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Complex structures on stratified Lie algebras

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Complex structures on stratified Lie algebras

Thesis Abstract

This thesis investigates some properties of complex structures on nilpotent Lie algebras. In particular, we focus on $\textit{nilpotent}$ $\textit{complex structures}$ that are characterized by a suitable J-invariant ascending or descending central series $\mathfrak{d}^j$ and $\mathfrak{d}_j$ respectively. In this thesis, we introduce a new descending series $\mathfrak{p}_j$ and use it to give a proof of a new characterization of nilpotent complex structures. We examine also whether nilpotent complex structures on stratified Lie algebras preserve the strata. We find that there exists a J-invariant stratification on a step 2 nilpotent Lie algebra with a complex structure.

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Abstract

This thesis investigates some properties of complex structures on nilpotent Lie algebras. In particular, we focus on *nilpotent complex structures* that are characterized by a suitable J-invariant ascending or descending central series $\mathfrak{d}^j$ and $\mathfrak{d}_j$ respectively. In this thesis, we introduce a new descending series $\mathfrak{p}_j$ and use it to give a proof of a new characterization of nilpotent complex structures. We examine also whether nilpotent complex structures on stratified Lie algebras preserve the strata. We find that there exists a J-invariant stratification on a step 2 nilpotent Lie algebra with a complex structure.

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Contents

Introduction

In recent years, complex structures on nilpotent Lie algebras have been shown to be very useful for understanding some geometric and algebraic properties of *nilmanifolds*, which are compact quotients of a simply connected nilpotent Lie group with a complex structure. Complex structures on 6 dimensional nilpotent Lie algebras were first investigated by Salamon in [26], who completed the classification of 6 dimensional nilpotent Lie algebras with complex structures. Unfortunately, his methods can not be extended to higher dimensional Lie algebras. Later on, Ovando made a classification of 4 dimensional solvable Lie algebras with complex structures in [22]. In [4], Cordero, Fernández, Gray and Ugarte introduced *nilpotent complex structures* and they studied 6 dimensional nilpotent Lie algebras with nilpotent complex structures in [5] and provided a classification. Meanwhile, as the ascending central series is not necessarily J-invariant, they introduced a J-invariant ascending central series to characterize nilpotent complex structures. More recently, Latorre, Ugarte and Villacampa defined the space of nilpotent complex structures on nilpotent Lie algebras and further studied complex structures on nilpotent Lie algebras with one dimensional center [15], [16]. They also provided a structure theorem, describing the ascending central series of 8 dimensional nilpotent Lie algebras with complex structures. In [12], Gao, Zhao and Zheng studied the relation between the step of a nilpotent Lie algebra and the smallest integer j_0 such that the J-invariant ascending central series stops. Furthermore, they introduced a J-invariant descending central series, which is another instrumental to characterize nilpotent complex structures. It is clear that a classification of nilpotent Lie algebras would help to study nilpotent complex structures. However, since little is known about the classification of real Lie algebras with dimensions higher than 7, it is an interesting question to study algebraic properties of higher dimensional real nilpotent Lie algebras with complex structures. In the paper referred to above, the language of differential forms is generally, we use the language of Lie algebras and provide a simple proof.

In this thesis, we consider a special type of nilpotent Lie algebras: *stratified Lie algebra*. Tanaka proposed the concept of stratified Lie algebras [27], [28] and this was further developed by Yamaguchi [29]. Recent results on nilpotent Lie algebras with a stratification can be found in [7], [8], [17]. A complex structure J on a stratified Lie algebra $\mathfrak{n}$ is said to be *strata-preserving* if it preserves each layer of the stratification. One of our goal is to examine the strata-preserving property of complex structures on stratified Lie algebras.

In Chapter 1, we introduce the notation and preliminary results. In particular, we will state the *Newlander–Nirenberg Theorem*, which is in the background throughout this thesis. In Section 1.1, we provide some definitions of smooth and complex manifolds, as well as the definition of complex structures on vector spaces. In particular, we show that a real vector space V with a complex structure J must admit a J-invariant inner product and a J-adapted orthonormal basis. Next, in Section 1.2 we introduce basic Lie theoretic tools and left-invariant complex structures on Lie groups, which leads to the integrability condition of a complex structure on a Lie algebra $\mathfrak{g}$.

Chapter 2 is the main chapter of the thesis and it is divided into several sections. Our main contributions are in this chapter. In the first 4 sections, we shall provide some general results on nilpotent Lie algebras with complex structures. Some applications of these results are studied in Section 2.5 and 2.6. We next explain the contents of each section in detail.

We first introduce, in Section 2.1, *stratified Lie algebras* and provide some examples. In particular, we show that not every even-dimensional nilpotent Lie algebra admits a complex structure, for instance, a class of $2n$-dimensional filiform algebras.

Next, in Section 2.2 we study the central series of nilpotent Lie algebras with complex structures. In particular, we show that there exists a stratification on a 2-step nilpotent Lie algebra with a complex structure J such that J is strata-preserving. The formal definition of *nilpotent complex structures* on Lie algebras appears in Section 2.3. The nilpotency of complex structures implies the nilpotency of Lie algebras. Our main objective here is to find a way to characterize nilpotent complex structures J by a J-invariant descending central series as the characterization of J is given by the J-invariant ascending central series $\mathfrak{d}^j$, which is studied in [4]. On the one hand, using the property of $\mathfrak{d}^j$, we deduce that if a $2n$ dimensional non-Abelian Lie algebra admits a nilpotent complex structure, then $2 \leq \dim \mathfrak{z} \leq 2n-2$. This implies that the Lie algebra of strictly upper triangular matrices does not admit nilpotent complex structures. One the other hand, we focus on the J-invariant descending central series $\mathfrak{d}_j$, which is first introduced in [12], and show that $\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{d}_j$, where $\mathfrak{c}_j(\mathfrak{n})$ is the descending central series of $\mathfrak{n}$. Meanwhile, we introduce a new type of descending central series $\mathfrak{p}_j$, which provides a method to show the following theorem and characterize nilpotent complex structures.

Theorem. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. The following are equivalent:*

(i) J is nilpotent of step j_0;

(ii) $\mathfrak{p}_{j_0} = \{0\}$ and $\mathfrak{p}_{j_0-1} \neq \{0\}$;

(iii) $\mathfrak{d}_{j_0} = \{0\}$ and $\mathfrak{d}_{j_0-1} \neq \{0\}$,

where $\mathfrak{d}_j$ and $\mathfrak{p}_j$ are as in Definition 2.3.2 *and Definition* 2.3.19.

One of the important consequence of this theorem is that strata-preserving complex structures are nilpotent complex structures of step k, where k is the nil-index of stratified Lie algebras. Conversely, nilpotent complex structures preserve the descending central series $\mathfrak{c}_j(\mathfrak{n})$ if $\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{c}^{k-j}(\mathfrak{n})$. Moreover, if J is nilpotent of step j_0, where $k \leq j_0$, the inclusion relations between the J-invariant descending and ascending central series are more explicit as follows

$$\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{c}_j(\mathfrak{n}) + J\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{p}_j + J\mathfrak{p}_j \subseteq \mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j} \subseteq \mathfrak{c}^{j_0-j}(\mathfrak{n}).$$

In Section 2.4 we investigate complex structures on the Lie algebra direct sum of two nilpotent Lie algebras.

In the last two sections, we focus on stratified Lie algebras with nilpotent complex structures. In particular, we study the strata-preserving properties of complex structures on 2-step stratified Lie algebras in Section 2.5. We will give a new proof that every complex structure structure on a 2-step nilpotent Lie algebra is nilpotent of step either 2 or 3, the original proof is in [12] and [25]. Moreover, under the assumption that $\dim \mathfrak{n}_2 = 2$, we have the following theorem.

Theorem. *Let* $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ *be a* 2*-step stratified Lie algebra with a complex structure* J *such that* $\dim \mathfrak{n}_2 = 2$*. Then*

(i) J *is nilpotent of step* 2;

(ii) if $\dim \mathfrak{d}^1 = 2$, *then* $J\mathfrak{n}_2 = \mathfrak{n}_2$.

In Subsection 2.5.1 we present a case study on 6 dimensional 2-step nilpotent Lie algebras with complex structures.

Finally, in the last section we will investigate step $k \geq 3$ stratified Lie algebras with complex structures. We show that if a 3-step stratified Lie algebra has a complex structure J that preserves the last layer, then J must be nilpotent of step 3. Furthermore, under the condition $\mathfrak{z} = \mathfrak{n}_k$, one can show that if J is nilpotent of step k, then J preserves $\mathfrak{z}$.

Chapter 1

Complex structures on real manifolds and Lie algebras

In this thesis, we will study and discuss a special class of complex manifolds. For the reader's convenience, we shall review some basic results about manifolds and Lie theory. Most of this background material can be found in [13], [14] and [18]. We start with the basic concept of manifolds in Section 1.1. There are many different types of manifolds. In this thesis, we are interested in manifolds that admit complex structures. Next, in Section 1.2 we provide some fundamental facts in Lie theory and investigate complex structures in real Lie algebras.

1.1 Complex structures on real smooth manifolds

Definition 1.1.1. A *topological space* is an ordered pair (M, τ), where M is a set and τ is a collection of subsets of M, satisfying the following conditions:

(a) The empty set $\emptyset$ and M itself belong to τ;

(b) if $G_i \in \tau$ for all $i \in I$, then $\bigcup_{i \in I} G_i \in \tau$, where I is the set of indexes which can be either finite or infinite;

(c) if $G_i \in \tau$ for $i = 1, \ldots, n$, then $\bigcap_{i=1}^{n} G_i \in \tau$.

If τ is clear from the context, then we often refer to M as a topological space.

If the topology has a countable basis, then a topological space M is called *second countable*; if distinct points can be separated by neighbourhoods, then M is *Hausdorff*. Let (M, τ_M) and (N, τ_N) be topological spaces with $m = \dim M$ and $n = \dim N$. A function $f : M \to N$ is *continuous* if $\mathcal{V} \in \tau_N$, then its inverse image $f^{-1}(\mathcal{V}) \in \tau_M$. Furthermore, a function $f : M \to N$ is a *homeomorphism* if

(a) f is bijective;

(b) both f and f^{-1} are continuous.

If f exists, then $m = n$.

Definition 1.1.2. Let M be a topological space.

(a) A *coordinate chart* on M is a pair $(\mathcal{U}, \phi)$, where $\mathcal{U}$ is an open subset of M and $\phi : \mathcal{U} \to \tilde{\mathcal{U}}$ is a homeomorphism from $\mathcal{U}$ to $\phi(\mathcal{U}) = \tilde{\mathcal{U}} \subseteq \mathbb{R}^n$.

(b) An *atlas* $\mathcal{A}$ is a collection of coordinate charts $\{(\mathcal{U}_\alpha, \phi_\alpha) : \phi_\alpha : \mathcal{U}_\alpha \to \mathbb{R}^n\}_{\alpha \in C}$ such that $\mathcal{U}_\alpha$ covers M, i.e., $\bigcup_{\alpha \in C} \mathcal{U}_\alpha = M$ and such that $\phi_\alpha \circ \phi_\beta^{-1}$ is a homeomorphism for all $\alpha, \beta \in C$, where C is an index set and $\phi_\alpha \circ \phi_\beta^{-1}$ is called a *transition map*.

(c) An atlas $\mathcal{A}$ is *maximal* if it is not properly contained in any larger atlas.

We say that a second countable, Hausdorff topological space is a *topological manifold* if it admits a maximal atlas.

Remark 1.1.3. (i) We say that M is a topological manifold of *dimension* n if each point of M has a neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$. We denote the dimension of M by $\dim M$.

(ii) Given a chart $(\mathcal{U}, \phi)$, the map ϕ is called a *coordinate map*. We define *local coordinates* on $\mathcal{U}$ by $\phi(p) = (x_1(p), \dots, x_n(p))$, where p is a point at $\mathcal{U}$ and $(x_1, \dots, x_n)$ are the component functions of ϕ.

(iii) By *Zorn's Lemma*, every atlas $\mathcal{A}$ is contained in a unique maximal atlas.

Let $\mathcal{U} \subseteq \mathbb{R}^n$ and $\mathcal{V} \subseteq \mathbb{R}^m$ be open subsets. In the sense of ordinary calculus, a real valued function $f : \mathcal{U} \to \mathcal{V}$ is called *smooth* if it is infinitely differentiable. A smooth function $f : \mathcal{U} \to \mathcal{V}$ is called a *diffeomorphism* if f is bijective and f^{-1} is smooth. Notice that the word *smooth* may be defined differently by some authors. Throughout this thesis, smooth is synonymous of C^∞.

Definition 1.1.4. Let M be a topological manifold. An atlas $\mathcal{A} = (\mathcal{U}_\alpha, \phi_\alpha)$ is called a *smooth atlas* if all transition maps

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \to \phi_\alpha(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$$

are diffeomorphism, where $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \{0\}$. A topological manifold with a maximal smooth atlas is a *smooth manifold*.

Remark 1.1.5. We say that two atlases are *equivalent* if their union is an atlas. In general, different atlases could give the same collection of smooth functions on M. For instance, [18, p13] the atlases on $\mathbb{R}^n$

$$\mathcal{A}_1 = \{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\} \text{ and } \mathcal{A}_2 = \{(B_1(x), \mathrm{Id}_{B_1(x)}) : x \in \mathbb{R}^n\}$$

are smooth, where Id is the identity map and $B_1(x)$ is the open ball of radius 1 around x, and their union is an atlas. We define a function $f : M \to \mathbb{R}$ to be smooth if and only if $f \cdot \varphi^{-1}$ is smooth in the sense of ordinary calculus for each coordinate chart (U, φ) in the atlas. Hence defining the maximal smooth atlas is an appropriate way of defining the equivalence class of smooth atlases. For more examples, please refer to [13] and [18].

Next, we provide some examples of smooth manifolds.

Example 1.1.6. (i) The Euclidean space $\mathbb{R}^n$ is a smooth manifold. The maximal atlas contains $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$.

(ii) Let $M(n, \mathbb{R})$ be the space of $n \times n$ real matrices. The general linear group is

$$GL(n, \mathbb{R}) = \{T \in M(n, \mathbb{R}) : \det T \neq 0\},$$

where $\det : GL(n, \mathbb{R}) \to \mathbb{R}$ is a continuous function. This is a smooth n^2-dimensional manifold since it is an open subset of the n^2-dimensional vector space $M(n, \mathbb{R})$.

There are still lots of interesting examples of smooth manifolds that we will not introduce here. See, e.g., [18, *Chapter* 1].

Definition 1.1.7. Let M, M' be smooth manifolds with atlases $\mathcal{A}$ and $\mathcal{A}'$ respectively. A function $f : M \to \mathbb{R}^n$ is *smooth* if for all charts $(U, \phi) \in \mathcal{A}$, the function

$$f \circ \phi^{-1} : \phi(U) \to \mathbb{R}^n$$

is smooth. A continuous function $F : M \to M'$ is a *smooth map* if for all charts $(U, \phi) \in \mathcal{A}$ of M and all charts $(V, \psi) \in \mathcal{A}'$ of M', the map

$$\psi \circ F \circ \phi^{-1} : \phi(F^{-1}(V) \cap U) \to \psi(V)$$

is smooth. Furthermore, F is a diffeomorphism between manifolds accordingly.

Remark 1.1.8. (i) Notice that $\phi(F^{-1}(V) \cap U)$ is open since F is continuous.

(ii) We have an equivalent characterization of smoothness between manifolds as follows:

Suppose that M and M' are smooth manifolds with atlases $\mathcal{A}$ and $\mathcal{A}'$ respectively. A continuous function $F : M \to M'$ is a smooth map if for all $p \in M$ and for all smooth charts $(U, \phi) \in \mathcal{A}$ around $p \in M$ and all charts $(V, \psi) \in \mathcal{A}'$ around $F(p) \in N$ such that $F(U) \subseteq V$, the composition

$$\psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V)$$

is smooth.

Next, we define the tangent space to smooth manifolds. Let $C^\infty(M)$ be the set of all real valued smooth functions on M. Notice that $C^\infty(M)$ is a vector space.

Definition 1.1.9. Let M be a smooth manifold, and let $p \in M$. A *tangent vector* to M at p is a linear map $X_p : C^\infty(M) \to \mathbb{R}$ that satisfies the following property

$$X_p(fg) = f(p)X_p(g) + g(p)X_p(f), \text{ for all } f, g \in C^\infty(M).$$

The set of all tangent vectors at p is called *tangent space to M at p* and it is denoted by T_pM.

Remark 1.1.10. (i) Notice that T_pM is a vector space with $\dim T_pM = \dim M$ for every point $p \in M$. For more details, please refer to [13, Section 8.3] and [18, Chapter 3].

(ii) In a coordinate chart, the tangent space T_pM has a natural basis

$$\left\{ \frac{\partial}{\partial x_1}\bigg|_p, \dots, \frac{\partial}{\partial x_n}\bigg|_p \right\},$$

where $n = \dim M$.

(iii) Suppose that $M = V$ is a vector space. The tangent space T_pV to V at $p \in V$ can be identified with V by a vector space isomorphism.

Definition 1.1.11. Let M, M' be smooth manifolds, and let $F : M \to M'$ be a smooth map. For each $p \in M$, the *differential of F at p* is the map $\mathrm{d}F_p : T_pM \to T_{F(p)}M'$, defined by

$$\mathrm{d}F_p(X_p)(f) = X_p(f \circ F).$$

Remark 1.1.12. (i) Notice that $\mathrm{d}F_p : T_pM \to T_{F(p)}M'$ is linear. If F is a diffeomorphism, then $\mathrm{d}F_p$ is an isomorphism.

(ii) Given $X_p \in T_pM$, $\mathrm{d}F_p(X_p)$ is a tangent vector at $F(p)$. Notice that if $f \in C^\infty(M')$, then $f \circ F \in C^\infty(M)$, hence $X_p(f \circ F)$ is well-defined.

We next look at the definition of tangent bundle on a smooth manifold.

Definition 1.1.13. Let M be a smooth manifold. We define the *tangent bundle* of M, denoted by TM, to be the disjoint union of tangent spaces at all points of M :

$$TM = \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{p\} \times T_pM = \bigcup_{p \in M} \{(p, q) : q = X_p \in T_pM\}.$$

A *global vector field* is a smooth map $A : M \to TM$ given by $p \mapsto (p, A_p)$ such that $\pi \circ A = \mathrm{Id}_M$, where $\pi : TM \to M$ is the projection. We denote by $\mathfrak{X}(M)$ the set of all such vector fields on M.

Remark 1.1.14. (i) For each $p \in M$, $\pi^{-1}(p) = T_pM$ is a real vector space.

(ii) For simplicity, we will omit the term 'global' in this thesis, since all vector fields we treat here are global.

(iii) The *rank* of a tangent bundle is $\dim T_pM$.

(iv) By combining together the differential of F at all $p \in M$, we get a globally defined map between tangent bundles, called the *global differential* and denoted by $\mathrm{d}F : TM \to TM'$. Notice that this map is smooth. [18, Proposition 3.21].

Notice that tangent bundles are a special case of vector bundles. For more details, please refer to [18, Chapter 10].

Definition 1.1.15. Let TM and TM' be tangent bundles, and let $\pi : TM \to M$ and $\pi' : TM' \to M'$ be the canonical projections. A diffeomorphism $F : TM \to TM'$ is a *tangent bundle homomorphism* if there exists a smooth map $f : M \to M'$ such that the following diagram commutes:

$$\begin{array}{ccc} TM & \xrightarrow{F} & TM' \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ M & \xrightarrow{f} & M' \end{array},$$

and such that $F_p : T_pM \to T_{f(p)}M'$ is a linear map for all $p \in M$.

Remark 1.1.16. A bijective tangent bundle homomorphism $F : TM \to TM'$ whose inverse is also a tangent bundle homomorphism is called a *tangent bundle iso--morphism*. Equivalently, for all $p \in M$, the map $F_p : T_pM \to T_{f(p)}M'$ is a vector space isomorphism.

We next define complex structures on vector spaces, which we always assume to be real, unless otherwise stated.

Definition 1.1.17. Let V be a vector space. A *complex structure* on V is a linear isomorphism $J : V \to V$ such that $J^2 = -I$.

Remark 1.1.18. (i) Notice that if V admits a complex structure J, then $\dim V \in 2\mathbb{N}$.

(ii) Let V be a vector space with a complex structure J. Defining the multiplication by a complex number by

$$(a + ib)v = a\ v + b\ J(v), \text{ for all } a, b \in \mathbb{R} \text{ and } v \in V,$$

gives a structure of complex vector space. Conversely, if V is a complex vector space with $\dim_{\mathbb{C}} V = n$, then define $J \in GL_{\mathbb{C}}(V)$ by $J(v) := i \cdot v$ for all $v \in V$. When V

is considered as a real $2n$ dimensional vector space, the isomorphism J induces a complex structure.

(iii) Let $V = T_pM$ for each $p \in M$. We usually denote the complex structure at p by J_p.

Lemma 1.1.19. *Let V be a finite-dimensional real vector space with a complex structure J. Then V admits a J-invariant inner product ψ, that is, $\psi(JX, JY) = \psi(X, Y)$ for all $X, Y \in V$. Consequently,*

(i) if V_1 is a J-invariant subspace of V, its orthogonal complement V_2 with respect to ψ is also J-invariant;

(ii) V admits a basis of the form $\{X_1, \ldots, X_n, JX_1, \ldots, JX_n\}$.

Proof. We first show that there exists a J-invariant inner product on V. Let ϕ be any inner product on V. Define $\psi : V \times V \to \mathbb{R}$ by

$$\psi(X, Y) = \phi(X, Y) + \phi(JX, JY), \text{ for all } X, Y \in V.$$

It is clear that ψ is an inner product. For every $X, Y \in V$,

$$\psi(JX, JY) = \phi(X, Y) + \phi(JX, JY) = \psi(X, Y).$$

Therefore ψ is a J-invariant inner product. Let $V_2 = V_1^{\perp} \subseteq V$. It is clear that $V = V_1 \oplus V_2$ since $V = V_1 \oplus V_1^{\perp}$. Finally, for all $Y \in V_2$,

$$\{0\} = \psi(V_1, Y) = \psi(JV_1, JY) = \psi(V_1, JY).$$

Hence $JY \in V_2$ and V_2 is J-invariant as required.

For part (ii), choose $0 \neq X_1 \in V$. Let $\beta_1 = \{X_1, JX_1\}$. Since

$$\psi(X_1, JX_1) = \phi(JX_1, X_1) - \phi(JX_1, X_1) = 0,$$

β_1 is a linearly independent subset. Next, suppose that there exist s independent non-zero vectors $X_1, \ldots, X_s$ such that $\beta_s = \{X_1, \ldots, X_s, JX_1, \ldots, JX_s\}$ is a linear independent subset for some $s \in \mathbb{N}$. Let $0 \neq X_{s+1} \in V$ such that $\psi(X_{s+1}, X_j) = \psi(X_{s+1}, JX_j) = 0$ for all $1 \leq j \leq s$. Then

$$\psi(JX_{s+1}, JX_j) = \phi(X_{s+1}, X_j) + \phi(JX_{s+1}, JX_j) = \psi(X_{s+1}, X_j) = 0;$$
$$\psi(X_{s+1}, JX_j) = \phi(JX_{s+1}, X_j) - \phi(X_{s+1}, JX_j) = -\psi(X_{s+1,JX_j}) = 0.$$

Therefore $\beta_{s+1} = \{X_1, \ldots, X_{s+1}, JX_1, \ldots, JX_{s+1}\}$ is a linearly independent set. By induction, β_n is a basis of V. □

We can now define complex manifolds. Roughly, complex manifolds can be thought of as topological spaces that are locally equivalent to a neighbourhood of $\mathbb{C}^n$. Let M be a real smooth manifold. A *maximal holomorphic atlas* is a holomorphic atlas that is not properly contained in any larger atlas. We say that M is a *complex manifold* if it has a maximal holomorphic atlas. There are many examples of complex manifolds, for instance, the Riemann Sphere, $\mathbb{CP}$, $\mathbb{C}^n$, etc.

Example 1.1.20. Let $M = \mathbb{R}^{2n}$ and $p \in M$. Clearly, $\mathbb{C}^n \cong \mathbb{R}^{2n}$. For all $z_j = x_j + iy_j$ with $1 \leq j \leq n$, the multiplication map $m_i : \mathbb{C}^n \to \mathbb{C}^n$ given by $z_j \mapsto iz_j$ induces a complex structure J on $\mathbb{R}^{2n}$ defined by $J = \xi \circ m_i \circ \xi^{-1}$, where $\xi : (\ldots, z_j, \ldots) \mapsto (\ldots, x_j, \ldots, y_j, \ldots)$. The action of J on M is given by

$$J = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}.$$

By Cayley-Hamilton theorem, the minimal polynomial of J is $x^2 + 1$. However, since $\mathbb{R}^{2n}$ is real, there are no real eigenspaces. We may extend J linearly to $J : (\mathbb{R}^{2n})^{\mathbb{C}} \to (\mathbb{R}^{2n})^{\mathbb{C}}$. Hence the eigenvalues of J are $\pm i$. The eigenspaces are $E_i = \operatorname{span}\{e_1, \ldots, e_n\}$ and $E_{-i} = \operatorname{span}\{f_1, \ldots, f_n\}$, where $e_j = (0, \ldots, 1, \ldots, 0, 0, \ldots, -i, \ldots, 0)$ and $f_j = (0, \ldots, 1, \ldots, 0, 0, \ldots, i, \ldots, 0)$. Since $\dim_{\mathbb{C}} E_{\pm i} = n$, we conclude that $(\mathbb{R}^{2n})^{\mathbb{C}} = T_p^{\mathbb{C}}\mathbb{R}^{2n} = E_i \oplus E_{-i}$.

