. We want to show that the map

$$\psi \in L^2([0,1]; U) \mapsto \left(t \mapsto \int_0^t G(u(s))\psi(s)\, ds\right) \in C([0,1]; H)$$

is a compact linear operator. It is straightforward to show that this map is a bounded linear operator, therefore we will only show that an arbitrary bounded sequence (ψ_n) in $L^2([0,1];U)$ is mapped to a sequence in $C([0,1];H)$ with a convergent subsequence.
Set $r := \sup_{n\in\mathbb{N}} \|\psi_n\|_{L^2([0,1];U)} < \infty$. For $\psi \in L^2([0,1];U)$ such that $\|\psi\|_{L^2([0,1];U)} \leq r$ and for $0 \leq t < s \leq 1$ we have

$$\begin{aligned}\left|\int_t^s G(u(\sigma))\psi(\sigma)\, d\sigma\right| &\leq \sup_{\sigma\in[0,1]} \|G(u(\sigma))\|_{L_2(U,H)} \int_t^s |\psi(\sigma)|_U\, d\sigma \\ &\leq r \sup_{\sigma\in[0,1]} \|G(u(\sigma))\|_{L_2(U,H)} \sqrt{s-t}.\end{aligned}$$

Thus the family of functions

$$\left\{ t \in [0,1] \mapsto \int_0^t G(u(s))\psi(s)\, ds \in H \quad : \ \psi \in L^2([0,1];U) \text{ and } \|\psi\|_{L^2([0,1];U)} \leq r \right\}$$

is uniformly equicontinuous. We will show that there is a subsequence of the sequence of continuous functions

$$\left(t \in [0,1] \mapsto \int_0^t G(u(s))\psi_n(s)\, ds \in H \right)$$

which converges pointwise on a dense subset of $[0,1]$; then, as this subsequence is uniformly equicontinuous, it is Cauchy in $C([0,1];H)$ and we will be done.
For $t \in (0,1]$ define the linear operator $A_t : L^2([0,1];U) \to H$ by

$$A_t\psi = \int_0^t G(u(s))\psi(s)\, ds \ , \quad \psi \in L^2([0,1];U).$$

We claim that A_t is Hilbert-Schmidt. Let (e_k) be an orthonormal basis of H and let (ϕ_k) be an orthonormal basis of $L^2([0,1];\mathbb{R})$. An orthonormal basis of U is (g_k) and the family of products $\{\phi_j g_k : \quad j,k \in \mathbb{N}\}$ is an orthonormal basis of $L^2([0,1];U)$. We have

$$\begin{aligned}
\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|A_t(\phi_j g_k)|^2 &= \sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\langle A_t(\phi_j g_k),\, e_i\rangle^2 \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\int_0^t \langle G(u(s))g_k,\, e_i\rangle \phi_j(s)\,ds\right)^2 \\
&= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\int_0^t \langle G(u(s))g_k,\, e_i\rangle^2\,ds \\
&= \int_0^t \sum_{k=1}^{\infty}\sum_{i=1}^{\infty}\langle G(u(s))g_k,\, e_i\rangle^2\,ds \\
&= \int_0^t \|G(u(s))\|^2_{L_2(U,H)}\,ds \quad < \infty.
\end{aligned}$$

For each $t_i \in (0,1]\cap\mathbb{Q}$, since A_{t_i} is a compact operator, the set $\{A_{t_i}\psi_n \;:\; n\in\mathbb{N}\}$ is relatively compact in H. We can apply the diagonal argument to the sequence of sequences

$$\begin{array}{ccccc}
A_{t_1}\psi_1 & A_{t_1}\psi_2 & A_{t_1}\psi_3 & A_{t_1}\psi_4 & \cdots \\
A_{t_2}\psi_1 & A_{t_2}\psi_2 & A_{t_2}\psi_3 & A_{t_2}\psi_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
A_{t_i}\psi_1 & A_{t_i}\psi_2 & A_{t_i}\psi_3 & A_{t_i}\psi_4 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

to conclude that there is a strictly increasing sequence of natural numbers (n_k) such that

$$\lim_{k\to\infty} A_{t_i}\psi_{n_k} \quad \text{exists for each } i\in\mathbb{N}.$$

Thus there is pointwise convergence of the subsequence

$$\left(t\in[0,1]\mapsto \int_0^t G(u(s))\psi_{n_k}(s)\,ds \in H\right)_{k=1}^{\infty}$$

on $[0,1]\cap\mathbb{Q}$.

Proof of Theorem 3.1(3). Let $B\subset L^2([0,1];U)$ be weakly sequentially compact and let $K\subset H$ be compact. We want to show that

$$\mathcal{C} := \{u\in C([0,1];H) \;:\; u = z_x^{\phi} \;\text{ for some }\; \phi\in B \;\text{ and some }\; x\in K\}$$

is compact. Set

$$q := \sup\{\|\psi\|_{L^2([0,1];U)} \,:\, \psi \in B\},$$

which is finite because weak sequential compactness of B implies that B is bounded. Let (u_n) be a sequence of elements of $\mathcal{C}$. For each $n \in \mathbb{N}$ there is $\phi_n \in B$ and $x_n \in K$ such that $u_n = z_{x_n}^{\phi_n}$, that is

$$u_n(t) = x_n + \int_0^t G(u_n(s))\phi_n(s)\,ds \quad \forall t \in [0,1].$$

By compactness of K and weak sequential compactness of B, there is a strictly increasing sequence of natural numbers (n_k) and there are vectors $x \in K$ and $\phi \in B$ such that x_{n_k} converges to x in H and ϕ_{n_k} converges to ϕ in the weak topology of $L^2([0,1];U)$ as k goes to infinity. We claim that $u_{n_k} \to u := z_x^\phi$ as $k \to \infty$.
Given $k \in \mathbb{N}$ we have for each $t \in [0,1]$

$$\begin{aligned}
|u(t) - u_{n_k}(t)| &\leq |x - x_{n_k}| + \left|\int_0^t G(u(s))\phi(s)\,ds - \int_0^t G(u_{n_k}(s))\phi_{n_k}(s)\,ds\right| \\
&\leq |x - x_{n_k}| + \left|\int_0^t G(u(s))(\phi(s) - \phi_{n_k}(s))\,ds\right| + \\
&\qquad \left|\int_0^t [G(u(s)) - G(u_{n_k}(s))]\phi_{n_k}(s)\,ds\right| \\
&\leq |x - x_{n_k}| + \sup_{r\in[0,1]}\left|\int_0^r G(u(s))(\phi(s) - \phi_{n_k}(s))\,ds\right| + \\
&\qquad \Lambda\int_0^t |u(s) - u_{n_k}(s)||\phi_{n_k}(s)|_U\,ds.
\end{aligned}$$

Thus by Gronwall's Lemma we have

$$\sup_{t\in[0,1]} |u(t) - u_{n_k}(t)| \leq \left(|x - x_{n_k}| + \sup_{r\in[0,1]}\left|\int_0^r G(u(s))(\phi(s) - \phi_{n_k}(s))\,ds\right|\right)\exp(\Lambda q) \quad \forall k \in \mathbb{N}$$

and the right hand side of the above inequality goes to zero as $k \to \infty$ because the compact linear operator

$$\psi \in L^2([0,1];U) \mapsto \left(t \mapsto \int_0^t G(u(s))\psi(s)\,ds\right) \in C([0,1];H)$$

maps the weakly convergent sequence (ϕ_{n_k}) in $L^2([0,1];U)$ to a norm convergent sequence in $C([0,1];H)$. This completes the proof of Theorem 3.1.

3.3 Reducing the problem to the case of bounded G

In this section we show that if Theorems 3.2 and 3.3 hold under the additional assumption that the function $G : H \to L_2(U,H)$ is bounded then they hold also for G which is not bounded. This idea is copied from Cerrai and Röckner [5, Theorem 6.4].

For each $R \in (0,\infty)$ define $G_R : H \to L_2(U,H)$ by

$$G_R(x) := \begin{cases} G(x) & \text{if } |x| \le R \\ G(\frac{R}{|x|}x) & \text{if } |x| > R; \end{cases}$$

it is straightforward to show that $\sup_{x\in H} \|G_R(x)\|_{L_2(U,H)} < \infty$ and that inequalities (3.3) and (3.4) also hold with G_R in place of G. For each $x \in H$ and $R \in (0,\infty)$ define $\mathcal{I}_{R,x} : C([0,1];H) \to [0,\infty]$ by

$$\mathcal{I}_{R,x}(u) := \\ \tfrac{1}{2}\inf\left\{ \int_0^1 |\phi(s)|_U^2\, ds \;:\; \phi \in L^2([0,1];U) \text{ and } u(t) = x + \int_0^t G_R(u(s))\phi(s)\, ds \quad \forall t \in [0,1] \right\},$$

for all $u \in C([0,1];H)$.
For each $x \in H$, $R \in (0,\infty)$ and $\epsilon \in (0,1]$ define $(X^\epsilon_{R,x}(t) : (\Omega,\mathcal{F}_t) \to (H,\mathcal{B}_H))_{t\in[0,1]}$ to be the continuous $(\mathcal{F}_t)$-predictable process satisfying

$$X^\epsilon_{R,x}(t) = S(\epsilon t)x + \epsilon \int_0^t S(\epsilon(t-s))F(\epsilon s, X^\epsilon_{R,x}(s))\, ds + \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-s))G_R(X^\epsilon_{R,x}(s))\, dW(s) \tag{3.14}$$

P a.e. for each $t \in [0,1]$ and let $X^\epsilon_{R,x} : \Omega \to C([0,1];H)$ be the corresponding trajectory-valued random variable:

$$X^\epsilon_{R,x}(\omega) := (t \mapsto X^\epsilon_{R,x}(t)(\omega)) \quad \forall \omega \in \Omega.$$

Recall that we defined $(X^\epsilon_x(t))_{t\in[0,1]}$ and X^ϵ_x in equations (3.11) and (3.12).

Lemma 3.5 *Let $\rho \in (0,\infty)$. Given $r \in (0,\infty)$ and $\delta \in (0,\infty)$ there exists $R \in (0,\infty)$ such that*

1. *for each $x \in B_H(0,\rho)$*
$$\{\mathcal{I}_{R,x} \le r\} = \{\mathcal{I}_x \le r\}$$

   and

2. for each $x \in B_H(0,\rho)$ and for each $\epsilon \in (0,1]$

$$P\{X_x^\epsilon \in B_{C([0,1];H)}(\{\mathcal{I}_x \leq r\},\delta)\} = P\{X_{R,x}^\epsilon \in B_{C([0,1];H)}(\{\mathcal{I}_x \leq r\},\delta)\}.$$

Proof. Let $r > 0$ and $\delta > 0$. Set

$$R := (\rho + \Lambda\sqrt{2r})\exp(\Lambda\sqrt{2r}) + \delta. \tag{3.15}$$

Let $x \in B_H(0,\rho)$.

We firstly prove part (1).

Suppose $u \in C([0,1];H)$ and $\mathcal{I}_x(u) \leq r$. Then there exists $\phi \in L^2([0,1];U)$ such that $\|\phi\|_{L^2([0,1];U)} \leq \sqrt{2r}$ and

$$u(t) = x + \int_0^t G(u(s))\,\phi(s)\,ds \quad \forall t \in [0,1]. \tag{3.16}$$

Taking the norm of both sides of this equation gives

$$|u(t)| \leq |x| + \Lambda\int_0^t (1+|u(s)|)|\phi(s)|_U\,ds \quad \forall t \in [0,1]$$

and by Gronwall's Lemma

$$\begin{aligned} \sup_{t\in[0,1]} |u(t)| &< (\rho + \Lambda\sqrt{2r})\exp(\Lambda\sqrt{2r}) & (3.17)\\ &< R. & (3.18) \end{aligned}$$

By definition of G_R, $G(x) = G_R(x)$ for all $x \in B_H(0,R)$; thus, from (3.16) and(3.18), u satisfies

$$u(t) = x + \int_0^t G_R(u(s))\phi(s)\,ds \quad \forall t \in [0,1]$$

and $\mathcal{I}_{R,x}(u) \leq \frac{1}{2}\int_0^1 |\phi(s)|_U^2\,ds \leq r$.

Suppose now that $v \in C([0,1];H)$ and $\mathcal{I}_{R,x}(v) \leq r$. Then for some $\psi \in L^2([0,1];U)$ such that $\|\psi\|_{L^2([0,1];U)} \leq \sqrt{2r}$ we have

$$v(t) = x + \int_0^t G_R(v(s))\psi(s)\,ds \quad \forall t \in [0,1].$$

By taking norms of both sides of the equation and then applying Gronwall's Lemma we conclude that

$$\sup_{t\in[0,1]} |v(t)| < R.$$

Thus v satisfies the equation

$$v(t) = x + \int_0^t G(v(s))\psi(s)\, ds \quad \forall t \in [0,1]$$

and it follows that $\mathcal{I}_x(v) \leq \frac{1}{2}\int_0^1 |\psi(s)|_U^2\, ds \leq r$.
We have shown that $\{\mathcal{I}_x \leq r\} = \{\mathcal{I}_{R,x} \leq r\}$.

We now prove part (2). Let $\epsilon \in (0,1]$.
Define the $(\mathcal{F}_t)$-stopping time

$$\tau(\omega) := \inf\{t \in [0,1] : |X_x^\epsilon(t)(\omega)| \geq R\}\ , \quad \omega \in \Omega,$$

where we take $\tau(\omega) = 1$ if $|X_x^\epsilon(t)(\omega)| < R$ for all $t \in [0,1]$. By our choice of R and inequality (3.17) we have

$$B_{C([0,1];H)}(\{\mathcal{I}_x \leq r\},\, \delta) \subset B_{C([0,1];H)}(0,R).$$

Thus we can tell if the trajectory $X_x^\epsilon(\omega)$ lies in $B_{C([0,1];H)}(\{\mathcal{I}_x \leq r\},\, \delta)$ by observing the trajectory just up to time $\tau(\omega)$; also for P a.e. $\omega \in \Omega$ we have $\sup_{t\in[0,\tau(\omega)]} |X_x^\epsilon(t)(\omega)| \leq R$ and if $\tau(\omega) < 1$ then $|X_x^\epsilon(\tau(\omega))(\omega)| = R$.
Let $t \in (0,1]$. We have

$$X_x^\epsilon(t) = S(\epsilon t)x + \epsilon\int_0^t S(\epsilon(t-s))F(\epsilon s, X_x^\epsilon(s))\, ds + \epsilon^{\frac{1}{2}}\int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\, dW(s) \quad P \text{ a.e..}$$

Multiplying both sides of this equation by the indicator of the stochastic interval $[0,\tau]$ we have

$$\begin{aligned} 1_{[0,\tau]}(t)X_x^\epsilon(t) \quad &= \quad 1_{[0,\tau]}(t)S(\epsilon t)x + 1_{[0,\tau]}(t)\epsilon\int_0^t S(\epsilon(t-s))F(\epsilon s, 1_{[0,\tau]}(s)X_x^\epsilon(s))\, ds\, + \\ &\qquad 1_{[0,\tau]}(t)\epsilon^{\frac{1}{2}}\int_0^{t\wedge\tau} 1_{[0,t]}(s)S(\epsilon(t-s))G(X_x^\epsilon(s))\, dW(s) \quad P \text{ a.e.} \end{aligned}$$

and, by the localization lemma [10, Lemma 4.9],

$$
\begin{aligned}
1_{[0,\tau]}(t)X_x^\epsilon(t) &= 1_{[0,\tau]}(t)S(\epsilon t)x + 1_{[0,\tau]}(t)\epsilon\int_0^t S(\epsilon(t-s))F(\epsilon s, 1_{[0,\tau]}(s)X_x^\epsilon(s))\,ds + \\
&\quad 1_{[0,\tau]}(t)\epsilon^{\frac{1}{2}}\int_0^t 1_{[0,\tau]}(s)1_{[0,t]}(s)S(\epsilon(t-s))G_R(1_{[0,\tau]}(s)X_x^\epsilon(s))\,dW(s)
\end{aligned}
$$

P a.e..

We also have from equation (3.14)

$$
\begin{aligned}
1_{[0,\tau]}(t)X_{R,x}^\epsilon(t) &= 1_{[0,\tau]}(t)S(\epsilon t)x + 1_{[0,\tau]}(t)\epsilon\int_0^t S(\epsilon(t-s))F(\epsilon s, 1_{[0,\tau]}(s)X_{R,x}^\epsilon(s))\,ds + \\
&\quad 1_{[0,\tau]}(t)\epsilon^{\frac{1}{2}}\int_0^t 1_{[0,\tau]}(s)1_{[0,t]}(s)S(\epsilon(t-s))G_R(1_{[0,\tau]}(s)X_{R,x}^\epsilon(s))\,dW(s)
\end{aligned}
$$

P a.e..

Therefore

$$
\begin{aligned}
&1_{[0,\tau]}(t)(X_x^\epsilon(t) - X_{R,x}^\epsilon(t)) = \\
&\quad 1_{[0,\tau]}(t)\epsilon\int_0^t S(\epsilon(t-s))[F(\epsilon s, 1_{[0,\tau]}(s)X_x^\epsilon(s)) - F(\epsilon s, 1_{[0,\tau]}(s)X_{R,x}^\epsilon(s))]ds + \\
&\quad 1_{[0,\tau]}(t)\epsilon^{\frac{1}{2}}\int_0^t 1_{[0,\tau]}(s)S(\epsilon(t-s))[G_R(1_{[0,\tau]}(s)X_x^\epsilon(s)) - G_R(1_{[0,\tau]}(s)X_{R,x}^\epsilon(s))]\,dW(s)
\end{aligned} \tag{3.19}
$$

P a.e..

By Theorem 3.18 we have

$$
\sup_{u\in[0,1]} E\left[|X_x^\epsilon(u)|^2\right] < \infty \quad \text{and} \quad \sup_{u\in[0,1]} E\left[|X_{R,x}^\epsilon(u)|^2\right] < \infty.
$$

Thus, taking norms on both sides of equation (3.19), then squaring both sides and taking expectations, we obtain

$$
\begin{aligned}
&E\left[|1_{[0,\tau]}(t)(X_x^\epsilon(t) - X_{R,x}^\epsilon(t))|^2\right] \\
&\leq\ 2(\epsilon^2+\epsilon)M^2\Lambda^2\int_0^t E\left[|1_{[0,\tau]}(s)(X_x^\epsilon(s) - X_{R,x}^\epsilon(s))|^2\right]\,ds \quad \text{for each } t\in[0,1].
\end{aligned} \tag{3.20}
$$

As the function $t\in[0,1]\mapsto E[|1_{[0,\tau]}(t)(X_x^\epsilon(t) - X_{R,x}^\epsilon(t))|^2]\in\mathbb{R}$ is measurable and bounded, Gronwall's Lemma can be used with inequality (3.20) to show that for arbitrarily small positive α we have

$$\sup_{t\in[0,1]} E[|1_{[0,\tau]}(t)(X_x^\epsilon(t) - X_{R,x}^\epsilon(t))|^2] \leq \alpha \exp(2(\epsilon^2+\epsilon)M^2\Lambda^2). \qquad (3.21)$$

Inequality (3.21) implies that

$$1_{[0,\tau]}(t)X_x^\epsilon(t) = 1_{[0,\tau]}(t)X_{R,x}^\epsilon(t) \quad P \text{ a.e. for each } t\in[0,1] \qquad (3.22)$$

and, since processes $(1_{[0,\tau]}(t)X_x^\epsilon(t))_{t\in[0,1]}$ and $(1_{[0,\tau]}(t)X_{R,x}^\epsilon(t))_{t\in[0,1]}$ have left continuous trajectories, equation (3.22) implies equality of trajectories

$$1_{[0,\tau]}(t)X_x^\epsilon(t) = 1_{[0,\tau]}(t)X_{R,x}^\epsilon(t) \quad \forall t\in[0,1] \quad P \text{ a.e.}.$$

We conclude that for P a.e. $\omega\in\Omega$,

1. if $\tau(\omega)=1$ then $X_x^\epsilon(\omega) = X_{R,x}^\epsilon(\omega)$ and
2. if $\tau(\omega)<1$ then $|X_x^\epsilon(\tau(\omega))(\omega)| = |X_{R,x}^\epsilon(\tau(\omega))(\omega)| = R$ and trajectories $X_x^\epsilon(\omega)$ and $X_{R,x}^\epsilon(\omega)$ do not belong to $B_{C([0,1];H)}(\{\mathcal{I}_x\leq r\},\,\delta)$.

Thus

$$P\{X_x^\epsilon \in B_{C([0,1];H)}(\{\mathcal{I}_x\leq r\},\delta)\} = P\{X_{R,x}^\epsilon \in B_{C([0,1];H)}(\{\mathcal{I}_x\leq r\},\delta)\}.$$

This completes the proof of Lemma 3.5.

In the rest of this section, given $x\in H$ and $\phi\in L^2([0,1];U)$ and $R>0$ we denote by $z_{R,x}^\phi$ the function $u\in C([0,1];H)$ such that $u(t) = x+\int_0^t G_R(u(s))\phi(s)\,ds$ for all $t\in[0,1]$; recall that z_x^ϕ is the function $v\in C([0,1];H)$ such that $v(t) = x+\int_0^t G(v(s))\phi(s)\,ds$ for all $t\in[0,1]$.

Lemma 3.6 *Let $K\subset H$ be compact. Given $\phi\in L^2([0,1];U)$ and $\delta>0$ there exists $R>0$ such that:*

1. *for all $x\in K$ we have*
$$z_x^\phi = z_{R,x}^\phi$$
and
2. *for all $x\in K$ and all $\epsilon\in(0,1]$ we have*
$$P\{X_x^\epsilon\in B_{C([0,1];H)}(z_x^\phi,\,\delta)\} = P\{X_{R,x}^\epsilon\in B_{C([0,1];H)}(z_x^\phi,\,\delta)\}.$$

Proof. Let $\phi \in L^2([0,1];U)$ and let $\delta > 0$. We know from Theorem 3.1(3) that

$$\mathcal{C} := \left\{ u \in C([0,1];H) \,:\, u(t) = x + \int_0^t G(u(s))\phi(s)\,ds \quad \forall t \in [0,1] \;\; \text{for some } x \in K \right\}$$

is compact and thus a bounded subset of $C([0,1];H)$. Set

$$R := \sup_{u \in \mathcal{C}} \sup_{t \in [0,1]} |u(t)| + \delta. \tag{3.23}$$

Then for each $x \in K$, $u \in C([0,1];H)$ satisfies the equation

$$u(t) = x + \int_0^t G(u(s))\phi(s)\,ds \quad \forall t \in [0,1]$$

only if it belongs to $\mathcal{C}$, in which case $\sup_{t \in [0,1]} |u(t)| < R$ and

$$u(t) = x + \int_0^t G_R(u(s))\phi(s)\,ds \quad \forall t \in [0,1];$$

that is, $u = z_x^\phi$ implies $u = z_{R,x}^\phi$.

Now we prove part (2). Let $x \in K$ and $\epsilon \in (0,1]$. Define the $(\mathcal{F}_t)$-stopping time

$$\tau(\omega) := \inf\{t \in [0,1] : |X_x^\epsilon(t)(\omega)| \geq R\}, \quad \omega \in \Omega,$$

where we take $\tau(\omega) = 1$ if $|X_x^\epsilon(t)(\omega)| < R$ for all $t \in [0,1]$. By our choice of R in equation (3.23) we have

$$B_{C([0,1];H)}(z_x^\phi,\, \delta) \subset B_{C([0,1];H)}(0, R).$$

Thus to see whether the trajectory $t \in [0,1] \mapsto X_x^\epsilon(t)(\omega)$ lies in $B_{C([0,1];H)}(z_x^\phi,\, \delta)$ it suffices to observe it just at times $t \in [0, \tau(\omega)]$. We proceed as in the proof of Lemma 3.5(2). For P a.e. $\omega \in \Omega$ we have $\sup_{t \in [0,\tau(\omega)]} |X_x^\epsilon(t)(\omega)| \leq R$, thus for each $t \in (0,1]$ we have

$$\begin{aligned} 1_{[0,\tau]}(t) X_x^\epsilon(t) &= 1_{[0,\tau]}(t) S(\epsilon t)x + 1_{[0,\tau]}(t)\epsilon \int_0^t S(\epsilon(t-s))F(\epsilon s,\, 1_{[0,\tau]}(s)X_x^\epsilon(s))\,ds + \\ &\quad 1_{[0,\tau]}(t)\epsilon^{\frac{1}{2}} \int_0^t 1_{[0,\tau]}(s) S(\epsilon(t-s)) G_R(1_{[0,\tau]}(s)X_x^\epsilon(s))\,dW(s) \quad P \text{ a.e.} \end{aligned}$$

and also

$$1_{[0,\tau]}(t)X^\epsilon_{R,x}(t) = 1_{[0,\tau]}(t)S(\epsilon t)x + 1_{[0,\tau]}(t)\epsilon\int_0^t S(\epsilon(t-s))F(\epsilon s, 1_{[0,\tau]}(s)X^\epsilon_{R,x}(s))\,ds + 1_{[0,\tau]}(t)\epsilon^{\frac{1}{2}}\int_0^t 1_{[0,\tau]}(s)S(\epsilon(t-s))G_R(1_{[0,\tau]}(s)X^\epsilon_{R,x}(s))\,dW(s) \quad P \text{ a.e..}$$

Just as in the proof of Lemma 3.5(2) we obtain the equality of trajectories

$$1_{[0,\tau]}(t)X^\epsilon_x(t) = 1_{[0,\tau]}(t)X^\epsilon_{R,x}(t) \quad \forall t \in [0,1] \quad P \text{ a.e.}$$

and from this we have

$$P\{X^\epsilon_x \in B_{C([0,1];H)}(z^\phi_x,\, \delta)\} = P\{X^\epsilon_{R,x} \in B_{C([0,1];H)}(z^\phi_x,\, \delta)\}.$$

This completes the proof of Lemma 3.6.

Corollary 3.7 *Suppose that Theorem 3.2 holds under the additional assumption that the diffusion function $G : H \to L_2(U,H)$ is bounded. Then it also holds if the function G is not bounded.*

Proof. Let $K \subset H$ be compact and let $\phi \in L^2([0,1];U)$ and let $\delta > 0$. Take $R > 0$ as in Lemma 3.6. For any $x \in K$ we have $z^\phi_x = z^\phi_{R,x}$ by Lemma 3.6(1).
Let $\gamma > 0$. Since G_R is a bounded function, by Theorem 3.2 there exists $\epsilon_0 > 0$ such that for all $x \in K$ and for all $\epsilon \in (0, \epsilon_0]$

$$\exp\left(\frac{-\frac{1}{2}\int_0^1 |\phi(s)|_U^2\,ds - \gamma}{\epsilon}\right) \le P\{X^\epsilon_{R,x} \in B_{C([0,1];H)}(z^\phi_{R,x}, \delta)\} = P\{X^\epsilon_x \in B_{C([0,1];H)}(z^\phi_x, \delta)\},$$

where the equality on the right is from Lemma 3.6(2).

Corollary 3.8 *Suppose that Theorem 3.3 holds under the additional assumption that the diffusion function $G : H \to L_2(U,H)$ is bounded. Then it also holds if the function G is not bounded.*

Proof. Let $K \subset H$ be compact. Take $\rho \in (0,\infty)$ such that $K \subset B_H(0,\rho)$. Let $r \in (0,\infty)$ and let $\delta \in (0,\infty)$. Take $R > 0$ as in Lemma 3.5. Let $\gamma > 0$. Since the function G_R is bounded, by Theorem 3.3 there exists $\epsilon_0 > 0$ such that for all $x \in K$ and for all $\epsilon \in (0, \epsilon_0]$

$$\exp\left(\frac{-r+\gamma}{\epsilon}\right) \geq P\{X^\epsilon_{R,x} \notin B_{C([0,1];H)}(\{\mathcal{I}_{R,x} \leq r\}, \delta)\} = P\{X^\epsilon_x \notin B_{C([0,1];H)}(\{\mathcal{I}_x \leq r\}, \delta)\},$$

where the equality on the right is from Lemma 3.5.

Thanks to Corollaries 3.7 and 3.8 our task reduces to proving Theorems 3.2 and 3.3 under the additional assumption:

(**A3**) the function $G : H \to L_2(U, H)$ is bounded, that is:

$$\sup_{x\in H} \|G(x)\|_{L_2(U,H)} < \infty.$$

3.4 Exponential bounds

To prove Theorems 3.2 and 3.3 in the case of bounded G we shall need some exponential tail estimates for stochastic integrals and stochastic convolutions due to Chow and Menaldi [8] and Peszat [25]. The formulations we present without proof are Peszat's [25].

Let $\mathcal{P}_1$ denote the $(\mathcal{F}_t)$-predictable σ-algebra of $[0,1] \times \Omega$. Let $\xi : ([0,1] \times \Omega, \mathcal{P}_1) \to (L_2(U,H), \mathcal{B}_{L_2(U,H)})$ be a measurable function.

Theorem 3.9 (Chow's and Menaldi's bound for stochastic integrals) *If there exists a positive real number η_1 such that*

$$\int_0^1 \|\xi(s)\|^2_{L_2(U,H)}\, ds \leq \eta_1 \quad P \text{ a.e.}$$

then for any $\delta > 0$

$$P\left\{\sup_{t\in[0,1]} \left|\int_0^t \xi(s)\, dW(s)\right| \geq \delta\right\} \leq 3\exp\left(-\frac{\delta^2}{4\eta_1}\right).$$

Theorem 3.10 (Peszat's bound for stochastic convolutions) *Let $(T(t))$ be a strongly continuous semigroup of bounded linear operators on H. Suppose $\alpha_0 \in (0, \frac{1}{2})$ and $p_0 > 1$ are such that*

$$\kappa := \left(\int_0^1 t^{(\alpha_0-1)p_0} \|T(t)\|^{p_0}_{L(H,H)}\, dt\right)^{\frac{1}{p_0}} < \infty.$$

If there exists a positive real number η_2 *such that*

$$\sup_{t\in[0,1]}\int_0^t (t-s)^{-2\alpha_0}\|T(t-s)\xi(s)\|^2_{L_2(U,H)}\,ds \le \eta_2 \quad P\ a.e.$$

then the process $(\int_0^t T(t-s)\xi(s)\,dW(s))_{t\in[0,1]}$ *has a continuous version in* H *and for any* $\delta>0$

$$P\left\{\sup_{t\in[0,1]}\left|\int_0^t T(t-s)\xi(s)\,dW(s)\right|\ge\delta\right\}\le C\exp\left(-\frac{\delta^2}{\kappa^2\eta_2}\right)$$

where $C=4+\exp(4n_0!)^{\frac{1}{n_0}}$ *and* $n_0=\frac{p_0}{2p_0-2}+1$.

In the proof of Theorem 3.3 we also use a large deviation principle associated with the trajectory-valued random variable $W:(\Omega,\mathcal{F},P)\to(C([0,1];U_1),\mathcal{B}_{C([0,1];U_1)})$ defined by

$$W(\omega):=(t\in[0,1]\mapsto W(t)(\omega)\in U_1)\quad\forall\omega\in\Omega.$$

As shown in [32, Theorem 1 in Section 6.2], the distribution of W is symmetric Gaussian and its reproducing kernel Hilbert space is

$$H_W:=\left\{t\in[0,1]\mapsto J\int_0^t\psi(s)\,ds\ :\ \psi\in L^2([0,1];U)\right\},$$

with norm $\|\cdot\|_{H_W}$ defined by

$$\|y\|^2_{H_W}:=\int_0^1|\psi(s)|^2_U\,ds\ :\ \ \psi\in L^2([0,1];U)\text{ and }y(t)=J\int_0^t\psi(s)\,ds\quad\forall t\in[0,1].$$

Thus, by [10, Theorem 12.7], the family of Gaussian measures

$$\{\mathcal{L}(\epsilon^{\frac12}W\ :\ (\Omega,\mathcal{F},P)\to(C([0,1];U_1),\mathcal{B}_{C([0,1];U_1)}))\ \ :\ \ \epsilon\in(0,1]\}$$

satisfies a large deviation principle with rate function $\mathcal{I}_W:C([0,1];U_1)\to[0,\infty]$ defined by

$$\mathcal{I}_W(f):=\begin{cases}\frac12\|f\|^2_{H_W} & \text{if } f\in H_W,\\ \infty & \text{if } f\notin H_W.\end{cases}\tag{3.24}$$

3.5 The lower bound

In this section we prove Theorem 3.2 under the additional assumption (A3).

Proof of Theorem 3.2 assuming (A3). Let $K \subset H$ be compact and fix $\phi \in L^2([0,1];U)$. Recall that for each $x \in K$ $z_x^\phi \in C([0,1];H)$ satisfies

$$z_x^\phi(t) = x + \int_0^t G(z_x^\phi(s))\phi(s)\,ds \quad \forall t \in [0,1].$$

Fix $\delta > 0$ and $\gamma > 0$. For each $\epsilon \in (0,1]$ define the process $(W^\epsilon(t) : (\Omega, \mathcal{F}_t) \to (U_1, \mathcal{B}_{U_1}))_{t\in[0,1]}$ by

$$W^\epsilon(t) := W(t) - \epsilon^{-\frac{1}{2}} J \int_0^t \phi(s)\,ds \quad \forall t \in [0,1]. \tag{3.25}$$

By [10, Theorem 10.14] $(W^\epsilon(t))_{t\in[0,1]}$ is a Wiener process with respect to filtration $(\mathcal{F}_t)$ on probability space $(\Omega, \mathcal{F}, P^\epsilon)$ where

$$dP^\epsilon(\omega) = \exp\left(\epsilon^{-\frac{1}{2}} \int_0^1 \langle \phi(s), \cdot\rangle_U\, dW(s)(\omega) - \frac{1}{2\epsilon}\int_0^1 |\phi(s)|_U^2\,ds\right) dP(\omega) \tag{3.26}$$

and $P^\epsilon(W^\epsilon(1))^{-1} = P(W(1))^{-1}$.
Taking the reciprocal of the Radon-Nikodym derivative in equation (3.26) we have

$$dP(\omega) = \exp\left(-\epsilon^{-\frac{1}{2}} \int_0^1 \langle \phi(s), \cdot\rangle_U\, dW(s)(\omega) + \frac{1}{2\epsilon}\int_0^1 |\phi(s)|_U^2\,ds\right) dP^\epsilon(\omega)$$

and we use Lemma 3.20 to replace the Itô integral on the right hand side by one with respect to $(W^\epsilon(t))_{t\in[0,1]}$:

$$\int_0^1 \langle \phi(s), \cdot\rangle_U\, dW^\epsilon(s) = \int_0^1 \langle \phi(s), \cdot\rangle_U\, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^1 |\phi(s)|_U^2\,ds \quad P^\epsilon \text{ a.e..}$$

Thus we have

$$dP(\omega) = \exp\left(-\epsilon^{-\frac{1}{2}} \int_0^1 \langle \phi(s), \cdot\rangle_U\, dW^\epsilon(s)(\omega) - \frac{1}{2\epsilon}\int_0^1 |\phi(s)|_U^2\,ds\right) dP^\epsilon(\omega).$$

To shorten notation, for each $x \in K$ and each $\epsilon \in (0,1]$ set

$$\begin{aligned}
\mathcal{A}(\epsilon, x) &:= \left\{\omega \in \Omega : \sup_{t\in[0,1]} |X_x^\epsilon(t)(\omega) - z_x^\phi(t)| < \delta\right\} \quad \text{and} \\
\mathcal{D}(\epsilon) &:= \left\{\omega \in \Omega : \left|\epsilon^{\frac{1}{2}} \int_0^1 \langle \phi(t), \cdot\rangle_U\, dW^\epsilon(t)(\omega)\right| \le \frac{\gamma}{2}\right\}.
\end{aligned}$$

We have

$$\begin{aligned}
P(\mathcal{A}(\epsilon, x)) &= \int_\Omega 1_{\mathcal{A}(\epsilon,x)}\, dP \\
&= \int_\Omega 1_{\mathcal{A}(\epsilon,x)} \exp\left(-\epsilon^{-\frac{1}{2}} \int_0^1 \langle \phi(s), \cdot \rangle_U \, dW^\epsilon(s) - \frac{1}{2\epsilon} \int_0^1 |\phi(s)|_U^2 \, ds\right) dP^\epsilon \\
&\geq \int_\Omega 1_{\mathcal{A}(\epsilon,x)\cap\mathcal{D}(\epsilon)} \exp\left(-\epsilon^{-\frac{1}{2}} \int_0^1 \langle \phi(s), \cdot \rangle_U \, dW^\epsilon(s) - \frac{1}{2\epsilon} \int_0^1 |\phi(s)|_U^2 \, ds\right) dP^\epsilon \\
&\geq \exp\left(-\frac{\gamma}{2\epsilon} - \frac{1}{2\epsilon}\int_0^1 |\phi(s)|_U^2 \, ds\right) P^\epsilon(\mathcal{A}(\epsilon, x) \cap \mathcal{D}(\epsilon)).
\end{aligned}$$

It remains to show that there exists $\epsilon_0 > 0$ such that $P^\epsilon(\mathcal{A}(\epsilon, x) \cap \mathcal{D}(\epsilon)) \geq \exp(-\frac{\gamma}{2\epsilon})$ for all $x \in K$ and for all $\epsilon \in (0, \epsilon_0]$. We will actually show something more:

$$P^\epsilon(\mathcal{A}(\epsilon, x)^c \cup \mathcal{D}(\epsilon)^c) \to 0 \quad \text{as} \quad \epsilon \to 0 \quad \text{uniformly in } x \in K.$$

Let $\epsilon \in (0, 1]$ and let $x \in H$.

For each $t \in (0, 1]$ we have

$$
\begin{aligned}
&|X_x^\epsilon(t) - z_x^\phi(t)| \\
&= \Bigg| S(\epsilon t)x - x + \epsilon \int_0^t S(\epsilon(t-s))(F(\epsilon s, X_x^\epsilon(s)) - F(\epsilon s, z_x^\phi(s)))\, ds \\
&\quad + \ \epsilon \int_0^t S(\epsilon(t-s))F(\epsilon s, z_x^\phi(s))\, ds \\
&\quad + \int_0^t S(\epsilon(t-s))(G(X_x^\epsilon(s)) - G(z_x^\phi(s)))\phi(s)\, ds \\
&\quad + \int_0^t (S(\epsilon(t-s)) - I_H)G(z_x^\phi(s))\phi(s)\, ds \\
&\quad + \epsilon^{\frac{1}{2}} \left(\int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\phi(s)\, ds \right) \Bigg| \\
&\le \sup_{r\in[0,\epsilon]} |S(r)x - x| + \epsilon M\Lambda \int_0^t |X_x^\epsilon(s) - z_x^\phi(s)|\, ds \\
&\quad + \epsilon M\Lambda \int_0^1 (1 + |z_x^\phi(s)|)\, ds \\
&\quad + M\Lambda \int_0^t |X_x^\epsilon(s) - z_x^\phi(s)||\phi(s)|_U\, ds \\
&\quad + \sup\{\|(S(r) - I_H)G(z_x^\phi(s))\|_{L_2(U,H)} \ : \ r \in [0, \epsilon]\,, \ \ s \in [0, 1]\} \left(\int_0^1 |\phi(s)|_U^2\, ds \right)^{\frac{1}{2}} \\
&\quad + \sup_{r\in[0,1]} \epsilon^{\frac{1}{2}} \left| \int_0^r S(\epsilon(r-s))G(X_x^\epsilon(s))\, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^r S(\epsilon(r-s))G(X_x^\epsilon(s))\phi(s)\, ds \right| . \quad (3.27)
\end{aligned}
$$

The last term on the right of (3.27) can be written in terms of a stochastic convolution with respect to integrator $(W^\epsilon(t))_{t\in[0,1]}$. For each $t \in (0, 1]$ the function

$$(s, \omega) \in ([0, 1] \times \Omega, \mathcal{P}_1) \mapsto 1_{[0,t]}(s)S(\epsilon(t-s))G(X_x^\epsilon(s)(\omega)) \ \in (L_2(U, H), \mathcal{B}_{L_2(U,H)})$$

is measurable and bounded; thus Lemma 3.20 applies and we have for each $t \in [0, 1]$

$$\int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\, dW^\epsilon(s) = \int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\phi(s)\, ds$$

P a.e.. Considering continuous versions of the processes we have P a.e.

$$
\begin{aligned}
&\epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\, dW^\epsilon(s) \\
&= \epsilon^{\frac{1}{2}} \left(\int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\phi(s)\, ds \right) \quad \forall t \in [0, 1]. \quad (3.28)
\end{aligned}
$$

Squaring both sides of inequality (3.27) and using equation (3.28) we have

$$
\begin{aligned}
&|X_x^\epsilon(t) - z_x^\phi(t)|^2 \\
&\quad \le \; 6\Bigg[\sup_{r\in[0,\epsilon]}|S(r)x - x|^2 + M^2\Lambda^2\left(\epsilon^2 + \int_0^1 |\phi(s)|_U^2\,ds\right)\int_0^t |X_x^\epsilon(s) - z_x^\phi(s)|^2\,ds \\
&\qquad + \epsilon^2 M^2\Lambda^2\left(\int_0^1 (1 + |z_x^\phi(s)|)\,ds\right)^2 \\
&\qquad + \sup\{\|(S(r) - I_H)G(z_x^\phi(s))\|_{L_2(U,H)} \,:\, s\in[0,1] \text{ and } r\in[0,\epsilon]\}^2 \int_0^1 |\phi(s)|_U^2\,ds \\
&\qquad + \left(\sup_{r\in[0,1]}\left|\epsilon^{\frac{1}{2}}\int_0^r S(\epsilon(r-s))G(X_x^\epsilon(s))\,dW^\epsilon(s)\right|\right)^2\Bigg]
\end{aligned}
$$

for all $t\in[0,1]$ P^ϵ a.e.. Hence from Gronwall's Lemma,

$$
\begin{aligned}
&\sup_{t\in[0,1]}|X_x^\epsilon(t) - z_x^\phi(t)|^2 \\
&\le \; 6\Bigg[\sup_{r\in[0,\epsilon]}|S(r)x - x|^2 + \epsilon^2 M^2\Lambda^2\left(\int_0^1 (1 + |z_x^\phi(s)|)\,ds\right)^2 + \\
&\quad \sup\{\|(S(r) - I_H)G(z_x^\phi(s))\|_{L_2(U,H)} \,:\, r\in[0,\epsilon]\,,\;\; s\in[0,1]\}^2 \int_0^1 |\phi(s)|_U^2\,ds + \\
&\quad \sup_{r\in[0,1]}\left|\epsilon^{\frac{1}{2}}\int_0^r S(\epsilon(r-s))G(X_x^\epsilon(s))\,dW^\epsilon(s)\right|^2\Bigg]\exp\left(6M^2\Lambda^2\left(1 + \int_0^1 |\phi(s)|_U^2\,ds\right)\right) \quad P^\epsilon \text{ a.e..}
\end{aligned}
$$

Thus

$$
\begin{aligned}
&P^\epsilon\{\sup_{t\in[0,1]}|X_x^\epsilon(t) - z_x^\phi(t)| \ge \delta\} \\
&\le \; P^\epsilon\Bigg\{\sup_{r\in[0,\epsilon]}|S(r)x - x|^2 + \epsilon^2 M^2\Lambda^2\left(\int_0^1 (1 + |z_x^\phi(s)|)\,ds\right)^2 + \\
&\quad \sup\{\|(S(r) - I_H)G(z_x^\phi(s))\|_{L_2(U,H)} \,:\, r\in[0,\epsilon]\,,\;\; s\in[0,1]\}^2 \int_0^1 |\phi(s)|_U^2\,ds + \\
&\quad \sup_{r\in[0,1]}\left|\epsilon^{\frac{1}{2}}\int_0^r S(\epsilon(r-s))G(X_x^\epsilon(s))\,dW^\epsilon(s)\right|^2 \ge \frac{\delta^2}{6\exp(6M^2\Lambda^2(1 + \int_0^1 |\phi(s)|_U^2\,ds))}\Bigg\}
\end{aligned}
$$

and there exists $\epsilon_1 > 0$ such that for all $x \in K$ and for all $\epsilon \in (0, \epsilon_1]$ we have

$$
\begin{aligned}
&P^\epsilon\{\sup_{t\in[0,1]} |X_x^\epsilon(t) - z_x^\phi(t)| \geq \delta\} \\
\leq \;& P^\epsilon \left\{ \sup_{r\in[0,1]} \left| \int_0^r S(\epsilon(r-s))G(X_x^\epsilon(s))\, dW^\epsilon(s) \right| \geq \frac{\delta}{3\epsilon^{\frac{1}{2}} \exp(3M^2\Lambda^2(1 + \int_0^1 |\phi(s)|_U^2\, ds))} \right\}. \quad (3.29)
\end{aligned}
$$

Since we are assuming (A3), we can apply Peszat's tail estimate from Theorem 3.10 to the term on the right hand side of (3.29). Thus for all $x \in K$ and for all $\epsilon \in (0, \epsilon_1]$ we have

$$
\begin{aligned}
P^\epsilon(\mathcal{A}(\epsilon, x)^c) &\leq C_1 \exp\left(\frac{-\delta^2}{\epsilon K_1}\right) \qquad (3.30)\\
&\to 0 \quad \text{as } \epsilon \to 0,
\end{aligned}
$$

where the numbers C_1 and K_1 in Peszat's exponential estimate (3.30) are positive real constants that do not depend on ϵ or on x.
We also have from Theorem 3.9

$$
\begin{aligned}
P^\epsilon(\mathcal{D}(\epsilon)^c) &\leq P^\epsilon \left\{ \omega \in \Omega \,:\, \sup_{t\in[0,1]} \left| \int_0^t \langle \phi(s), \cdot \rangle_U \, dW^\epsilon(s)(\omega) \right| > \frac{\gamma}{2\epsilon^{\frac{1}{2}}} \right\} \\
&\leq 3 \exp\left(- \frac{\gamma^2}{16\epsilon \int_0^1 |\phi(s)|_U^2\, ds} \right) \\
&\to 0 \quad \text{as } \epsilon \to 0.
\end{aligned}
$$

This completes the proof of the theorem.

