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# Growth and trade models incorporating overlapping generations

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# GROWTH AND TRADE MODELS INCORPORATING OVERLAPPING GENERATIONS

by

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DECLARATION

I hereby declare that this thesis is my own work and has not been submitted, in part or in full, for a higher degree to any other University or Institution.

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ABSTRACT

The traditional two-by-two model of international trade assumes two primary factors of production, labour and capital. Labour services are supposed to be provided from a homogeneous stock of individuals in the sense that the services of a young person are equivalent to those of an old person. The rate of growth of the population has therefore no bearing on the pattern of production, demand and trade. The principal aim of the thesis is to develop growth and trade models which incorporate populations composed of overlapping, life-cycle-maximizing generations; each provides an economically distinguishable factor of production and is blessed with perfect foresight over its own lifetime.

As capital is assumed away the basic model is really a variant of the neoclassical two-sector economy. However, under the strong assumption of identical separable homothetic preferences, it is possible to obtain definite relationships between the rate of population growth on the one hand and the patterns of production and international trade in commodities on the other. In particular, it can be shown that technical progress is not necessarily Pareto-superior to no progress in a closed-barter economy without intergenerational borrowing and lending.

Adding exchange between generations to the basic model by the creation of an immortal clearing house, the "biological" theory of interest, which is an analogue of the familiar "golden-rule" proposition in production theory, is derived. For a class of utility functions, the social

contract of intergenerational borrowing and lending is shown to be dynamically stable in the "Classical" economy but not in the "Samuelson" economy.

Replacing indirect trade between generations by fiat money, the analysis reveals that, in general, maximizing behaviour by many short-lived economic agents blessed with perfect but myopic foresight does not lead to convergence. It is demonstrated that, for a broad class of utility functions, the world economy of one or many trading countries is unstable in the sense that, away from its unique steady state, either the real value of the stock of money approaches zero or the world monetary system breaks down altogether. In the latter event, the assumption of perfect lifetime knowledge is seen to be incompatible with the assumption that prices be non-negative.

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# CHAPTER I

# INTRODUCTION AND LITERATURE SURVEY

## 1.1 OBJECTIVE OF THE THESIS

The standard theorems of a conventional two-sector neoclassical economy are usually established in an atemporal context. In particular, one primary factor of production, labour services, is supposed to be provided from a stationary stock of homogeneous individuals. Implicitly assumed is therefore the fact that members of the population are infinitely long-lived. This highly implausible postulate is imposed in order to offset the intrinsic one-directional, open-ended nature of time. Provided that there are neither real externalities (in production and consumption) nor local satiation (in consumption) it can then be proved that every competitive equilibrium yields a Pareto-optimal allocation. Evidently it is not necessarily true that the "invisible hand" stretches over economies whose evolution extends towards an unbounded time horizon whereas individuals are short-lived.

This was first recognized, at least implicitly, in Malinvaud's landmark paper on capital theory in 1953. However, it only received its first explicit elaboration in a 1958 paper by Samuelson [20]. His classic contribution led to an interesting controversy which consequently resulted in a substantial expansion of the economic literature on overlapping generations and the intertemporal distribution of output.

The objective of this thesis is to extend Samuelson's argument to the two-by-two (closed or open) economy. The models being developed are different from the traditional two-factor-two-commodity model in two important respects. First, the population is a stream of short-lived agents each associated with a particular period or interval of time. Second, the young and the old provide distinguishable factors of production. (However, to stay within the two-by-two framework, capital is assumed away.) Before proceeding to present the analysis, the relevant literature on overlapping-generations models is briefly surveyed. This prepares the ground for the approach to be employed in the following chapters.

## 1.2 THE GENERAL FRAMEWORK

The early studies in this field mainly considered discrete-time, one-commodity models of a perfectly competitive and closed economy. (See, for example, [20], [5], [2], [21], [22], [7] and [3].) They may be characterized in the following general way:

(a) Population is not homogeneous but consists of individuals who live for a finite number (at least two) of time periods and belong to different but overlapping generations. Each generation consists of essentially identical agents who are blessed with perfect knowledge of the economy's structure over their lifetimes.

(b) The rate of growth of the population is constant but its value can be either negative, zero or positive.

(c) Production either does not occur or is assumed to be exogenously given. If production takes place then the labour force is assumed to be homogeneous in the sense that the services of a young person are equivalent to those of an old person. By allowing for heterogeneity in the labour force the models being considered in the present work deviate substantially from the early models.

(d) A typical person receives an amount of a nonstorable consumption good, either in return for labour expended or as a gift of the gods, in each period of his life. The amount received may depend on the age of the recipient and may be zero.

(e) Intergenerational borrowing and lending, either through a social contract or fiat money, may or may not be possible.

The authors of these early studies then concentrated mainly upon questions relating to the existence of equilibrium and to the properties of steady-state equilibria. However, with a few exceptions, not much effort has been made to examine the evolution of the economy from some arbitrary starting date and for arbitrary initial conditions. The following discussion is divided into four parts: pure exchange models, production models, international trade models and monetary models. (We shall take the liberty of deviating somewhat from the original notation in order to be consistent with those used throughout the thesis.)

## 1.3 PURE EXCHANGE MODELS

The pure exchange model of overlapping generations was first formulated in a pioneering paper by Samuelson [20]. A number of others have since then refined and extended his central argument. (See, for example, Cass and Yaari [2], Shell [21], Starrett [22] and Gale [7].)

The general formulation of the problem in these papers may be described as follows. Consider a population growing geometrically at a constant rate $\eta (\eta \gtreqless 0)$. Each member of the population lives for two (or more) periods by consuming a physical good which can neither be produced nor stored. A typical person receives or earns an endowment

$$w = (w^1, w^2)$$

representing his income in the two periods of his life. (In the early models $w^2$ is assumed to be equal to zero.) Let

$$C(t) = [C^1(t), C^2(t+1)]$$

be the lifetime consumption program of a person born in period t. Then feasibility in period t is given by the equation

$$(1+\eta)[w^1 - C^1(t)] + [w^2 - C^2(t)] = 0 \qquad (1\text{-}1)$$

Let $r(t)$ $(r(t) \gtreqless 0)$ be the t-th period rate of interest. Then the lifetime budget constraint of a typical man born in period t is

$$[1+r(t)][w^1 - C^1(t)] + [w^2 - C^2(t+1)] = 0 \qquad (1\text{-}2)$$

It is also assumed that each individual has identically stable preferences defined over his lifetime consumption profile.

There are at most two possible steady-state equilibria which can be characterized by (I) $r = \eta$ and (II) $C^1 = w^1$. In case (I) the program $\bar{C}$ is optimal stationary. Clearly, Samuelson's "biological" theory of interest is the analogue of the familiar proposition in production theory that golden rules are competitive, with the interest rate equal to the growth rate. The most striking feature of the "biological" rate of interest is the fact that it is a completely mechanistic construct, having no reference to impatience and time preference or, more generally, to the utility function. Let $r^o$ be the rate of interest associated with the "no-trade" equilibrium (II). In analyzing his case Samuelson noted that $r^o$ turns out to be less than $\eta$. In fact, it can be easily shown that $\bar{C}^1 > w^1$ ($\bar{C}^1 < w^1$) if and only if $r^o > \eta$ ($r^o < \eta$). (Since $w^2 = 0$ in Samuelson's original model, it is necessarily true that $\bar{C}^1 < w^1$.)

Another conclusion reached by Samuelson is the "Impossibility Theorem" which states that if the model is started at the biological time origin so that there are no old people but only young people with their endowment $w^1$, then the model will never, following a competitive path, approach the golden rule. For the two-period model this is a triviality since if initially both $w^2$ and $C^2(0)$ are zero then the only competitive equilibrium is the no-trade steady state given by $\{C(t) = (w^1, w^2): \ t=0,1,2,\ldots\}$. It is not too difficult to see that the no-trade equilibrium is Pareto-optimal in the "Classical" case ($\bar{C}^1 > w^1$) but not in the Samuelson case ($\bar{C}^1 < w^o$). Thus in the infinite-horizon model where a typical person wishes to save against old age the well-known welfare theorem claiming that every competitive equilibrium is Pareto-optimal certainly fails to hold. Most attempts at explaining this "paradoxical" result have concentrated on two general points:

(a) There is an infinite number of dated commodities and an infinite number of individuals engaged in Samuelson's endless economic process. This violates the standard axioms of general equilibrium theory. (As Shell [21] showed, the fact that all short-lived individuals cannot meet in a single market is not essential of the Samuelson result.) Suppose that time has a biological end, as well as a beginning and that births continue right up to Doomsday. Then it is evident that the no-trade equilibrium is also Pareto-optimal in the Samuelson case.

(b) Nonoptimality arises from the lack of intergenerational borrowing and lending. Cass and Yaari [2] constructed a closed-loop model which is an exact finite analogue of the infinite-horizontal model, demonstrating that without borrowing and lending between economic agents, the competitive mechanism is not necessarily Pareto-optimal despite the finiteness of the closed-loop model. Note also that in the absence of borrowing and lending the assumption of commodity nondurability is inessential to the Samuelson result where $w^2 = 0$.

Intergenerational borrowing and lending can take place by the creation of an immortal clearing house or the social contrivance of fiat money. The discussion of monetary models of overlapping generations will be postponed to 1.6. Through the clearing house indirect trade of commodities between generations can be conducted indefinitely into the future. However, the competitive equilibrium is again the no-trade steady state in the Classical case. Thus a Classical economy can never move from the no-trade to the golden-rule steady state without intervention from some kind of far-sighted authority.

The question of dynamic stability of the model has been examined by Gale [7] who found that the no-trade equilibrium is unstable in the Classical case but locally stable in the Samuelson case. Of course, Gale's result is based on the assumption that a typical person does not have perfect foresight over his lifetime. Otherwise, in the Samuelson case, a young man simply consumes $\bar{C}^1$ and saves $\bar{s} = w^1 - \bar{C}^1 > 0$ at the "biological" rate of interest $\eta$; and, as a consequence, there is no problem of nonstationary programs. Furthermore, what will happen to the dynamic stability of the economy if the relative age distribution of the population is variable? The evolution of a more general model with initially variable age distribution is considered by means of an example in Section 3.4 of this thesis.

## 1.4 PURE BARTER MODELS

An overlapping-generations model with capital and production has been studied by Diamond [5]. However, his line of approach will not be pursued in the present work.

Diamond's formulation may be stated in the following way. Consider a discrete-time, infinite-horizon economy with a single reproducible capital good. The population is made up of individuals who live for two periods, working in the first period and retired in the second. Denoting the subpopulations of young people and old people in period t by $N^1(t)$ and $N^2(t)$, respectively. Then

$$N^1(t) = N_0^1(1+\eta)^t \qquad (\eta \gtreqless 0) \qquad (1\text{-}3)$$

and

$$N^2(t) = N^1(t-1)$$

The economy's unchanging technology is represented by a constant-returns-to-scale production function,

$$Y(t) = F[K(t),\ N^1(t)] = N^1(t)f[k(t)] \qquad (1\text{-}4)$$

where $k(t) = K(t)/N^1(t)$ is the capital-labour ratio at time t. Each person has an ordinal utility function $U[C^1(t), C^2(t+1)]$ based on his consumption in the two periods of his life. (Since capital and output are the same commodity, one can consume one's capital.) Let

$$C(t) = N^1(t)C^1(t) + N^2(t)C^2(t)$$

be the aggregate consumption in period t. Then the division of resources on hand between the alternative uses can be stated algebraically:

$$Y(t) - [K(t+1)-K(t)] = C(t) = N^1(t)C^1(t) + N^2(t)C^2(t)$$

Suppose that the economy is controlled by a central planning authority who decides to preserve a constant capital-labour ratio, $k = K(t)/N^1(t)$ for all t. Then the above equation can be rewritten as

$$y - \eta k = C(t)/N^1(t) = C^1(t) + C^2(t)/(1+\eta) \qquad (1\text{-}5)$$

where $y = Y(t)/N^1(t) = f(k)$ is the output-labour ratio. The otpimal capital-labour ratio therefore satisifes the golden-rule-path condition that $f'(k^*) = \eta$. The problem of maximizing the social welfare is then given by

$$\text{Maximize } U[C^1(t), C^2(t)]$$

$$\text{subject to } C^1(t) + C^2(t)/(1+\eta) = f(k^*) - \eta k^* \qquad (1\text{-}6)$$

It is evident that the optimal allocation of consumption between individuals of different generations is the same as the "biological" optimum found by Samuelson. Thus the optimal rate of interest is determined by the rate of population growth.

In a competitive framework, a typical man allocates his wage $w(t) = \partial F[K(t), N^1(t)]/\partial N^1(t) = f[k(t)]-k(t)f'[k(t)]$ between current and future consumption. Let $r(t+1)$ be the rate of interest on one-period loans from t to t+1. Then his lifetime consumption program is $[C^1(t) = w(t)-s(t);$ $C^2(t) = (1+r(t+1))s(t)]$ where $s(t)$ is his saving when young. The optimal saving, given a wage level and a market interest rate, can be expressed as

$$\bar{s}(t) = \bar{s}[w(t), \bar{r}(t+1)] \tag{1-7}$$

Since the young people's savings are employed as capital for production, the equilibrium interest rate will equal

$$\begin{aligned}\bar{r}(t+1) &= \partial F[K(t+1), N^1(t+1)]/\partial K(t+1) = f'[K(t+1)/N^1(t+1)] \\ &= f'[\bar{s}[w(t), \bar{r}(t+1)]/(1+\eta)]\end{aligned} \tag{1-8}$$

($K(t+1) = N^1(t)\bar{s}(t)$, $N^1(t+1) = (1+\eta)N^1(t)$.) By altering the wage levels in period t, it is possible to trace out the equilibrium interest rates which would occur in period t+1,

$$\bar{r}(t+1) = \psi[\bar{w}(t)] = \psi[\phi(\bar{r}(t))] \tag{1-9}$$

where $w(t) = \phi(r(t))$ is the "factor reward frontier". Suppose that a long-run equilibrium exists, i.e.,

$$\lim_{t\to\infty} \bar{r}(t) = \bar{r} \tag{1-10}$$

There is no guarantee that $\bar{r} = \eta$. Thus it is seen that, despite the absence of all the usual sources of inefficiency, the competitive solution of an unbounded-horizon, consumption-loan economy can be golden-rule inefficient. This may be regarded as an analogue in production theory of Samuelson's "Impossibility Theorem".

## 1.5 INTERNATIONAL TRADE MODELS

The pattern of trade of a small open country of neo-Ricardian type has been considered by Kemp [14]. In his model, the population consists of individuals who live for two periods and work in each period of their lives. At any instant of time, the young and the old combine their distinguishable labour services to produce two goods under the conditions of constant returns with unequal factor intensities. Suppose further that the subpopulation of young people $N^1(t)$ and the subpopulation of old people $N^2(t)$ all grow at the same rate $\eta (\eta \gtreqless 0)$. Then the endowment ratio is

$$k(t) = N^1(t)/N^2(t) = N^1(t)/N^1(t-1) = 1+\eta \qquad (1\text{-}11)$$

for $t \geq 1$.

Given world prices there exist $\underline{k}$ and $\bar{k}$, $\underline{k} < \bar{k}$, such that for all $k \leq \underline{k}$ only the commodity relative intensive in its use of old labour is produced and for all $k \geq \bar{k}$ only the commodity relative intensive in its use of young labour is produced. (In the special case where each commodity is produced by means of just one kind of labour, $\underline{k} = 0$ and $\bar{k} = \infty$.) Further, if the community's preferences are homothetic then there exists a trade-switching rate $\eta^* = k^*-1$ ($\underline{k} < k^* < \bar{k}$) such that for rates of growth less than $\eta^*$ the commodity which is relative intensive in its use of young labour is imported and for rates of growth greater than $\eta^*$ the other commodity is imported. At $k = k^*$, production matches consumption and trade is extinguished.

Implicit in Kemp's argument is the assumption that $\eta = 1-k$ is not greater than the rest of the world's rate of growth. Otherwise, there exists a time such that the "small" country completely dominates the world market and, consequently, alone determines the international terms of trade.

The question of trading gains in an intertemporal context has been discussed by Kemp [13], Kareken and Wallace [9] and Fried [6]. Kemp studied trade gains in a pure consumption-loan model while Kareken and Wallace and Fried approached the problem in a Heckscher-Ohlin framework.

To consider the welfare effects of free commodity trade Fried constructed a special two-factor-two-commodity model of a small open economy. Each agent lives for two periods, providing the same labour services in both periods of his life. Population is constant at the level N and the number of members in each generation is N/2 at any time. The other factor of production is land which is fixed in supply. The production functions are linear in labour for the first industry and linear in land for the second industry. Each individual has identical time-separable logarithmic utility defined over his lifetime consumption profile. Then Fried concluded that a fall in the prices of the labour-intensive good (i.e., good one) due to the opening of the economy to free commodity trade will reduce the welfare of the young generation alive at the time and of all future generations. Only the old generation - the land holders at the time free trade is instituted - gains from trade.

Note that Fried's result only holds in the absence of intergenerational transfers. In such a situation his result can be proved under more general conditions as will be shown in 4.3.1. It is difficult, however, to accept his argument that "the costs of finding the incidence and providing compensation to those currently alive from every innovation that occurred in the past boggles the mind". ([6], p. 75.) What is really required is simply a social agreement that the old will transfer some of their gains from free trade to the young from period to period.

Kareken and Wallace formulated a model similar to that of Fried's, except that only the young people are productive. To prepare for the retirement years a young person must save by purchasing the other factor of production, i.e., land. There are two international economic policy regimes: a laissez-faire regime, characterized by free trade in goods and complete freedom in portfolio choice; and a portfolio autarky regime, characterized by free trade in goods and a world-wide prohibition on the ownership of foreign assets (land). Then the familiar welfare propositions of static trade theory carry over in the sense that the laissez-regime is Pareto-optimal and the portfolio autarky is not.

Kemp examined intertemporal gains from international trade in a very general pure exchange model in which the birth rate may be constant or variable and individuals may differ in preferences, life span, date of birth and income profile. There is one commodity, a perishable consumption good which can neither be produced nor stored. (The assumption of a single good is inessential to the analysis.) Further, each person knows with certainty his own life span, income profile and interest rates that will prevail during his lifetime. Intergenerational borrowing and lending is conducted by an immortal clearing house.

The essence of Kemp's argument can be stated as follows. Any closed-economy competitive equilibrium involves a market redistribution of the community's aggregate endowment for each period among the surviving individuals of the period. The rates of interest are therefore determined by the market-clearance condition and the given initial conditions. (The initial values of interest rate may be thought of as being arbitrarily chosen by the clearing house.) In any open economy it is possible to achieve the closed-economy competitive allocation by means of lump-sum taxes and subsidies. Any open-economy trading away from that allocation must then be to the advantage of each individual.

Two points deserve mention here. First, in a world of a single common commodity international trade means borrowing and lending between people of the various trading countries. Naturally, in a many-commodity world there can be both international exchange in commodities and international borrowing and lending. Second, the analysis is incomplete without a demonstration that the equilibrium international rate of interest exists in every period. As will be shown in 4.5, the existence of such a rate of interest is not always guaranteed.

## 1.6 MONETARY MODELS

The social contrivance of money is another way of removing the source of nonoptimality in unbounded overlapping-generations models. Even in a world of nondurable goods, there is nothing to stop men from printing money as a means of saving. Thus, suppose that fiat money is cleverly invented by the very first generation and subsequently accepted by each succeeding generation. Then the economy may move from the "no-trade" to the "biological" equilibrium forever. Models of overlapping generations incorporating money (or monies) explicitly have been studied by Kareken and Wallace [10], Cass, Okuno and Zilcha [3], Kemp and Long [16], and Chang, Kemp and Long [4].

Cass, Okuno and Zilcha showed that the consumption-loan model with heterogeneous individuals who differ in preferences and income endowments, yields the following counter-examples:

(a) Coexistence Example. There are both barter ("no-trade") and monetary equilibria which are Pareto-optimal. (Note that in Samuelson's basic model there is a monetary equilibrium if and only if there is no barter equilibrium which is Pareto-optimal.)

(b) Nonoptimality Example. There are both barter and monetary equilibria none of which is Pareto-optimal. (Note also that in the basic model with essentially identical individuals a monetary equilibrium, if it exists, is Pareto-optimal.)

In obtaining the above results they relied heavily on the technique of "reflected generational offer curves" defined by

$$\{[z(t),\, z(t+1)]:\ z(t) = -\sum_h (C_h^1(t)-w_h^1(t)),\ z(t+1) = \sum_h (C_h^2(t+1)-w_h^2(t+1))\}$$

where $C_h^j(t+j-1)$ and $w_h^j(t+j-1)$ are respectively the j-th period optimal consumption and given endowment of the h-th member of the t-th period generation ($j=1,2$; $t=0,1,2,\ldots$). Their approach is graphically orientated and will not be followed in the present work.

In [16] Kemp and Long consider the dynamic stability of an indefinitely-growing monetary closed economy. Their analysis can be restated in the following way. Each individual lives for two periods and receives an identical lifetime income profile $(w^1, w^2)$ in terms of a single good. Without loss, the total population is set to 2, so that each subpopulation is equal to 1. The stock of money in existence during period t is

$$M(t) = M_0\mu^t, \quad \mu \text{ constant}, \quad \mu > 0. \tag{1-12}$$

Of any increase in the stock of money, a constant proportion $\alpha$ ($0 \leq \alpha \leq 1$) is transferred to the young. Hence the monetary transfers to a typical person born in period t are

$$\alpha\Delta M(t) = \alpha[(\mu-1)/\mu]M_0\mu^t \tag{1-13}$$

when he is young and, when he is old,

$$(1-\alpha)\Delta M(t+1) = (1-\alpha)[(\mu-1)/\mu]M_0\mu^{t+1} \tag{1-14}$$

Let p(t) be the price of the commodity in terms of money in period t and m(t) the stock-demand for money by a young man in period t. Then his budget constraints are

$$C^1(t) = w^1 + (1/p(t))\{\alpha[(\mu-1)/\mu]M_0\mu^t - m(t)\} \tag{1-15}$$

and

$$C^2(t+1) = w^2 + (1/p(t+1))\{(1-\alpha)[(\mu-1)/\mu]M_0\mu^{t+1} + m(t)\} \tag{1-16}$$

In a competitive economy the equilibrium conditions resulting from maximizing $U[C^1(t),\ C^2(t+1)]$ subject to (1-15) and (1-16) reduce to

$$U_2[\overset{o}{C}{}^1(t),\ \overset{o}{C}{}^2(t+1)]/U_1[\overset{o}{C}{}^1(t),\ \overset{o}{C}{}^2(t+1)] = p^o(t+1)/p^o(t) \tag{1-17}$$

where $U_j = \partial U/\partial C^j$ $(j=1,2)$ and

$$\overset{o}{C}{}^1(t) = w^1 - M_0[1-\alpha(\mu-1)/\mu](\mu^t/p^o(t)) \tag{1-18-a}$$

$$\overset{o}{C}{}^2(t+1) = w^2 + M_0[1-\alpha(\mu-1)/\mu](\mu^{t+1}/p^o(t+1)) \tag{1-18-b}$$

Kemp and Long then specialized the model and showed that, for separable and logarithmic utility functions, the economy is unstable in the sense that, either the real value of the stock of money goes to zero or the monetary system breaks down altogether. (The monetary system breaks down when the demand by young people for money is less than the available supply.)

Chang, Kemp and Long [4] extended [16] by considering a world of several trading economies. The general formulation of their model is similar to that of [16] except a minor modification that money of the k-th country bears interest at a constant rate $r_k$ $(r_k \geq -1)$. Then it is possible to derive the time paths of the exchange rate between monies of any two countries and, under favourable circumstances, of money prices of individual countries. Suitably re-phrased, the earlier conclusions in a closed economy remain valid for a world economy. In particular it is shown that, unless the world economy is initially in a steady state, either the real value of each country's stock of money goes to zero or the world monetary system collapses. Furthermore, for each country, both the price level and the real stock of money may be non-monotone.

A crucial but implicit assumption of these models is that each generation has perfect foresight over its lifetime but is unable to revise the monetary agreement handed down to it from the older generation alive at the same time. Otherwise, the young people in any period could initiate a monetary "reform" which will immediately result in a golden-rule steady state. Such a reform can be achieved at a cost (or benefit) fully borne by the old generation alive at the time.

The organization of this thesis is as follows. Chapter II formulates and studies the basic model of a pure barter closed economy in which capital is assumed away but there are two factors of production, young labour and old labour. In this sense the basic model is simply a Samuelson variant of the traditional two-sector closed-economy model. Expanding the analysis - much in the manner of Gale - Chapter III examines the extended closed-barter-economy model in which intergenerational borrowing and lending is taken into consideration. Particular attention is given to the dynamic stability of the social contract of intergenerational trade. In Chapter IV international trade in commodities is formally introduced to the model. The barter trading world can therefore be thought of as a Heckscher-Ohlin generalization of the Samuelson overlapping-generations model. Chapter V is concerned with models of money and growth in the context of a closed economy whereas monetary open economies are considered in Chapter VI. The last two chapters are respectively straightforward extensions of the work of Kemp and Long [16] and Chang, Kemp and Long [4].

# CHAPTER II

# CLOSED BARTER ECONOMY WITH OVERLAPPING GENERATIONS

## 2.1 INTRODUCTION

In the traditional Ricardian-type model of a closed economy there is a single, homogeneous primary factor of production, labour. From the point of view of production, the services of a young person are equivalent to those of an old person. Thus, the rate of growth of the population has no bearing on the pattern of production. But suppose that the young and the old provide economically distinguishable factors of production. Then the rate of population growth determines the factor endowment ratio and therefore both the composition of total output and the pattern of consumption.

The purpose of this chapter is to formulate and examine the steady-state equilibria of a discrete-time, infinite-horizon, deterministic economy which can be characterized in the following way:

(i) The population is changing at a constant proportional rate over time and individuals live for two periods, working in both periods of their life.

(ii) The economy is perfectly competitive, producing two perishable consumption goods with the aid of two primary factors of production, young labour and old labour.

(iii) Each person has an ordinal, separable homothetic utility function based on his consumption of the two commodities in the two periods of his life.

(iv) The economy is completely closed, i.e., there is no trade with foreign countries.

(v) It is a barter economy, i.e., there is no money.

(vi) Intergenerational borrowing and lending and bequests are absent.

## 2.2 ASSUMPTIONS AND FORMULATION OF THE MODEL

The model considered here is of a completely closed, pure barter, neo-Ricardian economy. It begins operation in period 0 and continues over periods $t=1,2,3,...$ extending indefinitely into the future. Its characteristics may be described as follows.

### 2.2.1 Population

The population is made up of agents who are assumed to live for exactly two periods. Members of the same generation are thought of as being identical in all respects, so it is possible to speak of a typical member of a generation without specifying the individual. During any time period (except possibly at $t=0$), there are two types of people, the young and the old. Let the total population, the subpopulation of young people and the subpopulation of old people at time t be denoted by $N(t)$, $N^1(t)$ and $N^2(t)$, respectively. They are connected by the following basic relationships:

$$N(t) = N^1(t) + N^2(t) \qquad t=0,1,2,... \qquad (2\text{-}1)$$

$$N^2(t) = N^1(t-1) \qquad t=1,2,3,... \qquad (2\text{-}2)$$

Equation (2-2) incorporates the assumption that no one dies in midstream. Later in this section, the effect of removing this implicit assumption will be considered. There are at least three possible processes of population change.

Population Model One

Suppose that only the young people (i.e., those in the first period of their lives) are fertile. Then the subpopulation of young people in the present period consists of offspring of the corresponding subpopulation of young people in the immediately preceding period. Further, the subpopulation of young men changes geometrically at an exogeneously given constant rate $\eta_1$ over time, i.e.,

$$N^1(t) = (1+\eta_1)\ N^1(t-1) = \gamma_1\ N^1(t-1) \qquad t=1,2,3,... \qquad (2\text{-}3)$$

where $\eta_1 \gtreqless 0$, $\gamma_1 = 1+\eta_1 > 0$.

The population is growing or stationary or decaying according as $\eta_1$ is positive or zero or negative respectively. The solution to the linear, first-order difference equation (2-3) is given by

$$N^1(t) = N_0^1\ (1+\eta_1)^t = N_0^1\ \gamma_1^t \qquad t=0,1,2,... \qquad (2\text{-}4)$$

where $N_0^1$ is the initial subpopulation of young people. Utilizing relationships (2-2) and (2-1), it may be shown that

$$N^2(t) = N_0^1 \ (1+\eta_1)^{t-1} = N_0^1 \ \gamma_1^{t-1} \qquad t=1,2,3,\ldots \qquad (2\text{-}5)$$

and

$$N(t) = N_1 \ (1+\eta_1)^{t-1} = N_1 \ \gamma_1^{t-1} \qquad t=1,2,3,\ldots \qquad (2\text{-}6)$$

where $N_1 = N_1^1 + N_1^2 = (1+\gamma_1) \ N_0^1$ is the total population at time t=1.

It is obvious that the ratio of young to old people is one plus the growth rate of the population for $t \geq 1$:

$$N^1(t)/N^2(t) = 1+\eta_1 = \gamma_1 \qquad t=1,2,3,\ldots \qquad (2\text{-}7)$$

It is not necessary to assume that the old people existed at the beginning of the biological time. In general, the ratio $N^1(t)/N^2(t)$ at t=0 is given by

$$N^1(t)/N^2(t) = \begin{cases} N_0^1/N_0^2 \text{ if } N_0^2 \neq 0 \\ \\ \text{undefined if } N_0^2 = 0 \end{cases} \qquad \text{at } t=0 \qquad (2\text{-}8)$$

where $N_0^2$ is the initial subpopulation of old people. However, without loss of generality, it is assumed that both $N_0^1$ and $N_0^2$ are positive. In this population model, the young men-old men ratio takes at most one period to reach its steady-state value.

Population Model Two

In contrast to population model one, it may be assumed that only the old people (i.e., those in the second period of their lives) are fertile and the subpopulation of young people at the present is a constant multiple of the subpopulation of old people in the immediately preceding period. Let the growth rate be $\eta_2$. Then we have

$$N^1(t) = (1+\eta_2)\ N^2(t-1) = \gamma_2 N^2(t-1) \qquad t=1,2,3,\ldots \qquad (2\text{-}9)$$

where $\eta_2 \gtreqless 0$ and $\gamma_2 = 1+\eta_2 > 0$.

This population model leads to a second-order, linear difference equation in $N^1(t)$ whose solution is

$$N^1(t) = \{[(-1)^t(N_0^1-\gamma_2^{1/2}\ N_0^2) + (N_0^1+\gamma_2^{1/2}\ N_0^2)]\ \gamma_2^{t/2}\}/2 \qquad t=0,1,2,\ldots \qquad (2\text{-}10)$$

where $N_0^1$ and $N_0^2$ are the initial subpopulations of the young and the old, respectively. From (2-2) and (2-1),

$$N^2(t) = \{[(-1)^{t-1}(N_0^1-\gamma_2^{1/2}\ N_0^2) + (N_0^1+\gamma_2^{1/2}\ N_0^2)]\gamma_2^{(t-1)/2}\}/2 \qquad t=0,1,2,\ldots \qquad (2\text{-}11)$$

and

$$N(t) = \{[\{(-1)^t+1\}\gamma_2^{1/2}N_0 + \{1-(-1)^t\}N_1]\gamma_2^{(t-1)/2}\}/2 \qquad t=0,1,2,\ldots \qquad (2\text{-}12)$$

where $N_0$ and $N_1$ are respectively the total populations at time t=0 and t=1, i.e., $N_0 = N_0^1 + N_0^2$ and $N_1 = N_1^1 + N_1^2$.

From (2-10) and (2-11), it can be shown that

$$N^1(t)/N^2(t) = \begin{cases} N_0^1/N_0^2 & t=0,2,4,\ldots \\ (1+\eta_2)N_0^2/N_0^1 = \gamma_2 N_0^2/N_0^1 & t=1,3,5,\ldots \end{cases} \tag{2-13}$$

As will be seen later, this population model is not suitable for our purpose.

Population Model Three

More generally, it can be assumed that both the young and the old are fertile and that the two rates of fertility may differ one from the other. In symbols,

$$\begin{aligned} N^1(t) &= (1+\eta_1)\ N^1(t-1) + (1+\eta_2)\ N^2(t-1) \\ &= \gamma_1\ N^1(t-1) + \gamma_2\ N^2(t-1) \\ &\qquad t=1,2,3,\ldots \end{aligned} \tag{2-14}$$

where $\eta_i \gtreqless 0$, $\gamma_i \geq 0$ $(i=1,2)$.

This leads to a second-order, linear difference equation in $N^1(t)$ whose solution is

$$N^1(t) = A_1\lambda_1^t + A_2\lambda_2^t \qquad t=0,1,2,\ldots \tag{2-15}$$

where

$$\lambda_1 = (\gamma_1 + [\gamma_1^2 + 4\gamma_2]^{1/2})/2 > 0$$

$$\lambda_2 = (\gamma_1 - [\gamma_1^2 + 4\gamma_2]^{1/2})/2 \leq 0$$

$$A_1 = (\lambda_2 N_0^1 - N_1^1)/(\lambda_2 - \lambda_1) = (\lambda_1 N_0^1 + \gamma_2 N_0^2)/[\gamma_1^2 + 4\gamma_2]^{1/2} > 0$$

$$A_2 = (N_1^1 - \lambda_1 N_0^1)(\lambda_2 - \lambda_1) = -(\lambda_2 N_0^1 + \gamma_2 N_0^2)/[\gamma_1^2 + 4\gamma_2]^{1/2}$$

The non-negativity constraints on $\gamma_1$ and $\gamma_2$ ensure that $\lambda_1$ and $\lambda_2$ are real numbers. Therefore,

$$N^2(t) = N^1(t-1) = A_1\lambda_1^{t-1} + A_2\lambda_2^{t-1} \qquad t=1,2,3,.... \tag{2-16}$$

and

$$N(t) = N^1(t) + N^2(t)$$

$$= A_1(1+\lambda_1)\lambda_1^{t-1} + A_2(1+\lambda_2)\lambda_2^{t-1} \qquad t=1,2,3,... \tag{2-17}$$

It can be shown that as t approaches infinity, $\lambda_1^t$ will dominate $\lambda_2^t$ and the ratio $N^1(t)/N^2(t)$ will approach a finite limit which is equal to $\lambda_1$, i.e.,

$$\lim_{t\to\infty} N^1(t)/N^2(t) = \lambda_1 = (\gamma_1 + [\gamma_1^2 + 4\gamma_2]^{1/2})/2 \tag{2-18}$$

The usefulness of this result becomes apparent in the analysis of the steady-state equilibrium of the economic model. Note that

(a) if $\gamma_2=0$ then $\lim_{t\to\infty} N^1(t)/N^2(t) = N^1(t)/N^2(t) = \lambda_1 = \gamma_1$

which is the same as (2-7).

(b) if $\gamma_1=0$ then $|\lambda_1/\lambda_2| = 1$, $\lambda_1^t$ does not dominate $\lambda_2^t$ as t becomes large and $N^1(t)/N^2(t)$ does not approach a unique limit. In fact, as (2-13) shows, $N^1(t)/N^2(t)$ in this case alternates between two finite values, depending on whether t is odd or even.

Population Model Four

Reconsider the previous population model where

$$N^1(t) = \gamma_1 N^1(t-1) + \gamma_2 N^2(t-1) \qquad t=1,2,3,.....$$

Replacing $N^2(t-1)$ with $N^1(t-2)$ a second-order, linear difference equation in $N^1(t)$ is obtained. In this present model, the assumption that no one dies in midstream will be removed and the effect of the change analysed. Suppose that not all young men survive to become old men in the next period. Specifically, let us assume that death may occur to a person born at the start of period t at the end of the same period (or the beginning of the next period t+1) at a constant rate d $(0 \leq d < 1)$. Then we have

$N^2(t)$ = old people in period t = surviving young people of period t-1

$$= (1-d)\, N^1(t-1) = \pi N^1(t-1) \qquad t=1,2,3,... \qquad (2\text{-}19)$$

where $0 < \pi = 1-d \leq 1$.

Substituting (2-19) into (2-14) results in a second order, linear difference equation in $N^1(t)$ whose solution is given as follows.

$$N^1(t) = B_1\theta_1^t + B_2\,\theta_2^t \qquad t=0,1,2,... \qquad (2\text{-}20)$$

where

$$\theta_1 = (\gamma_1 + [\gamma_1^2 + 4\pi\gamma_2]^{1/2})/2 > 0$$

$$\theta_2 = (\gamma_1 - [\gamma_1^2 + 4\pi\gamma_2]^{1/2})/2 \leq 0$$

$$B_1 = (\theta_2 N_0^1 - N_1^1)/(\theta_2 - \theta_1) = (\theta_1 N_0^1 + \gamma_2 N_0^2)/[\gamma_1^2 + 4\pi\gamma_2]^{1/2}$$

$$B_2 = (N_1^1 - \theta_1 N_0^1)/(\theta_2 - \theta_1) = -(\theta_2 N_0^1 + \gamma_2 N_0^2)/[\gamma_1^2 + 4\pi\gamma_2]^{1/2}$$

The constants $B_1$ and $B_2$ are computed on the assumption that $N_1^1 = \gamma_1 N_0^1 + \gamma_2 N_0^2$, i.e., all the young people at time t=0 were fertile before some of them died off at the end of t=0. The subpopulation of old people and the total population at time t are then given respectively by

$$N^2(t) = \pi N^1(t-1) = \pi(B_1\theta_1^{t-1} + B_2\theta_2^{t-1}) \qquad t=1,2,3,\ldots \quad (2\text{-}21)$$

and

$$\begin{aligned} N(t) &= N^1(t) + N^2(t) \\ &= B_1(\theta_1+\pi)\theta_1^{t-1} + B_2(\theta_2+\pi)\theta_2^{t-1} \qquad t=1,2,3,\ldots \quad (2\text{-}22) \end{aligned}$$

Provided that $\gamma_1$ is not equal to zero, the ratio $N^1(t)/N^2(t)$ will approach a finite limit as t becomes infinitely large,

$$\lim_{t\to\infty} N^1(t)/N^2(t) = \theta_1/\pi = (\gamma_1 + [\gamma_1^2 + 4\pi\gamma_2]^{1/2})/2\pi \qquad (2\text{-}23)$$

Equation (2-23) is so far the most general formula. If $\pi = 1$, i.e., $d = 0$ then (2-23) reduces to (2-18). If $\pi = 1$ and $\gamma_2 = 0$ then (2-23) reduces to equation (2-7). If $\pi = 1$ and $\gamma_1 = 0$ then (2-23) reduces to (2-13), i.e., the limit in equation (2-23) does not exist.

### 2.2.2 The Production Sector

In each period, people subsist by consuming two perishable physical goods $X_i(t)$ $(i=1,2;\ \ t=0,1,2,\ldots)$ whose various quantities are subscripted by type and by the period in which they are available. The market for each good is perfectly competitive and firms (indefinite in number) produce the commodity with the aid of two primary factors of production, labour services provided by the young people and labour services by the old people. (The young are strong and masters of simple repetitive skills and the old have skills which can be obtained only by time spent on the job.)

Let $L_{ij}(t)$ be the amount of labour of the jth age group employed in the ith industry at time t. The ith production relationship is then written

$$X_i(t) = F_i[L_{i1}(t),\ L_{i2}(t)]$$

$$i=1,2;\quad t=0,1,2,\ldots \qquad (2\text{-}24)$$

Production functions are assumed to satisfy the following conditions.

(a) Both factors are indispensable.

$$F_i[0,\ L_{i2}(t)] = F_i[L_{i1}(t),\ 0] = 0$$

$$i=1,2;\quad t=0,1,2,\ldots (2\text{-}25\text{-}a)$$

(b) $F_i$ is homogeneous of the first degree in $L_{i1}(t)$ and $L_{i2}(t)$.

$$X_i(t) = L_{i2}(t)\ F_i[L_{i1}(t)/L_{i2}(t),\ 1]$$

$$= L_{i2}(t)\ f_i[k_i(t)]$$

$$i=1,2;\quad t=0,1,2,\ldots \qquad (2\text{-}25\text{-}b)$$

where $k_i(t) = L_{i1}(t)/L_{i2}(t)$ is the factor ratio in the ith industry at time t.

(c) Marginal products are positive but diminishing.

$$\partial F_i(t)/\partial L_{i1}(t) = f_i'[k_i(t)] > 0 \quad \text{if } k_i(t) > 0$$

$$i=1,2;\quad t=0,1,2,\ldots \qquad (2\text{-}25\text{-}c)$$

$$\partial F_i(t)/\partial L_{i2}(t) = f_i[k_i(t)] - k_i(t)\ f_i'[k_i(t)] > 0 \quad \text{if } k_i(t) > 0$$

$$i=1,2;\quad t=0,1,2,\ldots \qquad (2\text{-}25\text{-}d)$$

$$f_i''[k_i(t)] < 0 \qquad i=1,2;\quad t=0,1,2,\ldots \qquad (2\text{-}25\text{-}e)$$

(d) The Inada regularity conditions are satisfied.

$$\lim_{k_i(t)\to 0} f_i[k_i(t)] = \lim_{k_i(t)\to\infty} f'[k_i(t)] = 0$$

$$\lim_{k_i(t)\to\infty} f_i[k_i(t)] = \lim_{k_i(t)\to 0} f_i'[k_i(t)] = \infty$$

$$i=1,2;\quad t=0,1,2,\ldots \qquad (2\text{-}25\text{-}f)$$

Assuming perfect competition in the factor markets, let $w_1(t)$ and $w_2(t)$ be respectively the wage rates of young people and old people at time t. Competitive producers in the ith industry (i=1,2) solve the cost-minimizing problem

$$\underset{\{L_{i1}(t),L_{i2}(t)\}}{\text{Minimize}} \quad TC_i(t) = \sum_{j=1}^{2} w_j(t)L_{ij}(t)$$

$$\text{subject to } \bar{x}_i(t) = F_i[L_{i1}(t),L_{i2}(t)]$$

$$L_{ij}(t) \geq 0$$

$$j=1,2; \quad i=1,2; \quad t=0,1,2,\ldots \qquad (2\text{-}26)$$

where $\bar{x}_i(t)$ is a given level of output of good i at time t. Introducing a Lagrange multiplier $\mu(t)$, this is achieved by minimizing

$$\mathcal{L}[L_{i1}(t),L_{i2}(t),\mu(t)] = \sum_{j=1}^{2} w_j(t)L_{ij}(t) +$$

$$\mu(t)\{\bar{x}_i(t) - F_i[L_{i1}(t),L_{i2}(t)]\}$$

so that $L_{i1}(t)$ and $L_{i2}(t)$ are given as functions of $\bar{x}_i(t)$, $w_1(t)$ and $w_2(t)$ by the first-order necessary conditions for an interior solution.

$$w_j(t) = \mu(t)[\partial F_i(t)/\partial L_{ij}(t)] \text{ and } \bar{x}_i(t) = L_{i2}(t)f_i[k_i(t)]$$

$$j=1,2; \quad i=1,2; \quad t=0,1,2,\ldots$$

or

$$\omega(t) = w_2(t)/w_1(t) = [\partial F_i(t)/\partial L_{i2}(t)]/[\partial F_i(t)/\partial L_{i1}(t)] =$$

$$\{f_i[k_i(t)]/f'[k_i(t)]\}-k_i(t) \text{ and } \bar{x}_i(t) = L_{i2}(t)f_i[k_i(t)].$$

$$i=1,2; \quad t=0,1,2,\ldots \qquad (2\text{-}27)$$

For any given ratio of factor rewards at time t, $\omega(t) = w_2(t)/w_1(t)$, equations (2-27) determine uniquely a pair of cost-minimizing factor ratios at time t, $k_1(t)$ and $k_2(t)$. The minimum total cost $TC_i(t)$ of producing any given level of output $\bar{x}_i(t)$ is then also a function of $\bar{x}_i(t)$, $w_1(t)$ and $w_2(t)$. Define the average cost $AC_i(t)$ and the marginal cost $MC_i(t)$ by

$$AC_i(t) = \text{Min } TC_i(t)/\bar{x}_i(t) \qquad i=1,2; \quad t=0,1,2,\ldots$$

$$MC_i(t) = \partial \text{Min } TC_i(t)/\partial \bar{x}_i(t) \qquad i=1,2; \quad t=0,1,2,\ldots$$

Using the cost-minimizing conditions (2-27) and Euler's Theorem, it can then be shown that

$$\begin{aligned} \text{Min } TC_i(t)/\mu(t) &= \sum_{j=1}^{2} [w_j(t)/\mu(t)]\ L_{ij}(t) \\ &= \sum_{j=1}^{2} [\partial F_i(t)/\partial L_{ij}(t)]\ L_{ij}(t) = \bar{x}_i(t) \end{aligned}$$

$$i=1,2; \quad t=0,1,2,\ldots \qquad (2\text{-}28)$$

Hence,

$$AC_i(t) = \mu(t)$$

Also,

$$MC_i(t) = \partial \text{Min } TC_i(t)/\partial \bar{x}_i(t) = \sum_{j=1}^{2} w_j(t)[\partial L_{ij}(t)/\partial \bar{x}_i(t)]$$

$$= \frac{w_j(t)}{[\partial F_i(t)/\partial L_{ij}(t)]} \sum_{j=1}^{2} \left(\frac{\partial F_i(t)}{\partial L_{ij}(t)} \cdot \frac{\partial L_{ij}(t)}{\partial \bar{x}_i(t)}\right) = \mu(t)$$

$$i=1,2; \quad t=0,1,2,\ldots \qquad (2\text{-}29)$$

Equation (2-29) is derived from the cost-minimizing conditions (2-27) and the fact that $\partial \bar{x}_i(t)/\partial \bar{x}_i(t) = \sum_{j=1}^{2} [\partial F_i(t)/\partial L_{ij}(t)]\ [\partial L_{ij}(t)/\partial \bar{x}_i(t)] = 1$ (i=1,2; t=0,1,2,... )

Equilibrium in the product market requires that profit in each industry in any time period is driven to zero.

$$p_i(t) = MC_i(t) = AC_i(t) = \mu(t) = w_j(t)/[\partial F_i(t)/\partial L_{ij}(t)]$$
$$j=1,2; \quad i=1,2; \quad t=0,1,2,... \quad (2\text{-}30)$$

where $p_i(t)$, the price of the ith commodity at time t, is given by the interaction of market supply and demand. The equilibrium output of each individual firm is indeterminate but the total output of each industry is determined by demand conditions. Making use of the homogeneity of the first degree of $F_i$ (i=1,2) the necessary conditions for equilibrium become

$$w_1(t) = p_i(t)\ f_i'[k_i(t)]$$
$$w_2(t) = p_i(t)\ \{f_i[k_i(t)] - k_i(t)\ f_i'[k_i(t)]\}$$
$$i=1,2; \quad t=0,1,2,... \quad (2\text{-}31)$$

The equilibrium total output of the ith industry per old man at time t, $y_i(t) = X_i(t)/N^2(t)$, given commodity prices $p_1(t)$ and $p_2(t)$ and the overall endowment ratio $k(t) = N^1(t)/N^2(t)$, are

$$\begin{aligned} y_i(t) &= l_i(t)\ f_i[k_i(t)] \\ &= l_i[p_1(t),p_2(t),k(t)]f_i[k_i(p_1(t),p_2(t))] \\ &= y_i[p_1(t),p_2(t),k(t)] = y_i[p(t),k(t)] \end{aligned}$$
$$i=1,2; \quad t=0,1,2,... \quad (2\text{-}32)$$

where $l_i(t) = L_{i2}(t)/N^2(t)$ is the ratio between the old people employed in the ith industry and the subpopulation of old people and $p(t) = p_2(t)/p_1(t)$ is the price of the second good in terms of the first. We have already made use of the fact that $y_i(t)$ is homogeneous of degree zero in the two nominal prices $p_1(t)$ and $p_2(t)$.

The full-employment labour constraints of the system,

$$L_{1j}(t) + L_{2j}(t) = N^j(t) \qquad j=1,2; \quad t=0,1,2,\ldots$$

may be rewritten in the newly defined variables as follows.

$$\begin{aligned} k_1(t)l_1(t) + k_2(t)l_2(t) &= \sum_{i=1}^{2} k_i(t)l_i(t) \\ &= k(t) = N^1(t)/N^2(t) \\ l_1(t) + l_2(t) &= \sum_{i=1}^{2} l_i(t) = 1 \end{aligned}$$

$$t=0,1,2,\ldots \qquad (2\text{-}33)$$

The supply functions are illustrated by Figures 2.1 and 2.2. For a detailed treatment of how $y_i(t)$ are generated, refer to Kemp [11]. It is quite clear that

$$(-1)^i[\partial y_i(t)/\partial p(t)]_{k(t)=\text{constant}} \begin{cases} >0 \text{ if } p(t)\epsilon(\underline{p}^s(t),\bar{p}^s(t)) \\ =0 \text{ if } p(t)\not\epsilon(\underline{p}^s(t),\bar{p}^s(t)) \end{cases}$$

$$i=1,2; \quad t=0,1,2,\ldots \qquad (2\text{-}34)$$

where $\underline{p}^s(t)$ and $\bar{p}^s(t)$ are defined recursively as in [11].

$$(-1)^{i}[\partial y_i(t)/\partial k(t)]_{p(t)=\text{constant}} \quad = \quad f_i(t)/(k_2-k_1) \begin{cases} >0 \text{ if } k_1(t)<k(t)<k_2(t) \\ <0 \text{ if } k_1(t)>k(t)>k_2(t) \end{cases}$$

$$i=1,2; \quad t=0,1,2,... \qquad (2\text{-}35)$$

If we define S(t) as the ratio of the supply functions in terms of the first commodity at time t, i.e.,

$$S(t) = X_2(t)/X_1(t) = y_2(t)/y_1(t) = S[p(t), k(t)]$$

then it is quite evident that S(t) will respond to changes in p(t) and k(t) in much the same way as $y_2(t)$.

$$[\partial S(t)/\partial p(t)]_{k(t)=\text{constant}} \begin{cases} >0 \text{ if } p(t)\varepsilon(\underline{p}^s(t),\ \bar{p}^s(t)) \\ =0 \text{ if } p(t) \leq \underline{p}^s(t) \\ \text{undefined if } p(t) \geq \bar{p}^s(t) \end{cases}$$

$$t=0,1,2,... \qquad (2\text{-}36)$$

and

$$[\partial S(t)/\partial k(t)]_{p(t)=\text{constant}} \begin{cases} >0 \text{ if } k_1(t) < k(t) < k_2(t) \\ <0 \text{ if } k_1(t) > k(t) > k_2(t) \end{cases}$$

for $p(t)\epsilon(\underline{p}^s(t),\ \bar{p}^s(t))$ $\qquad t=0,1,2,... \qquad (2\text{-}37)$

The supply ratio S(t) is illustrated by Figure 2.3.

### 2.2.3 The Consumption Sector

#### The Two-period Utility Function

Consider a typical man who is born at time t (the start of period t). Suppose that his tastes can be summarized by an ordinal utility function

$$U(t) = U_t[C_1^1(t),\ C_2^1(t);\ \ C_1^2(t+1),\ C_2^2(t+1)] \qquad t = 0,1,2,\ldots \qquad (2\text{-}38)$$

where $C_i^j(t+j-1)$ is the quantity of the ith good that he purchases and consumes in the jth period of his life ($i=1,2;\ \ j=1,2$).

For simplicity, the following assumptions are made.

(a) The utility function U(t) is identical for all persons of the same age group and independent of time, i.e., it is the same for all individuals in every generation.

$$U_t = U \qquad t=0,1,2,\ldots \qquad (2\text{-}39)$$

(b) The utility function U is time-separable in the sense that the consumption of any good in one period does not affect in any way the satisfaction derived from the consumption of goods in the other period. More specifically, U is supposed to take the form

$$U(t) = \Omega[u(C_1^1,\ C_2^1);\ \ u(C_1^2,\ C_2^2)] \qquad t=0,1,2,\ldots \qquad (2\text{-}40)$$

where $\Omega_j > 0$, $\Omega_{jj} \le 0$ $(j=1,2)$, $\Omega_{12} = \Omega_{21} \ge 0$ and $\Omega_{11}\Omega_{22} - \Omega_{12}^2 \ge 0$.

(c) Furthermore, $u(C_1^j, C_2^j)$ $(j=1,2)$ is supposed to have all the usual indifference-curve properties. u is continuous and possesses continuous first and second order partial derivatives, is strictly concave and is strictly increasing.

(d) Finally, u is assumed to be homothetic in the sense that the indifference map $u(C_1^j, C_2^j) = \text{constant}$ $(j=1,2)$ consists of convex curves that are radial expansions or contractions of each other. It then follows that neither good is inferior and $[\partial u/\partial C_1^j]/[\partial u/\partial C_2^j]$ is a homogeneous function of degree zero in $C_1^j$ and $C_2^j$ and a decreasing function of the ratio $C_1^j/C_2^j$ alone $(j=1,2)$.

<u>The Budget Constraints</u>

Under perfectly competitive conditions, pure profit in each industry is driven to zero in equilibrium. Assuming that no person receives (or gives away) inherited wealth, each person's income is simply the wage rate which is equated to the value of the marginal product of his labour services. Thus, a person who is born in period t earns $[w_1(p(t)),\ w_2(t+1))]$. The trading mechanism of the model is as follows. In each period of his life, a typical man works in either industry, earns income and spends his earnings to purchase consumption goods. The economy is a barter system without money; each individual working in the ith industry will receive an amount of good i as the reward for his labour services and can use part of his income to trade for the other good according to the ruling terms of trade, i.e., $p(t)$ and $p(t+1)$. Implicit in the budget equations are the following assumptions.

(a) Income is never thrown away. This in turn follows from the assumption that marginal utility is positive.

(b) Goods are perishable and cannot be stored into the next period, i.e., intertemporal trade with Nature is impossible.

(c) An individual derives no utility by leaving bequests to his offspring.

(d) Intertemporal trade between generations is also non-existent.

Consequently, a person must consume all of his income in each period of his life and his lifetime consumption program $C(t) = [C_1^1(t),\ C_2^1(t);\ C_1^2(t+1),\ C_2^2(t+1)]$ is therefore constrained by

$$p_1(t)C_1^1(t) + p_2(t)C_2^1(t) = w_1(t)$$

$$p_1(t+1)C_1^2(t+1) + p_2(t+1)C_2^2(t+1) = w_2(t+1)$$

$$t=0,1,2,\ldots \qquad (2\text{-}41)$$

Dividing the two equations in (2-41) by $p_1(t)$ and $p_1(t+1)$, respectively, the budget constraints for an individual born in period t become

$$C_1^1(t) + p(t)C_2^1(t) = w^1(t)$$

$$C_1^2(t+1) + p(t+1)C_2^2(t+1) = w^2(t+1)$$

$$t=0,1,2,\ldots \qquad (2\text{-}42)$$

where

$$w^1(t) = w_1(t)/p(t) = w^1[p(t)] \text{ and}$$

$$w^2(t+1) = w_2(t+1)/p(t+1) = w^2[p(t+1)]$$

Consumer's Utility Maximization

Given perfect foresight about present and future wage rates and commodity price ratios, a rational consumer born in period t faces the following optimizing problem.

$$\underset{\{C(t)\}}{\text{Maximize}}\ U[C(t)] = \Omega[u(C_1^1, C_2^1);\ u(C_1^2, C_2^2)]$$

subject to the budget restraints (2-42) and

$$C_i^1(t) \geq 0, \quad C_i^2(t+1) \geq 0$$
$$i=1,2; \quad t=0,1,2,... \qquad (2\text{-}43)$$

(The problem confronting a typical old person in period $t=0$ is simply to maximize $u[C_1^2(0), C_2^2(0)]$ subject to $C_1^2(0) + p(0)C_2^2(0) = w^2(0)$.)

The choice problem described by (2-43) is equivalent to two separate maximizing problems. In the first period of his life, the consumer maximizes $u[C_1^1(t), C_2^1(t)]$ subject to his first budget equation. When he is old, the consumer maximizes $u[C_1^2(t), C_2^2(t)]$ subject to the second budget constraint. In the absence of uncertainty and of any intervening innovation, his lifetime consumption plan will be realized. The solutions to (2-43) are then his ordinary demand functions.

$$C_i^1(t) = C_i^1[p(t), w^1(p(t))] = C_i^1[p(t)]$$

$$C_i^2(t+1) = C_i^2[p(t+1), w^2(p(t+1)] = C_i^2[p(t+1)]$$

$$i=1,2; \quad t=0,1,2,... \qquad (2\text{-}44)$$

Since $w^1(t)$ is also a function of $p(t)$, $C_i^1(t)$ $(i=1,2)$ can be thought of as a function of $p(t)$ alone. Similarly, $C_i^2(t+1)$ depends upon $p(t+1)$ only. As the income vector of a consumer born in period t depends upon the commodity price ratios $p(t)$ and $p(t+1)$, the signs of $dC_i^1(t)/dp(t)$ and $dC_i^2(t+1)/dp(t+1)$ for $i=1,2$ are not clear. For example, consider $dC_2^1(t)/dp(t)$. Suppose that $p(t)$ rises, i.e., good 2 becomes relatively dearer than good 1 in period t. Then the consumer would consume less of good 2 provided that his income in period t, i.e., $w^1(t)$, remained unchanged. However, if the second industry is relatively young-labour intensive, i.e., $k_1(t) < k(t) < k_2(t)$ then from the Stolper-Samuelson Theorem an increase in $p(t)$ will result in an increase in the real reward of young labour, $w^1(t)$. Thus, if the increase in $w^1(t)$ is of sufficient magnitude to outweigh the effect of an increase in $p(t)$, the individual consumer would consume more rather than less of good 2 in period t. To demonstrate the idea more formally, we shall make use of a result due to Burk [1] who proved that, under the assumptions of homothetic utility function and interior equilibrium, demand functions exhibit expenditure proportionality.

$$C_i^1(t) \quad = \quad C_i^1[p(t),\ w^1(t)] \quad = \quad \Phi_i[p(t)]\ w^1[p(t)]$$
$$i=1,2; \quad t=0,1,2,\ldots \qquad (2\text{-}45)$$

$$C_i^2(t+1) = C_i^2[p(t+1),\ w^2(t+1)] = \Phi_i[p(t+1)]\ w^2[p(t+1)]$$

where $\phi_1'(.) \geq 0$ and $\phi_2'(.) < 0$.

Now,

$$dC_2^1(t)/dp(t) = \phi_2'[p(t)]w^1[p(t)] + \phi_2[p(t)]\{dw^1[p(t)]/dp(t)\}$$
$$t=0,1,2,\ldots \qquad (2\text{-}46)$$

Since $dw^1[p(t)]/dp(t) = f_2[k_2(t)]/[k_2(t)-k_1(t)] > 0$ for $k_1(t) < k_2(t)$, the two terms in the right hand side of equation (2-46) are of opposite sign and the sign of $dC_2^1(t)/dp(t)$ is therefore unambiguous.

Fortunately, under the assumption that everyone has identical homothetic preferences, community indifference curves and, hence, the total market demand curve for each good can be constructed. The curves will exhibit the usual "law of demand" in certain ranges of the commodity price ratio p(t). Let Y(t) be the value of the total outputs of both commodities in terms of the first good at time t, i.e.,

$$Y(t) = X_1(t) + p(t)X_2(t) \qquad t=0,1,2,\ldots \qquad (2\text{-}47)$$

Let $D_i(t)$ be the total market demand for good i at time t. Then we have

$$D_i(t) = N^1(t)C_i^1(t) + N^2(t)C_i^2(t)$$
$$= D_i[Y(t),\ p(t)] \quad i=1,2; \quad t=0,1,2,\ldots \qquad (2\text{-}48)$$

Since Y(t) is also a function of p(t), $D_i(t)$ (i=1,2) can be regarded as depending on p(t) alone. Taking the total derivative of $D_i(t)$ with respect to p(t),

$$\frac{dD_i(t)}{dp(t)} = \left[\frac{\partial D_i(t)}{\partial Y(t)}\right]_{p(t)} \frac{dY(t)}{dp(t)} + \left[\frac{\partial D_i(t)}{\partial p(t)}\right]_{Y(t)}$$

$$i=1,2; \quad t=0,1,2,... \qquad (2\text{-}49)$$

From (2-47),

$$dY(t)/dp(t) = [dX_1(t)/dp(t)]+p(t)[dX_2(t)/dp(t)]+X_2(t) = X_2(t)$$

because $[dX_1(t)/dp(t)]+p(t)[dX_2(t)/dp(t)] = 0$ along the transformation curve $X_2(t) = X_2[X_1(t)]$ (t=0,1,2,...)

The second term in the R.H.S. of (2-49) is given by the Slutsky equation with Y(t) held constant,

$$\frac{\partial D_i(t)}{\partial p(t)} = \left[\frac{\partial D_i(t)}{\partial p(t)}\right]_{u=\bar{u}} - D_2(t)\left[\frac{\partial D_i(t)}{\partial Y(t)}\right]_{p(t)=\bar{p}}$$

$$i=1,2; \quad t=0,1,2,...$$

Substituting these results into (2-49) yields

$$\frac{dD_i(t)}{dp(t)} = X_2(t)\left[\frac{\partial D_i(t)}{\partial Y(t)}\right]_{p(t)=\bar{p}} + \left[\frac{\partial D_i(t)}{\partial p(t)}\right]_{u=\bar{u}} - D_2(t)\left[\frac{\partial D_i(t)}{\partial Y(t)}\right]_{p(t)=\bar{p}}$$

$$= [X_2(t)-D_2(t)]\left[\frac{\partial D_i(t)}{\partial Y(t)}\right]_{p(t)=\bar{p}} + \left[\frac{\partial D_i(t)}{\partial p(t)}\right]_{u=\bar{u}}$$

$$i=1,2; \quad t=0,1,2,... \qquad (2\text{-}50)$$

In equilibrium, $X_2(t) = D_2(t)$, only the pure substitution effect is operative and (2-50) becomes

$$dD_i(t)/dp(t) = [\partial D_i(t)/\partial p(t)]_{u=\bar{u}} \begin{cases} > 0 & i=1 \\ < 0 & i=2 \end{cases}$$

$$t=0,1,2,... \qquad (2\text{-}51)$$

Equation (2-51) is a local result which is only valid in a sufficiently small neighbourhood of the equilibrium point. There is no guarantee that (2-51) will hold globally for all values of $p(t)$. Suppose now that the equilibrium price ratio in period t, $p^o(t)$, exists uniquely. Then it is clear that

If $p(t) \leq p^o(t)$ then $X_2(t)-D_2(t) \leq 0$, i.e., $dD_2(t)/dp(t) < 0$.

If $p(t) \geq p^o(t)$ then $X_2(t)-D_2(t) \geq 0$, i.e., $dD_1(t)/dp(t) > 0$.

$$t=0,1,2,... \qquad (2\text{-}52)$$

because $\partial D_i(t)/\partial Y(t) > 0$ $(i=1,2;\ t=0,1,2,...)$. This follows from the fact that neither commodity is inferior.

