Pseudodifferential equations on spheres with spherical radial basis functions and spherical splines

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Pseudodifferential equations on spheres with spherical radial basis functions and spherical splines

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Pseudodifferential equations on the unit sphere in $\mathbb{R}^n$, $n \geq 3$, are considered. The class of pseudodiffrential operators have long been used as a modern and powerful tool to tackle linear boundary-value problems. These equations arise in geophysics, where the sphere of interest is the earth. Efficient solutions to these equations on the sphere become more demanding when given data are collected by satellites.

In this dissertation, firstly we solve these equations by using spherical radial basis functions. The use of these functions results in meshless methods, which have recently become more and more popular. In this dissertation, the collocation and Galerkin methods are used to solve pseudodifferential equations. From the point of view of application, the collocation method is easier to implement, in particular when the given data are scattered. However, it is well-known that the collocation methods in general elicit a complicated error analysis. A salient feature of our work is that error estimates for collocation methods are obtained as a by-product of the analysis for the Galerkin method. This unified error analysis is thanks to an observation that the collocation equation can be viewed as a Galerkin equation, due to the reproducing kernel property of the space in use.

Secondly, we solve these equations by using spherical splines with Galerkin methods. Our main result is an optimal convergence rate of the approximation. The key of the analysis is the approximation property of spherical splines as a subset of Sobolev spaces. Since the pseudodifferential operators to be studied can be of any order, it is necessary to obtain an approximation property in Sobolev norms of any real order, negative and positive.

Solving pseudodifferential equations by using Galerkin methods with spherical splines results, in general, in ill-conditioned matrix equations. To tackle this ill-conditionedness arising when solving two special pseudodifferential equations, the Laplace–Beltrami and hypersingular integral equations, we solve them by using a preconditioner which is defined by using the additive Schwarz method. Bounds for condition numbers of the preconditioned systems are established.

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ABSTRACT

Pseudodifferential equations on the unit sphere in $\mathbb{R}^n$, $n \geq 3$, are considered. The class of pseudodiffrential operators have long been used as a modern and powerful tool to tackle linear boundary-value problems. These equations arise in geophysics, where the sphere of interest is the earth. Efficient solutions to these equations on the sphere become more demanding when given data are collected by satellites.

In this dissertation, firstly we solve these equations by using spherical radial basis functions. The use of these functions results in meshless methods, which have recently become more and more popular. In this dissertation, the collocation and Galerkin methods are used to solve pseudodifferential equations. From the point of view of application, the collocation method is easier to implement, in particular when the given data are scattered. However, it is well-known that the collocation methods in general elicit a complicated error analysis. A salient feature of our work is that error estimates for collocation methods are obtained as a by-product of the analysis for the Galerkin method. This unified error analysis is thanks to an observation that the collocation equation can be viewed as a Galerkin equation, due to the reproducing kernel property of the space in use.

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Solving pseudodifferential equations by using Galerkin methods with spherical splines results, in general, in ill-conditioned matrix equations. To tackle this ill-conditionedness arising when solving two special pseudodifferential equations, the Laplace–Beltrami and hypersingular integral equations, we solve them by using a preconditioner which is defined by using the additive Schwarz method. Bounds for condition numbers of the preconditioned systems are established.

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Chapter 1

Introduction

In approximation theory, objects that are approximated are various. They can be, for example, function values, curves and surfaces, integrals, and solutions to differential or integral equations, all of which have many applications in science and industry. The choices of approximation spaces and methods of approximation are also various. Spaces of splines, wavelets, and radial basis functions are some popular approximation spaces. Lagrange and Hermite interpolation, collocation, least-squares fitting and finite element methods exemplify methods of approximation. The quality of the approximation can also be very flexible in meaning. It can be described, for example, by the visual pleasant of a surface in concern, or it can be addressed by using some more quantitative measure such as Sobolev norms. This wide variation in the choice of approximated objects, approximation subspaces, the method of approximation and the ways of measuring the quality of approximation results in a diverse and rich theory; see [12, 17, 63].

In this dissertation, we focus on the approximation of solutions to a class of pseudodifferential equations on the sphere. Pseudodifferential operators have long been used [33, 39] as a modern and powerful tool to tackle linear boundary-value problems. Svensson [73] introduces this approach to geodesists who study [29, 31] these problems on the sphere which is taken as a model of the earth. Efficient solutions to these equation on the sphere have recently been becoming more and more demanding when more and more satellites have been lauched into the space to collect data around the globe.

Solving pseudodifferential equations on the sphere with spherical radial basis functions and spherical splines will be the main theme of this dissertation. The precise definitions of spherical radial basis functions and spherical splines will be introduced in Chapter 2. It is known [82] that radial basis functions are very suitable for approximation with scattered data, because they can be easily defined at each data point, and the resulting linear system are positive definite. However, since the supports of these functions are usually large, computational effort is wasted when the data points are clustered as large supports excessively overlap; see [37]. Scaling technique can be applied to reduce overlaps and to obtain better-conditioned systems [10, 42] but also reduce the approximation power. On the other hand, finite element methods with spherical splines appear to be accurate [40] and less costly, so they are widely used and trusted by practioners. However, the cost of mesh generation and refinement (to increase accuracy where necessary) is a major part in the total cost of the methods [37]. For problems on the sphere, in particular problems in weather forecasting and geodesy, the deficiency seems to aggravate in regions with densely clustered points.

In the last decades, radial basis functions (in Euclidean spaces) and spherical radial basis functions have been used successfully in interpolation and data fitting problems (see e.g. [6, 47, 81, 82, 69, 25, 32, 38]). The use of these functions in solving pseudodifferential equations on spheres by using collocation methods has been studied by Morton and Neamtu [50]. Error bounds have later been improved by Morton [48, 49] in which the error is estimated in Sobolev norm $\|\cdot\|_{2\alpha}$, where 2α is the order of the operator. The crux of the analysis in [49, 50] is the transformation of the collocation problem to a Lagrange interpolation problem.

Open questions here are that how Galerkin methods can be used to solve pseudodifferential equations with spherical radial basis functions and are there other ways to obtain error analysis for the collocation methods based on that for the Galerkin methods, which is generally simpler, following well-known knowledge on the Galerkin methods. The latter question is inspired by the works [19, 5, 16], which solve quasilinear parabolic equations, pseudodifferential equations on closed curves, and boundary integral equations, respectively. These approaches use either a special set of collocation points or the duality inner product. Answering these two questions will be the first contribution of the dissertation.

The space of spherical splines defined on a spherical triangulation seems particularly appropriate for use on the sphere. It consists of functions whose pieces are spherical harmonics joined together with global smoothness, and thus has both the smoothness and high degree of flexibility [24]. That flexibility makes spherical splines become a powerful tool. These splines have been used successfully in interpolation and data approximation on spheres [4]. Baramidze and Lai [7] use these functions to solve the Laplace–Beltrami equation on the unit sphere.

A highly promising application of spherical splines is to approximate the solutions of pseudodifferential equations. That use has some significant advantages. One of them is the ability to write the approximate solutions of the equations in the form of linear combinations of Bernstein–Bézier polynomials which play an extremely important role in computer aided geometric design, data fitting and interpolation, computer vision and elsewhere; see e.g. [23, 34]. Another advantage is the ability to control the smoothness of a function and its derivatives across edges of the triangulations; see [2].

Our second contribution in this dissertation is solving strongly elliptic pseudodifferential equations by using the Galerkin method with spherical splines. Error analysis with optimal convergence rate is obtained, where the key of the analysis is an approximation property of the spaces of spherical splines as a subspace of Sobolev spaces of negative or positive orders.

When solving pseudodifferential equations on the sphere with either spherical radial basis functions or spherical splines, ill-conditioned linear systems may arise. In the case of spherical radial basis functions, preconditioning by using additive Schwarz methods has been studied in [76]. Bounds for the condition numbers of the preconditioned systems are proved. However, these bounds depend on the number of subdomains and the angles between the subdomains [76].

In the case of spherical splines the question is still open. Our next contribution in this dissertation is to fill this gap. We will design additive Schwarz preconditioners for the Laplace–Beltrami and the hypersingular integral equations.

The additive Schwarz preconditioner is, as usual for finite and boundary methods (see e.g. [74, 78]), defined based on a subspace splitting of the finite dimensional space in which the solution is sought. This splitting is in turn defined by a decomposition of the sphere into subdomains. In our studies of this type of preconditioners, we design an overlapping decomposition method based on a two-level mesh as usual. However, our construction of overlapping subdomains is different from the construction that is generally used in finite element and boundary element literatures, where a fine mesh is created by refining a given coarse mesh. This approach is impossible in our studies because we work with scattered data. In this dissertation, we define the fine and coarse meshes independently from two sets of scattered points, in which the set defining the coarse mesh is reasonably chosen to be coarser than the other set. A subdomain is constructed from each triangle in the coarse mesh by taking the union of all triangles in the fine mesh which intersect this coarse triangle. This results in a set of overlapping subdomains which we use to define our additive Schwarz operator. Bounds for the condition numbers of the preconditioned system are proved, witnessing significant improvements in the condition numbers.

In summary, the contributions of this dissertation are as follows.

- Firstly, we solve, with spherical radial basis functions, pseudodifferential equations of any nature, eliptic or strongly elliptic, and of any order, negative or positive, by using the Galerkin and collocation methods. A unified error analysis for both methods is studied.

- Secondly, we use spherical splines to solve strongly elliptic pseudodifferential equations of any order, negative or positive, by using the Galerkin method. Error analysis is studied.

- Thirdly, we overcome the ill-conditionedness arising when solving, with spherical splines, the Laplace–Beltrami and the hypersingular integral equations on the sphere by using additive Schwarz methods. Bounds for the condition numbers of the preconditioned systems are proved, witnessing significant improvements in the condition numbers.

The dissertation consists of six chapters. Chapter 1 is the introduction. Chapter 2 reviews some important spaces and operators which will be used frequently throughout the dissertation. All the results in this chapter are well-known and can be found in different literatures. Some proofs are presented in this chapter for completeness.

Chapter 3 is devoted to our first contribution. In this chapter, we use spherical radial basis functions to approximate solutions of pseudodifferential equations by using the Galerkin and collocation methods. Error estimates of the collocation method are obtained as a by-product of the analysis for the Galerkin method. The results in this chapter for strongly elliptic pseudodifferential equations have been reported in our article [59]. The results for elliptic operators will be reported in another paper.

In Chapter 4, we present the second contribution of this dissertation. Spherical splines are used to approximate solutions of strongly elliptic pseudodifferential equations. The method to be used is the Galerkin method. The results in this chapter have been reported in our article [60].

In Chapters 5 and 6, we present the use of preconditioners by the additive Schwarz method in solving the Laplace–Beltrami and the hypersingular integral equations with spherical splines. Although the same preconditioner will be used for both equations, different analyses are required due to the difference in Sobolev norms in consideration. The results of Chapter 5 and 6 are reported in our articles [61] and [58].

Chapter 2

Preliminaries

In this chapter we review some important spaces and operators which are used frequently in the rest of the dissertation. We start by discussing spherical harmonics, then review Sobolev spaces on the unit sphere. We then introduce the definition of pseudodifferential operators on the sphere. We finish this chapter with introduction of spherical radial basis functions and spherical splines. We are not going to recall them as a list of statements but present them in the way that they are related to each other and to the problems we are working on. All the results are well-known and can be found in different literatures.

2.1 Spherical harmonics

In this section we introduce spherical harmonics, which are used frequently in this dissertation. Good references for this topic are [51, 55].

Throughout this dissertation, for $n \geq 3$ we denote by $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$, i.e., $\mathbb{S}^{n-1} := \{\boldsymbol{x} \in \mathbb{R}^n : |\boldsymbol{x}| = 1\}$ where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. Let Δ be the Laplace operator in $\mathbb{R}^n$. The *Laplace–Beltrami operator* $\Delta_{\mathbb{S}^{n-1}}$ on $\mathbb{S}^{n-1}$ is defined by

$$\Delta_{\mathbb{S}^{n-1}} v(\boldsymbol{x}) := \Delta v_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{S}^{n-1}, \tag{2.1.1}$$

where, for any $\ell \in \mathbb{N}$, v_ℓ is the *homogeneous extension* of degree ℓ of v to $\mathbb{R}^n$, i.e.,

$$v_\ell(\boldsymbol{x}) := |\boldsymbol{x}|^\ell v\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right), \quad \boldsymbol{x} \in \mathbb{R}^n \backslash \{\boldsymbol{0}\}. \tag{2.1.2}$$

In the case of the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$, it is natural to use spherical coordinates (r, θ, φ), where r is the radius and θ, φ the two Euler angles so that

$$\begin{cases} x_1 = r\sin\theta\cos\varphi, \\ x_2 = r\sin\theta\sin\varphi, \\ x_3 = r\cos\theta. \end{cases}$$

In these coordinates, the Laplace operator has the form

$$\Delta v = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial \varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial v}{\partial\theta}\big)\right), \tag{2.1.3}$$

and the Laplace–Beltrami operator can be represented as

$$\Delta_{\mathbb{S}^2} v = \frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial v}{\partial\theta}\big). \tag{2.1.4}$$

The area element on the sphere $\mathbb{S}^2$ is given by $d\sigma = \sin\theta\, d\theta\, d\varphi$. The operator $\Delta_{\mathbb{S}^2}$ is self–adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_{L_2(\mathbb{S}^2)}$ in $L_2(\mathbb{S}^2)$ given by

$$\langle v, w\rangle_{L_2(\mathbb{S}^2)} = \int_{\mathbb{S}^2} vw\, d\sigma = \int_0^{2\pi}\int_0^{\pi} v(\theta,\varphi)\, w(\theta,\varphi)\, \sin\theta\, d\theta\, d\varphi.$$

Indeed, using integration by parts, we have

$$\begin{aligned}
\langle \Delta_{\mathbb{S}^2} v, w\rangle_{L_2(\mathbb{S}^2)} &= \int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{\sin^2\theta}\frac{\partial^2 v}{\partial\varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial v}{\partial\theta}\big)\right) w\, \sin\theta\, d\theta\, d\varphi \\
&= \int_0^{\pi}\frac{1}{\sin\theta}\int_0^{2\pi}\frac{\partial^2 v}{\partial\varphi^2} w\, d\varphi\, d\theta + \int_0^{2\pi}\int_0^{\pi}\frac{\partial}{\partial\theta}\big(\sin\theta\frac{\partial v}{\partial\theta}\big) w\, d\theta\, d\varphi \\
&= \int_0^{\pi}\frac{1}{\sin\theta}\left(\frac{\partial v}{\partial\varphi} w\Big|_{\varphi=0}^{2\pi} - \int_0^{2\pi}\frac{\partial v}{\partial\varphi}\frac{\partial w}{\partial\varphi}\, d\varphi\right) d\theta \\
&+ \int_0^{2\pi}\left(\sin\theta\,\frac{\partial v}{\partial\theta} w\Big|_{\theta=0}^{\pi} - \int_0^{\pi}\sin\theta\frac{\partial v}{\partial\theta}\frac{\partial w}{\partial\theta}\, d\theta\right) d\varphi \\
&= -\int_0^{2\pi}\int_0^{\pi}\left(\frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi}\frac{\partial w}{\partial\varphi} + \sin\theta\frac{\partial v}{\partial\theta}\frac{\partial w}{\partial\theta}\right) d\theta\, d\varphi \\
&= -\int_0^{2\pi}\int_0^{\pi}\left[\left(\frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi}\frac{1}{\sin\theta}\frac{\partial w}{\partial\varphi}\right) + \left(\frac{\partial v}{\partial\theta}\frac{\partial w}{\partial\theta}\right)\right] \sin\theta\, d\theta\, d\varphi \\
&= \langle v, \Delta_{\mathbb{S}^2} w\rangle_{L_2(\mathbb{S}^2)}\,.
\end{aligned} \tag{2.1.5}$$

The *surface gradient* of function v, denote by $\nabla_{\mathbb{S}^2} v$, is defined by

$$\nabla_{\mathbb{S}^2} v := \frac{1}{\sin\theta}\frac{\partial v}{\partial\varphi}\,\boldsymbol{e}_\varphi + \frac{\partial v}{\partial\theta}\,\boldsymbol{e}_\theta, \tag{2.1.6}$$

where $\boldsymbol{e}_\phi$ and $\boldsymbol{e}_\theta$ are the two unit vectors corresponding to the two Euler angles. It follows from (2.1.5) that

$$-\int_{\mathbb{S}^2} \Delta_{\mathbb{S}^2} v\, w\, d\sigma = \int_{\mathbb{S}^2} (\nabla_{\mathbb{S}^2} v\cdot\nabla_{\mathbb{S}^2} w)\, d\sigma. \tag{2.1.7}$$

Definition 2.1. *A function v defined in $\mathbb{R}^n$ is called a homogeneous function of homogeinity degree ℓ, for $\ell\in\mathbb{N}$, if v satisfies*

$$v(t\boldsymbol{x}) = t^\ell v(\boldsymbol{x}) \quad \forall t\in\mathbb{R} \quad \textit{and} \quad \forall \boldsymbol{x}\in\mathbb{R}^n.$$

We note that if v is a homogeneous polynomial then its polynomial degree equals to its homogeinity degree.

Definition 2.2. *A spherical harmonic v_ℓ of order ℓ on the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ is the restriction of a homogeneous polynomial $\overline{v}_\ell$ of degree ℓ defined in $\mathbb{R}^n$ which is harmonic, i.e. which satisfies*

$$\Delta\overline{v}_\ell(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in\mathbb{R}^n.$$

From Definition 2.2 we have immediately the following corollary.

Corollary 2.3. *Let v_ℓ be a spherical harmonic of degree ℓ. There holds*

$$v_\ell(-\boldsymbol{x}) = (-1)^\ell v_\ell(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{S}^{n-1}.$$

Proposition 2.4. *Let k and ℓ be two distinct nonnegative integers. Let v_k and v_ℓ be two spherical harmonics of degree k and ℓ, respectively. There holds*

$$\langle v_k, v_\ell \rangle_{L_2(\mathbb{S}^{n-1}))} := \int_{\mathbb{S}^{n-1}} v_k(\boldsymbol{x}) v_\ell(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}} = 0.$$

Proof. We assume that $\overline{v_k}$ and $\overline{v_\ell}$ are two homogeneous harmonic polynomials of degree k and ℓ in $\mathbb{R}^n$ satisfying

$$\overline{v}_k(\boldsymbol{x}) = v_k(\boldsymbol{x}) \quad \text{and} \quad \overline{v}_\ell(\boldsymbol{x}) = v_\ell(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \mathbb{S}^{n-1}.$$

Then we have

$$\overline{v}_k(\boldsymbol{x}) = t^{-k}\overline{v}_k(t\boldsymbol{x}), \quad \overline{v}_\ell(\boldsymbol{x}) = t^{-\ell}\overline{v}_\ell(t\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \mathbb{S}^{n-1}, \quad \forall t \neq 0.$$

Because of the homogeneity of $\overline{v}_k$ and $\overline{v}_\ell$, the corresponding normal derivatives on $\mathbb{S}^{n-1}$ are

$$\frac{\partial \overline{v}_k}{\partial \boldsymbol{n}}(\boldsymbol{x}) = k\overline{v}_k(\boldsymbol{x}) \quad \text{and} \quad \frac{\partial \overline{v}_\ell}{\partial \boldsymbol{n}}(\boldsymbol{x}) = \ell\overline{v}_\ell(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \mathbb{S}^{n-1}. \tag{2.1.8}$$

Noting that $\overline{v}_k$ and $\overline{v}_\ell$ are harmonic polynomials, using (2.1.8) and Green's Theorem, we have

$$0 = \iiint_{|\boldsymbol{x}| \le 1} (\overline{v}_\ell \Delta \overline{v}_k - \overline{v}_k \Delta \overline{v}_\ell)\, dV = \int_{\mathbb{S}^{n-1}} \overline{v}_k\, \overline{v}_\ell\, (k-\ell)\, d\sigma.$$

Hence, in the case $k \neq \ell$, there holds

$$\int_{\mathbb{S}^{n-1}} v_k(\boldsymbol{x}) v_\ell(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}} = 0,$$

completing the proof. □

In this dissertation, we denote by $\mathcal{P}_\ell$ the space of polynomials in $\mathbb{R}^n$ of degree less than or equal to ℓ. Let Π_ℓ be the space of homogeneous polynomials of degree ℓ in $\mathbb{R}^n$ and let $\mathcal{H}_\ell$ be the space of homogeneous harmonic polynomials of degree ℓ in $\mathbb{R}^n$. Denote by $\widetilde{\Pi}_\ell$ and $\mathbb{H}_\ell$ the sets of restrictions of polynomials in Π_ℓ and $\mathcal{H}_\ell$, respectively, on the unit sphere $\mathbb{S}^{n-1}$.

Proposition 2.5. *The space $\mathbb{H}_\ell$ of spherical harmonics of degree ℓ on $\mathbb{S}^{n-1}$ has dimension $N(n,\ell)$ being*

$$N(n,0) = 1 \quad \textit{and} \quad N(n,\ell) = \frac{2\ell+n-2}{\ell}\binom{\ell+n-3}{\ell-1}, \quad \ell \neq 0. \tag{2.1.9}$$

Proof. It is clear that $N(n,0) = 1$. We consider the case $\ell \ge 1$. By the definition of spherical harmonics, the dimension of the space of spherical harmonics of degree ℓ is equal to the dimension of the space of harmonic homogeneous polynomials of degree ℓ in $\mathbb{R}^n$. Any homogeneous polynomial $\overline{v}_\ell$ of degree ℓ in $\mathbb{R}^n$ can be represented as

$$\overline{v}_\ell(\boldsymbol{x}) = \sum_{i=0}^{\ell} x_n^i\, H_{\ell-i}(x_1, \ldots, x_{n-1}),$$

where $H_{\ell-i}(x_1,\dots,x_{n-1})$ are homogeneous polynomials of degree $\ell-i$ with variables $x_1,\dots,x_{n-1}$.

In this proof only, we denote by Δ_n the Laplace operator corresponding to $\mathbb{R}^n$. Since $\Delta_n = \frac{\partial^2}{\partial x_n^2} + \Delta_{n-1}$, we have

$$\Delta_n \overline{v}_\ell(\boldsymbol{x}) = \sum_{i=2}^{\ell} i(i-1)x_n^{i-2}H_{\ell-i}(x_1,\dots,x_{n-1}) + \sum_{i=0}^{\ell-2} x_n^i \Delta_{n-1}H_{\ell-i}(x_1,\dots,x_{n-1}).$$

If $\overline{v}_\ell$ is a harmonic polynomial then $\Delta_n\overline{v}_\ell(\boldsymbol{x}) = 0$ for all $\boldsymbol{x}\in\mathbb{R}^n$. This happens when

$$\Delta_{n-1}H_{\ell-i} = -(i+2)(i+1)H_{\ell-i-2}, \quad i = 0,\dots,\ell-2.$$

Therefore all the polynomials H_i, for $i=0,\dots,\ell$, are determined by H_ℓ and $H_{\ell-1}$. Hence the number $N(n,\ell)$ of linearly independent homogeneous and harmonic polynomials of degree ℓ in $\mathbb{R}^n$ is equal to the number of coefficients of H_ℓ and $H_{\ell-1}$. Denote by $M(n,\ell)$ the number of coefficients in a homogeneous polynomial of degree ℓ and n variables. It is well–known that

$$M(n,\ell) = \frac{(\ell+n-1)!}{\ell!(n-1)!}. \tag{2.1.10}$$

It follows that

$$\begin{aligned} N(n,\ell) &= M(n-1,\ell) + M(n-1,\ell-1) = \frac{(\ell+n-2)!}{\ell!(n-2)!} + \frac{(\ell+n-3)!}{(\ell-1)!(n-2)!} \\ &= \frac{2\ell+n-2}{\ell}\binom{\ell+n-3}{\ell-1}, \end{aligned}$$

completing the proof. $\square$

The following proposition will be frequently used in Chapters 5 and 6.

Proposition 2.6. *The space $\widetilde{\Pi}_\ell$ of spherical homogenous polynomials of degree ℓ contains constant functions if and only if ℓ is even.*

Proof. If ℓ is even, then for any constant c, the polynomial

$$v(\boldsymbol{x}) = c(x_1^2 + x_2^2 + \ldots + x_n^2)^{\ell/2} \quad \forall \boldsymbol{x} = (x_1,x_2,\dots,x_n)\in\mathbb{S}^{n-1}$$

belongs to the space $\widetilde{\Pi}_\ell$ and $v(\boldsymbol{x}) = c$ for all $\boldsymbol{x}\in\mathbb{S}^{n-1}$. On the other hand, if ℓ is odd and if there exists $v\in\widetilde{\Pi}_\ell$ such that $v(\boldsymbol{x}) = 1$, then by homogeneity we have

$$1 = v(-\boldsymbol{x}) = (-1)^\ell v(\boldsymbol{x}) = -1.$$

This contradiction shows that when ℓ is odd, the space $\widetilde{\Pi}_\ell$ does not contain constant functions. $\square$

Recalling the dimension $N(n,\ell)$ of the space of spherical harmonic of degree ℓ as given by (2.1.9), we may choose for this space an orthonormal basis $\{Y_{\ell,m}\}_{m=1}^{N(n,\ell)}$, i.e.,

$$\langle Y_{\ell,m}, Y_{\ell,k}\rangle_{L_2(\mathbb{S}^{n-1})} = \delta_{mk}, \quad m,k = 1,\dots,N(n,\ell). \tag{2.1.11}$$

Proposition 2.7. *The collection of all spherical harmonics*

$$\{Y_{\ell,m} : m = 1, \ldots, N(n,\ell), \ell \geq 0\}$$

forms an orthonormal basis for $L_2(\mathbb{S}^{n-1})$. *Moreover, the space* $\mathbb{H}_\ell$ *of spherical harmonics of degree* ℓ *coincides with the subspace spanned by the eigenfunctions of the Laplace–Beltrami operator associated with the eigenvalue* $\lambda_\ell = -\ell(\ell+n-2)$, *i.e.,*

$$\Delta_{\mathbb{S}^{n-1}} Y_{\ell,m} + \lambda_\ell Y_{\ell,m} = 0. \tag{2.1.12}$$

The eigenvalue λ_ℓ *has multiplicity* $N(n,\ell)$.

Proof. The orthogonality in $L_2(\mathbb{S}^{n-1})$ results from the orthogonality of the subspaces $\mathbb{H}_k$ and $\mathbb{H}_\ell$ when $k \neq \ell$; see Proposition 2.4. It remains to show (2.1.12). In the spherical coordinates in n dimensions, with the parametrisation $\boldsymbol{x} = r\boldsymbol{\theta}$ with r representing a positive real radius and $\boldsymbol{\theta}$ an element of the unit sphere $\mathbb{S}^{n-1}$, the Laplace operator and the Laplace–Beltrami operator are related to each other by

$$\Delta v = \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\Delta_{\mathbb{S}^{n-1}} v;$$

see e.g. (2.1.3) and (2.1.4). For any spherical harmonic $Y_{\ell,m}$, we denote by $p_{\ell,m}$ the homogeneous harmonic polynomial in $\mathbb{R}^n$ which is associated with $Y_{\ell,m}$. We have

$$\Delta_{\mathbb{S}^{n-1}} Y_{\ell,m} + r^2 \frac{\partial^2 p_{\ell,m}}{\partial r^2} + r(n-1)\frac{\partial p_{\ell,m}}{\partial r} = 0. \tag{2.1.13}$$

Since $p_{\ell,m}$ is homogeneous of degree ℓ, we have

$$\frac{\partial p_{\ell,m}}{\partial r} = \ell\,\frac{p_{\ell,m}}{r} \quad \text{and} \quad \frac{\partial^2 p_{\ell,m}}{\partial r^2} = \ell(\ell-1)\,\frac{p_{\ell,m}}{r^2}.$$

This together with (2.1.13) yields

$$\Delta_{\mathbb{S}^{n-1}} Y_{\ell,m} + \ell(\ell+n-2) Y_{\ell,m} = 0,$$

completing the proof. □

Proposition 2.8. *The space* Π_ℓ *is the direct sum of the space* $\mathcal{H}_\ell$ *and the space* $|\boldsymbol{x}|^2\Pi_{\ell-2}$, *i.e., each* p *in* Π_ℓ *can be uniquely written as*

$$p = p_1 + |\boldsymbol{x}|^2 p_2, \quad p_1 \in \mathcal{H}_\ell, \tag{2.1.14}$$

Proof. It is obvious that $\mathcal{H}_\ell$ and $|\boldsymbol{x}|^2\Pi_{\ell-2}$ are subspaces of Π_ℓ. Noting that $\dim(\mathcal{H}_\ell) = \dim(\mathbb{H}_\ell)$, Proposition 2.5 gives

$$\dim(\mathcal{H}_\ell) = N(n,\ell) = \frac{2\ell+n-2}{\ell}\binom{\ell+n-3}{\ell-1},$$

when $\ell \neq 0$. It is also known (2.1.10) that

$$\dim(\Pi_\ell) = \frac{(\ell+\mathrm{n}-1)!}{\ell!(\mathrm{n}-1)!} \quad \text{and} \quad \dim(|\boldsymbol{x}|^2\Pi_{\ell-2}) = \frac{(\ell+\mathrm{n}-3)!}{(\ell-2)!(\mathrm{n}-1)!}$$

A simple calculation reveals

$$\dim(\Pi_\ell) = \dim(\mathcal{H}_\ell) + \dim(|\boldsymbol{x}|^2\Pi_{\ell-2}). \tag{2.1.15}$$

We now prove that

$$\mathcal{H}_\ell \ \cap \ |\boldsymbol{x}|^2\Pi_{\ell-2} = \{\mathbf{0}\}. \tag{2.1.16}$$

Let $p_\ell \in \mathcal{H}_\ell \ \cap \ |\boldsymbol{x}|^2\Pi_{\ell-2}$. Then $p_\ell(\boldsymbol{x}) = |\boldsymbol{x}|^2 p_{\ell-2}(\boldsymbol{x})$, where $p_{\ell-2}$ is some polynomial in $\Pi_{\ell-2}$. Elementary calculations give

$$\Delta p_\ell(\boldsymbol{x}) = 2np_{\ell-2}(\boldsymbol{x}) + 4\sum_{i=1}^{n} x_i \frac{\partial p_{\ell-2}}{\partial x_i}(\boldsymbol{x}) + |\boldsymbol{x}|^2 \Delta p_{\ell-2}(\boldsymbol{x}).$$

By writing

$$p_{\ell-2}(\boldsymbol{x}) = \sum_{i_1+\ldots+i_n=\ell-2} c_{i_1,\ldots,i_n}\, x_1^{i_1}\ldots x_n^{i_n},$$

we have

$$\begin{aligned}\Delta p_\ell(\boldsymbol{x}) = 2n &\sum_{i_1+\ldots+i_n=\ell-2} c_{i_1,\ldots,i_n}\, x_1^{i_1}\ldots x_n^{i_n} + 4\sum_{k=1}^{n} \sum_{\substack{i_1+\ldots+i_n=\ell-2\\ i_k\geq 1}} i_k\, c_{i_1,\ldots,i_n}\, x_1^{i_1}\ldots x_n^{i_n} \\ &+ |\boldsymbol{x}|^2 \sum_{k=1}^{n} \sum_{\substack{i_1+\ldots+i_n=\ell-2\\ i_k\geq 2}} i_k(i_k-1)\, c_{i_1,\ldots,i_n}\, x_1^{i_1}\ldots x_k^{i_k-2}\ldots x_n^{i_n}.\end{aligned} \tag{2.1.17}$$

The coefficient corresponding to the monomial $x_1^{i_1}\ldots x_n^{i_n}$ is given by

$$c_{i_1,\ldots,i_n}\Big(2n + 4\sum_{\substack{k=1,\ldots,n\\ i_k\geq 1}} i_k + \sum_{\substack{k=1,\ldots,n\\ i_k\geq 2}} i_k(i_k-1)\Big) \tag{2.1.18}$$

Since p_ℓ is a harmonic polynomial, all the coefficients defined in (2.1.18) are equal to zero. It is easy to see that the factor in the bracket is positive, thus $c_{i_1,\ldots,i_n}$ must be zero. Hence (2.1.16) is proved. This together with (2.1.15) assures that Π_ℓ is a direct sum of $\mathcal{H}_\ell$ and $|\boldsymbol{x}|^2\Pi_{\ell-2}$, completing the proof of the proposition. □

Proposition 2.8 immediately implies

$$\begin{cases} \widetilde{\Pi}_\ell = \mathbb{H}_0 \ \oplus \mathbb{H}_2 \ \oplus \ldots \oplus \mathbb{H}_\ell & \text{if } \ell \text{ is even} \\ \widetilde{\Pi}_\ell = \mathbb{H}_1 \ \oplus \mathbb{H}_3 \ \oplus \ldots \oplus \mathbb{H}_\ell & \text{if } \ell \text{ is odd.} \end{cases}$$

Definition 2.9. *For any $\ell \geq 0$, the Legendre polynomial P_ℓ of degree ℓ defined on the interval $[-1,1]$ is given by*

$$P_\ell(t) = \frac{(-1)^\ell}{2^\ell \ell!}\left(\frac{d}{dt}\right)^\ell (1-t^2)^\ell, \quad t \in [-1,1].$$

The following proposition will be used frequently in this dissertation. We omit here the proof which can be easily found in many literatures; see e.g. [55].

Proposition 2.10. *Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two unit vectors. The following addition formula holds:*

$$\sum_{m=1}^{N(n,\ell)} Y_{\ell,m}(\boldsymbol{x})Y_{\ell,m}(\boldsymbol{y}) = \frac{N(n,\ell)}{\omega_n} P_\ell(\boldsymbol{x}\cdot\boldsymbol{y}), \tag{2.1.19}$$

where $Y_{\ell,m}$ are the spherical harmonics and P_ℓ the Legendre polynomials of orders ℓ. In particular, we have

$$\sum_{m=1}^{N(n,\ell)} |Y_{\ell,m}(\boldsymbol{x})|^2 = \frac{N(n,\ell)}{\omega_n}.$$

Here, $N(n,\ell)$ is the dimension of the space of homogeneous harmonic polynomials of degree ℓ in $\mathbb{R}^n$; see (2.1.9), *and ω_n is the surface area of the unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$.*

2.2 Sobolev spaces on the unit sphere

In this section we discuss the definition of Sobolev spaces defined on the unit sphere $\mathbb{S}^{n-1}$. Recall that we denote by $\{Y_{\ell,m} : m = 1,\dots,N(n,\ell),\ \ell \ge 0\}$ the set of spherical harmonics which form an orthogonal basis for $L_2(\mathbb{S}^{n-1})$; see Proposition 2.7. For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{S}^{n-1})$ is defined by

$$H^s(\mathbb{S}^{n-1}) := \Big\{ v \in \mathscr{D}'(\mathbb{S}^{n-1}) : \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s} |\widehat{v}_{\ell,m}|^2 < \infty \Big\},$$

where $\mathscr{D}'(\mathbb{S}^{n-1})$ is the space of distributions on $\mathbb{S}^{n-1}$ and $\widehat{v}_{\ell,m}$ are the Fourier coefficients of v,

$$\widehat{v}_{\ell,m} = \langle v, Y_{\ell,m}\rangle_{L_2(\mathbb{S}^{n-1})}.$$

The space $H^s(\mathbb{S}^{n-1})$ is equipped with the following norm and inner product:

$$\|v\|_{H^s(\mathbb{S}^{n-1})} := \left(\sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s} |\widehat{v}_{\ell,m}|^2 \right)^{1/2} \tag{2.2.1}$$

and

$$\langle v, w\rangle_{H^s(\mathbb{S}^{n-1})} := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s} \widehat{v}_{\ell,m} \widehat{w}_{\ell,m}.$$

We note that the series on the right hand side also converges when $v \in H^{s+\sigma}(\mathbb{S}^{n-1})$ and $w \in H^{s-\sigma}(\mathbb{S}^{n-1})$ for any $\sigma > 0$. Therefore, in the following we use the same notation $\langle\cdot,\cdot\rangle_{H^s(\mathbb{S}^{n-1})}$ for the duality product between $H^{s+\sigma}(\mathbb{S}^{n-1})$ and $H^{s-\sigma}(\mathbb{S}^{n-1})$.

In the rest of the dissertation, we denote $H^s := H^s(\mathbb{S}^{n-1})$ and the corresponding norm and inner product $\|\cdot\|_s := \|\cdot\|_{H^s(\mathbb{S}^{n-1})}$ and $\langle\cdot,\cdot\rangle_s := \langle\cdot,\cdot\rangle_{H^s(\mathbb{S}^{n-1})}$, respectively. When $s = 0$ we write $\langle\cdot,\cdot\rangle$ instead of $\langle\cdot,\cdot\rangle_0$; this is in fact the L_2-inner product. In this dissertation, we will frequently use the Cauchy–Schwarz inequality

$$|\langle v, w\rangle_s| \le \|v\|_s \|w\|_s \quad \text{for all } v, w \in H^s, \text{ for all } s \in \mathbb{R}, \tag{2.2.2}$$

and the following identity which can be easily proved

$$\|v\|_{s_1} = \sup_{\substack{w \in H^{s_2} \\ w \neq 0}} \frac{\langle v, w \rangle_{\frac{s_1+s_2}{2}}}{\|w\|_{s_2}} \quad \text{for all } v \in H^{s_1}, \text{ for all } s_1, s_2 \in \mathbb{R}. \tag{2.2.3}$$

In the case k belongs to the set of nonnegative integers $\mathbb{Z}^+$, the Sobolev space $H^k(\Omega)$ on a subset $\Omega \subset \mathbb{S}^{n-1}$ can also be defined by using atlas for the unit sphere $\mathbb{S}^{n-1}$ [54]. Let $\{(\Gamma_j, \phi_j)\}_{j=1}^J$ be an atlas for Ω, i.e, a finite collection of charts (Γ_j, ϕ_j), where Γ_j are open subsets of Ω, covering Ω, and where $\phi_j : \Gamma_j \to B_j$ are infinitely differentiable mappings whose inverses ϕ_j^{-1} are also infinitely differentiable. Here B_j, $j = 1, \dots, J$, are open subsets in $\mathbb{R}^2$. Also, let $\{\alpha_j\}_{j=1}^J$ be a partition of unity subordinate to the atlas $\{(\Gamma_j, \phi_j)\}_{j=1}^J$, i.e., a set of infinitely differentiable functions α_j on Ω vanishing outside the sets Γ_j, such that $\sum_{j=1}^J \alpha_j = 1$ on Ω. For any $k \in \mathbb{Z}^+$, the Sobolev space $H^k(\Omega)$ on the unit sphere is defined as follows

$$H^k(\Omega) := \{v : (\alpha_j v) \circ \phi_j^{-1} \in H^k(B_j),\ j = 1, \dots, J\}, \tag{2.2.4}$$

which is equipped with a norm defined by

$$\|v\|_{H^k(\Omega)}^* := \sum_{j=1}^J \|(\alpha_j v) \circ \phi_j^{-1}\|_{H^k(B_j)}. \tag{2.2.5}$$

Here, $\|\cdot\|_{H^k(B_j)}$ denotes the usual H^k-Sobolev norm defined on the subset B_j of the plane $\mathbb{R}^2$. In the case $\Omega = \mathbb{S}^{n-1}$, this norm is equivalent to the norm defined in (2.2.1); see [43].

Let $v \in H^k(\Omega)$, $k \geq 1$. For each $l = 1, \dots, k$, let v_{l-1} denote the unique homogeneous extension of v of degree $l - 1$; see (2.1.2). Then

$$|v|_{H^l(\Omega)} := \sum_{|\alpha|=l} \|D^\alpha v_{l-1}\|_{L_2(\Omega)} \tag{2.2.6}$$

is a Sobolev-type seminorm of v in $H^k(\Omega)$. Here $\|D^\alpha v_{l-1}\|_{L_2(\Omega)}$ is understood as the L_2-norm of the restriction of the trivariate function $D^\alpha v_{l-1}$ to Ω. When $l = 0$ we define

$$|v|_{H^0(\Omega)} := \|v\|_{L_2(\Omega)},$$

which can now be used together with (2.2.6) to define another norm in $H^k(\Omega)$:

$$\|v\|'_{H^k(\Omega)} := \sum_{l=0}^k |v|_{H^l(\Omega)}. \tag{2.2.7}$$

This norm is equivalent to the norm $\|\cdot\|_{H^k(\Omega)}^*$ defined by (2.2.5); see [54]. Hence, in the case $\Omega = \mathbb{S}^{n-1}$, this norm turns out to be equivalent to the Sobolev norm defined in (2.2.1).

The following theorem, which will be frequently used in the whole dissertation, is true for Hilbert spaces. We recall here only the result for the Sobolev spaces. For the proof of the theorem, please refer to [46, Theorem B.2].

Theorem 2.11. *Let $s_1, s_2, t_1, t_2 \in \mathbb{R}$ be such that $s_1 \le s_2$ and $t_1 \le t_2$. Assume that $T : H^{s_i} \to H^{t_i}$, $i = 1, 2$, are bounded linear operators satisfying*

$$\|Tv\|_{t_i} \le M_i \|v\|_{s_i} \quad \forall v \in H^{s_i},$$

for some $M_i \ge 0$, $i = 1, 2$. Then for any $\theta \in [0, 1]$, the operator $T : H^{\theta s_1 + (1-\theta)s_2} \to H^{\theta t_1 + (1-\theta)t_2}$ is bounded, and there holds

$$\|Tv\|_{\theta t_1 + (1-\theta)t_2} \le M_1^{\theta} M_2^{1-\theta} \|v\|_{\theta s_1 + (1-\theta)s_2} \quad \forall v \in H^{\theta s_1 + (1-\theta)s_2}.$$

2.3 Pseudodifferential operators

Let $\{\widehat{L}(\ell)\}_{\ell \ge 0}$ be a sequence of real numbers. A pseudodifferential operator L is a linear operator that assigns to any $v \in \mathscr{D}'(\mathbb{S}^{n-1})$ a distribution

$$Lv := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell) \widehat{v}_{\ell,m} Y_{\ell,m}.$$

The sequence $\{\widehat{L}(\ell)\}_{\ell \ge 0}$ is referred to as the *spherical symbol* of L. Let $\mathcal{K}(L) := \{\ell \in \mathbb{N} : \widehat{L}(\ell) = 0\}$. Then

$$\ker L = \operatorname{span}\{Y_{\ell,m} : \ell \in \mathcal{K}(L),\ m = 1, \dots, N(n, \ell)\}.$$

Denoting $M := \dim \ker L$, we assume that $0 \le M < \infty$.