3.6 The upper bound

In this section we assume that (A3) holds and we prove Theorem 3.3 using the following proposition.

Proposition 3.11 *Let $K \subset H$ be compact. Given $a > 0$ and $\delta > 0$ and $\phi \in L^2([0,1]; U)$ there exists $\epsilon_0 > 0$ and $b > 0$ such that for all $\epsilon \in (0, \epsilon_0]$ and for all $x \in K$ we have*

$$
P\left\{ \sup_{t\in[0,1]} |X_x^\epsilon(t) - z_x^\phi(t)| \geq \delta, \; \sup_{t\in[0,1]} \left| \epsilon^{\frac{1}{2}} W(t) - J \int_0^t \phi(s)\, ds \right|_{U_1} \leq b \right\} \leq \exp\left(-\frac{a}{\epsilon}\right).
$$

The virtue of this proposition is that given positive δ the exponential bound on the right hand side has a, which we can choose to be as large as we please, in the numerator; the cost is the restriction on $\epsilon^{\frac{1}{2}}W$, but we have the large deviation principle of $\{\mathcal{L}(\epsilon^{\frac{1}{2}}W) : \epsilon \in (0,1]\}$ to describe how these distributions behave. There is some work involved in arriving at the proof of Proposition 3.11 and this is left till the end. We only remark that we need several lemmas which use assumptions (A1), (A2) and (A3).

Proof of Theorem 3.3 assuming (A3). Let K be a compact subset of H. Fix $r > 0$ and $\delta > 0$ and $\gamma > 0$. Let a be a positive real number, to be specified later. By Proposition 3.11, for each $\phi \in L^2([0,1];U)$ there exists $b_\phi > 0$ and $\epsilon_\phi > 0$ such that for all $\epsilon \in (0, \epsilon_\phi]$ and for all $x \in K$ we have

$$P\left\{\sup_{t\in[0,1]} |X_x^\epsilon(t) - z_x^\phi(t)| \geq \delta,\ \sup_{t\in[0,1]} \left|\epsilon^{\frac{1}{2}}W(t) - J\int_0^t \phi(s)\,ds\right|_{U_1} \leq b_\phi\right\} \leq \exp\left(-\frac{a}{\epsilon}\right). \tag{3.31}$$

Recall from equation (3.24) that $\mathcal{I}_W$ is the rate function of the large deviation principle satisfied by $\{\mathcal{L}(\epsilon^{\frac{1}{2}}W) : \epsilon \in (0,1]\}$. We have

$$\begin{aligned}
\{\mathcal{I}_W \leq r\} &= \Big\{u \in C([0,1];U_1) : u(t) = J\int_0^t \psi(s)\,ds \quad \forall t \in [0,1], \\
&\qquad\qquad \text{where } \psi \in L^2([0,1];U) \text{ and } \int_0^1 |\psi(s)|_U^2\,ds \leq 2r\Big\} \\
&\subset \bigcup_{\substack{\psi\in L^2([0,1];U):\\ \int_0^1 |\psi(s)|_U^2\,ds\leq 2r}} \left\{v \in C([0,1];U_1) : \sup_{t\in[0,1]} \left|v(t) - J\int_0^t \psi(s)\,ds\right|_{U_1} < b_\psi\right\}.
\end{aligned}$$

Since $\{\mathcal{I}_W \leq r\}$ is a compact subset of $C([0,1];U_1)$, there exists a natural number l and $\phi_1, \ldots, \phi_l \in L^2([0,1];U)$ such that $\int_0^1 |\phi_j(s)|_U^2\,ds \leq 2r$ for each $j \in \{1, \ldots, l\}$ and

$$\{\mathcal{I}_W \leq r\} \subset \bigcup_{j=1}^{l} \left\{v \in C([0,1];U_1) : \sup_{t\in[0,1]} \left|v(t) - J\int_0^t \phi_j(s)\,ds\right|_{U_1} < b_{\phi_j}\right\} =: \mathcal{C}. \tag{3.32}$$

For each $x \in H$ we may appeal to the definition of $\mathcal{C}$ in (3.32) and write

$$\begin{aligned}
&P\{X_x^\epsilon \notin B_{C([0,1];H)}(\{\mathcal{I}_x \leq r\}, \delta)\} \\
&\leq\ P\{X_x^\epsilon \notin B_{C([0,1];H)}(\{\mathcal{I}_x \leq r\}, \delta),\ \epsilon^{\frac{1}{2}}W \in \mathcal{C}\} + P\{\epsilon^{\frac{1}{2}}W \notin \mathcal{C}\} \\
&\leq\ \sum_{j=1}^{l} P\left\{X_x^\epsilon \notin B_{C([0,1];H)}(\{\mathcal{I}_x \leq r\}, \delta),\ \sup_{t\in[0,1]} \left|\epsilon^{\frac{1}{2}}W(t) - J\int_0^t \phi_j(s)\,ds\right|_{U_1} < b_{\phi_j}\right\} \\
&\qquad + P\{\epsilon^{\frac{1}{2}}W \notin \mathcal{C}\}.
\end{aligned} \tag{3.33}$$

Set $\epsilon_1 := \min\{\epsilon_{\phi_1}, \dots, \epsilon_{\phi_l}\}$. For each $j \in \{1, \dots, l\}$ we have from inequality (3.31) that for all $\epsilon \in (0, \epsilon_1]$ and for all $x \in K$

$$\begin{aligned} & P\left\{X_x^\epsilon \notin B_{C([0,1];H)}(\{\mathcal{I}_x \le r\}, \delta), \sup_{t\in[0,1]} \left|\epsilon^{\frac{1}{2}} W(t) - J\int_0^t \phi_j(s)\,ds\right|_{U_1} < b_{\phi_j}\right\} \\ \le\ & P\left\{X_x^\epsilon \notin B_{C([0,1];H)}(z_x^{\phi_j}, \delta), \sup_{t\in[0,1]} \left|\epsilon^{\frac{1}{2}} W(t) - J\int_0^t \phi_j(s)\,ds\right|_{U_1} < b_{\phi_j}\right\} \\ \le\ & \exp\left(-\frac{a}{\epsilon}\right). \end{aligned} \tag{3.34}$$

Since the open set $\mathcal{C}$ contains $\{\mathcal{I}_W \le r\}$, by the upper bound of the large deviation principle of the family $\{\mathcal{L}(\epsilon^{\frac{1}{2}} W) : \epsilon \in (0,1]\}$ there exists $\epsilon_2 > 0$ such that for all $\epsilon \in (0, \epsilon_2]$

$$P\{\epsilon^{\frac{1}{2}} W \notin \mathcal{C}\} \le \exp\left(\frac{-r + \frac{\gamma}{2}}{\epsilon}\right). \tag{3.35}$$

Set $\epsilon_3 := \epsilon_1 \wedge \epsilon_2$. Returning to inequality (3.33), we have for all $x \in K$ and for all $\epsilon \in (0, \epsilon_3]$

$$\begin{aligned} P\{X_x^\epsilon \notin B_{C([0,1];H)}(\{\mathcal{I}_x \le r\}, \delta)\} &\le l\exp\left(-\frac{a}{\epsilon}\right) + \exp\left(\frac{-r + \frac{\gamma}{2}}{\epsilon}\right) \\ &\le (l+1)\exp\left(\frac{-r + \frac{\gamma}{2}}{\epsilon}\right) \end{aligned}$$

when a is taken as $r - \frac{\gamma}{2}$.
Finally set $\epsilon_4 := \epsilon_3 \wedge \frac{\gamma}{2\ln(l+1)}$. Then for all $x \in K$ and for all $\epsilon \in (0, \epsilon_4]$

$$P\{X_x^\epsilon \notin B_{C([0,1];H)}(\{\mathcal{I}_x \le r\}, \delta)\} \le \exp\left(\frac{-r + \gamma}{\epsilon}\right).$$

This completes the proof of the theorem.

Now we work towards proving Proposition 3.11. In the following we make use of (A3):

$$\Gamma := \sup_{x\in H} \|G(x)\|_{L_2(U,H)} < \infty,$$

as well as (A1) and (A2).
Fix $\phi \in L^2([0,1]; U)$. For each $\epsilon \in (0,1]$ define $\tilde{F}_\epsilon : ([0,1] \times H, \mathcal{B}_{[0,1]} \otimes \mathcal{B}_H) \to (H, \mathcal{B}_H)$ by

$$\tilde{F}_\epsilon(s,x) := \epsilon F(\epsilon s, x) + G(x)\phi(s) \quad \forall s \in [0,1] \text{ and } \forall x \in H. \tag{3.36}$$

It is not difficult to show that for each $\epsilon \in (0,1]$ $\tilde{F}_\epsilon$ is measurable and

$$|\tilde{F}_\epsilon(s,x) - \tilde{F}_\epsilon(s,y)| \leq \theta(s)|x-y| \quad \forall x,y \in H \text{ and } \forall s \in [0,1] \tag{3.37}$$

and

$$|\tilde{F}_\epsilon(s,x)| \leq \theta(s)(1+|x|) \quad \forall s \in [0,1] \text{ and } \forall x \in H, \tag{3.38}$$

where $\theta(s) := \Lambda(1 + |\phi(s)|_U)$, $s \in [0,1]$, is a function in $L^2([0,1];\mathbb{R})$.

By Theorem 3.18, for each $\epsilon \in (0,1]$ and each $x \in H$ we may define $(Z_x^\epsilon(t))_{t\in[0,1]}$ as the continuous $(\mathcal{F}_t)$-predictable process such that

$$Z_x^\epsilon(t) = S(\epsilon t)x + \int_0^t S(\epsilon(t-s))\tilde{F}_\epsilon(s, Z_x^\epsilon(s))\,ds + \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s) \tag{3.39}$$

for all $t \in [0,1]$ P a.e.. To prove Proposition 3.11 we will need some lemmas concerning the processes $(Z_x^\epsilon(t))_{t\in[0,1]}$. In the proofs of these lemmas the only properties of $\tilde{F}_\epsilon$ we use are those in (3.37) and (3.38).

Lemma 3.12 *Given $a \in (0,\infty)$ and $R \in (0,\infty)$ there exists $D \in (0,\infty)$ such that for all $\epsilon \in (0,1]$ and for all $x \in B_H(0,R)$ we have*

$$P\{\sup_{t\in[0,1]} |Z_x^\epsilon(t)| \geq D\} \leq \exp\left(-\frac{a}{\epsilon}\right).$$

Proof. Let $x \in B_H(0,R)$ and let $\epsilon \in (0,1]$. For each ω in the set of P measure 1 where the trajectory $t \mapsto Z_x^\epsilon(t)(\omega)$ satisfies equation (3.39) we have for all $t \in [0,1]$:

$$\begin{aligned}
|Z_x^\epsilon(t)| &\leq |S(\epsilon t)x| + |\int_0^t S(\epsilon(t-s))\tilde{F}_\epsilon(s, Z_x^\epsilon(s))\,ds| + \epsilon^{\frac{1}{2}}|\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)| \\
&\leq M|x| + M\int_0^t \theta(s)(1+|Z_x^\epsilon(s)|)\,ds + \epsilon^{\frac{1}{2}}|\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)| \\
&\leq M|x| + M\left(\int_0^1 \theta(s)^2\,ds\right)^{\frac{1}{2}} + M\left(\int_0^1 \theta(s)^2\,ds\right)^{\frac{1}{2}}\left(\int_0^t |Z_x^\epsilon(s)|^2\,ds\right)^{\frac{1}{2}} \\
&\quad + \epsilon^{\frac{1}{2}} \sup_{r\in[0,1]} |\int_0^r S(\epsilon(r-s))G(Z_x^\epsilon(s))\,dW(s)|.
\end{aligned}$$

Squaring both sides of the last inequality and then applying Gronwall's Lemma yields

$$\begin{aligned}\sup_{t\in[0,1]}|Z_x^\epsilon(t)|^2 &\leq 4\left[M^2|x|^2+M^2\int_0^1\theta^2(s)\,ds+\epsilon\sup_{r\in[0,1]}|\int_0^r S(\epsilon(r-s))G(Z_x^\epsilon(s))\,dW(s)|^2\right]\\ &\qquad\times\exp\left(4M^2\int_0^1\theta(s)^2\,ds\right)\quad P\text{ a.e..}\end{aligned}$$

Set

$$\begin{aligned}D_1 &:= 4\exp\left(4M^2\int_0^1\theta(s)^2\,ds\right)\left(M^2R^2+M^2\int_0^1\theta(s)^2\,ds\right)\text{ and}\\ D_2 &:= 4\exp\left(4M^2\int_0^1\theta(s)^2\,ds\right).\end{aligned}$$

Then we have for each $x\in B_H(0,R)$ and for each $\epsilon\in(0,1]$:

$$\sup_{t\in[0,1]}|Z_x^\epsilon(t)|^2\leq D_1+D_2\epsilon\sup_{t\in[0,1]}|\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)|^2\quad P\text{ a.e.}$$

and for any $D\in(0,\infty)$ such that $D^2>D_1$ we have

$$P\{\sup_{t\in[0,1]}|Z_x^\epsilon(t)|^2\geq D^2\}\leq P\left\{\epsilon\sup_{t\in[0,1]}\left|\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right|^2\geq\frac{D^2-D_1}{D_2}\right\}. \tag{3.40}$$

We can apply Theorem 3.10 to the right hand side of inequality (3.40). Take $\alpha_0\in(0,\frac{1}{2})$ and $p_0>1$ such that $(\alpha_0-1)p_0>-1$; then for all $\epsilon\in(0,1]$

$$\left(\int_0^1 t^{(\alpha_0-1)p_0}\|S(\epsilon t)\|_{L(H)}^{p_0}\,dt\right)^{\frac{1}{p_0}}\leq M\left(\frac{1}{(\alpha_0-1)p_0+1}\right)^{\frac{1}{p_0}}=:\kappa;$$

also for all $t\in(0,1]$ and for all $\epsilon\in(0,1]$ and for all $x\in H$

$$\int_0^t(t-s)^{-2\alpha_0}\|S(\epsilon(t-s))G(Z_x^\epsilon(s))\|_{L_2(U,H)}^2\,ds\leq\frac{M^2\Gamma^2}{1-2\alpha_0}=:\eta.$$

Thus by Theorem 3.10, for any $\delta\in(0,\infty)$, for all $\epsilon\in(0,1]$ and for all $x\in H$ we have

$$P\left\{\sup_{t\in[0,1]}\left|\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right|\geq\delta\right\}\leq C\exp\left(-\frac{\delta^2}{\kappa^2\eta}\right), \tag{3.41}$$

where $C = 4 + \exp(4n_0!)^{\frac{1}{n_0}}$ and $n_0 = \frac{p_0}{2p_0-2} + 1$. From inequalities (3.40) and (3.41), for all $D \in (0, \infty)$ such that $D^2 > D_1$ and for all $x \in B_H(0, R)$ and for all $\epsilon \in (0, 1]$

$$\begin{aligned} P\{\sup_{t\in[0,1]} |Z_x^\epsilon(t)| \geq D\} &\leq P\left\{\sup_{t\in[0,1]} \left|\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \geq \left(\frac{D^2 - D_1}{\epsilon D_2}\right)^{\frac{1}{2}}\right\} \\ &\leq \exp\left(-\frac{D^2 - D_1}{\epsilon D_2 \kappa^2 \eta} + \ln C\right). \end{aligned}$$

Thus taking $D^2 = (a + \ln C)D_2\kappa^2\eta + D_1$ gives the desired result.

We introduce some notation to be used in the following lemmas. Set

$$t_{n,k} := \frac{k}{2^n} \quad \text{for } n \in \mathbb{N} \text{ and } k = 0, 1, \ldots, 2^n.$$

Lemma 3.13 *Given $a > 0$ and $\delta > 0$ there is a natural number N such that for each $n \geq N$ there exists $\epsilon_n > 0$ such that for all $\epsilon \in (0, \epsilon_n]$ and for all $x \in H$ we have*

$$P\left\{\sup_{k\in\{0,1,\ldots,2^n-1\}} \sup_{t\in[t_{n,k},t_{n,k+1}]} |\epsilon^{\frac{1}{2}} \int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)| \geq \delta\right\} \leq \exp\left(-\frac{a}{\epsilon}\right).$$

Proof. For the purpose of applying Theorem 3.10, fix $\alpha_0 \in (0, \frac{1}{2})$ and $p_0 > 1$ such that $(\alpha_0 - 1)p_0 > -1$.
Let $x \in H$, let $\epsilon \in (0, 1]$, let $n \in \mathbb{N}$ and let $k \in \{0, 1, \ldots, 2^n - 1\}$. For each $t \in [t_{n,k}, t_{n,k+1}]$ we have

$$\int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s) = \int_0^t S(\epsilon(t-s))1_{[t_{n,k},t_{n,k+1}]}(s)G(Z_x^\epsilon(s))\,dW(s) \quad P \text{ a.e.}$$

and, considering continuous versions,

$$\begin{aligned} &\sup_{t\in[t_{n,k},t_{n,k+1}]} |\int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)| \\ &\leq \sup_{t\in[0,1]} |\int_0^t S(\epsilon(t-s))1_{[t_{n,k},t_{n,k+1}]}(s)G(Z_x^\epsilon(s))\,dW(s)| \quad P \text{ a.e..} \end{aligned}$$

Thus

$$\begin{aligned}
&P\left\{\sup_{t\in[t_{n,k},t_{n,k+1}]}|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^{t}S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)|\geq\delta\right\}\\
&\leq\; P\left\{\sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}\int_{0}^{t}S(\epsilon(t-s))1_{[t_{n,k},t_{n,k+1}]}(s)G(Z_x^\epsilon(s))\,dW(s)|\geq\delta\right\}. \qquad (3.42)
\end{aligned}$$

The function $\xi : ([0,1]\times\Omega,\mathcal{P}_1)\to(L_2(U,H),\mathcal{B}_{L_2(U,H)})$ defined by

$$\xi(s,\omega):=1_{[t_{n,k},t_{n,k+1}]}(s)G(Z_x^\epsilon(s)(\omega))\quad\forall(s,\omega)\in[0,1]\times\Omega$$

is measurable and for each $t\in(0,1]$

$$\begin{aligned}
\int_0^t(t-s)^{-2\alpha_0}\|S(\epsilon(t-s))\xi(s)\|^2_{L_2(U,H)}\,ds &\leq M^2\Gamma^2\int_0^t(t-s)^{-2\alpha_0}1_{[t_{n,k},t_{n,k+1}]}(s)\,ds\\
&\leq \frac{M^2\Gamma^2}{1-2\alpha_0}2^{-(1-2\alpha_0)n}=:\eta_n.
\end{aligned}$$

Set $\kappa:=M(\frac{1}{(\alpha_0-1)p_0+1})^{\frac{1}{p_0}}$. By Theorem 3.10 we have

$$P\left\{\sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}\int_{0}^{t}S(\epsilon(t-s))1_{[t_{n,k},t_{n,k+1}]}(s)G(Z_x^\epsilon(s))\,dW(s)|\geq\delta\right\}\leq C\exp\left(-\frac{\delta^2}{\epsilon\kappa^2\eta_n}\right), \qquad (3.43)$$

where $C=4+\exp(4n_0!)^{\frac{1}{n_0}}$ and $n_0=\frac{p_0}{2p_0-2}+1$. From inequalities (3.42) and (3.43) we have

$$\begin{aligned}
&P\left\{\sup_{k\in\{0,1,\dots,2^n-1\}}\sup_{t\in[t_{n,k},t_{n,k+1}]}|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^{t}S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)|\geq\delta\right\}\\
&\leq\;\sum_{k=0}^{2^n-1}P\left\{\sup_{t\in[t_{n,k},t_{n,k+1}]}|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^{t}S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)|\geq\delta\right\}\\
&\leq\;\exp\left(-\frac{\delta^2}{\epsilon\kappa^2\eta_n}+\ln(2^nC)\right).
\end{aligned}$$

Now observe that there exists $N\in\mathbb{N}$ such that for each $n\geq N$ we have

$$-\frac{\delta^2}{\kappa^2\eta_n}<-a$$

and there exists $\epsilon_n > 0$ such that

$$-\frac{\delta^2}{\kappa^2 \eta_n} + \epsilon \ln(2^n C) \leq -a \quad \forall \epsilon \in (0, \epsilon_n].$$

This completes the proof of the lemma.

Lemma 3.14 *Let $R > 0$. Given $a > 0$ and $\delta > 0$ there is a natural number n_0 such that for each $n \geq n_0$ there exists $\epsilon_n > 0$ such that for all $\epsilon \in (0, \epsilon_n]$ and for all $x \in B_H(0, R)$*

$$P\{\sup_{k\in\{0,1,\ldots,2^n-1\}} \sup_{t\in[t_{n,k},t_{n,k+1}]} |Z_x^\epsilon(t) - S(\epsilon(t - t_{n,k}))Z_x^\epsilon(t_{n,k})| \geq \delta\} \leq \exp\left(-\frac{a}{\epsilon}\right).$$

Proof. Let $x \in B_H(0, R)$ and $\epsilon \in (0, 1]$ and $n \in \mathbb{N}$ and $k \in \{0, 1, \ldots, 2^n - 1\}$. For $t \in [t_{n,k}, t_{n,k+1}]$ we have

$$\begin{aligned}
&|Z_x^\epsilon(t) - S(\epsilon(t - t_{n,k}))Z_x^\epsilon(t_{n,k})| \\
\leq\;& \left|\int_0^t S(\epsilon(t-s))\tilde{F}_\epsilon(s, Z_x^\epsilon(s))\,ds - S(\epsilon(t-t_{n,k}))\int_0^{t_{n,k}} S(\epsilon(t_{n,k}-s))\tilde{F}_\epsilon(s, Z_x^\epsilon(s))\,ds\right| \\
&+ \left|\epsilon^{\frac{1}{2}}\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s) - \epsilon^{\frac{1}{2}}\int_0^{t_{n,k}} S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \quad P \text{ a.e.} \\
=\;& \left|\int_{t_{n,k}}^t S(\epsilon(t-s))\tilde{F}_\epsilon(s, Z_x^\epsilon(s))\,ds\right| + \left|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \quad P \text{ a.e.} \\
\leq\;& M\int_{t_{n,k}}^t \theta(s)(1 + |Z_x^\epsilon(s)|)\,ds + \left|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \quad P \text{ a.e.} \\
\leq\;& M(1 + \sup_{r\in[0,1]}|Z_x^\epsilon(r)|)2^{-\frac{n}{2}}\left(\int_0^1 \theta(s)^2\,ds\right)^{\frac{1}{2}} \\
&+ \sup_{r\in[t_{n,k},t_{n,k+1}]}\left|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^r S(\epsilon(r-s))G(Z_x^\epsilon(s))\,dW(s)\right|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\sup_{k\in\{0,1,\ldots,2^n-1\}} \sup_{t\in[t_{n,k},t_{n,k+1}]} |Z_x^\epsilon(t) - S(\epsilon(t - t_{n,k}))Z_x^\epsilon(t_{n,k})| \\
\leq\;& 2^{-\frac{n}{2}}M\left(\int_0^1 \theta(s)^2\,ds\right)^{\frac{1}{2}} + 2^{-\frac{n}{2}}M\left(\int_0^1 \theta(s)^2\,ds\right)^{\frac{1}{2}}\sup_{r\in[0,1]}|Z_x^\epsilon(r)| \\
&+ \sup_{k\in\{0,1,\ldots,2^n-1\}} \sup_{r\in[t_{n,k},t_{n,k+1}]}\left|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^r S(\epsilon(r-s))G(Z_x^\epsilon(s))\,dW(s)\right| \quad P \text{ a.e..}
\end{aligned}$$

From this we have

$$
\begin{aligned}
&P\{\sup_{k\in\{0,1,\ldots,2^n-1\}}\sup_{t\in[t_{n,k},t_{n,k+1}]}|Z_x^\epsilon(t)-S(\epsilon(t-t_{n,k}))Z_x^\epsilon(t_{n,k})|\geq\delta\}\\
&\leq\ P\left\{2^{-\frac{n}{2}}M\left(\int_0^1\theta(s)^2\,ds\right)^{\frac{1}{2}}\geq\frac{\delta}{3}\right\}+P\left\{\sup_{r\in[0,1]}|Z_x^\epsilon(r)|\geq\frac{\delta 2^{\frac{n}{2}}}{3M(\int_0^1\theta(s)^2\,ds)^{\frac{1}{2}}}\right\}\\
&\quad+P\left\{\sup_{k\in\{0,1,\ldots,2^n-1\}}\sup_{r\in[t_{n,k},t_{n,k+1}]}\left|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^r S(\epsilon(r-s))G(Z_x^\epsilon(s))\,dW(s)\right|\geq\frac{\delta}{3}\right\}.
\end{aligned}
$$

There exists a natural number N_1 such that for each $n\geq N_1$ the first probability on the right hand side vanishes. Set $\tilde{a}:=a+\ln 2$. By Lemma 3.12 there is a natural number N_2 such that for all $n\geq N_2$ and for all $\epsilon\in(0,1]$ and for all $x\in B_H(0,R)$

$$
P\left\{\sup_{t\in[0,1]}|Z_x^\epsilon(t)|\geq\frac{\delta 2^{\frac{n}{2}}}{3M(\int_0^1\theta(s)^2\,ds)^{\frac{1}{2}}}\right\}\leq\exp\left(-\frac{\tilde{a}}{\epsilon}\right).
$$

By Lemma 3.13 there is a natural number N_3 such that for each $n\geq N_3$ there exists $\epsilon_n>0$ such that for all $\epsilon\in(0,\epsilon_n]$ and for all $x\in H$ we have

$$
P\left\{\sup_{k\in\{0,1,\ldots,2^n-1\}}\sup_{t\in[t_{n,k},t_{n,k+1}]}\left|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right|\geq\frac{\delta}{3}\right\}\leq\exp\left(-\frac{\tilde{a}}{\epsilon}\right).
$$

Thus for each $n\geq\max\{N_1,N_2,N_3\}$, for all $\epsilon\in(0,\epsilon_n]$ and for all $x\in B_H(0,R)$ we have

$$
\begin{aligned}
P\{\sup_{k\in\{0,1,\ldots,2^n-1\}}\sup_{t\in[t_{n,k},t_{n,k+1}]}|Z_x^\epsilon(t)-S(\epsilon(t-t_{n,k}))Z_x^\epsilon(t_{n,k})|\geq\delta\}\ &\leq\ 2\exp\left(-\frac{\tilde{a}}{\epsilon}\right)\\
&\leq\ \exp\left(-\frac{a}{\epsilon}\right).
\end{aligned}
$$

This completes the proof of the lemma.

To simplify notation, for each natural number n define the function

$$
\pi_n(t):=\begin{cases}\frac{k}{2^n} & \text{if } t\in\left(\frac{k}{2^n},\,\frac{k+1}{2^n}\right]\ ,\quad k=0,1,\ldots,2^n-1\\ 0 & \text{if } t=0.\end{cases}
$$

Lemma 3.15 *Let $R>0$. Given $a>0$ and $\delta>0$ there is a natural number n_0 such that for each $n\geq n_0$ there exists $\epsilon_n>0$ such that for all $\epsilon\in(0,\epsilon_n]$ and for all $x\in B_H(0,R)$*

and for all $T \in [0,1]$ we have

$$
\begin{aligned}
&P\left\{\epsilon^{\frac{1}{2}}\left|\int_0^T S(\epsilon(T-s))G(Z_x^\epsilon(s))\,dW(s)\right.\right.\\
&\qquad\qquad \left.\left. - \int_0^T S(\epsilon(T-s))G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))\,dW(s)\right| \geq \delta\right\}\\
&\leq \ \exp\left(-\frac{a}{\epsilon}\right).
\end{aligned}
$$

Proof. Let $n \in \mathbb{N}$ and let $\epsilon \in (0,1]$ and let $x \in B_H(0,R)$. It is straightforward to check that the function

$$
\begin{aligned}
(s,\omega) \in ([0,1]\times\Omega, \mathcal{P}_1) \ \ \mapsto \ \ & S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s))(\omega) \ \in (H, \mathcal{B}_H)\\
&= 1_{\{0\}}(s)x + \sum_{k=0}^{2^n-1} 1_{(t_{n,k},t_{n,k+1}]}(s)S(\epsilon(s-t_{n,k}))Z_x^\epsilon(t_{n,k})(\omega)
\end{aligned}
$$

is measurable. Let $\rho > 0$ and define

$$\tau_\rho(\omega) := \inf\{t \in [0,1] : |Z_x^\epsilon(t)(\omega) - S(\epsilon(t-\pi_n(t)))Z_x^\epsilon(\pi_n(t))(\omega)| \geq \rho\},$$

where we set $\inf\emptyset = 1$. Since $(\mathcal{F}_t)$ is a right continuous filtration, τ_ρ is a $(\mathcal{F}_t)$-stopping time.

Let $T \in (0,1]$. We have

$$\begin{aligned}
&P\left\{\epsilon^{\frac{1}{2}}\left|\int_0^T S(\epsilon(T-s))G(Z_x^\epsilon(s))\,dW(s)\right.\right.\\
&\qquad\left.\left.-\int_0^T S(\epsilon(T-s))G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))\,dW(s)\right|\geq\delta\right\}\\
&\leq P\left\{\epsilon^{\frac{1}{2}}\sup_{t\in[0,1]}\left|\int_0^t 1_{[0,T]}(s)S(\epsilon(T-s))[G(Z_x^\epsilon(s))-G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))]\,dW(s)\right|\geq\delta\right\}\\
&\leq P\left\{\epsilon^{\frac{1}{2}}\sup_{t\in[0,1]}\left|\int_0^t 1_{[0,T]}(s)S(\epsilon(T-s))[G(Z_x^\epsilon(s))-G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))]\,dW(s)\right|\geq\delta,\right.\\
&\qquad\left.\sup_{t\in[0,1]}|Z_x^\epsilon(t)-S(\epsilon(t-\pi_n(t)))Z_x^\epsilon(\pi_n(t))|<\rho\right\}\\
&\quad+P\left\{\sup_{t\in[0,1]}|Z_x^\epsilon(t)-S(\epsilon(t-\pi_n(t)))Z_x^\epsilon(\pi_n(t))|\geq\rho\right\}\\
&\leq P\left\{\epsilon^{\frac{1}{2}}\sup_{t\in[0,1]}\left|\int_0^{t\wedge\tau_\rho} 1_{[0,T]}(s)S(\epsilon(T-s))[G(Z_x^\epsilon(s))-G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))]\,dW(s)\right|\geq\delta\right\}\\
&\quad+P\left\{\sup_{t\in[0,1]}|Z_x^\epsilon(t)-S(\epsilon(t-\pi_n(t)))Z_x^\epsilon(\pi_n(t))|\geq\rho\right\}\\
&= P\left\{\epsilon^{\frac{1}{2}}\sup_{t\in[0,1]}\left|\int_0^t 1_{[0,\tau_\rho\wedge T]}(s)S(\epsilon(T-s))[G(Z_x^\epsilon(s))-G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))]\,dW(s)\right|\geq\delta\right\}\\
&\quad+P\left\{\sup_{t\in[0,1]}|Z_x^\epsilon(t)-S(\epsilon(t-\pi_n(t)))Z_x^\epsilon(\pi_n(t))|\geq\rho\right\},
\end{aligned}\tag{3.44}$$

where the last equality follows from the localization lemma [10, Lemma 4.9].
We have

$$\begin{aligned}
&\int_0^1 1_{[0,\tau_\rho\wedge T]}(s)\|S(\epsilon(T-s))[G(Z_x^\epsilon(s))-G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))]\|^2_{L_2(U,H)}\,ds\\
&\quad\leq\; M^2\Lambda^2\int_0^1 1_{[0,\tau_\rho]}(s)|Z_x^\epsilon(s)-S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s))|^2\,ds\\
&\quad\leq\; M^2\Lambda^2\rho^2.
\end{aligned}$$

Thus, by Theorem 3.9,

$$\begin{aligned} & P\left\{\sup_{t\in[0,1]}\left|\int_0^t 1_{[0,\tau_\rho\wedge T]}(s)S(\epsilon(T-s))[G(Z_x^\epsilon(s))-G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))]\,dW(s)\right|\geq\frac{\delta}{\epsilon^{\frac{1}{2}}}\right\} \\ \leq\ & 3\exp\left(-\frac{\delta^2}{\epsilon 4M^2\Lambda^2\rho^2}\right). \end{aligned} \tag{3.45}$$

We now choose $\rho\in(0,\infty)$ such that

$$\frac{\delta^2}{4M^2\Lambda^2\rho^2}\geq a+\ln 6. \tag{3.46}$$

By Lemma 3.14 we can find $n_0\in\mathbb{N}$ such that for each $n\geq n_0$ there exists $\epsilon_n>0$ such that for all $\epsilon\in(0,\epsilon_n]$ and for all $x\in B_H(0,R)$

$$P\left\{\sup_{t\in[0,1]}|Z_x^\epsilon(t)-S(\epsilon(t-\pi_n(t)))Z_x^\epsilon(\pi_n(t))|\geq\rho\right\}\leq\exp\left(-\frac{a+\ln 2}{\epsilon}\right). \tag{3.47}$$

With ρ chosen to satisfy inequality (3.46), inequalities (3.45) and (3.47) combine in equation (3.44) to give the desired result.

Lemma 3.16 *Given $a>0$ and $\delta>0$ and $0\leq T_1<T_2\leq 1$ and $R>0$ there exists $b>0$ and there exists $\epsilon_0\in(0,1]$ such that for each $x\in B_H(0,R)$ and for each $\epsilon\in(0,\epsilon_0]$*

$$\begin{aligned} & P\{\,|\epsilon^{\frac{1}{2}}\int_{T_1}^{T_2}S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))\,dW(s)|\geq\delta,\ \sup_{T_1\leq t\leq T_2}\epsilon^{\frac{1}{2}}|W(t)|_{U_1}\leq b\,\} \\ \leq\ & \exp\left(\frac{-a}{\epsilon}\right). \end{aligned}$$

Proof. Recall that (g_k) is an orthonormal basis of U and for each $n\in\mathbb{N}$ we define the projection in U:

$$\Pi_n(u)=\sum_{k=1}^n\langle u,g_k\rangle_U g_k\quad\forall u\in U.$$

In the course of this proof we choose numbers $D\in(0,\infty)$, $n\in\mathbb{N}$, $T_1<\tilde{T}_1<\tilde{T}_2<T_2$ and a partition $\tilde{T}_1=t_0<t_1<\cdots<t_l=\tilde{T}_2$ as well as $b\in(0,\infty)$ in order to control the size

of the five terms on the right hand side of the inequality

$$
\begin{aligned}
&P\{\,\epsilon^{\frac{1}{2}}|\int_{T_1}^{T_2} S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))\,dW(s)| \geq \delta,\ \sup_{t\in[T_1,T_2]} \epsilon^{\frac{1}{2}}|W(t)|_{U_1} \leq b\} \\
\leq\ & P\{|Z_x^\epsilon(T_1)| \geq D\} \\
&\quad + P\{\,\epsilon^{\frac{1}{2}}|\int_{T_1}^{T_2} S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))(I_U-\Pi_n)\,dW(s)\,| \geq \frac{\delta}{4},\ |Z_x^\epsilon(T_1)|<D\} \\
&\quad + P\left\{\epsilon^{\frac{1}{2}}\left|\int_{T_1}^{\tilde{T}_1} S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))\Pi_n\,dW(s) + \right.\right. \\
&\qquad\qquad \left.\left.\int_{\tilde{T}_2}^{T_2} S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))\Pi_n\,dW(s)\right| \geq \frac{\delta}{4}\right\} \\
&\quad + P\left\{\epsilon^{\frac{1}{2}}\left|\int_{\tilde{T}_1}^{\tilde{T}_2} S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))\Pi_n\,dW(s)\right.\right. \\
&\qquad\qquad \left. - \int_{\tilde{T}_1}^{\tilde{T}_2}\sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s)S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))\Pi_n\,dW(s)\right| \geq \frac{\delta}{4}, \\
&\qquad\qquad\qquad \left. |Z_x^\epsilon(T_1)|<D\right\} \\
&\quad + P\left\{\epsilon^{\frac{1}{2}}|\int_{\tilde{T}_1}^{\tilde{T}_2}\sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s)S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))\Pi_n\,dW(s)\,| \geq \frac{\delta}{4},\right. \\
&\qquad\qquad\qquad \left.\sup_{t\in[T_1,T_2]} \epsilon^{\frac{1}{2}}|W(t)|_{U_1} \leq b\right\} \\
=\ & \text{term } 1 + \text{term } 2 + \text{term } 3 + \text{term } 4 + \text{term } 5.
\end{aligned}
\tag{3.48}
$$

Let $\tilde{a} > a$.