Furthermore, if the equilibrium point is an interior solution, it is evident that the ratio $C_2^j(t)/C_1^j(t)$ is independent of $w^j(t)$ $(j=1,2;\ t=0,1,2,...)$:

$$C_2^j(t)/C_1^j(t) = \{\phi_2[p(t)w^j[p(t)]\}/\{\phi_1[p(t)]w^j[p(t)]\}$$

$$= \phi_2[p(t)]/\phi_1[p(t)] = \sigma[p(t)]$$

$$j=1,2;\ t=0,1,2,... \qquad (2\text{-}53)$$

Now $d\sigma[p(t)]/dp(t) = \{\phi_2'[p(t)]\phi_1[p(t)]-\phi_2[p(t)]\phi_1'[p(t)]\}/\phi_1^2[p(t)]$ and since $\phi_1'(.) \geq 0$, $\phi_2'(.) < 0$, this expression is negative; hence the ratio $C_2^j(t)/C_1^j(t)$ $(j=1,2;\ t=0,1,2,..)$

is a strictly decreasing function of $p(t)$. The total demand curves $D_1(t)$, $D_2(t)$ and the ratio of total demands $D(t) = D_2(t)/D_1(t)$ are depicted in Figures 2.4, 2.5 and 2.6, respectively.

## 2.3 ANALYSIS OF THE MODEL

### 2.3.1 Equilibrium Conditions

Without trade with the outside world, the equilibrium of the economy described above, in any period of time, is attained by equating the aggregate supply and aggregate demand for each good. This may be expressed in one of the following ways.

#### Individual Commodities

Let $X_i(t)$ and $D_i(t)$ be respectively the total output of and demand for good i at time t. Then,

$$\begin{aligned} X_i(t) &= N^2(t)y_i(t) = N^2(t)l_i(t)f_i[k_i(t)] \\ &= N^2(t)l_i[p(t),k(t)]f_i[k_i(p(t))] \\ &\qquad\qquad i=1,2; \quad t=0,1,2,\ldots \end{aligned}$$

$$\begin{aligned} D_i(t) &= N^1(t)c_i^1(t) + N^2(t)c_i^2(t) \\ &= N^2(t)\{k(t)c_i^1[p(t)] + c_i^2[p(t)]\} \\ &\qquad\qquad i=1,2; \quad t=0,1,2,\ldots \end{aligned}$$

Equilibrium of the economy requires

$$X_i(t) = D_i(t) \qquad i=1,2; \quad t=0,1,2,\ldots$$

$$l_i[p(t),k(t)]\ f_i[k_i(p(t))] = k(t)C_i^1[p(t)] + C_i^2[p(t)]$$
$$i=1,2; \quad t=0,1,2,\ldots \qquad (2\text{-}54)$$

This is illustrated by Figures 2.7 and 2.8.

## Ratios of Supply and Demand

The equilibrium conditions may be equivalently expressed in terms of the ratios of supply $S(t)$ and demand $D(t)$, as follows.

$$\begin{cases} S(t) = D(t) \\ Y(t) = X_1(t) + p(t)X_2(t) = D_1(t) + p(t)D_2(t) \end{cases}$$
$$t=0,1,2,\ldots \qquad (2\text{-}55)$$

where $S(t) = X_2(t)/X_1(t) = y_2(t)/y_1(t)$

$$= \frac{l_2[p(t),k(t)]f_2[k_2(p(t))]}{l_1[p(t),k(t)]f_1[k_1(p(t))]} = S[p(t),k(t)]$$

and

$$D(t) = D_2(t)/D_1(t)$$

$$= \frac{k(t)C_2^1[p(t)]+C_2^2[p(t)]}{k(t)C_1^1[p(t)]+C_1^2[p(t)]} = \sigma[p(t)]$$

It should be noted that the second equation of (2-55) is redundant because it is simply the consumers' budget equation.

$$X_1(t)+p(t)X_2(t) = N^1(t)w^1(t)+N^2(t)w^2(t) \quad \text{(zero-profit condition)}$$

$$= N^1(t)[C_1^1(t)+p(t)C_2^1(t)] + N^2(t)[C_1^2(t)+p(t)C_2^2(t)] \quad \text{(budget equations)}$$

$$= [N^1(t)C_1^1(t)+N^2(t)C_1^2(t)] + p(t)[N^1(t)C_2^1(t)+N^2(t)C_2^2(t)]$$

$$= D_1(t) + p(t)D_2(t)$$

Therefore, the equilibrium condition expressed in terms of S(t) and D(t) becomes

$$\frac{l_2[p(t),k(t)]f_2[k_2(p(t))]}{l_1[p(t),k(t)]f_1[k_1(p(t))]} = \frac{k(t)C_2^1[p(t)]+C_2^2[p(t)]}{k(t)C_1^1[p(t)]+C_1^2[p(t)]} \qquad t=0,1,2,... \qquad (2\text{-}56)$$

This is illustrated by Figure 2.9.

## Community Indifference and Transformation Curves

An alternative way of deriving the equilibrium of the economy is to solve the following community-utility-maximizing problem.

$$\underset{\{D_1(t),D_2(t)\}}{\text{Maximize}} \quad u(t) = u[D_1(t),D_2(t)]$$

subject to $D_i(t) = X_i(t) \quad i=1,2; \quad t=0,1,2,...$

and $\quad X_2(t) = X_2[X_1(t)] \qquad t=0,1,2,... \qquad (2\text{-}57)$

The existence of the social indifference curve follows from the assumption that everyone has the same homothetic preferences. (See Samuelson [19] who showed that under the assumption of homotheticity, the whole community behaves exactly like a single individual.) A geometrical description of the solution to (2-57) is given by Figure 2.10. In this graph, the equilibrium price ratio $p^o(t)$ is determined by the slope of the common tangential line to the community indifference curve and the production possibilities curve.

### 2.3.2 Existence and Uniqueness of the Equilibrium

The existence of an equilibrium price ratio for a competitive economy has been well treated elsewhere. (See, for example, Negishi [17].) Furthermore, if $p^o(t)$ is an interior solution, i.e., $\underline{p}^s(t) < p^o(t) < \bar{p}^s(t)$ then the equilibrium price ratio $p^o(t)$ must be unique. This follows from the fact that the production possibilities curve is strictly concave to the origin while the social indifference curves are strictly convex to the origin. The equilibrium pattern of production in this case is characterized by incomplete specialization. The interior equilibrium is statically stable in the Walrasian sense that when $p(t)$ is greater (less) than $p^o(t)$, the commodity ratio supplied $S(t)$ exceeds (falls short of) the commodity ratio demanded $D(t)$. It should be noted that the interior equilibrium price ratio $p^o(t)$ is also statically stable in the Marshallian sense as supply and demand curves are respectively upward and downward sloping to the right.

If the problem (2-57) has a corner solution as depicted in Figure 2.11, the equilibrium price ratio is no longer necessarily unique. It may still be unique if the production possibilities and social indifference curves have the same slope at the corner intersection. However, the two curves need not have the same slope at the corner equilibrium and the corner equilibrium price ratio $p^o(t)$ is therefore not unique in general. In fact, it can take any value from the set of price ratios bounded by one over the absolute values of the slopes at the corner intersection of the two curves. In view of Figure 2.11, $p^o(t)$ can take any value from the closed interval $[p^*(t), \underline{p}^s(t)]$. The corner equilibrium is still statically stable in the Walrasian sense that if $p(t) < p^*(t)$, there is excess demand for good 2 and if $p(t) > \underline{p}^s(t)$, there is excess supply for good 2. Figure 2.12 illustrates this.

It is shown in the Appendix that the equilibrium commodity price ratio $p^o(t)$ of the model being considered is necessarily an interior solution and is therefore unique. Suppose further that the production technology and preferences are unchanging over time. Then the equilibrium price ratio $p^o(t)$ can be solved uniquely in terms of the overall endowment ratio $k(t)$, i.e.,

$$p^o(t) = p^o[k(t)] \qquad t=0,1,2,\ldots \qquad (2\text{-}58)$$

2.3.3 <u>Comparative Statics of Steady-state Equilibria</u>

Suppose now that the population is stable in the sense that the ratio of young men to old men converges to a finite constant, say k, in the long run, i.e.,

$$\lim_{t\to\infty} k(t) = k \qquad \text{(positive finite constant)} \tag{2-59}$$

Then, in stationary equilibria,

$$\ldots = k(t-1) = k(t) = k(t+1) = \ldots = k \tag{2-60-a}$$

and

$$\ldots = p^o(t-1) = p^o(t) = p^o(t+1) = \ldots = p^o(k) \tag{2-60-b}$$

It is evident that a change in the rate of growth of the population will result in a new steady-state equilibrium price ratio $p^o$ and, therefore, new equilibrium production and consumption programs. Before presenting the main theorems, the relationships between the steady-state overall endowment ratio k and the various parameters of population growth will be analyzed. Three population models are considered.

Population Model One

$$k = \gamma_1 = 1+\eta_1 \qquad (\eta_1 \gtreqless 0,\ 0 < \gamma_1 \gtreqless 1)$$

Therefore,

$$dk = d\gamma_1 = d\eta_1 \tag{2-61-a}$$

i.e., an increase (or a decrease) in $\eta_1$ will lead to an increase (or a decrease) of the same magnitude in k.

Population Model Three

$$k = \lambda_1 = (\gamma_1+[\gamma_1^2+4\gamma_2]^{1/2})/2$$

$$= \{(1+\eta_1)+[(1+\eta_1)^2+4(1+\eta_2)]^{1/2}\}/2 \quad (\eta_1 \gtreqless 0,\ \eta_2 \gtreqless 0)$$

Hence,

$$dk = (1+\gamma_1[\gamma_1^2+4\gamma_2]^{-1/2})(d\gamma_1/2)+[\gamma_1^2+4\gamma_2]^{-1/2}\ d\gamma_2 \tag{2-61-b}$$

Other things being equal, an increase (or a decrease) in $\eta_1$ will cause an increase (or a decrease) of lesser magnitude in k as $0 < \partial k/\partial\eta_1 = (1+\gamma_1[\gamma_1^2+4\gamma_2]^{-1/2})/2 < 1$. Similarly, other things being the same, a change in $\eta_2$ will result in a change of the same direction in k since $\partial k/\partial\eta_2 = [\gamma_1^2+4\gamma_2]^{-1/2} > 0$. A simultaneous change in $\eta_1$ and $\eta_2$ will leave k unchanged if $d\gamma_2/d\gamma_1 = -[\partial k/\partial\eta_1]/[\partial k/\partial\eta_2] = -(\gamma_1+[\gamma_1^2+4\gamma_2]^{1/2})/2 < -\gamma_1 < 0$.

Population Model Four

$$k = \theta_1/\pi = (\gamma_1+[\gamma_1^2+4\pi\gamma_2]^{1/2})/2\pi$$

$$= \{(1+\eta_1)+[(1+\eta_1)^2+4(1-d)(1+\eta_2)]^{1/2}\}/[2(1-d)]$$

where $0 \le d < 1$.

Other things being held constant, a change in $\eta_1$ will result in a change in k in the same direction and of magnitude less than $1/\pi = 1/(1-d)$ because $0 < \partial k/\partial\eta_1 = (1+\gamma_1[\gamma_k^2+4\pi\gamma_2{}^{-1/2}])/2 < 1/\pi$. Similarly, a change in $\eta_2$ (or d) will cause k to change in the same direction, other factors being held constant, i.e., $\partial k/\partial \eta_2 > 0$ and $\partial k/\partial d > 0$.

Theorem 2.1 An increase (or a decrease) in any of the population parameters $\eta_1$, $\eta_2$ or d will result in an increase (or a decrease) of the steady-state equilibrium price and a decrease (or an increase) of the equilibrium demand of the commodity which uses old labour relatively intensively in its production. In symbols,

$$(k_1-k_2)(\partial p^o/\partial k) > 0 \qquad (2\text{-}62)$$

$$(k_1-k_2)(\partial D^o/\partial k) < 0 \qquad (2\text{-}63)$$

Proof Consider the effect of an increase in the steady-state overall endowmnet ratio from $k^1$ to $k^2$. Let the variables associated with $k^1$ and $k^2$ be denoted by the superscripts 1 and 2 respectively. Then, defining $\omega_i^j = [f_i(k^j)/f_i'(k^j)]-k^j$ (j=1,2; i=1,2),

$$\bar{\omega}^2 = \mathrm{Max}(\omega_1^2,\ \omega_2^2) > \mathrm{Max}(\omega_1^1,\ \omega_2^1) = \bar{\omega}^1$$

$$\underline{\omega}^2 = \mathrm{Min}(\omega_1^2,\ \omega_2^2) > \mathrm{Min}(\omega_1^1,\ \omega_2^1) = \underline{\omega}^1$$

because $d\omega_i/dk = -f_i(k)f_i''(k)/[f_i'(k)]^2 > 0$. (See Kemp [11].) The supply ratio curve will move from $S^1$ to $S^2$ following the Rybczynski-Samuelson Theorem. There are two cases.

(a) $k_1 > k > k_2$, i.e., the first industry is relatively young-labour intensive in its production. Then the increase in the endowment of young people will result in an expansion of the first industry and in a contraction of the second industry. Thus, the supply ratio curve will shift to the right from $S^1$ to $S^2$ as illustrated by Figure 2.13. Obviously,

$$(\bar{p}^s)^2 > (\bar{p}^s)^1$$

$$(\underline{p}^s)^2 > (\underline{p}^s)^1$$

because $dp^s/d\omega > 0$. (See Kemp [11].)

(b) $k_1 < k < k_2$, i.e., the second industry is relatively young-labour intensive in its production. Then the increase in the endowment of young people will give rise to an expansion of the second industry and a contraction of the first industry. Thus, the supply ratio curve $S^1$ will shift to the left to $S^2$ as shown by Figure 2.14. Clearly,

$$(\bar{p}^s)^2 < (\bar{p}^s)^1$$

$$(\underline{p}^s)^2 < (\underline{p}^s)^1$$

because $dp^s/d\omega < 0$. (See Kemp [11].)

Furthermore, under the assumption of homotheticity of u, it can be shown that the steady-state consumption program satisfies $C_2^1(p)/C_1^1(p) = C_2^2(p)/C_1^2(p) = \sigma(p)$ where $\sigma(p)$ is a positive, decreasing function of p alone. This follows from the fact that the typical consumer's indifference curves are radial expansions or contractions of each other. Substituting the above result into $D = D_2/D_1$ we have

$$D = D_2/D_1 = [kC_2^1(p)+C_2^2(p)]/[kC_1^1(p)+C_1^2(p)] = \sigma(p)$$

which implies that the demand ratio curve is independent of the steady-state endowment ratio k. It should be quite naturally expected because, as mentioned previously, the whole community behaves exactly like a single individual. Thus, the demand ratio curve D remains unchanged as k increases from $k^1$ to $k^2$ and, consequently, an increase in k will result in a higher (lower) steady-state equilibrium price ratio $p^o$ and a lower (higher) steady-state equilibrium consumption and production ratios $D^o$ and $S^o$ if $k_1 > k > k_2$ ($k_1 < k < k_2$). The geometrical descriptions of these comparative equilibria are given by Figures 2.13 and 2.14. The case of a decrease in k can be similarly analyzed. Equation (2-63) is a natural consequence of (2-62) as $\partial D^o/\partial k = [\partial D^o/\partial p][\partial p/\partial k] = \sigma'(p)[\partial p/\partial k]$ where $\sigma'(p) < 0$.

Q.E.D.

Theorem 2.2 An increase (or a decrease) in any of the population parameters $\eta_1$, $\eta_2$ or d will give rise to an increase (or a decrease) in the real reward of old labour and to a decline (or a rise) in the real reward of young labour.

Proof Combining Theorem 2.1 and the Stolper-Samuelson Theorem the proof of Theorem 2.2 is obtained. Q.E.D.

The interpretation of Theorem 2.1 and 2.2 is rather simple. As the rate of population growth increases, the population mixture is more heavily weighted towards the side of young people and it will promote an expansion of the young-labour intensive industry. But the additional young labour can only be absorbed by increasing the ratio in which young labour is combined with old labour, which becomes relatively scarce. The result will be a decline in the marginal product (and real wage) of young labour and an increase in the marginal product (and real wage) of old labour.

Turning to the consumption side, the following results can be established.

Lemma 2.1 (Gorman) There exists a function $g$, $g'(.) > 0$, $g''(.) < 0$, a function $L(p_1,p_2)$, homogeneous of degree one in $p_1$ and $p_2$, and a function $l(p)$, $l'(p) > 0$, $l''(p) < 0$ such that the indirect utility function $v(p,w^j)$ $(j=1,2)$ defined over the price-expenditure space can be written

$$u(C_1^j, C_2^j) = v(p, w^j) = g[w_j/L(p_1, p_2)] = g[w^j/l(p)] \qquad j=1,2 \qquad (2\text{-}64)$$

Proof By definition of strict concavity and homotheticity of $u$, there exists a function $g$, $g'(.) > 0$, $g''(.) < 0$, and a positively linear homogeneous function $h(C_1^j, C_2^j)$ $(j=1,2)$ such that

$$u(C_1^j, C_2^j) = g[h(C_1^j, C_2^j)] \qquad j=1,2$$

Clearly the problem of maximizing $u(C_1^j, C_2^j)$ subject to the budget constraint is equivalent to that of maximizing $h(C_1^j, C_2^j)$ subject to $p_1C_1^j + p_2C_2^j = w_j$ $(j=1,2)$. As Gorman [8] has shown, the indirect utility resulting from this optimizing problem is given by

$$h(C_1^j, C_2^j) = w_j/L(p_1, p_2) \qquad j=1,2$$

where L is homogeneous of degree one and $\partial L/\partial p_i > 0$ $(i=1,2)$. Making use of the homogeneity of L,

$$h(C_1^j, C_2^j) = w_j/[p_1 L(1, p_2/p_1)] = w^j/l(p) \qquad j=1,2$$

where $w^j = w_j/p_1$, $p = p_2/p_1$, $l(p) = L(1, p_2/p_1)$, $l'(p) > 0$ and $l''(p) < 0$. Q.E.D.

Theorem 2.3 Assuming that $k_1 > k_2$ $(k_1 < k_2)$ for all factor price ratios. Then the satisfaction of the first (second) period of an individual's life is a strictly decreasing function of the equilibrium price ratio.

Proof Consider first the case where $k_1 > k_2$ for all factor price ratios. From Theorems 2.1 and 2.2, it is clear that $dw^1/dp < 0$. Thus, $w^1/l(p)$ must be a strictly decreasing function of p as $l'(p) > 0$ from Lemma 2.1. Since $g'(.) > 0$, $u(C_1^1, C_2^1) = v(w^1, p) = g[w^1/l(p)]$ is also a strictly decreasing function of p. For the case $k_1 < k_2$, $u(C_1^2, C_2^2) = v(w^2, p) = g[w^2/l(p)]$ is a strictly decreasing function of p because $dw^2/dp < 0$ from Theorems 2.1 and 2.2 while $l'(p)$ and $g'(.)$ are both positive from Lemma 2.1.

Q.E.D.

### 2.3.4 Welfare Implications of Technical Improvements

As noted by Fried [6], a feature of this kind of model is a seemingly paradoxical result that a technical improvement may cause everyone born after the improvement to have a lower level of welfare. This is brought about by the facts that neither trading with Mother Nature nor trading between generations are permitted. Therefore, a reallocation of consumption goods to the low-marginal-utility period from the high-marginal-utility period may reduce the total lifetime satisfaction of a typical member of the population. In his paper, Fried only considered a one-good economy in which the young and the old provide the same factor of production (but there is land in his model), and he used a highly specialized form of utility function. His finding can be easily shown to remain valid under the more general conditions assumed here. Before stating the main theorem regarding the welfare implications of technical improvements, a simple lemma will be given.

Lemma 2.2 Suppose that a technical innovation takes place at the beginning of period $t_0$. Let $u^j$ and $\hat{u}^j$ be respectively the welfare levels that a typical member of the j-th generation (j=1,2) alive during period $t_0$ would enjoy during $t_0$ before and after the technical progress. Then a typical person alive during period $t_0$ may be worse off in the j-th period of his life (j=1 or 2) after the technical change from before, i.e., either $\Delta u^1 = \hat{u}^1 - u^1$ or $\Delta u^2 = \hat{u}^2 - u^2$ may be negative.

Proof Consider a technical improvement in period $t_0$ which simply involves a new way of combining existing factors in the production of one or both of the two existing consumption products. To show that $\Delta u^1$ may be negative, suppose that a young-labour-saving improvement occurs in the first industry which is relatively young-labour intensive ($k_1 > k_2$). This is illustrated by Figure 2.15. It can be inferred that the postimprovement equilibrium price ratio $\hat{p}$ is greater than the preimprovement price ratio p. To examine the effect of technological change on $w^1$, let rewrite the production relationship for the first industry as

$$X_1 = F_1(\lambda L_{11}, L_{12})$$

where $\lambda$ is the shift parameter which is initially equal to unity. Then the infinitesimal change in $w^1$ due to a very small increase in $\lambda$ is

$$\begin{aligned} dw^1/d\lambda &= [\partial w^1/\partial\lambda]_p + [\partial w^1/\partial p]_\lambda \ (dp/d\lambda) \\ &= [k_1 f_1'/(k_1-k_2)] + [f_2/(k_2^*-k_1^*)] \ (dp/d\lambda) \qquad (2\text{-}65) \end{aligned}$$

where $k_i$ (i=1,2) are factor ratios at E and $k_i^*$ (i=1,2) are factor ratios at $E^*$. The first term of the R.H.S. of (2-65) represents the movement from E to $E^*$ (see Kemp [11, Chapter 2]) and the second term, given by the well-known Stolper-Samuelson Theorem, represents the movement from $E^*$ to $\hat{E}$. Now,

$$\begin{aligned} d[w^1(p)/l(p)]/d\lambda = \{&[\partial w^1/\partial\lambda]_p l(p)+[[\partial w^1/\partial p]_\lambda l(p)-w^1(p)l'(p)] \\ &(dp/d\lambda)\}/l^2(p) \qquad (2\text{-}66) \end{aligned}$$

where $[\partial w^1/\partial\lambda]_p > 0$, $[\partial w^1/\partial p]_\lambda < 0$ and $dp/d\lambda > 0$. (It is assumed that $k_1^* > k_2^*$.) Although there are two conflicting tensions in the numerator of the R.H.S. of (2-66), it is always possible to assume that the negative term outweighs the positive term and the L.H.S. of (2-66) is therefore negative, i.e.,

$$\hat{w}^1/l(\hat{p}) < w^1/l(p)$$

where a hat on a variable denotes its postimprovement value. From Lemma 2.1, it is evident that $\hat{u}^1$ may be smaller than $u^1$, i.e., $\Delta u^1$ may be negative. Following the same line of reasoning, it is possible that $\hat{u}^2 < u^2$ or $\Delta u^2 < 0$. Q.E.D.

Further, since each individual's lifetime utility is time-separable and intergenerational borrowing and lending is not possible, it does not matter whether or not the innovation is anticipated by those who were born just one period before the innovation. Even if they correctly foresee the technical improvement in their old age, all the young people in period $t_0-1$ can do is to maximize $u^1$ subject to the preimprovement budget constraint $C_1^1 + pC_2^1 = w^1(p)$. It is now possible to state

Theorem 2.4 At a constant value of the overall endowment ratio, the postimprovement equilibrium may be "Pareto-superior" or "Pareto-inferior" or neither in relation to the preimprovement equilibrium.

Proof Suppose that technical progress takes place at the beginning of period $t_0$. Let $\Delta U$ be the change in lifetime utility due to the technical improvement of a person burn during or after period $t_0$. Then, from equation (2-40), it can be shown that

$$\Delta U = \Omega_1 \Delta u^1 + \Omega_2 \Delta u^2 \tag{2-67}$$

where $\Omega_1$ and $\Omega_2$ are evaluated at $\tilde{u}^1$ and $\tilde{u}^2$. ($\tilde{u}^1$ is between $u^1$ and $\hat{u}^1$, and $\tilde{u}^2$ is between $u^2$ and $\hat{u}^2$.) For an old man who lives in period $t_0$, his lifetime utility change is simply $\Delta U = \Omega_2 \Delta u^2$.

Clearly, the sign of $\Delta U$ will depend upon the direction of the changes in $\Delta u^1$ and $\Delta u^2$ and upon the relative magnitudes of $\Omega_1$, $\Omega_2$, $\Delta u^1$ and $\Delta u^2$. If $\Delta u^2$ is negative then it is clear that the old people alive during period $t_0$ are worse off after the technical innovation than before. Suppose that $\Delta u^1$ is negative. Then if a typical person greatly values his satisfaction when young (or equivalently discounts greatly his satisfaction when old), $\Omega_1$ will be considerably larger than $\Omega_2$. Thus, it is possible that $\Omega_1 \Delta u^1$ will outweigh $\Omega_2 \Delta u^2$ and $\Delta U$ will be negative, i.e., the young's welfare and, by extension, all future generations' welfare will decrease. The effects of a technical improvement on welfare can now be summarized in the following four cases.

(a) Everybody who is alive during or born after $t_0$ is better off. This necessarily implies $\Delta u^2 > 0$. The postimprovement equilibrium is "Pareto-superior" to the preimprovement equilibrium.

(b) Only old people alive during period $t_0$ are worse off. Young people living in period $t_0$ and all future generations are better off. This requires $\Delta u^1 > 0$ and $\Delta u^2 < 0$.

(c) The old people alive during period $t_0$ are better off but young people living in the same time period and all future generations are worse off. This requires $\Delta u^1 < 0$ and $\Delta u^2 > 0$.

(d) Everybody who is alive during or born after $t_0$ is worse off. This necessarily implies $\Delta u^2 < 0$. The postimprovement equilibrium is "Pareto-inferior" to the preimprovement equilibrium. Q.E.D.

Suppose now that the decision to implement technical progresses is collectively made by the population alive in period $t_0$ but there is no central authority to conduct intergenerational income transfers. Then, having complete knowledge over their lifetime, the $t_0$-th population will only adopt a technical improvement in the polar case (a). It should also be noted that although the theorem is proved for a finite change in the equilibrium price ratio, it remains valid if the change in p is infinitesimal. Differentiating (2-40) totally, the infinitesimal change in lifetime utility due to a technical innovation is

$$dU = \Omega_1 du^1 + \Omega_2 du^2 \tag{2-68}$$

where $du^1 = du(C_1^1, C_2^1)$ and $du^2 = du(C_1^2, C_2^2)$ and $\Omega_1$ and $\Omega_2$ are evaluated at the preimprovement technology. Obviously, the above argument still applies in this case.

Theorem 2.3 is different from Fried's result in two important respects. Firstly, the technical innovation in period $t_0$ is here supposed to be foreseen by the young people in period $t_0$-1 whereas in Fried's analysis it is assumed that the innovation is not anticipated. (If the innovation were anticipated, the prices of assets in Fried's model would change prior to the introduction of the innovation and cause complications in making welfare comparisons.)

Secondly, because of Fried's choice of lifetime utility function, only case (b) or (c) is possible in his analysis. In this limited sense, Theorem 2.3 can be regarded as a generalization of Fried's finding.

The theorem remains valid in the extreme case where labour inputs are perfect substitutes, i.e., even when the production functions do not satisfy the conditions assumed in Subsection 2.2.2. Thus, assume that the production technology is linear in young labour and old labour, i.e.,

$$X_i = a_i L_{i1} + b_i L_{i2} \qquad i=1,2$$

where $a_i > 0$ and $b_i > 0$ $(i=1,2)$. The shape of the production possibilities curve depends on the ratios $a_1/b_1$ and $a_2/b_2$. Suppose now that $a_1/b_1 > a_2/b_2$. Then the production possibilities curve is the union of two linear segments having a kink at $(X_1 = a_1 N^1,\ X_2 = b_2 N^2)$. Along the first linear segment,

$$X_1 = a_1 L_{11}$$

where $L_{11}$ varies from 0 to $N^1$, and

$$X_2 = a_2 L_{21} + b_2 N^2 = (a_2 N^1 + b_2 N^2) - (a_2/a_1) X_1 .$$

Along the second linear segment,

$$X_2 = b_2 L_{22}$$

where $L_{22}$ varies from $N^2$ to 0, and

$$X_1 = a_1 N^1 + b_1 L_{12} = (a_1 N^1 + b_1 N^2) - (b_1/b_2) X_2 .$$

Suppose also that an individual's lifetime welfare can be summarized by an additively time-separable utility function

$$U = u^1 + \delta u^2 \qquad (0 < \delta)$$

where $u^j = u(C_1^j, C_2^j) = (C_1^j)^{\beta_1}(C_2^j)^{\beta_2}$

$$(0 < \beta_1,\ 0 < \beta_2 \text{ and } \beta_1+\beta_2 < 1)$$
$$j=1,2$$

His demand functions are then

$$C_1^j = \beta_1 w^j/(\beta_1+\beta_2) \qquad j=1,2$$

$$C_2^j = \beta_2 w^j/[(\beta_1+\beta_2)p] \qquad j=1,2$$

and the indirect utility function in terms of $w^j$ (j=1,2) and p becomes

$$v(p,\ w^j) = K(w^j)^{\beta_1+\beta_2}/p^{\beta_2} \qquad j=1,2$$

where $$K = \beta_1^{\beta_1}\beta_2^{\beta_2}(\beta_1+\beta_2)^{-(\beta_1+\beta_2)}.$$

It can be shown that for a suitable choice of the parameters $a_i$, $b_i$ and $\beta_i$ (i=1,2), the equilibrium of the economy is attained on the first linear segment. This is illustrated by Figure 2.16. Thus, the equilibrium price ratio is $p = a_1/a_2$ and the equilibrium wage rates are respectively $w^1 = a_1 = pa_2$ and $w^2 = pb_2$. In equilibrium, the second industry is relative old-labour intensive in the sense that it employs all old labour in its production. Then the preimprovement levels of welfare of a typical young and a typical old are respectively

$$u^1 = K(w^1)^{\beta_1+\beta_2}/p^{\beta_2} = Ka_2^{\beta_1+\beta_2}p^{\beta_1} \qquad (w^1=pa_2)$$

$$= Ka_1^{\beta_1}a_2^{\beta_2} \qquad (p=a_1/a_2)$$

$$u^2 = K(w^2)^{\beta_1+\beta_2}/p^{\beta_2} = Kb_2^{\beta_1+\beta_2}p^{\beta_1} \qquad (w^2=pb_2)$$

$$= Kb_2^{\beta_1+\beta_2}\,a_1^{\beta_1}a_2^{-\beta_1}$$

Consider the effect of a young-labour-saving improvement in the second industry which can be represented by an increment of the young-labour productive coefficient in the second industry from $a_2$ to $\hat{a}_2 (a_2 < \hat{a}_2)$. Suppose further that the technical change is sufficiently small to maintain the initial shape of the production possibilities curve, i.e., $a_1/b_1 > \hat{a}_2/b_2$ and that the postimprovement equilibrium still occurs in the first linear segment where $X_2 = \hat{a}_2L_{21}+b_2N^2 = (\hat{a}_2N^1+b_2N^2) - (\hat{a}_2/a_1)X_1$. In the new equilibrium, the price ratio is $\hat{p}=\hat{a}_1/a_2$ and the young-labour and old-labour wages are $\hat{w}^1=a_1=\hat{p}\hat{a}^2$ and $\hat{w}^2=\hat{p}b_2$, respectively. This is illustrated by Figure 2.17. Hence, the post-improvement levels of welfare of a typical young and a typical old are respectively

$$\hat{u}^1 = K(\hat{w}^1)^{\beta_1+\beta_2}/\hat{p}^{\beta_2} = K\hat{a}_2^{\beta_1+\beta_2}\hat{p}^{\beta_1} \qquad (\hat{w}^1=\hat{p}\hat{a}_2)$$

$$= Ka_1^{\beta_1}\hat{a}_2^{\beta_2} \qquad (\hat{p}=a_1/\hat{a}_2)$$

$$\hat{u}^2 = K(\hat{w}^2)^{\beta_1+\beta_2}/\hat{p}^{\beta_2} = Kb_2^{\beta_1+\beta_2}\hat{p}^{\beta_1} \qquad (\hat{w}^2=\hat{p}b_2)$$

$$= Kb_2^{\beta_1+\beta_2}a_1^{\beta_1}\hat{a}_2^{-\beta_1}$$

It is evident that

$$\Delta u^1 = \hat{u}^1 - u^1 = Ka_1^{\beta_1}(\hat{a}_2^{\beta_2} - a_2^{\beta_2}) > 0 \qquad (\hat{a}_2 > a_2)$$

and

$$\Delta u^2 = \hat{u}^2 - u^2 = Kb_2^{\beta_1+\beta_2}a_1^{\beta_1}(\hat{a}_2^{-\beta_1} - a_2^{-\beta_1}) < 0$$

Since $\Delta u^2 < 0$, the old people alive at the time of the technical innovation are worse off after the innovation than before. Let

$$\delta^* = -\Delta u^1/\Delta u^2 = b_2^{-(\beta_1+\beta_2)}(\hat{a}_2^{\beta_2} - a_2^{\beta_2})(a_2^{-\beta_1} - \hat{a}_2^{-\beta_1})^{-1}$$

Then, for $\delta < \delta^*$, the young people alive at the time of the technical progress and, by extension, all future generations are better off after the progress than before. For $\delta > \delta^*$, the young's welfare and all future generations' welfare will decline. In this polar case, everybody who is alive or born after the technical improvement is made worse off.

Two final points deserve mention. First, in post-improvement equilibrium society as a whole can consume more of both goods as shown in Figure 2.16. However, individuals are worse off because of the redistribution of income to the low-marginal-utility period from the high-marginal-utility period. Second, when neutral and biased innovations are indistinguishable, the postimprovement equilibrium is "Pareto-superior" to the preimprovement equilibrium. This flows from the fact that in such a case the postimprovement factor rewards $\hat{w}^1$ and $\hat{w}^2$ both increase at a faster rate than $l(\hat{p})$. To see that, consider the Cobb-Douglas production functions

$$X_i = \lambda_i L_{i1}^{\alpha_i} L_{i2}^{1-\alpha_i} \qquad i=1,2$$

where $0 < \alpha_i < 1$ and $\lambda_i$ are shift parameters each of which is initially equal to unity (i=1,2). Because of the simple multiplicative form of the Cobb-Douglas functions, improvements defined in this manner cannot be said to be "neutral" or "factor biased". The factor rewards in terms of the price ratio are

$$w^1 = \lambda_1 \alpha_1 k_1^{\alpha_1 - 1}$$

$$= \lambda_1^{(1-\alpha_2)/(\alpha_1-\alpha_2)} \lambda_2^{(\alpha_1-1)/(\alpha_1-\alpha_2)} Ap^{(\alpha_1-1)/(\alpha_1-\alpha_2)}$$

and

$$w^2 = \lambda_1 (1-\alpha_1) k_1^{\alpha_1} = \lambda_1^{-\alpha_2/(\alpha_1-\alpha_2)} \lambda_2^{\alpha_1/(\alpha_1-\alpha_2)} Bp^{\alpha_1/(\alpha_1-\alpha_2)}$$

where

$$A = \alpha_1 \left(\frac{\alpha_1}{\alpha_2}\right)^{\alpha_2(1-\alpha_1)/(\alpha_1-\alpha_2)} \left[\frac{(1-\alpha_1)}{(1-\alpha_2)}\right]^{(1-\alpha_1)(1-\alpha_2)/(\alpha_1-\alpha_2)}$$

$$B = (1-\alpha_1) \left(\frac{\alpha_1}{\alpha_2}\right)^{-\alpha_1\alpha_2/(\alpha_1-\alpha_2)} \left[\frac{(1-\alpha_1)}{(1-\alpha_2)}\right]^{-\alpha_1(1-\alpha_2)/(\alpha_1-\alpha_2)}$$

Suppose further that an individual's j-th period welfare can be represented by the same utility function as in the previous example. Then by equating supply and demand for each good, the equilibrium price ratio is found to be

$$p^o = (\lambda_1/\lambda_2) Dk^{\alpha_1-\alpha_2}$$

where

$$D = \alpha_1^{\alpha_1}\alpha_2^{\alpha_2}(1-\alpha_1)^{(1-\alpha_1)}(1-\alpha_2)^{-(1-\alpha_2)}E^{(\alpha_1-\alpha_2)}$$

and

$$E = [(1-\alpha_1)\beta_1+(1-\alpha_2)\beta_2]/(\alpha_1\beta_1+\alpha_2\beta_2).$$

In preimprovement equilibrium where $\lambda_1 = \lambda_2 = 1$,

$p^o = Dk^{(\alpha_1-\alpha_2)}$, $w^1 = A(p^o)^{(\alpha_1-1)/(\alpha_1-\alpha_2)}$ and

$w^2 = B(p^o)^{\alpha_1/(\alpha_1-\alpha_2)}$. Therefore,

$$u^1 = K(w^1)^{(\beta_1+\beta_2)}/(p^o)^{\beta_2}$$

$$= KA^{(\beta_1+\beta_2)}(p^o)^{-[(1-\alpha_1)\beta_1+(1-\alpha_2)\beta_2]/(\alpha_1-\alpha_2)}$$

$$u^2 = K(w^2)^{(\beta_1+\beta_2)}/(p^o)^{\beta_2}$$

$$= KB^{(\beta_1+\beta_2)}(p^o)^{(\alpha_1\beta_1+\alpha_2\beta_2)/(\alpha_1-\alpha_2)}$$

where $K = \beta_1^{\beta_1}\beta_2^{\beta_2}(\beta_1+\beta_2)^{-(\beta_1+\beta_2)}$.

In postimprovement where $\lambda_1 > 1$ or $\lambda_2 > 1$ or both,

$\hat{p}^o = (\lambda_1/\lambda_2)p^o$, $\hat{w}^1 = \lambda_1 w^1$ and $\hat{w}^2 = \lambda_2 w^2$. Therefore,

$$\hat{u}^1 = K(\hat{w}^1)^{(\beta_1+\beta_2)}/(\hat{p}^o)^{\beta_2} = \lambda_1^{\beta_1}\lambda_2^{\beta_2}u^1$$

$$\hat{u}^2 = K(\hat{w}^2)^{(\beta_1+\beta_2)}/(\hat{p}^o)^{\beta_2} = \lambda_1^{\beta_1}\lambda_2^{\beta_2}u^2$$

It is obvious that

$$\Delta u^1 = \hat{u}^1 - u^1 = (\lambda_1^{\beta_1}\lambda_2^{\beta_2} - 1)u^1 \quad \text{and}$$

$$\Delta u^2 = \hat{u}^2 - u^2 = (\lambda_1^{\beta_1}\lambda_2^{\beta_2} - 1)u^2$$

are both positive since $\beta_i > 0$, $\lambda_i \geq 1$ $(i=1,2)$ and $\lambda_1\lambda_2 > 1$. Thus, the postimprovement equilibrium is "Pareto-superior" in relation to the preimprovement equilibrium.

## 2.4 NON-STATIONARY EQUILIBRIUM ANALYSIS

Suppose that at some specific point in time the first births miraculously took place. It is quite natural to think that there were no old people at the beginning of "biological" time. Since both factors of production are assumed to be indispensable, no positive amount of either good could be produced at this point of time even if the production technology was already known to mankind. Clearly, the earlier analysis does not apply here. It may be agreed that the very first generation of young people survived by consuming some freely available goods and succeeded in reproduction. Suppose further that they reproduced at the end of their youth (or the beginning of their old age) so that after one period from the "biological" zero, there were young men as well as old men; and production suddenly became possible. Therefore, by a suitable translation of the time origin, one may simply assume that the initial subpopulation of old people $N_0^2$ is positive. The economy then began to operate from this arbitrarily chosen time origin to an unbounded future. But unless the age structure becomes stabilized, the economy will never approach a steady-state equilibrium.

We turn now to the study of all equilibrium price ratios $p^o(t)$. Without formulating and solving the difference equation that represents the pattern of change of $p^o(t)$ through time, the dynamical behaviour of $p^o(t)$ can be analyzed by using the unique correspondence between $p^o(t)$ and $k(t)$. Thus, the time path of $p^o(t)$ can be derived by a two-step method.

(a) Obtain the time path of $k(t)$ and show that population is stable in the sense that $\lim_{t\to\infty} k(t) =$ positive finite constant.

(b) For each value of $k(t)$, there exists a unique value of $p^o(t)$ which equates $X_i(t)$ and $D_i(t)$ $(i=1,2;\ t=0,1,2,\ldots)$. This was shown in 2.3.2.

Two models of population growth will be examined.

Population Model One

In a two-period model, if only young people can reproduce then the young labour-old labour ratio is given by

$$k(t) = \begin{cases} N_0^1/N_0^2 \quad (N_0^2 \neq 0) & t=0 \\ (1+\eta_1) = \gamma_1 & t=1,2,3,\ldots \end{cases}$$

where $\eta_1$ is the constant growth rate of population. Since $k(t)$ only takes one time period at the most to reach its stabilized value, the graphs of $k(t)$ and $p^o(t)$ as functions of time are as given by Figures 2.18 and 2.19.

Population Model Three

Suppose now that young people of this period are offspring of both young and old people of the preceding period. Then it has been shown that

$$k(t) = \begin{cases} N_0^1/N_0^2 \quad (N_0^2 \neq 0) & t=0 \\ (A_1\lambda_1^t + A_2\lambda_2^t)/(A_1\lambda_1^{t-1} + A_2\lambda_2^{t-1}) & t=1,2,3,\ldots \end{cases}$$

where $A_1$, $A_2$, $\lambda_1$ and $\lambda_2$ are as defined in (2-15).

Firstly, it can be seen that $\lambda_1^t$ will dominate $\lambda_2^t$ as t becomes infinitely large and $\lim_{t\to\infty} k(t) = A_1\lambda_1^t/A_1\lambda_1^{t-1} = \lambda_1$. Secondly, values of k(t) for t=0,1,2,... will oscillate about the limiting value $\lambda_1$ since $k(t) - \lambda_1 = -A_2(\lambda_1-\lambda_2)\lambda_2^{t-1}/(A_1\lambda_1^{t-1}+A_2\lambda_2^{t-1})$ is positive for odd values of t and is negative for even values of t ($-A_2(\lambda_1-\lambda_2) > 0$ and $\lambda_2 < 0$). Thirdly, the distance from k(t) to its limiting value $\lambda_1$, $dis(t) = |k(t) - \lambda_1|$ is strictly decreasing for t, t+2, t+4, ... because the ratio $dis(t+2)/dis(t) = (A_1\lambda_1^{t-1}\lambda_2^2+A_2\lambda_2^{t+1})/(A_1\lambda_1^{t+1}+A_2\lambda_2^{t+1})$ is less than one. Therefore the sequence $\{k(t): t=1,2,3,.....\}$ can be represented by a set of uniformly damped oscillatory points. They are illustrated by Figure 2.20 and the corresponding graph of $p^o(t)$ by Figure 2.21.

Using Figure 2.21 the time path of the non-stationary consumption programs can be traced out. Let $D^1(t)$ and $D^2(t)$ denote respectively the consumption ratios of a young man and an old man at time t, i.e.,

$$D^1(t) \;=\; C_2^1(t)/C_1^1(t) \;=\; \{\phi_2[p(t)]w^1(t)\}/\{\phi_1[p(t)]w^1(t)\}$$

$$= \sigma[p(t)] \qquad\qquad t=0,1,2,\ldots$$

$$D^2(t) \;=\; C_2^2(t)/C_1^2(t) \;=\; \{\phi_2[p(t)]w^2(t)\}/\{\phi_1[p(t)]w^2(t)\}$$

$$= \sigma[p(t)] \qquad\qquad t=0,1,2,\ldots$$

where $\sigma'(.) < 0$. Define $\tilde{D}(t) = [D^1(t), D^2(t)]$ and $D(t) = [D^1(t), D^2(t+1)]$, where $D(t)$ is the vector of lifetime consumption ratios for a typical man born in period t. Now, clearly, $\tilde{D}(t)$ is a point on the 45° straight line in the first quadrant of the $(D^1, D^2)$ Cartesian coordinate, being determined solely by $p^o(t)$ which in turn is determined by k(t). $D(t)$ is the intersection of a vertical line passing through $\tilde{D}(t)$ and a horizontal line passing through $\tilde{D}(t+1)$. This is shown by Figure 2.22 for $k_1 > k_2$.

The pattern in Figure 2.22 resembles a convergent cobweb model. Since the oscillation in values of k(t) is uniformly damped, the equilibrium of the model is dynamically stable. Given technology and preferences, it is possible to solve for $p^o[k(t)]$ explicitly in terms of t only. Then $p^o(t)$ may be written as follows.

$$p^o(t) \;=\; CF(t) + p^o(k)$$

where the complementary function CF(t) oscillates between zero and approaches zero as t approaches infinity.

In conclusion, for each value of t,

(a) a value of k(t) can be computed depending upon the structure of the population.

(b) this value determines the equilibrium commodity price ratio $p^o[k(t)]$ according to the equilibrium conditions of the model with production technology and preferences given.

(c) the individual consumption ratios $D^1(t)$ and $D^2(t)$ are then given by $\sigma[p^o[k(t)]]$ where $\sigma$ is derived from the utility function u.

Finally, it should be noted that the conclusions of this section depend on the very strong assumption of a homothetic separable utility function.

# APPENDIX

To prove the uniqueness of the equilibrium of the model, the following conventional assumptions are made.

A1. $u(x,y)$ is homothetic (which is necessary for the existence of the social utility function). But the homothetic $u(x,y)$ might just as well be given the first-degree-homogeneous form

$$u(x,y) = \frac{1}{\lambda} u(\lambda x, \lambda y) \text{ for any } \lambda > 0,\ x \geq 0 \text{ and } y \geq 0.$$

A2. $u(x,y)$ is strictly quasi-concave for any $x > 0$ and $y > 0$, i.e., $S = \{(x,y)/u(x,y) \geq \bar{u}\}$ is strictly convex. This assumption is slightly weaker than that of Sub-section 2.2.3.

A3. The marginal utilities are finitely positive for any $x > 0$ and $y > 0$, i.e.,

$$0 < u_1 = \partial u/\partial x < \infty \quad \text{and} \quad 0 < u_2 = \partial u/\partial y < \infty$$

for any $x > 0$ and $y > 0$.

Since the production possibilities curve is strictly concave to the origin, an interior solution is ensured if

(a) the social indifference curves never intersect the axes

or

(b) a typical indifference curve intersects the x axis with slope of zero and the y axis with slope of minus infinity.

Of course, if $\lim_{x\to 0} u_1 = \lim_{y\to 0} u_2 = \infty$ then either (a) or (b) holds true. However, it will be assumed that $\lim_{x \to 0} u_1 < \infty$ and $\lim_{y \to 0} u_2 < \infty$.

We shall prove that for any constant value $\bar{u}$, there exists no positive $x^*$ such that $0 = \phi(x^*)$ and $\psi(0) < 0$ where

$\phi$: $\bar{u} = u(x,y)$ implies $y = \phi(x)$ (This is ensured by Implicit Function Theorem.)

and $\psi$: $\phi' = -u_1/u_2 = \psi(y/x)$ (From linear homogeneity of u.)

By method of contradiction, suppose that

<u>B1</u>. There exists $x^* > 0$ such that $0 = \phi(x^*)$ and $\psi(0) < 0$.

For any $x > 0$ and $y > 0$,

$$\partial\psi(y/x)/\partial x = -\psi' \frac{y}{x^2} = \partial(-u_1/u_2)/\partial x = -(u_{11}u_2 - u_{21}u_1)/u_2^2$$

$$\partial\psi(y/x)/\partial y = -\psi' \frac{1}{x} = \partial(-u_1/u_2)/\partial y = -(u_{12}u_2 - u_{22}u_1)/u_2^2$$

Equating $\psi'$ yields

$$u_{11} - (u_1/u_2)u_{21} = -(y/x)\ [u_{12} - (u_1/u_2)u_{22}]$$

Making use of the fact that $u_2(x^*,0) > 0^{(\dagger)}$ and $0 = \phi(x^*)$ (from <u>B1</u>.) we have

$$u_{11}(x^*,0) - \frac{u_1(x^*,0)}{u_2(x^*,0)}\ u_{21}(x^*,0) = 0$$

or

$$u_{11}(x^*,0) + \psi(0)u_{21}(x^*,0) = 0$$

Since $u_{11} \leq 0$ and $u_{21} \geq 0$ by A2. and $\psi(0) < 0$ by B1., the above equation can only hold if

B2. $u_{11}(X^*,0) = u_{21}(x^*,0) = 0.$

But for any $\varepsilon > 0$,

$$u_1(x^*,\varepsilon) = u_1(x^*,0)+u_{12}(x^*,0)\theta\varepsilon$$

where $0 < \theta < 1$. If $u_{21}(x^*,0) = u_{12}(x^*,0) = 0$ then

$$u_1(x^*,\varepsilon) = u_1(x^*,0)$$

for any $\varepsilon > 0$. However, this implies

$$u_{12}(x^*,y) = 0$$

for any $y > 0$. From strict quasi-concavity and linear homogeneity of u, for any $x > 0$ and $y > 0$, we have

$$u_{11}x + u_{12}y = 0 \text{ and } u_{11} < 0.$$

Therefore,

$$u_{12} = -(y/x)u_{11} > 0.$$

This contradicts B2. which in turn implies that B1. cannot hold. Bearing mind that x and y are interchangeable, the proof is completed. Q.E.D.

(†) If $u_2(x^*,0) = 0$ then, by expanding $u_2(x^*,0)$ around y=0, there exists a $\theta (0 < \theta < 1)$ such that

$$u_2(x^*,0) = u_2(x^*,0) + u_{22}(x^*,0)\theta y$$
$$= u_{22}(x^*,0)\theta y \leq 0 \quad (u_{22}(x^*,0) \leq 0 \text{ by A2.})$$

This certainly contradicts A3.

