The critical dimension as an invariant of Type III odometers

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The Critical Dimension as an Invariant of Type III Odometers

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Metric entropy is a good invariant for a useful class of measure preserving dynamical systems. This is due to metric entropy's computability and invariance under isomorphism. Many have tried to generalise metric entropy to the larger class of dynamical systems that are null-measure preserving. The problem with these proposed definitions is that they are difficult to compute. In this thesis we take one such entropy, the critical dimension, and show that with certain assumptions it is preserved under the induced transformation. This has far reaching consequences as many transformations between null-measure preserving dynamical systems are induced transformations. Hence many familiar transformations preserve the critical dimension. This allows us to compute the critical dimension for a larger range of dynamical systems, including some ITPFI factors of bounded type.

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Abstract

Metric entropy is a good invariant for a useful class of measure preserving dynamical systems. This is due to metric entropy's computability and invariance under isomorphism. Many have tried to generalise metric entropy to the larger class of dynamical systems that are null-measure preserving. The problem with these proposed definitions is that they are difficult to compute. In this thesis we take one such entropy, the critical dimension, and show that with certain assumptions it is preserved under the induced transformation. This has far reaching consequences as many transformations between null-measure preserving dynamical systems are induced transformations. Hence many familiar transformations preserve the critical dimension. This allows us to compute the critical dimension for a larger range of dynamical systems, including some ITPFI factors of bounded type.

Contents

Chapter 1

Introduction

A dynamical system consists of a measure space $(X, \mathcal{B}, \mu)$ together with a transformation $T : X \mapsto X$, where T represents the discrete iteration of one unit of time. If $\forall A \in \mathcal{B}, \mu(TA) = \mu(A)$ then the dynamical system is called measure preserving. If T is null-measure preserving then the dynamical system is called nonsingular.

The problem of deciding if two measure preserving dynamical systems are isomorphic was first studied by Kolomogorv [25, 26], who showed that a notion called *metric entropy* could be used to distinguish between non-isomorphic measure preserving dynamical systems. The converse true only for special cases, such as the class of Bernoulli shifts [38]. In this case entropy is a complete invariant: two Bernoulli shifts are metrically isomorphic iff they have the same entropy.

Metric entropy is regarded as the "most successful invariant so far" [45, Chapter 4, p. 75]. Many authors have extended the definition of entropy into the realm of nonsingular dynamical systems, such as the Krengel entropy [28], Parry entropy [39], Silvia and Thieullen's entropy [43], and the critical dimension [35]. Unfortunately "these invariants are less informative than their classical counterparts and they are more difficult to compute" [6, Section 9].

Indeed, computation of the critical dimension is difficult. Under certain conditions, the critical dimension is equal to the Average Co-ordinate (AC) entropy for product odometers [13], and Markov odometers [7, 8, 12]. The significance of this result being that AC entropy is easily computed. This is re-proven in chapter 2 with a small improvement on the conditions.

The critical dimension is also more useful than previously thought. In chapter 3 an extension of orbit equivalence is explored, called *Hurewicz equivalence*. Some common orbit equivalences are shown to be Hurewicz equivalence. In particular, chapter 4 gives a sufficient condition for the induced odometer of type III_0 Markov odometers to be Hurewicz equivalence.

The importance of this result is that a large class of Markov odometers (called product-type odometers) are orbit equivalent to product odometers [5, 17, 19]. Under some assumptions, we can compute the critical dimension of product odometers. Under the same assumptions, the orbit equivalence preserves the critical dimension. This allows us to compute the critical dimension of the original Markov odometer.

There exist Markov odometers which are not of product type [30], such an odometer was constructed by [9]. Nevertheless, we can say something similar. Any Markov odometer is orbit equivalent to a full Markov odometer [10]. Under some assumptions, we can compute the critical dimension of a full Markov odometer. Under the same assumptions, the orbit equivalence preserves the critical dimension. This allows us to compute the critical dimension of the original (non product-type) Markov odometer.

1.1 Background from Measure Preserving Systems

For the purposes of providing context, we begin with a brief mathematical history of entropy for measure preserving dynamical systems based on [41]. Given a measure space $(X, \mathcal{B}, \mu)$ and a transformation $T : X \mapsto X$ which is measure preserving in the sense that for any measurable subset $A \in \mathcal{B}$ then $\mu(TA) = \mu(A)$. We define the *join* of two partition $\alpha = \{A_i\}_{i=1}^n$ and

$\beta = \{B_j\}_{j=1}^m$ as

$$\alpha \vee \beta = \{A_i \cap B_j : 1 \leq i \leq n, 1 \leq j \leq m\}$$

and the entropy of a partition α as

$$H(\alpha) = -\sum_{i=1}^{n} \mu(A_i) \log(\mu(A_i))$$

and the entropy of a partition with respect to T as

$$h(T, \alpha) = \lim_{n \mapsto \infty} \frac{1}{n} H\left(\vee_{i=0}^{n-1} T^{-i}\alpha\right)$$

The *entropy*, or what we shall call *metric entropy* to distinguish it from other definitions of entropy, is the supremum of $h(T, \alpha)$ over all partitions

$$h(T) = \sup_{\alpha} h(T, \alpha)$$

The supremum over all partitions makes this quantity difficult to calculate. However the same result holds if we restrict our attention to partitions that generate the σ-algebra $\mathcal{B}$ in the sense that $\mathcal{B}$ is the minimal σ-algebra that contains the all the sets $T^i\alpha$ for $i \in \mathbb{Z}$.

Example 1.1.1 (Bernoulli shifts)**.** For $I = [0, 1, \cdots, k-1] \subset \mathbb{N}$, let μ_j be a probability measure on I where $\sum_{i \in I} \mu_j(i) = 1$. Define the infinite product space $X = \prod_{j \in \mathbb{Z}} I$, infinite product measure $\mu = \otimes_{j \in \mathbb{Z}} \mu_j$, let $\mathcal{B}$ be σ-algebra generated by cylinders, and $T : X \mapsto X$ be the "left shift" defined by $(T(x))_n = x_{n+1}$ for $n \in \mathbb{Z}$. The dynamical systems $(X, \mathcal{B}, \mu, T)$ is called the *Bernoulli shift*. The entropy of the Bernoulli shift is

$$h(T) = \sum_{i=0}^{k-1} \mu(i) \log(\mu(i))$$

Two measure preserving dynamical systems $(X, \mathcal{B}, \mu, T)$ and $(X', \mathcal{B}', \mu', T')$ are *isomorphic* when there exists a bi-measurable bijection $\phi : X \mapsto X'$ such that $\phi(T(x)) = S(\phi(x))$ for μ-almost every $x \in X$. The claim that entropy is a good invariant is justified by

Theorem 1.1.2. *If the measure preserving dynamical systems* $(X, \mathcal{B}, \mu, T)$ *and* $(X', \mathcal{B}', \mu', T')$ *are isomorphic, then*

$$h(T) = h(S)$$

The claim that entropy is a complete invariant for Bernoulli shifts is justified by

Theorem 1.1.3 ([38])**.** *Two Bernoulli shifts with the same entropy are isomorphic*

There are three other theorems which are included for comparison

Theorem 1.1.4 (Birkohff ergodic theorem)**.** *Given a measure preserving dynamical systems* $(X, \mathcal{B}, \mu, T)$ *and an integrable function* f*, then*

$$\lim_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = \int_X f(x) d\mu$$

Theorem 1.1.5 (Kaç's theorem)**.** *Given a measure preserving dynamical systems* $(X, \mathcal{B}, \mu, T)$ *with* $\mu(X) = 1$ *and* $A \in \mathcal{B}$ *be a set of positive measure. Let* $n_A(x)$ *be the return time to* A*. Then*

$$\int_A n_A(x) d\mu = 1$$

Theorem 1.1.6 (Abramov's formula)**.** *Given a measure preserving dynamical systems* $(X, \mathcal{B}, \mu, T)$ *with* $\mu(X) = 1$ *and* $A \in \mathcal{B}$ *with* $\mu(X - \cup_{i=1}^{\infty} T^i A) = 0$*, the induced dynamical system* $(A, \mathcal{B}|_A, \mu|_A, T|_A)$ *has entropy*

$$h(T|_A) = \frac{1}{\mu(A)} h(T)$$

Abramov's formula shows that *metric entropy* is not preserved for the induced dynamical system. This should be contrasted with the earlier claim that the critical dimension *is* preserved for the induced odometer. As we shall see later, the critical dimension is always 1 for measure preserving dynamical systems: including $(X, \mathcal{B}, \mu, T)$ and $(A, \mathcal{B}|_A, \mu|_A, T|_A)$ regardless of their metric entropy.

This ends our brief summary of metric entropy as an invariant, and measure preserving dynamical systems in general. The beauty of nonsingular dynamical systems is that these theorems often have their own nonsingular analogy.

1.2 Terminology and Theorems

This section follows [18, Chapter 1]. Common use makes definitions into terminology; we revise some popular definitions and defer numbering our definitions until such practice becomes practical.

Two σ-finite measures μ, μ' on $(X, \mathcal{B})$ are *equivalent* when for $A \in \mathcal{B}$, $\mu(A) = 0$ iff $\mu'(A) = 0$. Given $(X, \mathcal{B}, \mu), (X', \mathcal{B}', \mu')$ and an invertible, measurable mapping $\phi : X \mapsto X'$, the mapping ϕ is called an *isomorphism.* In the special case of $X = X'$, the isomorphism is called an *automorphism.* When $\mu' \sim \mu \circ \phi$ then ϕ is called *nonsingular.* A countable group of nonsingular automorphisms is denoted G, the elements can be enumerated g_i for $i \in \mathbb{N}$. The *full group* of G, is denoted $[G]$ and consists of all automorphisms that can be written piecewise as functions of G: that is to say that $f \in [G]$ when for some partition A_i of X

$$f(x) = g_i(x) \forall x \in A_i$$

We consider the case where $g_i = T^i, i \in \mathbb{Z}$, for some automorphism T. The nonsingular transformation $g \in G$ is said to have a *periodic* point when $g^i x = x$ for some $x \in X, i \in \mathbb{N}$, and G is called *aperiodic* when no $g \in G$ has a periodic point. It is called *conservative* if for every $A \in \mathcal{B}$,

$$\mu(A - \cup_{g \in G}^{\infty} gA) = 0$$

The set $A \in \mathcal{B}$ is called *G-invariant* when $gA = A$ for some $1 \neq g \in G$. If the only G-invariant sets are $\emptyset$ and X then G is called *ergodic*. This is equivalent to saying that the only g-invariant functions are the constant functions.

We make the standing assumptions that the group action of G is amenable, aperiodic, conservative and ergodic. A measure μ is assumed to non-atomic $(\forall x \in X, \mu(x) = 0)$, σ-finite and $\mu(X) < \infty$ unless otherwise stated.

The orbit of a point $x \in X$ under the transformation T is $\mathrm{Orb}_T(x) = \{T^i x : i \in \mathbb{Z}\}$. The forward orbit is $\mathrm{Orb}_T^+(x) = \{T^i x : i > 0\}$. The forward orbit can be considered as an ordered sequence by using the natural ordering on i from T^i.

Theorem 1.2.1. *Given two nonsingular transformations $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ the following are equivalent:*

1. *They are orbit equivalent. Or sometimes called weakly equivalent.*

2. *There exists a null-measure preserving isomorphism ϕ such that*

$$[S] = \phi[T]\phi^{-1}.$$

3. *The T-orbits of x are mapped to the S-orbits of $\phi(x)$:*

$$Orb_S(\phi(x)) = \phi(Orb_T(x))$$

4. *For some cocycle* $\sigma : \mathbb{N} \times X \mapsto \mathbb{N}$

$$\phi T^{\sigma(n,x)} x = S^n \phi x$$

In the special case where $\phi T^n x = S^n \phi x$, (when $\sigma(n,x) = n$) then the transformations are said to be *isomorphic* or *strongly equivalent*.

Given a nonsingular ergodic transformation T and $n \in \mathbb{N}$, the measure $\mu \circ T^n$ is equivalent to μ by definition. Hence the Radon-Nikodym derivative exists, which we denote by $\omega_n(x) = \frac{d\mu \circ T^n}{d\mu}(x)$. Note that the cocycle relation $\omega_{i+j}(x) = \omega_i(x)\omega_j(T^i x)$ holds.

When there is more than one nonsingular transformation in our context, we distinguish between the derivatives of $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ by decorating ω_n with the transformation, as $\omega_n^S(x) = \frac{d\nu \circ S^n}{d\nu}$.

We are now in a position to cite the nonsingular analogy of Birkhoff's ergodic theorem [23].

Theorems

Theorem 1.2.2 (Hurewicz Ergodic Theorem)**.** *Let T be a ergodic and nonsingular transformation of $(X, \mathcal{B}, \mu)$. If f is an integrable function then*

$$\lim_{n \mapsto \infty} \frac{\sum_{i=0}^{n-1} f(T^i x)\omega_i(x)}{\sum_{i=0}^{n-1} \omega_i(x)} = \int_X f d\mu$$

Given a conservative nonsingular transformation T, and $A \in \mathcal{B}$ of positive measure. Then for $x \in A$ define the *return time* $n_A(x)$ as the smallest power k of T such that $T^k x \in A$. The transformation $T|_A(x) = T^{n_A(x)}(x)$ is an automorphism of the restricted measure space $(A, \mathcal{B}|_A, \mu)$ and is called the *induced transformation* or *induced odometer* when $(X, \mathcal{B}, \mu, T)$ is an odometer. This definition is repeated as 4.1.1. We shall abbreviate the m'th return time as $n_A^m(x)$. Return time also obeys the cocycle relation $n_A^m(x) = n_A(T^{n_A^{m-1}} x) + n_A^{m-1}(x)$ where $n_A^1(x) = n_A(x)$.

Theorem 1.2.3 (Nonsingular Kaç's Theorem)**.** *When* $\mu(X) = 1$ *and* T *is conservative and ergodic nonsingular transformation of* $(X, \mathcal{B}, \mu)$*. If* $A \in \mathcal{B}$ *has positive measure then*

$$\int_A \sum_{i=0}^{n_A(x)-1} \omega_i(x) d\mu(x) = 1$$

Proof. We give a different proof to that in [42, Section 5.2].

Since T is conservative, the function $n_A(x)$ is finite for μ-almost every $x \in A$. Define $f(x) = \sum_{i=0}^{n_A(x)-1} \omega_i(x)$ for $x \in A$ and $f(x) = 0$ otherwise. This function is measurable since for every $n \in \mathbb{N}$, $f_n(x) = \sum_{i=0}^{\min\{n, n_A(x)-1\}} \omega_i(x)$ is measurable and $f(x)$ is the pointwise limit of these functions [27, Theorem 2, Section 28]. If n is the m'th time $T^i x$ returns to A for $i \leq n$, written $m = k(n, x) = \left|\sum_{k=0}^{n-1} 1_A(T^i x)\right|$, then

$$\sum_{i=0}^{n-1} \omega_i(x) = \sum_{i=0}^{n_A(x)-1} \omega_i(x) + \sum_{i=n_A(x)}^{n_A^2(x)-1} \omega_i(x) + \cdots + \sum_{i=n_A^{m-1}(x)}^{n_A^m(x)-1} \omega_i(x)$$

$$= f(x) + \omega_{n_A(x)}(x) f(T^{n_A(x)}) + \cdots + \omega_{n_A^{m-1}(x)}(x) f(T^{n_A^{m-1}(x)})$$

$$= \sum_{i=0}^{n-1} 1_A(T^i x) f(T^i x) \omega_i(x)$$

By theorem 1.2.2

$$1 = \lim_{n \mapsto \infty} \frac{\sum_{i=0}^{n-1} \omega_i(x)}{\sum_{i=0}^{n-1} \omega_i(x)}$$

$$= \lim_{n \mapsto \infty} \frac{\sum_{i=0}^{n-1} 1_A(T^i x) f(T^i x) \omega_i(x)}{\sum_{i=0}^{n-1} \omega_i(x)}$$

$$= \int_X 1_A(x) f(x) d\mu = \int_A f(x) d\mu$$

□

The nonsingular Kaç's theorem can also be proven (again, differently to [42, Section 5.2]) by constructing the Kakutani tower with base sets $B_i = n_A^{-1}(i)$ and using the fact that, because of conservation, the tower covers the whole space X

$$\begin{aligned} 1 =& \mu(X) \\ &= \sum_{i\geq 0}\sum_{j=0}^{i}\mu(T^j B_i) \\ &= \sum_{i\geq 0}\int_{B_i}\sum_{j=0}^{i}\omega_j(x)d\mu \\ &= \int_A \sum_{j=0}^{n_A(x)}\omega_j(x) \end{aligned}$$

While the latter proof is shorter, the former proof is preferred as this is the style proof is used in later chapters.

Lemma 1.2.4 (Borel-Cantelli Lemma)**.** *Let $(X, \mathcal{B}, \mu)$ be a measure space, and $C_n \in \mathcal{B}$ be a sequence of sets. If $\sum_{n=1}^{\infty}\mu(C_n) < \infty$ then for almost every $x \in X$ there exists an N_x such that for all $n > N_x, x \notin C_n$.*

Theorem 1.2.5 ([34, Lemma 2.2])**.** *For any $p \in \mathbb{N}$*

$$\lim_{n\mapsto\infty}\frac{\sum_{i=n}^{n+p}\omega_i(x)}{\sum_{i=0}^{n-1}\omega_i(x)} = 0$$

Proof. Our proof is different to that of [34]. Instead we appeal to the Hurewicz ergodic theorem 1.2.2

$$\begin{aligned}
1 = \mu(X) &= \mu(T^p X) \\
&= \int_X \omega_p(x) d\mu \\
&= \lim_{n \mapsto \infty} \frac{\sum\limits_{i=0}^{n-1} \omega_p(T^i x)\omega_i(x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)} \\
&= \lim_{n \mapsto \infty} \frac{\sum\limits_{i=p}^{n+p-1} \omega_i(x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)} \\
&= \lim_{n \mapsto \infty} \frac{\sum\limits_{i=p}^{n-1} \omega_i(x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)} + \frac{\sum\limits_{i=n}^{n+p-1} \omega_i(x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)} \\
&= 1 + \lim_{n \mapsto \infty} \frac{\sum\limits_{i=n}^{n+p-1} \omega_i(x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)}
\end{aligned}$$

from which the conclusion follows. □

Corollary 1.2.6.

$$\lim_{n \mapsto \infty} \frac{\log(\sum\limits_{i=0}^{n} \omega_i(x))}{\log(n+1)} - \frac{\log(\sum\limits_{i=0}^{n-1} \omega_i(x))}{\log(n)} = 0$$

Proof. By theorem 1.2.5

$$\lim_{n \mapsto \infty} \frac{\sum\limits_{i=0}^{n} \omega_i(x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)} = \lim_{n \mapsto \infty} \frac{\omega_n(x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)} + 1 = 1$$

taking log

$$\lim_{n\mapsto\infty} \log(\sum_{i=0}^{n} \omega_i(x))) - \log(\sum_{i=0}^{n-1} \omega_i(x))) = 0$$

and using the fact that $\lim_{n\mapsto\infty} \log(n+1)/\log(n) = 1$

$$\begin{aligned}
&= \lim_{n\mapsto\infty} \frac{\log(\sum_{i=0}^{n} \omega_i(x))}{\log(n+1)} - \frac{\log(\sum_{i=0}^{n-1} \omega_i(x))}{\log(n)} \\
&= \lim_{n\mapsto\infty} \frac{\log(\sum_{i=0}^{n} \omega_i(x))}{\log(n)} - \frac{\log(\sum_{i=0}^{n-1} \omega_i(x))}{\log(n)} \\
&= \lim_{n\mapsto\infty} \frac{\log(\sum_{i=0}^{n} \omega_i(x)) - \log(\sum_{i=0}^{n-1} \omega_i(x))}{\log(n)} \\
&= 0
\end{aligned}$$

□

There are two theorems from infinite ergodic theory that are relevant for the purpose of comparison

Theorem 1.2.7. *Suppose $(X, \mathcal{B}, \mu, T)$ is a conservative, ergodic, measure preserving transformation, with $\mu(X) = \infty$, for every $f \in L^1(\mu)$*

$$\lim_{n\mapsto\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) = 0$$

Furthermore, any attempt to re-normalise this limit by replacing n with some sequence a_n will result in either being asymptotically too small, or too large, for every function $f \in L^1(\mu)$

Theorem 1.2.8 (Aaronson's Theorem [1, Theorem 2.4.2]). *Suppose* $(X, \mathcal{B}, \mu, T)$ *is a conservative, ergodic, measure preserving transformation, with* $\mu(X) = \infty$ *and let* $a_n > 0$*, then either*

1. $\liminf_{n \mapsto \infty} \frac{\sum_{i=0}^{n-1} f(T^i x)}{a_n} = 0$ *for all* $f \in L^1(\mu)$*, or*

2. $\limsup_{n \mapsto \infty} \frac{\sum_{i=0}^{n-1} f(T^i x)}{a_n} = \infty$ *for all* $f \in L^1(\mu)$

The Three Types

An nonsingular ergodic transformation T on measure space $(X, \mathcal{B}, \mu)$ is of

Type I if the measure μ is atomic.