By Lemma 3.12 we can take $D \in (0,\infty)$ such that for all $x \in B_H(0,R)$ and for all $\epsilon \in (0,1]$

$$
\text{term } 1 := P\{|Z_x^\epsilon(T_1)| \geq D\} \leq \exp\left(-\frac{\tilde{a}}{\epsilon}\right). \tag{3.49}
$$

Let $x \in B_H(0,R)$ and $\epsilon \in (0,1]$. Define the $(\mathcal{F}_t)$-stopping time

$$
\tau_{x,\epsilon}(\omega) := \begin{cases} T_1 & \text{if } |Z_x^\epsilon(T_1)(\omega)| \geq D \\ 1 & \text{otherwise.} \end{cases} \tag{3.50}
$$

The function

$$(s,\omega) \in ([0,1]\times\Omega, \mathcal{P}_1) \mapsto 1_{(T_1,T_2]}(s)S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1)(\omega)) \in (L_2(U,H), \mathcal{B}_{L_2(U,H)})$$

is measurable and belongs to $L^2([0,1]\times\Omega, \mathcal{P}_1, \lambda\times P; L_2(U,H))$. Thus, applying the localization lemma [10, Lemma 4.9], we have for arbitrary $n \in \mathbb{N}$

$$\begin{aligned}
&\int_0^t 1_{[0,\tau_{x,\epsilon}]}(s)\epsilon^{\frac{1}{2}}1_{(T_1,T_2]}(s)S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))(I_U-\Pi_n)\,dW(s) \\
&= \int_0^{t\wedge\tau_{x,\epsilon}} \epsilon^{\frac{1}{2}}1_{(T_1,T_2]}(s)S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))(I_U-\Pi_n)\,dW(s) \quad \forall t\in[0,1]
\end{aligned}$$

P a.e.. Using this fact we have

$$\begin{aligned}
&\text{term 2} \\
&:= P\{\,|\epsilon^{\frac{1}{2}}\int_{T_1}^{T_2} S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))(I_U-\Pi_n)\,dW(s)| \geq \frac{\delta}{4},\, |Z_x^\epsilon(T_1)|<D\,\} \\
&\leq P\left\{\sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}\int_0^{t\wedge\tau_{x,\epsilon}} 1_{(T_1,T_2]}(s)S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))(I_U-\Pi_n)\,dW(s)| \geq \frac{\delta}{4},\right. \\
&\qquad\qquad \left. |Z_x^\epsilon(T_1)|<D \right\} \\
&= P\left\{\sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}\int_0^{t} 1_{[0,\tau_{x,\epsilon}]}(s)1_{(T_1,T_2]}(s)S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))(I_U-\Pi_n)\,dW(s)| \geq \frac{\delta}{4},\right. \\
&\qquad\qquad \left. |Z_x^\epsilon(T_1)|<D \right\}. \qquad (3.51)
\end{aligned}$$

Since

$$\begin{aligned}
&\int_0^1 \epsilon 1_{[0,\tau_{x,\epsilon}]}(s)1_{(T_1,T_2]}(s)\|S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))(I_U-\Pi_n)\|^2_{L_2(U,H)}\,ds \\
&\leq \epsilon M^2 \sup_{h\in B_H(0,MD)} \|G(h)(I_U-\Pi_n)\|^2_{L_2(U,H)} \quad P \text{ a.e.},
\end{aligned}$$

Theorem 3.9 yields an estimate of the term on the right hand side of inequality (3.51):

$$\begin{aligned}
&\text{term 2}\\
&\le P\{\sup_{t\in[0,1]} |\epsilon^{\frac{1}{2}} \int_0^t 1_{[0,\tau_{x,\epsilon}]}(s) 1_{(T_1,T_2]}(s) S(\epsilon(T_2-s)) G(S(\epsilon(s-T_1)) Z_x^\epsilon(T_1))(I_U-\Pi_n)\, dW(s)| \ge \frac{\delta}{4}\}\\
&\le 3\exp\left(-\frac{\delta^2}{64\epsilon M^2 \sup_{h\in B_H(0,DM)} \|G(h)(I_U-\Pi_n)\|^2_{L_2(U,H)}}\right).
\end{aligned}$$

By assumption (A2) we can now choose $n \in \mathbb{N}$ such that

$$\ln 3 - \frac{\delta^2}{64 M^2 \sup_{h\in B_H(0,DM)} \|G(h)(I_U-\Pi_n)\|^2_{L_2(U,H)}} \le -\tilde{a}$$

and we obtain

$$\text{term 2} \le \exp\left(-\frac{\tilde{a}}{\epsilon}\right) \qquad \forall x \in B_H(0,R) \text{ and } \forall \epsilon \in (0,1]. \tag{3.52}$$

We choose $\tilde{T}_1$ and $\tilde{T}_2$ such that $T_1 < \tilde{T}_1 < \tilde{T}_2 < T_2$ and

$$\ln 3 - \frac{\delta^2}{64 M^2 \Gamma^2 (\tilde{T}_1 - T_1 + T_2 - \tilde{T}_2)} \le -\tilde{a}.$$

Then again by Theorem 3.9 we have

$$\begin{aligned}
&\text{term 3}\\
:=\;& P\left\{\epsilon^{\frac{1}{2}} \left| \int_{T_1}^{\tilde{T}_1} S(\epsilon(T_2-s)) G(S(\epsilon(s-T_1)) Z_x^\epsilon(T_1)) \Pi_n\, dW(s) \right.\right.\\
&\qquad\qquad \left.\left. + \int_{\tilde{T}_2}^{T_2} S(\epsilon(T_2-s)) G(S(\epsilon(s-T_1)) Z_x^\epsilon(T_1)) \Pi_n\, dW(s)\right| \ge \frac{\delta}{4}\right\}\\
\le\;& P\{\sup_{t\in[0,1]} | \int_0^t 1_{(T_1,\tilde{T}_1]\cup(\tilde{T}_2,T_2]}(s) \epsilon^{\frac{1}{2}} S(\epsilon(T_2-s)) G(S(\epsilon(s-T_1)) Z_x^\epsilon(T_1)) \Pi_n\, dW(s)| \ge \frac{\delta}{4}\}\\
\le\;& 3\exp\left(-\frac{\delta^2}{64\epsilon M^2 \Gamma^2 (\tilde{T}_1 - T_1 + T_2 - \tilde{T}_2)}\right)\\
\le\;& \exp\left(-\frac{\tilde{a}}{\epsilon}\right) \qquad \forall x \in B_H(0,R) \text{ and } \forall \epsilon \in (0,1].
\end{aligned} \tag{3.53}$$

Let $\mathcal{T} := \{\tilde{T}_1 = t_0 < t_1 < \cdots < t_l = \tilde{T}_2\}$ be a partition of $[\tilde{T}_1, \tilde{T}_2]$ and set $\Delta_{\mathcal{T}} := \max\{t_{j+1} - t_j : j = 0, 1, \ldots, l-1\}$.

For $x \in B_H(0,R)$ and $\epsilon \in (0,1]$ define the $(\mathcal{F}_t)$-stopping time $\tau_{x,\epsilon}$ as in equation (3.50). By the localization lemma [10, Lemma 4.9] we have

$$\begin{aligned}
&\int_0^t 1_{[0,\tau_{x,\epsilon}]}(s) \sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s)\, [S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1)) \\
&\qquad\qquad -S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))]\, \Pi_n\, dW(s) \\
&= \int_0^{t\wedge\tau_{x,\epsilon}} \sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s)\, [S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1)) \\
&\qquad\qquad -S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))]\, \Pi_n\, dW(s) \quad \forall t \in [0,1] \ P \text{ a.e..}
\end{aligned}$$

Thus

$$\begin{aligned}
&\text{term 4} \\
&:= P\Bigg\{ \epsilon^{\frac{1}{2}} \Bigg| \int_{\tilde{T}_1}^{\tilde{T}_2} S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))\Pi_n\, dW(s) \\
&\qquad - \int_{\tilde{T}_1}^{\tilde{T}_2} \sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s) S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))\Pi_n\, dW(s) \Bigg| \ge \frac{\delta}{4}, \\
&\qquad\qquad |Z_x^\epsilon(T_1)| < D \Bigg\} \\
&\le P\Bigg\{ \sup_{t\in[0,1]} \epsilon^{\frac{1}{2}} \Bigg| \int_0^{t\wedge\tau_{x,\epsilon}} \sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s)[S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1)) \\
&\qquad - S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))]\Pi_n\, dW(s) \Bigg| \ge \frac{\delta}{4}, \\
&\qquad\qquad |Z_x^\epsilon(T_1)| < D \Bigg\} \\
&\le P\Bigg\{ \sup_{t\in[0,1]} \epsilon^{\frac{1}{2}} \Bigg| \int_0^t 1_{[0,\tau_{x,\epsilon}]}(s) \sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s)[S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1)) \\
&\qquad - S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))]\Pi_n\, dW(s) \Bigg| \ge \frac{\delta}{4} \Bigg\}.
\end{aligned} \tag{3.54}$$

In order to apply Theorem 3.9 to the right hand side of (3.54) we observe that

$$\begin{aligned}
&1_{(t_j,t_{j+1}]}(s)1_{[0,\tau_{x,\epsilon}]}(s)\|[S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1)) \\
&\qquad\qquad - S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))]\Pi_n\|_{L_2(U,H)} \\
&\le 1_{(t_j,t_{j+1}]}(s)1_{[0,\tau_{x,\epsilon}]}(s)\|S(\epsilon(T_2-s))[G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1)) - G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))]\Pi_n\|_{L_2(U,H)} \\
&\quad + 1_{(t_j,t_{j+1}]}(s)1_{[0,\tau_{x,\epsilon}]}(s)\|[S(\epsilon(T_2-s)) - S(\epsilon(T_2-t_j))]G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))\Pi_n\|_{L_2(U,H)} \\
&\le M\Lambda D\|S(\epsilon(s-T_1)) - S(\epsilon(t_j-T_1))\|_{L(H,H)}1_{(t_j,t_{j+1}]}(s) \\
&\quad + \Gamma\|S(\epsilon(T_2-s)) - S(\epsilon(T_2-t_j))\|_{L(H,H)}1_{(t_j,t_{j+1}]}(s).
\end{aligned}$$

Thus we have the bound

$$\begin{aligned}
&\int_0^1 \sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s)1_{[0,\tau_{x,\epsilon}]}(s)\|[S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1)) \\
&\qquad\qquad - S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))]\Pi_n\|^2_{L_2(U,H)}\,ds \\
&\le\ 2\int_0^1 \sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s)[M^2\Lambda^2D^2\|S(\epsilon(s-T_1)) - S(\epsilon(t_j-T_1))\|^2_{L(H,H)} \\
&\qquad\qquad + \Gamma^2\|S(\epsilon(T_2-s)) - S(\epsilon(T_2-t_j))\|^2_{L(H,H)}]\,ds \\
&\le\ 2(M^2\Lambda^2D^2+\Gamma^2) \\
&\quad \times \Bigg(\sup\Bigg\{\|S(\eta r) - S(\eta s)\|_{L(H,H)}\ :\ r,s\in[(\tilde T_1 - T_1)\wedge(T_2-\tilde T_2),1] \\
&\qquad\qquad \text{and } |r-s|\le\Delta_{\mathcal T} \text{ and } \eta\in(0,1]\Bigg\}\Bigg)^2;
\end{aligned}$$

the last expression does not depend on $\epsilon\in(0,1]$ or $x\in B_H(0,R)$ and goes to 0 as $\Delta_{\mathcal T}\to 0$ since, by (A2), the family of functions

$$\{t\in[(\tilde T_1 - T_1)\wedge(T_2-\tilde T_2),1]\mapsto S(\eta t)\in L(H,H),\quad \eta\in(0,1]\}$$

is uniformly equicontinuous in the norm topology. For brevity set

$$\begin{aligned}
\zeta(\Delta_{\mathcal T})\ \ :=\ \ \sup\Bigg\{&\|S(\eta r) - S(\eta s)\|_{L(H,H)} : r,s\in[(\tilde T_1 - T_1)\wedge(T_2-\tilde T_2),1] \\
&\qquad \text{and } |r-s|\le\Delta_{\mathcal T} \text{ and } \eta\in(0,1]\Bigg\}.
\end{aligned}$$

We now choose partition $\mathcal{T} = \{\tilde{T}_1 = t_0 < t_1 < \cdots < t_l = \tilde{T}_2\}$ such that $\Delta_{\mathcal{T}}$ satisfies

$$\ln 3 - \frac{\delta^2}{128(M^2\Lambda^2 D^2 + \Gamma^2)(\zeta(\Delta_{\mathcal{T}}))^2} \leq \ -\tilde{a}.$$

Then from inequality (3.54) and Theorem 3.9 we have

$$\text{term } 4 \leq \exp\left(-\frac{\tilde{a}}{\epsilon}\right) \qquad \forall x \in B_H(0, R) \text{ and } \forall \epsilon \in (0, 1]. \tag{3.55}$$

Finally we consider term 5. Recall that by definition of the inner product $\langle \cdot, \cdot \rangle_{U_1}$ in U_1, the bounded linear operator from U_1 into U

$$\Pi_n^1 u := \sum_{k=1}^{n} \langle u, \lambda_k^{-2} J g_k \rangle_{U_1} \, g_k \ , \quad u \in U_1,$$

satisfies $\Pi_n^1 \, Ju = \Pi_n u \quad \forall u \in U$. We will use the result

$$\int_0^1 1_{(c,d]}(s) \Phi \circ J \, dW(s) = \Phi(W(d) - W(c)) \quad P \text{ a.e.} \tag{3.56}$$

when $0 \leq c < d \leq 1$ and $\Phi : (\Omega, \mathcal{F}_c) \to (L_2(U_1, H), \mathcal{B}_{L_2(U_1,H)})$ is $\mathcal{F}_c$ measurable and $E[\|\Phi\|^2_{L_2(U_1,H)}] < \infty$; this result is clear when Φ is simple and can be shown for general Φ by approximation in $L^2(\Omega, \mathcal{F}_c, P; L_2(U_1, H))$ with simple functions.
We have for each $\epsilon \in (0, 1]$ and $x \in H$

$$\begin{aligned}
&\epsilon^{\frac{1}{2}} \Big| \int_{\tilde{T}_1}^{\tilde{T}_2} \sum_{j=0}^{l-1} 1_{(t_j, t_{j+1}]}(s) S(\epsilon(T_2 - t_j)) G(S(\epsilon(t_j - T_1)) Z_x^{\epsilon}(T_1)) \Pi_n \, dW(s) \Big| \\
&= \ \epsilon^{\frac{1}{2}} \Big| \sum_{j=0}^{l-1} \int_0^1 1_{(t_j, t_{j+1}]}(s) S(\epsilon(T_2 - t_j)) G(S(\epsilon(t_j - T_1)) Z_x^{\epsilon}(T_1)) \Pi_n^1 J \, dW(s) \Big| \quad P \text{ a.e.} \\
&= \ \epsilon^{\frac{1}{2}} \Big| \sum_{j=0}^{l-1} S(\epsilon(T_2 - t_j)) G(S(\epsilon(t_j - T_1)) Z_x^{\epsilon}(T_1)) \Pi_n^1 (W(t_{j+1}) - W(t_j)) \Big| \quad P \text{ a.e.} \qquad (3.57) \\
&\leq \ 2lM\Gamma \|\Pi_n^1\|_{L(U_1,U)} \epsilon^{\frac{1}{2}} \sup_{t \in [T_1, T_2]} |W(t)|_{U_1}. \qquad (3.58)
\end{aligned}$$

Equation (3.57) follows from equation (3.56). We choose $0 < b < \frac{\delta}{8lM\Gamma\|\Pi_n^1\|_{L(U_1,U)}}$, then for each $\epsilon \in (0,1]$ and each $x \in B_H(0,R)$

$$\text{term 5} \;=\; P\left\{\epsilon^{\frac{1}{2}}|\int_{\tilde{T}_1}^{\tilde{T}_2}\sum_{j=0}^{l-1} 1_{(t_j,t_{j+1}]}(s)S(\epsilon(T_2-t_j))G(S(\epsilon(t_j-T_1))Z_x^\epsilon(T_1))\Pi_n\,dW(s)| \geq \frac{\delta}{4},\right.$$
$$\left.\sup_{t\in[T_1,T_2]}\epsilon^{\frac{1}{2}}|W(t)|_{U_1} \leq b\right\} = 0, \tag{3.59}$$

by inequality (3.58).

With b chosen as in the last paragraph, we combine inequalities (3.48), (3.49), (3.52), (3.53), (3.55) and (3.59) to obtain for all $x \in B_H(0,R)$:

$$\begin{aligned}
&P\{|\epsilon^{\frac{1}{2}}\int_{T_1}^{T_2} S(\epsilon(T_2-s))G(S(\epsilon(s-T_1))Z_x^\epsilon(T_1))\,dW(s)| \geq \delta, \sup_{t\in[T_1,T_2]}\epsilon^{\frac{1}{2}}|W(t)|_{U_1} \leq b\} \\
&\leq\; 4\exp\left(-\frac{\tilde{a}}{\epsilon}\right) \qquad \forall\epsilon\in(0,1] \\
&\leq\; \exp\left(-\frac{a}{\epsilon}\right) \qquad \forall\epsilon\in(0,\epsilon_0],
\end{aligned}$$

where $\epsilon_0 = \frac{\tilde{a}-a}{\ln 4} \wedge 1$.
This completes the proof of the lemma.

Proposition 3.17 *Let $R \in (0,\infty)$. Given $a > 0$ and $\delta > 0$ there exist $b > 0$ and $\epsilon_0 > 0$ such that for all $x \in B_H(0,R)$ and for all $\epsilon \in (0,\epsilon_0]$ we have*

$$P\left\{\sup_{t\in[0,1]}\left|\epsilon^{\frac{1}{2}}\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \geq \delta, \quad \sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}W(t)|_{U_1} \leq b\right\} \leq \exp\left(-\frac{a}{\epsilon}\right).$$

Proof. Let $\tilde{a} > a$. Let n be a natural number. For each $k \in \{0,1,\ldots,2^n-1\}$ and $t \in [t_{n,k}, t_{n,k+1}]$ we have

$$\begin{aligned}
\left|\epsilon^{\frac{1}{2}}\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \;\leq\; & \left|\epsilon^{\frac{1}{2}}S(\epsilon(t-t_{n,k}))\int_0^{t_{n,k}} S(\epsilon(t_{n,k}-s))G(Z_x^\epsilon(s))\,dW(s)\right| \\
& + \left|\epsilon^{\frac{1}{2}}\int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \quad P \text{ a.e.}.
\end{aligned} \tag{3.60}$$

Since the processes in inequality (3.60) are continuous on $[t_{n,k}, t_{n,k+1}]$, for each $k \in \{0, 1, \ldots, 2^n - 1\}$ we have

$$\begin{aligned}
&\sup_{t\in[t_{n,k},t_{n,k+1}]} \left| \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\, dW(s) \right| \\
&\leq \;\; M\epsilon^{\frac{1}{2}} \left| \int_0^{t_{n,k}} S(\epsilon(t_{n,k}-s))G(Z_x^\epsilon(s))\, dW(s) \right| \\
&\qquad + \epsilon^{\frac{1}{2}} \sup_{t\in[t_{n,k},t_{n,k+1}]} \left| \int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\, dW(s) \right| \quad P \text{ a.e..}
\end{aligned}$$

Consequently

$$\begin{aligned}
&\sup_{t\in[0,1]} \left| \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\, dW(s) \right| \\
&\leq \;\; M\epsilon^{\frac{1}{2}} \sup_{0<k\leq 2^n-1} \left| \int_0^{t_{n,k}} S(\epsilon(t_{n,k}-s))G(Z_x^\epsilon(s))\, dW(s) \right| \\
&\qquad + \epsilon^{\frac{1}{2}} \sup_{0\leq k\leq 2^n-1} \sup_{t\in[t_{n,k},t_{n,k+1}]} \left| \int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\, dW(s) \right| \quad P \text{ a.e.} \\
&\leq \;\; M\epsilon^{\frac{1}{2}} \sup_{0<k\leq 2^n-1} \left| \int_0^{t_{n,k}} S(\epsilon(t_{n,k}-s))G(Z_x^\epsilon(s))\, dW(s) \right. \\
&\qquad\qquad \left. - \int_0^{t_{n,k}} S(\epsilon(t_{n,k}-s))G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))\, dW(s) \right| \\
&\qquad + M\epsilon^{\frac{1}{2}} \sup_{0<k\leq 2^n-1} \left| \int_0^{t_{n,k}} S(\epsilon(t_{n,k}-s))G(S(\epsilon(s-\pi_n(s)))Z_x^\epsilon(\pi_n(s)))\, dW(s) \right| \\
&\qquad + \epsilon^{\frac{1}{2}} \sup_{0\leq k\leq 2^n-1} \sup_{t\in[t_{n,k},t_{n,k+1}]} \left| \int_{t_{n,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\, dW(s) \right| \quad P \text{ a.e..}
\end{aligned}$$

By Lemma 3.13 and Lemma 3.15 respectively, there exist a natural number n_0 and a positive number ϵ_0 such that

1. for all $x \in H$ and for all $\epsilon \in (0, \epsilon_0]$ we have

$$P\left\{ \sup_{0\leq k\leq 2^{n_0}-1} \sup_{t\in[t_{n_0,k},t_{n_0,k+1}]} \left| \epsilon^{\frac{1}{2}} \int_{t_{n_0,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\, dW(s) \right| \geq \frac{\delta}{3} \right\} \leq \exp\left(-\frac{\tilde{a}}{\epsilon}\right)$$

2. and for each $k \in \{1, \ldots, 2^{n_0} - 1\}$ and for all $x \in B_H(0, R)$ and for all $\epsilon \in (0, \epsilon_0]$ we

have

$$P\left\{\epsilon^{\frac{1}{2}}\left|\int_0^{t_{n_0,k}} S(\epsilon(t_{n_0,k}-s))G(Z_x^\epsilon(s))\,dW(s)\right.\right.$$
$$\left.\left. -\int_0^{t_{n_0,k}} S(\epsilon(t_{n_0,k}-s))G(S(\epsilon(s-\pi_{n_0}(s)))Z_x^\epsilon(\pi_{n_0}(s)))\,dW(s)\right| \geq \frac{\delta}{3M}\right\} \leq \exp\left(-\frac{\tilde{a}}{\epsilon}\right).$$

Hence for arbitrary $b>0$ and for all $x \in B_H(0,R)$ and for all $\epsilon \in (0,\epsilon_0]$ we have

$$\begin{aligned}
&P\left\{\sup_{t\in[0,1]}\left|\epsilon^{\frac{1}{2}}\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \geq \delta,\ \sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}W(t)|_{U_1} \leq b\right\}\\
\leq\ & P\left\{\epsilon^{\frac{1}{2}}\sup_{0<k\leq 2^{n_0}-1}\left|\int_0^{t_{n_0,k}} S(\epsilon(t_{n_0,k}-s))G(Z_x^\epsilon(s))\,dW(s)\right.\right.\\
&\qquad\left.\left. -\int_0^{t_{n_0,k}} S(\epsilon(t_{n_0,k}-s))G(S(\epsilon(s-\pi_{n_0}(s)))Z_x^\epsilon(\pi_{n_0}(s)))\,dW(s)\right| \geq \frac{\delta}{3M}\right\}\\
&+P\left\{\epsilon^{\frac{1}{2}}\sup_{0\leq k\leq 2^{n_0}-1}\ \sup_{t\in[t_{n_0,k},t_{n_0,k+1}]}\left|\int_{t_{n_0,k}}^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \geq \frac{\delta}{3}\right\}\\
&+P\left\{\epsilon^{\frac{1}{2}}\sup_{0<k\leq 2^{n_0}-1}\left|\int_0^{t_{n_0,k}} S(\epsilon(t_{n_0,k}-s))G(S(\epsilon(s-\pi_{n_0}(s)))Z_x^\epsilon(\pi_{n_0}(s)))\,dW(s)\right| \geq \frac{\delta}{3M},\right.\\
&\qquad\left.\sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}W(t)|_{U_1} \leq b\right\}\\
\leq\ & 2^{n_0}\exp\left(-\frac{\tilde{a}}{\epsilon}\right)\\
&+P\left\{\sup_{0<k\leq 2^{n_0}-1}\epsilon^{\frac{1}{2}}\left|\int_0^{t_{n_0,k}} S(\epsilon(t_{n_0,k}-s))G(S(\epsilon(s-\pi_{n_0}(s)))Z_x^\epsilon(\pi_{n_0}(s)))\,dW(s)\right| \geq \frac{\delta}{3M},\right.\\
&\qquad\left.\sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}W(t)|_{U_1} \leq b\right\}\\
\leq\ & 2^{n_0}\exp\left(-\frac{\tilde{a}}{\epsilon}\right)\\
&+\sum_{j=0}^{2^{n_0}-2}P\left\{\epsilon^{\frac{1}{2}}\left|\int_{t_{n_0,j}}^{t_{n_0,j+1}} S(\epsilon(t_{n_0,j+1}-s))G(S(\epsilon(s-t_{n_0,j}))Z_x^\epsilon(t_{n_0,j}))\,dW(s)\right| \geq \frac{\delta}{3M^2(2^{n_0}-1)},\right.\\
&\qquad\left.\sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}W(t)|_{U_1} \leq b\right\}, \quad (3.61)
\end{aligned}$$

where the last line follows from the observation that for each $k \in \{1, 2, \ldots, 2^{n_0} - 1\}$ we have

$$\begin{aligned}
&\epsilon^{\frac{1}{2}} \left| \int_0^{t_{n_0,k}} S(\epsilon(t_{n_0,k} - s))G(S(\epsilon(s - \pi_{n_0}(s)))Z_x^\epsilon(\pi_{n_0}(s)))\, dW(s) \right| \\
&\leq \sum_{j=0}^{k-1} \epsilon^{\frac{1}{2}} \left| \int_{t_{n_0,j}}^{t_{n_0,j+1}} S(\epsilon(t_{n_0,k} - s))G(S(\epsilon(s - t_{n_0,j}))Z_x^\epsilon(t_{n_0,j}))\, dW(s) \right| \quad P \text{ a.e.} \\
&\leq M \sum_{j=0}^{2^{n_0}-2} \epsilon^{\frac{1}{2}} \left| \int_{t_{n_0,j}}^{t_{n_0,j+1}} S(\epsilon(t_{n_0,j+1} - s))G(S(\epsilon(s - t_{n_0,j}))Z_x^\epsilon(t_{n_0,j}))\, dW(s) \right| \quad P \text{ a.e..}
\end{aligned}$$

According to Lemma 3.16 we can find $b_1 > 0$ and $\epsilon_1 > 0$ such that for all $x \in B_H(0, R)$ and for all $j \in \{0, 1, \ldots, 2^{n_0} - 2\}$ and for all $\epsilon \in (0, \epsilon_1]$ we have

$$\begin{aligned}
&P\left\{ \epsilon^{\frac{1}{2}} \left| \int_{t_{n_0,j}}^{t_{n_0,j+1}} S(\epsilon(t_{n_0,j+1} - s))G(S(\epsilon(s - t_{n_0,j}))Z_x^\epsilon(t_{n_0,j}))\, dW(s) \right| \geq \frac{\delta}{3M^2(2^{n_0} - 1)}, \right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. \sup_{t \in [0,1]} |\epsilon^{\frac{1}{2}} W(t)|_{U_1} \leq b_1 \right\} \\
&\leq \exp\left(-\frac{\tilde{a}}{\epsilon} \right).
\end{aligned}$$

Now returning to inequality (3.61) with b_1 in place of b we see that there exists $\epsilon_2 \in (0, \epsilon_0 \wedge \epsilon_1]$ such that for all $x \in B_H(0, R)$ and for all $\epsilon \in (0, \epsilon_2]$ we have

$$P\left\{ \sup_{t \in [0,1]} \left| \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t - s))G(Z_x^\epsilon(s))\, dW(s) \right| \geq \delta,\ \sup_{t \in [0,1]} |\epsilon^{\frac{1}{2}} W(t)|_{U_1} \leq b_1 \right\} \leq \exp\left(-\frac{a}{\epsilon} \right).$$

We can now prove Proposition 3.11.

Proof of Proposition 3.11. Let $K \subset H$ be compact. Fix $a > 0$ and $\delta > 0$ and $\phi \in L^2([0, 1]; U)$. For $\epsilon \in (0, 1]$ and $x \in K$ and b a positive real number which will be specified later, we set

$$\mathcal{D}(\epsilon, x, b) := \left\{ \sup_{t \in [0,1]} |X_x^\epsilon(t) - z_x^\phi(t)| \geq \delta,\ \sup_{t \in [0,1]} \left| \epsilon^{\frac{1}{2}} W(t) - J \int_0^t \phi(s)\, ds \right|_{U_1} \leq b \right\}.$$

As in equation (3.25), define the process

$$W^\epsilon(t) := W(t) - \epsilon^{-\frac{1}{2}} J \int_0^t \phi(s)\, ds \quad \forall t \in [0,1].$$

By [10, Theorem 10.14], $(W^\epsilon(t))_{t\in[0,1]}$ is a Wiener process with respect to filtration $(\mathcal{F}_t)$ on probability space $(\Omega, \mathcal{F}, P^\epsilon)$ where

$$dP^\epsilon(\omega) = \exp\left(\epsilon^{-\frac{1}{2}} \int_0^1 \langle \phi(s), \cdot\rangle_U \, dW(s) - \frac{1}{2\epsilon}\int_0^1 |\phi(s)|_U^2 \, ds\right) dP(\omega)$$

and $P^\epsilon(W^\epsilon(1))^{-1} = P(W(1))^{-1}$.
For $\lambda > 0$ set

$$\mathcal{M}(\epsilon, \lambda) := \left\{\omega \in \Omega : \int_0^1 \langle \phi(s), \cdot\rangle_U \, dW(s)(\omega) \geq -\frac{\lambda}{\epsilon^{\frac{1}{2}}}\right\}.$$

We have

$$P(\mathcal{D}(\epsilon, x, b)) \leq P(\mathcal{D}(\epsilon, x, b) \cap \mathcal{M}(\epsilon, \lambda)) + P(\mathcal{M}(\epsilon, \lambda)^c). \tag{3.62}$$

By Theorem 3.9 we have immediately

$$\begin{aligned} P(\mathcal{M}(\epsilon, \lambda)^c) &\leq P\left\{\sup_{t\in[0,1]} \left|\int_0^t \langle \phi(s), \cdot\rangle_U \, dW(s)\right| > \frac{\lambda}{\epsilon^{\frac{1}{2}}}\right\} \\ &\leq 3\exp\left(-\frac{\lambda^2}{4\epsilon \int_0^1 |\phi(s)|_U^2 \, ds}\right). \end{aligned} \tag{3.63}$$

The rest of the proof involves finding an exponential bound for the first term on the right hand side of (3.62). We have

$$\begin{aligned} & P(\mathcal{D}(\epsilon, x, b) \cap \mathcal{M}(\epsilon, \lambda)) \\ &= \int_\Omega 1_{\mathcal{D}(\epsilon,x,b)\cap\mathcal{M}(\epsilon,\lambda)}(\omega) \exp\left(-\epsilon^{-\frac{1}{2}}\int_0^1 \langle \phi(s), \cdot\rangle_U \, dW(s)(\omega) + \frac{1}{2\epsilon}\int_0^1 |\phi(s)|_U^2 \, ds\right) dP^\epsilon(\omega) \\ &\leq \exp\left(\frac{\lambda}{\epsilon} + \frac{1}{2\epsilon}\int_0^1 |\phi(s)|_U^2 \, ds\right) P^\epsilon \left\{\sup_{t\in[0,1]} |X_x^\epsilon(t) - z_x^\phi(t)| \geq \delta, \ \sup_{t\in[0,1]} \epsilon^{\frac{1}{2}} |W^\epsilon(t)|_{U_1} \leq b\right\}. \end{aligned} \tag{3.64}$$

By equation (3.28) we have

$$\begin{aligned} X_x^\epsilon(t) &= S(\epsilon t)x + \epsilon \int_0^t S(\epsilon(t-s))F(\epsilon s, X_x^\epsilon(s))\,ds + \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\,dW^\epsilon(s) \\ &\quad + \int_0^t S(\epsilon(t-s))G(X_x^\epsilon(s))\phi(s)\,ds \qquad\qquad \forall t \in [0,1] \quad P \text{ a.e..} \end{aligned}$$

Recall that in equation (3.36) we defined

$$\tilde{F}_\epsilon(s,y) := \epsilon F(\epsilon s, y) + G(y)\phi(s) \quad \forall (s,y) \in [0,1] \times H.$$

Thus $(X_x^\epsilon(t) : (\Omega, \mathcal{F}_t, P^\epsilon) \to (H, \mathcal{B}_H))_{t\in[0,1]}$ is the solution of the equation

$$\begin{cases} dX^\epsilon(t) = (\epsilon A X^\epsilon(t) + \tilde{F}_\epsilon(t, X^\epsilon(t)))\,dt + \epsilon^{\frac{1}{2}} G(X^\epsilon(t))\,dW^\epsilon(t) \quad t \in (0,1] \\ X^\epsilon(0) = x. \end{cases} \tag{3.65}$$

Let $(Z_x^\epsilon(t) : (\Omega, \mathcal{F}_t, P) \to (H, \mathcal{B}_H))_{t\in[0,1]}$ be the solution of the equation

$$\begin{cases} dZ^\epsilon(t) = (\epsilon A Z^\epsilon(t) + \tilde{F}_\epsilon(t, Z^\epsilon(t)))\,dt + \epsilon^{\frac{1}{2}} G(Z^\epsilon(t))\,dW(t) \quad t \in (0,1] \\ Z^\epsilon(0) = x. \end{cases} \tag{3.66}$$

By Proposition 3.19 we have the equality of the distributions on $(C([0,1]; H \oplus U_1), \mathcal{B}_{C([0,1];H\oplus U_1)})$:

$$P^\epsilon(X_x^\epsilon, W^\epsilon)^{-1} = P(Z_x^\epsilon, W)^{-1};$$

here trajectory-valued random variables are defined as in equation (3.74). Thus we have

$$\begin{aligned} &P^\epsilon\left\{\sup_{t\in[0,1]} |X_x^\epsilon(t) - z_x^\phi(t)| \geq \delta, \quad \sup_{t\in[0,1]} \epsilon^{\frac{1}{2}} |W^\epsilon(t)|_{U_1} \leq b\right\} \\ &= P\left\{\sup_{t\in[0,1]} |Z_x^\epsilon(t) - z_x^\phi(t)| \geq \delta, \quad \sup_{t\in[0,1]} \epsilon^{\frac{1}{2}} |W(t)|_{U_1} \leq b\right\}. \end{aligned} \tag{3.67}$$

For each $t \in [0,1]$ we have P a.e.:

$$
\begin{aligned}
&|Z_x^\epsilon(t) - z_x^\phi(t)| \\
= \ & \left| S(\epsilon t)x + \epsilon \int_0^t S(\epsilon(t-s))F(\epsilon s, Z_x^\epsilon(s))\,ds + \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s) \right. \\
&\qquad\qquad + \int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\phi(s)\,ds \\
&\qquad \left. - x - \int_0^t G(z_x^\phi(s))\phi(s)\,ds \right| \\
\le \ & |S(\epsilon t)x - x| + \left| \epsilon \int_0^t S(\epsilon(t-s))(F(\epsilon s, Z_x^\epsilon(s)) - F(\epsilon s, z_x^\phi(s)))\,ds \right| \\
& + \left| \epsilon \int_0^t S(\epsilon(t-s))F(\epsilon s, z_x^\phi(s))\,ds \right| \\
& + \left| \int_0^t S(\epsilon(t-s))(G(Z_x^\epsilon(s)) - G(z_x^\phi(s)))\phi(s)\,ds \right| \\
& + \left| \int_0^t (S(\epsilon(t-s)) - I_H)G(z_x^\phi(s))\phi(s)\,ds \right| \\
& + \left| \epsilon^{\frac{1}{2}} \int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s) \right| \\
\le \ & \sup_{r \in [0,\epsilon]} |S(r)x - x| + \epsilon M \Lambda \left(\int_0^t |Z_x^\epsilon(s) - z_x^\phi(s)|^2\,ds \right)^{\frac{1}{2}} \\
& + \epsilon M \Lambda \int_0^1 (1 + |z_x^\phi(s)|)\,ds \\
& + M\Lambda \left(\int_0^t |Z_x^\epsilon(s) - z_x^\phi(s)|^2\,ds \right)^{\frac{1}{2}} \left(\int_0^1 |\phi(s)|_U^2\,ds \right)^{\frac{1}{2}} \\
& + \sup\{ \|(S(r) - I_H)G(z_x^\phi(s))\|_{L_2(U,H)} :\ s \in [0,1],\ r \in [0,\epsilon] \} \left(\int_0^1 |\phi(s)|_U^2\,ds \right)^{\frac{1}{2}} \\
& + \sup_{r \in [0,1]} \left| \epsilon^{\frac{1}{2}} \int_0^r S(\epsilon(r-s))G(Z_x^\epsilon(s))\,dW(s) \right|.
\end{aligned}
$$

Thus, since process $(Z_x^\epsilon(t))_{t\in[0,1]}$ is continuous,

$$\begin{aligned}
&|Z_x^\epsilon(t) - z_x^\phi(t)|^2 \\
&\leq 6\left[\sup_{r\in[0,\epsilon]} |S(r)x - x|^2\right. \\
&\quad + M^2\Lambda^2\left(\epsilon^2 + \int_0^1 |\phi(s)|_U^2\,ds\right)\int_0^t |Z_x^\epsilon(s) - z_x^\phi(s)|^2\,ds \\
&\quad + \epsilon^2 M^2\Lambda^2\left(\int_0^1 (1 + |z_x^\phi(s)|)\,ds\right)^2 \\
&\quad + \sup\{\|(S(r) - I_H)G(z_x^\phi(s))\|_{L_2(U,H)} : s\in[0,1],\ r\in[0,\epsilon]\}^2\int_0^1 |\phi(s)|_U^2\,ds \\
&\quad \left. + \sup_{r\in[0,1]}\left|\epsilon^{\frac{1}{2}}\int_0^r S(\epsilon(r-s))G(Z_x^\epsilon(s))\,dW(s)\right|^2\right] \quad \forall t\in[0,1]\quad P\text{ a.e..}
\end{aligned}$$

Applying Gronwall's Lemma we have

$$\begin{aligned}
&\sup_{t\in[0,1]} |Z_x^\epsilon(t) - z_x^\phi(t)|^2 \\
&\leq 6\left[\sup_{r\in[0,\epsilon]} |S(r)x - x|^2 + \epsilon^2 M^2\Lambda^2\left(\int_0^1 (1 + |z_x^\phi(s)|)\,ds\right)^2\right. \\
&\quad + \sup\{\|(S(r) - I_H)G(z_x^\phi(s))\|_{L_2(U,H)} : s\in[0,1] \text{ and } r\in[0,\epsilon]\}^2\int_0^1 |\phi(s)|_U^2\,ds \\
&\quad \left. + \sup_{r\in[0,1]}\left|\epsilon^{\frac{1}{2}}\int_0^r S(\epsilon(r-s))G(Z_x^\epsilon(s))\,dW(s)\right|^2\right]\exp\left(6M^2\Lambda^2\left(1 + \int_0^1 |\phi(s)|_U^2\,ds\right)\right)
\end{aligned}$$

P a.e.. It follows that there exists $\epsilon_1 > 0$ such that for all $x\in K$ and for all $\epsilon\in(0,\epsilon_1]$ we have

$$\begin{aligned}
&P\left\{\sup_{t\in[0,1]} |Z_x^\epsilon(t) - z_x^\phi(t)| \geq \delta,\ \sup_{t\in[0,1]} \epsilon^{\frac{1}{2}}|W(t)|_{U_1} \leq b\right\} \\
&\leq P\left\{\sup_{r\in[0,1]}\left|\epsilon^{\frac{1}{2}}\int_0^r S(\epsilon(r-s))G(Z_x^\epsilon(s))\,dW(s)\right| \geq \frac{\delta}{c},\ \sup_{t\in[0,1]} \epsilon^{\frac{1}{2}}|W(t)|_{U_1} \leq b\right\},
\end{aligned}$$

where $c := 3\exp(3M^2\Lambda^2(1 + \int_0^1 |\phi(s)|_U^2\,ds))$.

Given $\tilde{a} > a$, by Proposition 3.17 there exist positive real numbers b and ϵ_2 such that for all $\epsilon \in (0, \epsilon_2]$ and for all $x \in K$ we have

$$P\left\{\sup_{t\in[0,1]}\left|\epsilon^{\frac{1}{2}}\int_0^t S(\epsilon(t-s))G(Z_x^\epsilon(s))\,dW(s)\right| \geq \frac{\delta}{c}\,,\;\; \sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}W(t)|_{U_1} \leq b\right\} \leq \exp\left(-\frac{\tilde{a}}{\epsilon}\right).$$

Thus for all $x \in K$ and for all $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2]$ we have

$$P\left\{\sup_{t\in[0,1]}|Z_x^\epsilon(t) - z_x^\phi(t)| \geq \delta\,,\;\; \sup_{t\in[0,1]}|\epsilon^{\frac{1}{2}}W(t)|_{U_1} \leq b\right\} \leq \exp\left(-\frac{\tilde{a}}{\epsilon}\right). \tag{3.68}$$

Now from inequalities (3.64) and (3.67) and (3.68) we have for all $x \in K$ and for all $\epsilon \in (0, \epsilon_1 \wedge \epsilon_2]$:

$$P(\mathcal{D}(\epsilon, x, b) \cap \mathcal{M}(\epsilon, \lambda)) \leq \exp\left(\frac{\lambda + \frac{1}{2}\int_0^1 |\phi(s)|_U^2\,ds - \tilde{a}}{\epsilon}\right). \tag{3.69}$$

By firstly choosing λ such that $\frac{-\lambda^2}{4\int_0^1 |\phi(s)|_U^2\,ds} + \ln 3 < -a$ and then choosing $\tilde{a}$ such that $\lambda + \frac{1}{2}\int_0^1 |\phi(s)|_U^2\,ds - \tilde{a} < -a$, we see, on combining inequalities (3.62), (3.63) and (3.69), that there exist $b > 0$ and $\epsilon_0 > 0$ such that for all $x \in K$ and for all $\epsilon \in (0, \epsilon_0]$

$$P(\mathcal{D}(\epsilon, x, b)) \leq \exp\left(-\frac{a}{\epsilon}\right).$$

This completes the proof of Proposition 3.11.

3.7 An example

In this section we consider an example where we can apply our large deviation principle. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$ with a C^∞ boundary $\partial\mathcal{O}$ (this means that for each $x \in \partial\mathcal{O}$ there is a positive number r and a real-valued C^∞ function ϕ defined on some open subset of $\mathbb{R}^{n-1}$ such that for some $i \in \{1, \ldots, n\}$ we have $y_i = \phi(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n)$ for all $(y_1, \ldots, y_n) \in B_{\mathbb{R}^n}(x, r) \cap \partial\mathcal{O}$).
Let $H = U = L^2(\mathcal{O})$. Let $A : D(A) = W^{2,2}(\mathcal{O}) \cap W_0^{1,2}(\mathcal{O}) \to H$ be defined by

$$Au := \sum_{|\alpha|\leq 2} a_\alpha \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} u\,, \quad \text{for all } u \in D(A);$$

in this definition $\alpha = (\alpha_1, \ldots, \alpha_n) \in \{0, 1, 2\}^n$ and $|\alpha| := \sum_{j=1}^n \alpha_j$ and the functions a_α

are in $C^\infty(\overline{\mathcal{O}})$ and there is a positive number θ such that

$$\sum_{|\alpha|=2} a_\alpha(x)\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n} \geq \theta\sum_{j=1}^{n}\xi_j^2 \quad \forall x\in\overline{\mathcal{O}} \text{ and } \forall(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n.$$

By [24, Theorem 2.7 in chapter 7], A generates an analytic semigroup $(S(t))_{t\geq 0}$ of operators on H. Thus condition (A2) is satisfied.

We have in mind the physical system of a lump of material occupying $\mathcal{O}\subset\mathbb{R}^3$ and containing a reactive component which undergoes an exothermic reaction. Our condition that $F:([0,1]\times H,\mathcal{B}_{[0,1]}\otimes\mathcal{B}_H)\to(H,\mathcal{B}_H)$ be Lipschitz continuous in H uniformly in time interval $[0,1]$ is rather restrictive. Differential equations describing energy conservation (and mass conservation) often involve a generation or consumption term which is a polynomial function of degree greater than one. However if $f:\mathbb{R}\to\mathbb{R}$ is a fixed Lipschitz continuous function then

$$F(t,x):=(\xi\in\mathcal{O}\mapsto f(x(\xi)))\ ,\quad x\in H,$$

is Lipschitz continuous in H uniformly in time. To be specific, we define

$$f(r):=\begin{cases}\beta\mathrm{e}^{\frac{-\gamma}{r+\delta}}\ , & r>-\delta\\ 0\ , & r\leq-\delta\,,\end{cases} \tag{3.70}$$

where β, γ and δ are positive real numbers; the function f in equation (3.70) is an Arrhenius function [23, page 212]. We think of H as the space of temperature functions on $\mathcal{O}$ while F represents the heat generation from the chemical reaction. The deterministic initial value problem in H:

$$\begin{aligned}\frac{dx}{dt} &= Ax(t)+F(t,x(t))\ ,\quad t>0,\\ x(0) &= x_0\in D(A),\end{aligned}$$

models the temperature evolution of the lump of material immersed in a constant temperature bath. We are interested in the short time asymptotics of the solution of the related stochastic differential equation.

