A Theory of Nonlinear Negative Imaginary Systems

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A Theory of Nonlinear Negative Imaginary Systems

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Thesis/Dissertation Sheet

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Abstract : In this thesis, we aim to generalize the negative imaginary systems theory to a broad class of nonlinear systems. A formal definition will be given for the negative imaginary property in the nonlinear domain by invoking a new dissipativity notion with an appropriate work rate. This formula is considerably more general than the existing classical dissipativity framework. Flexible structures with colocated force actuators and position sensors are dissipative according to this new definition. Having defined the nonlinear negative imaginary property in a time-domain dissipativity framework, we are able to extend some of the main existing results on negative imaginary systems from the linear to nonlinear domain. First, a Lyapunov-based approach will be used to establish the stability robustness of a positive feedback interconnection of negative imaginary systems in the linear case under a set of theoretical assumptions. Then, these assumptions will be adapted in the nonlinear setup to establish the stability robustness analysis of a positive feedback interconnection of nonlinear negative imaginary systems by making use of Lyapunov stability theory and dissipativity techniques. The applicability of this nonlinear stability result will be illustrated through an example of nonlinear mass spring damper system. Furthermore, the nonlinear negative imaginary systems theory will be extended to the case of free motion. It will be shown that, under suitable assumptions, a cascade connection of an affine nonlinear system and single integrator will lead to a nonlinear negative imaginary system (with integrator). Finally, this thesis is concluded by a summary of current progress and a discussion of possible future developments of the nonlinear negative imaginary systems theory.

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Abstract

Feedback control theory is aimed at controlling a system input to obtain a desired output and making the system robust in the face of unmodeled dynamics and external disturbances. In real-word applications, most physical and engineering systems exhibit nonlinear behaviour which in general makes controller design difficult. One of the most appealing tools in the field of nonlinear control design is the "classical" dissipativity and passivity theory which characterizes the dissipation of energy with respect to a supplied energy rate from the outside environment. However, many systems that dissipate energy in the physical sense don't fall into this classical framework. For instance, flexible structures with colocated force actuators and position sensors are passive (dissipative) from the input to the derivative of the output instead of the output as in the classical passivity theory. Furthermore, it is not always straightforward to analyze the system's performance when the supply rate involves derivatives of the input and output.

In this regard, negative imaginary systems theory has proven to be an effective tool in the analysis and control design of linear time invariant systems which are passive from the input to the derivative of the output. Negative imaginary systems theory has become a well established systems-theoretical tool which has been employed in a wide variety of control applications including robust vibration control of flexible structure, atomic force microscopy, and nano-positioning systems.

In this thesis, we aim to generalize the negative imaginary systems theory to a broad class of nonlinear systems. A formal definition will be given for the negative imaginary property in the nonlinear domain by invoking a new dissipativity notion with an appropriate work rate. This formula is considerably more general than the existing classical dissipativity framework. Flexible structures with colocated force actuators and position sensors are dissipative according to this new definition.

Having defined the nonlinear negative imaginary property in a time-

domain dissipativity framework, we are able to extend some of the main existing results on negative imaginary systems from the linear to nonlinear domain. First, a Lyapunov-based approach will be used to establish the stability robustness of a positive feedback interconnection of negative imaginary systems in the linear case under a set of theoretical assumptions. Then, these assumptions will be adapted in the nonlinear setup to establish the stability robustness analysis of a positive feedback interconnection of nonlinear negative imaginary systems by making use of Lyapunov stability theory and dissipativity techniques. The applicability of this nonlinear stability result will be illustrated through an example of nonlinear mass spring damper system. Furthermore, the nonlinear negative imaginary systems theory will be extended to the case of free motion. It will be shown that, under suitable assumptions, a cascade connection of an affine nonlinear system and single integrator will lead to a nonlinear negative imaginary system (with integrator). Finally, this thesis is concluded by a summary of current progress and a discussion of possible future developments of the nonlinear negative imaginary systems theory.

Contents

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Notations and Acronyms

NI	Negative imaginary
SNI	Strict Negative Imaginary
LMI	Linear Matrix Inequaity
LTI	Linear Time Invariant
SISO	Single Input Single Output
MIMO	Multi Input Multi Output
PR	Positive Real
SPR	Strictly Positive Real
PPF	Positive Position Feedback
IRC	Integral Resonant Control
CCW	Counter Clockwise
AFM	Atomic Force Microscopy
MSD	Mass Spring Damper
$\lambda_{max}(\cdot)$	the maximum eigenvalue of a matrix with only real eigenvalues
C^1	the space of continuous differentiable functions
Ω	omega limit set
$\mathbb{R}^n$	the n-dimensional real Euclidean space
$\mathbb{R}$	the field of real numbers
∇f	the gradient of the function f

Chapter 1

Introduction

1.1 Background and Motivation

Feedback control systems are becoming an essential component in many modern advanced technologies to achieve a high level of performance. For instance, precision technologies such as atomic force microscopy, nanopositioning, micro-robotics and hard disc drives require high precision and performance in controller design [13, 26, 49]. Also, large scale technologies such as electrical power systems, environmental systems such as irrigation systems, and transportation systems such as road networks are reliant on feedback control systems to achieve energy efficiency as well as reliability [85]. However, designing a robust control system in face of system uncertainties is still a major challenge for many control problems. Furthermore, since most physical and engineering systems are inherently nonlinear, the resulting feedback dynamical systems can exhibit a very rich dynamical behavior. To meet this challenge, feedback control theory provides a series of mathematical tools to for the analysis and design of feedback controllers that manipulate system inputs to obtain a desired output of the system in the face of uncertainty and disturbances found in the controlled system [12,78,117].

One of these appealing and effective tools is the *passive systems theory* [50,111] which provides systems-theoretic framework to analyze the stability of the system based on energy-related considerations [17,85]. A dynamical system is said to be passive if it always dissipates energy, and the energy supplied to the system, called the supply rate, is given by the product of system input u and system output y. Typical examples of such systems are mechanical systems with colocated force actuators and velocity sensors. A distinguished feature of the passive systems theory is that the passivity properties of a system will keep the system internally stable. In particular, by the *Passivity Theorem* [3], an interconnection of two passive systems is passive and,thus, stable in the absence of exogenous inputs.

Other important mathematical tool is the dissipative systems theory introduced by J. C. Willems [111,112] as a generalization of the passive systems theory. In the latter papers, a characterization of the dissipativity property of a general nonlinear dynamical system was given to allow for a more general supplied energies. Roughly speaking, in [111] a nonlinear dynamical system with input u and output y system is said to be dissipative if there exists a so-called *storage function*, denoted $V(x)$ where x is the state of the system, and a *supply rate function*, denoted $w(u(s),y(s))$, such that the following dissipation inequality

$$V(x(t)) \leq V(x(0)) + \int_0^t w(u(s),y(s))ds \tag{1.1}$$

holds along all possible trajectories starting at $x(0)$, and for all $t \geq 0$. The physical interpretation of the above dissipation inequality is that the increase in the system's stored energy over a given time interval is less than or equal to the energy supplied to the system during this time interval. The dissipation of energy in physical systems can be employed to establish the robust stability of feedback interconnections of dissipative systems, see [50]. It is worth noting that the definition of dissipativity/passivity of a systems and related results continue to hold in the case where no energy interpretation is available for the system.

In practical application, the dissipative/passive systems theory has been used in a wide range of control design problems; see, for instance, [6, 10, 25, 62]. However, many systems that dissipate energy in the physical sense cannot be captured by the above classical definition of dissipativity. In particular, a flexible structure with displacement outputs y and force inputs u is not dissipative in this sense; rather, it is dissipative with a supply rate function $\dot{y}(t)u(t)$; see [43]. As a matter of fact, the analysis and robust control design of systems which are dissipative/passive from the input u to the derivative of the output $\dot{y}$ cannot in general be handled in a straightforward manner using the classical dissipative/passive systems theory; see e.g., [5, 44, 82, 86, 87]. This motivated the need for a more general systems-theoretical framework in order to allow for dissipative/passive dynamical systems for which the supply rate function involves derivatives of the outputs and inputs.

In this regard, the negative imaginary (NI) systems theory [60, 90] has emerged as a theoretical systems framework which complement the applicability of the dissipative/passive systems theory in the linear case. Broadly speaking, for linear time-invariant (LTI) systems, if the system is passive from the input to the derivative of the output, then the system is negative imaginary system. In recent years, NI systems have attracted the attention of many researcher which has led to rich and fruitful results in the field of control theory; see e.g. [42, 58, 60, 75, 90]. These NI results have been applied in many control applications including, for instance, the robust control of highly resonant flexible structures with colocated position sensors and force actuators, nanopositioning in atomic force microscopy [14, 20, 71, 72, 90]. Also, NI systems theory has been employed in the stability analysis of positive feedback loops in a similar way that passivity theory does for negative feedback interconnections; see [42, 90].

In this thesis, taking into account the above mentioned factors, we seek a natural nonlinear generalization of the negative imaginary notion and the most general linear NI stability results. To achieve that, a generalized NI definition will be given for a class of general nonlinear systems based on the time-domain interpretation of the negative imaginary notion in the linear case. This would lead to a more general dissipativity/passivity definition than the classical one. Flexible structures with colocated force actuators and position sensors are dissipative according this new definition.

1.2 Objectives and Contributions

In this thesis, we aim to develop generalised energy methods for nonlinear robust stability analysis by building on the NI and passive systems theories and their physical interpretations. A generalization of the NI property of LTI systems to the nonlinear setup will be adopted using time-domain dissipativity framework. More explicitly, a general nonlinear dynamical system will be defined to be *nonlinear negative imaginary* if it is dissipative with respect to supply rate $\dot{y}(t)u(t)$. This time-domain definition is considerably more general than the classical definition of dissipativity. We shall seek extension of the most general results from the negative imaginary systems theory to a broader class of nonlinear dynamical systems using the Lyapunov stability theory [47, 57]. This in turn lead to an extension of the applicability of ddissipative/passive systems theory to allow for more general supply rates which involve derivatives of the system output and input.

In summary, this thesis makes the following contributions:

- First, we will introduce a Lyapunov-based proof of the internal stability robustness of a positive feedback interconnection of LTI NI systems in the multi-input- multi-output (MIMO) case. The feedback system comprises of a plant which is negative imaginary with poles on the imaginary axis except at the origin, and a controller which is strictly negative imaginary. The dc loop gain (the loop gain at zero frequency) will be used to construct a candidate Lyapunov function for the closed-loop and to provide a proof of internal stability.

- Next, a Lyapunov-based approach and an invariance principle will be employed, under a set of mild theoretical assumptions, to guarantee the robust stability of a positive feedback interconnection of general nonlinear negative imaginary systems. In order to handle these general nonlinear systems, a generalization of the dc loop gain in the nonlinear setting will be developed. This nonlinear stability result will be shown to reduce to the case of a feedback interconnection of SISO LTI negative imaginary systems, where the plant may have poles on the imaginary axis except at the origin. To illustrate this stability result, an example of a nonlinear mass-spring-damper (MSD) system will be provided. It will be shown that the nonlinear MSD system can be stabilized by a strictly negative imaginary controller provided that a nonlinear generalisation of the dc loop gain is still satisfied.

- Further, an extension of the definition of nonlinear negative imaginary systems will be given, which allows for flexible structure systems with colocated force actuators and position sensors, and with free motion. In this context, a cascade connection of nonlinear systems which is affine in the input and a single integrator will be shown to be nonlinear negative imaginary (with integrator). This is achieved by finding a nonnegative storage function of the cascade system such that the dissipativity inequality (3.8) holds with supply rate $\dot{y}(t)u(t)$.

1.3 Structure of the Thesis

The structure of this thesis is as follows:

In Chapter 2, a literature review on the theoretical development of the negative imaginary theory and its use in practical applications will be given. In addition to this, we present background material on the class of NI system and the main related results. Furthermore, we highlight two classes of systems, namely the class of positive real (PR) systems and the class of counter-clockwise (CCW) systems. Many of the existing results related to PR and CCW systems are closely related to that of negative imaginary systems.

Chapter 3 introduces some necessary mathematical tools which will be used throughout this thesis. Some basic results available from the Lyapunov stability theory and dissipative/passive systems theory will be reviewed. These tools will be used in Chapter 4 and Chapter 5 to establish the robust stability of positive feedback interconnections of linear and nonlinear negative imaginary systems, respectively.

In Chapter 4, a Lyapunov-based stability proof of a positive feedback interconnection of LTI negative imaginary systems will be given. The feedback system is composed of the plant which is assumed to be negative imaginary with poles on the imaginary axis and the controller which is strictly negative imaginary. The dc loop gain of the feedback system will be used to construct a candidate Lyapunov function and to provide proof of the internal stability of the feedback system.

In Chapter 5, we introduce a time-domain definition of the NI property for a general nonlinear system. Then, under a set of mild theoretical assumptions, Lyapunov stability theory and an invariance principle will be used to guarantee the robust stability of a positive feedback interconnection of nonlinear negative systems. A generalization of the dc loop gain of the feedback system in the nonlinear setting will be developed. An illustrative example will be presented to elucidate this nonlinear stability result.

Chapter 6 addresses the extension of the nonlinear negative imaginary notion to include the free motion case.

Chapter 7 is devoted to conclusions and possible future research directions on nonlinear negative imaginary systems theory.

Chapter 2

Literature Review and Preliminaries

This chapter presents an overview of the theoretical aspects of the negative imaginary systems theory and their practical relevance. In addition, we highlight the classes of positive real and counter clockwise systems and their close relation to the class of negative imaginary systems.

2.1 Negative Imaginary Systems Theory: Background

Negative imaginary systems naturally arise in problems of robust vibration control of flexible structures with colocated position sensors and force actuators [14, 20, 60, 69, 71, 72, 90]. These systems are stable linear systems with an equal number of inputs and outputs. For a stable LTI system with transfer function matrix $G(s)$, the NI property is defined by

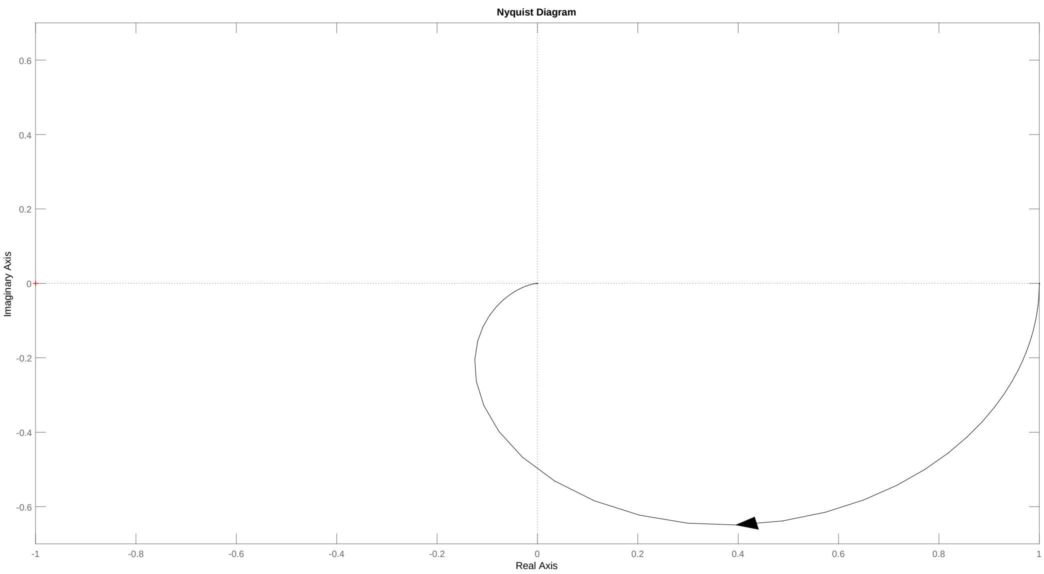

Figure 2.1: Nyquist plot of a typical negative imaginary transfer function for positive frequencies

the requirement that $j(G(j\omega) - G(j\omega)^*) \geq 0$ for all $\omega \in (0, \infty)$. For a SISO NI systems, the system has a phase-lag between 0 and $-\pi$, and hence the Nyquist plot of $G(j\omega)$ lies entirely below the real axis for all positive frequencies; see Figure 2.1. The theory of negative imaginary systems has proven itself as a powerful complement to positive real theory and passivity theory. In the SISO case, positive real systems have a phase shift in the interval $[\frac{-\pi}{2}, \frac{\pi}{2}]$ and therefore cannot have relative degree more than unity whereas NI systems can have a relative degree up to two.

In recent years, negative imaginary systems theory has attracted a lot of research interest which has led to a great deal of progress in the theory. This includes extensions to non-rational systems [37–40], descriptor systems [70, 114], controller synthesis for negative imaginary systems [100–102,113], a notion of strongly strict negative-imaginary systems [61], extensions to non-proper systems [65], and to infinite-dimensional systems [83]. Also, the class of NI systems has been shown to be closely related to the class of linear port-Hamiltonian input-output systems [105]. Moreover, in [75], the NI framework has been further extended to the case when the plant has free body motion where a new NI system definition has been given to allow for poles at the origin.

Applications of NI theories have seen increasing adoption in various control problems. For example, the NI systems theory has been applied in vibration control of a flexible robotic arm [75], in flexible robot manipulators [26], in ground and aerospace vehicles [49], in control of a DC servo motor [102], in vehicle platooning [20] and in position control of a swing-arm hard disk drive [64]. In addition, NI systems theory has been employed in nano-positioning control for atomic force microscopes (AFMs); e.g., see [27–30, 34, 79, 81, 96]. Also, NI feedback control schemes such as integral resonant control, and resonant feedback control have been used in the robust vibration control of flexible structures [48, 76, 88].