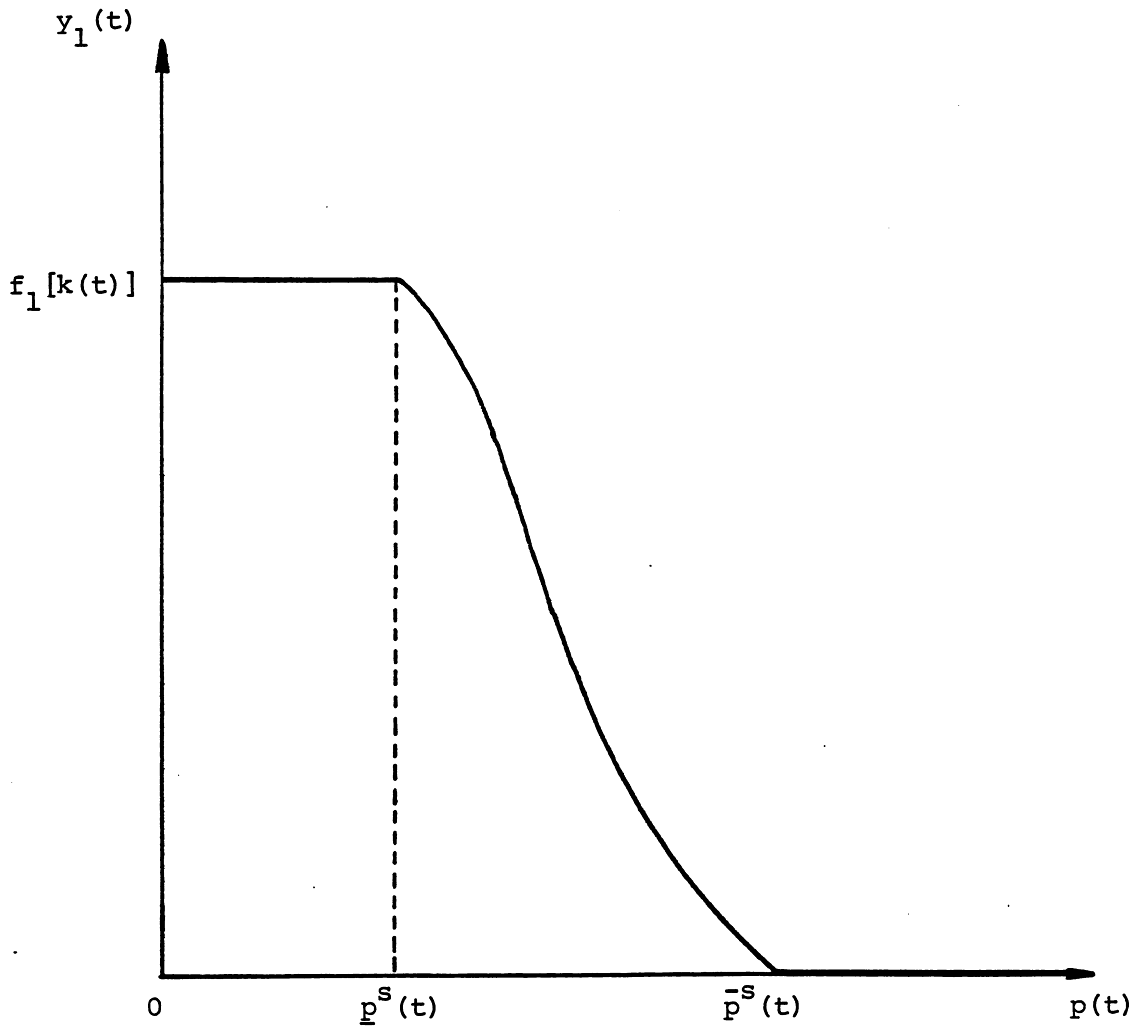



Figure 2.1: $y_1(t) = y_1[p(t), k(t)]$ with a given $k(t)$

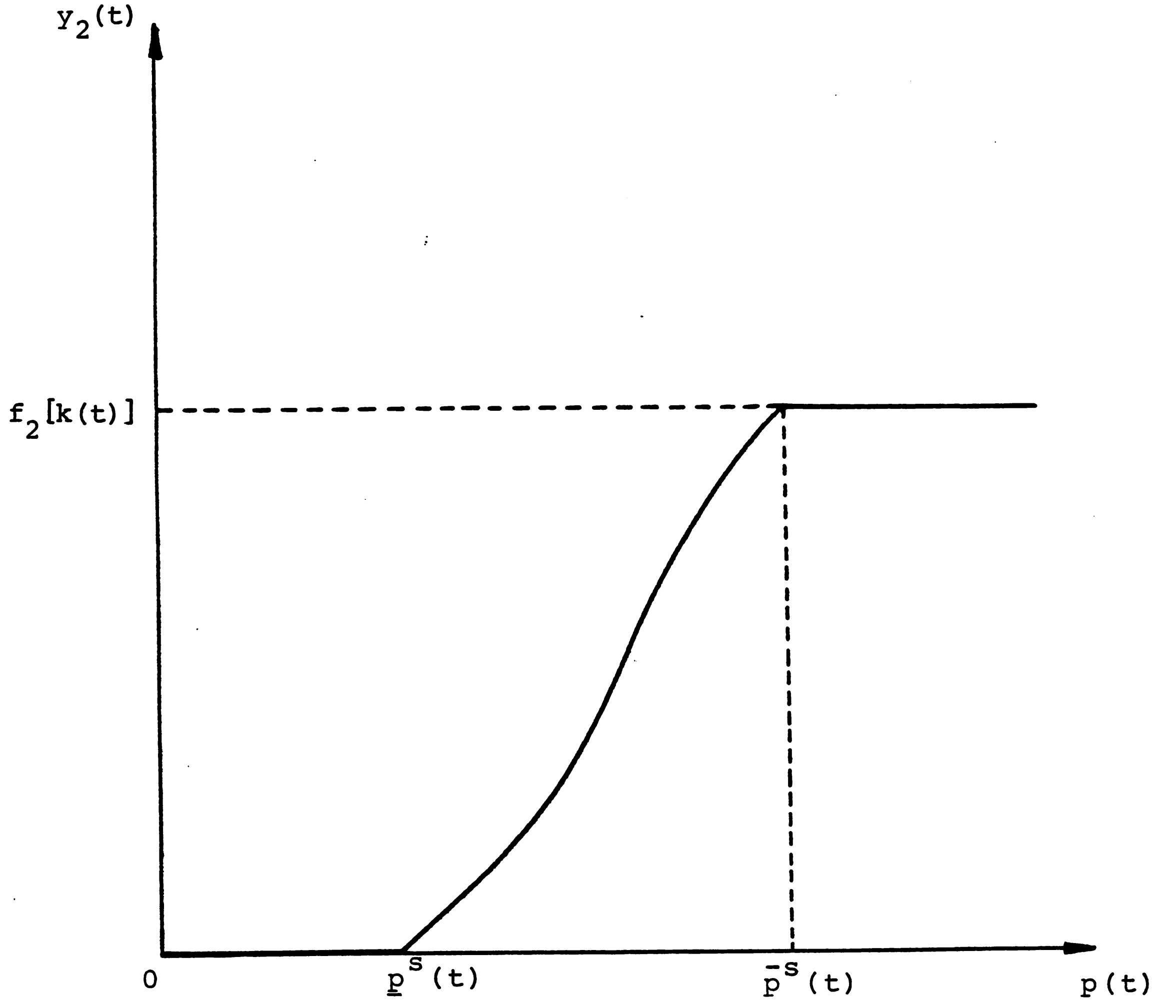



Figure 2.2: $y_2(t) = y_2[p(t), k(t)]$ with a given $k(t)$

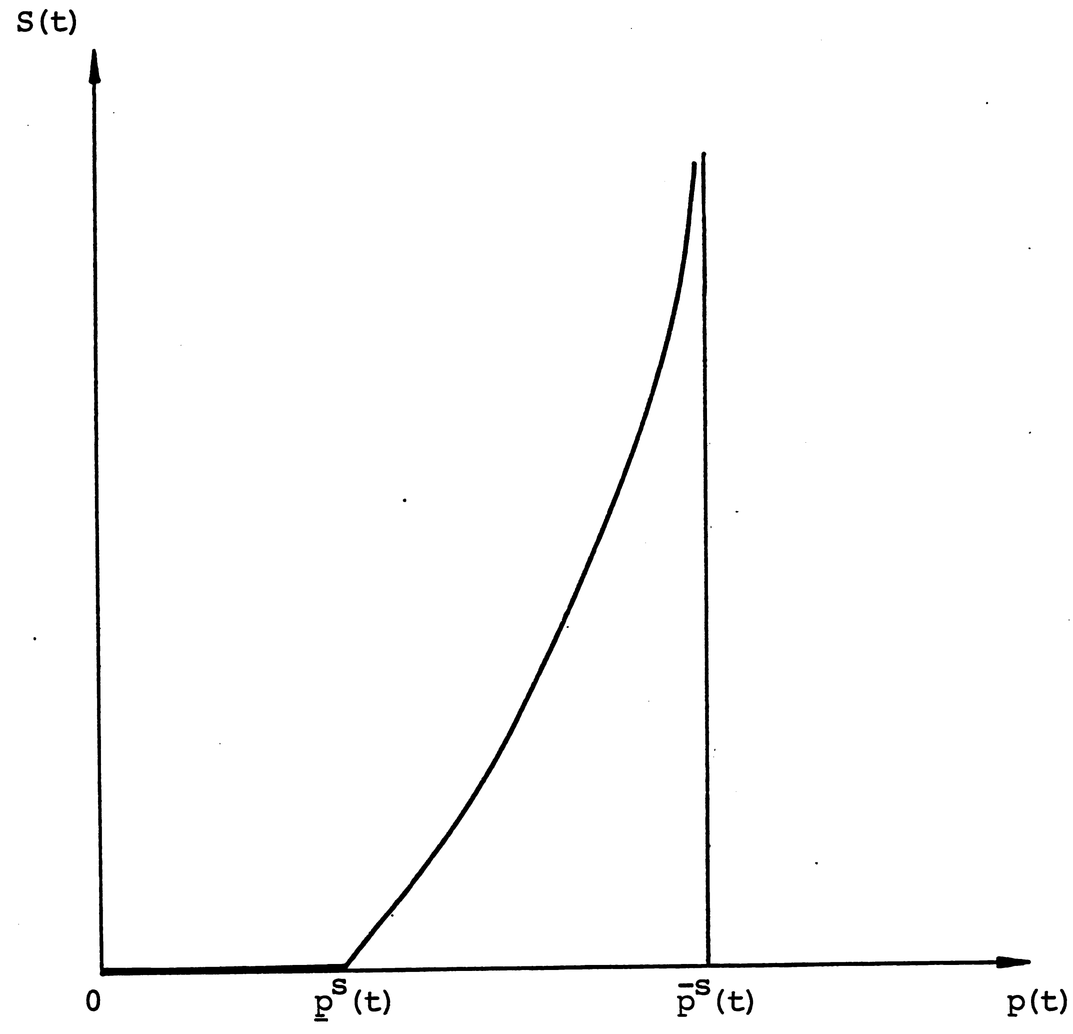



Figure 2.3: $S(t) = y_2(t)/y_1(t)$ with $k(t)$ given

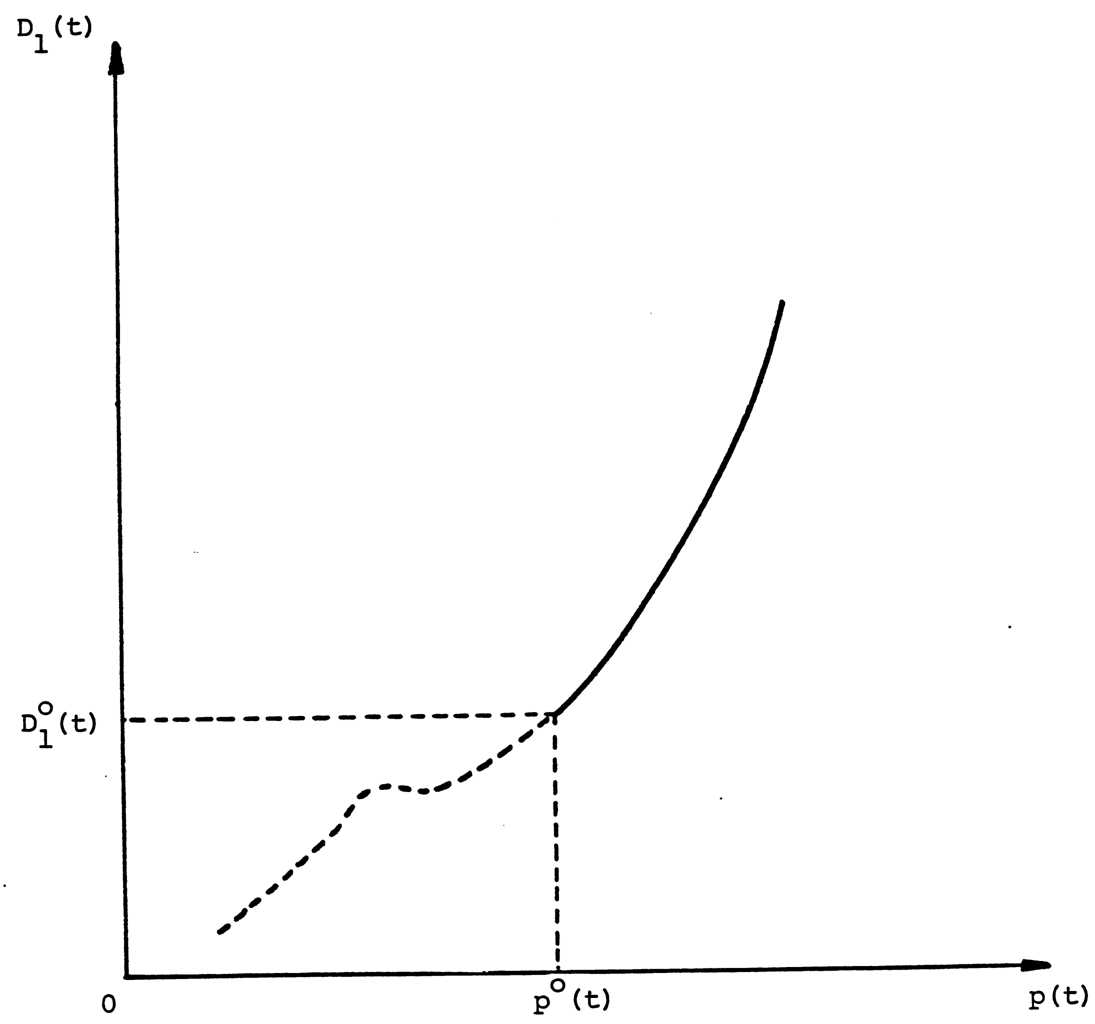



Figure 2.4: $D_1(t) = D_1[Y(t), p(t)]$ with a given $k(t)$

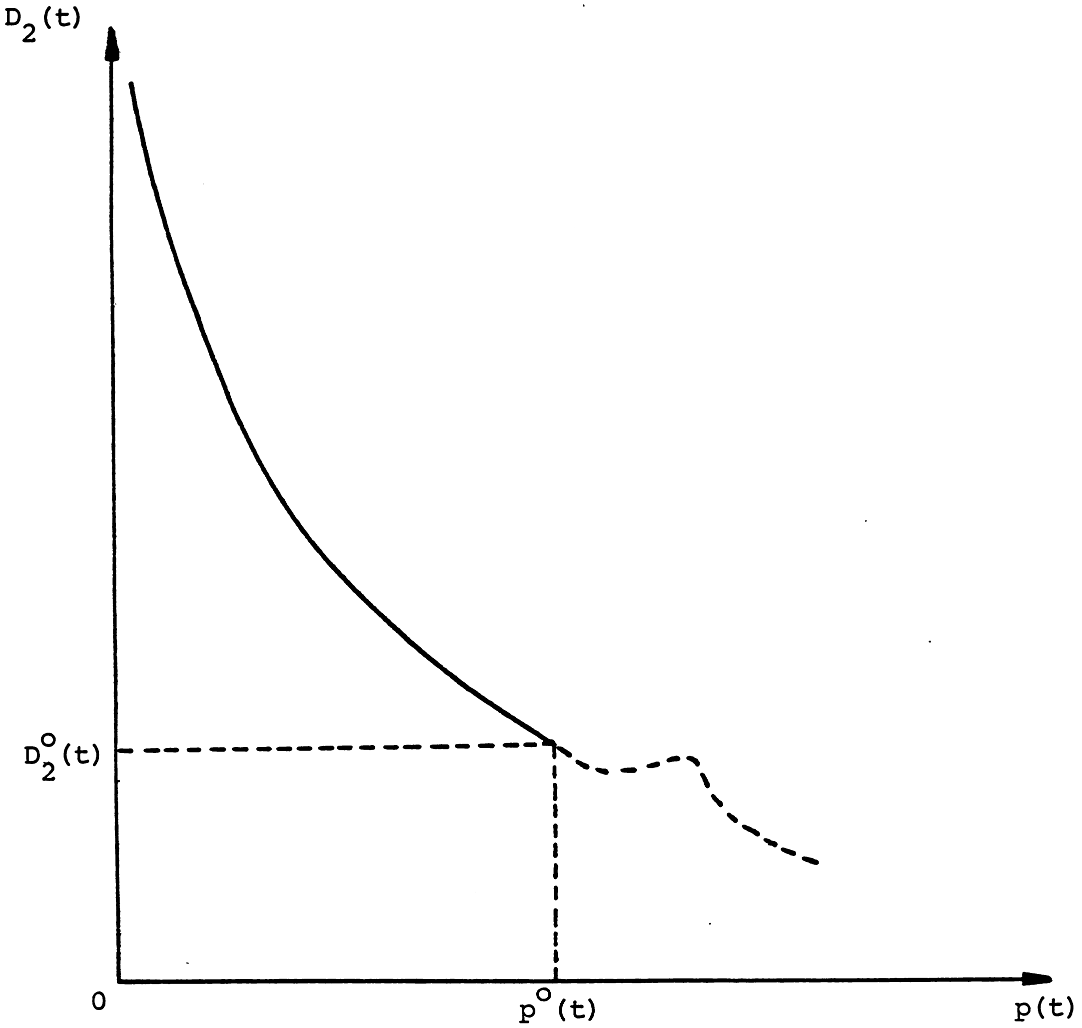



Figure 2.5: $D_2(t) = D_2[Y(t), p(t)]$ with a given $k(t)$

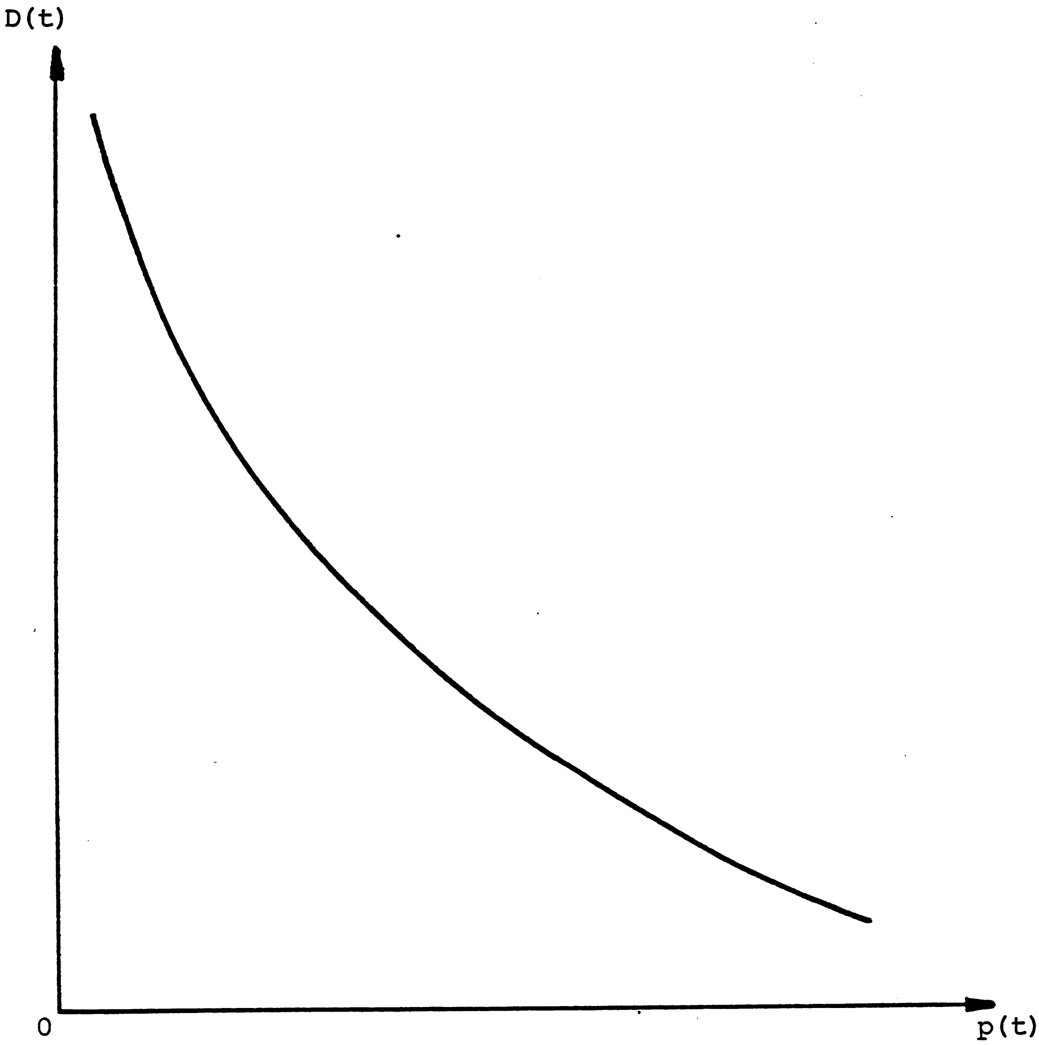



Figure 2.6: $D(t) = D_2(t)/D_1(t)$ with $k(t)$ given

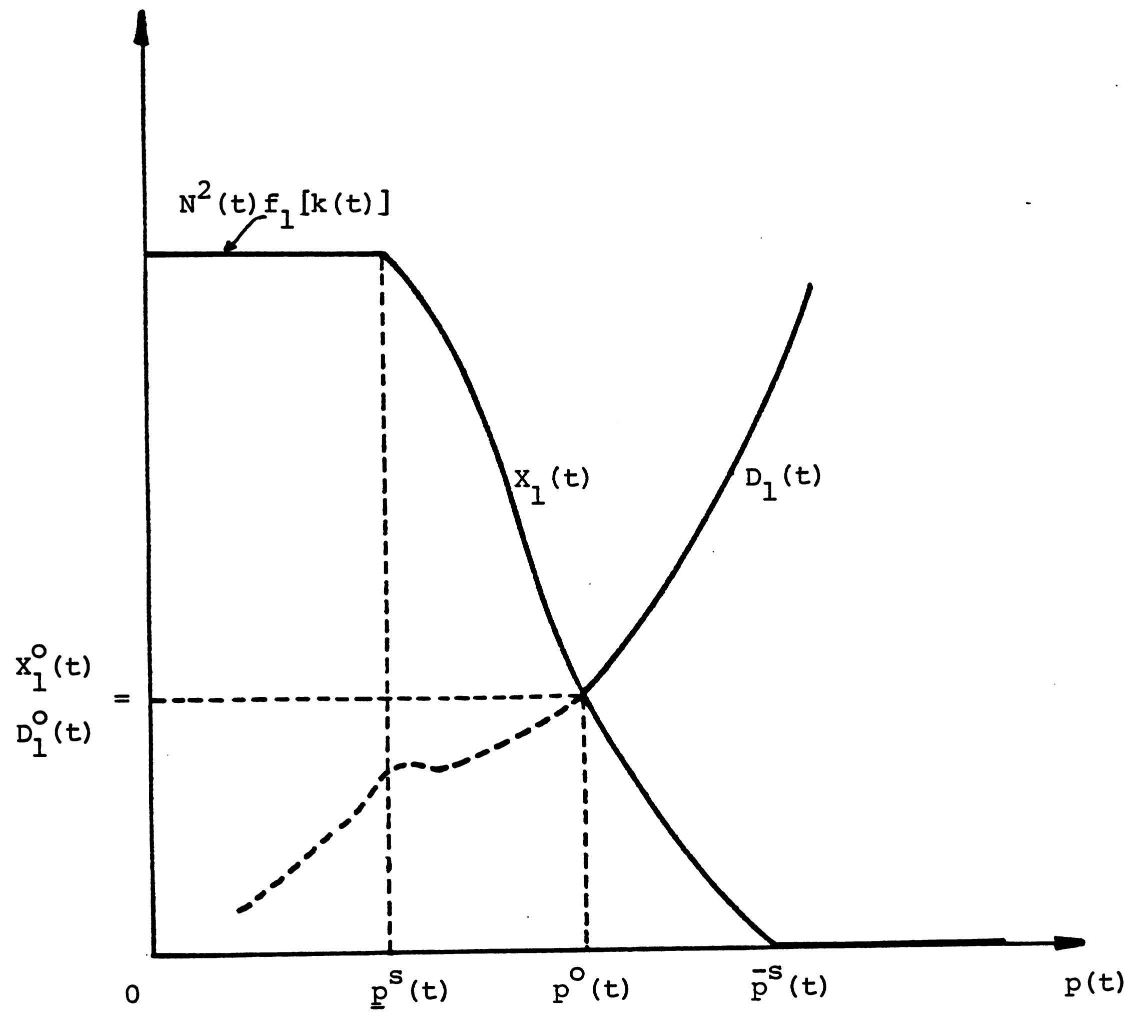
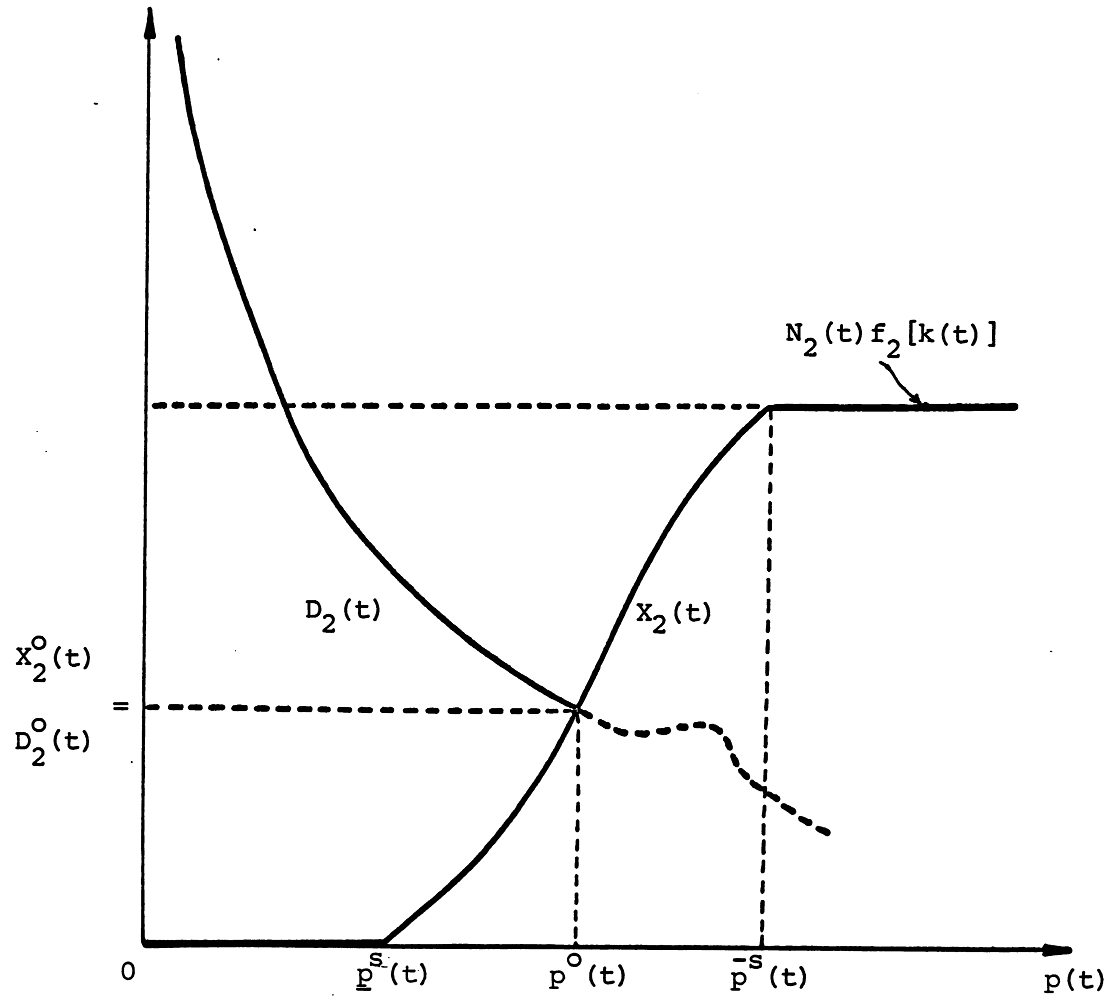



Figure 2.7: Equilibrium in the first market

$X_1[p(t), k(t)] = D_1[p(t), k(t)]$

with a given k(t)




<u>Figure 2.8</u>: Equilibrium in the second market
$X_2[p(t), k(t)] = D_2[p(t), k(t)]$
with a given $k(t)$

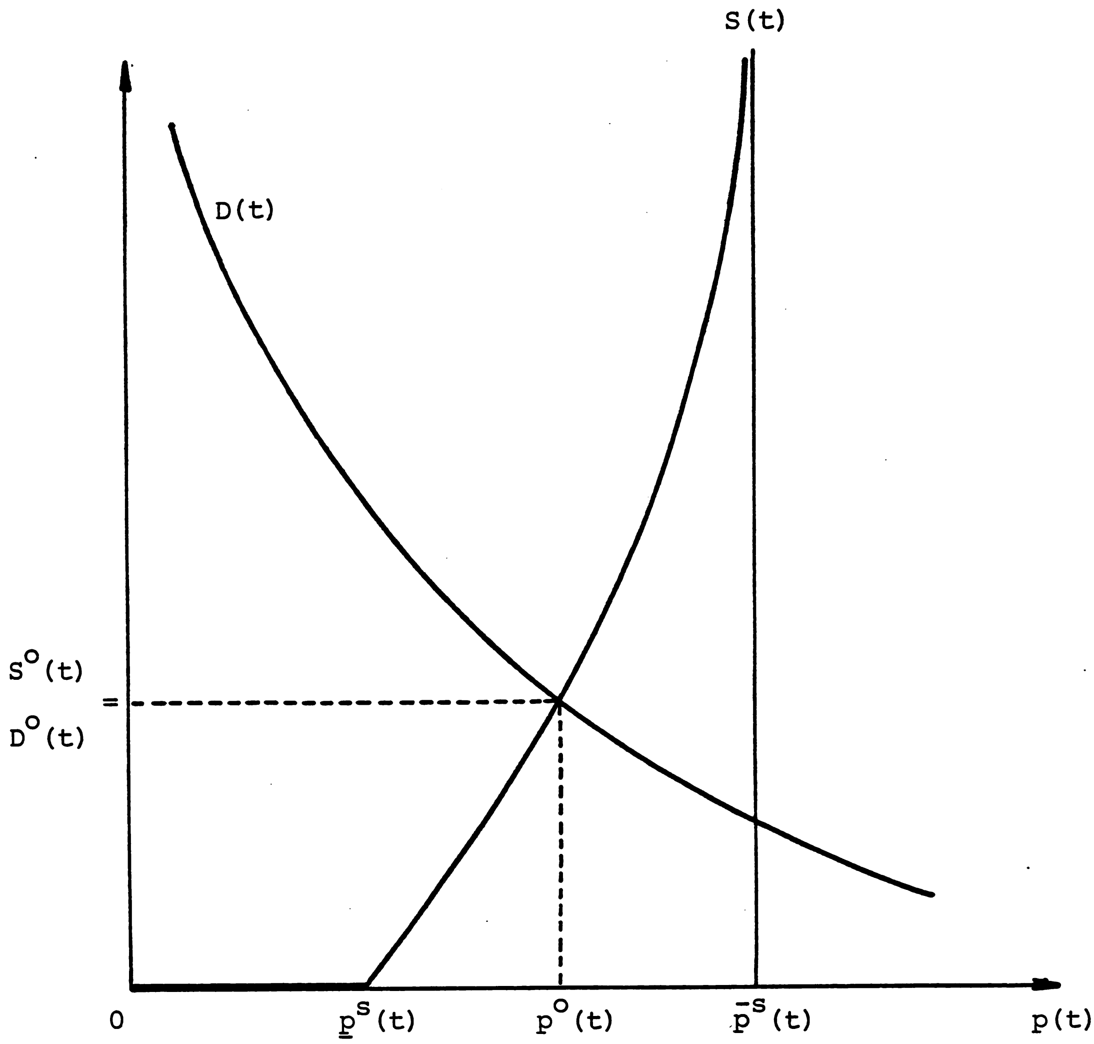



Figure 2.9: Equilibrium in the product markets $S[p(t), k(t)] = D[p(t), k(t)]$ with $k(t)$ given

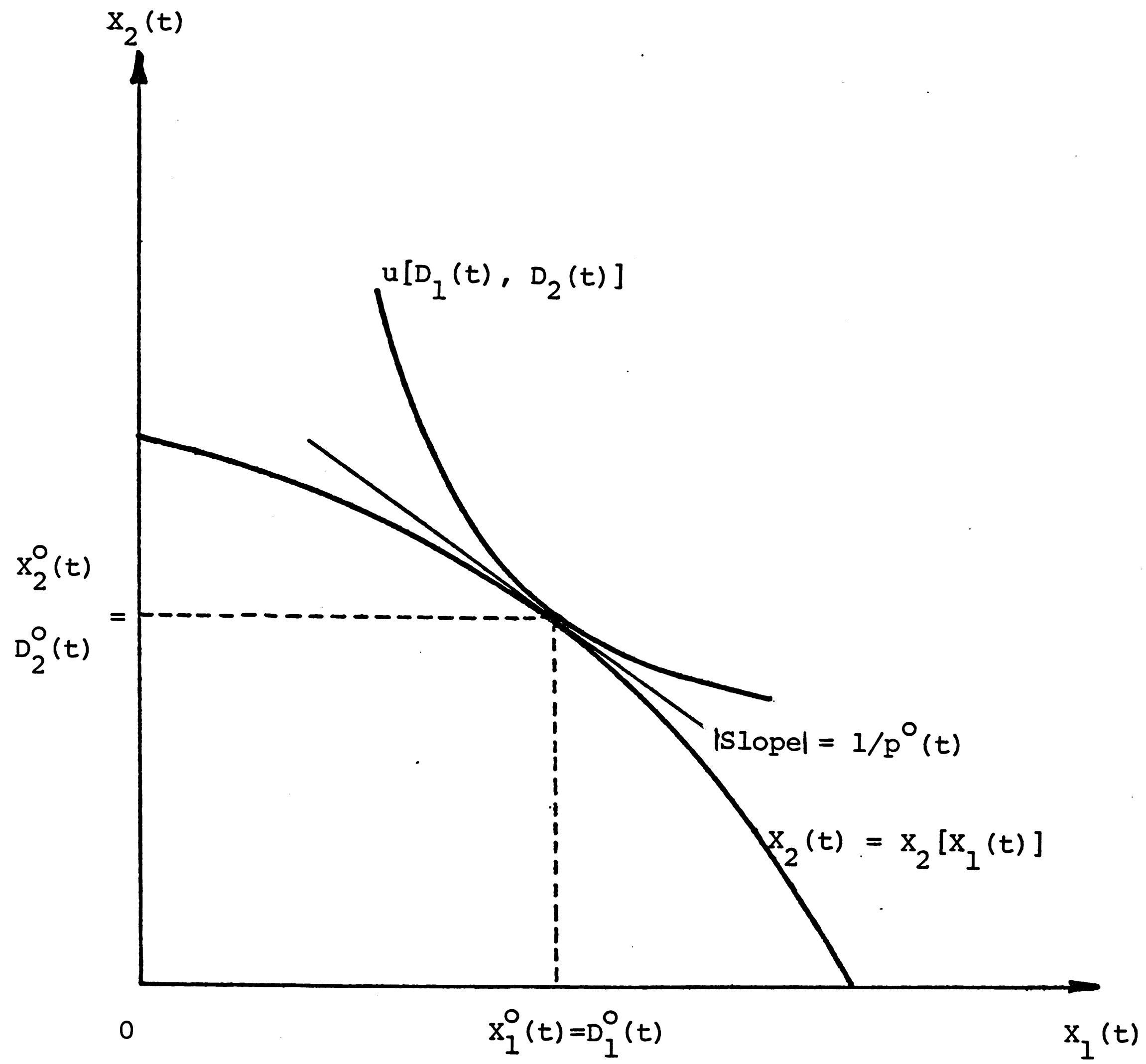
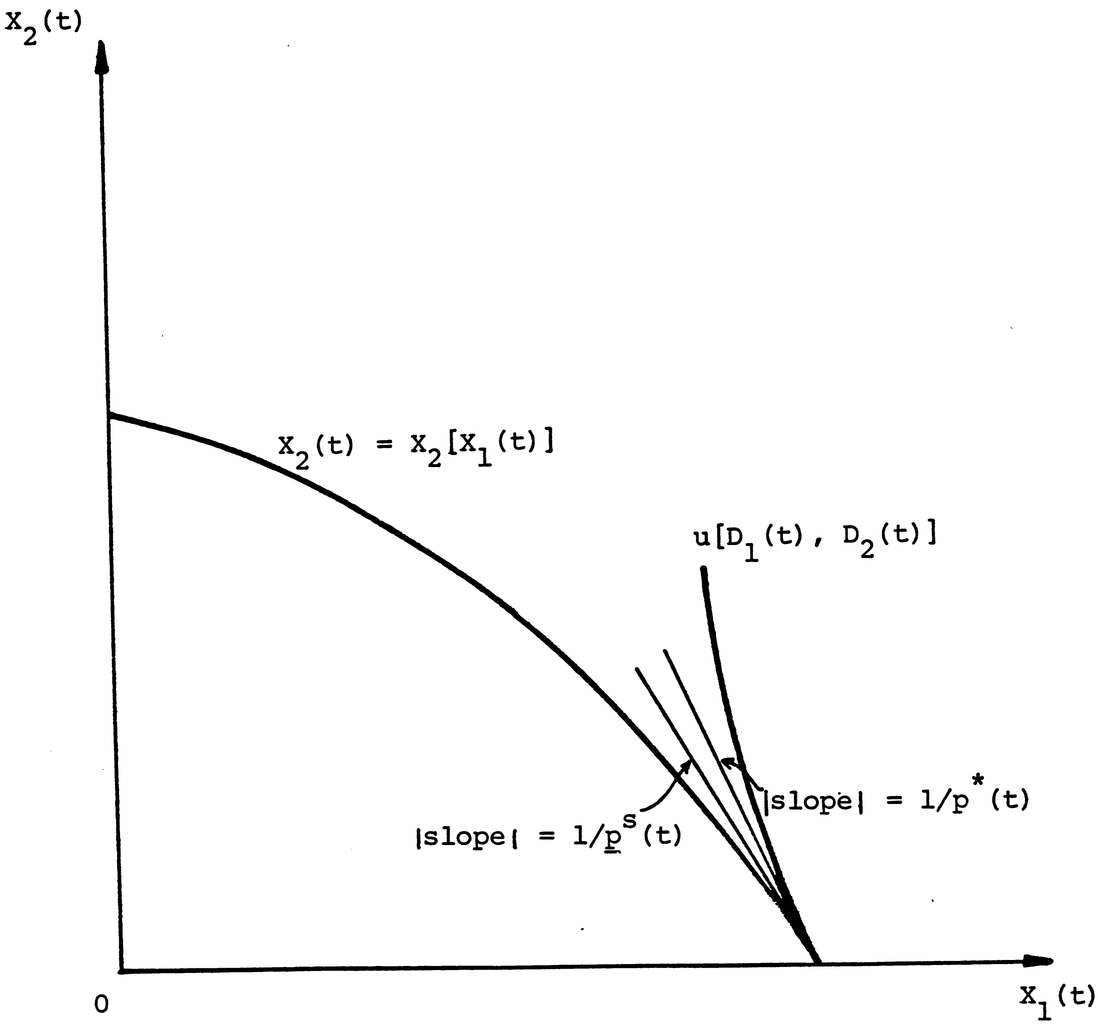



Figure 2.10: Equilibrium in the product markets by the constrained maximization of social welfare




<u>Figure 2.11</u>: Corner equilibrium

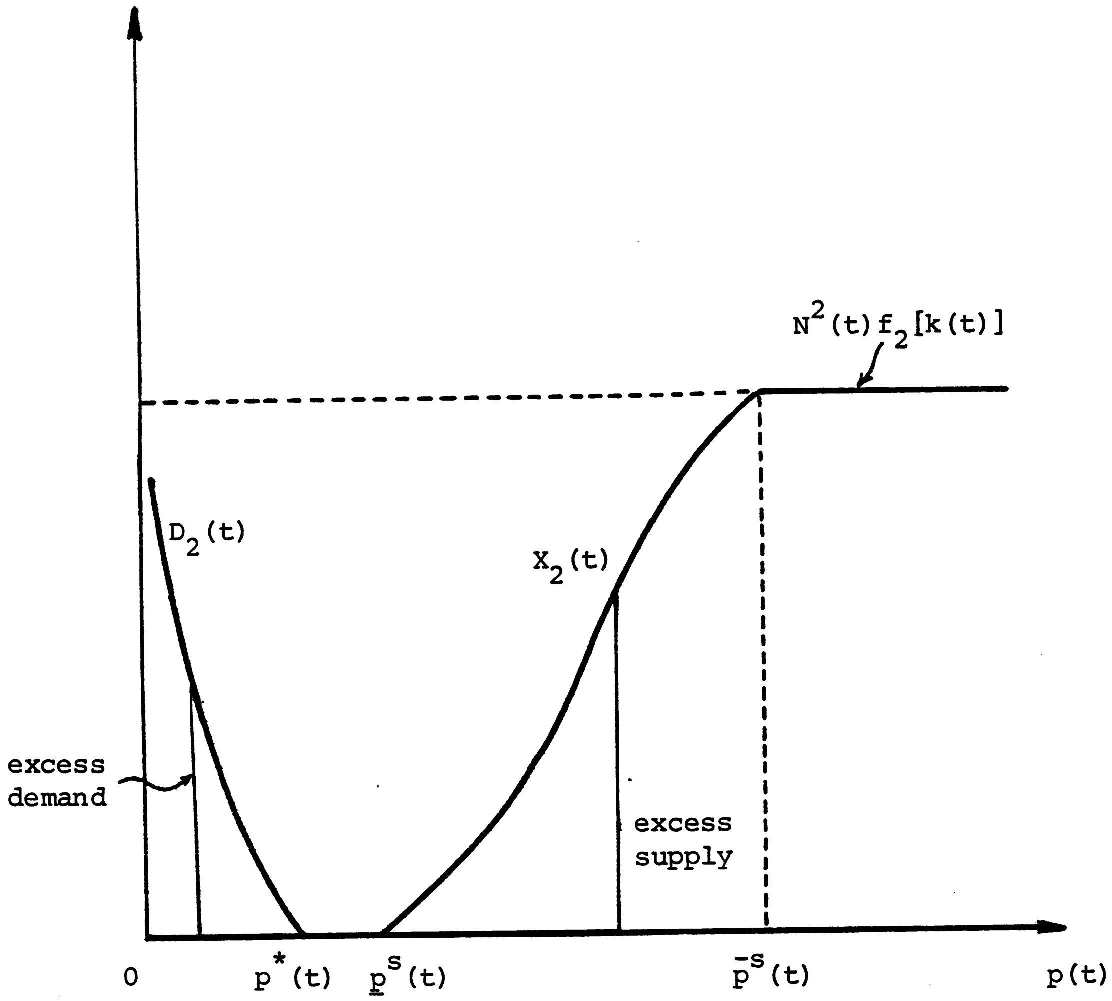
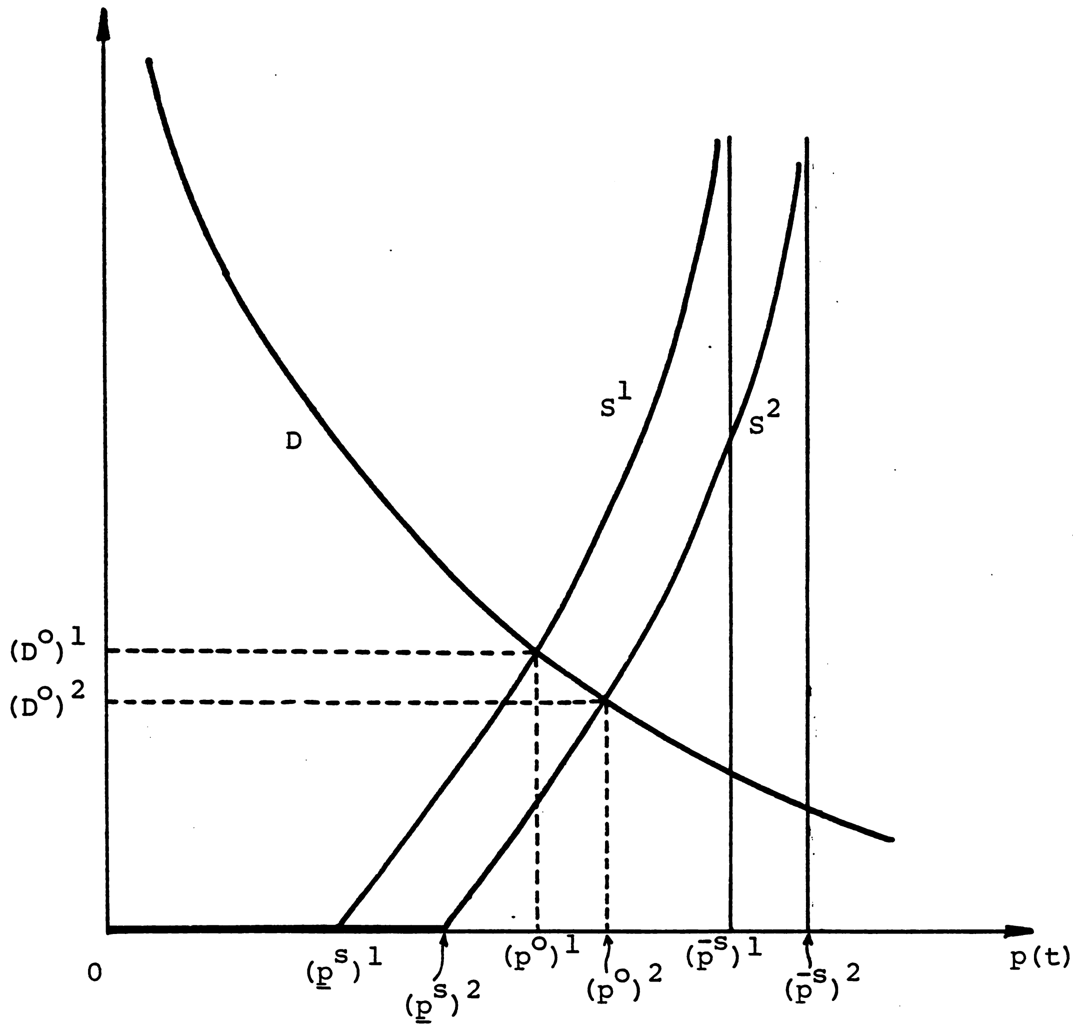



Figure 2.12: Static stability of corner equilibrium in the second product market




<u>Figure 2.13</u>: Effect of an increase in k where $k_1 > k > k_2$

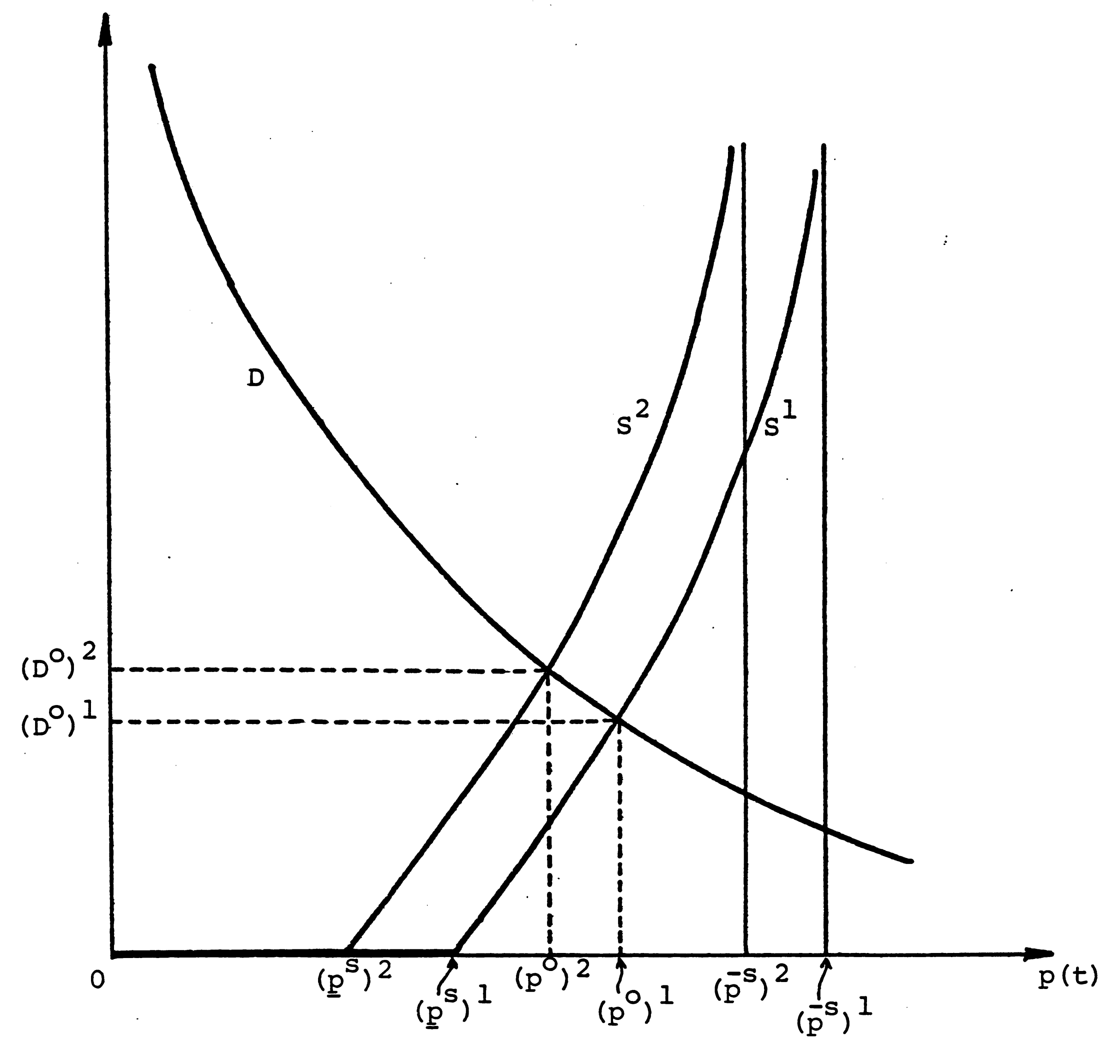
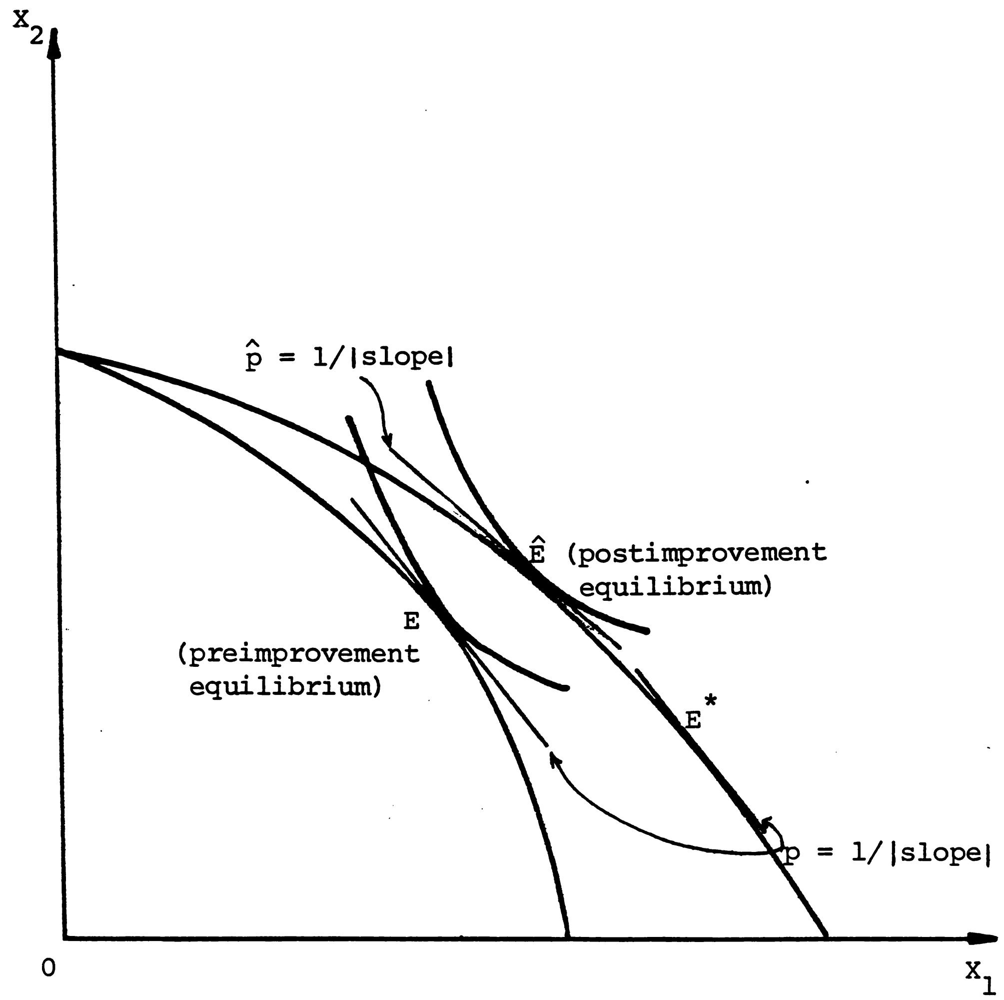



Figure 2.14: Effect of an increase in k where $k_1 < k < k_2$




<u>Figure 2.15</u>: Effect of a young-labour-saving improvement in the first industry where $k_1 > k_2$

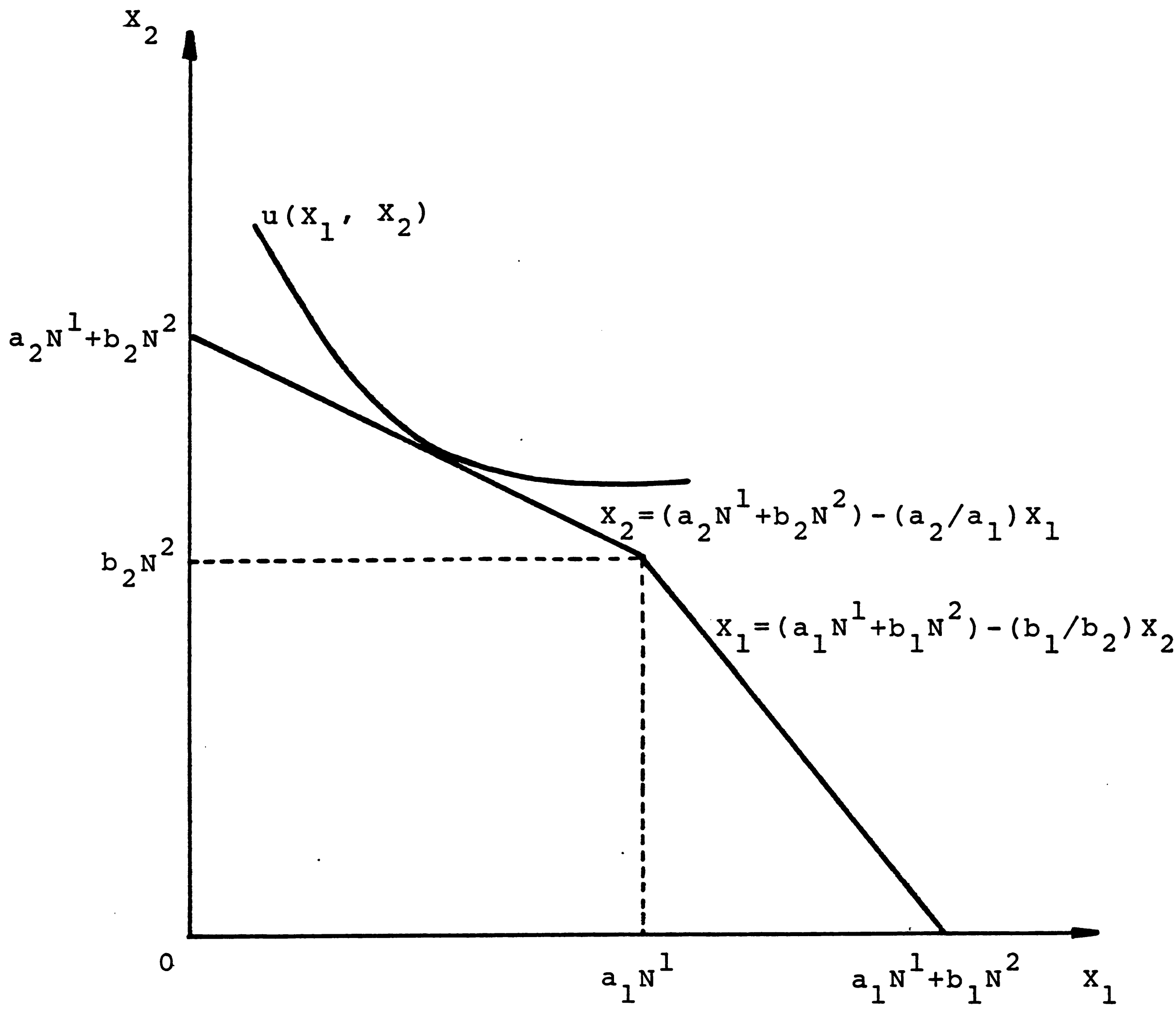



Figure 2.16: Preimprovement equilibrium with linear production functions where $a_1/b_1 > a_2/b_2$

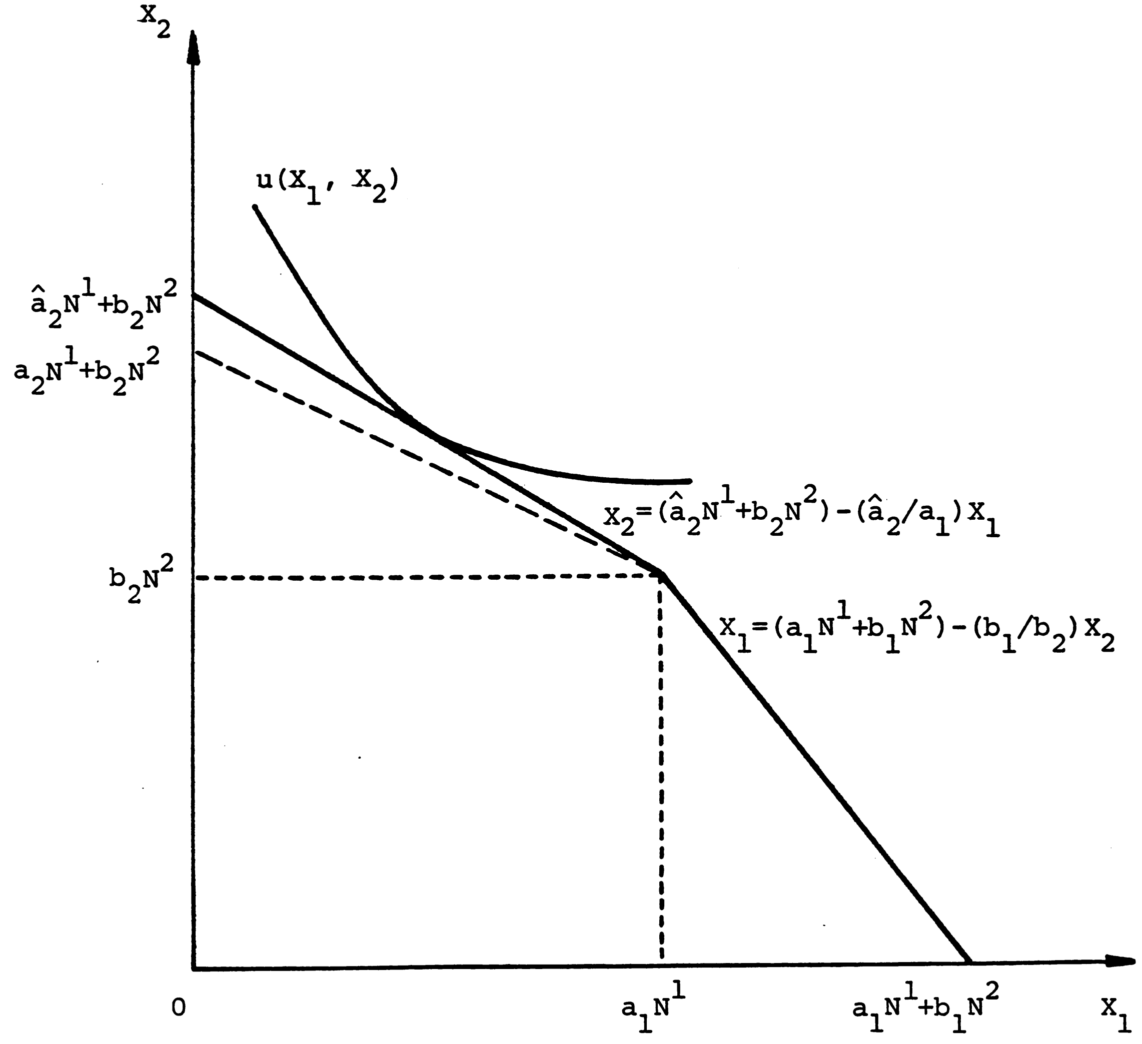
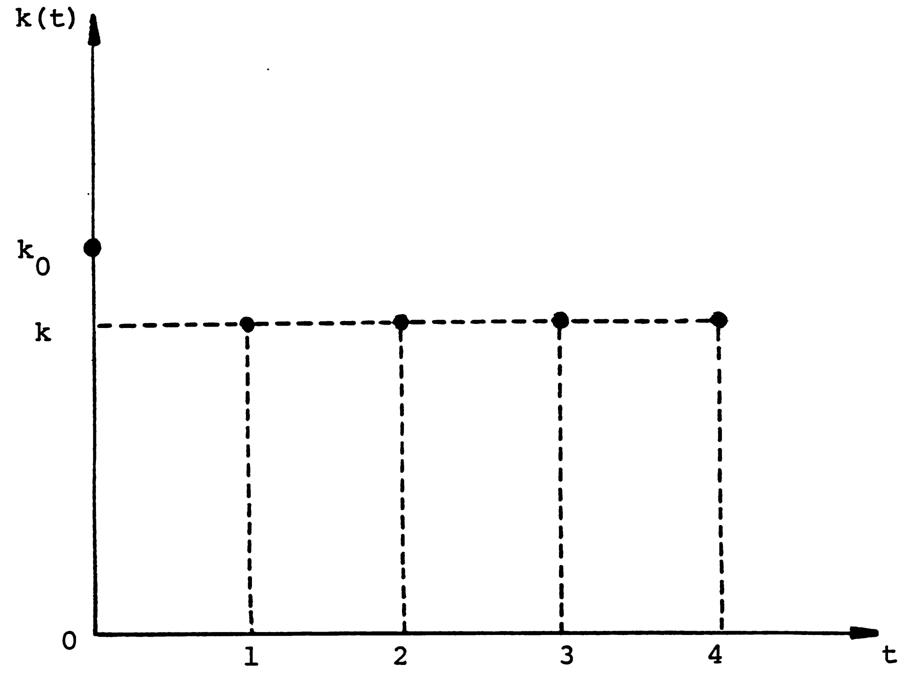
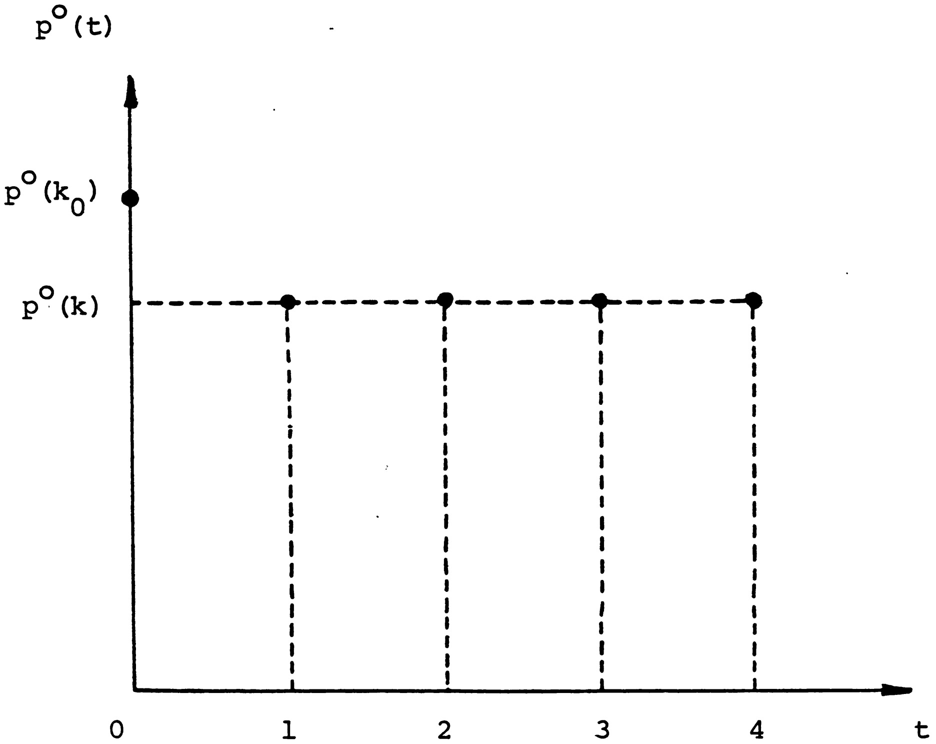



FIGURE 2.17: Postimprovement equilibrium with linear production functions where $a_1/b_1 > \hat{a}_2/b_2 > a_2/b_2$




<u>Figure 2.18</u>: Population model one where $k_0 = N_0^1/N_0^2 > k = \gamma_1$




<u>Figure 2.19</u>: The corresponding time path of $p^o(t)$ where $k_1 > k_2$

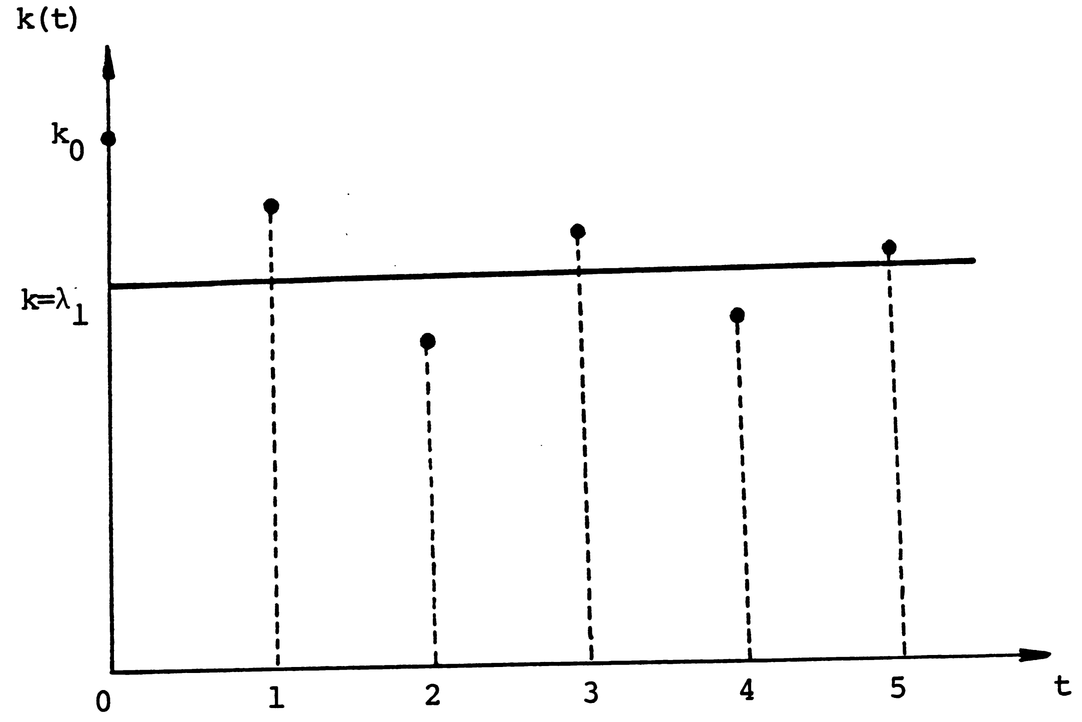



Figure 2.20: Population model three where $k_0 > k(1) > k = \lambda_1$

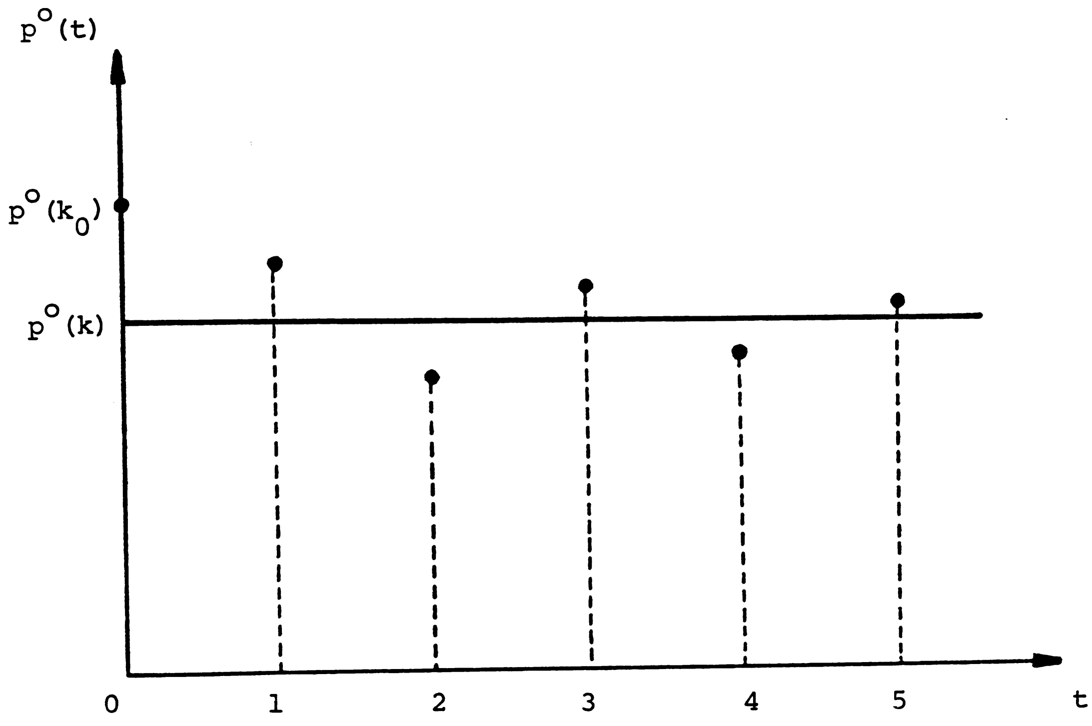
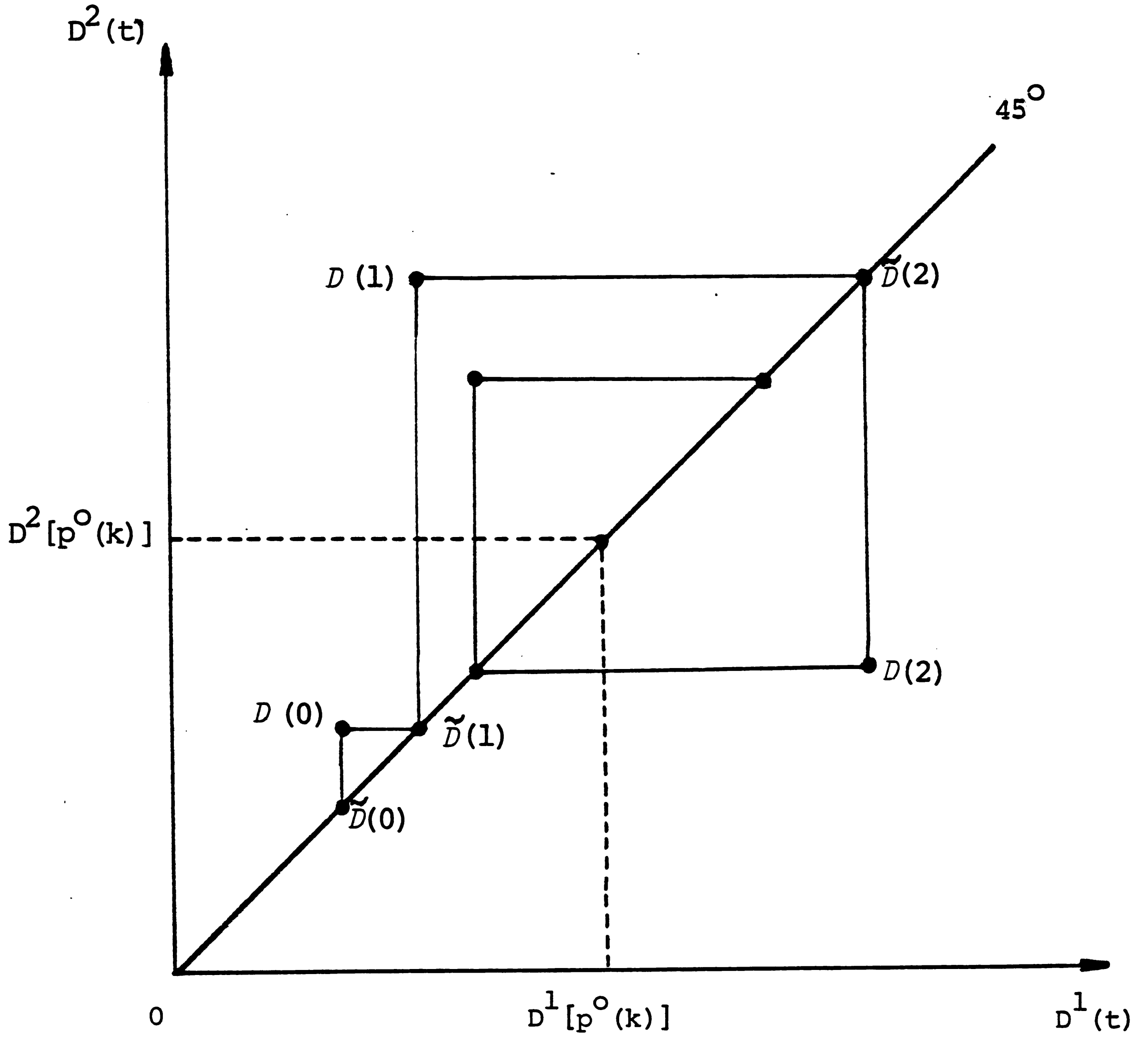



Figure 2.21: The corresponding time path of $p^o(t)$ where $k_1 > k_2$




<u>Figure 2.22</u>: Time path of the consumption ratios with population model three where $k_0 > k(1)$ and $k_1 > k_2$

# CHAPTER III

# CLOSED BARTER ECONOMY WITH OVERLAPPING GENERATIONS AND INTERGENERATIONAL TRADE

## 3.1 INTRODUCTION

One of the crucial assumptions of the basic model discussed previously is that no trade within or between generations takes place. Intragenerational borrowing and lending is not possible because the consumption patterns of all individuals of the same generation are exactly identical. Thus, a typical man will not be prepared at the equilibrium price to lend to or to borrow from another man of the same age group. The imposition of no direct intergenerational trade seems to be quite natural in the two-period model as developed so far. Young people are not willing to lend to old people living in the same time period; for when the young are old, the old who borrowed from them in the previous period will be dead and therefore unable to repay the loan. By similar reasoning, neither are the old people prepared to lend to young people.