Type II if there exists T-invariant σ-finite measure ν equivalent to μ.

Type III if no equivalent σ-finite measure is T-invariant.

Given a type II nonsingular transformation, any T-invariant measures ν, ν' equivalent to μ are necessarily different by a constant. Assume $\nu, \nu' \sim \mu$, since $\nu(TA) = \int_A \frac{d\nu \circ T}{d\nu} d\nu = \nu(A)$ then $d\nu \circ T / d\nu = 1$ (similarly with ν') and the function

$$\frac{d\nu \circ T}{d\nu' \circ T} = \frac{d\nu \circ T}{d\nu} \frac{d\nu}{d\nu'} \frac{d\nu'}{d\nu' \circ T} = \frac{d\nu}{d\nu'}$$

is a T-invariant function, hence constant by ergodicity. If $\nu(X) < \infty$ then so is every T-invariant measure equivalent to μ. So for a type II system, the T-invariant measures are either all finite (II_1) or infinite (II_∞).

The existence of type III odometers was foretold by [16] and the first example was given by [37].

Lemma 1.2.9 (The first type III mesaure)**.** *Define $A_n = \{0, \cdots, n\}$ and define a measure ν_n on A_n by*

$$\nu_n(i) = \begin{cases} \frac{1}{2} & \textit{if} \quad i = 0 \\ \frac{1}{2n} & \textit{if} \quad 0 < i \leq n \end{cases}$$

For the measure $\mu = \otimes_{i=0}^{\infty}$ on the space $X = \prod_{i=0}^{\infty} A(i)$. There exists no equivalent σ-finite measure.

Proof. Define $x_{\max}$ as the element of X such that $\forall i \in \mathbb{N}, (x_{\max})_i = i$. For any $x \in X, x \neq x_{\max}$, let $n_1(x)$ be the index of the first non-maximal digit

$$n_1(x) = \min \{i : x_i < i\}, n_1(x_{\max}) = \infty$$

An automorphism $T : X \mapsto X$ can be defined pointwise as

$$(Tx)_i = \begin{cases} 0 & \text{if} \quad i < n_1(x) \\ x_i + 1 & \text{if} \quad i = n_1(x) \\ x_i & \text{if} \quad i > n_1(x) \end{cases}$$

and $T(x_{\max}) = (0)_i$. This style of automorphism is called *the odometer action.* This is a nonsingular transformation since

$$\frac{d\mu \circ T}{d\mu}(x) = \prod_{n=0}^{n_1(x)} \frac{\nu_n((Tx)_n)}{\nu_n(x_n)} = \begin{cases} (n_1(x) - 1)!/n_1(x), & \text{if} \quad x_{n_1(x)} = 0 \\ (n_1(x) - 1)!, & \text{if} \quad x_{n_1(x)} \neq 0 \end{cases}$$

Suppose, by way of contradiction, that there exists a T invariant measure ν equivalent to μ and define $\phi(x) = \frac{d\mu}{d\nu}(x)$. By T-invariance $\omega_i^{\mu}(x) = \phi(T^i x)/\phi(x)$ and $0 < \phi(x) < \infty$. For a fixed $C > 1$ let $E_C = \phi^{-1}[C^{-1}, C]$ be a set of positive measure. We can approximate this set by cylinders: choose n so large that for some cylinder $[a_0, \cdots, a_n]$ with measure $\mu(E_C \cap [a_0, \cdots a_n]) > 0.9\mu([a_0, \cdots, a_n])$. Then

$$\begin{aligned}
\mu(E_C \cap [a_0, \cdots a_n]) &> \frac{9}{10}\left(\mu\left([a_0, \cdots, a_n, 0] \bigcup (\cup_{i=1}^{n}[a_0, \cdots, a_n, i])\right)\right) \\
&= \frac{9}{10}\left(\mu\left([a_0, \cdots, a_n, 0]\right) + \mu\left(\cup_{i=1}^{n}[a_0, \cdots, a_n, i]\right)\right) \\
&= \frac{9}{10}\left(\mu\left([a_0, \cdots, a_n, 0]\right) + \frac{1}{2n}\sum_{i=1}^{n}\mu\left([a_0, \cdots, a_n]\right)\right) \\
&= \frac{9}{10}\left(\mu\left([a_0, \cdots, a_n, 0]\right) + \frac{1}{2}\mu\left(\cup_{i=1}^{n}[a_0, \cdots, a_n]\right)\right) \\
&= \frac{9}{10}\left(\mu\left([a_0, \cdots, a_n, 0]\right) + \mu\left(\cup_{i=1}^{n}[a_0, \cdots, a_n, 0]\right)\right) \\
&= \frac{18}{10}\left(\mu\left([a_0, \cdots, a_n, 0]\right)\right)
\end{aligned}$$

and

$$\begin{aligned}\mu(E_C \cap [a_0, \cdots a_n]) &= \mu\left(E_C \cap [a_0, \cdots a_n, 0]\right) + \mu\left(E_C \cap \left(\cup_{i=1}^n [a_0, \cdots a_n, i]\right)\right) \\ &\le \mu\left(E_C \cap [a_0, \cdots a_n, 0]\right) + \mu\left(\left(\cup_{i=1}^n [a_0, \cdots a_n, i]\right)\right) \\ &= \mu\left(E_C \cap [a_0, \cdots a_n, 0]\right) + \mu\left(\left([a_0, \cdots a_n, 0]\right)\right)\end{aligned}$$

Combining the above equations

$$\mu\left(E_C \cap [a_0, \cdots a_n, 0]\right) \ge \frac{8}{10}\left(\mu\left([a_0, \cdots, a_n, 0]\right)\right) \tag{1.1}$$

Similarly

$$\sum_{i=1}^{n} \mu\left(E_C \cap [a_0, \cdots a_n, i]\right) \ge \frac{8}{10}\left(\sum_{i=1}^{n} \mu\left([a_0, \cdots, a_n, i]\right)\right)$$

So for at least one $i \in [1, \cdots, n]$

$$\mu\left(E_C \cap [a_0, \cdots a_n, i]\right) \ge \frac{8}{10}\left(\mu\left([a_0, \cdots, a_n, i]\right)\right)$$

Let $N_n > 0$ be the smallest odometer power which maps $[a_0, \cdots, a_n, 0]$ to $[a_0, \cdots, a_n, i]$. For all $x \in [a_0, \cdots, a_n, 0]$,

$$\begin{aligned}
\frac{d\mu \circ T^N}{d\mu}(x) &= \omega^{\mu}_{N_n}(x) \\
&= \prod_{j=1}^{\infty} \frac{\nu_j((T^{N_n}x)_j)}{\nu_j(x_j)} \\
&= \frac{\nu_{n+1}(i)}{\nu_{n+1}(0)} = \frac{1}{n+1}
\end{aligned} \tag{1.2}$$

Let $B \subset E_C \cap [a_0, \cdots, a_n, 0]$ be the elements of E_C not returned to E_C by T^{N_n}: $T^{N_n}B \not\subseteq E_C \cap [a_0, \cdots, a_n, i]$. So $T^{N_n}B \subseteq X - (E_C \cap [a_0, \cdots, a_n, i])$ which has measure

$$\mu(T^{N_n}B) \leq \frac{2}{10}\mu([a_0, \cdots, a_n, i]) = \frac{2}{10}\mu(T^{N_n}[a_0, \cdots, a_n, 0])$$

since $\omega^{\mu}_{N_n}(x) = \frac{1}{N_n+1}$ is constant on both $[a_0, \cdots, a_n, 0]$ and $B \subseteq [a_0, \cdots, a_n, 0]$, then by equation 1.2.

$$\begin{aligned}
\frac{1}{n+1}\mu(B) &= \mu(T^{N_n}B) \\
&\leq \frac{2}{10}\mu(T^{N_n}[a_0, \cdots, a_n, 0]) \\
&= \frac{2}{10}\mu([a_0, \cdots, a_n, 0])\frac{1}{n+1}
\end{aligned}$$

consequently

$$\mu(B) \leq \frac{2}{10}\mu([a_0, \cdots, a_n, 0]) \tag{1.3}$$

Together, equations 1.1 and 1.3 imply that the subset $E_0 = E_C \cap [a_0, \cdots, a_n, 0] - B$ has positive measure.

$$\mu(E_C \cap [a_0, \cdots, a_n]) - \mu(B) \geq \frac{6}{10}\mu([a_0, \cdots, a_n, 0])$$

For all $x \in E_0$, both $x, T^{N_n}x \in E_C$, hence

$$\frac{1}{n+1} = \omega^{\mu}_{N_n}(x) = \phi(x)\phi(T^{N_n}x) \geq C^{-2}$$

since n was arbitrary, this is a contradiction.

$\square$

In the case where μ is a product measure on the space of infinite binary strings, Moore's criteria [32] gives a less demanding method of determining the type according to the properties of the measure.

Theorem 1.2.10 (Moore's Criteria)**.** *An nonsingular ergodic transformation T on measure space $(X, \mathcal{B}, \mu)$, where $\mu = \otimes_{i=0}^{\infty}\mu_i$, and $X = \prod_{i=0}^{\infty} \mathbb{Z}_2$, and*

$$\mu_i(0) = \frac{1-a_i}{2}, \mu_i(1) = \frac{1+a_i}{2} \text{ where } a_i \in (0,1).$$

Then μ is

1. *type I iff $\sum_{i=0}^{\infty}(1-a_i) < \infty$*

2. *type II_1 iff $\sum_{i=0}^{\infty} a_i^2 < \infty$*

3. *type III iff $\sum_{i=0}^{\infty} \left((1-a_i)\left(\min\left(\frac{2a_i}{1-a_i}, 1\right)\right)\right) = \infty$*

4. type II_∞ otherwise.

Example 1.2.11 (Type II_∞ measure)**.** Take $X = \prod_{n=0}^{\infty} \mathbb{Z}_2$, and $A \subset \mathbb{N}$ of asymptotic density $d < 1$. Then define a product measure $\mu(x) = \prod_{n=0}^{\infty} \mu_n(x_n)$, where

$$\mu_n(0) = \begin{cases} \frac{1}{2} & \text{if} \quad n \in A \\ \frac{\lambda^j}{1+\lambda^j} & \text{if} \quad n \notin A, j = n - |A(n)| \end{cases}$$

where $j = n - |A(n)|$ means that n is the j'th element not in A. Since this is a binary odometer $\mu_n(1) = 1 - \mu_n(0)$.

Moore's criteria tells us that this measure is type II_∞.

Types II_1 and II_∞ are invariant under orbit equivalence [14], and they are the only orbit equivalence classes of type II. There are uncountably many orbit equivalence classes of type III, subclasses of type III are distinguished according to the ratio set and associated flow. The ratio set $R(T)$ is a closed multiplicative subgroup of $[0, \infty]$ [29]. Defined by $r \in R(T)$ iff for every $B \in \mathcal{B}$ and $\epsilon > 0$, there exists $k \in \mathbb{Z}^+$ and $C \subset B$ of positive measure such that $T^k C \subset B$ and for all $x \in C, |\omega_{-k}(x) - r| < \epsilon$. The ratio set allows us to subdivide type III systems because

Lemma 1.2.12. *The ratio set is an invariant of orbit equivalence*

Proof. Given $\epsilon > 0$ and $B \in \mathcal{B}$, take a subset of $C \subseteq B$ on which $\frac{d\mu}{d\nu}(x)$ is close to some non-zero constant a:

$$\exp(-\epsilon/3) < \left|\frac{d\mu}{d\nu}(x)/a\right| < \exp(\epsilon/3)$$

Let $0 \neq r \in R(T,\mu)$, we show that $r \in R(T,\nu)$. By definition there exists some $k \neq 0$ and a subset C' of C such that $T^{-k}C' \subset C$ and $e^{-\epsilon/3} < \omega^{\mu}_{-k}(x)/r < e^{-\epsilon/3}$ for all $x \in C' \subset C$.

Since both $x, T^k x \in C$

$$\exp(-\epsilon/3) < \frac{d\mu}{d\nu}(x)/a < \exp(\epsilon/3)$$

$$\exp(-\epsilon/3) < \frac{d\mu}{d\nu}(T^k x)/a < \exp(\epsilon/3)$$

so

$$\exp(-\epsilon) < \frac{\frac{d\nu}{d\mu}(T^k x)}{a} \frac{a}{\frac{d\nu}{d\mu}(x)} \frac{\omega^{\mu}_{-k}(x)}{r} < \exp(\epsilon)$$

Where the quantity in the middle is equal to $\frac{\omega^{\nu}_{-k}(x)}{r}$. Hence $r \in R(T,\nu)$. So the ratio set depends only on the equivalence class of μ rather than μ itself. The case for $r = 0$ is similar. □

All transformations in an orbit equivalence class share the same ratio set. The converse (transformations with the same ratio set are orbit equivalent) is true when $R(T) = \{1\}, \{0, \lambda^n : n\mathbb{Z}, \infty\}$ and $[0,\infty]$; called type II_∞, III_λ and III_1 respectively. But not when $R(T) = \{0, 1, \infty\}$; called type III_0.

The ratio set builds upon Moore's criteria, and allows us to further identify orbit equivalence classes within type III.

Example 1.2.13 (Type III_λ measure)**.** Take $X = \prod_{n=0}^{\infty} \mathbb{Z}_2$, and $A \subset \mathbb{N}$ of asymptotic density $d < 1$. For $\lambda \in (0, 1)$ define a product measure μ as $\mu(x) = \prod_{n=0}^{\infty} \mu_n(x_n)$, where

$$\mu_n(0) = \begin{cases} \frac{1}{2} & \text{if} \quad n \in A \\ \frac{\lambda}{1+\lambda} & \text{if} \quad n \notin A \end{cases}$$

and $\mu_n(1) = 1 - \mu_n(0)$.

By Moore's criteria 1.2.10, this is a type III product measure. Since the Radon-Nikodym derivatives are all of the form $\lambda^i, i \in \mathbb{Z}$, this is a type III_λ measure.

ITPFI transformations

A nonsingular transformation $(X, \mathcal{B}, \mu, T)$ is said to be an Infinite Tensor Product of Factors of type I (or just *ITPFI*) if it is orbit equivalent to a product odometer $(Y, \mathcal{C}, \nu, S)$ where $Y = \prod_{i=0}^{\infty}[0, \cdots, l_i - 1]$ and $\nu = \otimes_{i=0}^{\infty} \nu_i$ is a product measure. If the $l_i < M$ for some constant M then T is IPTFI of bounded type, and IPTFI$_2$ when $M = 2$.

Given a type III_0 nonsingular transformation $(X, \mathcal{B}, \mu, T)$, define a new measure ν on the space $X \times \mathbb{R}$ given by $d\nu(x, y) = d\mu(x)e^y dy$. Define a new transformation

$$T(x, y) = (Tx, y - \log(\omega_i(x)))$$

Since T is conservative, T is also conservative and commutes with the flow $S_t(x, y) = (x, y + t)$. However T is not always ergodic, so we restrict our attention to the space Z of T-ergodic components. The nonsingular action $(Z, \mathcal{B} \times \mathbb{R}|_Z, \nu, S_t)$ is called the *associated flow*. As with Moore's criteria, it is possible to classify T according to its associated flow

Proposition 1.2.14 ([15]). *T is of type*

1. *II iff the associated flow is $x \mapsto x + t, t \in \mathbb{R}$.*

2. *III_λ iff the associated flow is $x \mapsto x + t \pmod{(-\log(\lambda))}$.*

3. *III_1 iff T is ergodic.*

4. *III_0 iff S_t is not transitive.*

For T to be IPTFI, there is a necessary and sufficient condition on the associated flow, called *approximately transitive* flow or AT-flow. This was first proven in the context of von Neumann algebras by [5], and a measure theoretic proof was given later by [17, 19]. In particular [17, Prop. 6] constructs a subset $H \in \mathcal{B}$ of positive measure such that the induced odometer $(H, \mathcal{B}|_H, \nu, T|_H)$ is isomorphic to a product odometer.

Without the AT-flow assumption, the same construction can be performed [10]. What is lost is that the induced odometer $(H, \mathcal{B}|_H, \nu, T|_H)$ is no longer isomorphic to a product odometer, but is instead isomorphic to the more general Markov odometer.

We can compute the critical dimension of a product odometer $(Y, \mathcal{C}, \nu, S)$ where $Y = \prod_{i=0}^{\infty}[0, \cdots, l_i - 1]$ where each $l_i < M$ for some constant M, and if the orbit equivalence preserves the critical dimension, can we equate the computed critical dimensions with the critical dimension of any type III_0 IPTFI factor of bounded type.

Similarly we could compute the critical dimension of some non-IPTFI nonsingular transformations.

Sums of Radon-Nikodym derivatives

Notice that in both examples 1.2.11 and 1.2.13, the type was independent of the asymptotic density d. Changing d does not effect the type, but it does effect how quickly the Radon-Nikodym derivatives grow: as $d \mapsto 1$, more Radon-Nikodym derivatives are equal to 1 and the sum of derivatives grows in proportion to n.

Analysis of the asymptotic growth rates of the Radon-Nikodym derivatives belongs in the same mathematical toolbox as Moore's criteria and the ratio set. We begin by replicating Moore's criteria.

A nonsingular transformation $(X, \mathcal{B}, \mu, T)$ is said to have an equivalent T-invariant measure ν if $\nu \sim \mu$ and $\nu(TE) = \nu(E)$ for all $E \in \mathcal{B}$. According to [16, p. 571] a equivalent T-invariant measure ν exists iff there exists a measurable function f such that $f(T^n x)\omega_n(x) = f(x)$ and $0 < f(x) < \infty$. Indeed the measure ν can be constructed as

$$\nu(E) = \int_E f(x) d\mu$$

Which is T-invariant because

$$\nu(E) = \int_E f(x) d\mu = \int_E f(T^n x)\omega_n(x) d\mu = \int_{T^n E} f(x) d\mu = \nu(T^n E)$$

It is clear that f plays the role of the Radon-Nikodym derivative $d\mu/d\nu$. By the Hurewicz ergodic theorem 1.2.2

$$\begin{aligned}
\nu(B) &= \int_B f(x)\mu \\
&= \lim_{n\mapsto\infty} \frac{\sum\limits_{i=0}^{n-1} 1_B(T^i x) f(T^i x)\omega_i(x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)} \\
&= \lim_{n\mapsto\infty} \frac{\sum\limits_{i=0}^{n-1} 1_B(T^i x) f(x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)} \\
&= f(x) \lim_{n\mapsto\infty} \frac{\sum\limits_{i=0}^{n-1} 1_B(T^i x)}{\sum\limits_{i=0}^{n-1} \omega_i(x)}
\end{aligned}$$

Where the $B \in \mathcal{B}$ is necessary to handle the type II_∞ case. We rearrange the above equation to

$$0 < f(x) = \nu(B) \lim_{n\mapsto\infty} \frac{\sum\limits_{i=0}^{n-1} \omega_i(x)}{\sum\limits_{i=0}^{n-1} 1_B(T^i x)} < \infty$$

Notice that this equation is not within the jurisdiction of Aaronson's Theorem 1.2.8, as the normalising factor is a function of both n and x; not x alone. In this case the return time to B grows at the same rate as the sum of derivatives.

In the type II_1 case ($B = X, \nu(X) < \infty$), this equation can be simplified to

$$f(x) = \nu(X) \lim_{n\mapsto\infty} \frac{\sum\limits_{i=0}^{n-1} \omega_i(x)}{n} \tag{1.4}$$

As noted by [31], while the type II_1 Radon-Nikodym derivatives average nicely, great care must be taken while averaging the derivatives type II_∞. Nevertheless, collecting these results gives us a Moore's-criteria style theorem.

Theorem 1.2.15. *Given a nonsingular ergodic transformation T on the measure space $(X, \mathcal{B}, \mu)$.*

1. *if μ is an atomic measure, then μ is type I.*

2. *Define*
$$f_n(x) = \frac{\sum\limits_{i=0}^{n-1} \omega_i(x)}{n}$$
if $\lim_{n\mapsto\infty} f_n(x) = f(x)$ exists, $f(x) \in \mathcal{L}^1(\mu)$ and $0 < f(x) < \infty$ μ-almost everywhere, the μ is type II_1

3. *if for some subset $B \in \mathcal{B}$,*
$$f_n(x) = \frac{\sum\limits_{i=0}^{n-1} \omega_i(x)}{\sum\limits_{i=0}^{n-1} 1_B(T^i x)}$$
if $\lim_{n\mapsto\infty} f_n(x) = f(x)$ exists, $f(x) \in \mathcal{L}^1(\mu)$ and $0 < f(x) < \infty$ μ-almost everywhere, then μ is type II_∞

4. *type III otherwise*

It is doubtful that this theorem gives any advantage over existing methods for classifying measures according to types I, II and III. But it does serve to motivate our analysis $\sum_{i=0}^{n-1} \omega_i(x)$ as an object of interest.

