Small time asymptotics of implied volatility under local volatility models

Author:
Guo, Zhi Jun

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Small Time Asymptotics of Implied Volatility under Local Volatility Models

Zhi Jun Guo

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Abstract

Under a class of one dimensional local volatility models, this thesis establishes closed form small time asymptotic formulae for the gradient of the implied volatility, whether or not the options are at the money, and for the at the money Hessian of the implied volatility. Along the way it also partially verifies the statement by Berestycki, Busca and Florent (2004) that the implied volatility admits higher order Taylor series expansions in time near expiry. Both as a prelude to the presentation of these main results and as a highlight of the importance of the no arbitrage condition, this thesis shows in its beginning a Cox-Ingersoll-Ross type stock model where an equivalent martingale measure does not always exist.

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List of symbols

ATM: at the money, i.e., $s = k$ (stock price = strike)

$\mathcal{B}(s, \tau; \phi)$, $B(x, \tau; \varphi)$: the Black–Scholes call option price functionals

BUC(Ω): space of bounded and uniformly continuous functions defined on Ω

CIR: Cox-Ingersoll-Ross

"const": generic constant

$C(s, \tau) \equiv C(s, \tau; k)$: call price with stock price s, strike k and time to expiry τ

$c(s, t) \equiv c(s, t; k, T)$: time t call price, with stock price s, strike k and expiry T

$\mathbb{E}$: the expectation operator

EMM: equivalent martingale measure

$\mathbb{N}_0$, $\mathbb{N}$: nonnegative, strictly positive integers, respectively

ODE: ordinary differential equation

PDE: partial differential equation

$\phi(s, \tau)$: implied volatility in the (s, τ) coordinates

$\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{R}_{++}$: real, nonnegative, and strictly positive real numbers, respectively

SDE: stochastic (ordinary) differential equation

(S_t): stock price process

$(s - k)_+ = \max(s - k, 0)$

$\varphi(x, \tau)$: implied volatility in the (x, τ) coordinates

$Z \sim N(0, 1)$: Z is a standard normal random variable

$f(t) \sim g(t) \quad (t \to 0) \quad \Longleftrightarrow \quad \lim_{t \to 0} \frac{f(t)}{g(t)} = 1$

$f(t) = \mathrm{O}(g(t)) \quad (t \to 0) \quad \Longleftrightarrow \quad \lim_{t \to 0} \left| \frac{f(t)}{g(t)} \right| \leq \text{const}$

$f(t) = \mathrm{o}(g(t)) \quad (t \to 0) \quad \Longleftrightarrow \quad \lim_{t \to 0} \frac{f(t)}{g(t)} = 0$

Contents

Chapter 1

Introduction

As natural extensions of the Black–Scholes model, local volatility models aim to build an arbitrage free pricing framework that can account for implied volatility smiles (/skews/surfaces). By allowing the volatility of the stock to depend on stock price and time, instead of setting it constant, local volatility models can explain implied volatilities in terms of local volatilities, and vice versa. This is demonstrated by Berestycki et al. [4], who have extended the poineering work of Dupire [22] and Derman and Kani [20].

Unlike the Dupire formula, which connects local volatilities with European option prices, the PDE of Berestycki et al. [4, Eq. 15] links local and implied volatilities directly. In theory, this link affords a rationale for the use of implied smiles; in practice, it makes possible a theory of implied local volatility models, according to which exotic derivatives can be priced, without arbitrage, through knowledge of the implied smiles.

However, practicable numerical implementation of such a theory still faces three long standing hurdles: (i) the lack of understanding about the small time asymptotics of gradients and Hessians of implied volatilities, (ii) the ill posed inverse problem of inferring local volatilities from sparse real world implied volatility data, and (iii) the nonexistence of a martingale measure in real world markets.

Because overcoming the first hurdle will give a fuller picture of the implied-local volatility relation and help to overcome the other two hurdles, it is the design of this thesis to study, under a class of local volatility models, the small time asymptotics of gradients and Hessians of implied volatilities.

Except in Chapter 3, which is independent of the other chapters, we shall, for clarity, assume a zero risk-free interest rate in this thesis. This zero interest rate assumption allows us to focus on the main task in hand, which is the derivation of asymptotics for the implied volatility.

It is well known that by using forward prices or risk-free bonds as numeraire we can generalize the asymptotic results of this thesis to models with nonzero and deterministic interest rates. Yet, as shown in Chapter 3, care must be taken when imposing conditons on the local volatility function and the deterministic risk-free interest rate, because an equivalent martingale measure may not exist to ensure no arbitrage.

The organisation of this introduction is as follows. Section 1.1 introduces the concepts of implied and local volatilities. Section 1.2 presents some previous results on local volatility models. Section 1.3 lists for completeness some extensions of local volatility models and briefly reviews the related literature. Section 1.4 outlines the orgainisation of this thesis and its results. Section 1.5 explains some common notation and definitions.

1.1 Implied and local volatilities

The Black–Scholes model [7] assumes that in a frictionless one stock economy, the price (S_t) of the non-dividend paying stock follows the diffusion

$$\mathrm{d}S_t = \mu S_t \,\mathrm{d}t + \nu S_t \,\mathrm{d}W_t, \tag{1.1}$$

where the *volatility* $\nu > 0$ of the stock is assumed to be constant. Here, t denotes time, μ some constant appreciation rate, and (W_t) a standard Wiener process.

Assuming (1.1), the time t arbitrage free price of a European call option with stock price s, strike k, and expiry T, is given by the Black–Scholes formula

$$\mathcal{B}(s,\tau;k;\nu) = sN(\mathfrak{d}_1) - kN(\mathfrak{d}_2), \tag{1.2}$$

where

$$\begin{cases} N(\mathfrak{d}) = \displaystyle\int_{-\infty}^{\mathfrak{d}} n(z)\,\mathrm{d}z, \qquad n(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}, \\ \mathfrak{d}_1(s,\tau;k;\nu) = \dfrac{\ln(s/k)+\nu^2\tau/2}{\nu\sqrt{\tau}}, \quad \mathfrak{d}_2(s,\tau;k;\nu) = \mathfrak{d}_1 - \nu\sqrt{\tau}, \end{cases} \tag{1.3}$$

with $\tau = T - t$ being the (remaining) *time to expiry*.

Except ν, the volatility of the stock, all other parameters of the Black–Scholes formula can be directly observed in or substituted by observables of the market. Hence, having observed a call option price $C^\sharp$ and the values of s, τ, and k, one can find a unique number $\nu^\sharp$ such that $C^\sharp = \mathcal{B}(s,\tau;k;\nu^\sharp)$, as if the underlying price process is given by $\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \nu^\sharp S_t\,\mathrm{d}W_t$. It is in this sense that the volatility $\nu^\sharp$ is implied by the call price $C^\sharp$ and the parameters s, τ, and k.

More precisely, given a European call option price $C(s,\tau;k)$ with stock price s, strike k, and time to expiry τ, the (Black–Scholes) *implied volatility* $\phi(s,\tau;k)$ is defined as the unique solution to

$$C(s,\tau;k) = \mathcal{B}(s,\tau;k;\phi(s,\tau;k)).$$

Contrary to the constant volatility assumption of the Black–Scholes model, empirical evidence has shown that as a function of k and τ, implied volatilities do fluctuate and persistently exhibit smile/skew patterns (U/L shapes). Motivated by these smile phenomena are local volatility models, under which the volatility ν of the stock becomes a deterministic function of the stock price level and time, i.e. $\nu = \nu(s,t) : (0,\infty)\times[0,T] \to (0,\infty)$. With this modification, fluctuations in implied volatilities are explained by fluctuations in $\nu(s,t)$, and vice versa. To distinguish this price and time dependent volatility from the constant and the implied volatilities, $\nu(s,t)$ is called the *local volatility* of the stock.

1.2 Previous local volatility results

Local volatility models generally assume for the stock price process (S_t) the stochastic differential equation

$$\mathrm{d}S_t = \mu S_t\,\mathrm{d}t + \nu(S_t,t)S_t\,\mathrm{d}W_t,$$

where, as already mentioned, the local volatility ν is a deterministic function of (s,t). Under such a model, Dupire [22] and Derman and Kani [20] derived the Dupire formula

$$\nu^2(k,\tau) = \frac{2C_\tau(s,\tau;k)}{k^2 C_{kk}(s,\tau;k)}, \tag{1.4}$$

with $C_\tau = \partial_\tau C$ and so on.

However, as outlined in Berestycki et al. [4], pricing via the Dupire formula has two related shortcomings. Firstly, the Dupire formula lacks robustness. The limited number of option prices demands interpolation of the data. Consequently, numerical differentiations of C, for C_τ and C_{kk}, are extremely sensitive to the choice of the interpolation. Secondly, adding to the instability of the numerical differentiation is the indeterminacy of the formula (1.4) in the regions $\{T-t \ll 1\}$, $\{|\ln(s/k)| \gg 1\}$, and $\{T-t \gg 1\}$, where it assumes the form $0/0$.

Alert to these shortcomings of the Dupire approach, Berestycki et al. [4] opted for a different track. Under some mild conditions on the local volatility $\nu(s,t)$, they derived a direct link between implied and local volatilities. In the local Sobolev space $W^{2,1,p}_{\mathrm{loc}}((0,\infty)^2)$, $1 < p < \infty$ — see (1.17) below for the definition — they showed that for each $k \in (0, \infty)$, the implied volatility function $\phi(s, \tau) \equiv \phi(s, \tau; k)$ uniquely solves the time degenerate quasilinear parabolic PDE

$$\phi^2 + 2\tau\phi\phi_\tau - \nu^2(s,\tau)\left[\left(1 - \frac{s[\ln(s/k)]\phi_s}{\phi}\right)^2 - \left(1 - \frac{s\tau\phi\phi_s}{2}\right)^2 + 1 + s^2\tau\phi\phi_{ss}\right] = 0, \tag{1.5}$$

in $(0, \infty)^2$, with the initial condition

$$\lim_{\tau \searrow 0} \phi(s,\tau) = [\ln(s/k)] \left(\int_k^s \frac{\mathrm{d}z}{z\nu(z,0)}\right)^{-1}, \quad s \in (0,\infty). \tag{1.6}$$

Moreover, Berestycki et al. [4] proves the zero order Taylor expansion

$$\phi(s,\tau) = \phi^0(s) + \mathrm{O}(\tau), \quad \text{as } \tau \searrow 0, \tag{1.7}$$

where the order O depends on the local volatility but is independent of s and τ. The significance of this result rests with (1.6) and (1.7), which prove for the first time that at expiry, implied volatility exists as a limit and how fast it converges. In this thesis we will extend this result and go one step further to characterize the small time asymptotics of the gradient ϕ_s and Hessian ϕ_{ss} of the implied volatility ϕ.

1.3 Other extensions of local volatility models

For completeness we remark that apart from local volatility models, there are more complex stochastic volatility or stochastic volatility and jump diffusion models that can be used to produce implied volatility smiles. Some of these models can be found in Durrleman [23, 24], Fouque et al. [28], Gatheral [34], Hafner [39], Hagan et al. [40], Lee [59], and Lewis [60], Medvedev [65], Medvedev and Scaillet [66], Musiela and Rutkowski [67], to name a few references. Gatheral [34] in particular provides a comprehensive and practical discussion on both asymptotics and calibrations of the implied volatility surface under Heston type stochastic volatility models. On the other hand, Hagan et al. [40] have derived a small time series expansion for the implied volatility under the so-called SABR stochastic volatility model. The results in Hagan et al. [40] extend that of Hagan and Woodward [41], which is concerned with Black's (local volatility) model. Although Gatheral [34] gives small time asymptotic results for the gradient of the implied volatility, his derivations seem formal. While not as formal in their works, Durrleman [23, 24], Medvedev [65, Equation (10)], and Medvedev and Scaillet [66, see e.g. Proposition 1] all seem to suffer from the same shortcoming of making additional assumptions on assumptions that are already made. Alós et al. [1] appears to be the only work giving a rigorous proof of the small time asymptotics for the at the money gradient of the implied volatility. Their result was obtained under a fairly general stochastic volatility model, which presently we shall say a little more.

In general diffusion type stochastic volatility models, the stock price (X_t) and its volatility process (Y_t) typically follow SDEs of the form

$$\begin{cases} \mathrm{d}X_t = \mu(X_t, Y_t, t)\,\mathrm{d}t + \nu^{(1)}(X_t, Y_t, t)\,\mathrm{d}W_t^{(1)} + \mathrm{d}J_t, \\ \mathrm{d}Y_t = \theta(Y_t, t)\,\mathrm{d}t + \nu^{(2)}(Y_t, t)\,\mathrm{d}W_t^{(2)}, \end{cases}$$

where μ, θ, $\nu^{(1)}$ and $\nu^{(2)}$ are real valued functions, $(W_t^{(1)})$ and $(W_t^{(2)})$ are correlated standard Wiener processes, and (J_t) is some jump process. Here the process (Y_t) drives the volatility of (X_t). The processes (X_t) and (Y_t) can be $\mathbb{R}^n$ valued for any finite integer $n \geq 1$, provided that the corresponding drift and diffusion coefficient functions, the Wiener processes, and the jumps have suitable dimensions. An example of such a model without jumps can be found in Berestycki et al. [5], where the small time limit of the implield volatility under the stochastic volatilities is proved.

In more general stochastic volatility models, μ, ν and θ can themselves be stochastic processes, instead of being (deterministic) functions. See e.g. Alós et al. [1] for a one dimensional example with jumps.

Analogous to the modelling of yield curve term structure, Schönbucher [73], Brace et al. [9], and Schweizer and Wissel [75] directly model the implied volatility surface. In these works HJM type consistence results are discussed and small time asymptotics of the implied volatility are investigated within the framework of the market model.

Apart from the small time asymptotics, large time behaviour of the implied volatility has been studied by Rogers and Tehranchi [70]. Under some weak assumptions, they have found that the implied volatility surface flattens at long maturities. Complementing this large time result, Benaim and Friz [3, 2] have derived large strike asymptotics for implied volatilities.

1.4 Thesis outline

While Chapter 3 of this thesis deals with the existence of equivalent martingale measures, all other chapters are devoted to the main task of deriving small time asymptotics for the gradients and Hessians of implied volatility under a class of one dimensional local volatility models.

Subject to the smoothness assumptions on the local volatility $\nu = \nu(s)$ that are detailed in Chapter 2, we have found that the implied volatility ϕ can belong to

$C^{2,1}((0,\infty)\times[0,T])$, and together with its gradient ϕ_s and Hessian ϕ_{ss}, it possesses the properties that for $s, k \in (0,\infty)$,

$$\lim_{\tau\searrow 0}\phi(s,\tau) = \phi^0(s), \tag{1.8}$$

$$\lim_{\tau\searrow 0}\phi_s(s,\tau) = \phi^0_s(s), \tag{1.9}$$

$$\lim_{\tau\searrow 0}\{\phi_{ss}|_{s=k}\} \equiv \lim_{\tau\searrow 0}\phi_{ss}(k,\tau) = \phi^0_{ss}(k), \tag{1.10}$$

$$\lim_{\tau\searrow 0}\tau\phi_{ss}(s,\tau) = 0, \tag{1.11}$$

where

$$\phi^0(s) \equiv \phi^0(s;k) = [\ln(s/k)]\left(\int_k^s \frac{\mathrm{d}z}{z\nu(z)}\right)^{-1}. \tag{1.12}$$

Note that (1.10) holds for at the money options only, i.e., when $s = k$. In the process of deriving these results, we also partially verify the statement by Berestycki et al. [5, pp. 1356, 1370, cf. Equation (6.8)] that subject to certain regularity conditions on the diffusion coefficient of the underlying stock process, implied volatilities admit Taylor series expansions in time near expiry in arbitrary orders. We will demonstrate that the following first order Taylor expansion is valid:

$$\phi(s,\tau) = \phi(s,0) + \tau\phi_\tau(s,0) + O(\tau^2), \qquad \text{as} \qquad \tau\searrow 0, \tag{1.13}$$

where

$$\begin{aligned} \phi(s,0) &= \phi^0(s), \\ \phi_\tau(s,0) &= \phi^0(s)\left[\frac{\phi^0(s)}{\ln(s/k)}\right]^2 \ln\left(\frac{\sqrt{\nu(k)\nu(s)}}{\phi^0(s)}\right). \end{aligned} \tag{1.14}$$

The result in (1.8) and that $\phi \in C^{2,1}((0,\infty)\times(0,T])$ are straightforward extensions of Theorem 1 in Berestycki et al. [4]. However, the new result in (1.9) requires some work. Once (1.9) is obtained, (1.10)–(1.11) follow easily.

The organization of the thesis is as follows:

In Chapter 2, we will set up the local volatility model and state the main results of this thesis.

In Chapter 3, we will divert our attention to highlight the importance of imposing appropriate conditions on the stock process to ensure no arbitrage in local volatility models with nonzero interest rates. We will show that in a model where the stock price follows a Cox-Ingersoll-Ross process, an equivalent martingale measure does not always exist. Probabilistic in nature, the proof partly relies on the relationship between the

CIR and Bessel processes. A version of this chapter has been published as a note in [37].

Returning to our main task in Chapter 4, we will present some preliminary results about call option prices under the local volatility model, in particular, the convexity of the call price $C(s, \tau)$ in s, which is a well known fact. Moreover, we will modify the result of Berestycki et al. [4, Equations (15), (16)] and show that depending on the regularity of the local volatility ν, the implied volatility $\psi(s, \tau)$ can belong to $C^{2,1}((0, \infty) \times [0, T])$ and satisfy (1.5) and (1.6), with $\nu(s, t)$ in (1.5) and $\nu(s, 0)$ in (1.6) respectively replaced by $\nu(s)$ and $\nu(z)$.

In Chapters 5 and 6, we adapt the method of Berestycki et al. [4] to derive a zero and a first order Taylor expansion in time for the implied volatility. Although we work along the lines of Berestycki et al. [4, Theorem 1], we need stronger assumptions on the local volatility ν to derive the Taylor expansions, as now the implied volatility belongs $C^{2,1}((0, \infty) \times [0, T])$.

In Chapter 7, we apply probabilistic methods to prove a small time limit for the at the money gradient of the implied volatility, that is, to prove

$$\lim_{\tau \searrow 0} \Big\{ \phi_s(s, \tau)|_{s=k} \Big\} \equiv \lim_{\tau \searrow 0} \phi_s(k, \tau) = \nu_s(k)/2, \quad k \in (0, \infty). \tag{1.15}$$

Central to the proof is a representation formula for the gradient of call option prices, which is a consequence of the Bismut–Elworthy formula. The asymptotic formula obtained here for the at the money gradient of implied volatility is not new, although it is independently obtained by us under weaker conditions. Assuming different, and in some sense stronger regularity assumptions, Alòs et al. [1, Theorem 6.3] have proved (1.15) for more general stochastic volatility models with jumps.

In Chapter 8, we derive a small time asymptotic formula for the gradient of the implied volatility, whether or not the option is at the money. Playing a key role in the derivation is a series representation formula for solutions of second order parabolic equations. Coupled with the PDE characterization of the implied volatility, this gradient asymptotic result also sheds light on the asymptotics of the Hessian of the implied volatility. Represented by (1.9)—(1.11), the main results of this chapter are new and the proofs are mostly analytic.

In Chapter 9, we prove a technical theorem of Chapter 8, and in Chapter 10, we conclude the thesis with a list of future research directions.

1.5 Notation and definitions

In this thesis the standard notation and definitions for stochastic calculus can be found e.g. in Friedman [32], Karatzas and Shreve [53], or Revue and Yor [69]. We will use some PDE material from Krzyzanski [57] and Friedman [30]. Regarding solutions of PDEs, the definition below is standard, although there are other definitions requiring continuity of the solution at the boundary.

Let $D = \mathbb{R}$ or $(0, \infty)$. Consider the differential equation

$$\begin{cases} Lu \equiv a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u - u_t = 0, & (x,t) \in D \times (0,T], \\ u(x,0) = \rho(x), \quad x \in D. \end{cases}$$

where the coefficients a, b, and c are defined in $D \times [0,T]$.

Definition 1.1. A function $u = u(x,t)$ is a *solution* of $Lu = 0$ in $D \times [0,T]$ if

(i) all the derivatives of u that occur in Lu (i.e., u_x, u_{xx}, u_t) are continuous functions in $D \times (0,T]$,

(ii) $Lu(x,t) = 0$ at each point $(x,t) \in D \times (0,T]$,

(iii) $u(x,0) = \rho(x)$ for all $x \in D$.

According to this definition, a solution of the PDE does not need to be be continuous at the boundary $D \times \{0\}$.

We will let $\mathbb{N}_0$ denote the set of all nonnegative integers and $\mathbb{N}$ the strictly positive ones. In this thesis, $\mathit{\Omega}$ stands for the sample space, whereas Ω denotes a domain (bounded or unbounded) in $\mathbb{R}^d$, with $d \in \mathbb{N}$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multiindex and $|\alpha| = \sum_{i=1}^n \alpha_i$. For any function $f : \mathbb{R}^n \to \mathbb{R}$, define $\operatorname{dom}(f)$ to be the domain of f. Define the supremum norm $\|\cdot\|_m$ by

$$\|f\|_m = \sum_{|\alpha| \le m} \sup_{x \in \operatorname{dom}(f)} |\mathrm{D}^\alpha f(x)|, \quad \alpha \in \mathbb{R}^n, m \in \mathbb{N}_0. \tag{1.16}$$

We generally use "const" to denote a generic positive constant, whose value may depend on certain parameters, e.g. const = const(T).

Let $\Omega = (0,\infty)^2$. Then the Sobolev spaces $W^{2,1,p}(\Omega)$, $1 < p \le \infty$, are defined by

$$W^{2,1,p}(\Omega) = \left\{ w : \int_\Omega |w_{xx}|^p + |w_\tau|^p + |w|^p < \infty \right\}, \quad 1 < p < \infty, \tag{1.17}$$

$$W^{2,1,\infty}(\Omega) = \{ w : \|w_{xx}\|_\infty + \|w_\tau\|_\infty + \|w\|_\infty < \infty \}, \quad p = \infty, \tag{1.18}$$

where $\|\cdot\|_\infty$ denotes the essential supremum norm. The spaces $W^{2,1,p}(\Omega)$ are endowed with their natural norms and $W^{2,1,p}_{\text{loc}}(\Omega)$ are their local versions. In this thesis we do not use these Sobolev spaces to obtain our results. They are included here to explain the results of Berestycki et al. [4].

We will use the symbol $\sim$ in $Z \sim N(0,1)$ to signal that Z is a standard normal random variable. However, we will also use it in expressions for asymptotics like the following:

$$f(t) \sim g(t) \quad (t \to 0) \quad \iff \quad \lim_{t\to 0} \frac{f(t)}{g(t)} = 1.$$

The meaning of the symbol $\sim$ will be clear from the context.

The big O and the little o denote respectively

$$\begin{aligned} f(t) = \mathrm{O}(g(t)) \quad (t \to 0) \quad &\iff \quad \lim_{t\to 0} \left|\frac{f(t)}{g(t)}\right| \leq \text{const}, \\ f(t) = \mathrm{o}(g(t)) \quad (t \to 0) \quad &\iff \quad \lim_{t\to 0} \frac{f(t)}{g(t)} = 0. \end{aligned}$$

In particular, we have

$$\begin{aligned} f(t) = \mathrm{O}(1) \quad (t \to 0) \quad &\iff \quad \lim_{t\to 0} |f(t)| \leq \text{const}, \\ f(t) = \mathrm{o}(1) \quad (t \to 0) \quad &\iff \quad \lim_{t\to 0} f(t) = 0. \end{aligned}$$

In this thesis t, τ, and T all denote time and are nonnegative real numbers. So in the rest of this thesis we shall mostly simply write

$$t \to 0, \quad \tau \to 0, \quad T \to 0,$$

in lieu of the technically correct expressions

$$\begin{aligned} &t \to 0+, \quad t \searrow 0, \quad \text{or} \quad t \downarrow 0, \\ &\tau \to 0+, \quad \tau \searrow 0, \quad \text{or} \quad \tau \downarrow 0, \\ &T \to 0+, \quad T \searrow 0, \quad \text{or} \quad T \downarrow 0. \end{aligned}$$

Occasionally we will use the following equivalent expressions interchangeably:

$$X_\tau \xrightarrow{\tau \to 0} x \quad \iff \quad \lim_{\tau \to 0} X_\tau = x.$$

Chapter 2

Model setup and main results

In this chapter, we will set up the local volatility model of this thesis, state our main results as theorems, and comment on their extensions.

2.1 The local volatility model of this thesis

We assume a frictionless one stock economy in which the risk-free interest rate is zero and the price of the nondividend paying stock, denoted by (S_t), is governed by the stochastic (ordinary) differential equation

$$\mathrm{d}S_t = \nu(S_t)S_t\,\mathrm{d}W_t, \quad S_0 > 0, \quad 0 \le t \le T < \infty, \tag{2.1}$$

where (W_t) is a standard Wiener process with respect to a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ denoting the filtration generated by (S_t). This is the local volatility model of this thesis, where the local volatility $\nu(s) : (0,\infty) \to (0,\infty)$ is *always* assumed to satisfy (A_0) of the list of assumptions below. Further assumptions, selected from (A_1)–(A_4), will be imposed to obtain different results in this thesis.

List of assumptions on the local volatility

(A_0) The local volatility $\nu(\cdot)$ is locally Lipschitz continuous in $(0,\infty)$, and there exists a constant $\nu_0 > 1$ such that

$$0 < \underline{\nu} \equiv \frac{1}{\nu_0} \le \nu(s) \le \nu_0 \equiv \overline{\nu} < \infty, \quad \forall\, s \in (0,\infty); \tag{2.2}$$

(A_1) the first derivative $\nu_s(\cdot)$ is locally Lipschitz continuous in $(0,\infty)$ and

$$\nu_1 := \sup_{s\in(0,\infty)} |s\nu_s(s)| < \infty; \tag{2.3}$$

(A_2) the second derivative $\nu_{ss}(\cdot)$ exists and is continuous in $(0,\infty)$ and

$$\nu_2 := \sup_{s\in(0,\infty)} \left|s^2\nu_{ss}(s)\right| < \infty; \tag{2.4}$$

(A_3) the third derivative $\nu_{sss}(\cdot)$ exists and is continuous in $(0,\infty)$ and

$$\nu_3 := \sup_{s\in(0,\infty)} \left|s^3\nu_{sss}(s)\right| < \infty; \tag{2.5}$$

(A_4) the forth derivative $\nu_{ssss}(\cdot)$ exists and is continuous in $(0,\infty)$ and

$$\nu_4 := \sup_{s\in(0,\infty)} \left|s^4\nu_{ssss}(s)\right| < \infty. \tag{2.6}$$

For ease of notation, we define $\mathcal{V}_i$ to be the set

$$\mathcal{V}_i := \{\nu_0, \nu_1, \ldots, \nu_i\}, \quad i = 0,1,2,3,4, \tag{2.7}$$

with $\mathcal{V}_0 = \{\nu_0\}$.

Remark 2.1. The assumptions above will be used selectively for different results. We shall see in Theorems 2.5–2.8 below that if we want to know more about the implied volatility, we need to impose more assumptions on the local volatility, starting from (A_0)–(A_2) and then increasing to (A_0)–(A_4). Note however that under weaker assumptions, results similar to that of Theorem 2.5 have been obtained in the Sobolev space $W^{2,1,p}(\Omega)$, $1 < p < \infty$, by Berestycki et al. [4]. Without (A_2), Goldys and Roper [36] have also proved the small time limit of the implied volatility in (2.9), although they have not investigated whether (2.11) holds under their weaker assumptions. Note also that the local Lipschitz property in (A_1) is needed for the well-definedness of the first variation process of (S_t); see (7.14) and Protter [68, Theorem 49, p. 320].

Remark 2.2. In our model, the existence of an equivalent martingale measure is guaranteed by the assumptions of zero risk-free interest rate and zero diffusion drift in (2.1). It is none other than the probability measure $\mathbb{P}$ itself. Hence arbitrage is excluded by the fundamental theorem of asset pricing.

2.2 Call option price and implied volatility under the model

It is well known that if (A_0) and (A_1) hold, then the time t arbitrage free price of a European call option $c(s,t;k,T)$, with stock price s, strike k and expiry T, is given by

$$c(s,t;k,T) = \mathbb{E}_{s,t}[(S_T-k)_+] = \mathbb{E}[(S_T-k)_+|S_t=s],$$

for $(s,t;k,T) \in (0,\infty)\times[0,T]\times(0,\infty)\times(0,\infty)$.

Equivalently, using the time to expiry $\tau = T-t$, and the Markov property of (S_t), the arbitrage free price of a European call option $C(s,\tau;k)$, with stock price s, strike k and time to expiry τ, is given by

$$C(s,\tau;k) = \mathbb{E}_s[(S_\tau-k)_+] = \mathbb{E}[(S_\tau-k)_+|S_0=s],$$

for $(s,\tau;k) \in (0,\infty)\times[0,T]\times(0,\infty)$. The above call price formulas are proved in Lemma 4.2 below. We are now ready to rigorously define implied volatility.

Definition 2.3 (Implied volatility)**.** For each fixed $T \in (0,\infty)$, the unique function $\phi(s,\tau;k) : (0,\infty)\times(0,T]\times(0,\infty)\to(0,\infty)$ satisfying $\phi \in C^{2,1,2}((0,\infty)\times(0,T]\times(0,\infty))$ and

$$C(s,\tau;k) = \mathcal{B}(s,\tau;k;\phi(s,\tau;k)), \quad (s,\tau;k)\in(0,\infty)\times(0,T]\times(0,\infty),$$

is called the (Black–Scholes) implied volatility.

Remark 2.4. We will prove in Chapter 4 the existence and uniqueness of ϕ. Unless explicitly stated to the contrary, throughout the thesis we will treat $k,T \in (0,\infty)$ as parameters and define

$$c(s,t) := c(s,t;k,T), \quad C(s,\tau) := C(s,\tau;k), \quad \phi(s,\tau) := \phi(s,\tau;k).$$

2.3 Main results of the thesis

The following theorems are the main results of the thesis.

Theorem 2.5. *Let (2.1),* (A_0)–(A_2) *hold. Then for each* $k\in(0,\infty)$*, the implied volatility* ϕ *belongs to* $C^{2,1}((0,\infty)\times(0,T])$ *and satisfies the time degenerate quasilinear parabolic equation*

$$\phi^2 + 2\tau\phi\phi_\tau - \nu^2(s)\left[\left(1-\frac{s[\ln(s/k)]\phi_s}{\phi}\right)^2 - \left(1-\frac{s\tau\phi\phi_s}{2}\right)^2 + 1 + s^2\tau\phi\phi_{ss}\right] = 0, \quad (2.8)$$

in $(0,\infty)\times(0,T]$, *with the initial condition*

$$\phi(s,0) := \lim_{\tau\to 0}\phi(s,\tau) = \phi^0(s), \quad s\in(0,\infty), \tag{2.9}$$

where

$$\phi^0(s) \equiv \phi^0(s;k) := [\ln(s/k)]\left(\int_k^s \frac{\mathrm{d}z}{z\nu(z)}\right)^{-1}, \quad s\in(0,\infty). \tag{2.10}$$

Further, $\phi(s,\tau)$ *admits the zero order Taylor expansion in time*

$$\phi(s,\tau) = \phi^0(s) + \mathrm{O}(\tau) \quad \textit{as} \quad \tau\to 0, \tag{2.11}$$

where $\mathrm{O}=\mathrm{O}(\mathcal{V}_2)$.

Theorem 2.6 (First order Taylor expansion in time)**.** *Let (2.1),* (A_0)–(A_4) *hold. Then as* $\tau\to 0$,

$$\phi(s,\tau) = \phi(s,0)\left(1+\tau\frac{\phi^2(s,0)}{[\ln(s/k)]^2}\ln\left(\frac{\sqrt{\nu(k)\nu(s)}}{\phi(s,0)}\right)+\mathrm{O}(\tau^2)\right), \tag{2.12}$$

with $\mathrm{O}=\mathrm{O}(\mathcal{V}_4)$.

Theorem 2.7 (At the money gradient asymptotic)**.** *Let (2.1),* (A_0)–(A_2) *hold. Then*

$$\lim_{\tau\to 0}\left\{\phi_s(s,\tau)|_{s=k}\right\} \equiv \lim_{\tau\to 0}\phi_s(k,\tau) = \nu'(k)/2. \tag{2.13}$$

Theorem 2.8 (Gradient and Hessian asymptotics)**.** *Let (2.1),* (A_0)–(A_4) *hold. Then the implied volatility* ϕ *has the following properties:*

(i) For each $s\in(0,\infty)$,

$$\lim_{\tau\to 0}\phi_s(s,\tau) = \phi^0_s(s). \tag{2.14}$$

(ii) For each $k\in(0,\infty)$,

$$\lim_{\tau\to 0}\{\phi_{ss}|_{s=k}\} \equiv \lim_{\tau\to 0}\phi_{ss}(k,\tau) = \phi^0_{ss}(k) = \frac{\nu_{ss}(k)}{3} - \frac{\nu_s^2(k)}{6\nu(k)} - \frac{\nu_s(k)}{6k}. \tag{2.15}$$

(iii) For every $s\in(0,\infty)$,

$$\lim_{\tau\to 0}\tau\phi_{ss}(s,\tau) = 0. \tag{2.16}$$

Respectively, Theorems 2.5, 2.6, 2.7, and 2.8 will be proved in Chapters 5, 6, 7, and 8. However, before presenting the preliminary results and the proofs of these theorems, we shall in the next chapter divert our attention to the existence and nonexistence of equivalent martingale measures in CIR type local volatility models.

2.4 Comments on extensions of the main results

For a more direct illustration of the ideas we have set the local volatility $\nu = \nu(S_t)$ in (2.1). Nevertheless, the main results of this thesis can be extended to local volatility models where the local volatility depends on both space and time, i.e., $\nu = \nu(S_t, t)$ in (2.1). The caveat is that $\nu(S_t, t)$ and its derivatives must be regular enough. In fact, Alòs et al. [1, Theorem 6.3] have proved more general results than Theorem 2.7, not only for smooth enough deterministic local volatility $\nu = \nu(S_t, t)$ but also for general square integrable processes such as $\nu = \nu(S_t(\omega), t, \omega)$.

Now with $\nu = \nu(S_t, t)$ in (2.1), we make the following hypotheses:

(H_0) $\nu(s,t)$ is locally Lipschitz in $(0,\infty) \times [0,T]$, and for some constant $\nu_0 > 1$

$$0 < \underline{\nu} \equiv \frac{1}{\nu_0} \leq \nu(s,t) \leq \nu_0 \equiv \overline{\nu} < \infty, \quad \forall\ (s,t) \in (0,\infty) \times [0,T]; \tag{2.17}$$

(H_1) $\nu_s(s,t)$ and $\nu_t(s,t)$ are locally Lipschitz continuous in $(0,\infty) \times [0,T]$ and

$$\nu_1 := \sup_{(s,t)\in(0,\infty)\times[0,T]} \Big\{ |\nu_s(s,t)|, |\nu_t(s,t)|, |s\nu_s(s,t)|, |s\nu_t(s,t)| \Big\} < \infty; \tag{2.18}$$

(H_2) $\nu_{ss}(s,t)$ is continuous in $(0,\infty)$ and $\sup_{(s,t)\in(0,\infty)\times[0,T]} \left|s^2\nu_{ss}(s,t)\right| < \infty$;

(H_3) $\nu_{sss}(s,t)$ is continuous in $(0,\infty) \times [0,T]$ and $\sup_{s\in(0,\infty)\times[0,T]} \left|s^3\nu_{sss}(s,t)\right| < \infty$;

(H_4) $\nu_{ssss}(s,t)$ is continuous in $(0,\infty) \times [0,T]$ and $\sup_{s\in(0,\infty)\times[0,T]} \left|s^4\nu_{ssss}(s,t)\right| < \infty$.

Under (H_0)–(H_4), we conjecture that Theorems 2.5, 2.6, 2.7, and 2.8 all hold, with the following changes: in the theorems, $(A_i) \mapsto (H_i)$, $i = 0, \ldots, 4$; in (2.8) $v(s) \mapsto v(s,t)$; in (2.10) $v(z) \mapsto v(z,0)$; in (2.12) $v(k) \mapsto v(k,0)$, $v(s) \mapsto v(s,0)$; in (2.13) $v'(k) \mapsto v_s(s,0)|_{s=k}$; in (2.15) $v_s(k) \mapsto v_s(s,0)|_{s=k}$, $v_{ss}(k) \mapsto v_{ss}(s,0)|_{s=k}$.

Once this conjecture is proven, then the short time asymptotic results of this thesis can be further generalized to stochastic volatility models. This can be achieved because by Gyögy's theorem [38, Theorem 4.6], every stock price under a suitable stochastic model has the same marginal distribution as the stock price under the corresponding local volatility model.

Chapter 3

On the CIR Process and existence of equivalent martingale measures

3.1 Introduction

Published in Statistics and Probability Letters [37], this self-contained chapter shows that in a model where historical stock price follows a Cox–Ingersoll–Ross process, an equivalent martingale measure does not exist except when $k\theta = 0$. The symbols used in this chapter may differ in meaning from the same symbols appearing elsewhere in this thesis.

In the Cox–Ingersoll–Ross (**CIR**) interest rate model [16, 1985], instantaneous spot interest rates are modeled by the diffusion process (R_t),

$$\mathrm{d}R_t = k(\theta - R_t)\,\mathrm{d}t + \sigma\sqrt{R_t}\,\mathrm{d}W_t, \qquad R_0 > 0,$$

where the constants $k, \theta > 0$, $\sigma \in \mathbb{R}\backslash\{0\}$, and (W_t) is a standard Wiener process. In finance such a locally bounded semimartingale is called a CIR process; in mathematics, a square root process. Original studies of the CIR process can be traced back to Feller [25, 1951].

Well-known properties of the CIR process include that it is nonnegative and mean reverting. It can be reflective with respect to, absorbed by, or strictly away from the lower boundary point 0, depending on the values of the parameters k, θ and σ. Also well known is the fact that it is a space-time changed squared Bessel process, and as a result there are explicit formulas for its transition density, and for moments and various functionals of the process. See e.g. [35] or [52] and the references therein.

Although historically and still popularly associated with interest rate models, these known properties of the CIR process make it well suited for the modeling of stock price movements that display similar characteristics.

Indeed, for certain choices of the parameters, e.g. $k < 0$ and $\theta = 0$, the CIR process becomes a special case of the generalized Constant Elasticity of Variance (**CEV**) process that has been widely used to model stock prices (see e.g. [15], [17], [19], [63], [74] and Remark 3.8 below).

In most option pricing models, including those where the CIR process is used to model interest rates, stock prices, stochastic volatilities [48], or default times [11], it is crucial to ensure the existence of an Equivalent Martingale Measure (**EMM**) — see Definition 3.1 below. According to the (First) Fundamental Theorem of Asset Pricing, if the underlying interest rate or stock process is a locally bounded real-valued semimartingale, then the existence of an EMM is both necessary and sufficient to ensure that there is no arbitrage (see [18, Corollary 9.1.2]; cf. [77, p. 651; Theorem 1, p. 655; Corollary to Theorem 2, p. 657], [43, 44]).

Moreover, the existence of an EMM allows the *fair, arbitrage-free* price of the option to be represented by, and in many cases be simply calculated as, an expected value or a functional of the expected value under the EMM (see e.g. [18, p. 8], [77, Chapters VII & VIII], [43, 44]).

Nevertheless, the original CIR interest rate model [16] was essentially built on equilibrium arguments of economics, while more generalized versions of the fundamental theorem of asset pricing and the notion of no arbitrage were still taking shape with increasing mathematical rigor. See [18] for a summary of the development of the theorem.

For CIR-type interest rate models, the EMM question has been dealt with by Heath, Jarrow and Morton [45], Maghsoodi [64] and Shirakawa [76]. However, these studies do not investigate the question for CIR-type stock models.

On the other hand, Delbaen and Shirakawa [19] are concerned with CEV stock models, even though their result implies the existence of an EMM for a special case of our CIR stock model (see Remark 3.8 below).

In a different context, Wong and Heyde [79] discussed the martingale properties of stochastic exponentials and their relations to EMMs. Yet no CIR-type stock models were analyzed.

So far as we are aware, there is no published result directly dealing with the EMM question for CIR-type stock models. To fill this gap, we have studied a stock model in which the stock price follows a CIR process with a full range of parameters that are relevant to financial applications. We have found that except when $k\theta = 0$, an EMM does not exist.

Our main result, Theorem 3.3, differs from the work of Cherny and Urusov [14] in two important respects. Firstly, their work delineates the absolute continuity and singularity of measures for standard (squared) Bessel processes [14, Theorem 4.1, Corollary 4.1]. In contrast, our theorem answers the EMM question for CIR processes, which are space-time changed squared Bessel processes. Secondly, their theory about separating times for solutions of Stochastic Differential Equations is based on the assumption that the diffusion coefficients of the SDEs are never zero [14, Section 5; Equation (5.2)]. However our result also includes the situation where the diffusion coefficient in the SDE for the CIR process may become zero, because in our model the CIR process may hit 0 (see (3.1) and Lemma 3.4 below).

Similarly, the fact that the CIR process in our model may hit 0 distinguishes our result from that of Cheridito, Filipovic and Yor [13, Section 6], which examines equivalent and absolutely continuous measure changes for a class of extended, strictly positive CIR processes with jumps.

3.2 Equivalent martingale measures under the CIR model

The security market is assumed to consist of one risky stock and one risk-free bond. Without loss of generality, we will consider the model for the unit time interval $[0, 1]$, and let the stochastic stock price (X_t) and deterministic bond price B_t be given by

$$\begin{cases} X_t = \overline{X}_{t\wedge\tau}, \\ B_t = \mathrm{e}^{rt}, \qquad 0 \le t \le 1, \end{cases} \tag{3.1}$$

where

- $t \wedge \tau = \min(t, \tau)$;
- $\overline{X}_t = x + \int_0^t k(\theta - \overline{X}_s)\ \mathrm{d}s + \int_0^t \sigma\sqrt{|\overline{X}_s|}\ \mathrm{d}W_s, \qquad 0 \le t \le 1;$
- x, k, θ, σ and r are real constants with $x > 0$, $\sigma \neq 0$ and $r > 0$;
- (W_t) is a standard Wiener process defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le 1}$ denoting the filtration generated by $(\overline{X}_t)$;
- $\tau = \tau_0 \wedge 1$, with τ_0 being the first hitting time of 0 by the process $(\overline{X}_t)$, namely,

$$\tau_0 = \tau_0(\omega) := \inf\{0 \le t \le 1 :\ \overline{X}_t(\omega) = 0\}, \quad \inf\emptyset = \infty.$$

The stopped CIR process (X_t) is well defined (see e.g. [35, p. 313]) and by definition nonnegative. Further, since rarely in real life can share prices decrease to 0 and return to a positive value, the assumption that (X_t) remains at 0 once it hits 0 is economically sound.

Let $\mathbb{G} := (\mathcal{G}_t)_{0 \leq t \leq 1}$ be the filtration such that $\mathcal{G}_t = \mathcal{F}_{t \wedge \tau}$. Then for the CIR model (3.1), we have the following definition.

Definitions 3.1 (Equivalent Martingale Measure)**.** A probability measure $\widetilde{\mathbb{P}}$ is an **EMM** of $\mathbb{P}$ with respect to the CIR process (X_t), denoted $\widetilde{\mathbb{P}} \sim \mathbb{P}$, if

(a) on the stopping-time sigma field $\mathcal{F}_\tau$, $\widetilde{\mathbb{P}}_\tau \sim \mathbb{P}_\tau$, i.e. the measures $\widetilde{\mathbb{P}}_\tau$ and $\mathbb{P}_\tau$ are equivalent, and

(b) the discounted stock price process $(\mathrm{e}^{-rt} X_t)$ is a $(\widetilde{\mathbb{P}}_\tau, \mathbb{G})$-martingale.

Note that $\mathbb{P}_\tau$ denotes the restriction of $\mathbb{P}$ to $\mathcal{F}_\tau$.

Remark 3.2. Our definition of an EMM is nonstandard. When the parameters are set in such a way that (X_t) cannot hit 0, our EMM is the *usual* EMM, see e.g. [77, p. 652] and cf. [18, Theorem 8.2.1; Introduction, p. 207]. When (X_t) can hit 0, our EMM, strictly speaking, should be called ELMM, which is the *usual* Equivalent Local Martingale Measure with respect to τ (cf. [18, Theorem 8.2.1; Introduction, p. 207], [77, p. 652]). We include these two possibilities in our definition to streamline the proofs.

The following theorem constitutes the main result of this note.

Theorem 3.3. *For the CIR model (3.1), an EMM exists if and only if $k\theta = 0$.*

The proof of this theorem comprises four cases:

(I) if $\sigma^2 \leq 2k\theta, 0 < k\theta$, then an EMM does not exist;

(II) if $k\theta = 0$, then an EMM exists;

(III) if $\sigma^2 > 2k\theta > 0$, then an EMM does not exist;

(IV) if $k\theta < 0$, then an EMM does not exist.

To simplify the proof, we will present the preliminary results in the next section.

3.3 Preliminary results

Below are the preliminary results needed for the proof of Theorem 3.3.

Lemma 3.4 (Behaviour at 0)**.** *The CIR process (X_t) has the following boundary properties:*

(P1) if $\sigma^2 \leq 2k\theta$, then $\mathbb{P}[\tau_0 < \infty] = 0$;

(P2) if $\sigma^2 > 2k\theta > 0$ and $k > 0$, then $\mathbb{P}[\tau_0 < \infty] = 1$ and $\mathbb{P}[\tau_0 < 1] > 0$;

(P3) if $\sigma^2 > 2k\theta > 0$ and $k < 0$, then $\mathbb{P}[\tau_0 < \infty] \in (0,1)$ and $\mathbb{P}[\tau_0 < 1] > 0$.

Proof. All the facts can be directly found in [35, p. 315] or shown by using Feller's test for nonexplosion, except the claim in (P2) and (P3) that $\mathbb{P}[\tau_0 < 1] > 0$. That can be proved by noting that (X_t) is a space-time changed squared Bessel process (see e.g. [35, Equation (4)]). Under the conditions of (P2) or (P3), such a squared Bessel process admits a transition density for its first hitting time of zero, τ_0, like its corresponding Bessel process, meaning that $\mathbb{P}[\tau_0 < T] > 0$ for any $0 < T < \infty$ (see [56], [35], and [8, Section IV-44, p. 75]). ☐

The following lemma is also required.

Lemma 3.5. *For the CIR model (3.1), assume that there exists an EMM $\widetilde{\mathbb{P}}$. Then there exists a unique $\mathbb{G}$-adapted stopped process (γ_t) such that*

(i) (γ_t^2) is bounded $\mathbb{P}$-a.s., i.e.

$$\mathbb{P}\left[\int_0^\tau \gamma_t^2 \; \mathrm{d}t < \infty\right] = 1, \tag{3.2}$$

and the density (Z_t) of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$ is given by

$$Z_t = \exp\left(\int_0^t \gamma_s \; \mathrm{d}W_s - \frac{1}{2}\int_0^t \gamma_s^2 \; \mathrm{d}s\right), \quad 0 \leq t \leq \tau; \tag{3.3}$$

(ii) the process

$$\widetilde{W}_t := W_t - \int_0^t \gamma_s \; \mathrm{d}s, \quad 0 \leq t \leq \tau,$$

is a stopped standard $\widetilde{\mathbb{P}}$-Wiener process;

(iii) for all t, $0 \leq t \leq \tau$,

$$-rX_t + k(\theta - X_t) + \sigma\sqrt{X_t}\gamma_t = 0, \quad \mathbb{P}\text{-a.s.}. \tag{3.4}$$

Proof. Before proceeding, let us note that if statement (iii) holds for some $(\gamma_t^{(1)})$, then uniqueness follows from the observation that any two $(\gamma_t^{(1)})$ and $(\gamma_t^{(2)})$ satisfying (3.4) will be indistinguishable for all $t \leq \tau$. Now let us prove that statements (i) and (ii) hold for some $\mathbb{G}$-adapted (γ_t).

Since by assumption $\widetilde{\mathbb{P}} \sim \mathbb{P}$, the density (Z_t), defined by

$$Z_t := \mathbb{E}_{\mathbb{P}}\left[\frac{\mathrm{d}\,\widetilde{\mathbb{P}}_\tau}{\mathrm{d}\,\mathbb{P}_\tau}\middle|\mathcal{G}_t\right],$$

is a strictly positive, continuous, uniformly integrable $(\mathbb{P}, \mathbb{G})$-martingale such that $\mathbb{E}_{\mathbb{P}}[Z_t] = 1$ for all $0 \leq t \leq \tau$. Thus, statement (i) results from Proposition 1.5.1 of [78] and the integral representation theorem for local martingales [69, Theorem 3.5, p. 201].

Statement (ii) is a direct consequence of the Girsanov theorem, see e.g. [53, Theorem 5.1, p. 191].

Statement (iii) is proved by noting that by (ii), the discounted process $(\mathrm{e}^{-rt}X_t)$ can be expanded as, for $0 \leq t \leq \tau$,

$$\mathrm{d}(e^{-rt}X_t) = e^{-rt}\left(-rX_t + k(\theta - X_t) + \sigma\sqrt{X_t}\gamma_t\right)\,\mathrm{d}t + e^{-rt}\sigma\sqrt{X_t}\,\mathrm{d}\widetilde{W}_t.$$

By the definition of $\widetilde{\mathbb{P}} \sim \mathbb{P}$, the discounted process $(\mathrm{e}^{-rt}X_t)$ is a $(\widetilde{\mathbb{P}}_\tau, \mathbb{G})$-martingale; and it is also continuous and nonnegative. So its drift must vanish $\mathbb{P}$-a.s.., implying that for all $0 \leq t \leq \tau$,

$$-rX_t + k(\theta - X_t) + \sigma\sqrt{X_t}\gamma_t = 0, \quad \mathbb{P}\text{-a.s..}$$

The proof is thus complete. ☐

The following lemma will be needed to prove Case (III) of the main theorem. Let $\mathbb{N}_{++} := \mathbb{N} \setminus \{0\}$.

Lemma 3.6 ([62, Lemma 4.7 and its corollary p. 111]). *Let $f = f(t, \omega), t > 0$, be an adapted process relative to $(\Omega, \mathcal{F}, \mathbb{G}, \mathbb{P})$. Let $(\tau_n)_{n\in\mathbb{N}_{++}}$ be a nondecreasing sequence of Markov times with $\tau = \lim_{n\uparrow\infty} \tau_n$ and such that for each $n \in \mathbb{N}_{++}$,*

$$\mathbb{P}\left[\int_0^{\tau_n} f^2(s,\omega)\ \mathrm{d}s < \infty\right] = 1.$$

Then

$$\mathbb{P}\left[A \cap \left\{\omega : \sup_{n\in\mathbb{N}_{++}} \left|\int_0^{\tau_n} f(s,\omega)\ \mathrm{d}W_s\right| = \infty\right\}\right] = 0,$$

where

$$A = \left\{\omega : \int_0^{\tau} f^2(s,\omega)\ \mathrm{d}s < \infty\right\}.$$

3.4 Proof of the main theorem of the chapter

We are now ready to present the proof of Theorem 3.3.

Proof. **(I):** Suppose that there exists a probability measure $\widetilde{\mathbb{P}}$ such that $\widetilde{\mathbb{P}} \sim \mathbb{P}$. Then by Lemma 3.5, there exists a $\mathbb{G}$-adapted process (γ_t) such that (3.2)—(3.4) are satisfied. In particular, since by (P1) of Lemma 3.4, (X_t) does not reach 0, $\mathbb{P}$-a.s.., (3.4) gives for all $0 \le t \le \tau$,

$$\gamma_t = \frac{rX_t - k(\theta - X_t)}{\sigma\sqrt{X_t}}, \qquad \widetilde{\mathbb{P}}\text{-a.s.}$$

Hence, under $\widetilde{\mathbb{P}}$, the transformed CIR process is represented by

$$\begin{aligned} \mathrm{d}X_t &= k(\theta - X_t)\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}W_t \\ &= rX_t\,\mathrm{d}t + \sigma\sqrt{X_t}\,\mathrm{d}\widetilde{W}_t. \end{aligned}$$

By (P3) of Lemma 3.4, (X_t) hits 0 with a positive probability under $\widetilde{\mathbb{P}}$; yet under $\mathbb{P}$ it does not hit 0 a.s. This shows that $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ cannot be equivalent measures.