Definition 2.12. *A pseudodifferential operator L is said to be of order 2α for some $\alpha \in \mathbb{R}$ if there exists a positive constant C such that*

$$|\widehat{L}(\ell)| \le C(\ell + 1)^{2\alpha} \quad \textit{for all } \ell \ge 0. \tag{2.3.1}$$

A pseudodifferential operator L of order 2α is said to be elliptic if

$$C_1(\ell + 1)^{2\alpha} \le |\widehat{L}(\ell)| \le C_2(\ell + 1)^{2\alpha} \quad \textit{for all } \ell \notin \mathcal{K}(L), \tag{2.3.2}$$

and strongly elliptic if

$$C_1(\ell + 1)^{2\alpha} \le \widehat{L}(\ell) \le C_2(\ell + 1)^{2\alpha} \quad \textit{for all } \ell \notin \mathcal{K}(L), \tag{2.3.3}$$

for some positive constants C_1 and C_2.

More general pseudodifferential operators can be defined via Fourier transforms by using local charts; see e.g., [36, 57].

It can be easily seen that if L is a pseudodifferential operator of order 2α then $L : H^{s+\alpha} \to H^{s-\alpha}$ is bounded for all $s \in \mathbb{R}$.

The following commonly seen pseudodifferential operators are strongly elliptic; see [73].

(i) The Laplace–Beltrami operator (with the minus sign) is an operator of order 2 and has as symbol $\widehat{L}(\ell) = \ell(\ell + n - 2)$. Indeed, this can be derived directly from Proposition 2.7.

(ii) The weakly singular integral operator, which arises from the boundary-integral reformulation of the Dirichlet problem with the Laplacian in the interior or exterior of the sphere, is an operator of order -1 and has as symbol $\omega_n/(4\pi N(n,\ell))$, where ω_n is the surface area of the unit sphere $\mathbb{S}^{n-1}$.

(iii) The hypersingular integral operator (with the minus sign), which arises from the boundary-integral reformulation of the Neumann problem with the Laplacian in the interior or exterior of the sphere, is an operator of order 1 and has as symbol $\widehat{L}(\ell) = \ell(\ell+1)\omega_n/(4\pi N(n,\ell))$.

We define a bilinear form $a(\cdot,\cdot) : H^{\alpha+s}(\mathbb{S}^{n-1}) \times H^{\alpha-s}(\mathbb{S}^{n-1}) \to \mathbb{R}$, for any $s \in \mathbb{R}$, by

$$a(w,v) := \langle Lw, v\rangle \quad \text{for all } w \in H^{\alpha+s}(\mathbb{S}^{n-1}),\ v \in H^{\alpha-s}(\mathbb{S}^{n-1}). \tag{2.3.4}$$

In the sequel, for any $x, y \in \mathbb{R}$, $x \preceq y$ means that there exist a positive constant C satisfying $x \le Cy$, and $x \simeq y$ if $x \preceq y$ and $y \preceq x$.

The following simple results are often used in the rest of the dissertation.

Lemma 2.13. *Let s be any real number.*

1. *The bilinear form $a(\cdot,\cdot) : H^{\alpha+s} \times H^{\alpha-s} \to \mathbb{R}$ is bounded, i.e.,*

$$|a(w,v)| \le C\|w\|_{\alpha+s}\|v\|_{\alpha-s} \quad \text{for all } w \in H^{\alpha+s},\ v \in H^{\alpha-s}. \tag{2.3.5}$$

2. *If $w, v \in H^s$, then*

$$|\langle Lw, v\rangle_{s-\alpha}| \le C\|w\|_s\|v\|_s. \tag{2.3.6}$$

3. *Assume that L is strongly elliptic. If $v \in (\ker L)^{\perp}_{H^s}$, then*

$$\langle Lv, v\rangle_{s-\alpha} \simeq \|v\|_s^2. \tag{2.3.7}$$

In particular, setting $s = \alpha$, there holds $a(v,v) \simeq \|v\|_\alpha^2$ for all $v \in (\ker L)^{\perp}_{H^\alpha}$.

Here C is a constant independent of v and w.

Proof. Let $w \in H^{\alpha+s}$ and $v \in H^{\alpha-s}$. Noting (2.3.3) and using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}
|a(w,v)| &\le \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} |\widehat{L}(\ell)||\widehat{w}_{\ell,m}||\widehat{v}_{\ell,m}| \le C\sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{2\alpha}|\widehat{w}_{\ell,m}||\widehat{v}_{\ell,m}| \\
&= C\sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{\alpha+s}|\widehat{w}_{\ell,m}|(\ell+1)^{\alpha-s}|\widehat{v}_{\ell,m}| \\
&\le C\left(\sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{2(\alpha+s)}|\widehat{w}_{\ell,m}|^2\right)^{1/2}\left(\sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} (\ell+1)^{2(\alpha-s)}|\widehat{v}_{\ell,m}|^2\right)^{1/2} \\
&= C\|w\|_{\alpha+s}\|v\|_{\alpha-s},
\end{aligned}$$

proving (2.3.5). The proof for (2.3.6) and (2.3.7) can be done similarly, noting the definition (2.3.3) of strongly elliptic operators, and noting that $v \in (\ker L)^{\perp}_{H^s}$ if and only if $v \in H^s$ and $\widehat{v}_{\ell,m} = 0$ for all $\ell \in \mathcal{K}(L)$ and $m = 1,\dots,N(n,\ell)$. □

2.4 Spherical radial basis functions

Spherical radial basis function approximation has its roots in physically motivated problems [30]. It was independently developed by [26] and [79]. The use of these functions results in meshless methods which, over the past few years, become more and more popular [81, 82]. These methods are alternatives to finite-element methods. The list of applications of spherical radial basis functions in geophysics and physical geodesy is long (cf. [26], [27], [28], and many others). In this dissertation we shall use the spaces of spherical radial basis functions on spheres to solve pseudodifferential equations on the sphere.

In this section, we review the definition of spherical radial basis functions which are defined from kernels.

2.4.1 Positive-definite kernels

A continuous function $\Theta : \mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \to \mathbb{R}$ is called a *positive-definite kernel* on $\mathbb{S}^{n-1}$ if it satisfies

(i) $\Theta(\boldsymbol{x}, \boldsymbol{y}) = \Theta(\boldsymbol{y}, \boldsymbol{x})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^{n-1}$,

(ii) for any positive integer N and any set of distinct points $\{\boldsymbol{y}_1, \ldots, \boldsymbol{y}_N\}$ on $\mathbb{S}^{n-1}$, the $N \times N$ matrix $\boldsymbol{B}$ with entries $\boldsymbol{B}_{i,j} = \Theta(\boldsymbol{y}_i, \boldsymbol{y}_j)$ is positive-semidefinite.

If the matrix $\boldsymbol{B}$ is positive-definite then Θ is called a *strictly positive-definite kernel*; see [68, 84].

We characterise the kernel Θ by a *shape function* θ as follows. Let $\theta : [-1, 1] \to \mathbb{R}$ be a univariate function having a series expansion in terms of Legendre polynomials,

$$\theta(t) = \sum_{\ell=0}^{\infty} \omega_n^{-1} N(n, \ell) \widehat{\theta}(\ell) P_\ell(t), \tag{2.4.1}$$

where ω_n is the surface area of the sphere $\mathbb{S}^{n-1}$, and $\widehat{\theta}(\ell)$ is the *Fourier–Legendre coefficient*,

$$\widehat{\theta}(\ell) = \omega_{n-1} \int_{-1}^{1} \theta(t) P_\ell(t) (1 - t^2)^{(n-3)/2} \, dt.$$

Here, $P_\ell(t)$ denotes the degree ℓ normalised Legendre polynomial in n variables so that $P_\ell(1) = 1$, as described in [51]. Using this shape function θ, we define

$$\Theta(\boldsymbol{x}, \boldsymbol{y}) := \theta(\boldsymbol{x} \cdot \boldsymbol{y}) \quad \text{for all } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{S}^{n-1}, \tag{2.4.2}$$

where $\boldsymbol{x} \cdot \boldsymbol{y}$ denotes the scalar product between $\boldsymbol{x}$ and $\boldsymbol{y}$. We note that $\boldsymbol{x} \cdot \boldsymbol{y}$ is the cosine of the angle between $\boldsymbol{x}$ and $\boldsymbol{y}$, which is called the geodesic distance between the two points. Thus the kernel Θ is a zonal kernel. By using the addition formula (2.1.19), we can write

$$\Theta(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{\theta}(\ell) Y_{\ell,m}(\boldsymbol{x}) Y_{\ell,m}(\boldsymbol{y}). \tag{2.4.3}$$

Remark 2.14. In [11], a complete characterisation of strictly positive-definite kernels is established: the kernel Θ is strictly positive-definite if and only if $\widehat{\theta}(\ell) \geq 0$ for all $\ell \geq 0$, and $\widehat{\theta}(\ell) > 0$ for infinitely many even values of ℓ and infinitely many odd values of ℓ; see also [68] and [84].

In the remainder of this subsection, we will define a space of spherical radial basis functions from a given univariate shape function and a set of data points on the sphere.

2.4.2 Spherical radial basis functions

Given a shape function ϕ satisfying

$$\widehat{\phi}(\ell) \simeq (\ell+1)^{-2\tau} \quad \text{for all } \ell \geq 0, \tag{2.4.4}$$

for some $\tau \in \mathbb{R}$, the corresponding kernel Φ given by (2.4.2) is then strictly positive-definite; see Remark 2.14. The *native space* associated with ϕ is defined by

$$\mathcal{N}_\phi := \Big\{ v \in \mathscr{D}'(\mathbb{S}^{n-1}) : \|v\|_\phi^2 = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{|\widehat{v}_{\ell,m}|^2}{\widehat{\phi}(\ell)} < \infty \Big\}. \tag{2.4.5}$$

This space is equipped with an inner product and a norm defined by

$$\langle v, w\rangle_\phi = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{v}_{\ell,m}\widehat{w}_{\ell,m}}{\widehat{\phi}(\ell)} \quad \text{and} \quad \|v\|_\phi = \left(\sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{|\widehat{v}_{\ell,m}|^2}{\widehat{\phi}(\ell)} \right)^{1/2}.$$

Since $\widehat{\phi}(\ell)$ satisfies (2.4.4), the native space $\mathcal{N}_\phi$ can be identified with the Sobolev space H^τ, and the corresponding norms are equivalent.

Let $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ be a set of data points on the sphere. Two important parameters characterising the set X are the *mesh norm* h_X and *separation radius* q_X, defined by

$$h_X := \sup_{\boldsymbol{y}\in\mathbb{S}^{n-1}} \min_{1\leq i\leq N} \cos^{-1}(\boldsymbol{x}_i \cdot \boldsymbol{y}) \quad \text{and} \quad q_X := \frac{1}{2} \min_{\substack{i\neq j \\ 1\leq i,j\leq N}} \cos^{-1}(\boldsymbol{x}_i \cdot \boldsymbol{x}_j). \tag{2.4.6}$$

The *spherical radial basis functions* Φ_i, $i = 1, \dots, N$, associated with X and the kernel Φ are defined by (see (2.4.3))

$$\Phi_i(\boldsymbol{x}) := \Phi(\boldsymbol{x}, \boldsymbol{x}_i) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i) Y_{\ell,m}(\boldsymbol{x}). \tag{2.4.7}$$

We note that

$$\widehat{(\Phi_i)}_{\ell,m} = \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i), \quad i = 1, \dots, N. \tag{2.4.8}$$

It follows from (2.4.4) that, for any $s \in \mathbb{R}$,

$$\sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s} \left| \widehat{(\Phi_i)}_{\ell,m} \right|^2 \simeq \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2(s-2\tau)} |Y_{\ell,m}(\boldsymbol{x}_i)|^2.$$

By using (2.1.19) and noting $P_\ell(\boldsymbol{x}_i \cdot \boldsymbol{x}_i) = P_\ell(1) = 1$ we obtain, recalling that $N(n,\ell) = O(\ell^{n-2})$,

$$\sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2s} \left| (\widehat{\Phi_j})_{\ell,m} \right|^2 \simeq \sum_{\ell=0}^{\infty} (\ell+1)^{2(s-2\tau)+n-2}.$$

The series on the right hand side converges if and only if $2(s-2\tau)+n-2 < -1$. Hence,

$$\Phi_i \in H^s \quad \Longleftrightarrow \quad s < 2\tau + \frac{1-n}{2}. \tag{2.4.9}$$

The finite-dimensional subspace to be used in our approximation for strongly elliptic operators is defined by $\mathcal{V}_X^\phi := \operatorname{span}\{\Phi_1, \ldots, \Phi_N\}$. For brevity of notation we write $\mathcal{V}^\phi$ for $\mathcal{V}_X^\phi$ since there is no confusion. Due to (2.4.9), we have

$$\mathcal{V}^\phi \subset H^s \quad \text{for all } s < 2\tau + \frac{1-n}{2}. \tag{2.4.10}$$

We note that if $\tau > (n-1)/2$, then $\mathcal{V}^\phi \subset \mathcal{N}_\phi \simeq H^\tau \subset C(\mathbb{S}^{n-1})$, which is essentially the Sobolev embedding theorem.

2.5 Spherical splines

In this section we discuss the spaces of spherical splines defined on spherical triangulations of the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$; see [2, 3, 4]. We first review the key building blocks for spherical splines.

2.5.1 Spherical barycentric coordinates

Let $\boldsymbol{v}$ and $\boldsymbol{w}$ be two points on the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$ which do not lie on a line through the origin. The two points divide the great circle passing through $\boldsymbol{v}$ and $\boldsymbol{v}$ into two arcs in which the shorter is written as $\overline{\boldsymbol{v}\boldsymbol{w}}$.

Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be three unit vector (which span $\mathbb{R}^3$). The surface triangle generated by the vectors is called a *spherical triangle*

$$\tau = \{\boldsymbol{v} \in \mathbb{S}^2 : \boldsymbol{v} = b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 + b_3\boldsymbol{v}_3,\ b_i \geq 0\}.$$

It is clear that the boundary of τ consists of the three arcs $\overline{\boldsymbol{v}_1\boldsymbol{v}_2}$, $\overline{\boldsymbol{v}_2\boldsymbol{v}_3}$, $\overline{\boldsymbol{v}_3\boldsymbol{v}_1}$ which are called the three *edges* of the spherical triangle.

Definition 2.15. *Let τ be a spherical triangle with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ and let $\boldsymbol{v}$ be a point the sphere $\mathbb{S}^2$. The spherical barycentric coordinates of $\boldsymbol{v}$ relative to τ are the unique real numbers b_1, b_2, b_3 such that*

$$\boldsymbol{v} = b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 + b_3\boldsymbol{v}_3. \tag{2.5.1}$$

Equation (2.5.1) defining spherical barycentric coordinates can be written as a linear system of equations for b_1, b_2, b_3, supposing that $\boldsymbol{v}_i = (x_i, y_i, z_i)$ and $\boldsymbol{v} = (x, y, z)$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{2.5.2}$$

The matrix in (2.5.2) is nonsingular since the three vectors $\boldsymbol{v}_i$, $i = 1, 2, 3$, form a basis for $\mathbb{R}^3$. Using Cramer's rule, we have

$$b_1 = \frac{\det(\boldsymbol{v}, \boldsymbol{v}_2, \boldsymbol{v}_3)}{\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)}, \quad b_2 = \frac{\det(\boldsymbol{v}_1, \boldsymbol{v}, \boldsymbol{v}_3)}{\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)}, \quad b_3 = \frac{\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v})}{\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)}, \tag{2.5.3}$$

where

$$\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) := \det \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{bmatrix}$$

and so forth. Equation (2.5.3) shows that b_1 is the ratio of the volume of the trihedron formed by the origin and $\boldsymbol{v}, \boldsymbol{v}_2, \boldsymbol{v}_3$ over the volume of the trihedron formed by the origin and $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. Similar observations for b_2 and b_3.

The following proposition shows some basic properties of spherical barycentric coordinates.

Proposition 2.16. *Let τ be a spherical triangle generated by $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. The following statements are true.*

1. *The barycentric coordinates at the vertices of τ satisfy*

$$b_i(\boldsymbol{v}_j) = \delta_{ij}, \quad i, j = 1, 2, 3.$$

2. *For all $\boldsymbol{v}$ in the interior of τ, $b_i(\boldsymbol{v}) > 0$ for $i = 1, 2, 3$.*

3. *If the edges of τ are extended into great circles, the sphere $\mathbb{S}^2$ is divided into eight regions. The spherical barycentric coordinates have constant signs on each of these eight regions.*

4. *If a point $\boldsymbol{v}$ lies on an edge of τ, then the spherical coordinate corresponding to the vertex opposite the edge vanishes. The remaining two spherical coordinates are ratios of sines of geodesic distances.*

5. *Spherical barycentric coordinates are infinitely differentiable functions of $\boldsymbol{v}$.*

6. *The spherical barycentric coordinates of a point $\boldsymbol{v}$ on the sphere relative to one spherical triangle τ can be computed from those relative to another spherical triangle τ' by matrix multiplication.*

7. *The b_i are ratios of volumes of tetrahedra.*

8. *The spherical barycentric coordinates of a point $\boldsymbol{v}$ are invariant under rotation.*

9. *The span of the spherical barycentric coordinates $b_1(\boldsymbol{v}), b_2(\boldsymbol{v}), b_3(\boldsymbol{v})$ relative to any triangle is always the three dimensional linear space obtained by restricting the space of linear homogeneous polynomials on $\mathbb{R}^3$ to the sphere $\mathbb{S}^2$.*

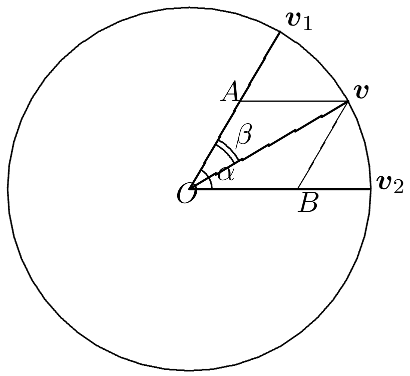

Figure 2.1: Spherical barycentric coordinates of $\boldsymbol{v}$ lying on $\overline{\boldsymbol{v}_1\boldsymbol{v}_2}$.

Proof. Statements 1, 2 and 3 can be immediately deduced from Definition 2.15. Statement 5 is true because the determinant in the denominators of (2.5.3) is nonzero. Statement 7 is deduced directly from (2.5.3). Statement 9 follows from the fact that the spherical barycentric coordinates of a point $\boldsymbol{v}$ on the sphere are homogeneous polynomial of degree 1 and the fact that they are linearly independent which can be seen easily from (2.5.3).

Assume that $\boldsymbol{M}$ is a 3×3 matrix which satisfies that for each $\boldsymbol{v} \in \mathbb{S}^2$, $\boldsymbol{M}\boldsymbol{v} \in \mathbb{S}^2$. Suppose that vector $\boldsymbol{v}$ is written as in (2.5.1). By multiplying (2.5.1) by $\boldsymbol{M}$, we obtain

$$\boldsymbol{M}\boldsymbol{v} = b_1\boldsymbol{M}\boldsymbol{v}_1 + b_2\boldsymbol{M}\boldsymbol{v}_2 + b_3\boldsymbol{M}\boldsymbol{v}_3,$$

proving Statement 6. Statement 8 is then a direct consequence since each rotation can be written as the multiplication of an orthogonal matrix.

For Statement 4, suppose that point $\boldsymbol{v}$ lies on the edge $\overline{\boldsymbol{v}_1, \boldsymbol{v}_2}$ (see Figure 2.1). It is easy that $b_3(\boldsymbol{v}) = 0$. From Figure 2.1, it is clear that

$$b_1(\boldsymbol{v}) = OA = \frac{\sin(\alpha - \beta)}{\sin\alpha} \quad \text{and} \quad b_2(\boldsymbol{v}) = OB = \frac{\sin\beta}{\sin\alpha}.$$

□

Proposition 2.16 shows that spherical barycentric coordinates have most of the properties of planar barycentric coordinates. There are some important differences though. Perhaps, the most significant difference is that spherical barycentric coordinates do not sum up to 1, except at the vertices of the spherical triangle in concern.

Proposition 2.17. *Let $\tau := \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ be a spherical triangle. There holds*

$$b_1(\boldsymbol{v}) + b_2(\boldsymbol{v}) + b_3(\boldsymbol{v}) > 1 \quad \forall \boldsymbol{v} \in \tau \backslash \{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}.$$

Proof. We denote by T the planar triangle with vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. Then we have

$$T = \{\boldsymbol{v} \in \mathbb{R}^3 : v = b_1\boldsymbol{v}_1 + b_2\boldsymbol{v}_2 + b_3\boldsymbol{v}_3, \quad b_i \geq 0 \quad \text{and} \quad b_1 + b_2 + b_3 = 1\}. \tag{2.5.4}$$

It is easy to see that for any $\boldsymbol{v} \in \tau$, we have $\boldsymbol{v} = \alpha \boldsymbol{w}$ for some $\alpha \geq 1$ and $\boldsymbol{w} \in T$. From this and (2.5.4), we deduce that the spherical barycentric coordinates of $\boldsymbol{v}$ sum up to a number greater than or equal to 1. The equality holds only when $\boldsymbol{v}$ coincides with the three vertices $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$. $\square$

2.5.2 Spherical Bernstein basis polynomials

Definition 2.18. *Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ be three linearly independent unit vectors in $\mathbb{R}^3$ and let $b_1(\boldsymbol{v}), b_2(\boldsymbol{v}), b_3(\boldsymbol{v})$ denote the spherical barycentric coordinates of $\boldsymbol{v}$ relative to the spherical triangle $\tau := \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$. Given a nonnegative integer d, the functions*

$$B_{ijk}^{d,\tau}(\boldsymbol{v}) := \frac{d!}{i!j!k!} b_1^i(\boldsymbol{v}) b_2^j(\boldsymbol{v}) b_3^k(\boldsymbol{v}), \quad i + j + k = d,$$

are called the spherical Bernstein basis polynomials of degree d.

It is clear from Definition 2.18 that the spherical Bernstein basis polynomials satisfy the following recurrence relation:

$$B_{ijk}^{d,\tau} = b_1 B_{i-1,j,k}^{d-1,\tau} + b_2 B_{i,j-1,k}^{d-1,\tau} + b_3 B_{i,j,k-1}^{d-1,\tau}, \quad i + j + k = d. \tag{2.5.5}$$

Here, we are using the convention that expressions with negative subscripts are defined as zero.

We have denoted by Π_d the space of homogeneous polynomials of degree d in $\mathbb{R}^3$ and by $\widetilde{\Pi}_d$ the set of restrictions of polynomials in Π_d to the unit sphere $\mathbb{S}^2$. It is easy to see that the dimension of the space $\widetilde{\Pi}_d$ is equal to that of Π_d and equal to $\binom{d+2}{2}$. In the following proposition we will show that the set $\{B_{ijk}^{d,\tau} : i + j + k = d\}$ forms a basis for $\widetilde{\Pi}_d$.

Proposition 2.19. *Let $\tau := \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ be a spherical triangle and let $\{B_{ijk}^{d,\tau}\}_{i+j+k=d}$ be the spherical Bernstein basis polynomials associated to τ. Then the polynomials $\{B_{ijk}^{d,\tau}\}_{i+j+k=d}$ form a basis for $\widetilde{\Pi}_d$.*

Proof. The above statement will be proved by using induction on d. We note that the number of spherical Bernstein polynomials $\{B_{ijk}^{d,\tau}\}_{i+j+k=d}$ is $\binom{d+2}{2}$, which is also the dimension of the space $\widetilde{\Pi}_d$ of homogeneous polynomials of degree d on $\mathbb{S}^2$. We will prove that

$$\operatorname{span}\{B_{ijk}^{d,\tau} : i + j + k = d\} = \widetilde{\Pi}_d. \tag{2.5.6}$$

It is clear that (2.5.6) is true when $d = 0$. The result 9 in Proposition 2.16 assures that (2.5.6) holds in the case $d = 1$. We now assume that (2.5.6) is true in the case of degree $\ell - 1$ with $\ell \geq 1$. Let $x^i y^j z^k$ be a monomial of degree ℓ. Since $\ell \geq 1$, there exists at least one of the numbers i,j, k greater zero. Without loss of generality, we assume that $i \geq 1$. Then we have

$$x^i y^j z^k = x(x^{i-1} y^j z^k). \tag{2.5.7}$$

Since (2.5.6) holds when $d = 1$ and $d = \ell - 1$, we have

$$x = c_1 b_1(x, y, z) + c_2 b_2(x, y, z) + c_3 b_3(x, y, z), \tag{2.5.8}$$

and

$$x^{i-1}y^j z^k = \sum_{i'+j'+k'=\ell-1} c^{\tau}_{i'j'k'} B^{\ell-1,\tau}_{i'j'k'}(x,y,z)$$

This together with (2.5.7) and (2.5.8) gives

$$\begin{aligned} x^i y^j z^k &= \big(c_1 b_1(x,y,z) + c_2 b_2(x,y,z) + c_3 b_3(x,y,z)\big) \sum_{i'+j'+k'=\ell-1} c^{\tau}_{i'j'k'} B^{\ell-1,\tau}_{i'j'k'}(x,y,z) \\ &= \sum_{i'+j'+k'=\ell-1} c^{\tau}_{i'j'k'} \Big(\frac{c_1(i'+1)}{\ell} B^{\ell,\tau}_{i'+1,j',k'} + \frac{c_2(j'+1)}{\ell} B^{\ell,\tau}_{i',j'+1,k'} \\ &\qquad + \frac{c_3(k'+1)}{\ell} B^{\ell,\tau}_{i',j',k'+1}\Big)(x,y,z). \end{aligned}$$

Hence, the monomial $x^i y^j z^k$ can be written as a linear combination of $\{B^{\ell,\tau}_{ijk} : i+j+k=\ell\}$. This confirms that (2.5.6) is true for $d=\ell$.

Hence we have shown that (2.5.6) is true for any degree d. Moreover, since the number of spherical Bernstein basis polynomials equals the dimension of $\widetilde{\Pi}_d$, the set $\{B^{d,\tau}_{ijk} : i+j+k=d\}$ is a basis for $\widetilde{\Pi}_d$, completing the proof of the proposition. □

Let $\tau = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ be a spherical triangle on the unit sphere $\mathbb{S}^2$. Proposition 2.19 shows us that any polynomial p in $\widetilde{\Pi}_d$ has a unique expansion of the form

$$p = \sum_{i+j+k=d} c_{ijk} B^{d,\tau}_{ijk}. \tag{2.5.9}$$

2.5.3 Derivatives and integration of spherical polynomials

Definition 2.20. *Let f be a sufficiently smooth function defined on the unit sphere $\mathbb{S}^2$. Suppose $\boldsymbol{w}$ is a given vector in $\mathbb{R}^3$. The directional derivative $D_{\boldsymbol{w}} f(\boldsymbol{v})$ of function f at a point $\boldsymbol{v}$ on $\mathbb{S}^2$ is defined by*

$$D_{\boldsymbol{w}} f(\boldsymbol{v}) := D_{\boldsymbol{w}} F(\boldsymbol{v}) = \boldsymbol{w}^T \nabla F(\boldsymbol{v}), \tag{2.5.10}$$

where F is some homogeneous extension of f and ∇F is the gradient of the trivariate function F.

Remark 2.21. *For each f defined on the unit sphere, there are an infinitely many homogeneous extensions of f. The value of its derivatives may depend on which extensions we use. However, if $\boldsymbol{w}$ is perpendicular to the vector $\boldsymbol{v}$ then the directional derivative $D_{\boldsymbol{w}}(\boldsymbol{v})$ does not depend on the extension F to be used as shown in the following proposition.*

Proposition 2.22. *Given a point $\boldsymbol{v}$ on $\mathbb{S}^2$. For any vector $\boldsymbol{w}$ perpendicular to vector $\boldsymbol{v}$, the directional derivative $D_{\boldsymbol{w}}(\boldsymbol{v})$ does not depend on the homogeneous extensions used in* (2.5.10).

Proof. Let $\mathscr{P}$ be the plane passing through the point $\boldsymbol{v}$ and the origin, and parallel to vector $\boldsymbol{w}$. This plane intersects the sphere $\mathbb{S}^2$ by a great circle arc $a^{\boldsymbol{v}}_{\boldsymbol{w}}$ which passes through the point $\boldsymbol{v}$. We assume that the arc $a^{\boldsymbol{v}}_{\boldsymbol{w}}$ is parametrised by arc length so that $a^{\boldsymbol{v}}_{\boldsymbol{w}}(0) = \boldsymbol{v}$.

Assume that $a_{\boldsymbol{w}}^{\boldsymbol{v}}(\beta) = (x(\beta), y(\beta), z(\beta))$. We have $\boldsymbol{w} = (x'(0), y'(0), z'(0))$. Let F be any homogeneous extension of f. By the chain rule, we have

$$\frac{df(a_{\boldsymbol{w}}^{\boldsymbol{v}}(\beta))}{d\beta}\Big|_{\beta=0} = \frac{dF(a_{\boldsymbol{w}}^{\boldsymbol{v}}(\beta))}{d\beta}\Big|_{\beta=0} = \boldsymbol{w}^T \nabla F(\boldsymbol{v}) = D_{\boldsymbol{w}} F(\boldsymbol{v}).$$

This shows that the definition of $D_{\boldsymbol{w}} f(\boldsymbol{v})$ does not depend on the choice of the extension F. □

Proposition 2.23. *Let $\tau = \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ be a spherical triangle on $\mathbb{S}^2$ and $\boldsymbol{w}$ be a given vector in $\mathbb{R}^3$. Then*

$$D_{\boldsymbol{w}} b_i = b_i(\boldsymbol{w}), \quad i = 1, 2, 3. \tag{2.5.11}$$

Proof. We prove (2.5.11) for $i = 1$. The others can be proved in the same manner. Recalling (2.5.3), we have $b_1(\boldsymbol{v}) = \det(\boldsymbol{v}, \boldsymbol{v}_2, \boldsymbol{v}_3)/\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$. Elementary calculations reveal that

$$D_{\boldsymbol{w}} b_1(\boldsymbol{v}) = \det(\boldsymbol{w}, \boldsymbol{v}_2, \boldsymbol{v}_3)/\det(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3) = b_1(\boldsymbol{w}),$$

completing the proof of the proposition. □

The chain rule and Proposition 2.23 imply the following formula for the directional derivative of an arbitrary homogeneous polynomial.

Proposition 2.24. *Let p be a spherical homogeneous polynomial of the form* (2.5.9). *For any given vector $\boldsymbol{w}$, there holds*

$$D_{\boldsymbol{w}} p(\boldsymbol{v}) = b(\boldsymbol{w})^T \nabla_b p,$$

where

$$\nabla_b p := \Big(\frac{\partial}{\partial b_1}, \frac{\partial}{\partial b_2}, \frac{\partial}{\partial b_3}\Big)^T.$$

We are now ready to compute higher order derivatives of spherical homogeneous polynomials of the form (2.5.9), i.e.,

$$p = \sum_{i+j+k=d} c_{ijk} B_{ijk}^{d,\tau}.$$

Let $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_m\}$, $1 \le m \le d$, be a set of direction vectors. Denote

$$c_{ijk}^0 := c_{ijk}, \quad i + j + k = d.$$

For each $1 \le \ell \le m$, let

$$c_{ijk}^\ell = b_1(\boldsymbol{w}_\ell) c_{i+1,j,k}^{\ell-1} + b_2(\boldsymbol{w}_\ell) c_{i,j+1,k}^{\ell-1} + b_3(\boldsymbol{w}_\ell) c_{i,j,k+1}^{\ell-1}, \quad \ell = 1, \ldots, m.$$

Proposition 2.25. *For any $0 \le m \le d$, we have*

$$D_{\boldsymbol{w}_1 \ldots \boldsymbol{w}_m} p(\boldsymbol{v}) := D_{\boldsymbol{w}_1} \ldots D_{\boldsymbol{w}_m} p(\boldsymbol{v}) = \frac{d!}{(d-m)!} \sum_{i+j+k=d-m} c_{ijk}^m B_{ijk}^{d-m}(\boldsymbol{v}). \tag{2.5.12}$$

Proof. We have

$$D_{\boldsymbol{w}_1} p(\boldsymbol{v}) = \sum_{i+j+k=d} c_{ijk} D_{\boldsymbol{w}_1} B^d_{ijk}(\boldsymbol{v}). \tag{2.5.13}$$

By Proposition 2.23, for $i+j+k=d$, we have

$$\begin{aligned} D_{\boldsymbol{w}_1} B^d_{ijk}(\boldsymbol{v}) &= \frac{d!}{i!j!k!}\Big(i b_1^{i-1} b_2^j b_3^k D_{\boldsymbol{w}_1} b_1 + j b_1^i b_2^{j-1} b_3^k D_{\boldsymbol{w}_1} b_2 + k b_1^i b_2^j b_3^{k-1} D_{\boldsymbol{w}_1} b_3 \Big) \\ &= d \Big(B^{d-1}_{i-1,j,k}(\boldsymbol{v}) b_1(\boldsymbol{w}_1) + B^{d-1}_{i,j-1,k}(\boldsymbol{v}) b_2(\boldsymbol{w}_1) + B^{d-1}_{i,j,k-1}(\boldsymbol{v}) b_3(\boldsymbol{w}_1) \Big). \end{aligned}$$

Substituting this into (2.5.13), we obtain

$$D_{\boldsymbol{w}_1} p(\boldsymbol{v}) = d \sum_{i+j+k=d} c_{ijk} \Big(B^{d-1}_{i-1,j,k}(\boldsymbol{v}) b_1(\boldsymbol{w}_1) + B^{d-1}_{i,j-1,k}(\boldsymbol{v}) b_2(\boldsymbol{w}_1) + B^{d-1}_{i,j,k-1}(\boldsymbol{v}) b_3(\boldsymbol{w}_1) \Big).$$

We then split the right hand side, rearrange it to obtain

$$\begin{aligned} D_{\boldsymbol{w}_1} p(\boldsymbol{v}) &= d \Big(\sum_{i+j+k=d-1} c_{i+1,j,k} B^{d-1}_{i,j,k}(\boldsymbol{v}) b_1(\boldsymbol{w}_1) + \sum_{i+j+k=d-1} c_{i,j+1,k} B^{d-1}_{i,j,k}(\boldsymbol{v}) b_2(\boldsymbol{w}_1) \\ &\qquad + \sum_{i+j+k=d-1} c_{i,j,k+1} B^{d-1}_{i,j,k}(\boldsymbol{v}) b_3(\boldsymbol{w}_1) \Big) \\ &= d \sum_{i+j+k=d-1} \Big(c_{i+1,j,k} b_1(\boldsymbol{w}_1) + c_{i,j+1,k} b_2(\boldsymbol{w}_1) + c_{i,j,k+1} b_3(\boldsymbol{w}_1) \Big) B^{d-1}_{ijk}(\boldsymbol{v}) \\ &= d \sum_{i+j+k=d-1} c^1_{ijk} B^{d-1}_{ijk}(\boldsymbol{v}), \end{aligned}$$

yielding (2.5.12) for $m=1$. The general result follows by induction. □

Evaluation of integrals of piecewise polynomial functions is of importance in many applications, e.g., in the finite element method or in minimal energy interpolation. Evaluating integrals of spherical polynomials is considerably more difficult than in the planar case. In the case of planar triangles, the integral of a Bernstein basis polynomial of degree d does not depend on the the particular basis polynomial or on the precise shape of the triangle. This property does not hold in the case of spherical Bernstein polynomial. In general, the integrals for two different spherical triangles are different, except in the case the two triangles are similar. Moreover, the integrals of the spherical Bernstein polynomials of degree d associated with a single triangle are also different in general.

There does not seem to be a convenient closed-form formula for integrals of spherical Bernstein polynomials. To compute the integrals of a spherical function over a spherical triangle, Alfeld, Neamtu and Schumaker propose a mapping of a spherical triangle τ to a planar triangle whose vertices are the same as the vertices of the spherical one. This enable us to use a standard technique of numerical integration for planar triangles.

Suppose that the spherical triangle τ have vertices being $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$. Let $\boldsymbol{A}$ be the matrix whose columns are the vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$ of τ. Then we have

$$\int_\tau f \, d\sigma = \int_0^1 \int_0^{1-u_1} f\left(\frac{\boldsymbol{A}\boldsymbol{u}^T}{|\boldsymbol{A}\boldsymbol{u}^T|} \right) \frac{|\det \boldsymbol{A}|}{|\boldsymbol{A}\boldsymbol{u}^T|^3} \, du_2 \, du_1, \tag{2.5.14}$$

where $\boldsymbol{u} = (u_1, u_2, 1-u_1-u_2)^T$; see [4].

Proposition 2.26. *Let p be a spherical homogeneous polynomial of degree d. Then*

$$\int_\tau p(\boldsymbol{v})\,d\sigma = |\det \boldsymbol{A}| \int_0^1 \int_0^{1-u_1} \frac{p(\boldsymbol{A}\boldsymbol{u})}{|\boldsymbol{A}\boldsymbol{u}^T|^{d+3}}\,du_2\,du_1.$$

By (2.5.14), we can now use numerical integration techniques for integral over planar triangles to evaluate integral for spherical functions.

2.5.4 Spaces of spherical splines

Definition 2.27. *A set of spherical triangles $\Delta := \{\tau_i : i = 1, \ldots, N\}$ is called a spherical triangulation if it satisfies:*

1. $\bigcup_{i=1}^N \tau_i = \mathbb{S}^2$,
2. *each triangle in Δ has only vertices or edges in common with other triangles.*

The definition above can be stated for a subset $\Omega \subset \mathbb{S}^2$ in which the condition 1 in the above definition is changed to Ω instead of $\mathbb{S}^2$. In this dissertation, we only work with spherical triangulation for the whole sphere. We next introduce the concept of quasi-uniform and regular triangulation on $\mathbb{S}^2$ which will be used in the rest of the dissertation.

For any spherical triangle τ, we denote by $|\tau|$ the diameter of the smallest spherical cap containing τ, and by ρ_τ the diameter of the largest spherical cap inside τ. Here the diameter of a cap is, as usual, twice its radius; see (4.3.1). We define

$$|\Delta| := \max\{|\tau|, \tau \in \Delta\},\ \rho_\Delta := \min\{\rho_\tau, \tau \in \Delta\},\ h_\tau := \tan\frac{|\tau|}{2} \text{ and } h_\Delta := \tan\frac{|\Delta|}{2}. \quad (2.5.15)$$

Definition 2.28. *A triangulation Δ is said to be quasi-uniform if for some $\beta > 1$, there holds*

$$|\tau| \le \beta\rho_\tau \quad \forall \tau \in \Delta,$$

and regular if for some positive number $\gamma < 1$, there holds

$$|\tau| \ge \gamma|\Delta| \quad \forall \tau \in \Delta.$$

The following proposition states the relationship between the number V of vertices, the number E of edges and the number T of spherical triangles in a spherical triangulation.

Proposition 2.29. *Let Δ be a spherical triangulation on the sphere $\mathbb{S}^2$. Then*

(i) $E = 3T/2$,

(ii) $T = 2V - 4$,

(iii) $E = 3V - 6$.

Proof. The equality (i) is obvious since every triangle has three edges and when we count these edges all, each edge will be counted twice. The proof for (ii) is just a matter of counting. The triangulation Δ can be obtained by starting with one triangle and adding one triangle a time until it covers the whole sphere. Assume that each time, a triangle is added so that it shares at least one common edge with the previous triangles already in. We denote by α_i, for $i = 1, 2, 3$, the number of times that the added triangles share i common edges with the other triangles. Since we are working on the sphere, we can assume that $\alpha_3 = 1$. It is obvious that

$$T = 1 + \alpha_1 + \alpha_2 + \alpha_3 = 2 + \alpha_1 + \alpha_2. \tag{2.5.16}$$

For the number of edges, we start with a single triangle which has 3 edges. Each time we add a triangle, the number of new edges is 2, 1, 0 when the number of edges shared by the new triangle with the others is 1, 2 and 3, respectively. Thus,

$$E = 3 + 2\alpha_1 + \alpha_2. \tag{2.5.17}$$

Similarly,

$$V = 3 + \alpha_1. \tag{2.5.18}$$

From (i), (2.5.16) and (2.5.17), we obtain $\alpha_1 = \alpha_2$. This together with (2.5.16) and (2.5.18) implies (ii). The equality (iii) can be deduced from (i) and (ii). □

We are now ready to define the space of spherical splines.

Definition 2.30. *Let Δ be a spherical triangulation on the unit sphere $\mathbb{S}^2$. The space of spherical splines of degree d and smoothness r is given by*

$$S_d^r(\Delta) := \{s \in C^r(\mathbb{S}^2) : s|_\tau \in \widetilde{\Pi}_d, \tau \in \Delta\}$$

For each spherical triangle $\tau := \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ and for any nonnegative integer d, we denote

$$\xi_{ijk} := \frac{i\boldsymbol{v}_1 + j\boldsymbol{v}_2 + k\boldsymbol{v}_3}{|i\boldsymbol{v}_1 + j\boldsymbol{v}_2 + k\boldsymbol{v}_3|}, \quad i + j + k = d.$$

The set

$$\mathcal{D}_{d,\tau} := \{\xi_{ijk} : i + j + k = d\}$$

is called the *set of domain points relative to* τ and the set $D_{d,\Delta} = \bigcup_{\tau \in \Delta} D_{d,\tau}$ is called the *set of domain points* of the triangulation Δ relative to the degree d.