Chapter 2

Average Co-Ordinate Entropy and the Critical Dimension

The critical dimension, loosely speaking, is the order of growth rate of the sum $\sum_{i=0}^{n-1} \omega_i(x)$. The previous chapter established this as an object of interest, and we were able to replicate a Moore's criteria style classification using this quantity.

Unfortunately, there no known method for computing the critical dimension directly. It was shown by [35], that under certain conditions it is equal the easily computable AC entropy.

In this section we re-prove the connection between AC entropy and the critical dimension, with a small improvement on the conditions under which these quantities are equal. This will be used in chapter 4, where the critical dimension is computed for a larger class of dynamical systems.

2.1 The Critical Dimension

We reiterate the standing assumptions that T is a nonsingular transformation on the σ-finite probability space $(X, \mathcal{B}, \mu)$. The transformation T is ergodic and conservative. We follow [13] and define

Definition 2.1.1 (The Lower Critical Dimension)**.** *The set*

$$X_{\alpha'} = \left\{ x \in X \,\middle|\, \liminf_{n \mapsto \infty} \frac{1}{n^{\alpha'}} \sum_{i=0}^{n-1} \omega_i(x) > 0 \right\}.$$

Is T-invariant, and hence has measure 0 or 1. Define the lower critical dimension α as

$$\alpha = \sup \left\{ \alpha' : \mu(X_{\alpha'}) = 1 \right\}.$$

Definition 2.1.2 (The Upper Critical Dimension)**.** *The set*

$$X_{\beta'} = \left\{ x \in X \,\middle|\, \limsup_{n \mapsto \infty} \frac{1}{n^{\beta'}} \sum_{i=0}^{n-1} \omega_i(x) = 0 \right\}.$$

Is T-invariant. Define the upper critical dimension β as

$$\beta = \inf \left\{ \beta' : \mu(X_{\beta'}) = 1 \right\}$$

As a direct consequence of these definitions

$$\underline{f}_\rho(x) = \liminf_{n\mapsto\infty} \frac{1}{n^\rho} \sum_{i=0}^{n-1} \omega_i(x) = \begin{cases} 0 & \text{when} \quad \rho > \alpha \\ \infty & \text{when} \quad \rho < \alpha \end{cases}$$

Similarly for limsup:

$$\overline{f}_\rho(x) = \limsup_{n\mapsto\infty} \frac{1}{n^\rho} \sum_{i=0}^{n-1} \omega_i(x) = \begin{cases} 0 & \text{when} \quad \rho > \beta \\ \infty & \text{when} \quad \rho < \beta \end{cases}$$

The definitions do not specify what happens when $\rho = \alpha$ or β. As [31] has shown, for a type II_1 odometer $\alpha = \beta = 1$ and $0 < \overline{f}_1(x) = \underline{f}_1(x) = f(x) < \infty$. We can also say something about this value for type III measures. Since

$$\begin{aligned} \underline{f}_\alpha(Tx)\omega_1(x) &= \liminf_{n\mapsto\infty} \frac{1}{n^\alpha} \sum_{i=0}^{n-1} \omega_i(Tx)\omega_1(x) \\ &= \liminf_{n\mapsto\infty} \frac{1}{n^\alpha} \sum_{i=0}^{n-1} \omega_{i+1}(x) \\ &= \liminf_{n\mapsto\infty} \frac{1}{n+1^\alpha} \sum_{i=0}^{n} \omega_i(x) - \frac{\omega_0(x)}{n^\alpha} \\ &= \underline{f}_\alpha(x) \end{aligned}$$

For a type III measure, the function $\underline{f}_\alpha(x)$ must be either zero or infinity μ-almost everywhere, otherwise by theorem 1.2.15 this is a type II measure. Similarly the function $\overline{f}_\alpha(x)$ must be either zero or infinity. Hence we have a result similar to Aaronson's Theorem 1.2.8: that for all ρ either

$\underline{f}_\rho(x) = 0$ or $\overline{f}_\rho(x) = \infty$. The value ρ at which this change occurs is the lower (for $\underline{f}_\rho$) and upper (for $\overline{f}_\rho$) critical dimension.

The critical dimensions can also be expressed in the language of Dirichlet series. For $a_n, s \in \mathbb{C}$, the ordinary Dirichlet series

$$\sum_{i=0}^{\infty} \frac{a_i}{i^s}$$

The abscissa of convergence is defined as

$$\sigma_c = \limsup_{n \mapsto \infty} \frac{\log\left(\sum_{i=0}^{n-1} a_i\right)}{\log(n)}$$

If $a_n = \omega_n(x)$, then

$$\sigma_c = \limsup_{n \mapsto \infty} \log\left(\sum_{i=0}^{n-1} \omega_i(x)\right) / \log(n) = \beta.$$

Similarly,

$$\alpha = \liminf_{n \mapsto \infty} \log\left(\sum_{i=0}^{n-1} \omega_i(x)\right) / \log(n)$$

This relationship means that the machinery of Dirichlet series may be brought to bear on the critical dimension. This relationship could provide an alternative method for computing the critical dimension directly.

Markov odometers

So far our examples have all been product odometers. We shall work in the more general setting of Markov odometers. The realm of Markov odometers is genuinely different from that of product odometers, and there exists type III_0 measures which are not even orbit equivalent to a product odometer [5, 30].

We begin with the usual definition of a Bratteli-Vershik system. This is adapted from [20], and is included here to establish notation.

Let $V = \cup_{i\geq 0} V^i$ be a vertex set, where each $V^{(i)}$ is considered disjoint and $V^{(0)} = \{v_0\}$ contains a single element. Let $E = \cup_{i\geq 1} E^{(i)}$ be a directed set of edges, where $(uv) \in E^{(i)}$ implies $u \in V^{(i-1)}, v \in V^{(i)}$. Multiple edges are permitted. Note that the graph $(V^{(i)} \cup V^{(i-1)}, E^{(i)})$ is bipartite. Define the source and range maps

$$s_n : E^{(n)} \mapsto V^{(n-1)}, r_n : E^{(n)} \mapsto V^{(n)}$$

which can also act on $x \in X$ by

$$s_n : X \mapsto V^{(n-1)} : s_n(x) = s_n(x_n)$$

$$r_n : X \mapsto V^{(n)} : r_n(x) = r_n(x_n)$$

Two edges $e, e' \in E^{(n)} \times E^{(n+1)}$ are connected iff $r_n(e) = s_{n+1}(e')$. Define for $v \in V^{(n)}$ let $E^{(n)}(v)$ be the set of all edges $e \in E^{(n)}$ with common range $r_n(e) = v$. If E is equipped with a partial order $\geq$ so that two edges $e, e' \in E^{(n)}$ are comparable iff they share a common range $r_n(e) = r_n(e')$ (i.e. the edges $E^{(n)}(v)$ are totally ordered), then (V, E) is called an *ordered*

Bratteli-Vershik diagram. Define the *Bratteli compactum* X as set of all infinite paths starting from v_0. The maximal and minimal paths are

$$x_{\max} = \left(e_i : \forall e \in E^{(i)}, r_i(e_i) = r_i(e) \implies e_i \geq e\right)$$

$$x_{\min} = \left(e_i : \forall e \in E^{(i)}, r_i(e_i) = r_i(e) \implies e_i \leq e\right)$$

The Bratteli compactum X is called *essentially simple* when there is a unique infinite maximal and minimal path. Denote the set of all paths from $V^{(m)}$ to $V^{(n)}$ by P_m^n, and call any such path $[e_m, \cdots, e_n] \in P_m^n, e_i \in E^{(i)}, m \leq i \leq n$ a *cylinder.* Let $\mathcal{B}$ be the σ-algebra generated by these cylinders.

For any $x \in X$, we define the number of cylinders from v_0 of length n by $s(n)$.

Given a sequence of stochastic matrices $\left\{P^{(n)}\right\}_n$, where the entries of $P^{(n)}$ are indexed by $(v, e) \in V^{(n-1)} \times E^{(n)}$ and:

1. $P^{(n)}_{(v,e)} > 0$ when $v = s_n(e)$, and

2. for all $v \in V^{(n-1)}, \sum_{e \in E^{(n)}}^{s_n(e)=v} P^{(n)}_{v,e} = 1$

so the edges leaving a vertex have weights summing to 1; whereas the edges entering a vertex are totally ordered. Define a *Markov measure* μ on X by

$$\mu([e_m, \cdots, e_n]) = \prod_{i=m}^{n} P^i_{(s_i(e_i), e_i)}$$

This measure is ergodic, conservative, non-atomic, and reduces to a product measure when the columns of the stochastic matrix $P^{(n)}$ are identical.

We define the *Vershik Transformation* $T : X \mapsto X$ as the odometer action on the path space. That is $Tx_{\max} = x_{\min}$, and otherwise Tx is the next element in the lexicographic (partial) ordering of X as defined in lemma 1.2.9.

The essentially simple Bratteli compactum X, σ-algebra $\mathcal{B}$, Markov measure μ and Vershik transformation T is called a *Markov odometer* and will be denoted by $(X, \mathcal{B}, \mu, T)$. When μ is a product measure this is called a *product odometer*.

Example 2.1.3 (The Full Product Odometer [10, Example 2.1])**.** Let each $V^{(n)} = \{v_n\}$ be singleton. Denote the edges $E^{(n)}$ by the numbers $1, \cdots, l_n$, where every edge $e \in E^{(n)}$ has the same source and range: for all $i \in E^{(n)}$, $s_n(i) = v_{n-1}, r_n(i) = v_n$. Then the Bratteli compactum is the product space $X = \prod_{i=1}^{\infty} \mathbb{Z}_{l_n}$. Together with the Vershik transformation T and Markov measure μ, call $(X, \mathcal{B}, \mu, T)$ the full product odometer.

Example 2.1.3 is easily seen to be a product odometer as there is only one v to index the stochastic matricies $P^{(n)}_{(v,e)}$, so all (one) columns are trivially identical.

Example 2.1.4 (The Full Markov Odometer [10, Example 2.2]). Let each $V^{(n)}$ consist of $l_n \in \mathbb{N}, l_n \geq 2$ vertices. Endow this graph with the full range of possible edges $E^{(n)} = V^{(n-1)} \times V^{(n)}$ so that every vertex at level $n-1$ is connected to every vertex at level n (this property will later be called BV1). Order all edges with common range according to the integer value of their source vertex. Then the Bratteli compactum is again the product space $X = \prod_{i=1}^{\infty} \mathbb{Z}_{l_n}$. Together with the Vershik transformation T and Markov measure μ, call $(X, \mathcal{B}, \mu, T)$ the full Markov odometer.

Example 2.1.4 can still reduce to a product odometer if on our choice of $P^{(n)}_{(v,e)}$ is independent of v.

Not every type III Markov odometers is orbit equivalent to a product odometer [30]. But, as was shown by [10] that every type III Markov odometer is orbit equivalent to a full Markov odometer, as in example 2.1.4. To be precise, it was proven that every type III Markov odometer there exists a set of positive measure A such that the induced odometer on A is orbit equivalent to the full odometer. The use of the induced odometer will become important in chapter 4.

Computing AC entropy of a Markov odometer

In this section, we show how to compute the AC entropy of a Markov odometer, we follow [8] and make some assumption on the connectivity between $V^{(n-1)}$ and $V^{(n)}$, and another assumption on the number of edges in $E^{(n)}$.

BV1 There exists some constant K such that for each $i, j \in \mathbb{N}$, if $|i-j| \geq K$ then every vertex in V^i is connected to every vertex in V^j by at least one path.

BV2 The number of edges at each level grows sub-exponentially: $\left|E^{(n)}\right| \leq a_n$ where $\lim_{n \mapsto \infty} \frac{\log(a_n)}{n} = 0$

Assumption BV1 is the same as that of [8, 13] in the case when $K = 1$, our BV2 assumption is weaker than that of [8], in as much as some growth is permitted. For example polynomial growth is permitted by BV2; but exponential growth, such as $a_n = 2^n$, is not permitted.

Since $s(n)$ is the number of cylinders of length n, the assumption BV1 also gives the following lower bound:

$$\frac{1}{K} = \frac{n}{nK} = \frac{\log\left(2^{\frac{n}{K}}\right)}{n} \leq \frac{\log(s(n))}{n} \tag{2.1}$$

Choosing a_n to be bound by some constant enables us further say that

$$\frac{\log(s(n)))}{n} \leq \frac{\sum_{i=0}^{n-1} n\log(N)}{n} = \log(N) < \infty$$

Under these assumptions, the quantity $\frac{\log(s(n)))}{n}$ is not bound from above. If we allow linear growth a_n of with n: say for example $a_n = n$. By Stirling's approximation

$$\frac{\sum_{i=0}^{n-1}\log(a_i)}{n} \sim \frac{(n-1)\log(n-1) - (n-1) + O(\log(n))}{n} \to \infty.$$

So unlike [8, 35] we can only say that $\frac{1}{K\log(s(n)))} \leq \frac{1}{n}$.

Definition 2.1.5 (Entropy of a Partition). *Let $\mathcal{P}_n$ be a partition of X by cylinders of length n. Then the entropy of this partition is*

$$H(\mathcal{P}_n) = \sum_{C\in P_n} -\mu(C)\log(\mu(C))$$

Definition 2.1.6 (Vertex Measure). *The push-forward measure $\nu^n : V^{(n)} \mapsto [0,1]$ is*

$$\nu^n(v) = \mu(\{(x_i)_{i\geq 0} \in X : r_n(x) = v\})$$

Definition 2.1.7 (Co-ordinate entropy). *For a given Bratelli-Vershik diagram, the entropy of the i'th co-ordinate is*

$$\begin{aligned} H^i_\mu(x) &= H\left(\left\{[e]_{i+1} : \text{ where } e \in E^{(i+1)}, r_i(x) = s_{i+1}(e)\right\}\right) \\ &= -\sum_{\substack{e\in E^{(i+1)} \\ r_i(x)=s_{i+1}(e)}} P^{i+1}_{s_{i+1}(e),e}\log(P^{i+1}_{s_{i+1}(e),e}) \end{aligned}$$

For example, $H_\mu^i(x)$ from figure 2.1 is

$$H_\mu^i(x) = -P^{i+1}_{r_i(x),e_1}\log(P^{i+1}_{r_i(x),e_1}) - P^{i+1}_{r_i(x),x_{i+1}}\log(P^{i+1}_{r_i(x),x_{i+1}}) - P^{i+1}_{r_i(x),e_2}\log(P^{i+1}_{r_i(x),e_2}).$$

Because we have a Markov measure, the entropy of the i'th co-ordinate depends the vertex: $r_i(x)$.

Lemma 2.1.8.

$$H(\mathcal{P}_n) = \sum_{i=0}^{n-1} E(H_\mu^i(x))$$

Proof. This proof sums over the paths in $\mathcal{P}_n$ in two ways. First, from the definition of $\nu^i(v)$

$$\nu^i(v)P^{(i+1)}_{v,e} = \mu(\{x \in X : x_{i+1} = e\})$$

so for any $e \in E^{(i+1)}$

$$-\nu_i(v)P^{i+1}_{s_{i+1}(e),e}\log(P^{i+1}_{s_{i+1}(e),e}) = -\log(P^{i+1}_{s_{i+1}(e),e})\mu(\{x \in X : x_{i+1} = e\}) \tag{2.2}$$

For i from 0 to $n-1$, sum the left hand side of equation 2.2 over all $V^{(i)}$, grouped by $v \in V^{(i)}$

$$\sum_{i=0}^{n-1}\sum_{v\in V^{(i)}} \nu_i(v)\left(\sum_{e\in E^{(i+1)}}^{s_{i+1}(e)=v} -P^{i+1}_{v,e}\log(P^{i+1}_{v,e})\right) = \sum_{i=0}^{n-1} E(H_\mu^i(x))$$

Rewrite the right hand side of 2.2 as

$$-\log(P^{i+1}_{s_{i+1}(e),e})\mu(\{x \in X : x_{i+1} = e\}) = -\log(P^{i+1}_{s_{i+1}(e),e}) \sum_{[e_1,\cdots,e_n]\in\mathcal{P}_n}^{e_{i+1}=e} \mu([e_1,\cdots,e_n])$$

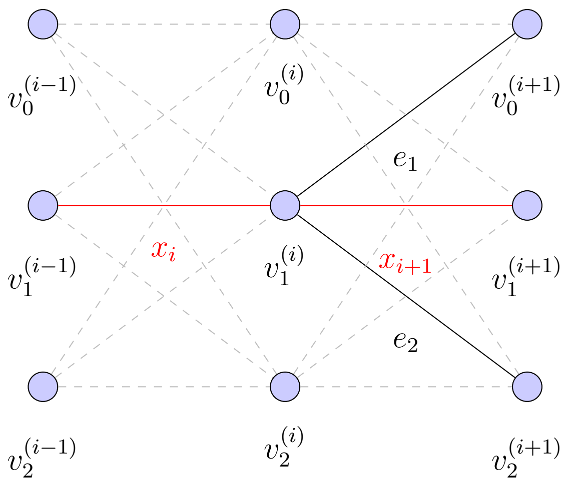

Figure 2.1: The middle path x follows edges x_i, x_{i+1}. Edges in $e_1, e_2 \in E^{(i+1)}$ share the same source as x_{i+1}

If this quantity is summed over paths $[e_1, \cdots, e_n]$ of length n

$$\sum_{[e_1,\cdots,e_n]\in\mathcal{P}_n} -\mu([e_1,\cdots,e_n]) \left(\sum_{i=1}^{n} \log(P^{(i)}_{s_{i+1}(e_i),e_i})\right)$$
$$= \sum_{[e_1,\cdots,e_n]\in\mathcal{P}_n} -\mu([e_1,\cdots,e_n]) \left(\log(\prod_{i=1}^{n} P^{(i)}_{s_{i+1}(e_i),e_i})\right)$$
$$= H(\mathcal{P}_n)$$

While the order of summation is different, these two quantities represent the same object, hence, they must be equal. $\square$

Definition 2.1.9 (The lower average co-ordinate entropy)**.** *Denote the lower average co-ordinate (AC) entropy by*

$$\underline{h}_{AC}(\mu) = \liminf_{n\mapsto\infty} -\frac{H(\mathcal{P}_n)}{\log(s(n))}$$

Definition 2.1.10 (The upper average co-ordinate entropy)**.** *Denote the upper average co-ordinate (AC) entropy by*

$$\overline{h}_{AC}(\mu) = \limsup_{n \mapsto \infty} -\frac{H(\mathcal{P}_n)}{\log(s(n))}$$

Definition 2.1.11 (The average co-ordinate entropy)**.** *If $\overline{h}_{AC}(\mu) = \underline{h}_{AC}(\mu)$, say the average co-ordinate (AC) entropy exists. Denoted by*

$$h_{AC}(\mu) = \lim_{n \mapsto \infty} -\frac{H(\mathcal{P}_n)}{\log(s(n))}$$

Our aim for the next two sections is to examine the conditions for which the AC entropy can be computed

$$\underline{h}_{AC}(\mu) = \liminf_{n \mapsto \infty} -\frac{\sum_{i=1}^{n} \log(P^i_{s_i(x_i),x_i})}{\log(s(n))}$$

and when it is equal to the critical dimension

$$\alpha = \liminf_{n \mapsto \infty} -\frac{\sum_{i=1}^{n} \log(P^i_{s_i(x_i),x_i})}{\log(s(n))}$$

this is summarised in theorem 2.1.26

Computing AC entropy

In this section we compute the AC entropy. Recall lemma 2.1.8,

$$H(\mathcal{P}_n) = \sum_{i=0}^{n-1} E(H^i_\mu(x))$$

To prove

$$\underline{h}_{AC}(\mu) = \liminf_{n \mapsto \infty} -\frac{\sum_{i=1}^{n} \log(P^i_{s_i(x_i),x_i})}{\log(s(n))}$$

it is sufficient to prove

$$\lim_{n\mapsto\infty} -\frac{\sum_{i=1}^{n}\log(P^i_{s_i(x_i),x_i}) - \sum_{i=1}^{n} E(H^i_\mu)}{\log(s(n))} = 0$$

because

$$\begin{aligned}
\underline{h}_{AC}(\mu) &= \liminf_{n\mapsto\infty} \frac{H(\mathcal{P}_n)}{\log(s(n))} \\
&= \liminf_{n\mapsto\infty} \frac{\sum_{i=0}^{n-1} E(H^i_\mu)}{\log(s(n))} \\
&= \liminf_{n\mapsto\infty} -\frac{\sum_{i=1}^{n}\log(P^i_{s_i(x_i),x_i}) - \sum_{i=1}^{n}\log(P^i_{s_i(x_i),x_i}) + \sum_{i=1}^{n} E(H^i_\mu)}{\log(s(n))} \\
&= \liminf_{n\mapsto\infty} -\frac{\sum_{i=1}^{n}\log(P^i_{s_i(x_i),x_i})}{\log(s(n))} + \lim_{n\mapsto\infty} -\frac{\sum_{i=1}^{n}\log(P^i_{s_i(x_i),x_i}) - \sum_{i=1}^{n} E(H^i_\mu)}{\log(s(n))} \\
&= \liminf_{n\mapsto\infty} -\frac{\sum_{i=1}^{n}\log(P^i_{s_i(x_i),x_i})}{\log(s(n))}
\end{aligned}$$

Two approaches have been taken to prove 2.1.18. Both invoke, as may be expected, the law of large numbers. The first approach [13] assumed μ was a product odometer, and hence the random variables $X_i = \log(P^i_{s_i(x_i),x_i})$ are independent because the quantity $(P^i_{s_i(x_i),x_i})$ does not depend on the source vertex $s(x_i)$.