(II): Let $(\overline{Z}_t)$ be the Doléans exponential given by

$$\overline{Z}_t = \exp\left(\int_0^t \overline{\gamma}_s\ \mathrm{d}W_s - \frac{1}{2}\int_0^t \overline{\gamma}_s^2\ \mathrm{d}s\right), \quad 0 \le t \le 1,$$

where $\overline{\gamma}_t = (r+k)\sqrt{X_t}/\sigma$, $0 \le t \le 1$. Then by Shirakawa [76, Theorem 3.2], $(\overline{Z}_t)$ is a continuous, uniformly integrable $(\mathbb{P},\mathbb{F})$-martingale, i.e., $\mathbb{E}[\overline{Z}_1] = 1$. As τ is an $\mathbb{F}$-stopping time and $Z_t = \overline{Z}_{t\wedge\tau}$, (Z_t) is a $(\mathbb{P},\mathbb{F})$- and $(\mathbb{P},\mathbb{G})$-martingale by Doob's optional sampling theorem (see e.g. [69, Theorem 3.2, p. 69; Corollary 3.6, p. 71]). In particular, $\mathbb{E}[Z_\tau] = 1$. Let $\widetilde{\mathbb{P}}[A] = \mathbb{P}[Z_\tau \mathbb{1}_A]$, $A \in \mathcal{F}_\tau$. Then $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent measures on $\mathcal{F}_\tau$.

Further, by appealing to the Girsanov theorem (see e.g. [53, Theorem 5.1, p. 191]), as in the proof of Lemma 3.5, it can be shown that the discounted stock price $(\mathrm{e}^{-rt}X_t)$ is a $(\widetilde{\mathbb{P}},\mathbb{G})$-martingale. Hence $\widetilde{\mathbb{P}}$ is an EMM with respect to (X_t).

(III): That in this case no EMM exists will be proved by contradiction. Suppose that an EMM $\widetilde{\mathbb{P}}$ exists. Then by Lemma 3.5, there exists a $\mathbb{G}$-adapted process (γ_t)

such that (3.2) and (3.4) are satisfied. Since $X_t > 0$ for each $t < \tau$, (3.4) gives, on the random time interval $[0, \tau)$,

$$\gamma_t = \frac{rX_t - k(\theta - X_t)}{\sigma\sqrt{X_t}}, \quad \widetilde{\mathbb{P}}\text{-a.s. and } \mathbb{P}\text{-a.s.}$$

By (3.2), this implies

$$\mathbb{P}\left[\int_0^\tau \frac{1}{X_s}\, \mathrm{d}s < \infty\right] = 1. \tag{3.5}$$

Let

$$A := \{\omega :\ \tau_0(\omega) < 1\} \cap \left\{\omega :\ \int_0^\tau \frac{1}{X_s}\, \mathrm{d}s < \infty\right\}.$$

By (P3) of Lemma 3.4, the condition $\sigma^2 > 2k\theta > 0$ guarantees that $\mathbb{P}[\tau_0 < 1] > 0$; consequently we have $\mathbb{P}[A] > 0$.

Put $\tau_n(\omega) := \inf\{0 \le t \le 1 :\ X_t(\omega) = 1/n\}$. Then $(\tau_n)_{n\in\mathbb{N}_{++}}$ is a nondecreasing sequence of Markov times such that $\tau = \lim_{n\uparrow\infty} \tau_n$. Clearly, $\ln X_{\tau_n} \downarrow -\infty$ as $n \uparrow \infty$ on A; and we have

$$\mathbb{P}\left[A \cap \left\{\omega : \lim_{n\uparrow\infty} \ln X_{\tau_n} \neq -\infty\right\}\right] = 0. \tag{3.6}$$

On the other hand, by the Itô formula, on the set A and for each $n \in \mathbb{N}_{++}$,

$$\ln X_{\tau_n} = \ln x - k\tau_n + \left(k\theta - \frac{\sigma^2}{2}\right) \int_0^{\tau_n} \frac{1}{X_s}\, \mathrm{d}s + \sigma \int_0^{\tau_n} \frac{1}{\sqrt{X_s}}\, \mathrm{d}W_s. \tag{3.7}$$

As (X_t) is nonnegative and τ_n is nondecreasing and $\tau_n \uparrow \tau$, (3.5) gives,

$$\mathbb{P}\left[\int_0^{\tau_n} \frac{1}{X_s}\, \mathrm{d}s < \infty\right] = 1, \qquad \text{for each } n \in \mathbb{N}_{++}.$$

Since the conditions of Lemma 3.6 are satisfied, we have

$$\mathbb{P}\left[A \cap \left\{\omega :\ \sup_{n\in\mathbb{N}_{++}} \left|\int_0^{\tau_n} \frac{1}{\sqrt{X_s}}\, \mathrm{d}W_s\right| = \infty\right\}\right] = 0. \tag{3.8}$$

And by (3.5), (3.7), and (3.8), we must also have

$$\mathbb{P}\left[A \cap \left\{\omega :\ \lim_{n\uparrow\infty} \ln X_{\tau_n} = -\infty\right\}\right] = 0. \tag{3.9}$$

However, (3.6) and (3.9) together imply that $\mathbb{P}[A] = 0$, which is a contradiction to $\mathbb{P}[A] > 0$. Hence $\widetilde{\mathbb{P}}$ cannot be equivalent to $\mathbb{P}$, and so an EMM does not exist in this case.

(IV): In this case (X_t) is a squared Bessel process with negative dimension $\delta = 4k\theta/\sigma^2 < 0$ under $\mathbb{P}$, meaning that 0 is reached in finite time and $\mathbb{P}[\tau_0 < 1] > 0$ [35, pp. 329-330]. Hence by the same arguments used in Case (III) above, we conclude that an EMM does not exist in this case either.

And this completes the proof. □

The following corollary follows immediately from the proofs of Cases (III) and (IV):

Corollary 3.7. *For the CIR process (X_t) defined in (3.1), if $\sigma^2 > 2k\theta > 0$ or $k\theta < 0$, then*

$$\mathbb{P}\left[\int_0^\tau \frac{1}{X_t}\,\mathrm{d}t = \infty\right] > 0.$$

Remark 3.8. The proof of Case (I) was inspired by Delbaen and Shirakawa [19, Theorem 4.2]. In Case (II), when $k < 0$ and $\theta = 0$, the CIR model (3.1) becomes a special case of a generalized CEV model, which is proved to have an EMM by Delbaen and Shirakawa [19, Equations (1.1) and (1.2) and Theorem 2.3].

Chapter 4

Preliminary results for the asymptotics

In this chapter, we will return to our main task of implied volatility asymptotics and prepare the ground for the proofs of the main results of this thesis, i.e. the theorems listed in Chapter 2. We will modify the results of Berestycki et al. [4, Equations (15), (16)], to show that under some further regularity assumptions on the local volatility ν, the implied volatility ϕ can belong to $C^{2,1}((0,\infty)\times[0,T])$ and it satisfies a similar time degenerate quasilinear parabolic equations.

This chapter is organised as follows. In Sections 4.1 and 4.2, we present some preliminary results about call option prices under the local volatility models and the convexity of the call price $C(s,\tau)$ in s. In Section 4.3, we prove the existence and uniqueness of the implied volatility. In Section 4.4, we derive the PDE for the implied volatility.

4.1 Properties of the stock price under the local model

Recall that under the local volatility model of this thesis, the stock price (S_t) is assumed to be governed by the stochastic differential equation

$$\mathrm{d}S_t = \nu(S_t)S_t\,\mathrm{d}W_t, \quad S_0 > 0, \quad 0 \le t \le T < \infty, \tag{4.1}$$

where (W_t) is a standard Wiener process with respect to a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ denoting the filtration generated by (S_t). Recall also that the local volatility $\nu(\cdot)$ is always assumed to satisfy (A_0).

Under these conditions, it is implicitly assumed that (4.1) has a unique (strong) solution. That this is indeed true is proved in the following Lemma.

Lemma 4.1. *Let* (A_0) *hold. Then (4.1) has a unique strong solution. Moreover,* (S_t) *is a continuous Markov process with transition density, a diffusion, a* $\mathbb{P}$*-martingale, and* $\mathbb{P}[0 < S_t < \infty,\ 0 \le t \le T < \infty \mid S_0 > 0] = 1$.

Proof. A definition of a strong solution can be found in e.g. Karatzas and Shreve [53, p. 285]. By (A_0), the coefficient $\nu(s)s$ is locally Lipschitz and grows linearly in s. Hence (4.1) admits a unique strong solution (S_t); see e.g., Friedman [32, Theorem 2.2, p. 104]. The same local Lipschitz and linear growth properties of $\nu(s)s$ also ensure that (S_t) is a continuous Markov process and in fact a diffusion; see e.g. Friedman [32, Theorem 4.2, p. 115]. By Friedman [32, Theorem 5.4, p. 149], (S_t) has a (differentiable) transition density.

It can be checked by applying Itô's lemma that the solution of (4.1) can be written as

$$S_t = S_0 \exp\left(\int_0^t \nu(S_r)\ \mathrm{d}W_r - \frac{1}{2}\int_0^t \nu^2(S_r)\ \mathrm{d}r\right). \tag{4.2}$$

Since $\nu(\cdot)$ is bounded, the Novikov condition is satisfied. So (S_t) is a $\mathbb{P}$-martingale; see e.g. Karatzas and Shreve [53, Corollary 5.13, p. 199]. From (4.2) we can infer that $\mathbb{P}[0 < S_t < \infty,\ 0 \le t \le T < \infty \mid S_0 > 0] = 1$. However, this can also be directly verified by using the scale function and speed measure to show that 0 and ∞ are natural boundaries. In other words, $\mathbb{P}[\tau_0 < \infty \,|S_0 > 0] = \mathbb{P}[\tau_\infty < \infty \,|S_0 > 0] = 0$, where τ_0 and τ_∞ denote respectively the first hitting time of 0 and ∞ by (S_t), i.e., $\tau_z := \inf\{t \in [0,\infty] :\ S_t = z\}$, $z \in [0,\infty]$, $\inf \emptyset = \infty$. See e.g. Karatzas and Shreve [53, p. 342 ff] or Karlin and Taylor [54, Chapter 15, p. 235]. $\square$

We are now ready to present some PDE and convexity results for call option prices under the local volatility model.

4.2 PDE and convexity results for the call option price

Since the risk-free interest rate is zero, trivially the discounted stock price (S_t) is again (S_t). By Lemma 4.1, (S_t) is a strictly positive exponential $\mathbb{P}$-martingale and $\mathbb{P}$ is the martingale measure of the local volatility model, since (A_0) is always assumed to hold. Hence, the local volatility model, with the fulfillment of (A_0), is complete, meaning that any $\mathbb{P}$-integrable contingent claim that is bounded from below is attainable. See e.g. Definition 10.2.1 and Proposition 10.2.1 of Musiela and Rutkowski [67].

As a result, under the local volatility model (and the martingale measure $\mathbb{P}$), the time t price of a call option option with stock price s, strike k, and expiry T is given by

$$c(s,t) \equiv c(s,t;k,T) = \mathbb{E}_{s,t}[(S_T - k)_+] = \mathbb{E}[(S_T - k)_+ | S_t = s], \tag{4.3}$$

where $(s,t;k,T) \in (0,\infty) \times [0,T] \times (0,\infty) \times (0,\infty)$. See e.g. Musiela and Rutkowski [67, Proposition 10.1.2].

Equivalently, using the time to expiry $\tau = T - t$, and the Markov property of (S_t), the arbitrage price of a European call option $C(s,\tau;k)$, with stock price s, strike k and time to expiry τ, is given by

$$C(s,\tau) \equiv C(s,\tau;k) = \mathbb{E}_s[(S_\tau - k)_+] = \mathbb{E}[(S_\tau - k)_+ | S_0 = s], \tag{4.4}$$

where $(s,\tau;k) \in (0,\infty) \times [0,T] \times (0,\infty)$. This gives the identity

$$C(s,\tau;k) = c(s, T-\tau; k, T). \tag{4.5}$$

By Lemma 4.1, (S_t) is a diffusion process and admits a transition density. This implies that

$$\begin{aligned} C(s,\tau;k) &= \int_{\mathbb{R}} (y-k)_+ p(s,0;y,\tau)\ \mathrm{d}y \\ &= \int_k^\infty (y-k) p(s,0;y,\tau)\ \mathrm{d}y, \end{aligned} \tag{4.6}$$

where $p(s,0;y,\tau)$ denotes the transition density of $(S_\tau^{s,0})$. Differentiating C with respect to k gives the following well known results of Breeden and Litzenberger [10]:

$$C_k(s,\tau;k) = -\int_k^\infty p(s,0;y,\tau)\ \mathrm{d}y, \quad C_{kk}(s,\tau;k) = p(s,0;k,\tau), \tag{4.7}$$

where $(s,t;k) \in (0,\infty) \times (0,T] \times (0,\infty)$.

On the other hand, if $k \in (0,\infty)$ is treated as a parameter, then by the reverse Feynman–Kac result of Janson and Tysk [51, Theorem 5.5], $c(s,t)$ uniquely solves the backward Kolmogorove equation

$$\begin{cases} c_t + \dfrac{1}{2}\nu^2(s) s^2 c_{ss} = 0, & \text{in} \quad \mathbb{R} \times [0,T), \\ \qquad c(s,T) = (s-k)_+, & s \in (0,\infty). \end{cases} \tag{4.8}$$

By identity (4.5), $C(s,\tau;k)$ uniquely solves the Cauchy problem

$$\begin{cases} C_\tau - \frac{1}{2}\nu^2(s)s^2 C_{ss} = 0, & \text{in } \mathbb{R}\times(0,T], \\ C(s,0) = (s-k)_+, & s\in(0,\infty). \end{cases} \tag{4.9}$$

In this thesis, derivatives valued at boundary points, e.g. $C_\tau(s,T;k)$ at $\tau = T$, are one sided derivatives. We now summarize the above PDE results for the call option price C in the following lemma, whose proof is given by the discussion above.

Lemma 4.2. *Let (4.1) and* (A_0) *hold. Then*

(i) the call price c is given by (4.3) and it solves (4.8);

(ii) the call price $C(s,\tau;k)$ is given by (4.4), it belongs to $C^{2,1,2}((0,\infty)\times[0,T)\times(0,\infty))$, and satisfies (4.6), (4.7), and (4.9).

Remark 4.3. The standard Feynman–Kac formula, see e.g. Friedman [32, Equations (5.22), Theorem 5.3, p. 148], does not seem to justify the stochastic representation formula (4.3) for the solution of (4.8). While there are equivalence results for martingales and PDEs, see e.g. Heath and Schweizer [46], it is more convenient to apply the reverse Feynman–Kac result of Janson and Tysk [51, Theorem 5.5].

Apart from satisfying the PDE (4.8), the call option price c is convex in the stock price s. That is, $c_{ss}(s,t)\ge 0$, for $(s,t)\in(0,\infty)\times[0,T]$. By (4.8), this convexity property is equivalent to the time decaying property of the call option price, as t tends to T, i.e., $c_t(s,t)\le 0$ for $(s,t)\in(0,\infty)\times[0,T]$. By (4.5) and (4.9), the convexity of $c(s,t)$ in s implies the convexity of $C(s,\tau)$ in s, namely, $C_{ss}(s,\tau)\ge 0$ and $C_\tau(s,\tau)\ge 0$, for $(s,\tau)\in(0,\infty)\times(0,T]$. These convexity results are stated in the following lemma.

Lemma 4.4 (Convexity of call prices)**.** *Let (4.1) and* (A_0) *hold. Then $c_t(s,t)\le 0$, $(s,t)\in(0,\infty)\times[0,T]$, and $C_\tau(s,\tau)\ge 0$, $(s,\tau)\in(0,\infty)\times[0,T]$.*

Proof. By (A_0), $\nu(s)s$ is locally Hölder continuous in $s\in(0,\infty)$ with exponent $1/2$. That is, for every $N>0$, there exists a $\mathrm{const}_N>0$ such that

$$|\nu(s_1)s_1-\nu(s_2)s_2| \le \mathrm{const}_N\,|s_1-s_2|^{1/2} \quad \forall\, s_1,s_2\le N.$$

Hence by Theorem 4 of Janson and Tysk [49], $c(s,t_1)\ge c(s,t_2)$ for all $s\in(0,\infty)$ and $0\le t_1\le t_2\le T$. This shows $c_t(s,t)\le 0$ for all $(s,t)\in(0,\infty)\times[0,T]$. Noting that $\tau = T-t$ and $C(s,\tau)=c(s,t)$, we also have $C_\tau(s,\tau)\ge 0$, $(s,\tau)\in(0,\infty)\times[0,T]$. ☐

Remark 4.5. The convexity properties of option prices have been investigated by many authors, including Bergman et al. [6], El Karoui et al. [55], Janson and Tysk [49, 50], and Lions and Musiela [61]. For uniformity of presentation, the result of Janson and Tysk [49] seems to fit our theorem best; it even allows (S_t) to hit and to be absorbed by the lower boundary point $\{0\}$. However, it is possible to prove the convexity property of c and C under weaker conditions.

Having shown the well definedness of the stock price process (S_t) and the PDE and convexity properties of the call option prices, we are ready to present the existence, uniqueness, and PDE results for the implied volatility.

Since we have presented the main results of this thesis in the (s, τ) coordinates, we will from now on use $C(s, \tau; k)$ rather than $c(s, t; k, T)$ to prove the existence, uniqueness, and PDE results for the implied volatility.

4.3 Implied volatility: existence and uniqueness

We will show the existence and uniqueness of the implied volatility.

Let us recall that, for a constant ϕ, the Black–Scholes pricing functional in the $(s, \tau; k)$ coordinates is given by

$$\mathcal{B}(s, \tau; k; \phi) = sN(\mathfrak{d}_1) - kN(\mathfrak{d}_2), \tag{4.10}$$

where

$$\begin{cases} N(\mathfrak{d}) = \displaystyle\int_{-\infty}^{\mathfrak{d}} n(z)\ \mathrm{d}z, \qquad n(z) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}, \\ \mathfrak{d}_1(s, \tau; k; \phi) = \dfrac{\ln(s/k) + \phi^2\tau/2}{\phi\sqrt{\tau}}, \\ \mathfrak{d}_2(s, t; k; \phi) = \mathfrak{d}_1 - \phi\sqrt{\tau}. \end{cases} \tag{4.11}$$

Bearing in mind the Black–Scholes functional, we have the following existence and uniqueness result.

Proposition 4.6 (Existence and uniqueness of the implied volatility)**.** *Let (4.1) and* (A_0) *hold. Then there exists a unique function* $\phi(s, \tau; k) : (0, \infty) \times (0, T] \times (0, \infty) \to (0, \infty)$ *such that it belongs to* $C^{2,1,2}((0, \infty) \times (0, T] \times (0, \infty))$ *and satisfies*

$$C(s, \tau; k) = \mathcal{B}(s, \tau; k; \phi(s, \tau; k)), \quad (s, \tau; k) \in (0, \infty) \times (0, T] \times (0, \infty), \tag{4.12}$$

and

$$0 < \frac{1}{\nu_0} \leq \phi(s, \tau; k) \leq \nu_0 < \infty, \quad (s, \tau; k) \in (0, \infty) \times (0, T] \times (0, \infty). \tag{4.13}$$

Proof. We will use the implicit function theorem to prove the existence and uniqueness of the implied volatility. Define $\digamma : (0, \infty) \times (0, T] \times (0, \infty)^2 \to \mathbb{R}$ by

$$\digamma(s, \tau, k; \psi) = \mathcal{B}(s, \tau; k; \psi) - C(s, \tau; k),$$

where $\mathcal{B}$ is the Black–Scholes price functional defined in (1.2). The Jacobian determinant $|J(\digamma)|$ is given by

$$|J(\digamma)| = |\digamma_\psi| = \mathcal{B}_\psi(s, \tau; k; \psi) = s\sqrt{\tau} n\Big(\mathfrak{d}_1(s, \tau; k; \psi)\Big) > 0, \tag{4.14}$$

in $(0, \infty) \times (0, T] \times (0, \infty)^2$. By definition, $\mathcal{B}$ is infinitely differentiable in all of its arguments in $(0, \infty) \times (0, T] \times (0, \infty)^2$. By Lemma 4.2, $C \in C^{2,1,2}((0, \infty) \times (0, T] \times (0, \infty))$. Hence, by the implicit function theorem, there exists a unique function

$$\phi : (0, \infty) \times (0, T] \times (0, \infty) \to (0, \infty)$$

and $\phi \in C^{2,1,2}((0, \infty) \times (0, T] \times (0, \infty))$ such that (4.12) holds.

To show (4.13) and that ϕ maps $(0, \infty) \times (0, T] \times (0, \infty)$ to $(1/\nu_0, \nu_0)$, rather than $(0, \infty)$, we put

$$\begin{aligned} \mathrm{d}\overline{S}_t &= \overline{\nu}\,\overline{S}_t \,\mathrm{d}W_t, \quad \overline{S}_0 = S_0 > 0, \quad 0 \leq t \leq T < \infty, \\ \mathrm{d}\underline{S}_t &= \underline{\nu}\,\underline{S}_t \,\mathrm{d}W_t, \quad \underline{S}_0 = S_0 > 0, \quad 0 \leq t \leq T < \infty, \end{aligned}$$

where, by (A_0), $\underline{\nu} = 1/\nu_0$ and $\overline{\nu} = \nu_0$ are distinct positive constants that $0 < \underline{\nu} \leq \nu(s) \leq \overline{\nu} < \infty$ for all $s \in (0, \infty)$. Now define the upper and lower call prices by

$$\begin{aligned} \overline{C}(s, \tau; k) &= \mathbb{E}_s[(\overline{S}_\tau - k)_+], \\ \underline{C}(s, \tau; k) &= \mathbb{E}_s[(\underline{S}_\tau - k)_+], \end{aligned}$$

where $(s, \tau; k) \in (0, \infty) \times [0, T] \times (0, \infty)$. By the same argument leading to Lemma 4.2, $(\overline{S}_\tau)$ and $(\underline{S}_\tau)$ are $\mathbb{P}$-martingales and the upper and lower call prices are well defined. Indeed, these upper and lower call prices are none other than the Black–Scholes prices with respectively the constant upper and lower volatilities $\overline{\nu}$ and $\underline{\nu}$, meaning that

$$\overline{C}(s,\tau;k) \equiv \mathcal{B}(s,\tau;k;\overline{\nu}),$$
$$\underline{C}(s,\tau;k) \equiv \mathcal{B}(s,\tau;k;\underline{\nu}).$$

Since the function $(s)_+ := \max(s,0)$ is convex, the mean stochastic comparison theorem of Hajek [42, Theorem 3 or Theorem 4.1] implies that

$$\underline{C}(s,\tau;k) \leq C(s,\tau;k) \leq \overline{C}(s,\tau;k),$$

for all $(s,\tau;k) \in (0,\infty) \times [0,T] \times (0,\infty)$. Since we have proved that for $(s,\tau;k) \in (0,\infty) \times (0,T] \times (0,\infty)$, $C(s,\tau;k) = \mathcal{B}(s,\tau;k;\phi(s,\tau;k))$, these call price inequalities imply that

$$\mathcal{B}(s,\tau;k;\underline{\nu}) \leq \mathcal{B}(s,\tau;k;\phi(s,\tau;k)) \leq \mathcal{B}(s,\tau;k;\overline{\nu})$$

for all $(s,\tau;k) \in (0,\infty) \times (0,T] \times (0,\infty)$. By (4.14), $\mathcal{B}(s,\tau;k;\psi)$ is monotonically increasing in ψ, other things being equal. From this (4.13) follows. The proof is now complete. □

Remark 4.7. We shall continue to write $\phi : (0,\infty) \times (0,T] \times (0,\infty) \to (0,\infty)$, to signify that the interval $(\underline{\nu},\overline{\nu}) \subset (0,\infty)$ can be made arbitrarily large.

It is important to note that the implicit function theorem does not guarantee the existence of the implied volatility $\phi(s,\tau;k)$ at $\tau = 0$, as the Jacobian determinant $|J(F)| = 0$ at $\tau = 0$.

Since the appearance of the Black–Scholes formula in 1973, implied volatility problems have attracted a great deal of interest from both practitioners and academics. Yet, it was only in 2002 that implied volatilities under a class of local volatility models were proved to have limits at expiry in the Sobolev spaces $W^{2,1,p}_{\mathrm{loc}}((0,\infty)^2)$, $1 < p < \infty$. See (1.17) for the definition of the Sobolev space and Berestycki et al. [4, Theorem 1] for details of the result.

In contrast to the result of Berestycki et al. [4], our implied volatility ϕ belongs to $C^{2,1,2}((0,\infty) \times (0,T] \times (0,\infty))$. This result comes at the cost of stronger regularity assumptions on the local volatility. Nevertheless, we also know more about the properties of the implied volatility, such as its first order Taylor expansion and gradient and Hessian asymptotics. In the rest of this and the next chapter, we will show that the Sobolev space results of Berestycki et al. [4, Theorem 1] can be carried over to $C^{2,1,2}((0,\infty) \times (0,T] \times (0,\infty))$, that is, ϕ solves a certain parabolic PDE and that

$\lim_{\tau\to 0}\phi(s,\tau;k)$ exists. The gradient and Hessian asymptotics are presented in Chapters 7 and 8.

4.4 PDE for the implied volatility

To derive the PDE for the implied volatility ϕ, we need the following basic identities for the Black–Scholes price functional $\mathcal{B}$.

Lemma 4.8. *For the Black–Scholes call option pricing function $\mathcal{B}(s,\tau;k;\psi)$ with ψ being a positive parameter, the following identities are valid:*

$$\frac{\partial\mathfrak{d}_1}{\partial\psi} = -\frac{\mathfrak{d}_2}{\psi}, \tag{4.15}$$

$$\frac{\partial\mathfrak{d}_2}{\partial\psi} = -\frac{\mathfrak{d}_1}{\psi}, \tag{4.16}$$

$$\mathcal{B}_\tau = \frac{\psi}{2\sqrt{\tau}}sn(\mathfrak{d}_1), \tag{4.17}$$

$$\mathcal{B}_s = N(\mathfrak{d}_1), \tag{4.18}$$

$$\mathcal{B}_{ss} = \frac{1}{s\psi\sqrt{\tau}}n(\mathfrak{d}_1), \tag{4.19}$$

$$\mathcal{B}_\psi = sn(\mathfrak{d}_1)\sqrt{\tau}, \tag{4.20}$$

$$\mathcal{B}_{\psi\psi} = \frac{s\sqrt{\tau}\mathfrak{d}_1\mathfrak{d}_2}{\psi}n(\mathfrak{d}_1), \tag{4.21}$$

$$\mathcal{B}_{s\psi} = n(\mathfrak{d}_1)\left[-\frac{\ln(s/k)}{\psi^2\sqrt{\tau}} + \frac{\sqrt{\tau}}{2}\right]. \tag{4.22}$$

Proof. Straightforward differentiation would yield these identities. □

Having these identities at our disposal, deriving the PDE for the implied volatility ϕ will involve little effort. Under uniform boundedness and continuity assumptions on the local volatility, Berestycki et al. [4, Equation (15)] derived the following PDE in some transformed coordinates for the implied volatility in $W^{2,1,p}_{\mathrm{loc}}((0,\infty)^2)$, $1 < p \le \infty$. Here, our PDE holds for $\phi \in C^{2,1}((0,\infty)\times(0,T])$.

Theorem 4.9. *Let (4.1) and* (A_0) *hold. Then the implied volatility ϕ satisfies the time degenerate quasilinear parabolic equation*

$$\phi^2 + 2\tau\phi\phi_\tau - \nu^2(s)\left[\left(1 - \frac{s[\ln(s/k)]\phi_s}{\phi}\right)^2 - \left(1 - \frac{s\tau\phi\phi_s}{2}\right)^2 + 1 + s^2\tau\phi\phi_{ss}\right] = 0 \tag{4.23}$$

in $(0,\infty)\times(0,T]$.

Proof. Since $C(s,\tau) = \mathcal{B}(s,\tau;\phi(s,\tau))$, by the chain rule we have

$$\begin{aligned} C_\tau &= \mathcal{B}_\tau + \mathcal{B}_\phi \phi_\tau, \\ C_s &= \mathcal{B}_s + \mathcal{B}_\phi \phi_s, \\ C_{ss} &= \mathcal{B}_{ss} + 2\mathcal{B}_{s\phi}\phi_s + \mathcal{B}_{\phi\phi}\phi_s^2 + \mathcal{B}_\phi \phi_{ss}. \end{aligned}$$

Substituting C_τ and C_{ss} into (4.9) and using the identities (4.15)–(4.22), we get

$$\begin{aligned} &\frac{\phi}{2\sqrt{\tau}} sn(\mathfrak{d}_1) + sn(\mathfrak{d}_1)\sqrt{\tau}\phi_\tau \\ &\quad - \frac{1}{2}\nu^2(s)s^2\left[\frac{1}{s\phi\sqrt{\tau}} n(\mathfrak{d}_1) - 2\frac{\mathfrak{d}_2}{\phi} n(\mathfrak{d}_1)\phi_s + \frac{s\sqrt{\tau}\mathfrak{d}_1\mathfrak{d}_2}{\phi} n(\mathfrak{d}_1)\phi_s^2 + sn(\mathfrak{d}_1)\sqrt{\tau}\phi_{ss}\right] = 0 \end{aligned}$$

in $\mathbb{R} \times (0,T]$. Multiplying $2\phi\sqrt{\tau}/[sn(\mathfrak{d}_1)]$ to both sides of the equation gives

$$\phi^2 + 2\tau\phi\phi_\tau - \nu^2(s)\left(1 - 2s\sqrt{\tau}\mathfrak{d}_2\phi_s + s^2\tau\mathfrak{d}_1\mathfrak{d}_2\phi_s^2 + s^2\tau\phi\phi_{ss}\right) = 0.$$

Grouping the terms inside the parentheses then gives (4.23), and the proof is complete. $\square$

Remark 4.10. Observe that in (4.23) if $\phi(s,0) := \lim_{\tau\to 0}\phi(s,\tau)$ exists, and if $\tau\phi\phi_\tau$, $\tau\phi\phi_s$, and $\tau\phi\phi_{ss}$ all tend to 0 as $\tau \to 0$, then as $\tau \to 0$, (4.23) becomes

$$\phi^2(s,0) - \nu^2(s)\left(1 - \frac{s[\ln(s/k)]\phi_s(s,0)}{\phi(s,0)}\right)^2 = 0. \tag{4.24}$$

Because a solution to (4.24) is given by the function

$$s \mapsto [\ln(s/k)]\left(\int_k^s \frac{\mathrm{d}z}{z\nu(z)}\right)^{-1}, \tag{4.25}$$

we conjecture that for all $s \in (0,\infty)$,

$$\phi(s,0) \equiv \lim_{\tau\to 0}\phi(s,\tau) = [\ln(s/k)]\left(\int_k^s \frac{\mathrm{d}z}{z\nu(z)}\right)^{-1}. \tag{4.26}$$

In the next chapter we will prove that (4.26) is indeed true.

Chapter 5

Implied volatility at expiry: zero order expansion

In this chapter we will show that the implied volatility admits a zero order Taylor expansion near expiry and hence at expiry the implied volatility $\phi(s,\tau)$ exists as a limit, uniformly in $s \in (0,\infty)$. Berestycki et al. [4, Theorem 1] have shown the same results for $\phi \in W^{2,1,p}_{\text{loc}}((0,\infty)^2)$. By adapting their argument we demonstrate that their asymptotic results remain valid for $\phi \in C^{2,1}((0,\infty)\times(0,T])$, provided that the local volatility is sufficiently smooth.

This chapter is organized as follows. Section 5.1 states the main results of this chapter. Section 5.2 explains the idea behind the proofs of the main results. Sections 5.3–5.8 detail how we are going to prove the main results in another coordinate system. Section 5.9 presents the proof of the zero order Taylor expansion first for the transformed implied volatility and then for the implied volatility. Section 5.10 summarizes the result for the transformed implied volatility in a PDE and limit theorem. This theorem will be used in the subsequent chapters.

Note that Goldys and Roper [36] have proved the small time limit in (5.4) under some weaker assumptions and for more general models. However, they have not dealt with convergence rates as in (5.1) below.

5.1 Main result of the chapter

Theorem 5.1. *Let (2.1),* (A_0)–(A_2) *hold. Then uniformly in* $s \in (0,\infty)$*, the implied volatility* ϕ *satisfies*

$$\phi(s,\tau) = \phi^0(s) + \mathrm{O}(\tau), \qquad as \qquad \tau \to 0, \tag{5.1}$$

where $\mathrm{O} = \mathrm{O}(\mathcal{V}_2)$ *and*

$$\phi^0(s) = [\ln(s/k)] \left(\int_k^s \frac{\mathrm{d}z}{z\nu(z)} \right)^{-1}. \tag{5.2}$$

We will prove this theorem in Section 5.9. Note that (5.1) indicates that as τ tends to zero, $\phi(s,\tau)$ converges to $\phi^0(s)$, uniformly in $s \in (0,\infty)$. Note also that once proven, Theorem 5.1, together with Theorem 4.9, implies the following result, whose proof we will omit:

Theorem 5.2. *Let (2.1),* (A_0)–(A_2) *hold. Then for each* $k \in (0,\infty)$*, the implied volatility* $\phi(s,\tau;k)$ *belongs to* $C^{2,1}((0,\infty)\times(0,T])$ *and satisfies the time degenerate quasilinear parabolic equation*

$$\phi^2 + 2\tau\phi\phi_\tau - \nu^2(s)\left[\left(1 - \frac{s[\ln(s/k)]\phi_s}{\phi}\right)^2 - \left(1 - \frac{s\tau\phi\phi_s}{2}\right)^2 + 1 + s^2\tau\phi\phi_{ss}\right] = 0, \tag{5.3}$$

in $(0,\infty)\times(0,T]$*, with the initial condition*

$$\phi(s,0) := \lim_{\tau\to 0}\phi(s,\tau) = \phi^0(s), \quad s \in (0,\infty). \tag{5.4}$$

Remark 5.3. After (5.1) is proved, we can extend the implied volatility $\phi(s,\tau)$ from $(0,\infty)\times(0,T]$ to $(0,\infty)\times[0,T]$. The extended implied volatility $\tilde{\phi}$ is defined as $\tilde{\phi}(s,\tau) = \phi(s,\tau)$ for $(s,\tau) \in (0,\infty)\times(0,T]$, and $\tilde{\phi}(s,0) = \phi^0(s)$. With some abuse of notation, we will write ϕ instead of $\tilde{\phi}$ in the rest of this thesis.

5.2 Idea of the proof

We will prove Theorems 5.1 as follows:

Firstly, we change the variable s to x by setting $x = \ln(s/k)$. In the (x,τ) coordinates, the *transformed implied volatility* φ is given by $\varphi(x,\tau) = \phi(s,\tau)$. Secondly, we prove that there exist upper and lower functions $\overline{\varphi}$ and $\underline{\varphi}$ such that for some small enough T,

$$\underline{\varphi}(x,\tau) \le \varphi(x,\tau) \le \overline{\varphi}(x,\tau), \quad (x,\tau) \in \mathbb{R}\times[0,T], \tag{5.5}$$

where, for some positive constant λ,

$$\begin{aligned} \overline{\varphi}(x,\tau) &= I(x)(1+\lambda\tau), \\ \underline{\varphi}(x,\tau) &= I(x)(1-\lambda\tau), \end{aligned} \tag{5.6}$$

with the *initial function* $I(x)$ given by

$$I(x) := x\left(\int_0^x \frac{\mathrm{d}z}{\sigma(z)}\right)^{-1}, \qquad x \in \mathbb{R}. \tag{5.7}$$

The upper and lower solutions $\overline{\varphi}$ and $\underline{\varphi}$ will force φ to be

$$\varphi(x,\tau) = I(x) + \mathrm{O}(\tau), \quad \forall\, x \in \mathbb{R}. \tag{5.8}$$

Thirdly, Theorem 5.1 is proved by simply transforming (5.8) back to the the (s,τ) coordinates and using the fact that as $x = \ln(s/k)$,

$$\phi^0(s) = I(x(s)). \tag{5.9}$$

That (5.9) holds can be checked by substitution and the details are omitted.

5.3 Change of variables: $(s,\tau) \to (x,\tau)$

As explained above, instead of working with the original variables (s,τ), we will work with the reduced variables (x,τ), where for each $k \in (0,\infty)$,

$$x = \ln(s/k), \quad s \in (0,\infty). \tag{5.10}$$

Note that in finance x is called the *log moneyness*. A call option is respectively in, at, and out of the money when respectively $x > 0$, $x = 0$, and $x < 0$. Let

$$\sigma(x) = \nu(k\mathrm{e}^x). \tag{5.11}$$

Then (A_0)–(A_4) respectively imply that

$(\bar{\mathrm{A}}_0)$ the local volatility $\sigma(\cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ and there exists a strictly positive constant ν_0 such that

$$0 < \underline{\nu} \equiv \frac{1}{\nu_0} \le \sigma(x) \le \nu_0 \equiv \overline{\nu} < \infty, \quad \forall\, x \in \mathbb{R}; \tag{5.12}$$

$(\bar{\mathrm{A}}_1)$ the first derivative $\sigma_x(\cdot)$ exists and is locally Lipschitz continuous in $\mathbb{R}$ and

$$\|\sigma_x\|_0 := \sup_{x\in\mathbb{R}} |\sigma_x(x)| \le \mathrm{const}(\mathcal{V}_1) < \infty; \tag{5.13}$$

($\bar{\mathrm{A}}_2$) the second derivative $\sigma_{xx}(\cdot)$ exists and is continuous in $\mathbb{R}$ and

$$\|\sigma_{xx}\|_0 \leq \mathrm{const}(\mathcal{V}_2) < \infty; \tag{5.14}$$

($\bar{\mathrm{A}}_3$) the third derivative $\sigma_{xxx}(\cdot)$ exists and is continuous in $\mathbb{R}$ and

$$\|\sigma_{xxx}\|_0 \leq \mathrm{const}(\mathcal{V}_3) < \infty; \tag{5.15}$$

($\bar{\mathrm{A}}_4$) the forth derivative $\sigma_{xxxx}(\cdot)$ exists and is continuous in $\mathbb{R}$ and

$$\|\sigma_{xxxx}\|_0 \leq \mathrm{const}(\mathcal{V}_4) < \infty. \tag{5.16}$$

Remark 5.4 (Warning)**.** In the rest of this thesis, when we state a result in the (x, τ) coordinates but impose (A_i), $i = 0, 1, 2, 3, 4$, as the assumptions, the reader should note that the ($\bar{\mathrm{A}}_i$)'s, as consequences of the (A_i)'s, would actually be used in the proof.

In the (x, τ) coordinates, we define the *transformed call option price* $v(x, \tau)$ and the *transformed Black–Scholes price* $B(x, \tau; \psi)$ by

$$v(x, \tau) := C(s, \tau; k)/k, \tag{5.17}$$

$$B(x, \tau; \psi) := \mathcal{B}(s, \tau; k; \psi)/k = \mathrm{e}^x N(d_1) - N(d_2), \tag{5.18}$$

where, as a parameter, ψ is any positive constant, and

$$\begin{cases} N(d) = \displaystyle\int_{-\infty}^{d} n(z)\,\mathrm{d}z, & n(z) = \dfrac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}, \\ d_1(x, \tau; \psi) = \dfrac{x}{\sqrt{\tau}\psi} + \dfrac{\sqrt{\tau}\psi}{2}, & d_2(x, \tau; \psi) = \dfrac{x}{\sqrt{\tau}\psi} - \dfrac{\sqrt{\tau}\psi}{2} = d_1 - \sqrt{\tau}\psi. \end{cases} \tag{5.19}$$

Moreover, in (x, τ), we defined the *transformed implied volatility* $\varphi(x, \tau)$ by

$$\varphi(x, \tau) := \phi(s(x), \tau; k). \tag{5.20}$$

The function φ is so named because it is precisely the volatility implied by the transformed call option price v, in the sense that

$$v(x, \tau) = B(x, \tau; \varphi(x, \tau)). \tag{5.21}$$

We will prove this equivalence relation between φ and ϕ in the following lemma.

Lemma 5.5 (Equivalence lemma)**.** *Let (2.1),* (A_0) *hold. Then* *(i) the transform implied volatility* φ *satisfies*

$$v(x,\tau) = B(x,\tau;\varphi(x,\tau)), \quad (x,\tau) \in \mathbb{R} \times (0,T], \tag{5.22}$$

if and only if the implied volatility ϕ *satisfies*

$$C(s,\tau;k) = \mathcal{B}(s,\tau;k;\phi(s,\tau;k)), \quad (x,\tau,k) \in (0,\infty) \times (0,T] \times (0,\infty); \tag{5.23}$$

(ii) φ *satisfies*

$$\varphi(x,\tau) = I(x) + \mathrm{O}(\tau) \quad \text{as} \quad \tau \to 0, \tag{5.24}$$

with $\mathrm{O} = \mathrm{O}(\mathcal{V}_2)$ *and*

$$I(x) = x\left(\int_0^x \frac{\mathrm{d}z}{\sigma(z)}\right)^{-1}, \tag{5.25}$$

if and only if ϕ *satisfies the conclusion of Theorem 5.1, i.e.*

$$\phi(s,\tau) = \phi^0(s) + \mathrm{O}(\tau) \quad \text{as} \quad \tau \to 0, \tag{5.26}$$

with $\mathrm{O} = \mathrm{O}(\mathcal{V}_2)$ *and*

$$\phi^0(s) = [\ln(s/k)]\left(\int_k^s \frac{\mathrm{d}z}{z\nu(z)}\right)^{-1}, \quad s \in (0,\infty). \tag{5.27}$$

Proof. Since (2.1), (A_0) hold, φ and ϕ are well defined. As a result, part (i) and (ii) follow from the change of variables $x = \ln(s/k)$, the definitions (5.17)–(5.20), and $C(s,\tau;k) = \mathcal{B}(s,\tau;k;\phi(s,\tau;k))$. □

This equivalence lemma shows that to prove Theorem 5.1, it is enough to prove

Theorem 5.6 (Zero order Taylor expansion)**.** *Let (2.1),* (A_0)–(A_2) *hold. Then*

$$\varphi(x,\tau) = I(x) + \mathrm{O}(\tau) \quad \text{as} \quad \tau \to 0, \tag{5.28}$$

where $\mathrm{O} = \mathrm{O}(\mathcal{V}_2)$*.*

The rest of this chapter is devoted to the proof of this theorem. We now list some basic results in the (x,τ) coordinates.

5.4 PDE for transformed call price and implied volatility

We have the following PDE result for the (transformed) call option price v.

Lemma 5.7. *Let (2.1),* (A_0) *hold. Then*

$$\begin{cases} v_\tau = \dfrac{1}{2}\sigma^2(x)(v_{xx} - v_x), \quad (x,\tau) \in \mathbb{R} \times (0,T], \\ v(x,0) = (\mathrm{e}^x - 1)_+, \quad x \in \mathbb{R}, \end{cases} \tag{5.29}$$

Further,

$$v_\tau(x,\tau) \geq 0 \quad \textit{for all} \quad (x,\tau) \in \mathbb{R} \times [0,T]. \tag{5.30}$$

Proof. Under the assumptions of this lemma, $C(s,\tau;k)$ satisfies PDE (4.9). Since by definition $v(x,\tau) = C(s,\tau;k)/k$, differentiating v and substituting the derivatives into (4.9) gives (5.29). By Lemma 4.4, $C_\tau(s,\tau) \geq 0$ for all $(s,\tau) \in (0,\infty) \times [0,T]$. This gives (5.30), the time decaying property of $v(x,\tau)$ in τ. The proof is thus complete. □

The following identities are needed to derive the PDE for the transformed implied volatility φ.

Lemma 5.8. *For the transformed Black–Scholes price $B(x,\tau;\psi)$ with ψ being a parameter, the following identities are valid:*

$$\mathrm{e}^x n(d_1) = n(d_2), \tag{5.31}$$

$$\frac{\partial d_1}{\partial \psi} = -\frac{x}{\sqrt{\tau}\psi^2} + \frac{\sqrt{\tau}}{2}, \tag{5.32}$$

$$\frac{\partial d_2}{\partial \psi} = -\frac{x}{\sqrt{\tau}\psi^2} - \frac{\sqrt{\tau}}{2}, \tag{5.33}$$

and

$$B_\tau = \frac{\psi}{2\sqrt{\tau}} n(d_2), \tag{5.34}$$

$$B_x = \mathrm{e}^x N(d_1), \tag{5.35}$$

$$B_{xx} = \mathrm{e}^x N(d_1) + \frac{n(d_2)}{\sqrt{\tau}\psi}, \tag{5.36}$$

$$B_\psi = \sqrt{\tau} n(d_2), \tag{5.37}$$

$$B_{\psi\psi} = d_2 \left[\frac{x}{\psi^2} + \frac{\tau}{2}\right] n(d_2), \tag{5.38}$$

$$B_{x\psi} = \mathrm{e}^x n(d_1) \left[-\frac{x}{\sqrt{\tau}\psi^2} + \frac{\sqrt{\tau}}{2}\right]. \tag{5.39}$$

Proof. An exercise of differentiation. □

In the (x,τ) coordinates, Theorem 4.9 is translated into the following result:

Lemma 5.9 (PDE for the transformed implied volatility). *Let (2.1),* (A_0) *hold. Then for each* $k \in (0,\infty)$*, the transformed implied volatility* $\varphi(x,\tau)$ *belongs to* $C^{2,1}((0,\infty)\times(0,T])$ *and satisfies the time degenerate quasilinear parabolic equation*

$$2\tau\varphi\varphi_\tau + \varphi^2 - \sigma^2(x)\left(1 - x\frac{\varphi_x}{\varphi}\right)^2 - \sigma^2(x)\tau\varphi\varphi_{xx} + \frac{1}{4}\sigma^2(x)\tau^2\varphi^2\varphi_x^2 = 0 \quad \text{in } \mathbb{R}\times(0,T]. \tag{5.40}$$

Proof. Under the assumptions (2.1) and (A_0), $v(x,\tau)$ solves (5.29). By definition, $\varphi(x,\tau) = \phi(s(x),\tau)$. Since $\phi \in C^{2,1}((0,\infty)\times(0,T])$, $\varphi \in C^{2,1}(\mathbb{R}\times(0,T])$. By (5.22),

$$v(x,\tau) = B(x,\tau;\varphi(x,\tau)), \tag{5.41}$$

so the chain rule gives

$$\begin{aligned} v_\tau &= B_\tau + B_\varphi\varphi_\tau, \\ v_x &= B_x + B_\varphi\varphi_x, \\ v_{xx} &= B_{xx} + 2B_{x\varphi}\varphi_x + B_{\varphi\varphi}\varphi_x^2 + B_\varphi\varphi_{xx}. \end{aligned} \tag{5.42}$$

By substituting v_τ, v_x, and v_{xx} into (5.29) and applying the identities (5.31)–(5.39), we get

$$\begin{aligned} 0 &= v_\tau - \frac{1}{2}\sigma^2(x)(v_{xx} - v_x) \\ &= (B_\tau + B_\varphi\varphi_\tau) - \frac{1}{2}\sigma^2(x)(B_{xx} + 2B_{x\varphi}\varphi_x + B_{\varphi\varphi}\varphi_x^2 + B_\varphi\varphi_{xx} - B_x - B_\varphi\varphi_x) \\ &= \frac{n(d_2)}{2\sqrt{\tau}\varphi}\left[2\tau\varphi\varphi_\tau + \varphi^2 - \sigma^2(x)\left(1 - x\frac{\varphi_x}{\varphi}\right)^2 - \sigma^2(x)\tau\varphi\varphi_{xx} + \frac{1}{4}\sigma^2(x)\tau^2\varphi^2\varphi_x^2\right] \end{aligned} \tag{5.43}$$

for all $(x,\tau) \in \mathbb{R}\times(0,T]$. Here $d_2 = d_2(x,\tau;\varphi)$. By (4.13), $0 < \underline{\nu} \le \phi(s,\tau;k) \le \overline{\nu} < \infty$ for all $(s,\tau;k) \in (0,\infty)\times(0,T]\times(0,\infty)$. By (5.20), $\varphi(x,\tau) = \phi(s,\tau;k)$. Hence $n(d_2)/(2\sqrt{\tau}\varphi)$ is bounded and strictly positive in $\mathbb{R}\times(0,T]$. Consequently the terms inside the square brackets in (5.43) must be zero in $\mathbb{R}\times(0,T]$. The proof is thus complete. □

Corollary 5.10. *Let* $\psi \in C^{2,1}(\mathbb{R}\times(0,T])$ *Put* $u(x,\tau) = B(x,\tau;\psi(x,\tau))$*. Then the* *following identity holds:*

$$\begin{aligned}
&u_\tau - \frac{1}{2}\sigma^2(x)(u_{xx} - u_x) \\
&= \frac{n(d_2)}{2\sqrt{\tau}\psi}\left[2\tau\psi\psi_\tau + \psi^2 - \sigma^2(x)\left(1 - x\frac{\psi_x}{\psi}\right)^2 - \sigma^2(x)\tau\psi\psi_{xx} + \frac{1}{4}\sigma^2(x)\tau^2\psi^2\psi_x^2\right]
\end{aligned} \tag{5.44}$$

in $\mathbb{R} \times (0, T]$.

Proof. The identity follows from (5.41)–(5.43). □

Remark 5.11. Note that (5.40) holds if and only if

$$\sigma(x) = \left(\frac{\varphi^2 + 2\tau\varphi\varphi_\tau}{\left(1 - x\frac{\varphi_x}{\varphi}\right)^2 + \tau\varphi\varphi_{xx} - \frac{1}{4}\tau^2\varphi^2\varphi_x^2}\right)^{1/2}, \quad (x,\tau) \in \mathbb{R} \times (0, T]. \tag{5.45}$$

This will be used in Section 5.6 below to define the associated local volatility functional $\Sigma[\cdot]$, see (5.54).

Remark 5.12. As explained in Remark 4.10, if φ, φ_τ, φ_x, and φ_{xx} are sufficiently regular, then as $\tau \to 0$, (5.40) would give the initial Ordinary Differential Equation (**ODE**)

$$\varphi^2(x,0) - \sigma^2(x)\left(1 - x\frac{\varphi_x(x,0)}{\varphi(x,0)}\right)^2 = 0, \quad x \in \mathbb{R}. \tag{5.46}$$

It can be checked that $I(x)$, defined by (5.7), solves (5.46). This motivates the conjecture that $\varphi(x,0) = I(x)$ and $\varphi(x,\tau) = I(x)(1 + \mathrm{O}(\tau))$.