In general, suppose that M is a $2n$-dimensional real smooth manifold and T_pM is a tangent space of M at $p \in M$. The complexification of the tangent space of M is denoted by $T_p^{\mathbb{C}}M$ for all $p \in M$. Set

$$T_p^{\mathbb{C}}M = \{X_p + iY_p | X_p, Y_p \in T_pM\} = T_pM \otimes \mathbb{C}.^1$$

Since $T_pM \cong \mathbb{R}^{2n}$, we can define a complex structure on T_pM, denoted by J_p, for all $p \in M$. Since J_p has no real eigenvalues on T_pM, we extend it to the complexification $J_p : T_p^{\mathbb{C}}M \to T_p^{\mathbb{C}}M$ as a $\mathbb{C}$-linear isomorphism defined by $J_p(X_p + iY_p) = J_pX_p + iJ_pY_p$. Hence J_p has eigenvalues $\pm i$. This allows us to define the following spaces:

$$T_p^{(1,0)}M = \{Z_p \in T_p^{\mathbb{C}}M | J_pZ_p = iZ_p\} = \{X_p - iJ_pX_p | X_p, Y_p \in T_pM\}$$
$$T_p^{(0,1)}M = \{Z_p \in T_p^{\mathbb{C}}M | J_pZ_p = -iZ_p\} = \{X_p + iJ_pX_p | X_p, Y_p \in T_pM\},$$

where $T_p^{(1,0)}M$ is the i-eigenspace and $T_p^{(0,1)}M$ is the $-i$-eigenspace. It is clear that $T_p^{(0,1)}M = \overline{T_p^{(1,0)}M}$, where $\bar{\cdot}$ is the complex conjugation. By the eigenspace decomposition,

$$T_p^{\mathbb{C}}M = T_p^{(1,0)}M \oplus T_p^{(0,1)}M, \text{ with } \dim_{\mathbb{C}} T_p^{(1,0)}M = \dim_{\mathbb{C}} T_p^{(0,1)}M = n.$$

For all $Z_p \in T_p^{\mathbb{C}}M$,

$$Z_p = \frac{1}{2}(Z_p + iJ_pZ_p) + \frac{1}{2}(Z_p - iJ_pZ_p).$$

The space $T_p^{(1,0)}M$ is called the *holomorphic tangent space*, $T_p^{(0,1)}M$ is the *anti -holomorphic tangent space*.

[1]Elements of $T_p^{\mathbb{C}}M$ are of the form $X_p \otimes 1 + Y_p \otimes i$, where $X_p, Y_p \in T_pM$. For the sake of simplicity, we will omit the tensor product and write $X_p + iY_p$ for a complex tangent vector on M.

Definition 1.1.21. Let M be a real smooth manifold. A tangent bundle isomorphism $J : TM \to TM$ such that $J^2 = -I$ is called an *almost complex structure*. In this case, M is called an *almost complex manifold*.

Remark 1.1.22. (i) Notice that if M is an almost complex manifold, then $\dim M \in 2\mathbb{N}$. Indeed, suppose that M is n-dimensional and let J be an almost complex structure. Since $J^2 = -I$, it follows that $(\det J)^2 = (-1)^n$. Since M is a real manifold, we observe that $\det J \in \mathbb{R}$. Therefore n must be an even number.

(ii) It is well-known that there is an almost complex structure J on all complex manifolds. See, e.g., [9].

(iii) Suppose that M admits an almost complex structure J. We can complexify TM to obtain $T^{\mathbb{C}}M$ and we call $T^{\mathbb{C}}M$ the *complex tangent bundle*.

An almost complex structure on M induces a decomposition of the complex tangent bundle. We define the holomorphic tangent subbundle of the complex tangent bundle $T^{\mathbb{C}}M$ as follows.

Definition 1.1.23. Let M be a real smooth manifold with an almost complex structure J. The *holomorphic tangent bundle* of M, $T^{(1,0)}M$ and the *anti-holomorphic tangent bundle* of M, $T^{(0,1)}M$, are given by

$$T^{(1,0)}M = \bigsqcup_{p\in M} T_p^{(1,0)}M \text{ and } T^{(0,1)}M = \bigsqcup_{p\in M} T_p^{(0,1)}M.$$

Furthermore, we have the tangent bundle decomposition $T^{\mathbb{C}}M = T^{(1,0)}M \oplus T^{(0,1)}M$. In other words, for each $p \in M$, $T_p^{\mathbb{C}}M = T_p^{(1,0)}M \oplus T_p^{(0,1)}M$.

For all $A, B \in \mathfrak{X}(M)$, we define the *Lie bracket of vector fields* to be the operator $[A, B] : C^\infty(M) \to C^\infty(M)$ such that

$$[A, B]f = A \circ Bf - B \circ Af.$$

It is clear that $[A, B] \in \mathfrak{X}(M)$. For each $p \in M$, $[A, B]_p$ is a tangent vector to M. Later, we will see that the Lie bracket gives $\mathfrak{X}(M)$ the structure of a *Lie algebra*. Next, we define an important tensor field.

Definition 1.1.24. Let M be a smooth manifold and J be an almost complex structure on M. We define the *Nijenhuis tensor* by

$$N_J(A, B) = [JA, JB] - [A, B] - J([JA, B] + [A, JB]),$$

for all $A, B \in \mathfrak{X}(M)$.

Remark 1.1.25. It is obvious that $N_J : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is bilinear.

Definition 1.1.26. Let M be a smooth manifold and let J be an almost complex structure on M. We say that J is *integrable* if for all $Z, W \in \mathfrak{X}^{(1,0)}(M)$,

$$[Z, W] \subseteq \mathfrak{X}^{(1,0)}(M), \tag{1.1}$$

where $\mathfrak{X}^{(1,0)}(M)$ is the set of all smooth vector fields $Z : M \to T^{(1,0)}M$. We will refer to (1.1) as the *integrability condition*. We say that J is a *complex structure* if it is an integrable almost complex structure.

The following theorem is an important tool that permits us to determine whether or not J is integrable, which is known as the *Newlander–Nirenberg Theorem*.

Theorem 1.1.27 ([20])**.** *Let J be an almost complex structure on a smooth manifold M. Then J is a complex structure if and only if $N_J = 0$.*

1.2 Complex structures on real Lie algebras

We will be interested in the particular case of manifolds that are nilpotent Lie groups. In what follows, we define Lie groups and Lie algebras.

Definition 1.2.1. A *Lie group* is a finite dimensional smooth manifold G equipped with a group operation $\mu : G \times G \to G$ such that

(a) μ is smooth;

(b) $\iota : G \to G$ defined by $\iota : x \mapsto x^{-1}$ is smooth.

Remark 1.2.2. (i) Notice that the two requirements can be combined into the single requirement that $\mu : (x, y) \mapsto x^{-1}y$ is smooth for every $x, y \in G$.

(ii) For all $x, y \in G$, let L_x be the left translation of G defined by $L_x(y) = xy$. One can define a *left-invariant vector field* $\tilde{X}$ on G by

$$\tilde{X}_x = \mathrm{d}(L_x)_e X, \tag{1.2}$$

where e is the identity of G, $\mathrm{d}(L_x)_e$ is the differential of L_x at e and X is a tangent vector to G at e. Since $\mathrm{d}(L_x)_y(a\tilde{X}_y + b\tilde{Y}_y) = a\ \mathrm{d}(L_x)_y\tilde{X}_y + b\ \mathrm{d}(L_x)_y\tilde{Y}_y$ for all $a, b \in \mathbb{R}$, the set of all smooth left-invariant vector fields on G is a linear subspace of $\mathfrak{X}(G)$.

Furthermore, if $\tilde{X}, \tilde{Y} \in \mathfrak{X}(G)$ are left-invariant, then $[\tilde{X}, \tilde{Y}]$ is also left-invariant. See, e.g., [18, Proposition 8.33].

Definition 1.2.3. A vector space $\mathfrak{g}$ over $\mathbb{R}$ equipped with a bilinear form $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called a *Lie algebra* if

(a) $[X, Y] = -[Y, X]$;

(b) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for all $X, Y, Z \in \mathfrak{g}$. This is called the *Jacobi identity*.

Remark 1.2.4. (i) An immediate consequence of (a) is that $[X, X] = 0$ for all $X \in \mathfrak{g}$.

(ii) The set of all smooth left-invariant vector fields on G, denoted $\mathrm{Lie}(G)$, is a Lie algebra over $\mathbb{R}$. There is a vector space isomorphism between $\mathrm{Lie}(G)$ and T_eG, namely, $\lambda : \mathrm{Lie}(G) \to T_eG$ given by $\lambda : \tilde{X} \mapsto X$ where $\tilde{X}$ is as in (1.2) and $X \in T_eG$. Hence the tangent space T_eG inherits a Lie algebra structure and we will denote it by $\mathfrak{g}$.

Euclidean spaces are the easiest example of Lie algebras.

Example 1.2.5. [13] For all $x, y \in GL(n, \mathbb{R})$, let the group operation $\mu : GL(n, \mathbb{R}) \times GL(n, \mathbb{R}) \to GL(n, \mathbb{R})$ be the matrix product xy. The smoothness of μ follows since the product of matrices has polynomial components. The smoothness of ι follows from that of the determinant function and the fact that $\det x \neq 0$ for all $x \in GL(n, \mathbb{R})$.

The vector space $M(n, \mathbb{R})$ of $n \times n$ real matrices becomes an n^2-dimensional Lie algebra with the Lie bracket given by $[X, Y] = XY - YX$. It is clear that bilinearity and antisymmetry hold and the Jacobi identity follows from a straightforward calculation. We denote this Lie algebra by $\mathfrak{gl}(n,\mathbb{R})$. It is known that we may identify $\mathrm{Lie}(GL(n,\mathbb{R}))$ and $\mathfrak{gl}(n,\mathbb{R})$. See, e.g., [18, Proposition 8.41].

Definition 1.2.6. Let $\mathfrak{g}$ be a Lie algebra. A *Lie subalgebra* is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[\mathfrak{h},\mathfrak{h}] \subseteq \mathfrak{h}$. An *ideal* $\mathfrak{i}$ of a Lie algebra is a subalgebra satisfying $[\mathfrak{i},\mathfrak{g}] \subseteq \mathfrak{i}$. In this case, we write $\mathfrak{i} \trianglelefteq \mathfrak{g}$ and the quotient space $\mathfrak{g}/\mathfrak{i} = \{X + \mathfrak{i} : \ \forall\, X \in \mathfrak{g}\}$ is a Lie algebra.

Remark 1.2.7. (i) A Lie algebra $\mathfrak{g}$ is *Abelian* if $[X,Y] = 0$ for all $X, Y \in \mathfrak{g}$. For instance, vector spaces are Abelian Lie algebras.

(ii) The projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ a surjective homomorphism of Lie algebras.

(iii) The *center* of $\mathfrak{g}$ is the ideal given by $\mathfrak{Z}(\mathfrak{g}) = \{X \in \mathfrak{g} : [X,\mathfrak{g}] = \{0\}\}$. We will denote $\mathfrak{Z}(\mathfrak{g})$ by $\mathfrak{z}$ in this thesis, if no confusion arises.

(iv) Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie algebras with Lie brackets $[\cdot,\cdot]$ and $[\cdot,\cdot]'$. A *Lie algebra homomorphism* is a linear map that preserves Lie brackets:

$$f : \mathfrak{g} \to \mathfrak{g}', \qquad f[X,Y] = [f(X), f(Y)]',$$

for all $X, Y \in \mathfrak{g}$.

We next define left-invariant almost complex structures on G.

Definition 1.2.8. An almost complex structure J on G is said to be *left-invariant* if $(\mathrm{d}L_x)_e \circ J_e = J_x \circ (\mathrm{d}L_x)_e$ for all $x \in G$.

Remark 1.2.9. Recall, from (1.2), that for all $\tilde{X} \in \mathrm{Lie}(G)$ and $x \in G$, $J_x\tilde{X} = (\mathrm{d}L_x)_e \circ J_e(X)$.

In the case of a Lie group G, there is a one to one correspondence between left-invariant almost complex structures on G and almost complex structures defined on $\mathfrak{g}$. In this context, Theorem 1.1.27 reads as follows.

Corollary 1.2.10. *A left invariant almost complex structure J on G is a complex structure if and only if J_e satisfies the integrability condition*

$$[J_eX, J_eY] - [X, Y] - J_e([J_eX, Y] + [X, J_eY]) = 0, \tag{1.3}$$

for all $X, Y \in \mathfrak{g}$.

Remark 1.2.11. (i) Formally, we should use J_e to represent the complex structure on $\mathfrak{g}$. However, since we are interested only on Lie algebras in this thesis, from now on, we will write J for J_e.

(ii) By abuse of notation, we denote the left hand side of (1.3) by $N_J(X, Y)$ and we will refer to (1.3) as the *Newlander–Nirenberg condition.*

(iii) Given a complex structure J on $\mathfrak{g}$, its complexification $\mathfrak{g}^{\mathbb{C}}$ splits into $\mathfrak{g}^{(1,0)} \oplus \mathfrak{g}^{(0,1)}$, where

$$\mathfrak{g}^{(1,0)} = \{Z \in \mathfrak{g}^{\mathbb{C}} : JZ = iZ\} \text{ and } \mathfrak{g}^{(0,1)} = \{Z \in \mathfrak{g}^{\mathbb{C}} : JZ = -iZ\}$$

are the $\pm i$-eigenspaces of J. By the integrability condition (1.1), J is a complex structure if and only if the $\mathfrak{g}^{(1,0)}$ and $\mathfrak{g}^{(0,1)}$ are complex subalgebras of $\mathfrak{g}^{\mathbb{C}}$.

There are different types of complex structures that can be defined on Lie algebras. We shall define some of them here. See, e.g., [2], [23].

Definition 1.2.12. A complex structure J on $\mathfrak{g}$ is called *bi-invariant* if $J[X, Y] = [JX, Y]$ for all $X, Y \in \mathfrak{g}$. A complex structure J is called *Abelian* if $[X, Y] = [JX, JY]$ for all $X, Y \in \mathfrak{g}$.

Remark 1.2.13. (i) Suppose that J is bi-invariant on $\mathfrak{g}$. It is easily seen that $J[X, Y] = [X, JY] = [JX, Y]$ for all $X, Y \in \mathfrak{g}$.

(ii) Suppose that J is bi-invariant over a Lie algebra $\mathfrak{g}$. The Lie brackets on $\mathfrak{g}$ are $\mathbb{C}$-linear. Conversely, if a Lie algebra $\mathfrak{g}$ admits a $\mathbb{C}$-linear Lie bracket, then J is bi-invariant.

Proposition 1.2.14. *Let $\mathfrak{g}$ be a Lie algebra with a complex structure J and $\mathfrak{g}^{\mathbb{C}}$ be its complexification. Then $\mathfrak{g}^{(1,0)}$ is Abelian if and only if J is Abelian. Furthermore, $\mathfrak{g}^{(1,0)}$ is a complex ideal of $\mathfrak{g}^{\mathbb{C}}$ if and only if J is bi-invariant.*

Proof. For all $X - iJX, Y - iJY \in \mathfrak{g}^{(1,0)}$, by the Newlander–Nirenberg condition,

$$\begin{aligned} [X - iJX, Y - iJY] &= ([X, Y] - [JX, JY]) - i([X, JY] + [JX, Y]) \\ &= ([X, Y] - [JX, JY]) + iJ([X, Y] - [JX, JY]). \end{aligned} \tag{1.4}$$

Suppose that $\mathfrak{g}^{(1,0)}$ is Abelian. Then (1.4) equals zero and $[X, Y] = [JX, JY]$ for all $X, Y \in \mathfrak{g}$ and J is Abelian. Conversely, assume that J is Abelian. Then again from (1.4), $[X - iJX, Y - iJY] = 0$. Therefore $\mathfrak{g}^{(1,0)}$ is Abelian.

Next, assume that $\mathfrak{g}^{(1,0)}$ is a complex ideal of $\mathfrak{g}^{\mathbb{C}}$. That is, $[\mathfrak{g}^{\mathbb{C}}, \mathfrak{g}^{(1,0)}] \subseteq \mathfrak{g}^{(1,0)}$. For all $Z = X - iJX \in \mathfrak{g}^{(1,0)}$ and for all $W \in \mathfrak{g}^{\mathbb{C}}$,

$$[W, X - iJX] = [W, X] - i[W, JX] \in \mathfrak{g}^{(1,0)}.$$

Let $W = A + iB$ where $A = \mathrm{Re}(W), B = \mathrm{Im}(W) \in \mathfrak{g}$. Then

$$[X - iJX, A + iB] = [X, A] + [JX, B] - i([JX, A] - [X, B]) \in \mathfrak{g}^{(1,0)}.$$

Since $[X, A] + [JX, B] - i([JX, A] - [X, B])$ is of the form $U - iJU$ for some $U \in \mathfrak{g}$, $[JX, A] - [X, B] = J([X, A] + [JX, B])$. By the Newlander–Nirenberg condition,

$$[X, B] + J[JX, B] = [JX, JB] - J[X, JB].$$

Hence

$$[JX, JB] - J[X, JB] = [JX, A] - J[X, A] \Rightarrow J[JX, A - JB] = [JX, A - JB].$$

By Definition 1.2.12, J is bi-invariant. Conversely, suppose that J is bi-invariant. Since the Lie bracket is $\mathbb{C}$-linear, for all $[W, X - iJX] \in [\mathfrak{g}^{(1,0)}, \mathfrak{g}^{\mathbb{C}}]$

$$\begin{aligned} J([W,X] - iJ[W,X]) &= J[W, X - iJX] = [JW, X - iJX] \\ &= [JW, X] - iJ[JW, X] \\ &= i([W,X] - iJ[W,X]). \end{aligned}$$

Hence $[W,X] - iJ[W,X] \in \mathfrak{g}^{(1,0)}$ and $\mathfrak{g}^{(1,0)}$ is a complex ideal of $\mathfrak{g}^{\mathbb{C}}$. □

Proposition 1.2.15. *Let $\mathfrak{g}$ be a Lie algebra with a complex structure J. Then J is both Abelian and bi-invariant if and only if $\mathfrak{g}$ is Abelian.*

Proof. Since J is both Abelian and bi-invariant, by definition, $[JX, JY] = [X, Y]$ and $J[X, Y] = [X, JY] = [JX, Y]$ for all $X, Y \in \mathfrak{g}$. Combining these two equalities, we have that

$$J[X,Y] = J[JX,JY] = -[JX,Y] = -J[X,Y],$$

which implies that $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. Hence $\mathfrak{g}$ is Abelian.

Conversely, suppose that $\mathfrak{g}$ is Abelian. Thus

$$J[X,Y] = [X,Y] = [JX,Y] = [X,JY] = [JX,JY] = 0$$

for all $X, Y \in \mathfrak{g}$. In conclusion, J is both Abelian and bi-invariant. □

Proposition 1.2.16. *Let $\mathfrak{g}$ be a Lie algebra with a complex structure J. Suppose that $\mathfrak{i}$ is a J-invariant ideal of $\mathfrak{g}$. Then J induces a complex structure $\hat{J}$ on the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$.*

Proof. Since $\mathfrak{i} \trianglelefteq \mathfrak{g}$, by definition, $\mathfrak{g}/\mathfrak{i} = \hat{\mathfrak{g}}$ is a quotient Lie algebra. For $\pi(X) = \hat{X} \in \mathfrak{g}/\mathfrak{i}$, define $\hat{J}(\hat{X}) = \pi(JX)$, where π the surjective Lie algebra homomorphism given by the projection $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$. Since $\mathfrak{i}$ is J-invariant, $\hat{J}$ is well defined. We now check that $\hat{J}$ is a complex structure on $\mathfrak{g}/\mathfrak{i}$. By a straightforward calculation,

$$\begin{aligned}
&\hat{J}^2(\hat{X}) = \hat{J}(\hat{J}(\hat{X})) = \hat{J}(\pi(JX)) = \pi(J^2(X)) = -\hat{X} \\
&[\hat{J}\hat{X}, \hat{J}\hat{Y}] - [\hat{X}, \hat{Y}] + \hat{J}[\hat{X}, \hat{J}\hat{Y}] + \hat{J}[\hat{J}\hat{X}, \hat{Y}] \\
&= [\pi(JX), \pi(JY)] - [\pi(X), \pi(Y)] + \hat{J}[\pi(X), \pi(JY)] + \hat{J}[\pi(JX), \pi(Y)] \\
&= \pi(N_J(X, Y)) = \pi(0) = \hat{0}.
\end{aligned}$$

By definition, $\hat{J}$ is a complex structure. □

Proposition 1.2.17. *Let $\mathfrak{g}$ be a Lie algebra with a complex structure J and assume that the induced map $\hat{J} \in GL(\mathfrak{g}/\mathfrak{i})$ is a linear isomorphism, where $\mathfrak{i}$ is a J-invariant ideal of $\mathfrak{g}$. If J is Abelian, then $\hat{J}$ is Abelian; if J is bi-invariant, then $\hat{J}$ is bi-invariant.*

Proof. By Proposition 1.2.16, $\hat{J}$ is a complex structure on $\mathfrak{g}/\mathfrak{i}$. Let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{i}$ be the surjective Lie algebra homomorphism as in Proposition 1.2.16. Then for all $\hat{X}, \hat{Y} \in \mathfrak{g}/\mathfrak{i}$,

$$\begin{aligned}
[\hat{J}\hat{X}, \hat{J}\hat{Y}] - [\hat{X}, \hat{Y}] &= [\pi(JX), \pi(JY)] - [\pi(X), \pi(Y)] \\
&= \pi\left([JX, JY] - [X, Y]\right) \\
\hat{J}[\hat{X}, \hat{Y}] - [\hat{J}\hat{X}, \hat{Y}] &= \hat{J}\pi([X, Y]) - \pi([JX, Y]) \\
&= \pi\left(J[X, Y] - [JX, Y]\right).
\end{aligned}$$

If J is Abelian, $\pi\left([JX, JY] - [X, Y]\right) = \pi(0) = \hat{0}$. Hence $\hat{J}$ is Abelian. If J is bi-invariant, $\pi\left(J[X, Y] - [JX, Y]\right) = \pi(0) = \hat{0}$. Therefore $\hat{J}$ is bi-invariant. □

Let $\mathfrak{g} = \mathrm{span}\{X_1, \ldots, X_{2n}\}$ be a Lie algebra. Whether or not an almost complex structure on $\mathfrak{g}$ is a complex structure is an interesting question to investigate. We first provide a necessary condition for $J \in GL(\mathfrak{g})$ to be an almost complex structure.

For $X_i \in \mathfrak{g}$ with $i \in \{1, \ldots, 2n\}$, suppose that $J \in GL(\mathfrak{g})$ is given by $JX_i = \sum_{j\geq 1}^{2n} a_{ij} X_j$ where $a_{ij} \in \mathbb{R}$.

Lemma 1.2.18. *Let $\mathfrak{g}$ be a Lie algebra and let J be a linear isomorphism on $\mathfrak{g}$. If any of the columns or any of the rows of the matrix representing J is zero except the diagonal term, then J is not an almost complex structure.*

Proof. Suppose, by contradiction, that J is an almost complex structure. By definition, $J^2 = -I$. Recall that $JX_i = \sum_{j\geq 1}^{2n} a_{ij} X_j$ where $a_{ij} \in \mathbb{R}$ for all $i, j \in \{1, \ldots, n\}$. Then

$$J^2(X_i) = \sum_{j=1}^{2n} \sum_{m\geq 1}^{2n} a_{ij} a_{jm} X_m = -X_i \Longrightarrow \sum_{j=1}^{2n} a_{ij} a_{ji} = -1. \tag{1.5}$$

Without loss of generality, taking $i = 1$, it follows that $\sum_{j=1}^{2n} a_{1j} a_{j1} = a_{11}^2 + a_{12}a_{21} + \ldots + a_{1,2n}a_{2n,1} = -1$. Suppose that either $a_{12} = a_{13} = \ldots = a_{1,2n} = 0$ or $a_{21} = a_{31} = \ldots = a_{2n,1} = 0$. By substituting on (1.5), $a_{11}^2 = -1$. This implies $a_{11} = \sqrt{-1}$, which contradicts the assumption that $a_{11} \in \mathbb{R}$. □

Lemma 1.2.19. *Let J be an almost complex structure on $\mathfrak{g}$ and let N_J be the Nijenhuis tensor. Then $N_J(X, Y) = -N_J(Y, X)$ and $N_J(JX, Y) = -JN_J(X, Y)$ for all $X, Y \in \mathfrak{g}$.*

Proof. For all $X, Y \in \mathfrak{g}$,

$$\begin{aligned}
N_J(Y, X) &= [Y, X] - [JY, JX] - J([JY, X] + [Y, JX]) \\
&= -([X, Y] - [JX, JY] - J([JX, Y] + [X, JY])) = -N_J(X, Y). \\
N_J(JX, Y) &= [JX, Y] + [X, JY] + J([X, Y] - [JX, JY]) \\
&= J([X, Y] - [JX, JY] - J([JX, Y] + [X, JY])) = -JN_J(X, Y).
\end{aligned}$$

In conclusion, $N_J(X, Y) = -N_J(Y, X)$ and $N_J(JX, Y) = -JN_J(X, Y)$ for all $X, Y \in \mathfrak{g}$. □

Proposition 1.2.20. *Let $J \in GL(\mathfrak{g})$ be an almost complex structure on $\mathfrak{g} = \mathrm{span}\{X_j, JX_j\}_{j=1}^n$. If $N_J(X_i, X_j) = 0$ for all $1 \le i < j \le n$, then J is a complex structure.*

Proof. Let $X = \sum_{i=1}^n (a_i X_i + b_i J X_i), Y = \sum_{j=1}^n (a'_j X_j + b'_j J X_j)$ for some $a_i, b_i, a'_j, b'_j \in \mathbb{R}$. Then

$$\begin{aligned} N_J(X, Y) = \sum_{i,j}^{n} \Big(& a_i a'_j N_J(X_i, X_j) + a_i b'_j N_J(X_i, JX_j) \\ & + b_i a'_j N_J(JX_i, X_j) + b_i b'_j N_J(JX_i, JX_j) \Big) . \end{aligned}$$

Recall, from Lemma 1.2.19, that

$$N_J(X_i, X_j) = -N_J(X_j, X_i) \text{ and } N_J(JX_i, X_j) = -JN_J(X_i, X_j).$$

Since $N_J(X_i, X_j) = 0$ for all $1 \le i < j \le n$, by Lemma 1.2.19, it is sufficient to show that

$$\begin{aligned} N_J(X_i, JX_j) &= JN_J(X_j, X_i) = 0 \text{ and} \\ N_J(JX_i, JX_j) &= -JN_J(X_i, JX_j) = N_J(X_i, X_j) = 0. \end{aligned}$$

Hence $N_J(X, Y) = 0$. By definition, J is a complex structure on $\mathfrak{g}$. □

Let J be an almost complex structure on $\mathfrak{g}$. We define the following subspace

$$\mathrm{Im} N_J = \mathrm{span}\{N_J(X, Y) : \ \forall \ X, Y \in \mathfrak{g}\}.$$

Corollary 1.2.21. *Let J be an almost complex structure on $\mathfrak{g} = \mathrm{span}\{X_j, JX_j\}_{j=1}^n$. Then $\dim \mathrm{Im} N_J \le 2\binom{n}{2}$.*

Proof. Since N_J is bilinear, Proposition 1.2.20 implies that

$$\mathrm{Im} N_J = \mathrm{span}\{N_J(X_i, X_j), JN_J(X_i, X_j) : 1 \le i < j \le n\}.$$

Hence $\dim \mathrm{Im} N_J \leq 2\binom{n}{2}$. □

Remark 1.2.22. Suppose that J is a complex structure on $\mathfrak{g}$. Then $\dim \mathrm{Im}(N_J) = 0$.