The second approach [8] assumed:

1. the probabilities in the stochastic matricies $P^i_{v,e}$ were bound below by some constant, and

2. the number of vertices at every level is bound by a constant.

then the law of large numbers applies due to theorems of [40] and [46].

One of the purposes of this thesis is to relax these assumptions, while still being able to compute the critical dimension. In this section we define a condition on Markov measures, which gives a sufficient criteria for the law of large numbers to hold. Loosely speaking, we require the Markov odometer to be a product odometer at regular intervals.

Define the random variable $X_i = -\log(P^{(i)}_{s_i(e),e})$, where $e \in E^{(i)}$ is chosen with probability $P^i_{s_i(e),e}$. Notice that in the general Markov measure setting, the random variable X_i is dependent on X_{i-1}. As in figure 2.1, choosing edge $x_i \in E^{(i)}$ limits the possible choices of edge at $E^{(i+1)}$ to e_1, x_{i+1} and e_2. Even with the connectivity assumption BV1, the stochastic matricies $P^{(i)}_{v,e}$ can be chosen in such a way as to make $E(X_k)E(X_l) \neq E(X_k X_l)$

The claim that the random variables X_l, X_k are (weakly) independent was first made by [10]. Here we do not claim that they are always independent, but instead give a sufficient condition for independence .

Definition 2.1.12. *A Markov Odometer has a bow at level n if the stochastic matrix P^n has all columns identical, and each entry non-zero.*

The intuition behind a bow at level n is that edge choices at level $i < n$ are independent of edge choices at level $j > n$. That each entry in the stochastic matrix is nonzero requires each vertex at level $n-1$ to be connected to each vertex at level n.

Lemma 2.1.13. *A product odometer has a bow at every level.*

Proof. This is the definition of a product measure, since the (single) columns of $P^{(n)}$ are always identical. □

The reason for definition 2.1.12 is the following lemma about of Radon derivatives.

Lemma 2.1.14. *Suppose $(X, \mathcal{B}, \mu, T)$ is a Markov odometer with a bow at level n. Let $E_{\min}$ be the unique infinite minimal path in the Bratteli compactum X. Let $C = [E_{\min}]_0^{n-1}$, and $r = r(x) = n_C(x)$ be the return time to C. Then for $i < r$*

$$\frac{\omega_i(T^r x)}{\omega_i(x)} = \frac{\omega_r(T^i x)}{\omega_r(x)} = 1$$

Proof. Denote the edges in $E^{(n)}(v)$ by integers $0, 1, \cdots, l_n - 1$. By the bow assumption for each $v, u \in V^{(n-1)}$, and $e \in \{0, 1, \cdots, l_n - 1\}$

$$P^n_{v,e} = P^n_{u,e}$$

Define

$$y = T^i x$$

$$z = T^r x$$

$$w = T^i z = T^{r+i} x$$

In general, the return time to a cylinder is not constant. However by the bow assumption the return time $r = r(x) = \left|P_0^{n-1}\right|$ is the number of cylinders of length $n-1$. Hence the abbreviation r is justified.

By assumption, every vertex at level $n-1$ is connected to every vertex at level n. So $i < r$ means that $T^i x$ can only change the first n edges; but $r_n(T^i x)$ is fixed. So the edges of $T^i x$ and x agree for all edges after n. So too do the edges $T^i z$ and z agree

$$\forall k > n, (y)_k = (x)_k, (z)_k = (w)_k$$

The edges of x and $T^r x$ agree for all $k < n$ since they are both members of the same cylinder C. So too are y and $T^r y$ both members of $T^i C$ and hence the edges are equal

$$\forall k < n, (z)_k = (x)_k, (y)_k = (w)_k$$

Denote by $u, v \in V^{(n-1)}$ the common source vertex of $s_n(x) = u = s_n(z)$ and $s_n(w) = v = s_n(y)$, and let

$$m_i(x) = \max\left\{k : x_k \neq (T^i x)_k\right\} < \infty.$$

be the index of the largest edge changed by $T^i x$. As already noted this must be less than n, furthermore

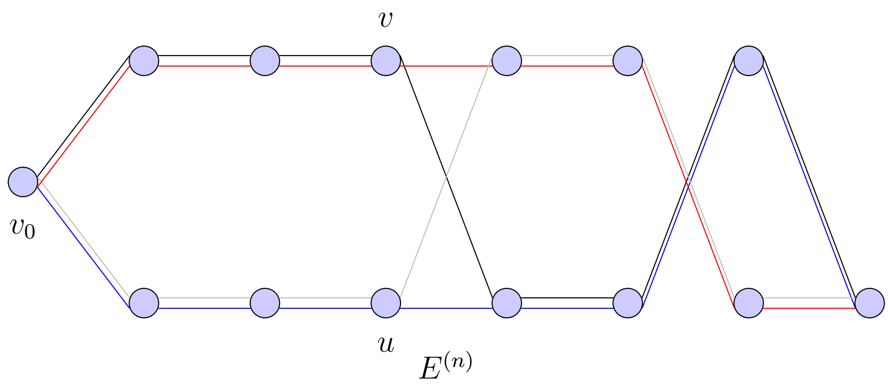

Figure 2.2: The paths x transitions from low to high at level n; $y = T^i x$ transitions from high to high; $z = T^r x$ transitions from low to low, and $w = T^{r+i} x$ transitions from high to low on the odometer

$$\begin{aligned}
\omega_i(x) &= \prod_{k=0}^{m_i(x)} \frac{P^k_{v,y_k}}{P^k_{u,x_k}} \\
&= \prod_{k=0}^{n} \frac{P^k_{v,y_k}}{P^k_{u,x_k}} \\
&= \frac{P^n_{v,y_n}}{P^n_{u,x_n}} \prod_{k=0}^{n-1} \frac{P^k_{v,w_k}}{P^k_{u,z_k}} \\
&= \frac{P^n_{v,y_n}}{P^n_{u,x_n}} \frac{P^n_{u,z_n}}{P^n_{v,w_n}} \prod_{k=0}^{n} \frac{P^k_{v,w_k}}{P^k_{u,z_k}} \\
&= \frac{P^n_{v,y_n}}{P^n_{u,x_n}} \frac{P^n_{u,z_n}}{P^n_{v,w_n}} \omega_i(T^r x)
\end{aligned}$$

While the edges x_n, y_n have different source, they have the same integer value $x_n = y_n \in \{0, 1, \cdots, l_n - 1\}$ and $z_n = w_n = x_n + 1 \pmod{l_n}$, we now use the fact that the columns of the stochastic matrix are independent of u, v

$$\omega_i(x) = \frac{P^n_{v,x_n}}{P^n_{u,x_n}} \frac{P^n_{u,z_n}}{P^n_{v,z_n}} \omega_i(T^r x)$$
$$= \omega_i(T^r x)$$

Hence $\omega_i(x) = \omega_i(T^r x)$. Using this equation and the cocycle relation

$$\omega_{i+r}(x) = \omega_i(x)\omega_r(T^i x)$$
$$= \omega_i(T^r x)\omega_r(x)$$
$$= \omega_i(x)\omega_r(x)$$

Hence $\omega_r(T^i x) = \omega_r(x)$.

□

The same can be said for multiples of r

Corollary 2.1.15. *Let $m \in \mathbb{N}$, and given a Markov odometer with a bow at level n, for μ-almost every $x \in X$. If $i < r$ then*

$$\frac{\omega_i(T^{mr}x)}{\omega_i(x)} = \frac{\omega_{mr}(T^i x)}{\omega_{mr}(x)} = 1$$

Proof. This is m applications of lemma 2.1.14

$$\frac{\omega_i(T^{mr}x)}{\omega_i(x)} = \prod_{j=1}^{m} \frac{\omega_i(T^{jr}x)}{\omega_i(T^{(j-1)r}x)}$$
$$= \prod_{j=1}^{m} 1 = 1$$

□

Lemma 2.1.16. *For $k < n < l$ if a Markov odometer has a bow at level n, and $X_k, X_l : X \mapsto \mathbb{R}$ functions that depend only on the k, l'th co-ordinate of $x \in X$ respectively.*

$$E(X_k X_l) = E(X_k)E(X_l)$$

Proof. For any $m \in \mathbb{N}$, collect the results of corollary 2.1.15, and the assumption that X_k depends only on co-ordinate $k < n$, and X_l is independent of co-ordinates $0, \ldots, n-1$. Again define $C = [E_{\min}]_0^{(n-1)}$ and $n_C(x) = r$ is the constant return time to C.

$$\omega_{mr}(T^i x) = \omega_r(x)$$

$$\omega_i(x) = \omega_i(T^{mr}x)$$

$$X_k(T^i x) = X_k(T^{i+mr}x)$$

$$X_l(T^i x) = X_l(x)$$

These are our tools. For $x \in C$

$$\begin{aligned}
\sum_{i=0}^{mr-1} \omega_i(x) &= \sum_{j=0}^{m-1} \sum_{i=0}^{r} \omega_{i+jr}(x) \\
&= \sum_{j=0}^{m-1} \sum_{i=0}^{r} \omega_i(T^{jr}x)\omega_{jr}(x) \\
&= \sum_{j=0}^{m-1} \sum_{i=0}^{r} \omega_i(x)\omega_{jr}(x) \\
&= \sum_{j=0}^{m-1} \left(\sum_{i=0}^{r} \omega_i(x) \right) \omega_{jr}(x) \\
&= \left(\sum_{i=0}^{r} \omega_i(x) \right) \left(\sum_{j=0}^{m-1} \omega_{jr}(x) \right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=0}^{mr-1} X_k(T^i x)\omega_i(x) &= \sum_{j=0}^{m-1} \sum_{i=0}^{r} X_k(T^{i+jr}x)\omega_{i+jr}(x) \\
&= \sum_{j=0}^{m-1} \sum_{i=0}^{r} X_k(T^i x)\omega_i(T^{jr}x)\omega_{jr}(x) \\
&= \sum_{j=0}^{m-1} \sum_{i=0}^{r} X_k(T^i x)\omega_i(x)\omega_{jr}(x) \\
&= \sum_{j=0}^{m-1} \left(\sum_{i=0}^{r} X_k(T^i x)\omega_i(x) \right) \omega_{jr}(x) \\
&= \left(\sum_{i=0}^{r} X_k(T^i x)\omega_i(x) \right) \left(\sum_{j=0}^{m-1} \omega_{jr}(x) \right)
\end{aligned}$$

similarly

$$\sum_{i=0}^{mr-1} X_l(T^i x)\omega_i(x) = \left(\sum_{i=0}^{r} \omega_i(x) \right) \left(\sum_{j=0}^{m-1} X_l(T^{jr}x)\omega_{jr}(x) \right)$$

and combine the previous three equations into

$$
\begin{aligned}
&\sum_{i=0}^{mr-1} X_k(T^i x) X_l(T^i x)\omega_i(x) \\
&= \sum_{j=0}^{m-1}\sum_{i=0}^{r} X_k(T^{i+jr}x) X_l(T^{i+jr}x)\omega_{i+jr}(x) \\
&= \sum_{j=0}^{m-1}\sum_{i=0}^{r} X_k(T^i x) X_l(T^{jr}x)\omega_i(T^{jr}x)\omega_{jr}(x) \\
&= \sum_{j=0}^{m-1}\sum_{i=0}^{r} X_k(T^i x) X_l(T^{jr}x)\omega_i(x)\omega_{jr}(x) \\
&= \sum_{j=0}^{m-1}\left(\sum_{i=0}^{r} X_k(T^i x)\omega_i(x)\right) X_l(T^{jr}x)\omega_{jr}(x) \\
&= \left(\sum_{i=0}^{r} X_k(T^i x)\omega_i(x)\right)\left(\sum_{j=0}^{m-1} X_l(T^{jr}x)\omega_{jr}(x)\right) \\
&= \frac{\left(\sum\limits_{i=0}^{mr-1} X_k(T^i x)\omega_i(x)\right)\left(\sum\limits_{i=0}^{mr-1} X_l(T^i x)\omega_i(x)\right)}{\left(\sum_{i=0}^{r}\omega_i(x)\right)\left(\sum_{j=0}^{m-1}\omega_{jr}(x)\right)} \\
&= \frac{\left(\sum\limits_{i=0}^{mr-1} X_k(T^i x)\omega_i(x)\right)\left(\sum\limits_{i=0}^{mr-1} X_l(T^i x)\omega_i(x)\right)}{\sum_{i=0}^{mr-1}\omega_i(x)}
\end{aligned}
$$

So, thanks to the bow, we have been able to separate $\sum X_l(T^i x)\omega_i(x)$ from $\sum X_k(T^i x)\omega_i(x)$. By the Hurewicz ergodic theorem 1.2.2, for all $x \in$

$C = [x_{\min}]_0^{n-1}$ (a set of positive measure)

$$
\begin{aligned}
E(X_k X_l) &= \lim_{m \mapsto \infty} \frac{\sum\limits_{i=0}^{mr-1} X_k(T^i x) X_l(T^i x)\omega_i(x)}{\sum\limits_{i=0}^{mr-1} \omega_i(x)} \\
&= \lim_{m \mapsto \infty} \frac{\sum\limits_{i=0}^{mr-1} X_k(T^i x)\omega_i(x) \sum\limits_{i=0}^{mr-1} X_l(T^i x)\omega_i(x)}{\left(\sum_{i=0}^{mr-1} \omega_i(x)\right)^2} \\
&= \lim_{m \mapsto \infty} \left(\frac{\sum\limits_{i=0}^{mr-1} X_k(T^i x)\omega_i(x)}{\sum_{i=0}^{mr-1} \omega_i(x)} \right) \left(\frac{\sum\limits_{i=0}^{mr-1} X_l(T^i x)\omega_i(x)}{\sum_{i=0}^{mr-1} \omega_i(x)} \right) \\
&= E(X_k)E(X_l)
\end{aligned}
$$

Hence $E(X_k X_l) = E(X_k)E(X_l)$ and the random variables X_l, X_k are independent. $\square$

This lemma allows us to apply the law of large numbers to a Markov odometer that contains a bows at regular intervals.

Lemma 2.1.17. *If $(X, \mathcal{B}, \mu, T)$ is a Markov odometer with Bratteli-Vershik diagram (V, E), and $X_i : X \mapsto \mathbb{R}$ a sequence of integrable functions that depend only on the i'th co-ordinate of $x \in X$. If for some $k \in \mathbb{N}$ the odometer has a bow at level jk for all $j \in \mathbb{N}$, then*

$$
\lim_{n \mapsto \infty} \left| \frac{1}{n} \left(\sum_{i=0}^{n-1} X_i - E(X_i) \right) \right| = 0
$$

Proof. Split the sequence $Y_i = X_i - E(X_i)$ into k subsequences: define $Y_j^{(m)} = Y_i$ for $i = jk + m, m, j \in \mathbb{N}$. Then for fixed $m \in [0, k-1]$ the random variables $\left\{Y_j^{(m)}\right\}_{j=0}^{\infty}$ are independent (by the bow assumption and lemma 2.1.16) and identically distributed ($E(Y_i) = 0$) then by the law of large numbers, the sample average converges to the expected value almost surely:

$$\lim_{n\mapsto\infty} \frac{\sum_{j=0}^{n} Y_j^{(m)}}{n} = 0$$

In the ergodic setting, almost sure convergence implies convergence almost everywhere[1]. We now recombine these sequences: for any $\epsilon > 0$ and for each of the k sequences, there exists an $N_{i,\epsilon}$ such that for all $n > N_{i,\epsilon}$

$$\left|\frac{\sum_{j=0}^{n} Y_j^{(m)}}{n}\right| < \epsilon/k$$

Choose $N_\epsilon = \max_{i\in[0,k-1]} N_{i,\epsilon}$. Then

$$\left|\frac{\sum_{j=0}^{n} Y_n}{n}\right| \leq \sum_{m=0}^{k-1} \left|\frac{\sum_{j=0}^{n} Y_j^{(m)}}{n}\right| < \epsilon$$

Hence the law of large numbers applies to the full sequence X_n.

$$\lim_{n\mapsto\infty} \frac{\sum_{i=0}^{n-1} X_i(x) - E(X_i(x))}{n} = 0 \tag{2.3}$$

$\square$

Corollary 2.1.18. *If $(X, \mathcal{B}, \mu, T)$ is a Markov odometer with bows at every k'th level, and P^n is a sequence of stochastic matricies,*

$$\lim_{n\mapsto\infty} \frac{1}{n}\left(\sum_{i=1}^{n} -\log(P^i_{s_i(x_i),x_i}) - H(\mathcal{P}_n)\right) = 0$$

[1] which is not always the case, as shown by Riesz [21, Theorem (11.26)]

Proof. Let $f_i(x) = -\log(P^i_{s_i(x_i),x_i})$, apply 2.1.17 to the functions $f_i = X_i$, where the expected value is

$$\begin{aligned}
E(f_i(x)) &= -\sum_{v \in V^i} \sum_{\substack{e \in E^{i+1} \\ s_{i+1}(e)=v}} \mu(\{x : x_{i+1} = e\}) \log(P^i_{v,e}) \\
&= -\sum_{v \in V^i} \sum_{\substack{e \in E^{i+1} \\ s_{i+1}(e)=v}} \mu(\{x : r(x_i) = v\}) P^i_{v,e} \log(P^i_{v,e}) \\
&= E(H^i_\mu(x))
\end{aligned}$$

By linearity of expectation, $E(\sum_{i=1}^n f_i(x)) = \sum_{i=1}^n E(f_i(x)) = \sum_{i=1}^n H^i_\mu(x) = H(\mathcal{P}_n)$. The result follows by application of lemma 2.1.17 $\square$

Corollary 2.1.19.

$$\lim_{n \mapsto \infty} \frac{1}{\log(s(n))} \left(\sum_{i=1}^n -\log(P^i_{s_i(x_i),x_i}) - H(\mathcal{P}_n) \right) = 0$$

Proof. By 2.1, $\frac{1}{K \log(s(n))} \leq \frac{1}{n}$

$$\begin{aligned}
&\frac{1}{K} \lim_{n \mapsto \infty} \frac{1}{\log(s(n))} \left(\sum_{i=1}^n -\log(P^i_{s_i(x_i),x_i}) - H(\mathcal{P}_n) \right) \\
&\leq \lim_{n \mapsto \infty} \frac{1}{n} \left(\sum_{i=1}^n -\log(P^i_{s_i(x_i),x_i}) - H(\mathcal{P}_n) \right) = 0
\end{aligned}$$

$\square$

In summary, the equation

$$\underline{h}_{AC}(\mu) = \liminf_{n \mapsto \infty} -\frac{\sum_{i=1}^n \log(P^i_{s_i(x_i),x_i})}{\log(s(n))}$$

is true if any of the following sufficient conditions hold

Corollary 2.1.20. *For the Markov odometer $(X, \mathcal{B}, \mu, T)$, the upper and lower AC entropy can be computed when*

1. *μ is a product measure [35], or*
2. *the stochastic matricies are bound below by a constant, and there are finitely many vertices at each level of the Bratteli-Vershik diagram [8], or*
3. *the Bratteli-Vershik diagram contains a bow at every k'th level for some fixed constant k by corollary 2.1.19. This generalises the case for product measures.*

Computing the Critical Dimension

In this section we look at a sufficient condition to compute the critical dimension as the quantity

$$\alpha = \liminf_{n \mapsto \infty} -\frac{\sum_{i=1}^{n} \log(P^i_{s_i(x_i), x_i})}{\log(s(n))}$$

Let $n_p(x)$ be the index of the pth non-maximal edge of x, and $I_p(x)$ be the integer k such that each $(T^k x)_j$ is maximal for $1 \leq j \leq n_p$. The link between the sum of derivatives and co-ordinate measures is given by

$$\sum_{i=I_{p-1}(x)+1}^{I_p(x)} \omega_i(x) = \sum_{\substack{e\in E^{(n_p)} \\ x_{n_p}<e}} \frac{\mu([e]_{n_p}^{(n_p)})}{\mu([x]_1^{(n_p)})} \tag{2.4}$$

As observed by [8, 13], this allows us to compute the sum of derivatives $\sum_{i=0}^{n-1} \omega_i(x)$ whenever $n-1 = I_p(x)$ for some $p \in \mathbb{Z}^+$.