5.5 Properties of the initial function

In what follows we will often suppress the arguments of the functions when presenting results. Let us rewrite the initial function $I(x)$ as

$$I(x) = \frac{x}{J(x)}, \quad J(x) := \int_0^x \frac{\mathrm{d}z}{\sigma(z)}. \tag{5.47}$$

Then as mentioned earlier in Remark 5.12 (c.f. Remark 4.10), $I(x)$ solves ODE (5.46), namely,

$$I^2 = \sigma^2(x)\left(1 - x\frac{I_x}{I}\right)^2. \tag{5.48}$$

Moreover, with the arguments suppressed we have the following identities:

$$\begin{aligned}
I &= \frac{x}{J}, \\
I_x &= \frac{1}{J}\left(1 - \frac{x}{\sigma J}\right) = \frac{I}{x} - \frac{I^2}{x\sigma}, \\
I_{xx} &= -\frac{2}{J^2\sigma} + \frac{2x}{J^3\sigma^2} + \frac{x\sigma_x}{J^2\sigma^2} = -\frac{2I^2}{x^2\sigma} + \frac{2I^3}{x^2\sigma^2} + \frac{\sigma_x I^2}{x\sigma^2}, \\
I_{xxx} &= \frac{6}{J^3\sigma^2} + \frac{3\sigma_x}{J^2\sigma^2} - \frac{6x}{J^4\sigma^3} - \frac{6x\sigma_x}{J^3\sigma^3} - \frac{2x\sigma_x^2}{J^2\sigma^3} + \frac{x\sigma_{xx}}{J^2\sigma^2} \\
&= \frac{6I^3}{x^3\sigma^2} + \frac{3I^2\sigma_x}{x^2\sigma^2} - \frac{6I^4}{x^3\sigma^3} - \frac{6I^3\sigma_x}{x^2\sigma^3} - \frac{2I^2\sigma_x^2}{x\sigma^3} + \frac{I^2\sigma_{xx}}{x\sigma^2}.
\end{aligned} \tag{5.49}$$

Further,

$$1 - x\frac{I_x}{I} = \frac{I}{\sigma}, \quad x \in \mathbb{R}. \tag{5.50}$$

Also, by the L'Hospital rule,

$$\begin{aligned}
I(0) &= \sigma(0), \\
I_x(0) &= \frac{1}{2}\sigma_x(0), \\
I_{xx}(0) &= \frac{2\sigma_{xx}(0)\sigma(0) - \sigma_x^2(0)}{6\sigma(0)}, \\
I_{xxx}(0) &= \frac{\sigma_{xxx}(0)\sigma^2(0) - 2\sigma_x(0)\sigma_{xx}(0)\sigma(0) + \sigma_x^3(0)}{4\sigma^2(0)}.
\end{aligned} \tag{5.51}$$

Keeping in mind that condition $(\mathrm{A}_i) \Rightarrow (\bar{\mathrm{A}}_i)$, $i = 0, \dots 4$, we get the following lemma.

Lemma 5.13. *If* (A_i), $i = 0, \dots, 4$, *holds, then*

$$\|I\|_i \le \mathrm{const}(\mathcal{V}_i), \quad i = 0, \dots, 4. \tag{5.52}$$

In particular, (A_0) *implies that*

$$0 < \underline{\nu} = \frac{1}{\nu_0} \le I(x) \le \nu_0 = \overline{\nu} < \infty, \quad \forall\, x \in \mathbb{R}. \tag{5.53}$$

Proof. The bounds follow from (5.49) and (5.51). □

5.6 Associated local volatilities and their Taylor series

To show existence of the upper and lower functions $\overline{\varphi}$ and $\underline{\varphi}$, we will define and use certain associated volatilities. Here we will closely follow Berestycki et al. [4]. For

$\psi \in C^{2,1}(\mathbb{R} \times (0,T])$, we define a local volatility functional $\Sigma[\cdot]$ by

$$\Sigma[\psi](x,\tau) = \left(\frac{G[\psi]}{H[\psi]}\right)^{1/2}, \tag{5.54}$$

where

$$\begin{aligned} G[\psi] &= (\tau\psi^2)_\tau = \psi^2 + 2\tau\psi\psi_\tau, \\ H[\psi] &= \left(1 - x\frac{\psi_x}{\psi}\right)^2 + \tau\psi\psi_{xx} - \frac{1}{4}\tau^2\psi^2\psi_x^2. \end{aligned} \tag{5.55}$$

Further, we let $\mathcal{I}[0,T]$ to be the class of those functions ψ in $C^{2,1}(\mathbb{R}\times(0,\infty))$ such that $\Sigma[\psi](x,\tau)$ is well defined, continuous in $\mathbb{R}\times[0,T]$, and on this region satisfies

$$0 < \mathrm{const}_1 \le \Sigma[\psi](x,\tau) \le \mathrm{const}_2 < \infty, \tag{5.56}$$

and

$$\lim_{\tau\to 0} \tau\psi^2(x,\tau) = 0 \quad \text{uniformly in } x \in \mathbb{R}. \tag{5.57}$$

Definition 5.14 (Associated local volatility)**.** If $\psi \in \mathcal{I}[0,T]$ for some $T > 0$, then $\Sigma[\psi](x,\tau)$ is called an associated local volatility, associated with the (transformed) local volatility $\sigma(x)$.

Note that $\mathcal{I}[0,T]$ is not empty. Any positive real constant will be an element of $\mathcal{I}[0,T]$. Note also that by Remark 5.11, if $\psi \in \mathcal{I}[0,T]$, then it satisfies

$$2\tau\psi\psi_\tau + \psi^2 - \Sigma^2[\psi](x,\tau)\left(1 - x\frac{\psi_x}{\psi}\right)^2 - \Sigma^2[\psi](x,\tau)\tau\psi\psi_{xx} + \frac{1}{4}\Sigma^2[\psi](x,\tau)\tau^2\psi^2\psi_x^2 = 0 \tag{5.58}$$

in $\mathbb{R}\times(0,T]$. To see this, replace $\sigma(x)$ with $\Sigma[\psi](x,\tau)$ in (5.45). Moreover, we have the following lemma.

Lemma 5.15. *Let $\psi \in \mathcal{I}[0,T]$ and $u(x,\tau) = B(x,\tau;\psi(x,\tau))$. Then*

$$\begin{cases} u_\tau = \frac{1}{2}\Sigma^2[\psi](x,\tau)(u_{xx} - u_x) & \textit{in} \quad \mathbb{R}\times(0,T], \\ u(x,0) = (\mathrm{e}^x - 1)_+, & x \in \mathbb{R}. \end{cases} \tag{5.59}$$

Proof. Since $\psi \in \mathcal{I}[0,T]$ and $u(x,\tau) = B(x,\tau;\psi(x,\tau))$, by (5.44)

$$\begin{aligned} u_\tau - \frac{1}{2}\Sigma^2[\psi](x,\tau)(u_{xx} - u_x) \\ = \frac{n(d_2)}{2\sqrt{\tau}\psi} \Bigg[2\tau\psi\psi_\tau + \psi^2 - \Sigma^2[\psi](x,\tau)\left(1 - x\frac{\psi_x}{\psi}\right)^2 \\ -\Sigma^2[\psi](x,\tau)\tau\psi\psi_{xx} + \frac{1}{4}\Sigma^2[\psi](x,\tau)\tau^2\psi^2\psi_x^2 \Bigg] \end{aligned}$$

in $\mathbb{R} \times (0,T]$. Then (5.59) follows from the definition of $\Sigma[\cdot]$ and (5.58). The initial condition follows from the initial condition of $B(x,\tau;\psi(x,\tau))$ and (5.56). The proof is therefore complete. □

Let

$$\psi(x,\tau) = I(x)(1+\lambda\tau), \quad (x,\tau) \in \mathbb{R} \times [0,T], \tag{5.60}$$

with λ being some arbitrarily fixed real constant. Assume for the moment that $\sigma(\cdot)$ is sufficiently bounded and smooth. (In Proposition 5.19 below we will specify some sufficient conditions on σ.) Then ψ belongs to $\mathcal{I}[0,T]$, and a first order Taylor expansion of $\Sigma[\psi](x,\tau)$ about $\tau = 0$ is given by

$$\Sigma[\psi](x,\tau) = \Sigma[\psi](x,0) + \tau\Sigma_\tau[\psi](x,0) + R_1[\psi](x,\tau), \tag{5.61}$$

where the remainder

$$R_1[\psi](x,\tau) = \int_0^\tau \{\Sigma_{\eta\eta}[\psi](x,\eta)\}\,(\tau-\eta)\;\mathrm{d}\eta. \tag{5.62}$$

By definition,

$$\Sigma_\tau[\psi](x,\tau) = \frac{1}{2\Sigma[\psi]H^2[\psi]}\left(G_\tau[\psi]H[\psi] - G[\psi]H_\tau[\psi]\right). \tag{5.63}$$

Suppressing the functional input ψ and the arguments x and τ, this becomes

$$\Sigma_\tau[\psi](x,\tau) = \frac{1}{2\Sigma H^2}(G_\tau H - GH_\tau). \tag{5.64}$$

Further,

$$\begin{aligned} \Sigma_{\tau\tau}[\psi](x,\tau) &= \frac{1}{2\Sigma^2 H^4}\left[(G_\tau H - GH_\tau)_\tau \Sigma H^2 - (G_\tau H - GH_\tau)(\Sigma H^2)_\tau\right], \\ &= \frac{1}{2\Sigma^2 H^4}F, \end{aligned} \tag{5.65}$$

where

$$F[\psi](x,\tau) := (G_{\tau\tau}H - GH_{\tau\tau})\Sigma H^2 - (G_\tau H - GH_\tau)(\Sigma_\tau H^2 + 2\Sigma H H_\tau). \tag{5.66}$$

Although $\tau \in [0,T]$, the above Taylor expansion about $\tau = 0$ can be justified by extending the time domain to $[-T,T]$ and setting $\tilde{\Sigma}[\psi](x,-\tau) = \Sigma[\psi](x,\tau)$ for all $\tau \geq 0$. In that case, the standard Taylor's theorem can be applied in an interval $(-T_1,T_1) \subset [-T,T]$, subject to the regularity of $\Sigma[\psi]$.

Now let us calculate the various derivatives of G and H and express them in terms of ψ. For G, we have

$$\begin{aligned} G[\psi] &= \psi^2 + 2\tau\psi\psi_\tau, \\ G_\tau[\psi] &= 4\psi\psi_\tau + 2\tau\psi_\tau^2 + 2\tau\psi\psi_{\tau\tau}, \\ G_{\tau\tau}[\psi] &= 6\psi_\tau^2 + 6\psi\psi_{\tau\tau} + 6\tau\psi_\tau\psi_{\tau\tau} + 2\tau\psi\psi_{\tau\tau\tau}. \end{aligned} \tag{5.67}$$

For H, we have

$$\begin{aligned} H[\psi] &= \left(1 - x\frac{\psi_x}{\psi}\right)^2 + \tau\psi\psi_{xx} - \frac{1}{4}\tau^2\psi^2\psi_x^2, \\ H_\tau[\psi] &= 2\left(1 - \frac{x\psi_x}{\psi}\right)\left(-\frac{x\psi_{x\tau}}{\psi} + \frac{x\psi_x\psi_\tau}{\psi^2}\right) \\ &\quad + \psi\psi_{xx} + \tau\psi_\tau\psi_{xx} + \tau\psi\psi_{xx\tau} - \frac{1}{2}\tau\psi^2\psi_x^2 - \frac{1}{2}\tau^2\psi\psi_x^2\psi_\tau - \frac{1}{2}\tau^2\psi^2\psi_x\psi_{x\tau}, \end{aligned} \tag{5.68}$$

and

$$\begin{aligned} &H_{\tau\tau}[\psi] \\ &= 2\left(-\frac{x\psi_{x\tau}}{\psi} + \frac{x\psi_x\psi_\tau}{\psi^2}\right)^2 \\ &\quad + 2\left(1 - \frac{x\psi_x}{\psi}\right)\left(-\frac{x\psi_{x\tau\tau}}{\psi} + \frac{2x\psi_{x\tau}\psi_\tau}{\psi^2} - \frac{2x\psi_x\psi_\tau^2}{\psi^3} + \frac{x\psi_x\psi_{\tau\tau}}{\psi^2}\right) \\ &\quad + 2\psi_\tau\psi_{xx} + 2\psi\psi_{xx\tau} + \tau\psi_{\tau\tau}\psi_{xx} + 2\tau\psi_\tau\psi_{xx\tau} + \tau\psi\psi_{xx\tau\tau} - \frac{1}{2}\psi^2\psi_x^2 - 2\tau\psi\psi_x^2\psi_\tau \\ &\quad - 2\tau\psi^2\psi_x\psi_{x\tau} - \frac{1}{2}\tau^2\psi_\tau^2\psi_x^2 - 2\tau^2\psi\psi_x\psi_\tau\psi_{x\tau} - \frac{1}{2}\tau^2\psi\psi_x^2\psi_{\tau\tau} - \frac{1}{2}\tau^2\psi^2\psi_{x\tau}^2 - \frac{1}{2}\tau^2\psi^2\psi_x\psi_{x\tau\tau}. \end{aligned} \tag{5.69}$$

Lemma 5.16. *Let* (A_0)–(A_2) *hold. Let* ψ *be as in (5.60). Then*

$$\begin{aligned}
\psi &= I(1+\lambda\tau),\\
\psi_x &= I_x(1+\lambda\tau),\\
\psi_{xx} &= I_{xx}(1+\lambda\tau),\\
\psi_\tau &= \lambda I,\\
\psi_{\tau\tau} &\equiv 0,\\
\psi_{\tau\tau\tau} &\equiv 0,\\
\psi_{x\tau} &= \lambda I_x,\\
\psi_{xx\tau} &= \lambda I_{xx},\\
\psi_{x\tau\tau} &\equiv 0,\\
\psi_{xx\tau\tau} &\equiv 0.
\end{aligned}$$

Moreover, as $\tau \to 0$*,*

$$\begin{aligned}
\psi &\to I,\\
\psi_x &\to I_x,\\
\psi_{xx} &\to I_{xx},\\
\psi_\tau &\to \lambda I,\\
\psi_{\tau\tau} &\equiv 0,\\
\psi_{\tau\tau\tau} &\equiv 0,\\
\psi_{x\tau} &\to \lambda I_x,\\
\psi_{xx\tau} &\to \lambda I_{xx},\\
\psi_{x\tau\tau} &\equiv 0,\\
\psi_{xx\tau\tau} &\equiv 0,
\end{aligned}$$

where the convergence is uniform in $x \in \mathbb{R}$*.*

Proof. By Lemma 5.13, I, I_x, and I_{xx} are well defined and bounded. Differentiating ψ and taking the limits gives the results. □

The lemma above leads to the following corollaries.

Corollary 5.17. *Let* (A_0)–(A_2) *hold. Let* $\lambda \in \mathbb{R}$ *be fixed and* $\psi(x,\tau) = I(x)(1+\lambda\tau)$*,* *for* $(x,\tau) \in \mathbb{R} \times [0,T]$. *Then uniformly in* $x \in \mathbb{R}$, *as* $\tau \to 0$,

$$\begin{aligned} G[\psi] &\to I^2, \\ G_\tau[\psi] &\to 4\lambda I^2, \\ G_{\tau\tau}[\psi] &\to 6\lambda^2 I^2, \end{aligned} \tag{5.70}$$

and

$$\begin{aligned} H[\psi] &\to \frac{I^2}{\sigma^2}, \\ H_\tau[\psi] &\to -II_{xx}, \\ H_{\tau\tau}[\psi] &\to 4\lambda II_{xx} - \frac{1}{2}I^2 I_{xx}^2, \end{aligned} \tag{5.71}$$

and

$$\begin{aligned} \Sigma[\psi] &\to \sigma, \\ \Sigma_\tau[\psi] &\to \sigma\left(\lambda\frac{2I^2}{\sigma^2} + \frac{II_{xx}}{2}\right). \end{aligned} \tag{5.72}$$

Proof. These uniform limits follow from the definitions of $G[\cdot]$, $H[\cdot]$, and $\Sigma[\cdot]$, and Lemma 5.16. $\square$

Recall that by (1.16),

$$\|\Sigma_{\tau\tau}[\psi]\|_0 = \sup_{(x,\tau)\in\mathbb{R}\times[0,T]} |\Sigma_{\tau\tau}[\psi](x,\tau)|.$$

Then we have the following corollary.

Corollary 5.18. *Let* (A_0)–(A_2) *hold. Let* $\lambda \in \mathbb{R}$ *be fixed and* $\psi(x,\tau) = I(x)(1+\lambda\tau)$, *for* $(x,\tau) \in \mathbb{R} \times [0,T]$. *Then the following statements are true:*

(i) There exist positive constants $T = T(\lambda, \mathcal{V}_2)$ *and* $\mathrm{const}_{1,2} = \mathrm{const}_{1,2}(T,\lambda,\mathcal{V}_2)$ *such that*

$$0 < \mathrm{const}_1 \le \psi(x,\tau), G[\psi](x,\tau), H[\psi](x,\tau), \Sigma[\psi](x,\tau) \le \mathrm{const}_2 < \infty,$$

for all $(x,\tau) \in [0,T]$.

(ii) $\psi \in \mathcal{I}[0,T]$ *for some positive* $T = T(\lambda, \mathcal{V}_2)$.

(iii) $T\,\|\Sigma_{\tau\tau}[\psi]\|_0 \to 0$ *as* $T \to 0$.

Proof. By Lemma 5.16, $\psi(x,\tau) \to I(x)$ as $\tau \to 0$, uniformly in $x \in \mathbb{R}$. Since $I(x)$ is strictly positive and bounded, ψ is the same. By (5.55), Lemma 5.13, and Corollary 5.17, we know that for sufficiently small $T = T(\lambda, \mathcal{V}_2)$, $G[\psi]$ and $H[\psi]$ are continuous, strictly positive, and bounded. Hence by (5.54), $\Sigma[\psi]$ is also continuous, strictly positive, and bounded for the same sufficiently small T. This proves statement (i). Statement (ii) then follows from the definition of $\mathcal{I}[0,T]$. Statement (iii) follows from (5.65), Corollary 5.17, and statement (i). The proof is thus complete. □

Proposition 5.19. *Let* (A_0)–(A_2) *hold. Let* $\lambda \in \mathbb{R}$ *be fixed and* $\psi(x,\tau) = I(x)(1+\lambda\tau)$, *for* $(x,\tau) \in \mathbb{R} \times [0,T]$. *Then for all sufficiently small* $T = T(\lambda, \mathcal{V}_2)$, *the associated local volatility* $\Sigma[\psi]$ *admits the first order Taylor expansion*

$$\Sigma[\psi](x,\tau) = \sigma(x) + \tau\sigma(x)\left(\lambda\frac{2I^2(x)}{\sigma^2(x)} + \frac{I(x)I_{xx}(x)}{2}\right) + R_1[\psi](x,\tau) \tag{5.73}$$

in $\mathbb{R} \times [0,T]$, *where* $R_1[\psi]$ *is defined by (5.62) and*

$$R_1[\psi](x,\tau) = \mathrm{O}(\tau^2) \qquad as \qquad \tau \to 0, \tag{5.74}$$

with $\mathrm{O} = \mathrm{O}(T, \lambda, \mathcal{V}_2)$.

Proof. We will prove the order property of $R_1[\psi]$ first. By the definition of ψ, (A_0)–(A_2), and Corollary 5.18, we know that $H[\psi](x,\tau)$ and $\Sigma[\psi](x,\tau)$ are continuous, strictly positive, and bounded in $\mathbb{R}\times[0,T_1]$, provided $T_1 = T_1(\lambda,\mathcal{V}_2)$ is sufficiently small. Similarly, by (5.66) and Corollary 5.18, we know that $F[\psi](x,\tau)$ is continuous and bounded in $\mathbb{R} \times [0,T_2]$, provided $T_2 = T_2(\lambda,\mathcal{V}_2)$ is sufficiently small. Let $T_3 = \min(T_1,T_2)$. Then $\Sigma_{\tau\tau}[\psi](x,\tau)$ is continuous and bounded in $\mathbb{R} \times [0,T_3]$ by (5.65); and so is $R_1[\psi]$ by (5.62). Moreover,

$$\begin{aligned} |R_1[\psi](x,\tau)| &= \left| \int_0^\tau \left\{\Sigma_{\eta\eta}[\psi](x,\eta)\right\}(\tau-\eta)\ \mathrm{d}\eta \right| \\ &\le \left(\sup_{(x,\tau)\in\mathbb{R}\times[0,T_3]} |\Sigma_{\tau\tau}[\psi](x,\tau)| \right) \int_0^\tau |\tau-\eta|\ \mathrm{d}\eta \\ &\le \mathrm{const} \times \tau^2, \end{aligned} \tag{5.75}$$

where $\mathrm{const} = \mathrm{const}(T_3,\lambda,\mathcal{V}_2)$. This shows (5.74). Next, by substituting (5.72) into (5.61) we obtain the Taylor expansion (5.73). The argument above also shows that the Taylor expansion (5.73) holds for all $0 < T \le T_3$, and the proof is complete. □

5.7 Upper and lower functions: local volatility bounds

We now show that the (transformed) local volatility $\sigma(\cdot)$ is bounded by upper and lower solutions $\overline{\varphi}$ and $\underline{\varphi}$ of the following form:

$$\begin{aligned}\overline{\varphi}(x,\tau) &= I(x)(1+\lambda\tau),\\ \underline{\varphi}(x,\tau) &= I(x)(1-\lambda\tau), \quad (x,\tau)\in\mathbb{R}\times[0,T],\end{aligned} \tag{5.76}$$

for some positive constants λ and T.

Lemma 5.20. *Let* (A_0)–(A_2) *hold. Then there exist positive constants* $\lambda = \lambda(\mathcal{V}_2)$ *and* $T = T(\lambda, \mathcal{V}_2)$ *such that*

$$\Sigma[\underline{\varphi}](x,\tau) \le \sigma(x) \le \Sigma[\overline{\varphi}](x,\tau), \quad \forall\ (x,\tau)\in\mathbb{R}\times[0,T], \tag{5.77}$$

where $\overline{\varphi}$ *and* $\underline{\varphi}$ *are defined by (5.76).*

Proof. Apply Proposition 5.19 to $\overline{\varphi}$ and use (5.73) to get

$$\Sigma[\overline{\varphi}](x,\tau) = \sigma(x)\left[1+\tau\left(\lambda\frac{2I^2(x)}{\sigma^2(x)} + \frac{I(x)I_{xx}(x)}{2}\right) + \frac{R_1[\overline{\varphi}](x,\tau)}{\sigma(x)}\right]. \tag{5.78}$$

By (5.12), $0 < 1/\nu_0 \le \sigma(x) \le \nu_0 < \infty$ for all $x\in\mathbb{R}$. By Lemma 5.13,

$$\begin{gathered}0 < 1/\nu_0 \le I(x) \le \nu_0 < \infty, \quad x\in\mathbb{R},\\ \|I_{xx}\|_0 \le \mathrm{const}(\mathcal{V}_2).\end{gathered}$$

Thus for any large enough positive constant $\lambda_1 = \lambda_1(\mathcal{V}_2)$,

$$\tau\left(\lambda_1\frac{2I^2(x)}{\sigma^2(x)} + \frac{I(x)I_{xx}(x)}{2}\right) \ge \tau\nu_0 \ge 0, \quad \forall\ (x,\tau)\in\mathbb{R}\times[0,T], \tag{5.79}$$

where T is arbitrary. Next, since $R_1[\overline{\varphi}] = \mathrm{O}(\tau^2)$, for any small enough positive constant $T_1 = T_1(\lambda_1, \mathcal{V}_2)$,

$$\frac{|R_1[\overline{\varphi}](x,\tau)|}{\sigma(x)} \le \mathrm{const}(T_1,\lambda_1,\mathcal{V}_2)\times\nu_0\tau^2 \le \nu_0\tau, \quad \forall\ (x,\tau)\in\mathbb{R}\times[0,T_1]. \tag{5.80}$$

Hence for such a large λ_1 and small T_1 we can define $\overline{\varphi}_1(x,\tau) = I(x)(1+\lambda_1\tau)$, and by (5.78), (5.79), and (5.80), we have

$$\Sigma[\overline{\varphi}_1](x,\tau) \ge \sigma(x), \quad \forall\ (x,\tau)\in\mathbb{R}\times[0,T_1].$$

Now replace λ with $-\lambda$ in (5.78). Then applying the same analysis shows that for some large enough positive $\lambda_2 = \lambda_2(\mathcal{V}_2)$ and small enough positive $T_2 = T_2(\lambda_2, \mathcal{V}_2)$,

$$\begin{aligned}\Sigma[\underline{\varphi}](x,\tau) &= \sigma(x)\left[1+\tau\left(-\lambda_2\frac{2I^2(x)}{\sigma^2(x)}+\frac{I(x)I_{xx}(x)}{2}\right)+\frac{R_1[\overline{\varphi}](x,\tau)}{\sigma(x)}\right]\\ &\le \sigma(x)\left(1-\tau\nu_0+\tau\nu_0\right)\\ &\le \sigma(x), \quad \forall\ (x,\tau)\in\mathbb{R}\times[0,T_2].\end{aligned} \tag{5.81}$$

By setting first $\lambda = \max(\lambda_1, \lambda_2)$ and then $T = \min(T_1, T_2)$, we have (5.77), and this completes the proof. □

5.8 A comparison principle

In this section we will adapt the idea of Berestycki et al. [4, Lemma 6] to prove a comparison principle. Like Berestycki et al., we will make use of the following well known result on positive solutions of the Cauchy problem, which is a consequence of the maximum principle for second order parabolic PDEs.

A theorem on positive solutions of the Cauchy problem

Let

$$\Omega_0 = \mathbb{R}\times(0,T], \quad \Omega = \mathbb{R}\times[0,T]. \tag{5.82}$$

Put

$$Lu = a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u - u_t = 0, \quad (x,t)\in\Omega_0. \tag{5.83}$$

Assume that in Ω_0 the coefficients $a(x,t)$, $b(x,t)$, and $c(x,t)$ are continuous and

$$|a(x,t)| \le \text{const}, \quad |b(x,t)| \le \text{const}\times(|x|+1), \quad c(x,t)\le \text{const}\times(|x|^2+1). \tag{5.84}$$

Here we call L a parabolic operator in Ω_0 if $a(x,t) > 0$ for all $(x,t)\in\Omega_0$. Then we have the following theorem:

Theorem 5.21 (Friedman [30, Theorem 9, p. 43])**.** *Assume that L is a parabolic operator in Ω_0, $Lu \le 0$ in Ω_0, and that*

$$u(x,t) \ge -\beta_1 \exp(\beta_2 |x|^2) \quad \text{in } \Omega \tag{5.85}$$

for some positve constants β_1, β_2. If $u(x,0)\ge 0$ in $\mathbb{R}$, then $u(x,t)\ge 0$ in Ω.

A comparison principle

Lemma 5.22 (Comparison principle, cf. Berestycki et al. [4, Lemma 6]). *Let* $\underline{\psi}, \overline{\psi} \in \mathcal{I}[0,T]$ *and suppose that*

$$\Sigma[\underline{\psi}](x,\tau) \leq \sigma(x) \leq \Sigma[\overline{\psi}](x,\tau)$$

for all $(x,\tau) \in \mathbb{R} \times [0,T]$. *Then* $\underline{\psi} \leq \varphi \leq \overline{\psi}$ *in* $\mathbb{R} \times (0,T]$.

Proof. We will modify the arguments of Berestycki et al. [4, Lemma 6] and prove the second inequality $\varphi \leq \overline{\psi}$ first. Let

$$\overline{v}(x,\tau) = B(x,\tau;\overline{\psi}).$$

Then by Lemma 5.15, $\overline{v}$ solves

$$\begin{cases} \overline{v}_\tau = \dfrac{1}{2}\Sigma^2[\overline{\psi}](x,\tau)(\overline{v}_{xx} - \overline{v}_x) & \text{in} \quad \mathbb{R} \times (0,T], \\ \overline{v}(x,0) = (\mathrm{e}^x - 1)_+, & x \in \mathbb{R}, \end{cases}$$

By (5.22), $v(x,\tau) = B(x,\tau;\varphi(x,\tau))$; by Lemma 5.7, v solves

$$\begin{cases} v_\tau = \dfrac{1}{2}\sigma^2(x)(v_{xx} - v_x) & \text{in} \quad \mathbb{R} \times (0,T], \\ v(x,0) = (\mathrm{e}^x - 1)_+, & x \in \mathbb{R}, \end{cases}$$

with the time-τ increasing property that $v_\tau(x,\tau) \geq 0$ for all $(x,\tau) \in \mathbb{R} \times [0,T]$. Putting $\overline{w}(x,\tau) = \overline{v}(x,\tau) - v(x,\tau)$ gives

$$\begin{cases} \overline{w}_\tau - \dfrac{1}{2}\Sigma^2[\overline{\psi}](x,\tau)(\overline{w}_{xx} - \overline{w}_x) = \left(\dfrac{\Sigma^2[\overline{\psi}](x,\tau)}{\sigma^2(x)} - 1\right) v_\tau & \text{in} \quad \mathbb{R} \times (0,T], \\ \overline{w}(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

By assumption, $\Sigma^2[\overline{\psi}]/\sigma^2 \geq 1$, and so $(\Sigma^2[\overline{\psi}]/\sigma^2 - 1) \geq 0$. Noting that $v_\tau \geq 0$, we have

$$\begin{cases} \overline{w}_\tau - \dfrac{1}{2}\Sigma^2[\overline{\psi}](x,\tau)(\overline{w}_{xx} - \overline{w}_x) \geq 0 & \text{in} \quad \mathbb{R} \times (0,T], \\ \overline{w}(x,0) = 0, & x \in \mathbb{R}. \end{cases}$$

Further, by the definition of the Black–Scholes functional $B(\cdot,\cdot;\cdot)$, we have

$$\begin{aligned}\overline{w}(x,\tau) &= \overline{v}(x,\tau) - v(x,\tau)\\ &= B(x,\tau;\overline{\psi}(x,\tau)) - B(x,\tau;\varphi(x,\tau))\\ &\geq -\beta_1 \exp\left(\beta_2 \left|x\right|^2\right), \quad \text{in} \quad \mathbb{R}\times[0,T],\end{aligned}$$

for some positive β_1 and β_2. Hence by Theorem 5.21, we have $\overline{w}(x,\tau)\geq 0$ on $\mathbb{R}\times[0,T]$. That implies that $v(x,\tau)\leq\overline{v}(x,\tau)$ in $\mathbb{R}\times[0,T]$, i.e.,

$$B(x,\tau;\varphi(x,\tau)) \leq B(x,\tau;\overline{\psi}(x,\tau)) \qquad \forall\ (x,\tau)\in\mathbb{R}\times[0,T].$$

By (5.37), $B(x,\tau;\phi)$ is strictly increasing in ϕ for $(x,\tau)\in\mathbb{R}\times(0,T]$. This implies that $\varphi\leq\overline{\psi}$ for all $(x,\tau)\in\mathbb{R}\times(0,T]$.

The proof of the first inequality $\underline{\psi}\leq\varphi$ is similar.

Let $\underline{v}(x,\tau)=B(x,\tau;\underline{\psi})$ and $\underline{w}(x,\tau)=v(x,\tau)-\underline{v}(x,\tau)$. By assumption, $\Sigma[\underline{\psi}](x,\tau)\leq\sigma(x)$. This gives

$$\begin{cases} \underline{w}_\tau - \dfrac{1}{2}\Sigma^2[\underline{\psi}](x,\tau)(\underline{w}_{xx}-\underline{w}_x) = \left(1-\dfrac{\Sigma^2[\underline{\psi}](x,\tau)}{\sigma^2(x)}\right)v_\tau \geq 0 \qquad \text{in} \quad \mathbb{R}\times(0,T],\\ \underline{w}(x,0)=0, \qquad x\in\mathbb{R}.\end{cases}$$

Then the inequality $\underline{\psi}\leq\varphi$ follows from Theorem 5.21 and the monotonicity of $B(x,\tau,\phi)$ in ϕ. And the proof is complete. □

Remark 5.23. Our comparison principle is weaker than that of Berestycki et al. [4, Lemma 6]. For example, we do not claim that if $\sigma(x)\leq\Sigma[\psi_1](x,\tau)\leq\Sigma[\psi_2](x,\tau)$ for some $\psi_1,\psi_2\in\mathcal{I}[0,T]$, then $\psi_1\leq\psi_2$.

Remark 5.24. In the proof of the comparison principle, we solely rely on the nonnegativity of v_τ to show that $\overline{w},\underline{w}\geq 0$. If the $\overline{v}_\tau$ and $\underline{v}_\tau$'s are nonnegative in $\mathbb{R}\times(0,T]$, then $\Sigma[\underline{\psi}]\leq\Sigma[\overline{\psi}]$ would imply $\underline{\psi}\leq\overline{\psi}$ for all $\underline{\psi},\overline{\psi}\in\mathcal{I}[0,T]$. However, we did not attempt to prove the nonnegativity of the $\overline{v}_\tau$ and $\underline{v}_\tau$'s. The comparison principle in its current form is sufficient for our purpose, which is to derive a zero order and a first order Taylor expansion for the implied volatility.

5.9 Proof of the zero order Taylor expansion

We are ready to prove Theorem 5.6, the zero order Taylor expansion theorem for the transformed implied volatility $\varphi(x,\tau)$.

Proof of Theorem 5.6. As in (5.76), let $\overline{\varphi}(x,\tau) = I(x)(1+\lambda\tau)$ and $\underline{\varphi}(x,\tau) = I(x)(1-\lambda\tau)$. Then by Lemma 5.20, there exist $\lambda = \lambda(\mathcal{V}_2)$ and $T = T(\lambda, \mathcal{V}_2)$ such that

$$\Sigma[\underline{\varphi}](x,\tau) \le \sigma(x) \le \Sigma[\overline{\varphi}](x,\tau), \quad \forall\ (x,\tau) \in \mathbb{R} \times [0,T].$$

By the comparison principle, Lemma 5.22,

$$\underline{\varphi}(x,\tau) \le \varphi(x,\tau) \le \overline{\varphi}(x,\tau), \quad \forall\ (x,\tau) \in \mathbb{R} \times (0,T].$$

This shows $\varphi(x,\tau) = I(x) + \mathrm{O}(\tau)$. And the proof is complete. □

We can now prove Theorem 5.1.

Proof of Theorem 5.1. This follows from Theorem 5.6 and the equivalence lemma, Lemma 5.5. □

5.10 A PDE and limit theorem for implied volatility

Having shown that $\varphi(x,\tau) \to I(x)$ as $\tau \to 0$, we follow the discussion in Remark 5.3 and define

$$\varphi(x,0) := I(x), \quad x \in \mathbb{R},$$

thus extending $\varphi(x,\tau)$ from $\mathbb{R} \times (0,T]$ to $\mathbb{R} \times [0,T]$. By Lemma 5.9 and Theorem 5.6, we obtain the following PDE and limit theorem for the transformed implied volatility:

Theorem 5.25. *Let (2.1),* (A_0)–(A_2) *hold. Then the (transformed) implied volatility* $\varphi(x,\tau)$ *belongs to* $C^{2,1}((0,\infty) \times (0,T])$ *and satisfies the time degenerate quasilinear parabolic equation*

$$2\tau\varphi\varphi_\tau + \varphi^2 - \sigma^2(x)\left(1 - x\frac{\varphi_x}{\varphi}\right)^2 - \sigma^2(x)\tau\varphi\varphi_{xx} + \frac{1}{4}\sigma^2(x)\tau^2\varphi^2\varphi_x^2 = 0, \tag{5.86}$$

in $\mathbb{R} \times (0,T]$*, with the initial condition* $\varphi(x,0) = I(x)$*. Further, as* $\tau \to 0$*,*

$$\varphi(x,\tau) = I(x) + \mathrm{O}(\tau), \tag{5.87}$$

where $\mathrm{O} = \mathrm{O}(\mathcal{V}_2)$*.*

Chapter 6

Implied volatility: first order expansion

Berestycki et al. [5, p. 1356 and Section 6.3] have pointed out that implied volatilities, under quite general multidimensional stochastic volatility models, can be expanded as Taylor series in time. However, they have not proved their conjecture. In different contexts, Medvedev [65] and Medvedev and Scaillet [66] have derived asymptotic formulas for implied volatilities under the assumption that the implied volatility admits Taylor expansions in both the space and time variables.

In this chapter, by producing a first order Taylor expansion of the implied volatility in time, we partially verify the statement of Berestycki et al. [5, pp. 1356, 1370]. Although we adapt the argument of Berestycki et al [4, Theorem 1], we give sufficient conditions on the local volatility for the first order Taylor expansion of the implied volatility to hold. Such sufficient conditions are not given in Berestycki et al. [5]. Moreover, our proof makes clear where the difficulties may lie if one wishes to adopt the same method to obtain higher order Taylor expansions for the implied volatility.

This chapter is organised as follows. In Section 6.1, we state the main theorems of this chapter. In Section 6.2, we explain the idea of the proofs, which we adopt from Berestycki et al. [4, 5]. In Sections 6.3 and 6.4, we first formally derive and then study the properties of the first order term of the Taylor expansion for the implied volatility. In Section 6.5, we deduce second order Taylor expansions for a class of associated local volatilities $\Sigma[\psi]$ that will be used to bound the local volatility σ. In Section 6.6, we show that there are upper and lower functions bounding the implied volatility. In Section 6.7, we prove the main theorems of this chapter.

6.1 Main result of the chapter

Theorem 6.1 (First order Taylor expansion in time). *Let (2.1),* (A$_0$)–(A$_4$) *hold. Then as* $\tau \to 0$,

$$\phi(s,\tau) = \phi(s,0)\left(1 + \tau \frac{\phi^2(s,0)}{[\ln(s/k)]^2} \ln\left(\frac{\sqrt{\nu(k)\nu(s)}}{\phi(s,0)}\right) + \mathrm{O}(\tau^2)\right), \tag{6.1}$$

with $\mathrm{O} = \mathrm{O}(\mathcal{V}_4)$.

Provided the diffusion coefficient $\nu(\cdot)$ is sufficiently smooth, we conjecture that higher order Taylor expansions in time can be obtained by the same method.

Similar to the proof of Theorem 5.1, instead of proving Theorem 6.1 directly, we will again use the (x,τ) coordinates and prove the desired result for the transformed implied volatility φ. In the (x,τ) coordinates, we will prove the following theorem:

Theorem 6.2. *Let (2.1),* (A$_0$)–(A$_4$) *hold. Then the transformed implied volatility* φ *admits the first order Taylor expansion*

$$\varphi(x,\tau) = I(x)(1 + f(x)\tau + \mathrm{O}(\tau^2)), \tag{6.2}$$

where $\mathrm{O} = \mathrm{O}(\mathcal{V}_4)$,

$$I(x) = \frac{x}{J}, \quad J(x) = \int_0^x \frac{\mathrm{d}z}{\sigma(z)}, \tag{6.3}$$

$$f(x) = \frac{I^2(x)}{x^2} \ln\left(\frac{\sqrt{\sigma(0)\sigma(x)}}{I(x)}\right). \tag{6.4}$$

Further, in the limit as $\tau \to 0$,

$$\varphi_\tau(x,0) := \lim_{\tau\to 0} \varphi_\tau(x,\tau) = I(x)f(x), \quad \textit{uniformly in } x \in \mathbb{R}. \tag{6.5}$$

6.2 Idea of the proof

The method we adopt here is similar to that used to prove the zero order Taylor series in Chapter 5. There, as explained in Remark 5.12, we formally sent τ to zero in the PDE for $\varphi(x,t)$; then we solved the resulting ODE to obtain a formal initial function $\varphi(x,0)$; after that we proved that this formal initial function was in fact the initial function. The proof was accomplished by constructing upper and lower functions that would force the implied volatility φ to converge to the desired formal initial function.

This method for deriving implied volatility asymptotics originates from Berestycki et al. [4].

Here we carry the idea of Berestycki et al. [4] a little further. First we will suppose that the implied volatility φ admits a second order Taylor expansion

$$\varphi(x,\tau) = I(x)(1 + f(x)\tau + g(x)\tau^2 + \mathrm{O}(\tau^3)), \tag{6.6}$$

for some smooth function f and g. Next, we will differentiate with respect to τ the PDE for φ, (5.86), to obtain the PDE for φ_τ. Then we will formally send τ to zero to obtain an ODE for f. Having solved the ODE and found f, we will show that there exist large constants λ and small constants T such that

$$\underline{\varphi}(x,\tau) \leq \varphi(x,\tau) \leq \overline{\varphi}(x,\tau), \quad \forall\ (x,\tau) \in \mathbb{R} \times (0,T], \tag{6.7}$$

where the upper and lower functions $\overline{\varphi}$ and $\underline{\varphi}$ are respectively defined by

$$\begin{aligned} \overline{\varphi}(x,\tau) &= I(x)(1 + f(x)\tau + \lambda\tau^2), \\ \underline{\varphi}(x,\tau) &= I(x)(1 + f(x)\tau - \lambda\tau^2), \qquad (x,\tau) \in \mathbb{R} \times [0,T]. \end{aligned} \tag{6.8}$$

Once (6.7) is proved, Theorem 6.2 follows from it. In principle, we can continue this process to derive the second order term g and so on. However, this conjecture needs a proof, and we will not pursue it in our thesis.

Remark 6.3. It appears that Berestycki et al. [5] are the first authors using this method to derive first order Taylor expansions in time for implied volatilities. In [5, Equations (6.8)–(6.10)], they give a first order expansion for the implied volatility under a two factor stochastic volatility model. However, they do not detail their derivation or prove that the term $\mathrm{O}(\tau^2)$ in their expansion was genuinely of second order. Nor do they touch upon first order expansions for implied volatilities under local volatility models. Here we verify that their method can be used to derive a first order Taylor series in time for the implied volatility under the local volatility model (2.1).

6.3 Derivation of the first order term of the Taylor series

In this section we *formally* derive the first order term of the Taylor series for the implied volatility φ. We assume that for some functions $f, g : \mathbb{R} \to \mathbb{R}$ and $f, g \in C^2(\mathbb{R})$, the implied volatility φ admits the Taylor expansion

$$\varphi(x,\tau) = I(x)(1 + f(x)\tau + g(x)\tau^2 + O(\tau^3)),$$

where $I(x)$ is defined by (6.3) and $\mathrm{O} = \mathrm{O}(\mathcal{V}_4)$. We also assume that $\|f\|_2$, $\|g\|_2 < \infty$, and

$$\partial_x^i \partial_\tau^j \mathrm{O}(\tau^3) = \partial_\tau^j \partial_x^i \mathrm{O}(\tau^3) = \mathrm{O}(\tau^{3-j}), \quad i,j = 0,1,2, \quad i+j \leq 3.$$

(Note that we impose these assumptions for the formal derivation only; they are not used in the actual proofs.) Before formally using PDE (5.86) to solve for f, let us list some properties of φ under the above assumptions. Suppressing the arguments x and τ, we have

$$\begin{aligned}
\varphi &= I(1 + f\tau + g\tau^2 + \mathrm{O}(\tau^2)),\\
\varphi_x &= I_x(1 + f\tau + g\tau^2 + \mathrm{O}(\tau^3)) + I(f_x\tau + g_x\tau^2 + \mathrm{O}(\tau^3)),\\
\varphi_{xx} &= I_{xx}(1 + f\tau + g\tau^2 + \mathrm{O}(\tau^3)) + 2I_x(f_x\tau + g_x\tau^2 + \mathrm{O}(\tau^3)) + I(f_{xx}\tau + g_{xx}\tau^2 + \mathrm{O}(\tau^3)),\\
\varphi_\tau &= I(f + 2g\tau + \mathrm{O}(\tau^2)),\\
\varphi_{\tau\tau} &= I(2g + \mathrm{O}(\tau)),\\
\varphi_{x\tau} &= I_x(f + 2g\tau + \mathrm{O}(\tau^2)) + I(f_x + 2g_x\tau + \mathrm{O}(\tau^2)),\\
\varphi_{xx\tau} &= I_{xx}(f + 2g\tau + \mathrm{O}(\tau^2)) + 2I_x(f_x + 2g_x\tau + \mathrm{O}(\tau^2)) + I(f_{xx} + 2g_{xx}\tau + \mathrm{O}(\tau^2)).
\end{aligned}$$

Moreover, as $\tau \to 0$,

$$\begin{aligned}
\varphi &\to I,\\
\varphi_x &\to I_x,\\
\varphi_{xx} &\to I_{xx},\\
\varphi_\tau &\to If,\\
\varphi_{\tau\tau} &\to 2Ig,\\
\varphi_{x\tau} &\to I_x f + I f_x,\\
\varphi_{xx\tau} &\to I_{xx} f + 2I_x f_x + I f_{xx}.
\end{aligned}$$

Recall from (5.86) that

$$2\tau\varphi\varphi_\tau + \varphi^2 - \sigma^2(x)\left(1 - x\frac{\varphi_x}{\varphi}\right)^2 - \sigma^2(x)\tau\varphi\varphi_{xx} + \frac{1}{4}\sigma^2(x)\tau^2\varphi^2\varphi_x^2 = 0. \tag{6.9}$$

Then differentiating both sides of this equation with respect to τ gives

$$\begin{aligned}2\varphi\varphi_\tau + 2\tau\varphi_\tau^2 + 2\tau\varphi\varphi_{\tau\tau} + 2\varphi\varphi_\tau - 2\sigma^2\left(1 - x\frac{\varphi_x}{\varphi}\right)\left(-x\frac{\varphi_{x\tau}\varphi - \varphi_x\varphi_\tau}{\varphi^2}\right)&\\ - \sigma^2\varphi\varphi_{xx} - \sigma^2\tau\varphi_\tau\varphi_{xx} - \sigma^2\tau\varphi\varphi_{xx\tau}&\\ + \frac{1}{2}\sigma^2\tau\varphi^2\varphi_x^2 + \frac{1}{2}\sigma^2\tau^2\varphi\varphi_\tau\varphi_x^2 + \frac{1}{2}\sigma^2\tau^2\varphi^2\varphi_x\varphi_{x\tau} &= 0.\end{aligned}$$

Grouping the terms, we get

$$\begin{aligned}4\varphi\varphi_\tau + 2\tau\varphi_\tau^2 + 2\tau\varphi\varphi_{\tau\tau} - 2\sigma^2\left(1 - x\frac{\varphi_x}{\varphi}\right)\left(-x\frac{\varphi_{x\tau}\varphi - \varphi_x\varphi_\tau}{\varphi^2}\right)&\\ - \sigma^2\varphi\varphi_{xx} - \sigma^2\tau\varphi_\tau\varphi_{xx} - \sigma^2\tau\varphi\varphi_{xx\tau}&\\ + \frac{1}{2}\sigma^2\tau\varphi^2\varphi_x^2 + \frac{1}{2}\sigma^2\tau^2\varphi\varphi_\tau\varphi_x^2 + \frac{1}{2}\sigma^2\tau^2\varphi^2\varphi_x\varphi_{x\tau} &= 0.\end{aligned}$$

Letting $\tau \to 0$, we have,

$$\begin{aligned}4IIf + 0 + 0 - 2\sigma^2\left(1 - x\frac{I_x}{I}\right)\left(-x\frac{(I_xf + If_x)I - I_xIf}{I^2}\right)&\\ - \sigma^2 II_{xx} - 0 - 0 + 0 + 0 + 0 &= 0.\end{aligned}$$

That is

$$4I^2f - 2\sigma^2\left(1 - x\frac{I_x}{I}\right)(-xf_x) - \sigma^2 II_{xx} = 0.$$

Rearranging the terms we get

$$f_x + \frac{2I^2}{x\sigma^2\left(1 - x\frac{I_x}{I}\right)}f = \frac{II_{xx}}{2x\left(1 - x\frac{I_x}{I}\right)}.$$

Since

$$I^2 = \sigma^2\left(1 - x\frac{I_x}{I}\right)^2, \tag{6.10}$$

the ODE can be simplified to

$$f_x + 2\left(\frac{1}{x} - \frac{I_x}{I}\right)f = \frac{\sigma I_{xx}}{2x}. \tag{6.11}$$

By the method of variation of the constant, a solution to this ODE is

$$f(x) = \frac{1}{J^2(x)}\ln\left(\frac{\sqrt{\sigma(0)\sigma(x)}}{I(x)}\right) = \frac{I^2(x)}{x^2}\ln\left(\frac{\sqrt{\sigma(0)\sigma(x)}}{I(x)}\right), \tag{6.12}$$

where the constant $\sqrt{\sigma(0)}$ is necessary to ensure that f is nonsingular at $x = 0$. Indeed, if (A_0)–(A_2) hold, then an application of the L'Hospital rule will show that

$$f(0) = \frac{1}{12}\sigma(0)\sigma_{xx}(0) - \frac{1}{24}\sigma_x^2(0) < \infty. \tag{6.13}$$

Hence by Lemma 5.13, $\|f\|_0 \leq \mathrm{const}(\mathcal{V}_2)$. We will present more boundedness properties of f in the next section.

Remark 6.4. Note that we have not claimed that f is the unique solution of ODE (6.11). Yet, this will not affect the limit of $\varphi_\tau(x,\tau)$ as τ tends to zero, for, the limit of $\varphi_\tau(x,\tau)$ must be unique, or otherwise there would be a contradiction. See (6.42) below.

6.4 Properties of the first order term of the Taylor series

In this section we study the properties of f as f constitutes part of the first order term in the first order Taylor expansion for the implied volatility.

Lemma 6.5. *Let* (A_0)–(A_4) *hold. Then*

$$\|f\|_i \leq \mathrm{const}(\mathcal{V}_{i+2}), \quad i = 0, 1, 2. \tag{6.14}$$

Proof. By straightforward differentiation, f, defined by (6.12), and its first two derivatives can be written as

$$\begin{aligned}
f(x) &= \frac{I^2(x)}{x^2}\ln\left(\frac{\sqrt{\sigma(0)\sigma(x)}}{I(x)}\right), \\
f_x(x) &= \frac{I^3(x)}{2x^3\sigma(x)}\left[\frac{x\sigma_x(x)}{I(x)} - \frac{2\sigma(x)}{I(x)} + 2 - 4\ln\left(\frac{\sqrt{\sigma(0)\sigma(x)}}{I(x)}\right)\right], \\
f_{xx}(x) &= \frac{I^4(x)}{2x^4\sigma^2(x)}\left[-\frac{x^2\sigma_x^2(x)}{I^2(x)} - \frac{6x\sigma_x(x)}{I(x)} + \frac{x^2\sigma(x)\sigma_{xx}(x)}{I^2(x)} + \frac{2\sigma^2(x)}{I^2(x)} + 8I(x)\sigma(x)\right. \\
&\quad \left. -10 + 12\ln\left(\frac{\sqrt{\sigma(0)\sigma(x)}}{I(x)}\right) + \frac{4x\sigma_x(x)}{I(x)}\ln\left(\frac{\sqrt{\sigma(0)\sigma(x)}}{I(x)}\right)\right].
\end{aligned}$$

An application of the L'Hospital rule then gives

$$\begin{aligned} f(0) &= \frac{1}{12}\sigma(0)\sigma_{xx}(0) - \frac{1}{24}\sigma_x^2(0), \\ f_x(0) &= \frac{\sigma_{xxx}(0)\sigma(0)}{24}, \\ f_{xx}(0) &= \frac{1}{1440\sigma^2}\left(-5\sigma_x^4 + 12\sigma_x\sigma_{xxx}\sigma^2 - 4\sigma\sigma_{xx}\sigma_x^2 + 4\sigma^2\sigma_{xx}^2 + 36\sigma^3\sigma_{xxxx}\right), \end{aligned} \tag{6.15}$$

where in the last identity the terms on the right hand side are evaluated at $x = 0$. Since by (A_0)–(A_4), σ, I, and their derivatives are continuous, f, f_x and f_{xx} are continuous in $\mathbb{R}$. Partition $\mathbb{R}$ into $[-1,1]$ and $\mathbb{R} \setminus [-1,1]$. By (A_0)–(A_4) and Lemma 5.13, σ, I, and their derivatives are bounded on $\mathbb{R}$; in particular, σ and I are are strictly positive. Hence the supremum norms of f, f_x, and f_{xx} are bounded on each of the sets $[-1,1]$ and $\mathbb{R} \setminus [-1,1]$. This then gives (6.14). □

Conjecture 6.6. *Let σ_0 be some positive constant such that $0 < 1/\sigma_0 \le \sigma(x) \le \sigma_0$ for all $x \in \mathbb{R}$. Assume $\sigma \in C^{n+2}(\mathbb{R})$ and $\|\sigma\|_{n+2} < \infty$ for $n = 2, 3, 4, \ldots$. Then*

$$\|f\|_i \le \mathrm{const}(\|\sigma\|_0, \ldots, \|\sigma\|_{i+2}), \quad i = 0, 1, 2, \ldots, n.$$

Remark 6.7. To derive an nth order Taylor series expansion in time for the implied volatility under our model and method, we need $\|f\|_n$ to be bounded. Proving this conjecture will help setting sufficient conditions on the local volatility σ for the higher order Taylor expansion.