Furthermore, the NI stability result provided in [60, 90] has been used in a number of practical applications [1, 13, 14, 21, 33, 77]. For example in [21], this stability result is applied to the problem of decentralized control of large vehicle platoons. In [13, 77], the NI stability result is applied to nano-positioning in an atomic force microscope. A positive position feedback control scheme based on the NI stability result provided in [60, 90] is used to design a novel compensation method for a coupled fuselage-rotor mode of a rotary wing unmanned aerial vehicle in [1]. In [33], an IRC scheme based on the NI stability result is used to design an active vibration control system for the mitigation of human induced vibrations in light-weight civil engineering structures, such as floors and footbridges via proof-mass actuators.

2.1.1 Negative-Imaginary Systems

We now review some of the main definitions and related results in the NI literature; see e.g, [58, 60, 89, 90] for detailed discussions.

Definition 2.1 (**Negative Imaginary Systems [59,60]**). *A square real-rational transfer function matrix $G(s)$ is negative imaginary if the following conditions are satisfied:*

1) $G(s)$ has no pole at the origin and in $\mathrm{Re}[s] > 0$.

2) For all $\omega > 0$ such that $j\omega$ is not a pole of $G(s)$, then $j(G(j\omega) - G(j\omega)^*) \geq 0$.

3) If $j\omega_0$, $\omega_0 \in (0, \infty)$, is a pole of $G(j\omega)$, it is at most a simple pole and the residue matrix $K_0 = \lim_{s \to j\omega_0}(s - j\omega_0)sG(s)$ is positive semi-definite Hermitian.

Remark 2.1.

- If $G(s)$ is SISO, then condition 2) in the above definition reduces to the condition $-2\,\mathrm{Im}\,G(j\omega) \geq 0$.
- A linear time invariant system is NI if its transfer function is NI.

A stronger version of the NI property, namely the strictly negative imaginary (SNI) property is given in the next definition.

Definition 2.2 (**Strictly Negative Imaginary [60,90]**). *A square real-rational transfer function matrix $G(s)$ is strictly negative imaginary (SNI) if:*

1) $G(s)$ has no poles in $\Re[s] \geq 0$;

2) $j[G(j\omega) - G^T(j\omega)] > 0$ *for* $\omega \in (0, \infty)$.

A linear time-invariant system is SNI if its transfer function matrix is SNI.

2.1.2 The Negative-Imaginary Lemma

The negative imaginary lemma provides a state-space characterization of NI systems in terms of a pair of linear matrix inequalities (LMIs). The NI lemma is as follows.

Lemma 2.1. *[73]. Let* (A, B, C, D) *be a minimal state-space realization of the* $m \times m$ *real-rational proper transfer function matrix* $G(s)$*, where* $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$ *with* $D = D^T$*. Then* $G(s)$ *is negative imaginary if and only if there exist matrices* $P = P^T > 0$*,* $W \in \mathbb{R}^{m\times m}$*, and* $L \in \mathbb{R}^{m\times n}$ *such that the following LMI is satisfied:*

$$\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix} = \begin{bmatrix} -L^T L & -L^T W \\ -W^T L & -W^T W \end{bmatrix} \leq 0. \quad (2.1)$$

Remark 2.2. The linear matrix equality (2.1) can be simplified to the following (see [60]),

$$AP + PA^T \leq 0, \quad \text{and} \quad B + APC^T = 0.$$

2.1.3 The Strict Negative-Imaginary Lemma

The following lemma gives a state space characterization of strictly negative imaginary systems.

Lemma 2.2. *[115]. Let (A,B,C,D) be a minimal state-space realization of the $m \times m$ real-rational proper transfer function matrix $G(s)$, where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$. Then $G(s)$ is strictly negative imaginary if and only if:*

1) *$det(A) \neq 0$, $D = D^T$;*

2) *there exists a matrix $P = P^T > 0, P \in \mathbb{R}^{n\times n}$, such that*

$$AP^{-1} + P^{-1}A^T \leq 0, \quad \textit{and} \quad B + AP^{-1}C^T = 0$$

3) *the transfer function matrix $M(s) \backsim \left[\begin{array}{c|c} A & B \\ \hline LPA^{-1} & 0 \end{array}\right]$ has full column rank at $s = jw$ for any $\omega \in (0,\infty)$ where $L^T L = -AP^{-1} - P^{-1}A^T$. That is, rank $M(j\omega) = m$ for any $\omega \in (0,\infty)$.*

2.1.4 Robust Stability of Negative Imaginary Systems

The stability robustness of a positive feedback interconnection of NI system is established in the following theorem:

Theorem 2.1. *[42, 60] Assume $G(s)$ is a negative imaginary system with no poles at the origin and $H(s)$ is a strictly negative imaginary system such that $G(\infty)H(\infty) = 0$ and $H(\infty) \geq 0$. Then, the positive feedback interconnection of $G(s)$ and $H(s)$ is internally stable if and only if*

$$\lambda_{max}(G(0)H(0)) < 1, \tag{2.2}$$

where $\lambda_{max}(\cdot)$ denotes the maximum eigenvalue of a matrix with only real eigenvalues.

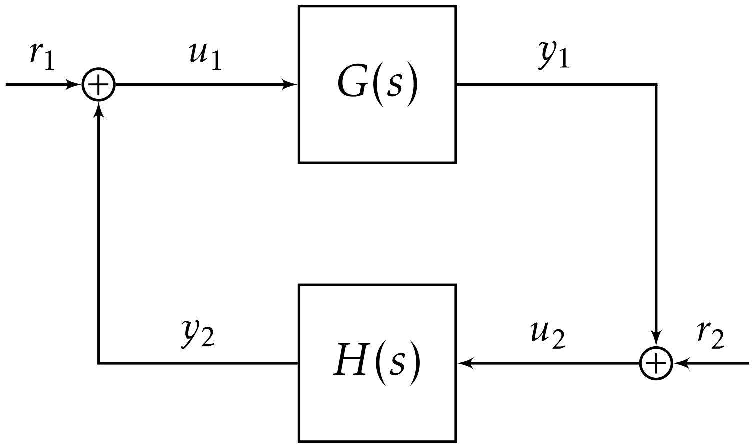

Figure 2.2: Feedback interconnection of NI systems.

The above NI stability result has been proved to remain valid for the case in which the plant has purely imaginary poles except at the origin [42,115]. Also, necessary and sufficient conditions are provided in [75] for the stability of positive feedback control systems where the plant is NI with poles on the imaginary axis including the origin, and the controller is strictly negative imaginary [75]. Moreover, the robust stability of interconnected NI systems has been established for various sub-classes of NI systems where the dc loop gain matrix information is adopted [15, 61]. Also, the absolute stability of a Lur'e system with positive feedback where the linear subsystem is NI is presented in [32]. Furthermore, the NI property and related stability results for discrete-time LTI systems have been considered; see [22, 38, 66].

2.1.5 Negative Imaginary Feedback Controllers

In many practical control applications, the aforementioned NI stability theorem is applied to ensure the robustness stability of NI feedback control systems as shown in Figure 2.2. In particular, flexible structures with colocated force actuators and position sensors are typically modelled as NI systems [90], and by Theorem 2.1, an SNI controller guarantees closed-loop internal stability if the dc-gain condition (2.2) is satisfied. Next, we review some examples where negative imaginary control schemes are applied to insure the internal stability of NI feedback systems.

2.1.6 Resonant Control

The resonant control scheme is used for the vibration control of flexible structures with colocated force actuators and position sensors [91], [48]. In the MIMO case, resonant controllers typically take one of the following two forms

$$C(s) = \sum_{i=1}^{M} \frac{-s^2}{s^2 + 2\xi_i \omega_i s + \omega_i^2} \alpha_i \alpha_i^T \tag{2.3}$$

and

$$C(s) = \sum_{i=1}^{M} \frac{-s(s + 2\xi_i \omega_i)}{s^2 + 2\xi_i \omega_i s + \omega_i^2} \beta_i \beta_i^T \tag{2.4}$$

where $\omega_i > 0$, $\xi_i > 0$ and α_i, β_i are $m \times 1$ vectors. These controllers have been shown to be SNI, see [90]. In light of Theorem 2.1, controllers of the form (2.3), (2.4) can be used to robustly stabilize any NI plant provided that the dc gain condition (2.2) is satisfied.

2.1.7 Positive Position Feedback

The positive position feedback (PPF) control strategy was introduced in [36] for vibration suppression in flexible structures. In the SISO case, a positive position feedback controller takes the form

$$C(s) = \sum_{i=1}^{M} \frac{k_i}{s^2 + 2\xi_i \omega_i s + \omega_i^2}, \tag{2.5}$$

where $\omega_i > 0$, $\xi_i > 0$, and $k_i > 0$. This controller has been shown to be SNI using a Nyquist argument; see [90]. For the MIMO case, the positive position feedback controllers of the form (2.5) can be extended to

$$C(s) = K^T (s^2 I + Ds + \Omega)^{-1} K, \tag{2.6}$$

where $D > 0$ and $\Omega > 0$.

The PPF method has been implemented in a diverse range of control applications to reduce vibrations in smart structures [19, 80, 93, 97, 99]. In [80], a PPF scheme has been used for vibration suppression of a flexible appendage by using embedded piezoceramic actuators. Also in [93], a PPF strategy for multi-modal vibration control in a composite plate with piezoelectric sensors and actuators has been applied, and in vibration control for a rotor-bearing system [19]. In [99] implemented PPF for single-mode vibration suppression and for multi-mode vibration suppression of a cantilevered beam. Moreover, the use of positive position feedback via piezoelectric actuators to suppress multi-mode vibrations, while slewing a single-link flexible manipulator has been investigated in [97].

2.1.8 Integral Resonant Control

Integral Resonant Control (IRC) is a simple and robust control scheme for vibration control of smart structures with colocated sensors and actuators [7, 92]. In the MIMO case, an integral resonant controller is a transfer function matrix of the form

$$C(s) = [sI + \Gamma\Phi]^{-1}\Gamma, \tag{2.7}$$

here, both Γ and Φ is positive-definite matrix. This controller has been shown to be SNI, see [90]. In [90], the use of an integral resonant controller to establish closed-loop internal stability for such systems has been shown in the SISO case.

2.1.9 State-Feedback Controller Synthesis

In [90], Theorem 2.1 has been used to robustly stabilize a feedback control system in the presence of SNI uncertainties as shown in Figure 2.3. It has been shown that applying a feedback control law to the nominal plant so that the closed-loop system is NI and the DC-gain condition is satisfied, then the stability robustness of the resulting feedback uncertain system is guaranteed. Indeed, suppose the uncertain system shown in Figure 2.3 is described by the following state equations

$$\dot{x} = Ax + B_1 w + B_2 u, \tag{2.8}$$

$$z = C_1 x, \tag{2.9}$$

$$w = \Delta(s) z, \tag{2.10}$$

where w is the disturbance entering the system and z is the output signal vector of the system and $\Delta(s)$ is the plant uncertainty matrix. The matrix $\Delta(s)$ is assumed to be SNI and satisfies $\lambda_{max}(\Delta(0)) \leq 1$ and $\Delta(\infty) \leq 0$.

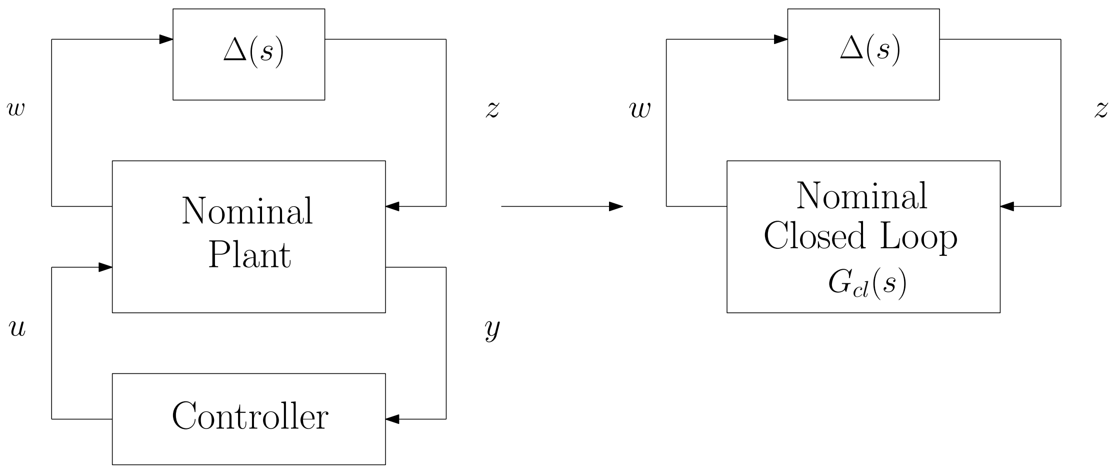

Figure 2.3: A feedback control system where the plant uncertainty $\Delta(s)$ is stable strictly negative imaginary, and satisfies $\Delta(\infty) \geq 0$ where the DC gain condition is satisfied. If the controller is chosen so that the nominal closed-loop transfer function matrix $G_{cl}(s)$ is strictly proper and negative-imaginary, then the closed loop system is robustly stable.

Applying a static state-feedback control law $u = Kx$, results in the following closed-loop uncertain system

$$\dot{x} = (A + B_2K)x + B_1w, \tag{2.11}$$

$$z = C_1x, \tag{2.12}$$

$$w = \Delta(s)z. \tag{2.13}$$

The corresponding nominal closed-loop transfer function matrix is

$$G_{cl}(s) = C_1(sI - A - B_2K)^{-1}B_1. \tag{2.14}$$

Now we have the following result which provides sufficient conditions to ensure the stability robustness of the above uncertain system.

Theorem 2.2. *[90]. Consider the uncertain system* (2.8)–(2.10) *and suppose* *that there exist matrices $Y > 0, M$, and a scalar $\varepsilon > 0$ such that*

$$\begin{bmatrix} AY + YA^T + B_2M + M^TB_2^T + \varepsilon I & B_1 + AYC_1^T + B_2MC_1^T \\ B_1^T + C_1YA^T + C_1M^TB_2^T & 0 \end{bmatrix} \leq 0, \tag{2.15}$$

$$C_1YC_1^T - I < 0. \tag{2.16}$$

Then the static state-feedback control law $u = MY^{-1}x$ is robustly stabilizing for the uncertain system (2.8)–(2.10).

For the sake of self-contained presentation, we introduce in the next two section two important classes of systems which are closely related to the class of negative imaginary systems, namely the class of *positive real systems* and the class of*counter clockwise systems*. As a matter of fact, these two class of systems represent the context from which negative imaginary systems emerged, both chronologically and conceptually.

2.2 Positive Real Systems

Positive real systems play a major role in system and control theory [2, 4, 107]. In [46, 54, 67, 95, 109], the stability robustness of positive real systems was studied. Here, we review some of the definitions of the PR class of positive real LTI systems in addition to the main stability result. Also, we highlight on the close relation between NI and PR systems.

2.2.1 Frequency-Domain and State-Space Representations

In the SISO case, PR systems have phase shift in the interval $[\frac{-\pi}{2}, \frac{\pi}{2}]$ (see Figure 2.4) and therefore cannot have a relative degree more than unity.

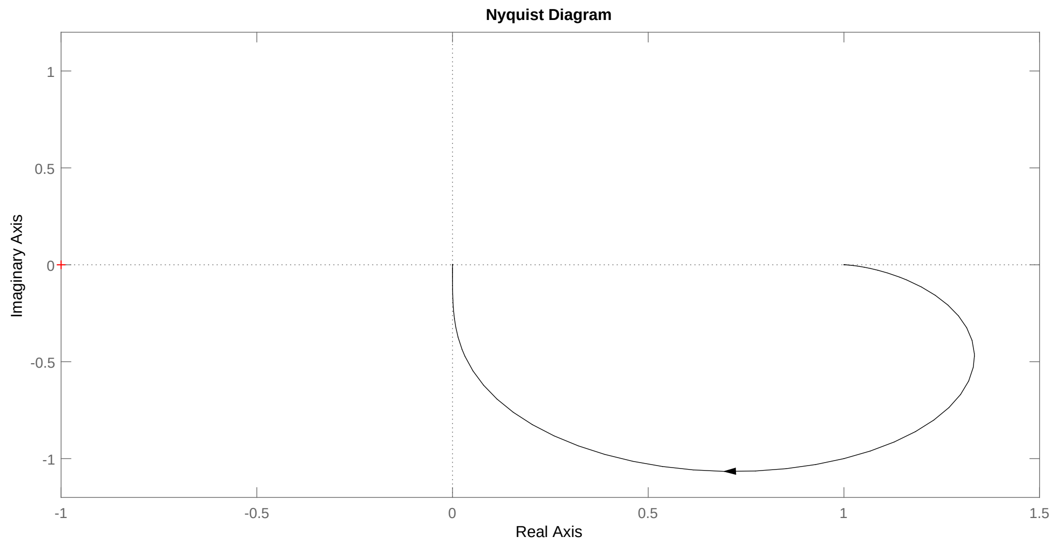

Figure 2.4: Typical Nyquist plot of a positive real transfer function

We have the following two definitions for positive real and strictly positive real systems in the MIMO case.

Definition 2.3. *[17] A rational transfer function matrix* $H : \mathbb{C} \to \mathbb{C}^{m\times m}$ *is positive real if*

1) $H(s)$ *is analytic in* $s \in \mathbb{C} : \mathrm{Re}[s] > 0$*;*

2) $H(s)$ *is real when s is real and positive;*

3) $H(s) + H(s)^* \geq 0$ *for all* $s \in s \in \mathbb{C} : \mathrm{Re}[s] > 0$*. Here,* $H(s)^*$ *denotes the complex conjugate transpose of* $H(s)$*.*

Definition 2.4. *[17] The transfer function* $H(s)$ *is called strictly positive real (SPR) if* $H(s-\varepsilon)$ *is positive real for some* $\varepsilon > 0$*.*

Remark 2.3. In the SISO case, the frequency domain condition in Definition 2.3 reduces to $\mathrm{Re}\, H(s) > 0$ for all $\mathrm{Re}[s] > 0$.