Our choice of diffusion function G is very important because G determines the rate function in equation (3.13). Unfortunately, justifying a choice of G on physical grounds is hard. Let Q be a positive definite, symmetric, trace class operator on H. Let $g:\mathbb{R}\to\mathbb{R}$ be a fixed Lipschitz continuous function and let c be a fixed function in H. We define

$G_1 : H \to L(H, H)$ by

$$G_1(u)v := g\left(\int_{\mathcal{O}} c(\xi)u(\xi)\,d\xi\right) v\ , \quad \forall u, v \in H.$$

Then the diffusion function

$$G(u) := G_1(u)\, Q^{\frac{1}{2}} \quad \forall u \in H$$

is Lipschitz continuous and satisfies (A1).

Alternatively, if H has an orthonormal basis (e_k) of eigenvectors of Q such that

$$\sup_{k\in\mathbb{N}} \sup_{\xi\in\mathcal{O}} |e_k(\xi)| < \infty$$

and $g : \mathbb{R} \to \mathbb{R}$ is a bounded and Lipschitz continuous function then we define $G_2 : H \to L(H, H)$ by

$$(G_2(u)v)(\xi) := g(u(\xi))v(\xi) \quad \forall \xi \in \mathcal{O} \text{ and } \forall u, v \in H;$$

the function

$$u \in H \mapsto G_2(u)Q^{\frac{1}{2}} \in L_2(H, H)$$

is another diffusion function we can use since it is Lipschitz continuous and satisfies (A1).

Let $(W(t))_{t\geq 0}$ be a $(\mathcal{F}_t)$-Wiener process with values in some Hilbert space U_1 such that $\mathcal{L}(W(1))$ has reproducing kernel Hilbert space H. The continuous mild solution $(X_x(t))_{t\in[0,1]}$ of the stochastic equation

$$\left.\begin{array}{rcl} dX(t) & = & (AX(t) + F(t, X(t)))\,dt + G(X(t))\,dW(t) \\ X(0) & = & x \in H \end{array}\right\} \tag{3.71}$$

may be loosely interpreted as the evolving temperature function in the lump of material when a source of noise is present. Recall that in equation (3.9) we set

$$\mu_x^\epsilon := \mathcal{L}(\omega \in \Omega \mapsto (t \in [0,1] \mapsto X_x(\epsilon t)(\omega))) \quad \text{for each } \epsilon \in (0,1].$$

Corollary 3.4 tells us that for each open subset G of $C([0,1]; H)$

$$\lim_{r\to 0} \inf_{\epsilon<r} \epsilon \log \mu_x^\epsilon(G) \geq -\inf_{u\in G} \mathcal{I}_x(u)$$

and for each closed subset F of $C([0,1];H)$

$$\lim_{r\to 0}\sup_{\epsilon<r}\epsilon\log\mu_x^\epsilon(F)\leq -\inf_{u\in F}\mathcal{I}_x(u),$$

where

$$\mathcal{I}_x(u)=\frac{1}{2}\inf\left\{\int_0^1|\phi(s)|^2\,ds\ :\phi\in L^2([0,1];H)\text{ and }u(t)=x+\int_0^t G(u(s))\phi(s)\,ds\ \ \forall t\in[0,1]\right\}.$$

3.8 Appendix

Let Hilbert spaces H, U and U_1 be as defined in Section 3.1.

Let $T\in(0,\infty)$. Let $(\Omega,\mathcal{F},P)$ be a probability space. Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration of sub σ-algebras of $\mathcal{F}$ such that all sets in $\mathcal{F}$ of P measure zero are in $\mathcal{F}_0$ and let $(W(t):(\Omega,\mathcal{F}_t,P)\to(U_1,\mathcal{B}_{U_1}))_{t\geq 0}$ be a U_1-valued $(\mathcal{F}_t)$-Wiener process such that $\mathcal{L}(W(1))$ has reproducing kernel Hilbert space U. Let $(S(t))_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on H. Let the function $F:([0,T]\times H,\mathcal{B}_{[0,T]}\otimes\mathcal{B}_H)\to(H,\mathcal{B}_H)$ be measurable and suppose there is a function $\theta\in L^2([0,T];\mathbb{R})$ such that

$$|F(t,x)-F(t,y)|\ \leq\ \theta(t)|x-y|\quad\forall t\in[0,T]\text{ and }\forall x,y\in H\text{ and}\tag{3.72}$$

$$|F(t,x)|\ \leq\ \theta(t)(1+|x|)\quad\forall t\in[0,T]\text{ and }\forall x\in H.\tag{3.73}$$

Let $G:H\to L_2(U,H)$ be Lipschitz continuous. Let $x\in H$.

Theorem 3.18 (Existence, uniqueness and continuity of solutions) *There exists a $(\mathcal{F}_t)$-predictable process $(X(t))_{t\in[0,T]}$, unique up to equivalence among processes satisfying*

$$P\{\int_0^T|X(t)|^2\,dt<\infty\}=1,$$

such that

$$X(t)=S(t)x+\int_0^t S(t-s)F(s,X(s))\,ds+\int_0^t S(t-s)G(X(s))\,dW(s)\quad P\ a.e.$$

for each $t\in[0,T]$. Moreover it has a continuous version and $\sup_{t\in[0,T]}E[|X(t)|^p]<\infty$ for each $p\in[2,\infty)$.

The proof of this theorem is omitted as it is almost identical to the proof of [10, Theorem 7.4].

Let P' be another probability measure on the σ-algebra $\mathcal{F}$ of subsets of Ω. Let $(\mathcal{G}_t)_{t\geq 0}$ be a filtration of sub σ-algebras of $\mathcal{F}$ such that all sets in $\mathcal{F}$ of P' measure zero are in $\mathcal{G}_0$. Let $(V(t) : (\Omega, \mathcal{G}_t, P') \to (U_1, \mathcal{B}_{U_1}))_{t\geq 0}$ be a U_1-valued $(\mathcal{G}_t)$-Wiener process such that $\mathcal{L}(V(1)) = \mathcal{L}(W(1))$.

Suppose that $(X(t))_{t\in[0,T]}$ is the continuous $(\mathcal{F}_t)$-predictable process in Theorem 3.18 and suppose that $(Y(t))_{t\in[0,T]}$ is the continuous $(\mathcal{G}_t)$-predictable process satisfying

$$Y(t) = S(t)x + \int_0^t S(t-s)F(s,Y(s))\,ds + \int_0^t S(t-s)G(Y(s))\,dV(s) \quad P' \text{ a.e.}$$

for each $t \in [0,T]$. Let $H \oplus U_1$ denote the Hilbert space $H \times U_1$ with componentwise addition and scalar multiplication and inner product

$$\langle (x_1, y_1), (x_2, y_2)\rangle_{H\oplus U_1} := \langle x_1, x_2\rangle + \langle y_1, y_2\rangle_{U_1} \quad \forall x_1, x_2 \in H \text{ and } \forall y_1, y_2 \in U_1.$$

We remark that the norm topology in $H \oplus U_1$ is the same as the product topology on $H \times U_1$. The trajectory-valued random variable $(X,W) : (\Omega, \mathcal{F}, P) \to C([0,T]; H \oplus U_1)$ is defined by

$$(X,W)(\omega) := t \in [0,T] \mapsto (X(t)(\omega), W(t)(\omega)) \quad \forall \omega \in \Omega \tag{3.74}$$

and $(Y,V) : (\Omega, \mathcal{F}, P') \to C([0,T]; H \oplus U_1)$ is defined analogously.

Proposition 3.19 *The trajectory-valued random variables (X,W) and (Y,V) have the same distribution.*

Proof. It is well known that the Borel σ-algebra of $C([0,T]; H \oplus U_1)$ is generated by the family of all finite linear combinations of continuous linear functionals on $C([0,T]; H\oplus U_1)$ of the form $\delta_t \otimes (u,v)$ where $t \in [0,T]$ and $(u,v) \in H \oplus U_1$ and

$$(\delta_t \otimes (u,v))(f,g) := \langle u, f(t)\rangle + \langle v, g(t)\rangle_{U_1} \quad \forall f \in C([0,T]; H) \text{ and } \forall g \in C([0,1]; U_1).$$

Thus we can conclude that $\mathcal{L}(X,W) = \mathcal{L}(Y,V)$ if

$$\begin{aligned} &\mathcal{L}\left(\langle u_1, X(t_1)\rangle + \langle v_1, W(t_1)\rangle_{U_1} + \cdots + \langle u_n, X(t_n)\rangle + \langle v_n, W(t_n)\rangle_{U_1}\right) = \\ &\qquad \mathcal{L}\left(\langle u_1, Y(t_1)\rangle + \langle v_1, V(t_1)\rangle_{U_1} + \cdots + \langle u_n, Y(t_n)\rangle + \langle v_n, V(t_n)\rangle_{U_1}\right) \end{aligned} \tag{3.75}$$

for arbitrary $n \in \mathbb{N}$ and $0 \leq t_1 < \cdots < t_n \leq T$ and $u_1, \ldots, u_n \in H$ and $v_1, \ldots, v_n \in U_1$.

Let $p \in (2, \infty)$. Define $\mathcal{H}_p((\mathcal{F}_t), P)$ to be the vector space of all processes $(U(t) : (\Omega, \mathcal{F}_t) \to (H, \mathcal{B}_H))_{t\in[0,T]}$ such that $\sup_{t\in[0,T]} \int_\Omega |U(t)|^p\,dP < \infty$ and $(U(t))_{t\in[0,T]}$ has a

$(\mathcal{F}_t)$-predictable version, with norm $\sup_{t\in[0,T]}(\int_\Omega |U(t)|^p\,dP)^{\frac{1}{p}}$; we identify processes that are equal P a.e. at each time $t \in [0,T]$. $\mathcal{H}_p((\mathcal{F}_t),P)$ is a Banach space. Define the Banach space $\mathcal{H}_p((\mathcal{G}_t),P')$ in the same way but with $\mathcal{G}_t$ taking the place of $\mathcal{F}_t$ and P' taking the place of P. Following the proof of [10, Theorem 7.4], one can show that the map $\mathcal{K} : \mathcal{H}_p((\mathcal{F}_t),P) \to \mathcal{H}_p((\mathcal{F}_t),P)$ defined by

$$(\mathcal{K}U)(t) := S(t)x + \int_0^t S(t-s)F(s,U(s))\,ds + \int_0^t S(t-s)G(U(s))\,dW(s) \quad \forall t \in [0,T] \tag{3.76}$$

for each $U \in \mathcal{H}_p((\mathcal{F}_t),P)$, is a contraction map provided that T is small enough. To simplify matters and avoid having to partition $[0,T]$ into shorter subintervals, we assume that $\mathcal{K}$ is a contraction with the given T. By [10, Proposition 7.3] the process $\mathcal{K}U$ defined in equation (3.76) has a continuous version and if $\sup_{t\in[0,T]}|U(t)| \in L^p(\Omega,\mathcal{F},P;\mathbb{R})$ then also $\sup_{t\in[0,T]}|(\mathcal{K}U)(t)| \in L^p(\Omega,\mathcal{F},P;\mathbb{R})$. Similar statements hold for the map $\mathcal{K}' : \mathcal{H}_p((\mathcal{G}_t),P') \to \mathcal{H}_p((\mathcal{G}_t),P')$ defined by

$$(\mathcal{K}'U)(t) := S(t)x + \int_0^t S(t-s)F(s,U(s))\,ds + \int_0^t S(t-s)G(U(s))\,dV(s) \quad \forall t \in [0,T] \tag{3.77}$$

for each $U \in \mathcal{H}_p((\mathcal{G}_t),P')$. Define $X_0(t) = x$ for each $t \in [0,T]$ and $Y_0(t) = x$ for each $t \in [0,T]$. Clearly for arbitrary $n \in \mathbb{N}$ and $u_1,\dots,u_n \in H$ and $v_1,\dots,v_n \in U_1$ and $0 \le t_1 < \dots < t_n \le T$ we have

$$\begin{aligned}\mathcal{L}(\langle u_1, X_0(t_1)\rangle + \langle v_1, W(t_1)\rangle_{U_1} + \dots + \langle u_n, X_0(t_n)\rangle + \langle v_n, W(t_n)\rangle_{U_1}) &= \\ \mathcal{L}(\langle u_1, Y_0(t_1)\rangle + \langle v_1, V(t_1)\rangle_{U_1} + \dots + \langle u_n, Y_0(t_n)\rangle + \langle v_n, V(t_n)\rangle_{U_1}).\end{aligned}$$

Thus

$$\mathcal{L}(X_0,W) = \mathcal{L}(Y_0,V). \tag{3.78}$$

Suppose for some $m \in \mathbb{N}$ we have continuous processes $(X_{m-1}(t))_{t\in[0,T]} \in \mathcal{H}_p((\mathcal{F}_t),P)$ and $(Y_{m-1}(t))_{t\in[0,T]} \in \mathcal{H}_p((\mathcal{G}_t),P')$ such that

$$\sup_{t\in[0,T]}|X_{m-1}(t)| \in L^p(\Omega,\mathcal{F},P;\mathbb{R}), \tag{3.79}$$

$$\sup_{t\in[0,T]}|Y_{m-1}(t)| \in L^p(\Omega,\mathcal{F},P';\mathbb{R}) \tag{3.80}$$

$$\text{and} \quad \mathcal{L}(X_{m-1},W) = \mathcal{L}(Y_{m-1},V). \tag{3.81}$$

Define

$$
\begin{aligned}
X_m(t) &:= (\mathcal{K}X_{m-1})(t) \quad \forall t \in [0,T] \text{ and} \\
Y_m(t) &:= (\mathcal{K}'Y_{m-1})(t) \quad \forall t \in [0,T].
\end{aligned}
$$

Given $n \in \mathbb{N}$ and $u_1, \dots, u_n \in H$ and $v_1, \dots, v_n \in U_1$ and $0 \le t_1 < \dots < t_n \le T$ we will show that

$$
\begin{aligned}
&\mathcal{L}(\langle u_1, X_m(t_1)\rangle + \langle v_1, W(t_1)\rangle_{U_1} + \dots + \langle u_n, X_m(t_n)\rangle + \langle v_n, W(t_n)\rangle_{U_1}) = \\
&\quad \mathcal{L}(\langle u_1, Y_m(t_1)\rangle + \langle v_1, V(t_1)\rangle_{U_1} + \dots + \langle u_n, Y_m(t_n)\rangle + \langle v_n, V(t_n)\rangle_{U_1}). \qquad (3.82)
\end{aligned}
$$

For this purpose we introduce some processes which approximate $(X_m(t))_{t\in[0,T]}$ and $(Y_m(t))_{t\in[0,T]}$ and are simpler in form. Let $\{T_jJ : j \in \mathbb{N}\}$ be an orthonormal basis of $L_2(U,H)$ where $T_j \in L(U_1,H)$ for each $j \in \mathbb{N}$. Then for each $R \in L_2(U,H)$ we have $\sum_{j=1}^{i} \langle R, T_jJ\rangle_{L_2(U,H)} T_jJ$ converges to R in $L_2(U,H)$ as i goes to ∞. For each $N \in \mathbb{N}$ and $i \in \mathbb{N}$ define the continuous processes

$$
\begin{aligned}
X_m^{(N,i)}(t) &:= S(t)x + \int_0^t S(t-s)F(s, X_{m-1}(s))\, ds + \\
&\quad \sum_{k=0}^{2^N-1} \sum_{j=1}^{i} \langle S(t(1-\tfrac{k}{2^N}))G(X_{m-1}(t\tfrac{k}{2^N})), T_jJ\rangle_{L_2(U,H)} T_j(W(t\tfrac{k+1}{2^N}) - W(t\tfrac{k}{2^N}))
\end{aligned}
$$

and

$$
\begin{aligned}
Y_m^{(N,i)}(t) &:= S(t)x + \int_0^t S(t-s)F(s, Y_{m-1}(s))\, ds + \\
&\quad \sum_{k=0}^{2^N-1} \sum_{j=1}^{i} \langle S(t(1-\tfrac{k}{2^N}))G(Y_{m-1}(t\tfrac{k}{2^N})), T_jJ\rangle_{L_2(U,H)} T_j(V(t\tfrac{k+1}{2^N}) - V(t\tfrac{k}{2^N}))
\end{aligned}
$$

for all $t \in [0,T]$. It is not too difficult to see that (3.81) implies $\mathcal{L}(X_m^{(N,i)}, W) = \mathcal{L}(Y_m^{(N,i)}, V)$ for arbitrary natural numbers N and i. Thus

$$
\begin{aligned}
&\mathcal{L}(\langle u_1, X_m^{(N,i)}(t_1)\rangle + \langle v_1, W(t_1)\rangle_{U_1} + \dots + \langle u_n, X_m^{(N,i)}(t_n)\rangle + \langle v_n, W(t_n)\rangle_{U_1}) = \\
&\quad \mathcal{L}(\langle u_1, Y_m^{(N,i)}(t_1)\rangle + \langle v_1, V(t_1)\rangle_{U_1} + \dots + \langle u_n, Y_m^{(N,i)}(t_n)\rangle + \langle v_n, V(t_n)\rangle_{U_1}) \qquad (3.83)
\end{aligned}
$$

for arbitrary natural numbers N and i.

Since relations (3.79) and (3.80) hold, for each $t \in [0,T]$, taking N and i sufficiently large makes $X_m^{(N,i)}(t)$ approach $X_m(t)$ as close as we please in $L^2(\Omega, \mathcal{F}, P; H)$ and also makes $Y_m^{(N,i)}(t)$ approach $Y_m(t)$ as close as we please in $L^2(\Omega, \mathcal{F}, P'; H)$. Thus from equation (3.83) there follows equation (3.82) and $\mathcal{L}(X_m, W) = \mathcal{L}(Y_m, V)$. By equation (3.78) and the induction principle

$$\mathcal{L}(X_m, W) = \mathcal{L}(Y_m, V) \quad \text{for each } m \in \{0, 1, 2, \ldots\}.$$

Since $\sup_{t\in[0,T]} \int_\Omega |X_m(t) - X(t)|^p \, dP \to 0$ and $\sup_{t\in[0,T]} \int_\Omega |Y_m(t) - Y(t)|^p \, dP' \to 0$ as $m \to \infty$, equation (3.75) follows from equation (3.82). This completes the proof of Proposition 3.19.

We conclude this appendix with a useful result which is not entirely obvious. It shows the relationship between Itô integrals with respect to two Wiener processes defined on related probability spaces. Let $\phi \in L^2([0,1]; U)$ and let $\epsilon \in (0,1]$. Define the probability measure P^ϵ on $(\Omega, \mathcal{F})$ by

$$dP^\epsilon(\omega) := \exp\left(\epsilon^{-\frac{1}{2}} \int_0^1 < \phi(s), \cdot >_U \, dW(s)(\omega) - \frac{1}{2\epsilon} \int_0^1 |\phi(s)|_U^2 \, ds\right) dP(\omega).$$

By [10, Theorem 10.14] the process

$$W^\epsilon(t) := W(t) - \epsilon^{-\frac{1}{2}} J \int_0^t \phi(s) \, ds \quad \forall t \in [0,1]$$

is a $(\mathcal{F}_t)$-Wiener process on the probability space $(\Omega, \mathcal{F}, P^\epsilon)$ and $P^\epsilon(W^\epsilon(1))^{-1} = P(W(1))^{-1}$. In the following lemma $\mathcal{P}_1$ denotes the $(\mathcal{F}_t)$-predictable σ-algebra of subsets of $[0,1] \times \Omega$.

Lemma 3.20 *Let $\Phi : ([0,1] \times \Omega, \mathcal{P}_1) \to (L_2(U,H), \mathcal{B}_{L_2(U,H)})$ be a measurable function such that for some positive real number C*

$$\int_0^1 \|\Phi(s,\omega)\|_{L_2(U,H)}^2 \, ds \le C \quad \textit{for } P \textit{ a.e. } \omega \in \Omega.$$

Then

$$\int_0^1 \Phi(s) \, dW^\epsilon(s) = \int_0^1 \Phi(s) \, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^1 \Phi(s)\phi(s) \, ds \quad P \textit{ a.e.}.$$

Proof. Suppose firstly that $\sup_{(t,\omega)\in[0,1]\times\Omega} \|\Phi(t,\omega)\|_{L_2(U,H)} = R < \infty$. Then we can find a sequence of elementary processes $(\Phi_n : ([0,1] \times \Omega, \mathcal{P}_1) \to L_2(U,H))$ such that

$\sup_{(t,\omega)\in[0,1]\times\Omega} \|\Phi_n(t,\omega)\|_{L_2(U,H)} \leq 2R$ for all $n \in \mathbb{N}$ and

$$\int_\Omega \int_0^1 \|\Phi(s) - \Phi_n(s)\|^2_{L_2(U,H)}\, ds\, dP \to 0 \quad \text{as } n \to \infty.$$

We can also assume, by taking a subsequence if necessary, that

$$\int_0^1 \|\Phi(s,\omega) - \Phi_n(s,\omega)\|^2_{L_2(U,H)}\, ds \to 0 \quad \text{as } n \to \infty \text{ for } P \text{ a.e. } \omega \in \Omega.$$

By definition of the Itô integral of an elementary process, for each $n \in \mathbb{N}$ we have

$$\int_0^1 \Phi_n(s)\, dW^\epsilon(s) = \int_0^1 \Phi_n(s)\, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^1 \Phi_n(s)\phi(s)\, ds.$$

For an appropriate subsequence (n_k), taking limits as k goes to infinity on both sides of this equation yields

$$\int_0^1 \Phi(s)\, dW^\epsilon(s) = \int_0^1 \Phi(s)\, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^1 \Phi(s)\phi(s)\, ds \quad P \text{ a.e.}.$$

Now suppose that Φ is not necessarily bounded but

$$\int_0^1 \|\Phi(s,\omega)\|^2_{L_2(U,H)}\, ds \leq C \quad \text{for } P \text{ a.e. } \omega \in \Omega,$$

where C is a positive real number. For each natural number N define $\rho_N : L_2(U,H) \to L_2(U,H)$ by

$$\rho_N(S) := \begin{cases} S & \text{if } \|S\|_{L_2(U,H)} \leq N, \\ \frac{N}{\|S\|_{L_2(U,H)}} S & \text{if } \|S\|_{L_2(U,H)} > N. \end{cases}$$

For each $N \in \mathbb{N}$, since $\rho_N(\Phi)$ is bounded we have

$$\int_0^1 \rho_N(\Phi(s))\, dW^\epsilon(s) = \int_0^1 \rho_N(\Phi(s))\, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^1 (\rho_N(\Phi(s)))\phi(s)\, ds \quad P \text{ a.e.}.$$

For an appropriate subsequence (N_k), taking limits as k goes to infinity on both sides of this equation yields the desired result:

$$\int_0^1 \Phi(s)\, dW^\epsilon(s) = \int_0^1 \Phi(s)\, dW(s) - \epsilon^{-\frac{1}{2}} \int_0^1 \Phi(s)\phi(s)\, ds \quad P \text{ a.e.}.$$

Chapter 4

Small time asymptotics when there is a nonlinear dissipative drift term and additive noise

4.1 Introduction

In this chapter we describe the exponential small time asymptotics of the mild solution of an equation having a nonlinear, dissipative drift function, $F : E \to E$, defined on a separable Banach space E. We only consider additive Wiener process noise to keep things simple. Our main results are Propositions 4.2 and 4.11 and Corollary 4.12. We mentioned in section 3.7 that differential equations for the evolution of temperature or concentration in a bounded domain $\mathcal{O}$ of $\mathbb{R}^n$ often contain polynomial functions of the dependent variable. If such a polynomial function $b : \mathbb{R} \to \mathbb{R}$ has degree greater than one, then for general $u \in L^2(\mathcal{O})$ the function $\xi \in \mathcal{O} \to b(u(\xi))$ need not belong to $L^2(\mathcal{O})$. However if E is a Banach space of continuous functions on $\mathcal{O}$ with the supremum norm then the function:

$$(F(u))(\xi) := b(u(\xi)) \quad \forall u \in E \text{ and } \forall \xi \in \mathcal{O},$$

may be a well defined mapping of E into E; if in addition b is a decreasing function then $F : E \to E$ is dissipative.

We now make definitions and summarize the contents of this chapter more precisely. Let $(H, |\cdot|)$ be a separable Hilbert space and let $A : D(A) \subset H \to H$ be the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on H. Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(W(t) : (\Omega, \mathcal{F}, P) \to H)_{t\geq 0}$ be an H-valued

Wiener process. The distribution of $W(1)$ on H is denoted by ν, its covariance operator by Q and its reproducing kernel Hilbert space by $(H_\nu, |\cdot|_{H_\nu})$. Let

$$k : H_\nu \hookrightarrow H$$

be the embedding of H_ν into H.

Our aim is to get a large deviation principle under the conditions of Da Prato's and Zabczyk's existence and uniqueness theorem [10, Theorem 7.13] for the case of nonlinear dissipative drift defined in a Banach space embedded in H. We now state those conditions. Let $(E, \|\cdot\|)$ be a separable Banach space, continuously and injectively embedded as a dense subset in H via the map

$$j : E \hookrightarrow H.$$

Sometimes to simplify notation we will omit the embedding , for example we may write E when we mean the subset $j(E)$ of H. Suppose that $S(t)(E) \subset E$ for each $t \geq 0$ and the linear operators on E defined by

$$S_E(t)\, x := S(t)\, x \quad \forall x \in E \text{ and } \forall t \geq 0$$

form a strongly continuous semigroup in $L(E, E)$. In addition, suppose that there is a positive real number θ such that

$$\|S_E(t)\|_{L(E,E)} \leq \mathrm{e}^{\theta t} \quad \text{for all } t \geq 0.$$

Let

$$F : E \to E$$

be a dissipative function which is uniformly continuous on bounded subsets of E; saying that F is dissipative means that

$$\|x - y - \lambda(F(x) - F(y))\| \geq \|x - y\| \quad \forall x, y \in E \text{ and } \forall \lambda > 0.$$

The following condition also comes from Da Prato's and Zabczyk's existence and uniqueness theorem.

(**B1**) *For each $\epsilon \in (0, 1]$ there is a version of the stochastic convolution process:*

$$W_{\epsilon A}(t) := \int_0^t S(\epsilon(t - s))k \, dW(s) \ , \quad t \in [0, 1], \tag{4.1}$$

whose trajectories are in $C([0,1];E)$*, the space of continuous functions mapping* $[0,1]$ *into* E*.*

Whenever (B1) holds, for each $\epsilon \in (0,1]$ we implicitly use the version of $(W_{\epsilon A}(t))_{t\in[0,1]}$ whose trajectories are in $C([0,1];E)$. We listed condition (B1) separately because we prove in Proposition 4.4 that this condition is implied by condition (B2) (see below).

Fix $\xi \in E$ throughout this chapter. By [10, Theorem 7.13], when (B1) holds, for each $\epsilon \in (0,1]$ there is a unique process $(X^\epsilon(t) : (\Omega, \mathcal{F}, P) \to E)_{t\in[0,1]}$ whose trajectories $t \to X^\epsilon(t)(\omega)$ are in $C([0,1];E)$ for all $\omega \in \Omega$ and which satisfies the equation

$$X^\epsilon(t) = S_E(\epsilon t)\xi + \epsilon^{\frac{1}{2}} W_{\epsilon A}(t) + \epsilon \int_0^t S_E(\epsilon(t-s))F(X^\epsilon(s))\, ds \quad \forall t \in [0,1]. \tag{4.2}$$

When ϵ is 1 we write $W_A(t) := W_{1A}(t)$ and $X(t) := X^1(t)$ for all $t \in [0,1]$.

In section 4.2 we assume that (B1) holds and we show in Proposition 4.1 that the problem of finding the exponential small time asymptotics of $(X(t))_{t\in[0,1]}$ is solved if we can find a large deviation principle for the random variables $X^\epsilon : (\Omega, \mathcal{F}, P) \to C([0,1];E)$ defined by

$$X^\epsilon(\omega) := t \mapsto X^\epsilon(t)(\omega) \quad \forall \omega \in \Omega\,, \quad \epsilon \in (0,1].$$

For each $\epsilon \in (0,1]$ define the process $(Z^\epsilon(t) : (\Omega, \mathcal{F}, P) \to E)_{t\in[0,1]}$ by

$$Z^\epsilon(t) := S_E(\epsilon t)\xi + \epsilon^{\frac{1}{2}} W_{\epsilon A}(t) \quad \forall t \in [0,1]. \tag{4.3}$$

In Proposition 4.2 we show that if the random variables $Z^\epsilon : (\Omega, \mathcal{F}, P) \to C([0,1];E)$ defined by

$$Z^\epsilon(\omega) := t \mapsto Z^\epsilon(t)(\omega) \quad \forall \omega \in \Omega\,, \quad \epsilon \in (0,1],$$

satisfy a large deviation principle then the family $\{X^\epsilon : \epsilon \in (0,1]\}$ satisfies the same large deviation principle. Thus our focus becomes proving a large deviation principle for $\{\mathcal{L}(Z^\epsilon) : \epsilon \in (0,1]\}$.

If $\nu(E) = 1$ then H_ν is continuously embedded in E (see Lemma 4.3 and the paragraph following it) and we denote the embedding by

$$i : H_\nu \hookrightarrow E.$$

In section 4.3 we assume that the following condition holds.

(**B2**)

1. *We have* $\nu(E) = 1$.

2. *There is a number* $\alpha \in (0, \frac{1}{2})$ *and an operator* $G \in L(E^*, E)$*, which is the covariance operator of a symmetric Gaussian measure on* E*, such that*

$$\int_0^1 \sigma^{-2\alpha} |i^* S_E^*(\epsilon\sigma) l^*|_{H_\nu}^2 \, d\sigma \le {}_E\langle G l^*, l^* \rangle_{E^*} \quad \forall l^* \in E^* \text{ and } \forall \epsilon \in (0,1].$$

Remark If $\nu(E) = 1$ and there is a positive real number c such that

$$S(t) Q^{\frac{1}{2}}(\overline{B}_H(0,1)) \subset Q^{\frac{1}{2}}(\overline{B}_H(0,c)) \quad \forall t \in [0,1]$$

then we have

$$|i^* S_E^*(t) l^*|_{H_\nu} \le c |i^* l^*|_{H_\nu} \quad \forall l^* \in E^* \text{ and } \forall t \in [0,1].$$

In this case condition (B2)(2) holds with any $\alpha \in (0, \frac{1}{2})$ and $G = \frac{c^2}{1-2\alpha} i i^*$ (here, to simplify notation, we identify H_ν and H_ν^*).

In Proposition 4.4 we show, assuming (B2), that (B1) holds. In Proposition 4.11 we show, assuming (B2), that $\{Z^\epsilon : \epsilon \in (0,1]\}$ satisfies a large deviation principle.
In section 4.4 we present an example.

Our problem is different from the small noise asymptotics problem studied by Fantozzi [14] because in our problem the dependence of the stochastic convolution process $(W_{\epsilon A}(t))_{t\in[0,1]}$ on ϵ complicates matters. We introduce condition (B2) to ensure convergence of paths of $(W_{\epsilon A}(t))_{t\in[0,1]}$ in $C([0,1];E)$ to those of the Wiener process as ϵ goes to zero. Unlike condition (B1), condition (B2) is not needed in Da Prato's and Zabczyk's existence and uniqueness theorem and it would have been preferable to obtain results assuming just (B1) instead of (B2).

4.2 Reduction to the linear problem

Throughout this section we assume that condition (B1) holds.
Let $\epsilon \in (0,1]$. The processes $(Z^\epsilon(t))_{t\in[0,1]}$ and $(X^\epsilon(t))_{t\in[0,1]}$ are related by the equation:

$$X^\epsilon(t) = Z^\epsilon(t) + \epsilon \int_0^t S_E(\epsilon(t-s)) \, F(X^\epsilon(s)) \, ds \quad \forall t \in [0,1] \tag{4.4}$$

and have continuous trajectories in E. We can write equation (4.4) as

$$X^\epsilon(t) - Z^\epsilon(t) = \epsilon \int_0^t S_E(\epsilon(t-s))\, F(X^\epsilon(s) - Z^\epsilon(s) + Z^\epsilon(s))\, ds \quad \forall t \in [0,1]. \tag{4.5}$$

For any continuous function $z : [0,1] \to E$ and $\alpha \in (0,\infty)$, Da Prato and Zabczyk [10, inequality (7.46)] provide a bound for the unique continuous solution $v_\epsilon^\alpha : [0,1] \to E$ of the integral equation

$$v_\epsilon^\alpha(t) = \epsilon \int_0^t S_E(\epsilon(t-s))\, F_\alpha(v_\epsilon^\alpha(s) + z(s))\, ds \quad \forall t \in [0,1], \tag{4.6}$$

where $F_\alpha : E \to E$ is a Lipschitz continuous function defined by

$$F_\alpha(x) := F((I_E - \alpha\epsilon F)^{-1} x) \quad \forall x \in E$$

which approximates F (see [10, Proposition D.11]); we have

$$\|v_\epsilon^\alpha(t)\| \le \epsilon e^{\epsilon\omega t} \int_0^t e^{-\epsilon\omega s} \|F(z(s))\|\, ds \quad \forall t \in [0,1] \text{ and } \forall \alpha \in (0,\infty). \tag{4.7}$$

The proof of [10, Theorem 7.13] shows that as $\alpha \searrow 0$, v_ϵ^α converges uniformly to the unique continuous function v_ϵ which satisfies the equation

$$v_\epsilon(t) = \epsilon \int_0^t S_E(\epsilon(t-s))\, F(v_\epsilon(s) + z(s))\, ds \quad \forall t \in [0,1] \tag{4.8}$$

and, from equation (4.7),

$$\sup_{t\in[0,1]} \|v_\epsilon(t)\| \le \epsilon e^{\epsilon\omega} \sup_{t\in[0,1]} \|F(z(t))\|. \tag{4.9}$$

Proposition 4.1 *The random variables in $C([0,1];E)$*

$$t \in [0,1] \mapsto X(\epsilon t)(\omega)\ , \quad \omega \in \Omega$$

and

$$t \in [0,1] \mapsto X^\epsilon(t)(\omega)\ , \quad \omega \in \Omega$$

have the same distribution.

Proof. Let $\alpha \in (0, \infty)$. We claim that the map $T_\alpha : C([0,1];E) \to C([0,1];E)$ defined by

$$(T_\alpha f)(t) = \epsilon \int_0^t S_E(\epsilon(t-s))\, F_\alpha((T_\alpha f)(s) + f(s))\, ds \quad \forall t \in [0,1] \text{ and } \forall f \in C([0,1];E)$$

is continuous. For f and $g \in C([0,1];E)$ we have

$$\begin{aligned} \|(T_\alpha f)(t) - (T_\alpha g)(t)\| &= \epsilon \left\| \int_0^t S_E(\epsilon(t-s))[F_\alpha((T_\alpha f)(s) + f(s)) - F_\alpha((T_\alpha g)(s) + g(s))]\, ds \right\| \\ &\leq K_\alpha \epsilon \mathrm{e}^{\epsilon\omega} \left(\int_0^t \|(T_\alpha f)(s) - (T_\alpha g)(s)\|\, ds + \sup_{r \in [0,1]} \|f(r) - g(r)\| \right) \end{aligned}$$

for all $t \in [0,1]$, where K_α denotes the Lipschitz constant of F_α. Now Gronwall's inequality yields

$$\sup_{t \in [0,1]} \|(T_\alpha f)(t) - (T_\alpha g)(t)\| \leq K_\alpha \epsilon \mathrm{e}^{\epsilon\omega} \sup_{r \in [0,1]} \|f(r) - g(r)\| \mathrm{e}^{K_\alpha \epsilon \mathrm{e}^{\epsilon\omega}}.$$

Hence T_α is Lipschitz continuous. The map $T : C([0,1];E) \to C([0,1];E)$ defined by

$$(Tf)(t) = \epsilon \int_0^t S_E(\epsilon(t-s))\, F((Tf)(s) + f(s))\, ds \quad \forall t \in [0,1] \text{ and } \forall f \in C([0,1];E)$$

is the pointwise limit of continuous functions:

$$Tf = \lim_{\alpha \searrow 0} T_\alpha f \quad \forall f \in C([0,1];E)$$

and is thus Borel measurable. From equation (4.5) we have

$$(t \in [0,1] \mapsto X^\epsilon(t)(\cdot)) = T(t \in [0,1] \mapsto Z^\epsilon(t)(\cdot)) + (t \in [0,1] \mapsto Z^\epsilon(t)(\cdot)). \tag{4.10}$$

Recall our notation $X(s) := X^1(s)$ for all $s \in [0,1]$. We have

$$\begin{aligned} X(\epsilon t) &= S_E(\epsilon t)\xi + W_A(\epsilon t) + \int_0^{\epsilon t} S_E(\epsilon t - s) F(X(s))\, ds \quad \forall t \in [0,1] \\ &= S_E(\epsilon t)\xi + W_A(\epsilon t) + \epsilon \int_0^t S_E(\epsilon(t-u)) F(X(\epsilon u))\, du \quad \forall t \in [0,1]. \end{aligned}$$

Thus we have

$$(t \in [0,1] \mapsto X(\epsilon t)(\cdot)) = T(t \in [0,1] \mapsto S_E(\epsilon t)\xi + W_A(\epsilon t)(\cdot)) + (t \in [0,1] \mapsto S_E(\epsilon t)\xi + W_A(\epsilon t)(\cdot)). \tag{4.11}$$

Let

$$\tilde{j} : C([0,1];E) \hookrightarrow C([0,1];H)$$

be the continuous embedding of $C([0,1];E)$ into $C([0,1];H)$. As in section 3.1 of chapter 3, we have equality of the distributions of the $C([0,1];H)$-valued random variables

$$\tilde{j}(t \in [0,1] \mapsto Z^\epsilon(t)(\cdot)) \text{ and } \tilde{j}(t \in [0,1] \mapsto S_E(\epsilon t)\xi + W_A(\epsilon t)(\cdot)).$$

By [30, Theorem 1.1] we have $\{\tilde{j}(B) : B \in \mathcal{B}_{C([0,1];E)}\} \subset \mathcal{B}_{C([0,1];H)}$ and it follows that the $C([0,1];E)$-valued random variables

$$(t \in [0,1] \mapsto Z^\epsilon(t)(\cdot)) \text{ and } (t \in [0,1] \mapsto S_E(\epsilon t)\xi + W_A(\epsilon t)(\cdot))$$

also have the same distribution. Hence, by equations (4.10) and (4.11), the random variables $(t \in [0,1] \mapsto X^\epsilon(t)(\cdot))$ and $(t \in [0,1] \mapsto X(\epsilon t)(\cdot))$ have the same distribution. This completes the proof of Proposition 4.1.

By Proposition 4.1 the family of $C([0,1];E)$-valued random variables $\{t \in [0,1] \mapsto X(\epsilon t)(\cdot) : \epsilon \in (0,1]\}$ satisfies a large deviation principle if and only if it is satisfied by the family of random variables $\{X^\epsilon : \epsilon \in (0,1]\}$. The following proposition shows that our problem may now be reduced to finding a large deviation principle for the family of random variables $\{Z^\epsilon : \epsilon \in (0,1]\}$.

Proposition 4.2 *Suppose that the family of $C([0,1];E)$-valued random variables*

$$\{Z^\epsilon : Z^\epsilon(\omega) := t \mapsto Z^\epsilon(t)(\omega) \quad \forall \omega \in \Omega\}_{\epsilon \in (0,1]}$$

satisfies a large deviation principle. Then for any $\delta > 0$ we have

$$\lim_{\epsilon \longrightarrow 0} \epsilon \log P \left\{ \sup_{t \in [0,1]} \|X^\epsilon(t) - Z^\epsilon(t)\| \geq \delta \right\} = -\infty, \tag{4.12}$$

which implies that the family of random variables in $C([0,1];E)$

$$\{X^\epsilon : X^\epsilon(\omega) := t \mapsto X^\epsilon(t)(\omega) \quad \forall \omega \in \Omega\}_{\epsilon \in (0,1]}$$

satisfies the same large deviation principle as $\{Z^\epsilon\}_{\epsilon \in (0,1]}$ does.

Proof. By comparing equations (4.5) and (4.8) we see that inequality (4.9) yields the bound

$$\sup_{t\in[0,1]}\|X^\epsilon(t)-Z^\epsilon(t)\| \le \epsilon \mathrm{e}^{\omega} \sup_{t\in[0,1]}\|F(Z^\epsilon(t))\| \quad \forall \epsilon\in(0,1].$$

Fix $\delta>0$. Take arbitrary $r\in(0,\infty)$. We have

$$\begin{aligned}
P\left\{\sup_{t\in[0,1]}\|X^\epsilon(t)-Z^\epsilon(t)\|\ge\delta\right\} &\le P\left\{\sup_{t\in[0,1]}\|F(Z^\epsilon(t))\|\ge\frac{\delta}{\epsilon\mathrm{e}^{\omega}}\right\}\\
&\le P\left\{\sup_{t\in[0,1]}\|F(Z^\epsilon(t))\|\ge\frac{\delta}{\epsilon\mathrm{e}^{\omega}}, \quad \sup_{t\in[0,1]}\|Z^\epsilon(t)\|<r\right\}\\
&\quad + P\left\{\sup_{t\in[0,1]}\|Z^\epsilon(t)\|\ge r\right\}.
\end{aligned}$$

For all sufficiently small $\epsilon>0$ the first probability on the right hand side vanishes because F is bounded on bounded subsets of E. Thus there is an $\epsilon_r>0$ such that

$$P\left\{\sup_{t\in[0,1]}\|X^\epsilon(t)-Z^\epsilon(t)\|\ge\delta\right\}\le P\left\{\sup_{t\in[0,1]}\|Z^\epsilon(t)\|\ge r\right\} \quad \forall\epsilon<\epsilon_r.$$

Let $\mathcal{I}:C([0,1];E)\to[0,\infty]$ be the rate function of the large deviation principle of $\{Z^\epsilon\}_{\epsilon\in(0,1]}$. Let $R\in(0,\infty)$. The set $\{\mathcal{I}\le R\}$ is compact and we now choose $r\in(0,\infty)$ such that

$$\{\mathcal{I}\le R\}\subset\{x\in C([0,1];E):\sup_{t\in[0,1]}\|x(t)\|<r\}.$$

By the Freidlin-Wentzell formulation (see [10, Proposition 12.2]) of the upper bound of the large deviation principle of $\{Z^\epsilon\}_{\epsilon\in(0,1]}$, given $\gamma\in(0,\infty)$ there is a number $\epsilon_{R,r,\gamma}>0$ such that

$$P\left\{\sup_{t\in[0,1]}\|Z^\epsilon(t)\|\ge r\right\}\le \mathrm{e}^{\frac{-R+\gamma}{\epsilon}} \quad \text{for all } \epsilon<\epsilon_{R,r,\gamma}.$$

Hence

$$P\left\{\sup_{t\in[0,1]}\|X^\epsilon(t)-Z^\epsilon(t)\|\ge\delta\right\}\le \mathrm{e}^{\frac{-R+\gamma}{\epsilon}} \quad \text{for all } \epsilon<(\epsilon_r\wedge\epsilon_{R,r,\gamma}).$$

Since $R\in(0,\infty)$ and $\gamma\in(0,\infty)$ are arbitrary we have

$$\lim_{\epsilon\to0}\epsilon\log P\left\{\sup_{t\in[0,1]}\|X^\epsilon(t)-Z^\epsilon(t)\|\ge\delta\right\}=-\infty.$$

It follows from [18, Lemma 27.13] that $\{X^\epsilon\}_{\epsilon\in(0,1]}$ satisfies the same large deviation principle as $\{Z^\epsilon\}_{\epsilon\in(0,1]}$. This completes the proof of Proposition 4.2.