To end this section, we have the following observation for arbitrary 2-dimensional Lie algebras.

Lemma 1.2.23. *Let $\mathfrak{g}$ be a 2-dimensional Lie algebra with an almost complex structure J. Then J is a complex structure on $\mathfrak{g}$.*

Proof. By Lemma 1.1.19, there exists a basis $\{X_1, JX_1\}$ such that $\mathfrak{g} = \mathrm{span}\{X_1, JX_1\}$. Suppose that $[X_1, JX_1] = 0$. It follows that $[\mathfrak{g}, \mathfrak{g}] = \{0\}$ and therefore $\mathfrak{g} = \mathrm{span}\{X_1, JX_1\} \cong \mathbb{R}^2$. Hence $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is a complex structure.

Next, suppose that $[\mathfrak{g}, \mathfrak{g}] \neq \{0\}$. Let $X = a_1X_1 + a_2JX_1$ and $Y = b_1X_1 + b_2JX_1 \in \mathfrak{g}$ for some $a_i, b_i \in \mathbb{R}$. Then

$$\begin{aligned} [JX, Y] &= [a_1JX_1 - a_2X_1, b_1X_1 + b_2JX_1] = -(a_1b_1 + a_2b_2)[X_1, JX_1] \\ [X, JY] &= [a_1X_1 + a_2JX_1, b_1JX_1 - b_2X_1] = (a_1b_1 + a_2b_2)[X_1, JX_1]. \end{aligned}$$

Hence $[JX, Y] = -[X, JY]$. By replacing Y by JY, $[X, Y] = [JX, JY]$. It follows that $N_J(X, Y) = 0$. By definition, J is a complex structure. □

Remark 1.2.24. Recall that a 2 dimensional Lie algebra is isomorphic to either $\mathfrak{g}_1 = \mathbb{R}^2$ or $\mathfrak{g}_2 = \{X_1, X_2 \in \mathfrak{g} : [X_1, X_2] = X_1\}$. Complex structures on each of those algebras must be Abelian.

Chapter 2

Complex structures on nilpotent Lie algebras

In this chapter, we first introduce some notation and provide the proof of some results obtained by different authors in different papers. Next, we consider some properties of the central series of nilpotent Lie algebras with complex structures J and define J-invariant central series. In particular, we introduce a J-invariant descending central series $\mathfrak{p}_j + J\mathfrak{p}_j$ that provides a new characterization of a special type of complex structure, called *nilpotent*. We next show that strata-preserving complex structures on stratified Lie algebras have to be nilpotent and provide a Lie theoretic proof of the connection between the J-invariant central series and nilpotent complex structures. Finally, we will study nilpotent complex structures on stratified Lie algebras and investigate how the nilpotency of J impacts the strata-preserving property.

2.1 Stratified Lie algebras

In Chapter 1, we introduced the general notation and provided the definition of Lie groups and Lie algebras. In this section, we focus on a special class of Lie algebras.

Definition 2.1.1. [14] Let $\mathfrak{g}$ be a Lie algebra. The *descending central series* and *ascending central series* of $\mathfrak{g}$ are denoted by $\mathfrak{c}_j(\mathfrak{g})$ and $\mathfrak{c}^j(\mathfrak{g})$ respectively, for all $j \geq 0$, and defined inductively by

$$\mathfrak{c}_0(\mathfrak{g}) = \mathfrak{g}, \ \ \mathfrak{c}_j(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{c}_{j-1}(\mathfrak{g})]; \tag{2.1}$$

$$\mathfrak{c}^0(\mathfrak{g}) = \{0\}, \quad \mathfrak{c}^j(\mathfrak{g}) = \{X \in \mathfrak{g} : [X, \mathfrak{g}] \subseteq \mathfrak{c}^{j-1}(\mathfrak{g})\}. \tag{2.2}$$

Remark 2.1.2. (i) Notice that $\mathfrak{c}^1(\mathfrak{g}) = \mathfrak{Z}(\mathfrak{g})$, $\mathfrak{c}_1(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}]$, and

$$\mathfrak{c}^j(\mathfrak{g})/\mathfrak{c}^{j-1}(\mathfrak{g}) = \mathfrak{Z}\left(\mathfrak{g}/\mathfrak{c}^{j-1}(\mathfrak{g})\right) \;\; \forall j \geq 1,$$

where $\mathfrak{Z}(\cdot)$ is the center of a Lie algebra. Furthermore, $\mathfrak{c}^j(\mathfrak{g})$ and $\mathfrak{c}_j(\mathfrak{g})$ are ideals of $\mathfrak{g}$ for all $j \geq 0$.

(ii) For all $k, l \in \mathbb{N}$, notice that $[\mathfrak{c}_k(\mathfrak{g}), \mathfrak{c}_l(\mathfrak{g})] \subseteq \mathfrak{c}_{k+l}(\mathfrak{g})$. Let $k = l = 1$. By definition, $[\mathfrak{c}_1(\mathfrak{g}), \mathfrak{c}_1(\mathfrak{g})] \subseteq [\mathfrak{c}_1(\mathfrak{g}), \mathfrak{g}] = \mathfrak{c}_2(\mathfrak{g})$. Assume that $[\mathfrak{c}_t(\mathfrak{g}), \mathfrak{c}_s(\mathfrak{g})] \subseteq \mathfrak{c}_{t+s}(\mathfrak{g})$ for some $s, t \in \mathbb{N}$. We first show that $[\mathfrak{c}_t(\mathfrak{g}), \mathfrak{c}_{s+1}(\mathfrak{g})] \subseteq \mathfrak{c}_{s+t+1}(\mathfrak{g})$. Note that

$$[\mathfrak{c}_t(\mathfrak{g}), \mathfrak{c}_{s+1}(\mathfrak{g})] = [\mathfrak{c}_t(\mathfrak{g}), [\mathfrak{c}_s(\mathfrak{g}), \mathfrak{g}]]. \tag{2.3}$$

For all $Y \in \mathfrak{g}$, $U_s \in \mathfrak{c}_s(\mathfrak{g})$ and $U_t \in \mathfrak{c}_t(\mathfrak{g})$, using the Jacobi identity,

$$\underbrace{[U_t, [U_s, Y]]}_{\in [\mathfrak{c}_t(\mathfrak{g}), [\mathfrak{g}, \mathfrak{c}_s(\mathfrak{g})]]} + \underbrace{[U_s, [Y, U_t]]}_{\in [\mathfrak{c}_s(\mathfrak{g}), [\mathfrak{g}, \mathfrak{c}_t(\mathfrak{g})]]} = - \underbrace{[Y, [U_t, U_s]]}_{\in [\mathfrak{g}, [\mathfrak{c}_t(\mathfrak{g}), \mathfrak{c}_s(\mathfrak{g})]] \subseteq \mathfrak{c}_{t+s+1}(\mathfrak{g})}.$$

Hence, returning to (2.3), $[\mathfrak{c}_t(\mathfrak{g}), \mathfrak{c}_{s+1}(\mathfrak{g})] \subseteq \mathfrak{c}_{t+s+1}(\mathfrak{g})$. Similarly, $[\mathfrak{c}_{t+1}(\mathfrak{g}), \mathfrak{c}_s(\mathfrak{g})] \subseteq \mathfrak{c}_{t+s+1}(\mathfrak{g})$. By induction, $[\mathfrak{c}_k(\mathfrak{g}), \mathfrak{c}_l(\mathfrak{g})] \subseteq \mathfrak{c}_{k+l}(\mathfrak{g})$ for all $k, l \in \mathbb{N}$.

Proposition 2.1.3. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{c}_j(\mathfrak{g})/\mathfrak{c}_{j+1}(\mathfrak{g}) \subseteq \mathfrak{Z}\left(\mathfrak{g}/\mathfrak{c}_{j+1}(\mathfrak{g})\right)$ for all $j \geq 0$.*

Proof. By definition, for all $U \in \mathfrak{c}_j(\mathfrak{g})$ and $Y \in \mathfrak{g}$ with all $j \geq 0$, $[U, Y] \in \mathfrak{c}_{j+1}(\mathfrak{g})$. Then

$$[U + \mathfrak{c}_{j+1}(\mathfrak{g}), Y + \mathfrak{c}_{j+1}(\mathfrak{g})] = [U, Y] + \mathfrak{c}_{j+1}(\mathfrak{g}) \subseteq \mathfrak{c}_{j+1}(\mathfrak{g}).$$

Hence $\mathfrak{c}_j(\mathfrak{g})/\mathfrak{c}_{j+1}(\mathfrak{g}) \subseteq \mathfrak{Z}\left(\mathfrak{g}/\mathfrak{c}_{j+1}(\mathfrak{g})\right)$. $\square$

Definition 2.1.4. A Lie algebra $\mathfrak{g}$ is called *nilpotent of step k*, for some $k \in \mathbb{N}$, if $\mathfrak{c}_k(\mathfrak{g}) = \{0\}$ and $\mathfrak{c}_{k-1}(\mathfrak{g}) \neq \{0\}$. We will denote nilpotent Lie algebras by $\mathfrak{n}$. A connected Lie group N is called *nilpotent* if its Lie algebra is nilpotent.

Remark 2.1.5. The number k is also called the *nil-index* or *nilpotent length* of $\mathfrak{n}$. Notice that $\mathfrak{n}$ is Abelian if and only if $k = 1$. Furthermore, if $\mathfrak{n}$ is nilpotent of step k, then $\mathfrak{c}^k(\mathfrak{n}) = \mathfrak{n}$ and $\mathfrak{c}^{k-1}(\mathfrak{n}) \subset \mathfrak{n}$. See, e.g., [13, Section 5.2] or [14].

Proposition 2.1.6. *Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra and $\mathfrak{i} \trianglelefteq \mathfrak{n}$. Then the quotient Lie algebra $\mathfrak{n}/\mathfrak{i}$ is nilpotent and every subalgebra of $\mathfrak{n}/\mathfrak{i}$ is nilpotent.*

Proof. Since $\pi : \mathfrak{n} \to \mathfrak{n}/\mathfrak{i}$ is a surjective Lie algebra homomorphism, it follows that $\mathfrak{c}_k(\mathfrak{n}/\mathfrak{i}) = \pi(\mathfrak{c}_k(\mathfrak{n})) = \pi(\{0\})$. Hence $\mathfrak{n}/\mathfrak{i}$ is nilpotent. Furthermore, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{n}/\mathfrak{i}$. By definition, $\mathfrak{c}_1(\mathfrak{h}) \subseteq \mathfrak{c}_1(\mathfrak{n}/\mathfrak{i})$. Next, assume that $\mathfrak{c}_s(\mathfrak{h}) \subseteq \mathfrak{c}_s(\mathfrak{n}/\mathfrak{i})$ for some $s \in \mathbb{N}$, Then

$$\mathfrak{c}_{s+1}(\mathfrak{h}) = [\mathfrak{c}_s(\mathfrak{h}), \mathfrak{n}] \subseteq [\mathfrak{c}_s(\mathfrak{n}/\mathfrak{i}), \mathfrak{n}] = \mathfrak{c}_{s+1}(\mathfrak{n}/\mathfrak{i}).$$

By induction, $\mathfrak{c}_j(\mathfrak{h}) \subseteq \mathfrak{c}_j(\mathfrak{n}/\mathfrak{i})$ for all $j \geq 1$. Hence there exists $k \in \mathbb{N}$ such that $\mathfrak{c}_k(\mathfrak{h}) \subseteq \mathfrak{c}_k(\mathfrak{n}/\mathfrak{i}) = \{0\}$. By definition, $\mathfrak{h}$ is nilpotent. $\square$

Remark 2.1.7. Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra and let $\mathfrak{z}$ be the center of $\mathfrak{n}$. It is clear that $\mathfrak{n}/\mathfrak{z}$ is nilpotent of step $k - 1$.

Definition 2.1.8. Let V be a vector space and let $\mathfrak{gl}(V)$ be the Lie algebra consisting of all linear endomorphisms of V. A *representation* of a Lie algebra $\mathfrak{g}$ on V is a Lie algebra homorphism $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$. The *adjoint representation*

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

is defined by $\mathrm{ad}(X)(Y) = [X, Y]$ for all $X, Y \in \mathfrak{g}$.

Remark 2.1.9. Definition 2.1.4 is equivalent to the following statement:

A Lie algebra $\mathfrak{n}$ is nilpotent of step k if

$$[X_1, [X_2, \ldots, [X_k, Y]], \ldots]] = \mathrm{ad}(X_1)\,\mathrm{ad}(X_2) \ldots \mathrm{ad}(X_k)(Y) = 0$$

for all $X_1, \ldots, X_k, Y \in \mathfrak{n}$ and there exist $X_1', \ldots, X_k', Y' \in \mathfrak{n}$ such that $\mathrm{ad}(X_1')\,\mathrm{ad}(X_2') \ldots \mathrm{ad}(X_{k-1}')(Y') \neq 0$.

Definition 2.1.10. A nilpotent Lie algebra $\mathfrak{n}$ is said to admit a *step k stratification* if it has a decomposition as a vector space direct sum of the form $\mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \ldots \oplus \mathfrak{n}_k$, where $\mathfrak{n}_k \neq \{0\}$, satisfying the bracket generating property

$$[\mathfrak{n}_1, \mathfrak{n}_{j-1}] = \mathfrak{n}_j \qquad \forall j \in \{2, \ldots, k\} \text{ and } [\mathfrak{n}_1, \mathfrak{n}_k] = \{0\}.$$

A Lie algebra $\mathfrak{n}$ that admits a stratification is called a *stratified Lie algebra*. A connected and simple connected Lie group N is called *stratified* if its Lie algebra is stratified.

Remark 2.1.11. Suppose that $\mathfrak{n}$ is a stratified Lie algebra. The Lie bracket generating property can be written as $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$ for all $i, j \geq 1$.

Lemma 2.1.12. *Let $\mathfrak{n}$ be a k-step stratified Lie algebra. Then*

$$\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1 \leq l \leq k} \mathfrak{n}_l \ \ \forall\ j \geq 0. \tag{2.4}$$

Proof. We shall prove (2.4) by induction. By definition, $\mathfrak{c}_0(\mathfrak{n}) = \mathfrak{n}$. Next, suppose that $\mathfrak{c}_s(\mathfrak{n}) = \bigoplus_{s+1 \leq j \leq k} \mathfrak{n}_j$ for some $s \in \mathbb{N}$. Notice that

$$\mathfrak{c}_{s+1}(\mathfrak{n}) = [\mathfrak{n}, \mathfrak{n}_{s+1} \oplus \ldots \oplus \mathfrak{n}_k] = \mathrm{span}\{[X, \sum_{j=s+1}^{k} X_j] : X \in \mathfrak{n}, X_j \in \mathfrak{n}_j\}$$

For all $X = \sum_{i=1}^{k} Y_i \in \mathfrak{n}$ with each $Y_i \in \mathfrak{n}_i$, since $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{j+i}$,

$$\sum_{j=s+1}^{k} [X, X_j] = \sum_{j=s+1}^{k} \sum_{i=1}^{k} \underbrace{[Y_i, X_j]}_{\in\, [\mathfrak{n}_i, \mathfrak{n}_j] \,\subseteq\, \mathfrak{n}_{i+j}} \in \bigoplus_{s+2 \leq j \leq k} \mathfrak{n}_j.$$

Hence $\mathfrak{c}_{s+1}(\mathfrak{n}) = \bigoplus_{s+2 \leq j \leq k} \mathfrak{n}_j$. By induction, $\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1 \leq l \leq k} \mathfrak{n}_l$ for all $j \geq 0$. $\square$

Proposition 2.1.13. *Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra. Then $\mathfrak{c}_{k-j}(\mathfrak{n}) \subseteq \mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 0$.*

Proof. We show the statement by induction. First, by assumption, $\mathfrak{c}_{k-1}(\mathfrak{n}) \subseteq \mathfrak{z} = \mathfrak{c}^1(\mathfrak{n})$. Next, for some $s \in \mathbb{N}$, assume that $\mathfrak{c}_{k-s}(\mathfrak{n}) \subseteq \mathfrak{c}^s(\mathfrak{n})$. Then

$$\mathfrak{c}_{k-s}(\mathfrak{n}) = [\mathfrak{c}_{k-s-1}(\mathfrak{n}), \mathfrak{n}] \subseteq \mathfrak{c}^s(\mathfrak{n}) \; \Rightarrow \mathfrak{c}_{k-s-1}(\mathfrak{n}) \subseteq \mathfrak{c}^{s+1}(\mathfrak{n}).$$

By induction, $\mathfrak{c}_{k-j}(\mathfrak{n}) \subseteq \mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 0$. □

Remark 2.1.14. Let $\mathfrak{n}$ be a k-step stratified Lie algebra. Then $\mathfrak{c}_{k-j}(\mathfrak{n}) \subseteq \mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 0$.

Proposition 2.1.15. *Let $\mathfrak{n}$ be a $2n$-dimensional step n nilpotent Lie algebra for some $n \in \mathbb{N}$. Suppose that $\dim \mathfrak{c}_j(\mathfrak{n}) = 2n - 2j$ for all $1 \leq j \leq n$. Then $\mathfrak{n}$ does not admit a stratification.*

Proof. Suppose, by contradiction, that $\mathfrak{n}$ admits a stratification. It follows, from Lemma 2.1.12, that $\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1 \leq l \leq n} \mathfrak{n}_l$ and $\dim \mathfrak{c}_1(\mathfrak{n}) = 2n - 2$, then $\dim \mathfrak{n}_1 = 2$. Furthermore, notice that $[\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{n}_2$. Thus $\dim \mathfrak{n}_2 = 1$ and $\dim \mathfrak{c}_2(\mathfrak{n}) = 2n - 3 > 2n - 4$. This is a contradiction. □

Remark 2.1.16. (i) Let $\mathfrak{n}$ be a stratified Lie algebra with a complex structure J. Suppose that $\dim \mathfrak{n}_1 = 2$. Then J is not strata-preserving. Indeed, suppose, by contradiction, that there exists a strata-preserving complex structure J. Then $\dim \mathfrak{n}_j \in 2\mathbb{N}$ for all $j \geq 1$. However, since $\dim \mathfrak{n}_1 = 2$, by definition,

$$\begin{aligned} [\mathfrak{n}_1, \mathfrak{n}_1] &= \operatorname{span}\{[\alpha X + \beta Y, \gamma X + \delta Y] : \alpha, \beta, \gamma, \delta \in \mathbb{R}\} \\ &= \operatorname{span}\{(\alpha\delta - \beta\gamma)[X, Y]\} = \mathfrak{n}_2. \end{aligned}$$

Hence $\dim \mathfrak{n}_2 = 1$, which contradicts the assumption that $\dim \mathfrak{n}_2 \in 2\mathbb{N}$. Hence $\mathfrak{n}$ does not have a strata-preserving complex structure.

(ii) Let $\mathfrak{n}$ be a 3-step stratified Lie algebra with a strata-preserving complex structure. Arguing in a similar way as in part (i), we conclude that $\dim \mathfrak{n} \neq 4$ or 6.

In the remainder of this section, we provide some examples of stratified Lie algebras and study one example of an even-dimensional stratified Lie algebra that does not admit a complex structure.

Example 2.1.17. [3] The Heisenberg groups $\mathbb{H}_n \cong \mathbb{C}^n \times \mathbb{R}$ are the easiest examples of non-Abelian stratified groups. Denote the coordinates on $\mathbb{H}_n$ by (z,t), where $z = (z_1, \dots, z_n)$ and $z_j = x_j + iy_j \in \mathbb{C}^n$ for all $j \geq 1$ and $t \in \mathbb{R}$. The group law on $\mathbb{H}_n$ is

$$(z,t) \cdot (w,s) = (z+w, t+s+2\mathrm{Im}\langle z,w\rangle),$$

where $\langle \cdot, \cdot \rangle$ is a Hermitian inner product. A basis for the set of left-invariant vector fields is $\{\tilde{X}_1 \dots, \tilde{X}_n, \tilde{Y}_1, \dots, \tilde{Y}_n, \tilde{T}\}$, where

$$\tilde{X}_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t},\ \tilde{Y}_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},\ \tilde{T} = \frac{\partial}{\partial t}$$

for all $j = 1, \dots, n$. It follows that $\mathfrak{h}_n = T_e\mathbb{H}_n \cong \mathrm{span}\{\tilde{X}_1, \dots, \tilde{X}_n, \tilde{Y}_1, \dots, \tilde{Y}_n, \tilde{T}\}$ with non-zero Lie brackets $[\tilde{X}_j, \tilde{Y}_j] = \tilde{T}$, where e is the identity element in $\mathbb{H}_n$. The Lie algebras $\mathfrak{h}_n$ are called *Heisenberg algebras*. They admit a stratification $\mathfrak{h}_n = \mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1$ is the $2n$-dimensional vector space generated by $\{(\tilde{X}_1)_e, \dots, (\tilde{X}_n)_e, (\tilde{Y}_1)_e, \dots, (\tilde{Y}_n)_e\}$ and $\mathfrak{n}_2$ is generated by $\{\tilde{T}_e\}$.

Next, we provide examples of nilpotent Lie algebras with stratifications of higher steps.

Example 2.1.18. [13] From Example 1.2.5, $\mathfrak{gl}(n,\mathbb{R})$ is a Lie algebra. Let $\mathfrak{n}(n,\mathbb{R}) = \{X \in \mathfrak{gl}(n,\mathbb{R}) : X_{ij} = 0 \text{ if } i \geq j\}$. This is the Lie algebra of *strictly upper triangular matrices* and it is nilpotent of step $n-1$. Notice that $\mathfrak{n}$ admits a stratification, where

$$\mathfrak{n}_l = \{X \in \mathfrak{n} : X_{ij} = 0,\ \text{unless } j - i = l\},$$

for all $1 \leq l \leq n-1$, are the strata of $\mathfrak{n}$. It is easy to see that

$$\mathfrak{z} = \{Z \in \mathfrak{z} : Z_{ij} = 0,\ \text{unless } i = 1 \text{ and } j = n\}.$$

It follows that $\mathfrak{n}_{n-1} = \mathfrak{z}$ and $\dim \mathfrak{z} = 1$.

Example 2.1.19. Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{n} \subseteq \mathfrak{g}$ be the nilpotent component of the Iwasawa decomposition of $\mathfrak{g}$. Then $\mathfrak{n}$ is a stratified Lie algebra. See [6] and [21] for more details on the decomposition of semisimple Lie algebras.

Example 2.1.20. [24] A $n+1$-dimensional Lie algebra $\mathfrak{n}$ is called *filiform* if $\dim \mathfrak{c}^i(\mathfrak{n}) = i$ for all $0 \leq i \leq n-1$. Filiform algebras are characterized by

$$\mathfrak{c}^i(\mathfrak{n}) = \mathfrak{c}_{n-i}(\mathfrak{n}), \text{ for all } 0 \leq i \leq n. \tag{2.5}$$

It follows that $\mathfrak{n}$ is nilpotent of step n because $\mathfrak{c}_n(\mathfrak{n}) = \mathfrak{c}^0(\mathfrak{n}) = \{0\}$ and $\dim \mathfrak{c}_{n-1}(\mathfrak{n}) = \dim \mathfrak{z} = 1$.

A class of examples of filiform algebras is characterized by a basis $\{X_1, \ldots, X_{n+1}\}$ of $\mathfrak{n}$ such that the non-zero Lie brackets are

$$[X_1, X_j] = X_{j+1}, \text{ for all } 1 \leq j \leq n.$$

The stratification of $\mathfrak{n}$ is of the form $\mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \ldots \oplus \mathfrak{n}_{n-1}$, where $\mathfrak{n}_1 = \operatorname{span}\{X_1, X_2\}$ and $\mathfrak{n}_j = \operatorname{span}\{X_{j+1}\}$ for all $j \geq 2$. Notice that $\mathfrak{z} = \mathfrak{n}_{n-1}$.