The state-space characterizations of PR and SPR transfer function matrices are given in the next two lemmas which are known as *positive real lemma* and *Kalman-Yakubovich-Popov lemma,* respectively.

Lemma 2.3 (**Positive Real Lemma** [2]). *Let* (A, B, C, D) *be minimal state realization of an* $m \times m$ *transfer function matrix* $H(s)$ *with* $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$. *Then,* $H(s)$ *is positive real if and only if there exists real symmetric positive define matrix* P, $P \in \mathbb{R}^{n\times n}$ *,and real matrices* $L \in \mathbb{R}^{n\times m}$, $W \in \mathbb{R}^{m\times m}$*, such that:*

$$\begin{aligned} PA + A^T P &= -L^T L; \\ PB - C^T &= -L^T W; \\ D + D^T &= W^T W. \end{aligned} \tag{2.17}$$

Lemma 2.4 (**Kalman-Yakubovich-Popov** [2]). *Let* (A, B, C, D) *be minimal state realization of an* $m \times m$ *transfer function matrix* $H(s)$ *with* $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$. *Then,* $H(s)$ *is strictly positive real if and only if there exists real symmetric positive define matrix* P, $P \in \mathbb{R}^{n\times n}$ *, and real matrices* $L \in \mathbb{R}^{n\times m}$, $W \in \mathbb{R}^{m\times m}$*, and a positive constant* ε *such that:*

$$\begin{aligned} PA + A^T P &= -L^T L - \varepsilon P; \\ PB - C^T &= -L^T W; \\ D + D^T &= W^T W. \end{aligned} \tag{2.18}$$

2.2.2 Feedback Stability of Positive Real Systems

Here we introduce one of the main theorems concerning the stability robustness of LTI positive real or strictly positive real systems when they are connected in negative feedback as shown in Figure 2.5 below.

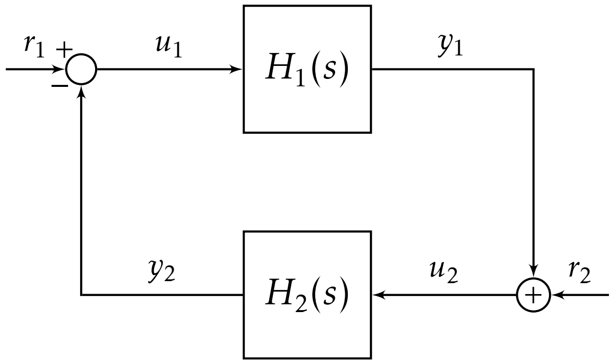

Figure 2.5: Negative feedback interconnection of the positive real systems $H_1(s)$ and $H_2(s)$.

The following theorem is one of the different versions of the passivity theorem for stability of LTI positive real systems.

Theorem 2.3. *[17]. Consider the negative feedback interconnection as shown in Figure 2.5. Suppose that $H_1(s)$ is positive real and $H_2(s)$ is strictly positive real. Then the feedback system is internally stable.*

2.2.3 Relation between PR and NI Systems

Positive real systems have a close relation to negative imaginary systems. The following two lemmas highlight on this relation in the frequency response domain.

Lemma 2.5. *[115]. Let (A,B,C,D) be a minimal state-space realization of a transfer function $G(s)$ where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times m}$ and $\tilde{G}(s) := G(s) - D$. Then $G(s)$ is NI if and only if $F(s) := s\tilde{G}(s)$ is positive real.*

Lemma 2.6. *[18]. Given a square proper positive real transfer function matrix $G(s)$, then $R(s) := \frac{G(s)}{s}$ is negative imaginary.*

2.3 Counter-Clockwise Input-Output Systems

In many dynamical systems, such as systems with hysteresis, the output tends to lag behind the input [5, 86], and in this case the corresponding plot of the system input $u(t)$ versus the system output $y(t)$ will have a counter-clockwise (CCW) orientation (see Figure 2.6). Dynamical systems with such property are termed *systems with counter-clockwise input-output dynamics*, or simply CCW systems. This property can be characterized using the classical Green's theorem [103] in the way that the enclosed area A is non-negative when the closed curve $\mathcal{C}$ has a counter-clockwise orientation.

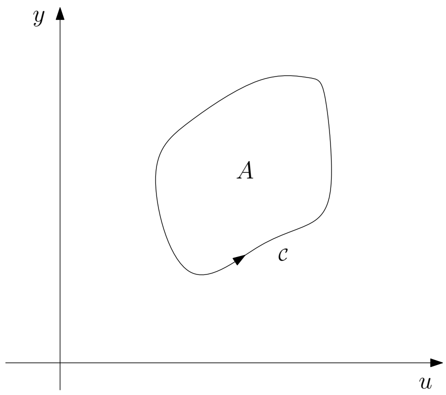

Figure 2.6: Input/Output responses of a system with CCW dynamics.

Let us now consider a SISO system, which has periodic (input $u(t)$ and output $y(t)$) signals with a time period T and the curve $(u(t), y(t))$ has counter-clockwise orientation as shown in Figure 2.6. The enclosed area A is evaluated as,

$$\begin{aligned} 0 \leq A = \frac{1}{2}\int_0^T u\,dy - y\,du &= \frac{1}{2}\int_0^T [\dot{y}(t)u(t) - \dot{u}(t)y(t)]dt \\ &= \int_0^T \dot{y}(t)u(t)dt - \frac{1}{2}u(T)y(T) + \frac{1}{2}u(0)y(0). \end{aligned}$$

Since $u(0) = u(T)$ and $y(0) = y(T)$, the area A enclosed by the curve $\mathcal{C}$ is given by:

$$A = \int_0^T \dot{y}(t)u(t)dt \geq 0, \quad \forall\ T > 0. \tag{2.19}$$

That means that the SISO system under consideration is CCW provided that the signed-area encircled by the curve $\mathcal{C}$ is non-negative. Moreover, by [5, Lemma 2.2], condition (2.19) was shown to be equivalent to

$$\liminf_{T\to\infty} \int_0^T \dot{y}(t)u(t)dt > -\infty, \quad \forall\ T > 0. \tag{2.20}$$

Based on the above argument, the definition of the CCW property for general nonlinear dynamical systems with bounded input and output according was given as follows,

Definition 2.5. *[5] A dynamical system $\Gamma : u(t) \longmapsto y(t)$, where $u(t), y(t) \in \mathbb{R}^m$, is said to have counter-clockwise input–output dynamics if for every $u(t)$ such that the corresponding output $y(t)$ is differentiable, the following inequality*

$$\liminf_{T\to\infty} \int_0^T \dot{y}(t)u(t)dt > -\infty \tag{2.21}$$

holds.

It is worth noting that the CCW property of a dynamical system can also be interpreted using the classical passivity theory: the system is passive from the input to the time derivative of the output (instead of the output in the passivity theory). In [5, 86], convergence analysis of a positive feedback interconnection of CCW systems was investigated in a similar way to that of a negative feedback interconnection of two passive systems.

2.3.1 Equivalence of NI and CCW LTI systems

In the case of LTI systems, CCW input–output dynamic systems is precisely related to the phase-lag introduced by the system over the whole range of possible frequencies (see Figure 2.7).

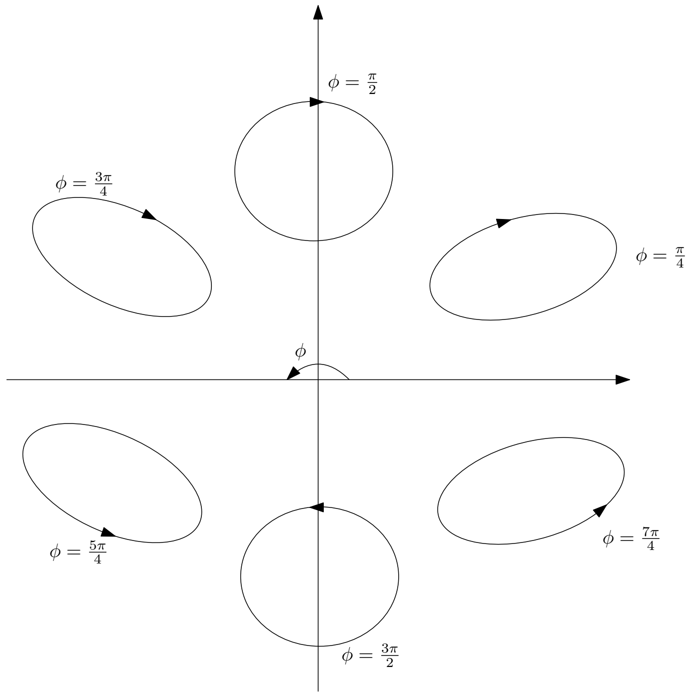

Figure 2.7: Relationship between the orientation of the input-output map and the phase angle $\phi = \angle G(j\omega)$ for linear systems (adapted from [86]).

This relation is made precise in the following theorem.

Theorem 2.4. *[86] Consider the following LTI systems*

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2.22}$$

$$y(t) = Cx(t) + Du(t) \tag{2.23}$$

with transfer function $G(s) = C(sI - A)^{-1}B + D$, where A, B, C, D are constant *matrices with suitable dimensions. Then the following statements are equivalent:*

- $G(s)$ *is negative imaginary*
- A *is Hurwitz and the LTI system is counter-clock wise.*

Chapter 3

Mathematical Background

This chapter presents some of the mathematical tools which will be used throughout the thesis. We review some of the basic concepts and results of Lyapunov stability theory in addition to the main tools of the dissipativity and passivity theory of nonlinear systems. These tools will be used in Chapter 4 and Chapter 5 to analyze the stability properties of positive feedback interconnections of linear/nonlinear NI systems. For detailed accounts on the analysis and control of nonlinear dynamical systems, see for example [47, 57, 107].

3.1 Stability Theory for Nonlinear Dynamical Systems

The stability of dynamical systems is one of the fundamental problems studied in control systems theory [11, 23, 41, 57, 68]. In this section, we review some technical tools which are utilized for investigating the stability of dynamical systems when its initial state is near an equilibrium state.

3.1.1 Basic Stability Notions

Consider the following autonomous dynamical system

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{3.1}$$

where $x \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a locally Lipschitz function. A point x_e is said to be an equilibrium point of (3.1) if $f(x_e) = \mathbf{0}$. We shall state the main definitions and theorems of stability of the equilibrium point (3.1) when it is at origin of the space $\mathbb{R}^n$, that is $x_e = \mathbf{0}$. If the equilibrium point $x_e \neq \mathbf{0}$, the stability analysis still can be performed by shifting it to the origin via a change of variables. The following definition characterizes the stability properties of the equilibrium point $x_e = \mathbf{0}$.

Definition 3.1 (**Stability of an Equilibrium Point** [**57**]). *The equilibrium point $x_e = \mathbf{0}$ of (3.1) is*

(i) stable (equivalently, Lyapunov stable) if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall\ t \geq 0; \tag{3.2}$$

(ii) unstable if it is not stable;

(iii) asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = \mathbf{0}. \tag{3.3}$$

The above Definition states that the equilibrium point x_e of the system

(3.1) is stable if all solutions which start nearby to x_e (in an ϵ-neighborhood of x_e) remain nearby (in a δ-neighborhood of x_e), otherwise it will be unstable. The equilibrium point is said to be asymptotically stable if it is stable and, furthermore, all solutions starting near to the equilibrium point x_e tend to x_e as $t \to \infty$.

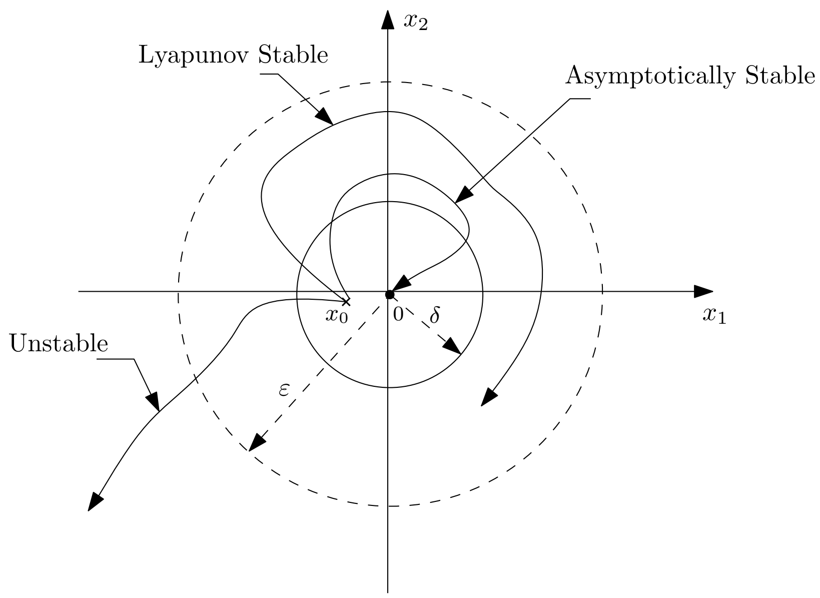

Figure 3.1: Visualization of the basic stability notions.

3.1.2 Lyapunov Stability Theory

In his seminal work entitled *"The General Problem of Stability of Motion"*, the Russian mathematician Aleksandr Mikhailovich Lyapunov introduced a method to analyze the stability of nonlinear dynamical systems without finding the system trajectories [68]. This approach depends on construction of a continuously differentiable positive definite function of the system's state, such a function is called a Lyapunov function. If the rate of

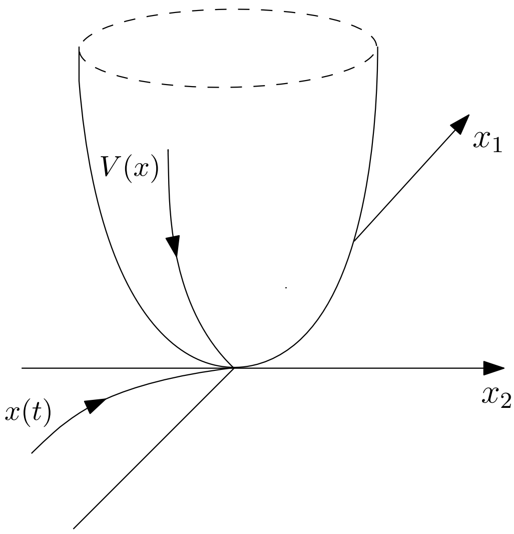

Figure 3.2: Graphical illustration of Lyapunov functions.

change of the Lyapunov function along the system trajectories is negative, this means that the system loses energy and will eventually come to rest.

Here, we review the main Lyapunov stability results needed for developing our results in Chapter 4 and Chapter 5. The Lyapunov idea is formalized in the next two (Lyapunov stability) theorems, in which sufficient conditions for the stability of the equilibrium point $x_e = 0$ are given.

Theorem 3.1 (Lyapunov's Theorem [47]). *Consider the system* (3.1) *with an equilibrium point* $x_e = \mathbf{0}$. *Let* $V : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuously differentiable function such that*

$$V(\mathbf{0}) = 0 \quad \textit{and} \quad V(x) > 0, \quad x \in \mathbb{R}^n - \{\mathbf{0}\}$$

$$\dot{V}(x) \leq 0, \quad x \in \mathbb{R}^n$$

then $x_e = 0$ *is stable. Moreover, if*

$$\dot{V}(x) < 0, \quad x \in \mathbb{R}^n - \{\mathbf{0}\} \tag{3.4}$$

then $x_e = \mathbf{0}$ *is asymptotically stable.*

The next theorem establishes the global asymptotic stability of the system (3.1) by imposing an extra condition that $V(\cdot)$ is radially unbounded, that is it satisfies $V(x) \to \infty$ as $x \to \infty$.

Theorem 3.2. *[47] Consider the system* (3.1) *with an equilibrium point* $x_e = \mathbf{0}$. *Let* $V : \mathbb{R}^n \to \mathbb{R}^n$ *be a continuously differentiable function such that*

$$V(\mathbf{0}) = 0 \quad \textit{and} \quad V(x) > 0, \quad x \in \mathbb{R}^n - \{\mathbf{0}\};$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty;$$

$$\dot{V}(x) < 0, \quad x \in \mathbb{R}^n - \{\mathbf{0}\}.$$

Then $x_e = \mathbf{0}$ *is globally asymptotically stable.*

3.1.3 LaSalle Invariance Principle

The asymptotic stability of the system (3.1) is established by Theorem 3.1 when $\dot{V}$ is negative definite along the trajectories of the system. It often happens that $\dot{V}$ is only negative semi-definite, not negative definite. In this case, LaSalle's invariance theorem, is utilized to guarantee the asymptotic stability of the system (3.1) and thus extends the applicability of Theorem 3.1. The LaSalle's invariance theorem is given in the following theorem.

Theorem 3.3. *[57] Consider the dynamical system* (3.1). *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable positive definite function such that* $\dot{V}(x) \leq 0$ *in* $\mathbb{R}^n$. *Let* $E := \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ *and let* M *be the largest invariant set in* E. *Then, all the solutions of* (3.1) *are bounded and approach* M *as* $t \to \infty$.

We conclude this section by stating the next two corollaries which provide a generalization of Theorem 3.1and Theorem 3.2, respectively.