4.3 Large deviation principle

In this section we show, assuming (B2), that (B1) holds and the family of $C([0,1];E)$-valued random variables $\{Z^\epsilon : \epsilon \in (0,1]\}$ satisfies a large deviation principle. We need a basic lemma.

Lemma 4.3 *Let U and V be separable Banach spaces such that U is continuously and injectively embedded in V via the map $j : U \hookrightarrow V$. If μ is a symmetric Gaussian measure on the Borel σ-algebra of V, $\mathcal{B}_V$, and $\mu(j(U)) = 1$ then there is a symmetric Gaussian measure μ_0 on the Borel σ-algebra of U, $\mathcal{B}_U$, such that $\mu = \mu_0 j^{-1}$.*

Proof. We have $\{j(A) : A \in \mathcal{B}_U\} \subset \mathcal{B}_V$ (see [30, Theorem 1.1 in chapter 1]) and $\{j^{-1}(B) : B \in \mathcal{B}_V\} \subset \mathcal{B}_U$ since j is measurable. Define a probability measure μ_0 on $\mathcal{B}_U$ by

$$\mu_0(A) := \mu(j(A)) \quad \forall A \in \mathcal{B}_U.$$

By [29, Theorem 4.12 Corollary (c)], $j^*(V^*)$ is a weak*-dense subspace of U^*. Since for each $h^* \in V^*$ we have

$$\begin{aligned}\mu_0(j^*h^*)^{-1} &= \mu_0\{x \in U : h^*(jx) \in \cdot\} = \mu(j\{x \in U : h^*(jx) \in \cdot\}) \\ &= \mu(\{y \in V : h^*(y) \in \cdot\} \cap j(U)) \\ &= \mu(h^*)^{-1}\end{aligned}$$

is symmetric Gaussian, [4, Corollary 1.3] tells us that μ_0 is a symmetric Gaussian measure on $\mathcal{B}_U$. For arbitrary $B \in \mathcal{B}_V$ we have

$$\mu(B) = \mu(B \cap j(U)) = \mu(j(j^{-1}(B))) = \mu_0(j^{-1}(B)).$$

This completes the proof of Lemma 4.3.

If condition (B2)(1) holds then by Lemma 4.3 there is a symmetric Gaussian measure ν' on the Borel σ-algebra of E such that $\nu = \nu' j^{-1}$. By [10, Proposition 2.8], ν' has the same reproducing kernel Hilbert space as ν and, as defined in section 4.1,

$$i : H_\nu \hookrightarrow E$$

is the inclusion map of H_ν into E, which is a compact operator (see [3, Corollary 3.2.4]). From the definition of reproducing kernel Hilbert space, the covariance operator of ν' is $ii^* : E^* \to E$ where, to simplify notation, we identify H_ν^* with H_ν. We also have $k = ji$ and $Q = kk^*$.

Proposition 4.4 *Assuming that (B2) holds, the Wiener process $(W(t))_{t\in[0,1]}$ has trajectories continuous in E, for each $\epsilon \in (0, 1]$ the process $(W_{\epsilon A}(t))_{t\in[0,1]}$ has a version whose trajectories are continuous in E and, considering such a version of $(W_{\epsilon A}(t))_{t\in[0,1]}$, we have*

$$E\left[\sup_{t\in[0,1]} \|W_{\epsilon A}(t) - W(t)\|^2\right] \to 0 \quad \text{as } \epsilon \to 0. \tag{4.13}$$

Before proving Proposition 4.4 we state four lemmas which we will use in the proof.

Lemma 4.5 ([21, Theorem 4.10]) *Let $(U, \|\cdot\|)$ be a separable Banach space and let $Q \in L(U^*, U)$ be the covariance operator of a symmetric Gaussian measure μ on U. Let $\mathcal{R}$ be a family of positive definite symmetric operators in $L(U^*, U)$ such that for some constant $K \in (0, \infty)$ and for all $R \in \mathcal{R}$ and for all $x^* \in U^*$ we have*

$$_U\langle Rx^*, x^*\rangle_{U^*} \le K^2\, _U\langle Qx^*, x^*\rangle_{U^*}.$$

Then each $R \in \mathcal{R}$ is the covariance operator of a symmetric Gaussian measure μ_R on U and the family $\{\mu_R : R \in \mathcal{R}\}$ is uniformly tight. Moreover, for all $R \in \mathcal{R}$ we have

$$\int_U \|x\|^2\, d\mu_R(x) \le K^2 \int_U \|x\|^2\, d\mu(x).$$

Lemma 4.6 ([21, Lemma 2.18]) *Let U be a separable Banach space and let (μ_n) be a uniformly tight sequence of probability measures on $(U, \mathcal{B}_U)$. Denote the characteristic function of a measure ν on U by $\widehat{\nu}$. Let F be a weak*-dense linear subspace of U^*. If for each $x^* \in F$ the sequence of complex numbers $(\widehat{\mu_n}(x^*))$ converges then (μ_n) converges weakly to some probability measure μ on $(U, \mathcal{B}_U)$ and*

$$\lim_{n\to\infty} \widehat{\mu_n}(x^*) = \widehat{\mu}(x^*) \quad \text{for all } x^* \in U^*.$$

Lemma 4.7 ([21, Theorem 3.25]) *Let $(U, \|\cdot\|)$ be a separable Banach space and let (μ_n) be a sequence of symmetric Gaussian measures on U that converges weakly to a* *symmetric Gaussian measure μ on U. Then*

$$\lim_{n\to\infty}\int_U \|x\|^2\,d\mu_n(x) = \int_U \|x\|^2\,d\mu(x).$$

Lemma 4.8 ([30, Corollary 1 of Theorem 5.7 in chapter 5]) *Let $m \in \mathbb{N}$. There is a positive real number C_m such that for any symmetric Gaussian measure μ on a separable Banach space $(U, \|\cdot\|)$ we have*

$$\int_U \|x\|^{2m}\,d\mu(x) \le C_m \left(\int_U \|x\|^2\,d\mu(x)\right)^m.$$

Proof of Proposition 4.4. Let $\alpha \in (0, \frac{1}{2})$ and $G \in L(E^*, E)$ be as in condition (B2)(2). Let m be a natural number such that $2m\alpha > 1$. For each $\epsilon \in [0,1]$ define

$$Y^\epsilon(s) := \int_0^s (s-\sigma)^{-\alpha} S(\epsilon(s-\sigma))k\,dW(\sigma) \quad \text{for all } s \in [0,1].$$

As shown in [10, Theorem 5.9], this process, which has a measurable version whose trajectories are in $L^{2m}([0,1];H)$ almost surely, provides us with a version of the stochastic convolution process whose trajectories are in $C([0,1];H)$:

$$W_{\epsilon A}(t) := \frac{\sin(\pi\alpha)}{\pi}\int_0^t S(\epsilon(t-s))(t-s)^{\alpha-1}Y^\epsilon(s)\,ds\ , \quad t \in [0,1]. \tag{4.14}$$

If $\epsilon = 0$ then the expression on the right hand side of equation (4.14) becomes the Wiener process $(W(t))_{t\in[0,1]}$. We will show that condition (B2)(1) implies that the process $(Y^0(t))_{t\in[0,1]}$ has a measurable version with trajectories in $L^{2m}([0,1];E)$; furthermore, condition (B2)(2) implies that for each $\epsilon \in (0,1]$ the process $(Y^\epsilon(t))_{t\in[0,1]}$ has a measurable version with trajectories in $L^{2m}([0,1];E)$. When we use this version of $(Y^\epsilon(t))_{t\in[0,1]}$ in equation (4.14), $(W_{\epsilon A}(t))_{t\in[0,1]}$ has trajectories in $C([0,1];E)$. We will then show that the convergence in (4.13) occurs.

Let $\epsilon \in [0,1]$. For any $t \in [0,1]$, it follows from the definition of the Itô integral that $Y^\epsilon(t)$ has symmetric Gaussian distribution on H whose covariance operator is

$$\tilde{P}_t x := \int_0^t s^{-2\alpha} S(\epsilon s)QS^*(\epsilon s)x\,ds\ , \quad x \in H.$$

We can define a symmetric, positive definite, bounded linear operator $P_t \in L(E^*, E)$ by

$$P_t l^* := \int_0^t s^{-2\alpha} S_E(\epsilon s) i i^* S_E^*(\epsilon s) l^* \, ds \ , \quad l^* \in E^*. \tag{4.15}$$

For any $l^* \in E^*$ we have

$$_E\langle P_t l^*, l^* \rangle_{E^*} \le {}_E\langle P_1 l^*, l^* \rangle_{E^*} = \int_0^1 s^{-2\alpha} |i^* S_E^*(\epsilon s) l^*|_{H_\nu}^2 \, ds \le {}_E\langle G l^*, l^* \rangle_{E^*}. \tag{4.16}$$

Thus, by Lemma 4.5, P_t is the covariance operator of a symmetric Gaussian measure ν_t on E. The measure $\nu_t j^{-1}$ on H has covariance operator $jP_t j^*$. For $x \in H$ and x^* the corresponding element of H^* we have

$$\begin{aligned} jP_t j^* x^* &= j \int_0^t s^{-2\alpha} S_E(\epsilon s) i i^* S_E^*(\epsilon s) j^* x^* \, ds \\ &= \int_0^t s^{-2\alpha} S(\epsilon s) k k^* S^*(\epsilon s) x \, ds \\ &= \tilde{P}_t x. \end{aligned}$$

Thus $\mathcal{L}(Y^\epsilon(t)) = \nu_t j^{-1}$ and in particular $P\{Y^\epsilon(t) \in j(E)\} = 1$.
Take a measurable version of the process $(Y^\epsilon(t))_{t\in[0,1]}$, that is, such that the function $Y^\epsilon : ([0,1] \times \Omega, \mathcal{B}_{[0,1]} \otimes \mathcal{F}) \to (H, \mathcal{B}_H)$ defined by

$$Y^\epsilon(t, \omega) := Y^\epsilon(t)(\omega) \ , \quad (t, \omega) \in [0,1] \times \Omega \, ,$$

is measurable, the domain having the product σ-algebra. Then the set

$$D := \{(t, \omega) \in [0,1] \times \Omega : Y^\epsilon(t)(\omega) \in j(E)\}$$

is measurable and for each $t \in [0,1]$ we have $P\{\omega \in \Omega : (t, \omega) \in D\} = 1$. Thus the E-valued process

$$Y_E^\epsilon(t) := 1_D(t, \cdot) Y^\epsilon(t) \ , \quad t \in [0,1]$$

is a measurable E-valued version of $(Y^\epsilon(t))_{t\in[0,1]}$.
We now show that the process $(Y_E^\epsilon(t))_{t\in[0,1]}$ has trajectories in $L^{2m}([0,1]; E)$. For each $t \in [0,1]$ the distribution of $Y_E^\epsilon(t)$ on E is ν_t and its covariance operator P_t satisfies inequality (4.16). Thus by Lemma 4.5 we have

$$\int_\Omega \|Y_E^\epsilon(t)\|^2 \, dP \le \int_\Omega \|Y_E^\epsilon(1)\|^2 \, dP \quad \forall t \in [0,1]. \tag{4.17}$$

From Lemma 4.8 and inequality (4.17) we have

$$\begin{aligned}
\int_\Omega \int_0^1 \|Y_E^\epsilon(s)\|^{2m}\, ds\, dP &= \int_0^1 \int_\Omega \|Y_E^\epsilon(s)\|^{2m}\, dP\, ds \\
&\leq \int_0^1 C_m \left(\int_\Omega \|Y_E^\epsilon(s)\|^2\, dP \right)^m ds \\
&\leq C_m \left(\int_\Omega \|Y_E^\epsilon(1)\|^2\, dP \right)^m < \infty \qquad (4.18)
\end{aligned}$$

and thus $\int_0^1 \|Y_E^\epsilon(s)\|^{2m}\, ds$ is finite P a.e..

If $f \in L^{2m}([0,1];E)$ then for each $\epsilon \in [0,1]$ the E-valued function

$$t \mapsto \int_0^t S_E(\epsilon(t-\sigma))(t-\sigma)^{\alpha-1} f(\sigma)\, d\sigma \ , \quad t \in [0,1] \qquad (4.19)$$

is continuous; this claim can be proved by following the same steps we use to get inequality (4.24) below. By [10, Theorem 5.9], for each $\epsilon \in (0,1]$ the process $(W_{\epsilon A}(t))_{t\in[0,1]}$ has the version

$$W_{\epsilon A}(t) := \frac{\sin(\pi\alpha)}{\pi} \int_0^t S(\epsilon(t-s))(t-s)^{\alpha-1} j Y_E^\epsilon(s)\, ds \quad \forall t \in [0,1]$$

whose trajectories are continuous in H; thus, by comparison with equation (4.19), we have the E-valued version

$$W_{\epsilon A}(t) := \frac{\sin(\pi\alpha)}{\pi} \int_0^t S_E(\epsilon(t-s))(t-s)^{\alpha-1} Y_E^\epsilon(s)\, ds \quad \forall t \in [0,1], \qquad (4.20)$$

which has trajectories continuous in E.

We now prove the convergence in (4.13).
For $0 < \epsilon \leq 1$ we have

$$\begin{aligned}
W(t) - W_{\epsilon A}(t) &= \frac{\sin(\pi\alpha)}{\pi} \int_0^t (I_E - S_E(\epsilon(t-s)))(t-s)^{\alpha-1} Y_E^0(s)\, ds \\
&\quad + \frac{\sin(\pi\alpha)}{\pi} \int_0^t S_E(\epsilon(t-s))(t-s)^{\alpha-1} (Y_E^0(s) - Y_E^\epsilon(s))\, ds \qquad (4.21) \\
&=: K_\epsilon(t) + J_\epsilon(t) \quad \text{for all } t \in [0,1], \qquad (4.22)
\end{aligned}$$

where $K_\epsilon(t)$ and $J_\epsilon(t)$ are defined to be the respective terms on the right hand side of equation (4.21). From the definition in equation (4.22) we have

$$\sup_{t\in[0,1]} \|W(t) - W_{\epsilon A}(t)\|^{2m} \leq 2^{2m} \left(\sup_{t\in[0,1]} \|K_\epsilon(t)\|^{2m} + \sup_{t\in[0,1]} \|J_\epsilon(t)\|^{2m} \right).$$

Thus it suffices to show that $\lim_{\epsilon\to 0} E[\sup_{t\in[0,1]} \|K_\epsilon(t)\|^{2m}] = 0$ and $\lim_{\epsilon\to 0} E[\sup_{t\in[0,1]} \|J_\epsilon(t)\|^{2m}] = 0$.

Step 1. We will show that

$$\lim_{\epsilon\to 0} E\left[\sup_{t\in[0,1]} \|K_\epsilon(t)\|^{2m} \right] = 0. \tag{4.23}$$

For all $0 < \epsilon \leq 1$ we have, by Hölder's inequality:

$$\begin{aligned}
&\sup_{t\in[0,1]} \|K_\epsilon(t)\|^{2m} \\
&\leq \left(\frac{\sin(\pi\alpha)}{\pi} \sup_{r\in[0,1]} \|I_E - S_E(r)\|_{L(E)} \right)^{2m} \left(\int_0^1 \sigma^{(\alpha-1)\frac{2m}{2m-1}}\, d\sigma \right)^{2m-1} \int_0^1 \|Y_E^0(s)\|^{2m}\, ds
\end{aligned}$$

and the right hand side is an integrable dominating function. If $\sup_{t\in[0,1]} \|K_\epsilon(t)\| \to 0$ as $\epsilon \to 0$ P a.e. then, by Lebesgue's dominated convergence theorem, equation (4.23) holds. Let $\omega \in \Omega$ be such that the path

$$s \in [0,1] \mapsto Y_E^0(s)(\omega)$$

belongs to $L^{2m}([0,1]; E)$; we will show that

$$\sup_{t\in[0,1]} \|K_\epsilon(t)(\omega)\| \to 0 \quad \text{as } \epsilon \to 0.$$

Let the sequence $(\epsilon_n) \subset (0,1]$ converge to 0 as n goes to infinity. Firstly note that for each fixed $t \in [0,1]$, $K_{\epsilon_n}(t)(\omega) \to 0$ as $n \to \infty$:

$$\begin{aligned}
\|K_{\epsilon_n}(t)(\omega)\| &\leq \frac{\sin(\pi\alpha)}{\pi} \left(\int_0^1 \sigma^{(\alpha-1)\frac{2m}{2m-1}}\, d\sigma \right)^{\frac{2m-1}{2m}} \\
&\qquad \times \left(\int_0^t \|(I_E - S_E(\epsilon_n(t-s)))Y_E^0(s)(\omega)\|^{2m}\, ds \right)^{\frac{1}{2m}} \\
&\to 0 \quad \text{as } n \to \infty
\end{aligned}$$

by Lebesgue's dominated convergence theorem. Uniform convergence to zero follows once we show that the family of functions from $C([0,1];E)$

$$t \mapsto K_\epsilon(t)(\omega) \ , \quad 0 < \epsilon \le 1,$$

is uniformly equicontinuous. For brevity, we set

$$\begin{aligned} y(s) &:= Y_E^0(s)(\omega) \quad \forall s \in [0,1] \quad \text{and} \\ z^\epsilon(s) &:= \frac{\pi}{\sin(\pi\alpha)} K_\epsilon(s)(\omega) \quad \forall s \in [0,1] \quad \text{for each } \epsilon \in (0,1]. \end{aligned}$$

By definition, for each $\epsilon \in (0,1]$

$$z^\epsilon(t) = \int_0^t (I_E - S_E(\epsilon(t-s)))(t-s)^{\alpha-1} y(s)\, ds \quad \forall t \in [0,1].$$

If $0 < \epsilon \le 1$ and $0 \le t < u \le 1$ then

$$\begin{aligned} &\|z^\epsilon(u) - z^\epsilon(t)\| \\ \le\ & \left\| \int_0^{u-t} (I_E - S_E(\epsilon(u-s)))(u-s)^{\alpha-1} y(s)\, ds \right\| + \\ & \left\| \int_{u-t}^{u} (I_E - S_E(\epsilon(u-s)))(u-s)^{\alpha-1} y(s)\, ds - \int_0^t (I_E - S_E(\epsilon(t-s)))(t-s)^{\alpha-1} y(s)\, ds \right\| \\ \le\ & \left\| \int_0^{u-t} (I_E - S_E(\epsilon(u-s)))(u-s)^{\alpha-1} y(s)\, ds \right\| + \\ & \left\| \int_0^{t} (I_E - S_E(\epsilon(t-s)))(t-s)^{\alpha-1} (y(s+u-t) - y(s))\, ds \right\| \\ \le\ & \sup_{r \in [0,1]} \|I_E - S_E(r)\|_{L(E)} \left(\int_0^1 \sigma^{(\alpha-1)\frac{2m}{2m-1}}\, d\sigma \right)^{\frac{2m-1}{2m}} \times \\ & \left[\left(\int_0^{u-t} \|y(s)\|^{2m}\, ds \right)^{\frac{1}{2m}} + \left(\int_0^{1-(u-t)} \|y(s+u-t) - y(s)\|^{2m}\, ds \right)^{\frac{1}{2m}} \right]. \end{aligned} \tag{4.24}$$

The expression on the right hand side of inequality (4.24) does not depend on ϵ and, since $y \in L^{2m}([0,1];E)$, both integrals inside the square brackets converge to 0 as $u - t \searrow 0$. Hence $\{z^\epsilon : 0 < \epsilon \le 1\}$ is uniformly equicontinuous.

Step 2. We now show that

$$\lim_{\epsilon\to 0} E\left[\sup_{t\in[0,1]} \|J_\epsilon(t)\|^{2m}\right] = 0. \tag{4.25}$$

We have

$$\sup_{t\in[0,1]} \|J_\epsilon(t)\|^{2m}$$

$$\leq \left(\frac{\sin(\pi\alpha)}{\pi}\sup_{r\in[0,1]}\|S_E(r)\|_{L(E)}\right)^{2m}\left(\int_0^1 \sigma^{(\alpha-1)\frac{2m}{2m-1}}\,d\sigma\right)^{2m-1}\int_0^1 \|Y_E^0(s)-Y_E^\epsilon(s)\|^{2m}\,ds. \tag{4.26}$$

Let $t\in[0,1]$ and $\epsilon\in(0,1]$. The random variable

$$Y^0(t)-Y^\epsilon(t) = \int_0^t (t-\sigma)^{-\alpha}(I_H - S(\epsilon(t-\sigma)))k\,dW(\sigma)$$

has symmetric Gaussian distribution on H with covariance operator

$$Q_t^\epsilon := \int_0^t \sigma^{-2\alpha}(I_H - S(\epsilon\sigma))Q(I_H-S(\epsilon\sigma))^*\,d\sigma.$$

Define the operator $P_t^\epsilon : E^* \to E$ by

$$P_t^\epsilon l^* := \int_0^t \sigma^{-2\alpha}(I_E - S_E(\epsilon\sigma))ii^*(I_E - S_E(\epsilon\sigma))^* l^*\,d\sigma \quad \forall l^*\in E^*.$$

The operator P_t^ϵ is a symmetric, positive definite, bounded linear operator and for all $l^*\in E^*$ we have

$$\begin{aligned}
{}_E\langle P_t^\epsilon l^*, l^*\rangle_{E^*} &= \int_0^t \sigma^{-2\alpha}|i^*(I_E - S_E(\epsilon\sigma))^* l^*|_{H_\nu}^2\,d\sigma \\
&\leq \int_0^1 \sigma^{-2\alpha}|i^*(I_E - S_E(\epsilon\sigma))^* l^*|_{H_\nu}^2\,d\sigma = {}_E\langle P_1^\epsilon l^*, l^*\rangle_{E^*} \qquad (4.27)\\
&\leq 2\int_0^1 \sigma^{-2\alpha}\left(|i^*l^*|_{H_\nu}^2 + |i^*S_E^*(\epsilon\sigma)l^*|_{H_\nu}^2\right)\,d\sigma \\
&\leq \frac{2}{1-2\alpha}|i^*l^*|_{H_\nu}^2 + 2\,{}_E\langle Gl^*, l^*\rangle_{E^*}. \qquad (4.28)
\end{aligned}$$

By Lemma 4.5, inequality (4.28) implies that P_t^ϵ is the covariance operator of a symmetric Gaussian measure ν_t^ϵ on E. For each $x\in H$ and the corresponding linear functional

$x^* \in H^*$ we have

$$jP_t^\epsilon j^* x^* = Q_t^\epsilon x.$$

Thus the distribution of $Y_E^0(t) - Y_E^\epsilon(t)$ on E is ν_t^ϵ. Moreover, by inequality (4.27) and Lemma 4.5 we have

$$\int_\Omega \|Y_E^0(t) - Y_E^\epsilon(t)\|^2 \, dP \leq \int_\Omega \|Y_E^0(1) - Y_E^\epsilon(1)\|^2 \, dP \ , \tag{4.29}$$

which holds for all $\epsilon \in (0,1]$ and for all $t \in [0,1]$. Inequality (4.28) and Lemma 4.5 also imply that the family of Gaussian measures on E

$$\{\mathcal{L}(Y_E^0(1) - Y_E^\epsilon(1)) = \nu_1^\epsilon \ : \epsilon \in (0,1]\}$$

is uniformly tight.

Let (ϵ_n) be a sequence of numbers from $(0,1]$ such that ϵ_n converges to 0 as n goes to infinity. We have

$$\begin{aligned} {}_E\langle P_1^{\epsilon_n} l^*, l^* \rangle_{E^*} &= \int_0^1 \sigma^{-2\alpha} |((I_E - S_E(\epsilon_n \sigma))i)^* l^*|^2_{H_\nu^*} \, d\sigma \\ &\leq \sup_{r \in [0,1]} \|(I_E - S_E(\epsilon_n r))i\|^2_{L(H_\nu, E)} \ \|l^*\|^2_{E^*} \frac{1}{1-2\alpha} \\ &\rightarrow 0 \quad \text{as } n \to \infty \end{aligned}$$

for all $l^* \in E^*$. Hence the sequence of characteristic functions $(\widehat{\nu_1^{\epsilon_n}})$ converges pointwise:

$$\widehat{\nu_1^{\epsilon_n}}(l^*) = \exp\left(-\frac{1}{2} {}_E\langle P_1^{\epsilon_n} l^*, l^* \rangle_{E^*}\right) \to 1 \quad \text{as } n \to \infty$$

for all $l^* \in E^*$. By Lemma 4.6 we have that $\nu_1^{\epsilon_n}$ converges weakly to δ_0, the point mass at 0, as n goes to infinity and, by Lemma 4.7, we have

$$\lim_{n \to \infty} \int_\Omega \|Y_E^0(1) - Y_E^{\epsilon_n}(1)\|^2 \, dP = 0. \tag{4.30}$$

By Lemma 4.8 we have

$$\begin{aligned} \int_0^1 \int_\Omega \|Y_E^0(s) - Y_E^{\epsilon_n}(s)\|^{2m} \, dP \, ds &\leq C_m \int_0^1 \left(\int_\Omega \|Y_E^0(s) - Y_E^{\epsilon_n}(s)\|^2 \, dP \right)^m \, ds \\ &\leq C_m \left(\int_\Omega \|Y_E^0(1) - Y_E^{\epsilon_n}(1)\|^2 \, dP \right)^m \\ &\rightarrow 0 \quad \text{as } n \to \infty, \end{aligned}$$

where the second line follows from inequality (4.29) and the third line from equation (4.30). Since (ϵ_n) is an arbitrary sequence from $(0,1]$ which converges to 0, we conclude that

$$\lim_{\epsilon\to 0}\int_0^1\int_\Omega \|Y_E^0(s)-Y_E^\epsilon(s)\|^{2m}\,dP\,ds=0.$$

Equation (4.25) now follows by taking expected values on both sides of equation (4.26). This completes the proof of Proposition 4.4.

Set

$$\delta_\epsilon^2 := E\left[\sup_{t\in[0,1]}\|W_{\epsilon A}(t)-W(t)\|^2\right] \quad \forall \epsilon\in(0,1].$$

It is straightforward to show, using [10, Proposition 2.9(i)], that for each $\epsilon\in(0,1]$ the random variable

$$\omega\mapsto(t\mapsto W_{\epsilon A}(t)(\omega)-W(t)(\omega)) \tag{4.31}$$

in $C([0,1];H)$ has symmetric Gaussian distribution; Lemma 4.3 tells us that the corresponding random variable in $C([0,1];E)$ also has symmetric Gaussian distribution. Consequently, except for differences of notation and context, the proof of the next lemma is from Fang and Zhang [13, Lemma 4.2].

Lemma 4.9 *Assuming that (B2) holds, there is a positive number c such that*

$$\sup_{0<\epsilon\le 1} E\left[\exp\left(c\,\frac{\sup_{t\in[0,1]}\|W_{\epsilon A}(t)-W(t)\|^2}{\delta_\epsilon^2}\right)\right]<\infty.$$

Proof. The proof makes clever use of Fernique's theorem as stated in [10, Theorem 2.6]. For each $\epsilon\in(0,1]$ define the symmetric Gaussian distribution μ_ϵ on $C([0,1];E)$ by

$$\mu_\epsilon:=\mathcal{L}\left(\omega\in\Omega\mapsto\ [t\mapsto \tfrac{1}{\delta_\epsilon}(W_{\epsilon A}(t)(\omega)-W(t)(\omega))]\right).$$

By Chebyshev's inequality, for any positive real number r we have

$$r^2\mu_\epsilon\{u\in C([0,1];E):\ \sup_{t\in[0,1]}\|u(t)\|>r\}\le 1 \quad \forall\epsilon\in(0,1].$$

Fix $r\in[2,\infty)$; then we have

$$\ln\left(\frac{1-\mu_\epsilon(\overline{B}_{C([0,1];E)}(0,r))}{\mu_\epsilon(\overline{B}_{C([0,1];E)}(0,r))}\right)\le\ln\left(\frac{1}{3}\right)<-1 \quad \forall\epsilon\in(0,1].$$

Hence we may choose $\lambda \in (0, \infty)$ such that

$$\ln\left(\frac{1 - \mu_\epsilon(\overline{B}_{C([0,1];E)}(0,r))}{\mu_\epsilon(\overline{B}_{C([0,1];E)}(0,r))}\right) + 32\lambda r^2 \leq -1 \quad \text{for all } \epsilon \in (0,1];$$

with this choice of λ we have from Fernique's theorem:

$$\int_{C([0,1];E)} \mathrm{e}^{\lambda \sup_{t\in[0,1]} \|u(t)\|^2} \, d\mu_\epsilon(u) \leq \mathrm{e}^{16\lambda r^2} + \frac{\mathrm{e}^2}{\mathrm{e}^2 - 1} \quad \forall \epsilon \in (0,1].$$

This completes the proof of the lemma.

Lemma 4.9 is used in the proof of the next lemma, which is the counterpart of [13, Lemma 4.3].

Lemma 4.10 *Assuming that (B2) holds, for any $\delta > 0$ we have*

$$\lim_{\epsilon \to 0} \epsilon \log P\left\{\epsilon^{\frac{1}{2}} \sup_{t\in[0,1]} \|W_{\epsilon A}(t) - W(t)\| \geq \delta\right\} = -\infty.$$

Proof: Let c be as in Lemma 4.9 and set

$$C := \sup_{0<\epsilon\leq 1} E\left[\exp\left(c\frac{\sup_{t\in[0,1]} \|W_{\epsilon A}(t) - W(t)\|^2}{\delta_\epsilon^2}\right)\right].$$

We have

$$\exp\left(c\frac{\delta^2}{\epsilon\delta_\epsilon^2}\right) 1_{\{\epsilon^{\frac{1}{2}} \sup_{t\in[0,1]} \|W_{\epsilon A}(t) - W(t)\| \geq \delta\}} \leq \exp\left(c\frac{\sup_{t\in[0,1]} \|W_{\epsilon A}(t) - W(t)\|^2}{\delta_\epsilon^2}\right)$$

for all $\epsilon \in (0,1]$. Taking expectations of both sides in this inequality, we have

$$P\left\{\epsilon^{\frac{1}{2}} \sup_{t\in[0,1]} \|W_{\epsilon A}(t) - W(t)\| \geq \delta\right\} \leq C \exp\left(-c\frac{\delta^2}{\epsilon\delta_\epsilon^2}\right) \quad \forall \epsilon \in (0,1].$$

Now we take logarithms and use $\delta_\epsilon \to 0$ as $\epsilon \to 0$, from Proposition 4.4. This completes the proof of Lemma 4.10.

Define the random variables in $C([0,1];E)$:

$$\begin{aligned} W_{\epsilon A}(\omega) &:= t \mapsto W_{\epsilon A}(t)(\omega) \quad \forall \omega \in \Omega \text{ and } \forall \epsilon \in (0,1] \text{ and} \\ W(\omega) &:= t \mapsto W(t)(\omega) \quad \forall \omega \in \Omega. \end{aligned}$$

Lemma 4.10 means that the two families of random variables in $C([0,1];E)$

$$\{Z^\epsilon = S_E(\epsilon\,\cdot)\xi + \epsilon^{\frac{1}{2}} W_{\epsilon A} : \epsilon \in (0,1]\} \quad \text{and} \quad \{\xi + \epsilon^{\frac{1}{2}} W : \epsilon \in (0,1]\}$$

are exponentially equivalent.
The distribution of $W : \Omega \to C([0,1];E)$ is symmetric Gaussian and, by [10, Proposition 2.8] and [32, Theorem 1 in section 6], its reproducing kernel Hilbert space is

$$H_W := \left\{ t \in [0,1] \mapsto \int_0^t u(s)\, ds \;:\; u \in L^2([0,1];H_\nu) \right\}, \tag{4.32}$$

whose norm $|\cdot|_{H_W}$ is defined by

$$|f|_{H_W}^2 := \int_0^1 |u(s)|_{H_\nu}^2\, ds \;:\; u \in L^2([0,1];H_\nu) \quad \text{and} \quad f(t) = \int_0^t u(s)\, ds \quad \forall t \in [0,1], \tag{4.33}$$

for each $f \in H_W$.

Proposition 4.11 *Assuming that condition (B2) holds, the family $\{Z^\epsilon\}_{\epsilon \in (0,1]}$ of random variables in $C([0,1];E)$ satisfies a large deviation principle with rate function $\mathcal{I}_\xi : C([0,1];E) \to [0,\infty]$ defined by*

$$\mathcal{I}_\xi(f) := \begin{cases} \frac{1}{2}|f-\xi|_{H_W}^2 & \textit{if } f - \xi \in H_W \\ \infty & \textit{otherwise.} \end{cases} \tag{4.34}$$

Proof. By [10, Theorem 12.7], the family of random variables $\{\epsilon^{\frac{1}{2}} W : \epsilon \in (0,1]\}$ in $C([0,1];E)$ satisfies a large deviation principle with rate function $\mathcal{I}_0 : C([0,1];E) \to [0,\infty]$ defined by

$$\mathcal{I}_0(f) := \begin{cases} \frac{1}{2}|f|_{H_W}^2 & \text{if } f \in H_W \\ \infty & \text{otherwise.} \end{cases}$$

Thus $\{\xi + \epsilon^{\frac{1}{2}} W : \epsilon \in (0,1]\}$ satisfies a large deviation principle with rate function $\mathcal{I}_\xi$. Since $\{Z^\epsilon : \epsilon \in (0,1]\}$ and $\{\xi + \epsilon^{\frac{1}{2}} W : \epsilon \in (0,1]\}$ are exponentially equivalent, they satisfy the same large deviation principle (see [18, Lemma 27.13]). This completes the proof of Proposition 4.11.

Corollary 4.12 *Assuming that condition (B2) holds, the family $\{\omega \in \Omega \mapsto [t \mapsto X(\epsilon t)(\omega)]\}_{\epsilon \in (0,1]}$ of random variables in $C([0,1];E)$ satisfies a large deviation principle with rate function $\mathcal{I}_\xi$ defined in equation (4.34).*

Proof. This follows from Proposition 4.11 and Proposition 4.2 and Proposition 4.1.

4.4 An example

We now present an example where condition (B2) holds and thus Corollary 4.12 provides the short time asymptotics.

Let $n \in \{1,2,3\}$. Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$ with C^∞ boundary $\partial\mathcal{O}$. Consider the second order elliptic operator

$$Lu := -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{i,j}\frac{\partial u}{\partial x_j}\right) + \sum_{k=1}^{n} b_k \frac{\partial u}{\partial x_k} + cu, \tag{4.35}$$

where the functions

$$a_{i,j} : \overline{\mathcal{O}} \to \mathbb{R} \quad , i,j = 1,\dots,n$$

are in $C^\infty(\overline{\mathcal{O}})$ and satisfy the conditions

1. $a_{i,j} = a_{j,i}$ for all $i,j \in \{1,\dots,n\}$ and
2. for some positive real number C

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq C\sum_{k=1}^{n}\xi_k^2 \quad \forall(\xi_1,\dots,\xi_n) \in \mathbb{R}^n \text{ and } \forall x \in \overline{\mathcal{O}} \tag{4.36}$$

and the functions

$$b_k : \overline{\mathcal{O}} \to \mathbb{R} \quad , \; k = 1,\dots,n \quad \text{and} \quad c : \overline{\mathcal{O}} \to \mathbb{R}$$

are also in $C^\infty(\overline{\mathcal{O}})$.
Define the operator $(A_2, D(A_2))$ on $L^2(\mathcal{O})$ by

$$D(A_2) := W^{2,2}(\mathcal{O}) \cap W_0^{1,2}(\mathcal{O})$$

and

$$A_2\, u := Lu \quad , u \in D(A_2).$$

By [24, Theorem 2.7 in chapter 7], $-A_2$ generates an analytic semigroup $(S(t))_{t\geq 0}$ on $L^2(\mathcal{O})$.
Let $E = C_0(\overline{\mathcal{O}})$, the continuous functions which vanish on $\partial\mathcal{O}$, with the supremum norm. Define an operator $(A_c, D(A_c))$ on $C_0(\overline{\mathcal{O}})$ by

$$D(A_c) := \{u : \; u \in W^{2,p}(\mathcal{O}) \text{ for all } p > n \text{ and } u = 0 \text{ on } \partial\mathcal{O} \text{ and } Lu \in C_0(\overline{\mathcal{O}})\}$$

and

$$A_c u := Lu \quad \forall u \in D(A_c).$$

The operator $-A_c$ generates an analytic semigroup on $C_0(\overline{\mathcal{O}})$ (see [24, Theorem 3.7 in chapter 7]) which we denote by $(S_E(t))_{t\geq 0}$. Since $C_0(\overline{\mathcal{O}})$ is continuously embedded in $L^2(\mathcal{O})$ and the graph of $-A_c$ is contained in the graph of $-A_2$ it follows that

$$S_E(t)u = S(t)u \quad \forall u \in C_0(\overline{\mathcal{O}}) \text{ and } \forall t \in [0,1].$$

According to [24, Remark 6.3 in chapter 7], if $u_0 \in C_c^\infty(\mathcal{O})$ then the function

$$u(t,x) := (S_E(t)\, u_0)(x) \quad , \ (t,x) \in [0,1] \times \overline{\mathcal{O}},$$

is in $C^\infty((0,1] \times \overline{\mathcal{O}})$ and is a classical solution of the initial value problem

$$\begin{cases} \frac{\partial u}{\partial t} + Lu = 0 & \text{in } [0,1] \times \mathcal{O} \\ u(t,x) = 0 & \text{on } [0,1] \times \partial\mathcal{O} \\ u(0,x) = u_0(x) & \text{in } \mathcal{O}. \end{cases}$$

From this, one can show by using the maximum principle (see, for example, [28, Theorem 4.26]) that there is a non-negative real number θ such that

$$\|S_E(t)\|_{L(E,E)} \leq \mathrm{e}^{\theta t} \quad \text{for all } t \geq 0.$$

Take any $a \geq 0$ such that the spectrum of $A_2 + aI$ lies in $\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > 0\}$. Let $\alpha > n$. We make two claims:

1. the operator $Q_{\alpha/2} := (A_2 + aI)^{-\frac{\alpha}{2}}((A_2 + aI)^{-\frac{\alpha}{2}})^*$ is the covariance operator of a symmetric Gaussian measure $\nu_{\alpha/2}$ on $L^2(\mathcal{O})$ such that $\nu_{\alpha/2}(E) = 1$;

2. there is $r \in (0,\infty)$ such that

$$S(t)(\overline{B}_{H_{\alpha/2}}(0,1)) \subset \overline{B}_{H_{\alpha/2}}(0,r) \quad \forall t \in [0,1],$$

   where $(H_{\alpha/2}, \|\cdot\|_{\alpha/2})$ is the reproducing kernel Hilbert space of $\nu_{\alpha/2}$ and for $s \geq 0$ $\overline{B}_{H_{\alpha/2}}(0,s) := \{x \in H_{\alpha/2} : \|x\|_{\alpha/2} \leq s\}$.

It follows from these two claims that if $\mathcal{L}(W(1)) = \nu_{\alpha/2}$ then condition (B2) holds and

Proposition 4.4 holds: for each $\epsilon \in (0,1]$ the process

$$W_{-\epsilon A_2}(t) = \int_0^t S(\epsilon(t-s))k\, dW(s) \quad , \ t \in [0,1],$$

has a version whose trajectories are continuous in E. Also if $b : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable, decreasing function such that $b(0) = 0$ and the function $F : E \to E$ is defined by

$$(F(\phi))(x) := b(\phi(x)) \quad \forall x \in \overline{\mathcal{O}} \text{ and } \forall \phi \in E,$$

then for any $\xi \in E$ Corollary 4.12 gives us the small time asymptotics of the continuous E-valued process $(X(t))_{t\in[0,1]}$ such that

$$X(t) = S_E(t)\xi + W_{-A_2}(t) + \int_0^t S_E(t-s)F(X(s))\, ds \quad \forall t \in [0,1].$$

Proof of claim 1. From [10, Corollary B.4] we have

$$H_{\alpha/2} := \operatorname{im}\left(Q_{\alpha/2}^{\frac{1}{2}}\right) = \operatorname{im}\left((A_2 + aI)^{-\frac{\alpha}{2}}\right) = D\left((A_2 + aI)^{\frac{\alpha}{2}}\right) \tag{4.37}$$

and for all $x \in \operatorname{im}(Q_{\alpha/2}^{\frac{1}{2}})$

$$\|x\|_{\alpha/2} := \left|Q_{\alpha/2}^{-\frac{1}{2}} x\right|_{L^2(\mathcal{O})} = \left|(A_2 + aI)^{\frac{\alpha}{2}} x\right|_{L^2(\mathcal{O})}. \tag{4.38}$$

By [16, Theorem 1.6.1], the Hilbert space

$$(H_{\alpha/4} := D((A_2 + aI)^{\frac{\alpha}{4}}),\ \|\cdot\|_{\alpha/4} := |(A_2 + aI)^{\frac{\alpha}{4}} \cdot |_{L^2(\mathcal{O})})$$

is continuously embedded in the space of continuous functions on $\overline{\mathcal{O}}$ with the supremum norm. Also, since $D(A_2) = W^{2,2}(\mathcal{O}) \cap W_0^{1,2}(\mathcal{O})$ consists of elements of $C_0(\overline{\mathcal{O}})$ and the set $D(A_2) \cap H_{\alpha/4}$ is dense in the space $H_{\alpha/4}$, it follows that

$$H_{\alpha/4} \hookrightarrow C_0(\overline{\mathcal{O}}),$$

where the symbol $\hookrightarrow$ denotes a continuous embedding.