Lemma 2.1.21 ([11, Lemma 5.3(i)])**.**

$$\alpha \leq \liminf_{p\mapsto\infty} -\frac{\sum_{i=1}^{n_p} \log(\mu([x]_1^i))}{\log(s(n_{p-1}))}$$

Proof. by equation 2.4

$$\sum_{i=I_1}^{I_p} \omega_i(x) = \sum_{j=1}^{p} \sum_{\substack{e\in E^{(n_j)} \\ x_{n_j}<e}} \frac{\mu([e]_{n_j}^{(n_j)})}{\mu([x]_1^{(n_j)})} \qquad \leq \sum_{j=1}^{p} \frac{1}{\mu([x]_1^{(n_j)})} \leq p\frac{1}{\mu([x]_1^{(n_p)})}$$

taking logs

$$\log\left(\sum_{i=I_1}^{I_p} \omega_i(x)\right) = \log(p) - \sum_{i=1}^{n_p} \log(\mu([x]_1^i))$$

and using the identity $s(n_{p-1}) \leq I_p(x)$

$$\begin{aligned}
\alpha &= \liminf_{n\mapsto\infty} \frac{\log\left(\sum_{i=0}^{n-1}\omega_i(x)\right)}{\log(n)} \\
&\le \liminf_{p\mapsto\infty} \frac{\log\left(\sum_{i=0}^{I_p}\omega_i(x)\right)}{\log(I_p))} \\
&= \liminf_{p\mapsto\infty} \frac{\log\left(\sum_{i=I_1}^{I_p}\omega_i(x)\right)}{\log(s(n_{p-1}))} \\
&\le \liminf_{p\mapsto\infty} \frac{\log(p)}{\log(s(n_{p-1}))} - \frac{\sum_{i=1}^{n_p}\log(\mu([x]_i^i))}{\log(s(n_{p-1}))} \\
&= \liminf_{p\mapsto\infty} -\frac{\sum_{i=1}^{n_p}\log(\mu([x]_i^i))}{\log(s(n_{p-1}))}
\end{aligned}$$

□

Lemma 2.1.22 ([8, Lemma 5.2]). *For μ-almost every $x \in X$*

$$\lim_{i\mapsto\infty} -\frac{\log(\nu^i(r(x_i)))}{i} = 0$$

Proof. Given $\epsilon > 0$, define $A_i = \{x \in X : -\log(\nu^i(r(x_i)))/i > \epsilon\}$. Then $\mu(A_i) \le N_i 2^{-\epsilon i}$, where N_i is the number of distinct edges in $E^{(i)}$ that share a common range with some $x \in A_i$. There are at most $\left|E^{(i)}\right| \le a_i$ such edges.

$$\mu(A_i) \le a_i 2^{-\epsilon i}$$

By assumption BV2, for the same $\epsilon > 0$ there exists some N_ϵ such that for all $n > N_\epsilon$

$$\frac{\log(a_n)}{n} < \epsilon$$

Then

$$\mu(A_n) \leq a_n 2^{-\epsilon n} = 2^{-n(\epsilon + \log(a_n)/n)} \leq 2^{-n(\epsilon+\epsilon)} \leq 2^{-2\epsilon n}$$

Which is summable. Hence the series $\mu(A_n)$ is summable. By the Borel-Cantelli lemma (1.2.4) $\log(\nu^n(r(x_n)))/i > \epsilon$ can only occur for finitely many i. Hence the limit exists and is equal to zero

$$\lim_{i \mapsto \infty} -\frac{\log(\nu^i(r(x_i)))}{i} = 0$$

$\square$

Lemma 2.1.23. *For μ-almost every $x \in X$*

$$\lim_{n \mapsto \infty} -\frac{\log\left(\sum_{\substack{e \in E^{(n)} \\ x_n < e}} \mu([e]_n^n)\right)}{n} = 0$$

Proof. Let b_i be a summable sequence $\sum_{i=0}^{\infty} b_i < \mu(X)$ for which $\log(b_i)/i \mapsto 0$, we first show that

$$\frac{1}{\nu^n(r(x_n))} \sum_{\substack{e \in E^{(n)} \\ x_n < e}} \mu([e]_n^{(n)}) < b_n \tag{2.5}$$

holds for all but finitely many n.

For $v \in V^{(n-1)}$, define $e_{\max}$ as the largest element in the total ordering $E^{(n)}(v)$, and $f : V^{(n-1)} \mapsto E^{(n)}$ as the smallest edge e' such that

$$\sum_{e' < e \leq e_{\max}} \mu([e]_n^n) < b_n$$

notice that $f(v) = e_{\max}$ when $\sum_{e' < e \leq e_{\max}} \mu([e]_n^{(n)}) \geq b_n$ for all $e' \in E^{(n)}(v)$

Define the set

$$E(n) = \bigcup_{v \in V^{(n-1)}} \bigcup_{f(v) < e \leq e_{\max}} [e]_n^n$$

this set has measure

$$\begin{aligned}
\mu(E(n)) &= \sum_{v \in V^{(n)}} \sum_{f(v) < e \leq e_{\max}} \mu([e]_n^n) \\
&= \sum_{v \in V^{(n)}} \nu^n(v) \sum_{f(v) < e \leq e_{\max}} \frac{\mu([e]_n^n)}{\nu^n(v)} \\
&\leq \sum_{v \in V^{(n)}} \nu^n(v) b_n \\
&= b_n
\end{aligned}$$

which is summable by assumption. Equation 2.5 follows as a consequence of the Borel-Cantelli lemma.

Rewrite the sum

$$\begin{aligned}
\sum_{\substack{e \in E^{(n)} \\ x_n < e}} \mu([e]_n^n) &= \frac{1}{\nu^n(r(x_n))} \sum_{\substack{e \in E^{(n)} \\ x_n < e}} \mu([e]_n^n) \nu^n(r(x_n)) \\
&\leq b_n \nu^n(r(x_n))
\end{aligned}$$

then, using lemma 2.1.22, equation 2.5 and our assumption about b_n

$$\lim_{n\mapsto\infty} -\frac{\log\left(\sum\limits_{\substack{e\in E^{(n)}\\ x_n<e}} \mu\left([e]_n^{(n)}\right)\right)}{n} = \lim_{n\mapsto\infty} -\frac{\log(\nu^n(s_n(x_n)))}{n} - \frac{\log(b_n)}{n}$$

$$= 0 + 0 = 0$$

□

The purpose of these lemmas is to prove

Lemma 2.1.24 ([11, Lemma 5.3(ii)]).

$$\alpha \geq \liminf_{p\mapsto\infty} -\frac{\sum\limits_{i=1}^{n_{p-1}} \log(\mu([x]_1^i))}{\log(s(n_p))}$$

Proof. For any n there exists a p such that $I_{p-1} < n < I_p \leq s(n)$. Then

$$\sum_{i=0}^{n} \omega_i(x) \geq \sum_{i=I_{p-2}+1}^{I_{p-1}} \omega_i(x)$$
$$= \sum_{\substack{e\in E^{(n_{p-1})}\\ x_{n_{p-1}}<e}} \frac{\mu([e]_{n_{p-1}}^{(n_{p-1})})}{\mu([x]_1^{(n_{p-1})})}$$

taking logs,

$$\frac{\log(\sum\limits_{i=0}^{n} \omega_i(x))}{\log(n)} \geq \frac{\log\left(\sum\limits_{\substack{e\in E^{(n_{p-1})}\\ x_{n_{p-1}}<e}} \mu([e]_{n_{p-1}}^{(n_{p-1})})\right)}{\log(s(n_p))} - \frac{\sum\limits_{i=1}^{n_{p-1}} \log(\mu([x]_i^i))}{\log(s(n_p))}$$

and using lemma 2.1.23

$$\alpha = \liminf_{n\mapsto\infty} \sum_{i=0}^{n} \omega_i(x)$$
$$\geq \liminf_{n\mapsto\infty} -\frac{\sum_{i=1}^{n_{p-1}} \log(\mu([x]_i^i))}{\log(s(n_p))}$$

□

So far lemmas 2.1.21 and 2.1.24 have proven that

$$\liminf_{n\mapsto\infty} -\frac{\sum_{i=1}^{n_{p-1}} \log(\mu([x]_i^i))}{\log(s(n_p))} \leq \alpha \leq \liminf_{p\mapsto\infty} -\frac{\sum_{i=1}^{n_p} \log(\mu([x]_i^i))}{\log(s(n_{p-1}))}$$

We are now in a position to prove

Theorem 2.1.25 ([8, proof of Theorem 5.1])**.**

$$\alpha = \liminf_{n\mapsto\infty} -\frac{\sum_{i=1}^{n} \log([x]_i^i)}{\log(s(n))}$$

Proof. The remainder of this proof is largely identical to that of [8, Theorem 5.1] and [13, Theorem 3.2]. By lemmas 2.1.21 and 2.1.24

$$\liminf_{n\mapsto\infty} -\frac{\sum_{i=1}^{n_{p-1}} \log(\mu([x]_i^i))}{\log(s(n_p))} \leq \alpha \leq \liminf_{p\mapsto\infty} -\frac{\sum_{i=1}^{n_p} \log(\mu([x]_1^{n_p}))}{\log(s(n_p))}$$

All that needs to be done is to show that

$$\lim_{n\mapsto\infty} \frac{\sum_{i=n_{p-1}}^{n_p} \log(P^i_{s_i(x_i),x_i})}{\log(s(n))} = 0 \qquad (2.6)$$

Given any $\epsilon > 0$, define $D_{u,v}$ as the set of all $x \in X$ such that x_i is maximal in the total edge ordering of $E^{(i)}(x_i)$, and

$$-\frac{\sum_{i=u}^{u+v} \log(P^i_{s_i(x_i),x_i})}{u+v} > \epsilon > 0$$

Here u plays the role of n_{p-1}, and v is the distance to the next non-maximal edge. Now $\mu(D_{v,u}) \leq 2^{-\epsilon(u+v)}$, and summing over all v gives

$$\begin{aligned}\mu(D_u) &\leq 2^{-\epsilon u}\sum_{v=1}^{\infty} 2^{-\epsilon v}\\ &= 2^{-\epsilon u}\frac{2^{\epsilon}}{1-2^{\epsilon}}\end{aligned}$$

which is itself a summable sequence. By the Borel-Cantelli lemma 1.2.4. Equation 2.6 is greater than ϵ for only finitely many values of u. The limit must be zero. $\square$

Recall that definition of cylinder sets: $\mu[x]^i_i = P^i_{s_i(x_i),x_i}$, and assumptions BV1 and BV2 were required to hold. Theorem 2.1.25 and corollary 2.1.20 can be summarised as

Theorem 2.1.26. *If the Markov odometer $(X, \mathcal{B}, \mu, T)$ satisfies BV1 and BV2, then the lower critical dimension is given by the formula*

$$\alpha = \liminf_{n\mapsto\infty} -\frac{\sum_{i=1}^{n} \log(P^i_{s_i(x_i),x_i})}{\log(s(n))}$$

If, in addition, the Markov odometer satisfies any of the equivalent conditions of 2.1.20 then this quantity can be computed, as it is equal to the lower

AC entropy

$$\alpha = \liminf_{n \mapsto \infty} -\frac{\sum_{i=1}^{n} \log(P^i_{s_i(x_i),x_i})}{\log(s(n))} = \underline{h}_{AC}(\mu)$$

2.2 Katok's Lemma

For measure preserving transformations, Katok's lemma gives a connection between the number of balls of size δ required to cover all but $1-\delta$ of the space [24]. An analogous result was proven for product odometers by [13, Corollary 3.1], and their proof ports seamlessly to the more general context of Markov odometers. We present a different proof to that of [13].

Definition 2.2.1 (n-covering number)**.** *Given a measure space* $(X, \mathcal{B}, \mu)$ *and a set* $A \in \mathcal{B}$ *of positive measure, the* n*-covering number is the smallest number of cylinders of length* n *required to cover* A

$$c_n(A) = \min\left\{k : x^{(i)} \subset X, A \subset \bigcup_{i=1}^{k-1} [x^{(i)}]_1^{n-1}\right\}$$

Recall that in the setting of Markov odometers if $x_j^{(i)} = e_j, 1 \leq j \leq n-1$

$$\log(\mu([x^{(i)}]_1^{n-1}) = \log(\mu([e_1 e_2 \cdots e_{n-1}]_1^{n-1}) = \sum_{j=1}^{n-1} \log(P^j_{s_j(e_j),e_j})$$

Proposition 2.2.2. *Given a Markov odometer* $(X, \mathcal{B}, \mu, T)$

1. *If*

$$\alpha = \liminf_{n \mapsto \infty} -\frac{\log(\mu[x]_1^{n-1})}{\log(s(n))}$$

then for μ-almost every $x \in X$

$$\alpha \leq \liminf_{n \mapsto \infty} \frac{1}{\log(s(n))} \log\left(\inf_{\mu(A)>1-\delta} c_n(A)\right)$$

for all $\delta \in (0,1)$

2. *If*

$$\beta = \limsup_{n \mapsto \infty} -\frac{\log(\mu[x]_1^{n-1})}{\log(s(n))}$$

then for μ-almost every $x \in X$

$$\beta \geq \limsup_{n \mapsto \infty} \frac{1}{\log(s(n))} \log\left(\inf_{\mu(A)>1-\delta} c_n(A)\right)$$

for all $\delta \in (0,1)$

Proof. Given $\delta \in (0,1)$ choose any set A of measure $1-\delta \leq \mu(A) \leq 1$. Then suppose for each n that A can be covered by $c_n(A)$ cylinders

$$A \subseteq \bigcup_{i=0}^{c_n(A)} C_i^{(n)}$$

Because $c_n(A)$ is the minimal number of cylinders required to cover A, the cylinders $C_i^{(n)}$ are pairwise disjoint and

$$\sum_{i=0}^{c_n(A)-2} \mu(C_i^{(n)}) < \mu(A) \leq \sum_{i=0}^{c_n(A)-1} \mu(C_i^{(n)})$$

At least one of these cylinders $C_i^{(n)}, 0 \leq i < c_n(A)-1$ has measure less than or equal to $\mu(A)/(c_n(A)-1)$, otherwise

$$\forall i < c_n(A) - 1, \mu(C_i^{(n)}) > \frac{\mu(A)}{c_n(A) - 1} \text{ and } \sum_{i=0}^{c_n(A)-2} \mu(C_i^{(n)}) > \mu(A)$$

and at least one cylinder has measure greater than or equal to $\mu(A)/c_n(A)$, otherwise

$$\forall i < c_n(A), \mu(C_i^{(n)}) < \frac{\mu(A)}{c_n(A)} \text{ and } \sum_{i=0}^{c_n(A)-1} \mu(C_i^{(n)}) < \mu(A)$$

Call these cylinders $C_{\min}^{(n)}$ and $C_{\max}^{(n)}$:

$$\mu(C_{\min}^{(n)}) \leq \frac{\mu(A)}{c_n(A) - 1} \text{ and } \frac{\mu(A)}{c_n(A)} \leq \mu(C_{\max}^{(n)})$$

then

$$\log(\mu(C_{\min}^{(n)})) \leq \log(\mu(A)) - \log(c_n(A) - 1)$$

$$\log(\mu(A)) - \log(c_n(A)) \leq \log(\mu(C_{\max}^{(n)}))$$

dividing through by $\log(s(n))$, and using the fact that $\lim_{n \mapsto \infty} \log(\mu(A))/\log(s(n)) = 0$

$$\liminf_{n \mapsto \infty} -\frac{\log(\mu(C_{\max}^{(n)}))}{\log(s(n))} \leq \liminf_{n \mapsto \infty} \frac{\log(c_n(A))}{\log(s(n))}$$

$$\limsup_{n \mapsto \infty} -\frac{\log(\mu(C_{\min}^{(n)}))}{\log(s(n))} \geq \limsup_{n \mapsto \infty} \frac{\log(c_n(A) - 1)}{\log(s(n))}$$

by theorem 2.1.26

$$\begin{aligned}
\alpha &= \liminf_{n \mapsto \infty} -\frac{\log(\mu([x]_{i=1}^{n}))}{\log(s(n))} \\
&\leq \liminf_{n \mapsto \infty} -\frac{\log(\mu(C_{\max}^{(n)}))}{\log(s(n))} \\
&\leq \liminf_{n \mapsto \infty} \frac{\log(c_n(A))}{\log(s(n))}
\end{aligned}$$

and

$$\begin{aligned}
\beta &= \limsup_{n \mapsto \infty} -\frac{\log(\mu([x]_{i=1}^{n}))}{\log(s(n))} \\
&\geq \limsup_{n \mapsto \infty} -\frac{\log(\mu(C_{\min}^{(n)}))}{\log(s(n))} \\
&\geq \limsup_{n \mapsto \infty} \frac{\log(c_n(A) - 1)}{\log(s(n))} \\
&= \limsup_{n \mapsto \infty} \frac{\log(c_n(A))}{\log(s(n))}
\end{aligned}$$

since this is true for any set A of measure $\mu(A) > 1 - \delta$, we have the result.

$\square$

Chapter 3

Entropy Preserving Transformations

In the previous chapter we (re)introduced the notion of AC entropy, the critical dimension, and how they can be computed. In this chapter we look at transformations which preserve the critical dimension. The original motivation was [33, Prop 2.6.3(a)], which invited us to consider the class of permutations of a product measure which preserved AC entropy. The construction is then repeated for the more general Markov measures, and a notion of equivalence is explored in section 3.3.

3.1 The Lévy Group

Our study of transformations which preserve entropy begins with the study of asymptotic density. Particularly, transformations that preserve asymptotic density. The term "Lévy Group" appears after work on the critical dimension by [31, 33], where the latter uses this idea all but in name [33, Proposition 2.5.2]. Much has been said about the Lévy Group and invariant measures on the integers [3, 4, 44], which we highlight now.

Definition 3.1.1 (Lévy Group). *The Lévy Group $\mathcal{G}$ is the group of all permutations π of $\mathbb{N}$ such that*

$$\lim_{n\mapsto\infty} \frac{|k : k \leq n < \pi(k)|}{n} = 0$$

For $A \subseteq \mathbb{N}$, let $A(n) = A \cap (1, \cdots, n)$. Then the Lévy group can also be characterised as the set of all permutations such that

$$\lim_{n\mapsto\infty} \frac{A(n) \triangle \pi A(n)}{n} = 0$$

for every $A \subseteq \mathbb{N}$

Definition 3.1.2 (asymptotic density). *The asymptotic density of a set $A \subseteq \mathbb{N}$ is defined as*

$$d(A) = \lim_{n\mapsto\infty} \frac{A(n)}{n}$$

The set of all sets for which d is defined is denoted by $\mathcal{D}$.

Definition 3.1.3 (density measure)**.** *A density measure is a finitely additive measure on* $\mathbb{N}$ *which extends asymptotic density. That is, for some set* $\mathcal{P}$ $\mathcal{D} \subseteq \mathcal{P} \subseteq P(\mathbb{N})$ *and* $\lambda : \mathcal{P} \mapsto [0, 1]$ *such that*

1. $\lambda(\mathbb{N}) = 1$*;*

2. $\lambda(A \cup B) = \lambda(A) + \lambda(B)$ *for all disjoint* $A, B \subseteq \mathbb{N}$*;*

3. $\lambda\,|_{\mathcal{D}} = d$*. Which is to say* λ *coincides with* d *whenever* d *is defined.*

We shall cite two theorems about the Lévy Group.

Theorem 3.1.4 ([36, Theorem 2])**.** *For any injective function* $f : \mathbb{N} \mapsto \mathbb{N}$ *which preserves the existence of asymptotic density, i.e.* $A \in \mathcal{D} \implies f(A) \in \mathcal{D}$*, then* $d(f(A)) = \lambda d(A)$ *where* $\lambda = d(f(\mathbb{N}))$*.*

This says that constant multiples of asymptotic density are the *only* functions which preserve the existence of asymptotic density.

In our case the functions are permutations and $d(f(\mathbb{N})) = \lambda = 1$. So the permutations which preserve the existence of asymptotic density must preserve the actual value of the asymptotic density as well. A permutation which preserves the existence of asymptotic density must also preserve its value.

The next theorem characterises members of the Lévy Group as those permutations which preserve asymptotic density for all sets.

Theorem 3.1.5 ([3, Lemma 2]). *The following are equivalent*

1. $\pi \in \mathcal{G}$

2. *For any* $f \in \ell^\infty(\mathbb{N})$ *(bounded real functions on* $\mathbb{N}$*)*

$$\lim_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} (f(i) - f(\pi(i))) = 0$$

3. $\forall A \subset \mathbb{N}$,

$$\lim_{n \mapsto \infty} \frac{A(n) - (\pi A)(n)}{n} = 0$$

Let $f(i) = H(\mu_i) \leq 1$, then item 2 of theorem 3.1.5 connects the AC entropy with the Lévy group. Hence we can say

Corollary 3.1.6. *Suppose* $(X, \mathcal{B}, \mu, T)$ *is a product odometer with AC entropy* $\underline{h}_{AC}(\mu) = \overline{h}_{AC}(\mu)$. *A permutation* $\pi : \mathbb{N} \mapsto \mathbb{N}$ *preserves AC entropy for every product odometer iff* $\pi \in \mathcal{G}$

This says that if a permutation π preserves AC entropy for a product odometer - regardless of the co-ordinate measures - then π is a member of the Lévy group and vice versa. Theorem 3.1.4 can be used to extend this result, and say that no permutation can preserves the existence of AC entropy and change the value of AC entropy. As opposed to the previous similar statement for the density of integers.