Corollary 3.1. *[57] Let $x_e = 0$ be an equilibrium point for the system* (3.1). *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable positive definite function such that $\dot{V}(x) \leq 0$ in $\mathbb{R}^n$. Assume that the set $E := \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ contains no invariant set other than the origin. Then, the origin is asymptotically stable.*

Corollary 3.2. *[57] Let $x_e = 0$ be an equilibrium point for system* (3.1). *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable, radially unbounded, positive definite function such that $\dot{V}(x) \leq 0$ in $\mathbb{R}^n$. Assume that the set $E := \{x \in \mathbb{R}^n : \dot{V}(x) = 0\}$ contains no invariant set other than the origin. Then, the origin is globally asymptotically stable.*

3.2 Dissipativity and Passivity Analysis

Another fundamental theoretical framework in the area of analysis and design of control systems is the dissipativity and passivity theory [50, 110, 111]. This theory has been employed in many applications such as large space structures [10], multi-agent systems [25], and cyber-physical systems [6, 62]. In this section, we review some of the basic approaches used to characterizing the dissipativity/passivity property for a nonlinear dynamical system, described by the following state-space representation affine in the input:

$$\dot{x}(t) = f(x) + g(x)u, \tag{3.5}$$

$$y(t) = h(x) + k(x)u \tag{3.6}$$

where x, u, and y are from finite-dimensional real Euclidean spaces. The involved function $f(\cdot), G(\cdot), h(\cdot)$, and $k(\cdot)$ are real-valued functions of x with appropriate dimensions and $f(0) = h(0) = 0$. It is assumed that these function satisfy the standard conditions for the existence and uniqueness of the solutions [50].

3.2.1 Dissipative Systems

The dissipation of energy is a very common phenomenon in many real-world physical systems [17]. Typical examples of dissipative systems are the electrical circuits, in which the supplied energy is partially dissipated as heat in the resistors. The key mathematical foundation in developing dissipativity theory for general nonlinear dynamical systems was presented by J. C. Willems [110, 111] using input-output properties based on energy-related considerations. In particular, Willems [110] introduced the definition of dissipativity for general dynamical systems in terms of a dissipation inequality involving two energy-like functions: *the storage function*, which is the energy stored by the system and the *supply function*, which represents the energy entered to the system from the external environment. The dissipation inequality implies that any increase in the stored energy over a given time interval cannot exceed the external energy delivered to the system during this time interval.

To begin with the definition in [111] of the dissipativity for the nonlinear dynamical system (3.5), (3.6), we consider a function, called the *supply function*, denoted by $w(u(s), y(s))$, with $w : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, satisfying

$$\int_0^t |w(u(s), y(s))| ds < \infty, \quad \forall\ t \geq 0. \tag{3.7}$$

Definition 3.2 (**Dissipative System, [111]**). *The nonlinear system* (3.5), (3.6) *is said to be dissipative with respect to supply function $w(u(t), y(t))$ if there exists a nonnegative real-valued smooth function called the storage function $V(x) \geq 0$ such that that the following dissipation inequality holds:*

$$V(x(t)) \leq V(x(0)) + \int_0^t w(u(s), y(s))ds \tag{3.8}$$

along all possible trajectories of the system starting at $x(0)$, for all $x(0), t \geq 0$.

The above definition of dissipativity requires existence of a possible storage function such that (3.8) holds. In [111], it was shown that the storage function is bounded from below by the available storage and from above by the required supply, which are defined as follows,

Definition 3.3 (**Available Storage [17]**). *The available storage $V_a(x)$ of the system* (3.5), (3.6) *is given by*

$$V_a(x) = \sup_{x=x(0),\ u(\cdot),\ t\geq 0} -\int_0^t w(u(s), y(s))ds \tag{3.9}$$

where $V_a(x)$ is the maximum amount of energy which can be extracted from the system with initial state $x = x(0)$.

Definition 3.4 (**Required Supply [17]**). *The required supply $V_r(x)$ of the system* (3.5), (3.6) *is given by*

$$V_r(x) = \inf_{u(\cdot),\ t\geq 0} -\int_{-t}^0 w(u(s), y(s))ds \tag{3.10}$$

where $V_r(x)$ is the is the required amount of energy to be injected in the system to go from $x(-t)$ to $x(0)$.

Both the available storage and required supply functions satisfy the dissipation inequality (3.8). Moreover, the class of possible storage functions is a convex set, and thus there exist a continuum of possible storage functions ranging between the available storage and the required supply. This is made precise in the following theorem:

Theorem 3.4. *[111, 112] The system* (3.5), (3.6) *is dissipative in the sense of Definition 3.2 if and only if the required supply satisfies* $V_r(x) \geq -K > -\infty$ *for all* $x \in \mathbb{X}$ *and some* $K \in \mathbb{R}$. *Moreover,* $0 \leq V_a(x) \leq V(x) \leq V_r(x)$ *for all* $x \in \mathbb{X}$ *for dissipative systems.*

The above approach of characterizing the dissipativity property of nonlinear systems assumes that a storage function exists. In [50], Hill and Moylan introduced another definition of dissipativity for a dynamical system where the existence of a storage function of the system's state is not required. More clearly, we have the following definition from [50].

Definition 3.5. *The system* (3.5), (3.6) *is dissipative with respect to the supply function* $w(u, y)$ *if for all admissible* $u(\cdot)$ *and all* $t > 0$

$$\int_0^t w(u(s), y(s))ds \geq 0. \tag{3.11}$$

with $x(0) = 0$ *and along the trajectories of the system.*

Remark 3.1. It is worth noting that the above definitions of dissipativity continue to hold in the case where no physical energy interpretation is available.

We conclude this subsection by the following stability result of interconnected dissipative systems. Consider two nonlinear systems H_1 and

H_2 of the form (3.5), (3.6), that are given by

$$\dot{x}_i(t) = f_i(x_i) + g_i(x_i)u_i, \tag{3.12}$$

$$y_i(t) = h_i(x_i) + k_i(x_i)u_i \tag{3.13}$$

for $i = 1,2$, where $f(0) = h(0) = 0$. The two systems are assumed to be connected in a negative feedback interconnection where $u_1 = -y_2$ and $u_2 = y_1$. Also, it is assumed that the feedback system is well defined; that is,

$$I + k_2(x_2)k_l(x_1) \text{ to be nonsingular }, \; \forall \; x_1, x_2.$$

Then we have the following theorem from [51].

Theorem 3.5. *Suppose that two subsystems H_1 and H_2 are dissipative with respect to the supply functions*

$$w_i\,(u_i, y_i) = y_i^\top Q_i y_i + 2y_i^\top S_i u_i + u_i^\top R_i u_i, \quad i = 1,2,$$

where Q, S, and R are constant matrices with Q and R symmetric. Then the negative feedback interconnection of H_1 and H_2 is stable (asymptotically stable) if the matrix

$$\hat{Q} = \begin{bmatrix} Q_1 + \alpha R_2 & -S_1 + \alpha S_2^\top \\ -S_1^\top + \alpha S_2 & R_1 + \alpha Q_2 \end{bmatrix}$$

is negative semi-definite (negative definite) for some $0 < \alpha \in \mathbb{R}$.

3.2.2 Passive Systems

A notable special class of dissipative systems are the *passive systems* [9, 17, 106]. The notion of passivity emerged from studying electrical networks where passive components are known to be stable and form stable feedback loops [4, 16]. Applications of passivity-based control methods are found in robotics [84, 98, 106], systems biology [8], large-scale systems analysis [50, 56], and chemical process control [116].

In the following, we briefly review some of the standard passivity definitions of the nonlinear dynamical system (3.5), and (3.6).

Definition 3.6 (**Passive Systems [57]**). *The system* (3.5), (3.6) *is passive if it is dissipative with supply function* $w(u(s), y(s)) = y^T u$*; i.e., there exists a storage function* $V(\cdot)$ *of the system's state such that*

$$V(x(t)) \leq V(x(0)) + \int_0^t y^T(s)u(s)ds, \quad \forall\ t \geq 0. \tag{3.14}$$

Definition 3.7 (**Strictly Passive Systems [57]**). *The system* (3.5), (3.6) *is strictly passive if it is dissipative with supply function* $w(u(t), y(t)) = y(t)^T u(t) - \psi(x)$*, and storage function* $V(x)$ *with* $V(0) = 0$*, such that*

$$V(x(t)) \leq V(x(0)) + \int_0^t u^T(s)y(s)ds - \int_0^t \psi(x(t))dt, \quad \forall\ t > 0, \tag{3.15}$$

where $\psi(x)$ *is a positive definite function. If the equality holds with* $\psi(x) \equiv 0$ *then the system is said to be lossless.*

Remark 3.2. For LTI system, the passivity property and the positive realness are equivalent notions. In [57], it was shown that an LTI system is passive (strictly passive, respectively) if and only if the system is positive real (strictly positive real, respectively).

The following definition concerns the general supply function which is useful to distinguish different types of strictly passive systems and will be useful in the Passivity Theorems [51] presented at the end of this chapter.

Definition 3.8 (**General Supply Function**). *Consider the system* (3.5), (3.6) *with supply function*

$$w(u,y) = y^T Qy + 2y^T Su + u^T Ru \tag{3.16}$$

with $Q = Q^T, R = R^T$. *If* $Q = 0,\ R = -\varepsilon I_m,\ \varepsilon > 0,\ S = \frac{1}{2} I_m$, *the system is said to be input strictly passive (ISP), i.e.*

$$\int_0^t y^T(s)u(s)ds \geq \beta + \varepsilon \int_0^t u^T(s)u(s)ds \tag{3.17}$$

If $R = 0,\ Q = -\delta I_m,\ \delta > 0,\ S = \frac{1}{2} I_m$, *the system is said to be output strictly passive* (OSP), *i.e.*

$$\int_0^t y^T(s)u(s)ds \geq \beta + \delta \int_0^t y^T(s)y(s)ds \tag{3.18}$$

If $Q = -\delta I_m,\ \delta > 0,\ R = -\varepsilon I_m,\ \varepsilon > 0,\ S = \frac{1}{2} I_m$, *the system is said to be very strictly passive (VSP), i.e.*

$$\int_0^t y^T(s)u(s)ds \geq \beta + \delta \int_0^t y^T(s)y(s)ds + \varepsilon \int_0^t u^T(s)u(s)ds. \tag{3.19}$$

A fundamental property of passive systems is that the parallel and feedback interconnection of (strictly) passive systems is again (strictly) passive, see [17]. The next result, known as the Passivity Theorem, concerns two connected passive systems $H_1 : u_1 \mapsto y_1$ and $H_2 : u_2 \mapsto y_2$ of the form (3.5), (3.6) where H_1 is in the feedforward path and H_2 is in the feedback path (*i.e.* $u_1 = -y_2$ and $u_2 = y_1$). The stability of the closed loop system of different types of passive systems is summarized in the following theorem.

Theorem 3.6. *[51] Assume that H_1 and H_2 are passive systems, then the* *feedback system is stable. Moreover, Asymptotic stability follows if, in addition, any one of the following (nonequivalent) conditions is satisfied:*

(I) One of H_1 and H_2 is VSP.

(II) Both H_1 and H_2 are ISP.

(III) Both H_1 and H_2 are OSP.

(IV) $H_1(-H_2)$ is zero-state detectable, and either

- *H_2 is ISP, or*
- *H_1 is OSP.*

(V) H_2H_1 is zero-state detectable, and either

- *H_2 is OSP,*
- *H_1 is ISP.*

Chapter 4

Lyapunov-Based Stability of Feedback Interconnections of Negative Imaginary Systems

The work, reported in this chapter, has been partially published in the following article:

Ahmed G. Ghallab, Mohamed A. Mabrok, and Ian R. Petersen (2017), *Lyapunov-based Stability of Feedback Interconnections of Negative Imaginary Systems*. IFAC-PapersOnLine 50 (1), 3424-3428.

4.1 Introduction

The stability of the feedback interconnection of negative imaginary systems is established in [60, 75, 90, 115]. However, the proofs of the stability results which is used in [75, 115] have a shortcoming due to a matrix invertibility issue for the case in which the plant has poles on the imaginary axis. In [115], Theorem 5 establishes the internal stability of a positive feedback interconnection comprising of a plant, with transfer function

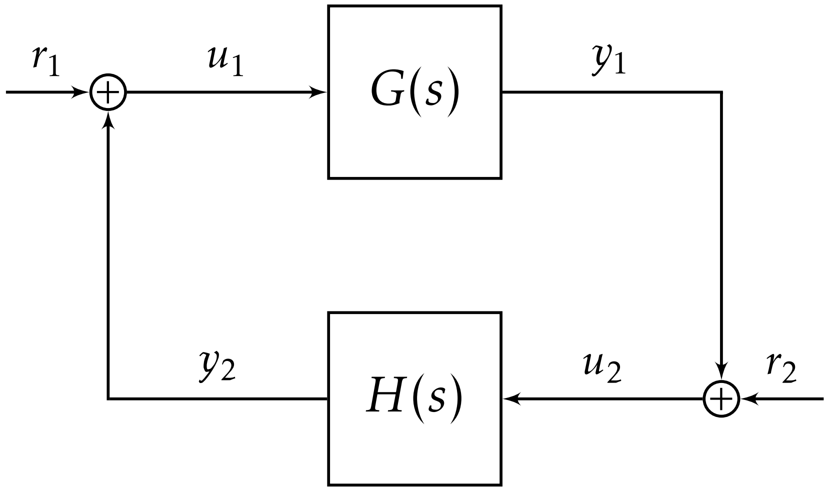

Figure 4.1: Feedback interconnection of NI systems. The plant is NI with transfer function $G(s)$ and the controller is SNI with transfer function $H(s)$

$G(s)$, and a controller with transfer function $H(s)$ (see Figure 4.1). However, the proof of Theorem 5 makes use of the following condition

$$\det(I - G(j\omega)H(j\omega)) \neq 0, \quad \omega \in (0, \infty)$$

which is not defined for values of the frequency ω corresponding to poles on the imaginary axis. The same issue appears in the proof of Theorem 5 in [75].

In this chapter, we use a Lyapunov-based stability approach to provide a correct proof of the result of [115]. We would like to remark that the result of [94] does provide a correct proof in the case of plant poles on imaginary axis but requires an extra condition in the definition of the NI property that a certain residue matrix is positive-definite. That extra condition is not required in our approach. We will consider the positive feedback interconnection of a linear NI system with a linear SNI system as shown in Figure 4.1. Consider a minimal state-space representation for the NI transfer function $G(s)$,

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u_1(t), \tag{4.1}$$

$$y_1(t) = C_1 x_1(t) + D_1 u_1(t), \tag{4.2}$$

where $A_1 \in \mathbb{R}^{n\times n}, B_1 \in \mathbb{R}^{n\times m}, C_1 \in \mathbb{R}^{m\times n}, D_1 \in \mathbb{R}^{m\times m}$.

Also, we consider a minimal state-space representation for the SNI transfer function $H(s)$,

$$\dot{x}_2(t) = A_2 x_2(t) + B_2 u_2(t), \tag{4.3}$$

$$y_2(t) = C_2 x_2(t) + D_2 u_2(t), \tag{4.4}$$

where $A_2 \in \mathbb{R}^{n\times n}, B_2 \in \mathbb{R}^{n\times m}, C_2 \in \mathbb{R}^{m\times n}, D_2 \in \mathbb{R}^{m\times m}$.

Since $G(s)$ is NI, Lemma 2.1 implies that there exists a symmetric matrix $P_1 > 0$, and a matrix L_1 such that

$$\begin{aligned} A_1 P_1^{-1} + A_1^T P_1^{-1} &= L_1^T L_1; \\ B_1 + A_1 P_1^{-1} C_1^T &= 0, \end{aligned} \tag{4.5}$$

which leads to the set of equations

$$\begin{aligned} P_1 A_1 + A_1^T P_1 &= -P_1 L_1^T L_1 P_1; \\ B_1^T P_1 - C_1 L_1^T L_1 P_1 &= C_1 A_1; \\ C_1 B_1 + (C_1 B_1)^T &= (L_1 C_1^T)^T (L_1 C_1^T). \end{aligned}$$

Also, since $H(s)$ is SNI, Lemma 2.2 implies there exists a symmetric matrix

$P_2 > 0$, and a matrix L_2 such that

$$
\begin{aligned}
A_2 P_2^{-1} + A_2^T P_2^{-1} &= L_2^T L_2; \\
B_2 + A_2 P_2^{-1} C_2^T &= 0,
\end{aligned}
\tag{4.6}
$$

which gives the following

$$
\begin{aligned}
P_2 A_2 + A_2^T P_2 &= -P_2 L_2^T L_2 P_2; \\
B_2^T P_2 - C_2 L_2^T L_2 P_2 &= C_2 A_2; \\
C_2 B_2 + (C_2 B_2)^T &= (L_2 C_2^T)^T (L_2 C_2^T).
\end{aligned}
$$

4.2 Preliminary result

The following lemma will be useful in establishing the Lyapunov-based stability of the positive feedback interconnection of $G(s)$ and $H(s)$. We show positive definiteness of a certain matrix which in turn will be used in constructing a Lyapunov function candidate of the closed-loop system.

Lemma 4.1. *Given a negative imaginary $G(s)$ and strictly negative imaginary $H(s)$. Assume that $G(\infty)H(\infty) = 0$ and $H(\infty) \geq 0$. Let P_1 and P_2 be the corresponding matrices defined in* (4.5) *and* (4.6). *Then, the matrix*

$$
\begin{bmatrix}
P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\
-C_2^T C_1 & P_2 - C_2^T D_1 C_2
\end{bmatrix}
$$

is positive definite if and only if $\lambda_{max}(G(0)H(0)) < 1$.