6.5 Associated local volatilities: second order expansions

Recall from Section 5.6 that the associated local volatility functional $\Sigma[\cdot]$ is defined by

$$\Sigma[\psi](x,\tau) = \left(\frac{G[\psi]}{H[\psi]}\right)^{1/2}, \tag{6.16}$$

where

$$\begin{aligned} G[\psi] &= (\tau\psi^2)_\tau = \psi^2 + 2\tau\psi\psi_\tau, \\ H[\psi] &= \left(1 - x\frac{\psi_x}{\psi}\right)^2 + \tau\psi\psi_{xx} - \frac{1}{4}\tau^2\psi^2\psi_x^2, \end{aligned} \tag{6.17}$$

for $\psi \in C^{2,1}(\mathbb{R} \times (0,T])$. Recall also that $\mathcal{I}[0,T]$ is the class of those functions $\psi \in C^{2,1}(\mathbb{R} \times (0,\infty))$ for which $\Sigma[\psi](x,\tau)$ is well defined, continuous in $\mathbb{R} \times [0,T]$, and satisfies there

$$0 < \text{const}_1 \leq \Sigma[\psi](x,\tau) \leq \text{const}_2 < \infty, \tag{6.18}$$

and

$$\lim_{\tau\to 0} \tau\psi^2(x,\tau) = 0 \quad \text{ uniformly in } x \in \mathbb{R}. \tag{6.19}$$

Assuming that the function $\psi \in \mathcal{I}[0,T]$ is regular enough, then a formal second order Taylor expansion of the associated volatility $\Sigma[\psi]$ is given by

$$\Sigma[\psi](x,\tau) = \Sigma[\psi](x,0) + \tau\Sigma_\tau[\psi](x,0) + \frac{\tau^2}{2}\Sigma_{\tau\tau}[\psi](x,0) + R_2[\psi](x,\tau), \tag{6.20}$$

where

$$R_2[\psi](x,\tau) = \frac{1}{2}\int_0^\tau \{\Sigma_{\eta\eta\eta}[\psi](x,\eta)\}\,(\tau-\eta)^2 \,\mathrm{d}\eta. \tag{6.21}$$

As already detailed in Section 5.6, suppressing the functional input ψ and the arguments x and τ, we have

$$\Sigma_\tau[\psi](x,\tau) = \frac{1}{2\Sigma H^2}(G_\tau H - GH_\tau). \tag{6.22}$$

Further,

$$\begin{aligned}\Sigma_{\tau\tau}[\psi](x,\tau) &= \frac{1}{2\Sigma^2 H^4}\left[(G_\tau H - GH_\tau)_\tau \Sigma H^2 - (G_\tau H - GH_\tau)(\Sigma H^2)_\tau\right] \\ &= \frac{1}{2\Sigma^2 H^4}F,\end{aligned} \tag{6.23}$$

where

$$F[\psi](x,\tau) := (G_{\tau\tau}H - GH_{\tau\tau})\Sigma H^2 - (G_\tau H - GH_\tau)(\Sigma_\tau H^2 + 2\Sigma H H_\tau). \tag{6.24}$$

Differentiating once more with respect to τ gives

$$\begin{aligned}\Sigma_{\tau\tau\tau}[\psi](x,\tau) &= \frac{1}{2\Sigma^4 H^8}[F_\tau \Sigma^2 H^4 - F(\Sigma^2 H^4)_\tau] \\ &= \frac{1}{2\Sigma^4 H^8}[F_\tau \Sigma^2 H^4 - F(2\Sigma\Sigma_\tau H^4 + 4\Sigma^2 H^3 H_\tau)],\end{aligned} \tag{6.25}$$

where in more detail

$$\begin{aligned}F_\tau[\psi](x,\tau) = {} & (G_{\tau\tau\tau}H + G_{\tau\tau}H_\tau - G_\tau H_{\tau\tau} - GH_{\tau\tau\tau})\Sigma H^2 \\ & + 2(G_{\tau\tau}H - GH_{\tau\tau})(\Sigma_\tau H^2 + 2\Sigma H H_\tau) \\ & + (G_\tau H - GH_\tau)(\Sigma_{\tau\tau}H^2 + 4\Sigma_\tau H H_\tau + 2\Sigma H_\tau^2 + 2\Sigma H H_{\tau\tau}).\end{aligned} \tag{6.26}$$

Let us list some derivatives of G and H and express them in terms of ψ. Note that some of them have already appeared in Section 5.6. For G, we have

$$\begin{aligned} G[\psi] &= \psi^2 + 2\tau\psi\psi_\tau, \\ G_\tau[\psi] &= 4\psi\psi_\tau + 2\tau\psi_\tau^2 + 2\tau\psi\psi_{\tau\tau}, \\ G_{\tau\tau}[\psi] &= 6\psi_\tau^2 + 6\psi\psi_{\tau\tau} + 6\tau\psi_\tau\psi_{\tau\tau} + 2\tau\psi\psi_{\tau\tau\tau}, \\ G_{\tau\tau\tau}[\psi] &= 24\psi_\tau\psi_{\tau\tau} + 8\psi\psi_{\tau\tau\tau} + 6\tau\psi_{\tau\tau}^2 + 8\tau\psi_\tau\psi_{\tau\tau\tau} + 2\tau\psi\psi_{\tau\tau\tau\tau}. \end{aligned} \tag{6.27}$$

For H, we have

$$\begin{aligned} H[\psi] &= \left(1 - x\frac{\psi_x}{\psi}\right)^2 + \tau\psi\psi_{xx} - \frac{1}{4}\tau^2\psi^2\psi_x^2, \\ H_\tau[\psi] &= 2\left(1 - \frac{x\psi_x}{\psi}\right)\left(-\frac{x\psi_{x\tau}}{\psi} + \frac{x\psi_x\psi_\tau}{\psi^2}\right) \\ &\quad + \psi\psi_{xx} + \tau\psi_\tau\psi_{xx} + \tau\psi\psi_{xx\tau} - \frac{1}{2}\tau\psi^2\psi_x^2 - \frac{1}{2}\tau^2\psi\psi_x^2\psi_\tau - \frac{1}{2}\tau^2\psi^2\psi_x\psi_{x\tau}, \end{aligned} \tag{6.28}$$

and

$$\begin{aligned} &H_{\tau\tau}[\psi] \\ &= 2\left(-\frac{x\psi_{x\tau}}{\psi} + \frac{x\psi_x\psi_\tau}{\psi^2}\right)^2 \\ &\quad + 2\left(1 - \frac{x\psi_x}{\psi}\right)\left(-\frac{x\psi_{x\tau\tau}}{\psi} + \frac{2x\psi_{x\tau}\psi_\tau}{\psi^2} - \frac{2x\psi_x\psi_\tau^2}{\psi^3} + \frac{x\psi_x\psi_{\tau\tau}}{\psi^2}\right) \\ &\quad + 2\psi_\tau\psi_{xx} + 2\psi\psi_{xx\tau} + \tau\psi_{\tau\tau}\psi_{xx} + 2\tau\psi_\tau\psi_{xx\tau} + \tau\psi\psi_{xx\tau\tau} - \frac{1}{2}\psi^2\psi_x^2 - 2\tau\psi\psi_x^2\psi_\tau \\ &\quad - 2\tau\psi^2\psi_x\psi_{x\tau} - \frac{1}{2}\tau^2\psi_\tau^2\psi_x^2 - 2\tau^2\psi\psi_x\psi_\tau\psi_{x\tau} - \frac{1}{2}\tau^2\psi\psi_x^2\psi_{\tau\tau} - \frac{1}{2}\tau^2\psi^2\psi_{x\tau}^2 - \frac{1}{2}\tau^2\psi^2\psi_x\psi_{x\tau\tau}. \end{aligned} \tag{6.29}$$

Moreover,

$$
\begin{aligned}
&H_{\tau\tau\tau}[\psi](x,\tau)\\
&= \tau\psi\psi_{xx\tau\tau\tau} + 3\tau\psi_{\tau\tau}\psi_{xx\tau} - 3\psi\psi_x^2\psi_\tau + \tau\psi_{\tau\tau\tau}\psi_{xx} - 3\psi^2\psi_x\psi_{x\tau}\\
&\quad + 6\left(-\frac{x\psi_{x\tau}}{\psi} + \frac{x\psi_x\psi_\tau}{\psi^2}\right)\left(-\frac{x\psi_{x\tau\tau}}{\psi} + \frac{2x\psi_{x\tau}\psi_\tau}{\psi^2} - \frac{2x\psi_x\psi_\tau^2}{\psi^3} + \frac{x\psi_x\psi_{\tau\tau}}{\psi^2}\right)\\
&\quad - 3\tau\psi_\tau^2\psi_x^2 - 3\tau\psi\psi_x^2\psi_{\tau\tau} - 3\tau\psi^2\psi_{x\tau}^2 - 3\tau\psi^2\psi_x\psi_{x\tau\tau} + 3\tau\psi_\tau\psi_{xx\tau\tau}\\
&\quad + 2\left(1 - \frac{x\psi_x}{\psi}\right)\\
&\quad \times\left(-\frac{x\psi_{x\tau\tau\tau}}{\psi} + \frac{3x\psi_{x\tau\tau}\psi_\tau}{\psi^2} - \frac{6x\psi_{x\tau}\psi_\tau^2}{\psi^3} + \frac{3x\psi_{x\tau}\psi_{\tau\tau}}{\psi^2} + \frac{6x\psi_x\psi_\tau^3}{\psi^4} - \frac{6x\psi_x\psi_\tau\psi_{\tau\tau}}{\psi^3} + \frac{x\psi_x\psi_{\tau\tau\tau}}{\psi^2}\right)\\
&\quad + 3\psi_{\tau\tau}\psi_{xx} + 6\psi_\tau\psi_{xx\tau} + 3\psi\psi_{xx\tau\tau} - 3\tau^2\psi\psi_{x\tau}^2\psi_\tau - 3\tau^2\psi\psi_x\psi_{\tau\tau}\psi_{x\tau} - 3\tau^2\psi\psi_x\psi_\tau\psi_{x\tau\tau}\\
&\quad - 12\tau\psi\psi_x\psi_\tau\psi_{x\tau} - \frac{3}{2}\tau^2\psi_\tau\psi_x^2\psi_{\tau\tau} - 3\tau^2\psi_\tau^2\psi_x\psi_{x\tau} - \frac{1}{2}\tau^2\psi\psi_x^2\psi_{\tau\tau\tau}\\
&\quad - \frac{3}{2}\tau^2\psi^2\psi_{x\tau}\psi_{x\tau\tau} - \frac{1}{2}\tau^2\psi^2\psi_x\psi_{x\tau\tau\tau}.
\end{aligned}
\tag{6.30}
$$

To construct upper and lower functions that bound the transformed implied volatility φ, we will use a subclass of $\mathcal{I}[0,T]$ such that the ψ's in this class are of the form

$$
\psi(x,\tau) = I(x)(1 + f(x)\tau + \lambda\tau^2), \quad (x,\tau) \in \mathbb{R}\times[0,T], \tag{6.31}
$$

where $I(x)$ and $f(x)$ are respectively defined by (6.3) and (6.4), and λ is some fixed real constant. Note that provided the positive constant T is sufficiently small, such ψ's do belong to $\mathcal{I}[0,T]$. This fact follows from the definition of $\mathcal{I}[0,T]$ — see the definition above (5.56) — and from Corollary 6.9 and Proposition 6.10 below.

We will study the the second order Taylor expansion in time of the associated volatility $\Sigma[\psi]$ when ψ is given by (6.31).

Lemma 6.8. *Let* (A_0)–(A_4) *hold. Let* $\lambda \in \mathbb{R}$ *be fixed and* $\psi(x,\tau) = I(x)(1 + f(x)\tau + \lambda\tau^2)$*, for* $(x,\tau) \in \mathbb{R}\times[0,T]$*. Then uniformly in* $x \in \mathbb{R}$*, as* $\tau \to 0$*,*

$$
\begin{aligned}
G[\psi] &\to I^2,\\
G_\tau[\psi] &\to 4I^2 f,\\
G_{\tau\tau}[\psi] &\to 6I^2f^2 + 12I^2\lambda,\\
G_{\tau\tau\tau}[\psi] &\to 48I^2 f\lambda,
\end{aligned}
\tag{6.32}
$$

and

$$
\begin{aligned}
H[\psi] &\to \frac{I^2}{\sigma^2},\\
H_\tau[\psi] &\to -\frac{2xIf_x}{\sigma} + II_{xx} = 4\frac{I^2}{\sigma^2}f,\\
H_{\tau\tau}[\psi] &\to 2x^2f_x^2 + \frac{4xIff_x}{\sigma} + 4II_{xx}f + 4II_xf_x + 2I^2f_{xx} - \frac{1}{2}I^2I_x^2,\\
H_{\tau\tau\tau}[\psi] &\to -6I^2I_x^2f - 3I^3I_xf_x - 12x^2ff_x^2\\
&\quad - \frac{I}{\sigma}\left(12xf_x\lambda - 12xf^2f_x\right) + 12II_{xx}\lambda + 6If(I_{xx}f + 2I_xf_x + If_{xx}),
\end{aligned}
\tag{6.33}
$$

and

$$
\begin{aligned}
\Sigma[\psi] &\to \sigma,\\
\Sigma_\tau[\psi] &\to 0,\\
\Sigma_{\tau\tau}[\psi] &\to \sigma\left[\frac{6I^2}{\sigma^2}\lambda + \frac{3I^2f^2}{\sigma^2} - \left(x^2f_x^2 + 2xff_x\frac{I}{\sigma} + 2II_{xx}f + 2II_xf_x + I^2f_{xx} - \frac{1}{4}I^2I_x^2\right)\right].
\end{aligned}
\tag{6.34}
$$

Proof. By assumption,

$$
\begin{aligned}
\psi &= I(1 + f\tau + \lambda\tau^2),\\
\psi_\tau &= I(f + 2\lambda\tau),\\
\psi_{\tau\tau} &= 2I\lambda,\\
\psi_{\tau\tau\tau} &\equiv 0,\\
\psi_x &= I_x(1 + f\tau + \lambda\tau^2) + If_x\tau,\\
\psi_{x\tau} &= I_x(f + 2\lambda\tau) + If_x,\\
\psi_{x\tau\tau} &= 2I_x\lambda,\\
\psi_{x\tau\tau\tau} &\equiv 0,\\
\psi_{xx} &= I_{xx}(1 + f\tau + \lambda\tau^2) + 2I_xf_x\tau + If_{xx}\tau,\\
\psi_{xx\tau} &= I_{xx}(f + 2\lambda\tau) + 2I_xf_x + If_{xx},\\
\psi_{xx\tau\tau} &= 2I_{xx}\lambda,\\
\psi_{xx\tau\tau\tau} &\equiv 0.
\end{aligned}
$$

By the uniform bounds of (5.52), (5.53), and (6.14), we have, as $\tau \to 0$,

$$\begin{aligned}
\psi &\to I, \\
\psi_\tau &\to If, \\
\psi_{\tau\tau} &\to 2I\lambda, \\
\psi_{\tau\tau\tau} &\equiv 0, \\
\psi_x &\to I_x, \\
\psi_{x\tau} &\to I_x f + I f_x, \\
\psi_{x\tau\tau} &\to 2I_x\lambda, \\
\psi_{x\tau\tau\tau} &\equiv 0, \\
\psi_{xx} &\to I_{xx}, \\
\psi_{xx\tau} &\to I_{xx} f + 2I_x f_x + I f_{xx}, \\
\psi_{xx\tau\tau} &\to 2I_{xx}\lambda, \\
\psi_{xx\tau\tau\tau} &\equiv 0,
\end{aligned}$$

uniformly in $x \in \mathbb{R}$. These uniform limits, together with (6.27)–(6.29), then imply (6.32) and (6.33), where the equality in the limit for $H_\tau[\psi]$ in (6.33) follows from the fact that by (6.10) and (6.11),

$$\begin{aligned}
f_x &= \frac{\sigma I_{xx}}{2x} - 2\left(\frac{1}{x} - \frac{I_x}{I}\right) f \\
&= \frac{\sigma I_{xx}}{2x} - \frac{2}{x}\left(1 - x\frac{I_x}{I}\right) f \\
&= \frac{\sigma I_{xx}}{2x} - \frac{2I}{x\sigma} f.
\end{aligned}$$

So we have proved (6.32) and (6.33).

To prove (6.34), we combine (6.32), (6.33), (6.4), and (5.48), to obtain $\Sigma[\psi] \to \sigma$, which is uniform in $x \in \mathbb{R}$. In turn, this shows that

$$\Sigma[\psi]H^2[\psi] \to I^4/\sigma^3. \tag{6.35}$$

On the other hand, by (6.32) and (6.33), we have $G_\tau[\psi]H[\psi] - G[\psi]H_\tau[\psi] \to 0$ as $\tau \to 0$, uniformly in $x \in \mathbb{R}$. Hence, by (6.22), we have $\Sigma_\tau[\psi] \to 0$ as $\tau \to 0$, uniformly in $x \in \mathbb{R}$. Then, applying the uniform limits obtained so far to (6.23) will yield the uniform limit for $\Sigma_{\tau\tau}[\psi]$ in (6.34). The proof is thus complete. □

Corollary 6.9. *Let* (A_0)–(A_4) *hold. Let* $\lambda \in \mathbb{R}$ *be fixed and* $\psi(x, \tau) = I(x)(1 + f(x)\tau +$

$\lambda\tau^2$), for $(x,\tau) \in \mathbb{R} \times [0,T]$. Then there exists $T_* = T_*(\lambda, \mathcal{V}_4)$, such that for all $T \le T_*$,

$$0 < \text{const}_1 \le \psi(x,\tau), G[\psi](x,\tau), H[\psi](x,\tau), \Sigma[\psi](x,\tau) \le \text{const}_2 < \infty, \tag{6.36}$$

uniformly on $\mathbb{R} \times [0,T]$, and that

$$\sup_{(x,\tau)\in\mathbb{R}\times[0,T]} \left\{ \begin{array}{l} \left|\partial_\tau^i \partial_x^j \psi(x,\tau)\right|, \left|\partial_\tau^i G[\psi](x,\tau)\right|, \left|\partial_\tau^i H[\psi](x,\tau)\right|, \left|\partial_\tau^i \Sigma[\psi](x,\tau)\right|, \\ \quad i = 0,1,2,3, \quad j = 0,1,2 \end{array} \right\} < \text{const}_3, \tag{6.37}$$

where const_ι, $\iota = 1,2,3$, depend on T_1, λ, and $\mathcal{V}_4$ only.

Proof. The existence of such a T_* follows from the uniform limits in Lemma 6.8 and the uniform bounds for σ, I, f, and their derivatives, which are given by (A_0)–(A_4), Lemma 5.13, and Lemma 6.5. □

Recall from (6.34) that

$$\begin{aligned} \Sigma[\psi](x,0) &= \sigma, \\ \Sigma_\tau[\psi](x,0) &= 0, \\ \Sigma_{\tau\tau}[\psi](x,0) &= \sigma \left[\frac{6I^2}{\sigma^2}\lambda + \frac{3I^2 f^2}{\sigma^2} \right. \\ &\qquad \left. - \left(x^2 f_x^2 + 2x f f_x \frac{I}{\sigma} + 2II_{xx} f + 2II_x f_x + I^2 f_{xx} - \frac{1}{4} I^2 I_x^2 \right) \right]. \end{aligned}$$

Then we have the following proposition for second order Taylor expansion of $\Sigma[\psi]$.

Proposition 6.10. *Let (A_0)–(A_4) hold. Let $\lambda \in \mathbb{R}$ be fixed and $\psi(x,\tau) = I(x)(1 + f(x)\tau + \lambda\tau^2)$, for $(x,\tau) \in \mathbb{R} \times [0,T]$. Then there exists $T_* = T_*(\lambda, \mathcal{V}_2)$ such that the associated local volatility $\Sigma[\psi]$ admits the second order Taylor expansion*

$$\Sigma[\psi](x,\tau) = \sigma(x) + \frac{\tau^2}{2}\Sigma_{\tau\tau}[\psi](x,0) + R_2[\psi](x,\tau) \tag{6.38}$$

in $\mathbb{R} \times [0,T_*]$, where $R_2[\psi]$ is defined by (6.21) and

$$R_2[\psi](x,\tau) = \mathrm{O}(\tau^3), \qquad \text{as } \tau \to 0, \tag{6.39}$$

with $\mathrm{O} = \mathrm{O}(T_*, \lambda, \mathcal{V}_2)$.

Proof. The second order Taylor expansion results from substituting $\Sigma[\psi](x,0)$, $\Sigma_\tau[\psi](x,0)$, $\Sigma_{\tau\tau}[\psi](x,0)$, and $\Sigma_{\eta\eta\eta}[\psi](x,\eta)$ into the formal expansion (6.20)–(6.30). For some $T_* = T_*(\lambda, \mathcal{V}_4)$, this second order expansion is valid by Lemma 6.8 and Corollary 6.9. In particular, (6.37) implies that $R_2[\psi](x,\tau) = \mathrm{O}(\tau^3)$, with $\mathrm{O} = \mathrm{O}(T_*, \lambda, \mathcal{V}_4)$. □

6.6 Upper and lower solutions and their corresponding associated volatilities

Let λ be some strictly positive constant and define

$$\begin{aligned}\overline{\varphi}(x,\tau) &= I(x)\Big(1 + f(x)\tau + \lambda\tau^2\Big),\\ \underline{\varphi}(x,\tau) &= I(x)\Big(1 + f(x)\tau - \lambda\tau^2\Big).\end{aligned} \tag{6.40}$$

Proposition 6.11. *Let (2.1),* (A_0)–(A_4) *hold. Let* $\overline{\varphi}$ *and* $\underline{\varphi}$ *be defined as in (6.40). Then there exist positive constants* $\lambda = \lambda(\mathcal{V}_4)$ *and* $T = T(\lambda, \mathcal{V}_4)$ *such that*

$$\Sigma[\underline{\varphi}](x,\tau) \le \sigma(x) \le \Sigma[\overline{\varphi}](x,\tau), \quad \forall\ (x,\tau) \in \mathbb{R} \times [0,T].$$

Proof. We will prove the second inequality first. By Proposition 6.10, $\Sigma[\overline{\varphi}](x,\tau)$ has the second order expansion

$$\Sigma[\overline{\varphi}](x,\tau) = \sigma(x)\left[1 + \frac{\tau^2}{2}\Lambda(\lambda,\sigma,I,f) + \frac{R_2[\psi](x,\tau)}{\sigma(x)}\right],$$

in $\mathbb{R} \times [0,T_0]$, for some $T_0 = T_0(\lambda, \mathcal{V}_4)$. Here

$$\begin{aligned}&\Lambda(\lambda,\sigma,I,f)\\ &\quad= \frac{6I^2}{\sigma^2}\lambda + \frac{3I^2f^2}{\sigma^2}\left(x^2f_x^2 + 2xff_x\frac{I}{\sigma} + 2II_{xx}f + 2II_xf_x + I^2f_{xx} - \frac{1}{4}I^2I_x^2\right).\end{aligned}$$

By (A_0)–(A_4), Lemma 5.13, and Lemma 6.5, the functions σ, I, f, and their derivatives are uniformly bounded. So for sufficiently large $\lambda_1 = \lambda_1(\mathcal{V}_4)$, we have $\Lambda(\lambda_1(\mathcal{V}_4),\sigma,I,f) > 4$, uniformly for all $x \in \mathbb{R}$. On the other hand, Corollary 6.9 shows that $\Sigma_{\tau\tau\tau}[\overline{\varphi}](x,\tau)$ is uniformly bounded on $\mathbb{R} \times [0,T_0]$, and hence we can find a $T_1 = T_1(\lambda_1, \mathcal{V}_4) < T_0$ such that

$$
\begin{aligned}
\left|\frac{R[\overline{\varphi}](x,\tau)}{\sigma(x)}\right| &\leq \nu_0 \left|R[\overline{\varphi}](x,\tau)\right| \\
&= \nu_0 \left|\frac{1}{2}\int_0^{\tau} \left\{\Sigma_{\eta\eta\eta}[\psi](x,\eta)\right\}(\tau-\eta)^2 \, \mathrm{d}\eta\right| \\
&\leq \frac{\nu_0}{2}\left(\sup_{(x,\tau)\in\mathbb{R}\times[0,T_1]} \left|\Sigma_{\tau\tau\tau}[\overline{\varphi}](x,\tau)\right|\right)\int_0^{\tau}(\tau-\eta)^2 \, \mathrm{d}\eta \\
&\leq \frac{\nu_0}{6}\left(\sup_{(x,\tau)\in\mathbb{R}\times[0,T_1]} \left|\Sigma_{\tau\tau\tau}[\overline{\varphi}](x,\tau)\right|\right)\tau^3 \\
&\leq \frac{\nu_0}{6}\left(\sup_{(x,\tau)\in\mathbb{R}\times[0,T_1]} \left|\Sigma_{\tau\tau\tau}[\overline{\varphi}](x,\tau)\right|\right)T_1\tau^2 \\
&\leq \tau^2.
\end{aligned}
$$

For such a large positive $\lambda_1 = \lambda_1(\mathcal{V}_4)$ and small positive $T_1 = T_1(\lambda_1, \mathcal{V}_4)$, we have

$$\Sigma[\overline{\varphi}_1](x,\tau) \geq \sigma(x)(1+2\tau^2-\tau^2) \geq \sigma(x)$$

for all $(x,\tau)\in\mathbb{R}\times[0,T_1]$, where $\overline{\varphi}_1(x,\tau) = I(x)(1+f(x)\tau+\lambda_1\tau^2)$. Now replace λ with $-\lambda$ in the analysis above, we can similarly find large positive $\lambda_2 = \lambda_2(\mathcal{V}_4)$ and small positive $T_2 = T_2(\lambda_2, \mathcal{V}_4)$ such that

$$\Sigma[\underline{\varphi}_2](x,\tau) \leq \sigma(x)(1-2\tau^2+\tau^2) \leq \sigma(x)$$

for all $(x,\tau)\in\mathbb{R}\times[0,T_2]$, where $\underline{\varphi}_2(x,\tau) = I(x)(1+f(x)\tau-\lambda_2\tau^2)$. Finally, the desired λ and T can be obtained by setting $\lambda = \max(\lambda_1,\lambda_2)$ and $T = \min(T_1,T_2)$. In fact, any larger λ and smaller T will work. □

6.7 Proof of the main theorems of the chapter

Here we will present the proofs of Theorems 6.2 and 6.1.

Proof of Theorem 6.2. Let $\overline{\varphi}$ and $\underline{\varphi}$ be defined as in (6.40). Then by Proposition 6.11, there exist $\lambda = \lambda(\mathcal{V}_4)$ and $T = T(\lambda, \mathcal{V}_4)$ such that

$$\Sigma[\underline{\varphi}](x,\tau) \leq \sigma(x) \leq \Sigma[\overline{\varphi}](x,\tau), \quad \forall\, (x,\tau)\in\mathbb{R}\times[0,T].$$

By the comparison principle, Lemma 5.22, we have

$$\underline{\varphi}(x,\tau) \le \varphi(x,\tau) \le \overline{\varphi}(x,\tau), \quad \forall\ (x,\tau) \in \mathbb{R} \times (0,T]. \tag{6.41}$$

This shows that as $\tau \to 0$,

$$\varphi(x,\tau) = I(x)(1 + f(x)\tau + \mathrm{O}(\tau^2)),$$

with $\mathrm{O} = \mathrm{O}(\mathcal{V}_4)$. It remains to prove the convergence of $\varphi_\tau(x,\tau)$. By (6.41),

$$\frac{\underline{\varphi}(x,\tau) - \varphi(x,0)}{\tau} \le \frac{\varphi(x,\tau) - \varphi(x,0)}{\tau} \le \frac{\overline{\varphi}(x,\tau) - \varphi(x,0)}{\tau},$$

for all $(x,\tau) \in \mathbb{R} \times (0,T]$. Since $\varphi(x,0) \equiv I(x)$, taking the limit yields

$$\lim_{\tau \to 0} \varphi_\tau(x,\tau) = I(x)f(x), \quad \text{uniformly in } x \in \mathbb{R}. \tag{6.42}$$

The proof is thus complete. □

Proof of Theorem 6.1. The conclusion of the theorem is a consequence of Theorem 6.2 and the identity $\phi(s,\tau) = \varphi(x(s),\tau)$, where $x = \ln(s/k)$. □

Chapter 7

At the money gradient asymptotics

In this chapter we employ probabilistic methods to prove a small time limit for the at the money (**ATM**) gradient of the implied volatility. Central to the proof is a gradient representation formula of Fournie et al. [29, Proposition 3.2] for call option price, which is a variant of the Bismut–Elworthy formula.

We repeat that the asymptotic formula obtained here for the ATM gradient of implied volatility is not new, although it is independently obtained by us under weaker conditions. Assuming different, and in some sense stronger regularity conditions, Alòs et al. [1, Theorem 6.3] have proved (7.1) for more general stochastic volatility models with jumps.

This chapter is organised as follows. In Section 7.1, we state the main result of this chapter. In Section 7.2, we explain the idea behind the proof. In Section 7.3, we derive a representation for the ATM gradient of the implied volatility. In Sections 7.4 and 7.5, we present a stochastic formula for and study the small time asymptotics of the gradient of the call option price. In Section 7.6, we prove the main theorem of the chapter.

7.1 Main result: the ATM theorem

The following theorem represents the main result of this chapter.

Theorem 7.1 (At the money gradient asymptotic). *Let (2.1),* (A_0)–(A_2) *hold. Then*

$$\lim_{\tau\to 0}\Big\{\phi_s(s,\tau)|_{s=k}\Big\} \equiv \lim_{\tau\to 0}\phi_s(k,\tau) = \nu'(k)/2. \tag{7.1}$$

The proof of this theorem is given in the last section of this chapter.

7.2 Idea of the proof

We will show that

$$\frac{\partial \phi}{\partial s} = \left(\frac{\partial C}{\partial s} - \frac{\partial \mathcal{B}}{\partial s}\right) \Big/ \left(\frac{\partial \mathcal{B}}{\partial \phi}\right), \qquad (s,\tau) \in (0,\infty) \times (0,T].$$

After that, we will use a variant of the Bismut–Elworthy formula to break the difference in the numerator into smaller parts so that the fraction would converge to the desired limit.

For ease of notation, we put

$$h(s) = s\nu(s), \quad s \in (0,\infty). \tag{7.2}$$

The stock process can be written as

$$\mathrm{d}S_t = h(S_t)\,\mathrm{d}W_t, \quad S_0 > 0, \quad 0 \le t \le T < \infty. \tag{7.3}$$

Let $\tau = T - t$ and assume $S_t = s > 0$. Then by the time homogeneity of the Markov process, $(S^{s,t}_{t+\tau})$ and the process $(S^{s,0}_\tau)$ have the same distribution, where in integral form

$$\begin{aligned} S^{s,t}_{t+\tau} &= s + \int_t^{t+\tau} h(S^{s,t}_r)\,\mathrm{d}W_r, \\ S^{s,0}_{\tau} &= s + \int_0^{\tau} h(S^{s,0}_r)\,\mathrm{d}W_r, \end{aligned} \tag{7.4}$$

with (W_r) being a standard Wiener process with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. Hence we will use (S_τ), where $S_\tau \equiv S^{s,0}_\tau$, in the derivation of the asymptotic formula for the ATM gradient of the implied volatility.

7.3 Representation for ATM gradient of implied volatility

Proposition 7.2. *Let (2.1),* (A_0)–(A_2) *hold. Then as* $\tau \to 0$*,*

$$\left.\frac{\partial \phi(s,\tau)}{\partial s}\right|_{s=k} = \frac{\sqrt{2\pi}}{k\sqrt{\tau}} \left(\left.\frac{\partial C(s,\tau)}{\partial s}\right|_{s=k} - \frac{1}{2} - \frac{\nu(k)\sqrt{\tau}}{2\sqrt{2\pi}} + \mathrm{O}(\tau^{3/2}) \right), \tag{7.5}$$

where $\mathrm{O} = \mathrm{O}(k, \mathcal{V}_2)$ *and the convergence is uniform in* k *on compact subsets of* $(0,\infty)$*.*

Proof. By (4.12) and (5.1),

$$C(s,\tau) = \mathcal{B}(s,\tau;\phi(s,\tau)), \quad (s,\tau) \in (0,\infty) \times [0,T]. \tag{7.6}$$

Differentiating with respect to s gives

$$\frac{\partial C(s,\tau)}{\partial s} = \frac{\partial \mathcal{B}(s,\tau;\phi)}{\partial s} + \frac{\partial \mathcal{B}(s,\tau;\phi)}{\partial \phi} \times \frac{\partial \phi(s,\tau)}{\partial s}, \qquad (s,\tau) \in (0,\infty) \times (0,T]. \tag{7.7}$$

With the arguments suppressed, this implies

$$\frac{\partial \phi}{\partial s} = \left(\frac{\partial C}{\partial s} - \frac{\partial \mathcal{B}}{\partial s}\right) \Big/ \left(\frac{\partial \mathcal{B}}{\partial \phi}\right), \qquad (s,\tau) \in (0,\infty) \times (0,T]. \tag{7.8}$$

At $s = k$, the small time asymptotics of $\partial \mathcal{B}/\partial \phi$ and $\partial \mathcal{B}/\partial s$ can be obtained as follows. By (4.11)

$$\mathfrak{d}_\sharp(k,\tau) := \mathfrak{d}_1(s,\tau;\phi(s,\tau))\Big|_{s=k} = \sqrt{\tau}\phi(k,\tau)/2. \tag{7.9}$$

Hence by (4.11) and (4.20),

$$\begin{aligned} \frac{\partial \mathcal{B}(s,\tau;\phi(s,\tau))}{\partial \phi}\Bigg|_{s=k} &= sn(\mathfrak{d}_1)\sqrt{\tau}\Big|_{s=k} \\ &= \frac{k\sqrt{\tau}}{\sqrt{2\pi}} \mathrm{e}^{-\tau\phi^2(k,\tau)/8}, \qquad (k,\tau) \in (0,\infty) \times [0,T]. \end{aligned} \tag{7.10}$$

On the other hand, (4.18) gives $\partial \mathcal{B}/\partial s = N(\mathfrak{d}_1)$. So by a second order Taylor expansion of $N(\cdot)$ about 0 and the Lagrange formula for the remainder, we obtain, as $\tau \to 0$,

$$\begin{aligned} \frac{\partial \mathcal{B}(s,\tau;\phi)}{\partial s}\Bigg|_{s=k} &= N(\mathfrak{d}_1)\Big|_{s=k} \\ &= N(\mathfrak{d}_\sharp) \\ &= N(0) + N'(0)\mathfrak{d}_\sharp(k,\tau) + \frac{1}{2}N''(0)\mathfrak{d}_\sharp^2(k,\tau) + \mathrm{O}\left(\mathfrak{d}_\sharp^3(k,\tau)\right) \\ &= \frac{1}{2} + \frac{\phi(k,\tau)\sqrt{\tau}}{2} n(0) + \mathrm{O}(\tau^{3/2}) \\ &= \frac{1}{2} + \frac{\sqrt{\tau}}{2\sqrt{2\pi}}\phi(k,\tau) + \mathrm{O}(\tau^{3/2}) \\ &= \frac{1}{2} + \frac{\sqrt{\tau}}{2\sqrt{2\pi}}\left(\phi^0(k) + \mathrm{O}(\tau)\right) + \mathrm{O}(\tau^{3/2}) \\ &= \frac{1}{2} + \frac{\sqrt{\tau}}{2\sqrt{2\pi}}\phi^0(k) + \mathrm{O}(\tau^{3/2}), \end{aligned} \tag{7.11}$$

where $\mathrm{O} = \mathrm{O}(k, \mathcal{V}_2)$. The third and the penultimate equalities are both justified by

Theorem 5.1, which gives $\phi(s,\tau) = \phi^0(s) + \mathrm{O}(\tau)$, with $\mathrm{O} = \mathrm{O}(\mathcal{V}_2)$. In particular, Theorem 5.1 implies that $\mathfrak{d}_\sharp(k,\tau) = \mathrm{O}(\tau^{1/2})$, $\mathrm{O} = \mathrm{O}(k,\mathcal{V}_2)$, and this order property justifies the Taylor expansion of $N(\mathfrak{d}_\sharp)$. Now noting that $\phi^0(k) = \nu(k)$, we get

$$\left.\frac{\partial \mathcal{B}(s,\tau;\phi)}{\partial s}\right|_{s=k} = \frac{1}{2} + \frac{\nu(k)\sqrt{\tau}}{2\sqrt{2\pi}} + \mathrm{O}(\tau^{3/2}). \tag{7.12}$$

Finally, the conclusion of the proposition follows from (7.8), (7.10), (7.12), and the fact that $\tau\phi^2(k,\tau) \to 0$ as $\tau \to 0$. $\square$

7.4 Formula for gradient of call option price

In view of the representation for the ATM gradient of implied volatility, to prove Theorem 7.1 it remains to show that in (7.5) the small-time asymptotic properties of $\frac{\partial C}{\partial s}|_{s=k}$ do lead to the desired result. Note that the gradient or space derivative of the call option price is also called the *delta* of the call option. We now present a variant of the Bismut–Elworthy formula for the delta, which we shall call the *delta formula* for short. This formula was originally derived by Fournie et al. [29].

Proposition 7.3 (Delta formula)**.** *Let (2.1),* (A_0)–(A_1) *hold. Then*

$$\frac{\partial C(s,\tau)}{\partial s} = \mathbb{E}_s\left[(S_\tau - k)_+ \frac{1}{\tau}\int_0^\tau \frac{Y_r}{h(S_r)}\,\mathrm{d}W_r\right], \qquad (s,\tau) \in (0,\infty)\times[0,T], \tag{7.13}$$

where the first variation process (Y_τ) *is given by*

$$Y_\tau = 1 + \int_0^\tau Y_r h'(S_r)\,\mathrm{d}W_r. \tag{7.14}$$

Proof. A proof for the well-definedness of the first variation process (Y_τ) can be found in e.g. Protter [68, Theorem 49, p. 320]. Then (7.13) follows from Proposition 3.2 of Fournie et al. [29], noting that their proof can be applied here under the assumptions (2.1) and (A_0)–(A_1), instead of their Assumption 3.1. $\square$

Remark 7.4. Let (2.1), (A_0)–(A_1) hold. Define (γ_τ) to be

$$\gamma_\tau = \int_0^\tau h'(S_r)\,\mathrm{d}W_r, \quad \tau \in [0,T].$$

Then (γ_τ) is a continuous $\mathbb{P}$-martingale with $\gamma_0 = 0$; see e.g. Friedman [32, Theorem

3.1, p. 67]. Further, by Doléans' theorem, (Y_τ) is the exponential martingale given by

$$Y_\tau = \exp\left(\gamma_\tau - \frac{1}{2}\langle\gamma\rangle_\tau\right), \tag{7.15}$$

where $\langle\gamma\rangle_\tau$ denotes the quadratic variation of (γ_τ). See Rogers and Williams [71, Theorem 37.1, p. 75] or Protter [68, p. 321].

We now present some useful bounds for (X_τ) and (Y_τ) and the associated Wiener integral used in the delta formula.

Lemma 7.5. *Let (2.1),* (A_0)–(A_1) *hold. Then for* $1 \le p < \infty$*,*

(i)

$$\mathbb{E}_s\left[S_\tau^{-p}\right] \le \mathrm{const}(s, \mathcal{V}_0, T, p), \qquad \tau \in [0,T]; \tag{7.16}$$

(ii)

$$\mathbb{E}_s\left[Y_\tau^{p}\right] \le \mathrm{const}(s, \mathcal{V}_1, T, p), \qquad \tau \in [0,T]; \tag{7.17}$$

(iii)

$$\mathbb{E}_s\left[\int_0^\tau \frac{Y_r^p}{h^p(S_r)}\,\mathrm{d}r\right] \le \mathrm{const}(s, \mathcal{V}_1, T, p) \times \tau, \qquad \tau \in [0,T]. \tag{7.18}$$

Proof. We will follow Protter [68, pp. 314–315] to prove properties (i) and (ii). Let

$$\vartheta_\tau = \int_0^\tau \nu(S_r)\,\mathrm{d}W_r.$$

Then assumption (A_0) ensures that (ϑ_τ) is a continuous $\mathbb{P}$-martingale that is null at $\tau = 0$; see e.g. Friedman [32, Theorem 3.1, p. 67]. Further (A_0) implies that

$$\sup_{\tau\in[0,T]} |\langle\vartheta\rangle_\tau| \le \nu_0^2 T, \quad \mathbb{P}\text{-a.s.} \tag{7.19}$$

Hence

$$\begin{aligned}
\mathbb{E}_s\left[S_\tau^{-p}\right] &= \mathbb{E}_s\left[\left\{s\exp\left(\vartheta_\tau - \frac{1}{2}\langle\vartheta\rangle_\tau\right)\right\}^{-p}\right] \\
&= \mathbb{E}_s\left[\frac{1}{s^p}\exp\left(-p\vartheta_\tau + \frac{1}{2}p\langle\vartheta\rangle_\tau\right)\right] \\
&\le \frac{1}{s^p}\exp\left(\frac{1}{2}p\nu_0^2 T\right)\mathbb{E}_s\left[\exp\left(p\vartheta_\tau^*\right)\right],
\end{aligned} \tag{7.20}$$

where $\vartheta_\tau^* = \sup_{r\in[0,\tau]}|\vartheta_r|$. Since (ϑ_τ) is a continuous martingale, $\vartheta_\tau = \tilde{W}_{\langle\vartheta\rangle_\tau}$ is a standard Wiener process on a different filtration; see e.g. Protter [68, Theorem 42, p.

88]. Taking into account (7.19), we have

$$\vartheta_\tau^* \leq \vartheta_T^* = \tilde{W}_{\nu_0^2 T}.$$

This shows $\mathbb{E}_s\left[\exp(p\vartheta_\tau^*)\right] \leq \mathbb{E}_s[\exp(p\tilde{W}^*_{\nu_0^2 T})]$. Applying the reflection principle (see e.g. Protter [68, Theorem 33, p. 23]), we have

$$\begin{aligned}\mathbb{E}_s\left[\exp\left(p\tilde{W}^*_{\nu_0^2 T}\right)\right] &= 2\,\mathbb{E}_s\left[\exp\left(p\tilde{W}_{\nu_0^2 T}\right)\right]\\ &= 2\exp\left(p^2\nu_0^2 T/2\right).\end{aligned} \tag{7.21}$$

Combining (7.20) and (7.21) then gives property (i). Noting that (Y_τ) is the exponential martingale given by (7.15), property (ii) can be similarly proved. For property (iii), we have

$$\begin{aligned}\mathbb{E}_s\left[\int_0^\tau \frac{Y_r^p}{h^p(S_r)}\,\mathrm{d}r\right] &= \mathbb{E}_s\left[\int_0^\tau \frac{Y_r^p}{S_r^p\nu^p(S_r)}\,\mathrm{d}r\right]\\ &= \nu_0^p\,\mathbb{E}_s\left[\int_0^\tau \frac{Y_r^p}{S_r^p}\,\mathrm{d}r\right]\\ &= \nu_0^p\int_0^\tau \mathbb{E}_s\left[\frac{Y_r^p}{S_r^p}\right]\,\mathrm{d}r\\ &\leq \nu_0^p\int_0^\tau \left(\mathbb{E}_s\left[Y_r^{2p}\right]\right)^{1/2}\left(\mathbb{E}_s\left[S_r^{-2p}\right]\right)^{1/2}\,\mathrm{d}r\\ &\leq \mathrm{const}(s,\mathcal{V}_1,T,p)\times\tau,\end{aligned}$$

where in the second equality we use (A_0); in the third equality, the Fubini theorem; in the first inequality, the Cauchy–Schwarz inequality; and in the last inequality, properties (7.16) and (7.17). And the proof is complete. □

7.5 Small time asymptotics of the delta

By Proposition 7.3, the delta formula, and by adding and subtracting the term $\Big\{(S_\tau - s)_+ W_\tau/\big(\tau h(s)\big)\Big\}$, we can write the ATM delta as

$$\left.\frac{\partial C(s,\tau)}{\partial s}\right|_{s=k} = I_1(k,\tau) + I_2(k,\tau), \tag{7.22}$$

where

$$I_1(s,\tau) = \mathbb{E}_s\left[(S_\tau - s)_+ \frac{1}{\tau}\int_0^\tau \left(\frac{Y_r}{h(S_r)} - \frac{1}{h(s)}\right)\, \mathrm{d}W_r\right], \tag{7.23}$$

$$I_2(s,\tau) = \mathbb{E}_s\left[(S_\tau - s)_+ \frac{W_\tau}{\tau h(s)}\right]. \tag{7.24}$$

We now study the small-time asymptotics of I_1 and I_2.

Small time asymptotic of I_1

For I_1, we have the following lemma.

Lemma 7.6. *Let (2.1),* (A_0)–(A_2) *hold. Then uniformly in s on compact subsets of* $(0,\infty)$.

$$I_1(s,\tau) = \mathrm{O}(\tau), \qquad \text{as } \tau \to 0, \tag{7.25}$$

where $\mathrm{O} = \mathrm{O}(s, \mathcal{V}_2, T)$.

Proof. Let

$$\Psi(y,x) = \frac{y}{s\nu(s)}.$$

Then

$$\begin{aligned}
\frac{\partial \Psi}{\partial y} &= \frac{1}{s\nu(s)}, \\
\frac{\partial \Psi}{\partial s} &= -y\left(\frac{1}{s^2\nu(s)} + \frac{\nu'(s)}{s\nu^2(s)}\right), \\
\frac{\partial^2 \Psi}{\partial y \partial s} &= -\frac{1}{s^2\nu(s)} - \frac{\nu'(s)}{s\nu^2(s)}, \\
\frac{\partial^2 \Psi}{\partial y^2} &= 0, \\
\frac{\partial^2 \Psi}{\partial s^2} &= y\left(\frac{2}{s^3\nu(s)} + \frac{2\nu'(s)}{s^2\nu^2(s)} + \frac{2\big(\nu'(s)\big)^2}{s\nu^3(s)} - \frac{\nu''(s)}{s\nu^2(s)}\right).
\end{aligned}$$

Note that $\nu'(s) = \mathrm{d}\nu(s)/\,\mathrm{d}s$. By Itô's formula,

$$\begin{aligned}
\frac{Y_r}{h(S_r)} - \frac{1}{h(s)} &= \int_0^r \frac{\partial \Psi}{\partial Y}(Y_\rho, S_\rho)\, \mathrm{d}Y_\rho + \int_0^r \frac{\partial \Psi}{\partial S}(Y_\rho, S_\rho)\, \mathrm{d}S_\rho \\
&\quad + \frac{1}{2}\int_0^r \frac{\partial^2 \Psi}{\partial S^2}(Y_\rho, S_\rho)\, \mathrm{d}\left\langle S\right\rangle_\rho + \int_0^r \frac{\partial^2 \Psi}{\partial Y \partial S}(Y_\rho, S_\rho)\, \mathrm{d}\left\langle Y, X\right\rangle_\rho .
\end{aligned}$$

Simplifying the terms gives

$$\frac{Y_r}{h(S_r)} - \frac{1}{h(s)} = \int_0^r g_\rho \, \mathrm{d}\rho, \tag{7.26}$$

$$g_\rho = -Y_\rho\Big(\nu'(S_\rho) + S_\rho \nu''(S_\rho)/2\Big). \tag{7.27}$$

Hence, by the Cauchy-Schwarz inequality and the definition of I_1, (7.23), we get

$$I_1^2(s,\tau) \le \frac{1}{\tau^2}\,\mathbb{E}_s\big[(S_\tau - s)_+^2\big]\,\mathbb{E}_s\left[\left(\int_0^\tau \int_0^r g_\rho \, \mathrm{d}\rho \, \mathrm{d}W_r\right)^2\right]. \tag{7.28}$$

It is well known that

$$\mathbb{E}_s\left[(S_\tau - s)^2\right] \le \mathrm{const}(s) \times \tau; \tag{7.29}$$

see e.g. Friedman [32, Theorem 2.3, p. 107]. So it remains to get the desired bound for the second expectation.

Using firstly Itô's formula and then the fact that

$$\left|\int_0^r g_\rho \, \mathrm{d}\rho\right| \le \int_0^r |g_\rho| \, \mathrm{d}\rho \le r \sup_{\rho\in[0,r]} |g_\rho|,$$

we obtain

$$\begin{aligned}
\mathbb{E}_s\left[\left(\int_0^\tau \int_0^r g_\rho \, \mathrm{d}\rho \, \mathrm{d}W_r\right)^2\right] &= \mathbb{E}_s\left[\int_0^\tau \left(\int_0^r g_\rho \, \mathrm{d}\rho\right)^2 \mathrm{d}r\right] \\
&\le \mathbb{E}_s\left[\int_0^\tau r^2 \left(\sup_{\rho\in[0,r]} |g_\rho|\right)^2 \mathrm{d}r\right] \\
&= \int_0^\tau r^2\,\mathbb{E}_s\left[\left(\sup_{\rho\in[0,r]} |g_\rho|\right)^2\right] \mathrm{d}r \\
&\le \mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]} |g_r|\right)^2\right] \int_0^\tau r^2 \, \mathrm{d}r \\
&= \frac{\tau^3}{3}\,\mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]} |g_r|\right)^2\right].
\end{aligned} \tag{7.30}$$

We will show that the expectation involving g_r is bounded, whereby the use of the Itô isometry and the Fubini theorem in the first and third steps above are justified. By

(A_0)–(A_2),

$$\begin{aligned}|g_\rho| &= \left|\frac{Y_\rho}{S_\rho}\right| \times \left|S_\rho \nu'(S_\rho) + \frac{S_\rho^2 \nu''(S_\rho)}{2}\right| \\ &= \mathrm{const}(\mathcal{V}_2)\left|Y_\rho/S_\rho\right|.\end{aligned}$$

As a result,

$$\begin{aligned}\mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|g_r|\right)^2\right] &\le \mathrm{const}(\mathcal{V}_2)\times \mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|Y_r/S_r|\right)^2\right] \\ &\le \mathrm{const}(\mathcal{V}_2)\times \mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|Y_r|\right)^2\left(\sup_{r\in[0,\tau]}|1/S_r|\right)^2\right] \\ &\le \mathrm{const}(\mathcal{V}_2)\times \left\{\mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|Y_r|\right)^4\right]\mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|1/S_r|\right)^4\right]\right\}^{1/2},\end{aligned} \tag{7.31}$$

where in the last step we have used the Cauchy–Schwarz inequality. Recall from (7.15) that (Y_τ) is a continuous $\mathbb{P}$-martingale. Applying firstly the martingale moment inequalities, see e.g. Karatzas and Shreve [53, Proposition 3.26, p. 163], and secondly Lemma 7.5, we obtain for all $\tau \in [0,T]$,

$$\begin{aligned}\mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|Y_r|\right)^4\right] &\le \mathrm{const}(2)\times \mathbb{E}_s\left[\langle Y\rangle_\tau^2\right] \\ &\le \mathrm{const}(2)\times \mathbb{E}_s\left[|Y_\tau|^4\right] \\ &\le \mathrm{const}(s,\mathcal{V}_1,T,4),\end{aligned} \tag{7.32}$$

where const(2) and $\mathrm{const}(s,\mathcal{V}_1,T,4)$ are respectively constants depending on the number 2, and on s, $\mathcal{V}_1$, T, and the number 4. Similarly, we can rewrite $(1/S_r)$ as a product of a martingale and a bounded process and apply the martingale inequalities to bound the expectation involving $(1/S_r)$. Rewrite $(1/S_\tau)$ as

$$\begin{aligned}S_\tau^{-1} &= \left\{s\exp\left(\vartheta_\tau - \frac{1}{2}\langle\vartheta\rangle_\tau\right)\right\}^{-1} \\ &= M_\tau N_\tau,\end{aligned}$$

where

$$
\begin{aligned}
M_\tau &= \exp\left(-\vartheta_\tau - \frac{1}{2}\langle\vartheta\rangle_\tau\right), \\
N_\tau &= \frac{1}{s}\exp\left(\langle\vartheta\rangle_\tau\right).
\end{aligned}
$$

Note that (M_τ) is a martingale; see e.g. Rogers and Williams [71, Theorem 37.8, p. 77]. On the other hand, by (7.19) and the definition of (N_τ), we have for all $\tau \in [0, T]$,

$$
0 < 1/s \leq N_\tau \leq \mathrm{const}(s, \mathcal{V}_0, T), \quad \mathbb{P}\text{-a.s.} \tag{7.33}
$$

Hence, for all $\tau \in [0, T]$,

$$
\begin{aligned}
\mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|1/S_r|\right)^4\right] &= \mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|M_r N_r|\right)^4\right] \\
&\leq \mathrm{const}(s, \mathcal{V}_0, T) \times \mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|M_r|\right)^4\right] \\
&\leq \mathrm{const}(s, \mathcal{V}_0, T, 2) \times \mathbb{E}_s\left[\langle M\rangle_\tau^2\right] \\
&\leq \mathrm{const}(s, \mathcal{V}_0, T, 2) \times \mathbb{E}_s\left[|M_\tau|^4\right] \\
&= \mathrm{const}(s, \mathcal{V}_0, T, 2) \times \mathbb{E}_s\left[\left|S_\tau^{-1} N_\tau^{-1}\right|^4\right] \\
&\leq \mathrm{const}(s, \mathcal{V}_0, T, 2) \times \left\{\mathbb{E}_s\left[\left|S_\tau^{-1}\right|^8\right]\right\}^{1/2}\left\{\mathbb{E}_s\left[\left|N_\tau^{-1}\right|^8\right]\right\}^{1/2} \\
&\leq \mathrm{const}(s, \mathcal{V}_0, T, 2) \times \left\{\mathbb{E}_s\left[\left|S_\tau^{-1}\right|^8\right]\right\}^{1/2} \\
&= \mathrm{const}(s, \mathcal{V}_0, T, 2) \times \left\{\mathbb{E}_s\left[S_\tau^{-8}\right]\right\}^{1/2} \\
&\leq \mathrm{const}(s, \mathcal{V}_0, T, 8),
\end{aligned} \tag{7.34}
$$

where the first inequality results from (7.33); the second and third inequalities from the martingale inequalities; the forth inequality from the Cauchy–Schwarz inequality; and the last inequality from Lemma 7.5. Now combining (7.31), (7.32) and (7.34) gives

$$
\mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}|g_r|\right)^2\right] \leq \mathrm{const}(s, \mathcal{V}_1, T, 8). \tag{7.35}
$$

This inequality, together with (7.28), (7.29) and (7.30), implies that for all $\tau \in [0, T]$,

$$I_1^2(s, \tau) \leq \frac{1}{\tau^2} \times \tau \times \tau^3 \times \mathrm{const}(s, \mathcal{V}_1, T, 8) = \tau^2 \times \mathrm{const}(s, \mathcal{V}_1, T, 8).$$

And the proof is complete. $\square$

Small-time asymptotic of I_2

It is a little more involved to derive the small-time asymptotic of I_2. We shall need a preliminary lemma.