Proposition 2.31. *There is an one-to-one correspondence between the linear space $S_d^0(\Delta)$ and the set $\mathbb{R}^{\mathcal{D}_{d,\Delta}}$.*

Proof. Given $s \in S_d^0(\Delta)$, Proposition 2.19 show that for each triangle $\tau \in \Delta$, there exists a unique set of coefficients $\{c_\xi : \xi \in \mathcal{D}_{d,\tau}\}$ such that

$$s|_\tau = \sum_{\xi \in \mathcal{D}_{d,\tau}} c_\xi B_\xi^{d,\tau}, \tag{2.5.19}$$

where $B_\xi^{d,\tau}$ are the spherical Bernstein basis polynomials of degree d associated with the spherical triangle τ. Since s is continuous, if ξ is a common vertex or lies on a common edge of two triangles τ and τ', then the coefficients c_ξ for τ and τ' are the same. Thus, for each $s \in S_d^0(\Delta)$, there is a unique associated set of coefficients $\{c_\xi : \xi \in \mathcal{D}_{d,\Delta}\}$. The converse also holds, i.e., given any $\{c_\xi : \xi \in \mathcal{D}_{d,\Delta}\}$ there is a unique spline $s \in S_d^0(\Delta)$ defined by (2.5.19). □

Proposition 2.31 shows that the dimension of $S_d^0(\Delta)$ is equal to the cardinality of $\mathcal{D}_{d,\Delta}$.

Proposition 2.32. *Let Δ be a spherical triangulation on the unit sphere $\mathbb{S}^2$. Then*

$$\dim S_d^0(\Delta) = \#\mathcal{D}_{d,\Delta} = V + (d-1)E + \binom{d-1}{2}T, \tag{2.5.20}$$

where V, E and T are the numbers of vertices, edges, and triangles in Δ.

Proof. The set of domain points associated with the triangulation Δ and degree d includes the vertices of the triangulation, the other domain points on edges which are not vertices and the domain points inside each triangle.

Let $\tau := \langle \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \rangle$ be a triangle in Δ. Domain points lie on the edge $\overline{\boldsymbol{v}_1\boldsymbol{v}_2}$ are

$$\boldsymbol{v}_{ij}^{\boldsymbol{v}_1\boldsymbol{v}_2} := \frac{i\boldsymbol{v}_1 + j\boldsymbol{v}_2}{|i\boldsymbol{v}_1 + j\boldsymbol{v}_2|}, \quad i + j = d,$$

which are $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ when $i = 1$ and $i = d$, respectively. The other domain points on $\overline{\boldsymbol{v}_1\boldsymbol{v}_2}$ are $\boldsymbol{v}_{ij}^{\boldsymbol{v}_1\boldsymbol{v}_2}$ when $i = 2, \dots, d-1$. This argument shows that the number of domain point on each edge is a constant and it is equal to $d-1$. Hence, the number of domain points lying on edges of the triangulation which are not vertices is $(d-1)E$.

Domain points inside the triangle τ are

$$\boldsymbol{v}_{ijk}^{\tau} := \frac{i\boldsymbol{v}_1 + j\boldsymbol{v}_2 + k\boldsymbol{v}_3}{|i\boldsymbol{v}_1 + j\boldsymbol{v}_2 + k\boldsymbol{v}_3|}, \quad i + j + k = d, \text{ and } i \geq 1, j \geq 1, k \geq 1.$$

A simple count gives the number of domain points inside τ is $\binom{d-1}{2}$.

Combining the above arguments, we deduce (2.5.20). □

We now construct simple locally supported basis functions for $S_d^0(\Delta)$. For each $\xi \in \mathcal{D}_{d,\Delta}$, let ψ_ξ be a spline in $S_d^0(\Delta)$ satisfying

$$\nu_\eta \psi_\xi = \delta_{\eta,\xi} \quad \forall \eta \in \mathcal{D}_{d,\Delta}, \tag{2.5.21}$$

where ν_η is a linear functional which picks off the coefficient associated with the domain point ξ. By construction, ψ_ξ has all zero coefficients except for $c_\xi = 1$.

Since for each triangle τ, the associated spherical Bernstein polynomials are nonnegative on τ, it follows immediately that $\psi_\xi(\boldsymbol{v}) \geq 0$ for all $\boldsymbol{v} \in \mathbb{S}^2$. Moreover, since ψ_ξ is identically zero on all triangles which do not contain ξ, it follows that the support of ψ_ξ is

- a single triangle τ, if ξ is in the interior of τ,
- the union of triangle τ and τ', if ξ lies on a common edge between τ and τ', and ξ is not a vertex.

- the union of all triangles sharing the vertex ξ, if ξ is a vertex.

Proposition 2.33. *The set of splines $\mathcal{B} := \{\psi_\xi : \xi \in \mathcal{D}_{d,\Delta}\}$ forms a basis for $S_d^0(\Delta)$.*

Proof. Since $\dim S_d^0(\Delta) = \#\mathcal{D}_{d,\Delta}$, it suffices to show that the ψ_ξ, $\xi \in \mathcal{D}_{d,\Delta}$, are linearly independent. Suppose that

$$s := \sum_{\xi \in \mathcal{D}_{d,\Delta}} c_\xi \psi_\xi = 0 \quad \text{on } \mathbb{S}^2.$$

Then on each triangle $\tau \in \Delta$, the restriction $s|_\tau$ is a spherical Bernstein polynomial of degree d which is identically zero. Proposition 2.19 implies that $c_\xi = 0$ for all $\xi \in \mathcal{D}_{d,\tau}$. This holds for all triangles in Δ, thus all coefficients must be equal to zero. This has shown the linear independence of $\mathcal{B}$, completing the proof of the proposition. $\square$

We now briefly discuss the construction of a quasi-interpolation operator $\widetilde{I} : L_2(\mathbb{S}^2) \to S_d^r(\Delta)$ which is defined in [54]. This operator will be used frequently in the rest of the dissertation. Assume that $\mathcal{D}_{d,\Delta} := \{\xi_1, \ldots, \xi_D\}$, where $D = \dim S_d^0(\Delta)$. Let $\{B_l : l = 1, \ldots, D\}$ be a basis for $S_d^0(\Delta_h)$ such that the restriction of B_l on each triangle containing ξ_l is the spherical Bernstein polynomial of degree d associated with this point, and that B_l vanishes on other triangles.

A set $\mathcal{M} := \{\zeta_l\}_{l=1}^N \subset \mathcal{D}_{d,\Delta}$ is called a *minimal determining set* for $S_d^r(\Delta)$ if, for every $s \in S_d^r(\Delta)$, all the coefficients $\nu_l(s)$ in the expression $s = \sum_{l=1}^D \nu_l(s) B_l$ are uniquely determined by the coefficients corresponding to the basis functions which are associated with points in $\mathcal{M}$. Given a minimal determining set $\mathcal{M}$, we construct a basis $\{B_l^*\}_{l=1}^N$ for $S_d^r(\Delta)$ by requiring

$$\nu_{l'}(B_l^*) = \delta_{l,l'}, \quad 1 \le l, l' \le N.$$

The use of Hahn–Banach Theorem extends the linear functionals ν_l, $l = 1, \ldots, N$, to all functions in $L_2(\mathbb{S}^2)$. We continue to use the same symbol for these extensions.

The quasi-interpolation operator $\widetilde{I} : L_2(\mathbb{S}^2) \to S_d^r(\Delta)$ is now defined by

$$\widetilde{I}v := \sum_{l=1}^N \nu_l(v) B_l^*, \quad v \in L_2(\mathbb{S}^2). \tag{2.5.22}$$

In this dissertation, we always assume that the integers d and r defining $S_d^r(\Delta)$ satisfy

$$\begin{cases} d \ge 3r + 2 & \text{if } r \ge 1, \\ d \ge 1 & \text{if } r = 0. \end{cases} \tag{2.5.23}$$

The following proposition, which will be used to prove an approximation property of the spaces of spherical splines in Chapter 4, is a special case of a result established in [7].

Proposition 2.34. *[7, Theorem 2] Assume that Δ is a quasi-uniform spherical triangulation with $|\Delta| \le 1$, and that (2.5.23) holds. Then for any $v \in H^m$ there holds*

$$|v - \widetilde{I}v|_k \le C h_\Delta^{m-k} |v|_m,$$

for all $k = 0, \ldots, \min\{m-1, r+1\}$, *and*

$$m = \begin{cases} 1, 3, \ldots, d+1 & \text{if } d \text{ is even,} \\ 2, 4, \ldots, d+1 & \text{if } d \text{ is odd.} \end{cases}$$

Here, C *is a positive constant depending only on* d *and the smallest angle in* Δ.

Remark 2.35. *(i) The condition* $k \le r+1$ *is to ensure that* $\widetilde{I}v \in H^k$, *which will be proved in Proposition 4.2.*

(ii) Theorem 2 in [7] proves the result for $r \ge 0$ *and* $d \ge 3r+2$. *In fact, the result can also be proved when* $r = 0$ *and* $d = 1$ *by using the same argument.*

Chapter 3

Pseudodifferential equations with spherical radial basis functions

3.1 Introduction

In this chapter, we solve the pseudodifferential equation $Lu = g$ with spherical radial basis functions. The operator L can be of any nature, elliptic or strongly elliptic, and of any order, negative or positive. To assure the unique existence of the solution to the equation, side conditions are introduced. The methods to be used are the Galerkin and collocation methods. From the point of view of application, the collocation method is easier to implement, in particular when the given data are scattered. However, it is well-known that collocation methods in general elicit a complicated error analysis.

In this chapter, first we solve strongly elliptic and elliptic pseudodifferential equations on the sphere by the Galerkin method. Error analysis is performed with well-known knowledge on Galerkin methods. When the pseudodifferential operator is strongly elliptic, a Bubnov–Galerkin method is used. However, a Petrov–Galerkin method is required for elliptic pseudodifferential operators to ensure that the resulting matrices are positive-definite. As a consequence, it is necessary to prove some inf-sup (or Ladyzenskaya–Babuška–Brezzi) condition involving spherical radial basis functions. This result is of interest in its own right. To the best of our knowledge, a first inf-sup condition involving radial basis functions is proved by Sloan and Wendland [71] for a hybrid of radial basis functions and polynomials. The inf-sup condition we prove in this chapter involves two spaces of spherical radial basis functions defined from two different shape functions. The proof technique is different from that used in [71].

Next, we solve the equations by collocation methods. A salient feature of this chapter is that error estimates for collocation methods (as considered in References [48, 49, 50]) are obtained as a by-product of the analysis for the Galerkin method. This unified error analysis is thanks to an observation that the collocation equation can be viewed as a Galerkin equation, due to the reproducing kernel property of the space in use. Efforts to perform error analysis for the collocation method based on that for the Galerkin method have been made by several authors to solve quasilinear parabolic equations [19], pseudodifferential equations on closed curves [5], and boundary integral equations [16]. These approaches use either a special set of collocation points or the duality inner product.

Our error estimates, as compared to those by Morton and Neamtu [49, 50], cover a wider range of Sobolev norms. Indeed, these authors only provide error estimates in the Sobolev norm $\|\cdot\|_{2\alpha}$, where 2α is the order of the operator. In the case of elliptic pseudodifferential operators, we also relax on the smoothness condition of the right-hand side of the equation, as compared to [49, 50].

3.2 The problem

The problem we are solving in this chapter is posed as follows.

Problem A: *Let L be a pseudodifferential operator of order 2α. Let $\mathcal{K}(L) := \{\ell \in \mathbb{N} : \widehat{L}(\ell) = 0\}$. Assume that the cardinality of $\mathcal{K}(L)$ is M. Given, for some $\sigma \geq 0$,*

$$g \in H^{\sigma-\alpha} \quad \text{satisfying} \quad \widehat{g}_{\ell,m} = 0 \quad \text{for all } \ell \in \mathcal{K}(L),\ m = 1, \ldots, N(n,\ell), \tag{3.2.1}$$

find $u \in H^{\sigma+\alpha}$ satisfying

$$\begin{aligned} Lu &= g, \\ \langle \mu_i, u \rangle &= \gamma_i, \quad i = 1, \ldots, M, \end{aligned} \tag{3.2.2}$$

where $\gamma_i \in \mathbb{R}$ and $\mu_i \in H^{-\sigma-\alpha}$ are given. Here $\langle \cdot, \cdot \rangle$ denotes the duality inner product between $H^{-\sigma-\alpha}$ and $H^{\sigma+\alpha}$, which coincides with the H^0-inner product when μ_i and u belong to H^0.

An explanation for the inclusion of σ in (3.2.1) is in order. For the Galerkin approximation, the energy space is H^α. Thus it suffices to assume (3.2.1) with $\sigma = 0$. However, for the collocation approximation, it is required that g be at least continuous. Moreover, we will reformulate the collocation equation into a Galerkin equation which requires $g \in H^\tau$ for some $\tau > 0$ to be specified in Section 3.6. Therefore, we include the constant σ in (3.2.1).

Problem A is uniquely solvable under the following assumption.

Assumption B: *The functionals $\mu_1, \ldots, \mu_M$ are assumed to be unisolvent with respect to $\ker L$, i.e., for any $v \in \ker L$ if $\langle \mu_i, v \rangle = 0$ for all $i = 1, \ldots, M$, then $v = 0$.*

The following theorem is proved in [50]. We include the proof here for completeness.

Theorem 3.1. *Under Assumption B, Problem A has a unique solution.*

Proof. Since $\ker L$ is a finite-dimensional subspace of $H^{\sigma+\alpha}$, we can represent $H^{\sigma+\alpha}$ as

$$H^{\sigma+\alpha} = \ker L \oplus (\ker L)^{\perp}_{H^{\sigma+\alpha}},$$

where $(\ker L)^{\perp}_{H^{\sigma+\alpha}}$ is the orthogonal complement of $\ker L$ with respect to the $H^{\sigma+\alpha}$-inner product. Writing the solution u in the form

$$u = u_0 + u_1 \quad \text{where} \quad u_0 \in \ker L \quad \text{and} \quad u_1 \in (\ker L)^{\perp}_{H^{\sigma+\alpha}}, \tag{3.2.3}$$

and noting that $L|_{(\ker L)^{\perp}_{H^{\sigma+\alpha}}}$ is injective, we can define u_1 by

$$u_1 = L^{-1}g \tag{3.2.4}$$

and find $u_0 \in \ker L$ by solving

$$\langle \mu_i, u_0 \rangle = \gamma_i - \langle \mu_i, u_1 \rangle, \quad i = 1, \ldots, M. \tag{3.2.5}$$

Since $u_0 \in \ker L$, it can represented as

$$u_0 = \sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} c_{\ell,m} Y_{\ell,m}.$$

Substituting this into (3.2.5) yields

$$\sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} c_{\ell,m} \langle \mu_i, Y_{\ell,m} \rangle = \gamma_i - \langle \mu_i, u_1 \rangle, \quad i = 1, \ldots, M. \tag{3.2.6}$$

Recalling that $M = \dim \ker L$, we note that there are M unknowns $c_{\ell,m}$. The unisolvency assumption B assures us that equation (3.2.5) with zero right-hand side has a unique solution $u_0 = 0$. Therefore, the matrix arising from (3.2.6) is invertible, which in turn implies unique existence of $c_{\ell,m}$, $m = 1, \ldots, N(n,\ell)$ and $\ell \in \mathcal{K}(L)$. The theorem is proved. $\square$

Recalling (2.3.4), we define a bilinear form $a(\cdot,\cdot) : H^{\alpha+s} \times H^{\alpha-s} \to \mathbb{R}$, for any $s \in \mathbb{R}$, by

$$a(w, v) := \langle Lw, v \rangle \quad \text{for all } w \in H^{\alpha+s},\ v \in H^{\alpha-s}. \tag{3.2.7}$$

In particular, when $s = \sigma$ we have by noting (3.2.4)

$$a(u_1, v) = \langle g, v \rangle \quad \text{for all } v \in H^{\alpha-\sigma}. \tag{3.2.8}$$

In the next section, we shall define finite-dimensional subspaces in which approximate solutions are sought for.

3.3 Approximation subspaces

In order to assure the positive definiteness of stiffness matrices arising from solving elliptic and strongly elliptic pseudodifferential equations, different spaces of spherical radial basis functions are used. In this section, we first review the definition of the space of spherical radial basis functions defined from a shape function ϕ which will be used to solve strongly elliptic equations. An approximation property of this space as a subspace of Sobolev spaces will be proved in a wide range of Sobolev norms. We then introduce the spaces of spherical radial basis functions which will be used to solve elliptic equations.

3.3.1 Strongly elliptic case

Let $\phi : [-1,1] \to \mathbb{R}$ be a univariate shape function satisfying (2.4.4). Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$ be a set of scattered points on the unit sphere $\mathbb{S}^{n-1}$. For strongly elliptic operators, we shall use the space $\mathcal{V}^\phi := \mathcal{V}_X^\phi$ of spherical radial basis functions defined as in Subsection 2.4.2.

It is noted that for any function v in the native space $\mathcal{N}_\phi$ defined by (2.4.5), there holds

$$v(\boldsymbol{x}_i) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{v}_{\ell,m}\widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i)}{\widehat{\phi}(\ell)} = \langle v, \Phi_i \rangle_\phi, \quad i = 1, \ldots, N; \tag{3.3.1}$$

see [82, page 134]. This property is crucial in our analysis for the collocation method in Section 3.6.

We finish this subsection by proving the approximation property of $\mathcal{V}^\phi$ as a subspace of Sobolev spaces. This property is obtained by using the interpolation error which is derived in [52, Theorem 5.5]. This theorem states that if $v \in H^{s^*}$ for some s^* satisfying $(n-1)/2 < s^* \le \tau$ then for $0 \le t^* \le s^*$ there holds

$$\|v - I_X v\|_{t^*} \le C \rho_X^{\tau - s^*} h_X^{s^* - t^*} \|v\|_{s^*}. \tag{3.3.2}$$

Here, $\rho_X = h_X/q_X$, and $I_X v \in \mathcal{V}^\phi$ is the interpolant of v at $\boldsymbol{x}_i$, $i = 1, \ldots, N$, given by

$$I_X v(\boldsymbol{x}_i) = v(\boldsymbol{x}_i), \quad i = 1, \ldots, N.$$

(In fact, it is required that $v \in \mathcal{N}_\phi$ so that $I_X v$ is well-defined.) When solving pseudodifferential equations of order 2α by the Galerkin method, it is natural to carry out error analysis in the energy space H^α. Since the order 2α may be negative (as in the case of the weakly-singular integral equation discussed after Definition 2.12) it is necessary to show an approximation property of the form (3.3.2) for a wider range of t^* and s^*, including negative real values.

Proposition 3.2. *Assume that* (2.4.4) *holds for some* $\tau > (n-1)/2$. *For any* $s^*, t^* \in \mathbb{R}$ *satisfying* $t^* \le s^* \le 2\tau$ *and* $t^* \le \tau$, *if* $v \in H^{s^*}$ *then there exists* $\eta \in \mathcal{V}^\phi$ *such that*

$$\|v - \eta\|_{t^*} \le C h_X^{s^* - t^*} \|v\|_{s^*} \tag{3.3.3}$$

for $h_X \le h_0$, *where* C *and* h_0 *are independent of* v *and* h_X.

Proof. For $k = 0, 1, 2, \ldots$, we denote $\mathcal{I}_k = [-k\tau, -(k-1)\tau]$ and prove by induction on k that (3.3.3) holds for $t^* \in \mathcal{I}_k$ for all k.

• We first prove that (3.3.3) is true when $t^* \in \mathcal{I}_0$. Indeed, let $t^* \in \mathcal{I}_0$. In this step, we consider two cases when s^* belongs to $[\tau, 2\tau]$ and $[t^*, \tau)$, respectively.

Case 1.1. $\tau \le s^* \le 2\tau$.

Let t and s be real numbers satisfying $0 \le t \le \tau \le s \le 2\tau$. Let $I_X v \in \mathcal{V}^\phi$ be the interpolant of v at $\boldsymbol{x}_i$, $i = 1, \ldots, N$. Then, by using (3.3.1), we deduce

$$\langle v - I_X v, w \rangle_\phi = 0 \quad \text{for all } w \in \mathcal{V}^\phi.$$

Hence, by using (2.4.4) and the Cauchy–Schwarz inequality, we obtain for $v \in H^{2\tau}$

$$\begin{aligned}
\|v - I_X v\|_\tau^2 \simeq \|v - I_X v\|_\phi^2 &= \langle v - I_X v, v - I_X v\rangle_\phi = \langle v - I_X v, v\rangle_\phi \\
&\le \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{|\widehat{v}_{\ell,m} - \widehat{(I_X v)}_{\ell,m}||\widehat{v}_{\ell,m}|}{\widehat{\phi}(\ell)} \\
&\simeq \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2\tau} |\widehat{v}_{\ell,m} - \widehat{(I_X v)}_{\ell,m}||\widehat{v}_{\ell,m}| \le \|v - I_X v\|_0 \|v\|_{2\tau} \qquad (3.3.4)
\end{aligned}$$

Proposition 3.5 in [75] gives

$$\|v - I_X v\|_0 \le C h_X^{2\tau} \|v\|_{2\tau}, \qquad (3.3.5)$$

which, together with (3.3.4), implies

$$\|v - I_X v\|_\tau \le C h_X^{\tau} \|v\|_{2\tau}. \qquad (3.3.6)$$

Noting the inequalities (3.3.5), (3.3.6), and applying Theorem 2.11 with $T = I - I_X$, $s_1 = s_2 = 2\tau$, $t_1 = 0$, $t_2 = \tau$, and $\theta = (\tau - t)/\tau$, we obtain

$$\|v - I_X v\|_t \le C h_X^{2\tau - t} \|v\|_{2\tau}, \quad 0 \le t \le \tau.$$

On the other hand, by using (3.3.2) with t^* and s^* replaced by t and τ, respectively, we obtain

$$\|v - I_X v\|_t \le C h_X^{\tau - t} \|v\|_{\tau}, \quad 0 \le t \le \tau.$$

Using Theorem 2.11 again with $T = I - I_X$, $t_1 = t_2 = t$, $s_1 = \tau$, $s_2 = 2\tau$, and $\theta = 2\tau - s/\tau$, we deduce

$$\|v - I_X v\|_t \le C h_X^{s - t} \|v\|_{s}, \quad 0 \le t \le \tau.$$

Hence, we have proved

$$\begin{cases} 0 \le t^* \le \tau \le s^* \le 2\tau, \\ \forall v \in H^{s^*},\ \exists \eta_v = I_X v \in \mathcal{V}^\phi : \|v - \eta_v\|_{t^*} \le C h_X^{s^* - t^*} \|v\|_{s^*}. \end{cases} \qquad (3.3.7)$$

Case 1.2. $t^* \le s^* < \tau$.

Let s and t be real numbers such that $0 \le s < \tau$ and $2s - 2\tau \le t \le s$. Let $P_s : H^s \to \mathcal{V}^\phi$ be defined by

$$\langle P_s v, w\rangle_s = \langle v, w\rangle_s \quad \forall w \in \mathcal{V}^\phi. \qquad (3.3.8)$$

It is easily seen that

$$\|v - P_s v\|_s \le \|v\|_s. \qquad (3.3.9)$$

If $2s - 2\tau \le t \le 2s - \tau$ so that $\tau \le 2s - t \le 2\tau$ then we apply (3.3.7) with t^* and s^* replaced by s and $2s - t$, respectively, to deduce that for any $w \in H^{2s-t}$, there exists $\eta_w \in \mathcal{V}^\phi$ such that

$$\|w - \eta_w\|_s \le C h_X^{s - t} \|w\|_{2s - t}. \qquad (3.3.10)$$

Since $\langle v - P_s v, \eta_w \rangle_s = 0$, it follows from (2.2.3), (2.2.2), (3.3.9) and (3.3.10) that

$$\|v - P_s v\|_t = \sup_{\substack{w \in H^{2s-t} \\ w \neq 0}} \frac{\langle v - P_s v, w \rangle_s}{\|w\|_{2s-t}} = \sup_{\substack{w \in H^{2s-t} \\ w \neq 0}} \frac{\langle v - P_s v, w - \eta_w \rangle_s}{\|w\|_{2s-t}}$$
$$\leq \|v - P_s v\|_s \sup_{\substack{w \in H^{2s-t} \\ w \neq 0}} \frac{\|w - \eta_w\|_s}{\|w\|_{2s-t}} \leq C h_X^{s-t} \|v\|_s.$$

In particular, for $t = 2s - \tau$ we have

$$\|v - P_s v\|_{2s-\tau} \leq C h_X^{-s+\tau} \|v\|_s. \tag{3.3.11}$$

If $2s - \tau < t \leq s$ then by noting (3.3.9) and (3.3.11), and applying Theorem 2.11 with $T = I - P_s$, $s_1 = s_2 = s$, $t_1 = 2s - \tau$, $t_2 = s$, and $\theta = (t - s)/(s - \tau)$ we obtain $\|v - P_s v\|_t \leq C h_X^{s-t} \|v\|_s$.

Combining both cases 1.1 and 1.2, we have proved that

$$\begin{cases} t^* \in \mathcal{I}_0, \ t^* \leq s^* \leq 2\tau, \\ \forall v \in H^{s^*}, \exists \eta_v \in \mathcal{V}^\phi : \|v - \eta_v\|_{t^*} \leq C h_X^{s^*-t^*} \|v\|_{s^*}. \end{cases} \tag{3.3.12}$$

• Assume that for some $k_0 \geq 0$, (3.3.3) is true when $t^* \in \mathcal{I}_k$, for all $k = 0, 1, \ldots, k_0$, i.e., the following statement holds,

$$\begin{cases} t^* \in \bigcup_{k=0}^{k_0} \mathcal{I}_k, \ t^* \leq s^* \leq 2\tau, \\ \forall v \in H^{s^*}, \ \exists \eta_v \in \mathcal{V}^\phi : \|v - \eta_v\|_{t^*} \leq C h_X^{s^*-t^*} \|v\|_{s^*}. \end{cases} \tag{3.3.13}$$

• We now prove that (3.3.3) is also true when $t^* \in \mathcal{I}_{k_0+1}$. Analogously to the case when $t^* \in \mathcal{I}_0$, we consider two cases when s^* belongs to $[-k_0\tau, 2\tau]$ and $[t^*, -k_0\tau)$, respectively.

Case 2.1. $-k_0\tau \leq s^* \leq 2\tau$.

Let t and s be real numbers satisfying $t \in \mathcal{I}_{k_0+1}$ and $s \in [-k_0\tau, 2\tau]$. Let $P_{-k_0\tau} : H^{-k_0\tau} \to \mathcal{V}^\phi$ be the projection defined by

$$P_{-k_0\tau} v \in \mathcal{V}^\phi : \quad \langle P_{-k_0\tau} v, w \rangle_{-k_0\tau} = \langle v, w \rangle_{-k_0\tau} \quad \forall w \in \mathcal{V}^\phi.$$

Then $P_{-k_0\tau} v$ is the best approximation of v from $\mathcal{V}^\phi$ in the $H^{-k_0\tau}$-norm. It follows from (3.3.13) with $-k_0\tau$ and s in place of t^* and s^*, respectively, that

$$\|v - P_{-k_0\tau} v\|_{-k_0\tau} \leq C h_X^{s+k_0\tau} \|v\|_s \quad \forall v \in H^s. \tag{3.3.14}$$

Since $t \in \mathcal{I}_{k_0+1}$ so that $-k_0\tau \leq -t - 2k_0\tau \leq 2\tau$, statement (3.3.13) with t^* and s^* replaced by $-k_0\tau$ and $-t - 2k_0\tau$, respectively, assures that for any $w \in H^{-t-2k_0\tau}$, there exists $\eta_w \in \mathcal{V}^\phi$ such that

$$\|w - \eta_w\|_{-k_0\tau} \leq C h_X^{-t-k_0\tau} \|w\|_{-t-2k_0\tau}. \tag{3.3.15}$$

Since $\langle v - P_{-k_0\tau} v, \eta_w \rangle_{-k_0\tau} = 0$, it follows from (2.2.3) and (2.2.2) that

$$\|v - P_{-k_0\tau} v\|_t = \sup_{\substack{w \in H^{-t-2k_0\tau} \\ w \neq 0}} \frac{\langle v - P_0 v, w \rangle_{-k_0\tau}}{\|w\|_{-t-2k_0\tau}} = \sup_{\substack{w \in H^{-t-2k_0\tau} \\ w \neq 0}} \frac{\langle v - P_0 v, w - \eta_w \rangle_{-k_0\tau}}{\|w\|_{-t-2k_0\tau}}$$

$$\leq \|v - P_{-k_0\tau}v\|_{-k_0\tau} \sup_{\substack{w\in H^{-t-2k_0\tau}\\ w\neq 0}} \frac{\|w-\eta_w\|_{-k_0\tau}}{\|w\|_{-t-2k_0\tau}}.$$

Inequalities (3.3.14) and (3.3.15) imply $\|v - P_{-k_0\tau}v\|_t \leq Ch_X^{s-t}\|v\|_s$.

Hence, we have proved that

$$\begin{cases} -(k_0+1)\tau \leq t^* \leq -k_0\tau, \ -k_0\tau \leq s^* \leq 2\tau, \\ \forall v \in H^{s^*}, \exists \eta_v \in \mathcal{V}^\phi : \|v-\eta_v\|_{t^*} \leq Ch_X^{s^*-t^*}\|v\|_{s^*}. \end{cases} \tag{3.3.16}$$

Case 2.2. $t^* \leq s^* < -k_0\tau$.

Let s and t be real numbers such that $-(k_0+1)\tau \leq s < -k_0\tau$ and $2s - 2\tau \leq t \leq s$. Let $P_s : H^s \to \mathcal{V}^\phi$ be defined by (3.3.8) with this new value of s.

If $2s-2\tau \leq t \leq 2s+k_0\tau$ so that $-k_0\tau \leq 2s-t \leq 2\tau$ then we can use the same argument as in Case 1.2 with (3.3.7) replaced by (3.3.16) to obtain $\|v - P_sv\|_t \leq Ch_X^{s-t}\|v\|_s$.

If $2s + k_0\tau < t \leq s$ then we use Theorem 2.11 in the same manner as in Case 1.2 to obtain the same estimate.

Combining both cases 2.1 and 2.2 we obtain the result for $k = k_0 + 1$, completing the proof. □

3.3.2 Elliptic case

In order to ensure positive definiteness of the resulting matrix (see Section 3.5 for detail), a different shape function ψ is required for elliptic operators L. Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_N\}$ be a set of points on the sphere. Let ϕ be the shape function given in Subsection 3.3.1. We define $\psi : [-1,1] \to \mathbb{R}$ by

$$\psi(t) = \sum_{\ell=0}^{\infty} \omega_n^{-1} N(n,\ell)\widehat{L}(\ell)\widehat{\phi}(\ell)P_\ell(n;t).$$

The corresponding kernel Ψ and spherical radial basis functions Ψ_i, $i = 1,\dots,N$, are defined by (see (2.4.2) and (2.4.7))

$$\Psi(\boldsymbol{x},\boldsymbol{y}) := \psi(\boldsymbol{x}\cdot\boldsymbol{y}) \quad \text{for all } \boldsymbol{x},\boldsymbol{y} \in \mathbb{S}^{n-1},$$

and

$$\Psi_i(\boldsymbol{x}) := \Psi(\boldsymbol{x},\boldsymbol{x}_i) = \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell)\widehat{\phi}(\ell)Y_{\ell,m}(\boldsymbol{x}_j)Y_{\ell,m}(\boldsymbol{x}), \tag{3.3.17}$$

for all $i = 1,\dots,N$. Note that

$$\Psi_i = L\Phi_i, \quad i = 1,\dots,N, \tag{3.3.18}$$

which are the basis functions used by Morton and Neamtu [50]. Noting (2.4.10) and $\widehat{\psi}(\ell) \simeq (1+\ell)^{-2(\tau-\alpha)}$, we have for any $i = 1,\dots,N$,

$$\Psi_i \in H^s \iff s < 2(\tau-\alpha) + \frac{1-n}{2}. \tag{3.3.19}$$

The approximation space to be used for elliptic operators is the span of these functions:

$$\mathcal{V}_X^\psi := \operatorname{span}\{\Psi_1,\dots,\Psi_N\}.$$

In the rest of the dissertation, we use $\mathcal{V}^\psi := \mathcal{V}_X^\psi$.

3.4 Approximate solutions

In this section we shall use $\mathcal{V}$ to stand for $\mathcal{V}^\phi$ or $\mathcal{V}^\psi$.

3.4.1 Approach

Noting (3.2.3), we shall seek an approximate solution $\widetilde{u} \in H^{\sigma+\alpha}$ in the form

$$\widetilde{u} = \widetilde{u}_0 + \widetilde{u}_1 \quad \text{where} \quad \widetilde{u}_0 \in \ker L \quad \text{and} \quad \widetilde{u}_1 \in \mathcal{V}.$$

The solution $\widetilde{u}_1$ will be found by the Galerkin or collocation method. Having found $\widetilde{u}_1$, we will find $\widetilde{u}_0 \in \ker L$ by solving the equations (cf. (3.2.5))

$$\langle \mu_i, \widetilde{u}_0 \rangle = \gamma_i - \langle \mu_i, \widetilde{u}_1 \rangle, \quad i = 1, \dots, M,$$

so that

$$\langle \mu_i, \widetilde{u} \rangle = \langle \mu_i, u \rangle, \quad i = 1, \dots, M. \tag{3.4.1}$$

The unique existence of $\widetilde{u}_0$ follows from Assumption B in exactly the same way as that of u_0; see Theorem 3.1.

We postpone, for a moment, the issue of finding $\widetilde{u}_1$. It is noted that in general $\mathcal{V} \not\subseteq (\ker L)^\perp_{H^{\sigma+\alpha}}$. However, $\widetilde{u}$ can be rewritten in a form similar to (3.2.3) as follows. Let

$$u_0^* := \widetilde{u}_0 + \sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \widehat{(\widetilde{u}_1)}_{\ell,m} Y_{\ell,m} \tag{3.4.2}$$

and

$$u_1^* = \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \widehat{(\widetilde{u}_1)}_{\ell,m} Y_{\ell,m}. \tag{3.4.3}$$

Then

$$\widetilde{u} = u_0^* + u_1^* \quad \text{with} \quad u_0^* \in \ker L \quad \text{and} \quad u_1^* \in (\ker L)^\perp_{H^{\sigma+\alpha}}. \tag{3.4.4}$$

It should be noted that, in general, u_1^* does not belong to $\mathcal{V}$, and that this function is introduced purely for analysis purposes. We do not explicitly compute u_1^*, nor u_0^*.

3.4.2 Preliminary error analysis

Assume that the exact solution u and the approximate solution $\widetilde{u}$ of Problem A belong to H^t for some $t \in \mathbb{R}$, and assume that $\mu_i \in H^{-t}$ for $i = 1, \dots, M$. Comparing (3.2.3) and (3.4.4) suggests that $\|u - \widetilde{u}\|_t$ can be estimated by estimating $\|u_0 - u_0^*\|_t$ and $\|u_1 - u_1^*\|_t$. It turns out that an estimate for the latter is sufficient, as shown in the following two lemmas.

Lemma 3.3. *Let u_0, u_1, u_0^* and u_1^* be defined by* (3.2.3), (3.4.2) *and* (3.4.3). *For $i = 1, \dots, M$, if $\mu_i \in H^{-t}$ for some $t \in \mathbb{R}$, then*

$$\|u_0 - u_0^*\|_t \le C \|u_1 - u_1^*\|_t,$$

where C is independent of u.

Proof. For $i = 1, \dots, M$, it follows from (3.4.1) that

$$\langle \mu_i, u_0 \rangle + \langle \mu_i, u_1 \rangle = \langle \mu_i, u_0^* \rangle + \langle \mu_i, u_1^* \rangle ,$$

implying $\langle \mu_i, u_0 - u_0^* \rangle = \langle \mu_i, u_1^* - u_1 \rangle$. Inequality (2.2.3) yields

$$|\langle \mu_i, u_0 - u_0^* \rangle| = |\langle \mu_i, u_1 - u_1^* \rangle| \le \|\mu_i\|_{-t} \|u_1 - u_1^*\|_t.$$

This result holds for all $i = 1, \dots, M$, implying

$$\|u_0 - u_0^*\|_\mu \ \le \ \mathcal{M} \, \|u_1 - u_1^*\|_t,$$

where $\mathcal{M} := \max_{i=1,\dots,M} \|\mu_i\|_{-t}$, and $\|v\|_\mu := \max_{i=1,\dots,M} |\langle \mu_i, v \rangle|$ for all $v \in \ker L$. (The unisolvency assumption assures us that the above norm is well-defined.) The subspace $\ker L$ being finite-dimensional, we deduce

$$\|u_0 - u_0^*\|_t \le C \, \|u_1 - u_1^*\|_t,$$

proving the lemma. □

Lemma 3.4. *Under the assumptions of Lemma 3.3, there holds*

$$\|u - \widetilde{u}\|_t \le C \|u_1 - u_1^*\|_t.$$

Proof. Noting (3.2.3) and (3.4.4), the norm $\|u - \widetilde{u}\|_t$ can be rewritten as

$$\begin{aligned}
\|u - \widetilde{u}\|_t^2 &= \sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2t} \, |\widehat{u}_{\ell,m} - \widehat{(\widetilde{u})}_{\ell,m}|^2 + \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2t} \, |\widehat{u}_{\ell,m} - \widehat{(\widetilde{u})}_{\ell,m}|^2 \\
&= \sum_{\ell \in \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2t} \, |\widehat{(u_0)}_{\ell,m} - \widehat{(u_0^*)}_{\ell,m}|^2 \\
&\quad + \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2t} \, |\widehat{(u_1)}_{\ell,m} - \widehat{(u_1^*)}_{\ell,m}|^2 = \|u_0 - u_0^*\|_t^2 + \|u_1 - u_1^*\|_t^2.
\end{aligned}$$

The required result now follows from Lemma 3.3. □

In the following sections, we describe methods to construct $\widetilde{u}_1$, and estimate $\|u_1 - u_1^*\|_t$ accordingly.

3.5 Galerkin approximation

3.5.1 Strongly elliptic case

In this subsection, we consider the case when L satisfies the strongly elliptic condition (2.3.3). Recalling (2.4.10), we choose the shape functions ϕ such that

$$\tau > \frac{1}{2} \left(\alpha + \frac{n-1}{2} \right), \tag{3.5.1}$$

so that $\mathcal{V}^\phi \subset H^\alpha$. We find $\widetilde{u}_1 \in \mathcal{V}^\phi$ by solving the Bubnov–Galerkin equation

$$a(\widetilde{u}_1, v) = \langle g, v \rangle \quad \text{for all } \ v \in \mathcal{V}^\phi. \tag{3.5.2}$$

By writing $\widetilde{u}_1 = \sum_{i=1}^N c_i \Phi_i$ we derive from (3.5.2) the matrix equation $\boldsymbol{A}^{(SG)}\boldsymbol{c} = \boldsymbol{b}$, where

$$\boldsymbol{A}_{ij}^{(SG)} = a(\Phi_i, \Phi_i) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell)\, [\widehat{\phi}(\ell)]^2\, Y_{\ell,m}(\boldsymbol{x}_i)\, Y_{\ell,m}(\boldsymbol{x}_i), \tag{3.5.3}$$

$\boldsymbol{c} = (c_1, \ldots, c_N)$, and $\boldsymbol{b} = (\langle g, \Phi_1 \rangle, \ldots, \langle g, \Phi_N \rangle)$.

Lemma 3.5. *The matrix $\boldsymbol{A}^{(SG)}$ is symmetric positive-definite.*

Proof. Let θ be a shape function whose Fourier–Legendre coefficients are given by

$$\widehat{\theta}(\ell) = \begin{cases} \widehat{L}(\ell)[\widehat{\phi}(\ell)]^2 & \text{if } \ell \notin \mathcal{K}(L) \\ 0 & \text{if } \ell \in \mathcal{K}(L). \end{cases}$$

Then $\boldsymbol{A}_{ij}^{(SG)} = \Theta(\boldsymbol{x}_i, \boldsymbol{x}_i)$ where Θ is the kernel defined from θ. Since $\widehat{\theta}(\ell) \ge 0$ for all $\ell \ge 0$, and $\widehat{\theta}(\ell) = 0$ only for a finite number of ℓ, it follows from Remark 2.14 that $\boldsymbol{A}^{(SG)}$ is symmetric positive-definite. $\square$

As a consequence of this lemma, there exists a unique solution $\widetilde{u}_1$ to (3.5.2). With $\widetilde{u}_1$ given by (3.5.2), u_1^* defined by (3.4.3) satisfies $u_1^* \in H^\alpha$ and

$$a(u_1^*, v) = \langle g, v \rangle \quad \text{for all } v \in \mathcal{V}^\phi. \tag{3.5.4}$$

Even though in general u_1^* does not belong to $\mathcal{V}^\phi$, the following result is essentially Céa's Lemma.

Lemma 3.6. *If u_1 and u_1^* are defined by* (3.2.8) *and* (3.4.3) *with $\widetilde{u}_1$ given by* (3.5.2)*, then*

$$\|u_1 - u_1^*\|_\alpha \le C\|u_1 - v\|_\alpha \quad \textit{for all } v \in \mathcal{V}^\phi.$$

Proof. It follows from the definition (3.4.3) of u_1^* that

$$a(w, u_1^*) = a(w, \widetilde{u}_1) \quad \text{for all } w \in H^\alpha. \tag{3.5.5}$$

Moreover, since $\mathcal{V}^\phi \subset H^\alpha \subset H^{\alpha-\sigma}$ (noting $\sigma \ge 0$) we infer from (3.2.8) and (3.5.4)

$$a(u_1 - u_1^*, v) = 0 \quad \text{for all } v \in \mathcal{V}^\phi. \tag{3.5.6}$$

Since $u_1 - u_1^* \in (\ker L)^\perp_{H^\alpha}$, Lemma 2.13 yields

$$\|u_1 - u_1^*\|_\alpha^2 \simeq a(u_1 - u_1^*, u_1 - u_1^*) = a(u_1 - u_1^*, u_1) - a(u_1 - u_1^*, u_1^*).$$

It follows from (3.5.5) and (3.5.6), noting $u_1 - u_1^* \in H^\alpha$ and $\widetilde{u}_1 \in \mathcal{V}^\phi$, that

$$\|u_1 - u_1^*\|_\alpha^2 \simeq a(u_1 - u_1^*, u_1) - a(u_1 - u_1^*, \widetilde{u}_1) = a(u_1 - u_1^*, u_1).$$

Hence, using again (3.5.6), we obtain for any $v \in \mathcal{V}^\phi$

$$\|u_1 - u_1^*\|_\alpha^2 \simeq a(u_1 - u_1^*, u_1 - v) \le C\|u_1 - u_1^*\|_\alpha \, \|u_1 - v\|_\alpha,$$

where in the last step we used Lemma 2.13. By cancelling similar terms we obtain the required result. $\square$

The above lemma and Proposition 3.2 will be used to estimate the error $u_1 - u_1^*$.