Now we want to show that the inclusion map

$$H_{\alpha/2} \hookrightarrow H_{\alpha/4}$$

is Hilbert-Schmidt. Let (e_k) be an orthonormal basis of $L^2(\mathcal{O})$. Then $((A_2 + aI)^{-\frac{\alpha}{2}} e_k)$ is an orthonormal basis of $H_{\alpha/2}$. We have

$$\sum_{k=1}^{\infty} \|(A_2 + aI)^{-\frac{\alpha}{2}} e_k\|_{\alpha/4}^2 = \sum_{k=1}^{\infty} |(A_2 + aI)^{-\frac{\alpha}{4}} e_k|_{L^2(\mathcal{O})}^2.$$

Thus it is equivalent to show that the inclusion map

$$H_{\alpha/4} \hookrightarrow L^2(\mathcal{O})$$

is Hilbert-Schmidt. To show this, we consider the self-adjoint operator $(A_{2s}, D(A_{2s}))$ on $L^2(\mathcal{O})$ defined by

$$D(A_{2s}) := W^{2,2}(\mathcal{O}) \cap W_0^{1,2}(\mathcal{O})$$

and

$$A_{2s}\, u := -\sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(a_{i,j} \frac{\partial u}{\partial x_j} \right) \quad , \ \forall u \in D(A_{2s}).$$

The ellipticity condition in inequality (4.36) ensures that the spectrum of A_{2s} is contained in $(0, \infty)$ and, as shown in [24, Theorem 3.6 in chapter 7], $-A_{2s}$ generates an analytic semigroup of contractions on $L^2(\mathcal{O})$. Thus Agmon [1, Theorem 13.6, Corollary] gives us bounds for the eigenvalues (λ_j) of the symmetric compact operator A_{2s}^{-1}, in order of decreasing modulus:

$$0 < \lambda_j \leq K j^{-\frac{2}{n}} \quad \text{for all } j \in \mathbb{N}, \tag{4.39}$$

where K is a positive real constant. Define the Hilbert space

$$H_{\alpha/4,s} := D(A_{2s}^{\frac{\alpha}{4}}) \quad \text{with norm } \|x\|_{\alpha/4,s} := |A_{2s}^{\frac{\alpha}{4}} x|_{L^2(\mathcal{O})}.$$

Let (e_j) be an orthonormal basis of $L^2(\mathcal{O})$ consisting of eigenvectors of A_{2s}^{-1}:

$$A_{2s}^{-1} e_j = \lambda_j e_j \quad \text{for all } j \in \mathbb{N}.$$

Using Agmon's bounds in inequality (4.39) we see that the embedding of $H_{\alpha/4,s}$ into $L^2(\mathcal{O})$

is Hilbert-Schmidt:

$$\begin{aligned}\sum_{j=1}^{\infty} |A_{2s}^{-\frac{\alpha}{4}} e_j|^2_{L^2(\mathcal{O})} &= \sum_{j=1}^{\infty} \lambda_j^{\frac{\alpha}{2}} \\ &\leq K^{\frac{\alpha}{2}} \sum_{j=1}^{\infty} j^{-\frac{\alpha}{n}} \\ &< \infty\end{aligned}$$

since $\alpha > n$. By [16, Theorem 1.4.8],

$$H_{\alpha/4} = H_{\alpha/4,s}$$

with equivalent norms. Thus the embedding $H_{\alpha/4} \hookrightarrow L^2(\mathcal{O})$ is also Hilbert-Schmidt and it follows that the embedding $H_{\alpha/2} \hookrightarrow H_{\alpha/4}$ is also Hilbert-Schmidt.

We have the following embeddings:

$$H_{\alpha/2} \stackrel{\text{H−S}}{\longrightarrow} H_{\alpha/4} \hookrightarrow E \hookrightarrow L^2(\mathcal{O}) \ , \tag{4.40}$$

where $\stackrel{\text{H−S}}{\rightarrow}$ denotes a Hilbert-Schmidt embedding. The Hilbert-Schmidt embedding in expression (4.40) implies that if (f_j) is an orthonormal basis of $H_{\alpha/2}$ and $(\gamma_j : (\Omega, \mathcal{F}, P) \to \mathbb{R})$ is a sequence of independent standard normal random variables then the series $\sum_{j=1}^{\infty} \gamma_j f_j$, which converges in $L^2(\Omega, \mathcal{F}, P; H_{\alpha/4})$, has symmetric Gaussian distribution on $H_{\alpha/4}$ whose reproducing kernel Hilbert space is $H_{\alpha/2}$. By [10, Proposition 2.8], the induced symmetric Gaussian measure on $L^2(\mathcal{O})$ also has reproducing kernel $H_{\alpha/2}$. By the definition of $Q_{\alpha/2}$ and the definition of $H_{\alpha/2}$ in equations (4.37) and (4.38) and equation (4.40), $Q_{\alpha/2}$ is of trace class and thus it is the covariance operator of a symmetric Gaussian measure $\nu_{\alpha/2}$ on $L^2(\mathcal{O})$ whose reproducing kernel Hilbert space is $H_{\alpha/2}$. Since a symmetric Gaussian measure on $L^2(\mathcal{O})$ is uniquely determined by its reproducing kernel Hilbert space (see [3, Corollary 3.2.6]), we have completed the proof of claim 1.

Proof of claim 2. We have for any x in $H_{\alpha/2}$ and t in $[0, 1]$:

$$\|S(t)x\|_{\alpha/2} = |(A_2{+}aI)^{\frac{\alpha}{2}} S(t)x|_{L^2(\mathcal{O})} = |S(t)(A_2{+}aI)^{\frac{\alpha}{2}} x|_{L^2(\mathcal{O})} \leq \sup_{r\in[0,1]} \|S(r)\|_{L(L^2(\mathcal{O}),L^2(\mathcal{O}))} \|x\|_{\alpha/2}.$$

This completes the proof of claim 2.

Remark Suppose in this example that b is a decreasing polynomial function, say $b(s) := -s^3$ for all $s \in \mathbb{R}$. Da Prato [9, Section 3.2] has proved existence and uniqueness of mild solutions for systems with dissipative nonlinear drift; his approach is well suited to reaction-diffusion equations with decreasing polynomial reaction terms and additive noise. Taking Da Prato's approach one can reformulate our example with the Banach space $L^6(\mathcal{O})$ replacing $C_0(\overline{\mathcal{O}})$ everywhere. The domain of the function F becomes the subspace $L^6(\mathcal{O})$ of $H = L^2(\mathcal{O})$. If $\mathcal{L}(W(1))(L^6(\mathcal{O})) = 1$ and $\xi \in L^{18}(\mathcal{O})$ then the exponential small time asymptotics of the unique process $(V(t) : (\Omega, \mathcal{F}, P) \to L^6(\mathcal{O}))_{t\in[0,1]}$ whose trajectories are continuous in H and bounded in $L^6(\mathcal{O})$ and which satisfies the equation

$$V(t) = S(t)\xi + W_{-A_2}(t) + \int_0^t S(t-s)F(V(s))\,ds \quad \forall t \in [0,1],$$

is described by a large deviation principle in $C([0,1];H)$ with rate function $\mathcal{I}_\xi : C([0,1];H) \to [0,\infty]$ defined by equation (4.34).
The advantage of working in L^p spaces which are of Banach type 2 is that [22, Theorem 5.5] takes care of convergence of Gaussian random variables without condition (B2)(2).

Chapter 5

Small time asymptotics for a linear equation with additive fractional Brownian motion noise

5.1 Introduction

In this chapter we find the small time asymptotics of the solution of a stochastic equation having only linear drift and additive fractional Brownian motion noise in a Hilbert space. This digression from our study of equations with Wiener process noise is to show that the method we used for the linear equation in the previous chapter also works when there is additive fractional Brownian motion noise. We again use the factorization method (as in the proof of [10, Theorem 5.12]) to show that trajectories of stochastic convolution processes converge to those of the noise process. This is Lemma 5.5, which corresponds to Proposition 4.4 in the previous chapter. Lemma 5.6 is proved in the same way as Lemma 4.9 in the previous chapter. Our main result in this chapter is the large deviation principle in Theorem 5.1.

5.2 Background

Let $(U, \langle\cdot,\cdot\rangle_U, |\cdot|_U)$ and $(V, \langle\cdot,\cdot\rangle_V, |\cdot|_V)$ be separable Hilbert spaces. Let Q be a positive definite symmetric trace class linear operator on U and let ν be the symmetric Gaussian measure on U with covariance operator Q. Let

$$(U_\nu := Q^{\frac{1}{2}}(U), \langle\cdot,\cdot\rangle_{U_\nu} := \langle Q^{-\frac{1}{2}}\cdot, Q^{-\frac{1}{2}}\cdot\rangle_U, |\cdot|_{U_\nu} := |Q^{-\frac{1}{2}}\cdot|_U)$$

be the reproducing kernel Hilbert space of ν. Fix $H \in (\frac{1}{2}, 1)$. Let $(B_Q^H(t) : (\Omega, \mathcal{F}, P) \to U)_{t \geq 0}$ be a fractional Q-Brownian motion with Hurst parameter H; this means that $(B_Q^H(t))_{t \geq 0}$ is a U-valued Gaussian process and

1. $E[B_Q^H(t)] = 0$ for all $t \geq 0$;
2. for all non-negative real numbers s and t and for all x and $y \in U$ we have

$$E\left[\langle B_Q^H(t), x\rangle_U \langle B_Q^H(s), y\rangle_U\right] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})\langle Qx, y\rangle_U;$$

3. the sample paths $t \mapsto B_Q^H(t)(\omega)$ are continuous U-valued functions for P a.e. $\omega \in \Omega$.

Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of U_ν and let $((\beta_n^H(t) : (\Omega, \mathcal{F}, P) \to \mathbb{R})_{t \geq 0})$ be a sequence of independent real-valued fractional Brownian motions with Hurst parameter H. Duncan, Maslowski and Pasik-Duncan [12, Proposition 2.1] showed that we can define

$$B_Q^H(t) := \sum_{j=1}^{\infty} \beta_j^H(t) e_j \quad \text{for all } t \geq 0,$$

where the series converges in $L^2(\Omega, \mathcal{F}, P; U)$. Duncan, Maslowski and Pasik-Duncan also defined the stochastic integral of a deterministic vector-valued function with respect to a fractional Brownian motion in [12]. This is done in three stages. Firstly define the stochastic integral of a V-valued step function with respect to a real-valued fractional Brownian motion $(\beta^H(t))_{t \geq 0}$ with Hurst parameter H: if $T \in (0, \infty)$ and $n \in \mathbb{N}$ and $t_0 = 0 < t_1 < t_2 < \cdots < t_n \leq T$ and $v_1, \ldots, v_n \in V$ then

$$\int_0^T \sum_{j=1}^{n} 1_{[t_{j-1}, t_j)} v_j \, d\beta^H := \sum_{j=1}^{n} (\beta^H(t_j) - \beta^H(t_{j-1})) v_j. \tag{5.1}$$

Since $(\beta^H(t_0), \ldots, \beta^H(t_n))$ has symmetric Gaussian distribution on $\mathbb{R}^{n+1}$, the random variable on the right hand side of equation (5.1) has symmetric Gaussian distribution in V. Let $p > \frac{1}{H}$. The stochastic integral defined in equation (5.1) gives a bounded linear operator which maps the dense subspace of $L^p([0,T]; V)$ consisting of step functions into $L^2(\Omega, \mathcal{F}, P; V)$ and the domain of this operator is then extended to all of $L^p([0,T]; V)$: if $f \in L^p([0,T]; V)$ and the sequence of step functions (f_n) converges to f in $L^p([0,T]; V)$ then

$$\int_0^T f \, d\beta^H := \lim_{n \to \infty} \int_0^T f_n \, d\beta^H \quad \text{in } L^2(\Omega, \mathcal{F}, P; V);$$

as the limit of a sequence of symmetric Gaussian random variables in V, $\int_0^T f\,d\beta^H$ is itself a symmetric Gaussian random variable in V and we also have

$$E\left[\left|\int_0^T f\,d\beta^H\right|_V^2\right] = \int_0^T \int_0^T \langle f(s), f(t)\rangle_V\,\phi(t-s)\,ds\,dt \tag{5.2}$$

where

$$\phi(r) = H(2H-1)|r|^{2H-2} \quad \text{for all } r \in \mathbb{R}.$$

Finally the stochastic integral of a Hilbert-Schmidt operator-valued function with respect to $(B_Q^H(t))_{t\geq 0}$ is defined using the definition of the stochastic integral of $f \in L^p([0,T];V)$ with respect to $(\beta^H(t))_{t\geq 0}$. Let $(L_2(U_\nu,V), \|\cdot\|_{L_2(U_\nu,V)})$ denote the Hilbert-Schmidt operators mapping U_ν into V. For $G \in L^p([0,T]; L_2(U_\nu,V))$, that is, $G:[0,T] \to L_2(U_\nu,V)$ is Borel measurable and $\int_0^T \|G(t)\|_{L_2(U_\nu,V)}^p\,dt < \infty$, define

$$\int_0^T G\,dB_Q^H := \sum_{n=1}^{\infty} \int_0^T G(s)e_n\,d\beta_n^H(s), \tag{5.3}$$

where the series on the right hand side converges in $L^2(\Omega,\mathcal{F},P;V)$. Equation (5.3) defines a bounded linear operator $\int_0^T \cdot dB_Q^H$ mapping $L^p([0,T];L_2(U_\nu,V))$ into $L^2(\Omega,\mathcal{F},P;V)$ and $\int_0^T G\,dB_Q^H$ has a symmetric Gaussian distribution in V and we have

$$\begin{aligned} E\left[\left|\int_0^T G\,dB_Q^H\right|_V^2\right] &\leq \int_0^T \int_0^T \|G(u)\|_{L_2(U_\nu,V)}\|G(v)\|_{L_2(U_\nu,V)}\,\phi(u-v)\,du\,dv \\ &\leq H(2H-1)\left(\frac{2(p-1)}{Hp-1}\right)^{\frac{2p-2}{p}} T^{\frac{2(Hp-1)}{p}} \left(\int_0^T \|G(s)\|_{L_2(U_\nu,V)}^p\,ds\right)^{\frac{2}{p}} \end{aligned} \tag{5.4}$$

if $\frac{1}{H} < p \leq 2$.

5.3 The small time asymptotics via a large deviation principle

Let $A: D(A) \subset V \to V$ be the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $(S(t))_{t\geq 0}$ on V. Let Φ be a bounded linear operator mapping U into V and let

$$i: U_\nu \hookrightarrow U$$

be the Hilbert-Schmidt embedding of U_ν into U. We define the mild solution of the stochastic initial value problem:

$$\begin{aligned} dX &= AX\,dt + \Phi\,dB_Q^H \\ X(0) &= x, \end{aligned}$$

where $x \in V$, to be

$$X(t) = S(t)x + \int_0^t S(t-r)\Phi i\,dB_Q^H(r) \tag{5.5}$$

for $t \geq 0$, where the stochastic integral on the right hand side of equation (5.5) is defined as in equation (5.3). We remark that for each $t \in (0,\infty)$ the function

$$r \in [0,t] \mapsto S(t-r)\Phi i \in L_2(U_\nu, V)$$

is continuous and hence the stochastic integral $\int_0^t S(t-r)\Phi i\,dB_Q^H(r)$ is well defined. Duncan, Maslowski and Pasik-Duncan [12, Proposition 3.2] have shown that there is a version of $(X(t))_{t\geq 0}$ with continuous sample paths P a.e.: specifically, for any $\alpha \in (0, \frac{1}{2})$ take

$$\int_0^t S(t-u)\Phi i\,dB_Q^H(u) = \frac{\sin(\pi\alpha)}{\pi}\int_0^t (t-s)^{\alpha-1}S(t-s)Y(s)\,ds \quad \text{for all } t \geq 0, \tag{5.6}$$

where

$$Y(s) = \int_0^s (s-u)^{-\alpha}S(s-u)\Phi i\,dB_Q^H(u) \quad \text{for all } s \geq 0. \tag{5.7}$$

For each $\epsilon \in (0,1]$ and $t \in [0,1]$ we have

$$\begin{aligned} X(\epsilon t) &= S(\epsilon t)x + \int_0^{\epsilon t} S(\epsilon t - s)\Phi i\,dB_Q^H(s) \\ &= S(\epsilon t)x + \sum_{j=1}^\infty \int_0^{\epsilon t} S(\epsilon t - s)\Phi i e_j\,d\beta_j^H(s) \\ &= S(\epsilon t)x + \sum_{j=1}^\infty \epsilon^H \int_0^t S(\epsilon(t-s))\Phi i e_j\,d\beta_j^{H,\epsilon}(s) \\ &= S(\epsilon t)x + \epsilon^H \int_0^t S(\epsilon(t-s))\Phi i\,dB_Q^{H,\epsilon}(s), \end{aligned} \tag{5.8}$$

where $((\beta_j^{H,\epsilon}(t))_{t\geq 0})$ is a sequence of independent fractional Brownian motions with Hurst parameter H defined by

$$\beta_j^{H,\epsilon}(t) := \epsilon^{-H}\beta_j^H(\epsilon t) \quad \text{for all } t \geq 0$$

and

$$B_Q^{H,\epsilon}(t) := \sum_{j=1}^{\infty} \beta_j^{H,\epsilon}(t) e_j \quad \text{for all } t \geq 0.$$

Notice from equation (5.8) and Lemma 5.7(2) in the appendix that the distribution of the random variable

$$(\omega \in \Omega \mapsto (t \mapsto X(\epsilon t)(\omega)))$$

in $C([0,1];V)$ is the same as the distribution of the continuous trajectories of

$$X^\epsilon(t) := S(\epsilon t)x + \epsilon^H \int_0^t S(\epsilon(t-s))\Phi i \, dB_Q^H(s) \ , \quad t \in [0,1].$$

The process $(X^\epsilon(t))_{t\in[0,1]}$ is the mild solution of the initial value problem

$$\begin{aligned} dX^\epsilon &= \epsilon A X^\epsilon \, dt + \epsilon^H \Phi \, dB_Q^H \ , \quad t \in [0,1], \\ X^\epsilon(0) &= x. \end{aligned}$$

For each $\epsilon \in (0,1]$ we set

$$W_{\epsilon A}^H(t) := \int_0^t S(\epsilon(t-s))\Phi i \, dB_Q^H(s) \quad \text{for all } t \in [0,1]$$

and assume that this is a continuous version of the process. We also set

$$W_0^H(t) := \Phi B_Q^H(t) \quad \text{for all } t \in [0,1].$$

In the following we denote by $W_{\epsilon A}^H$, W_0^H and X^ϵ the $C([0,1];V)$-valued random variables corresponding to the continuous processes $(W_{\epsilon A}^H(t))_{t\in[0,1]}$, $(W_0^H(t))_{t\in[0,1]}$ and $(X^\epsilon(t))_{t\in[0,1]}$, respectively. We also abuse notation and denote the constant function

$$t \in [0,1] \mapsto x$$

by x.

Define the function $K_H : [0,1] \times [0,1] \to \mathbb{R}$ by

$$K_H(t,s) := \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}} 1_{(0,\infty)}(s) s^{\frac{1}{2}-H} \, 1_{(0,\infty)}(t-s) \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} \, du \tag{5.9}$$

for all $(t,s) \in [0,1] \times [0,1]$, where β denotes the beta function. Define the function $\mathcal{I} : C([0,1];V) \to [0,\infty]$ by

$$\mathcal{I}(y) := \frac{1}{2} \inf \left\{ \int_0^1 |f(t)|^2_{U_\nu} \, dt : f \in L^2([0,1];U_\nu) \text{ and } y(t) = \int_0^t K_H(t,s) \Phi i f(s) \, ds \ \ \forall t \in [0,1] \right\}, \tag{5.10}$$

taking the infimum of the empty set to be ∞.

We now state our main result in this chapter.

Theorem 5.1 *For any closed set $F \subset C([0,1];V)$ we have*

$$\lim_{r \to 0} \sup_{\epsilon < r} \epsilon^{2H} \log P\{X^\epsilon \in F\} \le - \inf_{y \in F} \mathcal{I}(y-x) \tag{5.11}$$

and for any open set $G \subset C([0,1];V)$ we have

$$\lim_{r \to 0} \inf_{\epsilon < r} \epsilon^{2H} \log P\{X^\epsilon \in G\} \ge - \inf_{y \in G} \mathcal{I}(y-x). \tag{5.12}$$

Remark If we substitute $\delta = \epsilon^{2H}$ in inequalities (5.11) and (5.12) then we get the usual form of a large deviation principle for $\{X^{\delta^{\frac{1}{2H}}} : \delta \in (0,1]\}$:

$$\lim_{r \to 0} \sup_{\delta < r} \delta \log P\{X^{\delta^{\frac{1}{2H}}} \in F\} \le - \inf_{y \in F} \mathcal{I}(y-x)$$

for all closed sets $F \subset C([0,1];V)$ and

$$\lim_{r \to 0} \inf_{\delta < r} \delta \log P\{X^{\delta^{\frac{1}{2H}}} \in G\} \ge - \inf_{y \in G} \mathcal{I}(y-x)$$

for all open sets $G \subset C([0,1];V)$.

We prove Theorem 5.1 using several lemmas. The following lemma and its corollary are the basic results underlying Theorem 5.1.

Lemma 5.2 *The distribution*

$$\mu := \mathcal{L}(W_0^H : (\Omega, \mathcal{F}, P) \to C([0,1];V))$$

is symmetric Gaussian and its reproducing kernel $(H_\mu, |\cdot|_{H_\mu})$ *is*

$$H_\mu = \left\{ t \in [0,1] \mapsto \int_0^t K_H(t,s)\Phi i f(s)\, ds : f \in L^2([0,1]; U_\nu) \right\},$$

where $K_H : [0,1] \times [0,1] \to \mathbb{R}$ *is defined in equation (5.9) and for* $g \in H_\mu$ *we have*

$$|g|_{H_\mu}^2 = \inf \left\{ \int_0^1 |f(t)|_{U_\nu}^2\, dt : f \in L^2([0,1]; U_\nu) \ \ \textit{and}\ g(t) = \int_0^t K_H(t,s)\Phi i f(s)\, ds \ \ \forall t \in [0,1] \right\}.$$

Proof. There are three steps in the proof:

1. show that $(H_\mu, |\cdot|_{H_\mu})$ is a Hilbert space;
2. show that the embedding $j : (H_\mu, |\cdot|_{H_\mu}) \to C([0,1]; V)$ is continuous;
3. show that for every continuous linear functional l on $C([0,1]; V)$ we have

$$\mathcal{L}(l \circ W_0^H : (\Omega, \mathcal{F}, P) \to \mathbb{R}) = \mathcal{N}(0, \|l \circ j\|_{H_\mu^*}^2),$$

   that is, the symmetric Gaussian distribution with variance the square of the operator norm of $l \circ j$.

Step 1.
Define $F : [0,1]^3 \to \mathbb{R}$ by

$$F(u,s,t) := 1_{(0,\infty)}(s)\, s^{\frac{1}{2}-H}\, 1_{(0,\infty)}(u-s)\, 1_{(0,\infty)}(t-u)\, (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}$$

for all $(u,s,t) \in [0,1]^3$. The function F is measurable and non-negative and

$$K_H(t,s) = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}} \int_0^1 F(u,s,t)\, du \quad \forall (t,s) \in [0,1]^2$$

is measurable by Tonelli's theorem. Set

$$c_H = \left(\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})} \right)^{\frac{1}{2}}.$$

We have

$$|K_H(t,s)| \le c_H\, s^{\frac{1}{2}-H}\, 1_{(0,\infty)}(s) \frac{(t-s)^{H-\frac{1}{2}}}{H-\frac{1}{2}} 1_{(0,\infty)}(t-s) \quad \forall (t,s) \in [0,1]^2. \tag{5.13}$$

In particular, $K_H(t,\cdot) \in L^2([0,1];\mathbb{R})$ for each $t \in [0,1]$.
Let $f \in L^2([0,1];V)$. We shall show that the function

$$g(t) := \int_0^t K_H(t,s) f(s)\, ds \quad \text{for all } t \in [0,1] \tag{5.14}$$

belongs to $C([0,1];V)$. Let $0 \le t_1 < t_2 \le 1$. We have

$$|g(t_2)-g(t_1)|_V \le \left| \int_0^{t_1} (K_H(t_2,s) - K_H(t_1,s)) f(s)\, ds \right|_V + \left| \int_{t_1}^{t_2} K_H(t_2,s) f(s)\, ds \right|_V . \tag{5.15}$$

For $s \in (0,t_1)$ we have

$$\begin{aligned} K_H(t_2,s) - K_H(t_1,s) &= c_H s^{\frac{1}{2}-H} \int_{t_1}^{t_2} (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}}\, du \\ &\le c_H s^{\frac{1}{2}-H} \frac{1}{H-\frac{1}{2}} \left[(t_2-s)^{H-\frac{1}{2}} - (t_1-s)^{H-\frac{1}{2}} \right] \end{aligned}$$

and the factor in brackets on the right hand side can be made as small as we please, uniformly in s, by taking t_1 and t_2 sufficiently close together. This observation and inequality (5.13) imply that the terms on the right hand side of inequality (5.15) go to zero as $t_2 - t_1 \to 0$. Thus $g \in C([0,1];V)$.
We can now define a bounded linear operator $T : L^2([0,1];U_\nu) \to C([0,1];V)$ by

$$(Tf)(t) := \int_0^t K_H(t,s)\, \Phi i f(s)\, ds \quad \text{for all } t \in [0,1] \tag{5.16}$$

and for all $f \in L^2([0,1];U_\nu)$. Let N be the kernel of T and let $N^\perp$ be the orthogonal complement of N. By the projection theorem (for example, see [26, Theorem II.3]), each element of $L^2([0,1];U_\nu)$ can be written uniquely as the sum of an element of N and an element of $N^\perp$. Thus we have

$$T(N^\perp) = T(L^2([0,1];U_\nu))$$

and the function $\tilde{T} : N^\perp \to T(L^2([0,1];U_\nu))$ defined by

$$\tilde{T}v := Tv \quad \forall v \in N^\perp$$

is onto as well as one to one. For each $v \in N^{\perp}$ we have

$$|v|_{L^2([0,1];U_\nu)} = \inf\{|u|_{L^2([0,1];U_\nu)} \,:\, u \in L^2([0,1];U_\nu) \text{ and } Tu = Tv\}.$$

Define

$$\begin{aligned} H_\mu &:= T(L^2([0,1];U_\nu)) \quad \text{and} \\ |f|_{H_\mu} &:= |(\tilde{T})^{-1} f|_{L^2([0,1];U_\nu)} \quad \text{for all } f \in H_\mu. \end{aligned}$$

Then $\tilde{T}$ is an isometric isomorphism from the closed subspace $N^{\perp}$ of $L^2([0,1];U_\nu)$ onto H_μ. Thus H_μ is itself a Hilbert space.

Step 2.

We now prove that the embedding $j : H_\mu \to C([0,1];V)$ is continuous. Let $f \in H_\mu$. For $g = (\tilde{T})^{-1} f$ we have

$$\begin{aligned} |f(t)|_V &= \left| \int_0^t K_H(t,s) \Phi i g(s)\, ds \right|_V \quad \text{for all } t \in [0,1] \\ &\leq \|\Phi i\|_{L(U_\nu,V)} \frac{c_H}{(H-\frac{1}{2})(2-2H)^{\frac{1}{2}}} |g|_{L^2([0,1];U_\nu)}. \end{aligned}$$

Thus

$$\sup_{t\in[0,1]} |f(t)|_V \leq \|\Phi i\|_{L(U_\nu,H)} \frac{c_H}{(H-\frac{1}{2})(2-2H)^{\frac{1}{2}}} |f|_{H_\mu} \quad \forall f \in H_\mu.$$

Step 3.

For each $t \in [0,1]$ and $v \in V$ let $\delta_t \otimes v$ be the continuous linear functional on $C([0,1];V)$ defined by

$$(\delta_t \otimes v) f = \langle v, f(t)\rangle_V \quad \text{for all } f \in C([0,1];V).$$

By [10, Proposition 2.9] it suffices to show that

$$\mathcal{L}\left(l \circ W_0^H\right) = \mathcal{N}(0, \|l \circ j\|^2_{H_\mu^*})$$

for all $l \in M := \{\sum_{j=1}^n \delta_{t_j} \otimes v_j :\ n \in \mathbb{N} \text{ and } 0 \leq t_1 < \cdots < t_n \leq 1 \text{ and } v_1, \ldots, v_n \in V\}$, since this subspace of the continuous linear functionals on $C([0,1];V)$ separates points and generates the Borel σ-algebra of $C([0,1];V)$.

Let $n \in \mathbb{N}$ and $0 \le t_1 < \cdots < t_n \le 1$ and $v_1, \ldots, v_n \in V$. The distribution of

$$\sum_{j=1}^{n} (\delta_{t_j} \otimes v_j)(W_0^H) = \sum_{j=1}^{n} \langle v_j, W_0^H(t_j) \rangle_V = \sum_{j=1}^{n} \langle \Phi^* v_j, B_Q^H(t_j) \rangle_U$$

is symmetric Gaussian since $(B_Q^H(t))_{t \in [0,1]}$ is a Gaussian process and its variance is

$$\begin{aligned} E\left[\left(\sum_{j=1}^{n} \langle \Phi^* v_j,\, B_Q^H(t_j) \rangle_U\right)^2\right] &= \sum_{j=1}^{n}\sum_{i=1}^{n} E\left[\langle \Phi^* v_j,\, B_Q^H(t_j) \rangle_U \langle \Phi^* v_i,\, B_Q^H(t_i) \rangle_U\right] \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{1}{2}(t_j^{2H} + t_i^{2H} - |t_j - t_i|^{2H}) \langle Q\Phi^* v_j,\, \Phi^* v_i \rangle_U. \quad (5.17) \end{aligned}$$

Each element of H_μ can be written as

$$t \in [0,1] \mapsto \int_0^t K_H(t,s) \Phi i f(s)\, ds,$$

where $f \in N^\perp$ and we have

$$\begin{aligned} \sum_{k=1}^{n} (\delta_{t_k} \otimes v_k) j \left(t \in [0,1] \mapsto \int_0^t K_H(t,s)\Phi i f(s)\, ds\right) &= \sum_{k=1}^{n} \langle v_k, \int_0^{t_k} K_H(t_k, s)\Phi i f(s)\, ds \rangle_V \\ &= \sum_{k=1}^{n} \int_0^{t_k} K_H(t_k, s) \langle v_k,\, \Phi i f(s) \rangle_V\, ds \\ &= \int_0^1 \langle \sum_{k=1}^{n} K_H(t_k, s) Q\Phi^* v_k,\, f(s) \rangle_{U_\nu}\, ds. \end{aligned}$$

Notice that $\sum_{k=1}^{n} K_H(t_k, \cdot) Q\Phi^* v_k \in N^\perp$. Thus

$$\begin{aligned} \left\| \sum_{k=1}^{n} (\delta_{t_k} \otimes v_k) \circ j \right\|_{H_\mu^*}^2 &= \left| \sum_{k=1}^{n} K_H(t_k, \cdot) Q\Phi^* v_k \right|_{L^2([0,1];U_\nu)}^2 \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} \langle Q\Phi^* v_k, Q\Phi^* v_i \rangle_{U_\nu} \int_0^1 K_H(t_k, s) K_H(t_i, s)\, ds \\ &= \sum_{k=1}^{n}\sum_{i=1}^{n} \langle Q\Phi^* v_k, \Phi^* v_i \rangle_U \frac{1}{2}(t_k^{2H} + t_i^{2H} - |t_k - t_i|^{2H}), \quad (5.18) \end{aligned}$$

where the last line follows from [2, equation (6)]. Since the right hand sides of equations (5.17) and (5.18) are the same this completes the proof of Lemma 5.2.

From [10, Propositions 12.4 and 12.6] we have the following corollary.

Corollary 5.3 *The family of $C([0,1];V)$-valued random variables $\{\delta^{\frac{1}{2}}W_0^H : \delta \in (0,1]\}$ satisfies a large deviation principle with rate function $\mathcal{I} : C([0,1];V) \to [0,\infty]$ defined by*

$$\mathcal{I}(g) = \begin{cases} \frac{1}{2}|g|^2_{H_\mu} & \text{if } g \in H_\mu, \\ \infty & \text{if } g \in C([0,1];V)\backslash H_\mu. \end{cases} \tag{5.19}$$

From Corollary 5.3 we have immediately that $\{x + \delta^{\frac{1}{2}}W_0^H : \delta \in (0,1]\}$ satisfies a large deviation principle with rate function $\mathcal{I}(\cdot - x)$.

Our goal in most of the remainder of this chapter is to prove the following lemma, which is crucial in our proof of Theorem 5.1.

Lemma 5.4 *Let $r > 0$. We have*

$$\lim_{\epsilon \to 0} \epsilon^{2H} \log P\{\epsilon^H \sup_{t\in[0,1]} |W^H_{\epsilon A}(t) - W^H_0(t)|_V \geq r\} = -\infty. \tag{5.20}$$

Proof of Theorem 5.1. Lemma 5.4 is equivalent (just substitute $\delta = \epsilon^{2H}$ in equation (5.20)) to saying that the families of random variables

$$\{x + \delta^{\frac{1}{2}}W_0^H : \delta \in (0,1]\} \text{ and } \{x + \delta^{\frac{1}{2}}W^H_{\delta^{\frac{1}{2H}}A} : \delta \in (0,1]\}$$

are exponentially equivalent; hence both families satisfy the same large deviation principle (see for example [18, Lemma 27.13]). Since $S(t)x \to x$ as $t \to 0$ we have that

$$\{x + \delta^{\frac{1}{2}}W^H_{\delta^{\frac{1}{2H}}A} : \delta \in (0,1]\} \text{ and } \{S(\delta^{\frac{1}{2H}}\cdot)x + \delta^{\frac{1}{2}}W^H_{\delta^{\frac{1}{2H}}A} : \delta \in (0,1]\}$$

are exponentially equivalent. Thus Lemma 5.4 implies that $\{S(\delta^{\frac{1}{2H}}\cdot)x + \delta^{\frac{1}{2}}W^H_{\delta^{\frac{1}{2H}}A} : \delta \in (0,1]\}$ satisfies a large deviation principle with rate function $\mathcal{I}(\cdot - x)$, which completes the proof of Theorem 5.1.

We shall need two other lemmas in order to prove Lemma 5.4.

Lemma 5.5 *If $m \in \{2,3,4,\ldots\}$ then*

$$\lim_{\epsilon \to 0} E\left[\sup_{t\in[0,1]} |W^H_0(t) - W^H_{\epsilon A}(t)|^{2m}_V\right] = 0.$$

Remark Using this lemma and Hölder's inequality we get

$$
\begin{aligned}
E\left[\sup_{t\in[0,1]} |W_0^H(t) - W_{\epsilon A}^H(t)|_V^2\right] &\leq \left(E\left[\sup_{t\in[0,1]} |W_0^H(t) - W_{\epsilon A}^H(t)|_V^{2m}\right]\right)^{\frac{1}{m}} \\
&\rightarrow 0 \text{ as } \epsilon \rightarrow 0.
\end{aligned}
$$

Proof. Let m be a natural number greater than 1. Take $\alpha \in (\frac{1}{2m}, \frac{1}{2})$. For $0 \leq \epsilon \leq 1$ define

$$
Y^\epsilon(s) := \int_0^s (s-\sigma)^{-\alpha} S(\epsilon(s-\sigma))\Phi i \, dB_Q^H(\sigma) \quad \text{for all } s \in [0,1].
$$

By Lemma 5.8 in the appendix, the process $(Y^\epsilon(s))_{s\in[0,1]}$ has a measurable version whose sample paths are in $L^{2m}([0,1];V)$ almost surely.
If $f \in L^{2m}([0,1];V)$ then, as shown in [10, Theorem 5.9], the V-valued function defined by

$$
t \mapsto \int_0^t S(\epsilon(t-\sigma))(t-\sigma)^{\alpha-1} f(\sigma)\, d\sigma \quad \text{for all } t \in [0,1]
$$

is continuous. A continuous version of $(W_{\epsilon A}^H(t))_{t\in[0,1]}$ is:

$$
W_{\epsilon A}^H(t) = \frac{\sin(\pi\alpha)}{\pi} \int_0^t S(\epsilon(t-s))(t-s)^{\alpha-1} Y^\epsilon(s)\, ds \quad \text{for all } t \in [0,1].
$$

For $0 < \epsilon \leq 1$ we have

$$
\begin{aligned}
W_0^H(t) - W_{\epsilon A}^H(t) &= \frac{\sin(\pi\alpha)}{\pi} \int_0^t (I_V - S(\epsilon(t-s)))(t-s)^{\alpha-1} Y^0(s)\, ds + \\
&\quad \frac{\sin(\pi\alpha)}{\pi} \int_0^t S(\epsilon(t-s))(t-s)^{\alpha-1}(Y^0(s) - Y^\epsilon(s))\, ds \quad (5.21) \\
&=: K_\epsilon(t) + J_\epsilon(t) \quad \text{for all } t \in [0,1],
\end{aligned}
$$

where $K_\epsilon(t)$ and $J_\epsilon(t)$ are defined to be the respective terms on the right hand side of equation (5.21). Thus

$$
\sup_{t\in[0,1]} |W_0^H(t) - W_{\epsilon A}^H(t)|_V^{2m} \leq 2^{2m} \left(\sup_{t\in[0,1]} |K_\epsilon(t)|_V^{2m} + \sup_{t\in[0,1]} |J_\epsilon(t)|_V^{2m}\right).
$$

We will show that

$$
\lim_{\epsilon\rightarrow 0} E\left[\sup_{t\in[0,1]} |K_\epsilon(t)|_V^{2m}\right] = 0
$$

and

$$\lim_{\epsilon\to 0} E\left[\sup_{t\in[0,1]} |J_\epsilon(t)|_V^{2m}\right] = 0,$$

from which it will follow that $\lim_{\epsilon\to 0} E[\sup_{t\in[0,1]} |W_0^H(t) - W_{\epsilon A}^H(t)|_V^{2m}] = 0$.

Step 1.

We show that

$$\lim_{\epsilon\to 0} E\left[\sup_{t\in[0,1]} |K_\epsilon(t)|_V^{2m}\right] = 0. \tag{5.22}$$

For each $0 < \epsilon \leq 1$ we have, by Hölder's inequality:

$$\sup_{t\in[0,1]} |K_\epsilon(t)|_V^{2m} \leq \left(\frac{\sin(\pi\alpha)}{\pi}\right)^{2m} \sup_{r\in[0,1]} \|I_V - S(r)\|_{L(V,V)}^{2m} \left(\int_0^1 s^{(\alpha-1)\frac{2m}{2m-1}}\, ds\right)^{2m-1} \int_0^1 |Y^0(s)|_V^{2m}\, ds$$

and the right hand side is an integrable dominating function, as shown in the proof of Lemma 5.8. If $\sup_{t\in[0,1]} |K_\epsilon(t)|_V \to 0$ as $\epsilon \to 0$ almost surely, then equation (5.22) will follow by Lebesgue's dominated convergence theorem. Let $\omega \in \Omega$ be such that the sample path $s \in [0,1] \mapsto Y^0(s)(\omega)$ belongs to $L^{2m}([0,1];V)$; we will show that $\sup_{t\in[0,1]} |K_\epsilon(t)(\omega)|_V \to 0$ as $\epsilon \to 0$.

Let the sequence $(\epsilon_n) \subset (0,1]$ converge to 0 as n goes to infinity. Firstly note that for each fixed $t \in [0,1]$, $K_{\epsilon_n}(t)(\omega) \to 0$ as $n \to \infty$:

$$\begin{aligned} |K_{\epsilon_n}(t)(\omega)|_V &\leq \left(\frac{\sin(\pi\alpha)}{\pi}\right)\left(\int_0^1 s^{(\alpha-1)\frac{2m}{2m-1}}\, ds\right)^{\frac{2m-1}{2m}} \left(\int_0^t |(I_V - S(\epsilon_n(t-s)))Y^0(s)(\omega)|_V^{2m}\, ds\right)^{\frac{1}{2m}} \\ &\to 0 \text{ as } n \to \infty \text{ by Lebesgue's dominated convergence theorem.} \end{aligned}$$

Next we claim that the family of continuous functions

$$t \mapsto K_\epsilon(t)(\omega)\,, \quad 0 < \epsilon \leq 1,$$

is uniformly equicontinuous. For brevity, set

$$\begin{aligned} y(s) &= Y^0(s)(\omega) \quad \text{for all } s \in [0,1] \text{ and} \\ z^\epsilon(s) &= \frac{\pi}{\sin(\pi\alpha)} K_\epsilon(s)(\omega) \quad \text{for all } s \in [0,1] \text{ and } 0 < \epsilon \leq 1. \end{aligned}$$

By definition:

$$z^\epsilon(t) = \int_0^t (I_V - S(\epsilon(t-s)))(t-s)^{\alpha-1} y(s)\, ds \quad \text{for all } t \in [0,1] \text{ and } 0 < \epsilon \leq 1.$$

If $0 < \epsilon \le 1$ and $0 \le t < u \le 1$ then

$$
\begin{aligned}
&|z^\epsilon(u) - z^\epsilon(t)|_V \\
\le\ & \left| \int_0^{u-t} (I_V - S(\epsilon(u-s)))(u-s)^{\alpha-1} y(s)\, ds \right|_V \\
& + \left| \int_{u-t}^{u} (I_V - S(\epsilon(u-s)))(u-s)^{\alpha-1} y(s)\, ds - \int_0^t (I_V - S(\epsilon(t-s)))(t-s)^{\alpha-1} y(s)\, ds \right|_V \\
=\ & \left| \int_0^{u-t} (I_V - S(\epsilon(u-s)))(u-s)^{\alpha-1} y(s)\, ds \right|_V \\
& + \left| \int_0^{t} (I_V - S(\epsilon(t-s)))(t-s)^{\alpha-1} (y(s+u-t) - y(s))\, ds \right|_V \\
\le\ & \sup_{r \in [0,1]} \|I_V - S(r)\|_{L(V,V)} \left(\int_0^1 s^{(\alpha-1)\frac{2m}{2m-1}}\, ds \right)^{\frac{2m-1}{2m}} \\
& \times \left[\left(\int_0^{u-t} |y(s)|_V^{2m} ds \right)^{\frac{1}{2m}} + \left(\int_0^{1-(u-t)} |y(s+u-t) - y(s)|_V^{2m}\, ds \right)^{\frac{1}{2m}} \right]. \qquad (5.23)
\end{aligned}
$$

One can show that

$$\left(\int_0^{1-\delta} |y(s+\delta) - y(s)|_V^{2m}\, ds \right)^{\frac{1}{2m}} \to 0 \text{ as } \delta \searrow 0$$

so inequality (5.23) establishes that $\{z^\epsilon : 0 < \epsilon \le 1\}$ is uniformly equicontinuous.
We know that $z^{\epsilon_n}(t) \to 0$ as $n \to \infty$ for each t in $[0,1]$. Uniform equicontinuity of the sequence (z^{ϵ_n}) implies that there is uniform convergence to 0. This completes the proof of equation (5.22).