Proof. We have

$$
\begin{aligned}
&\lambda_{max}(G(0)H(0)) < 1 \\
&\quad \Leftrightarrow H(0)^{-1} - G(0) > 0 \\
&\quad \Leftrightarrow H(0)^{-1} - D - C_1 P_1 C_1^T > 0 \\
&\quad \Leftrightarrow \begin{bmatrix} P_1 & -C_1^T \\ C_1 & H(0)^{-1} - D_1 \end{bmatrix} > 0 \\
&\quad \Leftrightarrow H(0)^{-1} - D > 0, \quad \text{and} \\
&\quad\quad P_1 - C_1(H(0)^{-1} - D_1)^{-1} C_1 > 0 \\
&\quad \Leftrightarrow \lambda_{max}[D_1 H(0)] < 1, \quad \text{and} \\
&\quad\quad P_1 - C_1(H(0)^{-1} - D_1)^{-1}[D_2 + (H(0) - D_2)]C_1 > 0 \\
&\quad \Leftrightarrow \lambda_{max}[D_1 C_2 P_2^{-1} C_2^T] < 1, \quad \text{and} \\
&\quad\quad P_1 - C_1^T D_2 C_1 - C_1^T (I - H(0) D_1)^{-1}(H(0) - D_2) C_1 > 0 \\
&\quad \Leftrightarrow \lambda_{max}[P_2^{\frac{1}{2}} C_2^T D_1 C_2 P_2^{\frac{1}{2}}] < 1, \quad \text{and} \\
&\quad\quad P_1 - C_1^T D_2 C_1 - C^T (I - C_2 P_2 C_2^T D_1) C_2 P_2 C_2^T C_1 > 0 \\
&\quad \Leftrightarrow P_2 - C_2^T D_1 C_2 > 0, \quad \text{and} \\
&\quad\quad (P_1 - C_1^T D_2 C_1) - C_1^T C_2 (P_2 - C_2^T D_1 C_2)^{-1} C_2^T C_1 > 0 \\
&\quad \Leftrightarrow \begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 - C_2^T D_1 C_2 \end{bmatrix} > 0.
\end{aligned}
$$

□

4.3 Main results

In this section, we introduce the main result on the internal stability of the positive feedback interconnection of $G(s)$ and $H(s)$ as shown in Figure 4.1.

Theorem 4.1. *Assume that $G(s)$ is negative imaginary with minimal realization* (6.2), (4.2) *and $H(s)$ is strictly negative imaginary with minimal realization* (4.3), (4.4) *such that $G(\infty)H(\infty) = 0$ and $H(\infty) \geq 0$. Also, assume that $\lambda_{max}(G(0)H(0)) < 1$. Then, the positive feedback interconnection of $G(s)$ and $H(s)$ as in Figure 4.1 is internally stable.*

Proof. Let $V_1(x_1) = x_1^T P_1 x_1$ and $V_2(x_2) = x_2^T P_2 x_2$ and consider the function

$$\begin{aligned} V(x_1,x_2) &= V_1(x_1) + V_2(x_2) - 2y_1^T y_2 \\ &= \begin{bmatrix} x_1^T & x_2^T \end{bmatrix} \begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 - C_2^T D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

as a Lyapunov candidate for the closed-loop system. Note that it follows from Lemma 4.1 that the function $V(x_1,x_2)$ is positive definite. Now for the closed loop system we have

$$\begin{aligned} \dot{V}(x_1,x_2) = &\begin{bmatrix} \dot{x}_1^T & \dot{x}_2^T \end{bmatrix} \begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 - C_2^T D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &+ \begin{bmatrix} x_1^T & x_2^T \end{bmatrix} \begin{bmatrix} P_1 - C_1^T D_2 C_1 & -C_1^T C_2 \\ -C_2^T C_1 & P_2 - C_2^T D_1 C_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \end{aligned}$$

$$= x_1^T P_1 \dot{x}_1 + \dot{x}_1^T P_1 x_1 - 2\dot{x}_1^T C_1^T D_2 C_1 x_1 - 2\dot{x}_2^T C_2^T C_1 x_1$$

$$- 2\dot{x}_1^T C_1^T C_2 x_2 + \dot{x}_2^T P_2 x_2 + x_2^T P_2 \dot{x}_2 - 2x_2^T C_2^T D_1 C_2 \dot{x}_2$$

$$= (x_1^T A_1^T + u_1^T B_1^T) P_1 x_1 + x_1^T P_1 (A_1 x_1 + B_1 u_1)$$

$$+ (x_2^T A_2^T + u_2^T B_2^T) P_2 x_2 + x_2^T P_2 (A_2 x_2 + B_2 u_2)$$

$$- 2(\dot{y}_1^T - \dot{u}_1^T D_1^T) D_2 (y_1 - D_1 u_1)$$

$$- 2(\dot{y}_2^T - \dot{u}_2^T D_2^T)(y_1 - D_1 u_1)$$

$$- 2(\dot{y}_1^T - \dot{u}_1^T D_1^T)(y_2 - D_2 u_2)$$

$$- 2(\dot{y}_2^T - \dot{u}_2^T D_2^T) D_1 (y_2 - D_2 u_2)$$

$$= x_1^T (A_1^T P_1 + P_1 A_1) x_1 + x_2^T (A_2^T P_2 + P_2 A_2) x_2$$

$$+ 2u_1^T B_1^T P_1 x_1 + 2u_2^T B_2^T P_2 x_2$$

$$- 2\dot{y}_1^T D_2 y_1 - 2\dot{y}_2^T y_1 + 2\dot{y}_2^T D_1 u_1 + 2\dot{u}_2^T D_2^T y_1$$

$$+ 2\dot{y}_1^T D_2 u_2 + 2\dot{u}_1^T D_1^T y_2 - 2\dot{y}_2^T D_1 y_2 - 2\dot{y}_1^T y_2$$

$$= -x_1^T P_1 L_1^T L_1 P_1 x_1 - x_2^T P_2 L_2^T L_2 P_2 x_2 - 2\dot{y}_2^T y_1 - 2\dot{y}_1^T y_2$$

$$+ 2u_1^T (B_1^T P_1 - C_1 L_1^T L_1 P_1) x_1 - 2u_1^T C_1 L_1^T L_1 P_1 x_1$$

$$+ 2u_2^T (B_2^T P_2 - C_2 L_2^T L_2 P_2) x_2 - 2u_2^T C_2 L_2^T L_2 P_2 x_2$$

$$= -x_1^T P_1 L_1^T L_1 P_1 x_1 - x_2^T P_2 L_2^T L_2 P_2 x_2 - 2\dot{y}_2^T y_1 - 2\dot{y}_1^T y_2$$

$$+ 2u_1^T (C_1 A_1 x_1 + C_1 B_1 u_1) - 2u_1^T C_1 B_1 u_1$$

$$
\begin{aligned}
&+2u_2^T(C_2A_2x_2+C_2B_2u_2)-2u_2^TC_2B_2u_2\\
&-2u_1^TC_1L_1^TL_1P_1x_1-2u_2^TC_2L_2^TL_2P_2x_2\\
=&-x_1^TP_1L_1^TL_1P_1x_1-x_2^TP_2L_2^TL_2P_2x_2-2\dot{y}_2^Ty_1-2\dot{y}_1^Ty_2\\
&+2u_1^T\dot{y}_1-2u_1^TC_1B_1u_1-2u_1^TC_1L_1^TL_1P_1x_1\\
&+2u_2^T\dot{y}_2-2u_2^TC_2B_2u_2-2u_2^TC_2L_2^TL_2P_2x_2\\
=&-(L_1P_1x_1-L_1C_1^Tu_1)^T(L_1P_1x_1-L_1C_1^Tu_1)\\
&\qquad-(L_2P_2x_2-L_2C_2^Tu_2)^T(L_2P_2x_2-L_2C_2^Tu_2)
\end{aligned}
$$

where we used the feedback equations $u_1 = y_2$ and $u_2 = y_1$, and

$$2u_i^TC_iB_iu_i = u_i^T(C_iB_i+(C_iB_i)^T)u_i = u_i^TC_iL_i^TL_iC_i^Tu_i.$$

Define $\tilde{y}_i = L_iP_ix_i - L_iC_i^Tu_i$, for $i = 1,2$. Then,

$$\dot{V}(x_1,x_2) = -\tilde{y}_1^T\tilde{y}_1 - \tilde{y}_2^T\tilde{y}_2 \leq -\tilde{y}_2^T\tilde{y}_2 \leq 0. \tag{4.7}$$

This implies that the closed loop systems is at least Lyapunov-stable; *i.e.*, the closed loop system poles can only be the closed left half of the complex plane. We now show that the closed loop system matrix has no eigen values on the imaginary axis. The closed loop matrix for the systems is

$$\breve{A} = \begin{bmatrix} A_1+B_1D_2C_1 & B_1C_2 \\ B_2C_1 & A_2+B_2D_1C_2 \end{bmatrix}.$$

Suppose that this matrix has an eigenvalue on the $j\omega$-axis. Then there exists a nonzero $x = \begin{bmatrix} x_1^T & x_2^T \end{bmatrix}^T$ such that

$$\begin{bmatrix} A_1 - j\omega I + B_1 D_2 C_1 & B_1 C_2 \\ B_2 C_1 & A_2 - j\omega I + B_2 D_1 C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0,$$

for $\omega \in \mathbb{R}$. So, we have

$$(A_1 - j\omega I + B_1 D_2 C_1)x_1 + B_1 C_2 x_2 = 0, \tag{4.8}$$

and

$$B_2 C_1 x_1 + (A_2 - j\omega I + B_2 D_1 C_2)x_2 = 0. \tag{4.9}$$

Then, we have

$$(j\omega I - A_1)x_1 - B_1 y_2 = 0, \tag{4.10}$$

and

$$(j\omega I - A_2)x_2 - B_2 y_1 = 0, \tag{4.11}$$

where we used the state-space equations (4.2), (4.4) and the equations $u_1 = y_2$ and $u_2 = y_1$.

Integrating (4.7), we get

$$-V(0) \le V(t) - V(0) \le -\int_0^t \tilde{y}_2^T(s)\tilde{y}_2(s)ds. \tag{4.12}$$

Then

$$\int_0^t \tilde{y}_2^T(s)\tilde{y}_2(s)ds \le V(0), \tag{4.13}$$

which implies

$$\tilde{y}_2 = L_2 P_2 x_2 - L_2 C_2^T u_2 = 0. \tag{4.14}$$

Combining (4.11), (4.14) in a matrix equation form we get

$$\begin{bmatrix} A_2 - j\omega I & B_2 \\ L_2 P_2 & -L_2 C_2^T \end{bmatrix} \begin{bmatrix} x_2 \\ u_2 \end{bmatrix} = 0.$$

Since the matrix on the left has full rank for $\omega \in (0, \infty)$, it follows that $x_2 = u_2 = 0$ and hence $y_1 = y_2 = 0$. This implies $C_1 x_1 = 0, x_1 \neq 0$; *i.e.*, (A, C) is non-observable, which contradicts the minimality of the system. Therefore, $\breve{A}$ is semistable (*i.e.* $j\omega \notin \mathrm{spec}(\breve{A})$, for nonzero $\omega \in \mathbb{R}$).

Now suppose that the matrix $\breve{A}$ has an eigenvalue at the origin ($\omega = 0$). From (4.10) we have

$$C_1 x_1 = -C_1 A_1^{-1} B_1 y_2 = (G(0) - D_1) y_2.$$

This implies

$$y_1 - D_1 u_1 = (G(0) - D_1) y_2,$$

and then

$$y_1 = G(0) y_2. \tag{4.15}$$

Similarly, from (4.11) we have

$$y_2 = H(0) y_1. \tag{4.16}$$

Combining (4.15) and (4.16) we get

$$y_1 = G(0) H(0) y_1. \tag{4.17}$$

Note that, from (4.11), if $y_1 = 0$ we get $x_2 = 0$, since A_2 is asymptotically stable and hence invertible. Also, we have $y_2 = C_2x_2 + D_2u_2 = 0$, and from (4.10) we have $x_1 = 0$ since A_1 has no have eigenvalue at the origin. That leads to $(x_1, x_2) = 0$ which is not allowed, thus y_1 must be nonzero. However, (4.17) contradicts with the DC gain condition $\lambda_{max}(G(0)H(0)) < 1$. Therefore, we have shown by contradiction that the closed loop system does not have eigenvalues on the imaginary axis. From that, we conclude that the feedback interconnection of $G(s)$ and $H(s)$ is internally stable.

□

The next corollary shows that the result in [115] for the internal stability of positive feedback interconnections of NI systems is still correct.

Corollary 4.1. *Given a NI transfer function matrix $G(s)$ and a SNI transfer function matrix $H(s)$ and assume that $G(\infty)H(\infty) = 0$ and $H(\infty) \geq 0$. Then, the feedback interconnection of $G(s)$ and $H(s)$ is internally stable if and only if $\lambda_{max}(G(0)H(0)) < 1$.*

Proof. This result follows from Theorem 1 and the necessity part of Theorem 5 of [115] for which a correct proof has already been given in [115].

□

4.4 Concluding Remarks

In this chapter, a positive feedback interconnection of NI system with transfer function $G(s)$ and SNI system with transfer function $H(s)$ is considered. A Lyapunov-based approach has been used to give a correct proof of the internal stability of the closed-loop system. A Lyapunov function has been constructed by making use of the dc loop gain condition $\lambda_{max}(G(0)H(0)) < 1$. The time derivative of this function has been shown to be negative semi-definite. Then the dc loop gain condition has been employed again to show that the closed-loop system matrix doesn't have poles on the imaginary axis which proves the internal stability of the closed-loop system.

Chapter 5

Extending Negative Imaginary Systems Theory to Nonlinear Systems

The work, reported in this chapter, has been partially published in the following article:

Ahmed G. Ghallab, Mohamed A. Mabrok, and Ian R. Petersen (2018), *Extending Negative Imaginary Systems Theory to Nonlinear Systems*. IEEE Conference on Decision and Control (CDC), pp 2348-2353.

5.1 Introduction

Many nonlinear systems that dissipate energy in a physical sense do not fall into the classical dissipativity framework. For example, certain systems with hysteretic behaviour are dissipative with supply rate $\dot{y}(t)u(t)$; that is, the derivative of the system output is involved in the supply rate $\dot{y}(t)$ instead of the output $y(t)$ as is the case in the classical dissipativity / passivity. Also, flexible structures with colocated force actuators and position sensors are passive/dissipative relative to the input and the derivative of the output. For more details see [43, 44, 86, 87].

Generally speaking, the classical passivity and dissipativity theory does not apply in a straightforward manner in the analysis and design of dissipative (passive) control systems where the supply rate involves derivatives of the input and outputs [5, 86]. In particular, systems which are passive from the input u to the derivative of the output $\dot{y}$ do not satisfy the sector condition for the classical passivity property. Adding to this, the presence of nonlinearities in these systems may lead to difficulties in analyzing system's performance.

In this regard, negative imaginary systems theory has been proven as an effective tool in the analysis and design of control systems for the class of LTI systems which are passive (positive real) from u to $\dot{y}$; e.g. see [58, 60, 90, 115]. The negative imaginary property can be interpreted in terms of dissipativity for the class of LTI systems, by saying an LTI NI system is dissipative/passive between the input and the derivative of the output. This time-domain definition of the NI property can be generalized to the general nonlinear case.

In this chapter, we extend the negative imaginary systems theory to nonlinear systems using a time-domain dissipativity framework. This enables us to extend some of the existing results from the linear negative imaginary systems framework to a broader class of nonlinear systems. In particular, the asymptotic stability of a positive feedback interconnection of nonlinear negative imaginary systems will be established under suitable assumptions using technical tools from Lyapunov and dissipativity theories.

5.2 Characterization of Nonlinear Negative Imaginary Systems

In this section, we characterize the negative imaginary property by using a time-domain dissipativity framework. We start by looking at the linear case and then generalize the notion to a broader class of nonlinear systems. Consider the following SISO negative imaginary system:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.1}$$

$$y(t) = Cx(t) \tag{5.2}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}$, and $C \in \mathbb{R}^{1\times n}$. The NI system (5.1), (5.2) can be characterized in terms of dissipation inequality according to the following lemma.

Lemma 5.1. *Suppose that (A,B,C) is minimal. Then, the system (5.1), (5.2) is NI if and only if there exists a nonnegative function V such that*

$$\dot{V}(x(t)) \leq \dot{y}(t)u(t) \tag{5.3}$$

for all $t \geq 0$, where $V = \frac{1}{2}x^T Px$, and $P = P^T > 0$ satisfies the following LMI:

$$\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix} \leq 0. \tag{5.4}$$

Proof. We have

$$\dot{V}(x(t)) \leq \dot{y}(t)u(t)$$

$$\Leftrightarrow$$

$$\frac{1}{2}x^T(PA + A^T P)x + x^T PBu \leq$$
$$u^T CAx + \frac{1}{2}u^T(CB + B^T C^T)u$$

$$\Leftrightarrow$$

$$\frac{1}{2}\begin{bmatrix} x^T & u^T \end{bmatrix}\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \leq 0.$$

This completes the proof .

□

Next, we aim to generalize the linear negative imaginary property by introducing a class of nonlinear dissipative systems known as the class of *nonlinear Negative Imaginary systems*. Consider the following general nonlinear system

$$\dot{x} = f(x, u) \tag{5.5}$$

$$y = h(x) \tag{5.6}$$

where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a Lipschitz continuous function and $h : \mathbb{R}^n \to \mathbb{R}$ is a class C^1 function.

Definition 5.1. *The system* (5.5), (5.6) *is **nonlinear negative imaginary** if there exists a nonnegative function $V : \mathbb{R}^n \to \mathbb{R}$ of class C^1 such that*

$$\dot{V}(x(t)) \leq \dot{y}(t)u(t), \tag{5.7}$$

for all $t \geq 0$. Here, the function $V(\cdot)$ is called the storage function.

For the purpose of designing a stable control system involving nonlinear NI systems, we introduce a subclass of nonlinear NI systems called marginally strict nonlinear NI systems following a similar argument as in [55].