Lemma 3.7. *Assume that the shape function ϕ is chosen to satisfy* (2.4.4), (3.5.1) *and* $\tau \ge \alpha$, $\tau > (n-1)/2$. *Let u_1 and u_1^* be defined as in Lemma 3.6. Assume that $u_1 \in H^s$ for some s satisfying $\alpha \le s \le 2\tau$. Let $t \in \mathbb{R}$ satisfy $2(\alpha - \tau) \le t \le \alpha$. Then for h_X sufficiently small there holds*

$$\|u_1 - u_1^*\|_t \ \le\ C h_X^{s-t} \|u_1\|_s. \tag{3.5.7}$$

The constant C is independent of u and h_X.

Proof. The result for the case when $t = \alpha$ is a direct consequence of Lemma 3.6 and Proposition 3.2 (with $t^* = \alpha$ and $s^* = s$).

The proof for the case $t < \alpha$ is standard, using Aubin–Nitsche's trick, and is included here for completeness. It follows from (2.2.3) and (2.3.3) that

$$\|u_1 - u_1^*\|_t \le \sup_{\substack{v \in H^{2\alpha - t}\\ v \ne 0}} \frac{\langle u_1 - u_1^*, v\rangle_\alpha}{\|v\|_{2\alpha - t}} \le C \sup_{\substack{v \in H^{2\alpha - t}\\ v \ne 0}} \frac{a(u_1 - u_1^*, v)}{\|v\|_{2\alpha - t}}.$$

By using successively (3.5.6), Lemma 2.13, (3.5.7) with t replaced by α, we deduce for any $\eta \in \mathcal{V}^\phi$

$$\begin{aligned} \|u_1 - u_1^*\|_t &\le C \sup_{\substack{v \in H^{2\alpha - t}\\ v \ne 0}} \frac{a(u_1 - u_1^*, v - \eta)}{\|v\|_{2\alpha - t}} \le C \|u_1 - u_1^*\|_\alpha \sup_{\substack{v \in H^{2\alpha - t}\\ v \ne 0}} \frac{\|v - \eta\|_\alpha}{\|v\|_{2\alpha - t}} \\ &\le C h_X^{s-\alpha} \|u_1\|_s \sup_{\substack{v \in H^{2\alpha - t}\\ v \ne 0}} \frac{\|v - \eta\|_\alpha}{\|v\|_{2\alpha - t}}. \end{aligned} \tag{3.5.8}$$

Since $2(\alpha - \tau) \le t < \alpha$, there holds $\alpha < 2\alpha - t \le 2\tau$. By invoking Proposition 3.2 again with t^* and s^* replaced by α and $2\alpha - t$, respectively, we can choose $\eta \in \mathcal{V}^\phi$ satisfying

$$\|v - \eta\|_\alpha \le C h_X^{\alpha - t} \|v\|_{2\alpha - t}.$$

This together with (3.5.8) yields the required estimate, proving the lemma. □

We are now ready to state and prove the main result of this section.

Theorem 3.8. *Assume that the shape function ϕ is chosen to satisfy* (2.4.4), (3.5.1) *and* $\tau \ge \alpha$, $\tau > (n-1)/2$. *Assume further that $u \in H^s$ for some s satisfying $\alpha \le s \le 2\tau$. If $\mu_i \in H^{-t}$ for $i = 1, \ldots, M$ with $t \in \mathbb{R}$ satisfying $2(\alpha - \tau) \le t \le \alpha$, then for h_X sufficiently small there holds*

$$\|u - \widetilde{u}\|_t \ \le\ C h_X^{s-t} \|u\|_s.$$

The constant C is independent of u and h_X.

Proof. Since $\mu_i \in H^{-t}$ for $i = 1, \ldots, M$, Lemma 3.4 gives

$$\|u - \widetilde{u}\|_t \le C \|u_1 - u_1^*\|_t.$$

The required result is a consequence of Lemma 3.7, noting that $\|u_1\|_s \le \|u\|_s$. □

3.5.2 Elliptic case

In this subsection, we consider the case when L satisfies the elliptic condition (2.3.2). If we choose the same trial space $\mathcal{V}^\phi$ as in the previous subsection, then the resulting matrix would be $\boldsymbol{A}^{(SG)}$ defined in (3.5.3). This matrix might be singular because the shape function θ defined in the proof of Lemma 3.5 may have negative Fourier–Legendre coefficients $\widehat{\theta}(\ell)$. To avoid this situation, instead of $\mathcal{V}^\phi$ we follow Morton and Neamtu [50] to use $\mathcal{V}^\psi$ as the trial space, see Subsection 3.3.2, and use $\mathcal{V}^\phi$ as the test space. The component $\widetilde{u}_1$ will be found in $\mathcal{V}^\psi$ by solving the Petrov–Galerkin equation

$$a(\widetilde{u}_1, v) = \langle g, v\rangle \quad \text{for all } v \in \mathcal{V}^\phi. \tag{3.5.9}$$

In this case, recalling (2.4.10) and (3.3.19) we choose the shape function ϕ such that

$$\tau > \max\left\{\frac{1}{2}\left(3\alpha + \frac{n-1}{2}\right), \frac{1}{2}\left(\alpha + \frac{n-1}{2}\right)\right\} \tag{3.5.10}$$

so that $\mathcal{V}^\phi$ and $\mathcal{V}^\psi$ are finite-dimensional subspaces of H^α. The matrix $\boldsymbol{A}^{(EG)}$ arising from (3.5.9) has entries given by

$$\boldsymbol{A}^{(EG)}_{ij} = a(\Psi_i, \Phi_i) = \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} [\widehat{L}(\ell)]^2\,[\widehat{\phi}(\ell)]^2\,Y_{\ell,m}(\boldsymbol{x}_i)\,Y_{\ell,m}(\boldsymbol{x}_i). \tag{3.5.11}$$

Positive definiteness of $\boldsymbol{A}^{(EG)}$ can be confirmed by using a similar argument as in the proof of Lemma 3.5, which yields the unique existence of $\widetilde{u}_1 \in \mathcal{V}^\psi$.

As is usually required for Petrov–Galerkin methods, inf-sup (or Ladyzenskaya–Babuška–Brezzi) conditions are in order. We prove this condition for infinite dimensional spaces.

Proposition 3.9. *Let L be an elliptic pseudodifferential operator of order 2α. For any $t \in \mathbb{R}$, there exists a positive constant C such that*

$$\sup_{\substack{v\in H^{2\alpha-t}\\ v\neq 0}} \frac{a(w,v)}{\|v\|_{2\alpha-t}} \ge C\|w\|_t \quad \textit{for all } w \in (\ker L)^{\perp}_{H^t}. \tag{3.5.12}$$

Proof. Let $w \in (\ker L)^{\perp}_{H^t}$. Then

$$\widehat{w}_{\ell,m} = 0 \quad \text{for all } \ell \in \mathcal{K}(L),\ m = 1, \dots, N(n,\ell).$$

We define

$$v := \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} \operatorname{sgn}(\widehat{L}(\ell))\frac{\widehat{w}_{\ell,m}}{(\ell+1)^{2\alpha-2t}} Y_{\ell,m}$$

where

$$\operatorname{sgn}(\widehat{L}(\ell)) := \begin{cases} 1 & \text{if } \widehat{L}(\ell) \ge 0,\\ -1 & \text{if } \widehat{L}(\ell) < 0.\end{cases}$$

Then, $\|v\|_{2\alpha-t} = \|w\|_t$ and

$$a(w,v) = \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell)\widehat{w}_{\ell,m}\operatorname{sgn}(\widehat{L}(\ell))\frac{\widehat{w}_{\ell,m}}{(\ell+1)^{2\alpha-2t}} = \sum_{\ell=0}^{\infty}\sum_{m=1}^{N(n,\ell)} \frac{|\widehat{L}(\ell)|}{(\ell+1)^{2\alpha-2t}}|\widehat{w}_{\ell,m}|^2$$

$$\geq C \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2t} |\widehat{w}_{\ell,m}|^2 = C\|w\|_t^2,$$

where C is the constant given from the elliptic condition (2.3.2). Hence

$$\frac{a(w,v)}{\|v\|_{2\alpha-t}} \geq C\|w\|_t,$$

so that

$$\sup_{\substack{v \in H^{2\alpha-t} \\ v \neq 0}} \frac{a(w,v)}{\|v\|_{2\alpha-t}} \geq C\|w\|_t,$$

proving the proposition. □

In the remainder of this section, we consider a family of sets of data points $X_k = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N_k}\}$ for $k \in \mathbb{N}$ where $N_k < N_{k+1}$ so that $X_k \subset X_{k+1}$ and the corresponding mesh norm h_k, see (2.4.6), satisfies $h_k \to 0$ when $k \to \infty$. The corresponding finite-dimensional spaces $\mathcal{V}^{\phi}_{X_k}$ and $\mathcal{V}^{\psi}_{X_k}$ are denoted by $\mathcal{V}^{\phi}_{h_k}$ and $\mathcal{V}^{\psi}_{h_k}$. Hence $\mathcal{V}^{\phi}_{h_k} \subset \mathcal{V}^{\phi}_{h_{k+1}}$ and $\mathcal{V}^{\psi}_{h_k} \subset \mathcal{V}^{\psi}_{h_{k+1}}$, for all $k \in \mathbb{N}$.

The proof of the inf-sup condition involving these finite-dimensional spaces requires the introduction of the projection

$$P_{h_k} : H^{\alpha} \to \mathcal{V}^{\phi}_{h_k}$$

which is defined by

$$a(P_{h_k} v, w) = a(v,w) \quad \forall w \in \mathcal{V}^{\psi}_{h_k}, \quad k \in \mathbb{N}. \tag{3.5.13}$$

We note that (3.5.13) is a Petrov–Galerkin equation, which results in a matrix $\boldsymbol{B}$ whose entries are

$$\boldsymbol{B}_{ij} = a(\Phi_i, \Psi_i) = \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell)^2 \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i) Y_{\ell,m}(\boldsymbol{x}_j),$$

for $i,j = 1, \ldots, N_k$. Using the same argument as in the proof of Lemma 3.5, we can show that $\boldsymbol{B}$ is positive definite. Hence P_{h_k} is well-defined.

Proposition 3.10. *There exists a positive constant C independent of h_k such that*

$$\|P_{h_k}\|_{\alpha} \leq C \quad \forall k \in \mathbb{N}. \tag{3.5.14}$$

Proof. The result is proved by using the Banach-Steinhaus Theorem. Thus we need to show that

(i) $P_{h_k} \in \mathscr{L}(H^{\alpha}, \mathcal{V}^{\phi}_{h_k})$, the space of continuous linear mappings from H^{α} to $\mathcal{V}^{\phi}_{h_k}$,

(ii) $\sup_{k \in \mathbb{N}} \|P_{h_k} v\|_{\alpha} \leq C(v)$ for each $v \in H^{\alpha}$ where $C(v)$ is a positive constant.

Proof of (i). It is obvious that P_{h_k} is linear. For any $v \in H^{\alpha}$, since $P_{h_k} v \in \mathcal{V}^{\phi}_{h_k}$, there exists $\boldsymbol{c}(v) := (c_1(v), \ldots, c_{N_k}(v)) \in \mathbb{R}^{N_k}$ such that $P_{h_k} v = \sum_{i=1}^{N_k} c_i(v) \Phi_i$. We have

$$\|P_{h_k} v\|_{\alpha} \leq \sum_{i=1}^{N_k} |c_i(v)| \|\Phi_i\|_{\alpha} \leq \left(\max_{i=1,\ldots,N_k} \|\Phi_i\|_{\alpha} \right) \|\boldsymbol{c}(v)\|_{\ell_1}. \tag{3.5.15}$$

Here, for any $\boldsymbol{c} = (c_1, \cdots, c_{N_k}) \in \mathbb{R}^{N_k}$, $\|\boldsymbol{c}\|_{\ell_1}$ is the ℓ_1-norm of $\boldsymbol{c}$, i.e., $\|\boldsymbol{c}\|_{\ell_1} = \sum_{i=1}^{N_k} |c_i|$. It follows from (3.5.13), using $\boldsymbol{c}(v) = \boldsymbol{B}^{-1}\boldsymbol{b}(v)$, that

$$\|\boldsymbol{c}(v)\|_{\ell_1} \le \|\boldsymbol{B}^{-1}\|_{\ell_1} \cdot \|\boldsymbol{b}(v)\|_{\ell_1}, \tag{3.5.16}$$

where $\boldsymbol{b}(v) = \big(a(v, \Psi_1), \ldots, a(v, \Psi_{N_k})\big)$. By using Lemma 2.13, we obtain

$$\|\boldsymbol{b}(v)\|_{\ell_1} = \sum_{i=1}^{N_k} |a(v, \Psi_i)| \le \sum_{i=1}^{N_k} \|v\|_\alpha \|\Psi_i\|_\alpha \le N_k \left(\max_{i=1,\ldots,N_k} \|\Psi_i\|_\alpha \right) \|v\|_\alpha.$$

This together with (3.5.15) and (3.5.16) implies

$$\|P_{h_k} v\|_\alpha \le C\|v\|_\alpha,$$

where C is a constant depending on h_k, confirming the continuity of P_{h_k}.

Proof of (ii). We will show $\sup_{k\in\mathbb{N}} \|P_{h_k} v\|_\alpha \le C(v)$ for any $v \in H^\alpha$ by showing that $\|P_{h_k} v - v\|_\alpha \to 0$ when $k \to \infty$. Let $\{\Phi_i^*\}_{i\in\mathbb{N}} \subset \bigcup_{k\in\mathbb{N}} \mathcal{V}_{h_k}^\phi$ and $\{\Psi_i^*\}_{i\in\mathbb{N}} \subset \bigcup_{k\in\mathbb{N}} \mathcal{V}_{h_k}^\psi$ be such that, for each $k \in \mathbb{N}$, the sets $\{\Phi_1^*, \ldots, \Phi_{N_k}^*\}$ and $\{\Psi_1^*, \ldots, \Psi_{N_k}^*\}$ are, respectively, bases for $\mathcal{V}_{h_k}^\phi$ and $\mathcal{V}_{h_k}^\psi$ which satisfy

$$a(\Phi_i^*, \Psi_j^*) = \delta_{ij}, \quad i, j \in \mathbb{N}. \tag{3.5.17}$$

The existence of $\{\Phi_i^*\}_{i\in\mathbb{N}}$ and $\{\Psi_i^*\}_{j\in\mathbb{N}}$ will be discussed later. Since $\overline{\bigcup_{k\in\mathbb{N}} \mathcal{V}_{h_k}^\phi} = H^\alpha$ due to Theorem 3.2, any $v \in H^\alpha$ can be represented as $v = \sum_{i\in\mathbb{N}} c_i \Phi_i^*$. It follows from (3.5.17) that the projection $P_{h_k} v$ is given by $P_{h_k} v = \sum_{i=1}^{N_k} c_i \Phi_i^*$. Hence $\|P_{h_k} v - v\|_\alpha \to 0$ when $k \to \infty$.

Estimate (3.5.14) is now a result of (i), (ii), and the Banach–Steinhaus Theorem. □

The sequences $\{\Phi_i^*\}_{i\in\mathbb{N}}$ and $\{\Psi_i^*\}_{j\in\mathbb{N}}$ can be constructed by induction as follows. First, Φ_1^* and Ψ_1^* are defined by

$$\Phi_1^* = \Phi_1 \quad \text{and} \quad \Psi_1^* = e_1^1 \Psi_1 \quad \text{where} \quad e_1^1 = \frac{1}{a(\Phi_1, \Psi_1)}.$$

Assume that Φ_m^* and Ψ_n^*, $m, n = 1, \ldots, i-1$, have been defined in the form

$$\Phi_m^* = \sum_{s=1}^{m-1} d_s^m \Phi_s + \Phi_m \quad \text{and} \quad \Psi_n^* = \sum_{s=1}^{n} e_s^n \Psi_s,$$

where $d_s^m, e_s^n \in \mathbb{R}$, so that $e_n^n \ne 0$ and

$$a(\Phi_m^*, \Psi_n^*) = \delta_{m,n}, \quad m, n = 1, \ldots, i-1.$$

The next functions Φ_i^* and Ψ_i^* are defined in the following lemma.

Lemma 3.11. *There exist d_s^i and $e_s^i \in \mathbb{R}$ with $e_i^i \ne 0$ such that if*

$$\Phi_i^* = \sum_{s=1}^{i-1} d_s^i \Phi_s + \Phi_i \quad \textit{and} \quad \Psi_i^* = \sum_{s=1}^{i} e_s^i \Psi_s,$$

then

$$a(\Phi_m^*, \Psi_n^*) = \delta_{m,n}, \quad m, n = 1, \ldots, i.$$

Proof. We first note that there exist $d_1^i, \dots, d_{i-1}^i \in \mathbb{R}$ such that

$$a(\Phi_i^*, \Psi_n^*) = 0, \quad n = 1, \dots, i-1.$$

Indeed, the corresponding $(i-1) \times (i-1)$-matrix $\boldsymbol{D}$ arising from the above system of linear equations has entries given by

$$\boldsymbol{D}_{ns} = a(\Phi_s, \sum_{t=1}^{n} e_t^n \Psi_t), \quad n, s = 1, \dots, i-1.$$

By using Gaussian elimination, it is not hard to see that this matrix has the same rank as the matrix whose $(n,s)^{th}$ entry is $a(\Phi_s, \Psi_n)$. The latter matrix is positive definite by using the same argument as in the proof of Lemma 3.5. Hence, the existence of $(d_1^i, \dots, d_{i-1}^i)$ is confirmed.

We next need to confirm the existence of $e_1^i, \dots, e_i^i \in \mathbb{R}$ with $e_i^i \neq 0$ such that

$$a(\Phi_m^*, \Psi_i^*) = \delta_{m,i}, \quad m = 1, \dots, i.$$

This system of linear equations results in a matrix equation $\boldsymbol{E}\boldsymbol{e} = \boldsymbol{\delta}$ in which $\boldsymbol{E}$ is an $i \times i$-matrix with entries given by

$$\boldsymbol{E}_{mt} = a(\sum_{s=1}^{m-1} d_s^m \Phi_s + \Phi_m, \Psi_t), \quad m, t = 1, \dots, i,$$

and $\boldsymbol{e} = (e_1^i, \dots, e_i^i) \in \mathbb{R}^i$ and $\boldsymbol{\delta} = (0, \dots, 0, 1) \in \mathbb{R}^i$. By using Gaussian elimination we transform the matrix equation into

$$\boldsymbol{B}\boldsymbol{e} = \boldsymbol{\delta},$$

where $\boldsymbol{B}$ is a $i \times i$-matrix whose entries are given by

$$\boldsymbol{B}_{mt} = a(\Phi_m, \Psi_t), \quad m, t = 1, \dots, i.$$

Since $\boldsymbol{B}$ is positive definite, the equation has a unique solution $(e_1^i, \dots, e_i^i)$. Moreover, if $e_i^i = 0$ then $\boldsymbol{e} = \boldsymbol{0}$, resulting in a contradiction. The lemma is proved. □

We are now ready to prove the inf-sup condition for our finite-dimensional subspaces.

Proposition 3.12. *Under the condition* (3.5.10)*, there exists a positive constant C such that*

$$\sup_{\substack{v \in \mathcal{V}^\phi \\ v \neq 0}} \frac{a(w,v)}{\|v\|_\alpha} \geq C \|w\|_\alpha \quad \textit{for all } w \in \mathcal{V}^\psi. \tag{3.5.18}$$

Proof. Noting that $\mathcal{V}^\psi \subset (\ker L)^\perp_{H^\alpha}$, Proposition 3.9 confirms the existence of a positive constant C such that

$$\|w\|_\alpha \leq C \sup_{v \in H^\alpha} \frac{a(w,v)}{\|v\|_\alpha} \quad \forall w \in \mathcal{V}^\psi. \tag{3.5.19}$$

By the definition of P_h, and noting (3.5.14), we obtain

$$\sup_{v \in H^\alpha} \frac{a(w,v)}{\|v\|_\alpha} = \sup_{v \in H^\alpha} \frac{a(w, P_h v)}{\|P_h v\|_\alpha} \cdot \frac{\|P_h v\|_\alpha}{\|v\|_\alpha}$$

$$\begin{aligned} &\leq C \sup_{v \in H^\alpha} \frac{a(w, P_h v)}{\|P_h v\|_\alpha} \\ &= C \sup_{v \in \mathcal{V}^\phi} \frac{a(w, v)}{\|v\|_\alpha}. \end{aligned}$$

This together with (3.5.19) implies (3.5.18). □

Similarly to Lemma 3.6, the following result is technically Céa's Lemma for the Petrov–Galerkin approximation (3.5.9).

Lemma 3.13. *There exists $C > 0$ such that with u_1, $\widetilde{u}_1$ and u_1^* given by* (3.2.8), (3.5.9) *and* (3.4.3), *there holds*

$$\|u_1 - u_1^*\|_\alpha \leq C\|u_1 - w\|_\alpha \quad \textit{for all } w \in \mathcal{V}^\psi.$$

Proof. It follows from the definition of u_1^* that

$$\|u_1 - u_1^*\|_\alpha \leq \|u_1 - \widetilde{u}_1\|_\alpha. \tag{3.5.20}$$

Let $w \in \mathcal{V}^\psi$. By noting that $\widetilde{u}_1 - w \in \mathcal{V}^\psi$ and using Proposition 3.12 we obtain

$$\|\widetilde{u}_1 - w\|_\alpha \leq C \sup_{\substack{v \in \mathcal{V}^\phi \\ v \neq 0}} \frac{a(\widetilde{u}_1 - w, v)}{\|v\|_\alpha}.$$

On the other hand, (3.2.8) and (3.5.9) give

$$a(u_1, v) = a(\widetilde{u}_1, v) \quad \text{for all } v \in \mathcal{V}^\phi. \tag{3.5.21}$$

Therefore,

$$\|\widetilde{u}_1 - w\|_\alpha \leq C \sup_{\substack{v \in \mathcal{V}^\phi \\ v \neq 0}} \frac{a(u_1 - w, v)}{\|v\|_\alpha} \leq C \sup_{\substack{v \in \mathcal{V}^\phi \\ v \neq 0}} \frac{\|u_1 - w\|_\alpha \|v\|_\alpha}{\|v\|_\alpha} = C\|u_1 - w\|_\alpha,$$

where in the penultimate step we use Lemma 2.13. This inequality, (3.5.20) and the triangle inequality,

$$\|u_1 - \widetilde{u}_1\|_\alpha \leq \|u_1 - w\|_\alpha + \|\widetilde{u}_1 - w\|_\alpha,$$

yield

$$\|u_1 - u_1^*\|_\alpha \leq C\|u_1 - w\|_\alpha \quad \text{for all } w \in \mathcal{V}^\psi,$$

proving the lemma. □

Theorem 3.14. *Let* (3.5.10) *hold. We choose the shape function ϕ such that* (2.4.4) *with $\tau \geq 3\alpha$ and $\tau > (n-1)/2$. Let $u \in H^s$ for some s satisfying $\alpha \leq s \leq 2\tau - 2\alpha$. If $\mu_i \in H^{-t}$, $i = 1, \ldots, M$ for some t satisfying $2(\alpha - \tau) \leq t < \alpha$, then for h_X sufficiently small there holds*

$$\|u - \widetilde{u}\|_t \leq C h_X^{s-t} \|u - \widetilde{u}\|_s.$$

The constant C is independent of u and h_X.

Proof. We first prove the result for the case when $t = \alpha$. From Lemmas 3.13 and 3.4 (with $t = \alpha$), there follows

$$\|u - \widetilde{u}\|_\alpha \le C \inf_{w \in \mathcal{V}^\psi} \|u_1 - w\|_\alpha. \tag{3.5.22}$$

We note that Proposition 3.2 cannot be directly used here because the shape function ψ defining $\mathcal{V}^\psi$ does not possess a property similar to (2.4.4), due to possible negative sign in $\widehat{L}(\ell)$. To circumvent this hindrance, we introduce a new function. In this proof only, let

$$U_1 := \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{(u_1)}_{\ell,m}}{\widehat{L}(\ell)} Y_{\ell,m}.$$

Then $U_1 \in H^{s+2\alpha}$ and $\|U_1\|_{s+2\alpha} \le C\|u_1\|_s$ if $u_1 \in H^s$. We also define for any $V \in \mathcal{V}^\phi$,

$$w := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell) \widehat{V}_{\ell,m} Y_{\ell,m}.$$

By using (3.3.17) one can prove that $w \in \mathcal{V}^\psi$. Moreover,

$$\begin{aligned}
\|u_1 - w\|_\alpha^2 &= \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2\alpha} \left| \widehat{(u_1)}_{\ell,m} - \widehat{w}_{\ell,m} \right|^2 \\
&= \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{2\alpha} \left| \widehat{L}(\ell) \right|^2 \left| \widehat{(U_1)}_{\ell,m} - \widehat{V}_{\ell,m} \right|^2 \\
&\le C \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} (\ell+1)^{6\alpha} \left| \widehat{(U_1)}_{\ell,m} - \widehat{V}_{\ell,m} \right|^2 \\
&= C\|U_1 - V\|_{3\alpha}^2.
\end{aligned}$$

This is true for all $V \in \mathcal{V}^\phi$. Hence,

$$\inf_{w \in \mathcal{V}^\psi} \|u_1 - w\|_\alpha \le C \inf_{V \in \mathcal{V}^\phi} \|U_1 - V\|_{3\alpha},$$

which implies, together with (3.5.22),

$$\|u - \widetilde{u}\|_\alpha \le C \inf_{V \in \mathcal{V}^\phi} \|U_1 - V\|_{3\alpha}.$$

Since $\tau > (n-1)/2$, $\tau \ge 3\alpha$ and $3\alpha \le s + 2\alpha \le 2\tau$, we can invoke Proposition 3.2 with t^* and s^* replaced by 3α and $s + 2\alpha$, respectively, to obtain

$$\|u - \widetilde{u}\|_\alpha \le C h_X^{s-\alpha} \|U_1\|_{s+2\alpha} \le C h_X^{s-\alpha} \|u_1\|_s \le C h_X^{s-\alpha} \|u\|_s.$$

In the case when $2(\alpha - \tau) \le t < \alpha$, by using Lemma 3.4 and noting the inf-sup condition (3.5.12), we obtain

$$\|u - \widetilde{u}\|_t \le C\|u_1 - u_1^*\|_t \le C \sup_{\substack{v \in H^{2\alpha - t} \\ v \ne 0}} \frac{a(u_1 - u_1^*, v)}{\|v\|_{2\alpha - t}}.$$

Since $a(u_1 - u_1^*, \eta) = 0$ for all $\eta \in \mathcal{V}^\phi$ due to (3.5.21), we can use the same argument as in the proof of Theorem 3.8 to obtain the required result, finishing the proof of the theorem. □

In the next section we shall use the analysis for Galerkin approximations to estimate errors in the collocation approximation. This is the novelty of our work.

3.6 Collocation approximation

3.6.1 Strongly elliptic case

Recall that for this method it is assumed that $g \in H^{\sigma-\alpha}$ for some positive σ so that $u \in H^{\sigma+\alpha}$; see Problem A. We will assume that

$$\max\{2\alpha, \alpha\} + \frac{n-1}{2} < \tau \le \min\{\sigma - \alpha, \sigma\}. \tag{3.6.1}$$

Recall that (2.4.4) implies $\mathcal{N}_\phi \simeq H^\tau$. Thus, the condition $\sigma - \alpha \ge \tau$ assures us that $g \in \mathcal{N}_\phi$. The condition $2\alpha + (n-1)/2 < \tau$ is to assure that $L\widetilde{u}_1 \in \mathcal{N}_\phi$. Indeed, this condition implies $\widetilde{u}_1 \in \mathcal{V}^\phi \subset H^{\tau+2\alpha}$ which is equivalent to $L\widetilde{u}_1 \in H^\tau \simeq \mathcal{N}_\phi$.

The functions $L\widetilde{u}_1$ and g are required to be in the native space $\mathcal{N}_\phi$ so that property (3.3.1) can be used. The conditions $\alpha + (n-1)/2 \le \tau$ and $\tau \le \sigma$ are purely technical requirements of our proof.

In this method we find $\widetilde{u}_1 \in \mathcal{V}^\phi$ by solving the collocation equation

$$L\widetilde{u}_1(\boldsymbol{x}_j) = g(\boldsymbol{x}_j), \quad j = 1, \dots, N. \tag{3.6.2}$$

By writing $\widetilde{u}_1 = \sum_{j=1}^N c_j \Phi_j$, we derive from (3.6.2) the matrix equation $\boldsymbol{A}^{(C)}\boldsymbol{c} = \boldsymbol{g}$ where

$$\boldsymbol{A}^{(C)}_{ij} = L\Phi_i(\boldsymbol{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \widehat{L}(\ell)\, \widehat{\phi}(\ell)\, Y_{\ell,m}(\boldsymbol{x}_i)\, Y_{\ell,m}(\boldsymbol{x}_j),$$

$\boldsymbol{c} = (c_1, \dots, c_N)$ and $\boldsymbol{g} = (g(\boldsymbol{x}_1, \dots, g(\boldsymbol{x}_N))$. The symmetry and positive definiteness of the matrix $\boldsymbol{A}^{(C)}$ can be proved in the same manner as Lemma 3.5.

Since the function Φ defined as in (2.4.3) is a reproducing kernel for the Hilbert space $\mathcal{N}_\phi$, see (3.3.1), the collocation equation (3.6.2) can be rewritten as a Galerkin equation. This allows us to carry out error analysis in the same manner as in Section 3.5.

Recalling (3.3.1) and noting that $L\widetilde{u}_1, g \in \mathcal{N}_\phi$, we rewrite (3.6.2) as

$$\langle L\widetilde{u}_1, \Phi_j \rangle_\phi = \langle g, \Phi_j \rangle_\phi, \quad j = 1, \dots, N. \tag{3.6.3}$$

In order to see that the above equation is a Galerkin equation, we introduce a new finite-dimensional subspace $\mathcal{V}^{\widetilde{\phi}}$:

$$\mathcal{V}^{\widetilde{\phi}} := \operatorname{span}\{\widetilde{\Phi}_1, \dots, \widetilde{\Phi}_N\},$$

where the spherical radial basis functions $\widetilde{\Phi}_j$ are defined by

$$\widetilde{\Phi}_j(\boldsymbol{x}) := \widetilde{\phi}(\boldsymbol{x} \cdot \boldsymbol{x}_j), \quad j = 1, \dots, N.$$

Here, $\widetilde{\phi}$ is a shape function given by

$$\widetilde{\phi}(t) := \sum_{\ell=0}^{\infty} \omega_n^{-1} N(n,\ell) \big[\widehat{\phi}(\ell)\big]^{1/2} P_\ell(n;t),$$

It is easily seen that (cf. (2.4.8))

$$\widehat{(\widetilde{\Phi}_j)}_{\ell,m} = [\widehat{\phi}(\ell)]^{1/2}\, Y_{\ell,m}(\boldsymbol{x}_j), \quad j = 1, \dots, N. \tag{3.6.4}$$

It should be noted that this space $\mathcal{V}^{\widetilde{\phi}}$ is introduced purely for analysis purposes; it is not to be used in the implementation. Since (cf. (2.4.4))

$$c_1(\ell+1)^{-\tau} \le \widehat{(\widetilde{\phi})}(\ell) \le c_2(\ell+1)^{-\tau},$$

we have (cf. (2.4.10))

$$\mathcal{V}^{\widetilde{\phi}} \subset H^s \quad \text{for all } s < \tau + \frac{1-n}{2}. \tag{3.6.5}$$

In particular, $\mathcal{V}^{\widetilde{\phi}} \subset H^\alpha$ due to $\alpha + (n-1)/2 < \tau$ (see (3.6.1)).

The following lemma defines a weak equation equivalent to equation (3.2.8).

Lemma 3.15. *Let*

$$U_1 := \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \frac{(\widehat{u}_1)_{\ell,m}}{\big[\widehat{\phi}(\ell)\big]^{1/2}} Y_{\ell,m}, \tag{3.6.6}$$

where u_1 is the solution to (3.2.8)*. Then U_1 belongs to $H^{\sigma+\alpha-\tau}$ and satisfies*

$$a(U_1, V) = \langle G, V\rangle \quad \textit{for all } V \in H^{\alpha-\sigma+\tau}, \tag{3.6.7}$$

where

$$G := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{g}_{\ell,m}}{\big[\widehat{\phi}(\ell)\big]^{1/2}} Y_{\ell,m}. \tag{3.6.8}$$

Proof. Since $u_1 \in H^{\sigma+\alpha}$, it is easily seen that $U_1 \in H^{\sigma+\alpha-\tau}$. For any $V \in H^{\alpha-\sigma+\tau}$ there holds

$$a(U_1, V) = a(u_1, v),$$

where

$$v := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{V}_{\ell,m}}{\big[\widehat{\phi}(\ell)\big]^{1/2}} Y_{\ell,m}.$$

Noting $v \in H^{\alpha-\sigma}$ we deduce from (3.2.8) that

$$a(U_1, V) = \langle g, v\rangle = \langle G, V\rangle\,,$$

finishing the proof of the lemma. □

Analogously, the next lemma defines an equivalent to (3.6.3). It will be seen later that this equivalent is the Galerkin approximation to (3.6.7).

Lemma 3.16. *Let*

$$\widetilde{U}_1 := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{(\widetilde{u}_1)}_{\ell,m}}{\big[\widehat{\phi}(\ell)\big]^{1/2}} Y_{\ell,m} \tag{3.6.9}$$

where $\widetilde{u}_1$ is given by (3.6.2)*. Then $\widetilde{U}_1$ belongs to $\mathcal{V}^{\widetilde{\phi}}$ and satisfies*

$$a(\widetilde{U}_1, \widetilde{\Phi}_j) = \left\langle G, \widetilde{\Phi}_j \right\rangle, \quad j = 1, \dots, N. \tag{3.6.10}$$

Proof. Since $\widetilde{u}_1 \in \mathcal{V}^\phi$ we have $\widetilde{u}_1 = \sum_{j=1}^N c_j \Phi_j$ for some $c_j \in \mathbb{R}$, which together with (2.4.8) implies

$$\widehat{(\widetilde{u}_1)}_{\ell,m} = \widehat{\phi}(\ell) \sum_{j=1}^N c_j Y_{\ell,m}(\boldsymbol{x}_j).$$

This in turn gives

$$\widehat{(\widetilde{U}_1)}_{\ell,m} = [\widehat{\phi}(\ell)]^{1/2} \sum_{j=1}^N c_j Y_{\ell,m}(\boldsymbol{x}_j),$$

so that (see (3.6.4))

$$\widetilde{U}_1 = \sum_{j=1}^N c_j \widetilde{\Phi}_j,$$

i.e., $\widetilde{U}_1 \in \mathcal{V}^{\widetilde{\phi}}$. By using successively (3.2.7), (3.6.4), (3.6.9), (3.6.3), (2.4.8) and (3.6.8), we deduce

$$a(\widetilde{U}_1, \widetilde{\Phi}_j) = \left\langle L\widetilde{U}_1, \widetilde{\Phi}_j \right\rangle = \langle L\widetilde{u}_1, \Phi_j\rangle_\phi = \langle g, \Phi_j\rangle_\phi = \left\langle G, \widetilde{\Phi}_j \right\rangle, \quad j = 1, \ldots, N,$$

completing the proof of the lemma. □

Using the two above lemmas we can now estimate the error in the collocation approximation in the same manner as for the Galerkin approximation.

Theorem 3.17. *Let* (3.6.1) *hold. We choose the shape function* ϕ *such that* (2.4.4) *holds with* $\tau > n - 1$. *Assume further that* $u \in H^s$ *for some* s *satisfying* $\tau + \alpha \le s \le 2\tau$. *If* $\mu_i \in H^{-t}$, $i = 1, \ldots, M$ *for some* t *satisfying* $2\alpha \le t \le \tau + \alpha$, *then for* h_X *sufficiently small there holds*

$$\|u - \widetilde{u}\|_t \le C h_X^{s-t} \|u\|_s.$$

The constant C *is independent of* u *and* h_X.

Proof. Recall that $\widetilde{U}_1 \in \mathcal{V}^{\widetilde{\phi}} \subset H^\alpha$ and $U_1 \in H^{\sigma+\alpha-\tau} \subset H^\alpha$ since $\tau \le \sigma$; see (3.6.1). Moreover, (3.6.7) and (3.6.10) imply

$$a(U_1 - \widetilde{U}_1, \widetilde{\Phi}_j) = 0, \quad j = 1, \ldots, N.$$

Hence, $\widetilde{U}_1 \in \mathcal{V}^{\widetilde{\phi}}$ is the Galerkin approximation to U_1.

Analogously to (3.4.3) we define

$$U_1^* = \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=1}^{N(n,\ell)} \widehat{(\widetilde{U}_1)} Y_{\ell,m}. \tag{3.6.11}$$

Lemma 3.7 with $\mathcal{V}^\phi$ replaced by $\mathcal{V}^{\widetilde{\phi}}$ (and therefore, τ replaced by $\widetilde{\tau} := \tau/2$) and u_1, u_1^* replaced by U_1, U_1^*, gives

$$\|U_1 - U_1^*\|_{\widetilde{t}} \le C h_X^{\widetilde{s}-\widetilde{t}} \|U_1\|_{\widetilde{s}}, \quad \alpha \le \widetilde{s} \le 2\widetilde{\tau},\ 2(\alpha - \widetilde{\tau}) \le \widetilde{t} \le \alpha. \tag{3.6.12}$$

By the definition of $U_1, \widetilde{U}_1$ and U_1^*, see (3.6.6), (3.6.9) and (3.6.11), we have

$$\|u_1 - u_1^*\|_t \simeq \|U_1 - U_1^*\|_{t-\tau} \text{ and } \|u_1\|_s \simeq \|U_1\|_{s-\tau}. \tag{3.6.13}$$

Since t and s satisfy $2\alpha \le t \le \tau + \alpha$ and $\tau + \alpha \le s \le 2\tau$ so that $t - \tau$ and $s - \tau$ satisfy

$$2(\alpha - \widetilde{\tau}) \le t - \tau \le \alpha \quad \text{and} \quad \alpha \le s - \tau \le 2\widetilde{\tau},$$

the inequality (3.6.12) with $\widetilde{t} = t - \tau$ and $\widetilde{s} = s - \tau$ gives

$$\|U_1 - U_1^*\|_{t-\tau} \le C h_X^{s-t} \|U_1\|_{s-\tau}.$$

This together with (3.6.13) implies

$$\|u_1 - u_1^*\|_t \le C h_X^{s-t} \|u_1\|_s$$

Since $\mu_i \in H^{-t}$, for $i = 1, \ldots, M$, by using Lemma 3.4 and noting that $\|u_1\|_s \le \|u\|_s$, we deduce

$$\|u - \widetilde{u}\|_t \le C\|u_1 - u_1^*\|_t \le C h_X^{s-t} \|u_1\|_s \le C h_X^{s-t} \|u\|_s,$$

completing the proof of the theorem.

□

3.6.2 Elliptic case

Recall that for this method it is assumed that $g \in H^{\sigma-\alpha}$ for some positive σ so that $u \in H^{\sigma+\alpha}$; see Problem A. We assume that

$$\max\{4\alpha, 3\alpha, \alpha\} + \frac{n-1}{2} < \tau \le \sigma - \alpha. \tag{3.6.14}$$

Here, since $g \in H^{\sigma-\alpha}$, the condition $\sigma - \alpha \ge \tau$ assures us that $g \in H^\tau \simeq \mathcal{N}_\phi$. The condition $4\alpha + (n-1)/2 < \tau$ is to assure that $L\widetilde{u}_1 \in \mathcal{N}_\phi$. Indeed, the condition $4\alpha + (n-1)/2 < \tau$ is equivalent to $\tau + 2\alpha < 2(\tau - \alpha) + (1-n)/2$, which assures that $\widetilde{u}_1 \in \mathcal{V}^\psi \subset H^{\tau+2\alpha}$; see (3.3.19). This in turn shows that $L\widetilde{u}_1 \in H^\tau \simeq \mathcal{N}_\phi$.

The requirements that $L\widetilde{u}_1, g \in \mathcal{N}_\phi$ give us the ability to use property (3.3.1).

We find $\widetilde{u}_1$ by solving the collocation equation (3.6.2), with $\widetilde{u}_1$ belonging to $\mathcal{V}^\psi$ instead of $\mathcal{V}^\phi$. The resulting matrix $\boldsymbol{A}^{(EC)}$ has entries given by

$$\boldsymbol{A}_{ij}^{(EC)} = L\Psi_i(\boldsymbol{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} [\widehat{L}(\ell)]^2 \, \widehat{\phi}(\ell) \, Y_{\ell,m}(\boldsymbol{x}_i) \, Y_{\ell,m}(\boldsymbol{x}_j).$$

The matrix $\boldsymbol{A}$ is positive-definite, and thus $\widetilde{u}_1$ exists uniquely.

As in the case of strongly elliptic operators, the collocation equation can be rewritten as a Petrov–Galerkin equation

$$a(\widetilde{U}_1, \widetilde{\Phi}_i) = \left\langle G, \widetilde{\Phi}_i \right\rangle, \quad i = 1, \ldots, N, \tag{3.6.15}$$

where $\widetilde{U}_1$ is defined from $\widetilde{u}_1$ by using (3.6.9). Note that $\widetilde{U}_1$ belongs to $\mathcal{V}^{\widetilde{\psi}}$, where

$$\mathcal{V}^{\widetilde{\psi}} := \operatorname{span}\{\widetilde{\Psi}_1, \ldots, \widetilde{\Psi}_N\}$$

with

$$\widetilde{\Psi}_i := \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(n,\ell)} \frac{\widehat{(\Psi_i)}}{\widehat{\phi}(\ell)^{1/2}} Y_{\ell,m}.$$

The finite-dimensional spaces $\mathcal{V}^{\widetilde{\psi}}$ and $\mathcal{V}^{\widetilde{\phi}}$ are subspaces of H^α due to $\max\{3\alpha,\alpha\}+(n-1)/2<\tau$; see (3.6.14). Therefore we can use the result in Subsection 3.5.2 to obtain the following theorem.