Step 2.
We show that

$$\lim_{\epsilon \to 0} E\left[\sup_{t \in [0,1]} |J_\epsilon(t)|_V^{2m} \right] = 0.$$

We have

$$
\begin{aligned}
\sup_{t \in [0,1]} |J_\epsilon(t)|_V^{2m} \le\ & \left(\frac{\sin(\pi\alpha)}{\pi} \right)^{2m} \left(\sup_{r \in [0,1]} \|S(r)\|_{L(V,V)} \right)^{2m} \left(\int_0^1 s^{(\alpha-1)\frac{2m}{2m-1}}\, ds \right)^{2m-1} \\
& \times \int_0^1 |Y^0(s) - Y^\epsilon(s)|_V^{2m}\, ds. \qquad (5.24)
\end{aligned}
$$

For each s in $(0,1]$ the random variable $Y^0(s) - Y^\epsilon(s)$ has a symmetric Gaussian distribution so that for some constant $C_m \in (0,\infty)$ we have

$$E\left[|Y^0(s) - Y^\epsilon(s)|_V^{2m}\right] \le C_m \left(E\left[|Y^0(s) - Y^\epsilon(s)|_V^2\right]\right)^m \tag{5.25}$$

for all $s \in (0,1]$. We have for each s in $[0,1]$

$$Y^0(s) - Y^\epsilon(s) = \int_0^s (s-\sigma)^{-\alpha}(I_V - S(\epsilon(s-\sigma)))\Phi i \, dB_Q^H(\sigma)$$

and thus, by inequality (5.4),

$$\begin{aligned} E\left[|Y^0(s) - Y^\epsilon(s)|_V^2\right] &\le 2H \sup_{r\in[0,1]} \|(I_V - S(\epsilon r))\Phi i\|_{L_2(U_\nu,V)}^2 \int_0^1 \sigma^{-2\alpha}\, d\sigma \\ &\to 0 \text{ as } \epsilon \to 0\ . \end{aligned} \tag{5.26}$$

Thus from inequalities (5.24) and (5.25) and (5.26):

$$\begin{aligned} E\left[\sup_{t\in[0,1]} |J_\epsilon(t)|_V^{2m}\right] &\le \left(\frac{\sin(\pi\alpha)}{\pi} \sup_{r\in[0,1]} \|S(r)\|_{L(V,V)}\right)^{2m} \left(\int_0^1 s^{(\alpha-1)\frac{2m}{2m-1}}\, ds\right)^{2m-1} \\ &\quad \times \int_0^1 C_m \left(E\left[|Y^0(s) - Y^\epsilon(s)|_V^2\right]\right)^m ds \\ &\to 0 \text{ as } \epsilon \to 0. \end{aligned}$$

This completes the proof of Lemma 5.5.

It will be convenient to denote the supremum norm in $C([0,1];V)$ by $|\cdot|_\infty$:

$$|f|_\infty := \sup_{t\in[0,1]} |f(t)|_V \quad \text{for all } f \in C([0,1];V).$$

Set

$$\delta_\epsilon^2 = E[|W_{\epsilon A}^H - W_0^H|_\infty^2] \quad \text{for all } \epsilon \in (0,1].$$

Lemma 5.6 *There is a positive real number c such that*

$$\sup_{0<\epsilon\le 1} E\left[\exp\left(c\frac{|W_{\epsilon A}^H - W_0^H|_\infty^2}{\delta_\epsilon^2}\right)\right] < \infty.$$

Proof. By Lemma 5.7, for each $\epsilon \in (0,1]$ the distribution $\nu_\epsilon := \mathcal{L}\left(\frac{1}{\delta_\epsilon}(W_{\epsilon A}^H - W_0^H)\right)$ is a symmetric Gaussian measure on $C([0,1];V)$. The rest of the proof is the same as that of

Lemma 4.9 in chapter 4, but with ν_ϵ in place of μ_ϵ everywhere. This completes the proof of Lemma 5.6.

We now prove Lemma 5.4.

Proof of Lemma 5.4. Let $r > 0$. For any positive real number c we have

$$\exp\left(c\frac{r^2}{\epsilon^{2H}\delta_\epsilon^2}\right) 1_{\{|\epsilon^H(W_{\epsilon A}^H - W_0^H)|_\infty \geq r\}} \leq \exp\left(c\frac{|W_{\epsilon A}^H - W_0^H|_\infty^2}{\delta_\epsilon^2}\right) \quad \text{for all } \epsilon \in (0,1]. \quad (5.27)$$

Choose c, as in Lemma 5.6, such that

$$C := \sup_{0<\epsilon\leq 1} E\left[\exp\left(c\frac{|W_{\epsilon A}^H - W_0^H|_\infty^2}{\delta_\epsilon^2}\right)\right] < \infty.$$

Then integrating both sides of equation (5.27) gives:

$$P\{|\epsilon^H(W_{\epsilon A}^H - W_0^H)|_\infty \geq r\} \leq \exp\left(-c\frac{r^2}{\epsilon^{2H}\delta_\epsilon^2}\right) C \quad \text{for all } \epsilon \in (0,1].$$

Thus

$$\begin{aligned} \epsilon^{2H} \log P\{|\epsilon^H(W_{\epsilon A}^H - W_0^H)|_\infty \geq r\} \quad &\leq \quad -c\frac{r^2}{\delta_\epsilon^2} + \epsilon^{2H}\log C \\ &\to \quad -\infty \quad \text{as } \epsilon \to 0 \end{aligned}$$

since Lemma 5.5 implies that $\delta_\epsilon^2 \to 0$ as $\epsilon \to 0$. This completes the proof of Lemma 5.4.

5.4 Appendix

In this section we have two lemmas whose proofs are technical but routine.

Let

$$G_t : ([0,1], \mathcal{B}_{[0,1]}) \to (L_2(U_\nu, V), \mathcal{B}_{L_2(U_\nu,V)}) \,, \quad t \in [0,1],$$

be a family of measurable functions such that

$$\int_0^1 \|G_t(s)\|_{L_2(U_\nu,V)}^2 \, ds < \infty \quad \text{for each } t \in [0,1]$$

and the process $(\int_0^1 G_t(s)\, dB_Q^H(s))_{t\in[0,1]}$ has continuous trajectories in V. In particular we have in mind continuous stochastic convolution processes, where

$$G_t(\cdot) = 1_{[0,t]}(\cdot)\, S(\epsilon(t-\cdot))\Phi i \quad \text{for some } \epsilon \in [0,1].$$

Lemma 5.7

1. *The distribution of the random variable in* $C([0,1];V)$

$$G(\omega) := \left(t \in [0,1] \mapsto \int_0^1 G_t(s)\, dB_Q^H(s)(\omega) \right), \quad \omega \in \Omega,$$

   is symmetric Gaussian.

2. *If* $(\tilde{B}_Q^H(t))_{t\geq 0}$ *is another fractional* Q*-Brownian motion with Hurst parameter* H *and* $(\int_0^1 G_t(s)\, d\tilde{B}_Q^H(s))_{t\in[0,1]}$ *has continuous trajectories in* V *then the random variable in* $C([0,1];V)$

$$\tilde{G}(\omega) := \left(t \in [0,1] \mapsto \int_0^1 G_t(s)\, d\tilde{B}_Q^H(s)(\omega) \right), \quad \omega \in \Omega,$$

   has the same distribution as G.

Proof. In section 5.2 we saw that for $(\beta^H(t))_{t\geq 0}$ a real-valued fractional Brownian motion with Hurst parameter H and arbitrary $f \in L^2([0,1];\mathbb{R})$ the distribution of $\int_0^1 f(s)\, d\beta^H(s)$ is symmetric Gaussian, with variance $\int_0^1 \int_0^1 f(s)f(t)\phi(t-s)\, ds\, dt$ given by equation (5.2). Let $\{G_t : [0,1] \to L_2(U_\nu, V)\}_{t\in[0,1]}$ be a family of functions as in the statement preceding the lemma. For each $t \in [0,1]$ and $v \in V$ we define the element of $C([0,1];V)^*$:

$$(\delta_t \otimes v) f := \langle f(t), v\rangle_V \quad \forall f \in C([0,1];V).$$

The subspace of $C([0,1];V)^*$

$$\mathcal{M} := \left\{ \sum_{j=1}^n \delta_{t_j} \otimes v_j \;:\; n \in \mathbb{N} \text{ and } 0 \leq t_1 < \cdots < t_n \leq 1 \text{ and } v_1, \ldots, v_n \in V \right\}$$

generates $\mathcal{B}_{C([0,1];V)}$ (for example, see the proof of [10, Proposition 1.3]) and it also separates points of $C([0,1];V)$. We will show that for an arbitrary element $\sum_{j=1}^n \delta_{t_j} \otimes v_j$ of $\mathcal{M}$ the distribution of

$$\sum_{j=1}^n (\delta_{t_j} \otimes v_j) G = \sum_{j=1}^n \langle \int_0^1 G_{t_j}(s)\, dB_Q^H(s), v_j\rangle_V$$

is symmetric Gaussian and we will compute its variance. We have

$$\begin{aligned}
&\sum_{j=1}^{n}\langle\int_0^1 G_{t_j}(s)\,dB_Q^H(s), v_j\rangle_V \\
&= \sum_{j=1}^{n}\langle\sum_{k=1}^{\infty}\int_0^1 G_{t_j}(s)e_k\,d\beta_k^H(s), v_j\rangle_V \\
&= \sum_{k=1}^{\infty}\sum_{j=1}^{n}\langle\int_0^1 G_{t_j}(s)e_k\,d\beta_k^H(s), v_j\rangle_V \quad \text{(the series converges in } L^2(\Omega,\mathcal{F},P;\mathbb{R})) \\
&= \sum_{k=1}^{\infty}\int_0^1 \langle e_k, \sum_{j=1}^{n} G_{t_j}^*(s)v_j\rangle_{U_\nu}\,d\beta_k^H(s).
\end{aligned}$$

In the last line the summands are independent random variables. The kth summand has symmetric Gaussian distribution on $\mathbb{R}$ with variance

$$\int_0^1\int_0^1 \langle e_k, \sum_{j=1}^{n} G_{t_j}^*(s)v_j\rangle_{U_\nu}\langle e_k, \sum_{j=1}^{n} G_{t_j}^*(t)v_j\rangle_{U_\nu}\phi(t-s)\,ds\,dt.$$

Thus $\sum_{j=1}^{n}\langle\int_0^1 G_{t_j}(s)\,dB_Q^H(s), v_j\rangle_V$ has symmetric Gaussian distribution with variance

$$\begin{aligned}
&\int_0^1\int_0^1 \sum_{k=1}^{\infty}\langle e_k, \sum_{j=1}^{n} G_{t_j}^*(s)v_j\rangle_{U_\nu}\langle e_k, \sum_{j=1}^{n} G_{t_j}^*(t)v_j\rangle_{U_\nu}\phi(t-s)\,ds\,dt \\
&= \int_0^1\int_0^1 \langle\sum_{j=1}^{n} G_{t_j}^*(s)v_j, \sum_{j=1}^{n} G_{t_j}^*(t)v_j\rangle_{U_\nu}\phi(t-s)\,ds\,dt. \qquad (5.28)
\end{aligned}$$

Since $\sum_{j=1}^{n}\delta_{t_j}\otimes v_j$ is an arbitrary element of $\mathcal{M}$, by [10, Proposition 2.9(i)] the distribution of G in $C([0,1];V)$ is symmetric Gaussian. Notice that the expression on the right hand side of equation (5.28) depends on the fractional Brownian motion integrator only through the values of ν and H. Thus the characteristic functions of G and $\tilde{G}$ agree on $\mathcal{M}$; it follows that the distributions of G and $\tilde{G}$ on $C([0,1];V)$ are equal. This completes the proof of Lemma 5.7.

Let $m \in \{2,3,4,\ldots\}$. Let $\alpha \in (\frac{1}{2m}, \frac{1}{2})$. Let $\epsilon \in [0,1]$.
Define

$$Y^\epsilon(s) := \int_0^s (s-\sigma)^{-\alpha} S(\epsilon(s-\sigma))\Phi i\,dB_Q^H(\sigma) \quad \text{for all } s \in [0,1].$$

Lemma 5.8 *The map $s \in [0,1] \mapsto Y^\epsilon(s)$ is continuous in $L^2(\Omega,\mathcal{F},P;V)$. The process $(Y^\epsilon(s))_{s\in[0,1]}$ has a measurable version whose sample paths are in $L^{2m}([0,1];V)$ P a.e..*

Proof. Let $0 \le s < t \le 1$. We have

$$
\begin{aligned}
|Y^\epsilon(t) - Y^\epsilon(s)|_V &= \bigg| \int_0^t (t-\sigma)^{-\alpha} S(\epsilon(t-\sigma))\Phi i \, dB_Q^H(\sigma) \\
&\quad - \int_0^s (s-\sigma)^{-\alpha} S(\epsilon(s-\sigma))\Phi i \, dB_Q^H(\sigma) \bigg|_V \\
&\le \left| \int_0^t 1_{[0,s)}(\sigma)(t-\sigma)^{-\alpha}(S(\epsilon(t-\sigma)) - S(\epsilon(s-\sigma)))\Phi i \, dB_Q^H(\sigma) \right|_V \\
&\quad + \left| \int_0^t 1_{[0,s)}(\sigma)((t-\sigma)^{-\alpha} - (s-\sigma)^{-\alpha}) S(\epsilon(s-\sigma))\Phi i \, dB_Q^H(\sigma) \right|_V \\
&\quad + \left| \int_0^t 1_{[s,t)}(\sigma)(t-\sigma)^{-\alpha} S(\epsilon(t-\sigma))\Phi i \, dB_Q^H(\sigma) \right|_V \qquad (5.29) \\
&= |T_1|_V + |T_2|_V + |T_3|_V,
\end{aligned}
$$

where T_1, T_2 and T_3 are the three stochastic integrals on the right hand side of equation (5.29). We now show, using inequality (5.4), that $E[|T_k|_V^2]$ goes to 0 as $t-s$ goes to 0 for $k = 1, 2, 3$.

We have

$$
\begin{aligned}
E[|T_1|_V^2] &\le 2H \int_0^t 1_{[0,s)}(\sigma)(t-\sigma)^{-2\alpha} \|S(\epsilon(t-\sigma))\Phi i - S(\epsilon(s-\sigma))\Phi i\|_{L_2(U_\nu,V)}^2 \, d\sigma \\
&\le 2H \sup_{\substack{0 \le u < v \le 1 \\ \text{and } v-u \le t-s}} \|S(v)\Phi i - S(u)\Phi i\|_{L_2(U_\nu,V)}^2 \int_0^1 \sigma^{-2\alpha} \, d\sigma. \qquad (5.30)
\end{aligned}
$$

Uniform continuity of the function $r \in [0,1] \mapsto S(r)\Phi i \in L_2(U_\nu, V)$ ensures that the right hand side of equation (5.30) goes to 0 as $t-s$ goes to 0.

We have

$$
\begin{aligned}
E[|T_2|_V^2] &\le 2H \int_0^t 1_{[0,s)}(\sigma)((t-\sigma)^{-\alpha} - (s-\sigma)^{-\alpha})^2 \|S(\epsilon(s-\sigma))\Phi i\|_{L_2(U_\nu,V)}^2 \, d\sigma \\
&\le 2H \sup_{r\in[0,1]} \|S(r)\Phi i\|_{L_2(U_\nu,V)}^2 \int_0^s ((s-\sigma)^{-2\alpha} - (t-\sigma)^{-2\alpha}) \, d\sigma \\
&= 2H \sup_{r\in[0,1]} \|S(r)\Phi i\|_{L_2(U_\nu,V)}^2 \frac{1}{1-2\alpha}(s^{1-2\alpha} + (t-s)^{1-2\alpha} - t^{1-2\alpha}). \qquad (5.31)
\end{aligned}
$$

Since the function $r \in [0,1] \mapsto r^{1-2\alpha}$ is uniformly continuous, the right hand side of equation (5.31) goes to 0 as $t-s$ goes to 0.

We have

$$\begin{aligned} E[|T_3|_V^2] &\leq 2H \sup_{r\in[0,1]} \|S(r)\Phi i\|^2_{L_2(U_\nu,V)} \int_s^t (t-\sigma)^{-2\alpha}\, d\sigma \\ &= 2H \sup_{r\in[0,1]} \|S(r)\Phi i\|^2_{L_2(U_\nu,V)} \frac{(t-s)^{1-2\alpha}}{1-2\alpha} \\ &\to 0 \quad \text{as } t-s\to 0. \end{aligned}$$

Thus $s\in[0,1]\mapsto Y^\epsilon(s)\in L^2(\Omega,\mathcal{F},P;V)$ is continuous and, as shown in [10, Proposition 3.2], this implies that $(Y^\epsilon(t))_{t\in[0,1]}$ has a measurable version.
For each $s\in(0,1]$ the random variable $Y^\epsilon(s)$ has a symmetric Gaussian distribution on V and

$$E[|Y^\epsilon(s)|_V^2] \leq 2H \sup_{r\in[0,1]} \|S(r)\Phi i\|^2_{L_2(U_\nu,V)} \int_0^1 \sigma^{-2\alpha}\, d\sigma \quad \text{for all } s\in(0,1]. \tag{5.32}$$

By [10, Corollary 2.17], there is a constant $C_m\in(0,\infty)$ such that

$$E[|Y^\epsilon(s)|_V^{2m}] \leq C_m(E[|Y^\epsilon(s)|_V^2])^m \quad \text{for all } s\in(0,1]. \tag{5.33}$$

It follows from inequalities (5.32) and (5.33) that for a measurable version of $(Y^\epsilon(s))_{s\in[0,1]}$ we have

$$E\left[\int_0^1 |Y^\epsilon(s)|_V^{2m}\, ds\right] = \int_0^1 E[|Y^\epsilon(s)|_V^{2m}]\, ds < \infty.$$

Thus the sample paths $s\mapsto Y^\epsilon(s)(\omega)$ are in $L^{2m}([0,1];V)$ for P a.e. $\omega\in\Omega$. This completes the proof of Lemma 5.8.

Chapter 6

Small time asymptotics for moving from one set to another

6.1 Introduction

In this chapter we return to studying the solution $(X(t))$ of a stochastic equation in a Hilbert space H with Wiener process noise. We now consider the probability of the event $\{X(0) \in C,\, X(t) \in E\}$ as time t goes to zero; here the distribution of $X(0)$ need not be a point mass and C and E are Borel subsets of H. This problem is important because an evolving system modelled by a stochastic equation may be expected to have a random variable as its initial condition. For example, if the equation has an invariant measure then the solution whose initial distribution is the invariant measure is of particular interest.

Let $(H, \langle\cdot,\cdot\rangle, |\cdot|)$ be a separable Hilbert space. We assume

(**C1**) *there exists a process* $(X_\xi(t) : (\Omega, \mathcal{F}, P) \to (H, \mathcal{B}_H))_{t\in[0,1]}$ *which satisfies the equation*

$$X_\xi(t) = S(t)\xi + \int_0^t S(t-s)F(s, X_\xi(s))\,ds + \int_0^t S(t-s)G(s, X_\xi(s))\,dW(s) \quad P \ a.e. \tag{6.1}$$

for each $t \in [0,1]$.

In equation (6.1):

1. $(S(t))_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on H, whose infinitesimal generator is $A : D(A) \subset H \to H$;

2. $(W(t))_{t\geq 0}$ is a separable Hilbert space-valued Wiener process on the probability space $(\Omega, \mathcal{F}, P)$ with associated filtration $(\mathcal{F}_t)_{t\geq 0}$; the distribution of $W(1)$ is denoted by

ν and the reproducing kernel Hilbert space of ν is $(H_\nu, |\cdot|_{H_\nu})$;

3. the drift function $F : ([0,1] \times H, \mathcal{B}_{[0,1]} \otimes \mathcal{B}_H) \to (H, \mathcal{B}_H)$ is measurable;

4. the diffusion function $G : ([0,1] \times H, \mathcal{B}_{[0,1]} \otimes \mathcal{B}_H) \to (L_2(H_\nu, H), \mathcal{B}_{L_2(H_\nu,H)})$ is measurable;

5. the H-valued random variable ξ is $\mathcal{F}_0$-measurable and has distribution Ξ.

If ξ takes the constant value x, a point in H, then we write $X_x(t) := X_\xi(t)$ for all t in $[0,1]$.

We also assume

(**C2**) *for each bounded Borel measurable function $\phi : H \to \mathbb{R}$ and each time $t \in (0,1]$ we have*

$$E[\phi(X_\xi(t))|\xi] = (R_t\phi)(\xi) \quad P \textit{ a.e.},$$

where $(R_t\phi)(x) := E[\phi(X_x(t))]$ for all x in H.

If there exists a positive real constant Λ such that

$$|F(t,x) - F(t,y)| + \|G(t,x) - G(t,y)\|_{L_2(H_\nu,H)} \leq \Lambda|x-y| \quad \forall x,y \in H \text{ and } \forall t \in [0,1]$$

and

$$|F(t,x)|^2 + \|G(t,x)\|^2_{L_2(H_\nu,H)} \leq \Lambda(1+|x|^2) \quad \forall x \in H \text{ and } \forall t \in [0,1]$$

then by [10, Theorem 7.4] (C1) is satisfied and by [10, Theorem 9.8] (C2) is satisfied.

In section 6.2 we show that if E is a Borel subset of H and for each point x in H we have $\liminf_{t\to 0} t \ln P\{X_x(t) \in E\} \geq -\frac{1}{2}d^2(x,E)$ for some non-negative number $d(x,E)$ then we have

$$\lim_{r\to 0} \inf_{t<r} t \ln P\{\xi \in C, X_\xi(t) \in E\} \geq -\frac{1}{2}\text{essinf}_\Xi\{d^2(x,E) : x \in C\} \tag{6.2}$$

for any Borel subset C of H. In equation (6.2) essinf_Ξ is the essential infimum with respect to measure Ξ. This result was proved for open E by Zhang [33, Theorem 4.4].

In section 6.3 we find an upper bound for $\limsup_{t\to 0} t \ln P\{\xi \in C, X_\xi(t) \in E\}$ when $(X_\xi(t))_{t\in[0,1]}$ is an Ornstein-Uhlenbeck process driven by H-valued Wiener process $(W(t))_{t\geq 0}$. Fang and Zhang [13] and Hino and Ramirez [17] found a solution for this problem when there is an invariant measure μ and $\Xi = \mu$ and the transition semigroup on $L^2(H,\mu)$ is symmetric. We consider what happens when the transition semigroup on $L^2(H,\mu)$ is holomorphic and Ξ is absolutely continuous with respect to μ with square integrable density.

Our upper bound is not optimal because it does not reduce for finite dimensional H to the upper bound one can obtain using large deviations arguments.

6.2 The lower bound

Let E be a Borel subset of H and suppose that we have

$$\liminf_{r\to 0}\inf_{t<r} t\ln P\{X_x(t)\in E\} \geq -\frac{1}{2}d^2(x,E) \quad \text{for each } x\in H, \tag{6.3}$$

where the numbers $d(x,E)$ are non-negative and depend on x and may be infinity. Inequality (6.3) arises naturally when for each x in H the family of H-valued random variables $\{X_x(t) : t\in(0,1]\}$ satisfies a large deviation principle and E is open; then $\frac{1}{2}d^2(x,E)$ is the infimum of the rate function over E and our notation appears more justified. Set

$$d(C,E) := \operatorname{essinf}_{\Xi}\{d(x,E) : x\in C\}$$

for all Borel subsets C of H. Our proof of the following theorem does not depend on E being open.

Theorem 6.1 *Let C be a Borel subset of H such that $\Xi(C)>0$. Thanks to inequality (6.3) we have*

$$\lim_{r\to 0}\inf_{t<r} t\ln P\{\xi\in C,\, X_\xi(t)\in E\} \geq -\frac{1}{2}d^2(C,E).$$

Proof. Assume $d(C,E)<\infty$. We have for $t\in(0,1]$

$$\begin{aligned} P\{\xi\in C\,,\ X_\xi(t)\in E\} &= \int_\Omega 1_C(\xi)\, E[1_E(X_\xi(t))|\xi]\, dP \\ &= \int_H 1_C(x) P\{X_x(t)\in E\}\, d\Xi(x). \end{aligned}$$

Choose $\epsilon>0$. We can write

$$P\{\xi\in C\,,\ X_\xi(t)\in E\} = \mathrm{e}^{\frac{-\frac{1}{2}d^2(C,E)-\epsilon}{t}}\int_H 1_C(x)\mathrm{e}^{\frac{\frac{1}{2}d^2(C,E)+\epsilon}{t}} P\{X_x(t)\in E\}\, d\Xi(x).$$

Thus

$$t\ln P\{\xi\in C\,,\ X_\xi(t)\in E\} = -\frac{1}{2}d^2(C,E)-\epsilon+t\ln\int_H 1_C(x)\mathrm{e}^{\frac{\frac{1}{2}d^2(C,E)+\epsilon}{t}} P\{X_x(t)\in E\}\, d\Xi(x). \tag{6.4}$$

We shall show that the integral on the right hand side of equation (6.4) is bounded below by a positive number for all sufficiently small $t > 0$.
By definition of $d(C,E)$, the set $U := \{z \in C \;:\; d(z,E) < \sqrt{d^2(C,E)+\epsilon}\}$ has positive Ξ measure. For each z in U we have

$$\frac{1}{2}d^2(z,E) < \frac{1}{2}d^2(C,E) + \frac{\epsilon}{2}\,,\ \text{or equivalently}\ -\frac{1}{2}d^2(z,E) - \frac{\epsilon}{2} > -\frac{1}{2}d^2(C,E) - \epsilon;$$

we also have, by inequality (6.3), $\liminf_{t\to 0} t\ln P\{X_z(t) \in E\} \geq -\frac{1}{2}d^2(z,E)$. Thus for each $z \in U$ there is an $s_z > 0$ such that

$$P\{X_z(t) \in E\} \geq \mathrm{e}^{\frac{-\frac{1}{2}d^2(z,E)-\frac{\epsilon}{2}}{t}} \geq \mathrm{e}^{\frac{-\frac{1}{2}d^2(C,E)-\epsilon}{t}} \quad \text{for all } t \leq s_z.$$

Define

$$C_t := \left\{ x \in U \;:\; P\{X_x(t) \in E\} \geq \mathrm{e}^{\frac{-\frac{1}{2}d^2(C,E)-\epsilon}{t}} \right\} \quad \text{for each } 0 < t \leq 1.$$

We will show that there is a positive number a such that $\Xi(C_t) \geq a$ for all small enough t.
Let $\delta = \Xi(U) > 0$. Suppose, to get a contradiction, $\lim_{n\to\infty} \inf_{0<t<1/n} \Xi(C_t) = 0$. Then for each $n \in \{1,2,\ldots\}$ there is $0 < t_n < \frac{1}{n}$ such that $\Xi(C_{t_n}) < \frac{\delta}{2^{n+1}}$.
For each $z \in U$ there is $s_z > 0$ such that $t \leq s_z$ implies $z \in C_t$, thus $z \in C_{t_n}$ for all large enough n.
Hence $U = \bigcup_{n=1}^{\infty} C_{t_n}$ and it follows that

$$\delta = \Xi(U) \leq \sum_{n=1}^{\infty} \Xi(C_{t_n}) \leq \frac{\delta}{2}\,,$$

a contradiction. Thus $\lim_{n\to\infty} \inf_{0<t<1/n} \Xi(C_t) > 0$, which implies that for some $m \in \{1,2,\ldots\}$

$$a := \inf_{0<t<\frac{1}{m}} \Xi(C_t) > 0.$$

It follows that for all $t < \frac{1}{m}$ we have

$$\int_H 1_C(x)\mathrm{e}^{\frac{\frac{1}{2}d^2(C,E)+\epsilon}{t}} P\{X_x(t) \in E\}\, d\Xi(x) \geq a.$$

Using this in equation (6.4) we get

$$\lim_{r\to 0} \inf_{t<r} t\ln P\{\xi \in C\,,\ X_\xi(t) \in E\} \geq -\frac{1}{2}d^2(C,E).$$

This completes the proof.

6.3 An upper bound for Ornstein-Uhlenbeck processes

6.3.1 Introduction

It is relatively difficult to find a good upper bound for $\limsup_{t\to 0} t \ln P\{\xi \in C,\ X_\xi(t) \in E\}$. When we have a large deviation principle for the family $\{X_x(t) : t \in (0,1]\}$ for each x in H, Theorem 6.1 yields a lower bound for open E; we are also motivated to seek an upper bound of the same form as the lower bound. From now on we restrict our attention to an Ornstein-Uhlenbeck process driven by an H-valued Wiener process $(W(t))_{t\geq 0}$:

$$X_\xi(t) := S(t)\xi + \int_0^t S(t-s) i \, dW(s) \quad \text{for all } t \in [0,1];$$

here the operator $i : H_\nu \hookrightarrow H$ is the embedding of H_ν into H. The trace class covariance operator of ν is denoted by Q and we assume that $\ker Q = \{0\}$.

By using an exponential equivalence argument like that in chapter 4 one can show that for each x in H the family $\{\omega \in \Omega \mapsto (t \mapsto X_x(\epsilon t)(\omega)) : \epsilon \in (0,1]\}$ of trajectory-valued random variables in $C([0,1]; H)$ satisfies a large deviation principle with rate function

$$\mathcal{I}_x(u) := \begin{cases} \frac{1}{2}\int_0^1 |\phi(s)|^2_{H_\nu} \, ds & \text{if } \phi \in L^2([0,1]; H_\nu) \text{ and } u(t) = x + \int_0^t \phi(s)\, ds \ \ \forall t \in [0,1], \\ \infty & \text{otherwise.} \end{cases}$$

The continuous mapping theorem (see [18, Theorem 27.11]) then tells us that for each x in H the family of H-valued random variables $\{X_x(t) : t \in (0,1]\}$ satisfies a large deviation principle with rate function

$$\begin{aligned} \mathcal{J}_x(y) &:= \inf\{\mathcal{I}_x(u) : u \in C([0,1]; H) \text{ and } u(1) = y\} \\ &= \begin{cases} \frac{1}{2}|y-x|^2_{H_\nu} & \text{if } y - x \in H_\nu, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

For any point x in H and Borel subsets C and E of H define

$$d(x, E) := \inf\{|x-y|_{H_\nu} : y \in E\}, \quad \text{where we take } |z|_{H_\nu} = \infty \text{ if } z \in H \backslash H_\nu$$

and define

$$d(C, E) := \operatorname{essinf}_\Xi \{d(z, E) : z \in C\}.$$

For Borel subsets C and E of H and arbitrary $L < d(C,E)$ we can write

$$t \ln P\{\xi \in C,\, X_\xi(t) \in E\} = -\frac{1}{2}L^2 + t \ln \int 1_C(x) \mathrm{e}^{\frac{\frac{1}{2}L^2}{t}}\, P\{X_x(t) \in E\}\, d\Xi(x) \qquad (6.5)$$

for each $t \in (0,1]$. If E is closed then by the upper bound of the large deviation principle of $\{X_x(t) : t \in (0,1]\}$ the integrand in equation (6.5) converges to zero for Ξ a.e. x in C as t goes to zero. Thus we suspect that $\limsup_{t\to 0} t \ln P\{\xi \in C,\, X_\xi(t) \in E\}$ is bounded above by $-\frac{1}{2}d^2(C,E)$, at least for closed E. Proving this is another matter.

We assume that

(**C3**) *there exists a symmetric Gaussian invariant measure μ on H.*

The covariance operator of μ is $Q_\infty := \int_0^\infty S(t)QS^*(t)\,dt$. We may define the strongly continuous semigroup of transition operators on $L^2(H,\mu)$ by setting for each $t \in [0,1]$

$$(R_t\phi)(x) := E[\phi(X_x(t))] \quad \text{for } \mu \text{ a.e. } x \in H \ \text{ and for each } \phi \in L^2(H,\mu).$$

When (R_t) consists of symmetric operators and $\Xi = \mu$ Fang and Zhang [13, Theorem 2.1] showed that

$$\lim_{r\to 0} \sup_{t<r} t \ln P\{\xi \in C,\, X_\xi(t) \in E\} \le -\frac{1}{2}(d(C,E) \vee d(E,C))^2 \quad \text{for all sets } C \text{ and } E \in \mathcal{B}_H; \qquad (6.6)$$

the symmetric nature of the Markov process $(X_\xi(t))_{t\in[0,1]}$ results in the upper bound being symmetric in E and C. Fang's and Zhang's proof used the Lyons-Zheng decomposition which applies only to such symmetric Markov processes. Hino and Ramirez [17, Theorem 2.8] showed that when (R_t) consists of symmetric operators and $\Xi = \mu$ we have

$$P\{\xi \in C,\, X_\xi(t) \in E\} \le \sqrt{\mu(C)\mu(E)}\, \mathrm{e}^{\frac{-\frac{1}{2}d^2(C,E)}{t}} \quad \text{for all } t > 0. \qquad (6.7)$$

The proof of Hino's and Ramirez's theorem may be adapted in a straightforward way to yield an upper bound under more general assumptions.

We assume that the following two conditions hold. Notice in particular that (R_t) need not consist of symmetric operators.

(**C4**) *The distribution Ξ is absolutely continuous with respect to μ and has Radon-Nikodym derivative $\rho \in L^2(H,\mu)$.*

(**C5**) *The semigroup $(R_t^{\mathbb{C}})$ of operators on $L^2(H,\mu;\mathbb{C})$ obtained from (R_t) by defining*

$$R_t^{\mathbb{C}}(f) := R_t(\mathrm{Re}(f)) + \mathrm{i}\, R_t(\mathrm{Im}(f)) \quad \textit{for } f \in L^2(H,\mu;\mathbb{C}) \ \textit{ and } t \in [0,1],$$

is a restriction of a holomorphic semigroup.

Our definition of *holomorphic semigroup* is from [20].

Definition 1 *Let $K \in (0, \infty)$ and define the sector $s(K) = \{z \in \mathbb{C} : |\mathrm{Im}(z)| \le K\mathrm{Re}(z)\}$. The family $\{T(z) : z \in s(K)\}$ of bounded linear operators on $L^2(H, \mu; \mathbb{C})$ is called a* holomorphic semigroup *on the sector $s(K)$ if:*

1. *$T(0) = I_{L^2(H,\mu;\mathbb{C})}$, the identity operator on $L^2(H, \mu; \mathbb{C})$;*
2. *$T(z_1)T(z_2) = T(z_1 + z_2)$ for all $z_1, z_2 \in s(K)$;*
3. *for each $f \in L^2(H, \mu; \mathbb{C})$*

$$T(z)f \to f \ \text{ in } L^2(H, \mu; \mathbb{C}) \ \text{ as } z \to 0 \ \text{ in } s(\tilde{K})\,, \ \text{ for each } \tilde{K} \in (0, K);$$

4. *the function $z \mapsto < T(z)f, g >_{L^2(H,\mu;\mathbb{C})}$ is analytic in the interior of $s(K)$ for all $f, g \in L^2(H, \mu; \mathbb{C})$.*

Goldys [15] has shown that $(R_t^{\mathbb{C}})$ is a restriction of a holomorphic semigroup if and only if there is a positive real number a such that

$$| < Q_\infty A^* x, y > | \le a|Q^{\frac{1}{2}}x||Q^{\frac{1}{2}}y| \ \text{ for all } x, y \in D(A^*); \tag{6.8}$$

furthermore, if inequality (6.8) is true then $s(\frac{1}{2a})$ is an analyticity sector and $\|R_z^{\mathbb{C}}\|_{L(L^2(H,\mu;\mathbb{C}),L^2(H,\mu;\mathbb{C}))} = 1$ for all $z \in s(\frac{1}{2a})$. We remark that in the special case when (R_t) consists of symmetric operators we have (see, for example, [11, Proposition 10.1.6])

$$Q_\infty A^* = -\frac{1}{2}Q|_{D(A^*)}$$

and then inequality (6.8) is satisfied with $a = \frac{1}{2}$. As a corollary of Goldys' result we have this lemma, whose proof is in the appendix of this section.

Lemma 6.2 *The semigroup $(R_t^{\mathbb{C}})$ is a restriction of a holomorphic semigroup if and only if there is a bounded linear operator $B_1 \in L(H, H)$ such that*

$$\begin{aligned} Q_\infty A^* x &= Q^{\frac{1}{2}} B_1 Q^{\frac{1}{2}} x \ \textit{ for all } x \in D(A^*) && (6.9)\\ \textit{and } \ AQ_\infty &= Q^{\frac{1}{2}} B_1^* Q^{\frac{1}{2}}. && (6.10) \end{aligned}$$

Hence we assume that equations (6.9) and (6.10) are true and we set

$$B := B_1 - B_1^*. \tag{6.11}$$

Our main result is the following upper bound. Keep in mind that our recent assumptions (C3), (C4) and (C5) are assumed to hold.

Theorem 6.3 *For any sets C and E in $\mathcal{B}_H$*

$$P\{\xi \in C, X_\xi(t) \in E\} \leq \left(\int 1_C \, \rho^2 \, d\mu \right)^{\frac{1}{2}} (\mu(E))^{\frac{1}{2}} \, \mathrm{e}^{-\frac{d^2(C,E)}{2\beta t}} \qquad \textit{for all } t > 0,$$

where $\beta := \|B\|^2_{L(H,H)} + 1$ and B is the operator defined in equation (6.11).

In the special case when (R_t) consists of symmetric operators we have $B = 0$ and $\beta = 1$ and if $\rho \equiv 1$ then Theorem 6.3 gives the upper bound shown in inequality (6.7). However the upper bound in Theorem 6.3 is not as good as might be hoped. If H is finite dimensional, one can use the equation

$$\lim_{t \to 0} t \ln P\{|S(t)\xi - \xi| > \delta\} = -\infty \quad \text{for each } \delta > 0,$$

together with the large deviation principle satisfied by the family of Gaussian random variables $\{t^{\frac{1}{2}} W(1) : t \in (0,1]\}$ and exponential equivalence to show that even when $B \neq 0$ we have $\limsup_{t \to 0} t \ln P\{\xi \in C, \, X_\xi(t) \in E\} \leq -\frac{1}{2} d^2(C,E)$.

In the next subsection we introduce a closed bilinear form $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ on $L^2(H,\mu)$ which is associated with the semigroup (R_t) and we express this form in terms of a closed derivative-like operator $\bar{\nabla} : D(\mathcal{E}) \to L^2(H,\mu;H)$. In the third subsection we prove Theorem 6.3 by following the steps in the proof of Hino's and Ramirez's theorem [17, Theorem 2.8] and working with the expression of $\mathcal{E}$ in terms of $\bar{\nabla}$.

6.3.2 The machinery

Let $(L, D(L))$ be the infinitesimal generator of (R_t). Define a bilinear form $\mathcal{E} : D(L) \times D(L) \to \mathbb{R}$ by

$$\mathcal{E}(u,v) := \langle -Lu, v \rangle_{L^2(H,\mu)} = \int (-Lu) v \, d\mu \quad \text{for } u, v \in D(L).$$

Definition 2 *Given a bilinear form $\mathcal{B} : D(\mathcal{B}) \times D(\mathcal{B}) \to \mathbb{R}$ defined on a subspace $D(\mathcal{B})$*

of $L^2(H,\mu)$, *we define its* symmetric part *by*

$$\tilde{\mathcal{B}}(u,v) := \frac{1}{2}(\mathcal{B}(u,v) + \mathcal{B}(v,u)) \quad \textit{for all } u,v \in D(\mathcal{B})$$

and its antisymmetric part *by*

$$\check{\mathcal{B}}(u,v) := \frac{1}{2}(\mathcal{B}(u,v) - \mathcal{B}(v,u)) \quad \textit{for all } u,v \in D(\mathcal{B}).$$

We define an inner product on $D(L)$ by

$$\tilde{\mathcal{E}}_1(u,v) := \langle u,v\rangle_{L^2(H,\mu)} + \tilde{\mathcal{E}}(u,v) \quad \text{for all } u,v \in D(L);$$

the corresponding norm on $D(L)$ is $\tilde{\mathcal{E}}_1^{\frac{1}{2}}(u) := (\tilde{\mathcal{E}}_1(u,u))^{\frac{1}{2}}$ for $u \in D(L)$. Following Ma and Röckner [20, Theorem 2.15], we denote the completion of the normed vector space $(D(L), \tilde{\mathcal{E}}_1^{\frac{1}{2}})$ by $(D(\mathcal{E}), \tilde{\mathcal{E}}_1^{\frac{1}{2}})$ and then there is a unique bilinear extension of $\mathcal{E} : D(L) \times D(L) \to \mathbb{R}$ to the domain $D(\mathcal{E}) \times D(\mathcal{E})$ such that the extension is continuous with respect to the norm $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$ on $D(\mathcal{E})$. The continuous extension $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is a closed bilinear form, that is, $D(\mathcal{E})$ is a Hilbert space with the inner product

$$\tilde{\mathcal{E}}_1(u,v) := \langle u,v\rangle_{L^2(H,\mu)} + \tilde{\mathcal{E}}(u,v) \quad \text{for all } u,v \in D(\mathcal{E})$$

and $\tilde{\mathcal{E}}_1^{\frac{1}{2}}(u) = (\tilde{\mathcal{E}}_1(u,u))^{\frac{1}{2}}$ for all $u \in D(\mathcal{E})$. We also have

$$\mathcal{E}(u,v) = \langle -Lu, v\rangle_{L^2(H,\mu)} \quad \text{for all } u \in D(L) \text{ and } v \in D(\mathcal{E}). \tag{6.12}$$

For $n \in \mathbb{N}$ let $C_b^\infty(\mathbb{R}^n)$ be the space of continuous and bounded real-valued functions on $\mathbb{R}^n$ whose partial derivatives of all orders exist and are continuous and bounded. Given $\mathcal{A}$, a vector space of continuous bounded real-valued functions on H, let $\mathcal{A}_\mu$ be the subspace of $L^2(H,\mu)$ consisting of the classes which contain the functions in $\mathcal{A}$. Define

$$\mathcal{F}C_b^\infty(D(A^*)) := \{\phi \circ (\langle l_1, \cdot\rangle, \dots, \langle l_n, \cdot\rangle) : n \in \mathbb{N} \text{ and } \phi \in C_b^\infty(\mathbb{R}^n) \text{ and } l_1, \dots, l_n \in D(A^*)\}.$$

As shown in [11, Proposition 10.2.1], $\mathcal{F}C_b^\infty(D(A^*))_\mu$ is a core for $(L, D(L))$. It is convenient to work with the space $\mathcal{F}C_b^\infty(D(A^*))_\mu$ because we can compute Lu for $u \in \mathcal{F}C_b^\infty(D(A^*))_\mu$ and since this space is a core for $(L, D(L))$ it follows that $(D(\mathcal{E}), \tilde{\mathcal{E}}_1^{\frac{1}{2}})$ is also the completion of $(\mathcal{F}C_b^\infty(D(A^*))_\mu,\, \tilde{\mathcal{E}}_1^{\frac{1}{2}})$. From [11, Proposition 10.2.2] we have

$$\mathcal{E}(u,v) = -\int \langle Dv, Q_\infty A^*\, Du\rangle\, d\mu \quad \text{for } u,v \in \mathcal{F}C_b^\infty(D(A^*))_\mu, \tag{6.13}$$

where Du and Dv are the Fréchet derivatives of the representatives of u and v, respectively, belonging to $\mathcal{F}C_b^\infty(D(A^*))$. Define the linear operator $\nabla : \mathcal{F}C_b^\infty(D(A^*))_\mu \subset L^2(H,\mu) \to L^2(H,\mu;H)$ by

$$\nabla u := Q^{\frac{1}{2}}\, Du \quad \text{for } u \in \mathcal{F}C_b^\infty(D(A^*))_\mu; \tag{6.14}$$

here again Du is the Fréchet derivative of the representative of u that belongs to $\mathcal{F}C_b^\infty(D(A^*))$. Then from equation (6.13) and equations (6.9), (6.10) and (6.11) we have for the antisymmetric part

$$\check{\mathcal{E}}(u,v) = \frac{1}{2}\int \langle \nabla u,\, B\,\nabla v\rangle\, d\mu \quad \text{for } u,v \in \mathcal{F}C_b^\infty(D(A^*))_\mu \tag{6.15}$$

and from equation (6.13) and the Lyapunov equation,

$$Q_\infty A^* x + A Q_\infty x = -Qx \quad \text{for } x \in D(A^*) \quad \text{(see for example [11, Proposition 10.1.4])},$$

we have for the symmetric part

$$\tilde{\mathcal{E}}(u,v) = \frac{1}{2}\int \langle \nabla u, \nabla v\rangle\, d\mu \quad \text{for } u,v \in \mathcal{F}C_b^\infty(D(A^*))_\mu. \tag{6.16}$$

The operator $(\nabla,\, \mathcal{F}C_b^\infty(D(A^*))_\mu)$ is closable because if (u_n) is a sequence from $\mathcal{F}C_b^\infty(D(A^*))_\mu$ and $u_n \to 0$ in $L^2(H,\mu)$ and $|\nabla u_n - \nabla u_m|_{L^2(H,\mu;H)} \to 0$ as $n,m\to\infty$ then, from equation (6.16), $\tilde{\mathcal{E}}_1^{\frac{1}{2}}(u_n - u_m) = (|u_n - u_m|^2_{L^2(H,\mu)} + \frac{1}{2}|\nabla(u_n - u_m)|^2_{L^2(H,\mu;H)})^{\frac{1}{2}} \to 0$ as $n,m\to\infty$; since the space $(D(\mathcal{E}), \tilde{\mathcal{E}}_1^{\frac{1}{2}})$ is complete we have $\tilde{\mathcal{E}}_1^{\frac{1}{2}}(u_n - 0) \to 0$ as $n\to\infty$ which implies that $\nabla u_n \to 0$ in $L^2(H,\mu;H)$ as $n\to\infty$.