We shall see later that $\pi \in \mathcal{G}$ preserves the upper and lower AC entropies too; but the converse is false by proposition 3.2.7.

There are also permutations, not in the Lévy group, that preserve AC entropy for a particular product measure μ - as opposed to every μ. For example, the permutations which are trivial in the sense that they only permute co-ordinates with the same co-ordinate measure: $\mu_i = \mu_{\pi i}$. Hence to include such permutations in our analysis, we must look beyond the Lévy group, and regard the actual value of the measure μ.

3.2 Permutations of a Product Measure

The following proposition proves, in a different way to corollary 3.1.6, that AC entropy is invariant under members of the Lévy Group. Unlike 3.1.6, it does not show that these are the only such permutations.

Proposition 3.2.1 (adapted from [33, Prop. 2.5.2]). *Suppose* $\mu = \otimes_{i=1}^{\infty}\mu_i$ *and let* ν *be the permuted measure* $\nu = \otimes_{i=1}^{\infty}\mu_{\pi(i)}$ *for some permutation* π. *If* $\pi \in \mathcal{G}$, *then* $\overline{h}_{AC}(\nu) = \overline{h}_{AC}(\mu)$ *and* $\underline{h}_{AC}(\nu) = \underline{h}_{AC}(\mu)$

Proof. Since π is a member of the Lévy group, for any $\epsilon > 0$, we can find N_ϵ such that for all $n > N_\epsilon$

$$\left|(1, \cdots, n)\triangle\pi^{-1}(1, \cdots, n)\right| < \epsilon n$$

define $A(n) = (1, \cdots, n) \triangle \pi^{-1}(1, \cdots, n)$. For any $n > N_\epsilon$

$$\begin{aligned}\left|\frac{1}{n}\left(\sum_{i=1}^{n} H(\mu_i) - H(\nu_i)\right)\right| &= \left|\frac{1}{n}\left(\sum_{i \in A(n)} H(\mu_i) - H(\nu_i)\right)\right| \\ &\leq \frac{|A(n)|}{n}\left(\max\left(H(\mu_i) - H(\nu_i)\right)\right) \\ &\leq \frac{|A(n)|}{n} \leq \epsilon\end{aligned}$$

So

$$\lim_{n \mapsto \infty} \left|\frac{1}{n}\left(\sum_{i=1}^{n} H(\mu_i) - H(\nu_i)\right)\right| = 0 \tag{3.1}$$

Because this limit exists, equation 3.1 can be separated from the lim inf

$$\begin{aligned}\underline{h}_{AC}(\mu) &= \liminf_{n \mapsto \infty} \frac{1}{n} H(\mu_i) \\ &= \liminf_{n \mapsto \infty} \frac{1}{n} H(\nu_i) + \frac{1}{n}\left(\sum_{i=1}^{n} H(\mu_i) - H(\nu_i)\right) \\ &= \liminf_{n \mapsto \infty} \frac{1}{n} H(\nu_i) + \lim_{n \mapsto \infty} \frac{1}{n}\left(\sum_{i=1}^{n} H(\mu_i) - H(\nu_i)\right) \\ &= \underline{h}_{AC}(\nu) + 0\end{aligned}$$

similarly the upper AC entropies are equal. □

Equation 3.1 can be seen as a weighted version of the definition of Lévy group 3.1.1. Indeed, equation 3.1 is sufficient to ensure equal entropy. We can extend this to a sufficient condition on permutations to preserve AC entropy.

Example 3.2.2. Let $X = \prod_{n=0}^{\infty} \mathbb{Z}_2$. Given some set $A \subseteq \mathbb{N}$ with lower and upper asymptotic density $\underline{d}$ and $\overline{d}$ respectively. Let μ be a product measure

$\mu = \prod_{n=0}^{\infty} \mu_n$, where $H(\mu_n) = 1$ when $n \in A$ and $H(\mu_n) = \frac{1}{2}$ for $n \notin A$. For any n,

$$\frac{1}{n}\sum_{i=0}^{n-1} H(\mu_i) = \frac{|A(n)|}{n}1 + \left(1 - \frac{|A(n)|}{n}\right)\frac{1}{2} \tag{3.2}$$
$$= \frac{|A(n)|}{n}\frac{1}{2} + \frac{1}{2}$$

Hence

$$\underline{h}_{AC}(\mu) = \liminf_{n \mapsto \infty} \frac{|A(n)|}{n}\frac{1}{2} + \frac{1}{2} = \underline{d}\frac{1}{2} + \frac{1}{2}$$

$$\overline{h}_{AC}(\mu) = \liminf_{n \mapsto \infty} \frac{|A(n)|}{n}\frac{1}{2} + \frac{1}{2} = \overline{d}\frac{1}{2} + \frac{1}{2}$$

Equation 3.2 shows that for any permutation, the upper and lower AC entropies must be a convex combination of the co-ordinate entropies.

Proposition 3.2.3 ([35, Proposition 4.4])**.** *For any $\lambda \in (0,1]$ and $c \in [0,1]$ there exists a type III_λ binary product odometer $(X, \mathcal{B}, \mu, T)$ with $h_{AC}(\mu) = c$*

Corollary 3.2.4. *For any $\lambda \in (0,1]$ and $\alpha, \beta \in [0,1], \alpha \leq \beta$ there exists a type III_λ binary product odometer $(X, \mathcal{B}, \mu, T)$ with $\underline{h}_{AC}(\mu) = \alpha$ and $\overline{h}_{AC}(\mu) = \beta$*

Proof. From proposition 3.2.3 there exist type III_λ product odometers with AC entropies α and β. Call them $(X, \mathcal{B}, \mu^\alpha, T)$ and $(X, \mathcal{B}, \mu^\beta, T)$

Now take any set of integers A with lower asymptotic density 0 and upper asymptotic density 1. Construct a new measure μ on X by $\mu_i = \mu_j^\alpha$ for $i \in A$ where $j = |A(i)|$ (i is the j'th member of A), and $\mu_i = \mu_j^\beta$ for $i \notin A$ where $j = i - |A(i)|$.

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} H(\mu_i) &= \left(\frac{1}{n}\sum_{i=0}^{n-1} 1_A(i) H(\mu_i)\right) + \left(\frac{1}{n}\sum_{i=0}^{n-1} 1_{\mathbb{N}-A}(i) H(\mu_i)\right) \\
&= \left(\frac{|A(n)|}{n}\right)\left(\frac{1}{|A(n)|}\sum_{i=0}^{|A(n)|} H(\mu_i^\alpha)\right) \\
&+ \left(1 - \frac{|A(n)|}{n}\right)\left(\frac{1}{n-|A(n)|}\sum_{i=0}^{n-|A(n)|} H(\mu_i^\beta)\right)
\end{aligned}$$

So the AC entropy is a convex combination of α and β. The extreme points of this interval are achieved. Take the sequence n_k such that $\lim\limits_{k\mapsto\infty} \frac{|A(n_k)|}{n_k} = 1$, then

$$\lim_{k\mapsto\infty} \frac{1}{n_k}\sum_{i=0}^{n_k-1} H(\mu_i) = \lim_{k\mapsto\infty}\left(\frac{1}{|A(n_k)|}\sum_{i=0}^{|A(n_k)|} H(\mu_i^\alpha)\right) = \alpha$$

The $\limsup$ of β is similarly achieved.

□

Proposition 3.2.5. *Suppose $\mu = \otimes_{i=1}^\infty \mu_i$ and permuted measure $\nu = \otimes_{i=1}^\infty \mu_{\pi(i)}$*

for some permutation π. If

$$\lim_{n\mapsto\infty}\left|\frac{1}{n}\left(\sum_{i=1}^{n}H(\mu_i)-H(\nu_i)\right)\right|=0$$

then $\overline{h}_{AC}(\nu)=\overline{h}_{AC}(\mu)$ and $\underline{h}_{AC}(\nu)=\underline{h}_{AC}(\mu)$

Proof. The proof is the same as the proof of proposition 3.2.1 from equation 3.1 onward.

$$\begin{aligned}\underline{h}_{AC}(\mu)&=\liminf_{n\mapsto\infty}\frac{1}{n}H(\mu_i)\\&=\liminf_{n\mapsto\infty}\frac{1}{n}H(\nu_i)+\frac{1}{n}\left(\sum_{i=1}^{n}H(\mu_i)-H(\nu_i)\right)\\&=\liminf_{n\mapsto\infty}\frac{1}{n}H(\nu_i)+\lim_{n\mapsto\infty}\frac{1}{n}\left(\sum_{i=1}^{n}H(\mu_i)-H(\nu_i)\right)\\&=\underline{h}_{AC}(\nu)+0\end{aligned}$$

□

There is a partial converse to proposition 3.2.5, however it requires that the upper and lower AC entropies to be equal.

Proposition 3.2.6. *Suppose $\mu=\otimes_{i=1}^{\infty}\mu_i$ and permuted measure $\nu=\otimes_{i=1}^{\infty}\mu_{\pi(i)}$ for some permutation π. If*

$$\overline{h}_{AC}(\mu)=\underline{h}_{AC}(\mu)=\overline{h}_{AC}(\nu)=\underline{h}_{AC}(\nu)$$

then

$$\lim_{n\mapsto\infty}\left|\frac{1}{n}\left(\sum_{i=1}^{n}H(\mu_i)-H(\nu_i)\right)\right|=0$$

Proof. By assumption, the limits

$$\lim_{n\mapsto\infty} \frac{1}{n}\sum_{i=1}^{n} H(\mu_i) = \lim_{n\mapsto\infty} \frac{1}{n}\sum_{i=1}^{n} H(\nu_i)$$

exist and are equal. By linearity of limits,

$$\lim_{n\mapsto\infty} \left| \frac{1}{n}\left(\sum_{i=1}^{n} H(\mu_i) - H(\nu_i)\right)\right| = 0$$

□

The next result shows that the condition that the AC entropies are equal cannot be omitted.

Proposition 3.2.7. *There are permutations which preserve the upper and lower AC entropies, but do not satisfy equation 3.1.*

Proof. We construct an example. Consider the measure in example 3.2.2, and choose $A \subset \mathbb{N}$ such that

$$\liminf_{n\mapsto\infty} \frac{|A(n)|}{n} = 0, \limsup_{n\mapsto\infty} \frac{|A(n)|}{n} = 1$$

hence the measure μ has upper and lower AC entropies of 1 and $\frac{1}{2}$ respectively.

Construct a new measure ν, again as in example 3.2.2, however using $A' = \mathbb{N} - A$ instead of A. This new measure clearly has the same upper and lower AC entropies. However equation 3.1 does not hold. By assumption

$\limsup_{n\mapsto\infty} \frac{A(n)}{n} = 1$, so $\forall\epsilon > 0$ there exists some sequence $\{n_k\}_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$

$$\frac{A(n_k)}{n_k} > 1 - \epsilon$$

$$\frac{1}{n_k}\sum_{i=1}^{n_k} H(\mu_i) = \frac{A(n_k)}{n_k}\frac{1}{2} + \frac{1}{2} > 1 - \frac{\epsilon}{2}$$

for that same sequence $\{n_k\}_{k\in\mathbb{N}}$

$$\frac{A'(n_k)}{n_k} = 1 - \frac{A(n_k)}{n_k} < \epsilon$$

$$\frac{1}{n_k}\sum_{i=1}^{n_k} H(\nu_i) = \frac{A'(n_k)}{n_k}\frac{1}{2} + \frac{1}{2} < \frac{1}{2} + \frac{\epsilon}{2}$$

so

$$\begin{aligned}\frac{1}{n_k}\sum_{i=1}^{n_k} H(\mu_i) - H(\nu_i) &> 1 - \frac{\epsilon}{2} - \frac{1}{2} - \frac{\epsilon}{2} \\ &= \frac{1}{2} - \epsilon\end{aligned}$$

Hence the limit of this sequence, if it exists, cannot be zero. Finally, since A and A' are both countable, there exists a bijection $\phi : A \mapsto A'$. Define a permutation $\pi : \mathbb{N} \mapsto \mathbb{N}$ by $\pi(i) = \phi(i)$ when $i \in A$ and $\pi(i) = \phi^{-1}(i)$ when $i \in A' = \mathbb{N} - A$. This proves that the product measure ν is a permutation of μ.

□

This example emphasises the fact that for two measures ν, μ the values of $\frac{\sum_{i=1}^{n} H(\mu_i)}{n}$ and $\frac{\sum_{i=1}^{n} H(\nu_i)}{n}$ may not grow together - even when the lim sup and lim inf are the same. This is important because it highlights the need to look not just at the values that are achieved by the sequence $\frac{\sum_{i=1}^{n} H(\mu_i)}{n}$, but also consider *when* they achieve them.

AC entropy of a randomly generated measure

We finish this section with one final example, which is an extension of example 3.2.2 with a finite number of distinct measures at each co-ordinate.

Example 3.2.8. Let $X = \prod_{n=0}^{\infty} \mathbb{Z}_2$ and let $\{H_k\}_{k\in K}, K \subseteq \mathbb{N}, |K| < \infty$ be the set of possible co-ordinate entropies for the product measure $\mu = \prod_{n=0}^{\infty} \mu_n$. Define

$$A_k = \{n : H(\mu_n) = H_k\}$$

then

$$\frac{1}{n}\sum_{i=0}^{n-1} H(\mu_i) = \sum_{k\in K} \frac{A_k(n)}{n} H_k$$

and

$$\liminf_{n\mapsto\infty} \sum_{k\in K} \frac{A_k(n)}{n} H_k = \underline{h}_{AC}(\mu) \leq \overline{h}_{AC}(\mu) = \limsup_{n\mapsto\infty} \sum_{k\in K} \frac{A_k(n)}{n} H_k \tag{3.3}$$

If the elements of $A_k \subseteq \mathbb{N}$ are chosen independently and at random with $i \in A_k$ with probability p, then by the law of large numbers $\lim_{n\mapsto\infty} \frac{|A_k(n)|}{n} = p_k$ and A_k has an asymptotic density: $A_k \in \mathcal{D}$.

Given a sequence $\{p_k\}_{k\in K}$, $K \subseteq \mathbb{N}$ of positive real numbers such that $\sum_{k\in K} p_k = 1$, independently allocate each integer to the set A_k with probability p_k. Then by the law of large numbers each set A_k has asymptotic density p_k. So equation 3.3 from example 3.2.8 becomes

$$\begin{aligned}
\underline{h}_{AC}(\mu) &= \liminf_{n\mapsto\infty} \sum_{k\in K} \frac{A_k(n)}{n} H_k \\
&= \lim_{n\mapsto\infty} \sum_{k\in K} p_k H_k \\
&= \limsup_{n\mapsto\infty} \sum_{k\in K} \frac{A_k(n)}{n} H_k \\
&= \overline{h}_{AC}(\mu)
\end{aligned}$$

from which we conclude

Lemma 3.2.9. *A product measure μ constructed by choosing co-ordinate measures $\{\mu_k\}_{k\in K}$ independently and at random according to some fixed probability distribution $\{p_k\}_{k\in K}$ has AC entropy*

$$h_{AC}(\mu) = \sum_{k\in K} p_k H(\mu_k)$$

3.3 Hurewicz Equivalence

In the previous section we saw that for product odometers, there exists a sub-class of permutations which preserve AC entropy. In this chapter we use the more general Markov odometers, and show that there exist a sub-class of orbit equivalence relations that preserve the critical dimension.

Loosely speaking, for large values of n the quantity

$$f_n^X(x) = \log(\sum_{i=0}^{n-1} \omega_i^X(x))/\log(n)$$

moves between α and β. The manner in which this is done is arbitrary. Two odometers $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ with the same critical dimensions

$$\liminf_{n\mapsto\infty} f_n^X(x) = \liminf_{n\mapsto\infty} f_n^Y(x) = \alpha$$

$$\limsup_{n\mapsto\infty} f_n^X(x) = \limsup_{n\mapsto\infty} f_n^Y(x) = \beta$$

may move between α and β in completely different ways. We define $k(n, x)$ as a scaling factor to bring one close to the other, so that $\lim_{n\mapsto\infty} f_n^X(x) - f_{k(n,x)}^Y(x) = 0$.

Definition 3.3.1. *Hurewicz less than:* $\leq_H$

An orbit equivalence $\phi : X \mapsto Y$ *between Markov odometers* $(X, \mathcal{B}, \mu, T)$ *and* $(Y, \mathcal{C}, \nu, S)$*, is "Hurewicz less than" denoted* $X \leq_H Y$*, if for all* $x \in X$ *there exists a function* $k(n, x)$ *such that* $\lim_{n\mapsto\infty} k(n, x) = \infty$ *and*

$$\lim_{n\mapsto\infty} \frac{\log(\sum_{i=0}^{n-1} \omega_i^X(x))}{\log(n)} - \frac{\log(\sum_{i=0}^{k(n,x)-1} \omega_i^Y(\phi(x)))}{\log(k(n,x))} = 0$$

Proposition 3.3.2. *Hurewicz less-than is reflexive and transitive.*

Proof. That $(X,\mathcal{B},\mu,T) \leq_H (X,\mathcal{B},\mu,T)$ is obvious, since $\phi(x) = x$ and $k(n,x) = n$ satisfy the definition. Furthermore, if $(X,\mathcal{B},\mu,T) \leq_H (Y,\mathcal{C},\nu,S)$ and $(Y,\mathcal{C},\nu,S) \leq_H (Z,\mathcal{D},\rho,U)$ then there exist orbit equivalences $\phi_1 : X \mapsto Y, \phi_2 : Y \mapsto Z$ and non-decreasing functions $k_1(n,x), k_2(n,y)$ such that

$$\lim_{n\mapsto\infty} \frac{\log(\sum_{i=0}^{n-1}\omega_i^X(x))}{\log(n)} - \frac{\log(\sum_{i=0}^{k_1(n,x)-1}\omega_i^Y(\phi_1(x)))}{\log(k_1(n,x))} = 0$$

$$\lim_{n\mapsto\infty} \frac{\log(\sum_{i=0}^{n-1}\omega_i^Y(y))}{\log(n)} - \frac{\log(\sum_{i=0}^{k_2(n,y)-1}\omega_i^Z(\phi_2(y)))}{\log(k_2(n,y))} = 0$$

Define the orbit equivalence $\phi = \phi_2 \circ \phi_1 : X \mapsto Z$ and $k(n,x) = k_2(k_1(n,x),x)$, this satisfies the definition of $\leq_H$ since

$$\begin{aligned}
&\lim_{n\mapsto\infty} \frac{\log(\sum_{i=0}^{n-1}\omega_i^X(x))}{\log(n)} - \frac{\log(\sum_{i=0}^{k_2(k_1(n,x),x)-1}\omega_i^Z(\phi_2(\phi_1(x))))}{\log(k_2(k_1(n,x),x))} \\
&\leq \lim_{n\mapsto\infty} \frac{\log(\sum_{i=0}^{n-1}\omega_i^X(x))}{\log(n)} - \frac{\log(\sum_{i=0}^{k_1(n,x)-1}\omega_i^Y(\phi_1(x)))}{\log(k_1(n,x))} \\
&+ \lim_{n\mapsto\infty} \frac{\log(\sum_{i=0}^{k_1(n,x)-1}\omega_i^Y(\phi_1(x)))}{\log(k_1(n,x))} - \frac{\log(\sum_{i=0}^{k_2(k_1(n,x),x)-1}\omega_i^Z(\phi_2(\phi_1(x))))}{\log(k_2(k_1(n,x),y))} \\
&= 0 + 0 = 0
\end{aligned}$$

□

The purpose of defining $\leq_H$ is made clear by the following lemma

Proposition 3.3.3. *Given two orbit equivalent Markov odometers* $(X, \mathcal{B}, \mu, T)$ *and* $(Y, \mathcal{C}, \nu, S)$. *If* $X \leq_H Y$ *then* $\alpha_Y \leq \alpha_X \leq \beta_X \leq \beta_Y$.

Proof. Since $\{k(n,x) : n \in \mathbb{N}\} \subseteq \mathbb{N}$

$$\begin{aligned}
\alpha_X &= \liminf_{n \mapsto \infty} \frac{\log(\sum_{i=0}^{n-1} \omega_i^X(x))}{\log(n)} \\
&= \liminf_{n \mapsto \infty} \frac{\log(\sum_{i=0}^{k(n,x)-1} \omega_i^Y(\phi(x)))}{\log(k(n,x))} \\
&\geq \liminf_{n \mapsto \infty} \frac{\log(\sum_{i=0}^{n-1} \omega_i^Y(\phi(x)))}{\log(n)} \\
&= \alpha_Y
\end{aligned}$$

the proof that $\beta_X \leq \beta_Y$ is similar, and $\alpha_X \leq \beta_X$ is true by definition. □

The converse is also true, although the construction of $k(n,x)$ is not useful beyond this proof.