Proposition 2.1.21. *Let $\mathfrak{n}$ be the real filiform Lie algebra with $\mathfrak{n} = \operatorname{span}\{X_1, \ldots, X_{2n}\}$ and non-zero Lie bracket relations $[X_1, X_i] = X_{i+1}$ for all $i \in \{2, \ldots, 2n-1\}$. Then $\mathfrak{n}$ does not admit a complex structure J.*

Proof. We shall prove this proposition by contradiction. Suppose that $J : \mathfrak{n} \to \mathfrak{n}$ is a linear isomorphism of $\mathfrak{n}$ given by

$$J : X_1 \mapsto \sum_{j=1}^{2n} a_{1j} X_j; \; X_2 \mapsto \sum_{j=1}^{2n} a_{2j} X_j; \ldots; \; X_{2n} \mapsto \sum_{j=1}^{2n} a_{2n,j} X_j$$

such that $J^2 = -I$ and $N_J(X_i, X_j) = 0$ for all $1 \le i \le j \le 2n$. Then for all $1 \le i, j \le 2n$,

$$[JX_i, X_m] = \begin{cases} a_{i1}[X_1, X_m] & \text{if } m > 1 \\ -\sum_{j=2}^{2n-1} a_{ij} X_{j+1} & \text{if } m = 1 \end{cases} \tag{2.6}$$

$$[JX_i, JX_m] = (a_{i1}a_{m2} - a_{i2}a_{m1})X_3 + \ldots + (a_{i1}a_{m,2n-1} - a_{i,2n-1}a_{m1})X_{2n}. \tag{2.7}$$

For all $i, m \neq 1$, since $[X_i, X_m] = 0$, it follows that $N_J(X_i, X_m) = J(a_{i1}[X_1, X_m] - a_{m1}[X_1, X_i]) - [JX_i, JX_m] = 0$. So

$$[JX_i, JX_m] = J(a_{i1}[X_1, X_m] - a_{m1}[X_1, X_i]) \qquad \forall\ i, m \in \{2, \ldots, 2n-1\}. \tag{2.8}$$

In particular, taking $X_i = X_{2n-1}$ and $X_m = X_{2n}$, from (2.6) and (2.8),

$$[JX_{2n-1}, JX_{2n}] = J(a_{2n-1,1}[X_1, X_{2n}] - a_{2n,1}[X_1, X_{2n-1}]) = -a_{2n,1}JX_{2n}.$$

Hence $\mathrm{ad}(JX_{2n-1})(JX_{2n}) = -a_{2n,1}JX_{2n}$ and $a_{2n,1}$ is an eigenvalue of $\mathrm{ad}(JX_{2n-1})$. By Engel's theorem, [14, Theorem 1.35] $(\mathrm{ad}(X))^s = 0$ for all $X \in \mathfrak{n}$, where $s \in \mathbb{N}$ is the nil-index. So $a_{2n,1} = 0$. In general, for all $i \in \{1, \ldots, 2n-1\}$, the equation (2.8) yields

$$[JX_i, JX_{2n}] = J[X_i, JX_{2n}] = \begin{cases} \sum_{j=2}^{2n-1} a_{2n,j} JX_{j+1} & \text{if } i = 1 \\ \sum_{j=2}^{2n} a_{2n,j} J[X_i, X_j] = 0 & \text{if } 2 \le i \le 2n \end{cases}.$$

Hence $[JX_i, JX_{2n}] = J[X_i, JX_{2n}] = 0$ for all $2 \le i \le 2n$. Notice that $JX_{2n} \notin \mathfrak{z} = \mathrm{span}\{X_{2n}\}$. Then $JX_{2n} = \sum_{j \ge 2} a_{2n,j} X_j$. Using (2.7),

$$[JX_i, JX_{2n}] = a_{i1} \sum_{j=2}^{2n-1} a_{2n,j} X_{j+1} = 0 \tag{2.9}$$

for all $2 \leq i \leq 2n-1$. The family of equations (2.9) are

$$a_{21}(\sum_{j\geq 2}^{2n-1} a_{2n,j}X_{j+1}) = 0, \dots, a_{2n-1,1}(\sum_{j\geq 2}^{2n-1} a_{2n,j}X_{j+1}) = 0.$$

One has the two following cases,

$$(a)\ a_{i1} = 0,\ \text{for all } 1 \leq i \leq 2n-1;\ (b)\ \sum_{j\geq 2}^{2n-1} a_{2n,j}X_{j+1} = 0.$$

If (a) holds, then $J^2X_1 = a_{11}X_1$ and $a_{1j} = 0$ for all $2 \leq j \leq n$. By Lemma 1.2.18, J is not an almost complex structure.

Assume now (b). Since $\{X_j\}_{j\geq 2}^{2n-1}$ is a subset of the basis of $\mathfrak{n}$, X_j are linearly independent. Hence $a_{2n,j} = 0$ for all $2 \leq j \leq 2n-1$ and therefore $JX_{2n} = a_{2n,2n}X_{2n}$. Again by Lemma 1.2.18, J is not an almost complex structure.

In conclusion, $\mathfrak{n}$ admits no complex structures. □

Remark 2.1.22. In [23], Remm proved that there are no complex structures on filiform algebras. In Proposition 2.1.21, we provided a simple proof in a particular case.

2.2 Central series of nilpotent Lie algebras with complex structures

In this section, we investigate the central series of nilpotent Lie algebras with complex structures. We will show that there always exists a stratification on a 2-step nilpotent Lie algebra with a complex structure J such that J is strata-preserving.

Theorem 2.2.1. *Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra with a complex structure J.*

(i) Suppose that J is bi-invariant. Then both central series are J-invariant and $\mathfrak{n}$ admits a J-invariant stratification when $k = 2$.

(ii) Suppose that J is Abelian. Then all $\mathfrak{c}^j(\mathfrak{n})$ are J-invariant.

Proof. We start with part (i). Suppose that J is bi-invariant. Using the fact $J\mathfrak{n} = \mathfrak{n}$,

$$\mathfrak{c}_j(\mathfrak{n}) = [\mathfrak{n}, \mathfrak{c}_{j-1}(\mathfrak{n})] = [J\mathfrak{n}, \mathfrak{c}_{j-1}(\mathfrak{n})] = J[\mathfrak{n}, \mathfrak{c}_{j-1}(\mathfrak{n})] = J\mathfrak{c}_j(\mathfrak{n}).$$

Hence J preserves $\mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$. Furthermore, we show that all $\mathfrak{c}^j(\mathfrak{n})$ are J-invariant if and only if $[JX, \mathfrak{n}] \subseteq \mathfrak{c}^{j-1}(\mathfrak{n})$ for all $X \in \mathfrak{c}^j(\mathfrak{n})$. By definition, $J\mathfrak{c}^0(\mathfrak{n}) = J\mathfrak{n} = \mathfrak{n}$. Next, suppose that $J\mathfrak{c}^{s-1}(\mathfrak{n}) = \mathfrak{c}^{s-1}(\mathfrak{n})$ for some $s \in \mathbb{N}$. Then for all $X \in \mathfrak{c}^s(\mathfrak{n})$,

$$[JX, \mathfrak{n}] = J[X, \mathfrak{n}] \subseteq J\mathfrak{c}^{s-1}(\mathfrak{n}) = \mathfrak{c}^{s-1}(\mathfrak{n}).$$

Hence $JX \in \mathfrak{c}^s(\mathfrak{n})$ and so $J\mathfrak{c}^s(\mathfrak{n}) \subseteq \mathfrak{c}^s(\mathfrak{n})$. The invertibility of J shows that $J\mathfrak{c}^s(\mathfrak{n}) = \mathfrak{c}^s(\mathfrak{n})$. By induction, $J\mathfrak{c}^j(\mathfrak{n}) = \mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 0$.

Now let $k = 2$. Define a J-invariant inner product ψ as in Lemma 1.1.19. We show that there exists a stratification on $\mathfrak{n}$ such that $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are J-invariant. Define $\mathfrak{n}_2 = [\mathfrak{n}, \mathfrak{n}]$ and $\mathfrak{n}_1 = \mathfrak{n}_2^{\perp}$, the orthogonal complement of $\mathfrak{n}_2$ with respect to ψ. From the above paragraph, $\mathfrak{n}_2 = \mathfrak{c}_1(\mathfrak{n})$ is J-invariant and by definition $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$. Also note that

$$\mathfrak{n}_2 = [\mathfrak{n}_1 \oplus \mathfrak{n}_2, \mathfrak{n}_1 \oplus \mathfrak{n}_2] = [\mathfrak{n}_1, \mathfrak{n}_1].$$

This implies that $\mathfrak{n}_1$ generates $\mathfrak{n}$. Thus by Lemma 1.1.19, J is a complex structure that preserves both $\mathfrak{n}_1$ and $\mathfrak{n}_2$.

For part (ii), suppose that J is Abelian. For all $X \in \mathfrak{c}^j(\mathfrak{n})$ and for all $j \geq 0$, it follows that

$$[JX, \mathfrak{n}] = [JX, J\mathfrak{n}] = [X, \mathfrak{n}] \subseteq \mathfrak{c}^{j-1}(\mathfrak{n})$$

and therefore $JX \in \mathfrak{c}^j(\mathfrak{n})$. Hence $J\mathfrak{c}^j(\mathfrak{n}) = \mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 0$. □

Remark 2.2.2. (i) Suppose that $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ admits a step 2 stratification and a complex structure J. If $\mathfrak{n}_2$ is J-invariant, then there exists $\mathfrak{n}_1'$ such that $\mathfrak{n}_1' \oplus \mathfrak{n}_2$ is a J-invariant stratification. In particular, $\dim \mathfrak{n}_1', \dim \mathfrak{n}_2 \in 2\mathbb{N}$.

(ii) Suppose that $\mathfrak{n}$ is a k-step nilpotent Lie algebra. If J is bi-invariant, J preserves both $\mathfrak{c}_1(\mathfrak{n})$ and $\mathfrak{z}$, while if J is Abelian, then J preserves the center $\mathfrak{z}$.

Corollary 2.2.3. *Let $\mathfrak{n}$ be a k-step stratified Lie algebra with a bi-invariant complex structure J. Then $\dim \mathfrak{n}_j \in 2\mathbb{N}$ for all $j \in \{1, \ldots, k\}$.*

Proof. Since J is bi-invariant, by Theorem 2.2.1, $\dim \mathfrak{c}_j(\mathfrak{n}) \in 2\mathbb{N}$. From Lemma 2.1.12, $\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1\leq l\leq k} \mathfrak{n}_l$. We deduce that

$$\dim \bigoplus_{j+1\leq l\leq k} \mathfrak{n}_l \in 2\mathbb{N} \text{ and } \dim \bigoplus_{j\leq l\leq k} \mathfrak{n}_l \in 2\mathbb{N}.$$

Hence $\dim \mathfrak{n}_j \in 2\mathbb{N}$ as required. □

Remark 2.2.4. A stratified Lie algebra with some odd dimensional layers does not admit bi-invariant complex structures.

Since the descending central series is not necessarily J-invariant, it is interesting to focus on $\mathfrak{c}_j(\mathfrak{n}) \cap J\mathfrak{c}_j(\mathfrak{n}) = \{0\}$. A similar study for ascending central series appears in [15, Section 3].

Proposition 2.2.5. *Let $\mathfrak{n}$ be a $2n$-dimensional nilpotent Lie algebra with a complex structure J. Suppose that $\mathfrak{c}_j(\mathfrak{n}) \cap J\mathfrak{c}_j(\mathfrak{n}) = \{0\}$ for some $j \geq 2$. Then*

(i) $\mathfrak{c}_{j-1}(\mathfrak{n}) \cap J\mathfrak{c}_j(\mathfrak{n}) = \{0\}$;

(ii) $n - j \leq \dim \mathfrak{c}_j(\mathfrak{n}) \leq n - 1$.

Proof. We prove part (i) by contradiction. Suppose that there exists a non-zero $X \in \mathfrak{c}_j(\mathfrak{n})$ such that $JX \in \mathfrak{c}_{j-1}(\mathfrak{n})$. On the one hand, since $JX \in \mathfrak{c}_{j-1}(\mathfrak{n})$, there exists Y such that $[JX, Y] \in \mathfrak{c}_j(\mathfrak{n}) \setminus \mathfrak{c}_{j+1}(\mathfrak{n})$. On the other hand, since $X \in \mathfrak{c}_j(\mathfrak{n})$, by definition, $[X, JY] \in \mathfrak{c}_{j+1}(\mathfrak{n}) \subset \mathfrak{c}_j(\mathfrak{n})$. Hence there exists $[X, JY] + [JX, Y] \in \mathfrak{c}_j(\mathfrak{n}) \setminus \{0\}$. By the Newlander–Nirenberg condition,

$$0 \neq \underbrace{[JX, JY] - [X, Y]}_{\in\, \mathfrak{c}_j(\mathfrak{n})} = \underbrace{J\left([X, JY] + [JX, Y]\right).}_{\in\, J\mathfrak{c}_j(\mathfrak{n})}$$

This is a contradiction. Hence $\mathfrak{c}_{j-1}(\mathfrak{n}) \cap J\mathfrak{c}_j(\mathfrak{n}) = \{0\}$.

Now, we prove part (*ii*). Let us first look at the upper bound. Since $\mathfrak{c}_j(\mathfrak{n}) \cap J\mathfrak{c}_j(\mathfrak{n}) = \{0\}$, we deduce that $\dim \mathfrak{c}_j(\mathfrak{n}) \leq n$. We next show that $\dim \mathfrak{c}_j(\mathfrak{n}) \neq n$ by contradiction. Suppose that $\dim \mathfrak{c}_j(\mathfrak{n}) = n$. Then $\dim J\mathfrak{c}_j(\mathfrak{n}) = n$. By part (i), $\dim \mathfrak{c}_{j-1}(\mathfrak{n}) \oplus J\mathfrak{c}_j(\mathfrak{n}) > 2n$. This is a contradiction. Hence $\dim \mathfrak{c}_j(\mathfrak{n}) \leq n-1$.

On the other hand, since $\mathfrak{c}_j(\mathfrak{n})$ is a strictly deceasing series,

$$0 = \dim \mathfrak{c}_k(\mathfrak{n}) < \ldots < \dim \mathfrak{c}_2(\mathfrak{n}) \leq n-1.$$

Hence the lower bound of $\dim \mathfrak{c}_j(\mathfrak{n})$ is $n-j$. □

2.3 J-invariant central series of nilpotent Lie algebras

Following [4, Definition 1], we define the *J-invariant ascending central series* $\mathfrak{d}^j$ for nilpotent Lie algebras and introduce *nilpotent complex structures* on nilpotent Lie algebras. Furthermore, the *J-invariant descending central series* $\mathfrak{d}_j$ was introduced by Gao, Zhao and Zheng in [12, Definition 2.7]. The inspiration of the definition of $\mathfrak{d}^j$ comes from the following lemma, which implies that $\mathfrak{d}^j$ is a J-invariant ideal of $\mathfrak{n}$.

Lemma 2.3.1. *Let $\mathfrak{n}$ be a nilpotent Lie algebra with a complex structure J, and let $\mathfrak{y}$ be a J-invariant ideal in $\mathfrak{n}$. Define*

$$\mathfrak{w}(\mathfrak{n}) = \{X \in \mathfrak{n} : [X, \mathfrak{n}] \subseteq \mathfrak{y}, [JX, \mathfrak{n}] \subseteq \mathfrak{y}\}.$$

Then $\mathfrak{y} \subseteq \mathfrak{w}(\mathfrak{n})$ and $\mathfrak{w}(\mathfrak{n})$ is a J-invariant ideal in $\mathfrak{n}$.

Proof. By definition of $\mathfrak{w}(\mathfrak{n})$, since $\mathfrak{y}$ is J-invariant, it is clear that $\mathfrak{w}(\mathfrak{n})$ is a J-invariant subspace of $\mathfrak{n}$ and $\mathfrak{y} \subseteq \mathfrak{w}(\mathfrak{n})$. For all $X \in \mathfrak{w}(\mathfrak{n})$, by definition, $[X, \mathfrak{n}] \subseteq \mathfrak{y} \subseteq \mathfrak{w}(\mathfrak{n})$. Since $JX \in \mathfrak{w}(\mathfrak{n})$, we deduce that $[JX, \mathfrak{n}] \subseteq \mathfrak{w}(\mathfrak{n})$. Thus $\mathfrak{w}(\mathfrak{n})$ is an ideal of $\mathfrak{n}$. □

Definition 2.3.2. Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Define a sequence of J-invariant ideals of $\mathfrak{n}$ by $\mathfrak{d}^0 = \{0\}$ and

$$\mathfrak{d}^j = \{X \in \mathfrak{n} : [X, \mathfrak{n}] \subseteq \mathfrak{d}^{j-1}, [JX, \mathfrak{n}] \subseteq \mathfrak{d}^{j-1}\} \tag{2.10}$$

for all $j \geq 1$. We call the sequence $\mathfrak{d}^j$ the *ascending J-invariant central series*. The complex structure J is called *nilpotent of step* j_0 if there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{d}^{j_0} = \mathfrak{n}$ and $\mathfrak{d}^{j_0 - 1} \subset \mathfrak{n}$.

We define inductively the *J-invariant descending central series* by:

$$\mathfrak{d}_0 = \mathfrak{n}, \quad \mathfrak{d}_j = [\mathfrak{d}_{j-1}, \mathfrak{n}] + J[\mathfrak{d}_{j-1}, \mathfrak{n}] \tag{2.11}$$

all for $j \geq 1$.

Remark 2.3.3. (i) For the ascending J-invariant central series $\mathfrak{d}^j$,

$$\mathfrak{d}^j / \mathfrak{d}^{j-1} = \mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j-1}) \cap J\mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j-1}). \tag{2.12}$$

In particular, $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z}$. Notice that $\mathfrak{d}^1$ is the largest J-invariant subspace of $\mathfrak{z}$ and, if J is nilpotent, then $\mathfrak{d}^1 \neq \{0\}$.

(ii) From (2.10), if J is nilpotent of step j_0, then

$$\{0\} = \underbrace{[\mathfrak{n}, [\mathfrak{n}, \ldots, [\mathfrak{n}, \mathfrak{d}^{j_0}]]]}_{j_0 - 1 \text{ terms}} \subseteq \ldots \subseteq [\mathfrak{n}, [\mathfrak{n}, \mathfrak{d}^3]] \subseteq [\mathfrak{n}, \mathfrak{d}^2] \subseteq \mathfrak{d}^1. \tag{2.13}$$

(iii) By definition, if $\mathfrak{n}$ admits a nilpotent complex structure, then $\mathfrak{n}$ is nilpotent. In Proposition 2.3.11, we will investigate the relation between the nil-index of $\mathfrak{n}$ and the nilpotent step of J.

(iv) For the J-invariant descending series, $\mathfrak{d}_j / \mathfrak{d}_{j+1} \subseteq \mathfrak{Z}(\mathfrak{n}/\mathfrak{d}_{j+1})$ for all $j \geq 0$. Indeed, since $[\mathfrak{d}_j, \mathfrak{n}] \subseteq \mathfrak{d}_{j+1}$,

$$[\mathfrak{d}_j / \mathfrak{d}_{j+1}, \mathfrak{n}/\mathfrak{d}_{j+1}] \subseteq [\mathfrak{d}_j, \mathfrak{n}] + \mathfrak{d}_{j+1} \subseteq \mathfrak{d}_{j+1}.$$

(v) The nilpotency of J implies that the ascending J-invariant central series $\mathfrak{d}^j$ of $\mathfrak{n}$ is strictly increasing until $\mathfrak{d}^{j_0} = \mathfrak{n}$.

(vi) Let $\mathfrak{n}$ be a Lie algebra with a nilpotent complex structure J. Suppose that $\mathfrak{z} \subset \mathfrak{c}_1(\mathfrak{n})$. We show that $\mathfrak{c}_1(\mathfrak{n}) \cap J\mathfrak{c}_1(\mathfrak{n}) \neq \{0\}$. Indeed, since J is nilpotent, $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z} \neq \{0\}$ by Definition 2.3.2. Then $\mathfrak{d}^1 \subseteq \mathfrak{z} \subset \mathfrak{c}_1(\mathfrak{n})$. Hence $\mathfrak{c}_1(\mathfrak{n}) \cap J\mathfrak{c}_1(\mathfrak{n}) \neq \{0\}$.

(vii) It is known that $\mathfrak{d}_{j_0} = \{0\}$ and $\mathfrak{d}_{j_0-1} \neq \{0\}$ if and only if J is a nilpotent complex structure of step j_0. See, e.g., [12, Theorem 1.2]. We will provide another proof of this in Theorem 2.3.31.

2.3.1 J-invariant ascending central series and nilpotent complex structures

In Definition 2.3.2, we saw that nilpotent complex structures J are related to the J-invariant ascending series $\mathfrak{d}^j$. In this subsection, we will discuss more properties of $\mathfrak{d}^j$ and their relation with nilpotent complex structures. Note that some of these properties can be found in [4] and [5].

Proposition 2.3.4 ([4, Corollary 5])**.** *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Let $\mathfrak{d}^j$ be the J-invariant ascending central series, as in* (2.10)*. Then J preserves all $\mathfrak{c}^j(\mathfrak{n})$ if and only if $\mathfrak{d}^j = \mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 0$.*

Proof. Suppose that J preserves $\mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 0$. By definition, $\mathfrak{c}^0(\mathfrak{n}) = \{0\} = \mathfrak{d}^0$. Next, assume that $\mathfrak{d}^{s-1} = \mathfrak{c}^{s-1}(\mathfrak{n})$ for some $s \geq 1$. On the one hand, take $X \in \mathfrak{d}^s$. By (2.10), $[X, \mathfrak{n}] \subseteq \mathfrak{d}^{s-1}$ and $[JX, \mathfrak{n}] \subseteq \mathfrak{d}^{s-1}$ and from (2.2), $X \in \mathfrak{c}^s(\mathfrak{n})$ and $JX \in \mathfrak{c}^s(\mathfrak{n})$, for all $X \in \mathfrak{d}^s$. Therefore $\mathfrak{d}^s \subseteq \mathfrak{c}^s(\mathfrak{n})$. On the other hand, for all $X \in \mathfrak{c}^s(\mathfrak{n})$, $[X, \mathfrak{n}] \subseteq \mathfrak{c}^{s-1}(\mathfrak{n}) = \mathfrak{d}^{s-1}$ and $[JX, \mathfrak{n}] \subseteq \mathfrak{d}^{s-1}$. Again using (2.10), $X, JX \in \mathfrak{d}^s$. Hence $\mathfrak{c}^s(\mathfrak{n}) \subseteq \mathfrak{d}^s$ and therefore $\mathfrak{c}^s(\mathfrak{n}) = \mathfrak{d}^s$. By induction, $\mathfrak{d}^j = \mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 0$.

Conversely, if $\mathfrak{c}^j(\mathfrak{n}) = \mathfrak{d}^j$ for all $j \geq 0$, by definition, all $\mathfrak{c}^j(\mathfrak{n})$ are J-invariant. □

Corollary 2.3.5. *Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra with a complex structure J. Suppose that J is bi-invariant or Abelian. Then J is nilpotent of step k.*

Proof. From *Theorem* 2.2.1, J preserves $\mathfrak{c}^j(\mathfrak{n})$ for all $j \geq 1$. Hence by Proposition 2.3.4, $\mathfrak{d}^j = \mathfrak{c}^j(\mathfrak{n})$. Since $\mathfrak{n}$ is nilpotent of step k, it is clear that $\mathfrak{d}^k = \mathfrak{c}^k(\mathfrak{n}) = \mathfrak{n}$ and $\mathfrak{d}^{k-1} = \mathfrak{c}^{k-1}(\mathfrak{n}) \subset \mathfrak{n}$. Thus J is nilpotent of step k. □

Remark 2.3.6. In [10], Dotti and Fino showed that a Lie algebra $\mathfrak{g}$ that admits an Abelian complex structure has to be solvable.

Since $\mathfrak{d}^j \trianglelefteq \mathfrak{n}$, by Proposition 1.2.16, $\mathfrak{n}/\mathfrak{d}^j$ is a quotient nilpotent algebra. We can find a sufficient condition for a complex structure to be nilpotent. The following lemma is stated in [4] and [15] without a proof, we provide one here.

Lemma 2.3.7. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Suppose that J is nilpotent of step j_0. Then $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian. Conversely, if there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian, then J is nilpotent of step at most j_0.*

Proof. Suppose first that J is nilpotent of step j_0. By definition, $\mathfrak{d}^{j_0} = \mathfrak{n}$ and $\mathfrak{d}^{j_0-1} \subset \mathfrak{n}$. Then by (2.12),

$$\mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j_0-1}) \cap J\mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j_0-1}) = \mathfrak{n}/\mathfrak{d}^{j_0-1}.$$

It is obvious that $\mathfrak{Z}(\mathfrak{n}/\mathfrak{d}^{j_0-1}) = \mathfrak{n}/\mathfrak{d}^{j_0-1}$. Hence $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian.

Conversely, suppose that $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian. Then $\{0\} \neq \mathfrak{c}_1(\mathfrak{n}) \subseteq \mathfrak{d}^{j_0-1}$. Hence $\mathfrak{d}^{j_0-1} \neq \{0\}$. For all $X \in \mathfrak{n}$,

$$[X, \mathfrak{n}] \subseteq \mathfrak{d}^{j_0-1} \text{ and } [JX, \mathfrak{n}] \subseteq \mathfrak{d}^{j_0-1}.$$

We deduce that $\mathfrak{n} = \mathfrak{d}^{j_0}$ and therefore J is nilpotent of step at most j_0. □

To end this subsection, we will investigate the possible range of $\dim \mathfrak{z}$ for a Lie algebra $\mathfrak{n}$ with a nilpotent complex structure J. We show the following theorem.

Theorem 2.3.8. *Let $\mathfrak{n}$ be a non-Abelian Lie algebra of dimension $2n$ with a nilpotent complex structure J. Then $2 \leq \dim \mathfrak{z} \leq 2n-2$.*

Proof. Recall that $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z}$, which is the largest J-invariant subspace of $\mathfrak{z}$. If $\mathfrak{d}^1 = \{0\}$, it is clear that $\mathfrak{d}^j = \{0\}$ for all $j \geq 1$. Then J is not nilpotent. Hence $\mathfrak{d}^1 \neq \{0\}$. Since $\mathfrak{d}^1$ is J-invariant, it follows that $2 \leq \dim \mathfrak{d}^1 \leq \dim \mathfrak{z}$. Then the lower bound of $\dim \mathfrak{z}$ is 2.

Next, we show that the upper bound of $\dim \mathfrak{z}$ is $2n-2$. Since $\mathfrak{n}$ is non-Abelian, it is possible to find $X, Y \in \mathfrak{n}$ such that $0 \neq [X, Y] \in \mathfrak{c}_1(\mathfrak{n})$. Then $\operatorname{span}\{X, Y\}$ is 2-dimensional and $\operatorname{span}\{X, Y\} \cap \mathfrak{z} = \{0\}$. Hence $\dim \mathfrak{z} \leq 2n-2$.

In conclusion, $2 \leq \dim \mathfrak{z} \leq 2n-2$. □

Remark 2.3.9. (i) Suppose that $\dim \mathfrak{z} = 2n-2$, then $\dim \mathfrak{c}_1(\mathfrak{n}) = 1$. In this case, $\mathfrak{n}$ is a 2-step nilpotent Lie algebra. Furthermore, if $\mathfrak{n}$ is Abelian, then $\mathfrak{z} = \mathfrak{n}$.

(ii) From Theorem 2.3.8, we can further conclude that if $\dim \mathfrak{z} = 1$, then a complex structure J on $\mathfrak{n}$ is non-nilpotent.

We immediately have the following corollary.

Corollary 2.3.10. *Let $\mathfrak{n}$ be the Lie algebra of $n \times n$ strictly upper triangular matrices. Then a complex structure J on $\mathfrak{n}$ is not nilpotent.*

Proof. Recall, from Example 2.1.18, that $\mathfrak{n} = \{X \in \mathfrak{gl}(n, \mathbb{R}) : X_{ij} = 0 \text{ if } 1 \leq j \leq i \leq n\}$. Then $\mathfrak{n}_{n-1} = \mathfrak{z}$ and $\dim \mathfrak{z} = 1$. By Theorem 2.3.8, J is not nilpotent. □

Next, we will look at a result which was proven in [4] and [12]. We will provide an alternative demonstration.