Definition 5.2. *The system* (5.5), (5.6) *is said to be* ***a marginally strictly nonlinear NI*** *system if the dissipative inequality* (5.7) *is satisfied, and for all* $u(\cdot)$ *and* $x(\cdot)$ *such that*

$$\dot{V}(x(t)) = \dot{y}(t)u(t) \tag{5.8}$$

for all $t > 0$*, then* $\lim_{t\to\infty} u(t) = 0$.

We have also the following definition that gives a nonlinear analog to the strictly negative imaginary property of LTI systems.

Definition 5.3. *The system* (5.5), (5.6) *is said to be* ***a weak strictly nonlinear NI*** *system if it is marginally strict nonlinear NI and globally asymptotically stable with* $u(t) = 0$.

5.2.1 Relation between Nonlinear CCW Systems and Nonlinear NI Systems

Systems with the negative imaginary property are closely related to systems with the counter clockwise (CCW) input-output property. In [5, 86], a dynamical system with input u and output y has a counter-clockwise property if

$$\liminf_{T\to\infty} \int_0^T \dot{y}(t)u(t)dt > -\infty, \tag{5.9}$$

for each bounded pair $(u(t), y(t))$. The CCW property is defined for both the linear and nonlinear case. In the linear case, it has been shown for LTI systems that the CCW property is equivalent to the negative imaginary property [86]. In the nonlinear setup, we introduce the following lemma on the relation between the negative imaginary property and the CCW property.

Lemma 5.2. *If the system* (5.5), (5.6) *is nonlinear NI, then it is CCW.*

Proof. If the nonlinear system (5.5), (5.6) is nonlinear Negative Imaginary, then there exits a nonnegative function $V(\cdot)$ such that

$$\dot{V}(x(t)) \leq \dot{y}(t)u(t), \tag{5.10}$$

for all $t \geq 0$. Integrating both sides of (5.10) from 0 to T, we obtain

$$V(x(T)) - V(x(0)) \leq \int_0^T \dot{y}(t)u(t)dt. \tag{5.11}$$

Since $V(x(T)) \geq 0$, we conclude that

$$\liminf_{T\to\infty} \int_0^T \dot{y}(t)u(t)dt \geq V(x(0)) > -\infty. \tag{5.12}$$

□

Remark 5.1. The converse of this lemma has not been yet investigated as pointed out in [5], and is left here for potential future work.

5.3 Stability of Interconnected Nonlinear NI Systems

Here, we use a Lyapunov-based technique and an invariance principle to investigate the stability robustness of a positive feedback interconnec-

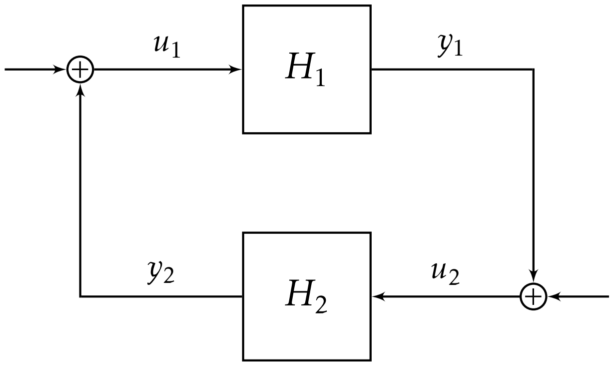

Figure 5.1: Feedback interconnection of nonlinear NI systems H_1 and H_2.

tion of two nonlinear Negative Imaginary systems H_1 and H_2 (see Figure 5.1) represented by

$$H_1\colon \quad \begin{cases} \dot{x}_1 = f_1(x_1, u_1) \\ y_1 = h_1(x_1) \end{cases} \quad \text{and} \quad H_2\colon \quad \begin{cases} \dot{x}_2 = f_2(x_2, u_2) \\ y_2 = h_2(x_2) \end{cases} \tag{5.13}$$

where $h_i : \mathbb{R}^n \to \mathbb{R}$ is a C^1 function with $h_i(0) = 0$, $f_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous and locally Lipschitz in x_i for bounded u_i, and where $f_i(0,0) = 0$.

5.3.1 Open-Loop System Result

We seek here to develop a nonlinear generalization of Lemma 4.1 in Chapter 4 which has been used (in the linear case) to find a Lyapunov function candidate of the feedback system comprising of NI and SNI systems. We consider the open-loop interconnection of systems H_1, and H_2 as shown in Figure 5.2. A Lyapunov function will be constructed for the purpose of investigating the robust stability of the feedback system shown in Figure 5.2.

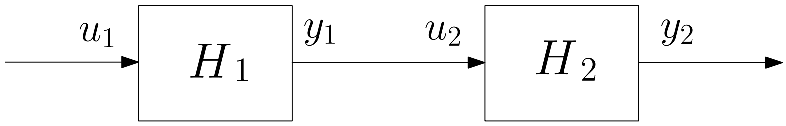

Figure 5.2: Open-loop interconnection of nonlinear NI systems H_1 and H_2.

Before stating our lemma, we make the following assumptions on the open-loop interconnection of H_1 and H_2.

Assumption 1. For any constant $\bar{u}_1$, there exists a unique solution $(\bar{x}_1, \bar{y}_1)$ to the equations

$$0 = f_1(\bar{x}_1, \bar{u}_1), \quad \bar{y}_1 = h_1(\bar{x}_1) \tag{5.14}$$

such that $\bar{u}_1 \neq 0$ implies $\bar{x}_1 \neq 0$ and the mapping $\bar{u}_1 \mapsto \bar{x}_1$ is continuous.

Assumption 2. For any constant $\bar{u}_2$, there exists a unique solution $(\bar{x}_2, \bar{y}_2)$ to the equations

$$0 = f_2(\bar{x}_2, \bar{u}_2), \quad \bar{y}_2 = h_2(\bar{x}_2). \tag{5.15}$$

Also, $\bar{u}_2 \neq 0$ implies $\bar{x}_2 \neq 0$ and the mapping $\bar{u}_2 \mapsto \bar{x}_2$ is continuous.

Assumption 3. $h_1(\bar{x}_1)h_2(\bar{x}_2) \geq 0$, for each $\bar{x}_1, \bar{x}_2$ as defined in Assumptions 1 and 2.

Assumption 4. There exits a constant $0 < \gamma < 1$ such that for any $\bar{u}_1$ and with $\bar{y}_2$ defined as in Assumption 2 the following sector bound condition:

$$\bar{u}_1 \bar{y}_2 \leq \gamma \bar{u}_1^2 \tag{5.16}$$

holds.

Lemma 5.3. *Referring to the open-loop interconnection as in Figure 5.2, suppose that H_1 is nonlinear NI with storage function $V_1(x_1) > 0$, and H_2 is weak*

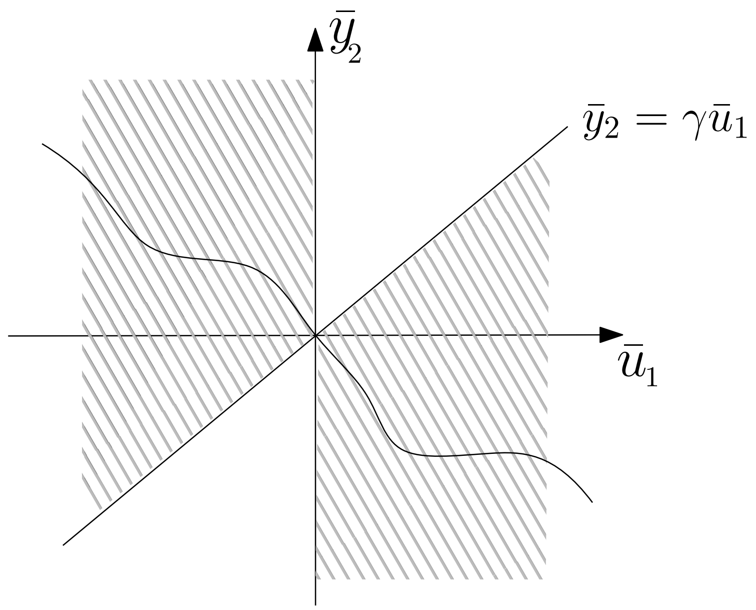

Figure 5.3: Graphical representation of the sector bound condition (5.16).

strict nonlinear NI with storage function $V_2(x_2) > 0$. Assume that the Assumptions 1 to 4 are satisfied. Then the function $W(x_1, x_2)$ defined as

$$W(x_1, x_2) := V_1(x_1) + V_2(x_2) - h_1(x_1)h_2(x_2), \tag{5.17}$$

is positive definite.

Proof. We have $W(0,0) = 0$. Fix any $x_1, x_2 \neq 0$, and
consider the following discrete iterations $\{\bar{u}_1^{[k]}\}$, $\{\bar{x}_1^{[k]}\}$, $\{\bar{y}_1^{[k]}\}$, $\{\bar{u}_2^{[k]}\}$, $\{\bar{x}_2^{[k]}\}$, such that

$$\bar{u}_1^{[1]} := h_2(x_2), \text{ and}$$
$$\bar{u}_1^{[k+1]} := \frac{1}{\gamma}\bar{y}_2^{[k]}, \ k = 1, 2, 3, \ldots$$

We have two cases:

Case 1) if $\bar{u}_1 = h_2(x_2) = 0$, then

$$W(x_1, x_2) = V_1(x_1) + V_2(x_2) > 0,$$

since $(x_1, x_2) \neq 0$ and $V_1(\cdot), V_2(\cdot)$ are positive definite.

Case 2) if $\bar{u}_1 = h_2(x_2) \neq 0$, then (5.16) implies $\bar{u}_1^{[k]}\bar{u}_1^{[k+1]} \leq (\bar{u}_1^2)^{[k]}$, that is;

$$\bar{u}_1^{[k]}(\bar{u}_1^{[k+1]} - \bar{u}_1^{[k]}) \leq 0.$$

When $\bar{u}_1^{[k]} \geq 0$ and $\bar{u}_1^{[n+1]} - \bar{u}_1^{[k]} \leq 0$, that is; $\bar{u}_1^{[k+1]} \leq \bar{u}_1^{[k]}$, then the sequence $\{\bar{u}_1^{[k]}\}$ converges to zero. On the other hand, if $\bar{u}_1^{[k]} \leq 0$ and $\bar{u}_1^{[k+1]} - \bar{u}_1^{[k]} \geq 0$, that is; $\bar{u}_1^{[k+1]} \geq \bar{u}_1^{[k]}$, it follows that $\{\bar{u}_1^{[k]}\}$ converges to zero. By the continuity of the map $\bar{u}_1 \mapsto \bar{x}_1$, we conclude that $\bar{x}_1^{[k]} \to 0$. Hence, $\bar{y}_1^{[k]} \to 0$, and since $\bar{y}_1 = \bar{u}_2$, the iteration $\{\bar{u}_2^{[k]}\}$ should converge to zero. By the continuity of the map $\bar{u}_2 \mapsto \bar{x}_2$, we have also $\bar{x}_2^{[k]} \to 0$.

Now we seek to find a lower bound for the function W. Each subsystem of the open loop interconnection satisfies

$$\dot{V}_i(x_i(t)) \leq \dot{y}_i(t)u_i(t) \tag{5.18}$$

For any constant inputs $u_i(t) \equiv u_i$ and integrating (5.18), we get

$$V_i(x_i(t)) - h_i(x_i(t))u_i \geq V_i(\xi_i) - h(\xi_i)u_i, \tag{5.19}$$

for all x_i, $\xi_i \in \mathbb{R}^n$. Let $u_1 = \bar{u}_1^{[1]} := h_2(x_2)$ and $\xi_1 = \bar{x}_1^{[1]}$ in (5.19) for the system H_1. Then

$$V_1(x_1) - h_1(x_1)h_2(x_2) \geq V_1(\bar{x}_1^{[1]}) - h_1(\bar{x}_1^{[1]})h_2(x_2).$$

It follows that

$$W(x_1,x_2) \geq V_1(\bar{x}_1^{[1]}) - h_1(\bar{x}_1^{[1]})h_2(x_2) + V_2(x_2).$$

Let $u_2 = \bar{u}_2^{[1]} = h_1(\bar{x}_1^{[1]})$ and $\xi_2 = \bar{x}_2^{[1]}$ in (5.19) for the system H_2. Then we get

$$V_2(x_2) - h_2(x_2)h_1(\bar{x}_1^{[1]}) \geq V_2(\bar{x}_2^{[1]}) - h_2(\bar{x}_2^{[1]})h_1(\bar{x}_1^{[1]}).$$

This leads to

$$W(x_1,x_2) \geq V_1(\bar{x}_1^{[1]}) + V_2(\bar{x}_2^{[1]}) - h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}).$$

From Assumption 3, we have three cases: Case i) $h_1(\bar{x}_1^{[1]}) = 0$ and $h_2(\bar{x}_2^{[1]}) \neq 0$. We have

$$W(x_1,x_2) \geq V_1(\bar{x}_1^{[1]}) + V_2(\bar{x}_2^{[1]}) \geq V_1(\bar{x}_1^{[1]}) > 0, \text{as } \bar{x}_1^{[1]} \neq 0.$$

Case ii) $h_1(\bar{x}_1^{[1]}) \neq 0$ and $h_2(\bar{x}_2^{[1]}) = 0$. In this case,

$$W(x_1,x_2) \geq V_1(\bar{x}_1^{[1]}) + V_2(\bar{x}_2^{[1]}) \geq V_2(\bar{x}_2^{[1]}) > 0, \text{as } \bar{x}_2^{[1]} \neq 0.$$

Case iii) $h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}) > 0$. In this case,

$$\begin{aligned} W(x_1,x_2) &\geq V_1(\bar{x}_1^{[1]}) + V_2(\bar{x}_2^{[1]}) - h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}) \\ &= V_1(\bar{x}_1^{[1]}) + V_2(\bar{x}_2^{[1]}) - \frac{1}{\gamma}h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}) \\ &\quad + (\frac{1}{\gamma} - 1)h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}). \end{aligned}$$

Let $u_1 = \bar{u}_1^{[2]} = \frac{1}{\gamma}h_2(\bar{x}_2^{[1]})$ and $\xi_1 = \bar{x}_1^{[2]}$ in (5.19) for the system H_1. Then we get

$$V_1(\bar{x}_1^{[1]}) - \frac{1}{\gamma}h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}) \geq V_1(\bar{x}_1^{[2]}) - \frac{1}{\gamma}h_1(\bar{x}_1^{[2]})h_2(\bar{x}_2^{[1]}),$$

which in turn leads to

$$\begin{aligned} W(x_1, x_2) \geq V_1(\bar{x}_1^{[2]}) - \frac{1}{\gamma}h_1(\bar{x}_1^{[2]})h_2(\bar{x}_2^{[1]}) + V_2(\bar{x}_2^{[1]}) + \\ + (\frac{1}{\gamma} - 1)h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}). \end{aligned}$$

Let $u_2 = \bar{u}_2^{[2]} = h_1(\bar{x}_1^{[2]})$ and $\xi_2 = \bar{x}_2^{[2]}$ in (5.19) for the system H_2. Then we get

$$V_2(\bar{x}_2^{[1]}) - \frac{1}{\gamma}h_2(\bar{x}_2^{[1]})h_1(\bar{x}_1^{[2]}) \geq V_2(\bar{x}_2^{[2]}) - \frac{1}{\gamma}h_1(\bar{x}_1^{[2]})h_2(\bar{x}_2^{[2]}),$$

which leads to

$$\begin{aligned} W(x_1, x_2) \geq V_1(\bar{x}_1^{[2]}) + V_2(\bar{x}_2^{[2]}) - \frac{1}{\gamma}h_1(\bar{x}_1^{[2]})h_2(\bar{x}_2^{[2]}) \\ + (\frac{1}{\gamma} - 1)h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}). \end{aligned}$$

Repeating the above process, we obtain

$$\begin{aligned} W(x_1, x_2) \geq V_1(\bar{x}_1^{[k]}) + V_2(\bar{x}_2^{[k]}) - \frac{1}{\gamma}h_1(\bar{x}_1^{[k]})h_2(\bar{x}_2^{[k]}) \\ + (\frac{1}{\gamma} - 1)h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}), \end{aligned}$$

Letting $k \to \infty$, we conclude that

$$\begin{aligned} W(x_1, x_2) &= V_1(0) + V_2(0) - \frac{1}{\gamma}h_1(0)h_2(0) \\ &\quad + (\frac{1}{\gamma} - 1)h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}) \\ &= 0 + (\frac{1}{\gamma} - 1)h_1(\bar{x}_1^{[1]})h_2(\bar{x}_2^{[1]}) > 0. \end{aligned}$$

Therefore, the function W is positive definite for all nonzero x_1, x_2. This completes the proof.

□

Remark 5.2. The aforementioned Assumptions 1 to 4 reduce to the equivalent conditions of Lemma 4.1 in Chapter 4. To show this, for the open loop systems as in Figure 5.2, consider the case where H_1 and H_2 are two LTI NI systems represented by

$$H_i\colon \quad \begin{cases} \dot{x}_i = A_i x_i + B_i u_i \\ y_i = C_i x_i, \end{cases} \tag{5.20}$$

where $G_1(s)$ and $G_2(s)$ are transfer functions for the systems H_1 and H_2, respectively. We can see that Assumptions 1 and 2 hold trivially. Also, Assumption 3 amounts to the condition $G_2(\infty) > 0$. The sector bound condition (5.16) reduces to the DC-gain condition $\lambda_{max}(G_1(0)G_2(0)) < 1$ which can be seen from the following

$$\begin{aligned} \bar{u}_1\bar{y}_2 \le \gamma\bar{u}_1^2 &\Rightarrow \bar{u}_1\bar{y}_2 < \bar{u}_1^2 \quad \text{as} \;\; \gamma < 1 \\ &\Rightarrow \frac{\bar{u}_2}{\bar{y}_2} - \frac{\bar{y}_1}{\bar{u}_1} > 0 \\ &\Rightarrow G_2(0)^{-1} - G_1(0) > 0 \\ &\Rightarrow \lambda_{max}(G_1(0)G_2(0)) < 1. \end{aligned}$$

5.3.2 Closed-Loop System Stability

The next result establishes the asymptotic stability of the positive feedback interconnection of the nonlinear NI systems H_1 and H_2, as shown in Figure 5.1. The stability of the feedback system is investigated using a Lyapunov function candidate constructed in Lemma 5.3. The proof follows along similar lines to proof of Theorem 3 in [52] on the absolute stability of a feedback interconnection of a weak strict positive real system and nonlinear passive system. The corresponding closed-loop system of H_1 and H_2 can be represented with the following state space representation:

$$\dot{z}(t) := \rho(z(t)), \quad z(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \in \mathbb{R}^{2n} \tag{5.21}$$

where $\rho : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is locally Lipschitz and $\rho(0) = 0$.