Let $\bar{\nabla} : D(\bar{\nabla}) \to L^2(H,\mu;H)$ be the closure of $(\nabla, \mathcal{F}C_b^\infty(D(A^*))_\mu)$. The domain of the closure is

$$\begin{aligned}
D(\bar{\nabla}) &= \big\{u \in L^2(H,\mu) : \exists (u_n) \subset \mathcal{F}C_b^\infty(D(A^*))_\mu \text{ such that } u_n \to u \text{ in } L^2(H,\mu) \text{ and} \\
&\qquad\qquad |\nabla u_n - \nabla u_m|_{L^2(H,\mu;H)} \to 0 \text{ as } n,m\to\infty\big\} \\
&= \big\{u \in L^2(H,\mu) : \exists (u_n) \subset \mathcal{F}C_b^\infty(D(A^*))_\mu \text{ such that } u_n \to u \text{ in } L^2(H,\mu) \text{ and} \\
&\qquad\qquad \tilde{\mathcal{E}}_1^{\frac{1}{2}}(u_n - u_m) \to 0 \text{ as } n,m\to\infty\ \big\} \\
&= D(\mathcal{E}).
\end{aligned}$$

Let $u \in D(\mathcal{E})$ and let the sequence $(u_n) \subset \mathcal{F}C_b^\infty(D(A^*))_\mu$ be such that $u_n \to u$ in $L^2(H,\mu)$ and $|\nabla u_n - \nabla u_m|_{L^2(H,\mu;H)} \to 0$ as $n, m \to \infty$; then $\bar{\nabla} u := \lim_{n\to\infty} \nabla u_n$ in $L^2(H,\mu;H)$ and also $u_n \to u$ in $(D(\mathcal{E}), \tilde{\mathcal{E}}_1^{\frac{1}{2}})$. Since $\mathcal{E} : D(\mathcal{E}) \times D(\mathcal{E}) \to \mathbb{R}$ is continuous with respect to the norm $\tilde{\mathcal{E}}_1^{\frac{1}{2}}$ on $D(\mathcal{E})$, we have

$$\tilde{\mathcal{E}}(u,u) = \lim_{n\to\infty} \tilde{\mathcal{E}}(u_n, u_n) = \lim_{n\to\infty} \frac{1}{2} \int \langle \nabla u_n, \nabla u_n \rangle \, d\mu = \frac{1}{2} \int \langle \bar{\nabla} u, \bar{\nabla} u \rangle \, d\mu.$$

Similarly we can show that for all $u, v \in D(\mathcal{E})$

$$\tilde{\mathcal{E}}(u,v) = \frac{1}{2} \int \langle \bar{\nabla} u, \bar{\nabla} v \rangle \, d\mu \tag{6.17}$$

and

$$\check{\mathcal{E}}(u,v) = \frac{1}{2} \int \langle \bar{\nabla} u, \, B \, \bar{\nabla} v \rangle \, d\mu. \tag{6.18}$$

If u and v belong to $D(\mathcal{E}) \cap L^\infty(H,\mu)$ and $F \in C^\infty(\mathbb{R})$ then we have the following rules of calculus:

$$\text{(i)} \quad uv \in D(\mathcal{E}) \quad \text{and} \quad \bar{\nabla}(uv) = u\bar{\nabla} v + v \bar{\nabla} u\,; \tag{6.19}$$

$$\text{(ii)} \quad F \circ u \in D(\mathcal{E}) \text{ and } \bar{\nabla}(F \circ u) = F'(u) \bar{\nabla} u. \tag{6.20}$$

These rules follow immediately for functions in $\mathcal{F}C_b^\infty(D(A^*))_\mu$. For general $u, v \in D(\mathcal{E}) \cap L^\infty(H,\mu)$ we can find uniformly bounded sequences of functions (u_n) and (v_n) from $\mathcal{F}C_b^\infty(D(A^*))_\mu$ such that $u_n \to u$ in $L^2(H,\mu)$ and $\nabla u_n \to \bar{\nabla} u$ in $L^2(H,\mu;H)$ and $v_n \to v$ in $L^2(H,\mu)$ and $\nabla v_n \to \bar{\nabla} v$ in $L^2(H,\mu;H)$ and then the rules of calculus follow by substituting u_n and v_n in equations (6.19) and (6.20) and taking limits.

6.3.3 Proof of Theorem 6.3

In this subsection we prove Theorem 6.3. Let C and E be Borel subsets of H such that $\Xi(C) = \int 1_C \, \rho \, d\mu > 0$ and $\mu(E) > 0$. We want to find an upper bound for $\int 1_C(R_t 1_E) \rho \, d\mu$ for each $t > 0$. Let $\hat{E}$ be a countable union of compact sets such that $\hat{E} \subset E$ and $\mu(E \backslash \hat{E}) = 0$. Since μ is a Radon measure, $\hat{E}$ exists. Define

$$v_t := R_t 1_E \quad \text{for } t \geq 0$$

and

$$w := d(\cdot, \hat{E}) \wedge d(C, E).$$

We assume that $d(C,E) < \infty$. From [27, Lemma 2.1] we have $w \in D(\mathcal{E})$ and $|\bar{\nabla} w| \leq 1$ μ a.e.; we give the proof of this important fact in the appendix. Note that for $t > 0$, $v_t \in D(L)$. This follows because, by assumption, for each $f \in L^2(H,\mu;\mathbb{C})$ the $L^2(H,\mu;\mathbb{C})$-valued function $z \mapsto R_z^{\mathbb{C}} f$ is weakly analytic and hence strongly analytic on the interior of a sector $s(K)$, for some $K \in (0,\infty)$. Thus $t \mapsto R_t^{\mathbb{C}} f$ is differentiable on $(0,\infty)$. This implies (see [24, chapter 2 Lemma 4.2]) that for each $t > 0$

$$R_t(L^2(H,\mu)) \subset D(L).$$

We now trace the steps of Hino and Ramirez [17, Theorem 2.8]. Let α be a real number and set

$$f(t) := \int v_t^2 \mathrm{e}^{2\alpha w}\, d\mu \quad \text{for } t \geq 0.$$

The function f is continuous on $[0,\infty)$ and differentiable on $(0,\infty)$. For $t > 0$ we have

$$\begin{aligned} f'(t) &= 2\int (Lv_t) v_t \mathrm{e}^{2\alpha w}\, d\mu \\ &= -2\mathcal{E}\left(v_t, v_t \mathrm{e}^{2\alpha w}\right) \quad \text{by equation (6.12)} \\ &= -2\left(\tilde{\mathcal{E}}\left(v_t, v_t\mathrm{e}^{2\alpha w}\right) + \check{\mathcal{E}}\left(v_t, v_t \mathrm{e}^{2\alpha w}\right)\right). \end{aligned} \tag{6.21}$$

We use equation (6.18) and the rules of calculus for $\bar{\nabla}$ to obtain for the antisymmetric part:

$$\begin{aligned} -2\check{\mathcal{E}}\left(v_t, v_t\mathrm{e}^{2\alpha w}\right) &= -\int \langle B\,\bar{\nabla}\left(\mathrm{e}^{2\alpha w} v_t\right), \bar{\nabla} v_t\rangle\, d\mu \\ &= -2\alpha \int v_t \mathrm{e}^{2\alpha w} \langle B\,\bar{\nabla} w, \bar{\nabla} v_t\rangle\, d\mu - \int \mathrm{e}^{2\alpha w} \langle B\,\bar{\nabla} v_t, \bar{\nabla} v_t\rangle > \, d\mu \\ &= -2\alpha \int v_t \mathrm{e}^{2\alpha w} \langle B\,\bar{\nabla} w, \bar{\nabla} v_t\rangle\, d\mu \quad \text{for } t > 0. \end{aligned}$$

Using equation (6.17), we also have for the symmetric part:

$$-2\tilde{\mathcal{E}}\left(v_t, \mathrm{e}^{2\alpha w} v_t\right) = -2\alpha \int v_t \mathrm{e}^{2\alpha w} \langle \bar{\nabla} w, \bar{\nabla} v_t\rangle\, d\mu - \int \mathrm{e}^{2\alpha w} \langle \bar{\nabla} v_t, \bar{\nabla} v_t\rangle\, d\mu \quad \text{for } t > 0.$$

Equation (6.21) now becomes

$$f'(t) = -2\alpha \int v_t \mathrm{e}^{2\alpha w} \langle (B + I_H)\bar{\nabla} w, \bar{\nabla} v_t\rangle\, d\mu - \int \mathrm{e}^{2\alpha w}\langle \bar{\nabla} v_t, \bar{\nabla} v_t\rangle\, d\mu \quad \text{for } t > 0. \tag{6.22}$$

Hölder's inequality provides an upper bound for the first term on the right hand side of equation (6.22):

$$4\alpha^2 \left(\int v_t e^{2\alpha w} \langle (B+I_H)\bar{\nabla} w, \bar{\nabla} v_t \rangle \, d\mu \right)^2$$

$$\begin{aligned}
&\leq 4\alpha^2 \int v_t^2 e^{2\alpha w} |(B+I_H)\bar{\nabla} w|^2 \, d\mu \int e^{2\alpha w} |\bar{\nabla} v_t|^2 \, d\mu \\
&\leq \alpha^4 \left(\int v_t^2 e^{2\alpha w} |(B+I_H)\bar{\nabla} w|^2 \, d\mu \right)^2 + \left(\int e^{2\alpha w} |\bar{\nabla} v_t|^2 \, d\mu \right)^2 \\
&\quad +2\alpha^2 \int v_t^2 e^{2\alpha w} |(B+I_H)\bar{\nabla} w|^2 \, d\mu \int e^{2\alpha w} |\bar{\nabla} v_t|^2 \, d\mu
\end{aligned}$$

thus

$$-2\alpha \int v_t e^{2\alpha w} \langle (B+I_H)\bar{\nabla} w, \bar{\nabla} v_t \rangle \, d\mu \leq \alpha^2 \int v_t^2 e^{2\alpha w} |(B+I_H)\bar{\nabla} w|^2 \, d\mu + \int e^{2\alpha w} |\bar{\nabla} v_t|^2 \, d\mu.$$

Substituting this bound into equation (6.22) we get

$$\begin{aligned}
f'(t) &\leq \alpha^2 \int v_t^2 e^{2\alpha w} |(B+I_H)\bar{\nabla} w|^2 \, d\mu \\
&= \alpha^2 \int v_t^2 e^{2\alpha w} (|B\bar{\nabla} w|^2 + |\bar{\nabla} w|^2) \, d\mu \\
&\leq \alpha^2 \beta \, f(t) \quad \text{for } t > 0,
\end{aligned} \tag{6.23}$$

where $\beta := \|B\|^2_{L(H,H)} + 1$. From inequality (6.23) it follows that

$$f(t) \leq f(0) e^{\alpha^2 \beta t} = \mu(E) e^{\alpha^2 \beta t} \quad \text{for all } t \geq 0.$$

Thus, by Hölder's inequality,

$$\begin{aligned}
\int 1_C (R_t 1_E) \rho \, d\mu &\leq \left(\int 1_C e^{-2\alpha w} \rho^2 \, d\mu \right)^{\frac{1}{2}} \left(\int e^{2\alpha w} v_t^2 \, d\mu \right)^{\frac{1}{2}} \\
&\leq e^{-\alpha d(C,E)} \left(\int 1_C \rho^2 \, d\mu \right)^{\frac{1}{2}} (\mu(E))^{\frac{1}{2}} e^{\frac{\alpha^2 \beta t}{2}} \quad \text{for all } t \geq 0.
\end{aligned}$$

Here the real number α is arbitrary; for each $t > 0$ we minimise the expression on the right hand side by taking $\alpha = \frac{d(C,E)}{\beta t}$. Then we have

$$\int 1_C(R_t 1_E)\rho\, d\mu \le \left(\int 1_C \rho^2\, d\mu\right)^{\frac{1}{2}} (\mu(E))^{\frac{1}{2}}\, \mathrm{e}^{-\frac{d^2(C,E)}{2\beta t}} \quad \text{for all } t > 0.$$

This completes the proof of Theorem 6.3.

Remark The fact that β appears in the argument of the exponential function in Theorem 6.3 suggests that, at least in the case of finite dimensional H, our use of Hölder's inequality for the term $\int v_t \mathrm{e}^{2\alpha w}\langle B\bar{\nabla} w, \bar{\nabla} v_t\rangle\, d\mu$ appearing in equation (6.22) is crude.

6.3.4 Appendix

Proof of Lemma 6.2

We now show that $(R_t^{\mathbb{C}})$ is a restriction of a holomorphic semigroup if and only if there is a bounded linear operator G on H such that

$$\begin{aligned} AQ_\infty &= Q^{\frac{1}{2}} G Q^{\frac{1}{2}} && \text{and} \\ Q_\infty A^* x &= Q^{\frac{1}{2}} G^* Q^{\frac{1}{2}} x && \text{for all } x \in D(A^*). \end{aligned}$$

According to Goldys [15, Theorem 2.2], $(R_t^{\mathbb{C}})$ is a restriction of a holomorphic semigroup if and only if there is a positive real number a such that

$$|\langle Q_\infty A^* x, y\rangle| \le a|Q^{\frac{1}{2}} x||Q^{\frac{1}{2}} y| \quad \text{for all } x, y \in D(A^*). \tag{6.24}$$

Let $x \in D(A^*)$. Since $D(A^*)$ is dense in H, we can take a sequence $(y_n) \subset D(A^*)$ such that $y_n \to Q_\infty A^* x$ in H as $n \to \infty$. Then substituting y_n for y in inequality (6.24) and taking limits on both sides gives

$$|Q_\infty A^* x| \le a\|Q^{\frac{1}{2}}\|_{L(H,H)}|Q^{\frac{1}{2}} x|.$$

This means that the operator $Q_\infty A^* Q^{-\frac{1}{2}} : Q^{\frac{1}{2}}(D(A^*)) \subset H \to H$ is bounded and since its domain $Q^{\frac{1}{2}}(D(A^*))$ is dense in H, there is a bounded linear operator E on H such that $Q_\infty A^* Q^{-\frac{1}{2}} = E|_{Q^{\frac{1}{2}}(D(A^*))}$ or, equivalently,

$$Q_\infty A^* = E Q^{\frac{1}{2}} \quad \text{on } D(A^*). \tag{6.25}$$

Let $y \in H$. For any $x \in D(A^*)$ we have

$$\langle A^* x, Q_\infty y\rangle = \langle Q_\infty A^* x, y\rangle = \langle E Q^{\frac{1}{2}} x, y\rangle = \langle x, Q^{\frac{1}{2}} E^* y\rangle.$$

Thus

$$Q_\infty(H) \subset D(A) \quad \text{and } A Q_\infty y = Q^{\frac{1}{2}} E^* y \quad \text{for all } y \in H. \tag{6.26}$$

Inequality (6.24) becomes

$$|\langle x, A Q_\infty y\rangle| = |\langle Q^{\frac{1}{2}} x, E^* y\rangle| \le a|Q^{\frac{1}{2}} x||Q^{\frac{1}{2}} y| \quad \text{for all } x, y \in D(A^*).$$

In this inequality let $y \in D(A^*)$ and replace x by a sequence $(x_n) \subset D(A^*)$ such that $Q^{\frac{1}{2}} x_n \to E^* y$ in H as $n \to \infty$. Taking limits we get

$$|E^* y| \le a|Q^{\frac{1}{2}} y|.$$

Thus $E^* Q^{-\frac{1}{2}} : Q^{\frac{1}{2}}(D(A^*)) \subset H \to H$ is bounded and there is a bounded linear operator G on H such that $E^* Q^{-\frac{1}{2}} = G|_{Q^{\frac{1}{2}}(D(A^*))}$ or, equivalently,

$$E^* = G Q^{\frac{1}{2}}.$$

Substituting this expression for E into equations (6.25) and (6.26) gives the desired results.

Proof that $w \in D(\mathcal{E})$ and $|\bar{\nabla} w| \le 1$

We now present Ren's and Röckner's proof [27, Lemma 2.1] that if F is a countable union of compact subsets of H and c is a positive real number then $d(\cdot, F) \wedge c$ is in $D(\mathcal{E})$ and $|\bar{\nabla}(d(\cdot, F) \wedge c)| \le 1$ μ a.e.. Ren and Röckner proved this result in the more general setting of a separable Banach space. In our application H is a separable Hilbert space, which simplifies the proof.

Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis for H composed of eigenvectors of Q:

$$Q e_j = \lambda_j e_j \quad \text{for all } j \in \mathbb{N}$$

and such that the sequence (λ_n) is non-increasing. The set $\{Q^{\frac{1}{2}} e_j = \lambda_j^{\frac{1}{2}} e_j : j \in \mathbb{N}\}$ is an orthonormal basis for H_ν. For each $n \in \mathbb{N}$, let $P_n : H \to H$ be the projection onto the linear span of $\{e_1, \ldots, e_n\}$:

$$P_n x = \sum_{j=1}^{n} \langle x, e_j \rangle e_j \quad \text{for } x \in H.$$

The hard work is in proving that if K is a compact subset of H and c is a positive real number then $d(\cdot, K) \wedge c \in D(\mathcal{E})$ and $|\bar{\nabla}(d(\cdot, K) \wedge c)| \leq 1$ μ a.e.. The steps of the proof are:

1. show that for any fixed $n \in \mathbb{N}$ and $y \in H$ the function

$$v_n(x) := |P_n x - P_n y|_{H_\nu} \wedge c\,, \quad x \in H,$$

   belongs to $D(\mathcal{E})$ and $|\bar{\nabla} v_n| \leq 1$ μ a.e.;

2. show that

$$x \mapsto d(P_n x, P_n(K)) \wedge c \in D(\mathcal{E})$$

   and $|\bar{\nabla}(d(P_n \cdot, P_n(K)) \wedge c)| \leq 1$ μ a.e.;

3. show that $d(\cdot, K) \wedge c \in D(\mathcal{E})$ and $|\bar{\nabla}(d(\cdot, K) \wedge c)| \leq 1$ μ a.e..

Step 1 Fix $n \in \mathbb{N}$ and $y \in H$. Define $v_n : H \to \mathbb{R}$ by

$$v_n(x) = |P_n x - P_n y|_{H_\nu} \wedge c \quad \text{for } x \in H.$$

Recall that $D(\mathcal{E})$ is the domain of the closure of the operator $\nabla : \mathcal{F}C_b^\infty(D(A^*))_\mu \subset L^2(H, \mu) \to L^2(H, \mu; H)$ defined in equation (6.14). Since $D(A^*)$ is dense in H, it is straightforward to show that $\mathcal{F}C_b^\infty(H)_\mu$, the set of elements of $L^2(H, \mu)$ which contain functions from

$$\mathcal{F}C_b^\infty(H) = \{\phi \circ (\langle l_1, \cdot \rangle, \ldots, \langle l_m, \cdot \rangle) : m \in \mathbb{N} \text{ and } \phi \in C_b^\infty(\mathbb{R}^m) \text{ and } l_1, \ldots, l_m \in H\},$$

is contained in $D(\mathcal{E})$ and for $u \in \mathcal{F}C_b^\infty(H)_\mu$, $\bar{\nabla} u = Q^{\frac{1}{2}} D u$ where Du is the Fréchet derivative of the corresponding function in $\mathcal{F}C_b^\infty(H)$. To show that $v_n \in D(\mathcal{E})$ we will find a sequence (u_m) from $\mathcal{F}C_b^\infty(H)_\mu$ such that $u_m \to v_n$ in $L^2(H, \mu)$ and $(Q^{\frac{1}{2}} D u_m)$ converges in $L^2(H, \mu; H)$ as $m \to \infty$. We have

$$v_n(x) = \left(\sum_{j=1}^{n} (\langle x, e_j \rangle - \langle y, e_j \rangle)^2 \lambda_j^{-1} \right)^{\frac{1}{2}} \wedge c \quad \text{for } x \in H.$$

Define $g : \mathbb{R}^n \to \mathbb{R}$ by

$$g(z) = \left(\sum_{j=1}^{n} (z_j - \langle y, e_j \rangle)^2 \lambda_j^{-1} \right)^{\frac{1}{2}} \quad \text{for } z = (z_1, \dots, z_n) \in \mathbb{R}^n.$$

Clearly $v_n(x) = g(Tx) \wedge c$, where

$$Tx := (\langle x, e_1 \rangle, \dots, \langle x, e_n \rangle) \quad \text{for } x \in H.$$

The partial derivatives $D_j g$ exist and are continuous and bounded on $\mathbb{R}^n \backslash \{Ty\}$:

$$D_j g(z) = \lambda_j^{-1}(z_j - \langle y, e_j \rangle) \left(\sum_{i=1}^{n} (z_i - \langle y, e_i \rangle)^2 \lambda_i^{-1} \right)^{-\frac{1}{2}} \quad \text{for } 1 \le j \le n.$$

By the C^∞-Urysohn's lemma, for each $m \in \mathbb{N}$ there is a function $\phi_m \in C_b^\infty(\mathbb{R})$ such that ϕ'_m has compact support and $0 \le \phi'_m \le 1$ and $\phi'_m(t) = 1$ for each $t \in [\frac{1}{m}, c]$ and the closure of the set $\{t \in \mathbb{R} : \phi'_m(t) \neq 0\}$ is contained in $(0, c + \frac{1}{m})$ and

$$\phi_m(t) = \begin{cases} t & \text{when } \frac{1}{m} \le t \le c, \\ c_1 & \text{when } t \le a_m, \\ c_2 & \text{when } t \ge c + \frac{1}{m}, \end{cases}$$

where $a_m = \min\{t \in \mathbb{R} : \phi'_m(t) \neq 0\} \in (0, \frac{1}{m})$ and c_1 and c_2 are constants such that $c_1 \in [0, \frac{1}{m}]$ and $c_2 \in [c, c + \frac{1}{m}]$. The sequence ϕ_m converges uniformly to $t \mapsto (0 \vee t) \wedge c$ and $\phi'_m(t)$ converges pointwise to $1_{(0,c]}(t)$.
For each $m \in \mathbb{N}$, $\phi_m \circ g \in C_b^\infty(\mathbb{R}^n)$ and thus $(\phi_m \circ g) \circ T \in \mathcal{F}C_b^\infty(H)$. We have

$$\begin{aligned} \int (\phi_m(g(Tx)) - v_n(x))^2 \, d\mu &= \int (\phi_m(g(Tx)) - g(Tx) \wedge c)^2 \, d\mu \\ &\to 0 \quad \text{as } m \to \infty. \end{aligned}$$

We also have

$$\begin{aligned} Q^{\frac{1}{2}} D((\phi_m \circ g) \circ T)(x) &= Q^{\frac{1}{2}} T^*(D(\phi_m \circ g)(Tx)) \quad \text{for } x \in H \\ &= Q^{\frac{1}{2}} T^* \phi'_m(g(Tx)) Dg(Tx) \quad \text{for } x \notin y + \ker T. \end{aligned}$$

Since $\ker Q_\infty = \{0\}$, $\mu(y + \ker T) = 0$.

Thus

$$\begin{aligned}\int \left|Q^{\frac{1}{2}}D((\phi_m\circ g)\circ T)(x) - 1_{(0,c]}(g(Tx))Q^{\frac{1}{2}}T^*Dg(Tx)\right|^2 d\mu \\ \leq \quad \|Q^{\frac{1}{2}}T^*\|^2_{L(\mathbb{R}^n,H)} \int \left(\phi_m'(g(Tx)) - 1_{(0,c]}(g(Tx))\right)^2 |Dg(Tx)|^2\,d\mu \\ \to \quad 0 \quad \text{as } m\to\infty.\end{aligned}$$

We have now shown that $v_n \in D(\mathcal{E})$ and for μ a.e. $x \in H$

$$\begin{aligned}|\bar{\nabla} v_n(x)| &\leq |Q^{\frac{1}{2}}T^*Dg(Tx)| \\ &= \left|\sum_{j=1}^n D_j g(Tx)Q^{\frac{1}{2}}e_j\right| = \left(\sum_{j=1}^n \lambda_j (D_j g(Tx))^2\right)^{\frac{1}{2}} = 1.\end{aligned}$$

This completes step 1 of the proof.

In the next step of the proof we will use a result from [20, chapter 4 Lemma 4.1]: if $u,v\in D(\mathcal{E})\cap L^\infty(H,\mu)$ then $u\wedge v\in D(\mathcal{E})$ and

$$\bar{\nabla}(u\wedge v) = 1_{\{u>v\}}\bar{\nabla} v + 1_{\{v>u\}}\bar{\nabla} u + 1_{\{u=v\}}\frac{1}{2}(\bar{\nabla} u + \bar{\nabla} v). \tag{6.27}$$

Step 2 Fix $n\in\mathbb{N}$. Let K be a compact subset of H. Let $\{y_1,y_2,y_3,\ldots\}$ be a countable subset of K whose closure in H is K. From step 1 we know that for each $m\in\mathbb{N}$ the function

$$v_{n,m}(x) := |P_n x - P_n y_m|_{H_\nu} \wedge c \quad \text{for } x\in H$$

belongs to $D(\mathcal{E})$ and $|\bar{\nabla} v_{n,m}| \leq 1$ μ a.e..
Suppose that for some $j\geq 1$ we have $v_{n,1}\wedge\ldots\wedge v_{n,j}\in D(\mathcal{E})$ and $|\bar{\nabla}(v_{n,1}\wedge\ldots\wedge v_{n,j})|\leq 1$ μ a.e.. Then by equation (6.27) we have $v_{n,1}\wedge\ldots\wedge v_{n,j}\wedge v_{n,j+1}\in D(\mathcal{E})$ and $|\bar{\nabla}(v_{n,1}\wedge\ldots\wedge v_{n,j}\wedge v_{n,j+1})|\leq 1$ μ a.e.. By induction $v_{n,1}\wedge\ldots\wedge v_{n,j}\in D(\mathcal{E})$ and $|\bar{\nabla}(v_{n,1}\wedge\ldots\wedge v_{n,j})|\leq 1$ μ a.e. for each $j\in\mathbb{N}$.
We have $v_{n,1}\wedge\ldots\wedge v_{n,j} \searrow d(P_n\cdot, P_n(K))\wedge c$ as $j\to\infty$. To see this, notice that given any $\delta>0$ there is a $y\in K$ such that $|P_n x - P_n y|_{H_\nu} < d(P_n x, P_n(K)) + \delta$ and there is a

sequence $\alpha : \mathbb{N} \to \mathbb{N}$ such that $y_{\alpha(i)}$ converges to y in H; thus

$$\begin{aligned} |P_n x - P_n y_{\alpha(i)}|_{H_\nu} &\leq |P_n x - P_n y|_{H_\nu} + |P_n y - P_n y_{\alpha(i)}|_{H_\nu} \\ &< d(P_n x, P_n(K)) + \delta + \lambda_n^{-\frac{1}{2}} |P_n y - P_n y_{\alpha(i)}| \\ &< d(P_n x, P_n(K)) + 2\delta \quad \text{for sufficiently large } i \in \mathbb{N}. \end{aligned}$$

Since all the functions are bounded by c we have $v_{n,1} \wedge \ldots \wedge v_{n,j} \to d(P_n \cdot, P_n(K)) \wedge c$ in $L^2(H, \mu)$.
Since $|\bar{\nabla}(v_{n,1} \wedge \ldots \wedge v_{n,j})|_{L^2(H,\mu;H)} \leq 1$ for all $j \in \mathbb{N}$ and the closed unit ball in $L^2(H, \mu; H)$ is weakly sequentially compact, there is a subsequence $(\bar{\nabla}(v_{n,1} \wedge \ldots \wedge v_{n,j_i}))_{i=1}^{\infty}$ which converges weakly to some V in the closed unit ball of $L^2(H, \mu; H)$. By the Banach-Saks theorem (see for example Theorem 2.2 in Appendix A of [20]) we may assume that the sequence of Cesaro means $(\frac{1}{N} \sum_{i=1}^{N} \bar{\nabla}(v_{n,1} \wedge \ldots \wedge v_{n,j_i}))_{N=1}^{\infty}$ converges to V in $L^2(H, \mu; H)$; we also have $\frac{1}{N} \sum_{i=1}^{N} (v_{n,1} \wedge \ldots \wedge v_{n,j_i}) \to d(P_n \cdot, P_n(K)) \wedge c$ in $L^2(H, \mu)$ as $N \to \infty$. Closedness of $(\bar{\nabla}, D(\mathcal{E}))$ now implies that $d(P_n \cdot, P_n(K)) \wedge c \in D(\mathcal{E})$ and $\bar{\nabla}(d(P_n \cdot, P_n(K)) \wedge c) = V$.
For each $N \in \mathbb{N}$,

$$\left| \frac{1}{N} \sum_{i=1}^{N} \bar{\nabla}(v_{n,1} \wedge \ldots \wedge v_{n,j_i}) \right| \leq \frac{1}{N} \sum_{i=1}^{N} |\bar{\nabla}(v_{n,1} \wedge \ldots \wedge v_{n,j_i})| \leq 1 \quad \mu \text{ a.e.}$$

and since some subsequence of this sequence of Cesaro means converges to $\bar{\nabla}(d(P_n \cdot, P_n(K)) \wedge c)$ pointwise μ a.e., it follows that $|\bar{\nabla}(d(P_n \cdot, P_n(K)) \wedge c)| \leq 1$ μ a.e.. This completes step 2 of the proof.

Step 3 For each $x \in H$ and $y \in K$, $|P_n x - P_n y|_{H_\nu} \nearrow |x-y|_{H_\nu}$, where we take $|x-y|_{H_\nu} = \infty$ if $x - y \in H \backslash H_\nu$. Thus

$$d(P_n x, P_n(K)) \nearrow \sup_{n \in \mathbb{N}} d(P_n x, P_n(K)) \leq d(x, K) \quad \text{for each } x \in H.$$

We shall show that actually $\sup_{n \in \mathbb{N}} d(P_n x, P_n(K)) = d(x, K)$. We can assume that $\sup_{n \in \mathbb{N}} d(P_n x, P_n(K)) < \infty$. Suppose that α is a real number such that

$$\sup_{n \in \mathbb{N}} d(P_n x, P_n(K)) < \alpha.$$

Then for each $n \in \mathbb{N}$ there is a vector $y_n \in K$ such that $|P_n x - P_n y_n|_{H_\nu} < \alpha$. Since the closed ball in H_ν centred at 0 and of radius α, $\bar{B}_{H_\nu}(0, \alpha)$, is a compact subset of H and

K is also compact, there is a subsequence (n_k) and $h \in \bar{B}_{H_\nu}(0,\alpha)$ and $y \in K$ such that $y_{n_k} \to y$ in H as $k \to \infty$ and $P_{n_k}x - P_{n_k}y_{n_k} \to h$ in H as $k \to \infty$. Thus $x - y = h$, which implies that $d(x, K) \leq \alpha$.
Since the sequence of functions converges pointwise and is uniformly bounded, we have

$$d(P_n\cdot, P_n(K)) \wedge c \to d(\cdot, K) \wedge c \quad \text{in } L^2(H,\mu) \text{ as } n \to \infty.$$

We also have that $|\bar{\nabla}(d(P_n\cdot, P_n(K)) \wedge c)|_{L^2(H,\mu;H)} \leq 1$ for all $n \in \mathbb{N}$. Now arguing in the same way as in step 2 we conclude that $d(\cdot, K) \wedge c \in D(\mathcal{E})$ and $|\bar{\nabla}(d(\cdot, K) \wedge c)| \leq 1$ μ a.e.. This completes step 3.

Finally, let $K_1 \subset K_2 \subset K_3 \subset \ldots$ be an increasing sequence of compact subsets of H and let $F = \cup_{j=1}^{\infty} K_j$. Then

$$d(x, K_j) \searrow d(x, F) \quad \text{for each } x \in H$$

and

$$d(\cdot, K_j) \wedge c \to d(\cdot, F) \wedge c \quad \text{in } L^2(H,\mu) \text{ as } j \to \infty.$$

We also have $|\bar{\nabla}(d(\cdot, K_j) \wedge c)|_{L^2(H,\mu;H)} \leq 1$ for all $j \in \mathbb{N}$. Again we can argue as in step 2 to conclude that $d(\cdot, F) \wedge c \in D(\mathcal{E})$ and $|\bar{\nabla}(d(\cdot, F) \wedge c)| \leq 1$ μ a.e..

Chapter 7

Conclusion

The large deviation principles in Corollary 3.4, Corollary 4.12 and Theorem 5.1 lead us to expect that for a given stochastic initial value problem the small time asymptotics of the continuous trajectories of the solution will be described by a large deviation principle whose rate function is determined only by the diffusion function and the noise process. Admittedly we have only dealt with drift functions and diffusion functions that are Lipschitz continuous or are in some sense close to being Lipschitz continuous.

From the point of view of the stochastic modeller, additive Wiener process noise is the simplest choice but it may not reflect the properties of the system being modelled. Observation of the small time asymptotics of the system of interest may provide useful information as to how to model the noise in the stochastic equation. On the other hand if we have a stochastic initial value problem, small time asymptotics estimates provide a relatively simple rough guide to the behaviour of the system as it moves from its initial state.

Continuous solutions of equations in a separable Hilbert space seem quite manageable compared to solutions of equations which are continuous in a more general separable Banach space. We got a taste of this in chapter 4. We faced the problem of getting a large deviation principle for the small time asymptotics of continuous trajectories in a general separable Banach space, without making overly restrictive assumptions. In particular it might be that our assumption that (B2)(2) holds is overly restrictive given the other assumptions we made in chapter 4.

Chapter 5 reminds us that stochastic integrals have been defined for integrators besides the familiar Wiener process. Much still remains to be done on small time asymptotics of solutions of stochastic equations with different types of noise process.

In chapter 6 we saw that getting a good upper bound for

$\limsup_{t\to 0} t \ln P\{X(0) \in C, X(t) \in E\}$ is a challenging problem for general $\mathcal{L}(X(0))$, even in the case of the solution of a linear equation with Wiener process noise in a separable Hilbert space H. A question that arises where we left off in chapter 6 is whether Theorem 6.3 gives a good upper bound for the small time asymptotics or whether we actually have $\limsup_{t\to 0} t \ln P\{X(0) \in C, X(t) \in E\} \leq -\frac{1}{2}d^2(C, E)$ when H is infinite dimensional and the transition semigroup on $L^2(H, \mu)$ is holomorphic and $\mathcal{L}(X(0))$ has integrable (rather than square integrable) density with respect to μ.

Bibliography

[1] S. Agmon. *Lectures on Elliptic Boundary Value Problems.* D. Van Nostrand Company, Inc., 1965.

[2] E. Alòs and D. Nualart: Stochastic integration with respect to the fractional Brownian motion. *Stochastics and Stochastics Reports*, **75**, No. 3, (2003), pp. 129–152.

[3] V.Bogachev. *Gaussian Measures.* American Mathematical Society, 1998.

[4] Z. Brzeźniak and J.M.A.M van Neerven. Stochastic convolution in separable Banach spaces and the stochastic linear Cauchy problem. *Studia Math.* **143**, (2000), pp.43–74.

[5] S. Cerrai and M. Röckner. Large deviations for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term. *The Annals of Probability* **32** no. 1B (2004), pp. 1100–1139.

[6] A. Chojnowska-Michalik and B. Goldys. On regularity properties of nonsymmetric Ornstein-Uhlenbeck semigroup in L^p spaces. *Stochastics and Stochastics Reports*, **59**, No. 3-4, (1996), pp. 183–209.

[7] A. Chojnowska-Michalik and B. Goldys. Symmetric Ornstein-Uhlenbeck semigroups and their generators. *Probab. Theory Relat. Fields*, **124**, (2002), pp. 459–486.

[8] P.-L. Chow and J.-L. Menaldi. Exponential estimates in exit probability for some diffusion processes in Hilbert spaces. *Stochastics and Stochastics Reports* **29** (1990), pp. 377–393.

[9] G. Da Prato. *Stochastic Evolution Equations by Semigroup Methods.* Center de Recerca Matemàtica, Barcelona, 1997.

[10] G. Da Prato and J. Zabczyk. *Stochastic Equations in Infinite Dimensions.* Cambridge University Press, 1992.

[11] G. Da Prato and J. Zabczyk. *Second Order Partial Differential Equations in Hilbert Spaces*. Cambridge University Press, 2002.

[12] T.E. Duncan and B. Maslowski and B. Pasik-Duncan. Fractional Brownian motion and stochastic equations in Hilbert spaces. *Stochastics and Dynamics*, **2**, No. 2, (2002), pp. 225–250.

[13] S. Fang and T.S. Zhang. On the small time behavior of Ornstein-Uhlenbeck processes with unbounded linear drifts. *Probab. Theory Relat. Fields*, **114**, (1999), pp. 487–504.

[14] M. Fantozzi. Large deviations for semilinear differential stochastic equations with dissipative non-linearities. *Stochastic Analysis and Applications*, **21**, No. 1, (2003), pp.127–139.

[15] B. Goldys. On analyticity of Ornstein-Uhlenbeck semigroups. *Rend. Mat. Acc. Lincei*, s.9, **10**, (1999), pp. 131–140.

[16] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*. Springer-Verlag, 1981.

[17] M. Hino and J. Ramirez. Small-time Gaussian behaviour of symmetric diffusion semigroups. *The Annals of Probability*, **31**, No. 3, (2003), pp.1254–1295.

[18] O. Kallenberg. *Foundations of Modern Probability*. Second Edition. Springer-Verlag, 2002.

[19] P. Lax. *Functional Analysis*. John Wiley and Sons, 2002.

[20] Z.-M. Ma and M. Röckner. *Introduction to the Theory of Non-symmetric Dirichlet Forms*. Springer-Verlag, 1992.

[21] J.M.A.M van Neerven. Gaussian sums in Banach spaces and γ-radonifying operators. *Lecture Notes of the Seminar "Stochastic Evolution Equations in Banach Spaces"*. Technical University of Delft, 2002/03.

[22] J.M.A.M van Neerven and L. Weis. Weak limits and integrals of Gaussian covariances in Banach spaces. *Probab. Math. Statist.*, **25**, (2005), pp. 55–74.

[23] J. Ockenden, S. Howison, A. Lacey, A. Movchan. *Applied Partial Differential Equations*. Revised edition. Oxford University Press, 2003.

[24] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, 1983.

[25] S. Peszat. Large deviation principle for stochastic evolution equations. *Probability Theory and Related Fields*, **98** (1994), pp. 113–136.

[26] M. Reed and B. Simon. *Methods of Modern Mathematical Physics Volume I: Functional Analysis*. Academic Press, 1980.

[27] J. Ren and M. Röckner. A remark on sets in infinite dimensional spaces with full or zero capacity. In: *Stochastic Analysis: Classical and Quantum. Perspectives of white noise theory.* T. Hida editor. World Scientific, (2005), pp.177–186.

[28] M.Renardy and R.Rogers. *An Introduction to Partial Differential Equations*. Second Edition. Springer-Verlag, 2004.

[29] W.Rudin. *Functional Analysis*. Second edition. McGraw-Hill, 1991.

[30] N.N. Vakhania, V.I. Tarieladze and S.A. Chobanyan. *Probability Distributions on Banach Spaces*. D.Reidel Publishing Company, 1987.

[31] S.R.S. Varadhan. Diffusion processes in a small time interval. *Communications on Pure and Applied Mathematics*, **20** (1967), pp. 659–685.

[32] J. Zabczyk. *Topics in Stochastic Systems. Asymptotics and Regularity.* Flight Systems Research Laboratory, University of California, Los Angeles. Technical Report No. 1-Z-4015-88, 1988.

[33] T.S. Zhang. On the small time asymptotics of diffusion processes on Hilbert spaces. *The Annals of Probability*, **28**, No. 2, (2000), pp. 537–557.