Theorem 3.18. *Let* (3.6.14) *hold. We choose the shape function ϕ such that $\tau \ge 6\alpha$ and $\tau > n-1$. Let $u \in H^s$ for some s satisfying $\tau+\alpha \le s \le 2\tau-2\alpha$. If $\mu_i \in H^{-t}$, $i=1,\dots,M$ for t satisfying $2\alpha \le t \le \tau+\alpha$, then for h_X sufficiently small there holds*

$$\|u-\widetilde{u}\|_t \le C h_X^{s-t}\|u\|_s.$$

The constant C is independent of u and h_X.

Proof. We can first use the same argument as used in the proof of Theorem 3.17 to transform the estimate of the collocation solution to the estimate of a Galerkin solution, and then apply the results in Theorem 3.14 to obtain the desired estimate. The details are omitted.

□

Remark 3.19. In comparison with the results obtained by Morton and Neamtu, our error estimates for the collocation approximation cover a wider range of Sobolev norms for both strongly elliptic and elliptic operators. In fact, these two authors only proved for strongly elliptic operators [48]

$$\|u-\widetilde{u}\|_{2\alpha} \le c h_X^{\lfloor 2(\tau-\alpha)\rfloor}\|u\|_{2\tau},$$

and for elliptic operators [49, 50]

$$\|u-\widetilde{u}\|_{2\alpha} \le c h_X^{2\tau-4\alpha}\|u\|_{2\tau-2\alpha}.$$

These are special cases of the results in Theorems 3.17 and 3.18.

3.7 Numerical experiments

In this section, we solved the Dirichlet problem

$$\begin{aligned} \Delta U &= 0 \text{ in } \mathbb{B}_e,\\ U &= U_D \text{ on } \mathbb{S}^2,\\ U(\boldsymbol{x}) &= O(1/|\boldsymbol{x}|) \text{ as } |\boldsymbol{x}|\to\infty, \end{aligned} \tag{3.7.1}$$

where $\mathbb{B}_e := \{\boldsymbol{x}\in\mathbb{R}^3 : |\boldsymbol{x}|>1\}$. It is well-known, see e.g. [62], that the problem (3.7.1) is equivalent to

$$Su = g \text{ on } \mathbb{S}^2, \tag{3.7.2}$$

where

$$g = -\frac{1}{2}U_D + DU_D, \tag{3.7.3}$$

and

$$Dv(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathbb{S}^2} v(\boldsymbol{y})\frac{\partial}{\partial\nu_{\boldsymbol{y}}}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\,d\sigma_{\boldsymbol{y}}.$$

Here, S is the weakly singular integral operator defined by

$$Sv(\boldsymbol{x}) = \frac{1}{4\pi}\int_{\mathbb{S}^2} \frac{v(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d\sigma_{\boldsymbol{y}},$$

which is a pseudodifferential operator of order -1 and $\widehat{S}(\ell) = 1/(2\ell+1)$; see the examples following Definition 2.12, noting that when $n=3$, $N(3,\ell) = 2\ell+1$ and $\omega_3 = 4\pi$.

We solved the problem (3.7.1) with the boundary data

$$U_D(\boldsymbol{x}) := U_D(x_1,x_2,x_3) = \frac{1}{(1.0625-0.5x_3)^{1/2}}$$

so that the exact solution to the Dirichlet problem (3.7.1) is given by

$$U(\boldsymbol{x}) = \frac{1}{|\boldsymbol{x}-\boldsymbol{q}|} \quad \text{with} \quad \boldsymbol{q} = (0,0,0.25),$$

and hence, the exact solution to the weakly singular integral equation (3.7.2) is $u(\boldsymbol{x}) = \partial_\nu U(\boldsymbol{x})$; see e.g. [62], i.e.,

$$u(\boldsymbol{x}) = \frac{-1+\boldsymbol{x}\cdot\boldsymbol{q}}{|\boldsymbol{x}-\boldsymbol{q}|^3} = \frac{0.25x_3-1}{(1.0625-0.5x_3)^{3/2}}.$$

For the approximation of (3.7.2), we use spherical radial basis functions suggested by Wendland [80, page 128]. The sets $X := \{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_N\}$ of points, which are chosen purely to observe the order of convergence, are generated by a simple algorithm [65] which partitions the sphere into equal areas. Experiments with real data can be found in [62].

The shape function $\phi : [-1,1] \to \mathbb{R}$ which is used to define the kernel Φ is given by

$$\phi(t) = \rho(\sqrt{2-2t}), \tag{3.7.4}$$

where ρ is Wendland's functions [82, page 128] defined by

$$\rho(r) = (1-r)_+^2.$$

Narcowich and Ward [53, Proposition 4.6] prove that $\widehat{\phi}(\ell) \sim (1+\ell)^{-2\tau}$ for all $\ell \geq 0$, where $\tau = 3/2$. The spherical radial basis functions Φ_i, $i = 1,\ldots,N$, are computed by

$$\Phi_i(\boldsymbol{x}) = \rho(\sqrt{2-2\boldsymbol{x}\cdot\boldsymbol{x}_i}), \quad \boldsymbol{x} \in \mathbb{S}^2. \tag{3.7.5}$$

We first found an approximate solution $u_X^G \in \mathcal{V}_X^\phi := \operatorname{span}\{\Phi_1,\Phi_2,\ldots,\Phi_N\}$ satisfying the Galerkin equation

$$a_S(u_X^G, v) := \langle S u_X^G, v\rangle = \langle g, v\rangle \quad \forall v \in \mathcal{V}_X^\phi. \tag{3.7.6}$$

The stiffness matrix arising from (3.7.6) has entries given by

$$a_S(\Phi_i,\Phi_j) = \sum_{\ell=0}^\infty \frac{|\widehat{\phi}(\ell)|^2}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell,m}(\boldsymbol{x}_i)Y_{\ell,m}(\boldsymbol{x}_j) = \frac{1}{4\pi}\sum_{\ell=0}^\infty |\widehat{\phi}(\ell)|^2 P_\ell(\boldsymbol{x}_i\cdot\boldsymbol{x}_j).$$

The right-hand side of (3.7.6) is computed by using (3.7.3), noting $\widehat{D}(\ell) = -1/(4\ell+2)$ (see [55, page 122]),

$$\langle g, \Phi_i \rangle = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \Big(-\frac{1}{2} - \frac{1}{2(2\ell+1)} \Big) \widehat{(U_D)}_{\ell,m} \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i)$$
$$= -\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{(\ell+1)}{2\ell+1} \widehat{(U_D)}_{\ell,m} \widehat{\phi}(\ell) Y_{\ell,m}(\boldsymbol{x}_i).$$

The errors are computed by

$$\|u - u_X^G\|_{-1/2} = \left(\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{|\widehat{u}_{\ell,m} - \widehat{(u_X^G)}_{\ell,m}|^2}{\ell+1} \right)^{1/2}. \tag{3.7.7}$$

Our theoretical result (Theorem 3.8) predicts an order of convergence of $2\tau + 1/2$ in the $H^{-1/2}$-norm. We carried out the experiment and observed some agreement between the experimented orders of convergence (EOC) and our theoretical results; see Tables 3.1.

Table 3.1: Galerkin method: Errors in $H^{-1/2}$-norm, $\tau = 1.5$. Expected order of convergence: 3.5.

N	h_X	$H^{-1/2}$-norm	EOC
20	0.65140	0.120349381	
30	0.51210	0.054895875	3.262
40	0.44180	0.025612135	5.163
51	0.37500	0.015883257	2.915
101	0.26720	0.006082010	2.832
200	0.19420	0.001977985	3.520
500	0.12370	0.000492078	3.084

The collocation solution $u_X^C \in \mathcal{V}_X^{\phi}$ is found by solving

$$S u_X^C(\boldsymbol{x}_i) = g(\boldsymbol{x}_i), \quad i = 1, \ldots, N. \tag{3.7.8}$$

By writing $u_X^C = \sum_{i=1}^N c_i \Phi_i$, we derive from (3.7.8) the matrix equation $\boldsymbol{S}^C \boldsymbol{c} = \boldsymbol{g}$, where $\boldsymbol{c} = (c_i)_{i=1,\ldots,N}$, $\boldsymbol{g} = (g(\boldsymbol{x}_i))_{i=1,\ldots,N}$ and

$$\boldsymbol{S}_{ij}^C = S\Phi_i(\boldsymbol{x}_j) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{\widehat{\phi}(\ell)}{2\ell+1} Y_{\ell,m}(\boldsymbol{x}_i) Y_{\ell,m}(\boldsymbol{x}_j), \quad i,j = 1, \ldots, N.$$

By using the addition formula (2.1.19), we obtain

$$\boldsymbol{S}_{ij}^C = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} \widehat{\phi}(\ell) P_\ell(\boldsymbol{x}_i \cdot \boldsymbol{x}_j). \tag{3.7.9}$$

The errors are then computed similarly as in (3.7.7). There is agreement between the experimented order of convergence (EOC) and our theoretical result (which is $2\tau + 1/2$); see Tables 3.2.

Table 3.2: Collocation method: Errors in $H^{-1/2}$-norm, $\tau = 1.5$. Expected order of convergence: 3.5.

N	h_X	$H^{-1/2}$-norm	EOC
20	0.65140	0.139479793	
30	0.51210	0.047806025	4.450
40	0.44180	0.020666895	5.679
51	0.37500	0.011785692	3.426
101	0.26720	0.003674365	3.439
400	0.12370	0.000277996	3.352

Chapter 4

Pseudodifferential equations on the sphere with spherical splines

4.1 Introduction

In this chapter, we solve strongly elliptic pseudodifferential equations on the sphere by the Galerkin method using spherical splines. This class of equations includes the Laplace–Beltrami equation, Stokes equation, weakly singular integral equations, and many others [29, 31, 73]. Our main result is an optimal convergence rate of the approximation. The key of the analysis is proving the approximation property of spherical splines as a subset of Sobolev spaces. Since the pseudodifferential operators to be studied can be of any real order, it is necessary to obtain an approximation property in Sobolev norms of real orders, negative and positive (see Theorem 4.3). The results in this chapter have been reported in our article [60].

4.2 The problem

The problem we are solving in this chapter is posed as follows.

Problem A: *Let L be a strongly elliptic pseudodifferential operator of order 2α. Given*

$$g \in H^{-\alpha} \quad \textit{satisfying} \quad \widehat{g}_{\ell,m} = 0 \quad \textit{for all } \ell \in \mathcal{K}(L),\ m = -\ell, \ldots, \ell, \tag{4.2.1}$$

find $u \in H^{\alpha}$ satisfying

$$\begin{aligned} Lu &= g, \\ \langle \mu_i, u \rangle &= \gamma_i, \quad i = 1, \ldots, M, \end{aligned} \tag{4.2.2}$$

where $\gamma_i \in \mathbb{R}$ and $\mu_i \in H^{-\alpha}$ are given.

We note here that the above problem is the problem introduced in Section 3.2, Chapter 3 when $\sigma = 0$. Problem A is uniquely solvable under the following assumption.

Assumption B: *The functionals $\mu_1, \dots, \mu_N$ are assumed to be unisolvent with respect to* $\ker L$.

The unisolvency assumption assures us that Problem A has a unique solution.

Lemma 4.1. *Under Assumption B, Problem A has a unique solution.*

Proof. The proof of this lemma employs similar argument as in the proof of Theorem 3.1 in which the solution u is written in the form

$$u = u_0 + u_1 \quad \text{where} \quad u_0 \in \ker L \quad \text{and} \quad u_1 \in (\ker L)^{\perp}_{H^\alpha}. \tag{4.2.3}$$

Here, $u_1 = L^{-1}g$ and u_0 is found in $\ker L$ by solving

$$\langle \mu_i, u_0 \rangle = \gamma_i - \langle \mu_i, u_1 \rangle, \quad i = 1, \dots, M. \tag{4.2.4}$$

Note here that

$$\langle Lu_1, v \rangle = \langle g, v \rangle \quad \forall v \in H^\alpha. \tag{4.2.5}$$

□

In the next section, we shall review the definition of the spaces $S^r_d(\Delta)$ of spherical splines and prove that $S^r_d(\Delta)$ is a subspace of the Sobolev space H^{r+1}.

4.3 Spherical splines

Given $X = \{\boldsymbol{x}_1, \dots, \boldsymbol{x}_N\}$ a set of points on $\mathbb{S}^2$, we can form a spherical triangulation Δ whose vertices are points in X. In the implementation, the spherical triangulations are generated from sets of data points by using STRIPACK free package. We briefly recall here that for any nonnegative integers r and d, the set of spherical splines of degree d and smoothness r associated with Δ is defined by

$$S^r_d(\Delta) := \{s \in C^r(\mathbb{S}^2) : s|_\tau \in \widetilde{\Pi}_d, \tau \in \Delta\},$$

where $\widetilde{\Pi}_d$ denotes the set of homogeneous polynomials of degree d; see Page 7.

We now use the definition of Sobolev spaces by local charts, see (2.2.4), to show the following necessary result.

Proposition 4.2. *Let Δ be a spherical triangulation on the unit sphere $\mathbb{S}^2$. Assume that* (2.5.23) *holds. There holds*

$$S^r_d(\Delta) \subset H^{r+1}.$$

Proof. We prove the result for $r = 0$ and use induction to obtain the result for $r > 0$. Here we use a specific atlas of charts to define the Sobolev space H^1. We denote by $\boldsymbol{n} = (0, 0, 1)$ and $\boldsymbol{s} = (0, 0, -1)$ the north and south poles of $\mathbb{S}^2$, respectively. For any point $\boldsymbol{x} \in \mathbb{S}^2$ and $R > 0$, we denote the spherical cap centred at $\boldsymbol{x}$ and having radius R by $C(\boldsymbol{x}, R)$, i.e.,

$$C(\boldsymbol{x}, R) := \{\boldsymbol{y} \in \mathbb{S}^2 : \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) \le R\}. \tag{4.3.1}$$

The interior of $C(\boldsymbol{x}, R)$ is denoted by $C^o(\boldsymbol{x}, R)$. Let

$$U_1 = C^o(\boldsymbol{n}, \theta_0) \quad \text{and} \quad U_2 = C^o(\boldsymbol{s}, \theta_0),$$

with $\theta_0 \in (\pi/2, 2\pi/3)$. Assume that Δ is fine enough such that $\mathbb{S}^2$ admits a simple cover $\mathbb{S}^2 = \Gamma_1 \cup \Gamma_2$, where

$$\Gamma_j = \bigcup_{\tau \subset U_j} \tau, \ j = 1, 2.$$

The stereographic projection ψ_1 from the punctured sphere $\mathbb{S}^2 \backslash \{\boldsymbol{n}\}$ onto $\mathbb{R}^2$ is defined as a mapping that maps $\boldsymbol{x} \in \mathbb{S}^2 \backslash \{\boldsymbol{n}\}$ to the intersection of the equatorial hyperplane $\{z = 0\}$ and the extended line that passes through $\boldsymbol{x}$ and $\boldsymbol{n}$. The stereographic projection ψ_2 based on $\boldsymbol{s}$ can be defined similarly. The scaled projections

$$\phi_1 := \frac{1}{\tan(\theta_0/2)}\, \psi_2|_{\Gamma_1} \quad \text{and} \quad \phi_2 := \frac{1}{\tan(\theta_0/2)}\, \psi_1|_{\Gamma_2}$$

map Γ_1 and Γ_2, respectively, onto the unit ball of $\mathbb{R}^2$. $B(0,1)$, It is clear that $\{\Gamma_j, \phi_j\}_{j=1}^2$ is a C^∞ atlas for $\mathbb{S}^2$.

Let $B_j = \phi_j(\Gamma_j)$, $j = 1, 2$, and let $\{\alpha_j : \mathbb{S}^2 \to \mathbb{R}\}_{j=1}^2$ be the partition of unity subordinate to $\{(\Gamma_j, \phi_j)\}_{j=1}^2$; see page 12. For any $v \in S_d^0(\Delta)$, it suffices to show that, see (2.2.4),

$$w_j := (\alpha_j v) \circ \phi_j^{-1} \in H^1(B_j) \quad \text{for } j = 1, 2.$$

This holds because w_j is continuous on $\overline{B_j}$ and $w_j|_\sigma$ belongs to $H^1(\sigma)$, where $\{\sigma := \phi_j(\tau) : \tau \subset \Gamma_j\}$ forms a partition of B_j; see [15, page 38].

□

4.4 Approximation property

When solving pseudodifferential equations of order 2α by the Galerkin method, it is natural to carry out error analysis in the energy space H^α. Since the order 2α may be negative (as in the case of the weakly-singular integral equation discussed after Definition 2.3.3) it is necessary to show an approximation property for a wide range of Sobolev norms, including negative real values.

Theorem 4.3. *Assume that Δ is a quasi-uniform spherical triangulation with $|\Delta| \le 1$, and that (2.5.23) holds. Then for any $v \in H^s$, there exists $\eta \in S_d^r(\Delta)$ satisfying*

$$\|v - \eta\|_{t^*} \le C h_\Delta^{s^*-t^*} \|v\|_{s^*}, \tag{4.4.1}$$

where $t^* \le r+1$ and $t^* \le s^* \le d+1$. Here C is a positive constant depending only on d and the smallest angle in Δ.

Proof. For $k = 0, 1, 2, \ldots$, we denote $\mathcal{I}_k = [-(r+1)k, -(r+1)(k-1)]$. We will prove that (4.4.1) holds with $t^* \in \mathcal{I}_k$ for all $k \ge 0$ by induction on k.

- We first prove that (4.4.1) is true when $t^* \in \mathcal{I}_0$.

Assume that $t^* \in \mathcal{I}_0$ and $t^* \le s^* \le d+1$. Let t and s satisfy $0 \le t \le r+1$ and $t \le s \le d+1$. Proposition 2.34 gives

$$|v - \widetilde{I}v|_k \le Ch_\Delta^{d+1-k}|v|_{d+1}, \quad k = 0, 1, \ldots, r+1. \tag{4.4.2}$$

For $k = 0$, we have

$$\|v - \widetilde{I}v\|_0 \le Ch_\Delta^{d+1}|v|_{d+1} \le Ch_\Delta^{d+1}\|v\|_{d+1} \quad \forall v \in H^{d+1}. \tag{4.4.3}$$

Summing (4.4.2) over $k = 0, 1, \ldots, r+1$ and noting that $h_\Delta < 1$, we obtain

$$\|v - \widetilde{I}v\|_{r+1} \le Ch_\Delta^{d-r}|v|_{d+1} \le Ch_\Delta^{d-r}\|v\|_{d+1} \quad \forall v \in H^{d+1}. \tag{4.4.4}$$

Noting (4.4.3), (4.4.4) and applying Theorem 2.11 with $T = I - \widetilde{I}$, $t_1 = 0$, $t_2 = r+1$, $s_1 = s_2 = d+1$ and $\theta = (t-r-1)/(-r-1)$, we deduce

$$\|v - \widetilde{I}v\|_t \le Ch_\Delta^{d+1-t}\|v\|_{d+1} \quad \forall v \in H^{d+1}. \tag{4.4.5}$$

Let $P_t : H^t \to S_d^r(\Delta)$ be the projection defined by

$$\langle P_t v, w\rangle_t = \langle v, w\rangle_t \quad \forall v \in H^t \text{ and } w \in S_d^r(\Delta).$$

It is well-known that $P_t v$ is the best approximation of v from $S_d^r(\Delta)$ in H^t-norm. We then have

$$\|v - P_t v\|_t \le \|v\|_t \quad \forall v \in H^t,$$

and

$$\|v - P_t v\|_t \le Ch_\Delta^{d+1-t}\|v\|_{d+1} \quad \forall v \in H^{d+1}, \tag{4.4.6}$$

noting (4.4.5). Applying Theorem 2.11 with $T = I - P_t$, $t_1 = t_2 = t$, $s_1 = t$, $s_2 = d+1$ and $\theta = (s-d-1)/(t-d-1)$, we obtain

$$\|v - P_t v\|_t \le Ch_\Delta^{s-t}\|v\|_s \quad \forall v \in H^s.$$

Hence, we have proved that

$$\begin{cases} t^* \in \mathcal{I}_0,\ t^* \le s^* \le d+1, \\ \forall v \in H^{s^*},\ \exists \eta_v \in S_d^r(\Delta) : \|v - \eta_v\|_{t^*} \le Ch_\Delta^{s^*-t^*}\|v\|_{s^*}. \end{cases} \tag{4.4.7}$$

• Assume that (4.4.1) is true for all $t^* \in \mathcal{I}_k$, for $k = 0, 1, \ldots, k_0$, i.e., the following statement holds

$$\begin{cases} t^* \in \bigcup_{k=0}^{k_0}\mathcal{I}_k,\ t^* \le s^* \le d+1, \\ \forall v \in H^{s^*},\ \exists \eta_v \in S_d^r(\Delta) : \|v - \eta_v\|_{t^*} \le Ch_\Delta^{s^*-t^*}\|v\|_{s^*}. \end{cases} \tag{4.4.8}$$

• We now prove that (4.4.1) holds for $t^* \in \mathcal{I}_{k_0+1}$ and $t^* \le s^* \le d+1$. We consider two cases when s^* belongs to $[-(r+1)k_0, d+1]$ and $[t^*, -(r+1)k_0)$.

Case 1. Consider $s^* \in [-(r+1)k_0, d+1]$.

Let $t \in \mathcal{I}_{k_0+1}$ and $s \in [-(r+1)k_0, d+1]$. Let $P_{-(r+1)k_0} \to S_d^r(\Delta)$ be the projection defined by

$$\langle P_{-(r+1)k_0} v, w \rangle_{-(r+1)k_0} = \langle v, w \rangle_{-(r+1)k_0} \quad \forall v \in H^{-(r+1)k_0} \text{ and } w \in S_d^r(\Delta).$$

Then $P_{-(r+1)k_0} v$ is the best approximation of v from $S_d^r(\Delta)$ in $H^{-(r+1)k_0}$-norm. This together with (4.4.8) with t^* and s^* replaced by $-(r+1)k_0$ and s, respectively, implies

$$\|P_{-(r+1)k_0} v - v\|_{-(r+1)k_0} \le C h_\Delta^{s+(r+1)k_0} \|v\|_s \quad \forall v \in H^s. \tag{4.4.9}$$

Applying (4.4.8) with with t^* and s^* replaced by $-(r+1)k_0$ and $-t-2(r+1)k_0$, respectively, noting that $-(r+1)k_0 \le -t-2(r+1)k_0$ as $t \in \mathcal{I}_{k_0+1}$, we deduce that for any $w \in H^{-t-2(r+1)k_0}$, there exists a $\eta_w \in S_d^r(\Delta)$ satisfying

$$\|w - \eta_w\|_{-(r+1)k_0} \le C h_\Delta^{-t-(r+1)k_0} \|w\|_{-t-2(r+1)k_0}.$$

Since $\langle P_{-(r+1)k_0} v - v, \eta \rangle_{-(r+1)k_0} = 0$ for any $\eta \in S_d^r(\Delta)$, applying (2.2.3), (2.2.2) and (4.4.9), we obtain

$$\begin{aligned}
\|P_{-(r+1)k_0} v - v\|_t &= \sup_{\substack{w \in H^{-t-2(r+1)k_0} \\ w \neq 0}} \frac{\langle P_{-(r+1)k_0} v - v, w \rangle_{-(r+1)k_0}}{\|w\|_{-t-2(r+1)k_0}} \\
&= \sup_{\substack{w \in H^{-t-2(r+1)k_0} \\ w \neq 0}} \frac{\langle P_{-(r+1)k_0} v - v, w - \eta_w \rangle_{-(r+1)k_0}}{\|w\|_{-t-2(r+1)k_0}} \\
&\le \|P_{-(r+1)k_0} v - v\|_{-(r+1)k_0} \sup_{\substack{w \in H^{-t-2(r+1)k_0} \\ w \neq 0}} \frac{\|w - \eta_w\|_{-(r+1)k_0}}{\|w\|_{-t-2(r+1)k_0}} \\
&\le C h_\Delta^{s-t} \|v\|_s.
\end{aligned}$$

Hence, we have proved that

$$\begin{cases} t^* \in \mathcal{I}_{k_0+1}, \ -(r+1)k_0 \le s^* \le d+1, \\ \forall v \in H^{s^*}, \ \exists \eta_v \in S_d^r(\Delta) : \|v - \eta_v\|_{t^*} \le C h_\Delta^{s^*-t^*} \|v\|_{s^*}. \end{cases} \tag{4.4.10}$$

Case 2. Consider $s^* \in [t^*, -(r+1)k_0)$.

Let $s, t \in \mathbb{R}$ satisfy $2s-(d+1) \le t \le s < -(r+1)k_0$. Let $P_s : H^s \to S_d^r(\Delta)$ be the projection defined by

$$\langle P_s v, w \rangle_s = \langle v, w \rangle_s \quad \forall v \in H^s \text{ and } w \in S_d^r(\Delta).$$

Then it is obvious that

$$\|P_s v - v\|_s \le \|v\|_s \quad \forall v \in H^s. \tag{4.4.11}$$

If $2s-(d+1) \le t \le 2s+(r+1)k_0$ so that $-(r+1)k_0 \le 2s-t \le d+1$, then (4.4.10) with with t^* and s^* replaced by s and $2s-t$, respectively, assures that for any $w \in H^{2s-t}$, there exists a $\eta_w \in S_d^r(\Delta)$ such that

$$\|w - \eta_w\|_s \le C h_\Delta^{s-t} \|w\|_{2s-t}. \tag{4.4.12}$$

Since $\langle P_s v - v, \eta_w \rangle = 0$, applying (2.2.3), (2.2.2), (4.4.11) and (4.4.12), we deduce

$$\begin{aligned}
\|P_s v - v\|_t &= \sup_{\substack{w \in H^{2s-t} \\ w \neq 0}} \frac{\langle P_s v - v, w\rangle_s}{\|w\|_{2s-t}} = \sup_{\substack{w \in H^{2s-t} \\ w \neq 0}} \frac{\langle P_s v - v, w - \eta_w\rangle_s}{\|w\|_{2s-t}} \\
&\le \|P_s v - v\|_s \sup_{\substack{w \in H^{2s-t} \\ w \neq 0}} \frac{\|w - \eta_w\|_s}{\|w\|_{2s-t}} \\
&\le C h_\Delta^{s-t} \|v\|_s .
\end{aligned}$$

In particular, when $t = 2s + (r+1)k_0$, there holds

$$\|P_s v - v\|_{2s+(r+1)k_0} \le C h_\Delta^{-s-(r+1)k_0} \|v\|_s \quad \forall v \in H^s. \tag{4.4.13}$$

If $2s + (r+1)k_0 < t < s$, then the required inequality can be obtained by applying Theorem 2.11 with $T = P_s - I$, $t_1 = 2s + (r+1)k_0$, $t_2 = s$, $s_1 = s_2 = s$ and $\theta = (t-s)/(s + (r+1)k_0)$, and noting (4.4.11) and (4.4.13).

Combining both cases, the following statement holds

$$\begin{cases} t^* \in \mathcal{I}_{k_0+1}, \ t^* \le s^* \le d+1, \\ \forall v \in H^{s^*}, \ \exists \eta_v \in S_d^r(\Delta) : \|v - \eta_v\|_{t^*} \le C h_\Delta^{s^*-t^*} \|v\|_{s^*}, \end{cases} \tag{4.4.14}$$

completing the proof. $\square$

In the next section we will use the result developed in this section to estimate the error of the Galerkin approximation.

4.5 Galerkin method

4.5.1 Approximate solution

Noting (4.2.3), we shall seek an approximate solution $\widetilde{u} \in H^{\sigma+\alpha}$ in the form

$$\widetilde{u} = \widetilde{u}_0 + \widetilde{u}_1 \quad \text{where} \quad \widetilde{u}_0 \in \ker L \quad \text{and} \quad \widetilde{u}_1 \in S_d^r(\Delta). \tag{4.5.1}$$

The solution $\widetilde{u}_1$ will be found by solving the Galerkin equation

$$\langle L^* \widetilde{u}_1, v \rangle = \langle g, v \rangle \quad \forall v \in S_d^r(\Delta), \tag{4.5.2}$$

in which L^* is a strongly elliptic pseudodifferential operator of order 2α whose symbol is given by

$$\widehat{L^*}(\ell) = \begin{cases} \widehat{L}(\ell) & \text{if } \ell \notin \mathcal{K}(L) \\ (1+\ell)^{2\alpha} & \text{if } \ell \in \mathcal{K}(L). \end{cases}$$

Noting (2.3.3), we have

$$\widehat{L^*}(\ell) \simeq (1+\ell)^{2\alpha} \quad \forall \ell \in \mathbb{N}.$$

This confirms that the bilinear form $a^* : H^\alpha \times H^\alpha \to \mathbb{R}$ defined by

$$a^*(v, w) := \langle L^* v, w \rangle, \quad v, w \in H^\alpha$$

is continuous and coercive in H^α. The well known Lax–Milgram Theorem confirms the unique existence of $\widetilde{u}_1 \in S_d^r(\Delta)$.

Having found $\widetilde{u}_1$, we will find $\widetilde{u}_0 \in \ker L$ by solving the equations (cf. (4.2.2))

$$\langle \mu_i, \widetilde{u}_0 \rangle = \gamma_i - \langle \mu_i, \widetilde{u}_1 \rangle, \quad i = 1, \dots, M,$$

so that

$$\langle \mu_i, \widetilde{u} \rangle = \langle \mu_i, u \rangle, \quad i = 1, \dots, M. \tag{4.5.3}$$

The unique existence of $\widetilde{u}_0$ follows from Assumption B in exactly the same way as that of u_0; see Lemma 4.1.

It is noted that in general $S_d^r(\Delta) \not\subseteq (\ker L)^\perp_{H^\alpha}$. However, $\widetilde{u}$ can be rewritten in a form similar to (4.2.3) as follows. Let

$$u_0^* := \widetilde{u}_0 + \sum_{\ell \in \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} \widehat{(\widetilde{u}_1)}_{\ell,m} Y_{\ell,m} \tag{4.5.4}$$

and

$$u_1^* = \sum_{\ell \notin \mathcal{K}(L)} \sum_{m=-\ell}^{\ell} \widehat{(\widetilde{u}_1)}_{\ell,m} Y_{\ell,m}. \tag{4.5.5}$$

Then

$$\widetilde{u} = u_0^* + u_1^* \quad \text{with} \quad u_0^* \in \ker L \quad \text{and} \quad u_1^* \in (\ker L)^\perp_{H^\alpha}. \tag{4.5.6}$$

It should be noted that, in general, u_1^* does not belong to $S_d^r(\Delta)$, and that this function is introduced purely for analysis purposes. We do not explicitly compute u_1^*, nor u_0^*.

4.5.2 Error analysis

Assume that the exact solution u and the approximate solution $\widetilde{u}$ of Problem A belong to H^t for some $t \in \mathbb{R}$, and assume that $\mu_i \in H^{-t}$ for $i = 1, \dots, M$. Comparing (4.2.3) and (4.5.6) suggests that $\|u - \widetilde{u}\|_t$ can be estimated by estimating $\|u_0 - u_0^*\|_t$ and $\|u_1 - u_1^*\|_t$. In fact, in this chapter we require that $\|u - \widetilde{u}\|_t \preceq \|u_1 - \widetilde{u}_1\|_t$ (see Lemma 4.4 and the proof is omitted), which can be proved by showing that $\|u - \widetilde{u}\|_t \preceq \|u_1 - u_1^*\|_t$ as in Lemmas 3.3 and 3.4, and noting that $\|u_1 - u_1^*\|_t \le \|u_1 - \widetilde{u}_1\|_t$.

Lemma 4.4. *Let u, u_1, $\widetilde{u}$ and $\widetilde{u}_1$ be defined by* (4.2.3), (4.5.1), *and* (4.5.2). *For $i = 1, \dots, M$, if $\mu_i \in H^{-t}$ for some $t \in \mathbb{R}$, there holds*

$$\|u - \widetilde{u}\|_t \le C \|u_1 - \widetilde{u}_1\|_t.$$

We now prove the main theorem of the chapter.

Theorem 4.5. *Assume that Δ is a quasi-uniform spherical triangulation with $|\Delta| \le 1$ and that* (2.5.23) *hold. If the order 2α of the pseudodifferential operator L satisfies $\alpha \le r + 1$, and if u and $\widetilde{u}$ satisfy, respectively,* (4.2.5) *and* (4.5.1), *then*

$$\|u - \widetilde{u}\|_t \le C h_\Delta^{s-t} \|u\|_s,$$

where $s \le d + 1$ and $2\alpha - d - 1 \le t \le \min\{s, \alpha\}$. Here C is a positive constant depending only on d and the smallest angle in Δ.

Proof. The proof is standard using Céa's Lemma and Nitsche's trick. We include it here for completeness. Noting that $u_1 \in (\ker L)^{\perp}_{H^\alpha}$ and (4.2.5), we have

$$a^*(u_1, v) = \langle g, v\rangle \quad \forall v \in H^\alpha.$$

Céa's Lemma gives

$$\|u_1 - \widetilde{u}_1\|_\alpha \le C \min_{v \in S_d^r(\Delta)} \|u_1 - v\|_\alpha,$$

which, together with Theorem 4.3, implies that there exists a constant C satisfying

$$\|u_1 - \widetilde{u}_1\|_\alpha \le C h_\Delta^{s-\alpha} \|u_1\|_s.$$

Noting that $\|u_1\|_s \le \|u\|_s$ and the result in Lemma 4.4 we have

$$\|u - \widetilde{u}\|_\alpha \le C h_\Delta^{s-\alpha} \|u\|_s. \tag{4.5.7}$$

Now consider $t < \alpha$. It follows from (2.2.3) and (2.3.3) that

$$\|u_1 - \widetilde{u}_1\|_t \le C \sup_{\substack{v \in H^{2\alpha - t}\\ v \ne o}} \frac{\langle u_1 - \widetilde{u}_1, v\rangle_\alpha}{\|v\|_{2\alpha - t}} \le C \sup_{\substack{v \in H^{2\alpha - t}\\ v \ne o}} \frac{a^*(u_1 - \widetilde{u}_1, v)}{\|v\|_{2\alpha - t}}.$$

By using successively (4.2.5), (4.5.2) we deduce that for arbitrary $\eta \in S_d^r(\Delta)$

$$\|u_1 - \widetilde{u}_1\|_t \le C \sup_{\substack{v \in H^{2\alpha - t}\\ v \ne o}} \frac{a^*(u_1 - \widetilde{u}_1, v - \eta)}{\|v\|_{2\alpha - t}} \le C \|u_1 - \widetilde{u}_1\|_\alpha \sup_{\substack{v \in H^{2\alpha - t}\\ v \ne o}} \frac{\|v - \eta\|_\alpha}{\|v\|_{2\alpha - t}}.$$

Since $2\alpha - d - 1 \le t < \alpha$, there holds $\alpha < 2\alpha - t \le d + 1$. By using Theorem 4.3 again, we can choose $\eta \in S_d^r(\Delta)$ satisfying

$$\|v - \eta\|_\alpha \le C h_\Delta^{\alpha - t} \|v\|_{2\alpha - t}.$$

Hence,

$$\|u_1 - \widetilde{u}_1\|_t \le C h_\Delta^{\alpha - t} \|u_1 - \widetilde{u}_1\|_\alpha \le C h_\Delta^{s-t} \|u\|_s.$$

Using Lemma 4.4, we obtain

$$\|u - \widetilde{u}\|_t \le C h_\Delta^{s-t} \|u\|_s.$$

□

4.6 Numerical experiments

In this section, we present the numerical results obtained from our experiments with different sets of points $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N\}$, where $\boldsymbol{x}_i = (x_{i,1}, x_{i,2}, x_{i,3})$, $i = 1, \ldots, N$, are points on the sphere generated by a simple algorithm [65] which partitions the sphere into equal areas. From these sets of data points, we used the FORTRAN package STRIPACK (which can be found at `http://www.netlib.org/toms/77`) to obtain the spherical triangulations Δ.

We solved pseudodifferential equations by using the space of spherical splines $S_1^0(\Delta)$, which is the space of continuous and piecewise homogeneous polynomial of degree 1. A set of basis functions for $S_1^0(\Delta)$ is

$$\{B_i : i = 1, \dots, N\},$$

where, for each $i = 1, \dots, N$, B_i is the nodal basis function corresponding to the vertex $\boldsymbol{x}_i$ defined as follows.

For any $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{S}^2$, there exists a spherical triangle $\tau \in \Delta$ containing $\boldsymbol{x}$. If $\boldsymbol{x}_i$ is not a vertex of τ then $B_i(\boldsymbol{x}) = 0$. Otherwise, assume that τ is formed by three vertices $\boldsymbol{x}_i$, $\boldsymbol{x}_j$ and $\boldsymbol{x}_k$. Then

$$B_i(\boldsymbol{x}) = \det \begin{bmatrix} x_1 & x_{j,1} & x_{k,1} \\ x_2 & x_{j,2} & x_{k,2} \\ x_3 & x_{j,3} & x_{k,3} \end{bmatrix} \cdot \left(\det \begin{bmatrix} x_{i,1} & x_{j,1} & x_{k,1} \\ x_{i,2} & x_{j,2} & x_{k,2} \\ x_{i,3} & x_{j,3} & x_{k,3} \end{bmatrix} \right)^{-1}, \tag{4.6.1}$$

see [2].

4.6.1 The Laplace–Beltrami equation

We solved the equation on the unit sphere of the form

$$Lu = g \quad \text{on } \mathbb{S}^2, \tag{4.6.2}$$

where $Lu = -\Delta_{\mathbb{S}^2} u + u$. This equation arises, for example, when one discretises in time the diffusion equation on the sphere.

We solved (4.6.2) with

$$g(\boldsymbol{x}) = 7x_3^3 - 6x_3(x_1^2 + x_2^2),$$

so that the exact solution is

$$u(\boldsymbol{x}) = x_3^3.$$

We carried out the experiment with various numbers of points N, namely $N = 101$, 200, 500, 1001, 2000, 4000, and 8001. Noting (2.1.7), the entry A_{ij} of the stiffness matrix A is computed by

$$\begin{aligned} A_{ij} &= \int_{\mathbb{S}^2} \Big(\nabla_{\mathbb{S}^2} B_i(\boldsymbol{x}) \cdot \nabla_{\mathbb{S}^2} B_j(\boldsymbol{x}) + B_i(\boldsymbol{x}) B_j(\boldsymbol{x}) \Big)\, d\sigma_{\boldsymbol{x}} \\ &= \sum_{\tau \in \Delta} \int_{\tau} \Big(\nabla_{\mathbb{S}^2} B_i(\boldsymbol{x}) \cdot \nabla_{\mathbb{S}^2} B_j(\boldsymbol{x}) + B_i(\boldsymbol{x}) B_j(\boldsymbol{x}) \Big)\, d\sigma_{\boldsymbol{x}}, \end{aligned}$$

where, for any $i = 1, \dots, N$, $B_i(\boldsymbol{x})$ is computed by using (4.6.1) and $\nabla_{\mathbb{S}^2} B_i$ is the surface gradient of B_i. The right hand side of the linear system has entries given by

$$\boldsymbol{b}_i = \int_{\mathbb{S}^2} B_i(\boldsymbol{x}) g(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}} = \sum_{\tau \in \Delta} \int_{\tau} B_i(\boldsymbol{x}) g(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}}, \quad i = 1, \dots, N. \tag{4.6.3}$$

The integral of a smooth function f over a spherical triangle τ with vertices $\boldsymbol{x}_i = (x_{i,1}, x_{i,2}, x_{i,3})$, $\boldsymbol{x}_j = (x_{j,1}, x_{j,2}, x_{j,3})$ and $\boldsymbol{x}_k = (x_{k,1}, x_{k,2}, x_{k,3})$ is computed as follows. Let T be the planar triangle (in $\mathbb{R}^3$) with the same vertices as τ, and let $\widehat{T}$ be the

Table 4.1: Errors in the L_2-norm for the Laplace–Beltrami equation. Expected order of convergence: 2.

N	h_Δ	$\|u_N - u\|_0$	EOC
101	0.2618	5.853E-2	
200	0.1819	2.713E-2	2.11
500	0.1136	1.093E-2	1.93
1001	0.0798	5.230E-3	2.09
2000	0.0570	2.675E-3	1.99
4000	0.0397	1.316E-3	1.96
8001	0.0283	6.627E-4	2.02

planar triangle of vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. A point on $\widehat{T}$ can be represented as $\boldsymbol{u} = (u_1, u_2, 1 - u_1 - u_2)$ where (u_1, u_2) satisfies $0 \le u_2 \le 1 - u_1$ and $0 \le u_1 \le 1$. It is shown in (2.5.14) that

$$\int_\tau f(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}} = |\det \mathcal{F}| \int_0^1 \int_0^{1-u_1} f\left(\frac{\mathcal{F}(\boldsymbol{u})}{|\mathcal{F}(\boldsymbol{u})|}\right) \frac{du_2\, du_1}{|\mathcal{F}(\boldsymbol{u})|^3}, \tag{4.6.4}$$

where $\mathcal{F} : \widehat{T} \to T$ is defined by

$$\mathcal{F}(\boldsymbol{u}) = \begin{bmatrix} x_{i,1} & x_{j,1} & x_{k,1} \\ x_{i,2} & x_{j,2} & x_{k,2} \\ x_{i,3} & x_{j,3} & x_{k,3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 1 - u_1 - u_2 \end{bmatrix}. \tag{4.6.5}$$

Having solved the linear system, we computed the L_2-norm of the error $u_N - u$, where in this section we denoted the approximate solution by u_N instead of $\widetilde{u}$. It is expected from our theoretical result (Theorem 4.5) that $\|u_N - u\|_0 = O(h_\Delta^2)$. The estimated orders of convergence (EOC) shown in Table 4.1 agree with our theoretical result.