Theorem 5.1. *Consider the positive feedback interconnection of systems H_1 and H_2 as in Figure 5.1. Suppose that the system H_1 is nonlinear NI and zero-state observable, and the system H_2 is weak strict nonlinear NI. Moreover, suppose that Assumptions 1 to 4 are satisfied. Then, the equilibrium point $z = 0$ of the corresponding closed-loop system* (5.21) *is asymptotically stable.*

Proof. Consider the function $W(x_1, x_2)$ as a Lyapunov function for the closed loop system (5.21). Differentiating W with respect to t and noting that $y_1 = u_2$ and $y_2 = u_1$, we get

$$\dot{W}(x_1, x_2) = \dot{V}_1(x_1) + \dot{V}_2(x_2) - \dot{y}_1 u_1 - \dot{y}_2 u_2 \leq 0.$$

The above inequality follows since the systems H_1 and H_2 are nonlinear NI, and the dissipation inequality $\dot{V}_i(x_i) \leq \dot{y}_i u_i$ holds for $i = 1, 2$. Hence, the closed-loop system of H_1 and H_2 is at least Lyapunov stable.

Next we show the asymptotic stability of the closed-loop system. Since the system H_2 is weak strict nonlinear NI, the system $\dot{x}_2 = f_2(x_2, 0)$ is a globally asymptotically stable. By using the result of [35], we have $\lim_{t\to\infty} x_2(t) =$

0 which in turn leads to $\lim_{t\to\infty} y_2(t) = 0$. For trajectories along which $\dot{W}(x_1, x_2) = 0$, we have

$$\dot{V}_2(x_2) - \dot{y}_2 u_2 = 0. \tag{5.22}$$

Thus, $\lim_{t\to\infty} u_2(t) = 0$ as H_2 is weak strict NI.

Now, let $\Omega(z_0)$ be an ω-limit set of a trajectory $z(t, z_0)$ with $\dot{W}(z) = W(x_1, x_2) = 0$. We show that $\Omega(z_0) = \{0\}$; *i.e.*, this set is a singleton. For any $\alpha \in \Omega(z_0)$, we write

$$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \in \mathbb{R}^{2n}. \tag{5.23}$$

Since $x_2(t) \to \infty$, $\alpha_2 = 0$. The limit set $\Omega(z_0)$ is an invariant set of the system (5.21). In other words, $z(t, \alpha) \in \Omega(z_0)$ for all $t \geq 0$. Then, $\alpha_2 = 0$ implies $z_2(t, \alpha) \equiv 0$ and

$$\dot{z}_1(t, \alpha) = f_1(z_1(t, \alpha), 0), \quad 0 \equiv h(z_1(t, \alpha)). \tag{5.24}$$

Since the system H_1 is zero-state observable, we conclude that $z_1(t, \alpha) \equiv 0$. Thus, we see $\alpha = 0$ and $\Omega(z_0) = \{0\}$. Hence, by Lemma 4.1 of [57], $z(t, z_0)$ approaches $\Omega(z_0) = \{0\}$ as $t \to \infty$. We have $W(z(t, z_0)) \equiv W(z_0) \to 0$, as $t \to \infty$ and hence $W(z_0) = 0$. It follows from the positive definiteness of W that $z_0 = 0$. It is concluded that any bounded trajectory $z(t)$ satisfying $\dot{W}(z) \equiv 0$ is the trajectory $z(t) \equiv 0$. Now from the Lasalle invariance principle, any bounded trajectory $z(t)$ tends to the origin. Therefore, the origin $z = 0$ is asymptotically stable.

□

5.4 Illustrative Example: Nonlinear Mass-Spring-Damper System

To illustrate the applicability of the above nonlinear NI stability result, we consider an example of nonlinear mass-spring-damper (MSD) system as shown in Figure 5.4. We aim at designing a controller using Theorem 5.1 to robustly stabilize the MSD system.

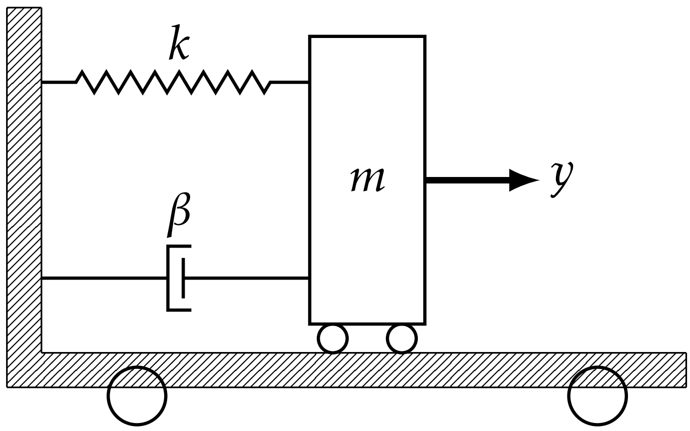

Figure 5.4: Nonlinear Mass-Spring-Damper System

In this spring-mass-damper system, the system is assumed to be nonlinear and obey the force law

$$f = k(x + x^3) \tag{5.25}$$

where f is the force applied to the spring and x is the displacement of the spring. Using Newton's second law, the dynamic equation of the mass-spring-damper system is described by

$$m\ddot{x} + \beta\dot{x} + k(x + x^3) = f(t) = u(t) \tag{5.26}$$

where m is the mass, β is the damper constant, k is the spring stiffness, x

is the position of the mass and f is the force acting on the mass. Define states $x_1 = x,\ x_2 = \dot{x}$, we obtain the following nonlinear state-space system

$$\begin{cases} \dot{x}_1 = x_2; \\ \dot{x}_2 = \frac{-k}{m}((x_1 + x_1^3)) - \frac{\beta}{m}x_2 + \frac{u(t)}{m}. \end{cases} \tag{5.27}$$

The output y of the considered mass-spring-damper system is the displacement x, that is

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus, the state space representation for the mass-spring-damper system is given by

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-k}{m}(x_1 + x_1^3) - \frac{\beta}{m}x_2 + \frac{u(t)}{m} \end{bmatrix}; \tag{5.28}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{5.29}$$

where the input of the system is the force f and the output is the displacement x. A natural Lyapunov function candidate for the mass-spring-damper system is the total energy of the system, given by

$$V(x(t), \dot{x}(t)) = \frac{1}{2}m\dot{x}^2 + kx(x + x^3). \tag{5.30}$$

The time derivative of this function along the system's trajectories satisfies

$$\dot{V}(x(t), \dot{x}(t)) \leq \dot{y}(t)u(t). \tag{5.31}$$

That is, the above mass-spring-damper system is nonlinear NI with a positive definite storage function $V(x,\dot{x})$. Now suppose the system is controlled with the SISO integral resonant controller

$$C(s) = \frac{\Gamma}{s + \Gamma\Phi} \tag{5.32}$$

where Γ, Φ are positive constants. The transfer function $C(s)$ is strictly negative imaginary [90]. We first check the assumptions of the open-loop interconnection required by Theorem 5.1. From the state equation (5.28) (setting $\beta = m = 1$ for simplicity), when $\dot{x} = 0$, we get

$$\bar{x}_2 = 0 \tag{5.33}$$

$$\bar{x}_1^3 + \bar{x} - \frac{\bar{u}}{k} = 0 \tag{5.34}$$

$$\bar{y} = \bar{x}_1 \tag{5.35}$$

Using the discriminant method to solve the cubic equation, since we have

$$\Delta = -4 - 27\left(\frac{\bar{u}}{k}\right)^2 < 0, \tag{5.36}$$

and therefore equation (5.33) has one real solution. Using Cardano's cubic formula, we find the equation has a unique real root which is

$$\bar{y} = \bar{x} = f(\bar{u}) = \sqrt[3]{\frac{\bar{u}}{2k} + \sqrt{\frac{1}{27} + \frac{\bar{u}^2}{4k^2}}} + \sqrt[3]{\frac{\bar{u}}{2k} - \sqrt{\frac{1}{27} + \frac{\bar{u}^2}{4k^2}}} \tag{5.37}$$

then we have

$$\bar{y}_c = \frac{1}{\Phi} f(\bar{u}) = \sqrt[3]{\frac{\bar{u}}{2k} + \sqrt{\frac{1}{27} + \frac{\bar{u}^2}{4k^2}}} + \sqrt[3]{\frac{\bar{u}}{2k} - \sqrt{\frac{1}{27} + \frac{\bar{u}^2}{4k^2}}} \tag{5.38}$$

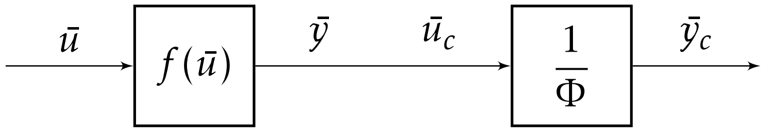

Figure 5.5: Open loop system ('c' refers to the controller)

Using a plotting tool, we can choose a value for the controller parameter Φ (for varying values of k) such that the sector bound condition

$$\bar{u}\bar{y}_c \leq \gamma \bar{u}^2 \tag{5.39}$$

is satisfied for $0 < \gamma < 1$ as shown in Figure 5.6.

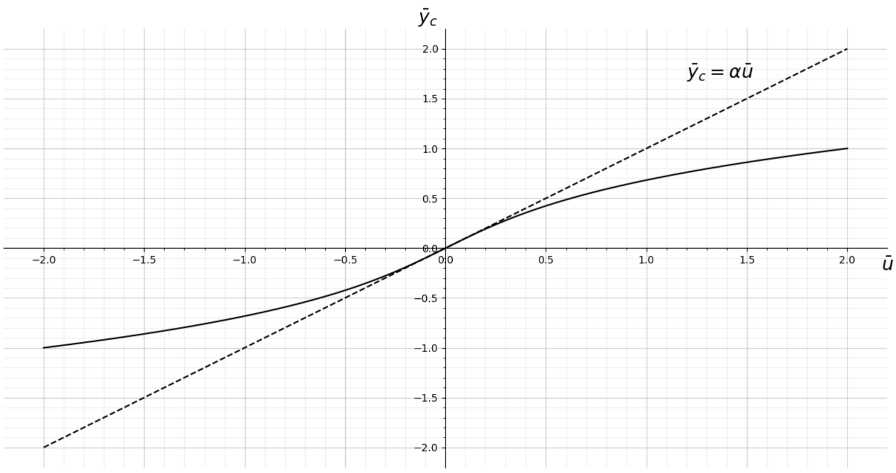

Figure 5.6: Sector bound condition (5.39) is satisfied.

5.5 Concluding Remarks

This Chapter generalizes the notion of negative imaginary systems to general nonlinear systems using a time-domain dissipativity framework and appropriate supply rate. The stability robustness of a positive feedback interconnection of nonlinear negative systems has been established. To achieve that, an open-loop connection of the subsystem has been considered to construct a Lyapunov function candidate for the closed-loop system. Then the time derivative of the constructed Lyapunov function has been shown to be negative semi-definite and by using the Lasalle invariance principle the asymptotic stability has been proved. This stability result has been shown to capture the result of the internal stability of a positive feedback interconnection of a negative imaginary plant with a strictly negative imaginary controller.

Chapter 6

Nonlinear Negative Imaginary System Theory for Systems with Free Motion

6.1 Introduction

The modelling of flexible structures with free body motion lead to models with poles at the origin. These systems arise in a number of practical applications including rotary cranes [45], dual-stage hard disk drives [31, 63, 104], robotics and flexible manipulators [24, 88], and rotating flexible spacecraft [53]. The notion of linear negative imaginary systems was extended in [74, 75] to include free body dynamics by allowing for up to two poles at the origin. This NI extension and related results allow for the analysis and feedback control of flexible structures with colocated force actuators and position sensors and with free body motion [75].

The main purpose of this chapter is to generalize some of the existing results on linear negative imaginary systems with poles at the origin to the nonlinear setting. In particular, we seek to extend the nonlinear negative imaginary notion presented in Chapter 5 to allow for the notion of a nonlinear negative imaginary systems containing a pure integration.

6.2 Preliminaries

Here, we recall the generalized definition of linear NI systems which allows for poles at the origin and the related generalized negative imaginary lemma. Consider the following LTI system:

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{6.1}$$

$$y(t) = Cx(t) + Du(t) \tag{6.2}$$

where the matrices $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. Assume that the system and has the $m \times m$ real-rational proper transfer function $G(s) := C(sI - A)^{-1}B + D$.

Definition 6.1. *[75] A square real-rational transfer function matrix $G(s)$ is negative imaginary if the following conditions are satisfied:*

1) $G(s)$ has no pole in $Re[s] > 0$.

2) For all $\omega > 0$, such that $j\omega$ is not a pole of $G(s)$, and $j(G(j\omega) - G(j\omega)^*) \geq 0$.

3) If $j\omega_0$, $\omega_0 \in (0,\infty)$, is a pole of $G(j\omega)$, it is at most a simple pole and the residue matrix $K_0 = \lim_{s\to j\omega_0}(s - j\omega_0)sG(s)$ is positive semi-definite Hermitian.

4) *If $s = 0$ is a pole of $G(s)$, then $\lim_{s\to\infty} s^k G(s)$ for all $k \geq 3$ and $\lim_{s\to\infty} s^2 G(s)$ is positive semi-definite Hermitian.*

The following lemma in [71], which is known as *the generalized negative imaginary lemma*, gives a state-space characterization of linear NI systems with free body motion.

Lemma 6.1. *[71] Let (A, B, C, D) be a minimal realization of a transfer function $G(s)$. Then $G(s)$ is NI if and only if $D = D^T$ and there exist matrices $P = P^T \geq 0$, $W \in \mathbb{R}^{m\times m}$, and $L \in \mathbb{R}^{m\times m}$ such that the following linear matrix inequality is satisfied*

$$\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix} = \begin{bmatrix} -L^T L & -L^T W \\ -W^T L & -W^T W \end{bmatrix} \leq 0. \quad (6.3)$$

Next, we seek to establish a time-domain dissipativity characterization of the notion of a linear NI system with free body motion.

Lemma 6.2. *Suppose that the system (6.1)-(6.2) (with $D = 0$) is controllable and observable. Then, $G(s)$ is negative imaginary if and only if there exists matrix P as in LMI (6.3) such that along the trajectories of the system, the function $V(x) = \frac{1}{2}x^T P x$ satisfies*

$$\dot{V}(x(t)) \leq \dot{y}(t)u(t), \quad \forall\ t \geq 0. \quad (6.4)$$

Proof. Differentiating the function V with respect to the time t, one has

$$\dot{V}(x(t)) = \frac{1}{2}x^T(PA + A^T P)x + x^T PBu.$$

Substituting into the dissipation inequality (6.4), we get

$$\frac{1}{2}x^T(PA+A^TP)x+x^TPBu \leq u^TCAx+\frac{1}{2}u^T(CB+B^TC^T)u.$$

In a matrix form, the above inequality is equivalent to

$$\frac{1}{2}\begin{bmatrix} x^T & u^T \end{bmatrix}\begin{bmatrix} PA+A^TP & PB-A^TC^T \\ B^TP-CA & -(CB+B^TC^T) \end{bmatrix}\begin{bmatrix} x \\ u \end{bmatrix} \leq 0.$$

By Lemma 2.1, the proof is complete.

□

6.3 Nonlinear Negative Imaginary System with a Single Integrator

In this section, we seek to establish a nonlinear generalization of the notion of NI systems with a pole at the origin. We consider a cascade interconnection of an input affine nonlinear system and a single integrator (as shown in Figure 6.1) represented by:

$$\Sigma : \quad \begin{cases} \dot{\eta} = f(\eta) + g(\eta)\xi; \\ \dot{\xi} = u; \\ y = h(\eta), \end{cases} \tag{6.5}$$

where $\eta \in \mathbb{R}^n$ and $u \in \mathbb{R}$ is the scalar control input. The functions $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $h : \mathbb{R}^n \to \mathbb{R}$, are continuously differentiable, $f(0) = 0$, and $g(\eta) \neq 0$ for all η.

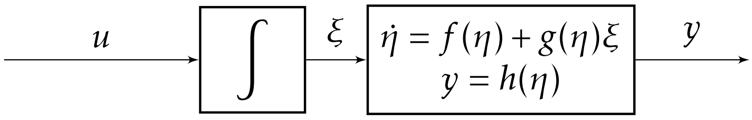

Figure 6.1: Cascade interconnection of an input affine nonlinear system with an integrator.