A simple device which can be used to allow for intergenerational borrowing and lending is an immortal central clearing house which acts as an intermediary between the young and the old. People will never trade directly with each other but work through this clearing house. Thus, suppose that, at time t, old people consume less than their income, having spent more than their income when young. The goods saved are accepted by the clearing house in discharge of the old's earlier debt. The new generation of young people wishes to consume this excess and must therefore buy it from the clearing house. Since they are not able to pay for it until their old age, they have to leave with the clearing house an I.O.U. for the goods they borrow. In the next period, they can pay off their earlier debt (with interest) and get back the I.O.U., which they can destroy. In the meantime, a new generation of young people has come to the clearing house with a new collection of I.O.U.'s, and so on. At each instant of time, the clearing house is holding I.O.U.'s in exactly the amount of debt owed by the population. In the other case, we have young people selling their excess income to the clearing house and this time it is the clearing house which gives the young people the I.O.U.'s to cash in their old age. Once again, the books balance because the population now holds the debt I.O.U.'s issued by the clearing house.

## 3.2 FORMULATION OF THE EXTENDED MODEL

### 3.2.1 The Budget Constraints Reconsidered

To rewrite the budget equations, consider a typical person born in period t. Suppose that he works in the i-th industry (i=1,2) during his lifetime. In the first period of his life, he receives an amount of good i which is worth $w_1(t)$ as the reward of his labour services. He can then consume directly a part of his income and trade some of his income for the other good with those who work in the other industry according to the ruling terms of trade p(t).

If the total value of his consumption is less than his income, this typical young man can deposit the remaining amount of good i in the central clearing house to earn interest. Otherwise, if the total consumption exceeds his income, the young man can now borrow from the clearing house some of good i to cover the difference. In the second period of his life, the old man received an amount of good i worth $w_2(t+1)$. He can consume part of it and trade some of it for the other good at the given terms of trade $p(t+1)$. Then his lifetime wage income vector is $w(t) = [w_1(t); w_2(t)] = [w_1(p(t)); w_2(p(t+1))]$. If he lent to the clearing house when he was young, the old man will receive, in addition to his wages, his saving with interest (in the form of good i); his gross income is therefore $w_2(t+1)$ plus the saving and interest. If, on the other hand, he borrowed as a young man then he will later have to repay the loan with interest (in terms of the same good) and his net income will be $w_2(t+1)$ minus the loan and interest.

Formally, let $C_i^j(t+j-1)$ be the consumption of good i in the j-th period of his life (i=1,2; j=1,2) and let s(t) be his saving (or dissaving) at time t. Then the budget constraint in period t becomes

$$w^1(t) - C_1^1(t) - p(t)C_2^1(t) = s(t) \qquad t=0,1,2,\ldots \qquad (3\text{-}1)$$

where $w^1(t) = w_1(t)/p_1(t)$ is the young man's wage rate in terms of the first good at time t. If $s(t) > 0$ ($< 0$) then the person saves (dissaves) in period t. Obviously, if he saves (dissaves) in period t then he will dissave (save) in period t+1. Let $r(t)$ ($-1 < r(t) \gtreqless 0$) and $\varepsilon(t) = 1+r(t)$ ($\varepsilon(t) > 0$)

denote the rate of interest and the interest factor at time t, respectively. Then the budget constraint in period t+1 is

$$C_1^2(t+1) + p(t+1)C_2^2(t+1) = w^2(t+1) + \varepsilon(t)s(t) \qquad t=0,1,2,\ldots \qquad (3\text{-}2)$$

where $w^2(t+1) = w_2(t+1)/p_1(t+1)$ is the wage rate of old labour in terms of the first good at time t+1.

Eliminating s(t) in (3-1) and (3-2), the two budget equations reduce to a single constraint

$$\varepsilon(t)\ [w^1(t)-C_1^1(t)-p(t)C_2^1(t)] + [w^2(t+1)-C_1^2(t+1)-p(t+1)C_2^2(t+1)] = 0 \qquad t=0,1,2,\ldots \qquad (3\text{-}3)$$

Since $C_1^1(t) \geq 0$, $C_2^1(t) \geq 0$ $(t=0,1,2,\ldots)$, equation (3-1) provides the upper bound value for s(t).

$$s(t) \leq w^1(t) \qquad t=0,1,2,\ldots$$

Before presenting some well-known results, it may be useful to introduce the following terminology.

### 3.2.2 Definitions

Competitive Programs Let $C(t) = [C_1^1(t);\ C_2^1(t);\ C_1^2(t+1), C_2^2(t+1)]$ denote the lifetime consumption program of a typical person born in period t. Then the sequence $\{C(t)\}$ is said to be competitive if C(t) is the solution of the utility-maximizing problem

$$\underset{\{C(t)\}}{\text{Maximize}}\ U[C(t)] = \Omega[u(C_1^1(t),\ C_2^1(t));\ u(C_1^2(t+1),\ C_2^2(t+1))]$$

subject to equation (3-3) and $C_i^j(t+j-1) \geq 0$

$$(i=1,2; \quad j=1,2)$$

In other words, if there exists

$$\tilde{C}(t) = [\tilde{C}_1^1(t),\ \tilde{C}_2^1(t);\quad \tilde{C}_1^2(t+1),\ \tilde{C}_2^2(t+1)] \neq C(t)$$

such that $U[\tilde{C}(t) > U[C(t)]$ then

$$\varepsilon(t)[w^1(t)-\tilde{C}_1^1(t)-p(t)\tilde{C}_2^1(t)]+[w^2(t+1)-\tilde{C}_1^2(t+1)-p(t+1)\tilde{C}_2^2(t+1)] < 0 \tag{3-4}$$

The choice problem faced by a typical old person in period $t=0$ is simply to maximize $u[C_1^2(0),\ C_2^2(0)]$ subject to

$$C_1^2(0)+p(0)C_2^2(0) = w^2(0)+k(0)s(0)$$

where $p(0)$ is the initial price ratio, $k(0) = N_0^1/N_0^2$ is the initial population endowment ratio and $s(0)$ is the saving (dissaving) of a representative young man at time $t=0$.

In a steady-state equilibrium, where $p(t)$, $p(t+1)$, $w^1(t)$, $w^2(t+1)$, $\varepsilon(t)$ and $C_i^j(t+j-1)$ $(i=1,2;\ \ j=1,2)$ are all independent of $t$, the competitive program satisfies

$$\varepsilon[w^1 - C_1^1 - pC_2^1] + [w^2 - C_1^2 - pC_2^2] = 0 \tag{3-5}$$

<u>Feasible Programs</u> A sequence of consumption program $\{C(t)\}$ is said to be feasible if, for each good, aggregate consumption equals aggregate output in each period. More specifically, $\{C(t)\}$ is feasible if it satisfies the following conditions.

(i) Total income is equal to the value of total consumption.

$$Y(t) = X_1(t) + p(t)X_2(t) = D_1(t) + p(t)D_2(t) \qquad t=0,1,2,... \tag{3-6}$$

where all variables are defined as in the previous chapter.

(ii) For each good, total supply is equal to total demand.

$$X_i(t) = D_i(t) \qquad t=0,1,2,\ldots \qquad (3\text{-}7)$$

By Walras' Law, if equation (3-7) holds for one good then it also holds for the other good provided that (3-6) is satisfied. Therefore, it is sufficient to require (3-7) to hold for one good. The market-clearing condition (3-6) may be rewritten as

$$X_1(t)+p(t)X_2(t) = N^1(t)w^1(t)+N^2(t)w^2(t) \qquad t=0,1,2,\ldots$$

and

$$D_1(t)+p(t)D_2(t) = [N^1(t)C_1^1(t)+N^2(t)C_1^2(t)] + p(t)[N^1(t)C_2^1(t)+N^2(t)C_2^2(t)] \qquad t=0,1,2,\ldots$$

Equating, rearranging and dividing by $N^2(t)$ we have

$$k(t)[w^1(t)-C_1^1(t)-p(t)C_2^1(t)] + [w^2(t)-C_1^2(t)-p(t)C_2^2(t)] = 0 \qquad t=0,1,2,\ldots \qquad (3\text{-}8)$$

where $k(t) = N^1(t)/N^2(t)$ is the overall endowment ratio at time t. The expression inside the first square bracket of the L.H.S. of (3-8) represents the saving (or dissaving) of a typical young man at time t and the expression inside the second square bracket represents the dissaving (or saving) of a typical old man at time t. Equation (3-8) can then be reinterpreted as requiring that the total saving (or dissaving) of one age group is exactly equal to the dissaving (or saving) of the other age group. Recalling that

$s(t) = w^1(t)=C_1^1(t)-p(t)C_2^1(t) \le w^1(t)$, equation (3-8) provides the lower bound value for $s(t)$. It is evident that

$$-w^2(t)/k(t) \le s(t) \le w^1(t) \qquad t=0,1,2,... \tag{3-9}$$

In steady-state equilibria,

$$k[w^1-C_1^1-pC_2^1] + [w^2-C_1^2-pC_2^2] = 0 \tag{3-10}$$

and

$$-w^2/k \le s \le w^1 \tag{3-11}$$

Equilibrium Programs A sequence of consumption programs $\{C(t)\}$ is said to be an equilibrium sequence if $\{C(t)\}$ is both competitive and feasible. In stationary equilibria, a consumption program $C = [C_1^1, C_2^1; C_1^2, C_2^2]$ satisfies equations (3-5) and (3-10). Our interest will be mainly focused on stationary equilibrium consumption programs.

In the early models of interest theory studied by "classical" writers such as Fisher and Bohm-Bawerk, people want to consume more than their income in their youth and pay it back in their old age. In contrast, Samuelson [20] formulated models in which people save in the early stage of their lives to prepare for their retirement years. Following Gale's paper [7], we introduce the following definitions. A program will be called Classical if, in steady-state equilibrium, a typical young man spends more than what he earns, i.e., if $s < 0$ or $C_1^1+pC_2^1 < w^1$. A program will be called Samuelson if, in steady-state equilibrium, a typical young man saves, i.e., if $s > 0$ or $C_1^1+pC_2^1 > w^1$. A program will be called neutral (or no-trade) if, in steady-state equilibrium, a typical young man consumes exactly his income, i.e., if $s = 0$ or $C_1^1 + pC_2^1 = w^1$.

## 3.3 STEADY-STATE ANALYSIS OF THE EXTENDED MODEL

### 3.3.1 Two Preliminary Results

Theorem 3.1 There are at most two possible steady-state equilibria. They are characterized by

(I) $\varepsilon = k$ and

(II) $w^1 = C_1^1 + pC_2^1$ or $s = 0$

Proof Subtracting equation (3-10) from equation (3-5),

$$(\varepsilon - k)\ (w^1 - C_1^1 - pC_2^1) = 0$$

and we have the desired result. Q.E.D.

In case (I), let the interest factor and the rate of interest be denoted by $\bar{\varepsilon}$ and $\bar{r}$ respectively. This steady-state equilibrium is "biological" in the sense that $\bar{\varepsilon} = k$. Let the corresponding consumption program be denoted by $\bar{C} = [\bar{C}_1^1,\ \bar{C}_2^1;\ \ \bar{C}_1^2,\ \bar{C}_2^2]$. Then $\bar{C}$ is optimal stationary in the sense that it is the most preferred stationary feasible program. For if there were some stationary program $\tilde{C} = [\tilde{C}_1^1,\ \tilde{C}_2^1;\ \ \tilde{C}_1^2,\ \tilde{C}_2^2]$ which is preferred to $\bar{C}$ then, from (3-4),

$$\bar{\varepsilon}(w^1 - \tilde{C}_1^1 - \tilde{C}_2^1) + (w^2 - \tilde{C}_1^2 - p\tilde{C}_2^2) = k(w^1 - \tilde{C}_1^1 - p\tilde{C}_2^1) + (w^2 - \tilde{C}_1^2 - p\tilde{C}_2^2) < 0$$

which implies that $\tilde{C}$ is not feasible. Let the optimal saving be denoted by $\bar{s} = w^1 - \bar{C}_1^1 - p\bar{C}_2^1$ ($\bar{s} \gtreqless 0$). Then the "biological" equilibrium is Classical, Samuelson or neutral depending on whether $\bar{s}$ is negative, positive or zero respectively.

In case (II), each person simply consumes exactly his own income in each period of his life, i.e., there is no trade between the two generations. Let $\overset{o}{C} = [\overset{o}{C}{}^1_1, \overset{o}{C}{}^1_2; \overset{o}{C}{}^2_1, \overset{o}{C}{}^2_2]$ denote the consumption program associated with the no-trade situation. Then it is clear that $U(\overset{o}{C}) \leq U(\bar{C})$. The stationary "no-trade" equilibrium is neutral because the corresponding saving $s^o = w^1 - \overset{o}{C}{}^1_1 - p\overset{o}{C}{}^1_2$ is equal to zero. The interest factor and the rate of interest associated with the "no-trade" equilibrium will be respectively denoted by $\varepsilon^o$ and $r^o$.

The "biological" equilibrium (I) and the "no-trade" equilibrium (II) will coincide if, in a steady state, the saving of a typical young man is zero at the interest factor $\bar{\varepsilon} = k$. The coincidental stationary equilibrium may be characterized by $\varepsilon^o = \bar{\varepsilon} = k$ and $s^o = \bar{s} = 0$.

<u>Theorem 3.2</u> An equilibrium is Classical ($\bar{s} < 0$) if and only if $\varepsilon^o > \bar{\varepsilon} = k$. An equilibrium is Samuelson ($\bar{s} > 0$) if and only if $\varepsilon^o < \bar{\varepsilon} = k$. An equilibrium is neutral ($\bar{s} = 0$) if and only if $\varepsilon^o = \bar{\varepsilon} = k$.

<u>Proof</u> The last statement of the theorem comes directly from the definition of the coincidental steady-state equilibrium. Consider the distinct case where $\overset{o}{C} \neq \bar{C}$. $\bar{C}$ is an optimal stationary program,

$$U(\bar{C}) > U(\overset{o}{C})$$

But $\overset{o}{C}$ is competitive, from inequality (3-4) with $\varepsilon = \varepsilon^o$,

$$\varepsilon^{o}(w^{1}-\bar{C}_{1}^{1}-p\bar{C}_{2}^{1}) + (w^{2}-\bar{C}_{1}^{2}-p\bar{C}_{2}^{2}) < 0$$

$\bar{C}$ is feasible, from equation (3-10) with $C = \bar{C}$,

$$k(w^{1}-\bar{C}_{1}^{1}-p\bar{C}_{2}^{1}) + (w^{2}-\bar{C}_{1}^{2}-p\bar{C}_{2}^{2}) = 0$$

Subtracting the second equation from the first inequality,

$$(\varepsilon^{o}-k)\ (w^{1}-\bar{C}_{1}^{1}-p\bar{C}_{2}^{1}) = (\varepsilon^{o}-k)\ \bar{s} < 0$$

Hence $(\varepsilon^{o}-k)$ and $\bar{s} = w^{1}-\bar{C}_{1}^{1}-p\bar{C}_{2}^{1}$ must have the opposite signs, which is what the theorem asserts. Q.E.D.

Theorems 3.1 and 3.2 are merely two accounting identities which owe nothing to the special features of the utility and production functions of the model.

### 3.3.2 Indirect Utility Function

In this subsection, the indirect utility function defined over the price-expenditure space will be derived. Two lemmas will be given here. The first is concerned with the constancy of the steady-state equilibrium price ratio and the second deals with the form of the indirect utility function. A theorem concerning the no-trade interest factor will then be stated in terms of the indirect function derived in the second lemma.

Lemma 3.1 The removal of the assumption of no borrowing and lending between generations does not alter the steady-state equilibrium commodity price ratio $p^{o}$.

Proof The existence of intergenerational borrowing and lending is equivalent to an intertemporal redistribution of income among the young and the old. However, since everyone has the same homothetic preferences, this does not alter the total demand for each good. The equilibrium commodity price ratio will, therefore, remain unchanged.

To demonstrate the result more rigorously, consider the demands for good i with and without trade between generations.

Without trade,

$$D_i^{wt} = N^1C_i^1 + N^2C_i^2 \qquad i=1,2$$

$$= N^2[k\phi_i(p)w^1+\phi_i(p)w^2] \qquad \text{(See equation (2-45).)}$$

With trade between generations,

$$D_i^t = N^1C_i^1 + N^2C_i^2 \qquad i=1,2$$

$$= N^2[k\phi_i(p)(w^1-s)+\phi_i(p)(w^2+\varepsilon s)]$$

$$= N^2[k\phi_i(p)w^1+\phi_i(p)w^2] + N^2[\phi_i(p)s(\varepsilon-k)]$$

But in a steady state, $s = 0$ or $\varepsilon = k$. Therefore, $D_i^{wt} = D_i^t$ $(i=1,2)$ and the equilibrium $p^o$ remains constant.

Q.E.D.

A direct consequence of this lemma is that the pre-trade production pattern will remain the same after the introduction of intergenerational borrowing and lending.

Lemma 3.2 (Gorman) Let $u = u(C_1^j,C_2^j)$ be a strictly increasing, strictly concave, homothetic function representing the satisfaction in the j-th (j=1,2) period of a typical consumer's life. Let $p_1$, $p_2$ and p be the prices of the first and second commodities and the price ratio $p_2/p_1$ respectively. Let $C_j$ and $C^j$ be respectively the consumer's expenditure and the ratio $C_j/p_1$ in the j-th (j=1,2) period of his life. Then there exist a function g, $g'(.) > 0$, $g''(.) < 0$, a function $L(p_1,p_2)$, homogeneous of degree one in $p_1$ and $p_2$, and a function $l(p)$, $l'(p) > 0$ such that the consumer's indirect utility can be written as follows.

$$u(C_1^j,C_2^j) = v(p,C^j) = g[C_j/L(p_1,p_2)] = g[C^j/l(p)] \qquad j=1,2 \qquad (3\text{-}12)$$

<u>Proof</u> See Lemma 2.1 and W.M. Gorman [8]. Q.E.D.

From Gorman [8], it has also been shown that

$$C_i^j = [\partial L(p_1,p_2)/\partial p_i]\ [C_j/L(p_1,p_2)]$$

$$= \begin{cases} [1-pl'(p)/l(p)]\ C^j & i=1 \\ [l'(p)/l(p)]\ C^j & i=2 \end{cases} \qquad j=1,2 \qquad (3\text{-}13)$$

where $C_i^j$ is the consumption of the i-th good in the j-th period (i=1,2; j=1,2). An example is

$$u(C_1^j,C_2^j) = \beta\log_e C_1^j + (1-\beta)\log_e C_2^j \qquad j=1,2$$

where $0 < \beta < 1$. It is evident that

$$g(.) = \log_e(.),\ g'(.) > 0,\ g''(.) < 0$$

$$h(C_1^j,C_2^j) = (C_1^j)^{\beta}(C_2^j)^{1-\beta} \qquad j=1,2$$

Now,

$$h(C_1^j,\ C_2^j) = C_j/(Kp_1^{\beta}p_2^{\ 1-\beta}) = C^j/(Kp^{1-\beta}) \qquad j=1,2$$

where $K = \beta^{-\beta}(1-\beta)^{\beta-1}$. That is,

$$L(p_1,p_2) = Kp_1^{\beta}p_2^{\ 1-\beta} \quad \text{and} \quad l(p) = Kp^{1-\beta}.$$

Therefore,

$$v(p,C^j) = \log_e[C_j/(Kp_1^{\beta}p_2^{\ 1-\beta})] = \log_e[C^j/(Kp^{1-\beta})] \quad j=1,2$$

Also,

$$C_1^j = [1-pl'(p)/l(p)]\ C^j = \beta C^j \qquad j=1,2$$

$$C_2^j = [l'(p)/l(p)]\ C^j = [(1-\beta)/p]\ C^j \qquad j=1,2$$

Theorem 3.3 For each given steady-state overall endowment ratio k, the steady-state "no-trade" interest factor $\varepsilon^o$ satisfies the following equation.

$$\varepsilon^o = \frac{\Omega_1}{\Omega_2}\,\frac{[\partial v(p^o,C^1)/\partial C^1]_{C^1=w^1}}{[\partial v(p^o,C^2)/\partial C^2]_{C^2=w^2}} = \frac{\Omega_1}{\Omega_2}\,\frac{g'[C^1/l(p^o)]_{C^1=w^1}}{g'[C^2/l(p^o)]_{C^2=w^2}} \qquad (3\text{-}14)$$

where $\Omega$, $w^1$, $w^2$, $p^o$, v, g, l, $C^1$ and $C^2$ are as defined previously and $\Omega_j$ (j=1,2) are evaluated at $p^o = p^o(k)$, $C^1 = w^1(p^o)$ and $C^2 = w^2(p^o)$.

Proof Consider the indirect-utility-maximizing problem

$$\underset{\{C^1,\ C^2\}}{\text{Maximize}}\ U = \Omega[v(p,\ C^1);\ v(p,\ C^2)] \qquad (3\text{-}15)$$

subject to $\varepsilon(w^1-C^1) + (w^2-C^2) = 0$ and $C^j \geq 0$ (j=1,2) where $\varepsilon$, $w^1$ and $w^2$ are all given. Forming the Lagrangean function

$$\mathcal{L}(C^1, C^2, \mu) = \Omega[v(p,C^1); v(p,C^2)] + \mu[\varepsilon(w^1-C^1)+(w^2-C^2)]$$

the necessary conditions for an interior solution are

$$\partial\mathcal{L}/\partial C^1 = \Omega_1[\partial v(p,C^1)/\partial C^1] - \mu\varepsilon = 0 \qquad (3\text{-}16\text{-}a)$$

$$\partial\mathcal{L}/\partial C^2 = \Omega_2[\partial v(p,C^2)/\partial C^2] - \mu = 0 \qquad (3\text{-}16\text{-}b)$$

$$\partial\mathcal{L}/\partial\mu = \varepsilon(w^1-C^1) + (w^2-C^2) = 0 \qquad (3\text{-}16\text{-}c)$$

Eliminating $\mu$ in (3-16-a) and (3-16-b) yields

$$\varepsilon = (\Omega_1/\Omega_2)[\partial v(p,C^1)/\partial C^1]/[\partial v(p,C^2)/\partial C^2] \qquad (3\text{-}17)$$

Equation (3-17) together with (3-16c) determine uniquely $C^1$ and $C^2$ that solves (3-14) provided that g is a strictly concave function of $[C^j/l(p)]$ $(j=1,2)$. Obviously, the interest factor that completely discourages trade between generations is given by

$$\varepsilon^o = \frac{\Omega_1}{\Omega_2}\,\frac{[\partial v(p/C^1)/\partial C^1]_{C^1=w^1}}{[\partial v(p,C^2)/\partial C^2]_{C^2=w^2}}$$

But

$$[\partial v(p,C^j)/\partial C^j]_{C^j=w^j} = [\partial[g(C^j/l(p))]/\partial C^j]_{C^j=w^j}$$

$$= [1/l(p)]\; g'[C^j/l(p)]_{C^j=w^j}$$

$$j=1,2$$

and $w^j$ is a function of the equilibrium commodity price ratio $p^o$, i.e.,

$$w^j = w^j(p^o) = w^j[p^o(k)] \qquad (j=1,2). \qquad \text{Q.E.D.}$$

To interpret the result, consider the simple case in which U takes the separable and additive form

$$U = u(C_1^1, C_2^1) + \delta u(C_1^2, C_2^2)$$

where $\delta = 1/(1+\rho) > 0$ is the constant factor of time preference and $(-1 < \rho \gtreqless 0)$ is the constant rate of time preference.

Suppose further that the production functions are chosen such that $w^1[p^o(k)] = w^2[p^o(k)]$. Then equation (3-14) reduces to

$$\varepsilon^o = 1/\delta$$

or $\quad r^o = \rho$

If men discount their future utility (i.e., $\rho > 0$), then the "no-trade" rate of interest $r^o$ is positive. If men overvalue their future utility (i.e., $\rho < 0$), then the "no-trade" rate of interest is negative. This is, of course, a well-known result in the theory of interest. Theorem 3.2 and 3.3 are consistent because

$$\bar{s} \gtreqless 0 \text{ is equivalent to } w^1 \gtreqless \bar{C}^1 \text{ and } w^2 \lesseqgtr \bar{C}^2$$

$$\text{is equivalent to } g'[w^1/l(p)] \lesseqgtr g'[\bar{C}^1/l(p)]$$

$$\text{and } g'[w^2/l(p)] \gtreqless g'[\bar{C}^2/l(p)]$$

$$\text{implies} \quad \varepsilon^o \lesseqgtr \bar{\varepsilon} = k$$

<u>Cases not covered by Theorem 3.3</u>

Theorem 3.3 holds for all strictly concave utility functions u. If u is a linear function of a homogeneous function of degree one in $C_1^j$ and $C_2^j$ $(j=1,2)$, i.e., u is not

strictly concave, then the transformation g in (3-12) is a linear mapping. Thus, $v(p,C^j) = a+b[C^j/l(p)]$ and equation (3-14) simplifies to

$$\varepsilon^o = \Omega_1/\Omega_2$$

where $\Omega_j$ (j=1,2) are evaluated at the equilibrium price ratio $p^o(k)$. Clearly, if $\Omega_j$ (j=1,2) are dependent on the equilibrium price ratio $p^o$ which in turn depends upon the overall endowment ratio k, consumers' preferences and the production technology, then Theorem 3.3 is still valid in its general form. However, in some special cases where $\Omega_1 = 1$ and $\Omega_2 = \delta$, the "no-trade" rate of interest is given by $r^o = \rho$, which is completely independent of the steady-state endowment ratio k and the economy's technology.

### 3.3.3 The Derivation of the Optimal "Biological" Saving $\bar{s}$

It is of particular interest to compare the magnitudes of the "no-trade" interest factor $\varepsilon^o$ and the "biological" interest factor $\bar{\varepsilon} = k$. Recalling theorem 3.2 that $\varepsilon^o \gtreqless \bar{\varepsilon} = k$ if and only if $\bar{s} \lesseqgtr 0$, it is sufficient to determine whether the optimal saving $\bar{s}$, associated with the "biological" interest factor $\bar{\varepsilon} = k$, is positive, zero or negative. This is not possible in general. However, with the technology of the economy and the consumers' preferences explicitly given, the optimal saving $\bar{s}$ can be derived by solving problem (3-15) with $\varepsilon = k$, $C^1 = w^1 - s$ and $C^2 = w^2+ks$.

#### Existence and Uniqueness of $\bar{s}$

An interior optimum of (3-15) necessarily implies

$$k = \frac{\Omega_1}{\Omega_2} \frac{g'[w^1-\bar{s})/l(p)]}{g'[(w^2+k\bar{s})/l(p)]} \tag{3-18}$$

where $\Omega_j$ $(j=1,2)$ are evaluated at $p^o$, $\bar{C}^1=w^1(p^o)-\bar{s}$ and $\bar{C}^2 = w^2(p^o)+k\bar{s}$, $g'[x/l(p)] = g'[C^j/l(p)]_{C^j=x}$ and $-w^2/k \le \bar{s} \le w^1$. Totally differentiating $v^j(p,C^j) = g[C^j/l(p)]$ $(j=1,2)$ with respect to s, keeping in mind $dp/ds = 0$, $C^1 = w^1-s$ and $C^2 = w^2+ks$, we have

$$dv^j/ds = [\partial v^j/\partial p](dp/ds) + [\partial v^j/\partial C^j](dC^j/ds)$$

$$= \begin{cases} -g'[C^1/l(p)]/l(p) & j=1 \\ kg'[C^2/l(p)]/l(p) & j=2 \end{cases} \tag{3-19-a}$$

Therefore,

$$d\Omega_j/ds = d\Omega_j(v^1,v^2)/ds = \Omega_{j1}dv^1/ds + \Omega_{j2}dv^2/ds$$

$$= [1/l(p)]\ \{-\Omega_{j1}g'[(w^1-s)/l(p)]+ \Omega_{j2}kg'[(w^2+ks)/l(p)]\} \tag{3-19-b}$$

$$\begin{cases} \ge 0 & j=1 \\ \le 0 & j=2 \end{cases}$$

because $\Omega_{11} \le 0$, $\Omega_{22} \le 0$ and $\Omega_{12} = \Omega_{21} \ge 0$.

Since $d\Omega_1/ds \ge 0$, $d\Omega_2/ds \le 0$ and g is strictly concave, a solution $\bar{s}$ to (3-18), if it exists, must be unique. The existence of such a solution will be ensured if $g'(x) \to \infty$ as $x \to 0$.

For a given k, the three possible optimal steady-state equilibria are diagrammatically illustrated by Figures 3.1, 3.2 and 3.3. When k changes, p, $w^1$, $w^2$ also change following Theorems 2.1 and 2.2 and both curves in each diagram will, therefore, move accordingly. It is interesting to note that if the "no-trade" rate of interest does not depend on k or the economy's technology, the optimal saving $\bar{s}$ may either take extreme values or not be unique.

Consider the following example where U has the additive form and u exhibits constant-returns-to-scale,

$$U(C) = u(C_1^1, C_2^1) + \delta u(C_1^2, C_2^2) \qquad 0 < \delta$$

$$\text{where } u(C_1^j, C_2^j) = (C_1^j)^{\beta}(C_2^j)^{1-\beta}. \qquad 0 < \beta < 1; \quad j=1,2$$

Then in the case of no trade between the young and the old, the optimal consumption program and the corresponding indirect utility function are given by

$$\overset{o}{C} = [\beta w^1, (1-\beta)w^1/p;\ \beta w^2, (1-\beta)w^2/p]$$

$$U^o = U(\overset{o}{C}) = (w^1+\delta w^2)/(Kp^{1-\beta})$$

where $K = \beta^{-\beta}(1-\beta)^{\beta-1}$.

In the presence of intergenerational borrowing and lending with $\bar{\varepsilon} = k$,

$$C = [\beta(w^1-s), (1-\beta)(w^1-s)/p;\ \beta(w^2+ks), (1-\beta)(w^2+ks)/p]$$

$$U = U(C) = [w^1+w^2+(\delta k-1)s]/(Kp^{1-\beta})$$

There are three cases:

(a) $\delta k-1 < 0$, i.e., $\varepsilon^o = 1/\delta > k = \bar{\varepsilon}$.

This is the Classical case, where $\bar{s} < 0$. Clearly $\bar{s} = -w^2/k$ and

$$\bar{C} = [\beta\{w^1+(w^2/k)\},\ (1-\beta)\ \{w^1+(w^2/k)\}/p;\ 0,\ 0]$$

$$\bar{U} = U(\bar{C}) = [w^1+(w^2/k)/(Kp^{1-\beta}) > U^o$$

(b) $\delta k-1 > 0$, i.e., $\varepsilon^o = 1/\delta < k = \bar{\varepsilon}$.

This is the Samuelson case, where $\bar{s} > 0$. Clearly

$\bar{s} = w^1$ and

$\bar{C} = [0,\ 0;\ \beta(w^2+kw^1),\ (1-\beta)(w^2+kw^1)/p]$

$\bar{U} = U(\bar{C}) = \delta(w^2+kw^1)/(Kp^{1-\beta}) > U^o$

(c) $\delta k-1 = 0$, i.e., $\varepsilon^o = 1/\delta = k = \bar{\varepsilon}$.

This is the coincidental case, where $\bar{U} = U^o$. However, $\bar{s}$ is not unique and $\bar{C}$ is not necessarily equal to $\overset{o}{C}$. Any value of $\bar{s}$ satisfying $-w^2/k \le \bar{s} \le w^1$ will correspond to the same level of welfare $U^o$, i.e., $U(C) = U(\overset{o}{C})$ for any s in the interval $[-w^2/k, w^1]$.

Note that, for $\delta k-1 \neq 0$, the optimal saving $\bar{s}$ takes boundary values only, i.e., a typical individual will give up completely the consumption of both goods in one period of his life.

### 3.3.4 The Neutral "Biological" Equilibrium

We now turn our attention to the neutral "biological" case ($\bar{s} = 0$) where the biological equilibrium ($\bar{\varepsilon} = k$) and the no-trade equilibrium ($\varepsilon^o$) coincide. In particular, we attempt to discover whether there is an endowment ratio $k^*$, say, such that if the interest factor equals $k^*$, there will be no trade between generations.

To answer this type of problem, recall that the coincidental equilibrium occurs when

$$\begin{aligned} k &= (\Omega_1/\Omega_2)[v_c(p,w^1[p(k)])/v_c(p,w^2[p(k)])] \\ &= (\Omega_1/\Omega_2)g'[w^1[p(k)]/l(p)]/g'[w^2[p(k)]/l(p)] \qquad (3\text{-}20) \end{aligned}$$

where $v_c(p,w^j) = [\partial v(p,C^j)/\partial C^j]_{C^j=w^j}$ and

$g'[w^j/l(p)] = g'[C^j/l(p)]_{C^j=w^j}$ $(j=1,2)$ or

$$\bar{s} = \bar{s}[k,p(k),w^1[p(k)],w^2[p(k)]] = \bar{s}(k) = 0 \tag{3-21}$$

For any given industrial technology and consumers' preferences, there are four possibilities regarding equation (3-20) or (3-21).

(a) The solution is non-existent, i.e., there is no $k \varepsilon (0,\infty)$ such that $\bar{s} = \bar{s}(k) = 0$.

(b) The solution exists and is unique, i.e., there exists uniquely a $k^* \varepsilon (0,\infty)$ such that $\bar{s} = \bar{s}(k^*) = 0$.

(c) The solution exists but is finitely non-unique, i.e., there exist finitely many k's such that $\bar{s} = \bar{s}(k) = 0$.

(d) The solution exists for infinitely many values of k.

Sufficient conditions for the existence of a solution to (3-20) or (3-21) for a particular class of utility functions U are given and discussed in the next theorem.

<u>Theorem 3.4</u> Assume that U takes the additively separable form $U(C) = u(C_1^1,C_2^1) + \delta u(C_1^2,C_2^2)$ where $\delta > 0$. Suppose further that

(i) $$\lim_{x \to 0} g'(x) < \infty \tag{3-22-a}$$

(ii) $$\lim_{x \to \infty} g'(x) > 0 \tag{3-22-b}$$

where g is as defined in Lemma 3.2 and $x \varepsilon (0,\infty)$. Then, for any technology and preferences, there exist a $k^* \varepsilon (0,\infty)$ such that (3-20) or (3-21) holds.

<u>Proof</u> Under Assumptions (i) and (ii) and strict concavity of g,

$$\lim_{k\to 0} \frac{g'[w^1[p(k)]/l(p)]}{g'[w^2[p(k)]/l(p)]}$$

and

$$\lim_{k\to\infty} \frac{g'[w^1[p(k)]/l(p)]}{g'[w^2[p(k)]/l(p)]}$$

are positive and finite. Let us rewrite (3-20) as

$$\delta = \Omega_2/\Omega_1 = (1/k)g'[w^1/l(p)]/g'[w^2/l(p)] \qquad (3\text{-}23)$$

Clearly, $\lim_{k\to 0}$ R.H.S. of (3-23) = $\infty$ and $\lim_{k\to\infty}$ R.H.S. of (3-23) = 0. On a graph with k as the independent variable, the L.H.S. of (3-23) can be represented by a positive horizontal line while the R.H.S. of (3-23) is a smooth continuous curve varying from $\infty$ to 0 as k varies from 0 to $\infty$. Thus, the two curves must intersect at least once, as illustrated by Figure 3.4.

Q.E.D.

Examples can be constructed to show that conditions (i) and (ii) are sufficient but not necessary for the existence of k. An interpretation of Assumption (i) is as follows. People subsist not only by consuming the two produced goods but by other free goods as well. And even if a person gives up his income completely for one period of his life, he can still survive with the free goods. Thus, the marginal utility of $[C^j/l(p)]$ $(j=1,2)$ is always finite as $[C^j/l(p)]$ $(j=1,2)$ approaches zero. The interpretation of Assumption (ii) is that no satiation consumption exists for a typical consumer, i.e., no matter what his consumption

$[c^j/l(p)]$ $(j=1,2)$ is, there is another which he prefers. This can be illustrated diagrammatically, as in Figure 3.5.

The shape of $g(x)$ suggests that it may be regarded as the sum of two components, a linear function and a concave function, i.e.,

$$g(x) = a + bx + G(x) \qquad x\varepsilon(0,\infty) \qquad (3\text{-}24)$$

where a and b are real parameters and $G(x)$ satisfies the conditions

$$g(0) = a + G(0) = 0 \qquad \text{(this is not critical)}$$

$$g'(x) = b + G'(x) > 0 \qquad x\varepsilon(0,\infty)$$

$$g''(x) = G''(x) < 0 \qquad x\varepsilon(0,\infty)$$

$$\lim_{x\to 0} g'(x) = b + \lim_{x\to 0} G'(x) < \infty$$

$$\lim_{x\to\infty} g'(x) = b + \lim_{x\to\infty} G'(x) > 0$$

These conditions are equivalent to

$$G(0) = -a \qquad (3\text{-}25\text{-}a)$$

$G''(x) < 0$, i.e. G is strictly concave for $x\varepsilon(0,\infty)$ (3-25-b)

$\lim_{x\to 0} G'(x) < \infty$, i.e., the slope of $G(x)$ at $x=0$ is finite.

(3-25-c)

$\lim_{x\to\infty} G'(x) > -b$, i.e., the slope of $G(x)$ as x approaches infinity is greater than $-b$. (3-25-d)

Now it should be clear that $G(x)$ need neither be increasing nor monotonic. Let the equation of the straight line asymptotic to $g(x)$ be $c + dx$ where $c > 0$ and $d > 0$. Then the sign of $G'(x)$ for sufficiently large x can be determined by the difference (d-b).

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} a+bx+G(x) = c + dx$$

$$\lim_{x\to\infty} G(x) = (c-a) + (d-b)x$$

d-b < 0 or b > d, $\lim_{x\to\infty} G(x) = -\infty$, $G'(x) < 0$ for large enough x.

d-b = 0 or b = d, $\lim_{x\to\infty} G(x) = c-a$, $G'(x) > 0$ for any $x\epsilon(0,\infty)$.

d-b > 0 or b < d, $\lim_{x\to\infty} G(x) = +\infty$, $G'(x) > 0$ for large enough x.

The three cases are graphically shown by Figures 3.6.a, 3.6.b and 3.6.c, respectively.

Without loss of generality, one may assume that

$a + bx = c + dx$, i.e., $a = c > 0$ and $b = d > 0$.

The function G(x) will then exhibit the properties

$$G(x) < 0 \ (x\epsilon(0,\infty)) \text{ with } G(0) = -a \text{ and } \lim_{x\to\infty} G(x) = 0 \qquad \text{(3-26-a)}$$

$$G'(x) > 0 \ (x\epsilon(0,\infty)) \text{ with } \lim_{x\to 0} G'(x) < \infty \text{ and } \lim_{x\to\infty} G'(x) > -d \qquad \text{(3-26-b)}$$

$$G''(x) < 0 \ (x\epsilon(0,\infty)) \qquad \text{(3-26-c)}$$

A particular G(x) satisfying (3-25-a) - (3-25-d) is

$$G(x) = \log_e(x+e^{-a}) \qquad (b > 0)$$

A few examples of G(x) satisfying (3-26-a) - (3-26-c) are

$$G(x) = -ae^{-x}$$

$$G(x) = -a/(ax+1)$$

### 3.3.5 Uniqueness of the "No-trade" Endowment Ratio $k^*$

Let the production technology and consumers' preference be fixed, with k variable. Suppose that there exists a non-empty subset $K^* \subset (0,\infty)$ such that any element k of $K^*$ is a solution to equation (3-20) or (3-21). The obvious problem to be tackled is whether or not $K^*$ consists of a single element only, i.e., whether the "no-trade" endowment ratio $k^*$ is unique or not.

In general, it is difficult to establish simple and meaningful conditions which are sufficient to ensure uniqueness of $k^*$. Such conditions would involve many quantities whose signs and magnitudes are either ambiguous or difficult to determine. However, it is possible to construct examples to show all possibilities regarding equation (3-20) or (3-21). U is now always assumed to be additively separable.

Case (a): No Solution

Consider an economy characterized by

$$X_i = (L_{i1})^{\alpha_i}(L_{i2})^{1-\alpha_i}$$

where $0 < \alpha_i < 1$ (i=1,2) and

$$u(C_1^j, C_2^j) = \beta_1 \log_e C_1^j + \beta_2 \log_e C_2^j$$

where $0 < \beta_1$, $0 < \beta_2$ and $\beta_1+\beta_2 < 1$ (j=1,2). Then, it can be shown that

$$\bar{s} = [\alpha_1^{\alpha_1}(1-\alpha_1)^{1-\alpha_1}/(1+\delta)]A^{-(1-\alpha_1)}(\delta-A)k^{-(1-\alpha_1)} \tag{3-27}$$

where $A = [(1-\alpha_1)\beta_1+(1-\alpha_2)\beta_2]/(\alpha_1\beta_1+\alpha_2\beta_2)$. Suppose further that $\delta \neq A$. Then $\bar{s} \neq 0$ for any $k \epsilon (0,\infty)$. In such cases, the sign of $\bar{s}$ is entirely independent of k. The graph of $\bar{s}$ as a function of k in (3-27) is drawn in Figure 3.7.a.

Case (b): Unique Solution

Consider an example in which the function g satisfies conditions (3-22-a) and (3-22-b),

$$u(C_1^j, C_2^j) = a + b(C_1^j)^{\beta}(C_2^j)^{1-\beta} + \log_e [(C_1^j)^{\beta}(C_2^j)^{1-\beta} + d]$$

where $b > 0$, $d = e^{-a} > 0$ and $0 < \beta < 1$ $(j=1,2)$. Then, by Lemma 3.2, the indirect utility function is

$$v(p, C^j) = a + b[C^j/l(p)] + \log_e\{[C^j/l(p)]+d\} = g[C^j/l(p)]$$

where $g(x) = a + bx + \log_e(x+d)$, $l(p) = Kp^{\beta-1}$ and $K = \beta^{-\beta}(1-\beta)^{\beta-1}$ $(j=1,2)$.

From Theorem 3.3, the "no-trade" endowment ratio $k^*$ solves

$$k = (1/\delta)\ g'[w^1/l(p)]/g'[w^2/l(p)]$$

$$= \frac{1}{\delta}\ \frac{b + \{[w^1/l(p)] + d\}^{-1}}{b + \{[w^2/l(p)] + d\}^{-1}} \tag{3-28}$$

To simplify (3-28), the production functions can be chosen in such a way that l(p) is independent of k. One possible choice is

$$X_i = (L_{i1})^{\alpha}(L_{i2})^{1-\alpha}$$

where $0 < \alpha < 1$ $(i=1,2)$. Then it follows that

$$w^1 = \alpha k^{\alpha-1}$$

$$w^2 = (1-\alpha)k^{\alpha}$$

$$l(p) = Kp^{1-\beta} = K$$

Equation (3-28 can now be rewritten as

$$\delta = (1/k)\ \{b+[1/D_1(k)]\}/\{b+[1/D_2(k)]\} = \psi(k)$$

where $D_1(k) = \alpha K^{-1}k^{\alpha-1} + d$ and $D_2(k) = (1-\alpha)K^{-1}k^{\alpha} + d$. Since $\psi(k)$ approaches $\infty$ as k approaches 0 and 0 as k approaches $\infty$, a solution must exist for any $\delta > 0$, $b > 0$, $d > 0$, $0 < \alpha < 1$ and $0 < \beta < 1$. Furthermore, the solution is unique if $\psi'(k) < 0$ for any $k \in (0,\infty)$. A simple calculation shows that

$$\psi'(k) = \frac{E(k)}{k^2 D_1^2(k) D_2^2(k)\{b+[1/D_1^2(k)]\}^2}$$

where

$$\begin{aligned} E(k) = {} & \{\alpha(1-\alpha)K^{-1}k^{\alpha-1}b[kD_1^2(k)+D_2^2(k)-D_1(k)D_2^2(k)-D_1^2(k)D_2(k)]\} + \\ & \{\alpha(1-\alpha)K^{-1}k^{\alpha-1}[D_1(k)+D_2(k)]-D_1(k)D_2(k)\} - \\ & \{b^2D_1^2(k)D_2^2(k)\} \\ = {} & E_1(k) + E_2(k) + E_3(k). \end{aligned}$$

Expanding $E_1(k)$ and $E_2(k)$,

$$\begin{aligned} E_1(k) = {} & -b[\alpha^2(1-\alpha)^2K^{-3}k^{3\alpha-2} + d\alpha^2K^{-2}k^{2\alpha-2} + \alpha^2(1-\alpha)^2K^{-3}k^{3\alpha-1} + \\ & 2d\alpha(1-\alpha)K^{-2}k^{2\alpha-1} + d^2(2+\alpha)\alpha K^{-1}k^{\alpha-1} + d^2(3-\alpha)(1-\alpha)K^{-1}k^{\alpha} + \\ & d(1-\alpha)^2K^{-2}k^{2\alpha} + 2d^3] < 0 \text{ and} \end{aligned}$$

$$E_2(k) = -d[\alpha^2K^{-1}k^{\alpha-1} + (1-\alpha)^2K^{-1}k^{\alpha} + d] < 0.$$

Also, $E_3(k) < 0$. Therefore, $\psi'(k) < 0$ for any $k \in (0,\infty)$. The graph of $\psi(k)$ is similar to that of Figure 3.4. In such cases, any endowment ratio $k \in (0,\infty)$ may be said to be "no-trade" (neutral) or Classical or Samuelson depending on whether k is equal to or less than or greater than $k^*$, respectively. This is shown in Figure 3.7.b.

<u>Case (c): Finitely Multiple Solutions</u>

Suppose that the individual's direct utility function is

$$u(C_1^j, C_2^j) = a + b(C_1^j)^{\beta}(C_2^j)^{1-\beta} - a\exp[-(C_1^j)^{\beta}(C_2^j)^{1-\beta}]$$

where $a > 0$, $b > 0$ and $0 < \beta < 1$ $(j=1,2)$. The indirect utility function is then

$$v(p, C^j) = a + b[C^j/l(p)] - a\exp[C^j/l(p)] = g[C^j/l(p)]$$

where $g(x) = a+bx-ae^{-x}$, $l(p) = Kp^{1-\beta}$ and $K = \beta^{-\beta}(1-\beta^{\beta-1}$ $(j=1,2)$.

The "no-trade" endowment ratio is the solution of

$$k = (1/\delta)\ g'[w^1/l(p)]/g'[w^2/l(p)]$$

$$= \frac{1}{\delta}\ \frac{b + a\ \exp[-w^1/l(p)]}{b + a\ \exp[-w^2/l(p)]} \qquad (3\text{-}29)$$

Suppose further that

$$X_i = (L_{i1})^{\alpha}(L_{i2})^{1-\alpha}$$

where $0 < \alpha < 1$ $(i=1,2)$. Then equation (3-29) can be rewritten as

$$\delta = (1/k)\ [b+a.\exp[-\alpha K^{-1}k^{\alpha-1}]]/[b+a.\exp[-(1-\alpha)K^{-1}k^{\alpha}]] = \psi(k)$$

Since $\lim_{k\to 0} \psi(k) = \infty$ and $\lim_{k\to\infty} \psi(k) = 0$, a solution must exist for any choice of $a > 0$, $b > 0$, $0 < \alpha < 1$ and $0 < \beta < 1$. It can also be shown that

$$\psi'(k) = F(k)/[k^2(b+a.\exp[-\alpha K^{-1}k^{\alpha}])^2]$$

where

$$\begin{aligned} F(k) = {} & a\alpha(1-\alpha)K^{-1}\{a(k^{\alpha}+k^{\alpha-1})\exp(-\alpha K^{-1}k^{\alpha}-(1-\alpha)K^{-1}k^{\alpha-1}) + \\ & b[k^{\alpha}\exp(-\alpha K^{-1}k^{\alpha}) + k^{\alpha-1}\exp(-(1-\alpha)K^{-1}k^{\alpha-1})]\} - \\ & \{a^2\exp(-\alpha K^{-1}k^{\alpha}-(1-\alpha)K^{-1}k^{\alpha-1}) + \\ & ab[\exp(-\alpha K^{-1}k^{\alpha}) + \exp(-(1-\alpha)K^{-1}k^{\alpha-1})] + b^2\}. \end{aligned}$$

It is quite clear that $\psi'(k) \to -\infty$ as $k \to 0$ and $\psi'(k) \to 0$ as $k \to \infty$. Assume that, by a suitable selection of a, b, $\alpha$ and $\beta$, the slope $\psi'(k)$ can be made positive for some $k > 0$. Then $\delta$ can be chosen accordingly to give multiple solutions to equation (3-29). This is illustrated by Figure 3.7.c. Now,

$$\begin{aligned} F(k) = {} & a^2[\alpha(1-\alpha)K^{-1}(k^{\alpha}+k^{\alpha-1})-1]\exp[-K^{-1}k^{\alpha}(\alpha+(1-\alpha)k^{-1})] + \\ & ab\{[\alpha(1-\alpha)K^{-1}k^{\alpha}-1]\exp(-\alpha K^{-1}k^{\alpha}) + \\ & [\alpha(1-\alpha)K^{-1}k^{\alpha-1}-1]\exp(-(1-\alpha)K^{-1}k^{\alpha-1})\} -b^2. \end{aligned}$$

Suppose that b is chosen sufficiently small that all factors involving b become negligible for those k's which are not close to 0 or $\infty$. F(k) can then be roughly approximated by

$$F(k) \cong a^2[\alpha(1-\alpha)K^{-1}(k^{\alpha}+k^{\alpha-1})\ -1]\exp\{-K^{-1}k^{\alpha}[\alpha+(1-\alpha)k^{-1}]\}$$

At $k = [\alpha(1-\alpha)K^{-1}]^{-1/\alpha}$ we have

$$\alpha(1-\alpha)K^{-1}\{[\alpha(1-\alpha)K^{-1}]^{-1} + [\alpha(1-\alpha)K^{-1}]^{(1-\alpha)/\alpha}\} = 1 + [\alpha(1-\alpha)K^{-1}]^{1/\alpha} > 1$$

Therefore, $F(k) > 0$ and $\psi'(k) > 0$ at $k = [\alpha(1-\alpha)K^{-1}]^{-1/\alpha}$. This shows that, by choosing the parameters appropriately, $F(k)$ and, consequently, $\psi'(k)$ can be made positive. Thus, it is possible to have finitely multiple solutions (at least three in the above case.)

Case (d): Infinitely Many Solutions

Returning to case (a), assume now that

$$\delta = A = [(1-\alpha_1)\beta_1+(1-\alpha_2)\beta_2]/(\alpha_1\beta_1+\alpha_2\beta_2)$$

Then it is obvious that $\bar{s} = 0$ for any $k\epsilon(0,\infty)$, i.e., the graph of optimal "biological" savings as a function of k is identical to the positive segment of the horizontal axis k as shown by Figure 3.7.d.

### 3.3.6 Comparative Statics of s(k) and $k^*$ in Special Cases

In this subsection, U is always supposed to be additively time-separable, i.e., $U = u^1+\delta u^2$ where $u^j = u(C_1^j, C_2^j)$ is homothetic $(j=1,2)$ and $\delta > 0$. For an infinitesimal change in k, the expression $d\bar{s}(k)/dk$ can be computed by using equation (3-18) with $\Omega_1 = 1$ and $\Omega_2 = \delta$. The sign of $d\bar{s}(k)/dk$ is generally ambiguous except for the very special case in which

$$u^j = a+b.h(C_1^j,\ C_2^j) \qquad j=1,2$$

where $a > 0$, $b > 0$ and h is homogeneous of degree one in $C_1^j$ and $C_2^j$. It can then be shown that the "no-trade" endowment ratio $k^*$ uniquely exists and is equal to $\delta^{-1}$. Furthermore, $\bar{s}$ either takes an extreme value or is indeterminate.

$$\bar{s}(k) = \begin{cases} -\,w^2/k < 0 & \text{for } k < k^* \text{ (Classical case)} \\ ? & \text{for } k = k^* \\ w^1 \quad > 0 & \text{for } k > k^* \text{ (Samuelson case)} \end{cases}$$

The graph of $\bar{s}(k)$ as a function of k with Cobb-Douglas production functions is illustrated by Figure 3.8.a. It is evident in this case that

$$\partial\bar{s}(k)/\partial k \begin{cases} > 0 & \text{for } k < k^* \\ \text{undefined} & \text{for } k = k^* \\ < 0 & \text{for } k > k^* \end{cases} \qquad (3\text{-}30)$$

As discussed previously in 2.3.4, the effect of a technical improvement on $[w^j(p)/l(p)]$ $(j=1,2)$ are ambiguous because of the two conflicting tensions in $d[w^j(p)/l(p)]/d\lambda_i$ where $\lambda_i$ is the i-th industry's shift parameter. The outcome for postimprovement $\bar{s}(k)$ is therefore indeterminate. However, it is possible to state

Theorem 3.5 Assume that the "no-trade" endowment ratio exists uniquely before and after an innovation. Let $k^*$ and $\hat{k}^*$ be respectively the preimprovement and postimprovement "no-trade" endowment ratios. Then if $k^* > \hat{k}^*$ $(k^* < \hat{k}^*)$, there exists an endowment ratio interval such that after the technical progress, the economy changes from the Classical (Samuelson) state to the Samuelson (Classical) state.

<u>Proof</u> Let $\varepsilon^o$ and $\hat{\varepsilon}^o$ be the preimprovement and postimprovement "no-trade" interest factors, respectively. Suppose that $k^* < \hat{k}^*$. Consider $k \in (\hat{k}^*, k^*)$. Before the improvement, the economy is Classical because $k = \bar{\varepsilon} < \varepsilon^o = k^*$ (See Theorem 3.2.) After the technical change, the economy is Samuelson because $k = \hat{\bar{\varepsilon}} > \hat{\varepsilon}^o = \hat{k}^*$. Suppose now that $k^* < \hat{k}^*$. For $k \in (k^*, \hat{k}^*)$, the economy is Samuelson before the technical change as $k = \bar{\varepsilon} > \varepsilon^o = k^*$ but becomes Classical after the change because $k = \hat{\bar{\varepsilon}} < \hat{\varepsilon}^o = \hat{k}^*$. Q.E.D.

It is, however, not possible to indicate precisely the direction of change in the "no-trade" endowment ratio $k^*$ due to a technical improvement without specifying the forms of the utility and production functions as well as the nature of the progress.

### 3.3.7 <u>Summary</u>

In the basic model with no intergenerational borrowing and lending, the steady-state equilibrium consumption program of a typical individual is $\overset{o}{C} = [\overset{o}{C}{}^1_1, \overset{o}{C}{}^1_2; \overset{o}{C}{}^2_1, \overset{o}{C}{}^2_2]$. The j-th period (j=1,2) consumptions are characterized by

$$\overset{o}{C}{}^j_1 = \overset{o}{w}{}^j/(1+p^o\sigma^o) \qquad j=1,2$$

$$\overset{o}{C}{}^j_2 = \sigma^o\overset{o}{w}{}^j/(1+p^o\sigma^o) \qquad j=1,2$$

where $p^o = p(k)$ is the unique steady-state equilibrium commodity price ratio, $\overset{o}{w}{}^j = w^j(p^o)$ is the j-th period (j=1,2) wage-income of a typical person and $\sigma^o = \sigma(p^o)$ is the value at $p^o$ of the inverse function of $[\partial u/\partial C_2^j]/[\partial u/\partial C_1^j] = \psi(C_2^j/C_1^j)$ (j=1,2). The rates of consumption in any time period exhibit expenditure proportionality, i.e.

$$C_2^j/C_1^j = \sigma^o = \sigma(p^o) \qquad j=1,2$$

where $\sigma'(.) < 0$. The indirect utility $v(p,w^j)$ (j=1,2) can be written as

$$v(p,w^j) = g[w^j/l(p)] \qquad j=1,2$$

where $g'(.) > 0$, $g''(.) < 0$ and $l'(p) > 0$.

In the extended model with intergenerational borrowing and lending, the equilibrium production pattern remains unchanged. However, there are now two types of steady-state equilibria.

I. <u>"No-trade" Equilibrium</u> There exists a rate of interest $r^o$ such that the consumption program is unaltered. The "no-trade" rate of interest $r^o$ satisfies

$$1+r^o = (\Omega_1/\Omega_2)\ g'[\overset{o}{w}{}^1/l(p^o)]/g'[\overset{o}{w}{}^2/l(p^o)]$$

where $\Omega_j$ (j=1,2) are evaluated at $p^o$.

II. <u>"Biological" Equilibrium</u> The "biological" rate of interest $\bar{r}$ is set at the steady-state rate of growth of the population k-1 and there are again three possibilities. A typical young man may consume more than or exactly the amount of or less than his wage-income, corresponding to $r^o \gtreqless \bar{r} = k-1$, respectively. The optimal "biological" steady-state consumpprogram is then $\bar{C} = [\bar{C}_1^1, \bar{C}_2^1; \bar{C}_1^2, \bar{C}_2^2]$ which is related to $\overset{o}{C}$ by

$$\bar{C}_1^j = \overset{o}{C}{}_1^j - [\bar{s}/(1+p^o\sigma^o)] \qquad j=1,2$$

$$\bar{C}_2^j = \overset{o}{C}{}_2^j - [\sigma^o\bar{s}/(1+p^o\sigma^o)] \qquad j=1,2$$

where the optimal "biological" saving $\bar{s} = \bar{s}[p^o(k)] = \bar{s}(k)$ is the unique solution of

$$k = (\Omega_1/\Omega_2)\ g'[(\overset{o}{w}{}^1-\bar{s})/l(p^o)]/g'[(\overset{o}{w}{}^2+k\bar{s})/l(p^o)]$$

with $\Omega_j$ $(j=1,2)$ evaluated at $p^o$, $\bar{C}^1 = \overset{o}{w}{}^1-\bar{s}$ and $\bar{C}^2 = \overset{o}{w}{}^2+k\bar{s}$.

There may exist an endowment ratio $k^*$ (not necessarily unique) such that $\bar{s}(k^*) = 0$, i.e., exchange between generations is extinguished in the "biological" equilibrium.

## 3.4 NON-STATIONARY ANALYSIS OF THE MODEL

### 3.4.1 Equilibrium Conditions of the Model

It is assumed that the economy is opened to trade between generations through the creation of an immortal clearing house in the beginning of mankind. (It does not matter when the clearing house is actually created. If it is created in period $t_0 > 0$ then by a translation of the time origin defined by $T = t-t_0$, we may as well agree that intergenerational borrowing and lending is possible from the new time origin.) The cost of setting up and maintaining such an institution is deliberately overlooked to avoid needless complications that would otherwise arise. Then the equilibrium of the economy in period $t \geq 1$ is completely described by $\{p^o(t), \overset{o}{C}(t), \bar{\varepsilon}(t), \varepsilon^o(t), \bar{s}(t)\}$.

The equilibrium commodity price ratio in period t, $p^o(t)$, is determined by the interaction of demand and supply in the product markets. It can be inferred from Lemma 3.1 that, under the strong assumption of homothetic preferences, $p^o(t)$ is independent of borrowing and lending between the young and the old. Thus, $p^o(t) = p^o[k(t)]$ for $t \geq 0$ and the time path of the consumption ratios remains unaltered.

The optimal interest factor $\bar{\varepsilon}(t)$ and optimal saving $\bar{s}(t)$ in period $t \geq 1$ are determined simultaneously by the feasibility condition (3-8) and a dynamic version of the competitiveness condition (3-17).

$$k(t)s(t) - \varepsilon(t-1)s(t-1) = 0 \qquad t=1,2,3,\ldots \qquad (3\text{-}8')$$

and

$$\varepsilon(t) = \frac{\Omega_1}{\Omega_2} \cdot \frac{l(p^o(t+1)]}{l[p^o(t)]} \cdot \frac{g'[\overset{o1}{w}(t)-s(t))/l(p^o(t))]}{g'[\overset{o2}{w}(t+1)+\varepsilon(t)s(t))/l(p^o(t+1))]}$$

$$t=1,2,3,\ldots \qquad (3\text{-}31)$$

where $\Omega_j$ (j=1,2) are evaluated at $p^o(t)$, $p^o(t+1)$, $C^1(t) = \overset{o1}{w}(t)-s(t)$ and $C^2(t+1) = \overset{o2}{w}(t+1)+\varepsilon(t)s(t)$. It follows from (3-8') and (3-31) that

$$\bar{s}(t) = \bar{\varepsilon}(t-1)\bar{s}(t-1)/k(t) = \frac{\bar{\varepsilon}(t-1)\bar{\varepsilon}(t-2)\ldots\bar{\varepsilon}(0)}{k(t)k(t-1)\ldots k(1)}\bar{s}(0)$$

$$t=1,2,3,\ldots \qquad (3\text{-}32)$$

and

$$\bar{\varepsilon}(t) = \frac{\Omega_1}{\Omega_2}\ \frac{l[p^o(t+1)]}{l[p^o(t)]}\ \frac{g'[(\overset{o}{w}{}^1(t)-\bar{s}(t))/l(p^o(t))]}{g'[\overset{o}{w}{}^2(t+1)+\bar{\varepsilon}(t)\bar{s}(t)/l(p^o(t+1))]}$$

$$= \psi[\bar{\varepsilon}(t)] \qquad t=1,2,3,\ldots \qquad (3\text{-}31')$$

The graph of $\psi[\bar{\varepsilon}(t)]$, given $p^o(t)$ and $p^o(t+1)$, is drawn for three different cases, $\bar{s}(t) \gtreqless 0$ in Figure 3.9. It is then evident that for a feasible $\bar{s}(t)$, there exists uniquely a positive $\bar{\varepsilon}(t)$ that solves (3-31'). The value indicated by $\varepsilon^o(t)$ in Figure 3.9 is the "no-trade" interest factor in period t which satisfies

$$\varepsilon^o(t) = \frac{\Omega_1}{\Omega_2}\ \frac{l[p^o(t+1)]}{l[p^o(t)]}\ \frac{g'[\overset{o}{w}{}^1(t)/l(p^o(t))]}{g'[\overset{o}{w}{}^2(t+1)/l(p^o(t+1))]}$$

$$t=0,1,2,\ldots \qquad (3\text{-}33)$$

where $\Omega_j$ $(j=1,2)$ are evaluated at $p^o(t)$, $p^o(t+1)$, $\overset{o}{w}{}^1(t)$ and $\overset{o}{w}{}^2(t+1)$. It is not too difficult to see that $\bar{s}(t) \gtreqless 0$ if and only if $\bar{\varepsilon}(t) \gtreqless \varepsilon^o(t)$. We note that along the solution path $(\bar{s}(0) = 0;\ \varepsilon^o(0))$ there is no intergenerational borrowing and lending. In each period of his life, therefore, the typical member of the t-th generation simply consumes his wage income. However, if $\bar{s}(0)$ is nonzero (or $\bar{\varepsilon}(t) \neq \varepsilon^o(t)$) then $\bar{s}(t)$ will continue to be nonzero indefinitely into the future.