4.6.2 Weakly singular integral equation

The equation

We also solved the weakly singular integral equation

$$Lu = g \quad \text{on} \quad \mathbb{S}^2, \tag{4.6.6}$$

where L is the operator given by

$$Lv(\boldsymbol{x}) = \frac{1}{4\pi} \int_{\mathbb{S}^2} v(\boldsymbol{y}) \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, d\sigma_{\boldsymbol{y}},$$

for any $v \in \mathscr{D}'(\mathbb{S}^2)$. This equation is the boundary integral reformulation of the Dirichlet problem for the Laplacian in the exterior or interior of the sphere [29]. Note that the weakly singular integral operator L is a pseudodifferential operator of order -1 with symbol $1/(2\ell + 1)$; see examples following Definition 2.12, noting that in the case $\mathbb{S}^2$, $\omega_3 = 4\pi$ and $N(3, \ell) = 2\ell + 1$.

Table 4.2: Errors in the $H^{-1/2}$-norm for the weakly singular integral equation. Expected order of convergence: 2.5.

N	h_Δ	$\|u - u_N\|_{-1/2}$	EOC
20	0.7432	0.4609E-01	
30	0.5153	0.1349E-01	3.36
40	0.4401	0.1110E-01	1.24
51	0.3757	0.8246E-02	1.87
80	0.2958	0.4337E-02	2.69
101	0.2617	0.3072E-02	2.82
125	0.2316	0.2371E-02	2.11
150	0.2104	0.1894E-02	2.35
175	0.1987	0.1602E-02	2.90
200	0.1819	0.1350E-02	1.95
226	0.1692	0.1179E-02	1.87
250	0.1625	0.1046E-02	2.96
275	0.1538	0.8856E-03	3.00

Equation (4.6.6) was solved with $g(\boldsymbol{x}) = x_1x_2/5$, so that the exact solution is $u(\boldsymbol{x}) = x_1x_2$. We solved this equation with various values of N, namely $N = 10$, 20, 30, 40, 51, 80, 101, 125, 150, 175 and 200. Each entry A_{ij} of the stiffness matrix A is given by

$$A_{ij} = \frac{1}{4\pi}\int_{\mathbb{S}}^{2}\int_{\mathbb{S}}^{2} \frac{B_i(\boldsymbol{x})B_j(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d\sigma_{\boldsymbol{y}}\, d\sigma_{\boldsymbol{x}}, \quad i,j \in \{1,\dots,N\}. \tag{4.6.7}$$

Computation of A_{ij} is explained in the subsection below. Computation of the right hand side is analogous to computation of (4.6.3) and is explained in Section 4.6.1.

To check the accuracy of the approximation, we computed the $H^{-1/2}$-norm of the error $u - u_N$, which was calculated as follows:

$$\begin{aligned}\|u-u_N\|_{-1/2}^2 &\simeq \langle L(u-u_N), u-u_N\rangle = \langle L(u-u_N), u\rangle \\ &= \langle Lu, u\rangle - \langle u_N, Lu\rangle = \int_{\mathbb{S}^2} gu\, d\sigma - \int_{\mathbb{S}^2} gu_N\, d\sigma.\end{aligned}$$

The estimated orders of convergence shown in Table 4.2 do not quite agree with the theoretical result, which states that $\|u - u_N\|_{-1/2} = O(h_\Delta^{5/2})$. A reason may be the fact that sets of Saff points used in this experiment are not nested. Indeed, the numbers shown in Table 4.3, which were obtained by using nested sets of points, seem to agree with the theoretical result. The starting mesh contains 8 equal spherical triangles with 6 nodes (4 on the equator and 2 at the poles). Every further refinement consists of partitioning every spherical triangle into 4 smaller spherical triangles by joining the midpoints of the edges (red refinement).

Table 4.3: Errors in the $H^{-1/2}$-norm for the weakly singular integral equation with nested triangulations. Expected order of convergence: 2.5.

N	h_Δ	$\|u-u_N\|_{-1/2}$	EOC
6	1.5707	4.0933E-01	0.00
18	0.7853	6.2962E-02	2.70
66	0.3926	7.6483E-03	3.04
258	0.1963	1.2607E-03	2.60

Computation of the stiffness matrix with numerical quadrature

Computation of elements of the stiffness matrix (4.6.7) requires evaluation of integrals of the type

$$I = \frac{1}{4\pi}\int_{\tau^{(1)}}\int_{\tau^{(2)}} \frac{f_1(\boldsymbol{x})f_2(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d\sigma_{\boldsymbol{y}}\, d\sigma_{\boldsymbol{x}}, \quad i,j \in \{1,\dots,N\} \tag{4.6.8}$$

where $\tau^{(1)}, \tau^{(2)}$ are spherical triangles from Δ and the basis functions $f_1(\boldsymbol{x})$ and $f_2(\boldsymbol{y})$ are analytic for all $\boldsymbol{x} \in \tau^{(1)}$ and $\boldsymbol{y} \in \tau^{(2)}$, cf. (4.6.1). Due to the nonlocal integral kernel $|\boldsymbol{x}-\boldsymbol{y}|^{-1}$ the integral value I is in general different from zero, even if f_1 and f_2 have disjoint supports. As a consequence, the stiffness matrix (4.6.7) is densely populated. We computed (4.6.8) approximately by a numerical quadrature. Note that an accurate numerical approximation of (4.6.8) becomes a challenging task if $\tau^{(1)}$ and $\tau^{(2)}$ share at least one point and hence $|\boldsymbol{x}-\boldsymbol{y}|^{-1}$ may become singular.

For ease of presentation, in this subsection we introduce the following technical notations. Suppose that the spherical triangle $\tau^{(q)}$, $q = 1,2$ has vertices $\{\boldsymbol{v}_1^{(q)}, \boldsymbol{v}_2^{(q)}, \boldsymbol{v}_3^{(q)}\}$. Extending the notations of Section 4.6.1 we denote by $T^{(q)}$ the planar triangle sharing its vertices with the spherical triangle $\tau^{(q)}$. Let $\widehat{T}$ be the planar triangle in $\mathbb{R}^3$ with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and K be the planar triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$ and $(0,1)$. Any two points $\boldsymbol{u}, \boldsymbol{v} \in \widehat{T}$ can be uniquely represented by two pairs $(u_1,u_2), (v_1,v_2) \in K$ by

$$\boldsymbol{u} = (u_1, u_2, 1-u_1-u_2) \quad \text{and} \quad \boldsymbol{v} = (v_1, v_2, 1-v_1-v_2).$$

Firstly, we describe a numerical approximation for (4.6.8) for two disjoint spherical triangles $\tau^{(1)}$ and $\tau^{(2)}$ (a so-called far field computation). In this case the kernel $|\boldsymbol{x}-\boldsymbol{y}|^{-1}$ and the integrand $f(\boldsymbol{x},\boldsymbol{y}) := \frac{f_1(\boldsymbol{x})f_2(\boldsymbol{y})}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}$ are analytic on $\tau^{(1)} \times \tau^{(2)}$. Similarly to (4.6.4), I can be represented as an integral over $K \times K$

$$\begin{aligned} I = \int_{\tau^{(1)}}\int_{\tau^{(2)}} f(\boldsymbol{x},\boldsymbol{y})\, d\sigma_{\boldsymbol{x}} d\sigma_{\boldsymbol{y}} &= |\det \mathcal{F}^{(1)}||\det \mathcal{F}^{(2)}| \\ \times \int_0^1\int_0^{1-u_1}\int_0^1\int_0^{1-v_1} f &\left(\frac{\mathcal{F}^{(1)}(\boldsymbol{u})}{|\mathcal{F}^{(1)}(\boldsymbol{u})|}, \frac{\mathcal{F}^{(2)}(\boldsymbol{v})}{|\mathcal{F}^{(2)}(\boldsymbol{v})|}\right) \frac{dv_2\, dv_1}{|\mathcal{F}^{(2)}(\boldsymbol{v})|^3} \frac{du_2\, du_1}{|\mathcal{F}^{(1)}(\boldsymbol{u})|^3}, \end{aligned} \tag{4.6.9}$$

where $\mathcal{F}^{(q)} : \widehat{T} \to T$, $q = 1, 2$ is defined by

$$\mathcal{F}^{(q)}(\boldsymbol{u}) = \begin{bmatrix} v_{1,1}^{(q)} & v_{2,1}^{(q)} & v_{3,1}^{(q)} \\ v_{1,2}^{(q)} & v_{2,2}^{(q)} & v_{3,2}^{(q)} \\ v_{1,3}^{(q)} & v_{2,3}^{(q)} & v_{3,3}^{(q)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ 1 - u_1 - u_2 \end{bmatrix}. \tag{4.6.10}$$

The mappings

$$\begin{cases} K & \to & T^{(q)}, \\ (u_1, u_2) & \mapsto & \mathcal{F}^{(q)}(\boldsymbol{u}), \end{cases} \qquad q = 1, 2$$

are affine, hence the mappings

$$\begin{cases} K & \to & \tau^{(q)}, \\ (u_1, u_2) & \mapsto & \dfrac{\mathcal{F}^{(q)}(\boldsymbol{u})}{|\mathcal{F}^{(q)}(\boldsymbol{u})|}, \end{cases} \qquad q = 1, 2$$

are analytic if and only if $T^{(q)}$ does not contain the origin, i.e. $|\mathcal{F}^{(q)}(\boldsymbol{u})| \neq 0$. This requirement is always satisfied if the underlying spherical triangulation Δ is sufficiently refined. Thus the integrand in (4.6.9) is a composition of analytic functions and hence is analytic. Therefore the integral I in (4.6.8) can be computed numerically by standard quadrature rules in case of disjoint spherical triangles $\tau^{(1)}$ and $\tau^{(2)}$.

Secondly, we consider numerical approximation of (4.6.8) for $\tau^{(1)}$ and $\tau^{(2)}$ sharing at least one point (a so-called near field computation). Depending on the mutual location of $\tau^{(1)}$ and $\tau^{(2)}$ we introduce an auxiliary parameter k as follows:

$$k = \begin{cases} 0 & \text{if } \tau^{(1)} \text{ and } \tau^{(2)} \text{ share a vertex}, \\ 1 & \text{if } \tau^{(1)} \text{ and } \tau^{(2)} \text{ share an edge}, \\ 2 & \text{if } \tau^{(1)} \text{ and } \tau^{(2)} \text{ are identical}. \end{cases}$$

Efficient and accurate quadratures for integrals of this type have been intensively developed over the last three decades mostly in the context of Boundary Element Methods [21, 35, 67, 70]. It is known that there exists a sequence of regularising coordinate transformations which removes the singularity in (4.6.8) completely. Note that this coordinate transformation is not unique. In this chapter we employ the approach in [13] which also extends to arbitrary dimension and noninteger singularity orders, cf. [14] for computational aspects in this approach and [22, 66] for alternative transformations. In the remainder of this section, we describe details of the computational process.

Without loss of generality we assume that $\tau^{(1)}$ and $\tau^{(2)}$ intersect at their first $k+1$ vertices, i.e.,

$$\boldsymbol{v}_j^{(1)} = \boldsymbol{v}_j^{(2)} \qquad j = 1, \dots, k+1. \tag{4.6.11}$$

and $\mathcal{F}^{(q)}$ is defined by (4.6.10) according to this order. The integrand in the right-hand side of (4.6.9) has a singularity on $K \times K$. However, location of singularity is easily described thanks to agreement (4.6.11): for $((u_1, u_2), (v_1, v_2)), \in K \times K$, the integrand in the right-hand side of (4.6.9) is singular if

$$\begin{cases} u_1 = 0 = v_1 & \text{and} \quad u_2 = 0 = v_2, \quad \text{if } k = 0, \\ u_1 = v_1 & \text{and} \quad u_2 = 0 = v_2, \quad \text{if } k = 1, \\ u_1 = v_1 & \text{and} \quad u_2 = v_2, \quad\quad\;\; \text{if } k = 2. \end{cases} \tag{4.6.12}$$

The algorithm in [13, 14] allows to approximate integrals of the form

$$\int_{u_1=0}^{1} \int_{u_2=0}^{1-u_1} \int_{v_1=0}^{1} \int_{v_2=0}^{1-v_1} g(u_1,u_2,v_1,v_2)\, dv_2 dv_1 du_2 du_1, \tag{4.6.13}$$

where $g(u_1,u_2,v_1,v_2)$ has singular support as in (4.6.12), by a quadrature rule

$$\sum_{l=1}^{N_k} g(u_{1,l},u_{2,l},v_{1,l},v_{2,l}) w_l \tag{4.6.14}$$

with weights $\{w_l\}_{l=1}^{N_k}$ and nodes $\{(u_{1,l},u_{2,l},v_{1,l},v_{2,l})\}_{l=1}^{N_k}$. Here N_k is the number of quadrature data used, which depends on the value of k. This quadrature rule is not of product type over $K \times K$, thus we enumerate the quadrature weights and nodes by a single index l. Furthermore, the quadrature error is bounded by $C\exp(-bN_k^{1/4})$ with $b, C > 0$ independent of N_k [13]. Based on this result and in view of the transformation (4.6.9) we approximate the integral I in (4.6.8) by the quadrature rule

$$Q_{N_k} = |\det \mathcal{F}^{(1)}||\det \mathcal{F}^{(2)}| \sum_{l=1}^{N_k} f\left(\frac{\mathcal{F}^{(1)}(\boldsymbol{u}_l)}{|\mathcal{F}^{(1)}(\boldsymbol{u}_l)|}, \frac{\mathcal{F}^{(2)}(\boldsymbol{v}_l)}{|\mathcal{F}^{(2)}(\boldsymbol{v}_l)|} \right) \frac{w_l}{|\mathcal{F}^{(2)}(\boldsymbol{v}_l)|^3 |\mathcal{F}^{(1)}(\boldsymbol{u}_l)|^3}.$$

Here we adopt the notation

$$\boldsymbol{u}_l = (u_{1,l}, u_{2,l}, 1-u_{1,l}-u_{2,l}) \quad \text{and} \quad \boldsymbol{v}_l = (v_{1,l}, v_{2,l}, 1-v_{1,l}-v_{2,l}) \quad \text{for} \quad l = 1,\dots,N_k.$$

Using again transformation (4.6.9) we express Q_{N_k} in a simpler form as a quadrature rule over $\tau^{(1)} \times \tau^{(2)}$:

$$Q_{N_k} = \sum_{l=1}^{N_k} \frac{f_1(\boldsymbol{x}_l) f_2(\boldsymbol{y}_l)}{|\boldsymbol{x}_l - \boldsymbol{y}_l|} W_l \tag{4.6.15}$$

where

$$\boldsymbol{x}_l := \frac{\mathcal{F}^{(1)}(\boldsymbol{u}_l)}{|\mathcal{F}^{(1)}(\boldsymbol{u}_l)|}, \qquad \boldsymbol{y}_l := \frac{\mathcal{F}^{(2)}(\boldsymbol{v}_l)}{|\mathcal{F}^{(2)}(\boldsymbol{v}_l)|}, \qquad W_l := \frac{|\det \mathcal{F}^{(1)}||\det \mathcal{F}^{(2)}| w_l}{4\pi |\mathcal{F}^{(1)}\boldsymbol{u}_l|^3 |\mathcal{F}^{(2)}\boldsymbol{v}_l|^3} \tag{4.6.16}$$

According to [13, 66] the quadrature error is bounded by

$$|I - Q_{N_k}| \le C\exp(-bN_k^{1/4}) \tag{4.6.17}$$

with $b, C > 0$ independent on N_k. This quadrature rule is used in our computations.

We illustrate performance of Q_{N_k} on the simplified integral

$$\int_{\tau^{(1)}} \int_{\tau^{(2)}} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|}\, d\sigma_{\boldsymbol{x}}\, d\sigma_{\boldsymbol{y}} \tag{4.6.18}$$

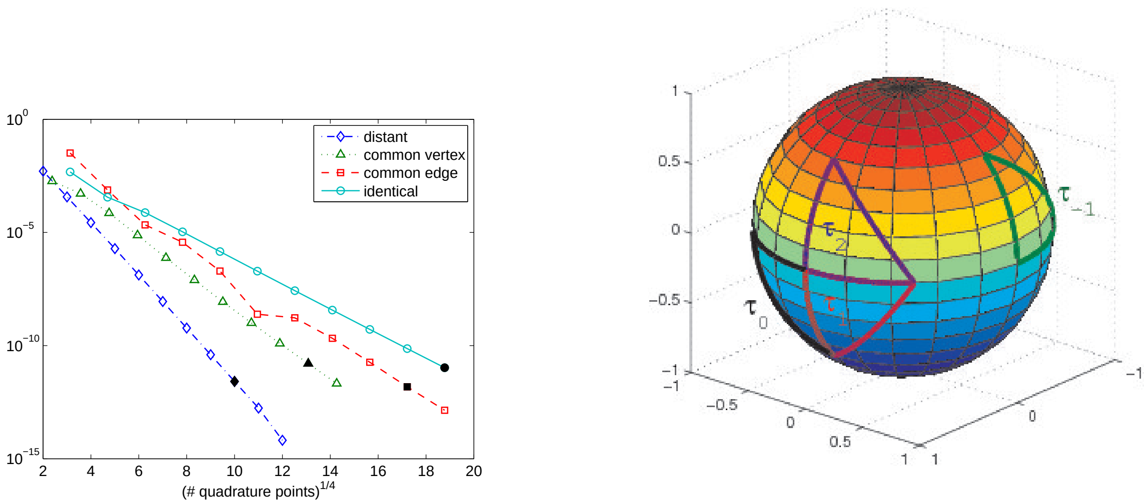

Figure 4.1: Relative error of the quadrature rule for approximation of (4.6.18) (left) and location of spherical triangles (right).

with $\tau^{(1)} = \tau_m$, $m = -1, 0, 1, 2$, $\tau^{(2)} = \tau_2$, and τ_m with vertices

$$\tau_{-1} = \langle (0,1,0), (-\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0), (0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}) \rangle, \quad \tau_1 = \langle (1,0,0), (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0), (\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}) \rangle,$$

$$\tau_0 = \langle (1,0,0), (\tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}, 0), (\tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}) \rangle, \quad \tau_2 = \langle (1,0,0), (\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}, 0), (\tfrac{1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}) \rangle,$$

as shown Fig. 4.1 (right). Thus, $\tau^{(1)}$ and $\tau^{(2)}$ are disjoint spherical triangles if $m = -1$, share a node if $m = 0$, share an edge if $m = 1$ and are identical triangles if $m = 2$. In case $m = -1$ we use a quadrature rule on $K \times K$ obtained from the Gauss-Legendre quadrature rule on $[0,1]^2 \times [0,1]^2$ by Duffy transformation

$$\begin{cases} [0,1]^2 \to K, \\ \boldsymbol{\zeta} \mapsto \begin{pmatrix} \zeta_1(1-\zeta_2) \\ \zeta_2 \end{pmatrix}. \end{cases} \tag{4.6.19}$$

In Fig. 4.1 (left) we give the relative error of the constructed quadrature approximation. In the numerical experiment presented in Table 4.2 and Table 4.3 we chose quadrature rules with the following total number of points: $N_{-1} = 10^4, N_0 = 2 \cdot 11^4, N_1 = 6 \cdot 11^4$ and $N_2 = 6 \cdot 12^4$. The corresponding relative errors are marked by solid symbols in Fig. 4.1 (left).

Remark 4.6. *The most time consuming part in the numerical simulation is the computation and assembly of the stiffness matrix, which is fully populated due to nonlocal nature of L in (4.6.6). It is possible to reduce the order of the quadrature rule if the distance between $\tau^{(1)}$ and $\tau^{(2)}$ is large, cf. [66, Section 5]. This reduces the number of function evaluations, but requires quadratures of different order in the far field computation.*

Chapter 5

Preconditioning for the Laplace–Beltrami equation

5.1 Introduction

We consider the model equation

$$-\Delta_{\mathbb{S}^2} u + \omega^2 u = g \quad \text{on } \mathbb{S}^2, \tag{5.1.1}$$

where $\Delta_{\mathbb{S}^2}$ is the Laplace-Beltrami operator defined in (2.1.1) and ω is some nonzero real constant. This elliptic equation arises, for example, when one discretises in time the diffusion equation on the sphere. When solving this equation on the unit sphere by the Galerkin method with spherical splines, as was done in Chapter 4, an ill-conditioned linear system will result; see Proposition 5.4 in the next section. Since the conjugate gradient method will be used to solve this system if the size of the matrix is large, a large number of iterations will be required.

In this chapter, we overcome this ill-conditionedness by preconditioning with additive Schwarz methods. This preconditioner has long been used in finite element and boundary element literatures [41, 74, 78].

As is usual for finite element or boundary element methods, the additive Schwarz preconditioner is defined based on a subspace splitting of the finite dimensional space in which the solution is sought. This splitting is in turn defined by a decomposition of $\mathbb{S}^2$ into subdomains. We here design an overlapping decomposition method based on a two-level mesh. A fine mesh and a coarse mesh are defined from two sets of data points on the sphere. The cardinality of the set defining the fine mesh is chosen to be larger than that of the set defining the coarse mesh. A subdomain is constructed from each triangle in the coarse mesh by taking the union of all triangles in the fine mesh which intersect this coarse triangle. This results in a set of overlapping subdomains which we use to define an additive Schwarz operator for solving (5.1.1).

This is our first step in studying overlapping additive Schwarz preconditioners for pseudodifferential equations on the sphere when spherical splines are used. The results in this chapter have been reported in our article [61]. This preconditioner will then be used when solving the hypersingular integral equation with spherical splines in Chapter 6. Preconditioners of other types have been studied in [44, 45].

5.2 The meshes

Let $X := \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_K\}$ be a set of points on $\mathbb{S}^2$. We denote by Δ_h the spherical triangulation generated by X, which will be referred to as the fine mesh. We construct a coarse mesh Δ_H using another set of points Y such that $|\Delta_H| > |\Delta_h|$. A triangle in the fine mesh will be denoted by τ whereas a triangle in the coarse mesh will be denoted by τ_H. For each $\tau \in \Delta_h$, we denote by A_τ the area of τ. We are interested in small values of $|\Delta_h|$. We assume that both Δ_h and Δ_H are regular and quasi-uniform (see page 24). Here, the mesh sizes of Δ_h and Δ_H are denoted by h and H, respectively, i.e.

$$h = \tan\left(|\Delta_h|/2\right) \quad \text{and} \quad H = \tan\left(|\Delta_H|/2\right).$$

To accompany results used in [7, 54] we also define (see (2.5.15))

$$\varrho = \tan(\rho_{\Delta_h}/2), \quad h_\tau = \tan(|\tau|/2), \quad \text{and} \quad \varrho_\tau = \tan(\rho_\tau/2) \quad \text{for } \tau \in \Delta_h$$

and, similarly for the coarse mesh,

$$\varrho_H = \tan(\rho_{\Delta_H}/2), \quad H_{\tau_H} = \tan(|\tau_H|/2), \quad \text{and} \quad \varrho_{\tau_H} = \tan(\rho_{\tau_H}/2) \quad \text{for } \tau_H \in \Delta_H.$$

It is straightforward to see that since Δ_h and Δ_H are regular and quasi-uniform, there hold

$$\begin{aligned} \rho_\tau \simeq |\tau| \simeq |\Delta_h| \simeq h \simeq h_\tau \simeq \varrho_\tau \simeq \varrho \\ \rho_{\tau_H} \simeq |\tau_H| \simeq |\Delta_H| \simeq H \simeq H_{\tau_H} \simeq \varrho_{\tau_H} \simeq \varrho_H. \end{aligned} \tag{5.2.1}$$

Denoting by $\mathrm{star}^1(\boldsymbol{v})$ the union of all triangles in Δ_h that share the vertex $\boldsymbol{v}$, we define

$$\mathrm{star}^k(\boldsymbol{v}) := \cup\{\mathrm{star}^1(\boldsymbol{w}) : \boldsymbol{w} \text{ is a vertex of } \mathrm{star}^{k-1}(\boldsymbol{v})\}, \quad k > 1,$$

and

$$\mathrm{star}^k(\tau) := \cup\{\mathrm{star}^k(\boldsymbol{w}) : \boldsymbol{w} \text{ is a vertex of } \tau\}, \quad k \geq 1.$$

The following result is proved in [7, 54].

Lemma 5.1. *Under the assumptions that Δ_h is a regular, quasi-uniform triangulation satisfying $|\Delta_h| \leq 1$ we have, for any $\tau \in \Delta_h$,*

1. $A_\tau \simeq h^2$,

2. $\nu_k(\tau) \preceq (2k+1)^2$

where A_τ denotes the area of τ, and $\nu_k(\tau)$ denotes the number of triangles in $\mathrm{star}^k(\tau)$. Here, the constants depend only on the smallest angle of the triangulation.

5.3 The problem

In this chapter, we will frequently use the following radial projection. Let Ω be a subset of $\mathbb{S}^2$. We denote by $\mathbf{r}_\Omega$ the centre of a spherical cap of smallest possible radius containing Ω, and by Π_Ω the tangential plane touching $\mathbb{S}^2$ at $\mathbf{r}_\Omega$. For each point $\boldsymbol{x} \in \Omega$, the intersection of Π_Ω and the ray passing through the origin and $\boldsymbol{x}$ is denoted by $\overline{\boldsymbol{x}}$. We define

$$R(\Omega) := \{\overline{\boldsymbol{x}} \in \Pi_\Omega : \boldsymbol{x} \in \Omega\}. \tag{5.3.1}$$

The radial projection $\mathcal{R}_\Omega$ is defined by

$$\begin{aligned} \mathcal{R}_\Omega : R(\Omega) &\to \Omega \\ \overline{\boldsymbol{x}} &\mapsto \boldsymbol{x} := \overline{\boldsymbol{x}}/|\overline{\boldsymbol{x}}|. \end{aligned} \tag{5.3.2}$$

It is clear that $\mathcal{R}_\Omega$ is invertible.

Recall the definition of Sobolev space $H^k(\Omega)$ defined on a subset Ω of the unit sphere $\mathbb{S}^2$ in Section 2.2. In this chapter, we use the following definition of the Sobolev space $H^1(\Omega)$ defined on a subset $\Omega \subset \mathbb{S}^2$ as follows:

$$H^1(\Omega) := \{v \in L_2(\Omega) : \|v\|_{H^1(\Omega)} < \infty\},$$

which is equipped with a seminorm

$$|v|^2_{H^1(\Omega)} := \int_\Omega |\nabla_{\mathbb{S}^2} v|^2 d\sigma$$

and a norm

$$\|v\|^2_{H^1(\Omega)} := \int_\Omega |v|^2 d\sigma + |v|^2_{H^1(\Omega)}. \tag{5.3.3}$$

Here $\nabla_{\mathbb{S}^2}$ is the surface gradient (see (2.1.6)), and $d\sigma$ is the Lebesgue measure on $\mathbb{S}^2$. The norm (5.3.3) is equivalent to that defined by (2.2.5).

To set up a weak formulation for (5.1.1), we introduce the bilinear form

$$a(u, v) := \left\langle -\Delta_{\mathbb{S}^2} u + \omega^2 u, v \right\rangle.$$

Noting (2.1.7) and (5.3.3) we deduce

$$a(v, v) \simeq \|v\|^2_{H^1(\mathbb{S}^2)} \quad \forall v \in H^1(\mathbb{S}^2). \tag{5.3.4}$$

A natural weak formulation of the equation (5.1.1) is

$$a(u, v) = \langle g, v \rangle \quad \forall v \in H^1(\mathbb{S}^2).$$

The bilinear form is clearly bounded and coercive (cf. [9]). The Galerkin approximation problem is: Find $u \in S^r_d(\Delta_h)$ satisfying

$$a(u, v) = \langle g, v \rangle \quad \forall v \in S^r_d(\Delta_h). \tag{5.3.5}$$

Denoting $\{\Phi_i : i = 1, \ldots, N\}$ a basis for $S_d^r(\Delta_h)$, the problem (5.3.5) reduces to the problem of solving the following linear system

$$\boldsymbol{A}c = \boldsymbol{g},$$

where for $i, j = 1, ..., N$, the entries of the matrix $\boldsymbol{A}$ are given as $A_{ij} = a(\Phi_i, \Phi_j)$ and the vector $\mathbf{f}$ is given as $\boldsymbol{g} = [g_i]_{i=1}^N$ in which $g_i = \langle g, \Phi_i \rangle$.

It is well known that the matrix $\boldsymbol{A}$ is ill-conditioned, namely, the condition number of $\boldsymbol{A}$, defined by $\kappa(\boldsymbol{A}) := \lambda_{\max}(\boldsymbol{A})/\lambda_{\min}(\boldsymbol{A})$, grows like h^{-2} as $h \to 0$ (i.e. $|\Delta_h| \to 0$). Since we cannot find a reference for this seemingly well-known result, we include the proof here for completeness. We will use the following two results from [7] to achieve this.

Lemma 5.2. *[7, Lemma 5] Let p be a homogeneous polynomial of degree d on a spherical triangle τ. If p is written in Bernstein-Bézier form as*

$$p(\boldsymbol{v}) = \sum_{i+j+k=d} c_{ijk} B_{ijk}^{d,\tau}(\boldsymbol{v}), \quad \boldsymbol{v} \in \tau,$$

then

$$\|p\|_{L_2(\tau)} \simeq A_\tau^{1/2} \|\mathbf{c}_\tau\|,$$

of τ and p. Here A_τ is the area of τ and $\mathbf{c}_\tau$ is the vector of components c_{ijk}, $i+j+k = d$.

Lemma 5.3. *[7, Lemma 6] Let p be a homogeneous polynomial of degree d on a spherical triangle τ in Δ_h. Then there holds*

$$|p|_{H^1(\tau)} \preceq \varrho_\tau^{-1} \|p\|_{L_2(\tau)}.$$

We can now prove a bound for the condition number of $\boldsymbol{A}$.

Proposition 5.4. *The condition number of the stiffness matrix $\boldsymbol{A}$ is bounded by*

$$\kappa(\boldsymbol{A}) \preceq h^{-2}.$$

Proof. Recall that $\{\Phi_i\}_{i=1}^N$ is a basis for $S_d^r(\Delta_h)$. Let $\mathbf{c} = [c_i]_{i=1}^N \in \mathbb{R}^N$. We define $u := \sum_{i=1}^N c_i \Phi_i \in S_d^r(\Delta_h)$. Noting (5.3.4), we have

$$\mathbf{c}^T \boldsymbol{A} \mathbf{c} = a(u, u) \simeq \|u\|_{H^1(\mathbb{S}^2)}^2. \tag{5.3.6}$$

From (5.3.6), Lemma 5.3 and (5.2.1), we have

$$\mathbf{c}^T \boldsymbol{A} \mathbf{c} \preceq \sum_{\tau \in \Delta_h} \left(|u|_{H^1(\tau)}^2 + \|u\|_{L_2(\tau)}^2 \right) \preceq h^{-2} \sum_{\tau \in \Delta_h} \|u\|_{L_2(\tau)}^2.$$

It follows from Lemmas 5.2 and 5.1 (i) that

$$\mathbf{c}^T \boldsymbol{A} \mathbf{c} \preceq h^{-2} \sum_{\tau \in \Delta_h} A_\tau \|\mathbf{c}_\tau\|^2 \preceq \sum_{\tau \in \Delta_h} \|\mathbf{c}_\tau\|^2.$$

Here, $\mathbf{c}_\tau$ is a vector whose components are that of the vector $\mathbf{c}$ corresponding to the basis functions Φ_j whose supports contain τ. By using Lemma 5.1 (ii) and noting (5.2.1) we obtain

$$\mathbf{c}^T \boldsymbol{A} \mathbf{c} \preceq \max_{\tau \in \Delta_h} \{\#\mathrm{star}^1(\tau)\} \|\mathbf{c}\|^2 \preceq \|\mathbf{c}\|^2 = \mathbf{c}^T \mathbf{c}$$

which gives

$$\lambda_{\max}(\boldsymbol{A}) \preceq 1.$$

Using (5.3.6), Lemma 5.2 and Lemma 5.1 (i) again, we have

$$\mathbf{c}^T \boldsymbol{A} \mathbf{c} \succeq \sum_{\tau \in \Delta_h} \|u\|^2_{L_2(\tau)} \succeq \sum_{\tau \in \Delta_h} A_\tau \|\mathbf{c}_\tau\|^2 \simeq h^2 \mathbf{c}^T \mathbf{c},$$

implying

$$\lambda_{\min}(\boldsymbol{A}) \succeq h^2.$$

The bound for $\kappa(\boldsymbol{A})$ can now be derived. □

5.4 Abstract framework of additive Schwarz methods

Additive Schwarz methods provide fast solutions to (5.3.5) by solving, at the same time, problems of smaller size. Let the space $V = S^r_d(\Delta_h)$ be decomposed as

$$V = V_0 + \cdots + V_J, \tag{5.4.1}$$

where V_i, $i = 0, ..., J$, are subspaces of V, and let $P_i : V \to V_i$, $i = 0, ..., J$, be projections defined by

$$a(P_i v, w) = a(v, w) \quad \forall v \in V, \quad \forall w \in V_i. \tag{5.4.2}$$

If we define

$$P := P_0 + \cdots + P_J, \tag{5.4.3}$$

then the additive Schwarz method for (5.3.5) consists of solving, by an iterative method, the equation

$$Pu = f, \tag{5.4.4}$$

where $f = \sum_{i=0}^J f_i$, with $f_i \in V_i$ being solutions of

$$a(f_i, w) = \langle g, w \rangle \quad \forall w \in V_i. \tag{5.4.5}$$

The equivalence of (5.3.5) and (5.4.4) was discussed in [77]. For completeness, we briefly explain that equivalence here. Let u_h be a solution of (5.3.5). From the definition of P_i and f_i we deduce

$$a(P_i u_h, v) = a(u_h, v) = \langle g, v \rangle = a(f_i, v) \quad \forall v \in V,$$

i.e. $P_i u_h = f_i$. Hence $Pu_h = f$. On the other hand, if $P : V \to V$ is invertible and u_h is a solution of (5.4.4), then by using successively the symmetry of P, (5.4.2) and (5.4.5), we obtain

$$\begin{aligned} a(u_h, v) = a(P^{-1} f, v) = a(f, P^{-1} v) &= \sum_{i=1}^J a(f_i, P^{-1} v) = \sum_{i=1}^J a(f_i, P_i P^{-1} v) \\ &= \sum_{i=1}^J \left\langle g, P_i P^{-1} v \right\rangle = \langle g, v \rangle \quad \text{for any } v \in V. \end{aligned}$$

A practical method to solve (5.4.4) is the conjugate gradient method; the additive Schwarz method can be viewed as a preconditioned conjugate gradient method.

Bounds for eigenvalues the condition number of the additive Schwarz operator P, can be obtained by using the following lemma; see [74].

Lemma 5.5. *Assume that for any $u \in V$ satisfying $u = \sum_{i=0}^{J} u_i$ with $u_i \in V_i$ for $i = 0, ..., J$ there holds*

$$a(u,u) \leq C_1 \sum_{i=0}^{J} a(u_i, u_i). \tag{5.4.6}$$

Assume further that for any $u \in V$, there exists a decomposition $u = \sum_{i=0}^{J} u_i'$ with $u_i' \in V_i$ for $i = 0, ..., J$ satisfying

$$\sum_{i=0}^{J} a(u_i', u_i') \leq C_2 a(u,u). \tag{5.4.7}$$

Then the extremal eigenvalues of the additive Schwarz operator P are bounded by

$$C_2^{-1} \leq \lambda_{\min}(P) \leq \lambda_{\max}(P) \leq C_1$$

and thus the condition number $\kappa(P) = \lambda_{\max}(P)/\lambda_{\min}(P)$ is bounded by

$$\kappa(P) \leq C_1 C_2.$$

5.5 Additive Schwarz method for the Laplace-Beltrami equation on the unit sphere

In this section we will define a subspace decomposition of the form (5.4.1), and in this way define the additive Schwarz operator for problem (5.3.5). Let $V_0' = S_d^r(\Delta_H)$ and $V_j = V \cap H_0^1(\Omega_j)$ for $j = 1, ..., J$, where Ω_j are overlapping subdomains (regarded as open) which will be defined below. Here $H_0^1(\Omega_j) := \{u \in\ L_2(\Omega_j) : \|u\|_{H^1(\Omega_j)} < \infty$ and $u = 0$ on $\partial\Omega_j\}$, where $\partial\Omega_j$ denotes the boundary of Ω_j. Since V_0' is not a subspace of V we define the coarse space $V_0 = \tilde{I}^h V_0'$ where $\tilde{I}^h$ is a quasi-interpolant over Δ_h which is defined as in (2.5.22). The use of a quasi-interpolant as opposed to a "regular" interpolant is to allow the use of results in [54]. We now have decomposed V as in (5.4.1) and can hence define the Schwarz operator P by (5.4.2) and (5.4.3).

We construct one subdomain for each triangle in Δ_H, hence the number of subdomains is J, the number of triangles in Δ_H. Consider a triangle $\tau_H^j \in \Delta_H$, $j = 1, \ldots, J$. The subdomain corresponding to τ_H^j is given by

$$\Omega_j = \cup\{\tau \in \Delta_h : \bar{\tau} \cap \bar{\tau}_H^j \neq \phi\}, \quad j = 1, ..., J. \tag{5.5.1}$$

Since triangles in the fine mesh can intersect more than one triangle in the coarse mesh, the subdomains are overlapping.

It will be assumed that the subdomains can be coloured using at most M colours in such a way that subdomains with the same colour are disjoint. It is clear that M depends on the smallest angle of the triangulation. We also assume, as in [74], that for $j = 1, ..., J$, there exists $\delta_j > 0$ such that for any $\boldsymbol{x} \in \Omega_j$ there exists $i \in \{1, ..., J\}$ satisfying

$$\boldsymbol{x} \in \Omega_i \quad \text{and} \quad \operatorname{dist}(\boldsymbol{x}, \partial\Omega_i) := \min_{\boldsymbol{y} \in \partial\Omega_i} \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) \geq \delta_j. \tag{5.5.2}$$

Here δ_j measures the amount of overlap in Ω_j and this assumption ensures that the overlap of the subdomains is small. We denote

$$\Omega_{j,\delta_j} := \{\boldsymbol{x} \in \Omega_j : \operatorname{dist}(\boldsymbol{x}, \partial\Omega_j) \leq \delta_j\}. \tag{5.5.3}$$

In the proceeding sections a lower bound for the minimum eigenvalue of P will be obtained by using the quasi-interpolation operators $\tilde{I}^h$ and $\tilde{I}^H$ relative to the fine mesh Δ_h and the coarse mesh Δ_H, and a family of functions associated with our set of overlapping subdomains that form a partition of unity. For the definition of the quasi-interpolation operator, please refer to (2.5.22).

The following two lemmas consist of stability and approximation results for these quasi-interpolants. They state the results for $\tilde{I}^h$ but similar results for $\tilde{I}^H$ will also be used in the remainder of the chapter.

Lemma 5.6. *For any $\tau \in \Delta_h$, let $\omega_\tau := \bigcup_{i \in I_\tau} \omega_i$, where $\omega_i := supp(\Phi_i)$ and $I_\tau := \big\{i \in \{1, \ldots, N\} : \tau \in \omega_i\big\}$. Then for $v \in H^k(\mathbb{S}^2)$, $k = 0, 1$, there hold*

(i) $|\tilde{I}^h v|_{H^k(\tau)} \preceq h^{-k} \|v\|_{L_2(\omega_\tau)}$,

(ii) $|\tilde{I}^h v|_{H^k(\mathbb{S}^2)} \preceq h^{-k} \|v\|_{L_2(\mathbb{S}^2)}$.

Here, the constants depend only on the smallest angle Θ_{Δ_h} of Δ_h and the polynomial degree d.

Proof. The proof for (i) can be found in [54, Proposition 5.2]. The following is the proof for (ii) when $k = 0$. By (i) we have

$$\|\tilde{I}^h v\|_{L_2(\tau)} \preceq \|v\|_{L_2(\omega_\tau)}.$$

It follows that

$$\begin{aligned}
\|\tilde{I}^h v\|^2_{L_2(\mathbb{S}^2)} &= \sum_{\tau \in \Delta_h} \|\tilde{I}^h v\|^2_{L_2(\tau)} \preceq \sum_{\tau \in \Delta_h} \|v\|^2_{L_2(\omega_\tau)} \\
&= \sum_{\tau \in \Delta_h} \sum_{\substack{\tau' \subset \omega_\tau \\ \tau' \in \Delta_h}} \|v\|^2_{L_2(\tau')} = \sum_{\tau' \in \Delta_h} \#\{\tau : \tau' \subset \omega_\tau\} \|v\|^2_{L_2(\tau')}.
\end{aligned}$$

Noting that $\#\{\tau : \tau' \subset \omega_\tau\} \leq \#\text{star}^2(\tau')$, by using Lemma 5.1 (ii) we deduce

$$\|\tilde{I}^h v\|^2_{L_2(\mathbb{S}^2)} \preceq \max_{\tau' \in \Delta_h} \{\#\text{star}^2(\tau')\} \|v\|^2_{L_2(\mathbb{S}^2)} \preceq \|v\|^2_{L_2(\mathbb{S}^2)},$$

proving (ii) for the case $k = 0$. A similar argument can be used to obtain the result for $k = 1$. $\square$

Lemma 5.7. *There exists a constant depending on Θ_{Δ_h}, the smallest angle in Δ_h, and the polynomial degree d such that for all $v \in H^1(\mathbb{S}^2)$*

$$\|v - \tilde{I}^h v\|_{L_2(\mathbb{S}^2)} \preceq h\|v\|_{H^1(\mathbb{S}^2)}.$$

Proof. Theorem 4.3 in Chapter 4 tells us that for all $v \in H^1(\mathbb{S}^2)$, there exists a spherical spline $\eta \in S_d^r(\Delta_h)$ such that

$$\|v - \eta\|_{L_2(\mathbb{S}^2)} \preceq h\|v\|_{H^1(\mathbb{S}^2)}. \tag{5.5.4}$$

where the constant depends on Θ_{Δ_h} and d. By the linearity of $\tilde{I}^h$ and the fact that $\tilde{I}^h$ reproduces functions in $S_d^r(\Delta_h)$, we can write

$$\|v - \tilde{I}^h v\|_{L_2(\mathbb{S}^2)} \le \|v - \eta\|_{L_2(\mathbb{S}^2)} + \|\tilde{I}^h(v - \eta)\|_{L_2(\mathbb{S}^2)}.$$

From Lemma 5.6 and (5.5.4) we deduce

$$\|v - \tilde{I}^h v\|_{L_2(\mathbb{S}^2)} \preceq \|v - \eta\|_{L_2(\mathbb{S}^2)} \preceq h\|v\|_{H^1(\mathbb{S}^2)},$$

proving the lemma. □

To construct a decomposition of each $u \in V$, we introduce a partition of unity as follows. Let

$$d_j(\boldsymbol{x}) = \begin{cases} \operatorname{dist}(\boldsymbol{x}, \partial\Omega_j), & \boldsymbol{x} \in \Omega_j \\ 0, & \boldsymbol{x} \notin \Omega_j \end{cases}.$$

The partition of unity $\{\theta_j\}_{j=1}^J$ is defined by

$$\theta_j(\boldsymbol{x}) = \frac{d_j(\boldsymbol{x})}{\sum_{k=1}^J d_k(\boldsymbol{x})}, \quad \boldsymbol{x} \in \mathbb{S}^2.$$

The following lemma proves an important property of our partition of unity.