Proposition 2.3.11. *Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra with a nilpotent complex structure J of step j_0. Then $k \leq j_0 \leq \frac{1}{2} \dim \mathfrak{n}$.*

Proof. Since J is nilpotent of step j_0, by definition, $\{0\} \subset \mathfrak{d}^1 \subset \ldots \subset \mathfrak{d}^{j_0} = \mathfrak{n}$. Then there exist J-invariant quotient spaces $\mathfrak{d}^j/\mathfrak{d}^{j-1}$ such that $\dim \mathfrak{d}^j/\mathfrak{d}^{j-1} = \dim \mathfrak{d}^j - \dim \mathfrak{d}^{j-1} \geq 2$ for all $1 \leq j \leq j_0$. By summing each term from 1 to j_0,

$$\sum_{j=1}^{j_0} \left(\dim \mathfrak{d}^j - \dim \mathfrak{d}^{j-1} \right) = \dim \mathfrak{n} \geq 2j_0.$$

Then $j_0 \leq \frac{1}{2} \dim \mathfrak{n}$.

Now, let us look at the lower bound of j_0. Suppose that $j_0 = k-1$. By definition, $\mathfrak{d}^{k-1} = \mathfrak{n}$. However, $\mathfrak{n} = \mathfrak{d}^{k-1} \subseteq \mathfrak{c}^{k-1}(\mathfrak{n})$, which means that $\mathfrak{n}$ is nilpotent of step less then k. This contradicts the assumption that $\mathfrak{n}$ is nilpotent of step k.

In conclusion, $k \leq j_0 \leq \frac{1}{2} \dim \mathfrak{n}$. □

2.3.2 New characterization of nilpotent complex structures

In this section, we investigate the relation between a J-invariant central series and nilpotent complex structures. We remind the reader that the J-invariant descending and ascending central series $\mathfrak{d}_j$ and $\mathfrak{d}^j$ are defined in Definition 2.3.2. We summaries some properties of $\mathfrak{d}_j$ in the following theorem.

Theorem 2.3.12. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Then*

(i) For all $j \geq 0$, $\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{d}_j$ and $J\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{d}_j$;

(ii) J preserves all $\mathfrak{c}_j(\mathfrak{n})$ if and only if $\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{d}_j$ for all $j \geq 0$. Furthermore, $\mathfrak{d}_j \trianglelefteq \mathfrak{n}$. In particular, the quotient algebra $\mathfrak{n}/\mathfrak{d}_1$ is Abelian.

Proof. We show part (i) by induction. By definition, $\mathfrak{n} + J\mathfrak{n} = \mathfrak{d}_0 = \mathfrak{n}$. Next, suppose that $\mathfrak{c}_{s-1}(\mathfrak{n}) + J\mathfrak{c}_{s-1}(\mathfrak{n}) \subseteq \mathfrak{d}_{s-1}$ for some $s \in \mathbb{N}$. Then

$$\begin{aligned}\mathfrak{c}_s(\mathfrak{n}) + J\mathfrak{c}_s(\mathfrak{n}) &= [\mathfrak{c}_{s-1}(\mathfrak{n}), \mathfrak{n}] + J[\mathfrak{c}_{s-1}(\mathfrak{n}), \mathfrak{n}] \\ &\subseteq [\mathfrak{d}_{s-1}, \mathfrak{n}] + J[\mathfrak{d}_{s-1}, \mathfrak{n}] \\ &= \mathfrak{d}_s.\end{aligned}$$

Hence by induction, $\mathfrak{c}_j(\mathfrak{n}) + J\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{d}_j$ for all $j \geq 0$. Furthermore, $\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{d}_j$ and $J\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{d}_j$.

For part (ii), we first show that J preserves all $\mathfrak{c}_j(\mathfrak{n})$ if and only if $\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{d}_j$ for all $j \geq 0$. On the one hand, suppose that $\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{d}_j$ for all j. By definition, all $\mathfrak{c}_j(\mathfrak{n})$ are J-invariant. On the other hand, suppose that J preserves all $\mathfrak{c}_j(\mathfrak{n})$. By definition, $\mathfrak{c}_0(\mathfrak{n}) = \mathfrak{n} = \mathfrak{d}_0$. Next, assume that $\mathfrak{c}_s(\mathfrak{n}) = \mathfrak{d}_s$ for some $s \in \mathbb{N}$. Then

$$\begin{aligned}\mathfrak{d}_{s+1} &= [\mathfrak{d}_s, \mathfrak{n}] + J[\mathfrak{d}_s, \mathfrak{n}] \\ &= [\mathfrak{c}_s(\mathfrak{n}), \mathfrak{n}] + J[\mathfrak{c}_s(\mathfrak{n}), \mathfrak{n}] = \mathfrak{c}_{s+1}(\mathfrak{n}).\end{aligned}$$

By induction, $\mathfrak{d}_j(\mathfrak{n}) = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$.

We now show that $\mathfrak{d}_j \trianglelefteq \mathfrak{n}$ for all $j \geq 0$. By definition, $\mathfrak{d}_0 = \mathfrak{n}$. Suppose that $\mathfrak{d}_{s-1} \trianglelefteq \mathfrak{n}$ for some $s \in \mathbb{N}$. That is, $[\mathfrak{d}_{s-1}, \mathfrak{n}] \subseteq \mathfrak{d}_{s-1}$. Then

$$\begin{aligned}
[\mathfrak{d}_s, \mathfrak{n}] &= [[\mathfrak{d}_{s-1}, \mathfrak{n}] + J[\mathfrak{d}_{s-1}, \mathfrak{n}], \mathfrak{n}] \\
&\subseteq [\mathfrak{d}_{s-1}, \mathfrak{n}] \subseteq \mathfrak{d}_s.
\end{aligned}$$

By induction, $\mathfrak{d}_j \trianglelefteq \mathfrak{n}$ for all $j \geq 0$. Moreover, since $\mathfrak{d}_j \trianglelefteq \mathfrak{n}$, $\mathfrak{n}/\mathfrak{d}_j$ is a quotient algebra from Proposition 2.1.6 for all j. We show that $\mathfrak{n}/\mathfrak{d}_1$ is Abelian. By definition, $\mathfrak{c}_1(\mathfrak{n}) \subseteq \mathfrak{d}_1$. Thus

$$[\mathfrak{n}/\mathfrak{d}_1, \mathfrak{n}/\mathfrak{d}_1] \subseteq [\mathfrak{n}, \mathfrak{n}] + \mathfrak{d}_1 = \mathfrak{c}_1(\mathfrak{n}) + \mathfrak{d}_1 \subseteq \mathfrak{d}_1,$$

and hence $\mathfrak{n}/\mathfrak{d}_1$ is Abelian. □

Remark 2.3.13. Since $\mathfrak{d}_1 = \mathfrak{c}_1(\mathfrak{n}) + J\mathfrak{c}_1(\mathfrak{n})$, if $\mathfrak{n}$ is non-Abelian, it follows that $\mathfrak{d}_1 \neq \{0\}$.

Lemma 2.3.14. *Let $\mathfrak{n}$ be a nilpotent Lie algebra with a complex structure J. Suppose that J is nilpotent of step j_0. Then $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$. Conversely, if there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$, then J is nilpotent of step at most j_0.*

Proof. We first suppose that J is nilpotent of step j_0. By definition, $\mathfrak{d}_0 = \mathfrak{n} = \mathfrak{d}^{j_0}$. Next, assume that $\mathfrak{d}_{s-1} \subseteq \mathfrak{d}^{j_0-s+1}$ for some $s \in \mathbb{N}$. Then

$$\begin{aligned}
\mathfrak{d}_s &= [\mathfrak{d}_{s-1}, \mathfrak{n}] + J[\mathfrak{d}_{s-1}, \mathfrak{n}] \\
&\subseteq [\mathfrak{d}^{j_0-s+1}, \mathfrak{n}] + J[\mathfrak{d}^{j_0-s+1}, \mathfrak{n}] \\
&\subseteq \mathfrak{d}^{j_0-s} + J\mathfrak{d}^{j_0-s} = \mathfrak{d}^{j_0-s}.
\end{aligned}$$

Hence by induction, $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$.

Conversely, suppose that there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$. Then $\mathfrak{d}_1 \subseteq \mathfrak{d}^{j_0-1}$. By definition, $\mathfrak{c}_1(\mathfrak{n}) \subseteq \mathfrak{d}_1$. It follows that

$$[\mathfrak{n}/\mathfrak{d}^{j_0-1}, \mathfrak{n}/\mathfrak{d}^{j_0-1}] \subseteq [\mathfrak{n}, \mathfrak{n}] + \mathfrak{d}^{j_0-1} = \mathfrak{c}_1(\mathfrak{n}) + \mathfrak{d}^{j_0-1} \subseteq \mathfrak{d}_1 + \mathfrak{d}^{j_0-1} \subseteq \mathfrak{d}^{j_0-1},$$

and thus $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian. By Lemma 2.3.7, J is nilpotent of step at most j_0. □

Remark 2.3.15. If J is nilpotent of step j_0, $\mathfrak{d}_{j_0-1} \subseteq \mathfrak{d}^1 \subseteq \mathfrak{z}$. Then $\mathfrak{d}_{j_0-1}$ is Abelian. Furthermore, there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{n}/\mathfrak{d}^{j_0-1}$ is Abelian if and only if $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$. This follows by induction from the proof of Lemma 2.3.14.

Corollary 2.3.16. *Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra with a complex structure J. Then J is nilpotent of step k if and only if $\mathfrak{d}_j \subseteq \mathfrak{d}^{k-j}$ for all $j \geq 0$.*

Proof. Suppose that J is nilpotent of step k. By Lemma 2.3.14, $\mathfrak{d}_j \subseteq \mathfrak{d}^{k-j}$ for all $j \geq 0$. Conversely, assume that $\mathfrak{d}_j \subseteq \mathfrak{d}^{k-j}$ for all j. Again by Lemma 2.3.14, J is nilpotent of step at most k. Furthermore, it follows, from Theorem 2.3.12, that $\{0\} \neq \mathfrak{c}_{k-1}(\mathfrak{n}) \subseteq \mathfrak{d}_{k-1}$. Therefore $\mathfrak{d}_{k-1} \neq \{0\}$ and J is nilpotent of step k. □

Remark 2.3.17. From Remark 2.3.15, we further conclude that J is nilpotent step k if and only if $\mathfrak{n}/\mathfrak{d}^{k-1}$ is Abelian.

Corollary 2.3.18. *Let $\mathfrak{n}$ be a Lie algebra with a nilpotent complex structure J of step j_0. Then $[\mathfrak{d}_{j_0-2}, \mathfrak{c}_j(\mathfrak{n})] = \{0\}$ for all $j \geq 0$. In particular, if $\mathfrak{n}$ is a k-step stratified Lie algebra, then $[\mathfrak{d}_{j_0-2}, \mathfrak{n}_{j+1}] = \{0\}$.*

Proof. Since J is nilpotent, by Lemma 2.3.14, $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$. Then by definition,

$$\{0\} = [\mathfrak{d}_{j_0-1}, \mathfrak{n}] \subseteq [\mathfrak{d}_{j_0-2}, \mathfrak{n}] \subseteq \mathfrak{d}_{j_0-1} \subseteq \mathfrak{d}^1.$$

For all $X, Y \in \mathfrak{n}$, $T \in \mathfrak{d}_{j_0-2}$, by the Jacobi identity,

$$\underbrace{[[T, X], Y]}_{\in [[\mathfrak{d}_{j_0-2}, \mathfrak{n}], \mathfrak{n}] = \{0\}} = \underbrace{[T, [X, Y]]}_{\in [\mathfrak{d}_{j_0-2}, \mathfrak{c}_1(\mathfrak{n})]} + \underbrace{[X, [Y, T]]}_{\in [\mathfrak{n}, [\mathfrak{d}_{j_0-2}, \mathfrak{n}]] = \{0\}}.$$

Hence $[\mathfrak{d}_{j_0-2}, \mathfrak{c}_1(\mathfrak{n})] = \{0\}$. Since $\mathfrak{c}_j(\mathfrak{n})$ is the descending central series, for all $T \in \mathfrak{d}_{j_0-2}$,

$$\{0\} = [T, \mathfrak{c}_k(\mathfrak{n})] \subseteq [T, \mathfrak{c}_{k-1}(\mathfrak{n})] \subseteq \ldots \subseteq [T, \mathfrak{c}_2(\mathfrak{n})] \subseteq [T, \mathfrak{c}_1(\mathfrak{n})] = \{0\}.$$

Hence $[\mathfrak{d}_{j_0-2}, \mathfrak{c}_j(\mathfrak{n})] = \{0\}$.

In particular, suppose that $\mathfrak{n}$ admits a stratification. Recall, from Lemma 2.1.12, that $\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{l\geq j} \mathfrak{n}_{l+1}$. From the above paragraph, $\{0\} = [\mathfrak{d}_{j_0-2}, \bigoplus_{l\geq j} \mathfrak{n}_{l+1}]$. Hence $[\mathfrak{d}_{j_0-2}, \mathfrak{n}_{j+1}] = \{0\}$ for all $j \geq 0$. □

In order to show that $\mathfrak{d}_{j_0-1} \neq \{0\}$ if J is nilpotent of step j_0, we need to introduces a new descending central series whose descending 'rate' is slower than that of $\mathfrak{c}_j(\mathfrak{n})$ but faster than that of $\mathfrak{d}_j$.

Definition 2.3.19. Let J be a complex structure on a Lie algebra $\mathfrak{n}$. We define a sequence inductively by

$$\mathfrak{p}_0 = \mathfrak{n} \text{ and } \mathfrak{p}_j = [\mathfrak{p}_{j-1}, \mathfrak{n}] + [J\mathfrak{p}_{j-1}, \mathfrak{n}] \text{ for all } j \geq 1. \tag{2.14}$$

Remark 2.3.20. It is clear that $\mathfrak{p}_1 = \mathfrak{c}_1(\mathfrak{n})$.

Lemma 2.3.21. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Then $\mathfrak{p}_j \trianglelefteq \mathfrak{n}$ and $\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{p}_j$ for all $j \geq 0$. Furthermore, $\mathfrak{p}_j \subseteq \mathfrak{d}_j$ and $J\mathfrak{p}_j \subseteq \mathfrak{d}_j$ for all $j \geq 0$, where $\mathfrak{d}_j$ is the J-invariant descending central series as in Definition* 2.3.2.

Proof. We first show that $\mathfrak{p}_{j+1} \subseteq \mathfrak{p}_j$ for all $j \geq 0$. By definition, $\mathfrak{p}_1 = \mathfrak{c}_1(\mathfrak{n}) \subseteq \mathfrak{n} = \mathfrak{p}_0$. Next, assume that $\mathfrak{p}_s \subseteq \mathfrak{p}_{s-1}$ for some $s \in \mathbb{N}$. Then by (2.14),

$$\mathfrak{p}_{s+1} = [\mathfrak{p}_s, \mathfrak{n}] + [J\mathfrak{p}_s, \mathfrak{n}] \subseteq [\mathfrak{p}_{s-1}, \mathfrak{n}] + [J\mathfrak{p}_{s-1}, \mathfrak{n}] = \mathfrak{p}_s.$$

By induction, $\mathfrak{p}_{j+1} \subseteq \mathfrak{p}_j$. Then

$$[\mathfrak{p}_j, \mathfrak{n}] \subseteq \mathfrak{p}_{j+1} \subseteq \mathfrak{p}_j.$$

Hence $\mathfrak{p}_j \trianglelefteq \mathfrak{n}$ for all $j \geq 0$. Now, we show that $\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{p}_j$ for all $j \geq 0$. By definition, $\mathfrak{c}_1(\mathfrak{n}) = \mathfrak{p}_1$. Assume next that $\mathfrak{c}_s(\mathfrak{n}) \subseteq \mathfrak{p}_s$ for some $s \in \mathbb{N}$. Then

$$\mathfrak{c}_{s+1}(\mathfrak{n}) = [\mathfrak{c}_s(\mathfrak{n}), \mathfrak{n}] \subseteq [\mathfrak{p}_s, \mathfrak{n}] \subseteq \mathfrak{p}_{s+1}.$$

By induction, $\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{p}_j$ for all $j \geq 0$.

Using (2.11), $[\mathfrak{d}_{j-1}, \mathfrak{n}] \subseteq \mathfrak{d}_j$. By definition, $\mathfrak{p}_0 = \mathfrak{n} = \mathfrak{d}_0$ and $J\mathfrak{p}_0 = J\mathfrak{n} = \mathfrak{n} = \mathfrak{d}_0$. Next, suppose that $\mathfrak{p}_s \subseteq \mathfrak{d}_s$ and $J\mathfrak{p}_s \subseteq \mathfrak{d}_s$ for some $s \in \mathbb{N}$. Then by (2.14),

$$\mathfrak{p}_{s+1} = [\mathfrak{p}_s, \mathfrak{n}] + [J\mathfrak{p}_s, \mathfrak{n}] \subseteq [\mathfrak{d}_s, \mathfrak{n}] \subseteq \mathfrak{d}_{s+1} \text{ and } J\mathfrak{p}_{s+1} \subseteq J[\mathfrak{d}_s, \mathfrak{n}] \subseteq \mathfrak{d}_{s+1}$$

By induction, $\mathfrak{p}_j \subseteq \mathfrak{d}_j$ and $J\mathfrak{p}_j \subseteq \mathfrak{d}_j$ for all $j \geq 0$. □

Remark 2.3.22. (i) Since $\mathfrak{p}_1 = \mathfrak{c}_1(\mathfrak{n})$, it is clear that $\mathfrak{n}/\mathfrak{p}_1$ is Abelian. Furthermore, if $\mathfrak{d}_j = \mathfrak{p}_j$ for all $j \geq 1$, then all $[\mathfrak{d}_{j-1}, \mathfrak{n}]$ are J-invariant. Indeed,

$$[\mathfrak{d}_j, \mathfrak{n}] = [\mathfrak{p}_j, \mathfrak{n}] = \mathfrak{p}_{j+1} = \mathfrak{d}_{j+1}.$$

(ii) Notice that $\mathfrak{p}_j/\mathfrak{p}_{j+1} \subseteq \mathfrak{Z}\left(\mathfrak{n}/\mathfrak{p}_{j+1}\right)$ for all $j \geq 0$. Indeed, for all $P \in \mathfrak{p}_j$ and $Y \in \mathfrak{n}$, since $[P, Y] \subseteq \mathfrak{p}_{j+1}$, it is enough to deduce

$$[P + \mathfrak{p}_{j+1}, Y + \mathfrak{p}_{j+1}] = [P, Y] + \mathfrak{p}_{j+1} \subseteq \mathfrak{p}_{j+1}.$$

This implies that $\mathfrak{p}_j$ is a central series.

(iii) By Lemma 2.3.21, $\mathfrak{p}_j + J\mathfrak{p}_j \subseteq \mathfrak{d}_j$. We show that $\mathfrak{p}_j + J\mathfrak{p}_j \trianglelefteq \mathfrak{n}$ for all $j \geq 0$. Indeed, for all $P, P' \in \mathfrak{p}_j$,

$$\underbrace{[P + JP', \mathfrak{n}]}_{\subseteq [\mathfrak{p}_j + J\mathfrak{p}_j, \mathfrak{n}]} \subseteq \underbrace{[P, \mathfrak{n}]}_{\subseteq \mathfrak{p}_{j+1}} + \underbrace{[JP', \mathfrak{n}]}_{\subseteq \mathfrak{p}_{j+1}} \subseteq \mathfrak{p}_{j+1} \subseteq \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1} \subseteq \mathfrak{p}_j + J\mathfrak{p}_j.$$

Hence $\mathfrak{p}_j + J\mathfrak{p}_j \trianglelefteq \mathfrak{n}$. From part (ii), we show that $\mathfrak{p}_j + J\mathfrak{p}_j$ is a J-invariant descending central series. Indeed, for all $T = P + JP' \in \mathfrak{p}_j + J\mathfrak{p}_j$ and $Y \in \mathfrak{n}$,

$$[T + \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1}, Y + \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1}] \subseteq [T, Y] + \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1} \subseteq \mathfrak{p}_{j+1} + J\mathfrak{p}_{j+1}.$$

(iv) Suppose that J is nilpotent of step j_0. By Corollary 2.3.18, $[\mathfrak{p}_{j_0-2}, \mathfrak{c}_j(\mathfrak{n})] \subseteq [\mathfrak{d}_{j_0-2}, \mathfrak{c}_j(\mathfrak{n})] = \{0\}$ for all $j \geq 0$. Hence $[\mathfrak{p}_{j_0-2}, \mathfrak{c}_j(\mathfrak{n})] = \{0\}$. We further assume that $\mathfrak{n}$ is a stratified Lie algebra. Then $[\mathfrak{p}_{j_0-2}, \mathfrak{n}_{j+1}] = \{0\}$ for all $j \geq 0$.

The contents from Definition 2.3.2 to here actually make sense for every Lie algebra $\mathfrak{g}$. In what follows, we will characterize nilpotent complex structures with descending central series $\mathfrak{p}_j$ and $\mathfrak{d}_j$.

Lemma 2.3.23. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Then there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{p}_{j_0} = \{0\}$ and $\mathfrak{p}_{j_0-1} \neq \{0\}$ if and only if J is nilpotent of step j_0.*

Proof. We first assume that J is nilpotent of step j_0. By Lemma 2.3.14, $\mathfrak{d}_{j_0-1} \subseteq \mathfrak{d}^1$. Hence by Lemma 2.3.21,

$$\mathfrak{p}_{j_0} \subseteq [\mathfrak{d}_{j_0-1}, \mathfrak{n}] \subseteq [\mathfrak{d}^1, \mathfrak{n}] = \{0\}.$$

Thus $\mathfrak{p}_{j_0} = \{0\}$. Assume, by contradiction, that $\mathfrak{p}_{j_0-1} = \{0\}$. We show that $\mathfrak{p}_{j_0-j-1} + J\mathfrak{p}_{j_0-j-1} \subseteq \mathfrak{d}^j$ for all $j \geq 0$ by induction. By definition, $\mathfrak{p}_{j_0-1} + J\mathfrak{p}_{j_0-1} = \{0\} = \mathfrak{d}^0$. Next, suppose that $\mathfrak{p}_{j_0-s-1} + J\mathfrak{p}_{j_0-s-1} \subseteq \mathfrak{d}^s$ for some $s \in \mathbb{N}$. Then for all $P, P' \in \mathfrak{p}_{j_0-s-2}$,

$$[P, \mathfrak{n}] \subseteq \mathfrak{p}_{j_0-s-1} \subseteq \mathfrak{d}^s \text{ and } [JP', \mathfrak{n}] \subseteq \mathfrak{p}_{j_0-s-1} \subseteq \mathfrak{d}^s.$$

This implies, using (2.10), $\mathfrak{p}_{j_0-s-2} + J\mathfrak{p}_{j_0-s-2} \subseteq \mathfrak{d}^{s+1}$. By induction, $\mathfrak{p}_{j_0-j-1} + J\mathfrak{p}_{j_0-j-1} \subseteq \mathfrak{d}^j$ for all $j \geq 0$. In particular, let $j = j_0 - 1$. Then $\mathfrak{n} \subseteq \mathfrak{d}^{j_0-1}$, which implies that J is nilpotent of step $j_0 - 1$ by definition. This is a contradiction. Therefore $\mathfrak{p}_{j_0-1} \neq \{0\}$.

Conversely, suppose that $\mathfrak{p}_{j_0} = \{0\}$ and $\mathfrak{p}_{j_0-1} \neq \{0\}$. We show that J is nilpotent of step j_0. By definition, $\mathfrak{p}_{j_0} + J\mathfrak{p}_{j_0} = \{0\} = \mathfrak{d}^0$. Next, suppose that $\mathfrak{p}_{j_0-s} + J\mathfrak{p}_{j_0-s} \subseteq \mathfrak{d}^s$ for some $s \in \mathbb{N}$. Then from Remark 2.3.22 part (iii),

$$[\mathfrak{p}_{j_0-s-1} + J\mathfrak{p}_{j_0-s-1}, \mathfrak{n}] \subseteq \mathfrak{p}_{j_0-s} + J\mathfrak{p}_{j_0-s} \subseteq \mathfrak{d}^s.$$

This implies, using (2.10), $\mathfrak{p}_{j_0-s-1} + J\mathfrak{p}_{j_0-s-1} \subseteq \mathfrak{d}^{s+1}$. By induction, $\mathfrak{p}_{j_0-j} + J\mathfrak{p}_{j_0-j} \subseteq \mathfrak{d}^j$ for all $j \geq 0$. Hence $\mathfrak{p}_{j_0-j} \subseteq \mathfrak{d}^j$. In particular, let $j = j_0 - 1$. Then

$$\mathfrak{p}_1 = [\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{d}^{j_0-1} \Rightarrow \mathfrak{n}/\mathfrak{d}^{j_0-1} \text{ is Abelian .}$$

By Lemma 2.3.7, J is nilpotent of step at most j_0.

We next show that $\mathfrak{d}^{j_0-1} \neq \mathfrak{n}$ by contradiction. Assume, by contradiction, that $\mathfrak{n} = \mathfrak{d}^{j_0-1}$. We show that $\mathfrak{p}_{j-1} \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 1$ by induction. By definition, $\mathfrak{p}_0 = \mathfrak{n} = \mathfrak{d}^{j_0-1}$. Next, suppose that $\mathfrak{p}_{s-1} \subseteq \mathfrak{d}^{j_0-s}$ for some $s \in \mathbb{N}$. Then

$$\begin{aligned}
\mathfrak{p}_s &= [\mathfrak{p}_{s-1}, \mathfrak{n}] + [J\mathfrak{p}_{s-1}, \mathfrak{n}] \\
&\subseteq [\mathfrak{d}^{j_0-s}, \mathfrak{n}] + [J\mathfrak{d}^{j_0-s}, \mathfrak{n}] \\
&\subseteq \mathfrak{d}^{j_0-s-1}.
\end{aligned}$$

By induction, $\mathfrak{p}_{j-1} \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 1$. In particular, let $j = j_0$. We deduce that $\mathfrak{p}_{j_0-1} \subseteq \mathfrak{d}^0 = \{0\}$. This implies that $\mathfrak{p}_{j_0-1} = \{0\}$ which is a contradiction. Hence $\mathfrak{d}^{j_0-1} \neq \mathfrak{n}$. By definition, J is nilpotent of step j_0. □

Remark 2.3.24. Suppose that a Lie algebra $\mathfrak{n}$ admits a nilpotent complex structure J of step j_0. Then from Lemma 2.3.21,

$$\mathfrak{c}_j(\mathfrak{n}) + J\mathfrak{c}_j(\mathfrak{n}) \subseteq \mathfrak{p}_j + J\mathfrak{p}_j \subseteq \mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j} \tag{2.15}$$

for all $j \geq 0$. In particular, the converse is true if j_0 is equal to the nil-index of $\mathfrak{n}$.