Let Z be a standard normal random variable, i.e. $Z \sim N(0, 1)$. Recall that $h(s) = s\nu(s)$ and $h'(s) = \nu(s) + s\nu'(s)$. Define for any $s \in (0, \infty)$,

$$H(s, \tau) := \mathbb{E}\left[\left(Z + \frac{\sqrt{\tau}}{2} h'(s)(Z^2 - 1)\right)_+ Z\right]. \tag{7.36}$$

Then we have the following lemma.

Lemma 7.7. *For any $(s, \tau) \in (0, \infty) \times [0, T]$, the following is true:*

(i) if $h'(s) = 0$, then

$$H(s, \tau) \equiv 1/2 \quad \forall\, \tau \in [0, T]; \tag{7.37}$$

(ii) if $h'(s) \neq 0$, then as $\tau \to 0$,

$$H(s, \tau) = \frac{1}{2} + \frac{\sqrt{\tau}}{2\sqrt{2\pi}} h'(s) + \mathrm{O}(\tau), \qquad \mathrm{O} = \mathrm{O}(s, \mathcal{V}_2, T). \tag{7.38}$$

Proof. **Case (i) $h'(s) = 0$.** In this case as $Z \sim N(0, 1)$, (7.36) implies

$$H(s, \tau) \equiv \mathbb{E}[Z_+ Z] = 1/2, \quad \forall\, \tau \in [0, T].$$

Case (ii) $h'(s) \neq 0$. There are two subcases: (a) $h'(s) > 0$, and (b) $h'(s) < 0$. For ease of exposition we will in what follows often suppress the arguments of the functions, e.g. writing h' for $h'(s)$. In both subcases (a) and (b), we will let $z_1(s, \tau)$ and $z_2(s, \tau)$ be the roots of the quadratic polynomial $q(z) = z + \sqrt{\tau} h'(z^2 - 1)/2$, i.e.

$$z_1(s, \tau) = \frac{-1 - \sqrt{1 + \tau(h'(s))^2}}{\sqrt{\tau} h'(s)}, \quad z_2(s, \tau) = \frac{-1 + \sqrt{1 + \tau(h'(s))^2}}{\sqrt{\tau} h'(s)}.$$

Subcase (ii)(a) $h'(s) > 0$. Here $z_1 < z_2$, and with s fixed, $z_{1,2}$ has the following properties as $\tau \to 0$:

$$\begin{cases} z_1 \searrow -\infty, \\ z_2 \searrow 0, \end{cases} \quad \begin{cases} \partial_\tau z_1 \nearrow +\infty, \\ \partial_\tau z_2 \nearrow 0, \end{cases} \quad \begin{cases} \partial_\tau^2 z_1 \searrow -\infty, \\ \partial_\tau^2 z_2 \searrow -\infty, \end{cases} \quad \begin{cases} \partial_\tau^3 z_1 \nearrow +\infty, \\ \partial_\tau^3 z_2 \nearrow +\infty. \end{cases} \tag{7.39}$$

Also, with s fixed, as a function of $\bar{\tau} = \sqrt{\tau}$, $z_{1,2}$ has the following properties when $h'(s) > 0$ and $\bar{\tau} \to 0$:

$$\begin{cases} z_1 \searrow -\infty, \\ z_2 \searrow 0, \end{cases} \quad \begin{cases} \partial_{\bar{\tau}} z_1 \nearrow +\infty, \\ \partial_{\bar{\tau}} z_2 \to h'/2, \end{cases} \quad \begin{cases} \partial_{\bar{\tau}}^2 z_1 \searrow -\infty, \\ \partial_{\bar{\tau}}^2 z_2 \searrow 0, \end{cases} \quad \begin{cases} \partial_{\bar{\tau}}^3 z_1 \nearrow +\infty, \\ \partial_{\bar{\tau}}^3 z_2 \to -3(h')^3/4. \end{cases} \tag{7.40}$$

By (7.36) and explicit integration we have

$$\begin{aligned} H(S,\tau) &= \int_{-\infty}^{z_1} \left(z + \frac{\sqrt{\tau}h'}{2}(z^2-1) \right) z \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}\, \mathrm{d}z \\ &\quad + \int_{z_2}^{\infty} \left(z + \frac{\sqrt{\tau}h'}{2}(z^2-1) \right) z \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}\, \mathrm{d}z \\ &= \left\{ 1 + \frac{1}{2}\operatorname{erf}\left(\frac{z_1}{\sqrt{2}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{z_2}{\sqrt{2}}\right) \right\} + \left\{ \frac{\sqrt{\tau}h'}{\sqrt{2\pi}} \left(\mathrm{e}^{-z_2^2/2} - \mathrm{e}^{-z_1^2/2} \right) \right\} \\ &= f_1(s,\tau) + g_1(s,\tau), \end{aligned}$$

where $\operatorname{erf}(\cdot)$ denotes the error function, and f_1 and g_1 denote respectively the terms inside the first and second set of braces. Then coupled with (7.40), a Taylor expansion of $f_1(s,\tau)$ in $\sqrt{\tau}$ about 0 gives, as $\tau \to 0$,

$$f_1(s,\tau) = \frac{1}{2} - \frac{\sqrt{\tau}h'}{2\sqrt{2\pi}} + \mathrm{O}(\tau),$$

where $\mathrm{O} = \mathrm{O}(s, \mathcal{V}_2, T)$. Similarly, by a Taylor expansion of $(\mathrm{e}^{-z_2^2/2} - \mathrm{e}^{-z_1^2/2})$ in τ about 0, the L'Hopital rule, and (7.39), we get, as $\tau \to 0$,

$$g_1(s,\tau) = \frac{\sqrt{\tau}h'}{\sqrt{2\pi}}\big(1 + \mathrm{O}(\tau)\big) = \frac{\sqrt{\tau}h'}{\sqrt{2\pi}} + \mathrm{O}(\tau),$$

where $\mathrm{O} = \mathrm{O}(s, \mathcal{V}_2, T)$.

Hence, when $h'(s) > 0$,

$$H(s,\tau) = \frac{1}{2} + \frac{\sqrt{\tau}h'}{2\sqrt{2\pi}} + \mathrm{O}(\tau).$$

Subcase (ii)(b) $h'(s) < 0$. Here $z_2 < z_1$, and with s fixed, $z_{1,2}$ has the following properties as $\tau \to 0$:

$$\begin{cases} z_1 \nearrow +\infty, \\ z_2 \nearrow 0, \end{cases} \quad \begin{cases} \partial_\tau z_1 \searrow -\infty, \\ \partial_\tau z_2 \searrow 0, \end{cases} \quad \begin{cases} \partial_\tau^2 z_1 \nearrow +\infty, \\ \partial_\tau^2 z_2 \nearrow +\infty, \end{cases} \quad \begin{cases} \partial_\tau^3 z_1 \searrow -\infty, \\ \partial_\tau^3 z_2 \searrow -\infty. \end{cases} \tag{7.41}$$

Also, with s fixed, as a function of $\bar{\tau} = \sqrt{\tau}$, $z_{1,2}$ has the following properties when $h'(s) < 0$ and $\bar{\tau} \to 0$:

$$\begin{cases} z_1 \nearrow +\infty, \\ z_2 \nearrow 0, \end{cases} \quad \begin{cases} \partial_{\bar{\tau}} z_1 \searrow -\infty, \\ \partial_{\bar{\tau}} z_2 \to h'/2, \end{cases} \quad \begin{cases} \partial_{\bar{\tau}}^2 z_1 \nearrow +\infty, \\ \partial_{\bar{\tau}}^2 z_2 \nearrow 0, \end{cases} \quad \begin{cases} \partial_{\bar{\tau}}^3 z_1 \searrow -\infty, \\ \partial_{\bar{\tau}}^3 z_2 \to -3(h')^3/4. \end{cases} \tag{7.42}$$

By (7.36) and explicit integration we have

$$\begin{aligned} H(s,\tau) &= \int_{z_2}^{z_1} \left(z + \frac{\sqrt{\tau} h'}{2}(z^2 - 1) \right) z \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}\,\mathrm{d}z \\ &= \left\{ \frac{1}{2} \operatorname{erf}\left(\frac{z_1}{\sqrt{2}} \right) - \frac{1}{2} \operatorname{erf}\left(\frac{z_2}{\sqrt{2}} \right) \right\} + \left\{ \frac{\sqrt{\tau} h'}{\sqrt{2\pi}} \left(\mathrm{e}^{-z_2^2} - \mathrm{e}^{-z_1^2/2} \right) \right\} \\ &= f_2(s,\tau) + g_2(s,\tau), \end{aligned}$$

where f_2 and g_2 denote respectively the terms inside the first and second set of braces. Then coupled with (7.42) and with s fixed, a Taylor expansion of $f_2(s,\tau)$ in $\sqrt{\tau}$ about 0 gives, as $\tau \to 0$,

$$f_2(s,\tau) = \frac{1}{2} - \frac{\sqrt{\tau} h'}{2\sqrt{2\pi}} + \mathrm{O}(\tau),$$

where $\mathrm{O} = \mathrm{O}(s, \mathcal{V}_2, T)$. Similarly, by a Taylor expansion of $(\mathrm{e}^{-z_2^2/2} - \mathrm{e}^{-z_1^2/2})$ in τ about 0, the L'Hopital rule, and (7.41), we get, as $\tau \to 0$,

$$g_2(s,\tau) = \frac{\sqrt{\tau} h'}{\sqrt{2\pi}} \big(1 + O(\tau)\big) = \frac{\sqrt{\tau} h'}{\sqrt{2\pi}} + \mathrm{O}(\tau),$$

where $\mathrm{O} = \mathrm{O}(s, \mathcal{V}_2, T)$. Hence, when $h'(s) < 0$, we again have

$$H(s,\tau) = \frac{1}{2} + \frac{\sqrt{\tau} h'}{2\sqrt{2\pi}} + \mathrm{O}(\tau). \tag{7.43}$$

Therefore, (7.43) holds whenever $h'(s) \neq 0$, and this completes the proof. □

We will need the following lemma.

Lemma 7.8. *Let (2.1),* (A_0)–(A_2) *hold. Then uniformly in s on compact subsets of* $(0,\infty)$,

$$I_2(s,\tau) = H(s,\tau) + \mathrm{o}(\tau^{1/2}), \qquad \text{as } \tau \to 0. \tag{7.44}$$

Proof. By (7.4), we have

$$
\begin{aligned}
I_2 &= \mathbb{E}_s\left[\left(\int_0^\tau h(S_r)\,\mathrm{d}W_r\right)_+ \frac{W_\tau}{\tau h(s)}\right] \\
&= \frac{1}{h(s)}\,\mathbb{E}_s\left[\left(h(s)\frac{W_\tau}{\sqrt{\tau}} + \frac{1}{\sqrt{\tau}}\int_0^\tau \Big(h(S_r) - h(s)\Big)\,\mathrm{d}W_r\right)_+ \frac{W_\tau}{\sqrt{\tau}}\right] \\
&= \frac{1}{h(s)}\,\mathbb{E}_s\left[\left(h(s)Z + \frac{1}{\sqrt{\tau}}\int_0^\tau \Big(h(S_r) - h(s)\Big)\,\mathrm{d}W_r\right)_+ Z\right],
\end{aligned}
\tag{7.45}
$$

where in the last equality we have used $(W_\tau/\sqrt{\tau}) \overset{\mathrm{d}}{=} Z \sim N(0,1)$. In this thesis, the symbol $\overset{\mathrm{d}}{=}$ means equality in distribution or law. Then by the definition of $h(\cdot)$, and by Itô's formula, we have

$$
\begin{aligned}
\mathfrak{J} &:= \frac{1}{\sqrt{\tau}}\int_0^\tau \Big(h(S_r) - h(s)\Big)\,\mathrm{d}W_u \\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau\int_0^r h'(S_\rho)h(S_\rho)\,\mathrm{d}W_\rho\,\mathrm{d}W_r + \frac{1}{2\sqrt{\tau}}\int_0^\tau\int_0^r h'(S_\rho)h^2(S_\rho)\,\mathrm{d}\rho\,\mathrm{d}W_r \\
&= \frac{1}{\sqrt{\tau}}\int_0^\tau\int_0^s h'(s)s\nu(s)\,\mathrm{d}W_\rho\,\mathrm{d}W_r + \frac{1}{2\sqrt{\tau}}\int_0^\tau\int_0^r h'(S_\rho)h^2(S_\rho)\,\mathrm{d}\rho\,\mathrm{d}W_r \\
&\quad + \frac{1}{\sqrt{\tau}}\int_0^\tau\int_0^r \Big(h'(S_\rho)h(S_\rho) - h'(s)h(s)\Big)\,\mathrm{d}W_\rho\,\mathrm{d}W_r \\
&= \mathfrak{J}_1 + \mathfrak{J}_2 + \mathfrak{J}_3,
\end{aligned}
\tag{7.46}
$$

where $\mathfrak{J}_i$, $i = 1,2,3$, denotes the ith integral term in the penultimate equality. We will show that the properties of the $\mathfrak{J}_i$'s give the desired result. We will check $\mathfrak{J}_1$ first. Since

$$
\int_0^\tau\int_0^r \mathrm{d}W_\rho\,\mathrm{d}W_r = \frac{1}{2}(W_\tau^2 - \tau),
$$

we have

$$
\begin{aligned}
\mathfrak{J}_1 &= \frac{\sqrt{\tau}}{2}h'(s)h(s)\left(\left(\frac{W_\tau}{\sqrt{\tau}}\right)^2 - 1\right) \\
&\overset{\mathrm{d}}{=} \frac{\sqrt{\tau}}{2}h'(s)h(s)(Z^2 - 1).
\end{aligned}
\tag{7.47}
$$

Next, we will check $\mathfrak{J}_2$. As in (7.30), we apply the Itô isometry and the Fubini theorem to get

$$\begin{aligned}\mathbb{E}_s\left[|J_2|^2\right] &= \frac{1}{4\tau}\,\mathbb{E}_s\left[\left(\int_0^\tau\int_0^r h'(S_\rho)h^2(S_\rho)\,\mathrm{d}\rho\,\mathrm{d}W_r\right)^2\right]\\ &= \frac{1}{4\tau}\,\mathbb{E}_s\left[\int_0^\tau\left(\int_0^r h'(S_\rho)h^2(S_\rho)\,\mathrm{d}\rho\right)^2\mathrm{d}r\right]\\ &\le \frac{1}{4\tau}\times\frac{\tau^3}{3}\,\mathbb{E}_s\left[\left(\sup_{r\in[0,\tau]}\left|h'(S_r)h^2(S_r)\right|\right)^2\right]\\ &\le \tau^2\times\mathrm{const}(s,\mathcal{V}_1,T,2),\end{aligned}\tag{7.48}$$

where the bound $\mathrm{const}(s,\mathcal{V}_1,T,2)$ is similarly obtained as the bound in (7.35). Lastly, we will check $\mathfrak{J}_3$. Let

$$\begin{aligned}\zeta_\rho &= h'(S_\rho)h(S_\rho)-h'(s)h(s),\\ \zeta^*_{\rho,\tau} &= \left(\sup_{\rho\in[0,\tau]}\mathbb{E}_s\left[\zeta_\rho^2\right]\right)^{1/2}.\end{aligned}$$

Since (S_τ) is a continuous process and $\nu\in C^2$, we have $\zeta^*_{\rho,\tau}\xrightarrow{\tau\to 0}0$. Then the Itô isometry and the Fubini theorem will give

$$\begin{aligned}\mathbb{E}_s\left[|\mathfrak{J}_3|^2\right] &= \frac{1}{\tau}\,\mathbb{E}_s\left[\left|\int_0^\tau\int_0^r \zeta_\rho\,\mathrm{d}W_\rho\,\mathrm{d}W_r\right|^2\right]\\ &= \frac{1}{\tau}\,\mathbb{E}_s\left[\int_0^\tau\left(\int_0^r\zeta_\rho\,\mathrm{d}W_\rho\right)^2\mathrm{d}r\right]\\ &= \frac{1}{\tau}\int_0^\tau\mathbb{E}_s\left[\left(\int_0^r\zeta_\rho\,\mathrm{d}W_\rho\right)^2\right]\mathrm{d}r\\ &= \frac{1}{\tau}\int_0^\tau\mathbb{E}_s\left[\int_0^r\zeta_\rho^2\,\mathrm{d}\rho\right]\mathrm{d}r\\ &= \frac{1}{\tau}\int_0^\tau\int_0^s\mathbb{E}_s\left[\zeta_\rho^2\right]\,\mathrm{d}\rho\,\mathrm{d}r\\ &\le \frac{(\zeta^*_{\rho,\tau})^2}{\tau}\int_0^\tau\int_0^r\mathrm{d}\rho\,\mathrm{d}r\\ &= (\zeta^*_{\rho,\tau})^2\tau/2.\end{aligned}\tag{7.49}$$

We now combine the small time asymptotics of the $\mathfrak{J}_i$'s to show the desired result. By

(7.45), (7.46), and (7.47) we can write I_2 as

$$I_2(s,\tau) = \frac{1}{h(s)}\,\mathbb{E}_s\left[\left(h(s)Z + \sum_{i=1}^{3}\mathfrak{J}_i\right)_+ Z\right].$$

Then by the definition of $H(s,\tau)$,

$$\begin{aligned}
d &:= |I_2(s,\tau) - H(s,\tau)| \\
&= \left| I_2 - \frac{1}{h(s)}\,\mathbb{E}_s\left[\left(h(s)Z + \mathfrak{J}_1\right)_+ Z\right]\right| \\
&= \frac{1}{h(s)}\left|\mathbb{E}_s\left[\left(\left(h(s)Z + \sum_{i=1}^{3}\mathfrak{J}_i\right)_+ - \left(h(s)Z + \mathfrak{J}_1\right)_+\right) Z\right]\right| \\
&\le \frac{1}{h(s)}\,\mathbb{E}_s\left[\left|\left(\left(h(s)Z + \sum_{i=1}^{3}\mathfrak{J}_i\right)_+ - \left(h(s)Z + \mathfrak{J}_1\right)_+\right) Z\right|\right] \\
&= \frac{1}{h(s)}\,\mathbb{E}_s\left[\left|\left(h(s)Z + \sum_{i=1}^{3}\mathfrak{J}_i\right)_+ - \left(h(s)Z + \mathfrak{J}_1\right)_+\right| |Z|\right] \\
&\le \frac{1}{h(s)}\,\mathbb{E}_s\left[\left|\left(h(s)Z + \sum_{i=1}^{3}\mathfrak{J}_i\right) - \left(h(s)Z + \mathfrak{J}_1\right)\right| |Z|\right] \\
&\le \frac{1}{h(s)}\,\mathbb{E}_s\left[|\mathfrak{J}_2 + \mathfrak{J}_3|\,|Z|\right] \\
&\le \frac{1}{h(s)}\Big(\mathbb{E}_s[|\mathfrak{J}_2|\,|Z|] + \mathbb{E}_s[|\mathfrak{J}_3|\,|Z|]\Big) \\
&\le \frac{1}{h(s)}\left(\left(\mathbb{E}_s\left[|\mathfrak{J}_2|^2\right]\right)^{1/2}\left(\mathbb{E}_s\left[|Z|^2\right]\right)^{1/2} + \left(\mathbb{E}_s\left[|\mathfrak{J}_3|^2\right]\right)^{1/2}\left(\mathbb{E}_s\left[|Z|^2\right]\right)^{1/2}\right) \\
&= \frac{1}{h(s)}\left(\left(\mathbb{E}_s\left[|\mathfrak{J}_2|^2\right]\right)^{1/2} + \left(\mathbb{E}_s\left[|\mathfrak{J}_3|^2\right]\right)^{1/2}\right) \\
&\le \frac{1}{h(s)}\left(\tau \times \mathrm{const}(s,\mathcal{V}_1,T,2) + \frac{\zeta^*_{\rho,\tau}\sqrt{\tau}}{2}\right),
\end{aligned}$$

where the fifth step follows from the inequality $|x_+ - y_+| \le |x - y|$; the second last step from the fact that $\mathbb{E}_s[|Z|^2] = 1$; the last step from (7.48) and (7.49). Since $\zeta^*_{\rho,\tau} \xrightarrow{\tau\to 0} 0$, we have $d/\sqrt{\tau} \xrightarrow{\tau\to 0} 0$, namely, $I_2(s,\tau) = H(s,\tau) + \mathrm{o}(\tau^{1/2})$ as required. □

Summarizing (7.22)–(7.25) and (7.44), we have the following proposition:

Proposition 7.9. *Let (2.1),* (A_0)–(A_2) *hold. Then uniform in k on compact subsets* *of* $(0,\infty)$*, we have, as* $\tau \to 0$*,*

$$\left.\frac{\partial C(s,\tau)}{\partial s}\right|_{s=k} = H(k,\tau) + \mathrm{O}(\tau) + \mathrm{o}(\tau^{1/2}), \tag{7.50}$$

where $\mathrm{O} = \mathrm{O}(k, \mathcal{V}_2, T)$.

7.6 Proof of the ATM theorem

We are now ready to prove Theorem 7.1.

Proof of Theorem 7.1. By (7.5) and (7.50), we have, as $\tau \to 0$,

$$\left.\frac{\partial \phi(s,\tau)}{\partial s}\right|_{s=k} = \frac{\sqrt{2\pi}}{k\sqrt{\tau}}\left(H(k,\tau) - \frac{1}{2} - \frac{\nu(k)\sqrt{\tau}}{2\sqrt{2\pi}} + \mathrm{O}(\tau) + \mathrm{o}(\tau^{1/2})\right), \tag{7.51}$$

where $\mathrm{O} = \mathrm{O}(k, \mathcal{V}_2, T)$. Then according to Lemma 7.7, there are two cases to check.

Case (i) $h'(k) = 0$**.** In this case, recalling from (7.2) that $h(s) = s\nu(s)$, we get

$$h'(k) = \nu(k) + k\nu'(k) = 0 \quad \Longrightarrow \quad \nu(k)/k = -\nu'(k).$$

Further, (7.37) gives $H(k,\tau) = 1/2$. Hence

$$\begin{aligned}\left.\frac{\partial \phi(s,\tau)}{\partial s}\right|_{s=k} &= -\frac{\nu(k)}{2k} + \mathrm{O}(\tau^{1/2}) + \mathrm{o}(1) \\ &= \frac{\nu'(k)}{2} + \mathrm{O}(\tau^{1/2}) + \mathrm{o}(1) \\ &\longrightarrow \frac{\nu'(k)}{2} \quad \text{as} \quad \tau \to 0.\end{aligned}$$

Case (ii) $h'(k) \neq 0$**.** From (7.51) and (7.38), and again recalling from (7.2) that $h(s) = s\nu(s)$, we get

$$\left.\frac{\partial \phi(s,\tau)}{\partial s}\right|_{s=k} = \frac{\nu'(k)}{2} + \mathrm{O}(\tau^{1/2}) + \mathrm{o}(1) \longrightarrow \frac{\nu'(k)}{2} \quad \text{as} \quad \tau \to 0.$$

As the convergence in both cases is uniform on compact subsets of $(0,\infty) \times [0,T]$, the conclusion of the theorem is proved. $\square$

Chapter 8

Small time asymptotics of gradient and Hessian of implied volatilities

In this chapter we will derive a small time asymptotic formula for the gradient of the implied volatility. This formula holds regardless whether the option is at the money. Playing a key role in the derivation is a series representation formula for solutions of second order parabolic equations. Coupled with the PDE characterization of the implied volatility, this gradient asymptotic result also brings forth some asymptotics of the Hessian of the implied volatility.

This chapter is organised as follows. In Section 8.1, we present the main results of the chapter. In Section 8.2, we lay out the ideas behind the proofs. In Section 8.3, we recall some facts about fundamental solutions for second order linear parabolic PDEs. In Sections 8.4 and 8.5, we derive representation formulas for the call option price and the gradient of the implied volatility. In Section 8.6, we prove some auxiliary limits. After stating a technical theorem in Section 8.7, we will prove the main result of the chapter in Section 8.8.

8.1 Main results of the chapter

Recall that under the conditions of Theorem 5.25 or 6.2, the implied volatility $\phi(s, \tau)$ tends to the initial function $\phi^0(s)$ as τ goes to zero, where from (1.12),

$$\phi^0(s) \equiv \phi^0(s; k) = [\ln(s/k)] \left(\int_k^s \frac{\mathrm{d}z}{z\nu(z)} \right)^{-1}, \quad s, k \in (0, \infty). \tag{8.1}$$

This gives, for $s \in (0, \infty)$,

$$\begin{aligned}
\phi_s^0(s) &= \frac{\phi^0(s)}{s\ln(s/k)} - \frac{[\phi^0(s)]^2}{s\nu(s)\ln(s/k)}, \\
\phi_{ss}^0(s) &= -\frac{\phi^0(s)}{s^2} - \frac{2[\phi^0(s)]^2}{s^2\nu(s)[\ln(s/k)]^2} \\
&\quad + \frac{2[\phi^0(s)]^3}{s^2\nu^2(s)[\ln(s/k)]^2} + \frac{[\phi^0(s)]^2}{s^2\nu(s)\ln(s/k)} + \frac{\nu_s(s)[\phi^0(s)]^2}{s\nu^2(s)\ln(s/k)}.
\end{aligned} \tag{8.2}$$

Bearing in mind these formulas, we are ready to state the main result of this chapter.

Theorem 8.1 (Gradient and Hessian asymptotics for ϕ)**.** *Let (2.1),* (A_0)–(A_4) *hold. Then the implied volatility ϕ has the following properties:*

(i) For each $s \in (0, \infty)$,

$$\lim_{\tau\to 0} \phi_s(s,\tau) = \phi_s^0(s). \tag{8.3}$$

(ii) For each $k \in (0, \infty)$,

$$\lim_{\tau\to 0} \phi_{ss}(k,\tau) = \phi_{ss}^0(k) = \frac{\nu_{ss}(k)}{3} - \frac{\nu_s^2(k)}{6\nu(k)} - \frac{\nu_s(k)}{6k}. \tag{8.4}$$

(iii) For every $s \in (0, \infty)$,

$$\lim_{\tau\to 0} \tau\phi_{ss}(s,\tau) = 0. \tag{8.5}$$

Instead of proving Theorem 8.1 directly in the (s,τ) coordinates, we shall prove an equivalent theorem in the (x,τ) coordinates. Recall that the (transformed) initial implied volatility function I is defined by

$$I(x) = x\left(\int_0^x \frac{\mathrm{d}z}{\sigma(z)}\right)^{-1}, \qquad x \in \mathbb{R}, \tag{8.6}$$

and

$$\begin{aligned}
I_x &= \frac{I}{x} - \frac{I^2}{x\sigma}, \\
I_{xx} &= -\frac{2I^2}{x^2\sigma} + \frac{2I^3}{x^2\sigma^2} + \frac{\sigma_x I^2}{x\sigma^2};
\end{aligned} \tag{8.7}$$

see (5.7) and (5.49). Recall also that $x = \ln(s/k)$; so "at the money" means $s = k$ and $x = 0$, and vice versa.

The following theorem is equivalent to Theorem 8.1. We will omit the proof of the equivalence of Theorems 8.1 and 8.2 as it is a direct consequence of the change of variables $x = \ln(s/k)$.

Theorem 8.2 (Gradient and Hessian asymptotics for φ)**.** *Let (2.1),* (A_0)–(A_4) *hold. Then the (transformed) implied volatility $\varphi(x,\tau)$ possesses the following properties:*

(i) For each $x \in \mathbb{R}$,

$$\lim_{\tau\to 0} \varphi_x(x,\tau) = I_x(x). \tag{8.8}$$

(ii) When $x = 0$,

$$\lim_{\tau\to 0} \varphi_{xx}(0,\tau) = I_{xx}(0) = \frac{\sigma_{xx}(0)}{3} - \frac{\sigma_x^2(0)}{6\sigma(0)}. \tag{8.9}$$

(iii) For every $x \in \mathbb{R}$,

$$\lim_{\tau\to 0} \tau\varphi_{xx}(x,\tau) = 0. \tag{8.10}$$

Remark 8.3. Property (i) is the core result of the theorem; properties (ii) and (iii) are simple consequences of it.

Remark 8.4 (Warning)**.** In this chapter, (2.1), (A_0)–(A_4) are always assumed to hold. So by Theorem 5.25 or 6.2, $\tau\varphi^2(x,\tau) \to 0$ as $\tau \to 0$ for all $x \in \mathbb{R}$. In the rest of this chapter this zero limit of $\tau\varphi^2$ will be repeatedly used in the proofs, as if it is a fact and mostly without explicit reference to Theorem 5.25 or 6.2.

8.2 Ideas of the proofs

As we shall prove Theorem 8.2 only, we will from now on switch to the (x,τ) coordinates. By Remark 8.3, we shall only outline our plan for the proof of property (i), the small time limit for the gradient φ_x.

By Lemma 5.5 and Theorem 5.25, the (transformed) implied volatility φ belongs to $C^{2,1}(\mathbb{R}\times[0,T])$ and satisfies

$$v(x,\tau) = B(x,\tau;\varphi(x,\tau)), \quad \forall\,(x,\tau) \in \mathbb{R}\times[0,T], \tag{8.11}$$

where B is the Black–Scholes formula defined by (5.18) and v is the call option price satisfying (5.29). Implicit differentiation of (8.11) and rearrangement of the derivatives would show

$$\varphi_x = (v_x - B_x)/B_\varphi, \quad (x,\tau) \in \mathbb{R}\times(0,T].$$

By (5.35) and (5.37),

$$\begin{aligned} B_x &= \mathrm{e}^x N(d_1), \\ B_\varphi &= \sqrt{\tau} n(d_2) = \frac{\sqrt{\tau}}{\sqrt{2\pi}} \mathrm{e}^{-x^2/(2\tau\varphi^2)} \mathrm{e}^{x/2} \mathrm{e}^{-\tau\varphi^2/8}. \end{aligned}$$

Hence explicitly φ_x can be written as

$$\begin{aligned} \varphi_x &= \frac{1}{B_\varphi}(v_x - B_x) \\ &= \frac{\sqrt{2\pi}}{\sqrt{\tau}} \mathrm{e}^{x^2/(2\tau\varphi^2)} \mathrm{e}^{x/2} \mathrm{e}^{\tau\varphi^2/8} \Big[\mathrm{e}^{-x} v_x - N(d_1) \Big], \quad (x,\tau) \in \mathbb{R} \times (0,T]. \end{aligned} \tag{8.12}$$

If (2.1), (A_0)–(A_4) hold, then by Theorem 5.25 or 6.2, $\tau\varphi^2(x,\tau) \to 0$ as $\tau \to 0$. This implies that when $x \neq 0$, the coefficient outside the parentheses grows exponentially to positive infinity as τ tends to zero, and when $x = 0$, the coefficient grows like $\tau^{-1/2}$. To cancel out the exponential or the $\tau^{-1/2}$ growth, we will decompose the difference inside the square brackets into equally fast decaying terms as follows.

By (5.19), we can decompose $N(d_1)$ into two terms:

$$\begin{aligned} N(d_1) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \mathrm{e}^{-z^2/2} \, \mathrm{d}z \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x/(\sqrt{\tau}\varphi)} \mathrm{e}^{-z^2/2} \, \mathrm{d}z + \frac{1}{\sqrt{2\pi}} \int_{x/(\sqrt{\tau}\varphi)}^{x/(\sqrt{\tau}\varphi)+\sqrt{\tau}\varphi/2} \mathrm{e}^{-z^2/2} \, \mathrm{d}z \\ &=: N_0(x,\tau) + N_1(x,\tau), \end{aligned} \tag{8.13}$$

where respectively, N_0 and N_1 denote the first and second integral terms in the second equality above.

To decompose the term $\mathrm{e}^{-x} v_x$ in (8.12), we set $u = \mathrm{e}^{-x} v_x$. We will show that u admits a decomposition $u = U_0 + U_1$. With this decomposition, (8.12), and (8.13), we can represent the gradient of the implied volatility as

$$\begin{cases} \varphi_x = \varphi_x^{(0)} - \varphi_x^{(1)} + \varphi_x^{(2)}, \qquad (x,\tau) \in \mathbb{R} \times (0,T], \\ \varphi_x^{(0)}(x,\tau) := \dfrac{\sqrt{2\pi}}{\sqrt{\tau}} \mathrm{e}^{x^2/(2\tau\varphi^2)} \mathrm{e}^{x/2} \mathrm{e}^{\tau\varphi^2/8} \Big(U_0 - N_0 \Big), \\ \varphi_x^{(1)}(x,\tau) := \dfrac{\sqrt{2\pi}}{\sqrt{\tau}} \mathrm{e}^{x^2/(2\tau\varphi^2)} \mathrm{e}^{x/2} \mathrm{e}^{\tau\varphi^2/8} N_1, \\ \varphi_x^{(2)}(x,\tau) := \dfrac{\sqrt{2\pi}}{\sqrt{\tau}} \mathrm{e}^{x^2/(2\tau\varphi^2)} \mathrm{e}^{x/2} \mathrm{e}^{\tau\varphi^2/8} U_1. \end{cases} \tag{8.14}$$

Finally, we will prove that the limits of the $\varphi_x^{(i)}$'s do add up to the desired gradient asymptotics in (8.8).

8.3 Facts about second order linear parabolic PDEs

With some abuse of notation, we will recall some facts about second order linear parabolic PDEs. The following is mostly taken from Krzyzanski [57, Section 65.4].

Let $Lu = 0$ be the parabolic equation

$$Lu \equiv a(x,t)u_{xx} + b(x,t)u_x + c(x,t)u - u_t = 0, \quad (x,t) \in \mathbb{R} \times (T_0, T_1), \tag{8.15}$$

where for some positive constants a_0 and a_1,

$$0 < a_0 \leq a(x,t), \quad (x,t) \in \mathbb{R} \times [T_0, T_1], \tag{8.16}$$

and that a_x, a_t, a_{xx} are bounded and of class C^1 in $\mathbb{R} \times [T_0, T_1]$. Let

$$\theta(x,t) = \int_0^x \frac{\mathrm{d}z}{\sqrt{a(z,t)}}, \tag{8.17}$$

and

$$\lambda(x,t) = b(x,t) - \frac{1}{2}a_x(x,t) - \sqrt{a(x,t)}\theta_t(x,t). \tag{8.18}$$

Further, assume that the functions $\theta_t(x,t)$, $b(x,t)$, $b_x(x,t)$ and $c(x,t)$ are of class C^1 and bounded in $\mathbb{R}$. Consequently $\sqrt{a(x,t)}\theta_t(x,t)$ and $\lambda(x,t)$ are of class C^1 and bounded, together with their first order derivatives. Let

$$K_0(x,t;y,s) = \frac{1}{\sqrt{4\pi(t-s)a(y,s)}} \exp\left(-\frac{[\theta(x,t) - \theta(y,s)]^2}{4(t-s)}\right). \tag{8.19}$$

Define, for $n = 0, 1, 2, \ldots$,

$$K_{n+1}(x,t;y,s) = \int_s^t \int_{-\infty}^{\infty} K_0(x,t;\xi,\tau) \left[\lambda(\xi,\tau)\frac{\partial K_n(\xi,\tau;y,s)}{\partial \xi} + c(\xi,\tau)K_n(\xi,\tau;y,s)\right] \mathrm{d}\xi\, \mathrm{d}\tau. \tag{8.20}$$

Then we have the following definition.

Definition 8.5. A fundamental solution of $Lu = 0$ (in $\mathbb{R} \times [T_0, T_1]$) is a function $\Gamma(x,t;y,s)$ defined for all $(x,t) \in \mathbb{R} \times [T_0, T_1]$, $(y,s) \in \mathbb{R} \times [T_0, T_1]$, $t > s$, which satisfies the following conditions:

(i) for fixed $(y,s) \in \mathbb{R} \times [T_0, T_1]$ it satisfies, as a function of (x,t) ($x \in \mathbb{R}, s < t \leq T_1$)

the equation $Lu = 0$;

(ii) for every continuous function $\rho(x)$, where

$$|\rho(x)| \leq \text{const}_1 \times \exp\left(\text{const}_2 \times |x|^2\right), \tag{8.21}$$

it holds that

$$\lim_{t \searrow s} \int_{-\infty}^{\infty} \Gamma(x, t; y, s)\rho(y) \, \mathrm{d}y = \rho(x). \tag{8.22}$$

See e.g. Friedman [30, p. 22] for information about how the well-definedness of the integral in (8.22) depends on const_2, a_0, a_1, T_0, and T_1.

Now define a function K by

$$K(x, t; y, s) = K_0(x, t; y, s) + \sum_{n=1}^{\infty} K_n(x, t; y, s), \tag{8.23}$$

where K_0 and K_n, $n = 1, 2, \ldots$, are defined by (8.19) and (8.19), for $(x, t) \in \mathbb{R} \times [0, T]$, and $(y, s) \in \mathbb{R} \times [0, t]$. Then we have the following lemma.

Lemma 8.6. *The series $\sum_{n=0}^{\infty} K_n$ converges uniformly for $x, y \in \mathbb{R}$ and $0 \leq s < t \leq T$, $t - s \geq \delta$ for each fixed $\delta > 0$. Moreover, the function $K(x, t; y, s)$ is a fundamental solution of (8.15).*

Proof. See e.g. Krzyzanski [57, p. 534]. □

Remark 8.7. The parametrix K_0 is slightly different from the standard one used in the literature. Alternative proofs for constructions of fundamental solutions using the standard parametrix can be found in standard texts such as Ladyzanskaja et al. [58, Section IV. §11, and p. 363], Friedman [30, p. 22, ff.], Friedman [31, Paragraph 1, p. 17, Theorem 10, p. 23], or Garroni and Menaldi [33, Lemma 3.1, p. 178]. The proofs in these standard texts can also be adapted to prove Lemma 8.6.

8.4 Representation for call option prices

Consider the PDE

$$\begin{cases} u_\tau = \dfrac{1}{2}\sigma^2 u_{xx} + \left(\sigma\sigma_x + \dfrac{1}{2}\sigma^2\right) u_x, & (x, \tau) \in \mathbb{R} \times (0, T], \\ u(x, 0) = \hbar(x), \qquad x \in \mathbb{R}, \end{cases} \tag{8.24}$$

where $\hbar(\cdot)$ is the Heaviside function

$$\hbar(x) = \begin{cases} 1, & x > 0, \\ 1/2, & x = 0, \\ 0, & x < 0. \end{cases} \tag{8.25}$$

We shall prove existence and uniqueness of u by constructing a fundamental solution according to (8.15)–(8.20). Let

$$\begin{aligned} a(x) &= \sigma^2(x)/2, \\ b(x) &= \sigma(x)\sigma_x(x) + \sigma^2(x)/2, \\ \theta(x) &= \int_0^x \frac{\mathrm{d}z}{\sqrt{a(z)}}, \\ \lambda(x) &= b(x) - a_x(x)/2. \end{aligned} \tag{8.26}$$

With these functions, the K_n's of (8.19) and (8.20) become

$$K_0(x,\tau;y,r) = \frac{1}{\sqrt{4\pi(\tau-r)a(y)}} \exp\left(-\frac{[\theta(x)-\theta(y)]^2}{4(\tau-r)}\right), \tag{8.27}$$

and for $n = 0, 1, 2, \ldots$,

$$K_{n+1}(x,\tau;y,r) = \int_r^\tau \int_{-\infty}^\infty K_0(x,\tau;\xi,\rho)\lambda(\xi)\frac{\partial K_n(\xi,\rho;y,r)}{\partial \xi}\ \mathrm{d}\xi\,\mathrm{d}\rho. \tag{8.28}$$

Then according to Lemma 8.6, a fundamental solution of (8.24) is given by

$$K(x,\tau;y,r) = K_0(x,\tau;y,r) + \sum_{n=1}^\infty K_n(x,\tau;y,r). \tag{8.29}$$

Let

$$\begin{aligned} u(x,\tau) &= \int_{-\infty}^\infty K(x,\tau;y,0)\hbar(y)\ \mathrm{d}y \\ &= \int_0^\infty K(x,\tau;y,0)\ \mathrm{d}y. \end{aligned} \tag{8.30}$$

Lemma 8.8. *Let* (A_0)–(A_4) *hold. Then u uniquely solves (8.24). Further, u admits the representation*

$$u(x,\tau) = \sum_{n=0}^\infty u_n(x,\tau), \tag{8.31}$$

with

$$u_n(x,\tau) = \int_0^\infty K_n(x,\tau;y,0)\ \mathrm{d}y, \quad n = 0,1,2,\ldots. \tag{8.32}$$

Proof. That u solves (8.24) follows from the arguments in Krzyzanski [57, Section 65.4]. For uniqueness, we suppose that $\tilde{u}$ is another solution of (8.24). Set $w_\tau = u - \tilde{u}$. Then w solves the PDE $w_\tau = a(x)w_{xx} + b(x)w_x$ in $\mathbb{R} \times (0,T]$ with $w(x,0) = 0$. It can be shown that the only function satisfying this PDE is the zero function, i.e., $w(x,\tau) \equiv 0$ in $\mathbb{R} \times [0,T]$. See e.g. Cerrai [12, Theroem 1.7.5] or Friedman [30, Theorem 16, p. 29]. This shows that $u(x,\tau) \equiv \tilde{u}(x,\tau)$ in $\mathbb{R} \times [0,T]$. Hence u uniquely solves (8.24). Lastly, the series representation of u results from Lemma 8.6, which shows that K is a uniformly convergent series. The proof is thus complete. □

Define ϱ to be

$$\varrho(\tau) = \frac{\sigma^2(0)}{2} \int_0^\tau u_x(0,r)\ \mathrm{d}r, \quad \tau \in [0,T], \tag{8.33}$$

where u is the solution of (8.24). Then the following lemma shows that ϱ is well defined and in $C^1((0,T))$.

Lemma 8.9. *Let* (A_0)–(A_4) *hold. Then* $|\varrho| \leq \mathrm{const}(\mathcal{V}_4, T)$ *and* ϱ *belongs to* $C^1((0,T))$.

Proof. If we can show that $|\varrho| < \infty$, then the C^1 property follows from Lemma 8.8, which shows that $u_x(0,\tau)$ is continuous in $(0,T)$. So it remains to prove the bound for ϱ. From (8.31), we get

$$u_x(x,\tau) = \int_0^\infty K_x(x,\tau;y,0)\ \mathrm{d}y.$$

By Eidelman's estimate, for $x, y \in \mathbb{R}$ and $0 < \tau \leq T$,

$$|K_x(x,\tau;y,0)| \leq \frac{\mathfrak{c}_1}{\tau} \exp\left[-\frac{\mathfrak{c}_2\,|x-y|^2}{\tau}\right],$$

where the positive constants $\mathfrak{c}_1$ and $\mathfrak{c}_2$ depend only on $\mathcal{V}_4$ and T. See e.g., Friedman [30, Equations (6.12), p. 24] or Krzyzanski [57, p. 539]. Hence

$$|\varrho(\tau)| \leq \mathfrak{c}_1 \int_0^\tau \int_0^\infty \frac{1}{r} \exp\left[-\frac{\mathfrak{c}_2\,|y|^2}{r}\right]\ \mathrm{d}y\,\mathrm{d}r.$$

Let

$$\varrho_1 = \int_0^\tau \int_0^1 \frac{1}{r} \exp\left[-\frac{\mathfrak{c}_2 |y|^2}{r}\right] \, \mathrm{d}y \, \mathrm{d}r,$$
$$\varrho_2 = \int_0^\tau \int_1^\infty \frac{1}{r} \exp\left[-\frac{\mathfrak{c}_2 |y|^2}{r}\right] \, \mathrm{d}y \, \mathrm{d}r.$$

Then following Friedman [30, Equations (3.2) and (3.3)], we have, for any $p > 3$,

$$\begin{aligned}
\frac{1}{r} \exp\left[-\frac{\mathfrak{c}_2 |y|^2}{r}\right] &= \frac{1}{\mathfrak{c}_2^{1/p} r^{1-1/p} y^{2/p}} \left(\frac{\mathfrak{c}_2 y^2}{r}\right)^{1/p} \exp\left[-\frac{\mathfrak{c}_2 |y|^2}{r}\right] \\
&\leq \mathrm{const}(\mathfrak{c}_2, p) \times \frac{1}{r^{1-1/p} y^{2/p}} \qquad \text{on} \qquad [0,\tau] \times [0,1],
\end{aligned}$$

This shows that

$$|\varrho_1| \leq \mathrm{const}(\mathfrak{c}_1, \mathfrak{c}_2, p) \times \int_0^\tau \int_0^1 \frac{1}{r^{1-1/p} y^{2/p}} \, \mathrm{d}y \, \mathrm{d}r \leq \mathrm{const}(\mathcal{V}_4, T, p).$$

We now bound $|\varrho_2|$. Making the change of variables $z = \mathfrak{c}_2 y^2 / r$ yields

$$\begin{aligned}
\varrho_2 &\leq \mathrm{const}(\mathfrak{c}_1, \mathfrak{c}_2) \times \int_0^\tau \frac{1}{\sqrt{r}} \int_{\mathfrak{c}_2/r}^\infty \frac{1}{\sqrt{z}} \mathrm{e}^{-z} \, \mathrm{d}z \, \mathrm{d}r \\
&= \mathrm{const}(\mathfrak{c}_1, \mathfrak{c}_2) \times \int_0^\tau \frac{1}{\sqrt{r}} \sqrt{\pi} \left[1 - \mathrm{erf}(\mathfrak{c}_2/r)\right] \, \mathrm{d}r \\
&\leq \mathrm{const}(\mathcal{V}_4, T),
\end{aligned}$$

where $\mathrm{erf}(\cdot)$ denotes the error function. The bound on $|\varrho|$ then follows from the bounds on $\varrho_{1,2}$. The proof is thus complete. □

We are now ready to state and prove a representation for the call price v. Recall that v satisfies the PDE

$$\begin{cases} v_\tau = \dfrac{1}{2}\sigma^2(x)(v_{xx} - v_x), & (x,\tau) \in \mathbb{R} \times (0,T], \\ v(x,0) = (\mathrm{e}^x - 1)_+, & x \in \mathbb{R}. \end{cases} \tag{8.34}$$

See Lemma 5.7.

Proposition 8.10. *Let (2.1),* (A$_0$)–(A$_4$) *hold. Then the call option price $v(x,\tau)$ admits the representation*

$$v(x,\tau) = \int_0^x \mathrm{e}^z u(z,\tau) \, \mathrm{d}z + \varrho(\tau), \quad (x,\tau) \in \mathbb{R} \times [0,T]. \tag{8.35}$$

Proof. The initial condition can be verified by taking the limit as $\tau \to 0$. Taking the limit inside the integral is permitted by the dominated convergence theorem. Further, for $\tau > 0$, and using the "differentiation under the integral sign" theorem, which can be found e.g. in Fleming [27, p. 197 ff.], we can differentiate under the integral sign to get

$$\begin{aligned} v_\tau &= \int_0^x \mathrm{e}^z u_\tau(z,\tau)\ \mathrm{d}z + \varrho_\tau \\ &= \int_0^x \mathrm{e}^z \left[a(z)u_{zz}(z,\tau) + b(z)u_z(z,\tau)\right]\ \mathrm{d}z + \frac{\sigma^2(0)}{2}u_x(0,\tau) \\ &= \frac{\sigma^2(x)}{2}\mathrm{e}^x u_x(x,\tau). \end{aligned}$$

Here the second equality follows from the PDE for u and the definition of ϱ (see (8.24) and (8.33)); the third equality is obtained by integrating by parts the term in the integrand involving u_{zz}, which cancels out with the term in the integrand involving u_z. The proof is then completed by noting that $v_{xx} - v_x = \mathrm{e}^x u_x$. □

Remark 8.11. In the proof above, in order to use the dominated convergence theorem and to differentiate under the integral sign, we rely on the differentiability and boundedness properties of the fundamental solution for the parabolic problem (8.24). For details of these properties see e.g. Sections 65.4 and 65.5 of the monograph by Krzyzanski [57] and note in particular Equations (19.65) and (20.65) on p. 535 and the Eidelman estimates on p. 539. For related comments in this thesis see Remark 9.6 below.

8.5 Representation for gradients of implied volatilities

We will state and prove a representation result for φ_x, the gradient of the implied volatility.

Proposition 8.12 (Gradient representation)**.** *Let (2.1),* (A$_0$)–(A$_4$) *hold. Let* u *and* u_n $n = 0, 1, \ldots$, *be as in Lemma 8.8 and put*

$$U_0(x,\tau) = u_0(x,\tau), \quad U_1(x,\tau) = \sum_{n=1}^{\infty} u_n(x,\tau). \tag{8.36}$$

Then φ_x, the gradient of the implied volatility, admits the representation

$$\begin{cases} \varphi_x = \varphi_x^{(0)} - \varphi_x^{(1)} + \varphi_x^{(2)}, \qquad (x,\tau) \in \mathbb{R} \times (0,T], \\ \varphi_x^{(0)}(x,\tau) := \dfrac{\sqrt{2\pi}}{\sqrt{\tau}} \mathrm{e}^{x^2/(2\tau\varphi^2)} \mathrm{e}^{x/2} \mathrm{e}^{\tau\varphi^2/8} \Big(U_0 - N_0 \Big), \\ \varphi_x^{(1)}(x,\tau) := \dfrac{\sqrt{2\pi}}{\sqrt{\tau}} \mathrm{e}^{x^2/(2\tau\varphi^2)} \mathrm{e}^{x/2} \mathrm{e}^{\tau\varphi^2/8} N_1, \\ \varphi_x^{(2)}(x,\tau) := \dfrac{\sqrt{2\pi}}{\sqrt{\tau}} \mathrm{e}^{x^2/(2\tau\varphi^2)} \mathrm{e}^{x/2} \mathrm{e}^{\tau\varphi^2/8} U_1. \end{cases} \tag{8.37}$$

Proof. By the explanation leading to (8.14) in Section 8.2, we only need to show that $\mathrm{e}^{-x} v_x = U_0 + U_1$. This is precisely what we have by Lemma 8.8, for, the lemma gives $v_x(x,\tau) = \mathrm{e}^x u(x,\tau) = \mathrm{e}^x (U_0 + U_1)$. ☐

8.6 Auxiliary small time limits

Recall that from (5.47), we have

$$I(x) = \frac{x}{J}, \quad J(x) = \int_0^x \frac{\mathrm{d}z}{\sigma(z)}. \tag{8.38}$$

Define

$$\mu(x,\tau) := \frac{1}{2\tau} \left(\frac{x^2}{\varphi^2} - J^2 \right), \quad (x,\tau) \in \mathbb{R} \times (0,T]. \tag{8.39}$$

Then we have the following lemma.