Lemma 5.8. *There exists a constant C such that for $j = 1, ..., J$*

$$|\nabla_{\mathbb{S}^2}\theta_j(\boldsymbol{x})| \le \frac{CM}{\delta_j}, \quad \forall \boldsymbol{x} \in \mathbb{S}^2, \tag{5.5.5}$$

where δ_j, which measures the amount of overlap in Ω_j, is given in (5.5.2).

Proof. This lemma was proved in [74] for a bounded polygonal or polyhedral domain. We employ a similar technique to prove the result for the sphere.

Consider an arbitrary but fixed j. Since $\nabla_{\mathbb{S}^2}\theta_j(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \notin \Omega_j$ we only need to prove (5.5.5) for $\boldsymbol{x} \in \Omega_j$. We will actually prove

$$|\theta_j(\boldsymbol{x}) - \theta_j(\boldsymbol{y})| \preceq \frac{M}{\delta_j}|\boldsymbol{x} - \boldsymbol{y}| \quad \forall \boldsymbol{x}, \boldsymbol{y} \in \Omega_j, \tag{5.5.6}$$

where $\boldsymbol{y}$ is sufficiently close to $\boldsymbol{x}$. To see that (5.5.6) implies (5.5.5), we extend θ_j to ψ_j defined on

$$\mathcal{A}_j := \left\{ \mathbf{u} \in \mathbb{R}^3 : \frac{1}{2} \le |\mathbf{u}| \le \frac{3}{2} \text{ and } \frac{\mathbf{u}}{|\mathbf{u}|} \in \Omega_j \right\}$$

as follows. Let $\psi_j(\mathbf{u}) = \theta_j(\boldsymbol{x})$ for all $\mathbf{u} \in \mathcal{A}_j$, where $\boldsymbol{x} = \mathbf{u}/|\mathbf{u}|$. In fact ψ_j is the homogeneous extension of θ_j of degree 0 to $\mathcal{A}_j$. Noting that $\nabla_{\mathbb{S}^2}\theta_j = \nabla_{\mathbb{S}^2}\left(\psi_j|_{\mathbb{S}}^2\right) = (\nabla\psi_j)|_{\mathbb{S}^2}$ we obtain (5.5.5) if we have

$$|\nabla\psi_j(\mathbf{u})| \preceq \frac{M}{\delta_j} \quad \forall \mathbf{u} \in \mathcal{A}_j, \tag{5.5.7}$$

or equivalently

$$|\psi_j(\mathbf{u}) - \psi_j(\mathbf{v})| \preceq \frac{M}{\delta_j}|\mathbf{u} - \mathbf{v}| \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{A}_j, \tag{5.5.8}$$

where $\mathbf{v}$ is sufficiently close to $\mathbf{u}$. Noting that $|\psi_j(\mathbf{u}) - \psi_j(\mathbf{v})| = |\theta_j(\boldsymbol{x}) - \theta_j(\boldsymbol{y})|$, where $\mathbf{u}, \mathbf{v} \in \mathcal{A}_j$, $\boldsymbol{x} = \mathbf{u}/|\mathbf{u}|$, and $\boldsymbol{y} = \mathbf{v}/|\mathbf{v}|$, and that $|\boldsymbol{x} - \boldsymbol{y}| \le 2|\mathbf{u} - \mathbf{v}|$, we see that (5.5.6) gives us (5.5.8) which in turn yields (5.5.7) and therefore (5.5.5).

We now prove (5.5.6). Recalling that $\boldsymbol{x}, \boldsymbol{y} \in \Omega_j$ we note that (5.5.6) holds if the following formulae are proved:

$$\theta_j(\boldsymbol{x}) - \theta_j(\boldsymbol{y}) = \frac{1}{\sum_{k=1}^J d_k(\boldsymbol{y})}\left(\theta_j(\boldsymbol{x})\tilde{\eta}_j(\boldsymbol{y}, \boldsymbol{x}) + \tilde{\theta}_j(\boldsymbol{x})\eta_j(\boldsymbol{x}, \boldsymbol{y})\right), \tag{5.5.9}$$

$$\sum_{k=1}^J d_k(\boldsymbol{x}) \ge \delta_j, \tag{5.5.10}$$

$$|\eta_k(\boldsymbol{x}, \boldsymbol{y})| \preceq |\boldsymbol{x} - \boldsymbol{y}|, \quad k = 1, ..., J, \tag{5.5.11}$$

and

$$|\tilde{\eta}_j(\boldsymbol{x}, \boldsymbol{y})| \preceq M|\boldsymbol{x} - \boldsymbol{y}|, \tag{5.5.12}$$

where

$$\eta_k(\boldsymbol{x}, \boldsymbol{y}) := d_k(\boldsymbol{x}) - d_k(\boldsymbol{y}), \quad \tilde{\eta}_j(\boldsymbol{x}, \boldsymbol{y}) := \sum_{\substack{k=1 \\ k \ne j}}^J \eta_k(\boldsymbol{x}, \boldsymbol{y}), \quad \tilde{\theta}_j(\boldsymbol{x}) := 1 - \theta_j(\boldsymbol{x}). \tag{5.5.13}$$

Indeed, assuming (5.5.9)–(5.5.12) hold we have

$$\begin{aligned}
|\theta_j(\boldsymbol{x}) - \theta_j(\boldsymbol{y})| &= \left|\frac{1}{\sum_{k=1}^J d_k(\boldsymbol{y})}\left(\theta_j(\boldsymbol{x})\tilde{\eta}_j(\boldsymbol{y}, \boldsymbol{x}) + \tilde{\theta}_j(\boldsymbol{x})\eta_j(\boldsymbol{x}, \boldsymbol{y})\right)\right| \\
&\le \frac{1}{\delta_j}\left(\theta_j(\boldsymbol{x})|\tilde{\eta}_j(\boldsymbol{y}, \boldsymbol{x})| + \tilde{\theta}_j(\boldsymbol{x})|\eta_j(\boldsymbol{x}, \boldsymbol{y})|\right) \\
&\preceq \frac{M}{\delta_j}\left(\theta_j(\boldsymbol{x})|\boldsymbol{x} - \boldsymbol{y}| + \tilde{\theta}_j(\boldsymbol{x})|\boldsymbol{x} - \boldsymbol{y}|\right) = \frac{M}{\delta_j}|\boldsymbol{x} - \boldsymbol{y}|.
\end{aligned}$$

We will now prove (5.5.9)–(5.5.12). From (5.5.13) we have

$$\begin{aligned}
\theta_j(\boldsymbol{x})\tilde{\eta}_j(\boldsymbol{y}, \boldsymbol{x}) + \tilde{\theta}_j(\boldsymbol{x})\eta_j(\boldsymbol{x}, \boldsymbol{y}) &= \theta_j(\boldsymbol{x})\sum_{\substack{k=1 \\ k \ne j}}^J (d_k(\boldsymbol{y}) - d_k(\boldsymbol{x})) + (1 - \theta_j(\boldsymbol{x}))(d_j(\boldsymbol{x}) - d_j(\boldsymbol{y})) \\
&= \theta_j(\boldsymbol{x})\sum_{k=1}^J (d_k(\boldsymbol{y}) - d_k(\boldsymbol{x})) + d_j(\boldsymbol{x}) - d_j(\boldsymbol{y}) \\
&= \sum_{k=1}^J d_k(\boldsymbol{y})(\theta_j(\boldsymbol{x}) - \theta_j(\boldsymbol{y})),
\end{aligned}$$

where in the final step we use $\theta_j(\boldsymbol{x})\sum_{k=1}^{J} d_k(\boldsymbol{x}) = d_j(\boldsymbol{x})$ twice. Hence (5.5.9) is proved. The property (5.5.10) is obvious from the assumption (5.5.2).

We now prove (5.5.11) recalling that $\boldsymbol{x}, \boldsymbol{y} \in \Omega_j$. Noting that our subdomains were defined as open, for the case when $\boldsymbol{x}, \boldsymbol{y} \notin \Omega_k$ we have $\eta_k(\boldsymbol{x}, \boldsymbol{y}) = 0$ and hence (5.5.11) is trivial. Consider now the case $\boldsymbol{x}, \boldsymbol{y} \in \Omega_k$. Let $\mathbf{z} \in \partial\Omega_k$ be such that $d_k(\boldsymbol{y}) = \cos^{-1}(\boldsymbol{y} \cdot \mathbf{z})$. Then

$$d_k(\boldsymbol{x}) \le \cos^{-1}(\boldsymbol{x} \cdot \mathbf{z}) \le \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) + \cos^{-1}(\boldsymbol{y} \cdot \mathbf{z}) = \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) + d_k(\boldsymbol{y})$$

which gives $\eta_k(\boldsymbol{x}, \boldsymbol{y}) \le \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y})$. Similarly $d_k(\boldsymbol{y}) \le \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) + d_k(\boldsymbol{x})$ and hence $|\eta_k(\boldsymbol{x}, \boldsymbol{y})| \le \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y})$. By simple geometry it can be shown for $\mathbf{a}, \mathbf{b} \in \mathbb{S}^2$ satisfying $\cos^{-1}(\mathbf{a} \cdot \mathbf{b}) \le \frac{\pi}{2}$ that

$$|\mathbf{a} - \mathbf{b}| \simeq \cos^{-1}(\mathbf{a} \cdot \mathbf{b}). \tag{5.5.14}$$

Since $\boldsymbol{y}$ is sufficiently close to $\boldsymbol{x}$ we have $|\eta_k(\boldsymbol{x}, \boldsymbol{y})| \preceq |\boldsymbol{x} - \boldsymbol{y}|$ and we obtain (5.5.11) for this case. Finally if $\boldsymbol{x} \notin \Omega_k$ and $\boldsymbol{y} \in \Omega_k$ we have

$$d_k(\boldsymbol{x}) = 0 \le \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) + d_k(\boldsymbol{y}) \quad \text{and} \quad d_k(\boldsymbol{y}) \le \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}).$$

Then

$$|\eta_k(\boldsymbol{x}, \boldsymbol{y})| = |d_k(\boldsymbol{y})| \le \cos^{-1}(\boldsymbol{x} \cdot \boldsymbol{y}) \preceq |\boldsymbol{x} - \boldsymbol{y}|$$

which gives us (5.5.11).

Since there are at most M values of k such that $d_k(\boldsymbol{x}) \ne 0$ (by the colouring assumption), we have at most M values of k such that $\eta_k(\boldsymbol{x}, \boldsymbol{y}) \ne 0$. Then (5.5.12) follows from (5.5.11). □

5.6 Main results

In this section we prove a bound on the condition number of P by using the abstract result in Lemma 5.5. We first prove (5.4.6).

Lemma 5.9. *There exists a positive constant C independent of Δ_h such that for any $u \in V$ satisfying $u = \sum_{j=0}^{J} u_j$ with $u_j \in V_j$ for $j = 0, \ldots, J$, there holds*

$$a(u, u) \le C \sum_{j=0}^{J} a(u_j, u_j),$$

where the constant C depends on the smallest angle of the triangulation.

Proof. By our assumption on the colouring of the subdomains there are at most M subdomains to which any $\boldsymbol{x} \in \mathbb{S}^2$ can belong. By a standard colouring argument we have

$$a(u, u) \preceq \|u\|_{H^1(\mathbb{S}^2)}^2 \le \|u_0\|_{H^1(\mathbb{S}^2)}^2 + \left\| \sum_{j=1}^{J} u_j \right\|_{H^1(\mathbb{S}^2)}^2$$

$$\preceq \|u_0\|^2_{H^1(\mathbb{S}^2)} + M\sum_{j=1}^J \|u_j\|^2_{H^1(\mathbb{S}^2)} \preceq \sum_{j=0}^J a(u_j,u_j).$$

The lemma is proved. □

We note that standard analysis as used in [74] cannot be directly used to prove (5.4.7). There are two levels of difficulties. First, we have to work with homogeneous polynomials on spherical triangles which do not have all properties of polynomials on planar triangles. Second, the space $S^r_d(\Delta_h)$ does not contain constant functions when d is odd.

In the following subsection, we prove (5.4.7) by using a different approach. The result is not as sharp as expected. In Subsection 5.6.2 we prove a better estimate when d is even so that the space $S^r_d(\Delta_h)$ contains constant functions.

5.6.1 A rough estimate for $\kappa(P)$

To prove (5.4.7), we need to introduce an operator P_H from $H^1(\mathbb{S}^2)$ into $S^r_d(\Delta_H)$ defined by

$$a(P_H u, v) = a(u,v) \quad \forall v \in S^r_d(\Delta_H)$$

for any $u \in H^1(\mathbb{S}^2)$. Standard finite arguments yield

$$\begin{aligned}
\|P_H u - u\|_{H^1(\mathbb{S}^2)} &\preceq \|u - v\|_{H^1(\mathbb{S}^2)} \quad \forall v \in S^r_d(\Delta_H)\\
\|P_H u - u\|_{H^1(\mathbb{S}^2)} &\preceq \|u\|_{H^1(\mathbb{S}^2)}\\
\|P_H u\|_{H^1(\mathbb{S}^2)} &\preceq \|u\|_{H^1(\mathbb{S}^2)}\\
\|P_H u - u\|_{L_2(\mathbb{S}^2)} &\preceq H\|u\|_{H^1(\mathbb{S}^2)}.
\end{aligned} \tag{5.6.1}$$

Lemma 5.10. *There exists a positive constant C depending on the smallest angle of Δ_h and the polynomial degree d such that for any $u \in V$ there exist $u_j \in V_j, j = 0, ..., J$, satisfying $u = \sum_{j=0}^J u_j$ and*

$$\sum_{j=0}^J a(u_j,u_j) \le C\left(\frac{H}{h}\right)^2 a(u,u). \tag{5.6.2}$$

Proof. Let $u_0' := P_H u$ and $u_0 = \tilde{I}^h u_0'$. We define $w := u - u_0$, and $u_j := \tilde{I}^h(\theta_j w)$ for $j = 1, ..., J$. Since $\mathrm{supp}(\theta_j) \subseteq \Omega_j$ there holds $u_j \in V_j$ for all $j = 1, \ldots, J$. Moreover,

$$u_0 + u_1 + \cdots u_J = u_0 + \tilde{I}^h\left(\sum_{j=1}^J \theta_j w\right) = u_0 + \tilde{I}^h u - \tilde{I}^h u_0 = u_0 + u - u_0 = u.$$

By Lemma 5.6, we have

$$\|\tilde{I}^h(P_H u - u)\|_{H^1(\mathbb{S}^2)} \preceq h^{-1}\|P_H u - u\|_{L_2(\mathbb{S}^2)}. \tag{5.6.3}$$

By using the triangle inequality, (5.6.3) and (5.6.1), we have, noting that $\tilde{I}^h u = u$,

$$\begin{aligned}
a(u_0,u_0) \simeq \|u_0\|^2_{H^1(\mathbb{S}^2)} &\le \left(\|\tilde{I}^h(P_H u - u)\|_{H^1(\mathbb{S}^2)} + \|u\|_{H^1(\mathbb{S}^2)}\right)^2\\
&\preceq \left(h^{-1}\|P_H u - u\|_{L_2(\mathbb{S}^2)} + \|u\|_{H^1(\mathbb{S}^2)}\right)^2\\
&\preceq \left(\frac{H}{h}\|u\|_{H^1(\mathbb{S}^2)}\right)^2 \simeq \left(\frac{H}{h}\right)^2 a(u,u).
\end{aligned} \tag{5.6.4}$$

Noting the support of θ_j and involving Lemma 5.6 again, we have

$$\begin{aligned} a(u_j, u_j) &\simeq \|\tilde{I}^h(\theta_j w)\|^2_{H^1(\mathbb{S}^2)} = \|\tilde{I}^h(\theta_j w)\|^2_{H^1(\Omega_j)} \\ &= \sum_{\substack{\tau\subset\Omega_j \\ \tau\in\Delta_h}} \|\tilde{I}^h(\theta_j w)\|^2_{H^1(\tau)} \preceq h^{-2} \sum_{\substack{\tau\subset\Omega_j \\ \tau\in\Delta_h}} \|\theta_j w\|^2_{L_2(\omega_\tau)} \\ &\preceq h^{-2} \sum_{\substack{\tau\subset\Omega_j \\ \tau\in\Delta_h}} \|w\|^2_{L_2(\omega_\tau)} \preceq h^{-2}\|w\|^2_{L_2(\Omega_j)}. \end{aligned}$$

Summing up over $j = 1, \ldots, J$, we obtain

$$\sum_{j=1}^{J} a(u_j, u_j) \preceq h^{-2}\|w\|^2_{L_2(\mathbb{S}^2)}. \tag{5.6.5}$$

Lemma 5.6 and (5.6.1) give

$$\begin{aligned} \|w\|_{L_2(\mathbb{S}^2)} &= \|u - \tilde{I}^h(P_H u)\|_{L_2(\mathbb{S}^2)} = \|\tilde{I}^h(u - P_H u)\|_{L_2(\mathbb{S}^2)} \\ &\preceq \|u - P_H u\|_{L_2(\mathbb{S}^2)} \preceq H\|u\|_{H^1(\mathbb{S}^2)}. \end{aligned}$$

This together with (5.6.5) gives

$$\sum_{j=1}^{J} a(u_j, u_j) \preceq \left(\frac{H}{h}\right)^2 a(u, u). \tag{5.6.6}$$

We now obtain from (5.6.4) and (5.6.6)

$$\sum_{j=0}^{J} a(u_j, u_j) \preceq \left(\frac{H}{h}\right)^2 a(u, u),$$

completing the proof. □

Combining the results in Lemmas 5.9, 5.10 and 5.5 we obtain a bound for the condition number $\kappa(P)$ of the additive Schwarz operator.

Theorem 5.11. *The condition number of the additive Schwarz operator P is bounded by*

$$\kappa(P) \le C\left(\frac{H}{h}\right)^2,$$

where C is a constant depending on the smallest angle in Δ_h and the polynomial degree d.

Recall that we have chosen the overlap δ to be proportional to h.

5.6.2 An improved estimate for $\kappa(P)$ for even degree splines

In this subsection we find a better upper bound for $\kappa(P)$ when d is even. First we introduce spherical versions of Poincaré and Friedrichs type inequalities, which have been proved for open subsets of $\mathbb{R}^n$; see e.g. [74]. These results provide tools to prove (5.4.7).

Proposition 5.12. *Let $\Omega \subseteq \mathbb{S}^2$ be a Lipschitz domain and let f_i, $i = 1, \ldots, L$, $L \geq 1$, be functionals (not necessarily linear) in $H^1(\Omega)$, such that, if v is constant in Ω,*

$$\sum_{i=1}^{L} |f_i(v)|^2 = 0 \quad \Longleftrightarrow \quad v = 0.$$

Then, there exist constants depending only on Ω and the functionals f_i, such that, for $u \in H^1(\Omega)$,

$$\|u\|^2_{L_2(\Omega)} \leq C_1 |u|^2_{H^1(\Omega)} + C_2 \sum_{i=1}^{L} |f_i(u)|^2.$$

Proof. For any $u \in H^1(\Omega)$, we denote by $\bar{u} \in H^1(R(\Omega))$ the restriction on $R(\Omega)$ (see (5.3.1)) of the homogeneous extension of degree 0 of u. It is obvious that $\bar{u} = u \circ R_\Omega$ where R_Ω is the radial projection defined in (5.3.2). For any $i = 1, \ldots, L$, we define the functional $\bar{f}_i : H^1(R(\Omega)) \to \mathbb{R}$ by

$$\bar{f}_i(\bar{u}) = f_i(u) \quad \forall \bar{u} \in H^1(R(\Omega)).$$

Let $\bar{v}$ be a constant function in $R(\Omega)$. Then $v = \bar{v} \circ R_\Omega^{-1}$ is a constant function in Ω. We have

$$\begin{aligned} \sum_{i=1}^{L} |\bar{f}_i(\bar{v})|^2 = 0 \quad &\Leftrightarrow \quad \sum_{i=1}^{L} |f_i(v)|^2 = 0 \\ &\Leftrightarrow \quad v = 0 \\ &\Leftrightarrow \quad \bar{v} = 0. \end{aligned}$$

By Theorem A.12 in [74], there exist constants C_1' and C_2' depending only on $R(\Omega)$ and functional $\bar{f}_i$, such that, for $\bar{u} \in H^1(R(\Omega))$,

$$\|\bar{u}\|^2_{L_2(R(\Omega))} \leq C_1' |\bar{u}|^2_{H^1(R(\Omega))} + C_2' \sum_{i=1}^{L} |\bar{f}_i(\bar{u})|^2.$$

By the definition of $\bar{f}_i$ and noting that $\|u\|_{L_2(\Omega)} \simeq \|\bar{u}\|_{L_2(R(\Omega))}$ and $|u|_{H^1(\Omega)} \simeq |\bar{u}|_{H^1(R(\Omega))}$, see [54], we have

$$\|u\|^2_{L_2(\Omega)} \leq C_1 |u|^2_{H^1(\Omega)} + C_2 \sum_{i=1}^{L} |f_i(u)|^2.$$

The lemma is proved. □

The above lemma, together with a scaling argument as in the case of open sets in $\mathbb{R}^n$ (see e.g. [74]), yields the spherical versions of Poincaré and Friedrichs type inequalities.

Lemma 5.13. *Let $\Omega \subseteq \mathbb{S}^2$ be a Lipschitz domain with diameter H. Then, there exist constants C, C_1 and C_2, depending only on the shape of Ω but not on H, such that*

$$\|u\|_{L_2(\Omega)} \leq CH|u|_{H^1(\Omega)}$$

for $u \in H^1(\Omega)$ with vanishing mean value on Ω (i.e. $\int_\Omega u\, d\sigma = 0$). Moreover, if $\Gamma \subseteq \partial\Omega$ has a length of order H, then

$$\|u\|^2_{L_2(\Omega)} \le C_1 H^2 |u|^2_{H^1(\Omega)} + C_2 H \|u\|^2_{L_2(\Gamma)}$$

for $u \in H^1(\Omega)$.

Using Lemma 5.13 we can prove the following result.

Lemma 5.14. *There exists a constant C depending on the smallest angle of the triangulation such that for all $u \in H^1(\Omega_j)$, $j = 1, ..., J$, there holds*

$$\|u\|^2_{L_2(\Omega_{j,\delta_j})} \le C\delta_j^2 \left(\left(1 + \frac{|\Omega_j|}{\delta_j}\right) |u|^2_{H^1(\Omega_j)} + \frac{1}{|\Omega_j|\delta_j} \|u\|^2_{L_2(\Omega_j)} \right).$$

Here, we recall that $|\Omega_j|$ is the size of Ω_j (the diameter of the smallest cap containing Ω), δ_j is the overlapping size of Ω_j and Ω_{j,δ_j} is the set of points in Ω_j that are within a distance δ_j of $\partial\Omega_j$ defined by (5.5.3).

Proof. We cover Ω_{j,δ_j} by shape-regular triangles with $O(\delta_j)$ diameters. By using Lemma 5.13 for each triangle and then summing over these triangles, we obtain

$$\|u\|^2_{L_2(\Omega_{j,\delta_j})} \le C \left(\delta_j^2 |u|^2_{H^1(\Omega_{j,\delta_j})} + \delta_j \|u\|^2_{L_2(\partial\Omega_j)} \right). \tag{5.6.7}$$

We can estimate the second term on the right by combining the trace result in [43] and the embedding $L_2(\partial\Omega_j) \subset H^1(\Omega_j)$ with a scaling argument to obtain

$$\begin{aligned} \|u\|^2_{L_2(\partial\Omega_j)} &\preceq H_j \|\widehat{u}\|^2_{L_2(\partial\widehat{\Omega}_j)} \preceq H_j \|\widehat{u}\|^2_{H^1(\widehat{\Omega}_j)} \simeq H_j \left(|\widehat{u}|^2_{H^1(\widehat{\Omega}_j)} + \|\widehat{u}\|^2_{L_2(\widehat{\Omega}_j)} \right) \\ &\simeq H_j |u|^2_{H^1(\Omega_j)} + 1/H_j \|u\|^2_{L_2(\Omega_j)}. \end{aligned} \tag{5.6.8}$$

Here, $\widehat{\Omega}_j$ is the subset in $\mathbb{S}^2$ with unit size and has the same shape with Ω_j, $\widehat{u}$ is the composition of u and the transformation which maps $\widehat{\Omega}_j$ onto Ω_j. Inequalities (5.6.7) and (5.6.8) complete the proof, noting that $H_j \simeq |\Omega_j|$. □

By using Lemma 5.13 again we will prove the boundedness of the quasi-interpolation operator $\tilde{I}^h$ in $H^1(\mathbb{S}^2)$ when d is even.

Lemma 5.15. *Let Δ_h be a regular and quasi-uniform spherical triangulation on $\mathbb{S}^2$. Then for any $v \in H^1(\mathbb{S}^2)$, there holds*

$$\|\tilde{I}^h v\|_{H^1(\mathbb{S}^2)} \le C \|v\|_{H^1(\mathbb{S}^2)}.$$

Proof. For any $\tau \in \Delta_h$, we define $\alpha_v := |\omega_\tau|^{-1} \int_{\omega_\tau} v\, d\sigma$ and $\widehat{v} := v - \alpha_v$. Since the quasi-interpolant $\tilde{I}^h$ reproduces any constant function, we have $\tilde{I}^h(\alpha_v) = \alpha_v$. Noting that $\int_{\omega_\tau} \widehat{v}\, d\sigma = 0$ and applying the results in Lemmas 5.6 and 5.13, we obtain

$$\begin{aligned} |\tilde{I}^h v|_{H^1(\tau)} &= |\tilde{I}^h v - \alpha_v|_{H^1(\tau)} = |\tilde{I}^h \widehat{v}|_{H^1(\tau)} \preceq h^{-1} |\widehat{v}|_{L_2(\omega_\tau)} \\ &\preceq |\widehat{v}|_{H^1(\omega_\tau)} = |v|_{H^1(\omega_\tau)}. \end{aligned}$$

Summing the above inequality over all $\tau \in \Delta_h$, we obtain $|\tilde{I}^h v|_{\mathbb{S}^2} \preceq |v|_{H^1(\mathbb{S}^2)}$. This together with the Lemma 5.6 implies the required inequality. □

We note that a similar result as in Lemma 5.15 is true for $\tilde{I}^H$ and $S_d^r(\Delta_H)$. We are now able to derive a better lower bound for $\lambda_{\min}(P)$ via (5.4.7).

Lemma 5.16. *There exists a positive constant C depending on the smallest angle of Δ_h and the polynomial degree d such that for any $u \in V$ there exist $u_j \in V_j, j = 0, ..., J$, satisfying $u = \sum_{j=0}^J u_j$ and*

$$\sum_{j=0}^{J} a(u_j, u_j) \le C \max_{1\le k\le J} \left(1 + \frac{H_k}{\delta_k}\right) a(u, u). \tag{5.6.9}$$

Proof. Let $u_0' := \tilde{I}^H u$ and $u_0 := \tilde{I}^h u_0'$. We define $w := u - u_0$ and for $j = 1, \dots, J$, let $u_j := \tilde{I}^h(\theta_j w)$. It is clear that $u_j \in V_j$ for $j = 0, \dots, J$ and

$$u = u_0 + \ldots + u_J.$$

The result in Lemma 5.15 gives

$$a(u_0, u_0) \simeq \|\tilde{I}^h(\tilde{I}^H u)\|^2_{H^1(\mathbb{S}^2)} \preceq \|u\|^2_{H^1(\mathbb{S}^2)} \simeq a(u, u). \tag{5.6.10}$$

For $j = 1, \dots, J$, noting that θ_j vanishes outside Ω_j, we have

$$a(u_j, u_j) \simeq \|\tilde{I}^h(\theta_j w)\|^2_{H^1(\Omega_j)} \preceq \|\theta_j w\|^2_{H^1(\Omega_j)} = |\theta_j w|^2_{H^1(\Omega_j)} + \|\theta_j w\|^2_{L_2(\Omega_j)}. \tag{5.6.11}$$

Since θ_j is not greater than 1, we have $\|\theta_j w\|_{L_2(\Omega_j)} \le \|w\|_{L_2(\Omega_j)}$. On the other hand,

$$\begin{aligned} |\theta_j w|^2_{H^1(\Omega_j)} &\le \int_{\Omega_j} |w \nabla_{\mathbb{S}^2} \theta_j|^2 \, d\sigma + \int_{\Omega_j} |\theta_j \nabla_{\mathbb{S}^2} w|^2 \, d\sigma \\ &\le \int_{\Omega_j} |w \nabla_{\mathbb{S}^2} \theta_j|^2 \, d\sigma + |w|^2_{H^1(\Omega_j)}. \end{aligned} \tag{5.6.12}$$

Noting that $\nabla_{\mathbb{S}^2}\theta_j = \mathbf{0}$ outside the strip Ω_{j,δ_j} and applying the results in Lemmas 5.8 and 5.14, we have

$$\begin{aligned} \int_{\Omega_j} |w \nabla_{\mathbb{S}^2} \theta_j|^2 \, d\sigma &= \int_{\Omega_{j,\delta_j}} |w \nabla_{\mathbb{S}^2} \theta_j|^2 \, d\sigma \preceq \frac{C}{\delta_j^2} \|w\|^2_{L_2(\Omega_{j,\delta_j})} \\ &\preceq C \left[\left(1 + \frac{H_j}{\delta_j}\right) |w|^2_{H^1(\Omega_j)} + \frac{1}{H_j \delta_j} \|w\|^2_{L_2(\Omega_j)} \right]. \end{aligned} \tag{5.6.13}$$

We obtain from (5.6.12) and (5.6.13)

$$|\theta_j w|^2_{H^1(\Omega_j)} \preceq C \left[\left(1 + \frac{H_j}{\delta_j}\right) |w|^2_{H^1(\Omega_j)} + \frac{1}{H_j \delta_j} \|w\|^2_{L_2(\Omega_j)} \right].$$

This together with (5.6.11) gives

$$a(u_j, u_j) \preceq C \max_{1\le k\le J} \left(1 + \frac{H_k}{\delta_k}\right) \left(\|w\|^2_{H^1(\Omega_j)} + \frac{1}{H_j^2} \|w\|^2_{L_2(\Omega_j)} \right).$$

Summing over j yields

$$\sum_{j=1}^{J} a(u_j, u_j) \preceq C \max_{1\le k\le J} \left(1 + \frac{H_k}{\delta_k}\right) \left(\|w\|^2_{H^1(\mathbb{S}^2)} + \frac{1}{H_j^2} \|w\|^2_{L_2(\mathbb{S}^2)} \right). \tag{5.6.14}$$

Since the quasi-interpolation operator is bounded in $H^1(\mathbb{S}^2)$, there holds

$$\|w\|^2_{H^1(\mathbb{S}^2)} = \|\tilde{I}^h(u-u_0')\|^2_{H^1(\mathbb{S}^2)} \preceq \|u-u_0'\|^2_{H^1(\mathbb{S}^2)} \preceq \|u\|^2_{H^1(\mathbb{S}^2)}. \tag{5.6.15}$$

The results in Lemma 5.6 and the approximate property of spherical splines give

$$\|w\|^2_{L_2(\mathbb{S}^2)} = \|\tilde{I}^h(u-u_0')\|^2_{L_2(\mathbb{S}^2)} \preceq \|u-u_0'\|^2_{L_2(\mathbb{S}^2)} \preceq H^2\|u\|^2_{H^1(\mathbb{S}^2)}. \tag{5.6.16}$$

Combining (5.6.14)–(5.6.16) yields

$$\sum_{j=1}^{J} a(u_j,u_j) \preceq C \max_{1\le k\le J}\left(1+\frac{H_k}{\delta_k}\right)\|u\|^2_{H^1(\mathbb{S}^2)} \preceq C \max_{1\le k\le J}\left(1+\frac{H_k}{\delta_k}\right) a(u,u).$$

Combining this with (5.6.10) completes the proof.

□

Combining the results in Lemmas 5.9, 5.16 and 5.5 we obtain a bound for the condition number $\kappa(P)$ of the additive Schwarz operator.

Theorem 5.17. *When the polynomial degree d is even, the condition number of the additive Schwarz operator P is bounded by*

$$\kappa(P) \le C \max_{1\le k\le J}\left(1+\frac{H_k}{\delta_k}\right),$$

where C is a constant depending on the smallest angle in Δ_h and on d.

5.7 Numerical results

We solved

$$-\Delta_{\mathbb{S}^2} u + u = g$$

with

$$g(x,y,z) = e^x(x^2+2x),$$

which has the solution

$$u(x,y,z) = e^x.$$

Data sets of various sizes were extracted from the large set of data collected by NASA's satellite MAGSAT. Using the software STRIPACK [64] developed by Robert Renka, we obtained Delaunay triangulations of these data sets. Also, the software SPARSKIT2 developed by Yousef Saad was used to deal with sparse matrices.

The computation of the stiffness matrix $\boldsymbol{A}$ and the load vector is discussed in Subsection 4.6.1, Chapter 4. We now discuss the overlapping additive Schwarz algorithm. The algorithm consists of constructing the subdomains, the subproblems for each subdomain, the stiffness matrix for the coarse mesh, the transformation matrices between the coarse and fine meshes, and finally solving the problem with the preconditioned conjugate gradient method. Recalling (5.5.1), a pseudo–code to construct the overlapping subdomains is as follows.

Input: Sets of triangles of the coarse mesh Δ_H and fine mesh Δ_h.

Output: A set of subdomains $\{\Omega_j : j = 1, \dots, J\}$ where each subdomain consists of triangles of the fine mesh.

foreach $\tau_H^j \in \Delta_H$ **do**
  $\Omega_j = \tau_H^j$
  foreach $\tau := \langle \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \rangle \in \Delta_h$ **do**
    if *at least one of* $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3$ *belongs to* $\bar{\tau}_H^j$ **then**
      $\Omega_j = \Omega_j \cup \tau$;
    end
  end
end

We note that this construction yields overlapping subdomains with overlap size δ proportional to h.

Before presenting results for the condition number of $\boldsymbol{A}$ we examine the L_2- and H^1-norms of the errors $u_h - u$ for different meshes and different degrees d of the splines to observe the accuracy of the solutions. From Theorem 4.5, we expect the convergence rates of the errors to decrease like $O(h^{d+1})$ in the L_2-norm and like $O(h^d)$ in the H^1-norm. This can be observed from Table 5.1. The same errors were observed when our preconditioner was used.

The condition numbers of the unpreconditioned matrices $\boldsymbol{A}$ (with $d = 1, 2, 3$) were computed and a log plot shows that $\kappa(\boldsymbol{A}) = O(h^{-2})$ as predicted by Proposition 5.4; see Figure 5.1.

We tested our overlapping method with different values of d, H and h. Tables 5.2, 5.3 and 5.4 present the results for $d = 1, 2$, and 3, respectively. In all cases, $\kappa(P)$ is smaller than $\kappa(\boldsymbol{A})$, as expected. Figures 5.2–5.4 seem to suggest that $\kappa(P)$ grows like $\frac{H}{h}$ for both odd and even polynomial degrees d.

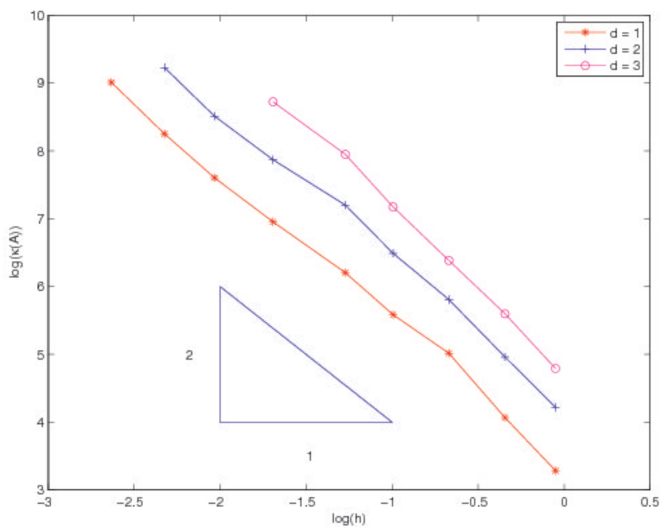

Figure 5.1: Log plot of $\kappa(\boldsymbol{A})$ vs h.

d	DoF	h	$\|u - u_h\|_{L_2(\mathbb{S}^2)}$	EOC	$\|u - u_h\|_{H^1(\mathbb{S}^2)}$	EOC
1	12	1.670	0.768124		1.813961	
	25	1.307	0.331639	3.43	1.219869	1.62
	50	0.951	0.182529	1.88	0.906867	0.93
	101	0.709	0.093200	2.29	0.648201	1.14
	204	0.511	0.044239	2.28	0.444733	1.15
	414	0.370	0.023030	2.02	0.322502	0.99
	836	0.280	0.011452	2.52	0.226659	1.27
	1635	0.184	0.005671	1.67	0.160397	0.82
	3250	0.131	0.002923	1.95	0.115241	0.97
	6423	0.098	0.001461	2.38	0.081315	1.20
	12865	0.072	0.000748	2.17	0.058055	1.09
2	42	1.670	0.115717		0.621473	
	94	1.307	0.029043	5.64	0.239582	3.89
	194	0.951	0.011152	3.01	0.126920	2.00
	398	0.709	0.004150	3.37	0.065421	2.26
	810	0.511	0.001360	3.42	0.030494	2.34
	1650	0.370	0.000508	3.04	0.015895	2.01
	3338	0.280	0.000183	3.69	0.007957	2.49
	6534	0.184	0.000062	2.59	0.003921	1.68
	12994	0.131	0.000023	2.84	0.002048	1.91
	25686	0.098	0.000008	3.60	0.001017	2.40
3	92	1.670	0.052294		0.374711	
	209	1.307	0.010358	6.60	0.112127	4.92
	434	0.951	0.002288	4.75	0.036183	3.56
	893	0.709	0.000616	4.47	0.013132	3.45
	1820	0.511	0.000148	4.37	0.004393	3.35
	3710	0.370	0.000039	4.13	0.001591	3.14
	7508	0.280	0.000010	4.90	0.000565	3.73
	14699	0.184	0.000002	3.62	0.000187	2.63

Table 5.1: Errors in the L_2- and H^1-norms for $d = 1, 2, 3$ (EOC: experimented order of convergence). Expected orders of convergence for degree $d = 1, 2, 3$ with respect to $L_2(\mathbb{S}^2)$-norm and $H^1(\mathbb{S}^2)$-norm are $d + 1$ and d, for $d = 1, 2, 3$.

DoF	h	$\kappa(\boldsymbol{A})$	H	$\kappa(P)$
101	0.709	58.4	0.951	50.6
			1.307	43.4
			1.670	38.9
			2.292	30.6
204	0.511	151.0	0.709	42.2
			0.951	58.9
			1.307	79.8
			1.670	64.6
			2.292	68.8
414	0.370	266.5	0.511	40.9
			0.709	44.3
			0.951	82.9
			1.307	93.4
			1.670	167.1
			2.292	133.1
836	0.280	494.7	0.370	48.1
			0.511	45.7
			0.709	73.3
			0.951	117.4
			1.307	255.6
			1.670	240.6
			2.292	284.0
1635	0.184	1048.5	0.280	47.5
			0.370	53.3
			0.511	74.7
			0.709	126.3
			0.951	307.2
			1.307	371.0
			1.670	444.9
3250	0.131	2005.7	0.184	46.8
			0.280	48.2
			0.370	93.2
			0.511	133.5
			0.709	242.1
			0.951	607.2
			1.307	707.9
			1.670	898.8

Table 5.2: Condition numbers when $d = 1$.

DoF	h	$\kappa(\boldsymbol{A})$	H	$\kappa(P)$
194	0.951	68.0	1.307	29.8
			1.670	27.6
			2.292	29.0
398	0.709	142.8	0.951	33.1
			1.307	32.0
			1.670	33.5
			2.292	40.0
810	0.511	333.2	0.709	37.3
			0.951	44.7
			1.307	53.3
			1.670	58.2
			2.292	84.8
1650	0.370	659.4	0.511	40.2
			0.709	49.4
			0.951	53.1
			1.307	80.3
			1.670	136.7
			2.292	155.7
3338	0.280	1341.1	0.370	49.7
			0.511	50.8
			0.709	61.0
			0.951	105.3
			1.307	165.0
			1.670	211.3
			2.292	323.4
6534	0.184	2616.3	0.280	45.6
			0.370	55.6
			0.511	64.1
			0.709	109.2
			0.951	208.3
			1.307	207.5
			1.670	331.2
			2.292	575.4

Table 5.3: Condition numbers when $d = 2$.

DoF	h	$\kappa(\boldsymbol{A})$	H	$\kappa(P)$
893	0.709	269.9	0.951	89.8
			1.307	97.9
			1.670	100.7
			2.292	91.1
1820	0.511	592.5	0.709	149.3
			0.951	133.2
			1.307	127.9
			1.670	119.3
			2.292	97.6
3710	0.370	1305.1	0.511	110.8
			0.709	152.6
			0.951	164.3
			1.307	150.2
			1.670	128.0
			2.292	145.7
7508	0.280	2832.8	0.370	178.6
			0.511	163.8
			0.709	171.7
			0.951	165.1
			1.307	185.0
			1.670	197.4
			2.292	312.0
14699	0.184	6154.1	0.280	158.4
			0.370	179.7
			0.511	193.1
			0.709	218.1
			0.951	222.7
			1.670	337.8
			2.292	498.4

Table 5.4: Condition numbers when $d = 3$.