Theorem 3.6 For any given feasible initial conditions $\bar{s}(0)$ and $\bar{\varepsilon}(0)$, (a) there is a "no-trade" equilibrium in which $\bar{s}(t) = 0$ for all $t \geq 0$, (b) any sequence of consumption programs with $\bar{s}(t) = 0$ for some but not all $t \geq 0$ is not an equilibrium sequence, and (c) $\bar{s}(t) \gtreqless 0$ if and only if $\bar{\varepsilon}(t) \gtreqless \varepsilon^{o}(t)$.

### 3.4.2 Evolution of the Economy

The description of the evolution of the economy is completed by considering the initial equilibrium of the model. A typical member of the oldest generation wishes to maximize $u[C_1^2(0), C_2^2(0)]$ subject to $C_1^2(0) + p(0)C_2^2(0) = w^2(0) + k(0)s(0)$ whereas a typical young man living in the same time period seeks to maximize $U[C(0)]$ by the choice of $C(0)$ subject to the $t = 0$ version of (3-1) and (3-2). The values of $s(0)$ and $\varepsilon(0)$ are clearly bounded by

$$-w^2(0)/k(0) \leq s(0) \leq w^1(0), \; -w^2(1)/\varepsilon(0) \leq s(0) \qquad (3\text{-}34)$$

and

$$\varepsilon(0) > 0 \qquad (3\text{-}35)$$

Any arbitrary choice of $s(0)$ and $\varepsilon(0)$ that satisfies (3-34), (3-35) and the $t = 0$ version of (3-31') is therefore optimal. It becomes evident that either $\bar{s}(0)$ or $\bar{\varepsilon}(0)$ is indeterminate.

Let $\varepsilon^{o}(0)$ be the value of $\varepsilon^{o}(t)$ at time $t = 0$. Then if the initial rate of interest $r(0)$ is set by the clearing house at $\varepsilon^{o}(0)-1$, intergenerational borrowing and lending will be annihilated. In this case $\bar{s}(0) = 0$ and the "no-trade" equilibrium prevails forever. If the economy is Classical (Samuelson) then $\bar{s}(0)$ must be chosen to be negative (positive) or, equivalently, $\bar{\varepsilon}(0)$ must be chosen to be smaller (greater) than $\varepsilon^{o}(0)$. However, the welfare implications of the two economies are not the same. In the Samuelson case, a typical young man saves for future consumption with the expectation of improving his lifetime satisfaction. Therefore, the initial generation of young people is quite prepared to leave its saving with the clearing house provided that it is to receive its savings plus interest in the next period $t = 1$. There is no reason why the clearing house should not pass the savings onto the initial old people. Thus, the optimal "biological" equilibrium of a Samuelson economy can be achieved in a decentralized manner from period $t = 0$ onwards. In contrast, a typical man of a Classical economy consumes more than he earns when young. Bearing in mind that bequests to his offspring give an old man no utility it is natural that the old generation in period $t = 0$ is not willing to lend to the young at any initial rate of interest. Hence, a Classical economy can never move from the "no-trade" to the "biological" equilibrium in a decentralized manner. Suppose that there now exists a central planning authority which requests the old at time $t = 0$ to give up some of their income for the young of the same time period. (The central authority needs not exist in any other period.) The amount of income which the old are forced to transfer to the young is determined by the $t = 0$ version of (3-31'). Although the initial old generation suffers the burden of the new social contract, each member of the population in the coming periods 1, 2, ... is better off than he would be in the "no-trade" equilibrium. In summary, one may state

<u>Theorem 3.7</u> The "no-trade" equilibrium is Pareto-optimal in the Classical economy but not in the Samuelson economy.

It is more plausible that $\bar{s}(0)$ rather than $\bar{\varepsilon}(0)$ be chosen arbitrarily. Consider the Classical case where

$$-w^2(0)/k(0) \leq \bar{s}(0) < 0.$$

and

$$-w^2(1)/\varepsilon(0) \leq \bar{s}(0).$$

From the point of view of a typical old man in period $t = 0$, his welfare loss can be minimized by choosing $\bar{s}(0)$ as close to 0 as feasible. By similar reasoning, his welfare gain in the Samuelson case can be maximized by selecting $\bar{s}(0)$ as close to $w^1(0)$ as possible. However, in neither case can $\bar{s}(0)$ take the extreme value (i.e., 0 or $w^1(0)$).

### 3.4.3 <u>Convergence of the Model</u>

To examine the dynamic stability of the model we assume without loss of generality that $k(t) = k$ for all $t \geq t_0$. If the equilibrium is "no-trade" then the economy attains a steady state from period $t_0$ onwards. In the Classical or Samuelson economy, if $\bar{s}(0)$ or $\bar{\varepsilon}(0)$ is by sheer accident chosen by the initial population in such a way that $\bar{s}(t_0) = \bar{s}$ or $\bar{\varepsilon}(t_0) = k$ then the economy achieves a "biological" stationary equilibrium for all $t \geq t_0$. Otherwise, there is no guarantee that the economy will ever reach a steady state.

We now specialize the model by supposing that the lifetime utility function takes the separable and logarithmic form

$$U[C(t)] = u[C_1^1(t),\ C_2^1(t)] + \delta u[C_1^2(t+1),\ C_2^2(t+1)],\ 0 < \delta \tag{3-36}$$

and

$$u[C_1^j(t),\ C_2^j(t)] = \beta \log_e C_1^j(t) + (1-\beta)\log_e C_2^j(t)\ , \quad 0 < \beta < 1 \tag{3-37}$$

for $j=1,2$ and $t=0,1,2,\ldots$ It is not necessary to introduce the production functions unless one is interested in deriving explicitly the conditions for the economy to be Classical or Samuelson. The indirect utility function is given by Lemma 3.2,

$$v[p(t),\ C^j(t)] = \log_e[\frac{C^j(t)}{Kp(t)^{1-\beta}}] \quad j=1,2; \quad t=0,1,2,\ldots$$

or

$$v[p, C^j(t)] = \log_e[\frac{C^j(t)}{Kp^{1-\beta}}] \qquad j=1,2; \quad t=t_0, t_0+1,\ldots \tag{3-38}$$

where $K = \beta^{-\beta}(1-\beta)^{\beta-1}$.

It is easy to see that the steady-state optimal "biological" saving $\bar{s}$ is

$$\bar{s} = (\delta k w^1 - w^2)/[(1+\delta)k] \qquad t \geq t_0. \tag{3-39}$$

Evidently, the economy is Classical or Samuelson if and only if $kw^1 < w^2$ or $kw^1 > w^2$, respectively. Given (3-38), equations (3-32) and (3-31') reduce to

$$\bar{s}(t) = \bar{\varepsilon}(t-1)\bar{s}(t-1)/k \qquad t \geq t_0 \tag{3-40}$$

and

$$\bar{\varepsilon}(t) = [w^2+\bar{\varepsilon}(t)\bar{s}(t)]/[\delta(w^1-\bar{s}(t))] \qquad t \geq t_0 \qquad (3\text{-}41)$$

(Implicit in (3-41) is the assumption that intergenerational borrowing and lending had been taking place from $t = 0$ to $t = t_0$.) Solving for $\bar{\varepsilon}(t)$ in terms of $\bar{s}(t)$, the time path of $\bar{\varepsilon}(t)$ for $t \geq t_0$ is characterized by

$$\bar{\varepsilon}(t) = w^2/[\delta w^1-(1+\delta)\bar{s}(t)] \qquad t \geq t_0 \qquad (3\text{-}42)$$

It is evident that $\bar{s}(t) \gtreqless \bar{s}$ if and only if $\bar{\varepsilon}(t) \gtreqless k$ in either the Classical or Samuelson case. However, the convergence properties of the two economies are not the same, as shown in the following analysis.

A. Classical Economy ($\bar{s} < 0$)

First, consider the case where

$\bar{s}(t_0) < \bar{s} < 0$ and $0 < \bar{\varepsilon}(t_0) < k$.

Then, by (3-40),

$$\bar{s}(t_0+1) = \bar{\varepsilon}(t_0)\bar{s}(t_0)/k > \bar{s}(t_0)$$

and, by (3-41),

$$\bar{\varepsilon}(t_0+1) = w^2/[\delta w^1-(1+\delta)\bar{s}(t_0+1)] > w^2/[\delta w^1-(1+\delta)\bar{s}(t_0)] = \bar{\varepsilon}(t_0).$$

Using the same method of argument, if $0 > \bar{s}(t_0) > \bar{s}$ and $\bar{\varepsilon}(t_0) > k > 0$ then $\bar{s}(t_0+1) < \bar{s}(t_0)$ and $\bar{\varepsilon}(t_0+1) < \bar{\varepsilon}(t_0)$.

In view of the above results, there exists a mechanism which brings $\bar{s}(t)$ and $\bar{\varepsilon}(t)$ to $\bar{s}$ and $k$, respectively, however large the deviation of $\bar{s}(t)$ (or $\bar{\varepsilon}(t)$) from $\bar{s}$ (or $k$) may be. Thus the Classical economy is dynamically stable in the sense that for any arbitrary, feasible initial conditions it always converges to the "biological" steady state $(\bar{s};\ k)$.

B. Samuelson Economy ($\bar{s} > 0$)

Consider first the case where

$$0 < \bar{s}(t_0) < \bar{s} \quad \text{and} \quad 0 < \bar{\varepsilon}(t_0) < k.$$

Then, by (3-40),

$$\bar{s}(t_0+1) = \bar{\varepsilon}(t_0)\bar{s}(t_0)/k < \bar{s}(t_0)$$

and, by (3-41),

$$\begin{aligned}\bar{\varepsilon}(t_0+1) &= w^2/[\delta w^1-(1+\delta)\bar{s}(t_0+1)] < w^2/[\delta w^1-(1+\delta)\bar{s}(t_0)] \\ &= \bar{\varepsilon}(t_0)\end{aligned}$$

Evidently, in this case,

$$\lim_{t\to\infty} \bar{s}(t) = 0$$

and

$$\lim_{t\to\infty} \bar{\varepsilon}(t) = \varepsilon^{o} = w^2/(\delta w^1)$$

i.e., the economy approaches the "no-trade" equilibrium as t grows indefinitely large.

Following the same line of reasoning, it is not difficult to see that, for $\bar{s}(t_0) > \bar{s} > 0$ and $\bar{\varepsilon}(t_0) > k > 0$,

$$\bar{s}(t+1) > \bar{s}(t) \quad \text{and} \quad \bar{\varepsilon}(t+1) > \bar{\varepsilon}(t) \quad \text{for all } t \geq t_0.$$

Then, for some $T > t_0$, either

$\bar{\varepsilon}(T)\bar{s}(T) \geq kw^1$, i.e.,

$$\bar{s}(T) \geq \delta k(w^1)^2/[(1+\delta)kw^1+w^2] \tag{3-43}$$

or

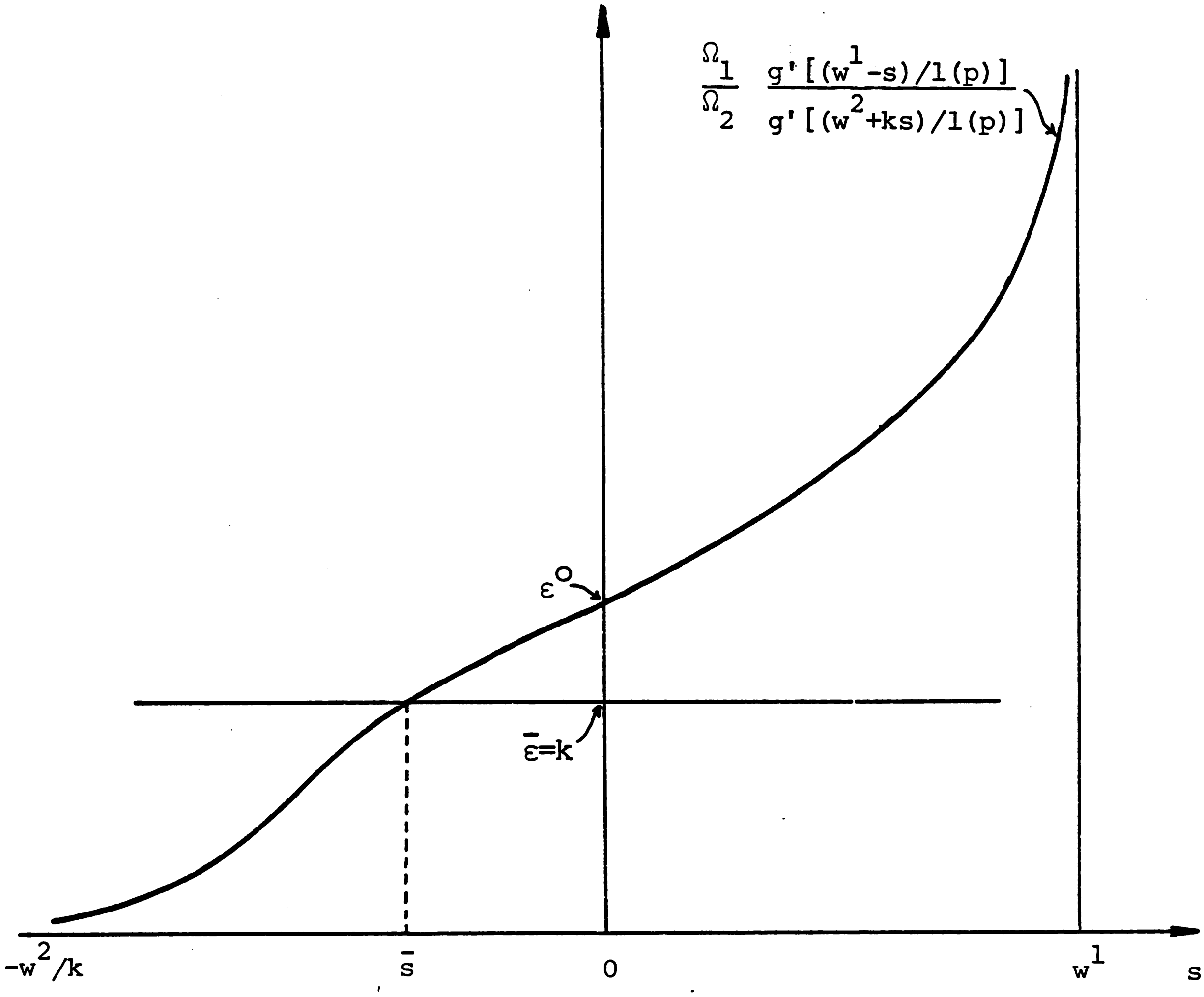
$$\bar{s}(T) \geq (\delta w^1)/(1+\delta) \tag{3-44}$$

In case (3-43), the feasibility condition in period T+1 is violated as $kw^1-\bar{\varepsilon}(t)\bar{s}(t) > 0$ for all $t \geq t_0$. In case (3-44) only a negative or infinite value of $\bar{\varepsilon}(T)$ satisfies (3-41). Thus, perfect but myopic foresight is not consistent with savings growing indefinitely at rates greater than one. Since

$$\delta k(w^1)^2/[(1+\delta)kw^1+w^2] = \delta w^1/[(1+\delta)+(w^2/ks^1)] < \delta w^1/(1+\delta),$$

once $\bar{s}(t)$ enters the region $[\delta k(w^1)^2/[(1+\delta)kw^1+w^2], \infty)$, the social contract of integenerational borrowing and lending collapses. The Samuelson economy is therefore dynamically unstable in the sense that, away from its "biological" steady state, either it approaches the "no-trade" economy or the system of trading between generations breaks down altogether.

The above analysis of a Samuelson economy is no longer valid if we allow the young people the power to revise the original intergenerational social contract at any time. Take the case $\bar{s}(t_0) < \bar{s}$ and $\bar{\varepsilon}(t_0) < k$. A typical old man in period $t_0+1$ only expects to receive $\bar{\varepsilon}(t_0)\bar{s}(t_0)$ from the clearing house as the payment for his saving in period $t_0$. However, being blessed with perfect foresight over his life-time, a typical young man in period $t_0+1$ will elect, if possible, to save $\bar{s}$ at the "biological" rate of interest $k-1$. Since $k\bar{s}$ is strictly greater than $\bar{\varepsilon}(t_0)\bar{s}(t_0)$, there is no reason why the old man at time $t_0+1$ should not accept $k\bar{s}$ instead of his expected payment of only $\bar{\varepsilon}(t_0)\bar{s}(t_0)$ from the clearing house. Now consider the case $\bar{s}(t_0) > \bar{s}$ and $\bar{\varepsilon}(t_0) > k$. Then $\bar{s}(T) \geq \delta k(w^1)^2/[(1+\delta)kw^1+w^2]$ for some $T > t_0$. If the young people in period T save $\bar{s}(T)$ at the rate of interest $\bar{\varepsilon}(T)-1$ then it is impossible for the clearing house in period T+1 to pay back the debt it owes the T-th young generation. Having complete knowledge over his lifetime, the best a typical young person in period T can do is to save $\bar{s}$ at the "biological" interest factor k. Since $k\bar{s}$ is positive but strictly less than $\bar{\varepsilon}(T-1)\bar{s}(T-1)$, this "reform" merely leads to a partial frustration of the T-th old generation's expectation.




<u>Figure 3.1</u>: The Classical equilibrium $\bar{s} < 0$ with k given

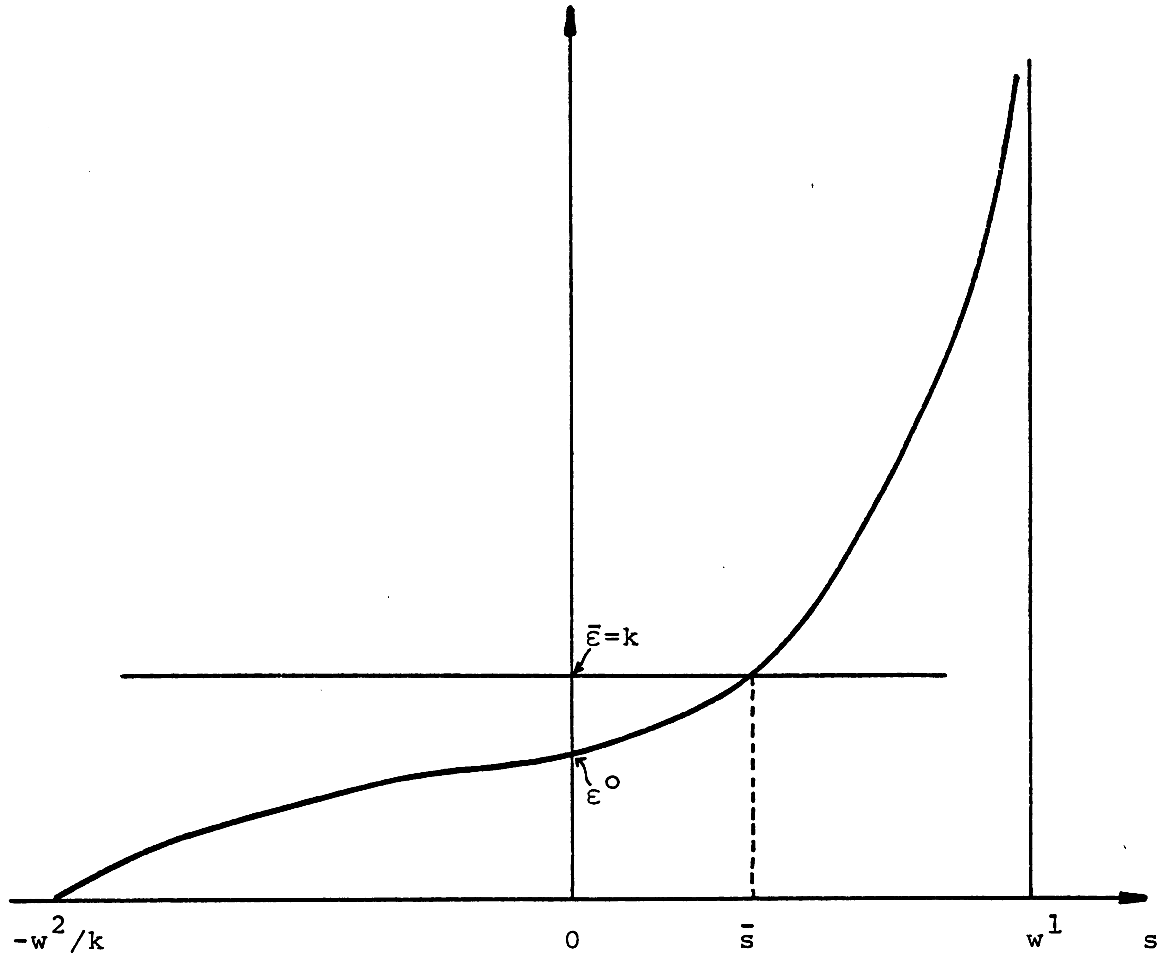



Figure 3.2: The Samuelson equilibrium $\bar{s} > 0$ with k given

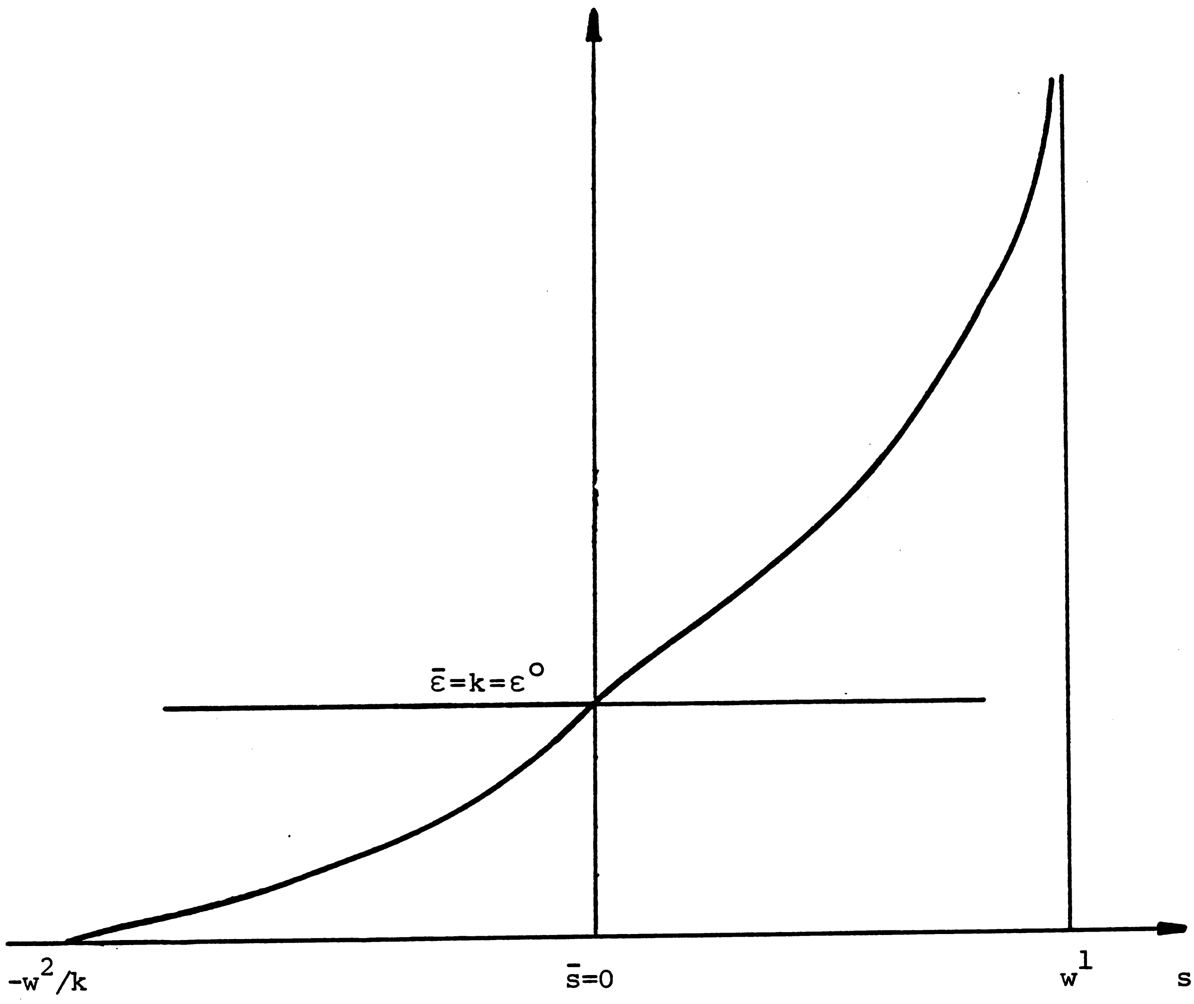



Figure 3.3: The Coincidental equilibrium $\bar{s}$ = 0 with k given

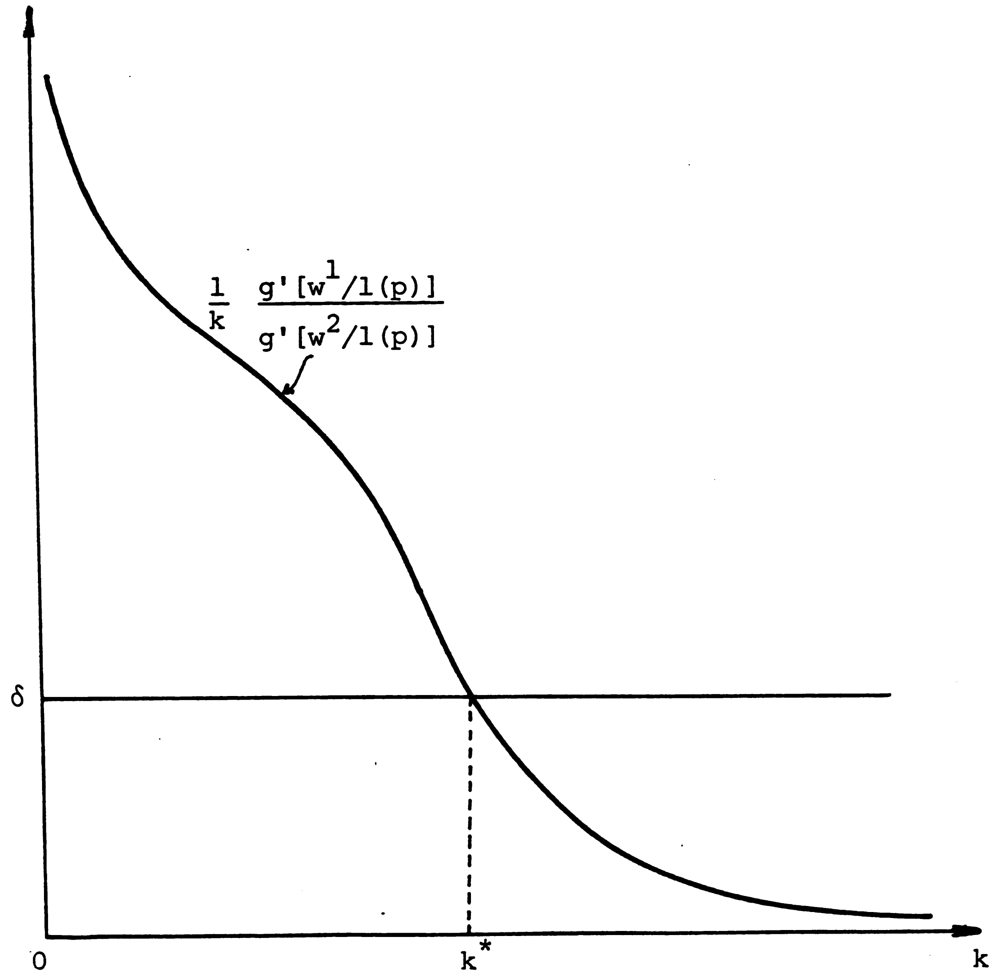



Figure 3.4: Existence of a "no-trade" endowment ratio $k^*$

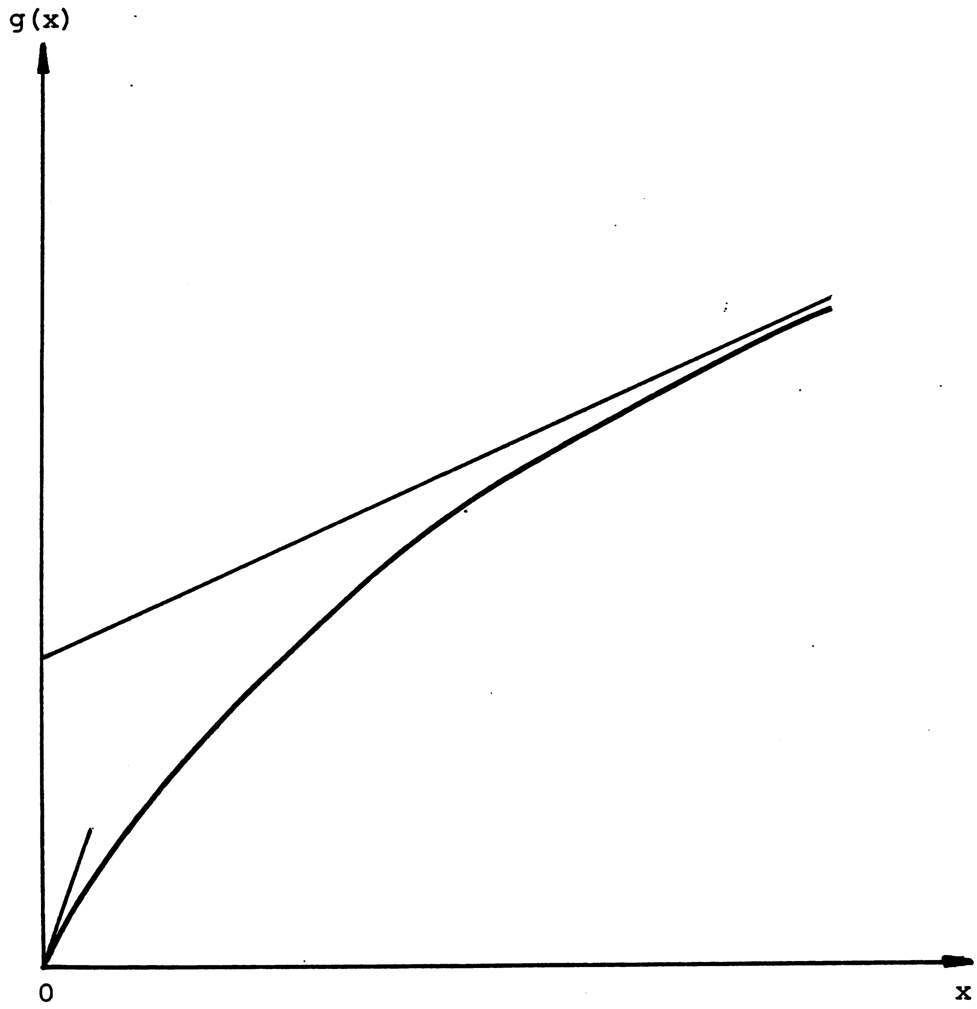



Figure 3.5: Graph of a function g satisfying (3-22-a) and (3-22-b)

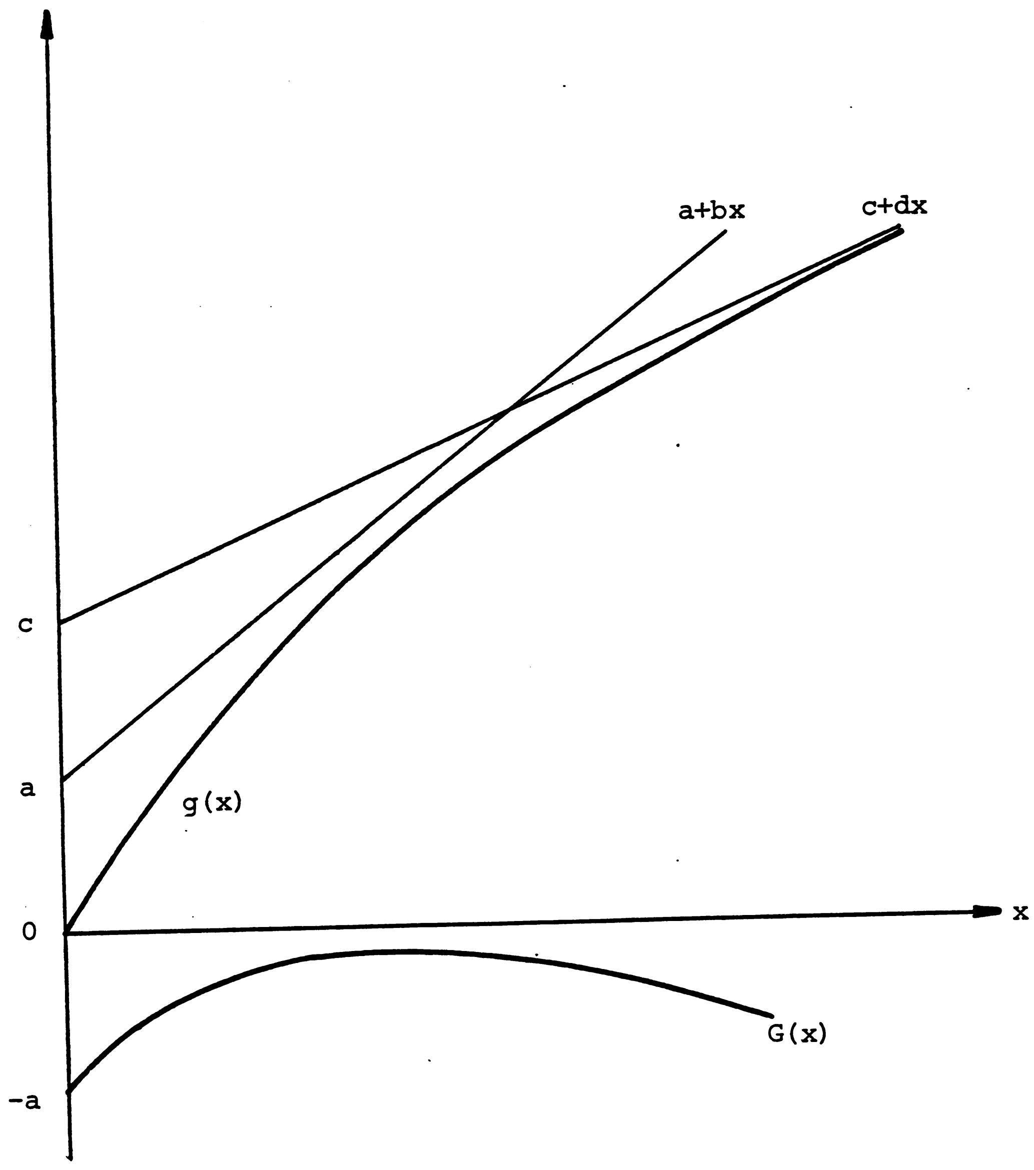
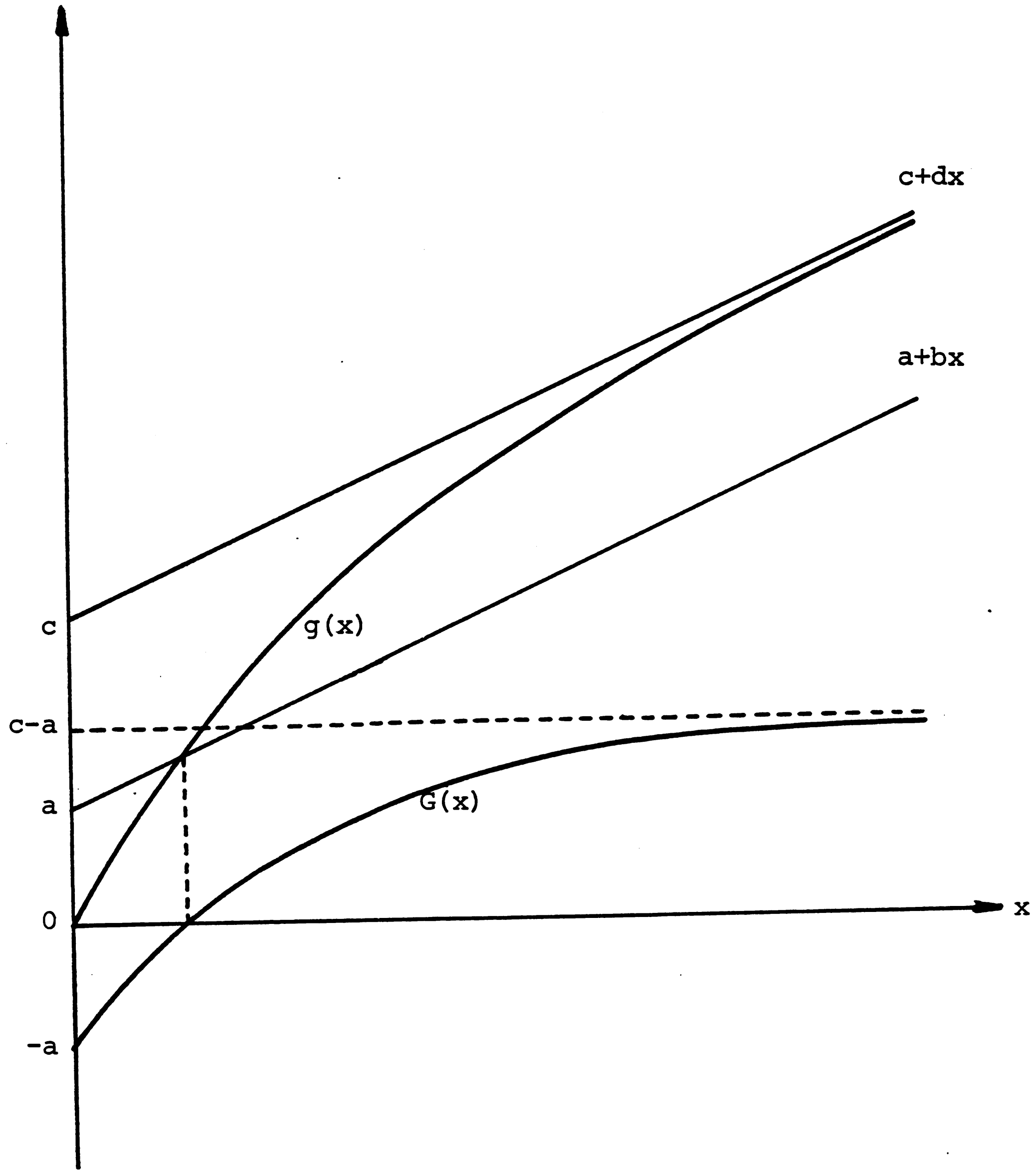



Figure 3.6.a: Case (a) with $b > d$ and $0 < a < c$




<u>Figure 3.6.b</u>: Case (b) with $b = d$ and $0 < a < c$

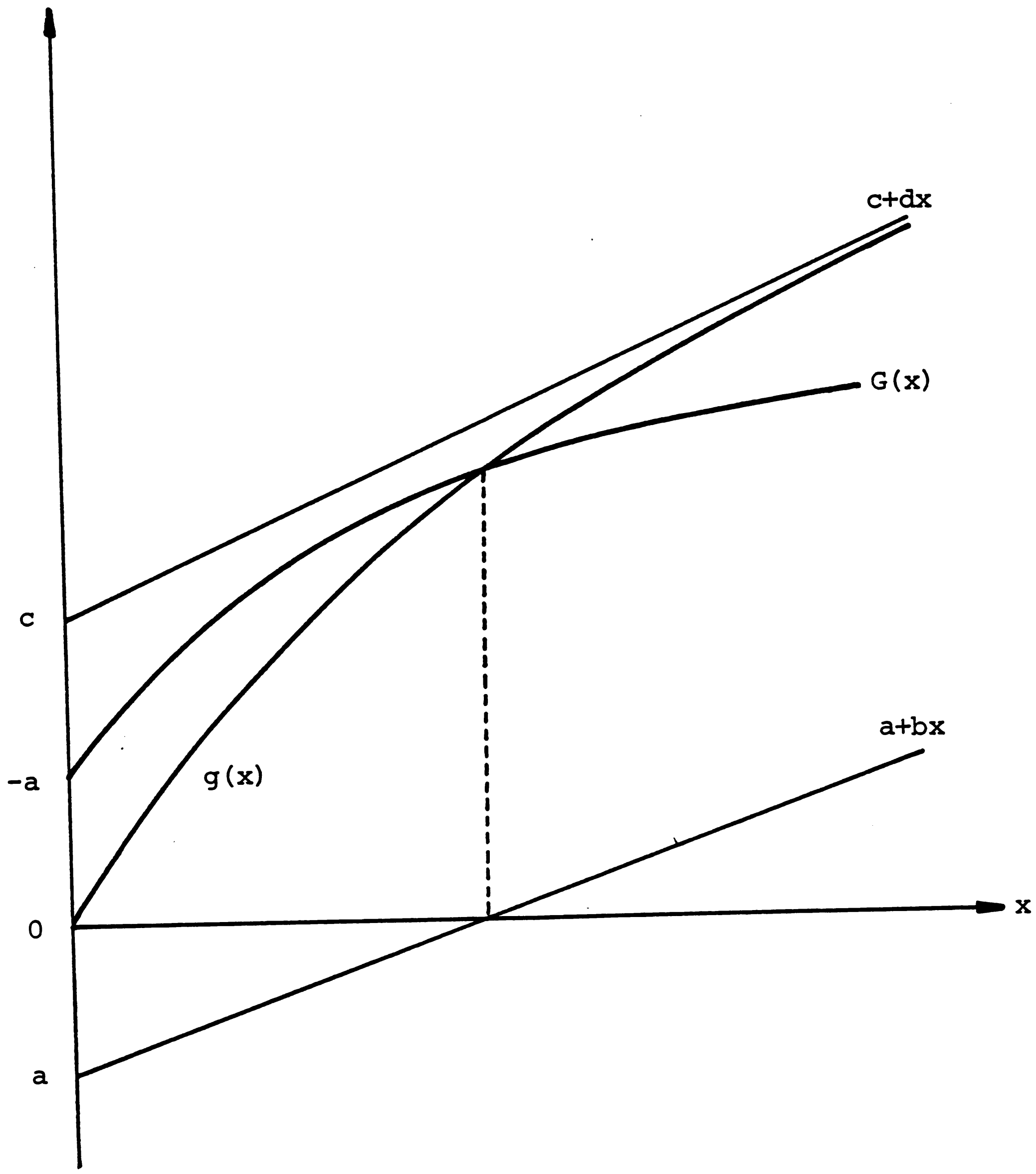



Figure 3.6.c: Case (c) with $b < d$ and $a < 0 < c$

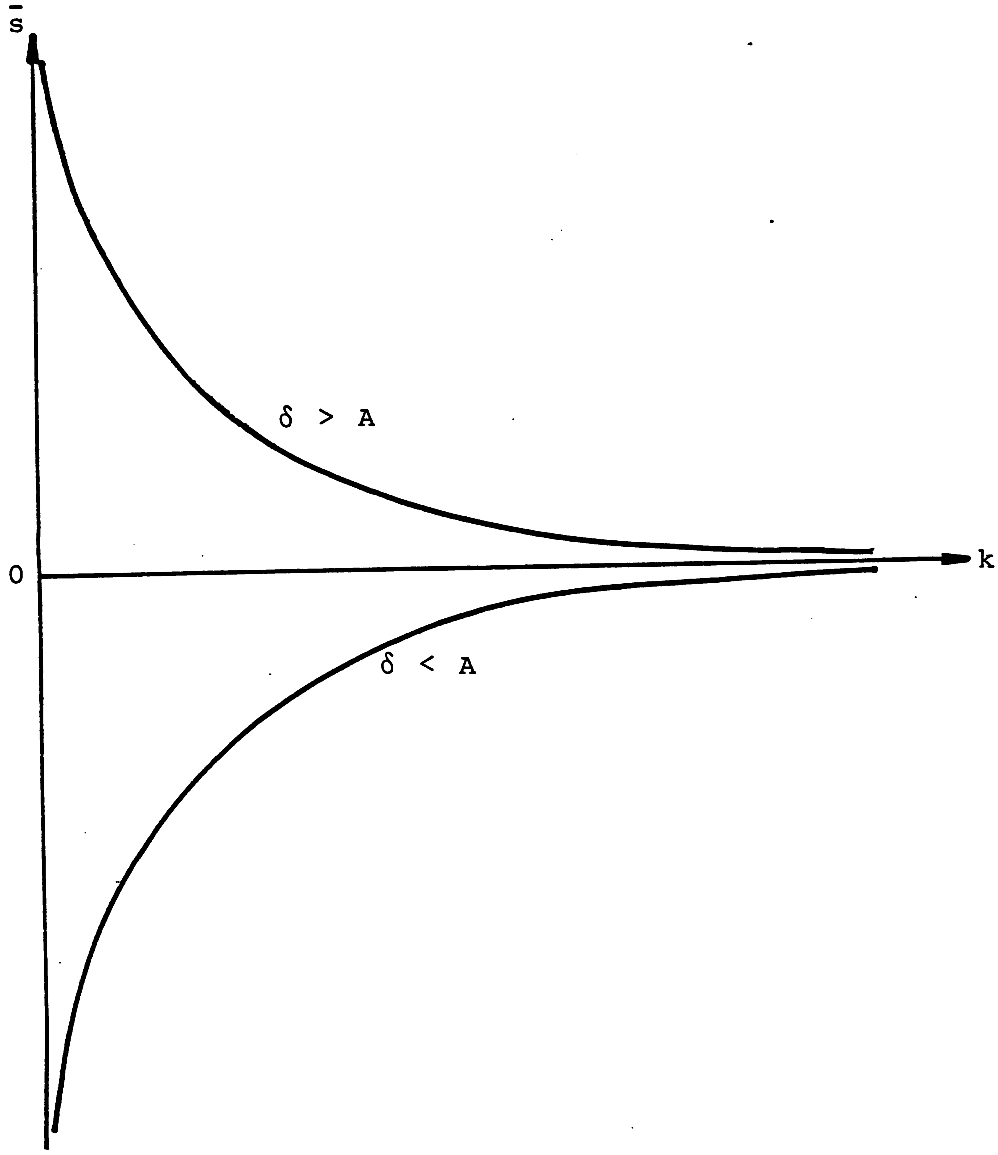



Figure 3.7.a: $\bar{s}(k) \neq 0$ for any $k \in (0,\infty)$ with technology and preferences given

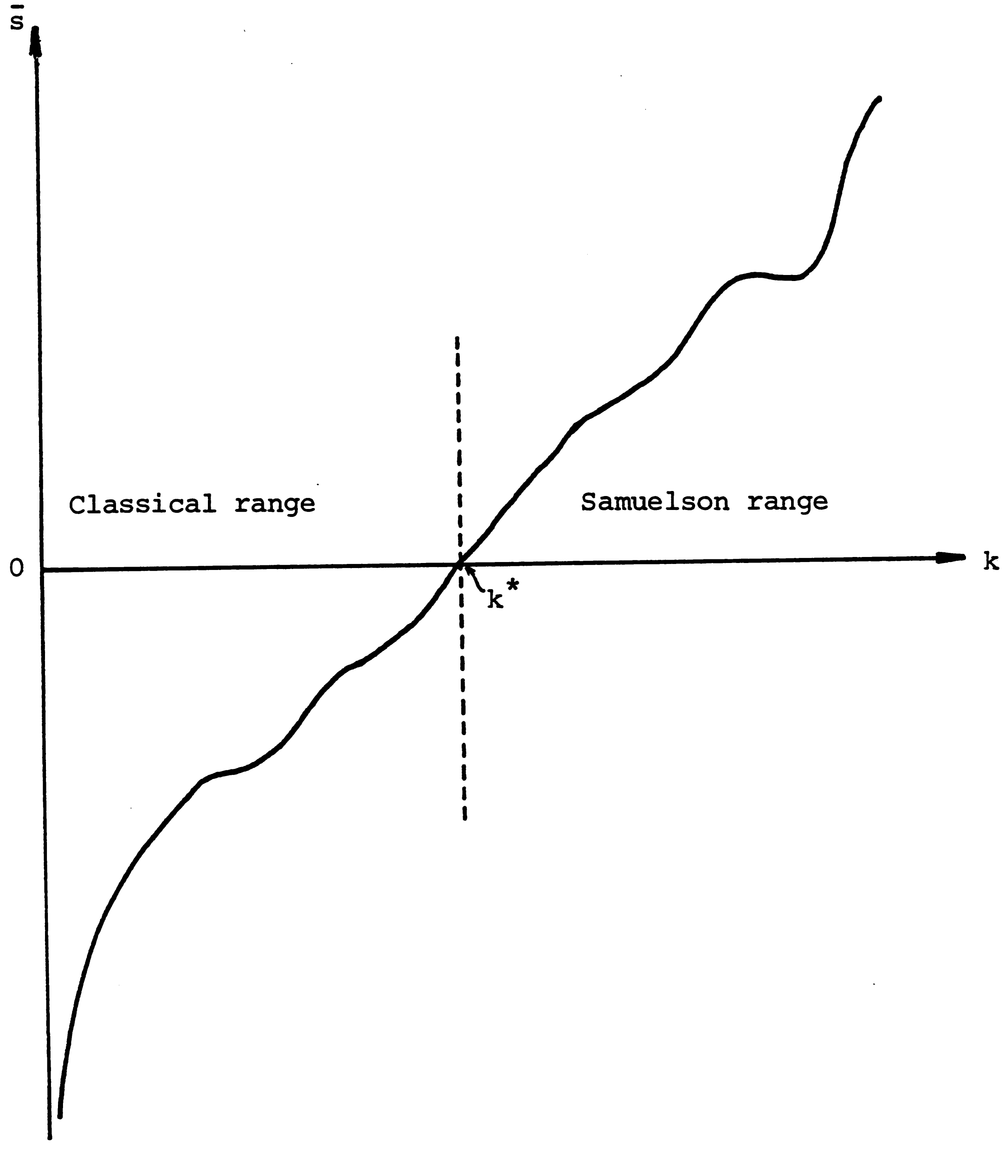



Figure 3.7.b: Unique "no-trade" endowment ratio $k^*$ with technology and preferences given

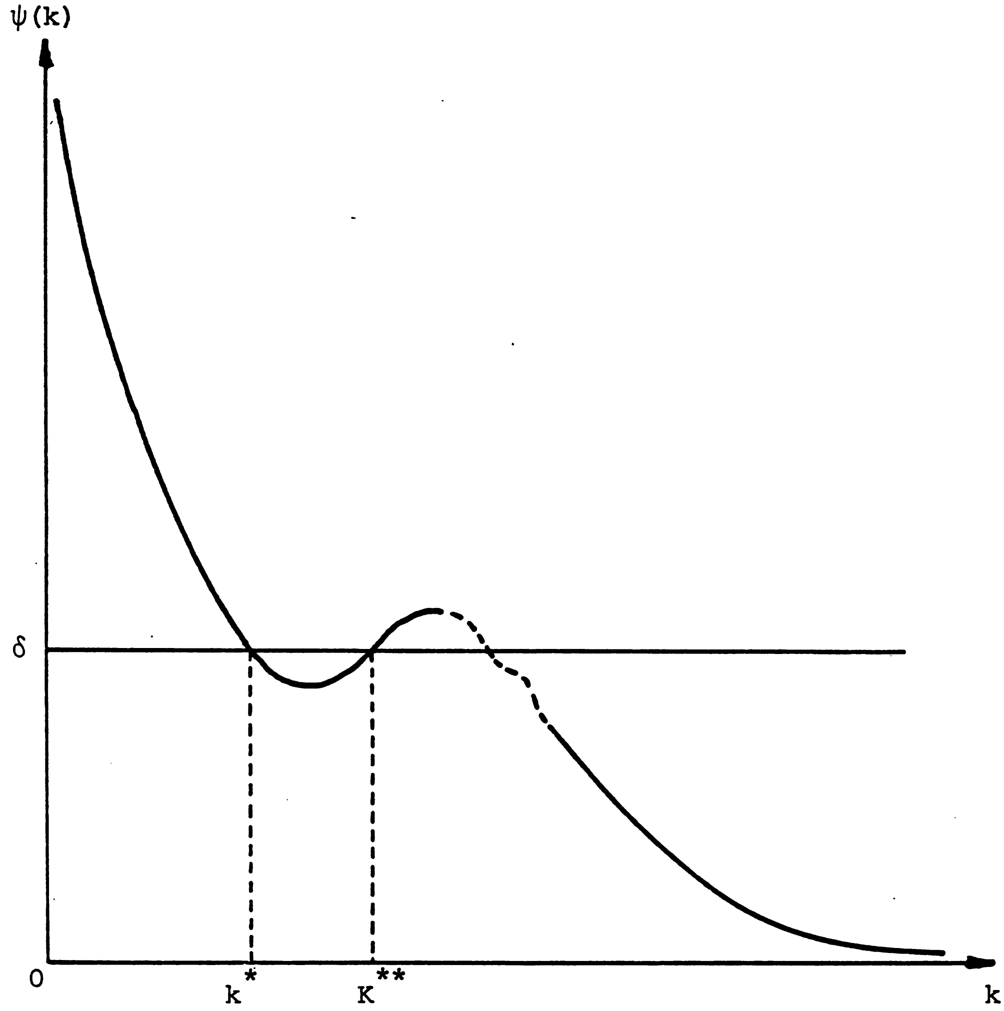



Figure 3.7.c: Finitely multiple "no-trade" endowment ratios with technology and preferences given

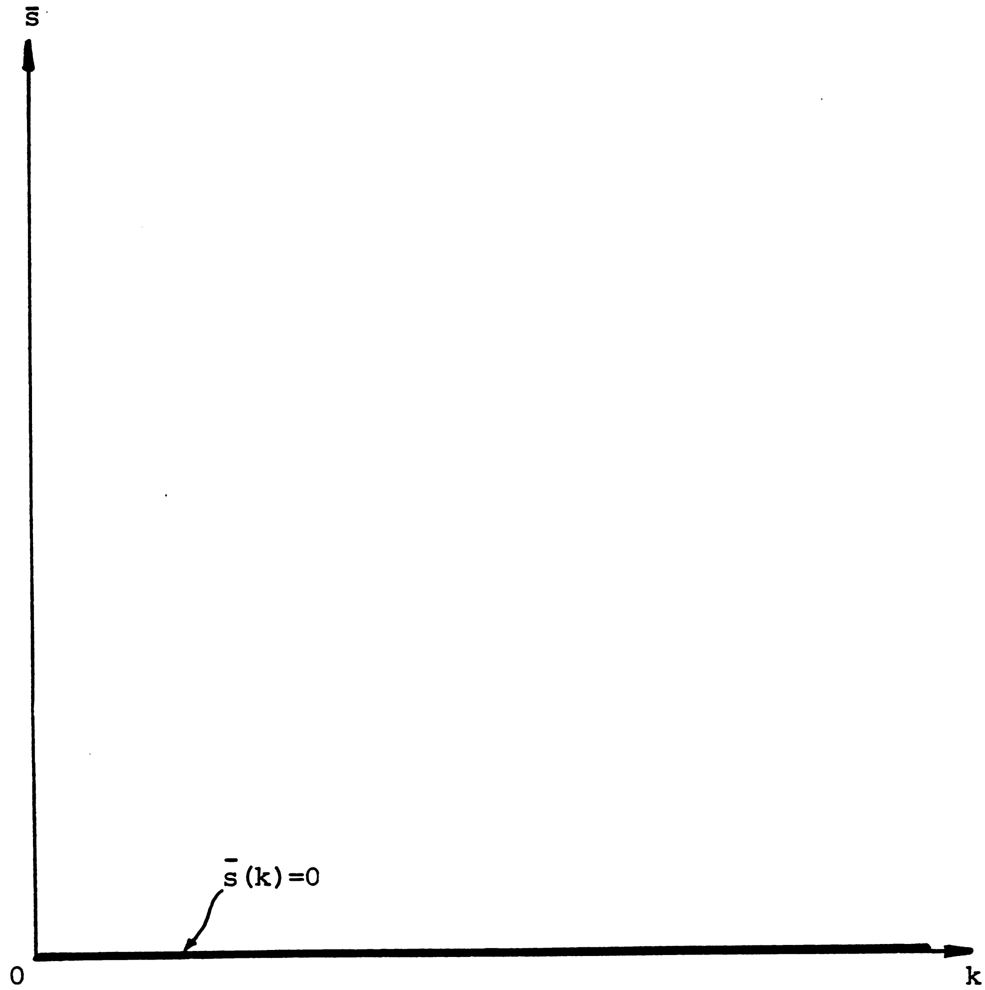
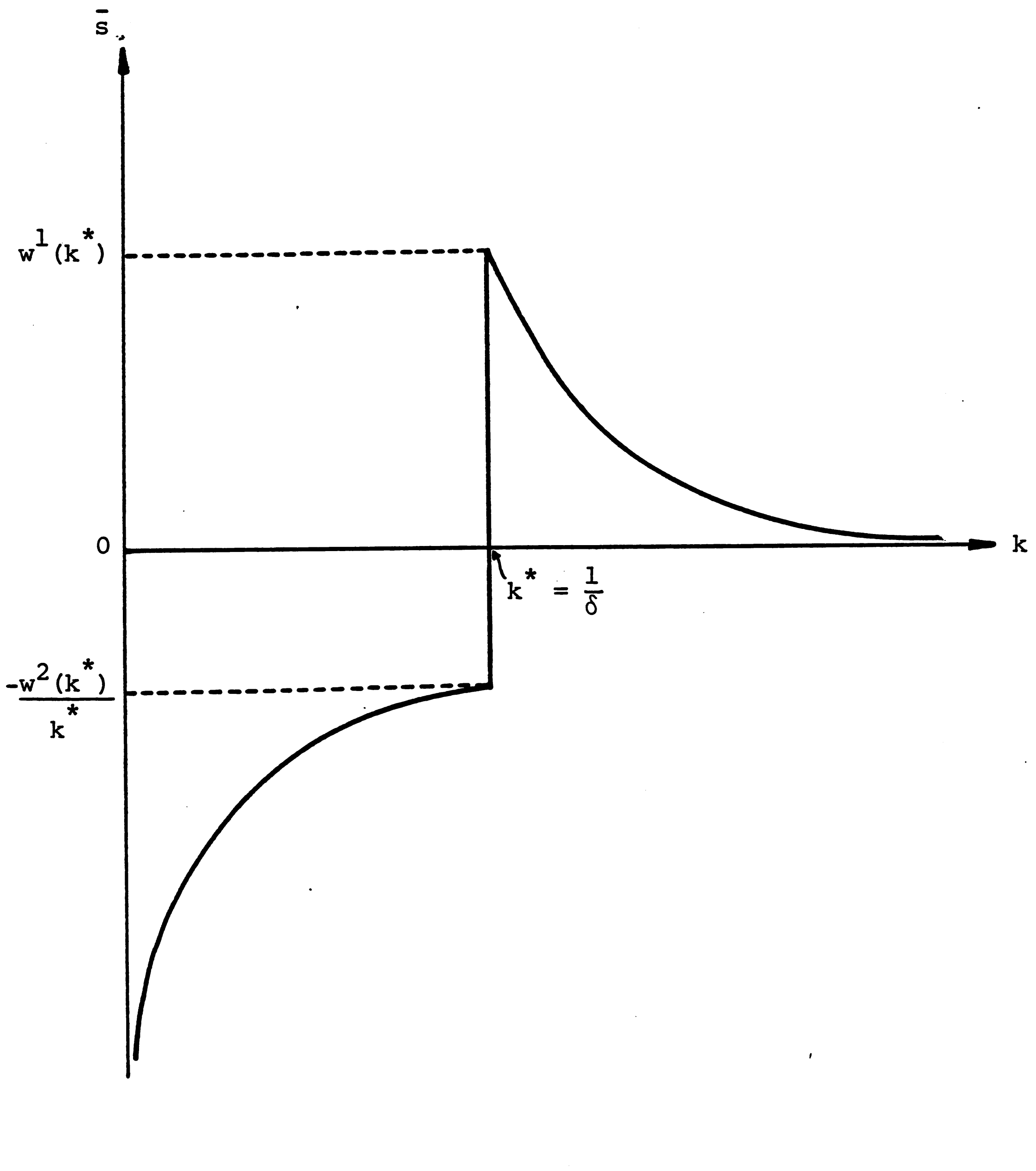



Figure 3.7.d: $\bar{s}(k) = 0$ for any $k \in (0,\infty)$ with technology and preferences given




<u>Figure 3.8.a</u>: $\bar{s} = \bar{s}(k)$ with separable constant-returns-to-scale utility and Cobb-Douglas production functions

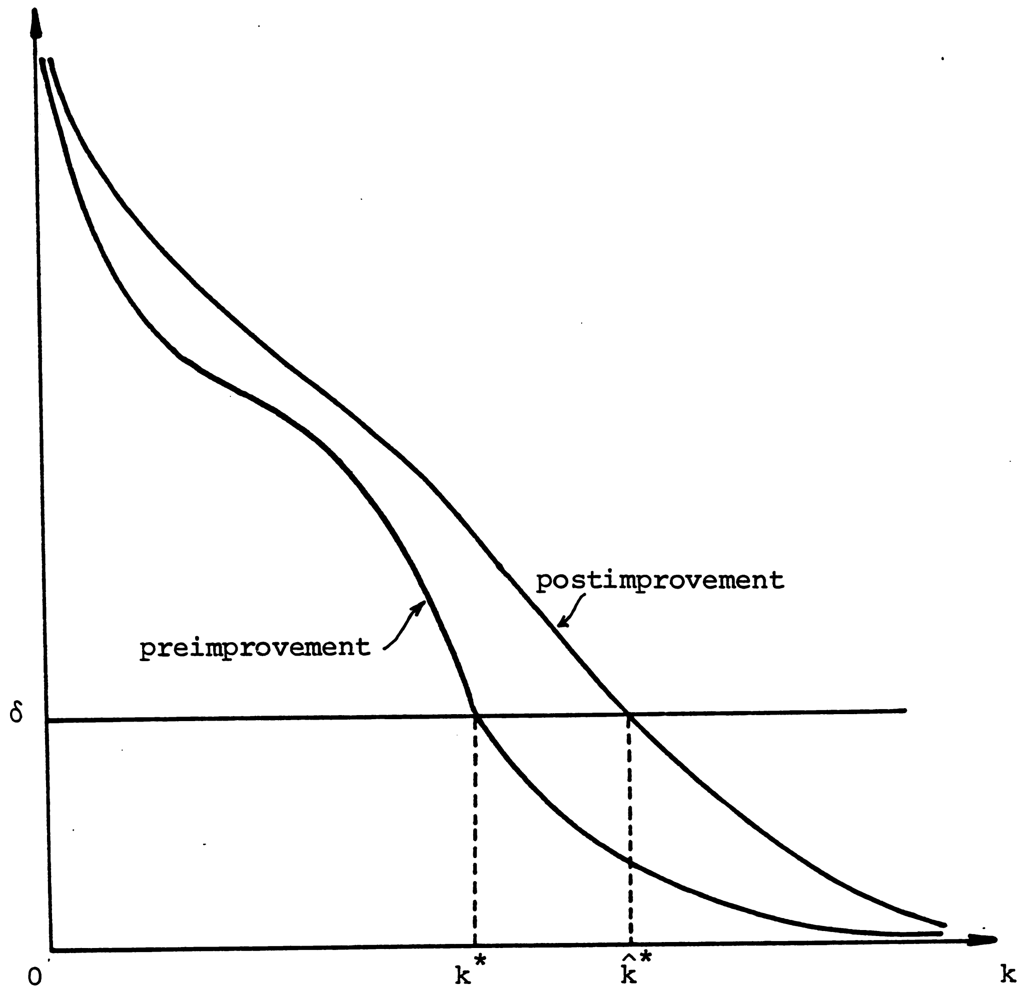
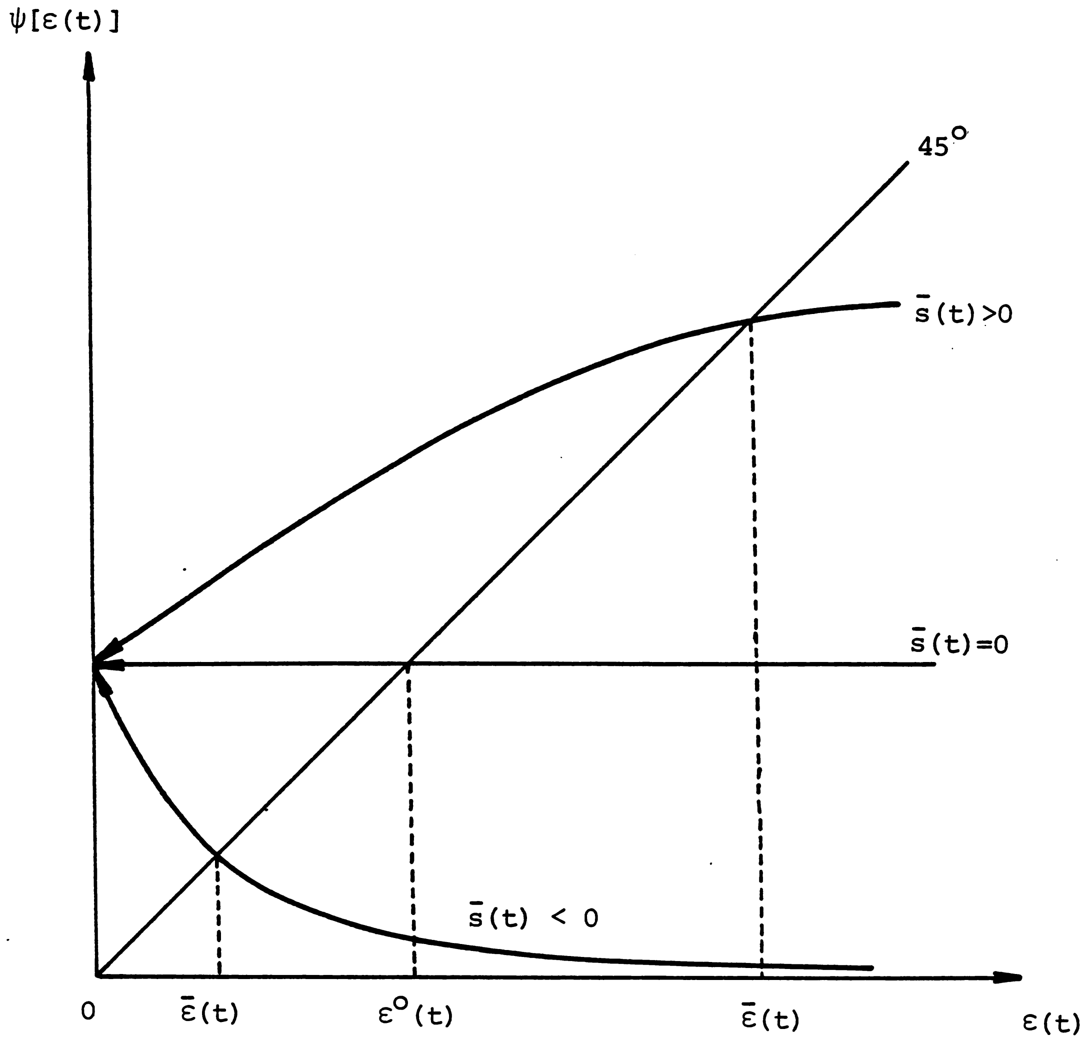



Figure 3.8.b: Effect of a technical improvement on the "no-trade" endowment ratio




<u>Figure 3.9</u>: Existence and Uniqueness of $\bar{\varepsilon}(t)$ for given $p^{o}(t)$, $p^{o}(t+1)$ and $\bar{s}(t)$

# CHAPTER IV

# OPEN BARTER ECONOMIES WITH OVERLAPPING GENERATIONS

## 4.1 INTRODUCTION

The analysis in Chapters II and III is confined to exchange between generations of the same country. The natural extension is to relax Assumption (iv) made in Chapter II to allow for trade between nations. The patterns of production and trade of an open economy which is very small, in the sense that its purchases and sales bear insignificantly on world prices, has been considered by Kemp [14]. The economy he examined is similar to those previously formulated and his results may be summarized as follows.