Proposition 3.3.4. *Given orbit equivalent Markov odometers* $(X, \mathcal{B}, \mu, T)$ *and* $(Y, \mathcal{C}, \nu, S)$. *If* $\alpha_Y \leq \alpha_X \leq \beta_X \leq \beta_Y$, *then* $X \leq_H Y$

Proof. Define the sum

$$f_n^X(x) = \frac{\log(\sum_{i=0}^{n-1} \omega_i^X(x))}{\log(n)}$$

Given the orbit equivalence $\phi : X \mapsto Y$, and $\alpha_Y \leq \alpha_X \leq \beta_X \leq \beta_Y$. We are required to define $k(n,x)$ such that

$$\lim_{n \mapsto \infty} f_n^X(x) - f_{k(n,x)}^Y(\phi(x)) = 0$$

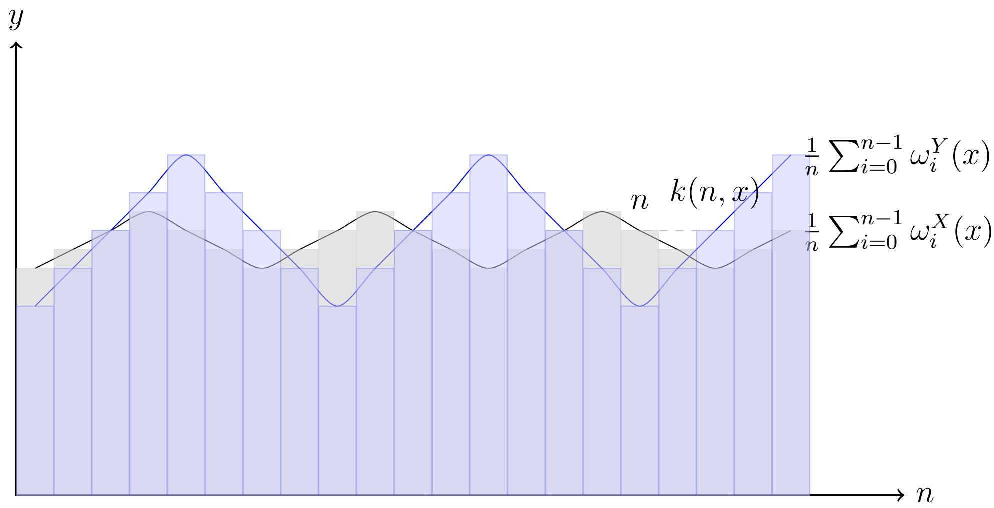

Figure 3.1: For each n, a corresponding $k(n, x)$ can be found. Note that in this picture $(Y, \mathcal{C}, \nu, S) \nleq_H (X, \mathcal{B}, \mu, T)$.

Take a non-increasing sequence $\epsilon_i > 0$ such that $\lim_{i \mapsto \infty} \epsilon_i = 0$. For each $i \in \mathbb{N}$ and μ-almost every $x \in X$ there exists some $N_{\epsilon_i,x}$ such that $\forall n > N_{\epsilon_i,x}$

$$\alpha_Y - \epsilon_i \leq f_n^Y(\phi(x)) \leq \beta_Y + \epsilon_i$$

and by corollary 1.2.6

$$\left| f_n^Y(\phi(x)) - f_{n+1}^Y(\phi(x)) \right| \leq \epsilon_i \tag{3.4}$$

Choose

$$n_i = \min \left\{ n > N_{\epsilon_i,x} : \left| f_n^Y(\phi(x)) - \alpha_Y \right| < \epsilon_i \right\}$$

$$m_i = \min \left\{ n > n_i > N_{\epsilon_i,x} : \left| f_n^Y(\phi(x)) - \beta_Y \right| < \epsilon_i \right\}$$

so $f^Y_{n_i}(\phi(x))$ is close to α_Y, and $f^Y_{m_i}(\phi(x))$ is close to β_Y and $n_i < m_i$. We can divide the interval $(\alpha - \epsilon_i, \beta + \epsilon_i)$ into M_i subintervals of length at most ϵ_i:

$$(\alpha - \epsilon_i, \beta + \epsilon_i) = \cup_{j=0}^{M_i - 1} I_j^{(i)}$$

where $I_j^{(i)} = (\alpha + (j-1)\epsilon_i, \alpha + j\epsilon_i]$ if $0 \leq j < M_i - 1$, and $I_{M_i-1}^{(i)} = (\alpha + (M_i - 2)\epsilon_i, \beta + \epsilon_i)$. Every interval contains at least one $f^Y_n(\phi(x))$ by 3.4 for some $n \in [n_i, m_i]$.

Define the function $k(i,x)$ as

$$k(i,x) = \begin{cases} k & \text{where} \quad f^X_i(x), f^Y_k(\phi(x)) \in I_j^{(i)} \text{ for some } k \in [n_i, m_i] \\ n_i & \text{if} \quad f^X_i(x) < \alpha_Y - \epsilon_i \\ m_i & \text{if} \quad f^X_i(x) > \beta_Y + \epsilon_i \end{cases}$$

It remains to be seen that this definition of $k(i,x)$ meets our requirements.

Given any $\epsilon > 0$ there exists an $N_{\epsilon,x}$ such that for all $i > N_{\epsilon,x}, \epsilon_i < \epsilon/2$, and using the fact that α_X, β_X are the critical dimensions

$$\alpha_Y - \epsilon/2 \leq \alpha_X - \epsilon/2 < f^X_i(x) < \beta_X + \epsilon/2 \leq \beta_Y + \epsilon/2$$

For each such i

1. either both $f^X_i(x), f^Y_{k(i,x)} \in I_j^{(i)}$ belong to the same interval of length $\epsilon_i < \epsilon/2 < \epsilon$, or

2. $k(i,x) = n_i$ and $\alpha_Y - \epsilon/2 < f_i^X(x) \leq \alpha_Y - \epsilon_i$, so

$$\begin{aligned}\left|f_i^X(x) - f_{k(i,x)}^Y(\phi(x))\right| &\leq \left|f_i^X(x) - \alpha\right| + \left|\alpha - f_{n_i}^Y(\phi(x))\right| \\ &\leq \epsilon/2 + \epsilon_i \\ &\leq \epsilon\end{aligned}$$

or

3. $k(i,x) = m_i$ and $\beta_Y + \epsilon_i \leq f_i^X(x) < \beta_Y + \epsilon/2$, so

$$\begin{aligned}\left|f_i^X(x) - f_{k(i,x)}^Y(\phi(x))\right| &\leq \left|f_i^X(x) - \beta\right| + \left|\beta - f_{m_i}^Y(\phi(x))\right| \\ &\leq \epsilon/2 + \epsilon_i \\ &\leq \epsilon\end{aligned}$$

Hence for all $i > N_{\epsilon,x}$

$$\left|f_i^X(x) - f_{k(i,x)}^Y(\phi(x))\right| < \epsilon$$

$\square$

Definition 3.3.5. *Hurewicz equivalence*

If $(X, \mathcal{B}, \mu, T) \leq_H (Y, \mathcal{C}, \nu, S)$ *and* $(Y, \mathcal{C}, \nu, S) \leq_H (X, \mathcal{B}, \mu, T)$ *then say the Markov odometers are Hurewicz equivalent*

Proposition 3.3.6. *Hurewicz equivalence is an equivalence relation.*

Proof. Since $\leq_H$ is a preorder (reflexive and transitive), the definition of Hurewicz equivalence makes this an equivalence relation. □

Theorem 3.3.7. *The Markov odometers* $(X, \mathcal{B}, \mu, T), (Y, \mathcal{C}, \nu, S)$ *are Hurewicz equivalent iff they are orbit equivalent and have the same upper and lower critical dimensions.*

Proof. This is a consequence of definition 3.3.5, proposition 3.3.3 and its converse 3.3.4.

□

The definition of Hurewicz equivalence emphasises *when* two odometers are similar. In the case where $\alpha_X = \beta_X = \alpha_Y = \beta_y$ it is easy to construct the required $k(n, x)$, as $k(n, x) = n$ will do. In some special cases this choosing $k(n, x) = n$ will also do when the critical dimensions are unequal. However this case is the exception, generally when the critical dimensions are unequal, $k(n, x)$ must be chosen different from n. Proposition 3.3.4 demonstrated that this can be done in theory, and theorem 4.2.1 demonstrates that this can sometimes be done in practice too. Indeed, the remainder of this thesis aims to demonstrate that definition 3.3.5 is a useful notion of equivalence through examples.

Metric Isomorphism

If $(X, \mathcal{B}, \mu, T)$ and $(Y, \mathcal{C}, \nu, S)$ are metrically isomorphic, then there exists a bi-measurable null-measure preserving map $\phi : X \mapsto Y$ such that for all $i \in \mathbb{N}$ and μ-almost every $x \in X$

$$\phi(T^i(x)) = S^i\phi(x)$$

Hence

$$\frac{d\nu \circ \phi}{d\mu}(T^i x)\omega_i^X(x) = \omega_i^Y(\phi(x))\frac{d\nu \circ \phi}{d\mu}(x)$$

and

$$\begin{aligned}\nu(\phi(X)) &= E(\frac{d\nu \circ \phi}{d\mu}) \\ &= \lim_{n \mapsto \infty} \frac{\sum\limits_{i=0}^{n-1} \frac{d\nu \circ \phi}{d\mu}(T^i x)\omega_i^X(x)}{\sum\limits_{i=0}^{n-1} \omega_i^X(x)} \\ &= \lim_{n \mapsto \infty} \frac{\frac{d\nu \circ \phi}{d\mu}(x) \sum\limits_{i=0}^{n-1} \omega_i^Y(\phi(x))}{\sum\limits_{i=0}^{n-1} \omega_i^X(x)}\end{aligned}$$

Taking logs, and dividing through by $\log(n)$, gives

$$\begin{aligned}&\lim_{n \mapsto \infty} \frac{1}{\log(n)} \left(\log\left(\sum_{i=0}^{n-1} \omega_i^Y(\phi(x))\right) - \log\left(\sum_{i=0}^{n-1} \omega_i^X(x)\right)\right) \\ &= \lim_{n \mapsto \infty} \frac{\log(\nu(\phi(X)))}{\log(n)} - \frac{\frac{d\nu \circ \phi}{d\mu}(x)}{\log(n)} = 0\end{aligned}$$

Choosing $k(n, x) = n$ shows that this is Hurewicz equivalence. This includes the case of Initial Co-ordinate Equivalence proposed in [33, Section 2.7].

Original Hurewicz Equivalence

Definition 3.3.5 originated from notion of equivalence proposed by [8], which stated that two orbit equivalent Markov odometers $(X, \mathcal{B}, \mu, T), (Y, \mathcal{C}, \nu, S)$ are *Original Hurewicz Equivalent*[1] iff for some $c, C \in \mathbb{R}$

$$0 < c \leq \liminf_{n \mapsto \infty} \frac{\sum_{i=0}^{n-1} \omega_i^X(x)}{\sum_{i=0}^{n-1} \omega_i^Y(\phi(x))} \leq \limsup_{n \mapsto \infty} \frac{\sum_{i=0}^{n-1} \omega_i^X(x)}{\sum_{i=0}^{n-1} \omega_i^Y(\phi(x))} \leq C < \infty$$

It was shown in [8] that two Original Hurewicz Equivalent Markov odometers have the same critical dimension. If orbit equivalent odometers have the same upper and lower critical dimensions, they are not necessarily Original Hurewicz Equivalent, which is a consequence of proposition 3.2.7. So Hurewicz Equivalence is genuinely different to Original Hurewicz Equivalence.

If $(X, \mathcal{B}, \mu, T), (Y, \mathcal{C}, \nu, S)$ are Original Hurewicz Equivalent, then setting $k(n, x) = n$ yields, for sufficiently large n

$$0 < \log(c) \leq \log(\sum_{i=0}^{n-1} \omega_i^X(x)) - \log(\sum_{i=0}^{n-1} \omega_i^Y(\phi(x))) \leq \log(C) < \infty$$

Dividing this equation through by $\log(n)$ shows that Original Hurewicz Equivalence is also Hurewicz Equivalent according to definition 3.3.5 with

[1] In this thesis we have have hijacked the name Hurewicz equivalence from [8], and refer to their definition as Original Hurewicz Equivalence

$k(n, x) = n$. Hence this definition generalises and extends Original Hurewicz Equivalence proposed by [8].

Canonical type III_λ odometer with critical dimensions α, β

We have seen in example 3.2.4 that the critical dimension is independent of the ratio set. That for each $\lambda \in (0, 1]$ there exists a odometer with arbitrary upper and lower critical dimension.

Definition 3.3.8 (The type $III_{\lambda,\alpha,\beta}$ odometer)**.** *Given λ, α, β such that $\lambda \in (0, 1]$ and $0 \leq \alpha \leq \beta \leq 1$, the product odometer constructed in example 3.2.4 with these parameters is type III_λ and has lower and upper critical dimensions α and β. Call this the canonical $III_{\lambda,\alpha,\beta}$ odometer.*

The canonical $III_{\lambda,\alpha,\beta}$ odometer is unique in the sense that

Proposition 3.3.9. *Any type III_λ odometer with critical dimensions α, β, is Hurewicz equivalent to the canonical type $III_{\lambda,\alpha,\beta}$ odometer.*

Proof. The odometers are orbit equivalent since they are of type III_λ. Since they have the same critical dimensions they are Hurewicz equivalent. □

By itself, this is an unexciting proposition. For any III_λ odometer, the critical dimensions cannot always be computed. All that has been shown so far is that the given odometer is Hurewicz equivalent to a non-specific canonical $III_{\lambda,\alpha,\beta}$ odometer. If the critical dimension can be computed, then this proposition becomes useful. Computing the critical dimension on a larger class of odometers is the purpose of the next chapter.

Chapter 4

The Induced Odometer

Perhaps the most important example of a Hurewicz equivalence is the induced odometer. It's importance stems from the role it plays in generating orbit equivalence of type III_0 odometers.

4.1 Orbit Equivalence and the Induced Odometer

Definition 4.1.1 (Induced Odometer)**.** *The induced odometer of a Markov odometer $(X, \mathcal{B}, \mu, T)$ is the odometer $(A, \mathcal{B}|_A, \nu, S)$, where $A \subseteq X$ is a set of positive measure, $\phi : X \mapsto A$ is a bi-measurable map, $\mathcal{B}|_A = \{A \cap B : B \in \mathcal{B}\}$ are measurable sets, the measure $\nu : A \mapsto [0,1]$ has derivative $0 < c \leq \frac{d\nu\circ\phi}{d\mu}(x) \leq C < \infty$ for some constants c, C and S is the induced transforma-*

tion $S(x) = T^{n_A(x)}(x)$.

For a type III Markov odometer $(X, \mathcal{B}, \mu, T)$ and for any set $A \in \mathcal{B}, \mu(A) > 0$ there exists a map $\phi : X \mapsto A$ which establishes an orbit equivalence between T and the induced transformation of T on A. Furthermore $\phi \in [\{T^i\}_{i \geq 1}]$. This often used property is seldom proved. In this section we re-prove this result, and in the following section extend the result to show that the orbit equivalence is in fact a Hurewicz equivalence.

The following is based on from [18] where it is presented in the more general context of a countable group of automorphisms $G = \{g_i\}_{i \in \mathbb{N}}$. Recall $[G] = [\{T^i\}_{i \in \mathbb{N}}]$, or $g \in [G]$ if there exists a countable partition A_i of X such that $\forall x \in A_i, g(x) = T^i x$. In this section we specialise the results of [18] to fit these standing assumptions.

Subsets $A, B \in \mathcal{B}$ are said to be *mutually* G*-equivalent* if there exists an isomorphism $g : A \mapsto B$ such that $gx = T^i x$ for all $x \in A_i$ for some countable partition A_i of A. The map g is called a G*-map* from A onto B. Notice that any G-map is a member of the full group $[\{T^i\}_{i \in \mathbb{N}}]$. Define a order relation $\leq$ on $\mathcal{B}$ by $A \leq B$ if there exists a G-map from A onto a subset of B. This relation is anti-symmetric since if $A \leq B$ and $B \leq A$ then let f map A into B, and g map B into A. Define $h : A \mapsto B$ by

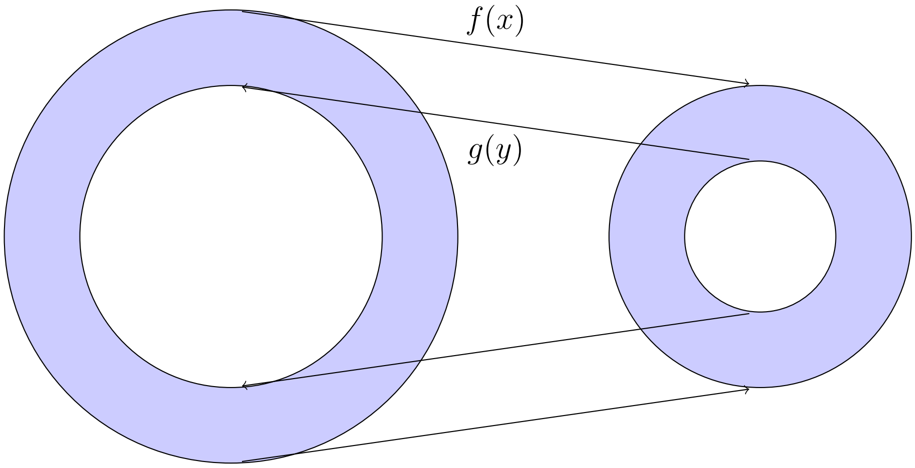

Figure 4.1: The Bernstien's map: the left shaded area represents $(\bigcup_{i=0}^{\infty}\{(gf)^iA - g(fg)^iB\}) \cup (\bigcap_{i=0}(gf)^i(x))$ and the enclosed white area represents $\bigcup_{i=0}^{\infty}\{g(fg)^iB - (gf)^iA\}$

$$h(x) = \begin{cases} f(x) & \text{if} \quad x \in \bigcup_{i=0}^{\infty}\{(gf)^iA - g(fg)^iB\} \\ f(x) & \text{if} \quad x \in \bigcap_{i=0}(gf)^i(x) \\ g^{-1}(x) & \text{if} \quad x \in \bigcup_{i=0}^{\infty}\{g(fg)^iB - (gf)^iA\} \end{cases}$$

Then h is a G-map and an isomorphism from A to B. Hence A and B are mutually G-equivalent. This construction is known as the *Bernstein's map* constructed by f and g. Since this relation is obviously reflexive and transitive, G-equivalence is an equivalence relation.

Any set $A \subset X$ is called G-infinite if it is G-equivalent to a proper subset of itself $A' \subset A$ and $\mu(A - A') > 0$. Otherwise A is said to be G-finite.

Lemma 4.1.2 ([18, Lemma 8]). *all G-infinite subsets, if they exist, are* *equivalent.*

Proof. Suffice to prove that a G-infinite subset is equivalent to the space X. Let g be a G-map from A onto a proper subset of A. By definition $B = A - gA$ has positive measure. By conservation $\cup_{i\geq 1} T^i B = X$ and we take the disjointification of these sets

$$\left\{ T^n B - \bigcup_{i=0}^{n-1} T^i B \right\}$$

as a partition of X. Define $f(x) = g^n T^{-n} x$ for $x \in \left\{ T^n B - \bigcup_{i=0}^{n-1} T^i B \right\}$. Then $f : X \mapsto A$ is a G-map from X into A. The Bernstien's map constructed from f and the identity map $1 : A \mapsto X$ gives a G-map from X onto A. □

Given a subset A of positive measure, the induced full group $[\{T^i\}_{i\in\mathbb{N}}]_A$ is the set of transformations $g : A \mapsto A$ such that g is a G-map from A to A.

Lemma 4.1.3 ([18, Lemma 9])**.** *For an ergodic nonsingular transformation T on measure space $(X, \mathcal{B}, \mu)$. If $A \subset X$ has positive measure and is G-infinite, then there exists an orbit equivalence $\phi : X \mapsto A$ between $(X, \mathcal{B}, \mu, T)$ and $(A, \mathcal{B}|_A, \mu \circ \phi^{-1}, T|_A)$, where the orbit equivalence $\phi \in [T]$.*

Proof. By lemma 4.1.2, there is a G-map ϕ from X onto A. Let A_i be the partition of X such that $\phi(x) = T^i x$ for all $x \in A_i$. Then $\cup_{i\geq 1} T^i A_i = A$.