Proposition 2.3.25. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. Then there exists $j_0 \in \mathbb{N}$ such that $\mathfrak{d}_{j_0} = \{0\}$ and $\mathfrak{d}_{j_0-1} \neq \{0\}$ if and only if J is nilpotent of step j_0.*

Proof. Since J is nilpotent of step j_0, it follows, from Lemma 2.3.14, that $\mathfrak{d}_j \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 0$. In particular, let $j = j_0$. By definition, $\mathfrak{d}_{j_0} = \mathfrak{d}^0 = \{0\}$. We show that $\mathfrak{d}_{j_0-1} \neq \{0\}$. By Lemma 2.3.21 and Lemma 2.3.23, $\{0\} \neq \mathfrak{p}_{j_0-1} + J\mathfrak{p}_{j_0-1} \subseteq \mathfrak{d}_{j_0-1}$. Hence $\mathfrak{d}_{j_0-1} \neq \{0\}$.

Conversely, assume that $\mathfrak{d}_{j_0} = \{0\}$ and $\mathfrak{d}_{j_0-1} \neq \{0\}$. By definition, $[\mathfrak{d}_{j_0-1}, \mathfrak{n}] \subseteq \mathfrak{d}_{j_0} = \{0\}$. Hence $\{0\} \neq \mathfrak{d}_{j_0-1} \subseteq \mathfrak{d}^1$. Next, assume that $\mathfrak{d}_{j_0-s} \subseteq \mathfrak{d}^s$ for some $s \in \mathbb{N}$. Then

$$[\mathfrak{d}_{j_0-s-1}, \mathfrak{n}] \subseteq \mathfrak{d}_{j_0-s} \subseteq \mathfrak{d}^s.$$

By (2.10), $\mathfrak{d}_{j_0-s-1} \subseteq \mathfrak{d}^{s+1}$. By induction, $\mathfrak{d}_{j_0-j} \subseteq \mathfrak{d}^j$ for all $j \geq 0$. Let $j = j_0$. We find that $\mathfrak{d}_0 = \mathfrak{n} \subseteq \mathfrak{d}^{j_0}$. Therefore $\mathfrak{d}^{j_0} = \mathfrak{n}$ and J is nilpotent of step at most j_0.

We next show that $\mathfrak{d}^{j_0-1} \neq \mathfrak{n}$. Suppose not, that is, $\mathfrak{n} = \mathfrak{d}^{j_0-1}$. We show that $\mathfrak{d}_{j-1} \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 1$ by induction. By definition, $\mathfrak{d}_0 = \mathfrak{n} = \mathfrak{d}^{j_0-1}$. Next assume that $\mathfrak{d}_{s-1} \subseteq \mathfrak{d}^{j_0-s}$ for some $s \in \mathbb{N}$. Then

$$\begin{aligned}
\mathfrak{d}_s &= [\mathfrak{d}_{s-1}, \mathfrak{n}] + J[\mathfrak{d}_{s-1}, \mathfrak{n}] \\
&\subseteq [\mathfrak{d}^{j_0-s}, \mathfrak{n}] + J[\mathfrak{d}^{j_0-s}, \mathfrak{n}] \\
&\subseteq \mathfrak{d}^{j_0-s-1}.
\end{aligned}$$

By induction, $\mathfrak{d}_{j-1} \subseteq \mathfrak{d}^{j_0-j}$ for all $j \geq 1$. In particular, let $j = j_0$. We find that $\mathfrak{d}_{j_0-1} \subseteq \{0\}$. This is a contradiction. Hence $\mathfrak{d}^{j_0-1} \neq \mathfrak{n}$.

In conclusion, J is nilpotent of step j_0. $\square$

Remark 2.3.26. Suppose that J is nilpotent of step j_0. This implies the following inclusion relations

$$[\mathfrak{p}_{j_0-1}, \mathfrak{n}] \subseteq [\mathfrak{d}_{j_0-1}, \mathfrak{n}] \subseteq \mathfrak{d}_{j_0} \subseteq \mathfrak{d}_{j_0-1} \subseteq \ldots \subseteq \mathfrak{d}_1 \subset \mathfrak{n}. \tag{2.16}$$

It is clear that $\mathfrak{d}_1 \subset \mathfrak{n}$. Otherwise, we have $\mathfrak{d}_j = \mathfrak{n}$ for all $j \geq 0$ which implies that J is not nilpotent. In general, if J is not nilpotent, it is still true that $\mathfrak{d}_1 \subset \mathfrak{n}$. The proof of this fact can be found in [19, Proposition 2.7] and the original source is [26].

In what follows, we show some observations from Lemma 2.3.23 and Proposition 2.3.25. It is shown that, in [4, Corollary 7], if $\mathfrak{c}_j(\mathfrak{n})$ is J-invariant for all $j \geq 0$, then J is nilpotent. We will provide a different approach to this.

Corollary 2.3.27. *Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra with a complex structure J. Suppose that all $\mathfrak{c}_j(\mathfrak{n})$ are J-invariant. Then $\mathfrak{p}_j = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$. Furthermore, J is nilpotent of step k.*

Proof. Since all $\mathfrak{c}_j(\mathfrak{n})$ are J-invariant, by definition, $\mathfrak{p}_0 = \mathfrak{n} = \mathfrak{c}_0(\mathfrak{n})$. Next, assume that $\mathfrak{p}_s = \mathfrak{c}_s(\mathfrak{n})$ for some $s \in \mathbb{N}$. Then

$$\mathfrak{p}_{s+1} = [\mathfrak{p}_s, \mathfrak{n}] + [J\mathfrak{p}_s, \mathfrak{n}] = [\mathfrak{n}, \mathfrak{c}_s(\mathfrak{n})] = \mathfrak{c}_{s+1}(\mathfrak{n}).$$

By induction, $\mathfrak{p}_j = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$. Therefore $\mathfrak{p}_k = \mathfrak{c}_k(\mathfrak{n}) = \{0\}$ and $\mathfrak{p}_{k-1} = \mathfrak{c}_{k-1}(\mathfrak{n}) \neq \{0\}$. By Lemma 2.3.23, J is nilpotent of step k. □

Remark 2.3.28. (i) Suppose that all $\mathfrak{c}_j(\mathfrak{n})$ are J-invariant. By Theorem 2.3.12, $\mathfrak{p}_j = \mathfrak{d}_j = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$.

(ii) Suppose that J is nilpotent of step k on a k-step nilpotent Lie algebra. It is not necessarily true that J always preserves all $\mathfrak{c}_j(\mathfrak{n})$. For instance, $\mathfrak{n} \cong \mathfrak{h}_1 \times \mathbb{R}$, a 4 dimensional 2-step nilpotent Lie algebra admits an Abelian complex structure such that $\dim \mathfrak{n}_2 = 1$, where $\mathfrak{h}_1$ is the 3-dimensional Heisenberg algebra. See, e.g., [2] and [11].

Corollary 2.3.29. *Let $\mathfrak{n}$ be a k-step nilpotent Lie algebra with a nilpotent complex structure J of step k. Suppose that $\mathfrak{c}_{k-1}(\mathfrak{n}) = \mathfrak{z}$. Then $\mathfrak{z}$ is J-invariant.*

Proof. Since J is nilpotent of step k, by Theorem 2.3.12 part (i) and Lemma 2.3.14,

$$\mathfrak{z} + J\mathfrak{z} \subseteq \mathfrak{d}_{k-1} \subseteq \mathfrak{d}^1 \subseteq \mathfrak{z} \Rightarrow [\mathfrak{z} + J\mathfrak{z}, \mathfrak{n}] = \{0\}.$$

Hence $J\mathfrak{z} = \mathfrak{z}$. Therefore $J\mathfrak{z} \subseteq \mathfrak{z}$ and $\mathfrak{z} = \mathfrak{d}_{k-1}$. □

Corollary 2.3.30. *Let $\mathfrak{n}$ be a k-step stratified Lie algebra with a strata-preserving complex structure J. Then $J\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$ and J is nilpotent of step k.*

Proof. We first show that $J\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{c}_j(\mathfrak{n})$ for all $j \geq 0$. Recall that $\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1 \leq l \leq k} \mathfrak{n}_l$ by Lemma 2.1.12. Then

$$J\mathfrak{c}_j(\mathfrak{n}) = \bigoplus_{j+1 \leq l \leq k} J\mathfrak{n}_l = \bigoplus_{j+1 \leq l \leq k} \mathfrak{n}_l = \mathfrak{c}_j(\mathfrak{n}).$$

Hence all $\mathfrak{c}_j(\mathfrak{n})$ are J-invariant. Finally, by Corollary 2.3.27, J is nilpotent of step k. □

We can write down the following theorem by combining Lemma 2.3.23 and Proposition 2.3.25.

Theorem 2.3.31. *Let $\mathfrak{n}$ be a Lie algebra with a complex structure J. The following are equivalent:*

(i) J is nilpotent of step j_0;

(ii) $\mathfrak{p}_{j_0} = \{0\}$ and $\mathfrak{p}_{j_0-1} \neq \{0\}$;

(iii) $\mathfrak{d}_{j_0} = \{0\}$ and $\mathfrak{d}_{j_0-1} \neq \{0\}$.

Proof. Since J is nilpotent of step j_0, by definition, $\mathfrak{d}^{j_0} = \mathfrak{n}$ and $\mathfrak{d}^{j_0-1} \subset \mathfrak{n}$. Then using Lemma 2.3.23, (i) and (ii) are equivalent. Furthermore, the equivalence between argument (i) and (iii) is given by Proposition 2.3.25. Hence (i), (ii) and (iii) are equivalent. □

Corollary 2.3.32. *Let $\mathfrak{n}$ be a Lie algebra with a nilpotent complex structure J of step j_0. Then for all $j \geq 1$, $\mathfrak{d}_{j_0-j}$ is not contained in $\mathfrak{d}^{j-1}$.*

Proof. Since J is nilpotent of step j_0, by Theorem 2.3.31, $\mathfrak{d}_{j_0-1} \neq \{0\} = \mathfrak{d}^0$. Hence $\mathfrak{d}_{j_0-1}$ is not contained in $\mathfrak{d}^0$. Next, suppose that $\mathfrak{d}_{j_0-s+1}$ is not contained in $\mathfrak{d}^{s-2}$ for some $\mathbb{N} \ni s \geq 2$. We show that $\mathfrak{d}_{j_0-s}$ is not contained in $\mathfrak{d}^{s-1}$. Suppose not. That is, $\mathfrak{d}_{j_0-s} \subseteq \mathfrak{d}^{s-1}$. Then

$$\begin{aligned}\mathfrak{d}_{j_0-s+1} &= [\mathfrak{d}_{j_0-s}, \mathfrak{n}] + J[\mathfrak{d}_{j_0-s}, \mathfrak{n}] \\ &\subseteq [\mathfrak{d}^{s-1}, \mathfrak{n}] + J[\mathfrak{d}^{s-1}, \mathfrak{n}] \subseteq \mathfrak{d}^{s-2}.\end{aligned}$$

It follows that $\mathfrak{d}_{j_0-s+1} \subseteq \mathfrak{d}^{s-2}$. This is a contradiction. Hence $\mathfrak{d}_{j_0-s}$ is not contained in $\mathfrak{d}^{s-1}$. By induction, $\mathfrak{d}_{j_0-j}$ is not contain in $\mathfrak{d}^{j-1}$ for all $j \geq 1$. □

2.4 Complex structures on decomposable nilpotent Lie algebras

We call a nilpotent Lie algebra $\mathfrak{n}$ *decomposable* if $\mathfrak{n} = \mathfrak{g}_1 \tilde{\oplus} \dots \tilde{\oplus} \mathfrak{g}_m$, where $\mathfrak{g}_i$ are nilpotent Lie algebras and $\tilde{\oplus}$ is a Lie algebra direct sum. Let $\mathcal{S}(\mathfrak{n}) = \{J \in GL(\mathfrak{n}) : J^2 = -I \text{ and } N_J = 0\}$ be the set of complex structures on a Lie algebra $\mathfrak{n}$. We observe the following lemmas about decomposable nilpotent Lie algebras with complex structures.

Lemma 2.4.1. *Let $\mathfrak{s} = \mathfrak{n} \tilde{\oplus} \mathfrak{n}'$ be the Lie algebra direct sum of nilpotent Lie algebras of step k and k' respectively.*

(i) $\mathfrak{s}$ is a nilpotent Lie algebra of step $k_0 = \max\{k, k'\}$;

(ii) suppose that $K \in \mathcal{S}(\mathfrak{n})$ and $K' \in \mathcal{S}(\mathfrak{n}')$. Then $J = (K, K')$ is a complex structure on $\mathfrak{s}$;

(iii) suppose that $K \in \mathcal{S}(\mathfrak{n})$ is nilpotent of step h and $K' \in \mathcal{S}(\mathfrak{n}')$ is nilpotent of step h'. Then $J = (K, K') \in \mathcal{S}(\mathfrak{s})$ is nilpotent of step $j_0 = \max\{h, h'\}$.

Proof. For part (i), since $[(X, X'), (Y, Y')] = ([X, Y], [X', Y'])$ for all $X, Y \in \mathfrak{n}$ and $X', Y' \in \mathfrak{n}'$ and $[\mathfrak{n}, \mathfrak{n}'] = \{0\}$, $[\mathfrak{s}, \mathfrak{s}] = [\mathfrak{n}, \mathfrak{n}] \tilde{\oplus} [\mathfrak{n}', \mathfrak{n}']$. Next, suppose that $\mathfrak{c}_{s-1}(\mathfrak{s}) =$

$\mathfrak{c}_{s-1}(\mathfrak{n}) \tilde{\oplus} \mathfrak{c}_{s-1}(\mathfrak{n}')$ for some $s \in \mathbb{N}$. Then

$$\mathfrak{c}_s(\mathfrak{s}) = [\mathfrak{c}_{s-1}(\mathfrak{s}), \mathfrak{s}] = [\mathfrak{c}_{s-1}(\mathfrak{n}) \tilde{\oplus} \mathfrak{c}_{s-1}(\mathfrak{n}'), \mathfrak{n} \tilde{\oplus} \mathfrak{n}'] = \mathfrak{c}_s(\mathfrak{n}) \tilde{\oplus} \mathfrak{c}_s(\mathfrak{n}')$$

By induction, $\mathfrak{c}_j(\mathfrak{s}) = \mathfrak{c}_j(\mathfrak{n}) \tilde{\oplus} \mathfrak{c}_j(\mathfrak{n}')$ for all $j \geq 0$. Without loss of generality, assume that $k \leq k'$. Then

$$\mathfrak{c}_k(\mathfrak{s}) = \mathfrak{c}_k(\mathfrak{n}) \tilde{\oplus} \mathfrak{c}_k(\mathfrak{n}') = \mathfrak{c}_k(\mathfrak{n}') \text{ and } \mathfrak{c}_{k'}(\mathfrak{s}) = \mathfrak{c}_{k'}(\mathfrak{n}') = \{0\}, \mathfrak{c}_{k'-1}(\mathfrak{s}) \neq \{0\}.$$

Hence $\mathfrak{s}$ is a nilpotent Lie algebra of step k.

For part (ii), consider J as defined in statement of the lemma by $J(X, X') = (KX, K'X')$, where $K \in \mathcal{S}(\mathfrak{n}), K' \in \mathcal{S}(\mathfrak{n}')$ and $X, Y \in \mathfrak{n}$, $X', Y' \in \mathfrak{n}'$. The Lie bracket on $\mathfrak{s}$ is given by $[(X, X'), (Y, Y')] = ([X, Y], [X', Y'])$. It is easy to see that this Lie bracket satisfies the Jacobi identity. Notice that J is an almost complex structure since

$$J^2(X, X') = J(KX, K'X') = (-X, -X') = -(X, X).$$

for all $X \in \mathfrak{n}$ and $X' \in \mathfrak{n}'$. Next, we show that J satisfies the Newlander–Nirenberg condition. For all $P = (X, X'), Q = (Y, Y') \in \mathfrak{s}$,

$$\begin{aligned}
[JP, JQ] &= [(KX, K'X'), (KY, K'Y')] = ([KX, KY], [K'Y, K'Y']);\\
[P, JQ] &= ([X, KY], [X', K'Y']);\ [JP, Q] = ([KX, Y], [K'X', Y']).
\end{aligned}$$

Then

$$\begin{aligned}
N_J(P, Q) &= [JP, JQ] - [P, Q] - J[JP, Q] - J[P, JQ]\\
&= ([KX, KY], [K'Y, K'Y']) - ([X, Y], [X', Y']) - (K[X, KX']\\
&\qquad + K[KX, X'], K'[Y, K'Y'] + K'[K'X', Y'])\\
&= (N_K(X, X'), N_{K'}(Y, Y')) = (0, 0)
\end{aligned}$$

since K and K' are complex structures. Hence $N_J(P,Q) = 0$ and therefore, by definition, J is a complex structure on $\mathfrak{s}$.

Now, we show part (iii). From part (ii), $J = (K, K')$ is a complex structure on $\mathfrak{s}$. Now, we show that $\mathfrak{d}_j(\mathfrak{s}) = \mathfrak{d}_j(\mathfrak{n}) \tilde{\oplus} \mathfrak{d}_j(\mathfrak{n}')$ for all $j \geq 0$, where $\mathfrak{d}_j(\mathfrak{n})$ and $\mathfrak{d}_j(\mathfrak{n}')$ are the J-invariant descending central series of $\mathfrak{n}$ and $\mathfrak{n}'$. By definition, $\mathfrak{d}_0(\mathfrak{s}) = \mathfrak{s} = \mathfrak{n} \tilde{\oplus} \mathfrak{n}' = \mathfrak{d}_0(\mathfrak{n}) \tilde{\oplus} \mathfrak{d}_0(\mathfrak{n}')$. Next, suppose that $\mathfrak{d}_{s-1}(\mathfrak{s}) = \mathfrak{d}_{s-1}(\mathfrak{n}) \tilde{\oplus} \mathfrak{d}_{s-1}(\mathfrak{n}')$ for some $s \in \mathbb{N}$. Using (2.11) we have

$$\begin{aligned}
\mathfrak{d}_s(\mathfrak{s}) &= ([\mathfrak{d}_{s-1}(\mathfrak{n}), \mathfrak{n}] + K[\mathfrak{d}_{s-1}(\mathfrak{n}), \mathfrak{n}], [\mathfrak{d}_{s-1}(\mathfrak{n}'), \mathfrak{n}'] + K'[\mathfrak{d}_{s-1}(\mathfrak{n}'), \mathfrak{n}']) \\
&= ([\mathfrak{d}_{s-1}(\mathfrak{n}), \mathfrak{n}], [\mathfrak{d}_{s-1}(\mathfrak{n}'), \mathfrak{n}']) + (K[\mathfrak{d}_{s-1}(\mathfrak{n}), \mathfrak{n}], K'[\mathfrak{d}_{s-1}(\mathfrak{n}'), \mathfrak{n}']) \\
&= [(\mathfrak{d}_{s-1}(\mathfrak{n}), \mathfrak{d}_{s-1}(\mathfrak{n}')), (\mathfrak{n}, \mathfrak{n}')] + (K, K')[(\mathfrak{d}_{s-1}(\mathfrak{n}), \mathfrak{d}_{s-1}(\mathfrak{n}')), (\mathfrak{n}, \mathfrak{n}')] \\
&= \mathfrak{d}_s(\mathfrak{n}) \tilde{\oplus} \mathfrak{d}_s(\mathfrak{n}').
\end{aligned}$$

By induction, $\mathfrak{d}_j(\mathfrak{s}) = \mathfrak{d}_j(\mathfrak{n}) \tilde{\oplus} \mathfrak{d}_j(\mathfrak{n}')$ for all $j \geq 0$.

Without loss of generality, suppose that $h \leq h'$. It is clear that $\mathfrak{d}_{h'-1}(\mathfrak{s}) = \{0\} \tilde{\oplus} \mathfrak{d}_{h'-1}(\mathfrak{n}') = \mathfrak{d}_{h'-1}(\mathfrak{n}') \neq \{0\}$ and $\mathfrak{d}_{h'}(\mathfrak{s}) = \{0\}$ by Theorem 2.3.31. Hence J is nilpotent of step h'. $\square$

Remark 2.4.2. Notice that the ascending central series of $\mathfrak{s}$ is

$$\mathfrak{c}^j(\mathfrak{s}) = \mathfrak{c}^j(\mathfrak{n}) \tilde{\oplus} \mathfrak{c}^j(\mathfrak{n}')$$

for all $j \geq 0$. Further, $\mathfrak{p}_j(\mathfrak{s}) = \mathfrak{p}_j(\mathfrak{n}) \tilde{\oplus} \mathfrak{p}_j(\mathfrak{n}')$ for all $j \geq 0$.

Corollary 2.4.3. *Let $\mathfrak{s} = \mathfrak{n} \tilde{\oplus} \mathfrak{n}'$ be a nilpotent be a Lie algebra sum of nilpotent Lie algebras $\mathfrak{n}$ and $\mathfrak{n}'$ with a complex structure $J = (K, K')$, where $K \in \mathcal{S}(\mathfrak{n})$ and $K' \in \mathcal{S}(\mathfrak{n}')$. Suppose that K and K' preserve $\mathfrak{z}(\mathfrak{n})$ and $\mathfrak{z}(\mathfrak{n}')$. Then J preserves $\mathfrak{z}(\mathfrak{s})$.*

Proof. Notice that $\mathfrak{z}(\mathfrak{s}) = \mathfrak{z}(\mathfrak{n} \tilde{\oplus} \mathfrak{n}') = \mathfrak{z}(\mathfrak{n}) \tilde{\oplus} \mathfrak{z}(\mathfrak{n}')$. For all $W \in \mathfrak{z}(\mathfrak{n})$ and $W' \in \mathfrak{z}(\mathfrak{n}')$,

$$J(W, W') = (KW, K'W') \in \mathfrak{z}(\mathfrak{n}) \tilde{\oplus} \mathfrak{z}(\mathfrak{n}') = \mathfrak{z}(\mathfrak{s}).$$

Hence J preserves $\mathfrak{z}(\mathfrak{s})$. □

Lemma 2.4.4. *Let $\mathfrak{s} = \mathfrak{n} \tilde{\oplus} \mathfrak{n}'$ be a nilpotent be a Lie algebra sum of nilpotent Lie algebras $\mathfrak{n}$ and $\mathfrak{n}'$ with a complex structure $J = (K, K')$, where $K \in \mathcal{S}(\mathfrak{n})$ and $K' \in \mathcal{S}(\mathfrak{n}')$. Then J is Abelian or bi-invariant if and only if both K and K' are. In particular, if $\mathfrak{n}$ and $\mathfrak{n}'$ are non-Abelian and K is Abelian and K' is bi-invariant, then J is neither Abelian nor bi-invariant.*

Proof. Now, suppose both K and K' are Abelian. Then for all $P = (X, X'), Q = (Y, Y') \in \mathfrak{s}$, where $X, Y \in \mathfrak{n}$ and $X', Y' \in \mathfrak{n}'$,

$$[JP, JQ] = ([KX, KY], [K'X', K'Y']) = ([X, Y], [X', Y']) = [P, Q],$$

which implies that J is Abelian. Similarly, J is bi-invariant if both K and K' are. Indeed,

$$J[P, Q] = (K[X, Y], K'[X', Y']) = ([X, KY], [X', K'Y']) = [P, JQ].$$

Conversely, if $J = (K, K')$ is Abelian, then

$$([KX, KY], [K'X', K'Y']) = [JP, JQ] = [P, Q] = ([X, Y], [X', Y']).$$

Hence both K and K' are Abelian. Similarly, if $J = (K, K')$ is bi-invariant, then

$$(K[X, Y], K'[X', Y']) = J[P, Q] = [P, JQ] = ([X, KY], [X', K'Y'])$$

Hence K and K' are bi-invariant.

Finally, suppose that $\mathfrak{n}$ and $\mathfrak{n}'$ are non-Abelian and K is Abelian and K' is bi-invariant. Notice that (K, K') is neither Abelian nor bi-invariant. Indeed,

$$[JP, JQ] = ([X, Y], -[X', Y']) \neq [P, Q];$$
$$J[P, Q] = (K[X, Y], [X', K'Y']) \neq [P, JQ]$$

as required. □

We present an example of a decomposable 2-step nilpotent Lie algebra with a complex structure that is neither Abelian nor bi-invariant.

Example 2.4.5. Let $\mathfrak{s} = \mathfrak{h}_1^{\mathbb{C}} \tilde{\oplus} \mathfrak{h}_1^{\mathbb{C}}$ where $\mathfrak{h}_1^{\mathbb{C}}$ is a complex Heisenberg algebra. It is a 12 dimensional 2-step nilpotent Lie algebra by Lemma 2.4.1 part (i). In [1, Theorem 3.3 and Remark 6], the authors show that there exists a family of Abelian complex structures K_t for all $t \in (0, 1]$ and a unique bi-invariant complex structure K' on $\mathfrak{h}_1^{\mathbb{C}}$. From Lemma 2.4.1 and Lemma 2.4.4, $J = (K_t, K')$ is nilpotent but neither bi-invariant nor Abelian.

2.5 2-step stratified Lie algebras with complex structures

In [17], the author showed that every 2-step nilpotent Lie algebra may be stratified. In this section we focus on 2-step nilpotent Lie algebras with complex structures. We start with the following proposition.

Proposition 2.5.1. *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be a 2-step stratified Lie algebra with a complex structure J.*

(i) Suppose that $J\mathfrak{z} = \mathfrak{z}$. Then J is nilpotent of step 2;

(ii) Suppose that $\mathfrak{n}_2 = \mathfrak{z}$ and $\mathfrak{z}$ is not J-invariant. Then J is nilpotent of step 3, where $\mathfrak{z}$ is the center of $\mathfrak{n}$.

Proof. We start with part (i). If $J\mathfrak{z} = \mathfrak{z}$, we conclude that $\mathfrak{d}^1 = \mathfrak{z}$. By definition, J is nilpotent of step 2.