(i) Given the world price p, there exist $\underline{k}(p)$ and $\bar{k}(p)$, $0 < \underline{k} < \bar{k}$, such that for all $k \leq \underline{k}$ only the commodity relatively intensive in its use of old labour is produced and such that for all $k \geq \bar{k}$ only the commodity relative intensive in its use of young labour is produced.

(ii) There exists a critical value of k, say $\hat{k}$ ($\underline{k} < \hat{k} < \bar{k}$), at which production matches consumption and trade is extinguished. For $k < \hat{k}$, the commodity which is relative intensive in its use of old labour is exported, for $k > \hat{k}$ the other commodity is exported. $\hat{k}$ is the trade-switching young labour-old labour ratio.

The primary objective of this chapter is to extend the above analysis to large trading economies under conditions of free trade, with no costs of transport. For simplicity, the world is supposed to consist of two countries, home and foreign. The two countries possess the same constant-returns-to-scale technology for each good and have identical homothetic preferences; however they have different factor endowment ratios. For notational convenience, variables associated with the foreign country will be all denoted by stars. Thus, suppose that $k > k^*$, i.e., the steady-state rate of growth of the population is higher at home than abroad. Then, in the sufficiently long run, the home country will completely dominate the foreign country which will have to take $p^o$, the home country's autarkic equilibrium price ratio, as the given terms of trade. Therefore, our attention will be focused only on intervals of time in which neither country is large enough to alone determine the international terms of trade.

## 4.2 PRELIMINARY RESULTS

The sole purpose of this section is to restate some well-known propositions of the static theory of international trade. Thus, it is concerned with excess demand functions, existence and uniqueness of an equilibrium terms of trade, the Heckscher-Ohlin and the Factor Price Equalization Theorems.

### 4.2.1 Excess (Net Import) Demand Functions

Consider the home country. Let $D_i(t)$ and $X_i(t)$ be respectively the aggregate local demand and supply of good i at time t. Then the excess (net import) demand for i-th commodity at time t may be defined as the difference between $D_i(t)$ and $X_i(t)$, i.e.,

$$E_i(t) = D_i(t) - X_i(t) \qquad i=1,2; \quad t=0,1,2,... \qquad (4\text{-}1)$$

In a similar manner, the excess demand ratio at time t, denoted by E(t), may be defined as the difference between the aggregate local demand ratio D(t) and supply ratio S(t), i.e.,

$$E(t) = D(t) - S(t) \qquad t=0,1,2,... \qquad (4\text{-}2)$$

where $D(t) = D_2(t)/D_1(t)$ and $S(t) = X_2(t)/X_1(t)$.

The condition of balance-of-payments equilibrium is

$$E_1(t) + p(t)E_2(t) = 0 \qquad t=0,1,2,... \qquad (4\text{-}3)$$

where $p(t) = p_2(t)/p_1(t)$ and the necessary condition for an interior equilibrium with u homothetic is

$$[\partial u/\partial D_2(t)]/[\partial u/\partial D_1(t)] = \psi[D_2(t)/D_1(t)] = p(t)$$
$$t=0,1,2,... \qquad (4\text{-}4)$$

where $\psi'(.) < 0$.

In (4-1)-(4-4), we have a system of 5 equations in 7 unknowns $D_i(t)$, $X_i(t)$, $E_i(t)$ (i=1,2) and E(t) with p(t) treated as a parameter. It is then possible to solve for $E_i(t)$ $(i=1,2)$ and $E(t)$ in terms of $X_i(t)$ $(i=1,2)$ alone.

$$E_1(t) = p(t)\{X_2(t) - \sigma[p(t)]X_1(t)\}/\{1+p(t)\sigma[p(t)]\}$$

$$E_2(t) = \{\sigma[p(t)]X_1(t) - X_2(t)\}/\{1+p(t)\sigma[p(t)]\}$$

$$E(t) = \sigma[p(t)] - [X_2(t)/X_1(t)]$$

$$t=0,1,2,\ldots \qquad (4\text{-}5)$$

where $\sigma = \psi^{-1}$ and $\sigma'(p) < 0$.

Given neo-classical production functions, young labour and old labour determine the region of commodity price ratios within which each country is completely specialized. Using Kemp's notation [11], we have

(a) $p(t) \leq \underline{p}^s(t)$:

$$X_1(t) = N^2(t)f_1[k(t)] = \bar{X}_1[k(t)] \text{ and } X_2(t) = 0$$

$$E_1(t) = -p(t)\{\sigma[p(t)]\bar{X}_1[k(t)]\}/\{1+p(t)\sigma[p(t)]\}$$

$$= E_1[p(t), k(t)] < 0 \qquad t=0,1,2,\ldots$$

$$E_2(t) = \sigma[p(t)]\bar{X}_1[k(t)]/\{1+p(t)\sigma[p(t)]\}$$

$$= E_2[p(t), k(t)] > 0 \qquad t=0,1,2,\ldots$$

$$E(t) = \sigma[p(t)] = E[p(t)] > 0$$

$$t=0,1,2,\ldots$$

(b) $\underline{p}^s(t) < p(t) < \bar{p}^s(t)$:

$$X_1(t) = X_1[p(t), k(t)] \text{ and } X_2(t) = X_2[p(t), k(t)]$$

$$E_1(t) = p(t)\{X_2[p(t), k(t)] - \sigma[p(t)]X_1[p(t), k(t)]\}/\{1+p(t)\sigma[p(t)]\} = E_1[p(t), k(t)] \qquad t=0,1,2,...$$

$$E_2(t) = \{\sigma[p(t)]X_1[p(t), k(t)] - X_2[p(t), k(t)]\}/\{1+p(t)\sigma[p(t)]\} = E_2[p(t), k(t)] \qquad t=0,1,2,...$$

$$E(t) = \sigma[p(t)] - \{X_2[p(t), k(t)]/X_1[p(t), k(t)]\} = E[p(t), k(t)] \qquad t=0,1,2,...$$

(c) $p(t) \geq \bar{p}^s(t)$:

$$X_1(t) = 0 \text{ and } X_2(t) = N^2(t)f_2[k(t)] = \bar{X}_2[k(t)]$$

$$E_1(t) = p(t)\bar{X}_2[k(t)]/\{1+p(t)\sigma[p(t)]\} = E_1[p(t), k(t)] > 0 \qquad t=0,1,2,...$$

$$E_2(t) = -\bar{X}_2[k(t)]/\{1+p(t)\sigma[p(t)]\} = E_2[p(t), k(t)] < 0 \qquad t=0,1,2,...$$

$E(t)$ undefined $\qquad t=0,1,2,...$

The global behaviour of $E_i[p(t), k(t)]$ $(i=1,2)$ at a constant factor endowment ratio $k(t)$ cannot be ascertained unless a specific form of utility function is assumed. However, it is possible to calculate the signs of $[\partial E_i[p(t), k(t)]/\partial p(t)]_{k(t)}$ in some intervals of $p(t)$. Now, differentiating (4-1) partially with respect to $p(t)$,

$$[\partial E_i(t)/\partial p(t)]_{k(t)} = [\partial D_i(t)/\partial p(t)]_{k(t)} - [\partial X_i(t)/\partial p(t)]_{k(t)} \qquad i=1,2; \quad t=0,1,2,... \qquad (4\text{-}6)$$

It is clear from (2-34) and (2-52) that $(-1)^i[\partial X_i(t)/\partial p(t)]_{k(t)}$ is positive for $\underline{p}^s(t) < p(t) < \bar{p}^s(t)$ and zero elsewhere $(i=1,2; \quad t=0,1,2,...)$, that $[\partial D_1(t)/\partial p(t)]_{k(t)} > 0$ for $p(t) \geq p^o(t)$ and that $[\partial D_2(t)/\partial p(t)]_{k(t)} < 0$ for $p(t) \leq p^o(t)$ $(t=0,1,2,...)$, where $p^o(t)$ is the home country's autarkic equilibrium price ratio. Combining the above inequalities yields

$$[\partial E_1(t)/\partial p(t)]_{k(t)} > 0, \; p(t) \geq p^o(t) \qquad t=0,1,2,... \qquad (4\text{-}7\text{-}a)$$

$$[\partial E_2(t)/\partial p(t)]_{k(t)} < 0, \; p(t) \leq p^o(t) \qquad t=0,1,2,... \qquad (4\text{-}7\text{-}b)$$

Because everyone has identical homothetic preferences, $E(t)$ is a strictly decreasing function of $p(t)$ at a constant value of $k(t)$. This can be shown by calculating $[\partial E(t)/ p(t)]_{k(t)}$. From (4-2), it is obvious that

$$\left[\frac{\partial E(t)}{\partial p(t)}\right]_{k(t)} = \sigma'[p(t)] - \frac{X_1(t)[\partial X_2(t)/\partial p(t)] - X_2(t)[\partial X_1(t)/\partial p(t)]}{X_1^2(t)}$$

$$< 0 \qquad t=0,1,2,... \qquad (4\text{-}8)$$

since $\sigma'(p) < 0$, $[\partial X_1(t)/\partial p(t)] \leq 0$, $[\partial X_2(t)/\partial p(t)] \geq 0$ and $X_i(t) \geq 0$ for $i=1,2$ and $t=0,1,2,\ldots$ The graphs of $E_i(t)$ $(i=1,2)$ and $E(t)$ at a constant level of $k(t)$ are drawn in Figures 4.1.a, 4.1.b, and 4.2, respectively.

### 4.2.2 Existence, Uniqueness and Stability of an Equilibrium Terms of Trade

Suppose now that from period $t = t_0$ onwards the home country is opened to free trade with the foreign country. Then there exists a unique value of $p(t)$ at which the sum $E(t)+E^*(t)$ is equal to zero for all $t \geq t_0$. Before showing this more formally, a simple lemma will be given.

Lemma 4.1 At a constant level of the commodity price ratio $p(t)$, an increase or a decrease in the overall endowment ratio $k(t)$ will result in a response of $E(t)$ in the same (opposite) direction if $k_1(t) > k_2(t)$ $(k_1(t) < k_2(t))$, i.e.,

$$[\partial E(t)/\partial k(t)]_{p(t)} \begin{cases} > 0 \text{ if } k_1(t) > k_2(t) \\ \\ < 0 \text{ if } k_1(t) < k_2(t) \end{cases}$$

$$t=0,1,2,\ldots \qquad (4\text{-}9)$$

Proof $E(t) = \sigma[p(t)] - [X_2(t)/X_1(t)] \qquad t=0,1,2,\ldots$

$$\left[\frac{\partial E(t)}{\partial k(t)}\right]_{p(t)} = \frac{[\partial X_1(t)/\partial k(t)]X_2(t) - [\partial X_2(t)/\partial k(t)]X_1(t)}{X_1^2(t)}$$

$$t=0,1,2,\ldots \qquad (4\text{-}10)$$

From Rybczynski's Theorem, $(-1)^i[\partial X_i(t)/\partial k(t)]_{p(t)} < 0$ ($> 0$) for $k_1(t) > k_2(t)$ ($k_1(t) < k_2(t)$) ($t=0,1,2,\ldots$). Equation (4-9) is then seen to follow by combining the above results. Q.E.D.

Now suppose that the home and foreign countries are alike in all respects except that $k(t) > k^*(t)$. (It should be noted that under the assumption of a homothetic social utility indicator, a difference in scale alone is an insufficient cause of international trade.) Suppose further that in each country the first commodity is relatively young-labour intensive, i.e., $k_1(t) > k_2(t)$ and $k_1^*(t) > k_2^*(t)$ for all factor price ratios. Then, from Lemma 4.1, it is obvious that $\underline{p}^{s*}(t) < \underline{\bar{p}}^s(t)$, $\bar{p}^{s*}(t) < p^s(t)$ and $p^{*o}(t) < p^o(t)$. Consider the sum of the two countries' differences between demand ratios and supply ratios in terms of the first commodity, defined by

$$E(t) + E^*(t) = E[p(t), k(t)] + E^*[p^*(t), k^*(t)] \qquad (4\text{-}11)$$

For $p(t) \leq p^{*o}(t)$, $E(t) > 0$ and $E^*(t) \geq 0$, i.e., the sum of demand ratios $[D_2(t)/D_1(t)] + [D_2^*(t)/D_1^*(t)]$ exceeds the sum of supply ratios $[X_2(t)/X_1(t)] + [X_2^*(t)/X_1^*(t)]$. For $p(t) \geq p^o(t)$, $E(t) \leq 0$ and $E^*(t) < 0$, i.e., the sum of the two countries' supply ratios exceeds the sum of their demand ratios. Since $E(t)$ and $E^*(t)$ are continuous functions of $p(t)$, with $k(t)$ and $k^*(t)$ given, there exists a commodity price ratio, say $\hat{p}(t)$ ($p^{*o}(t) < \hat{p}(t) < p^o(t)$ such that

$$E[\hat{p}(t), k(t)] + E^*[\hat{p}(t), k^*(t)] = 0 \qquad t=t_0, t_0+1, \ldots \qquad (4\text{-}12)$$

The existence of an equilibrium international terms of trade can be shown graphically by superimposing the curves of $E(t)$ and $-E^*(t)$ as in Figure 4.3.

Furthermore, since $E(t)$ and $-E^*(t)$ slope in opposite directions, the intersection is unique, i.e., there is only one $\hat{p}(t)$ satisfying equation (4-12). Also, $\hat{p}(t)$ is globally stable in the statical sense that if $p(t)$ deviates from $\hat{p}(t)$, there exists a mechanism which brings $p(t)$ to $\hat{p}(t)$ however large the deviation of $p(t)$ from $\hat{p}(t)$ may be. Such a mechanism is provided if $p(t)$ increases (decreases) according as $E(t) + E^*(t) > 0$ ($< 0$).

### 4.2.3 Two Well-known Results

The Heckscher-Ohlin Theorem A country will export the good whose production is relatively intensive in the factor of production in which that country is relatively abundant.

Proof Take the case $k(t) > k^*(t)$ and $k_1(t) > k_2(t)$ for all factor price ratios. From Figure 4.3, it can be seen that the home country will export good 1 and import good 2.

Q.E.D.

Because preferences are identically homothetic, the theorem is valid irrespective to how factor abundance is defined.

Corollary 4.1 Assume that $k(t) > k^*(t)$. Suppose further that the endowment ratio in the home country changes while that of the foreign country remains unchanged. Then the effects of it on international trade may be summarized as follows.

$$\partial\hat{p}(t)/\partial k(t) \begin{cases} > 0 \text{ if } k_1(t) > k_2(t) \\ < 0 \text{ if } k_1(t) < k_2(t) \end{cases} \qquad t=t_0,t_0+1,\ldots \qquad (4\text{-}13)$$

$$\partial E(t)/\partial k(t) \begin{cases} > 0 \text{ if } k_1(t) > k_2(t) \\ < 0 \text{ if } k_1(t) < k_2(t) \end{cases} \qquad t=t_0,t_0+1,\ldots \qquad (4\text{-}14)$$

Proof Take the case $k_1(t) > k_2(t)$. Suppose that $k(t)$ increases from $k^1(t)$ to $k^2(t)$ while $k^*(t)$ remains constant. Then from Lemma 4.1, the curve $E(t)$ will shift to the right while $E^*(t)$ remains unchanged. The results are a higher equilibrium terms of trade and a higher $E(t)$. Q.E.D.

Suppose now that both $k(t)$ and $k^*(t)$ vary. Differentiating $\hat{p}[k(t), k^*(t)]$ totally,

$$d\hat{p}(t) = [\partial\hat{p}(t)/\partial k(t)]dk(t) + [\partial\hat{p}(t)/\partial k^*(t)]dk^*(t) \qquad t=t_0,t_0+1,\ldots$$

Therefore, the sign of $d\hat{p}(t)$ depends on factor intensities and the direction of changes in $k(t)$ and $k^*(t)$. If both $k(t)$ and $k^*(t)$ increase, $d\hat{p}(t)$ is positive if $k_1(t) > k_2(t)$ and negative if $k_1(t) < k_2(t)$ $(t=t_0,t_0+1,\ldots)$. If both $k(t)$ and $k^*(t)$ decrease, $d\hat{p}(t)$ is negative if $k_1(t) > k_2(t)$ and positive if $k_1(t) < k_2(t)$ $(t=t_0,t_0+1,\ldots)$. A simultaneous change in $k(t)$ and $k^*(t)$ will leave $\hat{p}(t)$ unchanged if

$$dk(t)/dk^*(t) = -[\partial\hat{p}(t)/\partial k^*(t)]/[\partial\hat{p}(t)/\partial k(t)] \qquad t=t_0, t_0+1, \ldots$$

The Factor Price Equalization Theorem  Under free trade, there is a tendency to factor price equalization in the sense that $|\omega[p^o(t)] - \omega^*[p^{*o}(t)]| > |\omega[\hat{p}(t)] - \omega^*[\hat{p}(t)]|$ where $\omega$ and $\omega^*$ denote respectively the factor price ratios in the home and foreign countries. Furthermore, the inequality holds regardless of the choice of numeraire in the definition of $\omega$.

Proof  Consider the case of post-trading incomplete specialization in both countries. Clearly, then, $\omega[\hat{p}(t)] = \omega^*[\hat{p}(t)]$ and the theorem is valid in the very strict sense. In the case of post-trading specialization in one country (specialization in production after trade can never happen in both countries in the model being studied), assume that $k(t) > k^*(t)$ and $k_1(t) > k_2(t)$. Because of the symmetry of the problem, all other cases can be similarly analyzed. Suppose further that $\underline{p}^{s*}(t) < \hat{p}(t) \leq \underline{p}^s(t)$, i.e., the home country specializes in the production of good 1 in post-trading equilibrium. Two possible definitions of $\omega(t)$ will be examined.

(a) $\omega(t) = w^2(t)/w^1(t)$ where $w^j(t)$ $(j=1,2)$ are defined as before. Since $k(t) > k^*(t)$ (i.e., $p^o(t) > p^{*o}(t)$ and $k_1(t) > k_2(t)$),

$$|\omega[p^o(t)] - \omega^*[p^{*o}(t)]| = \omega[p^o(t)] - \omega^*[p^{*o}(t)]$$

$$|\omega[\hat{p}(t)] - \omega^*[\hat{p}(t)]| = \omega[\hat{p}(t)] - \omega^*[\hat{p}(t)]$$

Also, $\hat{p}(t) < p^{o}(t)$ implies that $\omega[\hat{p}(t)] < \omega[p^{o}(t)]$ and $\hat{p}(t) > p^{*o}(t)$ implies that $\omega^{*}[\hat{p}(t)] > \omega^{*}[p^{*o}(t)]$. Subtracting the second inequality from the first, $\omega[p^{o}(t)] - \omega^{*}[p^{*o}(t)] > \omega[\hat{p}(t)] - \omega^{*}[\hat{p}(t)]$. The theorem for the case $\omega(t) = w^{2}(t)/w^{1}(t)$ is illustrated by Figure 4.4.a.

(b) $\omega(t) = w^{1}(t)/w^{2}(t)$

Now,

$$|\omega[p^{o}(t)] - \omega^{*}[p^{*o}(t)]| = \omega^{*}[p^{*o}(t)] - \omega[p^{o}(t)]$$

$$|\omega[\hat{p}(t)] - \omega^{*}[\hat{p}(t)]| = \omega^{*}[\hat{p}(t)] - \omega[\hat{p}(t)]$$

Since $\omega^{*}[p^{*o}(t)] > \omega^{*}[\hat{p}(t)]$ and $-\omega[p^{o}(t)] > -\omega[\hat{p}(t)]$, $\omega^{*}[p^{*o}(t)] - \omega[p^{o}(t)] > \omega^{*}[\hat{p}(t)] - \omega[\hat{p}(t)]$. This is illustrated by Figure 4.4.b.

The case of complete specialization in the foreign country after trade, i.e., $p^{s*}(t) \leq \hat{p}(t) < p^{s}(t)$ can be analyzed in the same fashion. Q.E.D.

## 4.3 THE GAINS FROM FREE TRADE STATICALLY CONSIDERED

The prime motive of residents of a country to voluntarily engage in free trade is to improve their lifetime utilities. The classic question of the gains from international trade in a conventional atemporal world has been well considered elsewhere. (See, for example, Kemp and Wan [12].) In the context of an intertemporal, overlapping-generations model as developed so far, we shall distinguish two types of trade: uncompensated trade and compensated trade. It is assumed from now on that $k(t) = k$ and $k^*(t) = k^*$ for all $t \geq t_0$.

### 4.3.1 Uncompensated Free Trade

Uncompensated free trade may be defined as free trade without post-trading lump-sum taxes and subsidies. Then it is a well-known fact in the static theory of international trade that the opening up of uncompensated free trade will make some people worse off than under autarky. Using constant-returns-to-scale utility and linear production functions, J. Fried [6] has shown that for his particular overlapping-generations model,

(i) the old generation alive at the time of the opening up of uncompensated trade will gain and

(ii) the young generation alive at the time of the opening up of uncompensated trade as well as all future generations will be worse off compared to what they would have been under autarky.

This property is quite general. In fact, it can be shown that

<u>Theorem 4.1</u> The opening up of uncompensated free trade may be potentially Pareto-harmful to the population of one country.

<u>Proof</u> The method of argument here is analogous to that applied in Chapter II to the welfare implications of technical improvements. The introduction of international trade will shift the domestic equilibrium price ratio to $\hat{p}$

and, consequently, change the wage incomes of both generations of a country. If the wage income of a typical member of a generation is reduced and his lifetime utility is heavily biased towards the satisfactions he derives in the present period then it is conceivable that he is worse off than he would be under autarky. More formally, suppose that from period $t=t_0$, the home country is opened to free trade with the foreign country. Suppose further that $k > k^*$ and $k_1 < k_2$ for all factor price ratios. Then the equilibrium terms of trade is $\hat{p}$ ($p^o < \hat{p} < p^{*o}$). The welfare change of an old person alive in the home country at time $t_0$ is

$$\Delta U = \Omega_2 \Delta u(C_1^2, C_2^2)$$

where $\Omega_2$ is evaluated at a commodity price ratio which lies strictly between $p^o$ and $\hat{p}$. The welfare change of a young man alive in the home country at time $t_0$ and all the yet unborn is

$$\Delta U = \Omega_1 \Delta u^1 + \Omega_2 \Delta u^2$$

where $\Delta u^j = \Delta u(C_1^j, C_2^j)$ $(j=1,2)$ and $\Omega_j$ $(j=1,2)$ are evaluated somewhere between $p^o$ and $\hat{p}$. In terms of the indirect utility function, $\Delta u^j$ $(j=1,2)$ can be represented by

$$\Delta u^j = \Delta v^j(p, C^j) = \Delta g[C^j/l(p)] \qquad j=1,2$$

where $C^j$ is the j-th period expenditure of a typical person in the home country $(j=1,2)$. In the absence of lump-sum taxes and transfers, $C^j = w^j$ $(j=1,2)$. Then, $w^2(p^o) > w^2(\hat{p})$ and $l(p^o) < l(\hat{p})$ because $p^o < \hat{p}$, $k_1 < k_2$ and $l'(p) > 0$. Therefore, $g[w^2(p^o)/l(p^o)] > g[w^2(\hat{p})/l(\hat{p})]$ and

$\Delta u^2 = \Delta g[w^2/l(p)] < 0$ (since $g'(.) > 0$), i.e., all the old people alive at time $t_0$ are worse off than they would be under autarky. Further, if $\Omega_2(p)$ is considerably larger than $\Omega_1(p)$, it is possible that $\Delta U < 0$, i.e., all the young alive at time $t_0$ and all the future generations from $t_0$ onwards are worse off than under autarky. Q.E.D.

In general, there are four possible outcomes after the opening of uncompensated trade in the home country.

(a) All people from period $t_0$ onwards are worse off. This necessarily implies $\Delta u^2 < 0$. In such a case, uncompensated free trade is Pareto-inferior in relation to autarky.

(b) The old people at time $t_0$ are better off but the young people at time $t_0$ and all future generations are worse off. This implies $\Delta u^1 < 0$ and $\Delta u^2 > 0$. Uncompensated free trade is still Pareto-harmful in relation to autarky.

(c) The old people at time $t_0$ are worse off but the young people at time $t_0$ and all future generations are better off. This implies $\Delta u^1 > 0$ and $\Delta u^2 < 0$. Uncompensated free trade may then be thought of as potentially Pareto-beneficial in relation to autarky in this sense.

(d) All people from period $t_0$ onwards are better off, implying $\Delta u^2 > 0$. In such a case, uncompensated free trade is Pareto-superior to autarky.

Extending the analysis to the world economy, the following results can be established for uncompensated free trade.

A. $\underline{k_1 > k_2 \text{ and } k_1^* > k_2^* \text{ for all factor rewards}}$:

If $k > k^*$ then the young people in the home country gain and the young people in the foreign country are hurt by trade. If $k < k^*$ the young people in the home country are worse off whereas the young people in the foreign country are better off in relation to autarky.

B. $\underline{k_1 < k_2 \text{ and } k_1^* < k_2^* \text{ for all factor rewards}}$:

If $k > k^*$ then the old people in the home country are worse off but the old people in the foreign country are better off. If $k < k^*$ then the old people in the home country gain from trade while the old people in the foreign country are hurt by trade.

It is then evident that, for $k_1 < k_2$, uncompensated free trade between nations can never take place voluntarily in a decentralized world economy. Continuing to assume $k > k^*$ and $k_1 < k_2$ for all factor rewards, sufficient conditions for uncompensated free trade to be everywhere Pareto-beneficial in relation to autarky are for any $\bar{p} \epsilon (p^o, \hat{p})$ and $\bar{p}^* \epsilon (\hat{p}, p^{*o})$,

$$\Omega_2(\bar{p})/\Omega_1(\bar{p}) \leq -\Delta g[w^1/l(p)]/\Delta g[w^2/l(p)] \qquad \text{(4-15-a)}$$

and

$$\Omega_2(\bar{p}^*)/\Omega_1(\bar{p}^*) \geq -\Delta g[w^{1*}/l(p^*)]/\Delta g[w^{2*}/l(p^*)] \qquad \text{(4-15-b)}$$

where

$$\Delta g[w^j/l(p)] \quad = \quad g[w^j(\hat{p})/l(\hat{p})] - g[w^j(p^o)/l(p^o)]$$

and

$$\Delta g[w^{j*}/l(p^*)] \quad = \quad g[w^{j*}(\hat{p})/l(\hat{p})] - g[w^{j*}(p^{*o})/l(p^{*o})]$$

for j=1,2. In the simple case of separable-additive utility function $U = u^1+\delta u^2$, the above inequalities reduce to

$$-\Delta g[w^1/l(p)]/\Delta g[w^2/l(p)] \geq \delta \geq -\Delta g[w^{1*}/l(p^*)]/\Delta g[w^{2*}/l(p^*)] \tag{4-16}$$

4.3.2 Compensated Free Trade

The fundamental proposition of the gains from international trade in the conventional atemporal model is that "if an unanimous decision were required in order for trade to be permitted, it would always be possible for those who desired trade to buy off those opposed to trade, with the result that all could be made better off" (Samuelson [18]). Of course, the argument above is not inconsistent with this proposition. Suppose that there exists in each country a central authority which is entitled to impose taxes and subsidies on its residents. One may as well think of the central authorities as being responsible for trade between the two countries. Then the young people at time $t (t \geq t_0)$ in the home country will, in post-trading equilibrium, be forced by the government to transfer some of their income to the old to ensure each old person can at least enjoy the autarkic level of satisfaction after trade. Therefore, the standard theorem about the gainfulness of free trade is still valid in the intertemporal, overlapping-generations context.

<u>Theorem 4.2</u> Compensated free trade is Pareto-superior in relation to no trade.

<u>Proof</u> See Kemp [13]. Q.E.D.

### 4.3.3 Commodity Transfer Schemes

In the pure-barter-economy model, the system of lump-sum taxes and subsidies is equivalent to a commodity transfer scheme. Such a scheme is analogous to intergenerational borrowing and lending. Let f ($f \gtreqless 0$) be the amount of income that each individual of the young generation transfers to (or receives from) the old. Then the lifetime income vector of a typical person born in period $t \geq t_0$ in post-trading equilibrium with a transfer scheme is $[w^1(\hat{p})-f;\ w^2(\hat{p})+kf]$. For a typical old man at time $t=t_0$, his lifetime income vector is simply $[w^1(p^o);\ w^2(\hat{p})+kf]$. From a young man's standpoint, f is a tax if its value is positive and a subsidy if its value is negative. In this sense, a tax is analogous to saving and subsidy to dissaving, both at the "biological" rate of interest $r = k-1$.

Compensated free trade is Pareto-superior to autarky if there exists an f such that

$$\Omega[v(\hat{p},\ w^1(\hat{p})-f),\ v(\hat{p},\ w^2(\hat{p})+kf)] \geq$$

$$\Omega[v(p^o,\ w^1(p^o));\ \ v(p^o,\ w^2(p^o))] \qquad (4\text{-}17\text{-}a)$$

and

$$v(\hat{p},\ w^2(\hat{p})+kf) \geq v(p^o,\ w^2(p^o)) \tag{4-17-b}$$

subject to the side constraint

$$-w^2(\hat{p})/k \leq f \leq w^1(\hat{p}) \tag{4-18}$$

where the inequality holds strictly for either (4-17-a) or (4-17-b).

Although the existence of such an f is ensured by Theorem 4.2, it is evident that its value need not be unique. Of particular interest are the full compensation and optimal transfer schemes.

Under the full compensation transfer scheme, those who are hurt by free trade will be compensated by the other age group so that they can enjoy the same level of satisfaction as under autarky. For example, suppose that $k > k^*$ and $k_1 < k_2$ for all factor price ratios. Then to compensate the old people in the home country after the opening of international trade, each young man in the home country will have to pay a tax whose value is given by

$$[w^2(\hat{p})+k\tilde{f}]/l(\hat{p}) = w^2(p^o)/l(p^o)$$

i.e.,

$$\tilde{f} = k^{-1}\{[w^2(p^o)l(\hat{p})/l(p^o)]-w^2(\hat{p})\} > 0 \tag{4-19}$$

The final income of a young person in the home country after paying his post-trading tax is

$$w^1(\hat{p})-k^{-1}l(\hat{p})\{[w^2(p^o)/l(p^o)]-[w^2(\hat{p})/l(\hat{p})]\}.$$

The gainfulness of international trade can then be measured by the difference between $\{[w^1(\hat{p})+k^{-1}w^2(\hat{p})]/l(\hat{p})\}$ - $[k^{-1}w^2(p^o)/l(p^o)]$ and $w^1(p^o)/l(p^o)$. In other words, the larger the difference between $[kw^1(\hat{p})+w^2(\hat{p})]/l(\hat{p})$ and $[kw^1(p^o)+w^2(p^o)]/l(p^o)$ the more the gain from free trade.

Under the optimal transfer scheme, the amount of transfer is determined by a central authority which is endowed with perfect foresight to solve the following optimizing problem

$$\underset{\{f\}}{\text{Max}}\ U = \Omega[v(\hat{p},\ w^1(\hat{p})-f);\ v(\hat{p},\ w^2(\hat{p})+kf)] \qquad (4\text{-}20)$$

subject to (4-18) where k and $\hat{p}$ are all given

The optimal transfer $\bar{f}$ exists and is unique under the assumed regularity conditions as shown in Subsection 3.3.3. Further, it satisfies the condition

$$k = (\Omega_1/\Omega_2)\{g'[(w^1(\hat{p})-\bar{f})/l(\hat{p})]/g'[(w^2(\hat{p})+k\bar{f})/l(\hat{p})]\} \qquad (4\text{-}21)$$

where $g'(.) > 0$ and $g''(.) < 0$.

It is obvious that the optimal transfer $\bar{f}(k,\ \hat{p}) = \bar{f}(k,\ k^*)$ is simply the optimal "biological" saving $\bar{s}=\bar{s}(k)$ when inter-generational borrowing and lending is available to an individual having an income vector $[w^1(\hat{p});\ \ w^2(\hat{p})]$ and facing the equilibrium price ratio $\hat{p}$. However, the optimal transfer $\bar{f}$ that solves (4-20) subject to (4-18) need not satisfy (4-17-b), i.e., the optimal transfer scheme is not necessarily Pareto-superior to autarky. This is most likely to happen when a typical person greatly discounts his satisfaction in old age and therefore tends to "over-spend" in the first period of his life, i.e., the economy is Classical.

## 4.4 POST-TRADING EQUILIBRIA UNDER NO INTERNATIONAL BORROWING AND LENDING

The purpose of this section is to examine the lifetime consumption programs of a typical person born in the home or foreign country in post-trading equilibrium under various trading regimes without borrowing and lending between nations. It is assumed throughout this section that $k > k^*$ and $k_1 < k_2$ for all factor price ratios. The autarkic (with no intergenerational exchange) lifetime consumption profiles of a person born in the home country and that of his counterpart in the foreign country are respectively denoted by

$$C(p^o) = \{C_1^1(p^o),\ C_2^1(p^o);\quad C_1^2(p^o),\ C_2^2(p^o)\} \quad \text{and}$$

$$C^*(p^{*o}) = \{C_1^{1*}(p^{*o}),\ C_2^{1*}(p^{*o});\quad C_1^{2*}(p^{*o}),\ C_2^{2*}(p^{*o})\}.$$

### 4.4.1 Incomplete Specialization in Both Countries

Neither country will completely specialize in its production after trade if and only if $\underline{p}^{s*} < \hat{p} < \bar{p}^s$ because $\underline{p}^s < \underline{p}^{s*}$ and $\bar{p}^s < \bar{p}^{s*}$ from 2.3.3. A sufficient condition for incomplete specialization in both countries is therefore $\underline{p}^{s*} \leq p^o$ and $p^{*o} \leq \bar{p}^s$ as $p^o < \hat{p} < p^{*o}$ from Lemma 4.1. This will occur if k and $k^*$ are sufficiently close to each other. Under free trade and factor irreversibility, the Factor Price Equalization Theorem will hold in its strict sense, i.e.,

$w^j(\hat{p}) = w^{j*}(\hat{p})$ for $j=1,2$.

Uncompensated Free Trade

Suppose that there is no income transfer scheme and no intergenerational borrowing and lending in either country. As the old people alive in the home country during the opening of trade are worse off than under autarky, it is difficult to see how this form of trade can ever take place voluntarily in a decentralized home economy. To compare the consumption patterns under various trading regimes let us assume that uncompensated free trade takes place under a centrally planned home economy. Then, except for the old people in period $t_0$, the post-trading lifetime consumption program of a person in the home country is identical to that of his counterpart in the foreign country, for $\hat{p}$ prevails in both countries and the wage rates of the same generations are everywhere equalized. Let the lifetime consumption program after uncompensated free trade be $C(\hat{p}) = C^*(\hat{p}) = \{C_1^1(\hat{p}),\ C_2^1(\hat{p});\ \ C_1^2(\hat{p}),\ C_2^2(\hat{p})\}$.

Corollary 4.2 In post-uncompensated-trading equilibrium, a typical person born during or after $t_0$ in the home (foreign) country consumes more (less ) of good one when young and less (more) of good two than he would in autarkic equilibrium, i.e.,

$$C_1^1(p^o) < C_1^1(\hat{p}) < C_1^{1*}(p^{*o}) \tag{4-22-a}$$

and

$$c_2^2(p^o) > c_2^2(\hat{p}) > c_2^{2*}(p^{*o}) \qquad (4\text{-}22\text{-}b)$$

Proof An interior solution to the problem of maximizing $u(c_1^j, c_2^j)$ subject to $c_1^j + pc_2^j = w^j$ (j=1,2) with u homothetic necessarily implies that

$$c_i^j(p) = \phi_i(p)w^j$$

for i=1,2 and j=1,2 where $\phi_1'(p) \geq 0$ and $\phi_2'(p) < 0$. Since $p^o < \hat{p} < p^{*o}$ and $k_2 > k_1$ for all factor price ratios, it follows that $w^1(p^o) < w^1(\hat{p}) < w^{1*}(p^{*o})$ and $w^2(p^o) > w^2(\hat{p}) > w^{2*}(\hat{p}^{*o})$. Therefore,

$$c_1^1(p^o) = \phi_1(p^o)w^1(p^o) < \phi_1(\hat{p})w^1(\hat{p}) = c_1^1(\hat{p}) <$$

$$\phi_1(p^{*o})w^{1*}(p^{*o}) = c_1^{1*}(p^{*o}) \text{ and}$$

$$c_2^2(p^o) = \phi_2(p^o)w^2(p^o) > \phi_2(\hat{p})w^2(\hat{p}) = c_2^2(\hat{p}) >$$

$$\phi_2(p^{*o})w^{2*}(p^{*o}) = c_2^{2*}(p^{*o}) \qquad \text{Q.E.D.}$$

Full-Compensation Trade

Suppose now that the full-compensation income transfer scheme is in operation but there is still no intergenerational borrowing and lending. Let $\tilde{f}$ and $\tilde{f}^*$ be respectively the full-compensation income transfers in the home and foreign countries. Then it is evident that $\tilde{f}\tilde{f}^*$ is nonpositive as the old people alive in period $t_0$ in the home (foreign) country are worse (better) off than they would be under autarky. Consequently, the lifetime consumption profile of a person in the home country and that of his counterpart in the foreign country are no longer the same. Let

$$\tilde{C}(\hat{p}) = \{\tilde{C}_1^1(\hat{p}),\ \tilde{C}_2^1(\hat{p});\ \ \tilde{C}_1^2(\hat{p}),\ \tilde{C}_2^2(\hat{p})\}$$

be the post-trading full-compensation lifetime consumption program of a person born during or after the opening of free trade in the home country. Then the following result can be established.

<u>Corollary 4.3</u>

$$C_1^1(p^o) < \tilde{C}_1^1(\hat{p}) < C_1^1(\hat{p}) \tag{4-23-a}$$

$$C_2^2(p^o) > \tilde{C}_2^2(\hat{p}) > C_2^2(\hat{p}) \tag{4-23-b}$$

and

$$\tilde{C}_2^1(\hat{p}) < [\tilde{C}_1^1(\hat{p})/C_1^1(p^o)]C_2^1(p^o) \tag{4-24-a}$$

$$\tilde{C}_1^2(p) > C_1^2(p^o) \tag{4-24-b}$$

<u>Proof</u> It follows from the homotheticity of u that

$$u(C_1^j,\ C_2^j) = g[h(C_1^j,\ C_2^j)] = g[C_1^j . h(1,\ \sigma)] \quad j=1,2$$

where $g'(.)>0$, h is homogeneous of degree one in $C_i^j$ (i=1,2) and is strictly increasing in $\sigma = C_2^j/C_1^j$ and $\sigma'(p) < 0$. From the assumptions that $k > k^*$ and $k_1 < k_2$ for all factor price ratios, the old people in the home country will receive subsidies to the extent that

$$u[\tilde{C}_1^2(\hat{p}),\ \tilde{C}_2^2(\hat{p})] = u[C_1^2(p^o),\ C_2^2(p^o)]$$

This necessarily implies that

$$\tilde{C}_1^2(\hat{p}).h[1, \sigma(\hat{p})] = C_1^2(p^o).h[1, \sigma(p^o)]$$

Now, $p^o < \hat{p}$, and since $\sigma'(p) < 0$, it is evident that

$$\tilde{C}_1^2(\hat{p}) > C_1^2(p^o) \text{ and } \tilde{C}_2^2(\hat{p}) < C_2^2(p^o).$$

Furthermore,

$$\tilde{C}_2^2(\hat{p}) = \phi_2(\hat{p})[w^2(\hat{p})+k\tilde{f}] > \phi_2(\hat{p})w^2(\hat{p}) = C_2^2(\hat{p})$$

as $\tilde{f} > 0$ by (4-19).

Similarly, from the inequality

$$u[\tilde{C}_1^1(\hat{p}), \tilde{C}_2^1(\hat{p})] \geq u[C_1^1(p^o), C_2^1(p^o)]$$

it can be deduced that

$$\tilde{C}_1^1(\hat{p}) > C_1^1(p^o) \text{ and } \tilde{C}_2^1(\hat{p}) < [\tilde{C}_1^1(\hat{p})/C_1^1(p^o)]C_2^1(p^o).$$

Finally,

$$\tilde{C}_1^1(\hat{p}) = \phi_1(\hat{p})[w^1(\hat{p})-\tilde{f}] < \phi_1(\hat{p})w^1(\hat{p}) = C_1^1(\hat{p}). \qquad \text{Q.E.D.}$$

Optimal-Compensation Trade

There is now in each country an immortal clearing house that conducts intergenerational transfers. Denote the optimal-compensation income transfer that solves (4-20) subject to (4-18) by $\bar{f}$. If the value of $\bar{f}$ is negative (i.e., the home economy is Classical) then optimal-compensation trade can never take place unless the $t_0$-th old generation in the home country agrees to transfer some of its uncompensated income to the young people living in the same period. In view of the fact that leaving bequests to his offspring gives an old man no utility it is difficult to conceive this possibility. If the value of $\bar{f}$ is non-negative but less than $\check{f}$ then otpimal-compensation trade still cannot take place because the welfare loss of the old people in period $t_0$ due to free trade is not yet fully compensated by the optimal-compensation scheme. Therefore, optimal-transfer free trade will take place if and only if $\bar{f} \geq \check{f}$. In this case one may view $(\bar{f}-\check{f})$ as the amount of optimal saving by a typical young man in period $t \geq t_0$ after he has paid his full-compensation tax $\check{f}$. Let

$$\bar{c}(\hat{p}) = \{\bar{c}_1^1(\hat{p}),\ \bar{c}_2^1(\hat{p});\ \ \bar{c}_1^2(\hat{p}),\ \bar{c}_2^2(\hat{p})\}$$

be the optimal-transfer consumption profile of a person born in period $t \geq t_0$ in the home country. Then, as $\bar{f} \geq \check{f} > 0$, it can be easily seen that

$$c_i^1(\hat{p}) > \check{c}_i^1(\hat{p}) \geq \bar{c}_i^1(\hat{p}) \qquad i=1,2 \qquad \text{(4-25-a)}$$

$$c_i^2(\hat{p}) < \check{c}_i^2(\hat{p}) \leq \bar{c}_i^2(\hat{p}) \qquad i=1,2 \qquad \text{(4-25-b)}$$

### 4.4.2 Complete Specialization in One Country

It is evident that complete specialization in production can never occur in both countries after trade as $\underline{p}^s < p^o < \bar{p}^s$, $p^{s*} < p^{*o} < \bar{p}^{s*}$ and $\hat{p}$ lies strictly between $p^o$ and $p^{*o}$. Under the assumptions $k > k^*$ and $k_1 < k_2$ the home country will specialize in the production of the second commodity if $p^o < \bar{p}^s \leq \hat{p} < p^{*o}$ whereas the foreign country will specialize in the first industry if $p^o < \hat{p} \leq \underline{p}^{s*} < p^{*o}$. This in turn implies that the values of k and $k^*$ are much different from each other. Consequently, the home country will, in the sufficiently long run, dominate the world market completely and the foreign country will have to take $p^o$ as the given terms of trade. We are, effectively, in the small trading country case of Kemp [14]. Because of the symmetry of the complete specialization cases, we shall concentrate on the home country's specialization in the second industry when the home country is not large enough to alone determine the international terms of trade.

Suppose now that $\bar{p}^s \leq \hat{p} < p^{*o}$. Then the post-trading uncompensated wage rates in the home country are

$$w^1(\hat{p}) = \hat{p}f_2'(k) \tag{4-26-a}$$

$$w^2(\hat{p}) = \hat{p}[f_2(k)-kf_2'(k)] \tag{4-26-b}$$

In the foreign country, the corresponding wage rates are

$$w^{1*}(\hat{p}) = f_1'[k_1^*(\hat{p})] = \hat{p}f_2'[k_2^*(\hat{p})] \tag{4-27-a}$$

$$\begin{aligned} w^{2*}(\hat{p}) &= f_1[k_1^*(\hat{p})] - k_1^*(\hat{p})f_1'[k_1^*(\hat{p})] \\ &= \hat{p}\{f_2[k_2^*(\hat{p})] - k_2^*(\hat{p})f_2[k_2^*(\hat{p})]\} \end{aligned} \tag{4-27-b}$$

It is clear that if $\hat{p} = \bar{p}^s$ then $k = k_2^*(\hat{p})$ and the factor prices in the two countries are equalized. We are therefore only interested in the case $\bar{p}^s < \hat{p} < p^{*o}$ where $k > k_2^*(\hat{p})$ and, consequently, $w^1(\hat{p}) < w^{1*}(\hat{p})$ and $w^2(\hat{p}) > w^{2*}(\hat{p})$. Let

$$c^*(\hat{p}) = \{c_1^{1*}(\hat{p}),\ c_2^{1*}(\hat{p});\ \ c_1^{2*}(\hat{p}),\ c_2^2(\hat{p})\}$$

be the post-uncompensated-trading consumption program of a typical member of the t-th generation ($t \geq t_0$) in the foreign country. Then it is not difficult to show that

$$C_1^1(p^o) < C_1^1(\hat{p}) < C_1^{1*}(\hat{p}) < C_1^{1*}(p^{*o}) \tag{4-28-a}$$

and

$$C_2^2(p^o) > C_2^2(\hat{p}) > C_2^{2*}(\hat{p}) > C_2^{2*}(p^{*o}) \tag{4-28-b}$$

Let

$$\tilde{w}^1(\hat{p}) = w^1(\hat{p}) - \tilde{f} \quad \text{and}$$

$$\tilde{w}^2(\hat{p}) = w^2(\hat{p}) + k\tilde{f}$$

be respectively the full-compensation incomes of a representative young man and a representative old man in the home country's post-trading equilibrium. Similarly, denote the full-compensation incomes of a typical young person and a typical old person in the foreign country after trade by

$$\tilde{w}^{1*}(\hat{p}) = w^{1*}(\hat{p}) - \tilde{f}^* \quad \text{and}$$

$$\tilde{w}^{2*}(\hat{p}) = w^{2*}(\hat{p}) + k^*\tilde{f}^* ,$$

respectively. Then, since

$$0 < \tilde{f} < w^1(\hat{p}) \text{ and}$$

$$-w^{2*}(\hat{p})/k^* < \tilde{f}^* \leq 0,$$

it remains true that $\tilde{w}^1(\hat{p}) < \tilde{w}^{1*}(\hat{p})$ and $\tilde{w}^2(\hat{p}) > \tilde{w}^{2*}(\hat{p})$. Thus,

$$C_1^1(p^o) < \tilde{C}_1^1(\hat{p}) < C_1^1(\hat{p}) < C_1^{1*}(\hat{p}) \leq \tilde{C}_1^{1*}(\hat{p}) \tag{4-29-a}$$

and

$$C_2^2(p^o) > \tilde{C}_2^2(\hat{p}) > C_2^2(\hat{p}) > C_2^{2*}(\hat{p}) \geq \tilde{C}_2^{2*}(\hat{p}) \qquad (4\text{-}29\text{-}b)$$

Furthermore, domestic intergenerational borrowing and lending may take place in the home country if $\bar{f} > \tilde{f}$ or in the foreign country if $\bar{f}^* > \tilde{f}^*$.

## 4.5 POST-TRADING EQUILIBRIA UNDER FULL-COMPENSATION TRANSFERS AND INTERNATIONAL BORROWING AND LENDING

There is in each country an immortal clearing house that conducts full-compensation income transfers to ensure that free trade is Pareto-superior to autarky. After redistribution an individual in either country can, through the same clearing house, engage in borrowing and lending, consuming either more or less than under the post-full-compensation-trading equilibrium. In addition, assume that under a cooperative trading regime there exists an international body through which borrowing and lending between residents in the home and foreign countries can be conducted. It can then be shown that international borrowing and lending cannot take place as long as $k \neq k^*$.

Suppose that international borrowing and lending between the two countries takes place in addition to domestic borrowing and lending between generations. Then in equilibrium all savings or dissavings are willingly made, implying that the real rate of return from lending must be everywhere equalized. Let $\varepsilon(t) = 1+r(t) > 0$ be the equilibrium international interest factor in period t ($t \geq t_0$). A typical man born during period $t \geq t_0$ in the home country chooses s(t) to maximize

$$U = \Omega[u(C_1^1(t),\ C_2^1(t));\quad u(C_1^2(t+1),\ C_2^2(t+1))]$$

$$= \Omega[g[(\tilde{w}^1 - s(t))/l(\hat{p})];\ g[(\tilde{w}^2 + \varepsilon(t)s(t))/l(\hat{p})]] \qquad (4\text{-}30\text{-}a)$$

subject to

$$-\tilde{w}^2/\varepsilon(t) \leq s(t) \leq \tilde{w}^1 \qquad (4\text{-}30\text{-}b)$$

whereas his counterpart in the foreign country seeks to maximize

$$U^* = \Omega[u(C_1^{1*}(t),\ C_2^{1*}(t));\ u(C_1^{2*}(t+1),\ C_2^{2*}(t+1))]$$

$$= \Omega[g[(\tilde{w}^{1*} - s^*(t))/l(\hat{p})],\ g[(\tilde{w}^{2*} + \varepsilon(t)s^*(t))/l(\hat{p})]] \qquad (4\text{-}31\text{-}a)$$

by the choice of $s^*(t)$ subject to

$$-\tilde{w}^{2*}/\varepsilon(t) \leq s^*(t) \leq \tilde{w}^{1*} \qquad (4\text{-}31\text{-}b)$$

The feasibility condition in this world economy is

$$N^1(t)s(t)+N^{1*}(t)s^*(t) = N^2(t)[C^2(t)-\tilde{w}^2]+N^{2*}(t)[C^{2*}(t)-\tilde{w}^{2*}]$$

$$= \varepsilon(t)[N^1(t-1)s(t-1)+N^{1*}(t-1)s^*(t-1)]$$

$$t > t_0 \qquad (4\text{-}32)$$

Carrying out the maximizations described by (4-30-a) and (4-31-a), we obtain

$$\varepsilon(t) = \frac{\Omega_1}{\Omega_2} \frac{g'[(\tilde{w}^1 - s(t))/l(\hat{p})]}{g'[(\tilde{w}^2 + \varepsilon(t)s(t))/l(\hat{p})]} \qquad t \geq t_0 \qquad (4\text{-}33\text{-}a)$$

and

$$\varepsilon(t) = \frac{\Omega_1^*}{\Omega_2^*} \frac{g'[(\tilde{w}^{1*} - s^*(t))/l(\hat{p})]}{g'[(\tilde{w}^{2*} + \varepsilon(t)s^*(t))/l(\hat{p})]} \qquad t \geq t_0 \qquad (4\text{-}33\text{-}b)$$

Now, since $\partial\Omega_j/\partial C^j \leq 0$ $(j=1,2)$, $\partial\Omega_i/\partial C^j \geq 0$ for $i \neq j$ $(i=1,2;\ j=1,2)$ and $g''(.) < 0$, it is evident that the R.H.S. of (4-33-a) is, for a given $\varepsilon(t)$, a strictly increasing positive function of $s(t)$. Bearing the feasibility condition (4-32) in mind, the R.H.S. of (4-33-b) is clearly a strictly decreasing positive function of $s(t)$. Thus, (4-33-a) and (4-33-b) may hold only if

$$\tilde{w}^1 - s(t) = \tilde{w}^{1*} - s^*(t) \qquad t \geq t_0 \qquad (4\text{-}34\text{-}a)$$

and

$$\tilde{w}^2 + \varepsilon(t)s(t) = \tilde{w}^{2*} + \varepsilon(t)s^*(t) \qquad t \geq t_0 \qquad (4\text{-}34\text{-}b)$$

Evidently (4-34-a) and (4-34-b) together imply

$$\varepsilon(t) = (\tilde{w}^2 - \tilde{w}^{2*})/(\tilde{w}^{1*} - \tilde{w}^1) = \bar{\varepsilon} > 0 \qquad t \geq t_0 \qquad (4\text{-}35)$$

Substituting (4-35) back into (4-33-a) and (4-33-b), we have

$$\bar{\varepsilon} = \frac{\Omega_1}{\Omega_2} \frac{g'[(\tilde{w}^1 - s(t))/l(\hat{p})]}{g'[(\tilde{w}^2 + \bar{\varepsilon}s(t))/l(\hat{p})]} = \frac{\Omega_1^*}{\Omega_2^*} \frac{g'[(\tilde{w}^{1*} - s^*(t))/l(\hat{p})]}{g'[(\tilde{w}^{2*} + \bar{\varepsilon}s^*(t))/l(\hat{p})]} \qquad t \geq t_0 \qquad (4\text{-}36)$$

Equation (4-36) cannot hold for any general form of g. Even if (4-36) holds true, it necessarily requires that

$$s(t) = \bar{s}$$

and

$$s^*(t) = \bar{s}^*$$

for all $t \geq t_0$. The feasibility condition (4-32) then reduces to

$$N^1(t-1)[k-\bar{\varepsilon}]\bar{s} + N^{1*}(t-1)[k^*-\bar{\varepsilon}]\bar{s}^* = 0 \qquad (4\text{-}37)$$

Since k and $k^*$ are both different from $\bar{\varepsilon}$ in general, equation (4-37) is satisfied for all $t > t_0$ only in the special case $\bar{s} = \bar{s}^* = 0$. Hence international borrowing and lending cannot take place in this world economy. However, domestic intergenerational borrowing and lending may or may not take place in either country, according as the economy being considered is Samuelson or Classical, respectively. If it does take place then $\varepsilon(t)$ has to be set at the "biological" level k in the home country or $k^*$ in the foreign country.