We want to show that the system (6.5) is nonlinear negative imaginary system, according to Definition 5.1 of Chapter 5, by finding a nonnegative storage function of the system states, denoted $V(\eta,\xi))$, such that

$$\dot{V}(\eta,\xi) \leq \dot{y}(t)u(t) = \nabla h(\eta)[f(\eta) + g(\eta)\xi]u, \quad \forall\ u,\xi,\eta, \tag{6.6}$$

that is,

$$\frac{\partial V}{\partial \eta}(f(\eta) + g(\eta)\xi) + \frac{\partial V}{\partial \xi}u \leq u\nabla h(\eta)f(\eta) + u\nabla h(\eta)g(\eta)\xi, \quad \forall\ u,\xi,\eta.$$

Rearranging the terms of the above inequality, it follows that this inequality is equivalent to the inequalities

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\xi] + u[\frac{\partial V}{\partial \xi} - \nabla h(\eta)f(\eta) - \nabla h(\eta)g(\eta)\xi] \leq 0, \quad \forall\ u,\xi,\eta.$$

Then, $\forall\ t \in [0,\infty)$, the function V satisfies $\dot{V} \leq y(t)u(t)$ if and only if

$$\frac{\partial V}{\partial \xi} = \nabla h(\eta)[f(\eta) + g(\eta)\xi], \quad \text{and} \tag{6.7}$$

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\xi] \leq 0, \tag{6.8}$$

are satisfied $\forall\ \xi,\eta$.

Now, we aim to find conditions on the functions f,g and h for which the conditions (6.7), (6.8) are satisfied and which ensure at the same time that the function $V(\eta,\xi)$ is positive definite. Integrating both sides of (6.7) with respect to ξ, one has

$$V(\eta,\xi) = \bar{V}(\eta) + \nabla h(\eta)f(\eta)\xi + \frac{1}{2}\nabla h(\eta)g(\eta)\xi^2. \tag{6.9}$$

The variable $\bar{V}(\eta)$ can be freely chosen such that the function $V(\eta,\xi)$ is nonnegative and ensure at the same time that the inequality (6.8) holds. Thus, we make the following assumption:

Assumption 5. $\nabla h(\eta)g(\eta) > 0$, and $(\nabla h(\eta)f(\eta))^2 - 2\nabla h(\eta)g(\eta)\bar{V}(\eta) \leq 0,\ \forall \eta$.

From this assumption, we can choose

$$\bar{V}(\eta) = \frac{(\nabla h(\eta)f(\eta))^2}{2\nabla h(\eta)g(\eta)}. \tag{6.10}$$

Differentiating both sides of (6.9) with respect to η, gives

$$\begin{aligned}
\frac{\partial V}{\partial \eta} &= \frac{\partial \bar{V}}{\partial \eta} + \nabla^2 h(\eta)f(\eta)\xi + \nabla h(\eta)\nabla f(\eta)\xi + \frac{1}{2}\nabla^2 h(\eta)g(\eta)\xi^2 + \frac{1}{2}\nabla h(\eta)\nabla g(\eta)\xi^2 \\
&= \frac{\partial \bar{V}}{\partial \eta} + (\nabla^2 h(\eta)f(\eta) + \nabla h(\eta)\nabla f(\eta))\xi + \frac{1}{2}(\nabla^2 h(\eta)g(\eta) + \nabla h(\eta)\nabla g(\eta))\xi^2 \\
&= \alpha + \beta\xi + \frac{1}{2}\gamma\xi^2,
\end{aligned}$$

where we define

$$\begin{aligned}
\alpha &= \frac{\partial \bar{V}}{\partial \eta} = \nabla \bar{V}, \\
\beta &= \nabla^2 h(\eta)f(\eta) + \nabla h(\eta)\nabla f(\eta) = \nabla(\nabla h(\eta)f(\eta)), \\
\gamma &= \nabla^2 h(\eta)g(\eta) + \nabla h(\eta)\nabla g(\eta) = \nabla(\nabla h(\eta)g(\eta)).
\end{aligned}$$

Substituting this into (6.8) we get

$$(\alpha + \beta\xi + \frac{1}{2}\gamma\xi^2)(f(\eta) + g(\eta)\xi) \leq 0, \quad \forall\ \xi, \eta,$$

and by rearranging the terms, we obtain

$$\alpha f(\eta) + \xi[\beta f(\eta) + \alpha g(\eta)] + \frac{1}{2}\xi^2(\gamma f(\eta) + 2\beta g(\eta)) + \frac{1}{2}\xi^3\gamma g(\eta) \leq 0, \quad \forall\ \xi, \eta.$$

Thus the coefficient of ξ^3 must be set to zero; that is, $\gamma g(\eta) = 0$. So, we need the following assumption.

Assumption 6. $\nabla(\nabla h(\eta)g(\eta))g(\eta) = 0,$

To this end, we get

$$\alpha f(\eta) + (\beta f(\eta) + \alpha g(\eta))\xi + \beta g(\eta)\xi^2 \leq 0, \quad \forall\ \xi, \eta$$

Simplifying we get

$$\frac{\dot{\eta}^T(\nabla h)^T\beta\dot{\eta}}{\nabla h g} \leq 0. \tag{6.11}$$

Thus, one further assumption is needed here as follows:

Assumption 7.

$$(\nabla h)^T\beta \leq 0, \quad \forall\ \eta.$$

Based upon the above arguments, we have the storage function

$$V(\eta, \xi) = \frac{(\nabla h(\eta)f(\eta))^2}{2\nabla h(\eta)g(\eta)} + \nabla h(\eta)f(\eta)\xi + \frac{1}{2}\nabla h(\eta)g(\eta)\xi^2. \tag{6.12}$$

which is a non-negative function and satisfies the dissipative inequality $\dot{V}(\eta, \xi) \leq \dot{y}(t)u(t)$ for all $t \geq 0$. Thus the cascade system as shown in Figure 6.1 is nonlinear negative imaginary.

Now, we summarize this conclusion in the next theorem.

Theorem 6.1. *Consider the nonlinear system* (6.5)*. Assume that the following assumptions:*

1) $\nabla h(\eta) g(\eta) > 0$;

2) $\nabla[\nabla h(\eta) g(\eta)] g(\eta) = 0$;

3) $(\nabla h)^T \nabla(\nabla h(\eta) f(\eta)) \leq 0$,

are satisfied for all η*. Then the system* (6.5) *is nonlinear negative imaginary with the nonnegative storage function*

$$V(\eta, \xi) = \frac{(\nabla h(\eta) f(\eta))^2}{2 \nabla h(\eta) g(\eta)} + \nabla h(\eta) f(\eta) \xi + \frac{1}{2} \nabla h(\eta) g(\eta) \xi^2. \tag{6.13}$$

Example 6.1. *In this example, we show that the cascade connection of a positive real system with an integrator is nonlinear negative imaginary. Consider the following LTI system*

$$\begin{cases} \dot{\eta}(t) = a\, \eta(t) + b\, \xi; \\ \dot{\xi} = u; \\ y(t) = c\, \eta(t), \end{cases} \tag{6.14}$$

where $\eta \in \mathbb{R}^n, u \in \mathbb{R}, y \in \mathbb{R}$*, and* a, b, c *are real constants. The positive real subsystem has a transfer function of the form*

$$G(s) = \frac{cb}{s - a}. \tag{6.15}$$

The transfer function $G(s)$ *is positive real if* $cb > 0$ *and* $a < 0$*. We can see that* *the assumptions of Theorem 6.1 are satisfied. Then, we have*

$$\begin{aligned} V(\eta,\xi) &= \frac{1}{2}\frac{\nabla h(\eta) f(\eta)^2}{g(\eta)} + \nabla h(\eta) f(\eta)\xi + \frac{1}{2}\nabla h(\eta) g(\eta)\xi^2 \\ &= \frac{1}{2}\frac{ca^2}{b}\eta^2 + c\ a\ \eta\ \xi + \frac{1}{2}c\ b\ \xi^2, \end{aligned}$$

as the storage function, which has a time derivative evaluated as

$$\begin{aligned} \dot{V}(\eta,\xi) &\leq \frac{a^2 c}{b}\eta\ \dot{\eta} + c\ a\ \dot{\eta}\ \xi + c\ a\ \eta\ \dot{\xi} + c\ b\ \xi\ \dot{\xi} \\ &= \frac{ac}{b}(a\eta + b\xi)\dot{\eta} + c(a\eta + b\xi)\dot{\xi} = \frac{ac}{b}\dot{\eta}^2 + c\ \dot{\eta}\ \dot{\xi} \leq c\ \dot{\eta}\ \dot{\xi}. \end{aligned}$$

Hence, that the system (6.14) *is nonlinear negative imaginary.*

□

Example 6.2. *Consider the following positive real system*

$$G(s) = \frac{s+1}{s^2+s+1} \tag{6.16}$$

which has the following state-space representation

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D = 0. \tag{6.17}$$

This state-space realization can be put in the form of (6.5) *where*

$$f(\eta) = A\eta, \quad h(\eta) = C\eta, \quad g(\eta) = B. \tag{6.18}$$

We can see that,

$$\nabla h(\eta) g(\eta) = CB = 1 > 0,\ \nabla[\nabla h(\eta) g(\eta)] g(\eta) = 0,$$

and

$$\nabla h \nabla(\nabla h(\eta) f(\eta)) \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -1 \leq 0.$$

This shows that the assumptions of Theorem 6.1 are satisfied, and hence the system (6.16) *is nonlinear negative imaginary.*

□

Example 6.3. *Consider the following nonlinear system*

$$\Sigma : \begin{cases} \dot{\eta}_1 = \eta_2; \\ \dot{\eta}_2 = -\eta_1^3 - \eta_2 + \xi; \\ \dot{\xi} = u; \\ y = \eta_2, \end{cases} \tag{6.19}$$

which has the form of system (6.5) *where*

$$f(\eta) = \begin{bmatrix} \eta_2 \\ -\eta_1^3 - \eta_2 \end{bmatrix}, \quad g(\eta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad h(\eta) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}.$$

We have

$$\nabla h(\eta) g(\eta) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 > 0, \quad \text{and} \quad \nabla[\nabla h(\eta) g(\eta)] g(\eta) = 0,$$

and

$$\nabla h \nabla(\nabla h(\eta) f(\eta)) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -3\eta_1^2 \\ -1 \end{bmatrix} = -1 \leq 0.$$

By Theorem 6.1, we conclude that the system (6.19) *is nonlinear NI.*

□

6.4 Stability Analysis

In this section, we investigate the stability robustness of a positive feedback interconnection of system H_1 (of the form (6.5)) and system H_2, respectively represented by

$$H_1\colon \begin{cases} \dot{\eta}_1 &= f_1(\eta_1) + g_1(\eta_1)\xi \\ \dot{\xi} &= u_1 \\ y_1 &= h_1(\eta_1) \end{cases} \quad \text{and} \quad H_2\colon \begin{cases} \dot{\eta}_2 &= f_2(\eta_2) + g_2(\eta_2)u_2 \\ y_2 &= h_2(\eta_2) \end{cases} \tag{6.20}$$

where all the functions involved are sufficiently smooth.

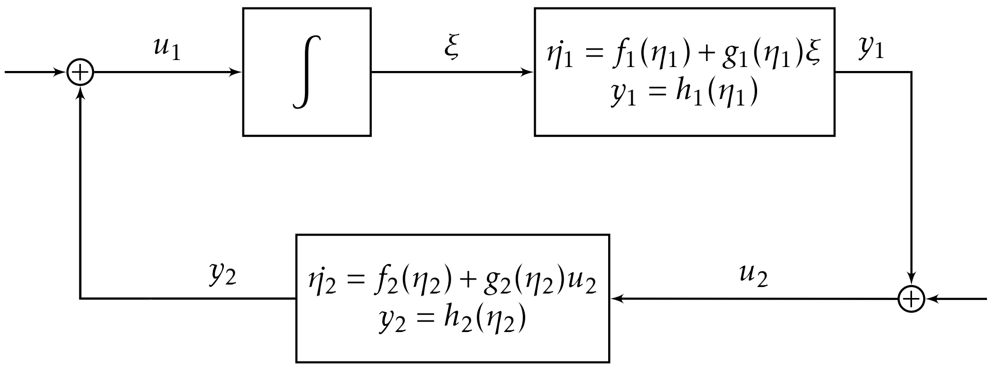

Figure 6.2: Feedback interconnection of nonlinear NI systems H_1 and H_2.

In the feedback system shown in Figure 6.2, the system H_1 is assumed to be nonlinear negative imaginary (with integrator); that is, by Theorem 6.1, there exists a continuously differentiable function non-negative function $V_{H_1}(\eta_1, \xi)$ of the form (6.13) such that

$$\dot{V}_{H_1}(\eta_1, \xi) \le \dot{y}_1(t)u_1(t), \ \forall t > 0. \tag{6.21}$$

The system H_2 is assumed to be nonlinear negative imaginary (without integrator); that is, from Definition 5.1, there exits a continuously differentiable positive definite function $V_{H_2}(\eta_2)$ such that

$$\dot{V}_{H_2}(\eta_2) \leq \dot{y}_2(t)u_2(t), \; \forall t > 0. \tag{6.22}$$

The stability robustness of the feedback interconnection of systems H_1 and H_2 (as shown in Figure 6.2) can be established in a similar way to that of Theorem 5.1. However, the proof of Theorem 5.1 is established using positive definite storage functions for the subsystems. From Theorem 6.1, the system H_1 is nonlinear NI with a storage function $V_{H_1}(\eta_1, \xi)$ evaluated as,

$$\begin{aligned} V(\eta_1, \xi) &= \frac{(\nabla h_1(\eta_1)f_1(\eta_1))^2}{2\nabla h_1(\eta_1)g_1(\eta_1)} + \nabla h_1(\eta_1)f_1(\eta_1)\xi + \frac{1}{2}\nabla h_1(\eta_1)g_1(\eta_1)\xi^2 \\ &= \frac{1}{2}\Big(\frac{(\nabla h_1(\eta_1)f_1(\eta_1))^2}{\sqrt{2}\sqrt{\nabla h_1(\eta_1)g_1(\eta_1)}} + \frac{1}{\sqrt{2}}\sqrt{\nabla h_1(\eta_1)g_1(\eta_1)\xi}\Big)^2 \\ &\quad + \frac{1}{4}\frac{(\nabla h_1(\eta_1)f_1(\eta_1))^2}{\nabla h_1(\eta_1)g_1(\eta_1)} + \frac{1}{4}\nabla h_1(\eta_1)f(\eta_1)\xi^2. \end{aligned}$$

Thus, to ensure that the function $V_{H_1}(\eta_1, \xi)$ is positive definite, the following additional assumption is made.

Assumption 8. $\nabla h_1(\eta_1)f_1(\eta_1) \neq 0, \quad \forall \eta_1 \neq 0.$

Now in order to state the stability theorem for the above feedback system, we represent the feedback interconnection of H_1 and H_2 by the following well-defined state-space representation:

$$\dot{z}(t) := \rho(z(t)), \quad z(t) := \begin{bmatrix} \eta_1(t) \\ \xi(t) \\ \eta_2(t) \end{bmatrix} \in \mathbb{R}^{2n+1}, \tag{6.23}$$

where $\rho : \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{2n+1}$ is assumed to be locally Lipschitz and $\rho(0) = 0$.

Theorem 6.2. *Consider the positive feedback interconnection of systems H_1 and H_2 as in Figure 6.2. Suppose that the system H_1 is nonlinear NI and zero-state observable, and the system H_2 is weak strict nonlinear NI. Moreover, suppose that the Assumptions 1 to 4 of Theorem 5.1 and Assumption 8 are satisfied. Then, the equilibrium point $z = 0$ of the corresponding closed-loop system* (6.23) *is asymptotically stable.*

Proof. The proof of this theorem is based on the same ideas as the proof of Theorem 5.1 in Chapter 5.

□

6.5 Concluding Remarks

In this chapter the notion of nonlinear negative imaginary systems has been extended to the case of free motion. A cascade connection of an input affine nonlinear system and a single integrator has been considered. Under suitable assumptions, this cascade connection has been shown to be a nonlinear negative imaginary system. Also, we have investigated the stability robustness of a positive feedback interconnection where the plant is nonlinear NI system with integrator and the controller is weak strict NI system. The stability proof in this case can be derived in a similar way to that of Theorem 5.1.

Chapter 7

Conclusion and Future Work

In this thesis, notion of the negative imaginary systems has been generalized from the linear case to the nonlinear case. This has been achieved by introducing a time-domain definition of the NI property in terms of dissipativity theory with an appropriate supply rate. A Lyapunov-based approach has been presented to prove the internal stability of a positive feedback interconnection comprising of a linear negative imaginary system and a linear strictly negative imaginary system. Next, this linear stability result has been generalized to establish the stability robustness of a positive feedback interconnection where the plant corresponds to a nonlinear negative imaginary system and the controller corresponds to a weak strict negative imaginary system.

The notion of nonlinear negative imaginary systems has been further extended to include the case of free motion. Under a set of assumptions, a cascade connection of a nonlinear system, affine in the input, has been shown to be nonlinear NI. Finally, we established the stability robustness of a positive feedback system where the plant is nonlinear NI with an integrator and the controller is a weak strict nonlinear Negative Imaginary system. Here is a summary of some of the possible future research directions which can extend the results in this thesis:

- The work in this thesis mainly focuses on nonlinear negative imaginary systems in the single-input-single-output case. It is worth addressing the case of multi-input-multi-output systems and related stability results.

- The Assumptions 1 and 2 of Theorem 5.1 can be weakened to include more general nonlinear systems where the steady-state solutions may not be unique such as in the case of a typical hysteretic system.

- Also, the results presented in Chapter 5 and 6 can be used for future investigation on the analysis and synthesis of nonlinear negative imaginary networked systems. For example, the work in [108] can be extended to nonlinear NI networked systems to investigate output feedback consensuses problems.

- Furthermore, the results of Chapter 5 can be generalized to the class of dissipative systems with more general supply rates than the ones considered in this thesis. For example, dissipative systems with quadratic supply rates which involves derivatives of the inputs and outputs of the system can be considered.

- Finally, as most real-world application are inherently nonlinear, another possible future research area could involve applying the nonlinear negative imaginary framework developed in this thesis to real-world applications for which the negative imaginary property naturally arises such as, for instance, robotics and flexible link manipulators [24].

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