Lemma 8.13. *Let (2.1),* (A_0)–(A_4) *hold. Then*

$$\mu_0(x) := \lim_{\tau \to 0} \mu(x,\tau) = \ln \left(\frac{I(x)}{\sqrt{\sigma(0)\sigma(x)}} \right), \quad x \in \mathbb{R}. \tag{8.40}$$

Proof. By (5.87) or (6.2), as $\tau \to 0$,

$$\left(\frac{x^2}{\varphi^2(x,\tau)} - J^2(x) \right) \to \left(\frac{x^2}{x^2/J^2} - J^2 \right) = 0.$$

Thus, by the L'Hospital rule and (6.5),

$$\begin{aligned}\mu_0 &= \lim_{\tau\to 0}\frac{1}{2\tau}\left(\frac{x^2}{\varphi^2(x,\tau)} - J^2(x)\right)\\ &= \lim_{\tau\to 0}\frac{1}{\partial_\tau\{2\tau\}}\left\{\partial_\tau\left(\frac{x^2}{\varphi^2(x,\tau)} - J^2(x)\right)\right\}\\ &= \lim_{\tau\to 0}\frac{1}{2}\left(-2\frac{x^2\varphi_\tau}{\varphi^3}\right)\\ &= -\frac{x^2 I(x) f(x)}{I^3(x)} = -\frac{x^2 f}{I^2} = -\frac{x^2 f}{x^2/J^2} = -J^2 f\\ &= \ln\left(\frac{I(x)}{\sqrt{\sigma(0)\sigma(x)}}\right).\end{aligned}$$

The proof is thus complete. □

Lemma 8.14. *Let (2.1),* (A_0)–(A_4) *hold. Then*

$$U_0(x,\tau) = \frac{1}{2} + \frac{1}{\sqrt{2\pi}}\int_0^{J/\sqrt{\tau}} \mathrm{e}^{-z^2/2}\,\mathrm{d}z = N\left(J/\sqrt{\tau}\right). \tag{8.41}$$

Proof. By Lemma 8.8, we have

$$\begin{aligned}u_0(x,\tau) &= \int_0^\infty K_0(x,\tau;y,0)\,\mathrm{d}y\\ &= \int_0^\infty \frac{1}{\sqrt{4\pi\tau a(y)}}\exp\left(-\frac{(\theta(x)-\theta(y))^2}{4\tau}\right)\,\mathrm{d}y,\end{aligned}$$

where $a(x) = \sigma^2(x)/2$ and $\theta(x) = \int_0^x [a(z)]^{-1/2}\,\mathrm{d}z$; see (8.26). Then, by making the change of variables $z = [\theta(x)-\theta(y)]/\sqrt{2\tau}$ and noting the definitions of $\theta(\cdot)$ and $J(\cdot)$, we get

$$\begin{aligned}u_0(x,\tau) &= \int_{-\infty}^{\theta(x)/\sqrt{2\tau}} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}\,\mathrm{d}z\\ &= \int_{-\infty}^{J/\sqrt{\tau}} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2}\,\mathrm{d}z\\ &= N(J/\sqrt{\tau}).\end{aligned}$$

The proof is thus complete. □

The following asymptotic result for the standard normal distribution function is well known. See e.g. Feller [26, Lemma 2, p. 175].

Lemma 8.15 (Tail estimate of standard normal distribution). *As $x \to \infty$,*

$$1 - N(x) \sim x^{-1} n(x). \tag{8.42}$$

We then have the following proposition:

Proposition 8.16. *Let (2.1),* (A_0)–(A_4) *hold. Then as $\tau \to 0$,*

$$\varphi_x^{(0)}(x,\tau) \to \frac{\mathrm{e}^{x/2}}{J(x)} \left(1 - \mathrm{e}^{\mu_0(x)}\right), \quad x \in \mathbb{R}. \tag{8.43}$$

Proof. By the definitions of $n(\cdot)$, U_0 and N_0, we have

$$\begin{aligned}
\varphi_x^{(0)} &= \frac{\mathrm{e}^{x/2}}{\sqrt{\tau} n\left(\frac{x}{\sqrt{\tau}\varphi(x,\tau)}\right)} \mathrm{e}^{\tau\varphi^2(x,\tau)/8} \left[N\left(\frac{J(x)}{\sqrt{\tau}}\right) - N\left(\frac{x}{\sqrt{\tau}\varphi(x,\tau)}\right)\right] \\
&= \frac{\mathrm{e}^{x/2}}{\sqrt{\tau} n(z_1)} \mathrm{e}^{\tau\varphi^2/8} \Big\{[1 - N(z_1)] - [1 - N(z_2)]\Big\} \\
&= \frac{\mathrm{e}^{x/2}}{\sqrt{\tau}} \mathrm{e}^{\tau\varphi^2/8} \left[\frac{1 - N(z_1)}{n(z_1)} - \frac{1 - N(z_2)}{n(z_1)}\right],
\end{aligned}$$

where $z_1 = x/(\sqrt{\tau}\varphi)$ and $z_2 = J/\sqrt{\tau}$. By Theorem 5.25 or 6.2, $\tau\varphi^2 \to 0$ as $\tau \to 0$. So it suffices to show that

$$\varphi_x^{(0)*} = \frac{1}{\sqrt{\tau}} \left[\frac{1 - N(z_1)}{n(z_1)} - \frac{1 - N(z_2)}{n(z_1)}\right] \to \frac{1}{J}\left(1 - \mathrm{e}^{\mu_0}\right) \quad \text{as} \quad \tau \to 0. \tag{8.44}$$

We will verify this in three separate cases: (i) $x = 0$, (ii) $x > 0$, and (iii) $x < 0$.

Case (i) $x = 0$**:** In this case we have

$$z_1(0,\tau) = 0 \quad \text{and} \quad z_2(0,\tau) = J(0)/\sqrt{\tau} = 0, \quad \forall\, \tau \in (0,T].$$

Hence $\varphi_x^{(0)*}(0,\tau) \equiv 0$ for all $\tau \in (0,T]$, implying that $\varphi_x^{(0)*}(0,\tau) \to 0$ as $\tau \to 0$. On the other hand, we note that

$$J(0) = 0, \quad I(0) = \sigma(0),$$

and so

$$\begin{aligned}\frac{1}{J}(1-\mathrm{e}^{\mu_0})\bigg|_{x=0} &= \lim_{x\to 0}\frac{1}{J}(1-\mathrm{e}^{\mu_0})\\ &= \lim_{x\to 0}\frac{1}{J}\left(1-\frac{I(x)}{\sqrt{\sigma(0)\sigma(x)}}\right)\\ &= \frac{1}{0}\,(1-1)\,.\end{aligned}$$

Hence, we apply the L'Hospital rule to get

$$\begin{aligned}\frac{1}{J}(1-\mathrm{e}^{\mu_0})\bigg|_{x=0} &= \lim_{x\to 0}\frac{1}{J_x}\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{I(x)}{\sqrt{\sigma(0)\sigma(x)}}\right)\\ &= \lim_{x\to 0}\frac{1}{1/\sigma(x)}\left(-\frac{1}{\sqrt{\sigma(0)}}\left(\frac{I_x(x)}{\sqrt{\sigma(x)}}-\frac{I(x)\sigma_x(x)}{2\sigma^{3/2}(x)}\right)\right)\\ &= 0,\end{aligned}$$

as $I(x) \xrightarrow{x\to 0} \sigma(0)$ and $I_x(x) \xrightarrow{x\to 0} \sigma_x(0)/2$; see (5.51). This proves (8.43) for $x = 0$.

Case (ii) $x > 0$**:** In this case we rewrite $\varphi_x^{(0)*}$ as

$$\varphi_x^{(0)*} = \frac{1}{z_1\sqrt{\tau}}\left[\frac{1-N(z_1)}{z_1^{-1}n(z_1)} - \frac{1-N(z_2)}{z_2^{-1}n(z_2)}\times\frac{z_2^{-1}n(z_2)}{z_1^{-1}n(z_1)}\right] = \frac{1}{z_1\sqrt{\tau}}\Big[\mathcal{R}_1 - \mathcal{R}_2\times\mathcal{R}_3\Big].$$

By Theorem 5.25 or 6.2, $\varphi(x,\tau)\to I(x)$ as $\tau\to 0$. This implies that as $\tau\to 0$,

$$\frac{1}{z_1\sqrt{\tau}} = \frac{\varphi(x,\tau)}{x} \to \frac{1}{J(x)} \qquad \text{and} \qquad z_1, z_2 \to \infty.$$

By (8.42), $\mathcal{R}_{1,2}\to 1$ as $\tau\to 0$. On the other hand,

$$\begin{aligned}\mathcal{R}_3 = \frac{z_2^{-1}n(z_2)}{z_1^{-1}n(z_1)} &= \frac{x/(\sqrt{\tau}\varphi)}{J/\sqrt{\tau}}\exp\left(\frac{x^2}{2\tau\varphi^2}-\frac{J^2}{2\tau}\right)\\ &= \frac{x}{J\varphi}\exp\left[\frac{1}{2\tau}\left(\frac{x^2}{\varphi^2}-J^2\right)\right] \longrightarrow \mathrm{e}^{\mu_0} \qquad \text{as} \qquad \tau\to 0.\end{aligned}$$

Thus for $x > 0$ we have, as $\tau\to 0$, $\varphi_x^{(0)*}\to(1-\mathrm{e}^{\mu_0})/J$ as required.

Case (iii) $x < 0$**:** In this case, we rewrite $\varphi_x^{(0)*}$ as

$$\begin{aligned}
\varphi_x^{(0)*} &= \frac{1}{\sqrt{\tau} n(z_1)} \Big(N(z_2) - N(z_1) \Big) \\
&= \frac{1}{\sqrt{\tau} n(z_1)} \Big([1 - N(-z_2)] - [1 - N(-z_1)] \Big) \\
&= \frac{1}{-z_1\sqrt{\tau}} \left[\frac{1 - N(-z_2)}{(-z_2)^{-1} n(-z_2)} \times \frac{(-z_2)^{-1} n(-z_2)}{(-z_1)^{-1} n(-z_1)} - \frac{1 - N(-z_1)}{(-z_1)^{-1} n(-z_1)} \right] \\
&= \frac{1}{z_1\sqrt{\tau}} \left[\frac{1 - N(-z_1)}{(-z_1)^{-1} n(-z_1)} - \frac{1 - N(-z_2)}{(-z_2)^{-1} n(-z_2)} \times \frac{(-z_2)^{-1} n(-z_2)}{(-z_1)^{-1} n(-z_1)} \right] \\
&= \frac{1}{z_1\sqrt{\tau}} \Big[\mathcal{R}_1 - \mathcal{R}_2 \times \mathcal{R}_3 \Big].
\end{aligned}$$

Since $\varphi(x,\tau) \to I(x)$ as $\tau \to 0$, we have

$$\frac{1}{z_1\sqrt{\tau}} = \frac{\varphi(x,\tau)}{x} \to \frac{1}{J(x)} \quad \text{and} \quad -z_1 \to \infty, \quad -z_2 \to \infty.$$

By (8.42), $\mathcal{R}_{1,2} \to 1$ as $\tau \to 0$. Also,

$$\begin{aligned}
\mathcal{R}_3 &= \frac{(-z_2)^{-1} n(-z_2)}{(-z_1)^{-1} n(-z_1)} = \frac{z_1 n(z_2)}{z_2 n(z_1)} = \frac{x/(\sqrt{\tau}\varphi)}{J/\sqrt{\tau}} \exp\left(\frac{x^2}{2\tau\varphi^2} - \frac{J^2}{2\tau} \right) \\
&= \frac{x}{J\varphi} \exp\left[\frac{1}{2\tau} \left(\frac{x^2}{\varphi^2} - J^2 \right) \right] \longrightarrow \mathrm{e}^{\mu_0} \quad \text{as} \quad \tau \to 0.
\end{aligned}$$

Hence, for $x < 0$, we also have $\varphi_x^{(0)*} \to (1 - \mathrm{e}^{\mu_0})/J$ as $\tau \to 0$. The proof is thus complete. □

Proposition 8.17. *Let (2.1),* (A_0)–(A_4) *hold. Then*

$$\varphi_x^{(1)} \to \frac{\mathrm{e}^{x/2}}{J} \left(1 - \mathrm{e}^{-x/2} \right) \quad \textit{as} \quad \tau \to 0. \tag{8.45}$$

Proof. Recall that

$$\begin{aligned}
\varphi_x^{(1)}(x,\tau) &= \frac{\sqrt{2\pi}}{\sqrt{\tau}} \mathrm{e}^{x^2/(2\tau\varphi^2)} \mathrm{e}^{x/2} \mathrm{e}^{\tau\varphi^2/8} N_1 \\
&= \frac{\mathrm{e}^{x/2}}{\sqrt{\tau} n\left(\frac{x}{\sqrt{\tau}\varphi} \right)} \mathrm{e}^{\tau\varphi^2/2} N_1.
\end{aligned}$$

Since $\tau\varphi^2 \to 0$ as $\tau \to 0$, it suffices to check that

$$\varphi_x^{(1)*} := \frac{1}{\sqrt{\tau}\, n\left(\frac{x}{\sqrt{\tau}\varphi}\right)} N_1 \to \frac{1}{J}\left(1 - \mathrm{e}^{-x/2}\right), \quad \text{as} \quad \tau \to 0,$$

when (i) $x > 0$, (ii) $x < 0$, and (iii) $x = 0$.

In cases (i) and (ii) the proof is similar to that of Lemma 8.16.

Case (i) $x > 0$**:** In this case, we put

$$z_1 = \frac{x}{\sqrt{\tau}\varphi}, \quad z_2 = \frac{x}{\sqrt{\tau}\varphi} + \frac{\sqrt{\tau}\varphi}{2},$$

and rewrite $\varphi_x^{(1)*}$ as

$$\begin{aligned}
\varphi_x^{(1)*} &= \frac{1}{\sqrt{\tau}\, n(z_1)}\Big(N(z_2) - N(z_1)\Big) \\
&= \frac{1}{\sqrt{\tau}\, n(z_1)}\Big([1 - N(z_1)] - [1 - N(z_2)]\Big) \\
&= \frac{1}{z_1\sqrt{\tau}}\left[\frac{1 - N(z_1)}{z_1^{-1} n(z_1)} - \frac{1 - N(z_2)}{z_2^{-1} n(z_2)} \times \frac{z_2^{-1} n(z_2)}{z_1^{-1} n(z_1)}\right] \\
&= \frac{1}{z_1\sqrt{\tau}}\Big[\mathcal{R}_1 - \mathcal{R}_2 \times \mathcal{R}_3\Big].
\end{aligned}$$

Since $\tau\varphi^2 \to 0$ as $\tau \to 0$, we have

$$z_1\sqrt{\tau} = x/\varphi \to J, \quad \text{and} \quad z_1, z_2 \to \infty \quad \text{as} \quad \tau \to 0.$$

Then by (8.42), $\mathcal{R}_{1,2} \to 1$ as $\tau \to 0$. One the other hand,

$$\begin{aligned}
\mathcal{R}_3 &= \frac{z_2^{-1} n(z_2)}{z_1^{-1} n(z_1)} = \frac{z_1 n(z_2)}{z_2 n(z_1)} = \frac{x/(\sqrt{\tau}\varphi)}{x/(\sqrt{\tau}\varphi) + \sqrt{\tau}\varphi/2} \exp\left(z_1^2/2 - z_2^2/2\right) \\
&= \frac{x}{x + \tau\varphi^2/2} \exp\left(-x/2 - \tau\varphi^2/4\right) \longrightarrow \mathrm{e}^{-x/2}, \quad \text{as} \quad \tau \to 0.
\end{aligned}$$

The desired result then follows from these limits.

Case (ii) $x < 0$**:** Since the proof is similar to that of case (i), we omit it.

Case (iii) $x = 0$**:** In this case we can write $\varphi_x^{(1)*}$ as

$$\varphi_x^{(1)*} = \frac{\sqrt{2\pi}}{\sqrt{\tau}} \int_0^{\sqrt{\tau}\varphi/2} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2}\, \mathrm{d}z = \frac{1}{\sqrt{\tau}} \int_0^{\sqrt{\tau}\varphi/2} \mathrm{e}^{-z^2/2}\, \mathrm{d}z.$$

Since $\tau\varphi^2 \to 0$ as $\tau \to 0$, we get $\varphi_x^{(1)*} \to 0/0$ as $\tau \to 0$. An application of the L'Hospital rule and the Leibniz formula for differentiation of the integral gives

$$\begin{aligned}\lim_{\tau\to 0}\varphi_x^{(1)*} &= \lim_{\tau\to 0}\left[2\sqrt{\tau}\mathrm{e}^{-\tau\varphi^2/8}\left(\frac{\varphi}{4\sqrt{\tau}}+\frac{\sqrt{\tau}\varphi_\tau}{2}\right)\right]\\ &= \lim_{\tau\to 0}\left[\mathrm{e}^{-\tau\varphi^2/8}\left(\frac{\varphi}{2}+\tau\varphi_\tau\right)\right]\\ &= \frac{\sigma(0)}{2},\end{aligned}$$

where the last equality results from Theorem 6.2. On the other hand, we have

$$\lim_{x\to 0}\frac{\mathrm{e}^{x/2}}{J}\left(1-\mathrm{e}^{-x/2}\right) = \sigma(0)/2.$$

This shows

$$\varphi_x^{(1)}\Big|_{x=0} \to \frac{\mathrm{e}^{x/2}}{J}\left(1-\mathrm{e}^{-x/2}\right)\Big|_{x=0} \qquad \text{as} \qquad \tau\to 0.$$

The proof is therefore complete. □

8.7 A technical theorem

We shall need the following technical theorem.

Theorem 8.18. *Let (2.1),* (A$_0$)–(A$_4$) *hold. Then as $\tau \to 0$,*

$$U_1 \sim \sqrt{\frac{\tau}{2\pi}}\mathrm{e}^{-J^2/(2\tau)}\frac{1}{J(x)}\left(1-\sqrt{\frac{\sigma(0)}{\sigma(x)}}\mathrm{e}^{-x/2}\right). \tag{8.46}$$

The proof of this theorem is a little involved, so we will prove it separately in Chapter 9. This technical theorem implies the following asymptotic result.

Proposition 8.19. *Let (2.1),* (A$_0$)–(A$_4$) *hold. Then as $\tau \to 0$,*

$$\varphi_x^{(2)} \to \mathrm{e}^{\mu_0}\frac{\mathrm{e}^{x/2}}{J(x)}\left(1-\sqrt{\frac{\sigma(0)}{\sigma(x)}}\mathrm{e}^{-x/2}\right). \tag{8.47}$$

Proof. Noting the definition of μ_0, see (8.40), and that $\tau\varphi^2 \to 0$ as $\tau \to 0$, this is a consequence of (8.46) and the definition of $\varphi_x^{(2)}$, which is given by (8.37). □

8.8 Proof of the main theorem of the chapter

We will present the proof of Theorem 8.2. The proof of Theorem 8.1, the equivalent result in the (s, τ) coordinates, will be omitted. Theorem 8.1 can be proved by applying the change of variables $x = \ln(s/k)$ to the result of Theorem 8.2.

Proof of Theorem 8.2. By (8.37), (8.43), (8.45), and (8.47), we have, as $\tau \to 0$,

$$\begin{aligned}
\varphi_x &= \varphi_x^{(0)} - \varphi_x^{(1)} + \varphi_x^{(2)} \\
&\to \frac{e^{x/2}}{J(x)}\left(1 - e^{\mu_0(x)}\right) - \frac{e^{x/2}}{J(x)}\left(1 - e^{-x/2}\right) + e^{\mu_0(x)}\frac{e^{x/2}}{J(x)}\left(1 - \sqrt{\frac{\sigma(0)}{\sigma(x)}}e^{-x/2}\right) \\
&= \frac{1}{J(x)}\left(1 - e^{\mu_0(x)}\sqrt{\frac{\sigma(0)}{\sigma(x)}}\right) \\
&= \frac{1}{J(x)}\left(1 - \frac{I(x)}{\sqrt{\sigma(0)\sigma(x)}}\sqrt{\frac{\sigma(0)}{\sigma(x)}}\right) \\
&= \frac{1}{J(x)}\left(1 - \frac{I(x)}{\sigma(x)}\right) \\
&= \frac{I(x)}{x} - \frac{I^2(x)}{x\sigma(x)} \\
&= I_x(x),
\end{aligned} \tag{8.48}$$

where the last equality follows from (8.7). Note that the convergence is uniform in (x, τ) on compact subsects of $\mathbb{R} \times [0, T]$. This proves (8.8).

It remains to prove the Hessian limits (8.9) and (8.10). To prove the ATM Hessian limit (8.9), we use (5.86), the PDE for φ. Since $x = \ln(s/k)$ and the option is at the money, we have $x = 0$. A rearrangement of the PDE for φ, see (5.86), gives

$$\varphi_{xx}(0, \tau) = \frac{2\varphi_\tau(0, \tau)}{\sigma^2(0)} + \frac{\varphi^2(0, \tau) - \sigma^2(0)}{\tau\sigma^2(0)\varphi(0, \tau)} + \frac{1}{4}\tau\varphi(0, \tau)\varphi_x^2(0, \tau). \tag{8.49}$$

By Theorem 6.2,

$$\begin{aligned}
&\lim_{\tau \to 0} \varphi(0, \tau) = \sigma(0), \\
&\lim_{\tau \to 0} \varphi_\tau(0, \tau) = I(0)f(0) = \sigma(0)\left(\frac{\sigma(0)\sigma_{xx}(0)}{12} - \frac{\sigma_x^2(0)}{24}\right),
\end{aligned} \tag{8.50}$$

where $I(0)$ and $f(0)$ are respectively given by (5.51) and (6.15). The first limit in

(8.50), together with (8.48), implies that uniformly in $x \in \mathbb{R}$,

$$\tau\varphi\varphi_x^2 \to 0 \quad \text{as} \quad \tau \to 0. \tag{8.51}$$

Further, we have

$$\frac{\varphi^2(0,\tau) - \sigma^2(0)}{\tau\sigma^2(0)\varphi(0,\tau)} \to \frac{0}{0} \quad \text{as} \quad \tau \to 0. \tag{8.52}$$

Applying L'Hospital's rule and (8.50) then gives

$$\begin{aligned}
\lim_{\tau\to 0} \frac{\varphi^2(0,\tau) - \sigma^2(0)}{\tau\sigma^2(0)\varphi(0,\tau)} &= \lim_{\tau\to 0} \frac{2\varphi(0,\tau)\varphi_\tau(0,\tau)}{\sigma^2(0)\varphi(0,\tau) + \tau\sigma^2(0)\varphi_\tau(0,\tau)} \\
&= \frac{2\varphi_\tau(0,0)}{\sigma^2(0)} \\
&= \frac{2I(0)f(0)}{\sigma^2(0)}.
\end{aligned} \tag{8.53}$$

By combining (8.49)–(8.51) and (8.53), we get

$$\lim_{\tau\to 0} \varphi_{xx}(0,\tau) = \frac{4I(0)f(0)}{\sigma^2(0)} = \frac{\sigma_{xx}(0)}{3} - \frac{\sigma_x^2(0)}{6\sigma(0)} = I_{xx}(0),$$

where the last equality follows from (5.51). This proves (8.9); and it remains to prove (8.10). We will again use the PDE for φ. From (5.86), we have

$$2\tau\varphi\varphi_\tau + \varphi^2 - \sigma^2(x)\left(1 - x\frac{\varphi_x}{\varphi}\right)^2 - \sigma^2(x)\tau\varphi\varphi_{xx} + \frac{1}{4}\sigma^2(x)\tau^2\varphi^2\varphi_x^2 = 0,$$

in $\mathbb{R} \times (0,T]$. By Theorem 6.2 and (8.48), we have, for each $x \in \mathbb{R}$, as $\tau \to 0$,

$$\begin{aligned}
2\tau\varphi\varphi_\tau &\to 0, \\
\varphi^2 - \sigma^2(x)\left(1 - x\frac{\varphi_x}{\varphi}\right)^2 &\to 0, \\
\frac{1}{4}\sigma^2(x)\tau^2\varphi^2\varphi_x^2 &\to 0, \\
\sigma^2(x)\varphi &\to \sigma^2(x)I(x) > 0.
\end{aligned}$$

This shows $\tau\varphi_{xx} \to 0$ as $\tau \to 0$, and the proof is complete. $\square$

Remark 8.20. It can be checked that $\lim_{x\to 0}\varphi_x(x,0) = \frac{1}{2}\sigma_x(0)$. Converting this limit to the (s,τ) coordinates shows that it agrees with the probabilistic result of Theorem 7.1.

Chapter 9

Proof of the technical theorem

In this chapter we will prove Theorem 8.18, the technical theorem in Chapter 8. For easy referencing we repeat the theorem here.

Theorem 9.1. *Let* (A_0)–(A_4) *hold. Then as* $\tau \to 0$*,*

$$U_1(x,\tau) \sim \sqrt{\frac{\tau}{2\pi}}\mathrm{e}^{-J^2(x)/(2\tau)}\frac{1}{J(x)}\left(1-\sqrt{\frac{\sigma(0)}{\sigma(x)}}\mathrm{e}^{-x/2}\right), \tag{9.1}$$

where $U_1 = \sum_{n=1}^{\infty} u_n(x,\tau)$ *is defined by (8.36).*

Remark 9.2. Note that in this chapter the letter n stands for an index; it does not denote the normal density function $n(\cdot)$ as in the previous chapters.

This chapter is organised as follows. In Section 9.1, we derive a series representation for the gradient of the call option price using a change of the space variables. In Section 9.2, we present some technical results. In Section 9.3, by calculating u_1 and u_2, the first and second term of the series U_1, we demonstrate how the technical results of Section 9.2 actually work. In Section 9.4, we first prove an auxiliary proposition and then Theorem 9.1. In short, this chapter shows that by some changes of the variables and interchange of integration and differentiation, each term of the series U_1 can be explicitly computed and hence summed to give (9.1).

9.1 Change of variables

From Lemma 8.8, we know that u solves the PDE

$$\begin{cases} u_\tau = \dfrac{1}{2}\sigma^2 u_{xx} + \left(\sigma\sigma_x + \dfrac{1}{2}\sigma^2\right)u_x, & (x,\tau) \in \mathbb{R}\times(0,T], \\ u(x,0) = \hbar(x), & x \in \mathbb{R}, \end{cases} \tag{9.2}$$

where $\hbar(\cdot)$ is the Heaviside function

$$\hbar(x) = \begin{cases} 1, & x > 0, \\ 1/2, & x = 0, \\ 0, & x < 0. \end{cases} \tag{9.3}$$

Moreover, we know that u admits the series representation

$$u(x,\tau) = \sum_{n=0}^{\infty} u_n(x,\tau), \tag{9.4}$$

with

$$u_n(x,\tau) = \int_0^{\infty} K_n(x,\tau;y,0)\,\mathrm{d}y, \quad n = 0,1,2,\ldots, \tag{9.5}$$

where the K_n's are defined by (8.27) and (8.28)

We make the following change of variables. Define

$$\bar{x} = \theta(x) = \sqrt{2}\int_0^x \frac{\mathrm{d}z}{\sigma(z)}, \tag{9.6}$$

and put

$$\bar{u}(\bar{x},\tau) = u(x,\tau). \tag{9.7}$$

Let

$$\bar{b}(\bar{x}) = \frac{1}{\sqrt{2}}\Big(\sigma'\big(\theta^{-1}(\bar{x})\big) + \sigma\big(\theta^{-1}(\bar{x})\big)\Big), \tag{9.8}$$

with $\theta^{-1}(\cdot)$ denoting the inverse function of $\theta(\cdot)$ and $\sigma'(z) = \mathrm{d}\sigma(z)/\,\mathrm{d}z$. As in (8.27) and (8.28), we define

$$K_0(\bar{x},\tau;\bar{y},r) = \frac{1}{\sqrt{4\pi(\tau-r)}}\exp\left(-\frac{(\bar{x}-\bar{y})^2}{4(\tau-r)}\right), \tag{9.9}$$

and for $n = 0,1,2,\ldots$,

$$K_{n+1}(\bar{x},\tau;\bar{y},r) = \int_r^{\tau}\int_{-\infty}^{\infty} K_0(\bar{x},\tau;\bar{\xi},\rho)\bar{b}(\bar{\xi})\frac{\partial K_n(\bar{\xi},\rho;\bar{y},r)}{\partial\bar{\xi}}\,\mathrm{d}\bar{\xi}\,\mathrm{d}\rho. \tag{9.10}$$

Further, let

$$K(\bar{x},\tau;\bar{y},r) = K_0(\bar{x},\tau;\bar{y},r) + \sum_{n=1}^{\infty} K_n(\bar{x},\tau;\bar{y},r). \tag{9.11}$$

Then we have the following lemma.

Proposition 9.3. *Let* (A_0)–(A_4) *hold and* $u(x,\tau)$ *and* $u_n(x,\tau)$, $n = 0,1,\ldots,$ *be as in*

Lemma 8.8. Then the following statements are true:

(i) The function $\bar{u}(\bar{x},\tau)$ *uniquely solves the PDE*

$$\begin{cases} \bar{u}_\tau = \bar{u}_{\bar{x}\bar{x}} + \bar{b}(\bar{x})\bar{u}_{\bar{x}}, \quad (\bar{x},\tau) \in \mathbb{R} \times (0,T], \\ \bar{u}(\bar{x},0) = \hbar(\bar{x}), \quad \bar{x} \in \mathbb{R}. \end{cases} \tag{9.12}$$

(ii) The function $\bar{u}$ *admits the representation*

$$\bar{u}(\bar{x},\tau) = \int_0^\infty K(\bar{x},\tau;\bar{y},0)\ \mathrm{d}\bar{y} = \sum_{n=0}^\infty \bar{u}_n(\bar{x},\tau), \tag{9.13}$$

where, for $n = 0, 1, 2, \ldots,$

$$\bar{u}_n(\bar{x},\tau) = \int_0^\infty K_n(\bar{x},\tau;\bar{y},0)\ \mathrm{d}\bar{y}. \tag{9.14}$$

(iii) The function $\bar{u}(\bar{x},\tau)$ *has the decomposition*

$$\bar{u}(\bar{x},\tau) = \bar{U}_0 + \bar{U}_1, \quad \bar{U}_0(\bar{x},\tau) = \bar{u}_0(\bar{x},\tau), \quad \bar{U}_1(\bar{x},\tau) = \sum_{n=1}^\infty \bar{u}_n(\bar{x},\tau). \tag{9.15}$$

(iv) The following identities hold:

$$\begin{aligned} &\bar{u}_n(\bar{x}(x),\tau) = u_n(x,\tau), \quad n = 0,1,2,\ldots, \\ &\bar{U}_0(\bar{x}(x),\tau) = U_0(x,\tau), \\ &\bar{U}_1(\bar{x}(x),\tau) = U_1(x,\tau). \end{aligned} \tag{9.16}$$

Proof. Statement (i) results from (9.7) and (9.2). Like the series representation of $u(x,\tau)$, statement (ii) follows from the explicit construction of the solution via the parametrix method; see e.g. Krzyzanski [57, Section 65.4]. Statement (iii) is given by Lemma 8.6. Statement (iv) can be verified by induction, using $\bar{x}(x) = \theta(x)$ to prove the equality of $\bar{u}_n$ and u_n's. The proof is now complete. □

9.2 Technical results

In this section we will be working towards proving the following proposition.

Proposition 9.4. *Let* (A_0)–(A_4) *hold. Then as* $\tau \to 0$,

$$\bar{u}_n(\bar{x},\tau) \sim \frac{\sqrt{\tau}}{\sqrt{\pi}}\left[\exp\left(-\frac{\bar{x}^2}{4\tau}\right)\right]\frac{1}{\bar{x}}\cdot\frac{(-1)^{n-1}}{n!}\left(\frac{1}{2}\left[\theta^{-1}(\bar{x})+\ln\frac{\sigma(\theta^{-1}(\bar{x}))}{\sigma(0)}\right]\right)^n, \quad n=1,2,\ldots \tag{9.17}$$

We defer the proof of this proposition to Section 9.4 as it requires the technical results that we now present.

Technical details

For $n = 2, 3, 4, \ldots$, and $\tau_n > 0$, define $E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0)$ to be

$$\begin{aligned}
&E_n(\bar{\xi}_n,\tau_n;\bar{y},0)\\
&=\int_0^{\tau_n}\int_{-\infty}^{\infty}K_0(\bar{\xi}_n,\tau_n;\bar{\xi}_{n-1},\tau_{n-1})\bar{b}(\bar{\xi}_{n-1})\\
&\qquad\times\int_0^{\tau_{n-1}}\int_{-\infty}^{\infty}\frac{\partial}{\partial\bar{\xi}_{n-1}}K_0(\bar{\xi}_{n-1},\tau_{n-1};\bar{\xi}_{n-2},\tau_{n-2})\bar{b}(\bar{\xi}_{n-2})\\
&\qquad\times\cdots\\
&\qquad\times\int_0^{\tau_2}\int_{-\infty}^{\infty}\frac{\partial}{\partial\bar{\xi}_2}K_0(\bar{\xi}_2,\tau_2;\bar{\xi}_1,\tau_1)\bar{b}(\bar{\xi}_1)\\
&\qquad\quad\times\int_0^{\tau_1}\int_{-\infty}^{\infty}\frac{\partial}{\partial\bar{\xi}_1}K_0(\bar{\xi}_1,\tau_1;\bar{\xi}_0,\tau_0)\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0,\tau_0;\bar{y},0)\,\mathrm{d}\bar{\xi}_0\,\mathrm{d}\tau_0\\
&\qquad\quad\times\,\mathrm{d}\bar{\xi}_1\,\mathrm{d}\tau_1\\
&\qquad\times\cdots\\
&\qquad\times\,\mathrm{d}\bar{\xi}_{n-2}\,\mathrm{d}\tau_{n-2}\\
&\qquad\times\,\mathrm{d}\bar{\xi}_{n-1}\,\mathrm{d}\tau_{n-1},
\end{aligned} \tag{9.18}$$

where $\bar{b}$ and K_0 are respectively defined by (9.8) and (9.9). To illustrate this definition, we set $n = 2$ to get

$$\begin{aligned}
&E_2(\bar{\xi}_2,\tau_2;\bar{y},0)\\
&=\int_0^{\tau_2}\int_{-\infty}^{\infty}K_0(\bar{\xi}_2,\tau_2;\bar{\xi}_1,\tau_1)\bar{b}(\bar{\xi}_1)\\
&\qquad\times\int_0^{\tau_1}\int_{-\infty}^{\infty}\frac{\partial}{\partial\bar{\xi}_1}K_0(\bar{\xi}_1,\tau_1;\bar{\xi}_0,\tau_0)\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0,\tau_0;\bar{y},0)\,\mathrm{d}\bar{\xi}_0\,\mathrm{d}\tau_0\\
&\quad\times\,\mathrm{d}\bar{\xi}_1\,\mathrm{d}\tau_1.
\end{aligned}$$

Taking into account this definition and that of the K_n's given by (9.9) and (9.10), we obtain the following lemma.

Lemma 9.5. *Let* (A_0)–(A_4) *hold. Then for* $n = 2, 3, \ldots,$

$$K_n(\bar{\xi}_n, \tau_n; \bar{y}, 0) = -\frac{\partial}{\partial \bar{y}} E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0). \tag{9.19}$$

Proof. Suppose this is true for some $n \in \{2, 3, \ldots\}$. Then by definition, see (9.10), we have

$$\begin{aligned}
&K_{n+1}(\bar{\xi}_{n+1}, \tau_{n+1}; \bar{y}, 0) \\
&= \int_0^{\tau_{n+1}} \int_{-\infty}^{\infty} K_0(\bar{\xi}_{n+1}, \tau_{n+1}; \bar{\xi}_n, \tau_n) \bar{b}(\bar{\xi}_n) \frac{\partial}{\partial \bar{\xi}_n} K_n(\bar{\xi}_n, \tau_n; \bar{y}, 0) \, \mathrm{d}\bar{\xi}_n \, \mathrm{d}\tau_n \\
&= \int_0^{\tau_{n+1}} \int_{-\infty}^{\infty} K_0(\bar{\xi}_{n+1}, \tau_{n+1}; \bar{\xi}_n, \tau_n) \bar{b}(\bar{\xi}_n) \frac{\partial}{\partial \bar{\xi}_n} \left\{ -\frac{\partial}{\partial \bar{y}} E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0) \right\} \, \mathrm{d}\bar{\xi}_n \, \mathrm{d}\tau_n \\
&= -\frac{\partial}{\partial \bar{y}} \int_0^{\tau_{n+1}} \int_{-\infty}^{\infty} K_0(\bar{\xi}_{n+1}, \tau_{n+1}; \bar{\xi}_n, \tau_n) \bar{b}(\bar{\xi}_n) \frac{\partial}{\partial \bar{\xi}_n} E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0) \, \mathrm{d}\bar{\xi}_n \, \mathrm{d}\tau_n.
\end{aligned} \tag{9.20}$$

The equality of

$$\frac{\partial}{\partial \bar{\xi}_n} \left\{ -\frac{\partial}{\partial \bar{y}} E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0) \right\} = -\frac{\partial}{\partial \bar{y}} \frac{\partial}{\partial \bar{\xi}_n} E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0) \tag{9.21}$$

is justified since the mixed derivatives are continuous for all $\bar{\xi}_n \in \mathbb{R}$, $\bar{y} \in \mathbb{R}$, and $\tau_n > 0$. Also, the differentiation under the integral sign is valid by the "differentiation under the integral sign" theorem, which can be found in Fleming [27, p. 197 ff.], for example.

Repeating the argument above shows that (9.19) holds for $n = 2$, and thus by induction it holds for all $n = 2, 3, 4, 5, \ldots$. And the proof is complete. □

Remark 9.6. In the proof above, to use the "differentiation under the integral sign" theorem in Fleming [27, p. 197 ff.], we have invoked some boundedness and integrability properties of the K_n's. These properties of the K_n's are well known in PDE theory. They can be found in, e.g., Krzyzanski [57, p. 534]. Note three things however. First, our K_n's are the U_n's in Krzyzanski [57, p. 534]. Second, in Krzyzanski [57, p. 534], there is a typographical error in the inequality

$$|U_n(x, t; y, s)| \leq \frac{H^{2n}}{\Gamma\left(\frac{n}{2}\right)} (t - s)^{n/2 - 1};$$

it should read

$$|U_n(x, t; y, s)| \leq \frac{H^{2n}}{\Gamma\left(\frac{n}{2}\right)}.$$

Third, Friedman [30, Chapter 1] provides a more detailed discussion on the boundedness and integrability properties of the K_n's. The inequalities in Krzyzanski [57, p. 534] can be derived by the same methods used in Friedman [30, Chapter 1, Section 4]. We quote Krzyzanski's results because they are stated in the form we want.

We now come to the following lemma:

Lemma 9.7. *Let* (A$_0$)–(A$_4$) *hold. Then for each* $\tau_n > 0$, $n = 2, 3, \ldots$,

$$\bar{u}_n(\bar{\xi}_n, \tau_n) = E_n(\bar{\xi}_n, \tau_n; 0, 0). \tag{9.22}$$

Proof. Put $\bar{\xi}_n = \bar{x}$. Then by Proposition 9.3, we have

$$\bar{u}_n(\bar{\xi}_n, \tau_n) = \int_0^\infty K_n(\bar{\xi}_n, \tau_n; \bar{y}, 0)\ \mathrm{d}\bar{y}. \tag{9.23}$$

By (9.19), we have

$$\begin{aligned}
\bar{u}_n(\bar{\xi}_n, \tau_n) &= \int_0^\infty -\frac{\partial}{\partial \bar{y}} E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0)\ \mathrm{d}\bar{y} \\
&= \lim_{R\to\infty} \int_0^R -\frac{\partial}{\partial \bar{y}} E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0)\ \mathrm{d}\bar{y} \\
&= \lim_{R\to\infty} \left[\lim_{\bar{y}\to 0} E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0) - \lim_{\bar{y}\to R} E_n(\bar{\xi}_n, \tau_n; \bar{y}, 0)\right] \\
&= \lim_{R\to\infty} \left[E_n(\bar{\xi}_n, \tau_n; 0, 0) - E_n(\bar{\xi}_n, \tau_n; R, 0)\right] \\
&= E_n(\bar{\xi}_n, \tau_n; 0, 0),
\end{aligned} \tag{9.24}$$

where the third equality follows from the fundamental theorem of calculus, see e.g. Rudin [72, Theorem 7.21, p. 149]; and the limits in the last two equalities follow from Lebesgue's dominated convergence theorem, see e.g. Rudin [72, Theorem 1.34, p. 26]. □

To study the small time properties of v_n, we make the following change of variables to normalize the τ_n's. Given any $n = 1, 2, \ldots$, we put

$$\begin{aligned}
\bar{\tau}_{n-1} &= \frac{\tau_{n-1}}{\tau_n}, \\
\bar{\tau}_{n-2} &= \frac{\tau_{n-2}}{\tau_n \bar{\tau}_{n-1}}, \\
\bar{\tau}_{n-3} &= \frac{\tau_{n-3}}{\tau_n \bar{\tau}_{n-1} \bar{\tau}_{n-2}}, \\
&\vdots \\
\bar{\tau}_0 &= \frac{\tau_0}{\tau_n \bar{\tau}_{n-1} \cdots \bar{\tau}_2 \bar{\tau}_1}.
\end{aligned} \tag{9.25}$$

Equivalently, we have, for any $n = 1, 2, \ldots$,

$$\begin{aligned}
\tau_{n-1} &= \tau_n \bar{\tau}_{n-1}, \\
\tau_{n-2} &= \tau_n \bar{\tau}_{n-1} \bar{\tau}_{n-2}, \\
\tau_{n-3} &= \tau_n \bar{\tau}_{n-1} \bar{\tau}_{n-2} \bar{\tau}_{n-3}, \\
&\vdots \\
\tau_0 &= \tau_n \prod_{j=0}^{n-1} \bar{\tau}_j.
\end{aligned} \tag{9.26}$$

Then by substituting the $\bar{\tau}_n$'s into $E_n(\bar{\xi}_n, \tau_n; 0, 0)$ and by (9.22), we obtain the following lemma:

Lemma 9.8. *Let* (A_0)–(A_4) *hold. Then*

$$\begin{aligned}
\bar{u}_n(\bar{\xi}_n, \tau_n) = \tau_n^n \int_0^1 \int_{-\infty}^{\infty} & K_0(\bar{\xi}_n, \tau_n; \bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1}) \bar{b}(\bar{\xi}_{n-1}) \\
& \times \bar{\tau}_{n-1}^{n-1} \int_0^1 \int_{-\infty}^{\infty} \frac{\partial}{\partial \bar{\xi}_{n-1}} K_0(\bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1}; \bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1} \bar{\tau}_{n-2}) \bar{b}(\bar{\xi}_{n-2}) \\
& \times \cdots \\
& \times \bar{\tau}_2^2 \int_0^1 \int_{-\infty}^{\infty} \frac{\partial}{\partial \bar{\xi}_2} K_0 \left(\bar{\xi}_2, \tau_n \prod_{j=2}^{n-1} \bar{\tau}_j; \bar{\xi}_1, \prod_{j=1}^{n-1} \bar{\tau}_j \right) \bar{b}(\bar{\xi}_1) \\
& \qquad \times \bar{\tau}_1 \int_0^1 \int_{-\infty}^{\infty} \frac{\partial}{\partial \bar{\xi}_1} K_0 \left(\bar{\xi}_1, \prod_{j=1}^{n-1} \bar{\tau}_j; \bar{\xi}_0, \prod_{j=0}^{n-1} \bar{\tau}_j \right) \bar{b}(\bar{\xi}_0) \\
& \qquad \qquad \times K_0 \left(\bar{\xi}_0, \prod_{j=0}^{n-1} \bar{\tau}_j; 0, 0 \right) \, \mathrm{d}\bar{\xi}_0 \, \mathrm{d}\tau_0 \\
& \qquad \times \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\tau}_1 \\
& \times \cdots \\
& \times \mathrm{d}\bar{\xi}_{n-2} \, \mathrm{d}\bar{\tau}_{n-2} \\
& \times \mathrm{d}\bar{\xi}_{n-1} \, \mathrm{d}\bar{\tau}_{n-1}.
\end{aligned} \tag{9.27}$$

We shall need the following K_0 recombination formula for the heat/Green kernels. Our proof uses an identity from Friedman [30, the last equation in p. 15]. The same identity also appeared in the earlier work of Dressel [21].

Lemma 9.9 (K_0 recombination formula)**.** *Let* (A_0)–(A_4) *hold. Then for* $n = 1, 2, 3, \ldots,$

$$K_0\left(\bar\xi_n, \tau_n; \bar\xi_{n-1}, \tau_n\bar\tau_{n-1}\right) K_0(\bar\xi_{n-1}, \tau_n\bar\tau_{n-1}; 0, 0) \\ = K_0(\bar\xi_n, \tau_n; 0, 0) K_0(\bar\xi_{n-1}, \tau_n\bar\tau_{n-1}; \bar\xi_n\bar\tau_{n-1}, \tau_n\bar\tau_{n-1}^2). \quad (9.28)$$

Further, for $n = 2, 3, \ldots$ *and* $i = 1, 2, 3, \ldots, n-1$*, we have*

$$K_0\left(\bar\xi_i, \tau_n\prod_{j=i}^{n-1}\bar\tau_j; \bar\xi_{i-1}, \tau_n\prod_{j=i-1}^{n-1}\bar\tau_j\right) K_0\left(\bar\xi_{i-1}, \tau_n\prod_{j=i-1}^{n-1}\bar\tau_j; 0, 0\right) \\ = K_0\left(\bar\xi_i, \tau_n\prod_{j=i}^{n-1}\bar\tau_j; 0, 0\right) K_0\left(\bar\xi_{i-1}, \tau_n\prod_{j=i-1}^{n-1}\bar\tau_j; \bar\xi_i\bar\tau_{i-1}, \tau_n\prod_{j=i-1}^{n-1}\bar\tau_j\bar\tau_{i-1}\right). \quad (9.29)$$

Proof. We will prove the first identity first. Let us note that

$$\begin{aligned}\frac{\bar\xi_n - \bar\xi_{n-1}}{4\tau_n(1-\bar\tau_{n-1})} + \frac{\bar\xi_{n-1}^2}{4\tau_n\bar\tau_{n-1}} &= \frac{1}{4\tau_n\bar\tau_{n-1}(1-\bar\tau_{n-1})}\left[(\bar\xi_n - \bar\xi_{n-1})^2\bar\tau_{n-1} + \bar\xi_{n-1}^2(1-\bar\tau_{n-1})\right] \\ &= \frac{1}{4\tau_n\bar\tau_{n-1}(1-\bar\tau_{n-1})}\left(\bar\xi_n^2 - 2\bar\xi_n\bar\xi_{n-1}\bar\tau_{n-1} + \bar\xi_{n-1}^2\right) \\ &= \frac{1}{4\tau_n\bar\tau_{n-1}(1-\bar\tau_{n-1})}\left[(\bar\xi_{n-1} - \bar\xi_n\bar\tau_{n-1})^2 + \bar\xi_n^2\bar\tau_{n-1}(1-\bar\tau_{n-1})\right] \\ &= \frac{(\bar\xi_{n-1} - \bar\xi_n\bar\tau_{n-1})^2}{4\tau_n\bar\tau_{n-1}(1-\bar\tau_{n-1})} + \frac{\bar\xi_n^2}{4\tau_n}.\end{aligned} \quad (9.30)$$

Together with the definition of K_0, see (9.9), this gives

$$\begin{aligned}&K_0\left(\bar\xi_n, \tau_n; \bar\xi_{n-1}, \tau_n\bar\tau_{n-1}\right) K_0(\bar\xi_{n-1}, \tau_n\bar\tau_{n-1}; 0, 0) \\ &= \frac{1}{\sqrt{4\pi\tau_n(1-\bar\tau_{n-1})}}\left[\exp\left(-\frac{(\bar\xi_n - \bar\xi_{n-1})^2}{4\tau_n(1-\bar\tau_{n-1})}\right)\right]\frac{1}{\sqrt{4\pi\tau_n\bar\tau_{n-1}}}\left[\exp\left(-\frac{(\bar\xi_n - \bar\xi_{n-1})^2}{4\tau_n(1-\bar\tau_{n-1})}\right)\right] \\ &= \frac{1}{\sqrt{4\pi\tau_n}}\left[\exp\left(-\frac{\bar\xi_n^2}{4\tau_n}\right)\right]\frac{1}{\sqrt{4\pi\tau_n\bar\tau_{n-1}(1-\bar\tau_{n-1})}}\left[\exp\left(-\frac{(\bar\xi_{n-1} - \bar\xi_n\bar\tau_{n-1})^2}{4\tau_n\bar\tau_{n-1}(1-\bar\tau_{n-1})}\right)\right] \\ &= K_0(\bar\xi_n, \tau_n; 0, 0)K_0(\bar\xi_{n-1}, \tau_n\bar\tau_{n-1}; \bar\xi_n\bar\tau_{n-1}, \tau\bar\tau_{n-1}^2).\end{aligned} \quad (9.31)$$

This proves the first identity. Now set

$$\tilde\tau_{n,i} = \tau_n\prod_{j=i}^{n-1}\bar\tau_j, \quad n = 2, 3, \ldots, i = 1, 2, 3, \ldots, n-1. \quad (9.32)$$

Then by a similar argument to the derivation of the first identity, we get

$$K_0(\bar{\xi}_i, \tilde{\tau}_{n,i}; \bar{\xi}_{i-1}, \tilde{\tau}_n \bar{\tau}_{i-1}) K_0(\bar{\xi}_{i-1}, \tilde{\tau}_n \bar{\tau}_{i-1}; 0, 0)$$
$$= K_0(\bar{\xi}_i, \tilde{\tau}_{n,i}; 0, 0) K_0(\bar{\xi}_{i-1}, \tilde{\tau}_n \bar{\tau}_{i-1}; \bar{\xi}_i \bar{\tau}_{i-1}, \tilde{\tau}_n \bar{\tau}_{i-1}^2). \quad (9.33)$$

And the proof is complete. □

To simplify the integrands in (9.27) we use the definition of K_0 and explicit differentiation to get

$$\frac{\partial}{\partial \bar{\xi}_i} K_0 \left(\bar{\xi}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j; \bar{\xi}_{i-1}, \tau_n \prod_{j=i-1}^{n-1} \bar{\tau}_j \right)$$
$$= \frac{\bar{\xi}_i - \bar{\xi}_{i-1}}{2\tau_n \prod_{j=i}^{n-1} \bar{\tau}_j (1 - \bar{\tau}_{i-1})} K_0 \left(\bar{\xi}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j; \bar{\xi}_{i-1}, \tau_n \prod_{j=i-1}^{n-1} \bar{\tau}_j \right), \quad (9.34)$$

for $n = 2, 3, 4, \ldots$, and $i = 1, 2, \ldots, n-1$. For the same n's and i's, we define

$$I^{(1)}_{\bar{u}_n}(\tau_n; \bar{\xi}_1, \bar{\tau}_1) = \bar{\tau}_1 \int_0^1 \int_{-\infty}^{\infty} \frac{\partial}{\partial \bar{\xi}_1} K_0 \left(\bar{\xi}_1, \tau_n \prod_{j=1}^{n-1} \bar{\tau}_j; \bar{\xi}_0, \tau_n \prod_{j=0}^{n-1} \bar{\tau}_j \right) \bar{b}(\bar{\xi}_0)$$
$$\times K_0 \left(\bar{\xi}_0, \tau_n \prod_{j=0}^{n-1} \bar{\tau}_j; 0, 0 \right) \, \mathrm{d}\bar{\xi}_0 \bar{\tau}_0,$$
$$I^{(2)}_{\bar{u}_n}(\tau_n; \bar{\xi}_2, \bar{\tau}_2) = \bar{\tau}_2 \int_0^1 \int_{-\infty}^{\infty} \frac{\partial}{\partial \bar{\xi}_2} K_0 \left(\bar{\xi}_2, \tau_n \prod_{j=2}^{n-1} \bar{\tau}_j; \bar{\xi}_1, \tau_n \prod_{j=1}^{n-1} \bar{\tau}_j \right) \bar{b}(\bar{\xi}_1)$$
$$\times I^{(1)}_{\bar{u}_n}(\tau_n; \bar{\xi}_1, \bar{\tau}_1) \, \mathrm{d}\bar{\xi}_1 \bar{\tau}_1,$$
$$\vdots \qquad \vdots \qquad \vdots$$
$$I^{(n-1)}_{\bar{u}_n}(\tau_n; \bar{\xi}_{n-1}, \bar{\tau}_{n-1}) = \bar{\tau}_{n-1}^{n-1} \int_0^1 \int_{-\infty}^{\infty} \frac{\partial}{\partial \bar{\xi}_{n-1}} K_0 \left(\bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1}; \bar{\xi}_{n-2}, \tau_n \bar{\tau}_{n-1} \bar{\tau}_{n-2} \right) \bar{b}(\bar{\xi}_{n-2})$$
$$\times I^{(n-2)}_{\bar{u}_n}(\tau_n; \bar{\xi}_{n-2}, \bar{\tau}_{n-2}) \, \mathrm{d}\bar{\xi}_{n-2} \bar{\tau}_{n-2},$$
$$I^{(n)}_{\bar{u}_n}(\bar{\xi}_n, \tau_n) \equiv \bar{u}_n(\bar{\xi}_n, \tau_n) = \tau_n^n \int_0^1 \int_{-\infty}^{\infty} \frac{\partial}{\partial \bar{\xi}_n} K_0 \left(\bar{\xi}_n, \tau_n; \bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1} \right) \bar{b}(\bar{\xi}_{n-1})$$
$$\times I^{(n-1)}_{\bar{u}_n}(\tau_n; \bar{\xi}_{n-1}, \bar{\tau}_{n-1}) \, \mathrm{d}\bar{\xi}_{n-1} \bar{\tau}_{n-1}. \quad (9.35)$$

Note that $\tau_i^i \equiv (\tau_i)^i$, and it is the ith power of the ith time variable, and that for each fixed n, the nth time variable τ_n is always unbarred. Note also that in the last identity we have merely expressed $\bar{u}_n$ in a different form.