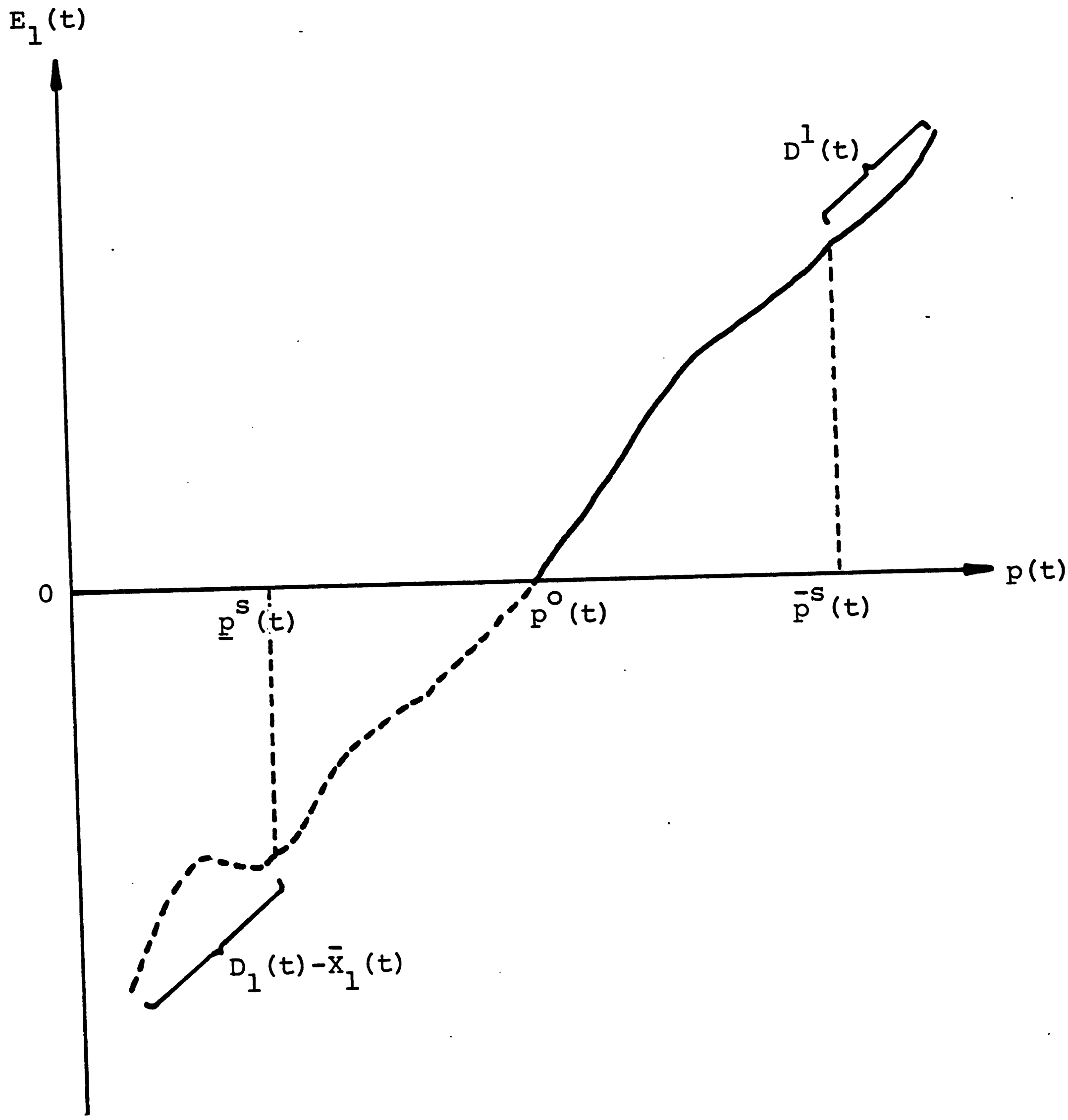



Figure 4.1.a: $E_1[p(t), k(t)] = D_1(t) - X_1(t)$ with a given $k(t)$

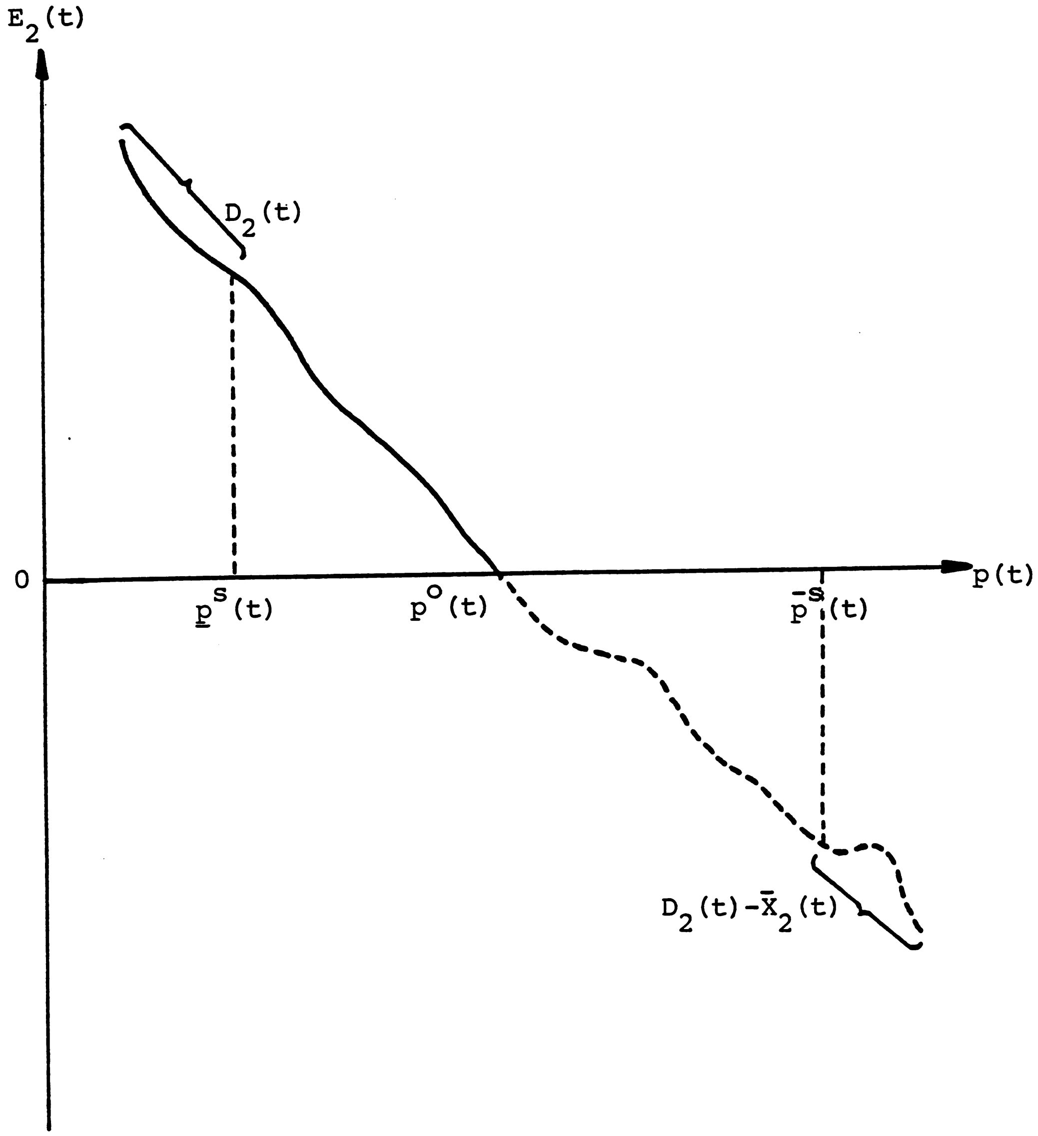



Figure 4.1.b: $E_2[p(t), k(t)] = D_2(t) - X_2(t)$ with a given k(t)

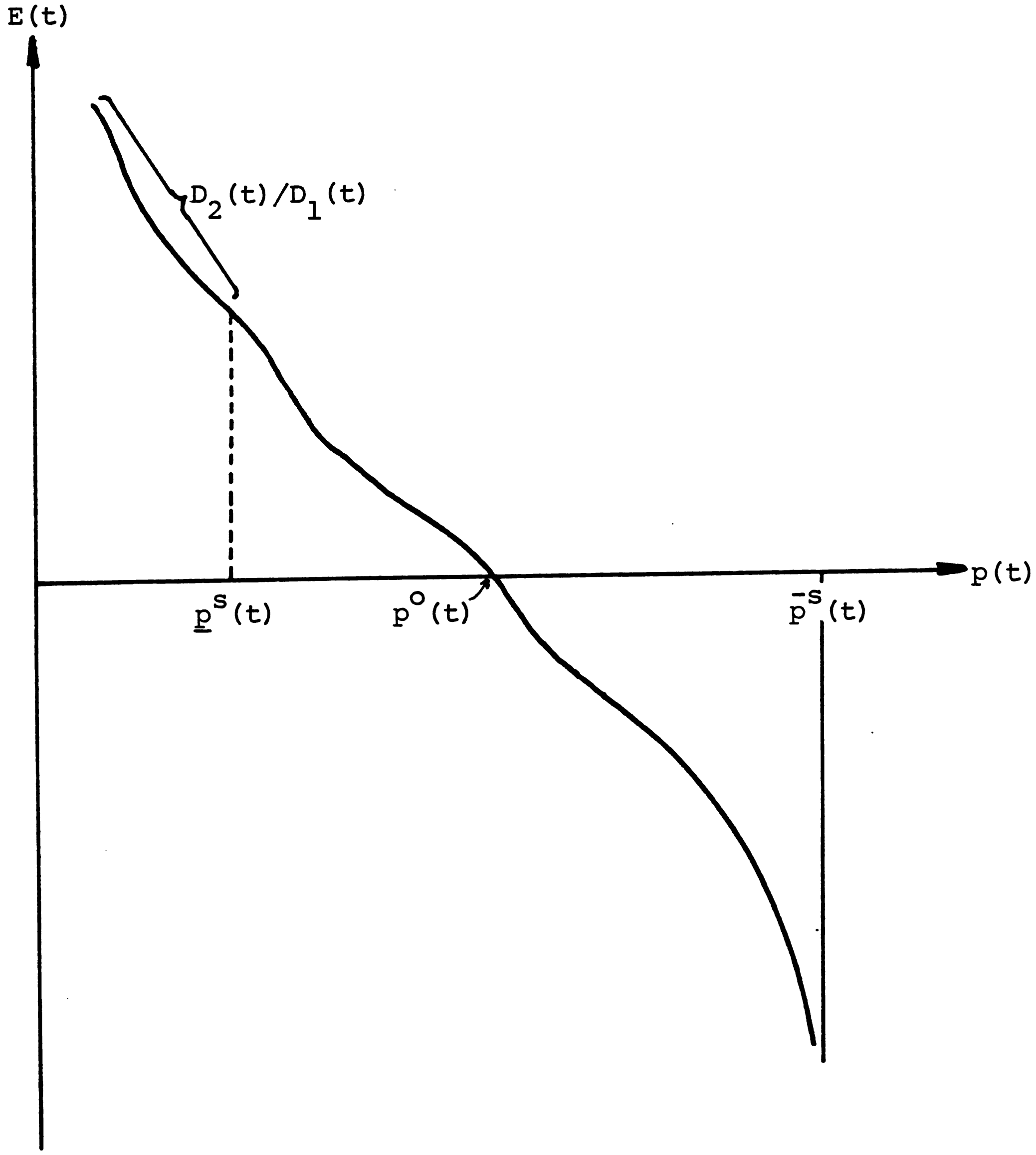



Figure 4.2: $E[p(t), k(t)] = \frac{D_2(t)}{D_1(t)} - \frac{X_2(t)}{X_1(t)}$ with a given k(t)

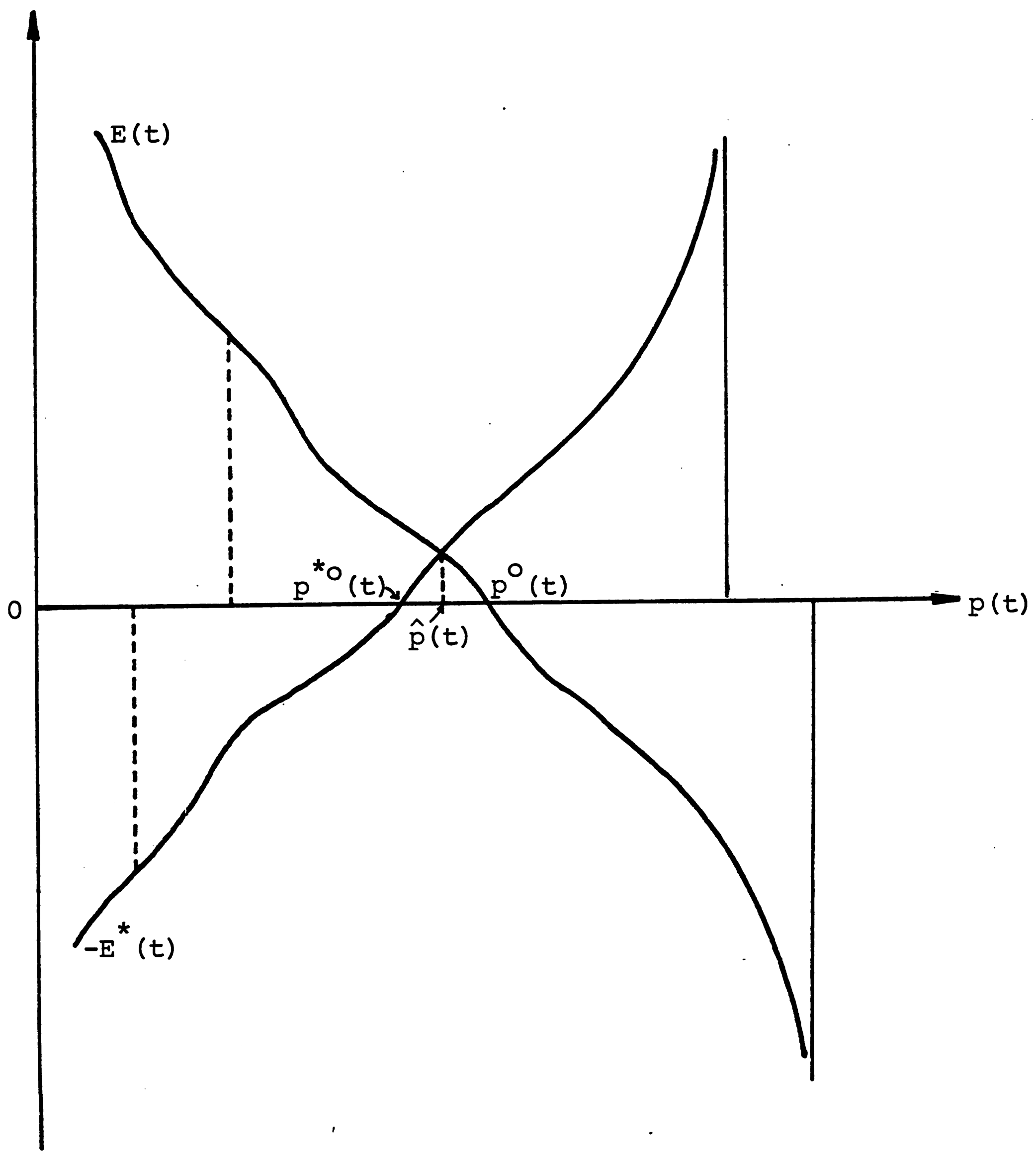



Figure 4.3: Existence of a unique international terms of trade $\hat{p}(t) = \hat{p}[k(t), k^*(t)]$

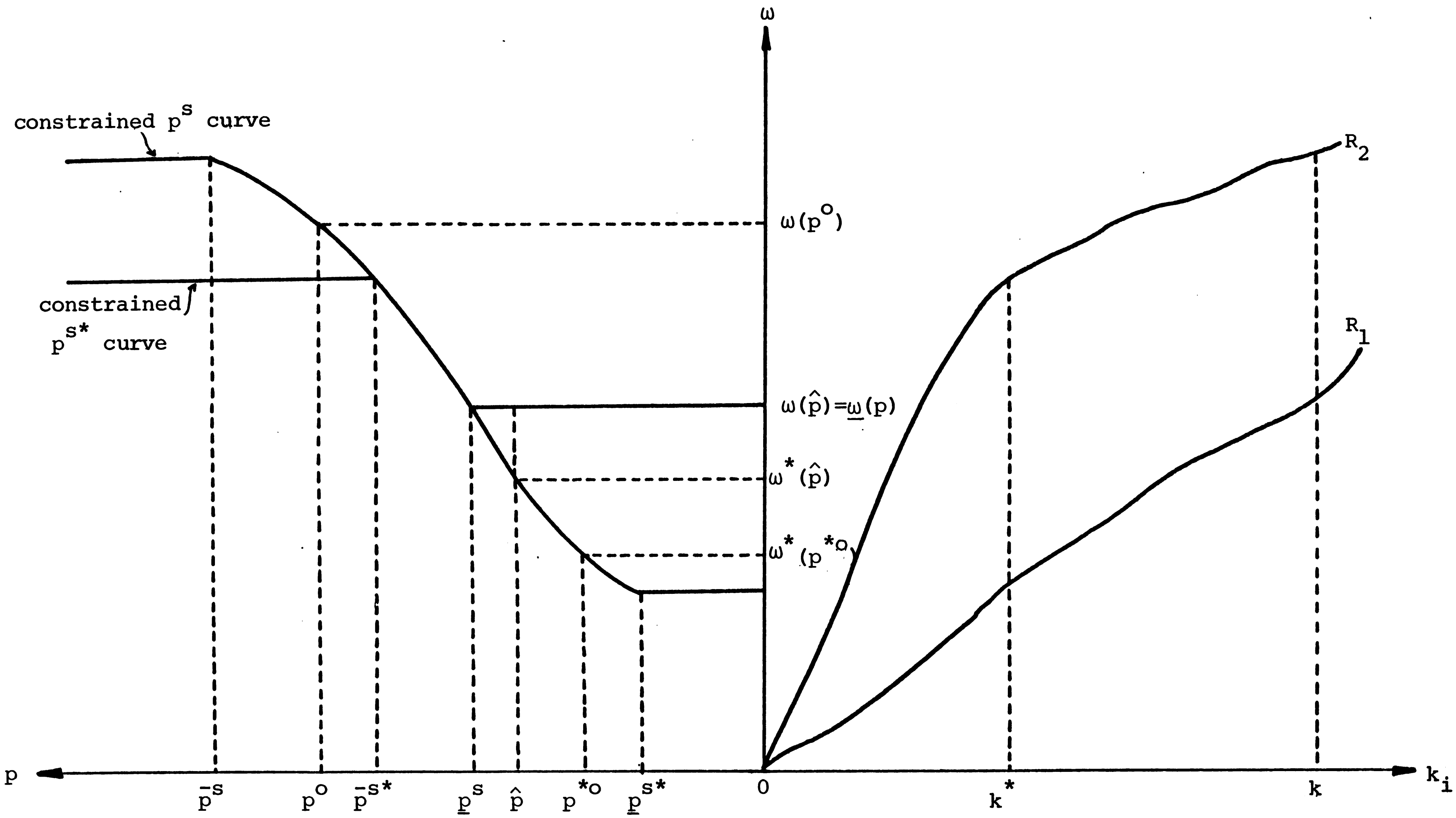



Figure 4.4.a: Factor price equalization with $\omega = w^2/w^1$, $k > k^*$ and $k_1 > k_2$

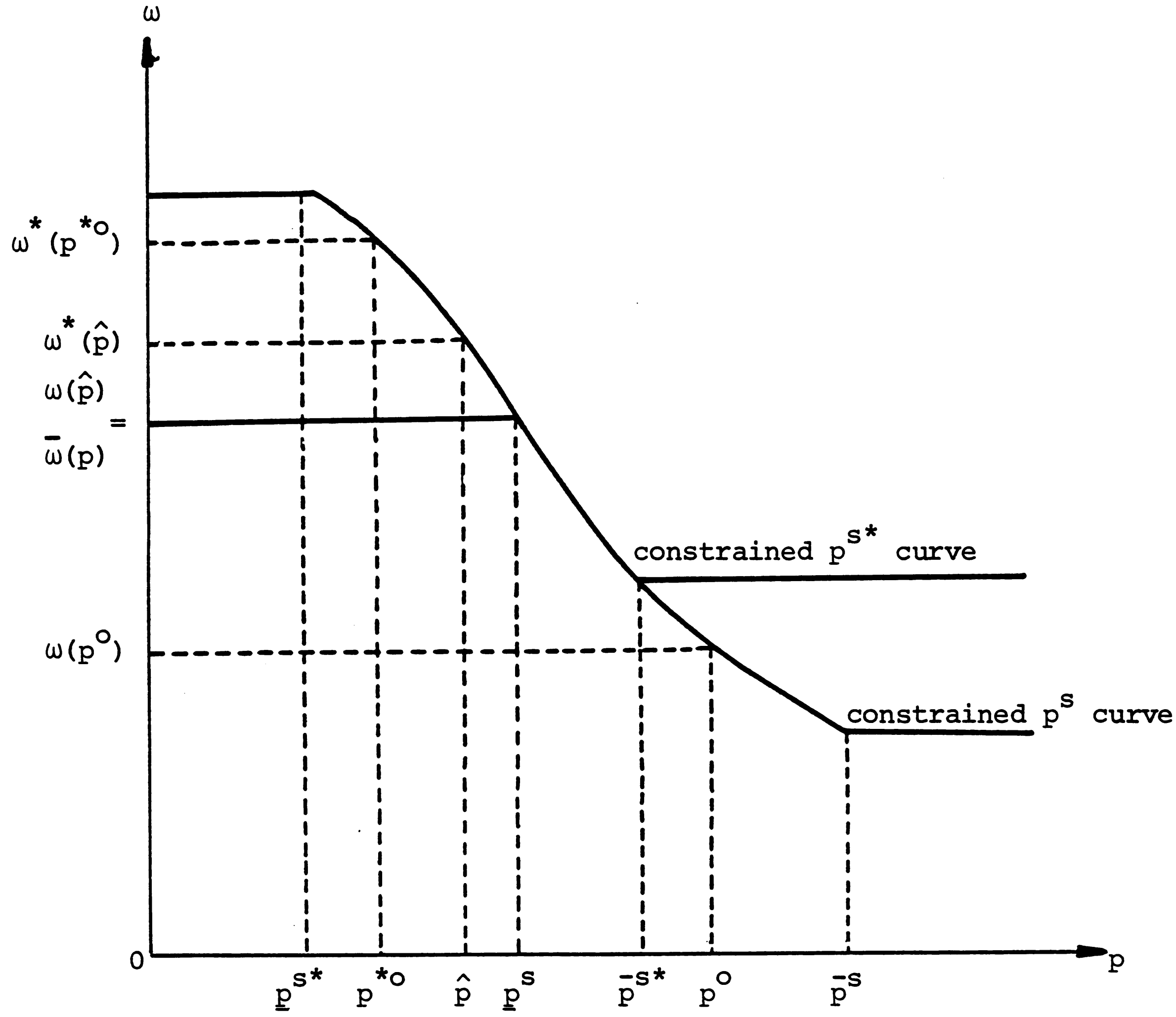



Figure 4.4.b: Factor price equalization with $\omega = w^1/w^2$, $k > k^*$ and $k_1 > k_2$

# CHAPTER V

# CLOSED ECONOMY WITH FIAT MONEY AND OVERLAPPING GENERATIONS

## 5.1 INTRODUCTION

As argued in Chapter III, closed-barter-economy society could move from the "no-trade" consumption program to the "biological" consumption program through inter-generational borrowing and lending. Since each individual lives for just two periods, exchange between generations must be necessarily conducted through an immortal clearing house. In assuming the existence and operation of such an institution, two critical issues have been overlooked. Firstly, the costs of creating and maintaining a clearing house that will conduct its transactions indefinitely into the future might be prohibitive. Secondly, it is difficult to see why the old people alive at the time of the creation of the clearing house in a Classical economy should give up some of their incomes for the sake of those who have not yet been born.

In the present chapter, a closed-monetary-economy model will be developed. Its characteristics are identical to those of the basic model considered in Chapter II except that Assumption II-v is now dropped. There is no arguing with Nature about the storability of produced consumption goods. However, there is nothing to prevent men from printing money which can be used as a store of value. Before proceeding to present the model, it should be noted that the new durable money is quite worthless for consumption, i.e., does not enter directly into the utility function. Consequently, only young men wish to hold money to exchange for consumption goods in their old age. Thus, money has no value in the Classical economy where a typical young man wants to consume more than he earns. Generally speaking, money of this kind has no role in overlapping-generations models where a typical person's lifetime utility can only be improved by reallocating consumption goods to the high-marginal-utility first period from the low-marginal-utility second period. The following analysis is mainly confined therefore to Samuelson economies where a representative young man saves against old age. In this sense, monetary exchange is less general than intergenerational borrowing and lending. Finally, the following analysis takes as its starting point the work of Kemp and Long [16].

## 5.2 ASSUMPTIONS AND FORMULATION OF THE MODEL

### 5.2.1 Money Stock and Monetary Transfers

The closed-monetary-economy model is identical to the basic model considered in Chapter II in all respects except that there now exists a stock of money which grows geometrically at the constant rate $\mu-1$, $\mu>0$. Let $M(t)$ denote the stock of money during the t-th period. Then

$$M(t) = M_0\mu^t \qquad t=0,1,2,.... \tag{5-1}$$

where $M_0$ is the stock of money at time $t=0$. (Of course, it does not matter when $M_0$ was created. Suppose that it was created at time $t=t_0$. Then by a suitable translation of the time origin, $T=t-t_0$, one may as well assume that money exists since time T=0.)

Money bears interest at the constant rate r, $r > -1$. Thus, from (5-1), the increase in the stock of money from period t-1 to period t is

$$\Delta M(t) = M(t) - M(t-1) = (\mu-1)M_0\mu^{t-1} \qquad t=1,2,3,\ldots \tag{5-2}$$

Of this amount,

$$rM(t-1) = rM_0\mu^{t-1} \qquad t=1,2,3,\ldots \tag{5-3}$$

represents interest payments and the remainder,

$$(\mu-1-r)M_0\mu^{t-1} = \eta M_0\mu^{t-1} \qquad t=1,2,3,\ldots \tag{5-4}$$

represents monetary transfers to the population in period $t \geq 1$. If $\eta > 0$, so that the stock of money grows at a rate greater than the rate of interest, the excess money created is distributed to the two overlapping generations. If $\eta < 0$, so that the excess is negative, poll taxes are levied. The distribution scheme is that a constant proportion $\alpha$, $0 \leq \alpha \leq 1$, goes to the young of period t except for t=0. Hence the monetary transfer to a typical young person in period t is

$$\alpha\eta M_0\mu^{t-1}/N^1(t) \qquad t=1,2,3,\ldots \tag{5-5}$$

and the transfer to the same person in period t+1 is

$$(1-\alpha)\eta M_0 \mu^t / N^2(t+1) \qquad t=1,2,3,\ldots \tag{5-6}$$

where $N^2(t+1) = N^1(t)$ is the subpopulation of young people at time t.

### 5.2.2 The Budget Constraints Reformulated

It is necessary to describe two sets of budget constraints: that faced by the population in period t=0 and that faced by succeeding generations. For $t \geq 1$, let m(t) be the stock-demand for money by a typical member of the subpopulation of young people in period t. Then in the first period of his life he faces the constraint

$$C_1^1(t)+p(t)\cdot C_2^1(t) \;=\; w^1(t)+q(t)\{[\alpha\eta M_0\mu^{t-1}/N^1(t)] - m(t)\}$$

$$t=1,2,3,\ldots \tag{5-7}$$

where q(t) is the price of money in terms of the first good in period t. In the next period, his budget constraint is

$$C_1^2(t+1)+p(t+1)C_2^2(t+1) = w^2(t+1)+q(t+1)\{[(1-\alpha)\eta M_0\mu^t/N^1(t)] + (1+r)m(t)\}$$

$$t=1,2,3,\ldots \tag{5-8}$$

Assuming that money is not worthless at any instant of time, i.e., $q(t) > 0$ for $t=0,1,2,\ldots$, m(t) can be eliminated from (5-7) and (5-8) to yield a single budget equation

$$(1+r)[q(t+1)/q(t)][w^1(t)-C_1^1(t)-p(t)C_2^1(t)]+[w^2(t+1)-C_1^2(t+1)-p(t+1)C_2^2(t+1)]+q(t+1)[\eta M_0\mu^t/N^1(t)]\{[(1+r)\alpha/\mu]+(1-\alpha)\} = 0$$

$$t=1,2,3,... \qquad (5\text{-}9)$$

From equation (5-7) the value of $m(t) \geq 0$ is bounded from above by

$$0 \leq m(t) \leq [w^1(t)/q(t)] + [\alpha\eta M_0\mu^{t-1}/N^1(t)]$$

$$t=1,2,3,... \qquad (5\text{-}10)$$

Comparing (5-10) with (3-9) it is clear that the social contrivance of money is not as general as the social contract of intergenerational borrowing and lending.

It is assumed that members of the oldest generation own (in equal amounts) all of the initial stock of money. Thus, the choice of a typical old person in period t=0 is subject to

$$C_1^2(0)+p(0)C_2^2(0) = w^2[p(0)]+q_0M_0/N_0^2 \qquad (5\text{-}11)$$

The budget constraints of a typical young man in period t=0 are

$$C_1^1(0)+p(0)C_2^2(0) = w^1[p(0)]-q_0m(0) \qquad (5\text{-}12)$$

where $m(0) = M_0/N_0^1$, and the t=0 version of equation (5-8). The initial nonnegative values of M(t) and q(t) can be chosen quite arbitrarily but they must satisfy

$$0 \leq q_0M_0 \leq N_0^1w^1[p(0)] \qquad (5\text{-}13)$$

The inequalities (5-13) simply state that the initial value of the stock of money is nonnegative but cannot exceed the total value of the income of all young people in period $t=0$, all measured in terms of the first commodity.

### 5.2.3 Definitions

Competitive Consumption Programs An infinite sequence of consumption programs $\{[C_1^2(0),\ C_2^2(0)];\quad C(t) = [C_1^1(t),\ C_2^1(t);\ C_1^2(t+1),\ C_2^2(t+1)]:\quad t=0,1,2,\ldots\}$ is said to be competitive if $[C_1^2(0),\ C_2^2(0)]$ maximizes $u[C_1^2(0),\ C_2^2(0)]$ subject to (5-11) and $C(t)$ maximizes $U[C(t)] = \Omega[u(C_1^1(t),\ C_2^1(t));\ u(C_1^2(t+1),\ C_2^2(t+1))]$ subject to the budget constraint (5-9). In other words, if $\tilde{C}(t) \neq C(t)$ and $U[\tilde{C}(t)] > U[C(t)]$ where $C(t)$ is competitive necessarily implies that the L.H.S. of equation (5-9) with $C(t)$ replaced by $\tilde{C}(t)$ is negative.

Feasible Consumption Programs An infinite consumption program sequence $\{[C_1^2(0),\ C_2^2(0)];\quad C(t) = [C_1^1(t),\ C_2^1(t);\ C_1^2(t+1),\ C_2^2(t+1)]:\quad t=0,1,2,\ldots\}$ is said to be feasible if, for each good, aggregate demand equals aggregate supply and if, in addition, the demand for money equals the available stock at each instant of time, i.e.,

$$D_i(t) = X_i(t) \qquad i=1,2; \qquad t=0,1,2,\ldots$$

and

$$N^1(t)m(t) = M(t) \qquad\qquad t=0,1,2,\ldots$$

It can be deduced from the above conditions that

$$k(t)[w^1(t)-C_1^1(t)-p(t)C_2^1(t)] + [w^2(t)-C_1^2(t)-p(t)C_2^2(t)] = 0$$

$$t=0,1,2,\ldots \qquad (5\text{-}14)$$

and

$$m(t) = M_0\mu^t/N^1(t)$$

$$t=0,1,2,\ldots \qquad (5\text{-}15)$$

Equilibrium Consumption Programs A consumption program $C(t)$ is an equilibrium program if it is both competitive and feasible. It is equilibrium consumption programs that our attention will be focused upon.

## 5.3 ANALYSIS OF EQUILIBRIUM CONSUMPTION PROGRAMS

### 5.3.1 Equilibrium Conditions

An equilibrium of the economy in period $t=1,2,3,\ldots$ can be completely characterized by a set of equilibrium values $p^o(t)$, $q^o(t)$, $m^o(t)$ and $\overset{o}{C}(t)$. Maximizing $U[C(t)]$ subject to (5-9) and assuming that money is not worthless, for an internal competitive consumption program it is necessary that

$$q(t+1)/q(t) = [1+\pi_1(t)]^{-1}$$

$$= \frac{1}{(1+r)} \frac{\Omega_1}{\Omega_2} \frac{l[p(t+1)]}{l[p(t)]} \frac{g'[C^1(t)/l[p(t)]]}{g'[C^2(t+1)/l[p(t+1)]]}$$

$$t=1,2,3,\ldots \qquad (5\text{-}16)$$

where $\Omega_j = \Omega_j[C^1(t),\ C^2(t+1),\ p(t),\ p(t+1)]$ $(j=1,2)$, $C^1(t) = C_1^1(t)+p(t)C_2^1(t)$, $C^2(t+1) = C_1^2(t+1)+p(t+1)C_2^2(t+1)$, $g(.)$ and $l(p)$ are as defined in 3.3.2, and $\pi_1(t)$ is the rate of monetary inflation in terms of the first good in period t.

In equilibrium the commodity price ratio $p^{o}(t)$ is determined by interaction between demand and supply in the product markets, i.e., $D_i(t) = X_i(t)$ $(i=1,2;\ t=1,2,3,\ldots)$. It has been shown in Lemma 3.1 that under the strong assumption of homothetic preferences the introduction of money does not alter the total demand for each good in any period of time. Hence, for given utility and production functions, $p^{o}(t)$ is dependent only on the population endowment ratio $k(t)$, i.e., $p^{o}(t) = p^{o}[k(t)]$. Furthermore, the equilibrium stock-demand for money by a typical young man of the t-th generation is given by (5-15), $m^{o}(t) = M_0\mu^{t}/N^{1}(t)$. Thus, (5-17) reduces to

$$q^{o}(t+1)/q^{o}(t) = [1+\pi_1^{o}(t)]^{-1}$$

$$= \frac{1}{(1+r)}\,\frac{\Omega_1^{o}}{\Omega_2^{o}}\,\frac{l[p^{o}(t+1)]}{l[p^{o}(t)]}\,\frac{g'[\overset{o}{C}{}^{1}(t)/l[p^{o}(t)]]}{g'[\overset{o}{C}{}^{2}(t+1)/l[p^{o}(t+1)]]}$$

$$t=1,2,3,\ldots \qquad (5\text{-}17)$$

where $\Omega_j^{o}$ are evaluated at $\overset{o}{C}{}^{j}(t+j-1)$, $p^{o}(t+j-1)$ $(j=1,2;\ t=1,2,3,\ldots)$ and

$$\overset{o}{C}{}^{j}(t+j-1) = w^{j}[p^{o}(t+j-1)]+(-1)^{j}q^{o}(t+j-1)[M_0\mu^{t+j-2}/N^{1}(t)]$$

$$[\alpha(1+r)+(1-\alpha)\mu]$$

$$j=1,2;\quad t=1,2,3,\ldots \qquad (5\text{-}18)$$

is the j-th period equilibrium expenditure in terms of the first commodity by the young man $(j=1,2;\ t=1,2,3,\ldots)$.

His lifetime equilibrium budget constraint then becomes

$$[\mu q^{o}(t+1)/q^{o}(t)]\,[\overset{o}{w}{}^{1}(t)-\overset{o}{C}{}^{1}(t)]+[\overset{o}{w}{}^{2}(t+1)-\overset{o}{C}{}^{2}(t+1)] \;=\; 0$$

$$(q^{o}(t) \neq 0) \qquad\qquad t=1,2,3,\ldots \qquad\qquad (5\text{-}19)$$

It can be seen from equation (5-18) that the value of $q^{o}(t)$ is bounded by

$$-\{w^{2}[p^{o}(t)]N^{2}(t)\}/\{M_{0}\mu^{t-1}[\alpha(1+r)+(1-\alpha)\mu]\} \;\leq\; q^{o}(t) \;\leq$$

$$\{w^{1}[p^{o}(t)]N^{1}(t)\}/\{M_{0}\mu^{t-1}[\alpha(1+r)+(1-\alpha)\mu]\}$$

$$t=1,2,3,\ldots \qquad\qquad (5\text{-}20)$$

However, we are only interested in solution paths along which $q^{o}(t) > 0$ for all $t \geq 0$. Equation (5-17) is a first-order, nonlinear difference equation in $q^{o}(t)$. It is useful to rewrite (5-17) as

$$q^{o}(t) \;=\; (1+r)q^{o}(t+1)\;\frac{\Omega_{2}^{o}}{\Omega_{1}^{o}}\;\frac{l[p^{o}(t)]}{l[p^{o}(t+1)]}\;\frac{g'[\overset{o}{C}{}^{2}(t+1)/l[p^{o}(t+1)]]}{g'[\overset{o}{C}{}^{1}(t)/l[p^{o}(t)]]}$$

$$t=1,2,3,\ldots \qquad\qquad (5\text{-}17')$$

There is for any $q^{o}(t+1) > 0$ one and only one positive value of $q^{o}(t)$ that satisfies equation (5-17') because for a given positive $q^{o}(t+1)$ the L.H.S. of (5-17') can be represented by a $45^{o}$ straight line in the first quadrant of the $q^{o}(t)$ coordinate whereas the R.H.S. of (5-17') can be represented by a negatively sloping curve in the same quadrant. (Note that $\partial\Omega_{1}/\partial q^{o}(t) > 0$, $\partial\Omega_{2}/\partial q^{o}(t) < 0$, $\partial\overset{o}{C}{}^{1}(t)/\partial q^{o}(t) < 0$ and $g''(.) < 0$.) Consequently, once the positive values $q^{o}(0)$ and $q^{o}(1)$ are supplied, the infinite sequence $\{q^{o}(t):\; t=2,3,4,\ldots\}$ can be determined recursively by equation (5-17). Given $q^{o}(t)$ and $q^{o}(t+1)$, the values of $\overset{o}{C}{}^{j}(t+j-1)$ $(j=1,2;\ t=1,2,3,\ldots)$ can be computed by (5-18) and the equilibrium consumption program $\overset{o}{C}(t)$ is then given by

$$\overset{o}{C}{}_{1}^{j}(t+j-1) = \{1+p^{o}(t+j-1)\sigma[p^{o}(t+j-1)]\}^{-1}\overset{o}{C}{}^{j}(t+j-1)$$

$$\overset{o}{C}{}_{2}^{j}(t+j-1) = \sigma[p^{o}(t+j-1)]\overset{o}{C}{}_{1}^{j}(t+j-1)$$

$$j=1,2;\quad t=1,2,3,\ldots \tag{5-21}$$

where $\sigma(p)$ is as defined in equation (2-53).

### 5.3.2 Evolution of the Closed Monetary Economy

To examine the evolution of the economy over time it is necessary to consider the initial equilibrium of the economy. A typical member of the old generation at time $t=0$ wishes to maximize $u[C_1^2(0),\ C_2^2(0)]$ by the choice of $[C_1^2(0),\ C_2^2(0)]$ subject to (5-11) whereas a typical young man in the same period chooses $C(0)$ to maximize $U[C(0)]$ subject to (5-12) and the $t=0$ version of (5-8). If the young man decides to save against old age by trading some of his income for some of the money invented by the oldest generation then $q^{o}(0)$ and $q^{o}(1)$ are connected by the $t=0$ version of (5-17),

$$q^{o}(1)/q^{o}(0) = \frac{1}{(1+r)}\,\frac{\Omega_1^{o}}{\Omega_2^{o}}\,\frac{l[p^{o}(1)]}{l[p^{o}(0)]}\,\frac{g'[\overset{o}{C}{}^{1}(0)/l[p^{o}(0)]]}{g'[\overset{o}{C}{}^{2}(1)/l[p^{o}(1)]]} \tag{5-22}$$

where $\Omega_j^{o}$ $(j=1,2)$ are evaluated at $w^1[p^{o}(0)]$, $w^2[p^{o}(1)]$, $p^{o}(0)$, $p^{o}(1)$, $q^{o}(0)$ and $q^{o}(1)$. For a given $q^{o}(0)$ satisfying

(5-13) it may be taken for granted that there exists a unique positive value of $q^{o}(1)$ that solves (5-22) and is bounded from above by $\{N^{1}(1)w^{o}[p^{o}(1)]\}/\{M_{0}[\alpha(1+r)+(1-\alpha)\mu]\}$. Note that in equilibrium the lifetime budget constraint of a typical young person of the initial population is

$$\{[\alpha(1+r)+(1-\alpha)\mu]q^{o}(1)/q^{o}(0)\}[\overset{o}{w}{}^{1}(0)-\overset{o}{C}{}^{1}(0)]+[\overset{o}{w}{}^{2}(1)-\overset{o}{C}{}^{2}(1)] = 0 \tag{5-23}$$

which cannot be obtained from (5-19) by replacing t=0.

The remaining problem is to determine a value for $q^{o}(0)=q_{0}$. For a given positive value of $M_{0}$, any arbitrary choice of $q_{0}$ satisfying (5-13) is feasible; for, since $k(0) = N_{0}^{1}/N_{0}^{2}$, $k(0)[\overset{o}{w}{}^{1}(0)-\overset{o}{C}{}^{1}(0)]+[\overset{o}{w}{}^{2}(0)-\overset{o}{C}{}^{2}(0)] = [q_{0}M_{0}/N_{0}^{1}][k(0)/N_{0}^{1})-(1/N_{0}^{2})] = 0$. However, there is a problem of indeterminacy of $q_{0}$, as explained below. From the typical old man's point of view, his utility is maximized when $q_{0}$ takes the maximum allowable value $N_{0}^{1}w^{1}[p^{o}(0)]/M_{0}$, i.e., the old generation at time t=0 consumes all the commodities produced in that period. Naturally, the assumed properties of lifetime utility do not permit such a possibility, except in special cases discussed in the next section. Thus, it must be true that $q_{0} < N_{0}^{1}w^{o}[p^{o}(0)]/M_{0}$. Furthermore, an example in 5.5 will show that the choice of $q_{0}$ determines the dynamic stability of the monetary economy.

It has so far been assumed that money is not worthless in equilibrium. For any positive $q_{0}$ satisfying (5-13) it is possible that $U = \Omega[u(C_{1}^{1}(0), C_{2}^{1}(0)); u(C_{1}^{2}(1), C_{2}^{2}(1))] < U = \Omega[u(\bar{C}_{1}^{1}(0), \bar{C}_{2}^{1}(0)); u(\bar{C}_{1}^{2}(1), \bar{C}_{2}^{2}(1))]$ where C(0) satisfies (5-12) and the t=0 version of (5-8), $q_{0}$ and q(1) are connected by equation (5-22) and $\bar{C}(0)$ is the optimal autarkic lifetime consumption vector of a typical young man in period t=0, i.e., $\bar{C}^1_1(0)+p^o(0)\bar{C}^1_2(0) = \overset{o}{w}{}^1(0) > \overset{o}{C}{}^1(0)$ and $\bar{C}^2_1(1)+p^o(1)\bar{C}^2_2(1) = \overset{o}{w}{}^2(1) < \overset{o}{C}{}^2(1)$. (It should be noted that equation (5-22) is only a necessary but not sufficient condition for utility maximization.) This could be true if a person discounts his future satisfaction to the extent that his lifetime utility can only be improved by reallocating consumption goods from the low-marginal-utility second period to the high-marginal-utility first period. Thus, having perfect knowledge about the present and the future, this typical man will not wish to hold any money in his youth at all, i.e., he will place zero value in $q_0$. Obviously, by allowing $q(t) < 0$ for all $t \geq 0$, so that money is equivalent to a kind of I.O.U., it is possible to trace out the time path of $q^o(t)$ and therefore the evolution of the economy. But if $q_0$ is negative, and keeping in mind that bequests to his offspring give him no utility, it is difficult to see why the typical old man in period t=0 should sacrifice some of his income to the younger generation alive at the same time. This may be summarized as follows.

Theorem 5.1 If a decentralized economy is Classical in the sense that a man can only improve his lifetime utility by consuming more than his income when young then money is worthless along the equilibrium time path of the economy, $q^o(t) = 0$ for all $t \geq 0$. It is therefore natural to term an economy Classical barter if the price of money is zero and Samuelson monetary if it is positive.

## 5.4 ANALYSIS OF STEADY-STATE EQUILIBRIA

The economy is said to have attained a stationary equilibrium from period $t_0$ onwards if the lifetime consumption profile of a person born in period $t \geq t_0$ is independent of the passage of time, i.e., $\overset{\circ}{C}^j(t+j-1) = \overset{\circ}{C}^j$ $(j=1,2;$ $t=t_0,t_0+1,t_0+2,\ldots)$. This necessarily requires that the age structure of the population has been stabilized, i.e., $k(t)=k$ for all $t \geq t_0$ and the amount of saving by a typical young man is a constant for all $t \geq t_0$. Then it is possible to prove

<u>Theorem 5.2</u> There are at most two steady-state equilibria. They are characterized by

(I) $\quad q^o(t+1) = (k/\mu)q^o(t) \qquad t=t_0,t_0+1,\ldots$

(II) $\quad \overset{\circ}{C}^j = \overset{\circ}{w}^j \qquad j=1,2;\ t=t_0,t_0+1,\ldots$

<u>Proof</u> Substracting equation (5-14) from equation (5-19), keeping in mind that $k(t)=k$, $w^j(t)=w^j$ and $C^j(t)=C^j$, we have

$$\{[\mu q^o(t+1)/q^o(t)]-k\}\ [\overset{\circ}{w}^1-\overset{\circ}{C}^1] = 0$$

which is the desired result. Q.E.D.

In case (II) we have a Classical barter equilibrium in which money is worthless and every person consumes exactly his income in each period of his life. In case (I), we have a Samuelson monetary equilibrium in which the price of money (in terms of either good) is changing geometrically at the constant rate $(k/\mu)-1$ over time. Thus,

$$q^o(t) = q^o(t_0)(k/\mu)^{t-t_0} \qquad t=t_0,t_0+1,\ldots \qquad (5\text{-}24)$$

and

$$z^{o}(t) = [q^{o}(t_0)/p^{o}(k)](k/\mu)^{t-t_0}$$

$$t=t_0, t_0+1, \ldots$$

where z(t) is the price of money in terms of the second commondity in period t and $p^{o}(k)$ is the stationary equilibrium commodity price ratio. This is simply a generalization of an earlier result found by Kemp and Long. (See, for example, [16].) Combining (I) and a stationary version of (5-17) yields

$$k = \frac{\mu}{(1+r)} \frac{\Omega_1^{o}}{\Omega_2^{o}} \frac{g'[\overset{o}{C}{}^{1}/l(p^{o})]}{g'[\overset{o}{C}{}^{2}/l(p^{o})]}$$

$$t=t_0, t_0+1, \ldots \qquad (5\text{-}25)$$

where $\Omega_j^{o}$ (j=1,2) are evaluated at $\overset{o}{w}{}^{j}$ (j=1,2), $p^{o}(k)$, $q^{o}(t)$ and $q^{o}(t+1)$. Equation (5-25) is in no way a novel finding as it simply expresses a necessary condition for lifetime utility maximization and a simpler version of (5-25) has been derived earlier in Theorem 3.3.

In the Samuelson monetary steady-state equilibrium the real value of the stock of money is $M(t_0)[q^{o}(t)/(k/\mu)^{t}] = M(t_0)q^{o}(t_0) = M_0\mu^{t_0}q^{o}(t_0)$ but the price of money remains unchanged only in the special case where $k=\mu$. In general the price of money will approach infinity or zero as time grows indefinitely large depending on whether k is greater or smaller than $\mu$.

<u>Theorem 5.3</u> The steady-state equilibrium rate of inflation (in terms of either commodity) is positive or negative as the rate of change of the stock of money exceeds or falls short of the rate of change of the population but is in

general not equal to the simple difference between $\mu$ and k.

<u>Proof</u> It is clear from (I) that

$$1+\pi_1^o(t) = q^o(t)/q^o(t+1) = \mu/k \qquad t=t_0,t_0+1,\ldots$$

Furthermore,

$$\begin{aligned} 1+\pi_2^o(t) &= [q^o(t)/p^o(t)]/[q^o(t+1)/p^o(t+1)] \\ &= 1+\pi_1^o(t) \qquad\qquad t=t_0,t_0+1,\ldots \end{aligned}$$

as $p^o(t) = p^o(k)$ for all $t \geq t_0$ where $\pi_2^o(t)$ is the equilibrium rate of inflation in terms of the second commodity in period t. Hence

$$\pi_1^o(t) = \pi_2^o(t) = \pi^o = (\mu-k)/k \qquad t=t_0,t_0+1,\ldots \qquad (5\text{-}26)$$

$\pi^o \gtreqless 0$ if and only if $\mu \gtreqless k$. Q.E.D.

When the population is growing, the rate of inflation is smaller in magnitude than $\mu$-k; and when the population is declining, the rate of inflation is greater in magnitude than $\mu$-k. In other words, population growth blunts the response of the rate of inflation to changes in the rate of monetary growth, and the decay of population sharpens the response. Only in the singular case of a stationary population (k=1) does the crude quantity formula $\pi = \mu-k$ hold. Figure 5.1 provides a simple illustration of the conclusion.

<u>Cases not covered by equation (5-25)</u>

It has been noted in 3.3 that if U takes the additive form $U = u(C_1^1,C_2^1)+\delta u(C_1^2,C_2^2)$ where $\delta > 0$ and u exhibits constant-returns-to-scale (i.e., u is not strictly concave) then the lifetime consumption profile of a typical person is an extreme solution in the sense that he will give up completely the consumption of both goods in one period of his life. In this case equation (5-25) is neither true nor necessary for the maximization of U. However, if the economy is Samuelson, i.e., $\overset{o1}{C} = 0$ and $\overset{o2}{C} = w^2(p^o)+kw^1(p^o)$ then part (I) of Theorem 5.2 still holds true. By equating $\overset{o1}{C} = w^1(p^o)-q^o(t)\{M_0\mu^{t-1}/N^1(t)\}[\alpha(1+r)+(1-\alpha)\mu]$ to zero one can show that

$$q^o(t) = \{[\mu N^1(t_0)w^1(p^o)]/[M(t_0)[\alpha(1+r)+(1-\alpha)\mu]]\}(k/\mu)^{t-t_0} \qquad t=t_0,t_0+1,\ldots \tag{5-27}$$

Equation (5-27) is actually a special case of (5-24) in which

$$q^o(t_0) = [\mu N^1(t_0)w^1(p^o)]/\{M(t_0)[\alpha(1+r)+(1-\alpha)\mu]\}.$$

<u>Theorem 5.4</u> Provided that the age structure will become stabilized a Samuelson monetary economy can always attain a steady-state equilibrium if $q_0$ is chosen appropriately.

<u>Proof</u> Suppose that $k(t)=k$ for all $t \geq t_0$. If a Samuelson monetary economy is stationary then the equilibrium amount of saving in terms of the first commodity by a person born in period $t \geq t_0$ is independent of time, i.e.,

$$\begin{aligned} s^o(t) &= s^o = q^o(t)[M_0\mu^{t-1}/N^1(t)][\alpha(1+r)+(1-\alpha)\mu] \\ &= q^o(t_0)[M(t_0)/(\mu N^1(t_0))][\alpha(1+r)+(1-\alpha)\mu] \end{aligned}$$

since $q^o(t) = q^o(t_0)(k/\mu)^{t-t_0}$. Therefore,

$$q^{o}(t_o) \quad = \quad [\mu N^{1}(t_0)s^{o}]/\{M(t_0)[\alpha(1+r)+(1-\alpha)\mu]\} \tag{5-28}$$

where $s^{o}$ is the optimal saving that solves

$$k \quad = \quad \frac{\mu}{(1+r)} \frac{\Omega_1}{\Omega_2} \quad \frac{g'[(\overset{o}{w}{}^{1}-s)/l(p^{o})]}{g'[(\overset{o}{w}{}^{2}+ks)/l(p^{o})]}$$

Knowing the value of $q^{o}(t_0)$ it is possible to use equation (5-17) recursively to determine $q^{o}(1)$ and then (5-22) to deduce the corresponding value for the initial price of money in terms of the first good $q_0$. Q.E.D.

If the population in period t=0 by sheer accident chooses $q_0$ in such a way that $q^{o}(t_0)$ is equal to the R.H.S. of (5-28) then the monetary system is stable in the sense that the economy will reach a steady state in which the real value of the stock of money remains constant. Otherwise, the monetary economy is unstable, as will be shown in Example 5.5.

Theorem 5.5 Let the steady-state optimal saving $s^{o}$ be dependent on k in sign. Then a steady-state economy may, in response to a change in k, change from being Samuelson monetary to being Classical barter and vice versa.

Proof Consider a steady-state Samuelson monetary economy. The optimal saving $s^{o}$ by a typical young man is dependent on k for given utility and production functions. Furthermore, if the sign of $s^{o}$ is determined by the magnitude of k then a change in k may cause the value of $s^{o}$ to become negative. Consequently, the economy may become Classical barter and money is no longer accepted by the young people as a means of saving. Q.E.D.

The following example is intended to provide a simple illustration of the above theorem. Suppose that

$$U(C) = (C_1^1)^{\beta_1}(C_2^1)^{\beta_2} + \delta(C_1^2)^{\beta_1}(C_2^2)^{\beta_2}$$

and

$$X_i = (L_{i1})^{\alpha_i}(L_{i2})^{1-\alpha_i}$$

where $\delta > 0$, $0 < \beta_i < 1$ $(i=1,2)$, $\beta_1+\beta_2 < 1$ and $0 < \alpha_i < 1$ $(i=1,2)$.

Then one can after some calculation show that

$$s^o = \{[\delta(1+r)k/\mu]^{1/(1-\beta_1-\beta_2)}\overset{o}{w}{}^1 - \overset{o}{w}{}^2\}/\{k+[\delta(1+r)k/\mu]^{1/(1-\beta_1-\beta_2)}\}$$

$$= \overset{o}{w}{}^1\{[\delta(1+r)k/\mu]^{1/(1-\beta_1-\beta_2)} - Kk\}/\{k+[\delta(1+r)k/\mu]^{1/(1-\beta_1-\beta_2)}$$

where $K = [(1-\alpha_1)\beta_1+(1-\alpha_2)\beta_2]/(\alpha_1\beta_1+\alpha_2\beta_2)$.

Let $k^* = \{\mu K^{(1-\beta_1-\beta_2)}/[\delta(1+r)]\}^{1/(\beta_1+\beta_2)}$. Then $q^o(t) > 0$ if and only if $k > k^*$ and $q^o(t) = 0$ if and only if $k \leq k^*$. Thus, $k^*$ is the monetary-barter-switching population endowment ratio.

<u>Theorem 5.6</u> In a Samuelson monetary steady state the level of lifetime utility enjoyed by a typical person reaches a maximum when the rate of monetary expansion is equal to the rate of interest, i.e., $\mu-1 = r$.

<u>Proof</u> Rewrite U in terms of the indirect utility function, $U = \Omega[g[(\overset{o}{w}{}^1-s)/l(p^o)];\ g[(\overset{o}{w}{}^2+ks)/l(p^o)]]$. It is evident from (5-25) that

$$dU/d\mu = [1/l(p^o)]\{\Omega_1^o g'[(\overset{o}{w}{}^1 - s)/l(p^o)]\}\{[\mu/(1+r)]-1\}(ds/d\mu) \tag{5-29}$$

Clearly, $dU/d\mu = 0$ if and only if $\mu = 1+r$ and a similar result can be obtained for $dU/dr$. Q.E.D.

Strictly speaking, it is necessary to show that U is concave in $\mu$. However, this result is not surprising in view of Theorem 3.3. For each feasible value of $\mu$, there is a different steady state. The steady states differ both in $q^o(t)$ and the profile of consumption enjoyed by a typical member of the population. When $\mu-1 = r$ in a steady state, the real rate of interest becomes $[(1+r)q^o(t+1)/q^o(1)]-1 = [(1+r)k/\mu]-1 = k-1$.

## 5.5 AN EXAMPLE

We now specialize the utility and production functions to derive explicitly the equilibrium time path of $q(t)$. Necessary and sufficient conditions for the existence of a Samuelson monetary steady-state solution as well as properties of nonsteady-state solutions will also be examined.

Let the lifetime utility U be additively separable and the j-th period utility function u take the logarithmic form,

$$U[C(t)] = u[C_1^1(t), C_2^1(t)] + \delta u[C_1^2(t+1), C_2^2(t+1)], \quad \delta > 0 \tag{5-30}$$

and

$$u[C_1^j(t), C_2^j(t)] = \beta \log_e C_1^j(t) + (1-\beta)\log_e C_2^j(t), \qquad 0<\beta<1$$

$$j=1,2; \; t=0,1,2,\ldots \tag{5-31}$$

The production functions are of Cobb-Douglas form

$$X_i(t) = [L_{i1}(t)]^{\alpha_i}[L_{i2}(t)]^{1-\alpha_i}$$

$$i=1,2;\ t=0,1,2,... \qquad (5\text{-}32)$$

The indirect utility function in the j-th period is then

$$v[p^o(t),c^j(t)] = \log_e[c^j/[A[p^o(t)]^{1-\beta}]]$$

$$j=1,2;\ t=0,1,2,... \qquad (5\text{-}33)$$

where $A = \beta^{-\beta}(1-\beta)^{\beta-1}$. Equations (5-17) and (5-22) then become respectively,

$$q^o(t+1)/q^o(t) = \frac{\mathring{w}^2(t+1)+[Dq^o(t+1)M(t)/N^1(t)]}{B\{\mathring{w}^1(t)-[Dq^o(t)M(t-1)/N^1(t)]\}}$$

$$t=1,2,3,... \qquad (5\text{-}34)$$

$$q^o(1)/q_0 = \frac{\mathring{w}^2(1) + [Dq^o(1)M_0/N^1{}_0]}{B\{\mathring{w}^1(0) - [q_0M_0/N_0^1]\}} \qquad (5\text{-}35)$$

where $B = (1+r)\delta$ and $D = \alpha(1+r)+(1-\alpha)\mu$. The equilibrium time path of $q(t)$ is characterized by

$$q^o(t+1) = \begin{cases} [q_0\mathring{w}^2(1)]/[B\mathring{w}^1(0)-(B+D)(q_0M_0/N_0^1)] & t=0 \\ [q^o(t)\mathring{w}^2(t+1)]/[B\mathring{w}^1(t)-(B+\mu)Dq^o(t)M(t-1)/N^1(t)] & \end{cases}$$

$$t=1,2,3,... \qquad (5\text{-}36)$$

The economy is Samuelson monetary, i.e., $q_0 > 0$ if and only if

$$\delta > \{\mu[1-\alpha_2+\beta(\alpha_2-\alpha_1)]\}/\{(1+r)[\alpha_2-\beta(\alpha_2-\alpha_1)]\} \qquad (5\text{-}37)$$

For a finite solution $q^o(t)$ must satisfy the following inequalities

$$q^o(t) < \begin{cases} [BN_0^1\overset{o}{w}^1(0)]/[(B+D)M_0] & t=0 \\ [BN^1(t)\overset{o}{w}^1(t)]/[(B+\mu)DM(t-1)] & t=1,2,3,\ldots. \end{cases} \qquad (5\text{-}38)$$

Suppose now that $k(t) = k$ for all $t \geq t_0 > 0$. Defining $\tau(t) = q^o(t)/(k/\mu)^t$, equation (5-34) reduces to

$$\tau(t+1)/\tau(t) = (\mu/B)[\overset{o}{w}^2+E\tau(t+1)]/[k\overset{o}{w}^1-E\tau(t)] \qquad t=t_0,t_0+1,\ldots \qquad (5\text{-}39)$$

where $E = D[M(t_0)/N^1(t_0)](k/\mu)^{t_0+1}$. The first-order, non-linear difference equation (5-39) has a constant solution

$$\bar{\tau} = [Bk\overset{o}{w}^1-\mu\overset{o}{w}^2]/[E(B+\mu)] \qquad (5\text{-}40)$$

It is not too difficult to see that $\bar{\tau} = s^o$, the optimal saving in terms of the first good of a young man born in period $t \geq t_0$, and that $\bar{\tau} > 0$ if and only if (5-37) holds true. Furthermore, if $\tau(t_0)$ is chosen equal to $\bar{\tau}$, i.e.,

$$q^o(t_0) = [\mu N^1(t_0)(Bk\overset{o}{w}^1-\overset{o}{w}^2)]/[D(B+\mu)kM(t_0)] \qquad (5\text{-}41)$$

then the economy attains a steady-state equilibrium from period $t_0$ onwards. In summary, the economy described by (5-30), (5-31) and (5-32) will reach a Samuelson monetary steady state if and only if the following three conditions are satisfied: (a) $k(t) = k$ for all $t \geq t_0 > 0$, (b) equation (5-37) is valid, and (c) $q_0$ is chosen in such a way that (5-41) holds true. In such a case the economy is stable in the sense that the real value of the stock of money $M(t_0)\tau(t)$ remains constant for all $t \geq t_0$. Note that in this particular example the sign of $\bar{\tau}$ is independent of the magnitude of k and Theorem 5.5 is therefore not applicable.

What happens to the monetary system if the economy is Samuelson and the age structure is stable but condition (c) is not satisfied?

From (5-39),

$$\tau(t+1) \;=\; [\mu\overset{o}{w}{}^2\tau(t)]/[Bk\overset{o}{w}{}^1 - E(B+\mu)\tau(t)] \;=\; \psi[\tau(t)]$$

$$t=t_0, t_0+1, \ldots \qquad (5\text{-}42)$$

Since $\psi(0) = 0$, $\delta\psi[\tau(t)]/\delta\tau(t) = (\mu Bk\overset{o}{w}{}^1\overset{o}{w}{}^2)/[Bk\overset{o}{w}{}^1 - E(B+\mu)\tau(t)]^2 > 0$ and $\delta^2\psi[\tau(t)]/\delta\tau(t)^2 = [2\mu B(B+\mu)Ek\overset{o}{w}{}^1\overset{o}{w}{}^2]/[Bk\overset{o}{w}{}^1 - E(B+\mu)\tau(t)]^3 > 0$, the graph of $\tau(t)$ has the properties displayed in Figure 5.2. It is obvious that $q^o(t_0)$ is not equal to the R.H.S. of (5-41) then the path of $\{\tau(t)\}$ diverges steadily from $\bar{\tau}$. If $\tau(t_0) < \bar{\tau}$, $\tau(t)$ goes to zero, implying that $(k/\mu)^t$ grows faster than $q^o(t)$. Otherwise, if $\tau(t) > \bar{\tau}$ then, for some $t > t_0$, $\tau(t)$ exceeds $(Bk\overset{o}{w}{}^1)/[E(B+\mu)]$ and only a negative value of $\tau(t+1)$ satisfies (5-42). Thus, perfect myopic foresight is consistent with money prices growing indefinitely at rates less than $\pi^o = (\mu-k)/k$ but not with money prices growing indefinitely at rates greater than $\pi^o$. In the latter case, the real value of the stock of money $M(t_0)\tau(t)$ eventually becomes so large (larger than or equal to $[Bk\overset{o}{w}{}^1 M(t_0)]/[E(B+\mu)]$) that the demand for money by the young people is less than the available supply for all positive $\tau(t+1)$. Once $\tau(t)$ enters the region $[(Bk\overset{o}{w}{}^1)/[E(B+\mu)], \infty)$, money becomes worthless, i.e., the monetary system collapses and the economy turns from being Samuelson monetary into being Classical barter.

Three points deserve mention here. First, the above results are of broader generality than the assumption of logarithmic utility might suggest. Second, the analysis can be easily extended to take into account the possibility of technical progress. For example, if both $\overset{o}{w}{}^1$ and $\overset{o}{w}{}^2$ grow autonomously at the same rate $\lambda-1 > 0$ then by redefining $\tau(t)$ as $q^o(t)/(\lambda k/\mu)^t$ equation (5-39) carries over. Finally, if condition (c) does not hold then the monetary system is dynamically stable if and only if $\psi[\tau(t)]$ is a positively increasing and concave function of $\tau(t)$.

## 5.6 CONCLUSIONS AND HISTORICAL NOTES

In summary, there exist in an overlapping-generations model two possible equilibria, Classical barter or Samuelson monetary, depending on the parameters and the rate of growth of the population of the model. Money is worthless along the equilibrium time path of a Classical barter economy and, consequently, no form of intergenerational borrowing and lending will ever take place in such an economy. On the other hand, the Samuelson monetary economy, for a class of utility functions, is unstable in the sense that, away from the steady state, either the real value of the stock of money goes to zero or the monetary system breaks down altogether. It is also shown that, in general, maximizing behaviour by many economic agents blessed with perfect but myopic foresight does not lead to stationarity. Finally, the results of the analysis in this chapter are in many ways straightforward generalizations of those discovered by Kemp [15] and Kemp and Long [16].