For part (ii), since $\mathfrak{z} = \mathfrak{n}_2$ is not J-invariant, by definition, $\mathfrak{d}_1 = \mathfrak{z} + J\mathfrak{z}$. Then for all $Z, Z' \in \mathfrak{z}$ and $X \in \mathfrak{n}$, by the Newlander–Nirenberg condition,

$$[\mathfrak{d}_1, \mathfrak{n}] \ni [Z + JZ', X] = -[JZ', X] = J[JZ', JX] \in J[J\mathfrak{z}, \mathfrak{n}] \subseteq \mathfrak{z}.$$

Hence by definition, $\{0\} \neq \mathfrak{p}_2 = [J\mathfrak{z}, \mathfrak{n}] \subseteq \mathfrak{z}$ and $\mathfrak{p}_3 = [\mathfrak{p}_2, \mathfrak{n}] + [J\mathfrak{p}_2, \mathfrak{n}] = \{0\}$. By Theorem 2.3.31, J is nilpotent of step 3. □

Suppose that $\mathfrak{n}$ is a 2-step nilpotent Lie algebra. Let ψ be a J-invariant inner product on $\mathfrak{n}$ as in Lemma 1.1.19. Suppose that $J\mathfrak{z} = \mathfrak{z}$. There exists a vector space decomposition $\mathfrak{n} = \mathfrak{u} \oplus \mathfrak{z}$ such that $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{z}$, where $\mathfrak{u}$ is a J-invariant orthogonal complement of $\mathfrak{z}$ with respect to ψ. We define a linear map $j : \mathfrak{z} \to \mathrm{End}(\mathfrak{u})$ by

$$\psi(j(Z)X, Y) = \psi(Z, [X, Y]), \quad \text{for all } X, Y \in \mathfrak{u},\ Z \in \mathfrak{z}. \tag{2.17}$$

Notice that, for every $Z \in \mathfrak{z}$, $j(Z) \in \mathrm{End}(\mathfrak{u})$ is skew symmetric. Indeed,

$$\psi(j(Z)^T X, Y) = \psi(X, j(Z)Y) = \psi(Z, [Y, X]) = -\psi(j(Z)X, Y),$$

where $j(Z)^T$ is the transpose of $j(Z)$. We deduce that $j(Z)^T = -j(Z)$. Hence $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{u})$ is skew symmetric. For any 2-step nilpotent Lie algebras, since $\mathfrak{c}_1(\mathfrak{n}) \subseteq \mathfrak{z}$, we denote the complement of $\mathfrak{c}_1(\mathfrak{n})$ in $\mathfrak{z}$ by $\mathfrak{z} \ominus \mathfrak{c}_1(\mathfrak{n})$. That is, $\mathfrak{z} \oplus \mathfrak{c}_1(\mathfrak{n}) \subseteq \mathfrak{z}$ and $(\mathfrak{z} \ominus \mathfrak{c}_1(\mathfrak{n})) \oplus \mathfrak{c}_1(\mathfrak{n}) = \mathfrak{z}$.

Proposition 2.5.2. *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be a 2-step stratified Lie algebra with a complex structure J. Suppose that $\mathfrak{z}$ is J-invariant. We further assume that $J\mathfrak{n}_2 = \mathfrak{n}_2$ and $\mathfrak{n}_2 = \mathfrak{n}_1^{\perp}$. Then $\mathfrak{n}$ has a J-invariant vector space decomposition of the form $\mathfrak{u} \oplus (\mathfrak{z} \ominus \mathfrak{n}_2) \oplus \mathfrak{n}_2$ and $\dim \mathfrak{n}_2 \leq \frac{1}{2} \dim \mathfrak{u}(\dim \mathfrak{u} - 1)$.*

Proof. Let ψ be an J-invariant inner product on $\mathfrak{n}$. Since $J\mathfrak{z} = \mathfrak{z}$, it follows, from Proposition 2.5.1, that J is nilpotent of step 2. Since $J\mathfrak{n}_2 = \mathfrak{n}_2$, there exists a vector space decomposition $\mathfrak{z} = (\mathfrak{z} \ominus \mathfrak{n}_2) \oplus \mathfrak{n}_2$ such that $\mathfrak{z} \ominus \mathfrak{n}_2$ is the J-invariant orthogonal complement of $\mathfrak{n}_2$ in $\mathfrak{z}$ with respect to ψ. Hence $\mathfrak{n} = \mathfrak{u} \oplus (\mathfrak{z} \ominus \mathfrak{n}_2) \oplus \mathfrak{n}_2$. Let $j : \mathfrak{z} \to \mathfrak{so}(\mathfrak{u})$ be the linear mapping defined in (2.17). Furthermore, define $j' : \mathfrak{n}_2 \to \mathfrak{so}(\mathfrak{u})$ by restricting j to $\mathfrak{n}_2$. We next show that j' is injective. Since j' is linear, it is sufficient to show that $\ker j' = \{0\}$. Suppose that $j'(Z) = 0$ for some $Z \in \mathfrak{n}_2$. Then for all $X, Y \in \mathfrak{u}$,

$$\psi(Z, [X, Y]) = 0 \Rightarrow \psi(Z, \mathfrak{n}_2) = \{0\} \Rightarrow Z \in \mathfrak{n}_2^{\perp}.$$

This implies that $Z \in \mathfrak{n}_1$. Since $\mathfrak{n}_1 \cap \mathfrak{n}_2 = \{0\}$, $Z = 0$ and so $\ker j' = \{0\}$. Hence j' is injective and therefore $\dim \mathfrak{n}_2 \leq \dim \mathfrak{so}(\mathfrak{u}) = \frac{1}{2} \dim \mathfrak{u}(\dim \mathfrak{u} - 1)$. $\square$

Corollary 2.5.3. *Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with a complex structure J and let $j(Z)$ be as in (2.17) for all $Z \in \mathfrak{z}$. If J is Abelian, then $J \circ j(Z) = j(Z) \circ J$; furthermore, if J is bi-invariant, then $J \circ j(Z) = -j(Z) \circ J$.*

Proof. Suppose that J is Abelian. Then using (2.17), for all $X, Y \in \mathfrak{u}$,

$$\begin{aligned} \psi((j(Z) \circ J)(X), JY) &= \psi(Z, [JX, JY]) \\ &= \psi(Z, [X, Y]) = \psi(j(Z)X, Y) = \psi(J \circ j(Z)X, JY). \end{aligned}$$

Hence $J \circ j(Z) = j(Z) \circ J$ for all $Z \in \mathfrak{z}$. Notice that this result is presented in [2, Proposition 4.2].

Now, assume that J is bi-invariant. For all $Z \in \mathfrak{z}$, $JZ \in \mathfrak{z}$. From (2.17), for all $X, Y \in \mathfrak{u}$,

$$\begin{aligned} \psi(Jj(Z)X, Y) &= -\psi(j(Z)X, JY) = -\psi(Z, [X, JY]) = -\psi(Z, J[X, Y]) \\ &= \psi(JZ, [X, Y]) = \psi(j(JZ)X, Y); \\ \psi(j(Z)JX, Y) &= \psi(Z, [JX, Y]) = \psi(Z, J[X, Y]) = -\psi(j(JZ)X, Y). \end{aligned}$$

It follows that $\psi(Jj(Z)X, Y) = -\psi(j(Z)JX, Y)$, so $J \circ j(Z) + j(Z) \circ J = 0$ as required. $\square$

It is known, and we will provide another proof in Proposition 2.5.4, that every complex structure on a 2-step stratified nilpotent Lie algebra is nilpotent of step 2 or 3. See, e.g., [12, Theorem 1.3] and [25, Proposition 3.3]. Instead of studying the J-invariant subspaces of $\mathfrak{z}$, we shall look at J-invariant subspaces of $\mathfrak{n}_2$. In what follows, we denote by $\mathfrak{k} = \mathfrak{n}_2 \cap J\mathfrak{n}_2$ the largest J-invariant subspace contained in $\mathfrak{n}_2$ and we remind the reader that $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z}$ is the largest J-invariant subspace contained in $\mathfrak{z}$.

Proposition 2.5.4. *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be a 2-step stratified Lie algebra with a complex structure J and a J-invariant inner product ψ.*

(i) Suppose that $\mathfrak{k} = \{0\}$. Then $\mathfrak{d}_1$ is Abelian and J is nilpotent of step 2.

(ii) Suppose that $\{0\} \neq \mathfrak{k} \subset \mathfrak{n}_2$. Then $\mathfrak{n}/\mathfrak{k}$ is a step 2 quotient algebra with an Abelian complex structure J' and J is nilpotent of step 3.

(iii) Suppose that $\mathfrak{n}_2 = \mathfrak{k}$. Then J is strata-preserving and nilpotent of step 2.

In conclusion, J is nilpotent of either step 2 or 3.

Proof. For part (i), by the Newlander–Nirenberg condition, for all $X, Y \in \mathfrak{n}$,

$$J\mathfrak{n}_2 \ni J([JX, JY] - [X, Y]) = [X, JY] + [JX, Y] \in \mathfrak{n}_2.$$

Since $\mathfrak{k} = \{0\}$, it follows that $[JX, JY] - [X, Y] = 0$. Hence J is Abelian and $\mathfrak{z}$ is J-invariant by Theorem 2.2.1. Using Corollary 2.3.5, J is nilpotent of step 2. Next, we show that $\mathfrak{d}_1$ is Abelian. By Definition, $\mathfrak{d}_1 = \mathfrak{n}_2 + J\mathfrak{n}_2$. Since $\mathfrak{k} = \{0\}$, $\mathfrak{d}_1 = \mathfrak{n}_2 \oplus J\mathfrak{n}_2$. Then $[\mathfrak{d}_1, \mathfrak{d}_1] \subseteq [J\mathfrak{n}_2, J\mathfrak{n}_2]$. By the Newlander–Nirenberg condition, for all $Z_1, Z_2 \in \mathfrak{n}_2$

$$[J\mathfrak{n}_2, J\mathfrak{n}_2] \ni [JZ_1, JZ_2] = [Z_1, Z_2] + J([Z_1, JZ_2] + [JZ_1, Z_2]) = 0. \tag{2.18}$$

Thus $[\mathfrak{d}_1, \mathfrak{d}_1] = \{0\}$ and $\mathfrak{d}_1$ is Abelian.

For part (ii), suppose that $\{0\} \neq \mathfrak{k} \subset \mathfrak{n}_2$. It is clear that $\mathfrak{k}$ is Abelian. Let $\hat{\mathfrak{n}} = \mathfrak{n}/\mathfrak{k}$ and let

$$\hat{\mathfrak{z}} = \{X + \mathfrak{k} \in \hat{\mathfrak{n}} : [X, \mathfrak{n}] \subseteq \mathfrak{k}\}$$

be the center of $\hat{\mathfrak{n}}$. By Proposition 1.2.16, $\hat{J} \in GL(\hat{\mathfrak{n}})$ is a complex structure. We show that $\hat{\mathfrak{n}}$ is a 2-step nilpotent Lie algebra. For all $\hat{X} = X + \mathfrak{k}, \hat{Y} = Y + \mathfrak{k} \in \hat{\mathfrak{n}}$,

$$[X + \mathfrak{k}, Y + \mathfrak{k}] = [X, Y] + \mathfrak{k} \in \mathfrak{n}_2 + \mathfrak{k} = \hat{\mathfrak{n}}_2.$$

Furthermore, $\hat{J}$ is strata-preserving if J is. Recall that $\hat{\mathfrak{n}}$ is a 2-step nilpotent Lie algebra if $\hat{\mathfrak{n}}/\hat{\mathfrak{z}}$ is Abelian. For all $\hat{X} + \hat{\mathfrak{z}}, \hat{Y} + \hat{\mathfrak{z}} \in \hat{\mathfrak{n}}/\hat{\mathfrak{z}}$, it follows that

$$[\hat{X} + \hat{\mathfrak{z}}, \hat{Y} + \hat{\mathfrak{z}}] = [\hat{X}, \hat{Y}] + \hat{\mathfrak{z}} \in \hat{\mathfrak{n}}_2 + \hat{\mathfrak{z}} \subseteq \hat{\mathfrak{z}}.$$

Hence $\hat{\mathfrak{n}}/\hat{\mathfrak{z}}$ is Abelian and $\hat{\mathfrak{n}}$ is a 2-step nilpotent Lie algebra. Now, we show that $\hat{J}$ is Abelian. We prove that $\hat{\mathfrak{k}} = \hat{\mathfrak{n}}_2 \cap \hat{J}\hat{\mathfrak{n}}_2 = \{\hat{0}\}$. By a direct calculation,

$$(\mathfrak{n}_2 + \mathfrak{k}) \cap (J\mathfrak{n}_2 + \mathfrak{k}) = \mathfrak{n}_2 \cap J\mathfrak{n}_2 + \mathfrak{k} = \mathfrak{k}.$$

Hence $\hat{J}$ is Abelian and therefore $\hat{J}$ is nilpotent of step 2. It follows that J is nilpotent of step 3.

For part (iii), suppose that $\mathfrak{n}_2 = \mathfrak{k}$. We find that J preserves $\mathfrak{n}_2$. By Theorem 2.2.1, J is strata-preserving. By Corollary 2.3.30, J is nilpotent of step 2.

In conclusion, J is either nilpotent of step 2 or 3. □

Remark 2.5.5. (i) From the equation (2.18), we can conclude that $\mathfrak{n}_1 \neq J\mathfrak{n}_2$. Indeed, suppose that $\mathfrak{n}_1 = J\mathfrak{n}_2$. Then $\mathfrak{n}_2 = [J\mathfrak{n}_2, J\mathfrak{n}_2] = \{0\}$. This is a contradiction. Moreover, if J is nilpotent of step 3, then there does not exist a J-invariant stratification.

(ii) Recall, from [12, Theorem 1.3], that if $\mathfrak{z}$ is not J-invariant, then J is nilpotent of step 3. Combining Proposition 2.5.1 and Proposition 2.5.4, we have the following table:

J	Strata-preserving	Non-strata-preserving
$J\mathfrak{z} = \mathfrak{z}$	J nilpotent of step 2	J nilpotent of step 2
$J\mathfrak{z} \neq \mathfrak{z}$	J nilpotent of step 2	J nilpotent of step 3

Table 2.1: nilpotency of J

From Table 2.1, if J is nilpotent of step 2, then J is either strata-preserving or center-preserving. More precisely, we observe the following corollary.

Corollary 2.5.6. *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be a 2-step stratified Lie algebra with a complex structure J. Suppose that J is nilpotent of step 2. Then either $\mathfrak{k} = \mathfrak{n}_2 \cap J\mathfrak{n}_2 = \{0\}$ or $J\mathfrak{n}_2 = \mathfrak{n}_2$. Furthermore, if $\mathfrak{n}_2 = \mathfrak{z}$, then J is nilpotent of step 2 if and only if $J\mathfrak{z} = \mathfrak{z}$.*

Proof. Assume that $\mathfrak{k} \neq \{0\}$ and $J\mathfrak{n}_2 \neq \mathfrak{n}_2$. By Proposition 2.5.4 part (ii), J is nilpotent of step 3. This is a contradiction.

Moreover, if $J\mathfrak{z} = \mathfrak{z}$, by Proposition 2.5.1 part (i), J is nilpotent of step 2. Conversely, if J is nilpotent of step 2, by Corollary 2.3.29, $\mathfrak{z}$ is J-invariant. □

Notice that an even-dimensional nilpotent Lie algebra with $\dim \mathfrak{c}_1(\mathfrak{n}) = 1$ is step 2. There does not exist a J-invariant stratification for dimensional reasons. Furthermore, the authors state, in [2, Proposition 3.4], the following result without a proof. We provide a simple one.

Corollary 2.5.7. *Let $\mathfrak{n}$ be an even-dimensional 2-step nilpotent Lie algebra such that $\dim \mathfrak{c}_1(\mathfrak{n}) = 1$. Then there exists an Abelian complex structure J on $\mathfrak{n}$ and every complex structure on $\mathfrak{n}$ is Abelian.*

Proof. It is known, (see, for instance, [2]), that an even-dimensional 2-step nilpotent Lie algebra with one dimensional commutator space is isomorphic to $\mathfrak{h}_k \oplus \mathbb{R}^m$, where $\mathfrak{h}_k$ is a $2k+1$-dimensional Heisenberg algebra for some $k, m \in \mathbb{N}$. Suppose that $\mathfrak{n}$ is generated by basis elements $X_1, \dots, X_k$, $Y_1, \dots, Y_k$, $E_1, \dots, E_m, T$ such that the Lie bracket relations are given by $[X_j, Y_j] = T$ for all $j \in \{1, \dots, k\}$ and the remaining undetermined commutators vanish. Thus the stratification of $\mathfrak{n}$ is of the form $\mathfrak{n}_1 \oplus \operatorname{span}\{T\}$, where $\dim \mathfrak{n}_1 \in 2\mathbb{N}+1$ and $\mathfrak{n}_1 = \operatorname{span}\{X_1, \dots, X_k, Y_1, \dots, Y_k, E_1, \dots, E_m\}$.

We first show that there exists an Abelian complex structure on $\mathfrak{n}$. Notice that $\mathfrak{z} = \mathbb{R}^m \oplus \operatorname{span}\{T\}$. Let J be the linear isomorphism on $\mathfrak{n}$ with $J^2 = -I$ defined by

$$JX_i = Y_i, JY_i = -X_i, JE_{2p} = E_{2p-1}, JE_{2p-1} = -E_{2p}, JE_m = T \tag{2.19}$$

for all $i \in \{1, \dots, k\}$ and $p \in \{1, \dots, \frac{m-1}{2}\}$. It is easy to check that J is a complex structure on $\mathfrak{n}$. One can also check that $[JX, JY] = [X, Y]$ for every $X, Y \in \mathfrak{n}$. Indeed, $\mathfrak{k} = \mathfrak{n}_2 \cap J\mathfrak{n}_2 = \{0\}$ since $\dim \mathfrak{n}_2 = 1$. By Proposition 2.5.4 part (i), every complex structure on $\mathfrak{n}$ is Abelian. □

Suppose that $\dim \mathfrak{c}_1(\mathfrak{n}) = 2$. We first have a look at the following example.

Example 2.5.8. Let $\mathfrak{s} = (\mathfrak{h}_1 \oplus \mathbb{R}) \tilde{\oplus} (\mathfrak{h}_1 \oplus \mathbb{R})$. Then $\mathfrak{s}$ is a 8 dimensional decomposable 2-step nilpotent Lie algebra. By Lemma 2.4.1 and Corollary 2.5.7, there exist Abelian complex structures $K, K' \in \mathcal{S}(\mathfrak{h}_1 \oplus \mathbb{R})$ such that $J = (K, K') \in \mathcal{S}(\mathfrak{s})$. Furthermore, J is Abelian from Lemma 2.4.4. Notice that $\mathfrak{s}$ has a stratification of the form $\mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\dim \mathfrak{s}_1 = 6$ and $\dim \mathfrak{s}_2 = 2$. By the definition of $\mathfrak{s}$, $J\mathfrak{s}_2 \neq \mathfrak{s}_2$.

Theorem 2.5.9. *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be a 2-step stratified Lie algebra with a complex structure J such that $\dim \mathfrak{n}_2 = 2$. Then*

(i) J is nilpotent of step 2;

(ii) if $\dim \mathfrak{d}^1 = 2$, then $J\mathfrak{n}_2 = \mathfrak{n}_2$.

Proof. By Proposition 2.5.4, J is nilpotent of either step 2 or 3.

For part (i), notice that J could be either strata-preserving or not. If J is strata-preserving, by Proposition 2.5.4 part (iii), J is nilpotent of step 2. Otherwise, J is not strata-preserving. Since $\dim \mathfrak{n}_2 = 2$, it follows that $\mathfrak{k} = \mathfrak{n}_2 \cap J\mathfrak{n}_2 = \{0\}$. Then from Proposition 2.5.4 part (i) and Corollary 2.3.5, J is Abelian and nilpotent of step 2.

Now, for part (ii), recall that $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z}$ is the largest J-invariant subspace of $\mathfrak{z}$. Suppose that $\mathfrak{n}_2$ is not J-invariant. Then $\mathfrak{k} = \{0\}$. From part (i), J is nilpotent of step 2. It follows, from Lemma 2.3.14, $\mathfrak{d}_1 \subseteq \mathfrak{d}^1$. However, $\dim \mathfrak{d}_1 = \dim \mathfrak{n}_2 \oplus J\mathfrak{n}_2 = 4 > \dim \mathfrak{d}^1$. This is a contradiction. Hence $J\mathfrak{n}_2 = \mathfrak{n}_2$. □

Corollary 2.5.10. *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be a 2-step stratified Lie algebra with a complex structure J such that $\dim \mathfrak{n}_2 = 2$. Then J is center-preserving or strata-preserving or both. Furthermore, suppose that $2 \leq \dim \mathfrak{z} \leq 3$, or $\dim \mathfrak{z} = 4$ and $J\mathfrak{z} \neq \mathfrak{z}$. Then there exists a J-invariant stratification.*

Proof. By Theorem 2.5.9, J is nilpotent of step 2. Then by Table 2.1, $J\mathfrak{n}_2 = \mathfrak{n}_2$ or $J\mathfrak{z} = \mathfrak{z}$ or both if $\mathfrak{n}_2 = \mathfrak{z}$.

Furthermore, $\dim \mathfrak{d}^1 = 2$ since $2 \leq \dim \mathfrak{z} \leq 3$ or $\dim \mathfrak{z} = 4$ and $J\mathfrak{z} \neq \mathfrak{z}$. By part (ii) of Theorem 2.5.9, $J\mathfrak{n}_2 = \mathfrak{n}_2$. Furthermore, by Theorem 2.2.1, there exists a J-invariant stratification. □

Example 2.5.11. Suppose that $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ is an 8 dimensional 2-step stratified Lie algebra with a complex structure J such that $\dim \mathfrak{n}_2 = 2$. By Theorem 2.5.9, J is nilpotent of step 2. Furthermore, assume that $\dim \mathfrak{z} = 4$ and $J\mathfrak{n}_2 \neq \mathfrak{n}_2$. Using Corollary 2.5.10, $J\mathfrak{z} = \mathfrak{z}$. By definition, $\mathfrak{d}_1 = \mathfrak{n}_2 \oplus J\mathfrak{n}_2 \subseteq \mathfrak{z}$. Furthermore, $\mathfrak{z} \cong \mathfrak{d}_1$ as both $\mathfrak{z}$ and $\mathfrak{d}_1$ are Abelian. Thus there exists a J-invariant orthogonal complement $\mathfrak{v}$ of $\mathfrak{d}_1$ such that $\mathfrak{n} = \mathfrak{v} \oplus \mathfrak{d}_1$ and J is Abelian, where the orthogonality is with respect to a J-invariant inner product ψ. Let $\mathfrak{v} = \operatorname{span}\{X, Y, JX, JY\}$. Since J is Abelian, the possible Lie brackets spanning $[\mathfrak{n}, \mathfrak{n}]$ are

$$\operatorname{span}\{[X, Y] = [JX, JY], [JX, Y] = -[X, JY], [X, JX], [Y, JY]\}.$$

Proposition 2.5.12. *Let $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ be a 2-step stratified Lie algebra with a complex structure J such that $\dim \mathfrak{n}_2 = 2l$ for some $l \in \mathbb{N}$. Suppose that $\dim \mathfrak{d}^1 \leq 4l - 2$ and $J\mathfrak{n}_2 \neq \mathfrak{n}_2$. Then J is nilpotent of step 3.*

Proof. Notice that $l \neq 1$. Otherwise $\dim \mathfrak{n}_2 = \dim \mathfrak{d}^1 = 2$. This implies that $J\mathfrak{n}_2 = \mathfrak{n}_2$. Suppose, by contradiction, that J is not nilpotent of step 3. By Proposition 2.5.4, J is nilpotent of step 2. Then from Corollary 2.5.6, $\mathfrak{k} = \{0\}$. By definition and Lemma 2.3.14, $\mathfrak{d}_1 = \mathfrak{n}_2 \oplus J\mathfrak{n}_2 \subseteq \mathfrak{d}^1$. However, $\dim \mathfrak{d}_1 = 4l > \dim \mathfrak{d}^1$. This is a contradiction. Hence $\{0\} \neq \mathfrak{k} \subset \mathfrak{n}_2$. By Proposition 2.5.4 part (ii), J is nilpotent of step 3. $\square$

To end of this section, we consider the strata-preserving property on decomposable Lie algebras.

Proposition 2.5.13. *Let $\mathfrak{s} = \mathfrak{n} \tilde{\oplus} \mathfrak{n}'$ be the Lie algebra direct sum of nilpotent Lie algebras of step 2 and let $J = (K, K')$, where $K \in \mathcal{S}(\mathfrak{n})$ and $K' \in \mathcal{S}(\mathfrak{n}')$. Suppose that $\mathfrak{n}$ and $\mathfrak{n}'$ admit K and K'-invariant stratifications. Then $\mathfrak{s}$ admits a J-invariant stratification.*

Proof. Define K and K'-invariant inner products ϕ and ϕ' on $\mathfrak{n}$ and $\mathfrak{n}'$. Since $\mathfrak{n}$ and $\mathfrak{n}'$ admit K and K'-invariant stratifications, K and K' preserve the strata $\mathfrak{n}_1$

and $\mathfrak{n}_2$ and $\mathfrak{n}_1'$ and $\mathfrak{n}_2'$, and further, $\mathfrak{n}_1 = \mathfrak{n}_2^\perp$ and $\mathfrak{n}_1' = {\mathfrak{n}_2'}^\perp$. By Corollary 2.3.30, both K and K' are nilpotent of step 2. It follows, from Lemma 2.4.1 part (iii), that $J = (K, K')$ is nilpotent of step 2. Define $\mathfrak{s}_2 = \mathfrak{n}_2 \tilde{\oplus} \mathfrak{n}_2' = [\mathfrak{s}, \mathfrak{s}]$ and take $\psi = (\phi, \phi')$ on $\mathfrak{n}$. For all $X_2 \in \mathfrak{n}_2$ and $X_2' \in \mathfrak{n}_2'$,

$$J(X_2, X_2') = (KX_2, K'X_2') \in \mathfrak{n}_2 \tilde{\oplus} \mathfrak{n}_2'.$$

Hence J preserves $\mathfrak{s}_2$. Define $\mathfrak{s}_1 = \mathfrak{n}_1 \oplus \mathfrak{n}_1'$. Then $\mathfrak{s}_1 = \mathfrak{s}_2^\perp$. By a direct calculation, $\mathfrak{s}_2$ generates $\mathfrak{n}$ and $\mathfrak{s}_1$ is J-invariant. Hence J preserves both the strata $\mathfrak{s}_1$ and $\mathfrak{s}_2$. □

Example 2.5.14. Consider Example 2.4.5, $\mathfrak{s} = \mathfrak{h}_1^{\mathbb{C}} \tilde{\oplus} \mathfrak{h}_1^{\mathbb{C}}$. Then $\mathfrak{s}$ has a stratification of the form $(\mathfrak{n}_1 \tilde{\oplus} \mathfrak{n}_1') \oplus (\mathfrak{n}_2 \tilde{\oplus} \mathfrak{n}_2') = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\dim \mathfrak{s}_1 = \dim \mathfrak{n}_1 \tilde{\oplus} \mathfrak{n}_1' = 8$ and $\dim \mathfrak{s}_2 = \dim \mathfrak{n}_2 \tilde{\oplus} \mathfrak{n}_2' = 4$. Recall, from [1, Theorem 3,3 and Remark 6] and Lemma 2.4.4, that there exist bi-invariant complex structures $K, K' \in \mathcal{S}(\mathfrak{h}_1^{\mathbb{C}})$ such that $J = (K, K')$ is bi-invariant. Hence by Theorem 2.2.1, K and K' are strata-preserving. Then by Proposition 2.5.13, J is strata-preserving.