Also, for $n = 2, 3, 4, \ldots$ and $i = 1, 2, \ldots, n-1$, we set

$$\begin{aligned}
I^{(i)}_{\bar{u}_n,\mathcal{R}}(\tau_n; \bar{\xi}_i, \bar{\tau}_i) := &\int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_i - \bar{\xi}_{i-1}}{1 - \bar{\tau}_{i-1}} K_0 \left(\bar{\xi}_{i-1}, \tau_n \prod_{j=i-1}^{n-1} \bar{\tau}_j; \bar{\xi}_i \bar{\tau}_{i-1}, \tau_n \prod_{j=i-1}^{n-1} \bar{\tau}_j \bar{\tau}_{i-1} \right) \bar{b}(\bar{\xi}_{i-1}) \\
&\times \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_{i-1} - \bar{\xi}_{i-2}}{1 - \bar{\tau}_{i-2}} K_0 \left(\bar{\xi}_{i-2}, \tau_n \prod_{j=i-2}^{n-1} \bar{\tau}_j; \bar{\xi}_{i-1} \bar{\tau}_{i-2}, \tau_n \prod_{j=i-2}^{n-1} \bar{\tau}_j \bar{\tau}_{i-2} \right) \bar{b}(\bar{\xi}_{i-2}) \\
&\times \cdots \\
&\times \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_2 - \bar{\xi}_1}{1 - \bar{\tau}_1} K_0 \left(\bar{\xi}_1, \tau_n \prod_{j=1}^{n-1} \bar{\tau}_j; \bar{\xi}_2 \bar{\tau}_1, \tau_n \prod_{j=1}^{n-1} \bar{\tau}_j \bar{\tau}_1 \right) \bar{b}(\bar{\xi}_1) \\
&\times \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_1 - \bar{\xi}_0}{1 - \bar{\tau}_0} K_0 \left(\bar{\xi}_0, \tau_n \prod_{j=0}^{n-1} \bar{\tau}_j; \bar{\xi}_1 \bar{\tau}_0, \tau_n \prod_{j=0}^{n-1} \bar{\tau}_j \bar{\tau}_1 \right) \bar{b}(\bar{\xi}_0) \\
&\times \, \mathrm{d}\bar{\xi}_0 \, \mathrm{d}\bar{\tau}_0 \, \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\tau}_1 \times \cdots \times \, \mathrm{d}\bar{\xi}_{i-2} \, \mathrm{d}\bar{\tau}_{i-2} \, \mathrm{d}\bar{\xi}_{i-1} \, \mathrm{d}\bar{\tau}_{i-1}.
\end{aligned} \tag{9.36}$$

Here $\mathcal{R}$ is not an index; it signifies that $I^{(i)}_{\bar{u}_n,\mathcal{R}}$ is the *residual* component of the term $I^{(i)}_{\bar{u}_n}$. This can be seen in the following lemma.

Lemma 9.10. *Let* (A$_0$)–(A$_4$) *hold. Then for* $n = 2, 3, 4, \ldots$ *and* $i = 1, 2, \ldots, n-1$,

$$I^{(i)}_{\bar{u}_n}(\tau_n; \bar{\xi}_i, \bar{\tau}_i) = \frac{(-1)^i}{\left(2\tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j \right)^i} K_0 \left(\bar{\xi}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j; 0, 0 \right) I^{(i)}_{\bar{u}_n,\mathcal{R}}(\tau_n; \bar{\xi}_i, \bar{\tau}_i), \tag{9.37}$$

where for $i = n-1$, $\prod_{j=n}^{n-1} \bar{\tau}_j := 1$.

Proof. We will prove the lemma by induction. For any fixed (and finite) $n = 2, 3, \ldots$, explicit calculation shows

$$I^{(1)}_{\bar{u}_n}(\tau_n; \bar{\xi}_1, \bar{\tau}_1) = \frac{(-1)^1}{\left(2\tau_n \prod_{j=1}^{n-1} \bar{\tau}_j \right)^1} K_0 \left(\bar{\xi}_1, \tau_n \prod_{j=1}^{n-1} \bar{\tau}_j; 0, 0 \right) I^{(1)}_{\bar{u}_n,\mathcal{R}}(\tau_n; \bar{\xi}_1, \bar{\tau}_1). \tag{9.38}$$

So the identity in question is true for $i = 1$ for any fixed $n = 2, 3, \ldots$. Now suppose that the identity is true for some (n, i), $i > 1$. We will show that it is also true for $(n, i+1)$. Recall that the identity in question would hold only for $i = 1, 2, \ldots, n-1$. Hence without loss of generality we assume that $i + 1 \leq n - 1$. By definition

$$\begin{aligned}
&I_{\bar{u}_n}^{(i+1)}(\tau_n; \bar{\xi}_{i+1}, \bar{\tau}_{i+1}) \\
&= \bar{\tau}_{i+1}^{i+1} \int_0^1 \int_{-\infty}^{\infty} \frac{\partial}{\partial \bar{\xi}_{i+1}} K_0 \left(\bar{\xi}_{i+1}, \tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j; \bar{\xi}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j \right) \bar{b}(\bar{\xi}_i) I_{\bar{u}_n}^{(i)}(\tau_n; \bar{\xi}_i, \bar{\tau}_i) \, \mathrm{d}\bar{\xi}_i \bar{\tau}_i.
\end{aligned} \tag{9.39}$$

Then, by explicit differentiation, see (9.34), and by (9.37), the identity for $I_{\bar{u}_n}^{(i)}$ — since we have supposed that it holds for some fixed pair (n, i) — we get

$$\begin{aligned}
&I_{\bar{u}_n}^{(i+1)}(\tau_n; \bar{\xi}_{i+1}, \bar{\tau}_{i+1}) \\
&= \bar{\tau}_{i+1}^{i+1} \int_0^1 \int_{-\infty}^{\infty} - \frac{\bar{\xi}_{i+1} - \bar{\xi}_i}{2\tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j (1 - \bar{\tau}_i)} K_0 \left(\bar{\xi}_{i+1}, \tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j; \bar{\xi}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j \right) \bar{b}(\bar{\xi}_i) \\
&\qquad \times \frac{(-1)^i}{\left(2\tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j \right)^i} K_0 \left(\bar{\xi}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j; 0, 0 \right) I_{\bar{u}_n, \mathcal{R}}^{(i)}(\tau_n; \bar{\xi}_i, \bar{\tau}_i) \, \mathrm{d}\bar{\xi}_i \, \mathrm{d}\bar{\tau}_i \\
&= \frac{(-1)^{i+1}}{\left(2\tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j \right)^{i+1}} \int_0^1 \int_{-\infty}^{\infty} - \frac{\bar{\xi}_{i+1} - \bar{\xi}_i}{1 - \bar{\tau}_i} K_0 \left(\bar{\xi}_{i+1}, \tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j; \bar{\xi}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j \right) \bar{b}(\bar{\xi}_i) \\
&\qquad \times K_0 \left(\bar{\xi}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j; 0, 0 \right) I_{\bar{u}_n, \mathcal{R}}^{(i)}(\tau_n; \bar{\xi}_i, \bar{\tau}_i) \, \mathrm{d}\bar{\xi}_i \, \mathrm{d}\bar{\tau}_i.
\end{aligned} \tag{9.40}$$

By recombining the K_0's with (9.29), we get

$$\begin{aligned}
&I^{(i+1)}_{\bar{u}_n}(\tau_n;\bar{\xi}_{i+1},\bar{\tau}_{i+1}) \\
&= \frac{(-1)^{i+1}}{\left(2\tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j\right)^{i+1}} K_0\left(\bar{\xi}_{i+1}, \tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j; 0, 0\right) \\
&\quad \times \int_0^1 \int_{-\infty}^{\infty} -\frac{\bar{\xi}_{i+1}-\bar{\xi}_i}{1-\bar{\tau}_i} K_0\left(\bar{\xi}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j; \bar{\xi}_{i+1}\bar{\tau}_i, \tau_n \prod_{j=i}^{n-1} \bar{\tau}_j \bar{\tau}_i\right) \bar{b}(\bar{\xi}_i) \\
&\qquad \times I^{(i)}_{\bar{u}_n,\mathcal{R}}(\tau_n;\bar{\xi}_i,\bar{\tau}_i)\, \mathrm{d}\bar{\xi}_i\, \mathrm{d}\bar{\tau}_i \\
&= \frac{(-1)^{i+1}}{\left(2\tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j\right)^{i+1}} K_0\left(\bar{\xi}_{i+1}, \tau_n \prod_{j=i+1}^{n-1} \bar{\tau}_j; 0, 0\right) I^{(i+1)}_{\bar{u}_n,\mathcal{R}}(\tau_n;\bar{\xi}_{i+1},\bar{\tau}_{i+1}).
\end{aligned} \tag{9.41}$$

This shows that if (9.37) holds for some (n,i), it holds also for $(n,i+1)$. Hence by induction, for a fixed n, (9.37) holds for all $i = 1, 2, \ldots, n-1$. Repeating the same argument shows that (9.37) holds for any fixed integer $n \geq 2$. So by induction again it indeed holds for all arbitrary integer $n \geq 2$. And the proof is complete. □

We now have another lemma.

Lemma 9.11. *Let* (A_0)–(A_4) *hold. Then for* $n = 2, 3, \ldots,$

$$\begin{aligned}
\bar{u}_n(\bar{\xi}_n,\tau_n) = &\left\{\frac{(-1)^{n-1}}{2^{n-1}} \tau_n K_0(\bar{\xi}_n,\tau_n;0,0)\right\} \times \int_0^1 \int_{-\infty}^{\infty} K_0\left(\bar{\xi}_{n-1}, \tau_n\bar{\tau}_{n-1}; \bar{\xi}_n\bar{\tau}_{n-1}, \tau_n\bar{\tau}_{n-1}^2\right) \bar{b}(\bar{\xi}_{n-1}) \\
&\times \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_{n-1}-\bar{\xi}_{n-2}}{1-\bar{\tau}_{n-2}} K_0\left(\bar{\xi}_{n-2}, \tau_n\bar{\tau}_{n-1}\bar{\tau}_{n-2}; \bar{\xi}_{n-1}\bar{\tau}_{n-2}, \tau_n\bar{\tau}_{n-1}\bar{\tau}_{n-2}^2\right) \bar{b}(\bar{\xi}_{n-1}) \\
&\times \cdots \\
&\times \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_2-\bar{\xi}_1}{1-\bar{\tau}_1} K_0\left(\bar{\xi}_1, \tau_n \prod_{j=1}^{n-1} \bar{\tau}_j; \bar{\xi}_2\bar{\tau}_1, \tau_n \prod_{j=1}^{n-1} \bar{\tau}_j \bar{\tau}_1\right) \bar{b}(\bar{\xi}_1) \\
&\times \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_1-\bar{\xi}_0}{1-\bar{\tau}_0} K_0\left(\bar{\xi}_0, \tau_n \prod_{j=0}^{n-1} \bar{\tau}_j; \bar{\xi}_1\bar{\tau}_0, \tau_n \prod_{j=0}^{n-1} \bar{\tau}_j \bar{\tau}_0\right) \bar{b}(\bar{\xi}_0) \\
&\times \mathrm{d}\bar{\xi}_0\, \mathrm{d}\bar{\tau}_0\, \mathrm{d}\bar{\xi}_1\, \mathrm{d}\bar{\tau}_1 \times \cdots \times \mathrm{d}\bar{\xi}_{n-2}\, \mathrm{d}\bar{\tau}_{n-2}\, \mathrm{d}\bar{\xi}_{n-1}\, \mathrm{d}\bar{\tau}_{n-1}.
\end{aligned} \tag{9.42}$$

Proof. By (9.35), we have

$$\bar{u}_n(\bar{\xi}_n, \tau_n) = \tau_n^n \int_0^1 \int_{-\infty}^{\infty} K_0(\bar{\xi}_n, \tau_n; \bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1}) \bar{b}(\bar{\xi}_{n-1}) I_{\bar{u}_n}^{(n-1)}(\tau_n; \bar{\xi}_{n-1}, \bar{\tau}_{n-1}) \, \mathrm{d}\bar{\xi}_{n-1} \, \mathrm{d}\bar{\tau}_{n-1}. \tag{9.43}$$

By (9.37),

$$\begin{aligned}
\bar{u}_n(\bar{\xi}_n, \tau_n) &= \tau_n^n \int_0^1 \int_{-\infty}^{\infty} K_0(\bar{\xi}_n, \tau_n; \bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1}) \bar{b}(\bar{\xi}_{n-1}) \\
&\quad \times \frac{(-1)^{n-1}}{\left(2\tau_n \prod_{j=n}^{n-1} \bar{\tau}_j\right)^{n-1}} K_0\left(\bar{\xi}_{n-1}, \tau_n \prod_{j=n-1}^{n-1} \bar{\tau}_j; 0, 0\right) \\
&\quad \times I_{\bar{u}_n, \mathcal{R}}^{(n-1)}(\tau_n; \bar{\xi}_{n-1}, \bar{\tau}_{n-1}) \, \mathrm{d}\bar{\xi}_{n-1} \, \mathrm{d}\bar{\tau}_{n-1} \\
&= \tau_n^n \int_0^1 \int_{-\infty}^{\infty} \frac{(-1)^{n-1}}{(2\tau_n)^{n-1}} K_0(\bar{\xi}_n, \tau_n; \bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1}) K_0\left(\bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1}; 0, 0\right) \\
&\quad \times \bar{b}(\bar{\xi}_{n-1}) I_{\bar{u}_n, \mathcal{R}}^{(n-1)} \, \mathrm{d}\bar{\xi}_{n-1} \, \mathrm{d}\bar{\tau}_{n-1} \\
&= \frac{(-1)^{n-1}}{2^{n-1}} \tau_n K_0(\bar{\xi}_n, \tau_n; 0, 0) \\
&\quad \times \int_0^1 \int_{-\infty}^{\infty} K_0(\bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1}; \bar{\xi}_n \bar{\tau}_{n-1}, \tau_n \bar{\tau}_{n-1}^2) \bar{b}(\bar{\xi}_{n-1}) I_{\bar{u}_n, \mathcal{R}}^{(n-1)} \, \mathrm{d}\bar{\xi}_{n-1} \, \mathrm{d}\bar{\tau}_{n-1},
\end{aligned} \tag{9.44}$$

where in the last equality we have used (9.29) to recombine the K_0's. The desired result then follows from the definition of $I_{\bar{u}_n, \mathcal{R}}^{(n-1)}$; see (9.36). The proof is thus complete. □

For each fixed $n = 2, 3, 4, \ldots$, we will change the variables $\bar{\xi}_i$, $i = 1, 2, \ldots, n-1$, in (9.42). For each fixed $n = 2, 3, 4, \ldots$, let

$$\begin{aligned}
\bar{u}_n^{(1)} &= \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_1 - \bar{\xi}_0}{1 - \bar{\tau}_0} K_0\left(\bar{\xi}_0, \tau_n \prod_{j=0}^{n-1} \bar{\tau}_j; \bar{\xi}_1 \bar{\tau}_0, \tau_n \prod_{j=0}^{n-1} \bar{\tau}_j \bar{\tau}_0\right) \bar{b}(\bar{\xi}_0) \, \mathrm{d}\bar{\xi}_0 \, \mathrm{d}\bar{\tau}_0, \\
\bar{u}_n^{(2)} &= \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_2 - \bar{\xi}_1}{1 - \bar{\tau}_1} K_0\left(\bar{\xi}_1, \tau_n \prod_{j=1}^{n-1} \bar{\tau}_j; \bar{\xi}_2 \bar{\tau}_1, \tau_n \prod_{j=1}^{n-1} \bar{\tau}_j \bar{\tau}_1\right) \bar{b}(\bar{\xi}_1) \bar{u}_n^{(1)} \, \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\tau}_1, \\
&\vdots
\end{aligned} \tag{9.45}$$

and

$$\bar{u}_n^{(i)} = \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_i - \bar{\xi}_{i-1}}{1 - \bar{\tau}_{i-1}} K_0 \left(\bar{\xi}_{i-1}, \tau_n \prod_{j=i-1}^{n-1} \bar{\tau}_j; \bar{\xi}_i \bar{\tau}_{i-1}, \tau_n \prod_{j=i-1}^{n-1} \bar{\tau}_j \bar{\tau}_{i-1} \right)$$
$$\times \bar{b}(\bar{\xi}_{i-1}) \bar{u}_n^{(i-1)} \, \mathrm{d}\bar{\xi}_{i-1} \, \mathrm{d}\bar{\tau}_{i-1},$$
$$\vdots$$
$$\bar{u}_n^{(n-1)} = \int_0^1 \int_{-\infty}^{\infty} \frac{\bar{\xi}_{n-1} - \bar{\xi}_{n-2}}{1 - \bar{\tau}_{n-2}} K_0 \left(\bar{\xi}_{n-2}, \tau_n \prod_{j=n-2}^{n-1} \bar{\tau}_j; \bar{\xi}_i \bar{\tau}_{n-2}, \tau_n \prod_{j=n-2}^{n-1} \bar{\tau}_j \bar{\tau}_{n-2} \right)$$
$$\times \bar{b}(\bar{\xi}_{n-2}) \bar{u}_n^{(n-2)} \, \mathrm{d}\bar{\xi}_{n-2} \, \mathrm{d}\bar{\tau}_{n-2},$$
$$\bar{u}_n^{(n)}(\bar{\xi}_n, \tau_n) = \int_0^1 \int_{-\infty}^{\infty} K_0 \left(\bar{\xi}_{n-1}, \tau_n \bar{\tau}_{n-1}; \bar{\xi}_n \bar{\tau}_{n-1}, \tau_n \bar{\tau}_{n-1}^2 \right) \bar{b}(\bar{\xi}_{n-1}) \bar{u}_n^{(n-1)} \, \mathrm{d}\bar{\xi}_{n-1} \, \mathrm{d}\bar{\tau}_{n-1}, \tag{9.46}$$

where $\bar{u}_n^{(i)} = \bar{u}_n^{(i)} \left(\bar{\xi}_i, \bar{\tau}_i, \bar{\tau}_{i+1}, \ldots, \bar{\tau}_{n-1}; \bar{\xi}_n, \tau_n \right)$, $i = 1, 2, \ldots, n-1$.

To illustrate how to change the $\bar{\xi}_n$ variables, let us put

$$\bar{y}_0 = \frac{\bar{\xi}_0 - \bar{\xi}_1 \bar{\tau}_0}{\sqrt{4\tau_n \prod_{j=0}^{n-1} \bar{\tau}_j (1 - \bar{\tau}_0)}}. \tag{9.47}$$

Then a rearrangement gives

$$\bar{\xi}_0(\bar{y}_0; \tau_n) = \bar{y}_0 \sqrt{4\tau_n \prod_{j=0}^{n-1} \bar{\tau}_j (1 - \bar{\tau}_0)} + \bar{\xi}_1 \bar{\tau}_0. \tag{9.48}$$

Notice that

$$\bar{\xi}_0|_{\tau_n=0} := \lim_{\tau_n \to 0} \bar{\xi}_0(\bar{y}_0; \tau_n) = (\bar{\xi}_1|_{\tau_n=0}) \bar{\tau}_0 \tag{9.49}$$

if in turn $\bar{\xi}_1$ also depends on τ_n.

Now by the definition of $\bar{u}_n^{(1)} \left(\bar{\xi}_1, \bar{\tau}_1, \bar{\tau}_2, \ldots, \bar{\tau}_{n-1}; \bar{\xi}_n, \tau_n \right)$, see (9.45), we get

$$\bar{u}_n^{(1)} = \int_0^1 \int_{-\infty}^{\infty} \frac{1}{1 - \bar{\tau}_0} \left[\bar{\xi}_1 - \bar{\xi}_1 \bar{\tau}_0 - \bar{y}_0 \sqrt{4\tau_n \prod_{j=0}^{n-1} \bar{\tau}_j (1 - \bar{\tau}_0)} \right]$$
$$\times \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_0^2} \bar{b}(\bar{\xi}_0(\bar{y}_0; \tau_n)) \, \mathrm{d}\bar{y}_0 \, \mathrm{d}\bar{\tau}_0 \tag{9.50}$$
$$= \int_0^1 \int_{-\infty}^{\infty} \left[\left(\bar{\xi}_1 - \bar{y}_0 \sqrt{\frac{4\tau_n \prod_{j=0}^{n-1} \bar{\tau}_j}{1 - \bar{\tau}_0}} \right) \bar{b}(\bar{\xi}_0(\bar{y}_0; \tau_n)) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_0^2} \right] \, \mathrm{d}\bar{y}_0 \, \mathrm{d}\bar{\tau}_0.$$

Hence

$$
\begin{aligned}
&\lim_{\tau_n\to 0} \bar{u}_n^{(1)}\left(\bar{\xi}_1, \bar{\tau}_1, \bar{\tau}_2, \ldots, \bar{\tau}_{n-1}; \bar{\xi}_n, \tau_n\right) \\
&= \lim_{\tau_n\to 0} \int_0^1 \int_{-\infty}^{\infty} \left[\left(\bar{\xi}_1 - \bar{y}_0\sqrt{\frac{4\tau_n \prod_{j=0}^{n-1}\bar{\tau}_j}{1-\bar{\tau}_0}}\right) \bar{b}(\bar{\xi}_0(\bar{y}_0;\tau_n))\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\right] \,\mathrm{d}\bar{y}_0\,\mathrm{d}\bar{\tau}_0 \\
&= \int_0^1 \int_{-\infty}^{\infty} \lim_{\tau_n\to 0} \left[\left(\bar{\xi}_1 - \bar{y}_0\sqrt{\frac{4\tau_n \prod_{j=0}^{n-1}\bar{\tau}_j}{1-\bar{\tau}_0}}\right) \bar{b}(\bar{\xi}_0(\bar{y}_0;\tau_n))\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\right] \,\mathrm{d}\bar{y}_0\,\mathrm{d}\bar{\tau}_0 \\
&= \int_0^1 \int_{-\infty}^{\infty} \bar{\xi}_1|_{\tau_n=0}\, \bar{b}(\bar{\xi}_0|_{\tau_n=0})\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\,\mathrm{d}\bar{y}_0\,\mathrm{d}\bar{\tau}_0 \\
&= \left(\int_0^1 \bar{\xi}_1|_{\tau_n=0}\, \bar{b}(\bar{\xi}_0|_{\tau_n=0})\,\mathrm{d}\bar{\tau}_0\right)\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\,\mathrm{d}\bar{y}_0\right) \\
&= \int_0^1 \bar{\xi}_1|_{\tau_n=0}\, \bar{b}(\bar{\xi}_0|_{\tau_n=0})\,\mathrm{d}\bar{\tau}_0
\end{aligned}
\tag{9.51}
$$

where the taking of the limit inside the integral is permitted by Lebesgue's dominated convergence theorem, see e.g. Rudin [72, Theorem 1.34, p. 26]; and that the splitting of the integrals are justified as $\bar{\xi}_0|_{\tau_n=0}$ and $\bar{\xi}_1|_{\tau_n=0}$ are independent of $\bar{y}_0$.

Having illustrated the idea of how to change the $\bar{\xi}_0$ variable, we will list the full transformation for all the $\bar{\xi}_i$'s, $i = 0, 1, \ldots, n-1$, for any fixed $n = 2, 3, 4, \ldots$ as follows:

$$
\begin{aligned}
\bar{y}_0 &= \frac{\bar{\xi}_0 - \bar{\xi}_1\bar{\tau}_0}{\sqrt{4\tau_n\prod_{j=0}^{n-1}\bar{\tau}_j(1-\bar{\tau}_0)}}, \quad \text{or} \quad \bar{\xi}_0 = \bar{y}_0\sqrt{4\tau_n\prod_{j=0}^{n-1}\bar{\tau}_j(1-\bar{\tau}_0)} + \bar{\xi}_1\bar{\tau}_0, \\
\bar{y}_1 &= \frac{\bar{\xi}_1 - \bar{\xi}_2\bar{\tau}_1}{\sqrt{4\tau_n\prod_{j=1}^{n-1}\bar{\tau}_j(1-\bar{\tau}_1)}}, \quad \text{or} \quad \bar{\xi}_1 = \bar{y}_1\sqrt{4\tau_n\prod_{j=1}^{n-1}\bar{\tau}_j(1-\bar{\tau}_1)} + \bar{\xi}_2\bar{\tau}_1, \\
&\vdots \\
\bar{y}_i &= \frac{\bar{\xi}_i - \bar{\xi}_{i+1}\bar{\tau}_i}{\sqrt{4\tau_n\prod_{j=i}^{n-1}\bar{\tau}_j(1-\bar{\tau}_i)}}, \quad \text{or} \quad \bar{\xi}_i = \bar{y}_i\sqrt{4\tau_n\prod_{j=i}^{n-1}\bar{\tau}_j(1-\bar{\tau}_i)} + \bar{\xi}_{i+1}\bar{\tau}_i, \\
&\vdots \\
\bar{y}_{n-2} &= \frac{\bar{\xi}_{n-2} - \bar{\xi}_{n-1}\bar{\tau}_{n-2}}{\sqrt{4\tau_n\prod_{j=n-2}^{n-1}\bar{\tau}_j(1-\bar{\tau}_{n-2})}}, \quad \text{or} \quad \bar{\xi}_{n-2} = \bar{y}_{n-2}\sqrt{4\tau_n\prod_{j=n-2}^{n-1}\bar{\tau}_j(1-\bar{\tau}_{n-2})} + \bar{\xi}_{n-1}\bar{\tau}_{n-2}, \\
\bar{y}_{n-1} &= \frac{\bar{\xi}_{n-1} - \bar{\xi}_n\bar{\tau}_{n-1}}{\sqrt{4\tau_n\bar{\tau}_{n-1}(1-\bar{\tau}_{n-1})}}, \quad \text{or} \quad \bar{\xi}_{n-1} = \bar{y}_{n-1}\sqrt{4\tau_n\bar{\tau}_{n-1}(1-\bar{\tau}_{n-1})} + \bar{\xi}_n\bar{\tau}_{n-1}.
\end{aligned}
\tag{9.52}
$$

Then, as $\tau_n \to 0$,

$$\begin{aligned}
\bar{\xi}_{n-1}|_{\tau_n=0} &= \bar{\xi}_n \bar{\tau}_{n-1}, \\
\bar{\xi}_{n-2}|_{\tau_n=0} &= (\bar{\xi}_{n-1}|_{\tau_n=0})\bar{\tau}_{n-1} = \bar{\xi}_n \bar{\tau}_{n-1}\bar{\tau}_{n-2}, \\
&\vdots \\
\bar{\xi}_2|_{\tau_n=0} &= \bar{\xi}_n \prod_{j=2}^{n-1} \bar{\tau}_j, \\
\bar{\xi}_1|_{\tau_n=0} &= \bar{\xi}_n \prod_{j=1}^{n-1} \bar{\tau}_j, \\
\bar{\xi}_0|_{\tau_n=0} &= \bar{\xi}_n \prod_{j=0}^{n-1} \bar{\tau}_j.
\end{aligned} \tag{9.53}$$

In general, for each fixed $n = 2, 3, \ldots,$

$$\bar{\xi}_i|_{\tau_n=0} = \bar{\xi}_n \prod_{j=i}^{n-1} \bar{\tau}_j, \quad i = 0, 1, 2, \ldots, n-1. \tag{9.54}$$

We then have the following lemma.

Lemma 9.12. *Let* (A_0)–(A_4) *hold. Then for all* $n = 2, 3, \ldots,$

$$\begin{aligned}
\bar{u}_n^{(1)} &= \int_0^1 \int_{-\infty}^{\infty} \left(\bar{\xi}_1(\bar{y}_1; \tau_n) - \bar{y}_0 \sqrt{\frac{4\tau_n \prod_{j=0}^{n-1} \bar{\tau}_j}{1 - \bar{\tau}_0}} \right) \bar{b}(\bar{\xi}_0(\bar{y}_0; \tau_n)) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_0^2} \, \mathrm{d}\bar{y}_0 \, \mathrm{d}\bar{\tau}_0, \\
\bar{u}_n^{(2)} &= \int_0^1 \int_{-\infty}^{\infty} \left(\bar{\xi}_2(\bar{y}_2; \tau_n) - \bar{y}_1 \sqrt{\frac{4\tau_n \prod_{j=1}^{n-1} \bar{\tau}_j}{1 - \bar{\tau}_1}} \right) \bar{b}(\bar{\xi}_1(\bar{y}_1; \tau_n)) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_1^2} \bar{u}_n^{(1)} \, \mathrm{d}\bar{y}_1 \, \mathrm{d}\bar{\tau}_1, \\
&\vdots \\
\bar{u}_n^{(n-1)} &= \int_0^1 \int_{-\infty}^{\infty} \left(\bar{\xi}_{n-1}(\bar{y}_{n-1}; \tau_n) - \bar{y}_{n-2} \sqrt{\frac{4\tau_n \prod_{j=n-2}^{n-1} \bar{\tau}_j}{1 - \bar{\tau}_{n-2}}} \right) \\
&\qquad \times \bar{b}(\bar{\xi}_{n-2}(\bar{y}_{n-2}; \tau_n)) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_{n-2}^2} \bar{u}_n^{(n-2)} \, \mathrm{d}\bar{y}_{n-2} \, \mathrm{d}\bar{\tau}_{n-2}, \\
\bar{u}_n^{(n)}(\bar{\xi}_n, \tau_n) &= \int_0^1 \int_{-\infty}^{\infty} \bar{b}(\bar{\xi}_{n-1}(\bar{y}_{n-1}; \tau_n)) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_{n-1}^2} \bar{u}_n^{(n-1)} \, \mathrm{d}\bar{y}_{n-1} \, \mathrm{d}\bar{\tau}_{n-1},
\end{aligned} \tag{9.55}$$

where $\bar{u}_n^{(i)} = \bar{u}_n^{(i)} \left(\bar{\xi}_i, \bar{\tau}_i, \bar{\tau}_{i+1}, \ldots, \bar{\tau}_{n-1}; \bar{\xi}_n, \tau_n \right)$, $i = 1, 2, \ldots, n-1$.

Proof. We have shown in (9.50) that for any arbitrary positive integer $n \geq 2$, the formula for $\bar{u}_n^{(1)}$ holds. Now by the definition (9.55) and the variable transform formulae (9.52) we can apply a similar argument to that leading to (9.50), to obtain the desire formulae for $\bar{u}_n^2, \ldots, \bar{u}_n^{(n)}$.

It remains to show that the formulae above indeed hold for all $n = 2, 3, \ldots$. This will be done by induction. Since we have shown that the formulae hold for any arbitrary integer $n \geq 2$, we only need to show that if the formula for $\bar{u}_n^{(n)}$ holds for some positive integer m, then it also holds for $m+1$. By definition,

$$\begin{aligned}\bar{u}_{m+1}^{(m+1)}(\bar{\xi}_{m+1}, \tau_{m+1}) = \int_0^1 \int_{-\infty}^{\infty} & K_0(\bar{\xi}_m, \tau_{m+1}\bar{\tau}_m; \bar{\xi}_{m+1}\bar{\tau}_{m+1}, \tau_{m+1}\bar{\tau}_m^2) \\ & \times \bar{b}(\bar{\xi}_m; \tau_{m+1})\bar{u}_m^{(m)} \, \mathrm{d}\bar{\xi}_m \, \mathrm{d}\bar{\tau}_m.\end{aligned} \tag{9.56}$$

Let

$$\bar{y}_m = \frac{\bar{\xi}_m - \bar{\xi}_{m+1}\bar{\tau}_m}{\sqrt{4\tau_{m+1}\bar{\tau}_m(1-\bar{\tau}_m)}}, \tag{9.57}$$

or equivalently

$$\bar{\xi}_m(\bar{y}_m; \tau_{m+1}) := \bar{\xi}_m(\bar{y}_m, \tau_m; \bar{\xi}_{m+1}, \tau_{m+1}) = \bar{y}_m\sqrt{4\tau_{m+1}\bar{\tau}_m(1-\bar{\tau}_m)} + \bar{\xi}_{m+1}\bar{\tau}_m. \tag{9.58}$$

Then we have

$$\bar{u}_{m+1}^{(m+1)}(\bar{\xi}_{m+1}, \tau_{m+1}) = \int_0^1 \int_{-\infty}^{\infty} \bar{b}\Big(\bar{\xi}_m(\bar{y}_m; \tau_{m+1})\Big) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_m^2} \bar{u}_m^{(m)} \, \mathrm{d}\bar{y}_m \, \mathrm{d}\bar{\tau}_m, \tag{9.59}$$

showing that the formula for $\bar{u}_n^{(n)}$ holds also for $n = m+1$ if it holds for $n = m$. Hence the proof is complete. □

We will use (9.55) to calculate the limit of $\bar{u}_n^{(i)}$, $i = 1, 2, \ldots, n$, as $\tau_n \to 0$. Note that by the dominated convergence theorem, we can take the limits of the integrands before integrating them, as in (9.51). For each $n = 2, 3, \ldots$, let

$$\begin{aligned}\bar{\xi}_i|_{\tau_n=0} &= \lim_{\tau_n \to 0} \bar{\xi}_i(\bar{y}_i; \tau_n), \quad i = 1, 2, \ldots, n-1, \\ \bar{u}_n^{(i)}|_{\tau_n=0} &= \lim_{\tau_n \to 0} \bar{u}_n^{(i)}\left(\bar{\xi}_i, \bar{\tau}_i, \bar{\tau}_{i+1}, \ldots, \bar{\tau}_{n-1}; \bar{\xi}_n, \tau_n\right), \quad i = 1, 2, \ldots, n-1, \\ \bar{u}_n^{(n)}|_{\tau_n=0} &= \lim_{\tau_n \to 0} \bar{u}_n^{(n)}(\bar{\xi}_n, \tau_n).\end{aligned} \tag{9.60}$$

Then by (9.54) and (9.55), we have

$$
\begin{aligned}
\bar{u}_n^{(1)}|_{\tau_n=0} &= \lim_{\tau_n\to 0}\int_0^1\int_{-\infty}^{\infty}\bar{\xi}_1(\bar{y}_1;\tau_n)\bar{b}\Big(\bar{\xi}_0(\bar{y}_0;\tau_n)\Big)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\,\mathrm{d}\bar{y}_0\,\mathrm{d}\bar{\tau}_0 \\
&= \int_0^1\int_{-\infty}^{\infty}\bar{\xi}_1(\bar{y}_1;\tau_n)|_{\tau_n=0}\bar{b}\Big(\bar{\xi}_0(\bar{y}_0;\tau_n)|_{\tau_n=0}\Big)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\,\mathrm{d}\bar{y}_0\,\mathrm{d}\bar{\tau}_0 \\
&= \int_0^1\int_{-\infty}^{\infty}\left(\bar{\xi}_n\prod_{j=1}^{n-1}\bar{\tau}_j\right)\bar{b}\left(\bar{\xi}_n\prod_{j=0}^{n-1}\bar{\tau}_j\right)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\,\mathrm{d}\bar{y}_0\,\mathrm{d}\bar{\tau}_0 \\
&= \left[\int_0^1\left(\bar{\xi}_n\prod_{j=1}^{n-1}\bar{\tau}_j\right)\bar{b}\left(\bar{\xi}_n\prod_{j=0}^{n-1}\bar{\tau}_j\right)\,\mathrm{d}\bar{\tau}_0\right]\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\,\mathrm{d}\bar{y}_0\right] \\
&= \int_0^1\left(\bar{\xi}_n\prod_{j=1}^{n-1}\bar{\tau}_j\right)\bar{b}\left(\bar{\xi}_n\prod_{j=0}^{n-1}\bar{\tau}_j\right)\,\mathrm{d}\bar{\tau}_0.
\end{aligned}
\tag{9.61}
$$

Similarly,

$$
\begin{aligned}
\bar{u}_n^{(2)}|_{\tau_n=0} &= \lim_{\tau_n\to 0}\int_0^1\int_{-\infty}^{\infty}\bar{\xi}_2(\bar{y}_2;\tau_n)\bar{b}\Big(\bar{\xi}_1(\bar{y}_1;\tau_n)\Big)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_1^2}\bar{u}_n^{(1)}\,\mathrm{d}\bar{y}_1\,\mathrm{d}\bar{\tau}_1 \\
&= \int_0^1\int_{-\infty}^{\infty}\bar{\xi}_2(\bar{y}_2;\tau_n)|_{\tau_n=0}\bar{b}\Big(\bar{\xi}_1(\bar{y}_1;\tau_n)|_{\tau_n=0}\Big)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_1^2}\bar{u}_n^{(1)}|_{\tau_n=0}\,\mathrm{d}\bar{y}_1\,\mathrm{d}\bar{\tau}_1 \\
&= \int_0^1\int_{-\infty}^{\infty}\left(\bar{\xi}_n\prod_{j=2}^{n-1}\bar{\tau}_j\right)\bar{b}\left(\bar{\xi}_n\prod_{j=1}^{n-1}\bar{\tau}_j\right)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_1^2}\bar{u}_n^{(1)}|_{\tau_n=0}\,\mathrm{d}\bar{y}_1\,\mathrm{d}\bar{\tau}_1 \\
&= \left[\int_0^1\left(\bar{\xi}_n\prod_{j=2}^{n-1}\bar{\tau}_j\right)\bar{b}\left(\bar{\xi}_n\prod_{j=1}^{n-1}\bar{\tau}_j\right)\bar{u}_n^{(1)}|_{\tau_n=0}\,\mathrm{d}\bar{\tau}_0\right]\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\,\mathrm{d}\bar{y}_0\right] \\
&= \int_0^1\left(\bar{\xi}_n\prod_{j=2}^{n-1}\bar{\tau}_j\right)\bar{b}\left(\bar{\xi}_n\prod_{j=1}^{n-1}\bar{\tau}_j\right)\bar{u}_n^{(1)}|_{\tau_n=0}\,\mathrm{d}\bar{\tau}_0.
\end{aligned}
\tag{9.62}
$$

Repeating the same process, we have

$$
\begin{aligned}
\bar{u}_n^{(n-1)}|_{\tau_n=0} &= \lim_{\tau_n\to 0}\int_0^1\int_{-\infty}^{\infty}\left[\bar{\xi}_{n-1}(\bar{y}_{n-1};\tau_n) - \bar{y}_{n-2}\sqrt{\frac{4\tau_n\prod_{j=n-2}^{n-1}\bar{\tau}_j}{1-\bar{\tau}_{n-2}}}\right] \\
&\qquad \times \bar{b}\Big(\bar{\xi}_{n-2}(\bar{y}_{n-2};\tau_n)\Big)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_{n-2}^2}\bar{u}_n^{(n-2)}\ \mathrm{d}\bar{y}_{n-2}\,\mathrm{d}\bar{\tau}_{n-2} \\
&= \int_0^1\int_{-\infty}^{\infty}\bar{\xi}_{n-1}(\bar{y}_{n-1};\tau_n)|_{\tau_n=0}\bar{b}\Big(\bar{\xi}_{n-2}(\bar{y}_{n-2};\tau_n)|_{\tau_n=0}\Big) \\
&\qquad \times \frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_{n-2}^2}\bar{u}_n^{(n-2)}|_{\tau_n=0}\ \mathrm{d}\bar{y}_{n-2}\,\mathrm{d}\bar{\tau}_{n-2} \\
&= \int_0^1\int_{-\infty}^{\infty}\left(\bar{\xi}_n\prod_{j=n-1}^{n-1}\bar{\tau}_j\right)\bar{b}\left(\bar{\xi}_n\prod_{j=n-2}^{n-1}\bar{\tau}_j\right)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_{n-2}^2}\bar{u}_n^{(n-2)}|_{\tau_n=0}\ \mathrm{d}\bar{y}_{n-2}\,\mathrm{d}\bar{\tau}_{n-2} \\
&= \int_0^1\int_{-\infty}^{\infty}\bar{\xi}_n\bar{\tau}_{n-1}\bar{b}\Big(\bar{\xi}_n\bar{\tau}_{n-1}\bar{\tau}_{n-2}\Big)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_{n-2}^2}\bar{u}_n^{(n-2)}|_{\tau_n=0}\ \mathrm{d}\bar{y}_{n-2}\,\mathrm{d}\bar{\tau}_{n-2} \\
&= \left[\int_0^1\bar{\xi}_n\bar{\tau}_{n-1}\bar{b}\Big(\bar{\xi}_n\bar{\tau}_{n-1}\bar{\tau}_{n-2}\Big)\bar{u}_n^{(n-2)}|_{\tau_n=0}\ \mathrm{d}\bar{\tau}_{n-2}\right]\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_{n-2}^2}\ \mathrm{d}\bar{y}_{n-2}\right] \\
&= \int_0^1\bar{\xi}_n\bar{\tau}_{n-1}\bar{b}\Big(\bar{\xi}_n\bar{\tau}_{n-1}\bar{\tau}_{n-2}\Big)\bar{u}_n^{(n-2)}|_{\tau_n=0}\ \mathrm{d}\bar{\tau}_{n-2}.
\end{aligned}
\tag{9.63}
$$

and

$$
\begin{aligned}
\bar{u}_n^{(n)}|_{\tau_n=0} &= \int_0^1\int_{-\infty}^{\infty}\bar{b}\Big(\bar{\xi}_{n-1}(\bar{y}_{n-1};\tau_n)|_{\tau_n=0}\Big)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_{n-1}^2}\bar{u}_n^{(n-1)}|_{\tau_n=0}\ \mathrm{d}\bar{y}_{n-1}\,\mathrm{d}\tau_{n-1} \\
&= \int_0^1\bar{b}(\bar{\xi}_n\bar{\tau}_{n-1})\bar{u}_n^{n-1}|_{\tau_n=0}\ \mathrm{d}\bar{\tau}_{n-1}.
\end{aligned}
\tag{9.64}
$$

In sum, we have obtained the following lemma for the $\bar{u}_n^{(n)}$'s.