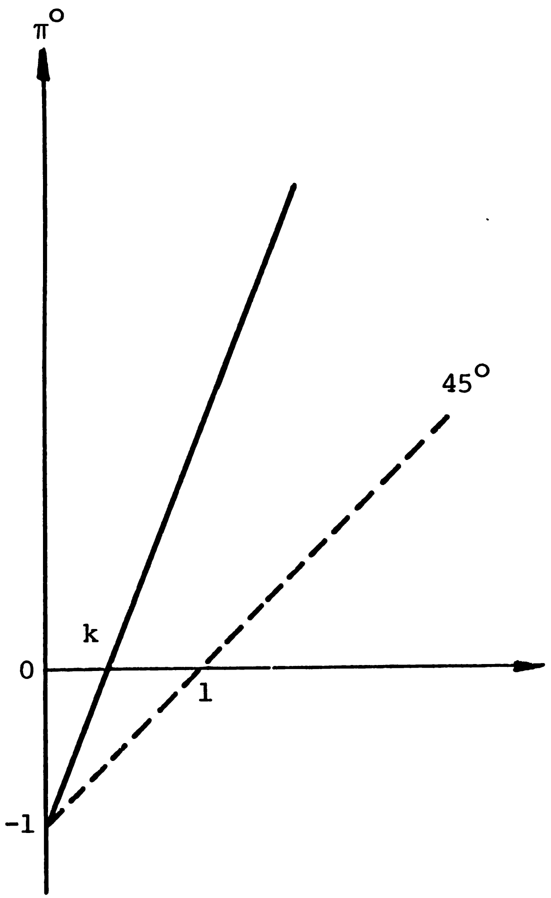



Figure 5.1.a: k < 1

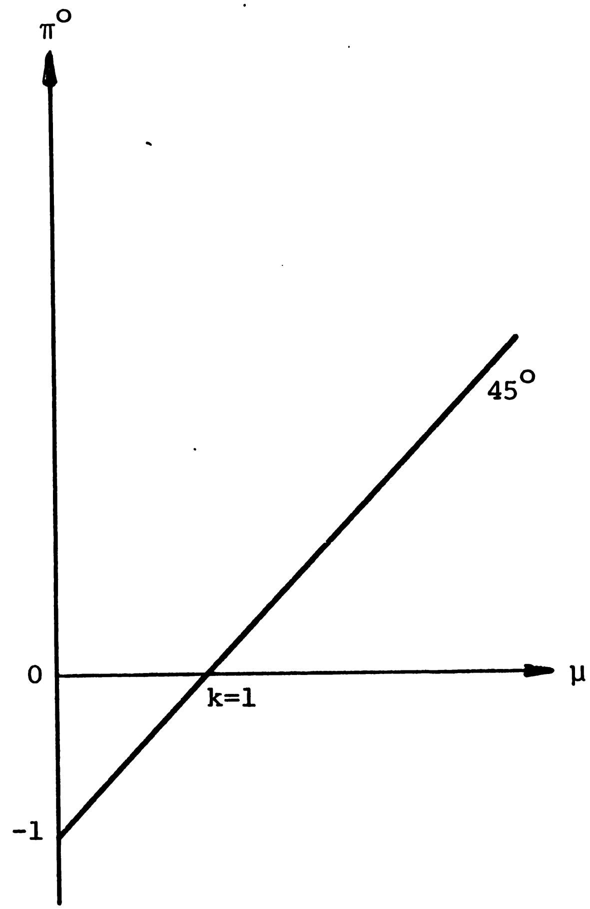



Figure 5.1.b: k = 1

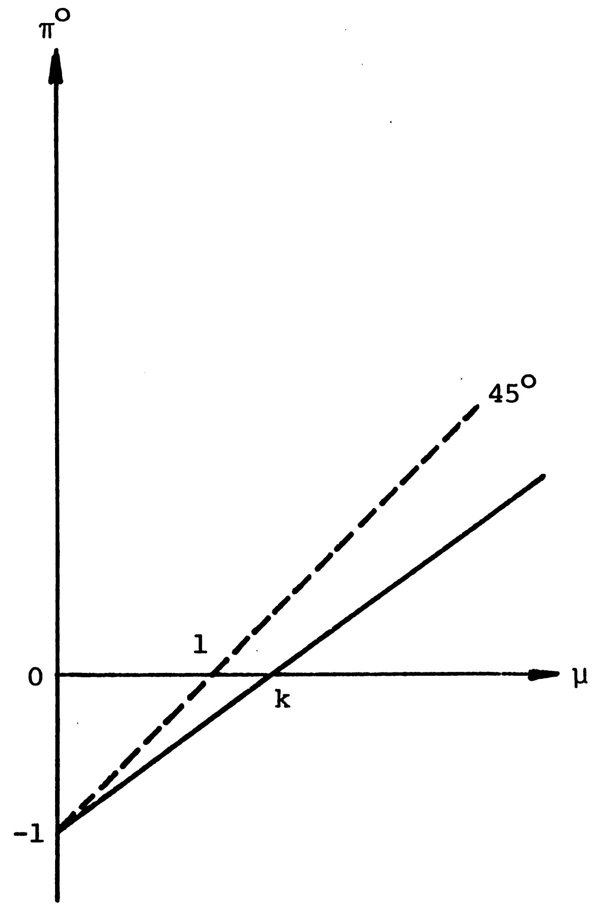



Figure 5.1.c: k > 1

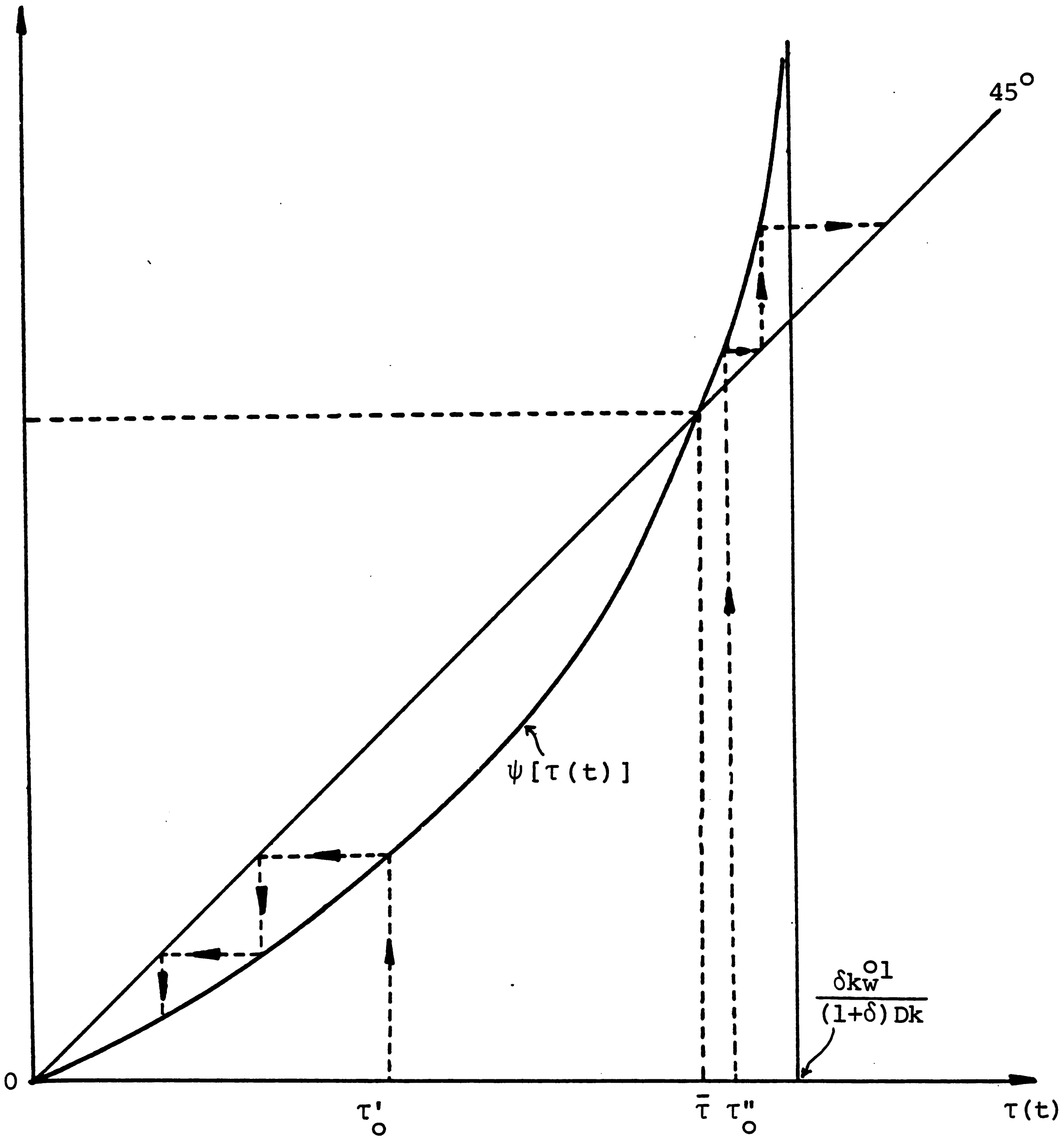



Figure 5.2: Dynamic stability of the monetary system

# CHAPTER VI

# OPEN ECONOMIES WITH FIAT MONIES AND OVERLAPPING GENERATIONS

## 6.1 INTRODUCTION

In Chapter V a model of a closed, monetary, two-good-two-factor economy was developed. The model incorporated a population of overlapping, life-cycle-maximizing generations, each blessed with complete knowledge over its own lifetime. It was shown that the monetary system is less general than intergenerational borrowing and lending and that the money price is positive if and only if the economy is Samuelson. Furthermore, for a class of utility functions, money prices are shown to be unstable. Away from the unique steady state, either the real value of the stock of money approaches zero or the monetary system collapses once the real money price becomes sufficiently high. In the latter event, the assumption of perfect myopic foresight is revealed to be incompatible with the assumption that money prices be nonnegative.

The analysis can be extended by considering a world economy made up of two (or any greater number of) trading countries, into which has been introduced the social contrivance of not one but two (or several) fiat monies. Each of the countries of the world economy has a government which controls the supply of its own fiat money and conducts its own monetary-fiscal policy. Under a world laissez-faire trading regime, every individual is free to purchase either or both of the two fiat monies. Thus, this world economy can be thought of as the Heckscher-Ohlin generalization of Samuelson's (1958) overlapping-generations model.

## 6.2 ASSUMPTIONS AND GENERAL FORMULATION

It suffices to assume that the world economy consists of only two countries, home and foreign. (For notational convenience, variables associated with the foreign country will be denoted by asterisks.) The two countries possess the same constant-returns-to-scale technology for each industry and have identical homothetic preferences; however they need not have the same population endowment ratios. Without loss of generality it is assumed that the age structures are stable in both countries, i.e.,

$$k(t) = N^1(t)/N^2(t) = k > 0 \qquad t=0,1,2,... \tag{6-1-a}$$

$$k^*(t) = N^{1*}(t)/N^{2*}(t) = k^* > 0 \qquad t=0,1,2,... \tag{6-1-b}$$

Suppose that international trade in commodities takes place from period t=0 onwards. Then a direct consequence of (6-1-a) and (6-1-b) is that the international terms of trade in the product markets and factor rewards in both countries are independent of the passage of time,

$$\hat{p}[k(t),k^*(t)] = \hat{p}(k,k^*) \qquad t=0,1,2,... \tag{6-2}$$

$$w^j[\hat{p}[k(t),k^*(t)]] = w^j[\hat{p}(k,k^*)] \qquad j=1,2;\ t=0,1,2,... \tag{6-3-a}$$

$$w^{j*}[\hat{p}[k(t),k^*(t)]] = w^{j*}[\hat{p}(k,k^*)] \qquad j=1,2;\ t=0,1,2,... \tag{6-3-b}$$

### 6.2.1 Money Stocks and Monetary Transfers

The monetary system and monetary-fiscal policy in each country are as described in 5.2.1. Let $M(t)$ and $M^*(t)$ denote respectively the stock of money in the home and foreign countries during the t-th period. Then,

$$M(t) = M_0\mu^t \qquad t=0,1,2,... \qquad (6\text{-}4\text{-}a)$$

$$M^*(t) = M_0^*(\mu^*)^t \qquad t=0,1,2,... \qquad (6\text{-}4\text{-}b)$$

where $\mu$ and $\mu^*$ are both positive. Let the rates of interest in the home and foreign countries be $r$ ($r > -1$) and $r^*$ ($r^* > -1$), respectively. Then the interest payments in the home and foreign countries from period t-1 to period t are, respectively

$$rM(t-1) = rM_0\mu^{t-1} \qquad t=1,2,3,... \qquad (6\text{-}5\text{-}a)$$

$$r^*M^*(t-1) = r^*M_0^*(\mu^*)^{t-1} \qquad t=1,2,3,... \qquad (6\text{-}5\text{-}b)$$

The home government's monetary-fiscal policy is such that the transfers (or taxes) to a typical person born in period t are

$$\alpha(\mu-1-r)M_0\mu^{t-1}/N^1(t) = \alpha\eta M_0\mu^{t-1}/N^1(t)$$
$$t=1,2,3,... \qquad (6\text{-}6\text{-}a)$$

when he is young and

$$(1-\alpha)(\mu-1-r)M_0\mu^t/N^1(t) = (1-\alpha)\eta M_0\mu^t/N^1(t)$$
$$t=1,2,3,... \qquad (6\text{-}6\text{-}b)$$

when he is old ($0 \leq \alpha \leq 1$). Similarly, a typical man born in period t in the foreign country receives (or pays)

$$\alpha^*(\mu^*-1-r^*)M_0^*(\mu^*)^{t-1}/N^{1*}(t) = \alpha^*\eta^*M_0^*(\mu^*)^{t-1}/N^{1*}(t) \qquad t=1,2,3,\ldots \qquad (6\text{-}7\text{-}a)$$

when young and

$$(1-\alpha^*)(\mu^*-1-r^*)M_0^*(\mu^*)^{t}/N^{1*}(t) = (1-\alpha^*)\eta^*M_0^*(\mu^*)^{t-1}/N^{1*}(t) \qquad t=1,2,3,\ldots \qquad (6\text{-}7\text{-}b)$$

when old from his own government ($0 \leq \alpha^* \leq 1$).

### 6.2.2 Budget Constraints Reformulated

Suppose that members of the oldest generation (i.e., old people in period t=0) in each country own (in equal amounts) all of the initial stock of the money of the country concerned. Then, for each country, it is necessary to describe the budget constraints faced by the initial population and those faced by the succeeding generations. Because of the symmetry of the home and foreign countries, we shall first concentrate on the population of the home country. Consider a typical person born in period $t \geq 1$ in the home country. In post-trading equilibrium, his first-period income is $w^1(t) = w^1[\hat{p}(k,k^*)]$ where $\hat{p}(k,k^*)$ is the international commodity price ratio. To ensure that free trade in commodities is Pareto non-inferior to autarky this typical man has to pay (or receive) a tax whose value in terms of the first good is $\tilde{f}$. (For a discussion of $\tilde{f}$, refer to 4.3.3.) Under a cooperative laissez-faire exchange regime, he is free to purchase the fiat money from his own government and/or the money from the foreign country. Let $m(t)$ and $m_f(t)$ be respectively his stock-demands for monies of his own and of the foreign government. Then in the first period of his life he faces the constraint

$$C_1^1(t)+p(t)C_2^1(t) = [w^1(t)-\tilde{f}]+q(t)\{[\alpha\eta M_0\mu^{t-1}/N^1(t)]-m(t)\}-q^*(t)m_f(t)$$
$$t=1,2,3,\ldots \qquad (6\text{-}8\text{-}a)$$

where $q(t)$ and $q^*(t)$ are the t-th period prices of home and foreign monies in terms of the first commodity, respectively. Defining $C^1(t) = C_1^1(t)+p(t)C_2^1(t)$, the total expenditure in terms of the first good by a typical young man of period t, $\tilde{w}^1(t) = w^1(t)-\tilde{f}$, his compensated income, and $e(t) = q^*(t)/q(t)$, the t-th period exchange rate that measures the price of the foreign-country money in units of the home-country money. Then (6-8-a) reduces to

$$\begin{aligned} C^1(t) &= \tilde{w}^1(t) + q(t)\{[\alpha\eta M_0\mu^{t-1}/N^1(t)]-[(m(t)+e(t)m_f(t)]\} \\ &= \tilde{w}^1(t) + q(t)\{[\alpha\eta M_0\mu^{t-1}/N^1(t)]-Z(t)\} \end{aligned}$$
$$t=1,2,3,\ldots \qquad (6\text{-}8'\text{-}a)$$

where $Z(t) = m(t)+e(t)m_f(t)$ is the total demand for money by a young man born in period t in the home country, measured in terms of his own country's money. In the next period, the same man is constrained by

$$\begin{aligned} C_1^2(t+1)+p(t+1)C_2^2(t+1) = {} & [w^2(t+1)+k\tilde{f}]+q(t+1)\{[(1-\alpha)\eta M_0\mu^t/N^1(t)] \\ & +(1+r)m(t)\}+q^*(t+1)(1+r^*)m_f(t) \end{aligned}$$
$$t=1,2,3,\ldots \qquad (6\text{-}9\text{-}a)$$

Defining $C^2(t+1) = C_1^2(t+1)+p(t+1)C_2^2(t+1)$, $\tilde{w}^2(t+1) = w^2(t+1)+k\tilde{f}$.

Then (6-9-a) becomes

$$\begin{aligned} C^2(t+1) = {} & \tilde{w}^2(t+1)+q(t+1)\{[(1-\alpha)\eta M_0\mu^t/N^1(t)]+(1+r)m(t)+(1+r^*) \\ & e(t+1)m_f(t)\} \qquad t=1,2,3,\ldots \qquad (6\text{-}9'\text{-}a) \end{aligned}$$

The choice of a typical old person in the home country at time t=0 is subject to

$$C^2(0) = w^2(0)+q_0M_0/N_0^2 \tag{6-10-a}$$

whereas a typical young man alive in the same period is constrained by

$$C^1(0) = w^1(0)-q_0m(0)-q_0^*m_f(0)$$

$$= w^1(0)-q_0Z(0) \tag{6-11-a}$$

and by the t=0 version of equation (6-9'-a). Having established the budget equations for the population of the home country, it is possible to deduce the same for the population of the foreign country. Let $m^*(t)$ and $m_h^*(t)$ be respectively the stock-demands for the monies of his own government and of the home government by a typical person born in the foreign country at time t $\geq$ 1. Then his budget constraints in period t and t+1 are, respectively,

$$C^{1*}(t) = \tilde{w}^{1*}(t)+q^*(t)\{[\alpha^*\eta^*M_0^*(\mu^*)^{t-1}/N^{1*}(t)-Z^*(t)\} \qquad t=1,2,3,... \tag{6-8-b}$$

$$C^{2*}(t+1) = \tilde{w}^{2*}(t+1)+q^*(t+1)\{[(1-\alpha^*)\eta^*M_0^*(\mu^*)^t/N^{1*}(t)] +(1+r^*)m^*(t)+(1+r)m_h^*(t)/e(t+1)\} \qquad t=1,2,3,... \tag{6-9-b}$$

where

$$C^{1*}(t) = C_1^{1*}(t)+p(t)C_2^{1*}(t) ,$$

$$C^{2*}(t+1) = C_1^{2*}(t+1)+p(t+1)C_2^{2*}(t+1) ,$$

$$\tilde{w}^{1*}(t) = w^{1*}(t)-\tilde{f}^* ,$$

$$\tilde{w}^{2*}(t+1) = w^{2*}(t+1)+k^*\tilde{f}^* , \text{ and}$$

$$Z^*(t) \quad = \quad m^*(t)+[m_h^*(t)/e(t)]$$

The typical member of the old generation in period t=0 in the foreign country wishes to maximize his second-period utility subject to

$$C^{2*}(0) = w^{2*}(0)+q_0^* M_0^*/N_0^{2*} \qquad \text{(6-10-b)}$$

while a young man living at the same time is subject to

$$C^{1*}(0) = w^{1*}(0)-q_0^* Z^*(0) \qquad \text{(6-11-b)}$$

and to the t=0 version of the (6-9-b).

### 6.2.3 Definitions

Competitive Consumption Programs An infinite sequence $\{[C_1^2(0), C_2^2(0)], [C_1^{2*}(0), C_2^{2*}(0)], C(t), C^*(t): t=0,1,2,...\}$ is said to be competitive if $[C_1^2(0), C_2^2(0)]$ maximizes $u[C_1^2(0), C_2^2(0)]$ subject to (6-10-a), $[C_1^{2*}(0), C_2^{2*}(0)]$ maximizes $u[C_1^{2*}(0), C_2^{2*}(0)]$ subject to (6-10-b), $C(t)$ maximizes $U[C(t)]$ subject to (6-8'-a) and (6-9'-a), and $C^*(t)$ maximizes $U(C^*(t)]$ subject to (6-8-b) and (6-9-b).

Feasible Consumption Programs An infinite sequence $\{[C_1^2(0), C_2^2(0)], [C_1^{2*}(0), C_2^{2*}(0)], C(t), C^*(t): t=0,1,2,...\}$ is said to be feasible if the following conditions are satisfied.

$$D_i(t)+D_i^*(t) = X_i(t)+X_i^*(t) \quad i=1,2; \ t=0,1,2,... \qquad \text{(6-12)}$$

$$N^1(t)m(t)+N^{1*}(t)m_h^*(t) = M(t) \qquad t=0,1,2,... \qquad \text{(6-13-a)}$$

$$N^1(t)m_f(t)+N^{1*}(t)m^*(t) = M^*(t) \qquad t=0,1,2,... \qquad \text{(6-13-b)}$$

It can be deduced from equations (6-12) that

$$N^1(t)[\tilde{w}^1(t)-C^1(t)] + N^{1*}(t)[\tilde{w}^{1*}(t)-C^{1*}(t)] + N^2(t)[\tilde{w}^2(t)-C^2(t)] + N^{2*}(t)[\tilde{w}^{2*}(t)-C^{2*}(t)] = 0 \qquad t=0,1,2,... \tag{6-14}$$

Furthermore, if Factor Price Equalization prevails then $w^j(t) = w^{j*}(t) = w^j(\hat{p})$ for $j=1,2$ and all $t \geq 0$. Conditions (6-13-a) and (6-13-b) together imply

$$N^1(t)Z(t)+N^{1*}(t)e(t)Z^*(t) = M(t)+e(t)M^*(t) \qquad t=0,1,2,... \tag{6-15}$$

Equilibrium Consumption Programs A consumption program $[C(t), C^*(t)]$ is an equilibrium program if it is both competitive and feasible.

## 6.3 ANALYSIS OF EQUILIBRIUM PROGRAMS

An equilibrium of the world economy in period $t=1,2,3,...$ is completely described by a set of equilibrium values $\{\hat{p}(t), q^o(t), q^{*o}(t), m^o(t), m^o_f(t), m_h^{*o}(t), m^{*o}(t), \overset{o}{C}(t), \overset{o}{C}{}^*(t)\}$.

### 6.3.1 Equilibrium in the Product Markets

The equilibrium international commodity price ratio $\hat{p}(t)$ is determined by world demand and supply conditions in the product markets. Under the strong assumption of identical homothetic preferences in both countries, $\hat{p}(t)$ is unaffected by the monetary-fiscal policy in either country. Thus, $\hat{p}(t)$ is, for given utility and production functions, is dependent only on $k$ and $k^*$, i.e., $\hat{p}(t) = \hat{p}(k,k^*)$.

### 6.3.2 Equilibrium in the Money Markets

Consider a typical member of the t-th generation in the home country. Suppose that he wishes to save in his youth in order to obtain extra consumption goods in his old age. For one unit of the first commodity he can obtain $1/q(t)$ units of his own country's money (or $1/q^*(t)$ units of the foreign currency) which, in the next period, will yield him $(1+r)/q(t)$ units of home money (or $(1+r^*)/q^*(t)$ of foreign money) or $(1+r)q(t+1)/q(t)$ units of the first good. Thus, the real rate of return on an investment in his own country's money measured in terms of the first commodity is $[(1+r)q(t+1)/q(t)]-1$. In equilibrium all money-stocks are willingly held, implying that the home and foreign rates of return are equal, i.e.,

$$(1+r)q(t+1)/q(t) = (1+r^*)q^*(t+1)/q^*(t) \qquad t=1,2,3,... \tag{6-16}$$

Making use of the fact that $e(t) = q^*(t)/q(t)$ one can obtain a first-order linear difference equation in $e(t)$,

$$(1+r^*)e(t+1) = (1+r)e(t) \qquad t=1,2,3,... \tag{6-17}$$

The solution to (6-17) is clearly

$$e(t) = e_0[(1+r)/(1+r^*)]^t \qquad t=1,2,3,... \tag{6-18}$$

where $e_0 = q_0^*/q_0$ is the initial rate of exchange between the two currencies. Evidently,

$$\Delta e(t)/e(t) = (r-r^*)/(1+r) \qquad t=1,2,3,... \tag{6-19}$$

i.e., the rate at which the foreign money appreciates (or depreciates) in terms of the home country money bears a relationship of constant proportionality to the difference in $r$ and $r^*$. This is a well-known result which owes nothing to the special assumptions of the model.

### 6.3.3 Equilibrium Time Paths of $q^o(t)$ and $q^{*o}(t)$

To determine the equilibrium time paths of $q(t)$ and $q^*(t)$ it is sufficient to pin down the behaviour over time of the money prices in the home country. To do so we substitute (6-17) into (6-9'-a) to yield

$$C^2(t+1) = \tilde{w}^2(t+1)+q(t+1)\{[(1-\alpha)\eta M_0\mu^t/N^1(t)]+(1+r)Z(t)\} \qquad t=1,2,3,\ldots \qquad (6\text{-}9''\text{-a})$$

Maximizing $U = \Omega[u(C_1^1(t),\ C_2^1(t));\ \ u(C_1^2(t+1),\ C_2^2(t+1))]$ subject to (6-8'-a) and (6-9"-a) necessarily implies

$$q^o(t+1)/q^o(t) = \frac{1}{(1+r)}\ \frac{\Omega_1}{\Omega_2}\ \frac{g'[\overset{o}{C}{}^1(t)/l(\hat{p})]}{g'[\overset{o}{C}{}^2(t+1)/l(\hat{p})]} \qquad t=1,2,3,\ldots \qquad (6\text{-}20)$$

where $\Omega_j$ $(j=1,2)$ are evaluated at $\overset{o}{C}{}^j(t+j-1)$ $(j=1,2;\ t=1,2,3,\ldots)$ and $\hat{p}$. Equation (6-20) is a first-order, non-linear difference equation in $q^o(t)$. It has been shown in 5.3.1 that, for given positive values of $q^o(t)$ and $Z^o(t)$, there exists a positive value of $q^o(t+1)$ that solves (6-20) uniquely. To obtain $Z^o(t)$ consider the utility-maximizing problem from the viewpoint of a person born in period $t \geq 1$ in the foreign country. For an internal solution in which money is not worthless it is necessary that

$$q^{*o}(t+1)/q^{*o}(t) = \frac{1}{(1+r^*)}\ \frac{\Omega_1^*}{\Omega_2^*}\ \frac{g'[\overset{o}{C}{}^{1*}(t)/l(\hat{p})]}{g'[\overset{o}{C}{}^{2*}(t+1)/l(\hat{p})]} \qquad t=1,2,3,\ldots \qquad (6\text{-}21)$$

Combining (6-16), (6-20) and (6-21) it is quite easy to see that

$$\frac{\Omega_1}{\Omega_2}\;\frac{g'[\overset{o}{C}{}^{1}(t)/l(\hat{p})]}{g'[\overset{o}{C}{}^{2}(t+1)/l(\hat{p})]} = \frac{\Omega_1^*}{\Omega_2^*}\;\frac{g'[\overset{o}{C}{}^{1*}(t)/l(\hat{p})]}{g'[\overset{o}{C}{}^{2*}(t+1)/l(\hat{p})]}$$

$$t=1,2,3,\ldots \qquad (6\text{-}22)$$

Finally, the world-wide aggregate demand for money must equal the total stock for money in equilibrium, i.e., equation (6-15) must hold true. The assumed properties of lifetime utility ensures that there exist unique values of $Z^o(t)$ and $Z^{*o}(t)$ that satisfy (6-22) and (6-15) simultaneously. Therefore, once the positive values of $q_0$ and $q^o(1)$ are supplied, the infinite sequence $\{q^o(t):\ t=2,3,4,\ldots\}$ can be determined recursively by equation (6-20) and the sequence $\{q^{*o}(t):\ t=1,2,3,\ldots.\}$ by

$$q^{*o}(t) = e(t)q^o(t) = e_0[(1+r)/(1+r^*)]^t\, q^o(t)$$

$$t=1,2,3,\ldots \qquad (6\text{-}23)$$

Given $q^o(t)$, $q^o(t+1)$ and $Z^o(t)$, the consumption profile of a typical member of the t-th generation in the home country can be calculated as follows.

$$\overset{o}{C}{}_1^{j}(t+j-1) = [1+\hat{p}\sigma(\hat{p})]^{-1}\overset{o}{C}{}^{j}(t+j-1)$$

$$\overset{o}{C}{}_2^{j}(t+j-1) = \sigma(\hat{p})\overset{o}{C}{}_1^{j}(t+j-1)$$

$$j=1,2;\quad t=1,2,3,\ldots \qquad (6\text{-}24)$$

where $\sigma(p)$ is as defined in (2-53).

### 6.3.4 Indeterminacy of $m^o(t)$, $m_f^o(t)$, $m^{*o}(t)$ and $m_h^{*o}(t)$

Note that although the equilibrium values of $Z(t)$ and $Z^*(t)$ are completely determined by equations (6-22) and (6-15), the holding by each country of the other country's money is indeterminate. Given perfect foresight about the equilibrium exchange rate between the two currencies in period t, a typical young man of the t-th generation in the home (foreign) country is indifferent to the actual distribution of the two fiat monies in his total demand for money. To show it more rigorously, consider the following linear system of four equations in four unknown,

$$\begin{aligned} m(t)+e(t)m_f(t) &= Z^o(t) \\ m^*(t)+m_h^*(t)/e(t) &= Z^{*o}(t) \\ N^1(t)m(t)+N^{1*}(t)m_h^*(t) &= M(t) \\ N^1(t)m_f(t)+N^{1*}(t)m^*(t) &= M^*(t) \end{aligned}$$

$$t=1,2,3,\ldots \qquad (6\text{-}25)$$

Since the last two feasibility conditions in the money markets have been used to compute $Z^o(t)$ and $Z^{*o}(t)$, the above system is consistent but clearly degenerate. A quick calculation shows that only three equations are linearly independent and there are, therefore, infinitely many solutions to (6-25). Taking $m(t)$ as the independent variable, a complete solution to (6-25) is

$$\begin{aligned} m_f^o(t) &= [Z^o(t)-m^o(t)]/e(t) \\ m^{*o}(t) &= Z^{*o}(t)+\{[N^1(t)m^o(t)-M(t)]/[N^{1*}(t)e(t)]\} \\ m_h^{*o}(t) &= [M(t)-N^1(t)m^o(t)]/N^{1*}(t) \end{aligned}$$

$$0 \leq m^{o}(t) \leq Z^{o}(t)$$

$$t=1,2,3,... \qquad (6\text{-}26)$$

It is obvious that the individual components $m(t)$, $m_f(t)$, $m^*(t)$ and $m_h^*(t)$ are determinate if one more constraint is added into the system described by (6-25). For example, one may assume that a typical member of the t-th generation in the home country wishes to hold the two fiat monies in a constant proportion, i.e., $m_f(t)/m(t) = h > 0$. Then $m^o(t) = Z^o(t)/[1+he(t)]$ and the solution to (6-25) is unique. However, it is difficult to find plausible arguments to justify such a restrictive assumption.

### 6.3.5 Evolution of the World Monetary Economy

To complete the description of the world monetary economy it is necessary to examine the initial equilibria in the home and foreign countries. A typical young man at time t=0 in the home country chooses $C(0)$ to maximize $U[C(0)]$ subject to (6-11-a) and the t=0 version of (6-9"-a) whereas a man of the same age living in the foreign country wishes to maximize $U[C^*(0)]$ by the choice of $C^*(0)$ subject to (6-11-b) and the t=0 version of (6-9'-b). Suppose that both of them decide to save against old age by exchanging some of their income for some of the fiat monies created by the oldest generations in the home and foreign countries. Then for given positive initial values of $M_0$, $q_0$, $M_0^*$ and $q_0^*$, the equilibrium values of $q(1)$, $q^*(1)$, $Z(0)$ and $Z^*(0)$ are determined by the t=0 versions of (6-20), (6-22), (6-15) and the t=1 version of (6-23). More explicitly, for given initial conditions, $\{q^o(1), q^{*o}(1), Z^o(0), Z^{*o}(0)\}$ is a solution to the system described by

$$q^{o}(1)/q_0 = \frac{1}{(1+r)} \frac{\Omega_1}{\Omega_2} \frac{g'[\overset{o}{C}{}^{1}(0)/1(\hat{p})]}{g'[\overset{o}{C}{}^{2}(1)/1(\hat{p})]}$$

$$\frac{\Omega_1}{\Omega_2} \frac{g'[\overset{o}{C}{}^{1}(0)/1(\hat{p})]}{g'[\overset{o}{C}{}^{2}(1)/1(\hat{p})]} = \frac{\Omega_1^*}{\Omega_2^*} \frac{g'[\overset{o}{C}{}^{1*}(0)/1(\hat{p})]}{g'[\overset{o}{C}{}^{2*}(1)/1(\hat{p})]}$$

$$N_0^1 Z^{o}(0)+N_0^{1*}e_0 Z^{*o}(0) = M_0+e_0 M_0^*$$

$$q^{*o}(1) = e(1)q^{o}(1) = e_0[(1+r)/(1+r^*)]q^{o}(1) \tag{6-27}$$

where $e_0=q_0/q_0^*$, $\Omega_j$ $(j=1,2)$ are evaluated at $\tilde{w}^1(0)$, $\tilde{w}^2(1)$, $q_0$, $q^{o}(1)$ and $Z^{o}(0)$, and $\Omega_j^*$ $(j=1,2)$ are evaluated at $\tilde{w}^{1*}(0)$, $\tilde{w}^{2*}(1)$, $q_0^*$, $q^{*o}(1)$ and $Z^{*o}(0)$. It is evident that $q_0$ and $q_0^*$, the initial money prices in the home and foreign countries respectively, are nonnegative but otherwise arbitrary constants. In fact, any arbitrary choice of $q_0$ and $q_0^*$ satisfying

$$q_0 M_0+q_0^* M_0^* \leq N_0^1 w^1(0)+N_0^{1*} w^{1*}(0)$$

is feasible; for

$$\begin{aligned} &N_0^1[\tilde{w}^1(0)-\overset{o}{C}{}^{1}(0)]+N_0^{1*}[\tilde{w}^{1*}(0)-\overset{o}{C}{}^{1*}(0)]+ \\ &N_0^2[\tilde{w}^2(0)-\overset{o}{C}{}^{2}(0)]+N_0^{2*}[\tilde{w}^{2*}(0)-\overset{o}{C}{}^{2*}(0)] = \\ &q_0\{[N_0^1 Z^{o}(0)+N_0^{1*}e_0 Z^{*o}(0)]-(M_0+e_0 M_0^*)\} = 0 \end{aligned}$$

by virtue of the third equation in (6-27). However, there is, as discussed in 5.3.2, an inevitable problem of indeterminacy of $q_0$ and $q_0^*$.

It should be clear that an initial world monetary equilibrium in which both $q_0$ and $q_0^*$ are positive is not the only possibility in general. As k and $k^*$ are not necessarily equal, there are four possible outcomes in period t=0.

(a) $q_0 > 0$ and $q_0^* > 0$, i.e., both a typical young member of the home population in period t=0 and his counterpart in the foreign country wish to save against old age.

(b) $q_0=0$ but $q_0^* > 0$, i.e., only the young people at time t=0 in the foreign country wish to hold their own currency.

(c) $q_0 > 0$ but $q_0^*=0$, i.e., only the young generation of period t=0 in the home country wishes to accept the home money as a means of saving.

(d) $q_0=q_0^*=0$, i.e., neither a typical young person living in the home country at time t=0 nor his counterpart in the foreign country desires to save against old age.

In either case (b) or (c) only a domestic monetary system is operative. It necessarily requires that the amount of saving by a typical young man in either country depends on the magnitudes of the population endowment ratios k and $k^*$. Moreover, $k < k^*$ in case (b) and $k > k^*$ in case (c). In the polar case (d) the international monetary system does not exist at all because a typical member of the world economy discounts his future satisfaction to the extent that his lifetime utility will decline if he saves when young. However, once $q_0$ and $q_0^*$ are both positive, as in case (a), the values of $q^o(t)$ and $q^{*o}(t)$ will continue to be positive indefinitely into the future. Let us define a world economy to be Samuelson monetary if $0 < e_0 < \infty$, locally monetary if either $e_0=0$ or $1/e_0=0$ exclusively, and non-monetary if $q_0=q_0^*=0$. Then it is possible to propose

<u>Theorem 6.1</u> For a world economy of overlapping generations, there are three possible equilibria, Samuelson monetary, locally monetary and nonmonetary. Furthermore, any money price sequence with $q(t)=0$ (or $q^*(t)=0$) for some but not all $t \geq 0$ is not an equilibrium sequence.

## 6.4 ANALYSIS OF STEADY-STATE EQUILIBRIA

6.4.1 <u>Definition</u> The world economy is said to have attained a stationary equilibrium from period $t_0$ onwards if the consumption profile of a typical man born in period $t \geq t_0$ in the home country and that of his counterpart in the foreign country are independent of the passage of time, i.e., $\overset{o}{C}{}^j(t)=\overset{o}{C}{}^j$ and $\overset{o}{C}{}^{j*}(t)=\overset{o}{C}{}^{j*}$ for $j=1,2$ and all $t \geq t_0$.

From this definition a nonmonetary equilibrium in which $q_0=q_0^*=0$ is a steady state from the time origin since $\overset{o}{C}{}^j(t)=w^j(t)=w^j$ and $\overset{o}{C}{}^{j*}(t)=w^{j*}$ for $j=1,2$ and $t \geq 0$. The only trade that takes place is the exchange of commodities between the two countries. Each person of the world then simply consumes his compensated income in each period of his life. Suppose that the home economy is monetary but the foreign economy is nonmonetary, i.e., $q_0 > 0$ but $q_0^*=0$. Then the home-country-monetary steady state is characterized by $q^o(t+1)/q^o(t)=k/\mu$, $\pi^o(t)=(\mu-k)/k$ and $q^{*o}(t)=0$ for all $t \geq t_0$. A symmetrical result applies to the foreign-country-monetary steady-state equilibrium. In the next sub-section an attempt will be made to derive the necessary condition for a Samuelson monetary steady state in a special case.

### 6.4.2 Necessary Condition for a Samuelson monetary steady state when $\eta/[(1+r)k] = \eta^*/[(1+r^*)k^*]$

In a Samuelson monetary stationary equilibrium, the feasibility condition (6-14) becomes

$$N^1(t)[\tilde{w}^1-\mathring{C}^1]+N^{1*}(t)[\tilde{w}^{1*}-\mathring{C}^{1*}]+N^2(t)[\tilde{w}^2-\mathring{C}^2]+N^{2*}(t)[\tilde{w}^{2*}-\mathring{C}^{2*}] = 0$$

$$t=t_0,t_0+1,\ldots \qquad (6\text{-}28)$$

Eliminating $Z^o(t)$ in (6-8'-a) and (6-9"-a), the lifetime budget constraint of a typical man born in period $t \geq t_0 > 0$ in the home country is

$$[(1+r)q^o(t+1)/q^o(t)][\tilde{w}^1-\mathring{C}^1]+[\tilde{w}^2-\mathring{C}^2]+\eta A[q^o(t+1)M(t)/N^1(t)] = 0$$

$$t=t_0,t_0+1,\ldots \qquad (6\text{-}29\text{-a})$$

where $A=[\alpha(1+r)+(1-\alpha)\mu]/\mu$. Similarly, the lifetime budget equation of his counterpart in the foreign country is

$$[(1+r^*)q^{*o}(t+1)/q^{*o}(t)][\tilde{w}^{1*}-\mathring{C}^{1*}]+[\tilde{w}^{2*}-\mathring{C}^{2*}]+$$
$$\eta^*A^*[q^{*o}(t+1)M^*(t)/N^{1*}(t)] \quad = \quad 0$$

$$t=t_0,t_0+1,\ldots \qquad (6\text{-}29\text{-b})$$

where $A^*=[\alpha^*(1+r^*)+(1-\alpha^*)\mu^*]/\mu^*$. Multiplying equations (6-29-a) and (6-29-b) by $N^2(t)$ and $N^{2*}(t)$, respectively, and adding them together yields

$$[(1+r)N^2(t)q^o(t+1)/q^o(t)][\tilde{w}^1-\mathring{C}^1]+N^2(t)[\tilde{w}^2-\mathring{C}^2]+$$
$$[(1+r^*)N^{2*}(t)q^{*o}(t+1)/q^{*o}(t)][\tilde{w}^{1*}-\mathring{C}^{1*}]+N^{2*}(t)[\tilde{w}^{2*}-\mathring{C}^{2*}]+$$
$$\eta A[N^2(t)q^o(t+1)M(t)/N^1(t)]+$$
$$\eta^*A^*[N^{2*}(t)q^{*o}(t+1)M^*(t)/N^{1*}(t)] \quad = \quad 0$$

$$t=t_0,t_0+1,\ldots \qquad (6\text{-}30)$$

Consider the last two terms in the R.H.S. of (6-30). In equilibrium, the world-wide total saving by the young generations must equal to the total value of the world's stock of money, i.e.,

$$N^1(t)[\tilde{w}^1-\overset{o}{C}{}^1]+N^{1*}(t)[\tilde{w}^{1*}-\overset{o}{C}{}^{1*}] = q^o(t)M(t)+q^{*o}(t)M^*(t)$$
$$-\alpha\eta q^o(t)M(t-1)-\alpha^*\eta^*q^{*o}(t)M^*(t-1) = Aq^o(t)M(t)+A^*q^{*o}(t)M^*(t)$$
$$t=t_0,t_0+1,\ldots \qquad (6\text{-}31)$$

Therefore,

$$\eta A[N^2(t)q^o(t+1)M(t)/N^1(t)]+\eta^*A^*[N^{2*}(t)q^{*o}(t+1)M^*(t)/N^{1*}(t)]=$$
$$\{[\eta N^2(t)q^o(t+1)]/[N^1(t)q^o(t)]\}\{N^1(t)[\tilde{w}^1-\overset{o}{C}{}^1]+N^{1*}(t)[\tilde{w}^{1*}-\overset{o}{C}{}^{1*}]-$$
$$[A^*q^{*o}(t)M^*(t)]\}+\eta^*A^*[N^{2*}(t)q^{*o}(t+1)M^*(t)/N^{1*}(t)]=$$
$$[\eta N^2(t)q^o(t+1)/q^o(t)][\tilde{w}^1-\overset{o}{C}{}^1]+[\frac{\eta k^*(1+r^*)}{k(1+r)}][N^{2*}(t)q^{*o}(t+1)/q^{*o}(t)]$$
$$[\tilde{w}^{1*}-\overset{o}{C}{}^{1*}]+A^*[(\frac{\eta^*}{k^*})-\frac{\eta(1+r^*)}{k(1+r)}]q^{*o}(t+1)M^*(t)$$
$$t=t_0,t_0+1,\ldots \qquad (6\text{-}32)$$

(We have already made use of the fact that $(1+r)q^o(t+1)/q^o(t) = (1+r^*)q^{*o}(t+1)/q^{*o}(t)$.) Nothing much can be said about (6-32) in general. However, if

$$\eta/[(1+r)k] = \eta^*/[(1+r^*)k^*] \qquad (6\text{-}33)$$

then (6-32) reduces to

$$[\eta N^2(t)q^o(t+1)/q^o(t)][\tilde{w}^1-\overset{o}{C}{}^1]+[\eta^*N^{2*}(t)q^{*o}(t+1)/q^{*o}(t)]$$
$$[\tilde{w}^{1*}-\overset{o}{C}{}^{1*}] \qquad (6\text{-}32')$$

Substituting (6-32') into (6-30) and keeping in mind that $\eta = \mu-1-r$ and $\eta^* = \mu^*-1-r^*$, equation (6-30) reduces into

$$[\mu N^2(t)q^o(t+1)/q^o(t)][\tilde{w}^1-\overset{o}{C}^1]+N^2(t)[\tilde{w}^2-\overset{o}{C}^2]+$$
$$[\mu^* N^{2*}(t)q^{*o}(t+1)/q^{*o}(t)][\tilde{w}^{1*}-\overset{o}{C}^{1*}]+N^{2*}(t)[\tilde{w}^{2*}-\overset{o}{C}^{2*}] = 0$$
$$t=t_0,t_0+1,\ldots \qquad (6\text{-}30')$$

Now, subtracting (6-28) from (6-30') yields

$$N^2(t)\{[\mu q^o(t+1)/q^o(t)]-k\}[\tilde{w}^1-\overset{o}{C}^1]+$$
$$N^{2*}(t)\{[\mu^* q(t+1)/q^{*o}(t)]-k^*\}[\tilde{w}^{1*}-\overset{o}{C}^{1*}] = 0$$
$$t=t_0,t_0+1,\ldots \qquad (6\text{-}34)$$

Since $\tilde{w}^1-\overset{o}{C}^1$ and $\tilde{w}^{1*}-\overset{o}{C}^{1*}$ are both positive in a Samuelson monetary steady state, equation (6-34) necessarily implies that

$$q^o(t+1)/q^o(t) = k/\mu \qquad t=t_0,t_0+1,\ldots \qquad (6\text{-}35\text{-a})$$

and that

$$q^{*o}(t+1)/q^{*o}(t) = k^*/\mu^* \qquad t=t_0,t_0+1,\ldots \qquad (6\text{-}35\text{-b})$$

Adding the equilibrium condition (6-16), it is evident that a necessary condition for a Samuelson monetary stationarity in this case is

$$\mu/[(1+r)k] = \mu^*/[(1+r^*)k^*] \qquad (6\text{-}36)$$

But, by combining (6-33) and (6-36), it can be shown that

$$\mu/(1+r) = \mu^*/(1+r^*) \qquad (6\text{-}37)$$

and, consequently, $k=k^*$. In this redundant case we are, effectively, in the one-country world discussed in the previous chapter. The steady-state-equilibrium price of the home money in terms of the first commodity grows at the rate $(k/\mu)-1$ and its rate of inflation is $(\mu-k)/k$. The equilibrium price of the foreign money in terms of the first good grows at the rate $(k^*/\mu^*)-1 = \{[(1+r)k]/[(1+r^*)\mu]\}-1$ and its rate of inflation is $(\mu^*-k^*)/k^* = [\mu(1+r^*)-(1+r)k]/[(1+r)k]$. Let $\beta(t)$ and $\beta^*(t)$ be respectively the shares of the home-country and foreign-country fiat monies in the world's stock of money at time t. Then,

$$\beta(t) = M(t)/[M(t)+e(t)M^*(t)] \qquad t=0,1,2,... \tag{6-38-a}$$

and

$$\beta^*(t) = [e(t)M^*(t)]/[M(t)+e(t)M^*(t)] \qquad t=0,1,2,... \tag{6-38-b}$$

It is evident that $\beta(t)$ and $\beta^*(t)$ are positive constants in a Samuelson monetary steady state.

$$\beta(t) = \frac{M(t_0)\mu^{t-t_0}}{M(t_0)\mu^{t-t_0}+e(t_0)M^*(t_0)[(1+r)\mu^*/(1+r^*)]^{t-t_0}}$$

$$= \{1+[e(t_0)M^*(t_0)/M(t_0)]\}^{-1} \qquad t=t_0,t_0+1,... \tag{6-39-a}$$

$$\beta^*(t) = [e(t_0)M^*(t_0)/M(t_0)]/\{1+[e(t_0)M^*(t_0)/M(t_0)]\} \qquad t=t_0,t_0+1,... \tag{6-39-b}$$

since $(1+r)\mu^* = (1+r^*)\mu$.

## 6.5 AN EXAMPLE

We now specialize the model by supposing that the lifetime utility function U is everywhere additively separable and the j-th period utility function u takes the logarithmic form

$$U[C(t)] = u[C_1^1(t), C_2^1(t)] + \delta u[C_1^2(t+1), C_2^2(t+1)], \qquad \delta>0 \qquad (6\text{-}40)$$

and

$$u[C_1^j(t), C_2^j(t)] = \beta \log_e C_1^j(t) + (1-\beta)\log_e C_2^j(t), \qquad 0<\beta<1 \qquad (6\text{-}41)$$

For simplicity, assume further that Factor Price Equalization holds true in the post-commodity-trading equilibrium so that $\overset{o}{w}{}^{j*} = \overset{o}{w}{}^{j}$ for $j=1,2$. Consequently, except when deriving necessary conditions for the world economy to be Samuelson monetary, it is immaterial whether or not the production functions are explicitly introduced into the model. Thus, countries now differ only in their population growth rates and money characteristics (initial stocks of money, rates of monetary expansion or decay, rates of money interest, and the shares of the two generations in any monetary transfers).

The first equation of (6-27) and equation (6-20) then become respectively,

$$q^o(1)/q_0 = \{w^2 + q^o(1)[((1-\alpha)\eta M_0/N_0^1) + (1+r)Z^o(0)]\}/\{B[w^1 - q_0 Z^o(0)]\} \qquad (6\text{-}42)$$

and

$$q^o(t+1)/q^o(t) = \frac{w^2 + q^o(t+1)[((1-\alpha)\eta M(t)/N^1(t)) + (1+r)Z^o(t)]}{B\{w^1 + q^o(t)[(\alpha\eta M(t-1)/N^1(t)) - Z^o(t)]\}} \qquad t=1,2,3... \qquad (6\text{-}43)$$

where $B = (1+r)\delta > 0$. The equilibrium time paths of $q(t)$ and $q^*(t)$ are characterized by

$$q^{o}(t+1) = \begin{cases} \dfrac{w^{2}q_{0}}{Bw^{1}-q_{0}[((1-\alpha)\eta M_{0}/N_{0}^{1})+(1+r)(1+\delta)Z^{o}(0)]} & t=0 \\ \\ \dfrac{w^{2}q^{o}(t)}{B\{w^{1}-q^{o}(t)[DM(t)/N^{1}(t))+((1+\delta)/\delta)Z^{o}(t)]\}} & \\ & t=1,2,3,\ldots \end{cases} \tag{6-44}$$

and

$$q^{*o}(t) = e(t)q^{o}(t) = e_{0}[(1+r)/(1+r^{*})]^{t}q^{o}(t)$$

where $D = \eta[((1-\alpha)/B)-(\alpha/\mu)]$. The conditions for the world economy to be Samuelson monetary ($0 < e_{0} < \infty$) involves $k$, $k^{*}$ and the utility and production parameters of the model. Now, solving for $Z^{o}(t)$ and $Z^{*o}(t)$ by taking the foreign country and the equilibrium conditions in the money markets into consideration, we obtain

$$Z^{o}(t) = \begin{cases} [(1-(PN_{0}^{1*}/N_{0}^{1}))M_{0}+(1+P^{*})e_{0}M_{0}^{*}]/(N_{0}^{1}+N_{0}^{1*}) & t=0 \\ \\ \dfrac{[1-(QN^{1*}(t)/N^{1}(t))]M(t)+(1+Q^{*})e(t)M^{*}(t)}{N^{1}(t)+N^{1*}(t)} & \\ & t=1,2,3,\ldots \end{cases} \tag{6-45}$$

and

$$Z^{*o}(t) = \begin{cases} [(1+P)(M_{0}/e_{0})+(1-(P^{*}N_{0}^{1}/N_{0}^{1*}))M_{0}^{*}]/(N_{0}^{1}+N_{0}^{1*}) & t=0 \\ \\ \dfrac{(1+Q)(M(t)/e(t))+[1-(Q^{*}N^{1}(t))]/N^{1*}(t)]M^{*}(t)}{N^{1}(t)+N^{1*}(t)} & \\ & t=1,2,3,\ldots \end{cases} \tag{6-46}$$

where

$P = [(1-\alpha)\eta]/[(1+r)(1+\delta)]$, $P^* = [(1-\alpha^*)\eta^*]/[(1+r^*)(1+\delta)]$,

$Q = \delta D/(1+\delta)$, $Q^* = \delta D^*/(1+\delta)$, $D^* = \eta^*[((1-\alpha^*)/B^*)-(\alpha^*/\mu^*)]$

and $B^* = (1+r^*)\delta$.

Replacing

$N^1(t) = N_0^1 k^t$, $N^{1*}(t) = N_0^{1*}(k^*)^t$, $M(t) = M_0\mu^t$, $M^*(t) = M_0^*(\mu^*)^t$

and $e(t) = e_0[(1+r)/(1+r^*)]^t$ into (6-45) and (6-46), the time paths of $Z^o(t)$ and $Z^{*o}(t)$ for $t \geq 1$ are given by

$$Z^o(t) = \frac{[M_0(k/k^*)^t - F](\mu k^*/k)^t + F^*[(1+r)\mu^*/(1+r^*)]^t}{N_0^1 k^t + N_0^{1*}(k^*)^t} \qquad t=1,2,3,\ldots \tag{6-47}$$

and

$$Z^{*o}(t) = \frac{G[(1+r^*)\mu/(1+r)]^t + [M_0^*(k^*/k)^t - G^*](\mu^* k/k^*)^t}{N_0^1 k^t + N_0^{1*}(k^*)^t} \qquad t=1,2,3,\ldots \tag{6-48}$$

where

$F = (QN_0^{1*}M_0)/N_0^1$, $F^* = (1+Q^*)e_0M_0^*$, $G = (1+Q)(M_0/e_0)$ and

$G^* = (Q^*N_0^1M_0^*)/N_0^{1*}$.

It is then evident from (6-47) and (6-48) that the international monetary system is dynamically unstable in general. For example, assume that $k < k^*$ and $\mu/(1+r) > \mu^*/(1+r^*)$. Then, as t grows indefinitely large, $(k/k^*)^t$ tends to zero, $(\mu k^*/k)^t$ grows at a rate faster than $\mu^t$ and $[(1+r)\mu^*/(1+r^*)]^t$ grows at a rate slower than $\mu^t$. Therefore, there exists a $T > 0$ such that $Z^o(t) < 0$. In the opposite case, suppose that $k > k^*$ and $\mu/(1+r) < \mu^*/(1+r^*)$. Then there exists a $T^* > 0$ such that $Z^{*o}(T^*) < 0$.

A necessary (but not sufficient) condition for the existence of a Samuelson monetary steady state is clearly $k=k^*$ and $\mu/(1+r) = \mu^*/(1+r^*)$. In this interesting special case the world economy simplifies, effectively, to a one-country economy, with (6-44) corresponding to equation (5-36) of the last chapter. Equations (6-47) and (6-48) then reduce respectively to

$$Z^o(t) = [(M_0-F+F^*)/(N_0^1+N_0^{1*})](\mu/k)^t \qquad t\geq 1 \tag{6-49-a}$$

and

$$Z^{*o}(t) = [(M_0^*+G-G^*)/(N_0^1+N_0^{1*})](\mu^*/k^*)^t \qquad t\geq 1 \tag{6-49-b}$$

Thus, the equilibrium time paths of $q(t)$ and $q^*(t)$ for $t \geq 1$ become

$$q^o(t+1) = [w^2q^o(t)]/[Bw^1-L\mu q^o(t)(\mu/k)^t], \quad t\geq 1 \tag{6-50-a}$$

and

$$q^{*o}(t+1) = [w^2q^{*o}(t)]/[B^*w^1-L^*\mu^*q^{*o}(t)(\mu^*/k^*)^t] \qquad t\geq 1 \tag{6-50-b}$$

where

$$L = \left(\frac{B}{\mu}\right)\left\{\frac{(DM_0/N_0^1)+[(1+\delta)(M_0-F+F^*)]}{\delta(N_0^1+N_0^{1*})}\right\}$$

and

$$L^* = \left(\frac{B^*}{\mu^*}\right)\left\{\frac{(D^*M_0^*/N_0^{1*})+[(1+\delta)(M_0^*+G-G^*)]}{\delta(N_0^1+N_0^{1*})}\right\}$$

Defining $\tau(t) = q^o(t)/(k/\mu)^t$ and $\tau^*(t) = q^{*o}(t)/(k^*/\mu^*)^t$, the difference equations (6-50-a) and (6-50-b) can be respectively rewritten as

$$\tau(t+1) = [w^2\tau(t)]/[(Bk/\mu)w^1-kL\tau(t)] = \psi[\tau(t)] \qquad t \geq 1 \qquad (6\text{-}51\text{-}a)$$

and

$$\tau^*(t+1) = [w^2\tau^*(t)]/[(B^*k^*/\mu^*)w^1-k^*L^*\tau^*(t)] = \psi^*[\tau^*(t)] \qquad t \geq 1 \qquad (6\text{-}51\text{-}b)$$

The first-order, nonlinear difference equations (6-51-a) and (6-51-b) have constant solutions,

$$\bar{\tau} = [(Bk/\mu)w^1-w^2]/(kL) \qquad (6\text{-}52\text{-}a)$$

and

$$\bar{\tau}^* = [(B^*k^*/\mu^*)w^1-w^2]/(k^*L^*) \qquad (6\text{-}52\text{-}b)$$

Since $Bk/\mu = (1+r)\delta k/\mu = (1+r^*)\delta k^*/\mu^* = B^*k^*/\mu^*$, $\bar{\tau}$ and $\bar{\tau}^*$ are both positive if and only if

$$(Bk/\mu)w^1-w^2 > 0 \qquad (6\text{-}53)$$

(Equation (6-53) is simply a necessary and sufficient condition for $0 < e_0 < \infty$ in this special world economy.) Furthermore, if $q_0$ and $q_0^*$ are by sheer accident chosen such that $q^o(1) = k\bar{\tau}/\mu$ and $q^{*o}(1) = k^*\bar{\tau}^*/\mu^*$, i.e.,

$$q_0 = (Bkw^1\bar{\tau})/(\mu w^2+Rk\bar{\tau}) \qquad (6\text{-}54\text{-}a)$$

and

$$q_0^* = (B^*k^*w^1\bar{\tau}^*)/(\mu^*w^2+R^*k^*\bar{\tau}^*) \qquad (6\text{-}54\text{-}b)$$

where

$$R = [(1-\alpha)\eta M_0/N_0^1]+(1+r)(1+\delta)Z^o(0) \quad \text{and}$$

$$R^* = [(1-\alpha^*)\eta^*M_0^*/N_0^{1*}]+(1+r^*)(1+\delta)Z^{*o}(0) \ ,$$

then the world economy attains a Samuelson monetary steady state from period t=1 onwards. The price of the home-country fiat money grows (decays) at the rate $(k/\mu)-1$ over time and the real value of the stock of money is $M(1)q^o(1) = [(Bk/\mu)w^1-w^2]M_0/L$. In the foreign country, the price of its money grows (decays) at the rate $(k^*/\mu^*)-1 = [(1+r)k]/[(1+r^*)\mu]-1$ and the real value of its stock of money $M^*(1)q^{*o}(1)=[(B^*k^*/\mu^*)w^1-w^2]M_0^*/L^* = [(Bk/\mu)w^1-w^2](q_0^*M_0^*/q_0)/L$. It is clear that the initial rate of exchange between the two currencies is

$$e_0 = q_0^*/q_0 = L/L^* \tag{6-55}$$

In summary, in a world where $k=k^*$ and $\mu/(1+r) = \mu^*/(1+r^*)$, (6-53), (6-54-a) and (6-54-b) are necessary and sufficient conditions for a Samuelson monetary stationary equilibrium.

It is easy to deduce from the properties of $\psi[\tau(t)]$ that if $q^o(1)\neq k\bar{\tau}/\mu$ then the path of $\{\tau(t)\}$ steadily diverges from $\bar{\tau}$. If $q^o(1) < k\bar{\tau}/\mu$, $\tau(t)$ goes to zero, implying that $q^o(t)$ grows slower than $(k/\mu)^t$. If, on the other hand, $q^o(1) > k\bar{\tau}/\mu$, then, for some $t>1$, $\tau(t)$ is greater than or equal to $Bw^1/(L\mu)$ so that either only a negative value or no finite value of $\tau(t+1)$ satisfies (6-51-a). At that point, the real value of the stock of home money has become so great that the demand for that money by young people in the world is less than the available supply for all positive $\tau(t+1)$; home money becomes worthless and the international monetary system collapses. Note also from (6-39-a) and (6-39-b) that all real monetary stocks move in step, all going to zero or all going to infinity or all remaining at their initial levels. Thus the necessary and sufficient condition for break down of the international monetary system is simply that $q^o(1) > k\bar{\tau}/\mu$.

To show that whereas the real stock of the home-country necessarily follows a monotone path, its price level may be nonmonotone, we define $\xi(t) = 1/\tau(t) = (k/\mu)^t/q^o(t)$ being the inverse of the real price of home money. Then equation (6-51-a) becomes

$$w^2\xi(t+1) - (Bk/\mu)w^1\xi(t) + kL = 0 \quad t\geq 1 \qquad (6\text{-}56)$$

Equation (6-56) is a first-order, linear difference equation in $\xi(t)$ whose solution is

$$\xi(t) = \{\xi(1) - [(kL)/((Bk/\mu)w^1 - w^2)]\}[(Bkw^1)/(\mu w^2)]^{t-1} + [(kL)/((Bk/\mu)w^1 - w^2)] \quad t\geq 1 \qquad (6\text{-}57)$$

Now, let $v(t) = 1/q^o(t)$ be the price of the first commodity in terms of the home-country money. Then (6-56) becomes

$$w^2v(t+1) - Bw^1v(t) + (L\mu)(\mu/k)^t = 0 \quad t\geq 1 \qquad (6\text{-}58)$$

Assume $Bw^1 \neq w^2$ then (6-58) has the solution

$$v(t) = [(L\mu)/(Bw^1-w^2)](\mu/k)^{t-1} + \{v(1) - [(L\mu)/(Bw^1-w^2)]\}(Bw^1/w^2)^{t-1} \quad t\geq 1 \qquad (6\text{-}59)$$

In view of (6-53), $(Bw^1)/w^2 > (\mu/k)$ and the second term in the R.H.S. of (6-59) eventually dominates the first term. Hence, after some finite time interval, the time path of $v(t)$ is monotone. However, if

$$v(1) < (L\mu)/(Bw^1-w^2) \qquad (6\text{-}60)$$

then the price of the first good in terms of home money may rise before entering its long-term decline. In this case, the price level ($q(t)$ or $v(t)$) may change direction once. Once we leave the special case where $k = k^*$ and $\mu/(1+r) = \mu^*/(1+r^*)$, neither the price level nor the real stock of the home-country (or foreign-country) money need be monotone. (See Chang, Kemp and Long [4].)

Two points deserve mention here. First, it is implicitly assumed in the above example that free uncompensated trade is Pareto non-inferior to autarky. Thus, $\tilde{f} = \tilde{f}^* = 0$ and, as a consequence, $\tilde{w}^j = \tilde{w}^{j*} = w^j$ for $j=1,2$. Second, in the special case $k = k^*$, international trade in commodities does not take place.

## 6.6 CONCLUSIONS

It has been shown that, suitably re-phrased, the earlier conclusions in Chapter V remain valid in a world of several trading economies. For a class of utility functions, it can be demonstrated that the international monetary system is dynamically unstable in general. Even if many restrictive conditions are satisfied, maximizing behaviour by many economic agents, each blessed with perfect foresight over its lifetime, does not necessarily lead to convergence. Unless the initial prices $q_0$ and $q_0^*$ are accidentally chosen so that the world economy is in steady state in period $t=1$, either the real value of each country's stock of money goes to zero or the world monetary system collapses. Furthermore, neither the price level nor the real stock of money need be monotone in general conditions. Finally, a similar model in which $k = k^* = 1$ has been analyzed by Chang, Kemp and Long [4].

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