2.5.1 A case study: 6 dimensional nilpotent Lie algebras

In Chapter 1, we found that an almost complex structure on a 2-dimensional Lie algebra is a complex structure. Moreover, 4 dimensional non-Abelian nilpotent Lie algebras admit complex structures J if and only if they are isomorphic to $\mathfrak{h}_1 \tilde{\oplus} \mathbb{R}$, where $\mathfrak{h}_1$ is the 3-dimensional Heisenberg algebra. See, e.g., [11]. Furthermore, such J are Abelian. See, e.g., [2]. In this subsection, we will focus on 6 dimensional 2-step nilpotent Lie algebras with complex structures. In [5, Table 1], there is a complete classification of complex structures on these algebras. However, no information is provided on whether or not J preserves the strata.

Proposition 2.5.15. *Let $\mathfrak{n}$ be a 6 dimensional 2-step nilpotent Lie algebra with a complex structure J. Then the quotient algebra $\hat{\mathfrak{n}} = \mathfrak{n}/\mathfrak{d}^1$ admits an Abelian complex structure.*

Proof. By Proposition 2.5.4, J is nilpotent. Hence $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z} \neq \{0\}$. By Theorem 2.3.8, $2 \leq \dim \mathfrak{z} \leq 4$. Hence $\dim \mathfrak{d}^1 \in \{2, 4\}$. Since $\mathfrak{d}^1 \trianglelefteq \mathfrak{n}$, by Proposition 2.1.6, $\hat{\mathfrak{n}}$

is nilpotent and J induces a complex structure $\hat{J} \in GL(\hat{\mathfrak{n}})$. Then $\dim \hat{\mathfrak{n}} = 4$. From the above paragraph, $\hat{\mathfrak{n}} \cong \mathfrak{h}_1 \tilde{\oplus} \mathbb{R}$ and therefore $\hat{J}$ is Abelian. $\square$

Suppose that $\mathfrak{n}$ is a 6 dimensional 2-step nilpotent Lie algebra with a complex structure J. One can ask if J always preserves each layer when each layer is even-dimensional. We start with $\dim \mathfrak{z} = 4$. In particular, since $\mathfrak{n}$ admits a stratification with $\dim \mathfrak{n}_2 = 1$, for dimensional reasons, $\mathfrak{n}$ does not admit a J-invariant stratification. Hence we omit this case.

Now, suppose that $\dim \mathfrak{z} \leq 3$. We have the following theorem.

Theorem 2.5.16 ([1, 5])**.** *Let $\mathfrak{n}$ be a 6 dimensional 2-step nilpotent Lie algebra with a complex structure J such that* $\dim \mathfrak{c}_1(\mathfrak{n}) = 2$. *Then $\mathfrak{n}$ admits a J-invariant stratification.*

Proof. This is a direct consequence of Corollary 2.5.10. $\square$

Remark 2.5.17. From [5, Table 1], $\mathfrak{n}$ is isomorphic to one of the following Lie algebras:

$$\operatorname{span}\{X_i : [X_1, X_2] = X_5,\ [X_3, X_4] = X_6, 1 \leq i \leq 6\} \cong \mathfrak{h}_1 \tilde{\oplus} \mathfrak{h}_1;$$

$$\operatorname{span}\{X_i : [X_1, X_3] = [X_2, X_4] = X_5,\ [X_1, X_4] = -[X_2, X_3] = X_6,\ 1 \leq i \leq 6\} \cong \mathfrak{h}_1^{\mathbb{C}};$$

$$\operatorname{span}\{X_i : [X_1, X_2] = X_5, [X_1, X_3] = [X_2, X_4] = X_6,\ 1 \leq i \leq 6\};$$

$$\operatorname{span}\{X_i : [X_1, X_2] = X_5, [X_1, X_3] = X_6,\ 1 \leq i \leq 6\}.$$

To end this subsection, we will present three examples, which illustrate Table 2.1 in detail and investigate 6 dimensional decomposable step 2 nilpotent algebras.

Example 2.5.18. Consider the Lie algebra $\mathfrak{n} = \operatorname{span}\{X_i : [X_1, X_2] = X_5, [X_1, X_3] = X_6,\ 1 \leq i \leq 6\}$ with a complex structure J. From Remark 2.5.17, notice that $\mathfrak{n}$ admits a J-invariant stratification of the form $\mathfrak{n}_1 \oplus \mathfrak{n}_2$, where $\mathfrak{n}_1 = \operatorname{span}\{X_1, X_2, X_3, X_4\}$ and $\mathfrak{n}_2 = \operatorname{span}\{X_5, X_6\}$. Furthermore, $\mathfrak{z} = \operatorname{span}\{X_4, X_5, X_6\}$. Define $J \in GL(\mathfrak{n})$ by

$$X_1 = JX_4,\ X_3 = JX_2 \text{ and } X_6 = JX_5.$$

Since $[JX_4, JX_2] = J[JX_4, X_2]$, $N_J(X_i, X_j) = 0$ for all $X_i, X_j \in \mathfrak{a}$. Furthermore, since $\mathfrak{d}^1 = \text{span}\{X_5, X_6\}$, by (2.10), $\mathfrak{d}^2 = \mathfrak{n}$. By Definition 2.3.2, J is nilpotent of step 2.

Example 2.5.19. Consider the Lie algebra

$$\mathfrak{b} = \text{span}\{X_i : [X_1, X_2] = X_6, [X_1, X_3] = X_4, [X_2, X_3] = X_5,\ 1 \leq i \leq 6\}.$$

Define $J \in GL(\mathfrak{b})$ by

$$X_2 = JX_1,\ X_3 = JX_6 \text{ and } X_4 = JX_5.$$

By a direct calculation, $J^2 = -I$. Since $[X_1, JX_6] = J[JX_1, JX_6], [X_1, JX_1] = X_6$, $N_J(X_i, X_j) = 0$ for all $X_i, X_j \in \mathfrak{b}$. By definition, J is a complex structure. Furthermore, since $\mathfrak{d}^1 \subseteq \mathfrak{z}$, $\mathfrak{d}^1 = \text{span}\{X_4, X_5\}$. Hence using (2.10), we conclude that $\mathfrak{d}^2 = \text{span}\{X_3, X_4, X_5, X_6\} \subset \mathfrak{b}$ and $\mathfrak{d}^3 = \mathfrak{b}$. By Definition 2.3.2, J is nilpotent of step 3. Notice that $\mathfrak{b} = \mathfrak{n}_1 \oplus \mathfrak{z}$, where $\mathfrak{n}_1 = \text{span}\{X_1, X_2, X_3\}$ and $\mathfrak{z} = \mathfrak{n}_2 = \text{span}\{X_4, X_5, X_6\}$. For dimensional reasons, J is not strata-preserving. Moreover, the J-invariant descending central series $\mathfrak{d}_j$ is

$$\mathfrak{d}_1 = \text{span}\{X_3, X_4, X_5, X_6\} = \mathfrak{d}^2,\ \mathfrak{d}_2 = \mathfrak{d}^1 \text{ and } \mathfrak{d}_3 = \{0\}.$$

Recall that $\mathfrak{k} = \mathfrak{n}_2 \cap J\mathfrak{n}_2$. In this case, $\mathfrak{k} = \text{span}\{X_4, X_5\}$. Then

$$\hat{\mathfrak{b}} = \mathfrak{b}/\mathfrak{k} = \text{span}\{\hat{X}_i \in \hat{\mathfrak{b}} : [\hat{X}_1, \hat{X}_2] = \hat{X}_6,\ i = 1, 2, 3, 6\}$$

is the quotient Lie algebra and $\mathfrak{Z}(\hat{\mathfrak{b}}) = \text{span}\{\hat{X}_3, \hat{X}_6\}$. It is clear that $\hat{\mathfrak{b}}$ is nilpotent of step 2. Define $\hat{J} \in GL(\hat{\mathfrak{b}})$ by

$$\hat{X}_2 = \hat{J}\hat{X}_1 \text{ and } \hat{X}_3 = \hat{J}\hat{X}_6.$$

By Proposition 1.2.16, $\hat{J}$ is a complex structure. Since $\dim \hat{\mathfrak{n}}_2 = 1$, by Corollary 2.5.7, we further deduce that $\hat{J}$ is Abelian.

Example 2.5.20. Let $\mathfrak{s} = \mathfrak{n} \tilde{\oplus} \mathfrak{n}'$ be a 6 dimensional step 2 decomposable nilpotent Lie algebra, where $\mathfrak{n}$ and $\mathfrak{n}'$ are indecomposable nilpotent Lie algebras. Then the possible choices of $(\dim \mathfrak{n}, \dim \mathfrak{n}')$ are $(5, 1)$, $(4, 2)$ and $(3, 3)$.

(i) For the choice $(5, 1)$, one can deduce that $\mathfrak{n}'$ is Abelian. Since $\mathfrak{n}$ is indecomposable, $\mathfrak{n} \cong \mathfrak{h}_2$, the 5-dimensional Heisenberg algebra. Therefore $\mathfrak{s} \cong \mathfrak{h}_2 \tilde{\oplus} \mathbb{R}$. By Corollary 2.5.7, $\mathfrak{s}$ admits an Abelian complex structure.

(ii) Consider the second choice $(4, 2)$. From the lower dimensional Lie algebra classification, $\mathfrak{n}' \cong \mathbb{R}^2$ otherwise $\mathfrak{n}'$ is solvable. Recall that a 4-dimensional nilpotent Lie algebra $\mathfrak{n}$ is isomorphic to one of $\mathbb{R}^4$ or $\mathfrak{h}_1 \tilde{\oplus} \mathbb{R}$ or $\mathfrak{e}$, the Engel algebra[1]. Since $\mathfrak{n}$ is indecomposable, $\mathfrak{n} \cong \mathfrak{e}$. Hence $\mathfrak{s} \cong \mathfrak{e} \tilde{\oplus} \mathbb{R}^2$. By Proposition 2.1.21, $\mathfrak{e}$ does not admit a complex structure. Hence $\mathfrak{s}$ does not admit a complex structure.

(iii) Finally, for the case $(3, 3)$, notice that every 3 dimensional 2-step nilpotent Lie algebra is isomorphic to $\mathfrak{h}_1$, the 3-dimensional Heisenberg algebra. For dimensional reasons, $\mathfrak{h}_1$ does not admit complex structures. Hence $\mathfrak{n} = \mathfrak{h}_1 \tilde{\oplus} \mathfrak{h}_1$ admits Abelian complex structures.

2.6 Higher step stratified Lie algebras with complex structures

In this section, we will investigate nilpotent complex structures on higher step stratified Lie algebras. We start with 3-step stratified Lie algebras and then proceed to the case of k-step stratified Lie algebras.

Proposition 2.6.1. *Let $\mathfrak{n}$ be a 3-step stratified Lie algebra. Then $\mathfrak{p}_1$ is Abelian. Furthermore, if $\mathfrak{n}$ admits a nilpotent complex structure J of step 3, then $\mathfrak{d}_1$ is Abelian.*

Proof. For all $X, Y, X' \in \mathfrak{c}_1(\mathfrak{n})$, let $X = X_2 + X_3$ and $X' = X'_2 + X'_3$ where $X_i, X'_i \in \mathfrak{n}_i$ for all $i = 2, 3$. Then since $[\mathfrak{n}_2, \mathfrak{n}_2] = [\mathfrak{n}_2, \mathfrak{n}_3] = \{0\}$, $[X, X'] = 0$. Therefore $[\mathfrak{c}_1(\mathfrak{n}), \mathfrak{c}_1(\mathfrak{n})] = \{0\}$. By definition, $\mathfrak{p}_1 = \mathfrak{c}_1(\mathfrak{n})$ and so $\mathfrak{p}_1$ is Abelian.

[1] The 4-dimensional filiform algebra.

Furthermore, since J is nilpotent of step 3, using Corollary 2.3.18, we deduce that $[\mathfrak{d}_1, \mathfrak{c}_1(\mathfrak{n})] = \{0\}$. Then

$$\underbrace{[X + JY, X']}_{\in [\mathfrak{d}_1, \mathfrak{c}_1(\mathfrak{n})] = \{0\}} = \underbrace{[X, X']}_{\in [\mathfrak{c}_1(\mathfrak{n}), \mathfrak{c}_1(\mathfrak{n})] = \{0\}} + \underbrace{[JY, X']}_{\in [J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{c}_1(\mathfrak{n})]}. \tag{2.20}$$

From (2.20), we deduce that $[JY, X'] = 0$ and so $[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{c}_1(\mathfrak{n})] = \{0\}$. Furthermore, for all $X, Y \in \mathfrak{c}_1(\mathfrak{n})$,

$$\underbrace{[JX, JY]}_{\in [J\mathfrak{c}_1(\mathfrak{n}), J\mathfrak{c}_1(\mathfrak{n})]} = \underbrace{[X, Y]}_{\in [\mathfrak{c}_1(\mathfrak{n}), \mathfrak{c}_1(\mathfrak{n})] = \{0\}} + \underbrace{J[JX, Y] + J[X, JY]}_{\in J[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{c}_1(\mathfrak{n})] = \{0\}} = 0.$$

This implies that $[J\mathfrak{c}_1(\mathfrak{n}), J\mathfrak{c}_1(\mathfrak{n})] = \{0\}$. Finally, for all $X + JY, X' + JY' \in \mathfrak{d}_1$,

$$\begin{aligned}\underbrace{[X + JY, X' + JY']}_{\in [\mathfrak{d}_1, \mathfrak{d}_1]} &= \underbrace{[X, X']}_{\in [\mathfrak{c}_1(\mathfrak{n}), \mathfrak{c}_1(\mathfrak{n})]} + \underbrace{[X, JY'] - [X', JY]}_{\in [\mathfrak{c}_1(\mathfrak{n}), J\mathfrak{c}_1(\mathfrak{n})]} + \underbrace{[JY, JY']}_{\in [J\mathfrak{c}_1(\mathfrak{n}), J\mathfrak{c}_1(\mathfrak{n})]} \\ &= 0.\end{aligned}$$

So $[\mathfrak{d}_1, \mathfrak{d}_1] = \{0\}$ and $\mathfrak{d}_1$ is Abelian. □

Lemma 2.6.2. *Let $\mathfrak{n}$ be a 3-step stratified Lie algebra with a complex structure J. Suppose that $J\mathfrak{n}_3 = \mathfrak{n}_3$. Then J is nilpotent of step 3.*

Proof. By the definition of the descending central series $\mathfrak{p}_j$ in (2.14), $\{0\} \neq \mathfrak{p}_2 = \mathfrak{n}_3 + [J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}]$. On the one hand, suppose that $[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}] = \{0\}$. We deduce that $\mathfrak{p}_2 = \mathfrak{n}_3$ and hence $\mathfrak{p}_3 = \{0\}$ by definition. Using Theorem 2.3.31, J is nilpotent of step 3. On the other hand, suppose that $[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}] \neq \{0\}$. Then by the Newlander–Nirenberg condition, for all $U \in \mathfrak{c}_1(\mathfrak{n})$ and $X, JX \in \mathfrak{n}$

$$0 \neq \underbrace{[JU, JX] - J[JU, X]}_{\in [J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}] + J[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}]} = \underbrace{[U, X] + J[U, JX]}_{\in \mathfrak{n}_3}.$$

Hence $[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}] \subseteq \mathfrak{n}_3$. This implies that $\mathfrak{p}_2 \subseteq \mathfrak{n}_3$ and therefore $J\mathfrak{p}_2 \subseteq \mathfrak{n}_3$. Then $\mathfrak{p}_3 = [\mathfrak{p}_2, \mathfrak{n}] + [J\mathfrak{p}_2, \mathfrak{n}] = \{0\}$. Again by Theorem 2.3.31, J is nilpotent of step 3. □

Remark 2.6.3. Under the condition of Lemma 2.6.2, we further assume that $[J\mathfrak{c}_1(\mathfrak{n})$ $, \mathfrak{n}] = \{0\}$. Then $\mathfrak{z}$ is not J-invariant. Suppose not, $J\mathfrak{z} = \mathfrak{z}$. We deduce that $J\mathfrak{c}_1(\mathfrak{n}) \subseteq$ $\mathfrak{z}$ and therefore $\mathfrak{c}_1(\mathfrak{n}) \subseteq J\mathfrak{z} = \mathfrak{z}$. This implies that $\mathfrak{n}$ is nilpotent of step 2, which is a contradiction.

Recall, from Corollary 2.3.27, that J is nilpotent of step k on a k-step nilpotent Lie algebra if $\mathfrak{c}_j(\mathfrak{n})$ are J-invariant for all j. Now, in the following proposition, we can loose our condition to get a nilpotent complex structure of step k on a k-step stratified Lie algebra.

Proposition 2.6.4. *Let $\mathfrak{n}$ be a k-step stratified Lie algebra with a complex structure J. Suppose that $J\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{c}_j(\mathfrak{n})$ for all $1 \leq j \leq k-2$ and $J\mathfrak{z} = \mathfrak{z}$. Then J is nilpotent of step k.*

Proof. Since $J\mathfrak{c}_j(\mathfrak{n}) = \mathfrak{c}_j(\mathfrak{n})$ for all $1 \leq j \leq k-2$, by Corollary 2.3.27, $\mathfrak{p}_j = \mathfrak{c}_j(\mathfrak{n})$ for all $1 \leq j \leq k-2$. Then

$$\{0\} \neq \mathfrak{p}_{k-1} = \mathfrak{n}_k + [\mathfrak{c}_{k-2}(\mathfrak{n}), \mathfrak{n}] = \mathfrak{n}_k \text{ and } \mathfrak{p}_k = [J\mathfrak{n}_k, \mathfrak{n}].$$

Since $J\mathfrak{z} = \mathfrak{z}$, $J\mathfrak{n}_k \subseteq J\mathfrak{z} = \mathfrak{z}$. This implies that $\mathfrak{p}_k = \{0\}$ and hence J is nilpotent of step k. □

Lemma 2.6.5. *Let $\mathfrak{n}$ be a k-step stratified Lie algebra with a nilpotent complex structure J of step k.*

(i) Suppose that $\mathfrak{n}_k = \mathfrak{z}$, then $J\mathfrak{z} = \mathfrak{z}$;

(ii) suppose that $\dim \mathfrak{n}_k = 2$ and $\dim \mathfrak{z} \leq 3$. Then $J\mathfrak{n}_k = \mathfrak{n}_k$.

Proof. Part (i) is a direct consequence of Corollary 2.3.29.

Next, for part (ii), since J is nilpotent of step k and $\dim \mathfrak{z} \leq 3$, $\mathfrak{d}^1 = \mathfrak{z} \cap J\mathfrak{z} \neq \{0\}$ and $\dim \mathfrak{d}^1 = 2$. By Theorem 2.3.12 part (i) and Lemma 2.3.14, $\{0\} \subset \mathfrak{n}_k = \mathfrak{c}_{k-1}(\mathfrak{n}) \subseteq \mathfrak{d}_{k-1} \subseteq \mathfrak{d}^1 \subseteq \mathfrak{z}$. Hence $\dim \mathfrak{d}_{k-1} = 2$. Since $\dim \mathfrak{d}_{k-1} = \dim \mathfrak{d}^1 = \dim \mathfrak{n}_k = 2$ and $\mathfrak{d}_{k-1}$, $\mathfrak{d}^1$ and $\mathfrak{n}_k$ are all Abelian, it follows that $\mathfrak{n}_k \cong \mathfrak{d}^1 \cong \mathfrak{d}_{k-1}$. Hence $J\mathfrak{n}_k = \mathfrak{n}_k$. □

Remark 2.6.6. Notice that if $\mathfrak{n}_k = \mathfrak{d}_{k-1}$, then J is nilpotent of step k. Indeed, by Definition 2.3.2, $\mathfrak{d}_k = [\mathfrak{d}_{k-1}, \mathfrak{n}] + J[\mathfrak{d}_{k-1}, \mathfrak{n}] = \{0\}$. Furthermore, $\mathfrak{d}_{k-1} \neq \{0\}$. By Theorem 2.3.31, J is nilpotent of step k.

Proposition 2.6.7. *Let $\mathfrak{n}$ be a 8 dimensional 3-step stratified Lie algebra with a complex structure J such that $2 \dim \mathfrak{n}_3 = \dim \mathfrak{c}_1(\mathfrak{n}) = 4$. Suppose that $J\mathfrak{n}_3 \neq \mathfrak{n}_3$ and $\dim \mathfrak{z} \leq 3$. Then J is nilpotent of step 4. Furthermore, $\mathfrak{d}_2 = \mathfrak{n}_3 \oplus J\mathfrak{n}_3$.*

Proof. Since $\mathfrak{n}_3 \subseteq \mathfrak{z}$, $\dim \mathfrak{z} \geq 2$. By [15, Corollary 3.12], J is nilpotent. Then using Proposition 2.3.11, $3 \leq j_0 \leq 4$, where j_0 is the nilpotent step of J. Suppose, by contradiction, that J is nilpotent of step 3. It follows, from Theorem 2.3.12 and Lemma 2.3.14, that $\mathfrak{n}_3 + J\mathfrak{n}_3 \subseteq \mathfrak{d}_2 \subseteq \mathfrak{d}^1 \subseteq \mathfrak{z}$. On the one hand, since $\dim \mathfrak{z} \leq 3$, $\dim \mathfrak{d}^1 = 2$. On the other hand, since $J\mathfrak{n}_3 \neq \mathfrak{n}_3$ and $\dim \mathfrak{n}_3 = 2$, $\mathfrak{n}_3 \cap J\mathfrak{n}_3 = \{0\}$ and therefore $\dim \mathfrak{n}_3 \oplus J\mathfrak{n}_3 = 4 > \dim \mathfrak{d}^1$. This is a contradiction. So J is nilpotent of step 4.

We now show that $\mathfrak{d}_2 = \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. It is sufficient to show that $\mathfrak{d}_2 \subseteq \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. By definition,

$$\begin{aligned}
\mathfrak{d}_2 &= [\mathfrak{d}_1, \mathfrak{n}] + J[\mathfrak{d}_1, \mathfrak{n}] \\
&= \operatorname{span}\left\{[T, X] + J[T', X'] : \ \forall\, T, T' \in \mathfrak{d}_1, \ \forall\, X, X' \in \mathfrak{n}\right\}.
\end{aligned}$$

For all $T, T' \in \mathfrak{d}_1$, we may write $T = U + JV$ and $T' = U' + JV'$ where $U, V, U', V' \in \mathfrak{c}_1(\mathfrak{n})$. Then

$$0 \neq [T, X] + J[T', X'] = \underbrace{[U, X] + J[U', X']}_{\in\, \mathfrak{n}_3 \oplus J\mathfrak{n}_3} + [JV, X] + J[JV', X']. \tag{2.21}$$

By the Newlander–Nirenberg condition,

$$0 \neq \underbrace{[JV, X] + J[JV, JX]}_{\in\, [J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}] + J[J\mathfrak{c}_1(\mathfrak{n}), \mathfrak{n}]} = J[V, X] - [V, X] \in \mathfrak{n}_3 \oplus J\mathfrak{n}_3.$$

Hence $[JV, X] + J[JV', X'] \in \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. From (2.21), $[T, X] + J[T', X'] \in \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. Hence $\mathfrak{d}_2 \subseteq \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. In conclusion, $\mathfrak{d}_2 = \mathfrak{n}_3 \oplus J\mathfrak{n}_3$. □

Remark 2.6.8. Suppose that $\dim \mathfrak{z} \leq 3$ and $J\mathfrak{n}_3 \neq \mathfrak{n}_3$. Since $\mathfrak{d}_2 = \mathfrak{n}_3 \oplus J\mathfrak{n}_3$, for all $Y_3 + JY_3, Y_3' + JY_3' \in \mathfrak{n}_3 \oplus J\mathfrak{n}_3$, by definition,

$$\mathfrak{d}_3 = \operatorname{span}\{[Y_3 + JY_3, X] + J[Y_3' + JY_3', X'] : \ \forall\ X, X' \in \mathfrak{n}\}.$$

Then

$$[Y_3 + JY_3, X] + J[Y_3' + JY_3', X'] = [JY_3, X] + J[JY_3', X'] \in [J\mathfrak{n}_3, \mathfrak{n}] + J[J\mathfrak{n}_3, \mathfrak{n}].$$

Furthermore, by the Newlander–Nirenberg condition,

$$0 \neq [JY_3, X] = J[JY_3, JX] \in J[J\mathfrak{n}_3, \mathfrak{n}].$$

Hence $\mathfrak{d}_3 \subseteq [J\mathfrak{n}_3, \mathfrak{n}]$.

List of Symbols

J	A complex structure on a Lie algebra
N_J	The Nijenhuis tensor
$\mathfrak{n}$	A real nilpotent Lie algebra
ad	The adjoint representation
$\mathcal{S}(\mathfrak{n})$	The set of complex structures on $\mathfrak{n}$
$\mathfrak{z}$	The center of a nilpotent Lie algebra
$\mathfrak{Z}(\cdot)$	The center of a Lie algebra
$\mathfrak{c}^j(\mathfrak{n})$	The ascending central series of $\mathfrak{n}$
$\mathfrak{c}_j(\mathfrak{n})$	The descending central series of $\mathfrak{n}$
$\mathfrak{k}$	The largest J-invariant subspace of $\mathfrak{c}_1(\mathfrak{n})$
$\mathfrak{h}_j$	The $2j+1$-dimensional Heisenberg algebra
$\mathfrak{d}^j$	The J-invariant ascending central series of $\mathfrak{n}$
$\mathfrak{d}_j$	The J-invariant descending central series of $\mathfrak{n}$
$\mathfrak{p}_j$	A new descending central series of $\mathfrak{n}$
$\trianglelefteq$	Inclusion of a Lie algebra ideal
$\tilde{\oplus}$	The Lie algebra direct sum
$\oplus$	The vector space direct sum
$\ominus$	The vector space complement

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