Lemma 9.13. *Let* (A_0)–(A_4) *hold. Then for all* $n = 2, 3, \ldots,$

$$\begin{aligned}\lim_{\tau_n \to 0} \bar{u}_n^{(n)}(\bar{\xi}_n, \tau_n) = &\int_0^1 \bar{b}(\bar{\xi}_n \bar{\tau}_{n-1}) \\ &\times \int_0^1 \bar{\xi}_n \bar{\tau}_{n-1} \bar{b}\Big(\bar{\xi}_n \bar{\tau}_{n-1} \bar{\tau}_{n-2}\Big) \\ &\times \cdots \times \int_0^1 \left(\bar{\xi}_n \prod_{j=2}^{n-1} \bar{\tau}_j\right) \bar{b} \left(\bar{\xi}_n \prod_{j=1}^{n-1} \bar{\tau}_j\right) \\ &\qquad \times \int_0^1 \left(\bar{\xi}_n \prod_{j=1}^{n-1} \bar{\tau}_j\right) \bar{b} \left(\bar{\xi}_n \prod_{j=0}^{n-1} \bar{\tau}_j\right) \, \mathrm{d}\bar{\tau}_0 \\ &\times \, \mathrm{d}\bar{\tau}_1 \times \cdots \times \, \mathrm{d}\bar{\tau}_{n-2} \, \mathrm{d}\bar{\tau}_{n-1}. \end{aligned} \tag{9.65}$$

For $n = 2, 3, \ldots,$ we make the following change of variables:

$$\begin{aligned} \bar{z}_0 &= \bar{\xi}_n \prod_{j=0}^{n-1} \bar{\tau}_j = \left(\bar{\xi}_n \prod_{j=1}^{n-1} \bar{\tau}_j\right) \bar{\tau}_0, \\ \bar{z}_1 &= \bar{\xi}_n \prod_{j=1}^{n-1} \bar{\tau}_j = \left(\bar{\xi}_n \prod_{j=2}^{n-1} \bar{\tau}_j\right) \bar{\tau}_1, \\ &\vdots \\ \bar{z}_i &= \bar{\xi}_n \prod_{j=i}^{n-1} \bar{\tau}_j = \left(\bar{\xi}_n \prod_{j=i+1}^{n-1} \bar{\tau}_j\right) \bar{\tau}_i, \\ &\vdots \\ \bar{z}_{n-1} &= \bar{\xi}_n \prod_{j=n-2}^{n-1} \bar{\tau}_j = \left(\bar{\xi}_n \bar{\tau}_{n-1}\right) \bar{\tau}_{n-2}, \\ \bar{z}_{n-2} &= \bar{\xi}_n \bar{\tau}_{n-1}. \end{aligned} \tag{9.66}$$

Recall that by definition $\bar{u}_n^{(n)}|_{\tau_n = 0} = \bar{u}_n^{(n)}(\bar{\xi}_n, 0)$. Then we have the following lemma:

Lemma 9.14. *Let* (A_0)–(A_4) *hold. Then for each* $n = 1, 2, 3, \ldots,$

$$\begin{aligned}\bar{u}_n^{(n)}(\bar{\xi}_n, 0) = \frac{1}{\bar{\xi}_n} \int_0^{\bar{\xi}_n} \bar{b}(\bar{z}_{n-1}) \int_0^{\bar{z}_{n-1}} \bar{b}(\bar{z}_{n-2}) \int_0^{\bar{z}_{n-2}} \bar{b}(\bar{z}_{n-3}) \times \cdots \times \int_0^{\bar{z}_2} \bar{b}(\bar{z}_1) \int_0^{\bar{z}_1} \bar{b}(\bar{z}_0) \\ \times \, \mathrm{d}\bar{z}_0 \, \mathrm{d}\bar{z}_1 \times \cdots \times \, \mathrm{d}\bar{z}_{n-3} \, \mathrm{d}\bar{z}_{n-2} \, \mathrm{d}\bar{z}_{n-1}. \end{aligned} \tag{9.67}$$

Proof. We will prove the lemma by induction. Suppose that the formula is true for some n. Then by (9.59),

$$\begin{aligned}
&\lim_{\tau_{n+1}\to 0} \bar{u}^{(n+1)}_{n+1}(\bar{\xi}_{n+1},\tau_{n+1}) \\
&= \lim_{\tau_{n+1}\to 0} \int_0^1 \int_{-\infty}^{\infty} \bar{b}\Big(\bar{\xi}_n(\bar{y}_n;\tau_{n+1})\Big) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_n^2} \bar{u}^{(n)}_{n+1}\, \mathrm{d}\bar{y}_n\, \mathrm{d}\bar{\tau}_n \\
&= \int_0^1 \int_{-\infty}^{\infty} \bar{b}\Big(\bar{\xi}_n(\bar{y}_n;\tau_{n+1})|_{\tau_{n+1}=0}\Big) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_n^2} \bar{u}^{(n)}_{n+1}|_{\tau_{n+1}=0}\, \mathrm{d}\bar{y}_n\, \mathrm{d}\bar{\tau}_n \\
&= \left[\int_0^1 \bar{b}\Big(\bar{\xi}_n(\bar{y}_n;\tau_{n+1})|_{\tau_{n+1}=0}\Big) \bar{u}^{(n)}_{n+1}|_{\tau_{n+1}=0}\, \mathrm{d}\bar{\tau}_n\right] \left[\int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_n^2}\, \mathrm{d}\bar{y}_n\right],
\end{aligned} \tag{9.68}$$

where

$$\begin{aligned}
\bar{u}^{(n)}_{n+1}(\bar{\xi}_n,\bar{\tau}_n;\bar{\xi}_{n+1},\tau_{n+1}) = \int_0^1 \int_{-\infty}^{\infty} &\left[\bar{\xi}_n(\bar{\xi}_n;\tau_{n+1}) - \bar{y}_{n-1}\sqrt{\frac{4\tau_{n+1}\prod_{j=n-1}^{n}\bar{\tau}_j}{1-\bar{\tau}_{n-1}}}\right] \\
&\times \bar{b}\Big(\bar{\xi}_{n-1}(\bar{y}_{n-1};\tau_{n+1})\Big) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_{n-1}^2} \bar{u}^{(n-1)}_{n+1}\, \mathrm{d}\bar{y}_{n-1}\, \mathrm{d}\bar{\tau}_{n-1}.
\end{aligned} \tag{9.69}$$

Then by repeating the variable transform (9.66), with the index n raised to $n+1$, we get

$$\begin{aligned}
\bar{u}^{(n+1)}_{n+1}(\bar{\xi}_{n+1},0) = \frac{1}{\bar{\xi}_{n+1}} \int_0^{\bar{\xi}_{n+1}} \bar{b}(\bar{z}_n) \int_0^{\bar{z}_n} \bar{b}(\bar{z}_{n-1}) \int_0^{\bar{z}_{n-1}} \bar{b}(\bar{z}_{n-2}) \times \cdots \times \int_0^{\bar{z}_2} \bar{b}(\bar{z}_1) \int_0^{\bar{z}_1} \bar{b}(\bar{z}_0) \\
\times\, \mathrm{d}\bar{z}_0\, \mathrm{d}\bar{z}_1 \times \cdots \times \mathrm{d}\bar{z}_{n-2}\, \mathrm{d}\bar{z}_{n-1}\, \mathrm{d}\bar{z}_n.
\end{aligned} \tag{9.70}$$

Hence (9.67) holds for $n+1$ if it holds for some positive integer n. It can be verified that (9.67) holds for $n = 1, 2$, so by induction (9.67) holds for all $n = 1, 2, 3, \ldots$. The proof is thus complete. □

It turns out (9.67) can be integrated explicitly. Recall that

$$\bar{b}(\bar{x}) = \frac{1}{\sqrt{2}} \left(\sigma(\theta^{-1}(\bar{x})) + \sigma'(\theta^{-1}(\bar{x}))\right), \tag{9.71}$$

$$\theta(x) = \sqrt{2} \int_0^x \frac{\mathrm{d}z}{\sigma(z)}. \tag{9.72}$$

Let

$$\begin{aligned} I_n^{(1)}(\bar{z}_1) &= \int_0^{\bar{z}_1} \bar{b}(\bar{z}_0)\ \mathrm{d}\bar{z}_0, \\ I_n^{(2)}(\bar{z}_2) &= \int_0^{\bar{z}_2} \bar{b}(\bar{z}_1) I_n^{(1)}(\bar{z}_1)\ \mathrm{d}\bar{z}_1, \\ &\vdots \\ I_n^{(n-1)}(\bar{z}_{n-1}) &= \int_0^{\bar{z}_{n-1}} \bar{b}(\bar{z}_{n-2}) I_n^{(1)}(\bar{z}_{n-2})\ \mathrm{d}\bar{z}_{n-2}. \end{aligned} \tag{9.73}$$

Then

$$\bar{u}_n^{(n)}(\bar{\xi}_n, 0) = \frac{1}{\bar{\xi}_n} \int_0^{\bar{\xi}_n} \bar{b}(\bar{z}_{n-1}) I_n^{(n-1)}(\bar{z}_{n-1})\ \mathrm{d}\bar{z}_{n-1}. \tag{9.74}$$

For each fixed $n \in \mathbb{N}_0$, implicitly define the new variables $z_0, z_1, \ldots, z_{n-1}$, by

$$\begin{aligned} \bar{z}_0 &= \theta(z_0), \\ \bar{z}_1 &= \theta(z_1), \\ &\vdots \\ \bar{z}_{n-1} &= \theta(z_{n-1}). \end{aligned} \tag{9.75}$$

Then in these new variables,

$$\begin{aligned} I_n^{(1)}(\bar{z}_1) &= \int_0^{\theta^{-1}(\bar{z}_1)} \frac{1}{\sqrt{2}} \big(\sigma(z_0) + \sigma'(z_0)\big) \frac{\sqrt{2}}{\sigma(z_0)}\ \mathrm{d}z_0 \\ &= \int_0^{\theta^{-1}(\bar{z}_1)} \left[1 + \frac{\sigma'(z_0)}{\sigma(z_0)}\right]\ \mathrm{d}z_0 \\ &= \int_0^{\theta^{-1}(\bar{z}_1)} \frac{\mathrm{d}}{\mathrm{d}z_0}\Big[z_0 + \ln\sigma(z_0)\Big]\ \mathrm{d}z_0 \\ &= \theta^{-1}(\bar{z}_1) + \ln\frac{\sigma(\theta^{-1}(\bar{z}_1))}{\sigma(0)}. \end{aligned} \tag{9.76}$$

This gives

$$I_n^{(1)}(z_1) = z_1 + \ln\frac{\sigma(z_1)}{\sigma(0)}. \tag{9.77}$$

Similarly,

$$
\begin{aligned}
I_n^{(2)}(\bar{z}_2) &= \int_0^{\theta^{-1}(\bar{z}_2)} \frac{1}{\sqrt{2}} \left(\sigma(z_0) + \sigma'(z_0)\right) \frac{\sqrt{2}}{\sigma(z_1)} \left[z_1 + \ln \frac{\sigma(z_1)}{\sigma(0)} \right] \, \mathrm{d}z_1 \\
&= \int_0^{\theta^{-1}(\bar{z}_2)} \left[1 + \frac{\sigma'(z_0)}{\sigma(z_0)} \right] \left[z_1 + \ln \frac{\sigma(z_1)}{\sigma(0)} \right] \, \mathrm{d}z_1 \\
&= \int_0^{\theta^{-1}(\bar{z}_2)} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}z_0} \left\{ \left[z_1 + \ln \frac{\sigma(z_1)}{\sigma(0)} \right]^2 \right\} \, \mathrm{d}z_1 \\
&= \frac{1}{2!} \left[\theta^{-1}(\bar{z}_2) + \ln \frac{\sigma(\theta^{-1}(\bar{z}_2))}{\sigma(0)} \right]^2 .
\end{aligned}
\tag{9.78}
$$

In other words,

$$
I_n^{(2)}(z_2) = \frac{1}{2!} \left[z_2 + \ln \frac{\sigma(z_2)}{\sigma(0)} \right]^2 . \tag{9.79}
$$

Noting these results, we are now ready to prove the following lemma:

Lemma 9.15. *Let* (A_0)–(A_4) *hold. Then for any* $n = 2, 3, \ldots$,

$$
I_n^{(i)}(\bar{z}_i) = \frac{1}{i!} \left[\theta^{-1}(\bar{z}_i) + \ln \frac{\sigma(\theta^{-1}(\bar{z}_i))}{\sigma(0)} \right]^i , \quad i = 1, 2, \ldots, n-1. \tag{9.80}
$$

Proof. Recall that we have shown in the above that (9.80) is true for $i = 1, 2$ for any fixed integer $n \geq 2$. Now suppose it is true for some i and an arbitrary integer n, we will show that it is also true for $i+1$, provided $i + 1 \leq n - 1$. By the definition (9.73),

$$
I_n^{(i+1)}(\bar{z}_{i+1}) = \int_0^{\bar{z}_{i+1}} \bar{b}(\bar{z}_i) I_n^{(i)}(\bar{z}_i) \, \mathrm{d}\bar{z}_i . \tag{9.81}
$$

By assumption, (9.80) holds for some i. Hence

$$
I_n^{(i+1)}(\bar{z}_{i+1}) = \int_0^{\bar{z}_{i+1}} \bar{b}(\bar{z}_i) \frac{1}{i!} \left[\theta^{-1}(\bar{z}_i) + \ln \frac{\sigma(\theta^{-1}(\bar{z}_i))}{\sigma(0)} \right]^i \, \mathrm{d}\bar{z}_i . \tag{9.82}
$$

Setting $\bar{z}_i = \theta(z_i)$, we have

$$
I_n^{(i+1)}(\bar{z}_{i+1}) = \int_0^{\theta^{-1}(\bar{z}_{i+1})} \frac{1}{\sqrt{2}} \left(\sigma(z_i) + \sigma'(z_i)\right) \frac{\sqrt{2}}{\sigma(z_i)} \frac{1}{i!} \left[z_i + \ln \frac{\sigma(z_i)}{\sigma(0)} \right]^i \, \mathrm{d}z_i . \tag{9.83}
$$

Simplifying the terms gives

$$
\begin{aligned}
I_n^{(i+1)}(\bar{z}_{i+1}) &= \int_0^{\theta^{-1}(\bar{z}_2)} \left[1+\frac{\sigma'(z_0)}{\sigma(z_0)}\right]\frac{1}{i!}\left[z_i+\ln\frac{\sigma(z_i)}{\sigma(0)}\right]^i \,\mathrm{d}z_i \\
&= \int_0^{\theta^{-1}(\bar{z}_{i+1})} \frac{1}{(i+1)!}\frac{\mathrm{d}}{\mathrm{d}z_0}\left\{\left[z_i+\ln\frac{\sigma(z_i)}{\sigma(0)}\right]^{i+1}\right\}\,\mathrm{d}z_i \qquad (9.84)\\
&= \frac{1}{(i+1)!}\left[\theta^{-1}(\bar{z}_{i+1})+\ln\frac{\sigma(\theta^{-1}(\bar{z}_{i+1}))}{\sigma(0)}\right]^{i+1}.
\end{aligned}
$$

Hence (9.80) holds for any arbitrarily fixed $n = 2, 3, \ldots$, and all $i = 1, 2, \ldots, n-1$. It holds for all $n = 2, 3, \ldots$ and all $i = 1, 2, \ldots, n-1$ by induction. The proof is thus complete. □

9.3 Calculation of the 1st and 2nd term of the series

Remember that in this chapter our aim is to prove Theorem 9.1. To this end we have proceeded to calculate term by term the time τ limit of the series $U_1(x,\tau) = \sum_{n=1}^{\infty} u_n(x,\tau)$. By (9.15) and (9.16), this is the same as to calculate term by term the time τ limit of the series $\bar{U}_1(\bar{x},\tau) = \sum_{n=1}^{\infty} \bar{u}_n(\bar{x},\tau)$. While yet to be proved, Proposition 9.4 gives the limits we are to obtain for the $\bar{u}_n$'s. The technical results given in Section 9.2 above have paved the way for the proof of these limits. Before proving these limits, i.e., before proving Proposition 9.4, and the main theorem of this chapter, let us demonstrate how those technical results of Section 9.2 actually work in the calculation of the time τ limits of the $\bar{u}_n$'s. In this demonstration we will calculate the limit as τ tends to 0 of $\bar{u}_1$ and $\bar{u}_2$, the first and second term of the series $\bar{U}_1(\bar{x},\tau)$.

Calculation of the time τ limit of the first term

We now calculate the limit of the first term $\bar{u}_1$ as τ tends to 0. Note that by (9.16) we have $\bar{u}_n = u_n$ for all $n \in \mathbb{N}_0$. By (9.5)–(9.10), (9.13), and with $(\bar{x},\tau)$ replaced by

$(\bar{\xi}_1, \tau_1)$, we have

$$\begin{aligned}
\bar{u}_1(\bar{\xi}_1, \tau_1) &= \int_{-\infty}^{\infty} K_1(\bar{\xi}_1, \tau_1; \bar{y}, 0)\hbar(\bar{y})\ \mathrm{d}\bar{y} \\
&= \int_{0}^{\infty} K_1(\bar{\xi}_1, \tau_1; \bar{y}, 0)\ \mathrm{d}\bar{y} \\
&= \int_{0}^{\infty} -\frac{\partial}{\partial \bar{y}}\left[\int_0^{\tau_1}\int_{-\infty}^{\infty} K_0(\bar{\xi}_1, \tau_1; \bar{\xi}_0, \tau_0)\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0, \tau_0; \bar{y}, 0)\ \mathrm{d}\bar{\xi}_0\, \mathrm{d}\tau_0\right] \mathrm{d}\bar{y} \\
&= \left[\int_0^{\tau_1}\int_{-\infty}^{\infty} K_0(\bar{\xi}_1, \tau_1; \bar{\xi}_0, \tau_0)\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0, \tau_0; \bar{y}, 0)\ \mathrm{d}\bar{\xi}_0\, \mathrm{d}\tau_0\right]_{\bar{y}=0}^{\infty} \\
&= \int_0^{\tau_1}\int_{-\infty}^{\infty} K_0(\bar{\xi}_1, \tau_1; \bar{\xi}_0, \tau_0)\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0, \tau_0; 0, 0)\ \mathrm{d}\bar{\xi}_0\, \mathrm{d}\tau_0,
\end{aligned} \tag{9.85}$$

where the third equality is justified by (9.19), the forth by (9.22), and the fifth by the dominated convergence theorem. This corresponds to (9.22) with $n = 1$. To normalize the time integration from the interval $(0, \tau_1)$ to the interval $(0, 1)$, we put $\bar{\tau}_0 = \tau_0/\tau_1$. This gives $\tau_0 = \tau_1\bar{\tau}_0$. Consequently, recombining the kernels gives

$$\begin{aligned}
&\bar{u}_1(\bar{\xi}_1, \tau_1) \\
&= \tau_1 \int_0^1\int_{-\infty}^{\infty} K_0(\bar{\xi}_1, \tau_1; \bar{\xi}_0, \tau_1\bar{\tau}_0)\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0, \tau_1\bar{\tau}_0; 0, 0)\ \mathrm{d}\bar{\xi}_0\, \mathrm{d}\bar{\tau}_0 \\
&= \tau_1 \int_0^1\int_{-\infty}^{\infty} K_0(\bar{\xi}_1, \tau_1; 0, 0)\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0, \tau_1\bar{\tau}_0; \bar{\xi}_1\bar{\tau}_0, \tau_1\bar{\tau}_0^2)\ \mathrm{d}\bar{\xi}_0\, \mathrm{d}\bar{\tau}_0 \\
&= \frac{\tau_1}{\sqrt{4\pi\tau_1}}\left[\exp\left(-\frac{\bar{\xi}_1^2}{4\tau_1}\right)\right]\int_0^1\int_{-\infty}^{\infty} \bar{b}(\bar{\xi}_0)\frac{1}{\sqrt{4\pi\tau_1\bar{\tau}_0(1-\bar{\tau}_0)}}\exp\left[-\frac{(\bar{\xi}_0 - \bar{\xi}_1\bar{\tau}_0)^2}{4\tau_1\bar{\tau}_0(1-\bar{\tau}_0)}\right]\ \mathrm{d}\bar{\xi}_0\, \mathrm{d}\bar{\tau}_0.
\end{aligned} \tag{9.86}$$

This corresponds to (9.42) with $n = 1$. Now as in (9.47) or (9.52), we let

$$\bar{y}_0 = \frac{\bar{\xi}_0 - \bar{\xi}_1\bar{\tau}_0}{\sqrt{4\tau_1\bar{\tau}_0(1-\bar{\tau}_0)}}. \tag{9.87}$$

Then a rearrangement of $\bar{y}_0$ gives

$$\bar{\xi}_0(\bar{y}_0; \tau_1) = \bar{y}_0\sqrt{4\tau_1\bar{\tau}_0(1-\bar{\tau}_0)} + \bar{\xi}_1\bar{\tau}_0. \tag{9.88}$$

By this change of variables we get

$$\bar{u}_1(\bar{\xi}_1, \tau_1) = \frac{\tau_1}{\sqrt{4\pi\tau_1}}\left[\exp\left(-\frac{\bar{\xi}_1^2}{4\tau_1}\right)\right]\int_0^1\int_{-\infty}^{\infty} \bar{b}\Big(\bar{\xi}_0(\bar{y}_0; \tau_1)\Big)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\ \mathrm{d}\bar{y}_0\, \mathrm{d}\bar{\tau}_0. \tag{9.89}$$

Noting that

$$\bar{\xi}_0(\bar{y}_0;\tau_1) \to \bar{\xi}_1\bar{\tau}_0 \quad \text{as} \quad \tau_1 \to 0, \tag{9.90}$$

we have, as $\tau_1 \to 0$,

$$\begin{aligned}
\bar{u}_1(\bar{\xi}_1,\tau_1) &\sim \frac{\tau_1}{\sqrt{4\pi\tau_1}}\left[\exp\left(-\frac{\bar{\xi}_1^2}{4\tau_1}\right)\right]\int_0^1\int_{-\infty}^{\infty}\bar{b}(\bar{\xi}_1\bar{\tau}_0)\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\,\mathrm{d}\bar{y}_0\,\mathrm{d}\bar{\tau}_0 \\
&= \frac{\sqrt{\tau_1}}{\sqrt{4\pi}}\left[\exp\left(-\frac{\bar{\xi}_1^2}{4\tau_1}\right)\right]\int_0^1\bar{b}(\bar{\xi}_1\bar{\tau}_0)\,\mathrm{d}\bar{\tau}_0\int_{-\infty}^{\infty}\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\bar{y}_0^2}\,\mathrm{d}\bar{y}_0 \\
&= \left\{\frac{\sqrt{\tau_1}}{\sqrt{4\pi}}\left[\exp\left(-\frac{\bar{\xi}_1^2}{4\tau_1}\right)\right]\right\}\int_0^1\bar{b}(\bar{\xi}_1\bar{\tau}_0)\,\mathrm{d}\bar{\tau}_0,
\end{aligned} \tag{9.91}$$

where in the last equality the integral corresponds to $\bar{u}_n^{(n)}$ in (9.65) with $n=1$, and the curly bracket term corresponds to the curly bracket term in (9.42), again with $n=1$. Following (9.66), we then set $\bar{z}_0 = \bar{\xi}_1\bar{\tau}_0$. By (9.91), as $\tau_1 \to 0$,

$$\bar{u}_1(\bar{\xi}_1,\tau_1) \sim \frac{\sqrt{\tau_1}}{\sqrt{4\pi}}\left[\exp\left(-\frac{\bar{\xi}_1^2}{4\tau_1}\right)\right]\int_0^{\bar{\xi}_1}\bar{b}(\bar{z}_0)\frac{1}{\bar{\xi}_1}\,\mathrm{d}\bar{z}_0, \tag{9.92}$$

where the integral corresponds to (9.67) with $n=1$. As in (9.75), we put $z_0 = \theta(\bar{z}_0)$, i.e., $\bar{z}_0 = \theta(z_0)$. Then

$$\begin{aligned}
\int_0^{\bar{\xi}_1}\bar{b}(\bar{z}_0)\frac{1}{\bar{\xi}_1}\,\mathrm{d}\bar{z}_0 &= \frac{1}{\bar{\xi}_1}\int_0^{\theta^{-1}(\bar{\xi}_1)}\bar{b}\Big(\theta(z_0)\Big)\frac{\sqrt{2}}{\sigma(z_0)}\,\mathrm{d}z_0 \\
&= \frac{1}{\bar{\xi}_1}\int_0^{\theta^{-1}(\bar{\xi}_1)}\frac{1}{\sqrt{2}}\left(\sigma(z_0)+\sigma'(z_0)\right)\frac{\sqrt{2}}{\sigma(z_0)}\,\mathrm{d}z_0 \\
&= \frac{1}{\bar{\xi}_1}\left[\theta^{-1}(\bar{\xi}_1) + \ln\frac{\sigma(\theta^{-1}(\bar{\xi}_1))}{\sigma(0)}\right].
\end{aligned} \tag{9.93}$$

This is nothing more than replacing $\bar{z}_1$ in (9.76) with $\bar{\xi}_1$. Together with (9.92), this shows that as $\tau_1 \to 0$,

$$\bar{u}_1(\bar{\xi}_1,\tau_1) \sim \frac{\sqrt{\tau_1}}{\sqrt{4\pi}}\left[\exp\left(-\frac{\bar{\xi}_1^2}{4\tau_1}\right)\right]\frac{1}{\bar{\xi}_1}\left[\theta^{-1}(\bar{\xi}_1) + \ln\frac{\sigma(\theta^{-1}(\bar{\xi}_1))}{\sigma(0)}\right]. \tag{9.94}$$

Replacing $(\bar{\xi}_1,\tau_1)$ with $(\bar{x},\tau)$, and recalling

$$\bar{x} \equiv \sqrt{2}\int_0^x\frac{\mathrm{d}z}{\sigma(z)} \equiv \sqrt{2}J(x), \tag{9.95}$$

we have, as $\tau \to 0$,

$$\begin{aligned}\bar{u}_1(\bar{x},\tau) &\sim \frac{\sqrt{\tau}}{\sqrt{4\pi}}\left[\exp\left(-\frac{\bar{x}^2}{4\tau}\right)\right]\frac{1}{\bar{x}}\left[\theta^{-1}(\bar{x})+\ln\frac{\sigma(\theta^{-1}(\bar{x}))}{\sigma(0)}\right]\\ &= \left\{\frac{\sqrt{\tau}}{\sqrt{\pi}}\left[\exp\left(-\frac{\bar{x}^2}{4\tau}\right)\right]\frac{1}{\bar{x}}\cdot\frac{(-1)^{n-1}}{n!}\left(\frac{1}{2}\left[\theta^{-1}(\bar{x})+\ln\frac{\sigma(\theta^{-1}(\bar{x}))}{\sigma(0)}\right]\right)^n\right\}_{n=1}.\end{aligned} \tag{9.96}$$

This completes the calculation of the time τ limit of $\bar{u}_1(\bar{x},\tau)$, the first term of the series $\bar{U}_1(\bar{x},\tau)=\sum_{n=1}^{\infty}\bar{u}_n(\bar{x},\tau)$.

Calculation of the time τ limit of the second term

We now calculate $\bar{u}_2$, the second term of the series $\bar{U}_1(\bar{x},\tau)=\sum_{n=1}^{\infty}\bar{u}_n(\bar{x},\tau)$. By (9.5)–(9.10), (9.13), and with $(\bar{x},\tau)$ replaced by $(\bar{\xi}_2,\tau_2)$, we have

$$\begin{aligned}\bar{u}_2(\bar{\xi}_2,\tau_2) &= \int_{-\infty}^{\infty} K_2(\bar{\xi}_2,\tau_2;\bar{y},0)\hbar(\bar{y})\,\mathrm{d}\bar{y}\\ &= \int_0^{\infty} K_2(\bar{\xi}_2,\tau_2;\bar{y},0)\,\mathrm{d}\bar{y}\\ &= \int_0^{\infty} -\frac{\partial}{\partial\bar{y}}\left\{\int_0^{\tau_2}\int_{-\infty}^{\infty} K_0(\bar{\xi}_2,\tau_2;\bar{\xi}_1,\tau_1)\bar{b}(\bar{\xi}_1)\right.\\ &\qquad\times\int_0^{\tau_1}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\bar{\xi}_1}K_0(\bar{\xi}_1,\tau_1;\bar{\xi}_0,\tau_0)\right]\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0,\tau_0;\bar{y},0)\,\mathrm{d}\bar{\xi}_0\,\mathrm{d}\tau_0\\ &\qquad\left.\times\,\mathrm{d}\bar{\xi}_1\,\mathrm{d}\tau_1\right\}\mathrm{d}\bar{y}\\ &= \int_0^{\tau_2}\int_{-\infty}^{\infty} K_0(\bar{\xi}_2,\tau_2;\bar{\xi}_1,\tau_1)\bar{b}(\bar{\xi}_1)\\ &\quad\times\int_0^{\tau_1}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\bar{\xi}_1}K_0(\bar{\xi}_1,\tau_1;\bar{\xi}_0,\tau_0)\right]\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0,\tau_0;0,0)\,\mathrm{d}\bar{\xi}_0\,\mathrm{d}\tau_0\,\mathrm{d}\bar{\xi}_1\,\mathrm{d}\tau_1.\end{aligned} \tag{9.97}$$

This corresponds to (9.22) with $n=2$. To normalize the time integration from the interval $(0,\tau_2)$ to the interval $(0,1)$, we put $\bar{\tau}_1=\tau_1/\tau_2$. This gives $\tau_1=\tau_2\bar{\tau}_1$ and

$$\begin{aligned}\bar{u}_2(\bar{\xi}_2,\tau_2) &= \tau_2\int_0^{1}\int_{-\infty}^{\infty} K_0(\bar{\xi}_2,\tau_2;\bar{\xi}_1,\tau_2\bar{\tau}_1)\bar{b}(\bar{\xi}_1)\\ &\quad\times\int_0^{\tau_2\bar{\tau}_1}\int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\bar{\xi}_1}K_0(\bar{\xi}_1,\tau_2\bar{\tau}_1;\bar{\xi}_0,\tau_0)\right]\bar{b}(\bar{\xi}_0)K_0(\bar{\xi}_0,\tau_0;0,0)\,\mathrm{d}\bar{\xi}_0\,\mathrm{d}\tau_0\,\mathrm{d}\bar{\xi}_1\,\mathrm{d}\bar{\tau}_1.\end{aligned} \tag{9.98}$$

Setting $\bar{\tau}_0 = \tau_0/(\tau_2\bar{\tau}_1)$ gives $\tau_0 = \tau_2\bar{\tau}_1\bar{\tau}_2$. This change of the variables then gives

$$\begin{aligned}\bar{u}_2(\bar{\xi}_2, \tau_2) = \tau_2^2 \int_0^1 \int_{-\infty}^{\infty} & K_0(\bar{\xi}_2, \tau_2; \bar{\xi}_1, \tau_2\bar{\tau}_1)\bar{b}(\bar{\xi}_1) \\ & \times \bar{\tau}_1 \int_0^1 \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial \bar{\xi}_1} K_0(\bar{\xi}_1, \tau_2\bar{\tau}_1; \bar{\xi}_0, \tau_2\bar{\tau}_1\bar{\tau}_0)\right] \bar{b}(\bar{\xi}_0) K_0(\bar{\xi}_0, \tau_2\bar{\tau}_1\bar{\tau}_0; 0, 0) \\ & \times \ \mathrm{d}\bar{\xi}_0 \, \mathrm{d}\bar{\tau}_0 \, \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\tau}_1.\end{aligned} \tag{9.99}$$

This equation corresponds to (9.27) with $n = 2$. Note that in this expression we have the $\bar{\tau}_0$ and $\bar{\tau}_1$, instead of τ_0 and τ_1, time integrals. Let f be the inner most integrand of $\bar{u}_2(\bar{\xi}_2, \tau_2)$ in (9.99); that is,

$$f = \left[\frac{\partial}{\partial \bar{\xi}_1} K_0(\bar{\xi}_1, \tau_2\bar{\tau}_1; \bar{\xi}_0, \tau_2\bar{\tau}_1\bar{\tau}_0)\right] \bar{b}(\bar{\xi}_0) K_0(\bar{\xi}_0, \tau_2\bar{\tau}_1\bar{\tau}_0; 0, 0). \tag{9.100}$$

By differentiating explicitly and applying the K_0 recombination formula (9.28), we get

$$f = -\frac{\bar{\xi}_1 - \bar{\xi}_0}{2\tau_2\bar{\tau}_1(1 - \bar{\tau}_0)} \bar{b}(\bar{\xi}_0) K_0(\bar{\xi}_1, \tau_2\bar{\tau}_1; 0, 0) K_0(\bar{\xi}_0, \tau_2\bar{\tau}_1\bar{\tau}_0; \bar{\xi}_1\bar{\tau}_0, \tau_2\bar{\tau}_1\bar{\tau}_0^2). \tag{9.101}$$

Note that in the application of the K_0 recombination formula (9.28), we could treat $\tau_2\bar{\tau}_1$ as a time point $\tilde{\tau}$. Moreover, (9.101) can be directly verified by using the definition of K_0, see (9.9). Now by (9.99) and (9.101) we can rewrite $\bar{u}_2(\bar{\xi}_2, \tau_2)$ as

$$\begin{aligned}\bar{u}_2(\bar{\xi}_2, \tau_2) = \tau_2^2 \int_0^1 \int_{-\infty}^{\infty} & K_0(\bar{\xi}_2, \tau_2; \bar{\xi}_1, \tau_2\bar{\tau}_1)\bar{b}(\bar{\xi}_1) \\ & \times \bar{\tau}_1 K_0(\bar{\xi}_1, \tau_2\bar{\tau}_1; 0, 0) \left[\int_0^1 \int_{-\infty}^{\infty} g \ \mathrm{d}\bar{\xi}_0 \, \mathrm{d}\bar{\tau}_0\right] \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\tau}_1,\end{aligned} \tag{9.102}$$

where

$$g = -\frac{\bar{\xi}_1 - \bar{\xi}_0}{2\tau_2\bar{\tau}_1(1 - \bar{\tau}_0)} \bar{b}(\bar{\xi}_0) K_0(\bar{\xi}_0, \tau_2\bar{\tau}_1\bar{\tau}_0; \bar{\xi}_1\bar{\tau}_0, \tau_2\bar{\tau}_1\bar{\tau}_0^2). \tag{9.103}$$

Applying the K_0 recombination formula (9.28) to $K_0(\bar{\xi}_2, \tau_2; \bar{\xi}_1, \tau_2\bar{\tau}_1)K_0(\bar{\xi}_1, \tau_2\bar{\tau}_1; 0, 0)$ then gives

$$\begin{aligned}\bar{u}_2(\bar{\xi}_2, \tau_2) = \tau_2^2 K_0(\bar{\xi}_2, \tau_2; 0, 0) & \int_0^1 \int_{-\infty}^{\infty} K_0(\bar{\xi}_1, \tau_2\bar{\tau}_1; \bar{\xi}_2\bar{\tau}_1, \tau_2\bar{\tau}_1^2)\bar{b}(\bar{\xi}_1) \\ & \times \bar{\tau}_1 \int_0^1 \int_{-\infty}^{\infty} g \ \mathrm{d}\bar{\xi}_0 \, \mathrm{d}\bar{\tau}_0 \, \mathrm{d}\bar{\xi}_1 \, \mathrm{d}\bar{\tau}_1.\end{aligned} \tag{9.104}$$

Starting with the inner most integral that now has g as its integrand, we will further simplify $\bar{u}_2(\bar{\xi}_2, \tau_2)$. As in (9.52), to simplify g we set

$$\bar{y}_0 = \frac{\bar{\xi}_0 - \bar{\xi}_1 \bar{\tau}_0}{\sqrt{4\tau_2 \bar{\tau}_1 \bar{\tau}_0 (1 - \bar{\tau}_0)}}. \tag{9.105}$$

This gives

$$\bar{\xi}_0(\bar{y}_0; \tau_2) = \bar{y}_0 \sqrt{4\tau_2 \bar{\tau}_1 \bar{\tau}_0 (1 - \bar{\tau}_0)} + \bar{\xi}_1 \bar{\tau}_0. \tag{9.106}$$

And we have

$$K_0(\bar{\xi}_0, \tau_2 \bar{\tau}_1 \bar{\tau}_0; \bar{\xi}_1 \bar{\tau}_0, \tau_2 \bar{\tau}_1 \bar{\tau}_0^2)\, \mathrm{d}\bar{\xi}_0 = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_0^2}\, \mathrm{d}\bar{y}_0. \tag{9.107}$$

Moreover, we get

$$\begin{aligned} \bar{\xi}_1 - \bar{\xi}_0 &= \bar{\xi}_1 - \bar{y}_0 \sqrt{4\tau_2 \bar{\tau}_1 \bar{\tau}_0 (1 - \bar{\tau}_0)} - \bar{\xi}_1 \bar{\tau}_0 \\ &= \bar{\xi}_1 (1 - \bar{\tau}_0) - \bar{y}_0 \sqrt{4\tau_2 \bar{\tau}_1 \bar{\tau}_0 (1 - \bar{\tau}_0)}, \end{aligned} \tag{9.108}$$

and

$$-\frac{\bar{\xi}_1 - \bar{\xi}_0}{2\tau_2 \bar{\tau}_1 (1 - \bar{\tau}_0)} = -\frac{1}{2\tau_2 \bar{\tau}_1} \left[\bar{\xi}_1 - \bar{y}_0 \sqrt{\frac{4\tau_2 \bar{\tau}_1 \bar{\tau}_0}{1 - \bar{\tau}_0}} \right]. \tag{9.109}$$

By (9.105)–(9.109) , $\bar{u}_2(\bar{\xi}_2, \tau_2)$, in the form expressed by (9.104), can be rewritten as

$$\begin{aligned} \bar{u}_2(\bar{\xi}_2, \tau_2) &= \tau_2^2 K_0(\bar{\xi}_2, \tau_2; 0, 0) \int_0^1 \int_{-\infty}^{\infty} K_0(\bar{\xi}_1, \tau_2 \bar{\tau}_1; \bar{\xi}_2 \bar{\tau}_1, \tau_2 \bar{\tau}_1^2) \bar{b}(\bar{\xi}_1) \\ &\quad \times \frac{-\bar{\tau}_1}{2\tau_2 \bar{\tau}_1} \int_0^1 \int_{-\infty}^{\infty} \bar{b}\Big(\bar{\xi}_0(\bar{y}_0; \tau_2)\Big) \left[\bar{\xi}_1 - \bar{y}_0 \sqrt{\frac{4\tau_2 \bar{\tau}_1 \bar{\tau}_0}{1 - \bar{\tau}_0}} \right] \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_0^2}\, \mathrm{d}\bar{y}_0\, \mathrm{d}\bar{\tau}_0\, \mathrm{d}\bar{\xi}_1\, \mathrm{d}\bar{\tau}_1 \\ &= -\frac{\tau_2}{2} K_0(\bar{\xi}_2, \tau_2; 0, 0) \int_0^1 \int_{-\infty}^{\infty} K_0(\bar{\xi}_1, \tau_2 \bar{\tau}_1; \bar{\xi}_2 \bar{\tau}_1, \tau_2 \bar{\tau}_1^2) \bar{b}(\bar{\xi}_1) \\ &\quad \times \int_0^1 \int_{-\infty}^{\infty} \bar{b}\Big(\bar{\xi}_0(\bar{y}_0; \tau_2)\Big) \left[\bar{\xi}_1 - \bar{y}_0 \sqrt{\frac{4\tau_2 \bar{\tau}_1 \bar{\tau}_0}{1 - \bar{\tau}_0}} \right] \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_0^2}\, \mathrm{d}\bar{y}_0\, \mathrm{d}\bar{\tau}_0\, \mathrm{d}\bar{\xi}_1\, \mathrm{d}\bar{\tau}_1, \end{aligned} \tag{9.110}$$

where $\bar{\xi}_0(\bar{y}_0; \tau_2)$ is given by (9.106). To simplify the remaining K_0 term in (9.110), we put

$$\bar{y}_1 = \frac{\bar{\xi}_1 - \bar{\xi}_2 \bar{\tau}_1}{\sqrt{4\tau_2 \bar{\tau}_1 (1 - \bar{\tau}_1)}}. \tag{9.111}$$

This gives

$$\bar{\xi}_1(\bar{y}_1; \tau_2) = \bar{y}_1 \sqrt{4\tau_2 \bar{\tau}_1 (1 - \bar{\tau}_1)} + \bar{\xi}_2 \bar{\tau}_1. \tag{9.112}$$

Further, we have

$$K_0(\bar{\xi}_1, \tau_2 \bar{\tau}_1; \bar{\xi}_2 \bar{\tau}_1, \tau_2 \bar{\tau}_1^2)\, \mathrm{d}\bar{\xi}_1 = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_1^2}\, \mathrm{d}\bar{y}_1. \tag{9.113}$$

By (9.111)–(9.113), $\bar{u}_2(\bar{\xi}_2, \tau_2)$, as expressed by the last equality of (9.110), can be rewritten as

$$\begin{aligned}\bar{u}_2(\bar{\xi}_2, \tau_2) = -\frac{\tau_2}{2} K_0(\bar{\xi}_2, \tau_2; 0, 0) &\int_0^1 \int_{-\infty}^{\infty} \bar{b}\Big(\bar{\xi}_1(\bar{y}_1; \tau_2)\Big) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_1^2} \\ &\times \int_0^1 \int_{-\infty}^{\infty} \bar{b}\Big(\bar{\xi}_0(\bar{y}_0; \tau_2)\Big) \left[\bar{y}_1 \sqrt{4\tau_2 \bar{\tau}_1 (1-\bar{\tau}_1)} + \bar{\xi}_2 \bar{\tau}_1 - \bar{y}_0 \sqrt{\frac{4\tau_2 \bar{\tau}_1 \bar{\tau}_0}{1-\bar{\tau}_0}} \right] \\ &\quad \times \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_0^2} \,\mathrm{d}\bar{y}_0 \,\mathrm{d}\bar{\tau}_0 \,\mathrm{d}\bar{y}_1 \,\mathrm{d}\bar{\tau}_1.\end{aligned} \tag{9.114}$$

By (9.106), we have $\bar{\xi}_1(\bar{y}_1; \tau_2) \xrightarrow{\tau_2 \to 0} \bar{\xi}_2 \bar{\tau}_1$. This, together with (9.112), then gives

$$\bar{\xi}_0(\bar{y}_0; \tau_2) \xrightarrow{\tau_2 \to 0} \lim_{\tau_2 \to 0} \bar{\xi}_1(\bar{y}_1; \tau_2) \bar{\tau}_0 = \bar{\xi}_2 \bar{\tau}_1 \bar{\tau}_0.$$

Hence, by taking into account (9.114) and the definition of K_0, and using the dominated convergence theorem, we get, as $\tau_2 \to 0$,

$$\begin{aligned}\bar{u}_2(\bar{\xi}_2, \tau_2) &\sim -\frac{\tau_2}{2} \frac{1}{\sqrt{4\pi\tau_2}} \exp\left(-\frac{\bar{\xi}_2^2}{4\tau_2}\right) \\ &\qquad \times \int_0^1 \int_{-\infty}^{\infty} \bar{b}(\bar{\xi}_2 \bar{\tau}_1) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_1^2} \left[\int_0^1 \int_{-\infty}^{\infty} \bar{\xi}_2 \bar{\tau}_1 \bar{b}(\bar{\xi}_2 \bar{\tau}_1 \bar{\tau}_0) \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_0^2} \,\mathrm{d}\bar{y}_0 \,\mathrm{d}\bar{\tau}_0 \right] \\ &\qquad\quad \times \mathrm{d}\bar{y}_1 \,\mathrm{d}\bar{\tau}_1 \\ &= -\frac{\tau_2}{2} \frac{1}{\sqrt{4\pi\tau_2}} \exp\left(-\frac{\bar{\xi}_2^2}{4\tau_2}\right) \\ &\qquad \times \int_0^1 \bar{b}(\bar{\xi}_2 \bar{\tau}_1) \left[\int_0^1 \bar{\xi}_2 \bar{\tau}_1 \bar{b}(\bar{\xi}_2 \bar{\tau}_1 \bar{\tau}_0) \,\mathrm{d}\bar{\tau}_0 \right] \mathrm{d}\bar{\tau}_1 \\ &\qquad \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_1^2} \,\mathrm{d}\bar{y}_1 \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\bar{y}_0^2} \,\mathrm{d}\bar{y}_0 \\ &= \left\{ -\frac{\tau_2}{2} \frac{1}{\sqrt{4\pi\tau_2}} \exp\left(-\frac{\bar{\xi}_2^2}{4\tau_2}\right) \right\} \times \int_0^1 \bar{b}(\bar{\xi}_2 \bar{\tau}_1) \left[\int_0^1 \bar{\xi}_2 \bar{\tau}_1 \bar{b}(\bar{\xi}_2 \bar{\tau}_1 \bar{\tau}_0) \,\mathrm{d}\bar{\tau}_0 \right] \mathrm{d}\bar{\tau}_1.\end{aligned} \tag{9.115}$$

Here the (double) integral term corresponds to the right hand side of (9.65) with $n = 2$, and the curly bracket term corresponds to the curly bracket term in (9.42), also with $n = 2$.

Following (9.66), we set $\bar{z}_0 = \bar{\xi}_2\bar{\tau}_1\bar{\tau}_0$ and $\bar{z}_1 = \bar{\xi}_2\bar{\tau}_1$. Then (9.115) becomes

$$\begin{aligned}\bar{u}_2(\bar{\xi}_2, \tau_2) &\sim \left\{-\frac{\tau_2}{2}\frac{1}{\sqrt{4\pi\tau_2}}\exp\left(-\frac{\bar{\xi}_2^2}{4\tau_2}\right)\right\} \times \int_0^1 \bar{b}(\bar{\xi}_2\bar{\tau}_1)\left[\int_0^{\bar{\xi}_2\bar{\tau}_1} \bar{b}(\bar{z}_0)\ \mathrm{d}\bar{z}_0\right]\mathrm{d}\bar{\tau}_1 \\ &= \left\{-\frac{\tau_2}{2}\frac{1}{\sqrt{4\pi\tau_2}}\exp\left(-\frac{\bar{\xi}_2^2}{4\tau_2}\right)\right\} \times \frac{1}{\bar{\xi}_2}\int_0^{\bar{\xi}_2} \bar{b}(\bar{z}_1)\left[\int_0^{\bar{z}_1} \bar{b}(\bar{z}_0)\ \mathrm{d}\bar{z}_0\right]\mathrm{d}\bar{z}_1.\end{aligned} \tag{9.116}$$

We will calculate first the integral inside the square brackets. Recall from (9.6) that

$$\theta(x) = \sqrt{2}\int_0^x \frac{\mathrm{d}z}{\sigma(z)}.$$

As in (9.75), we can implicitly (and uniquely) define z_0 by

$$\bar{z}_0 = \theta(z_0). \tag{9.117}$$

Then

$$\begin{aligned}\int_0^{\bar{z}_1} \bar{b}(\bar{z}_0)\ \mathrm{d}\bar{z}_0 &= \int_0^{\theta^{-1}(\bar{z}_1)} \bar{b}(\theta(z_0))\frac{\sqrt{2}}{\sigma(z_0)}\ \mathrm{d}z_0 \\ &= \int_0^{\theta^{-1}(\bar{z}_1)} \frac{1}{\sqrt{2}}\left[\sigma'\Big(\theta^{-1}(\theta(z_0))\Big) + \sigma\Big(\theta^{-1}(\theta(z_0))\Big)\right]\frac{\sqrt{2}}{\sigma(z_0)}\ \mathrm{d}z_0 \\ &= \int_0^{\theta^{-1}(\bar{z}_1)} \left[\sigma'(z_0) + \sigma(z_0)\right]\frac{1}{\sigma(z_0)}\ \mathrm{d}z_0 \\ &= \int_0^{\theta^{-1}(\bar{z}_1)} \left[1 + \frac{\sigma'(z_0)}{\sigma(z_0)}\right]\ \mathrm{d}z_0 \\ &= \left[z_0 + \ln\sigma(z_0)\right]_{z_0=0}^{\theta^{-1}(\bar{z}_1)} \\ &= \theta^{-1}(\bar{z}_1) + \ln\frac{\sigma(\theta^{-1}(\bar{z}_1))}{\sigma(0)},\end{aligned} \tag{9.118}$$

where in the second equality we have used (9.8), the definition of $\bar{b}(\cdot)$. Now we implicitly define z_1 by

$$\bar{z}_1 = \theta(z_1). \tag{9.119}$$

Then by (9.118), the (double) integral in (9.116) becomes

$$\begin{aligned}\int_0^{\bar{\xi}_2} \bar{b}(\bar{z}_1)\left[\int_0^{\bar{z}_1} \bar{b}(\bar{z}_0)\ \mathrm{d}\bar{z}_0\right]\mathrm{d}\bar{z}_1 &= \int_0^{\bar{\xi}_2} \bar{b}(\bar{z}_1)\left[\theta^{-1}(\bar{z}_1) + \ln\frac{\sigma(\theta^{-1}(\bar{z}_1))}{\sigma(0)}\right]\ \mathrm{d}\bar{z}_1 \\ &= \int_0^{\theta^{-1}(\bar{\xi}_2)} \bar{b}(\theta(z_1))\left[\theta^{-1}(\theta(z_1)) + \ln\frac{\sigma(\theta^{-1}(\theta(z_1)))}{\sigma(0)}\right]\frac{\sqrt{2}}{\sigma(z_1)}\ \mathrm{d}z_1.\end{aligned} \tag{9.120}$$

Applying (9.8), the definition of $\bar{b}(\cdot)$, we then get

$$\begin{aligned}
&\int_0^{\bar{\xi}_2} \bar{b}(\bar{z}_1)\left[\int_0^{\bar{z}_1} \bar{b}(\bar{z}_0)\,\mathrm{d}\bar{z}_0\right]\mathrm{d}\bar{z}_1 \\
&= \int_0^{\theta^{-1}(\bar{\xi}_2)} \frac{1}{\sqrt{2}}\left[\sigma'(z_1)+\sigma(z_1)\right]\times\left[z_1+\ln\frac{\sigma(z_1)}{\sigma(0)}\right]\frac{\sqrt{2}}{\sigma(z_1)}\,\mathrm{d}z_1 \\
&= \int_0^{\theta^{-1}(\bar{\xi}_2)} \left[1+\frac{\sigma'(z_1)}{\sigma(0)}\right]\times\left[z_1+\ln\frac{\sigma(z_1)}{\sigma(0)}\right]\,\mathrm{d}z_1 \qquad (9.121)\\
&= \frac{1}{2}\int_0^{\theta^{-1}(\bar{\xi}_2)} \frac{\mathrm{d}}{\mathrm{d}z_1}\left\{\left[z_1+\ln\frac{\sigma(z_1)}{\sigma(0)}\right]^2\right\}\,\mathrm{d}z_1 \\
&= \frac{1}{2}\left[\theta^{-1}(\bar{\xi}_2)+\ln\frac{\sigma(\theta^{-1}(\bar{\xi}_2))}{\sigma(0)}\right]^2.
\end{aligned}$$

This, coupled with (9.116), then shows that as $\tau_2 \to 0$,

$$\begin{aligned}
\bar{u}_2(\bar{\xi}_2,\tau_2) &\sim \left\{-\frac{\tau_2}{2}\frac{1}{\sqrt{4\pi\tau_2}}\exp\left(-\frac{\bar{\xi}_2^2}{4\tau_2}\right)\right\}\times\frac{1}{\bar{\xi}_2}\times\frac{1}{2}\left[\theta^{-1}(\bar{\xi}_2)+\ln\frac{\sigma(\theta^{-1}(\bar{\xi}_2))}{\sigma(0)}\right]^2 \\
&= \frac{\sqrt{\tau_2}}{\sqrt{\pi}}\left[\exp\left(-\frac{\bar{\xi}_2^2}{4\tau_2}\right)\right]\frac{1}{\bar{\xi}_2}\cdot\frac{(-1)^{2-1}}{2!}\left(\frac{1}{2}\left[\theta^{-1}(\bar{\xi}_2)+\ln\frac{\sigma(\theta^{-1}(\bar{\xi}_2))}{\sigma(0)}\right]\right)^2.
\end{aligned} \qquad (9.122)$$

Relabelling $(\bar{\xi}_2,\tau_2)$ with $(\bar{x},\tau)$, we have, as $\tau \to 0$,

$$\bar{u}_2(\bar{x},\tau) \sim \left\{\frac{\sqrt{\tau}}{\sqrt{\pi}}\left[\exp\left(-\frac{\bar{x}^2}{4\tau}\right)\right]\frac{1}{\bar{x}}\cdot\frac{(-1)^{n-1}}{n!}\left(\frac{1}{2}\left[\theta^{-1}(\bar{x})+\ln\frac{\sigma(\theta^{-1}(\bar{x}))}{\sigma(0)}\right]\right)^n\right\}_{n=2}, \qquad (9.123)$$

which what is given by (9.17) in Proposition 9.4. This completes the calculation of the small time asymptotic limit of $\bar{u}_2$, the second term of the series $\bar{U}_1(\bar{x},\tau) = \sum_{n=1}^{\infty}\bar{u}_n(\bar{x},\tau)$. The other $\bar{u}_n$ terms in the series can be calculated accordingly, as we have already proved in Section 9.2. We are now ready to present the main proofs of this chapter.

9.4 The main proofs of the chapter

In this section we will prove firstly Proposition 9.4 and secondly Theorem 9.1.

Proof of Proposition 9.4. Lemma 9.15 shows that for any $n = 2, 3, \ldots$,

$$I_n^{(n-1)}(\bar{z}_{n-1}) = \frac{1}{(n-1)!}\left[\theta^{-1}(\bar{z}_{n-1})+\ln\frac{\sigma(\theta^{-1}(\bar{z}_{n-1}))}{\sigma(0)}\right]^{n-1}. \qquad (9.124)$$

Then by (9.74), and by setting $\bar{z}_{n-1} = \theta(z_{n-1})$,

$$\begin{aligned} \bar{u}_n^{(n)}(\bar{\xi}_n, 0) &= \frac{1}{\bar{\xi}_n} \int_0^{\bar{\xi}_n} \bar{b}(\bar{z}_{n-1}) I_n^{(n-1)}(\bar{z}_{n-1}) \, \mathrm{d}\bar{z}_{n-1} \\ &= \frac{1}{\bar{\xi}_n} \cdot \frac{1}{n!} \left[\theta^{-1}(\bar{\xi}_n) + \ln \frac{\sigma(\theta^{-1}(\bar{\xi}_n))}{\sigma(0)} \right]^n . \end{aligned} \tag{9.125}$$

By (9.42), (9.55) and (9.125), as $\tau_n \to 0$,

$$\begin{aligned} \bar{u}_n(\bar{\xi}_n, \tau_n) &\sim \frac{(-1)^{n-1}}{2^{n-1}} \tau_n K_0(\bar{\xi}_n, \tau_n; 0, 0) \bar{u}_n^{(n)}(\bar{\xi}_n, 0) \\ &= \frac{(-1)^{n-1}}{2^{n-1}} \tau_n \frac{1}{\sqrt{4\pi\tau_n}} \left[\exp\left(-\frac{\bar{\xi}_n^2}{4\tau_n} \right) \right] \frac{1}{\bar{\xi}_n} \cdot \frac{1}{n!} \left[\theta^{-1}(\bar{\xi}_n) + \ln \frac{\sigma(\theta^{-1}(\bar{\xi}_n))}{\sigma(0)} \right]^n . \end{aligned} \tag{9.126}$$

Replacing $(\bar{\xi}_n, \tau_n)$ with $(\bar{x}, \tau)$, we have, as $\tau \to 0$,

$$\begin{aligned} \bar{u}_n(\bar{x}, \tau) &\sim \frac{(-1)^{n-1}}{2^{n-1}} \tau \frac{1}{\sqrt{4\pi\tau}} \left[\exp\left(-\frac{\bar{x}^2}{4\tau} \right) \right] \frac{1}{\bar{x}} \cdot \frac{1}{n!} \left[\theta^{-1}(\bar{x}) + \ln \frac{\sigma(\theta^{-1}(\bar{x}))}{\sigma(0)} \right]^n \\ &= \frac{(-1)^{n-1}\sqrt{\tau}}{2^n\sqrt{\pi}} \left[\exp\left(-\frac{\bar{x}^2}{4\tau} \right) \right] \frac{1}{\bar{x}} \cdot \frac{1}{n!} \left[\theta^{-1}(\bar{x}) + \ln \frac{\sigma(\theta^{-1}(\bar{x}))}{\sigma(0)} \right]^n , \\ &= \frac{\sqrt{\tau}}{\sqrt{\pi}} \left[\exp\left(-\frac{\bar{x}^2}{4\tau} \right) \right] \frac{1}{\bar{x}} \cdot \frac{(-1)^{n-1}}{n!} \left(\frac{1}{2} \left[\theta^{-1}(\bar{x}) + \ln \frac{\sigma(\theta^{-1}(\bar{x}))}{\sigma(0)} \right] \right)^n , \end{aligned} \tag{9.127}$$

for $n = 2, 3, \ldots$. The proof is completed by invoking (9.96), which shows that the formula also works for $n = 1$. □

We now prove Theorem 9.1.

Proof of Theorem 9.1. Applying the change of variable formula $\bar{x} = \theta(x) = \sqrt{2}J(x)$ to (9.17) gives

$$u_n(x, \tau) \sim \frac{\sqrt{\tau}}{\sqrt{2\pi}} \left[\exp\left(-\frac{J^2(x)}{2\tau} \right) \right] \frac{1}{J(x)} \cdot \frac{-1}{n!} \left(-\frac{1}{2} \left[x + \ln \frac{\sigma(x)}{\sigma(0)} \right] \right)^n \quad n = 1, 2, \ldots . \tag{9.128}$$

The desired result then follows from this and the fact that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{-1}{n!} \left\{ -\frac{1}{2} \left[x + \ln \frac{\sigma(x)}{\sigma(0)} \right] \right\}^n &= 1 - \exp\left(-\frac{1}{2} \left[x + \ln \frac{\sigma(x)}{\sigma(0)} \right] \right) \\ &= 1 - \frac{\sqrt{\sigma(0)}}{\sqrt{\sigma(x)}} \mathrm{e}^{-x/2} . \end{aligned} \tag{9.129}$$

□

Chapter 10

Future research

The field of implied volatilities in option pricing is rapidly expanding, involving more branches of mathematics and more advanced techniques. Already, stochastic models are going infinite dimensional and applications of large deviations and differential geometry are appearing fast, headed by the work of Henry-Labordere [47]. So, what is left for the local volatility model and where to go from here? Here are some plausible and incomplete answers:

- A modest forward step would be to develop numerical schemes for solving the degenerate quasilinear parabolic PDE linking local and implied volatilities; see (2.8). An efficient scheme would provide a basis for practicable exotic derivative pricing with implied volatilities.

- A more ambitious step would be to derive the known asymptotic results under the assumption that the diffusion coefficient, e.g., $\nu(\cdot)$ or $\sigma(\cdot)$ in this thesis, can hit zero or blow up to infinity. The asymptotics can be of small or large time or large strike. We note that the main tricks used in this thesis are not likely to work.

- On a different scale of complexity, one could investigate the small time properties of the gradient and Hessian of the implied volatility in stochastic volatility models that, unlike the one dimensional local volatility model in my thesis, allow for an arbitrary number of stochastic volatility factors. This would extend the work of Berestycki et al. [5].

- Of course, to reach a more admirable pinnacle, one can borrow tools from geometric analysis, after the fashion of yield curve modelling, to break new paths in the field of implied volatilities. More admirable still, if the new paths can be statistically tested and can stand the test of time.

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