For μ-almost every $x \in X$, there exists some $k \in \mathbb{N}$ such that $x \in A_k$. Then

$$\begin{aligned}
\phi(\mathrm{Orb}_T(x)) &= \phi\left(\left\{(T^j x)) : j \in \mathbb{Z}\right\}\right) \\
&= \bigcup_{i=1}^{\infty} \left(\phi\left(\left\{T^j(x) : j \in \mathbb{Z}\right\} \cap A_i\right)\right) \\
&= \bigcup_{i=1}^{\infty} \left(\left\{T^{j+i}(x) : j \in \mathbb{Z}\right\} \cap T^i(A_i)\right) \\
&= \bigcup_{i=1}^{\infty} \left(\left\{T^{j+i-k}(\phi(x)) : j \in \mathbb{Z}\right\} \cap T^i(A_i)\right) \\
&= \bigcup_{i=1}^{\infty} \left(\left\{T^j(\phi(x)) : j \in \mathbb{Z}\right\} \cap T^i(A_i)\right) \\
&= \bigcup_{i=1}^{\infty} \left(\left\{T|^j_{T^i A_i}(\phi(x)) : j \in \mathbb{Z}\right\}\right) \\
&= \left\{T|^j_A(\phi(x)) : j \in \mathbb{Z}\right\} \\
&= \mathrm{Orb}_{T|_A}(\phi(x))
\end{aligned}$$

It only remains to be said that ϕ is a null measure preserving isomorphism. That ϕ is an isomorphism is by definition of G-map. Suppose $\mu(B) = 0$, then $\mu(T^k(B \cap A_i)) = 0$ for all k and $\mu(\phi(B)) = \sum_{i=1}^{\infty} \mu(T^i(B \cap A_i)) = \sum_{i=1}^{\infty} \mu(B \cap A_i) = \mu(B) = 0$. The proof that $\mu(\phi(B)) = 0$ implies $\mu(B) = 0$ is similar. $\square$

Notice that because we chose the push-forward measure for the induced odometer, the derivatives $\frac{d\nu \circ \phi}{d\mu}(x) = 1$. However we only require the derivative to be bound away from zero and infinity.

The next theorem was proven by [16, 22]. Cited below is the version provided by [18].

Theorem 4.1.4 ([18, Theorem 11]). *Given a nonsingular, conservative, ergodic dynamical system* $(X, \mathcal{B}, \mu, T)$*, then* μ *is of type*

1. II_1 *iff* X *is* G*-finite.*
2. II_∞ *iff* X *is* G*-infinite, and contains a* G*-finite subset of positive measure.*
3. III *iff every subset with positive measure is* G*-infinite.*

The combination of lemma 4.1.3 and theorem 4.1.4 says that for any type III nonsingular system $(X, \mathcal{B}, \mu, T)$, and any subset of positive measure A, there is an orbit equivalence ϕ between $(X, \mathcal{B}, \mu, T)$ and $(A, \mathcal{B}|_A, \mu \circ \phi^{-1}, T|_A)$ which is a member of the full group. None of this is new, except for the emphasis on $\phi \in [G] = [\{T^i\}_{i \in \mathbb{N}}]$.

Control of the orbit equivalence relation

Let ω_i^X denote the i'th Radon-Nikodym derivative of $(X, \mathcal{B}, \mu, T)$ and ω_i^A denote the i'th Radon-Nikodym derivative of $(A, \mathcal{C}, \nu, S)$. For a set $Y \in \mathcal{B}$, define $n_Y^T(x)$ as the first return time of x to Y under the automorphism T (to distinguish it from the return time under S).

Lemma 4.1.5. *For a type III_0 Markov odometer $(X, \mathcal{B}, \mu, T)$ and induced odometer $(A, \mathcal{C}, \nu, S)$ where $\phi : X \mapsto A, \phi \in [T^i{}_{i\in\mathbb{N}}]$. There exists a subset $Y \subseteq X$ of positive measure such that for μ-almost every $x \in Y$*

$$S^{n^S_{\phi(Y)}(\phi(x))}\phi(x) = \phi T^{n^T_Y(x)}(x) \tag{4.1}$$

Proof. Since $\phi \in \left[\{T^i\}_{i\in\mathbb{N}}\right]$, then X can be partitioned into sets X_i such that $\forall x \in X_i, \phi(x) = T^i x$. At least one of these sets has positive measure, say X_k. Take this X_k as Y.

For any $y \in A \subset X$

$$\mathrm{Orb}^+_S(y) = \left\{S^i y : i > 0\right\} = A \cap \left\{T^i y : i > 0\right\} = \mathrm{Orb}^+_{T|_A}(y).$$

If we consider only those elements in $\phi(Y) \subseteq A$

$$\mathrm{Orb}^+_{S|_{\phi(Y)}}(y) = \mathrm{Orb}^+_{T|_{\phi(Y)}}(y).$$

Equate the first elements these ordered sets

$$S^{n^S_{\phi(Y)}(y)}(y) = T^{n^T_{\phi(Y)}(y)}(y)$$

or, written in terms of $x \in Y$.

$$S^{n^S_{\phi(Y)}(\phi(x))}\phi(x) = T^{n^T_{\phi(Y)}(\phi(x))}(\phi(x))$$

The tricky part is to disentangle ϕ from x, which we can do because both

$x, T^{n_Y^T(x)}(x) \in Y$ and we know how ϕ behaves on elements of Y.

$$\begin{aligned}
S^{n^S_{\phi(Y)}(\phi(x))}\phi(x) &= T^{n^T_{\phi(Y)}(\phi(x))}(\phi(x)) \\
&= T^{n_Y^T(x)}(\phi(x)) \\
&= T^{n_Y^T(x)+k}(x) \\
&= T^{k+n_Y^T(x)}(x) \\
&= \phi(T^{n_Y^T(x)}(x))
\end{aligned}$$

Where the last equality again uses the fact that $\phi = T^k$ for $T^{n_Y^T(x)}x \in Y$ $\square$

There is nothing sacred about the first return time, and this result can be extended to equate the n'th return times.

The set $A_k = \{x : \phi(x) = T^k x\}$ gave us a workable relation between the derivatives, but we also need a relation between the number of derivatives.

Mean Sojurn Time

A consequence of the conservation assumption is that for any set A of positive measure, μ-almost any $x \in X$ will return to A after finitely many steps. It is natural to ask, how often do the points $\{T^i x\}_{i\in\mathbb{N}}$ appear in the set A.

To make this question more precise. Define the upper and lower means sojurn times as the limit superior and limit inferior of the sequence

$$a_n(x) = \frac{1}{n}\sum_{k=0}^{n-1} 1_A(T^k x)$$

$$\overline{a}(x) = \limsup a_n(x)$$

$$\underline{a}(x) = \liminf a_n(x)$$

In regards to the asymptotic behavior of these quantities, the following is known.

- In the case of a measure preserving dynamical systems with $\mu(X) < \infty$, these quantities coincide as a consequence of the famous Birkhoff ergodic theorem (1.1.4). Furthermore they converge to the measure of the set $\mu(A)$.
- In the case of a measure preserving dynamical system with $\mu(X) = \infty$, these quantities coincide. Furthermore they converge to zero when $\mu(A) < \infty$.

For the odometers considered in chapter 2, we have the following lemma

Lemma 4.1.6. *If $(X, \mathcal{B}, \mu, T)$ is a Markov odometer satisfying assumption BV1, then for any set A of positive measure*

$$0 < \liminf_{n\mapsto\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_A(T^k x) \leq \limsup_{n\mapsto\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_A(T^k x) < 1$$

Proof. Since the σ-algebra $\mathcal{B}$ is generated by cylinders, there exists a cylinder $C \subset A$ of positive measure. If C has length N then by assumption BV1 the return time to C is bound above by $s(N+K)$. The average time that $T^k x$ spends in A is larger than average time that $T^k x$ spends in C, and if $n = iS(N+K) + j$, for some $i \in \mathbb{N}$ and $j < S(N+K)$ then

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1} 1_A(T^k x) &\geq \frac{1}{n}\sum_{k=0}^{n-1} 1_C(T^k x) \\
&\geq \frac{1}{(i+1)S(N+K)} \sum_{k=0}^{iS(N+K)-1} 1_C(T^k x) \\
&\geq \frac{i}{(i+1)S(N+K)}
\end{aligned}$$

hence we have proven the left hand side inequality

$$0 < \frac{1}{S(N+K)} \leq \liminf_{n\mapsto\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_C(T^k x) \leq \liminf_{n\mapsto\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_A(T^k x)$$

If we replace A with $X - A$, then this becomes

$$0 < \liminf_{n\mapsto\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_{X-A}(T^k x) = 1 - \limsup_{n\mapsto\infty} \frac{1}{n}\sum_{k=0}^{n-1} 1_A(T^k x)$$

which is the right hand side inequality.

$\square$

In general, these bounds are not known to hold [11, p. 6]. Furthermore, an extension of Rokhlin's lemma 4.1.7 was proven by [2, Theorem 1], which shows that there exist sets with unbounded return times provided the expected value of the return times is finite. Hence the assumption BV1 is a non-trivial assumption.

Theorem 4.1.7. *Given probability vector* $\pi = (\pi_1, \pi_2, \cdots)$ *with the property that the integers* $\{k : \pi_k > 0\}$ *are relatively prime and* $\sum_{i=1}^{\infty} i\pi_i < \infty$*, then there exists a measurable set* B*,* $\mu(B) > 0$ *such that* $\pi_i = \mu(x \in B : i = n_B(x))/\mu(B)$*.*

While assumption BV1 is familiar from computation of the critical dimension, it would be of interest to know if the same computation can be performed under the weaker assumption that the average sojurn time is bound away from 0 and 1.

Corollary 4.1.8. *Define* $k(n,x) = \sum_{k=0}^{n-1} 1_A(T^k x))$*, where the set* A *has measure* $0 < \mu(A) < 1$*, If BV1 holds, then*

$$\lim_{n \mapsto \infty} \frac{\log(k(n,x))}{\log(n)} = 1$$

Proof. Since $k(n,x) \leq n$, we have that $\frac{\log(k(n,x))}{\log(n)} \leq 1$. By lemma 4.1.6 for some $\delta > 0$ there exists an $N_{\delta,x}$ such that for all $n > N_{\delta,x}$

$$\begin{aligned} 0 < \delta &\leq \frac{k(n,x)}{n} \\ \log(\delta) &\leq \log(k(n,x)) - \log(n) \\ \log(n) + \log(\delta) &\leq \log(k(n,x)) \\ 1 + \frac{\log(\delta)}{\log(n)} &\leq \frac{\log(k(n,x))}{\log(n)} \end{aligned}$$

Hence

$$1 = \lim_{n\mapsto\infty} 1 + \frac{\log(\delta)}{\log(n)} \leq \lim_{n\mapsto\infty} \frac{\log(k(n,x))}{\log(n)} \leq 1$$

□

If we define $K(m,x)$ as the odometer power such that $T^{K(m,x)}x \in A$ for the m'th time. Then there is a useful relation between $k(n,x)$ and $K(m,x)$:

$$k(K(n,x),x) = n \tag{4.2}$$

4.2 Orbit Equivalence as Hurewicz Equivalence

Let us summarise what we have so far: $(X, \mathcal{B}, \mu, T)$ is a type III Markov odometer satisfying assumption BV1 and BV2 (see section 2.1) and $A \subset X$ a set of positive measure, the induced odometer is $(A, \mathcal{B}|_A, \nu, T|_A)$ is orbit equivalent to the Markov odometer X with orbit equivalence relation $\phi : X \mapsto A$. As we saw in the previous section, $\phi \in [T]$. Denote $Sx = T|_A x = T^{n_A(x)}x$ as the induced odometer, and there exists constants c, C such that $0 < c \leq \frac{d\nu\circ\phi}{d\mu}(x) \leq C < \infty$.

We also have some control over the orbit equivalence: for some set $A_k = \left\{x : \phi(x) = T^k x\right\}$ of positive measure

$$S^{n^S_{\phi(A_k)}(\phi(x))}\phi(x) = \phi T^{n^T_{A_k}(x)}(x)$$

Theorem 4.2.1. *If the Markov odometer $(X, \mathcal{B}, \mu, T)$ satisfies assumption BV1, then it is Hurewicz Equivalent to the induced Markov odometer $(A, \mathcal{C}, \nu, S)$.*

This proof is an application of the chain rule to equation 4.1, followed by two applications of the Hurewicz ergodic theorem 1.2.2. Corollary 4.1.8 makes an appearance at the end.

Proof. Applying the chain rule to equation 4.1 gives

$$\begin{aligned}
\omega^S_{n^S_{\phi(A_k)}(\phi(x))}(\phi(x)) &= \frac{d\nu \circ S^{n^S_{\phi(A_k)}(\phi(x))}}{d\nu}(\phi(x)) \\
&= \frac{d\nu \circ S^{n^S_{\phi(A_k)}(\phi(x))} \circ \phi}{d\nu \circ \phi}(x) \\
&= \frac{d\nu \circ \phi T^{n^T_{A_k}(x)}}{d\nu \circ \phi}(x) \\
&= \frac{d\nu \circ \phi}{d\mu}(T^{n^T_{A_k}(x)}(x)) \frac{d\mu \circ T^{n^T_{A_k}(x)}}{d\mu}(x) \frac{d\mu}{d\nu \circ \phi}(x)
\end{aligned}$$

Because $c = \frac{d\nu \circ \phi}{d\mu}(x)$

$$\omega^X_{n^T_{A_k}(x)}(x) = \omega^A_{n^S_{\phi(A_k)}(\phi(x))}(\phi(x)) \tag{4.3}$$

As remarked above, lemma 4.1.5 can be extended to equate the n'th return times. Let $k(n,x)$ be the number of times $T^i(x)$ returns to A_k for $i \leq n$. Then summing equation 4.3 over the first $k(n,x)$ elements

$$\frac{c}{C} \sum_{i=0}^{n-1} 1_{A_k}(T^i x)\omega^X_i(x) \tag{4.4}$$

$$\leq \sum_{i=0}^{k(n,x)-1} 1_{\phi(A_k)}(S^i \phi(x))\omega^A_i(\phi(x)) \tag{4.5}$$

$$\leq \frac{C}{c} \sum_{i=0}^{n-1} 1_{A_k}(T^i x)\omega^X_i(x) \tag{4.6}$$

According to the Hurewicz ergodic theorem, equation 4.5 grows in proportion to $\nu(\phi(A_k)) \sum_{i=0}^{k(n,x)-1} \omega^A_i(\phi(x))$, and equations 4.4, 4.6 grow in proportion to $\mu(A_k)\sum_{i=0}^{n-1}\omega_i^X(x)$. Hence they grow at the same rate as each other. More formally

$$\begin{aligned}
\frac{c}{C} &\leq \liminf_{n\mapsto\infty}\frac{\sum_{i=0}^{k(n,x)-1} 1_{\phi(A_k)}(S^i\phi(x))\omega_i^A(\phi(x))}{\sum_{i=0}^{n-1} 1_{A_k}(T^ix)\omega_i^X(x)} \\
&= \frac{\nu(\phi(A_k))}{\mu(A_k)}\liminf_{n\mapsto\infty}\frac{\sum_{i=0}^{k(n,x)-1}\omega_i^A(\phi(x))}{\sum_{i=0}^{n-1}\omega_i^X(x)} \\
&\leq \frac{\nu(\phi(A_k))}{\mu(A_k)}\limsup_{n\mapsto\infty}\frac{\sum_{i=0}^{k(n,x)-1}\omega_i^A(\phi(x))}{\sum_{i=0}^{n-1}\omega_i^X(x)} \\
&\leq \limsup_{n\mapsto\infty}\frac{\sum_{i=0}^{k(n,x)-1} 1_{\phi(A_k)}(S^i\phi(x))\omega_i^A(\phi(x))}{\sum_{i=0}^{n-1} 1_{A_k}(T^ix)\omega_i^X(x)} \\
&\leq \frac{C}{c}
\end{aligned}$$

Taking logs, and dividing through by $\log(n)$ shows that the limit

$$\lim_{n\mapsto\infty}\frac{\log\left(\sum_{i=0}^{n-1}\omega_i^X(x)\right)-\log\left(\sum_{i=0}^{k(n,x)-1}\omega_i^A(\phi(x))\right)}{\log(n)}=0$$

exists and is equal to zero. By lemma 4.1.6

$$\lim_{n\mapsto\infty}\frac{\log\left(\sum_{i=0}^{n-1}\omega_i^X(x)\right)}{\log(n)}-\frac{\log\left(\sum_{i=0}^{k(n,x)-1}\omega_i^A(\phi(x))\right)}{\log(k(n,x))}=0$$

So $X \leq_H A$. Choose $n = K(m,x)$ to be the m'th return time to A, that is to say the odometer power K such that $|A \cap \{T^i x : i < K\}| = m$. Then by equation 4.2, $k(K(n,x),x) = n$ and

$$\lim_{m \mapsto \infty} \frac{\log\left(\sum_{i=0}^{K(m,x)-1} \omega_i^T(x)\right)}{\log(K(m,x))} - \frac{\log\left(\sum_{i=0}^{m-1} \omega_i^S(\phi(x))\right)}{\log(m)} = 0$$

So $A \leq_H X$. Hence the measures are Hurewicz equivalent. □

Example 4.2.2 (Transformation induced on a cylinder)**.** Continuing from example 3.2.2, given a (full) binary odometer $(X, \mathcal{B}, \mu, T)$ with $X = \prod_{n=0}^{\infty} \mathbb{Z}_2$ with product measure[1] $\mu = \prod_{i \geq 0} \mu_i$ and AC entropies

$$\underline{h}_{AC}(\mu) = \liminf_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} H(\mu_i)$$

$$\overline{h}_{AC}(\mu) = \limsup_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} H(\mu_i)$$

If we take A to be a cylinder of length l and consider the induced odometer $(A, \mathcal{B}|_C, \nu, S)$ where $S = T|_A$ and ν is the normalised push-forward measure of μ. Then this measure has AC entropies

$$\begin{aligned}
\underline{h}_{AC}(\nu) &= \liminf_{n \mapsto \infty} \frac{1}{n} \sum_{i=l}^{n-1} H(\mu_i) \\
&= \liminf_{n \mapsto \infty} \frac{1}{n} \sum_{i=0}^{n-1} H(\mu_i) \\
&= \underline{h}_{AC}(\mu)
\end{aligned}$$

[1] Markov measure works equally well

Similarly $\underline{h}_{AC}(\nu) = \underline{h}_{AC}(\mu)$

Example 4.2.3 (Kakutani equivalence)**.** Two ergodic transformations $(X, \mathcal{B}, \mu, T)$ and $(X', \mathcal{B}', \mu', T')$ are *Kakutani Equivalent* if there exists subsets A,B such that $T|_A$ and $T'|_B$ are isomorphic. This is also a Hurewicz Equivalence if we assume that both X, X' satisfy BV1. By theorem 4.2.1 both μ and μ' are Hurewicz equivalent to their induced odometers. Since isomorphism is also a Hurewicz equivalence (see section 3.3) the odometers on the spaces X, A, B, X' are all Hurewicz equivalent.

4.3 Applications

Theorem 4.2.1 allows us to say a great many things about odometers that satisfy BV1

Corollary 4.3.1. *Given a type III_0 nonsingular measure $(X, \mathcal{B}, \mu, T)$. If the odometer of the associated flow is conservative, aperiodic and approximately transitive then it is orbit equivalent to a product odometer. If, in addition, $(X, \mathcal{B}, \mu, T)$ satisfies BV1, then the orbit equivalence is a Hurewicz equivalence.*

Proof. The orbit equivalence between $(X, \mathcal{B}, \mu, T)$ and a induced odometer has been shown to be a Hurewicz equivalence by theorem 4.2.1. The induced odometer is isomorphic to a a product odometer [17, Prop. 6] and therefore $(X, \mathcal{B}, \mu, T)$ is Hurewicz equivalent to a product odometer. $\square$

If in addition the product odometer from corollary 4.3.1 satisfies BV2 (see section 2.1), then the critical dimension is equal to the AC entropy [13]. Notice that ITPFI_2 factors trivially satisfy BV2. Because Hurewicz equivalence preserves the critical dimension, this lemma allows us to compute the critical dimensions of measures of product type. This is to be contrasted with corollary 2.1.20 which permitted computation of AC entropy for product odometers, and some Markov odometers. By corollary 4.3.1, we have been able to extend this to include Markov odometers which satisfy BV1 and are ITPFI_2.

Of the measures that are not of product type, we can say

Corollary 4.3.2. *Every type III_0 Markov odometer $(X, \mathcal{B}, \mu, T)$ is orbit equivalent to the full Markov odometer (as in example 2.1.4). If, in addition, $(X, \mathcal{B}, \mu, T)$ satisfies BV1, then the orbit equivalence is a Hurewicz equivalence.*

Proof. It was shown in [10, Theorem 1.1] that every nonsingular measure $(X, \mathcal{B}, \mu, T)$ is orbit equivalent to a full Markov odometer. Again this orbit equivalence was born of an induced odometer [10, p. 121] which is the full

Markov odometer. By theorem 4.2.1 the orbit equivalence is an Hurewicz equivalence. □

Again, we have from corollary 2.1.20 sufficient conditions for computing the critical dimension of a Markov odometer. If the induced odometer can be chosen to satisfy the conditions of corollary 4.3.1, then the critical dimension can be computed for the induced odometer and is equal to the critical dimensions of $(X, \mathcal{B}, \mu, T)$ by theorem 2.1.26.

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