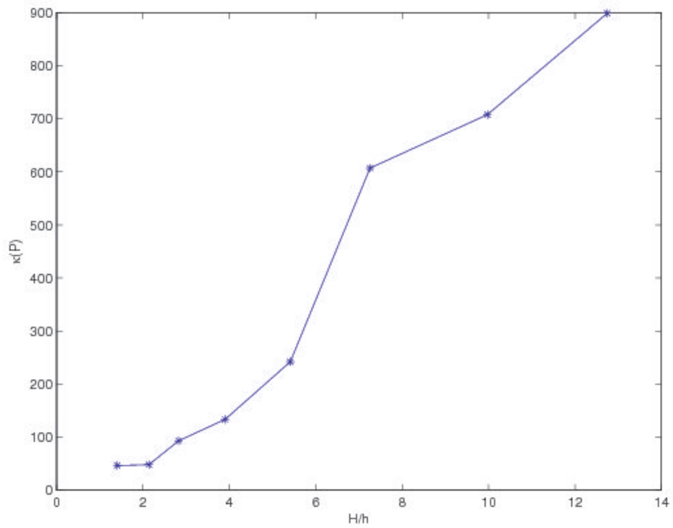

Figure 5.2: Condition number vs H/h for $d = 1$ and $h = 0.131$.

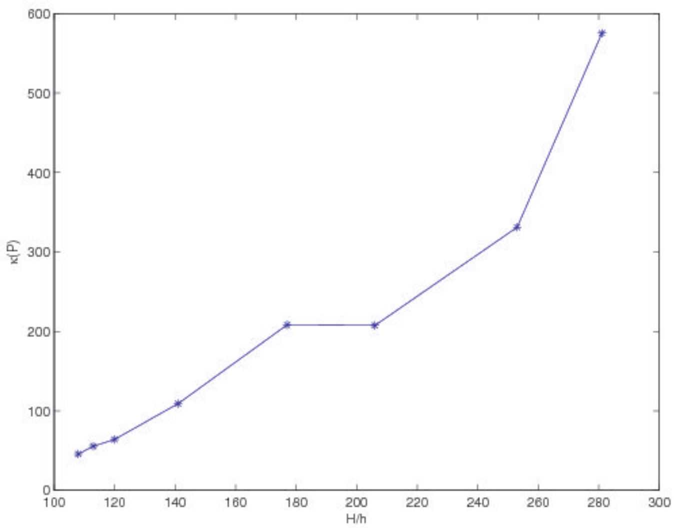

Figure 5.3: Condition number vs H/h for $d = 2$ and $h = 0.184$.

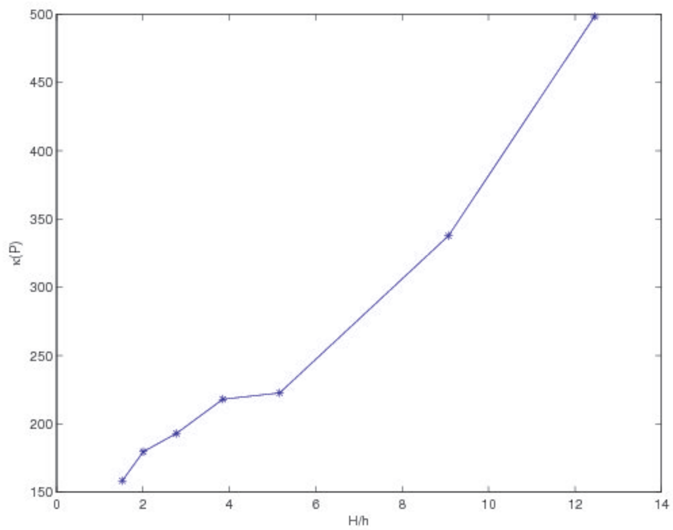

Figure 5.4: Condition number vs H/h for $d = 3$ and $h = 0.184$.

Chapter 6

Preconditioning for the hypersingular integral equation

6.1 Introduction

Hypersingular integral equations have many applications, for example in acoustics, fluid mechanics, elasticity and fracture mechanics [18]. Together with physical problems and their resulting mathematical models from these areas, a wide range of numerical methods for solving this equation have been proposed and developed during the last few decades; see for example [55, 72, 83]. In this chapter, we study the hypersingular integral equation of the form

$$-Nu + \omega^2 \int_{\mathbb{S}^2} u \, d\sigma = g \quad \text{on } \mathbb{S}^2, \tag{6.1.1}$$

where N is the hypersingular integral operator given by

$$Nv(\boldsymbol{x}) := \frac{1}{4\pi} \frac{\partial}{\partial \nu_{\boldsymbol{x}}} \int_{\mathbb{S}^2} v(\boldsymbol{y}) \frac{\partial}{\partial \nu_{\boldsymbol{y}}} \frac{1}{|\boldsymbol{x} - \boldsymbol{y}|} d\sigma_{\boldsymbol{y}}, \tag{6.1.2}$$

ω is some nonzero real constant. Here $\partial/\partial\nu_{\boldsymbol{x}}$ is the normal derivative with respect to $\boldsymbol{x}$.

Equation (6.1.1) arises from the boundary-integral reformulation of the Neumann problem with the Laplacian in the interior or exterior of the sphere; see e.g. [55, 72, 83]. When solving the equation by using the Galerkin method with spherical splines, an ill-conditioned linear system arises as seen in Proposition 6.3 in the next section. The purpose of this chapter is to overcome this ill-conditionedness by preconditioning with additive Schwarz methods as used in the previous chapter.

We prove that the condition number of the preconditioned system is bounded by $O(H/\delta)$ for all parities of polynomial degrees, and by $O\big(1+\log^2(H/\delta)\big)$ in the case of even degree polynomials. Here, H is the mesh size of the coarse mesh and δ is the size of the overlapping which is proportional to the mesh size of the fine mesh. The weaker estimate in the case of odd degree might be only a technical obstacle that we could not overcome. We note that the overlapping subdomains are, in general, not spherical triangles. In the analysis, we have to use another set of artificial subdomains which are triangles.

6.2 Preliminaries

Let $\Omega \subset \mathbb{S}^2$ be a Lipschitz domain. Recall that the space $H^{1/2}(\Omega)$ is defined by Hilbert space interpolation [8] so that

$$H^{1/2}(\Omega) := [L_2(\Omega), H^1(\Omega)]_{1/2} \tag{6.2.1}$$

with the norm

$$\|v\|^2_{H^{1/2}(\Omega)} = \int_0^\infty K(t,v)^2 \frac{dt}{t^2}, \tag{6.2.2}$$

where the K-functional is defined, for $v \in L_2(\Omega) + H^1(\Omega)$, by

$$K(t,v)^2 = \inf_{v=v_0+v_1} \left(\|v_0\|^2_{L_2(\Omega)} + t^2 \|v_1\|^2_{H^1(\Omega)} \right).$$

Similarly, we define the subspace $\widetilde{H}^{1/2}(\Omega) \subset H^{1/2}(\Omega)$ by

$$\widetilde{H}^{1/2}(\Omega) := [L_2(\Omega), H^1_0(\Omega)]_{1/2}.$$

The spaces $H^{-1/2}(\Omega)$ and $\widetilde{H}^{-1/2}(\Omega)$ are defined as the dual spaces of $\widetilde{H}^{1/2}(\Omega)$ and $H^{1/2}(\Omega)$, respectively, with respect to the L_2 duality which is the usual extension of the L_2 inner product on Ω.

For the analysis in this chapter we also define the following norms:

$$|||v|||^2_{H^{1/2}(\Omega)} := \frac{1}{\operatorname{diam}(\Omega)} \|v\|^2_{L_2(\Omega)} + |v|^2_{H^{1/2}(\Omega)} \tag{6.2.3}$$

and

$$|||v|||^2_{\widetilde{H}^{1/2}(\Omega)} := |v|^2_{H^{1/2}(\Omega)} + \int_\Omega \frac{v^2(\boldsymbol{x})}{\operatorname{dist}(\boldsymbol{x}, \partial\Omega)}\, d\boldsymbol{x}, \tag{6.2.4}$$

where

$$|v|^2_{H^{1/2}(\Omega)} := \int_\Omega \int_\Omega \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^3}\, d\boldsymbol{x}\, d\boldsymbol{y}. \tag{6.2.5}$$

For a subset R of $\mathbb{R}^2$, the Sobolev spaces $H^{1/2}(R)$ and $\widetilde{H}^{1/2}(R)$ can be defined similarly to the case of $\Omega \subset \mathbb{S}^2$, with norms and seminorms given by (6.2.3)–(6.2.5) accordingly. In particular, when $R = I \times J$, where I, J are intervals in $\mathbb{R}$, there hold

$$\|v\|_{\widetilde{H}^{1/2}(R)} \simeq |||v|||_{\widetilde{H}^{1/2}(R)} \quad \forall v \in \widetilde{H}^{1/2}(R) \tag{6.2.6}$$

and

$$|v|^2_{H^{1/2}(R)} \simeq \int_I \int_I \frac{\|v(x,\cdot) - v(x',\cdot)\|^2_{L_2(J)}}{|x - x'|^2}\, dx\, dx' + \int_J \int_J \frac{\|v(\cdot,y) - v(\cdot,y')\|^2_{L_2(I)}}{|y - y'|^2}\, dy\, dy'. \tag{6.2.7}$$

Here, the constants in the equivalences are independent of the sizes of I and J. The result (6.2.6) is proved in [1, Lemma 2] and (6.2.7) in [56, Lemma 5.3] (see also Exercise 5.1 following that lemma).

In this chapter, we solve the hypersingular integral equation by using Galerkin method with the space of spherical splines $S^r_d(\Delta_h)$. Preconditioning by additive Schwarz method as introduced in Section 5.4, Chapter 5 will be used. A fine mesh Δ_h and a coarse mesh Δ_H as defined in the previous chapter will be employed. The following lemma states the stability for the quasi-interpolation operators $\tilde{I}^h$ and $\tilde{I}^H$ with respect to the spaces of spherical splines corresponding to the fine and coarse meshes.

Lemma 6.1. *For any $\tau \in \Delta_h$, let $\omega_\tau := \bigcup_{i \in I_\tau} \omega_i$, where ω_i is the support of the basis function Φ_i and $I_\tau := \{i \in \{1, \ldots, N\} : \tau \subset \omega_i\}$. For $v \in L_2(\mathbb{S}^2)$ and $k = 0, 1/2, 1$, there holds*

$$\|\tilde{I}^h v\|_{H^k(\mathbb{S}^2)} \le C h^{-k} \|v\|_{L_2(\mathbb{S}^2)}. \tag{6.2.8}$$

Here, the constants depend only on the smallest angle Θ_{Δ_h} of Δ_h and the polynomial degree d.

Proof. The proof for $k = 0, 1$ can be found in Lemma 5.6. We now use the interpolation inequality

$$\|w\|_{H^{1/2}(\mathbb{S}^2)} \preceq \|w\|_{H^1(\mathbb{S}^2)}^{1/2} \|w\|_{L_2(\mathbb{S}^2)}^{1/2} \quad \forall w \in H^1(\mathbb{S}^2),$$

see e.g. [43, Proposition 2.3], to obtain

$$\|\tilde{I}^h v\|_{H^{1/2}(\mathbb{S}^2)} \preceq h^{-1/2} \|v\|_{L_2(\mathbb{S}^2)},$$

completing the proof of the lemma. □

We will next prove the boundedness of the quasi-interpolation operator $\tilde{I}^h$ in $H^{1/2}(\mathbb{S}^2)$ when d is even.

Lemma 6.2. *Let Δ_h be a regular and quasi-uniform spherical triangulation on $\mathbb{S}^2$ and let $\tilde{I}^h : L_2(\mathbb{S}^2) \to S_d^r(\Delta_h)$ be the quasi-interpolation operator defined by (2.5.22) with d even. Then for any $v \in H^{1/2}(\mathbb{S}^2)$, there holds*

$$\|\tilde{I}^h v\|_{H^{1/2}(\mathbb{S}^2)} \le C \|v\|_{H^{1/2}(\mathbb{S}^2)}. \tag{6.2.9}$$

Proof. Since the degree d is even, Lemma 5.15 proves that

$$\|\tilde{I}^h v\|_{H^1(\mathbb{S}^2)} \le C \|v\|_{H^1(\mathbb{S}^2)}. \tag{6.2.10}$$

Inequality (6.2.9) is then obtained by applying Theorem 2.11 with $t_1 = 0$, $t_2 = 1$, $s_1 = 0$, $s_2 = 1$ and $\theta = 1/2$, noting (6.2.10) and the results in Lemma 6.1 for $k = 0$. □

6.3 The hypersingular integral equation

Recall the hypersingular integral equation (6.1.1) where g is some given smooth function and the hypersingular integral operator N is given by (6.1.2). To set up a weak formulation, we introduce the bilinear form

$$a(u, v) := -\langle Nu, v\rangle + \omega^2 \langle u, 1\rangle \langle v, 1\rangle, \quad u, v \in H^{1/2}(\mathbb{S}^2).$$

We note that (see [55])

$$a(v, v) \simeq \|v\|_{H^{1/2}(\mathbb{S}^2)}^2 \quad \forall v \in H^{1/2}(\mathbb{S}^2). \tag{6.3.1}$$

A natural weak formulation of equation (6.1.1) is: Find $u \in H^{1/2}(\mathbb{S}^2)$ satisfying

$$a(u, v) = \langle g, v\rangle \quad \forall v \in H^{1/2}(\mathbb{S}^2).$$

This bilinear form is clearly bounded and coercive (cf. [9]). This guarantees the unique solvability of the equation. The Ritz-Galerkin approximation problem is: Find $u_h \in S_d^r(\Delta_h)$ satisfying

$$a(u_h, v_h) = \langle g, v_h \rangle \quad \forall v_h \in S_d^r(\Delta_h). \tag{6.3.2}$$

Denoting $\{\Phi_i : i = 1, \dots, N\}$ a basis for $S_d^r(\Delta_h)$, the problem (6.3.2) reduces to the problem of solving the following linear system

$$\boldsymbol{A}\boldsymbol{c} = \boldsymbol{g}, \tag{6.3.3}$$

where for $i, j = 1, ..., N$, the entries of the matrix $\boldsymbol{A}$ are given by $A_{ij} = a(\Phi_i, \Phi_j)$, $\boldsymbol{c} = (c_i)_{i=1}^N$ where $u_h = \sum_{i=1}^N c_i \Phi_i$, and the vector $\boldsymbol{g}$ is given as $\boldsymbol{g} = (g_i)_{i=1}^N$ in which $g_i = \langle g, \Phi_i \rangle$.

It is well known that the matrix $\boldsymbol{A}$ is ill-conditioned, namely, the condition number of $\boldsymbol{A}$, denoted by $\kappa(\boldsymbol{A})$, grows like h^{-1} as $h \to 0$ (i.e. $|\Delta_h| \to 0$). Since we cannot find a reference for this seemingly well-known result, we include the proof here for completeness.

Proposition 6.3. *The condition number of the stiffness matrix $\boldsymbol{A}$ is bounded by*

$$\kappa(\boldsymbol{A}) \preceq h^{-1}.$$

Proof. This proposition can be proved in the same manner as in the proof of Proposition 5.4 in which the result in Lemma 6.1 is employed instead of Lemma 5.3. □

This behaviour of $\kappa(\boldsymbol{A})$ subjected to the change of h is corroborated by the numerical results in Table 6.1.

In this chapter we use the same subspace decomposition as used in Chapter 5 in which the subspaces V_i, $i = 1, \dots, J$ and the set of overlapping subdomains $\{\Omega_i, i = 1, \dots, J\}$ are defined in Section 5.5, Chapter 5. It is also assumed that the subdomains can be coloured using at most M colours in such a way that subdomains with the same colour are disjoint.

6.4 Main results

In this section we prove a bound on the condition number of P by using the abstract result in Lemma 5.5. We first prove (5.4.6).

Lemma 6.4. *There exists a positive constant C independent of Δ_h such that for any $u \in V$ satisfying $u = \sum_{i=0}^J u_i$ with $u_i \in V_i$ for $i = 0, ..., J$,*

$$a(u, u) \le C \sum_{i=0}^J a(u_i, u_i),$$

where the constant C depends on the smallest angle of the triangulation.

Proof. This lemma can be proved by using a standard colouring argument as in the proof of Lemma 5.9. □

In the following subsection, we prove (5.4.7) for both odd and even polynomial degrees d. In Subsection 6.4.2, a better estimate of $\kappa(P)$ is established for even degrees d when the quasi-interpolation operators $\tilde{I}^h$ and $\tilde{I}^H$ reproduce constant functions.

6.4.1 A general result for both odd and even degrees

To prove (5.4.7), we need to introduce an operator P_H from $H^{1/2}(\mathbb{S}^2)$ into $S_d^r(\Delta_H)$ defined by

$$a(P_H u, v) = a(u, v) \quad \forall v \in S_d^r(\Delta_H)$$

for any $u \in H^{1/2}(\mathbb{S}^2)$. Standard finite element arguments yield

$$\begin{aligned}
\|P_H u - u\|_{H^{1/2}(\mathbb{S}^2)} &\preceq \|u - v\|_{H^{1/2}(\mathbb{S}^2)} \quad \forall v \in S_d^r(\Delta_H) \\
\|P_H u - u\|_{H^{1/2}(\mathbb{S}^2)} &\preceq \|u\|_{H^{1/2}(\mathbb{S}^2)} \\
\|P_H u\|_{H^{1/2}(\mathbb{S}^2)} &\preceq \|u\|_{H^{1/2}(\mathbb{S}^2)} \\
\|P_H u - u\|_{L_2(\mathbb{S}^2)} &\preceq H^{1/2}\|u\|_{H^{1/2}(\mathbb{S}^2)}.
\end{aligned} \tag{6.4.1}$$

Lemma 6.5. *There exists a positive constant C depending on the smallest angle of Δ_h and the polynomial degree d such that for any $u \in V$ there exist $u_i \in V_i, i = 0, ..., J$, satisfying $u = \sum_{i=0}^J u_i$ and*

$$\sum_{i=0}^J a(u_i, u_i) \preceq \frac{H}{h} a(u, u).$$

Proof. The proof of this lemma can be done in the same manner as in Lemma 5.10 in which $u_0 := \tilde{I}^h(P_H u)$ and $u_i := \tilde{I}^h(\theta_i w)$, $i = 1, \ldots, J$, where $w := u - u_0$,. Here, $\{\theta_i\}_{i=1}^J$ is a partition of unity defined on $\mathbb{S}^2$ satisfying $\operatorname{supp}(\theta_i) = \overline{\Omega}_i$, for $i = 1, ..., J$. Then we can split $u \in V$ by $u = u_0 + u_1 + \ldots + u_J$. It is clear that $u_i \in V_i$ for all $i = 0, \ldots, J$. By Lemma 6.1, we have

$$\|\tilde{I}^h(P_H u - u)\|_{H^{1/2}(\mathbb{S}^2)} \preceq h^{-1/2}\|P_H u - u\|_{L_2(\mathbb{S}^2)}. \tag{6.4.2}$$

By writing $u_0 = \tilde{I}^h(P_H u - u) + u$ and using the triangular inequality, (6.4.2) and (6.4.1), we have

$$\begin{aligned}
a(u_0, u_0) \simeq \|u_0\|^2_{H^{1/2}(\mathbb{S}^2)} &\leq \Big(\|\tilde{I}^h(P_H u - u)\|_{H^{1/2}(\mathbb{S}^2)} + \|u\|_{H^{1/2}(\mathbb{S}^2)}\Big)^2 \\
&\preceq \Big(h^{-1/2}\|P_H u - u\|_{L_2(\mathbb{S}^2)} + \|u\|_{H^{1/2}(\mathbb{S}^2)}\Big)^2 \\
&\preceq \frac{H}{h}\|u\|^2_{H^{1/2}(\mathbb{S}^2)} \simeq \frac{H}{h} a(u, u).
\end{aligned} \tag{6.4.3}$$

Applying Lemma 6.1 and noting that θ_i vanishes outside Ω_i, we obtain

$$a(u_i, u_i) \simeq \|\tilde{I}^h(\theta_i w)\|^2_{H^{1/2}(\mathbb{S}^2)} \preceq h^{-1}\|\theta_i w\|^2_{L_2(\mathbb{S}^2)} \simeq h^{-1}\|\theta_i w\|^2_{L_2(\Omega_i)}. \tag{6.4.4}$$

This together with the fact that $\|\theta_i w\|_{L_2(\Omega_i)} \leq \|w\|_{L_2(\Omega_i)}$ implies

$$a(u_i, u_i) \preceq h^{-1}\|w\|^2_{L_2(\Omega_i)}.$$

Summing up the above inequality over all subdomains, we obtain

$$\sum_{i=1}^J a(u_i, u_i) \preceq h^{-1}\|w\|^2_{L_2(\mathbb{S}^2)}. \tag{6.4.5}$$

Noting that $w \in S_d^r(\Delta_h)$ then $\tilde{I}^h w = w$ and by applying Lemma 6.1 and the last inequality in (6.4.1), we infer

$$\begin{aligned}\|w\|_{L_2(\mathbb{S}^2)} &= \|\tilde{I}^h(u - \tilde{I}^h P_H u)\|_{L_2(\mathbb{S}^2)} = \|\tilde{I}^h(u - P_H u)\|_{L_2(\mathbb{S}^2)}\\ &\preceq \|u - P_H u\|_{L_2(\mathbb{S}^2)} \preceq H^{1/2}\|u\|_{H^{1/2}(\mathbb{S}^2)}.\end{aligned}$$

This together with (6.4.5) gives

$$\sum_{i=1}^{J} a(u_i, u_i) \preceq \frac{H}{h}\|u\|^2_{H^{1/2}(\mathbb{S}^2)} \simeq \frac{H}{h} a(u, u).$$

From this and (6.4.3) we infer

$$\sum_{i=0}^{J} a(u_i, u_i) \preceq \frac{H}{h} a(u, u),$$

completing the proof. □

Combining the results in Lemmas 6.4, 6.5 and 5.5 we obtain a bound for the condition number $\kappa(P)$ of the additive Schwarz operator.

Theorem 6.6. *The condition number of the additive Schwarz operator P is bounded by*

$$\kappa(P) \preceq \frac{H}{h},$$

where the constant depends on the smallest angle in Δ_h and the polynomial degree d.

Remark 6.7. *Recall that we have chosen the overlap δ to be proportional to h.*

6.4.2 A better estimate for even degrees

In this subsection we assume that the degree d of the spherical splines is even. We now define the triangle $\tau^i_{H,h}$ satisfying

$$\tau^i_H \subset \tau^i_{H,h} \subset \Omega_i, \quad i = 1, \ldots, J. \tag{6.4.6}$$

Without loss of generality we can assume that the edges of $\tau^i_{H,h}$ are parallel and of distance h from those of τ^i_H (see Figure 6.1). (We can always choose Ω_i to be a bigger domain so that the assumption holds.)

For simplicity of presentation, we assume that the spherical triangles τ^i_H, $i = 1, \ldots, J$, are equilateral triangles. Then so are the spherical triangles $\tau^i_{H,h}$. The set $\{\tau^i_{H,h} : i = 1, \ldots, J\}$ is a set of overlapping spherical triangles which covers the sphere $\mathbb{S}^2$ (Figure 6.2). We will define a partition of unity $\{\theta_i\}_{i=1}^J$, whose definition is given in Appendix, satisfying

$$\operatorname{supp}(\theta_i) \subset \overline{\tau^i_{H,h}}, \quad i = 1, \ldots, J. \tag{6.4.7}$$

In the sequel, we denote

$$T_i := \operatorname{supp}(\theta_i), \quad i = 1, \ldots, J, \tag{6.4.8}$$

and W_i the smallest rectangle which contains T_i and shares a common edge with T_i (see Figure 6.4).

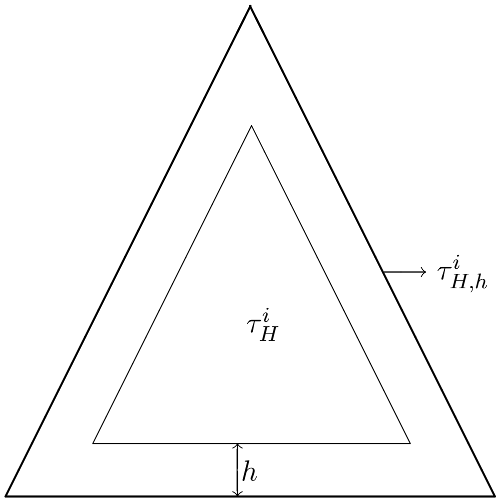

Figure 6.1: Extended spherical triangle $\tau^i_{H,h}$.

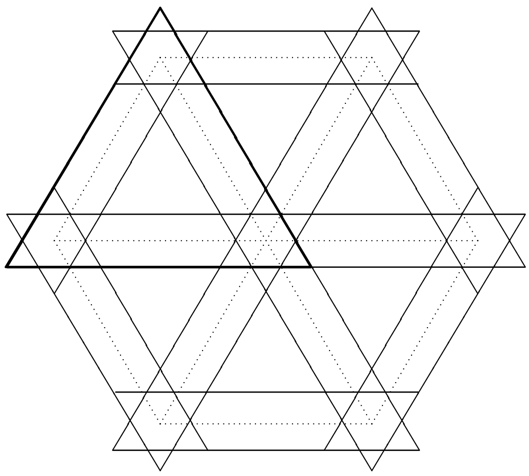

Figure 6.2: Six overlapping extended spherical triangles (Triangles with dotted edges: τ^i_H).

Lemma 6.8. *For any $v \in H^{1/2}(\mathbb{S}^2)$ there holds*

$$\sum_{i=1}^{J} |||\theta_i v|||^2_{\widetilde{H}^{1/2}(T_i)} \preceq \sum_{i=1}^{J} \left(1 + \log \frac{H_i}{\delta}\right)^2 |||v|||^2_{H^{1/2}(W_i)}. \tag{6.4.9}$$

Proof. See Appendix. □

Lemma 6.9. *Assume that the polynomial degree d is even. For any $u \in V$, there exist $u_i \in V_i$ for $i = 0, \dots, J$ satisfying $u = \sum_{i=0}^{J} u_i$ and*

$$\sum_{i=0}^{J} a(u_i, u_i) \preceq \left(1 + \log^2 \frac{H}{\delta}\right) a(u,u), \tag{6.4.10}$$

where the constant depends only on the smallest angle of the triangulations.

Proof. Recall that the support T_i of the partition of unity function θ_i satisfies $T_i \subset \Omega_i$ (see (6.4.6)). We now define a decomposition of $u \in V$ such that (6.4.10) holds. For any $u \in V$ let $u_0 := \tilde{I}^h P_H u \in V_0$ and $u_i = \tilde{I}^h(\theta_i w) \in V_i$, $i = 1, \dots, J$, where $w = u - u_0$. It is clear that $u = \sum_{i=1}^{J} u_i$. By using (6.2.9) and (6.4.1), we obtain

$$a(u_0, u_0) \simeq \|u_0\|^2_{H^{1/2}(\mathbb{S}^2)} = \|\tilde{I}^h P_H u\|^2_{H^{1/2}(\mathbb{S}^2)} \preceq \|u\|^2_{H^{1/2}(\mathbb{S}^2)} \simeq a(u,u) \tag{6.4.11}$$

$$\|w\|_{H^{1/2}(\mathbb{S}^2)} = \|\tilde{I}^h(u - P_H u)\|_{H^{1/2}(\mathbb{S}^2)} \preceq \|u - P_H u\|_{H^{1/2}(\mathbb{S}^2)} \preceq \|u\|_{H^{1/2}(\mathbb{S}^2)} \tag{6.4.12}$$

and

$$\|w\|_{L_2(\mathbb{S}^2)} = \|\tilde{I}^h(u - P_H u)\|_{L_2(\mathbb{S}^2)} \preceq \|u - P_H u\|_{L_2(\mathbb{S}^2)} \preceq H^{1/2} \|u\|_{H^{1/2}(\mathbb{S}^2)}. \tag{6.4.13}$$

By Lemma 6 in [1] there holds

$$\|\theta_i w\|_{H^{1/2}(\mathbb{S}^2)} \simeq |||\theta_i w|||_{\widetilde{H}^{1/2}(T_i)}. \tag{6.4.14}$$

By using successively (6.2.9), (6.4.14), (6.4.9) and (6.2.3), we obtain

$$\begin{aligned} \sum_{i=1}^{J} a(u_i, u_i) &\simeq \sum_{i=1}^{J} \|u_i\|^2_{H^{1/2}(\mathbb{S}^2)} \preceq \sum_{i=1}^{J} \|\theta_i w\|^2_{H^{1/2}(\mathbb{S}^2)} \simeq \sum_{i=1}^{J} |||\theta_i w|||^2_{\widetilde{H}^{1/2}(T_i)} \\ &\preceq \sum_{i=1}^{J} \left(1 + \log \frac{H_i}{\delta}\right)^2 |||w|||^2_{H^{1/2}(W_i)} \\ &\simeq \left(1 + \log \frac{H}{\delta}\right)^2 \sum_{i=1}^{J} \left(\frac{1}{H} \|w\|^2_{L_2(W_i)} + |w|^2_{H^{1/2}(W_i)}\right). \end{aligned} \tag{6.4.15}$$

It is obvious that $\sum_{i=1}^{J} \|w\|^2_{L_2(W_i)} \simeq \|w\|_{L_2(\mathbb{S}^2)}$ and by the definition of the seminorm $|\cdot|_{H^{1/2}(W_i)}$, it is clear that

$$\sum_{i=1}^{J} |w|^2_{H^{1/2}(W_i)} \preceq |w|^2_{H^{1/2}(\mathbb{S}^2)}. \tag{6.4.16}$$

This together with (6.4.15), (6.4.13) and (6.4.12) implies

$$\sum_{i=1}^{J} a(u_i, u_i) \preceq \left(1 + \log \frac{H}{\delta}\right)^2 \big(\frac{1}{H} \|w\|^2_{L_2(\mathbb{S}^2)} + |w|^2_{H^{1/2}(\mathbb{S}^2)}\big)$$

$$\preceq \left(1 + \log \frac{H}{\delta}\right)^2 \|u\|^2_{H^{1/2}(\mathbb{S}^2)}$$
$$\simeq \left(1 + \log \frac{H}{\delta}\right)^2 a(u, u),$$

completing the proof of the lemma. □

Combining the results in Lemmas 6.4, 6.9 and 5.5 we obtain a bound for the condition number $\kappa(P)$ of the additive Schwarz operator.

Theorem 6.10. *Assume that the polynomial degree d is even. The condition number of the additive Schwarz operator P is bounded by*

$$\kappa(P) \preceq \left(1 + \log^2 \frac{H}{\delta}\right)$$

where the constant depends on the smallest angle in Δ_h and the polynomial degree d.

Remark 6.11. *It can be seen that the boundedness in the $H^{1/2}(\mathbb{S}^2)$-norm of the quasi-interpolation operators $\tilde{I}^h$ and $\tilde{I}^H$ is crucial for the proof of Lemma 6.9. The proof of this boundedness (Lemma 6.2) requires the property that $\tilde{I}^h$ and $\tilde{I}^H$ reproduce constant functions, which does not hold in the case of odd degree splines.*

6.5 Numerical results

We solved (6.1.1),

$$-Nu + \omega^2 \int_{\mathbb{S}^2} u \, d\sigma = g \quad \text{on} \quad \mathbb{S}^2, \tag{6.5.1}$$

with

$$g(\boldsymbol{x}) = g(x, y, z) = e^x(2x - xz).$$

by using spherical spline spaces $S^0_d(\Delta_h)$, in which Δ_h are spherical triangulations of the following types:

- Type 1: Uniform triangulations which are generated as follows. First, we start with a spherical triangulation whose vertices are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(-1, 0, 0)$, $(0, -1, 0)$, and $(0, 0, -1)$. The following finer meshes are obtained by dividing each triangle in the previous triangulation into four equal equilateral triangles (by connecting midpoints of three edges), resulting in triangulations with number of vertices being 18, 66, 258, 1026, and 4098.
- Type 2: Triangulations with vertices obtained from MAGSAT satellite data. The free STRIPACK package is used to generate the triangulations from these vertices. Number of vertices to be used are 204, 414, 836, and 1635.

Assume that $B_1, B_2, \dots, B_M$ are the basis functions for the approximation space $S^0_d(\Delta)$. The entry A_{ij}, for $i, j = 1, \dots, M$, of the stiffness matrix $\boldsymbol{A}$ in (6.3.3) is computed by

$$A_{ij} = -\int_{\mathbb{S}^2} (NB_i)(\boldsymbol{x}) B_j(\boldsymbol{x}) \, d\sigma_{\boldsymbol{x}} + \omega^2 \int_{\mathbb{S}^2} B_i(\boldsymbol{x}) \, d\sigma_{\boldsymbol{x}} \int_{\mathbb{S}^2} B_j(\boldsymbol{x}) \, d\sigma_{\boldsymbol{x}}. \tag{6.5.2}$$

The first integral in (6.5.2) is computed by using the following formula

$$-\int_{\mathbb{S}^2} (Nu)v\, d\sigma = \frac{1}{4\pi} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \frac{\overrightarrow{\mathrm{curl}}_{\mathbb{S}^2} u(\boldsymbol{x}) \cdot \overrightarrow{\mathrm{curl}}_{\mathbb{S}^2} v(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d\sigma_{\boldsymbol{x}}\, d\sigma_{\boldsymbol{y}}$$

for any smooth functions u and v; see [55, Theorem 3.3.2]. Here, $\overrightarrow{\mathrm{curl}}_{\mathbb{S}}^2 v$ is the vectorial surfacic rotation defined by

$$\overrightarrow{\mathrm{curl}}_{\mathbb{S}}^2 v = -\frac{\partial v}{\partial \theta}\boldsymbol{e}_\varphi + \frac{1}{\sin\theta}\frac{\partial v}{\partial \varphi}\boldsymbol{e}_\theta,$$

where $\boldsymbol{e}_\varphi$, $\boldsymbol{e}_\theta$ are the two unit vectors corresponding to the Euler angles. Therefore

$$\begin{aligned} -\int_{\mathbb{S}^2} (NB_i)B_j\, d\sigma &= \frac{1}{4\pi} \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} \frac{\overrightarrow{\mathrm{curl}}_{\mathbb{S}^2} B_i(\boldsymbol{x}) \cdot \overrightarrow{\mathrm{curl}}_{\mathbb{S}^2} B_j(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d\sigma_{\boldsymbol{x}}\, d\sigma_{\boldsymbol{y}} \\ &= \frac{1}{4\pi} \sum_{\tau\in\Delta} \sum_{\tau'\in\Delta} \int_\tau \int_{\tau'} \frac{\overrightarrow{\mathrm{curl}}_{\mathbb{S}^2} B_i(\boldsymbol{x}) \cdot \overrightarrow{\mathrm{curl}}_{\mathbb{S}^2} B_j(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d\sigma_{\boldsymbol{x}}\, d\sigma_{\boldsymbol{y}}. \end{aligned} \tag{6.5.3}$$

Computation of the double integrals in (6.5.3) requires evaluation of integrals of the type

$$\int_{\tau^{(1)}} \int_{\tau^{(2)}} \frac{f_1(\boldsymbol{x})\, f_2(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\, d\sigma_{\boldsymbol{x}} d\sigma_{\boldsymbol{y}},$$

where $\tau^{(1)}$ and $\tau^{(2)}$ are spherical triangles in Δ_h and the functions f_1 and f_2 are analytic for all $\boldsymbol{x} \in \tau^{(1)}$ and $\boldsymbol{y} \in \tau^{(2)}$. For more details about the above evaluation, please refer to Section 4.6, Chapter 4.

The right hand side of the linear system (6.3.3) has entries given by

$$\boldsymbol{b}_i = \int_{\mathbb{S}^2} B_i(\boldsymbol{x}) g(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}} = \sum_{\tau\in\Delta} \int_\tau B_i(\boldsymbol{x}) g(\boldsymbol{x})\, d\sigma_{\boldsymbol{x}}, \quad i = 1, \ldots, M. \tag{6.5.4}$$

The computation of the right hand side as seen in (6.5.4) includes the evaluation of integrals of a smooth function f over a spherical triangle τ. This computation was discussed in Section 4.6, Chapter 4. For the overlapping additive Schwarz algorithm please refer to Section 5.7, Chapter 5.

The results for uniform triangulations are presented in Tables 6.1, 6.2, 6.3 and 6.4. In Table 6.1 we computed the experimented rate of increasing $\kappa(\boldsymbol{A}) = O(h^\alpha)$, and the numbers show that $\alpha \approx -1$ as predicted by Proposition 6.3. Tables 6.2, 6.3 and 5.4 show the significant improvement of our preconditioner.

The results for triangulations generated from satellite data are presented in Tables 6.5, 6.6 and 6.7. Again the advantage of the preconditioner can be observed.

6.6 Appendix

We first define a partition of unity $\{\theta_i\}_{i=1}^J$. Note here that the partition of unity will be defined for the extended spherical triangles $\{\tau_{H,h}^i : i = 1, \ldots, J\}$ and it is also a partition of unity for the overlapping subdomains $\{\Omega_i : i = 1, \ldots, J\}$.

Let $\boldsymbol{x}$ be a point on the sphere $\mathbb{S}^2$. Then $\theta_i(\boldsymbol{x})$, for $i = 1, \ldots, J$, are defined as follows:

- Case 1: $\boldsymbol{x}$ belongs to only one triangle $\tau_{H,h}^{i_0}$ for some $i_0 \in \{1,\dots,J\}$ (e.g. $\boldsymbol{x} \in A$ in Figure 6.3). Then

$$\theta_{i_0}(\boldsymbol{x}) := 1 \quad \text{and} \quad \theta_i(\boldsymbol{x}) := 0, \quad \text{for all } i \neq i_0.$$

- Case 2: $\boldsymbol{x}$ belongs to exactly two extended triangles $\tau_{H,h}^{i_0}$ and $\tau_{H,h}^{i_1}$ (e.g. $\boldsymbol{x} \in B$ in Figure 6.3). Then

$$\begin{aligned}
\theta_{i_0}(\boldsymbol{x}) &:= \delta^{-1} \operatorname{dist}(\boldsymbol{x}, \partial\tau_{H,h}^{i_0}) \\
\theta_{i_1}(\boldsymbol{x}) &:= \delta^{-1} \operatorname{dist}(\boldsymbol{x}, \partial\tau_{H,h}^{i_1}) \\
\theta_i(\boldsymbol{x}) &:= 0, \quad i \notin \{i_0, i_1\},
\end{aligned}$$

where $\partial\tau_{H,h}^{i}$ is the boundary of $\tau_{H,h}^{i}$.

- Case 3: $\boldsymbol{x}$ belongs to the intersection of more than two extended triangles (i.e. $\boldsymbol{x} \in U^\star$ in Figure 6.3). For simplicity of notation we denote the six triangles in Figure 6.3 by $\tau_1,\dots,\tau_6$. Then

$$\theta_i(\boldsymbol{x}) := 0, \quad i \notin \{1,2,3,4,5,6\}$$

$$\theta_1(\boldsymbol{x}) = \begin{cases} \delta^{-1} \operatorname{dist}(\boldsymbol{x}, \partial\tau_1), & \text{if } \boldsymbol{x} \in U_3 \cup U_7 \\ \delta^{-2} \operatorname{dist}(\boldsymbol{x}, \partial\tau_1 \cap \operatorname{int}(\tau_5)) \operatorname{dist}(\boldsymbol{x}, \partial\tau_1 \cap \operatorname{int}(\tau_3)), & \text{if } \boldsymbol{x} \in U_1 \cup U_2 \cup U_5 \\ 0, & \text{if } \boldsymbol{x} \in U_4 \cup U_6, \end{cases}$$

$$\theta_2(\boldsymbol{x}) = \begin{cases} \delta^{-2} \operatorname{dist}(\boldsymbol{x}, \partial\tau_2 \cap \operatorname{int}(\tau_6)) \operatorname{dist}(\boldsymbol{x}, \partial\tau_2 \cap \operatorname{int}(\tau_4)), & \text{if } \boldsymbol{x} \in U_3 \\ \delta^{-2} \operatorname{dist}(\boldsymbol{x}, \partial\tau_2) \operatorname{dist}(\boldsymbol{x}, \partial\tau_1 \cap \operatorname{int}(\tau_5)), & \text{if } \boldsymbol{x} \in U_2 \cup U_4 \\ \delta^{-3} \operatorname{dist}(\boldsymbol{x}, \partial\tau_2 \cap \operatorname{int}(\tau_6)) \times & \\ \qquad \operatorname{dist}(\boldsymbol{x}, \partial\tau_2 \cap \operatorname{int}(\tau_4)) \operatorname{dist}(\boldsymbol{x}, \partial\tau_1 \cap \operatorname{int}(\tau_5)), & \text{if } \boldsymbol{x} \in U_1 \\ 0, & \text{if } \boldsymbol{x} \in U_5 \cup U_6 \cup U_7, \end{cases}$$

Here, $\operatorname{int}(\tau_i)$ denotes the interior of τ_i. The function $\theta_4(\boldsymbol{x})$ is defined similarly to $\theta_1(\boldsymbol{x})$, and $\theta_3(\boldsymbol{x})$, $\theta_5(\boldsymbol{x})$, $\theta_6(\boldsymbol{x})$ similarly to $\theta_2(\boldsymbol{x})$.

The supports T_i of these functions θ_i can be one of the four shapes in Figure 6.4. The rectangles W_i mentioned in Lemma 6.8 are also depicted in Figure 6.4.

We will frequently use the following results; see [20].

Lemma 6.12. *Let $0 < \alpha < \min\{\beta, \beta'\}$. If $v \in H^{1/2}([0,\beta'] \times [0,\beta])$ then*

$$\int_0^\beta \left(\int_\alpha^{\beta'} \frac{|v(x,y)|^2}{x}\, dx \right) dy \preceq \left(1 + \log^2 \frac{\beta'}{\alpha}\right) |||v|||_{H^{1/2}([0,\beta']\times[0,\beta])}, \tag{6.6.1}$$

and

$$\frac{1}{\alpha} \|v\|^2_{L_2([0,\alpha]\times[0,\beta])} \preceq \left(1 + \log \frac{\beta'}{\alpha}\right)^2 |||v|||^2_{H^{1/2}([0,\beta']\times[0,\beta])}. \tag{6.6.2}$$

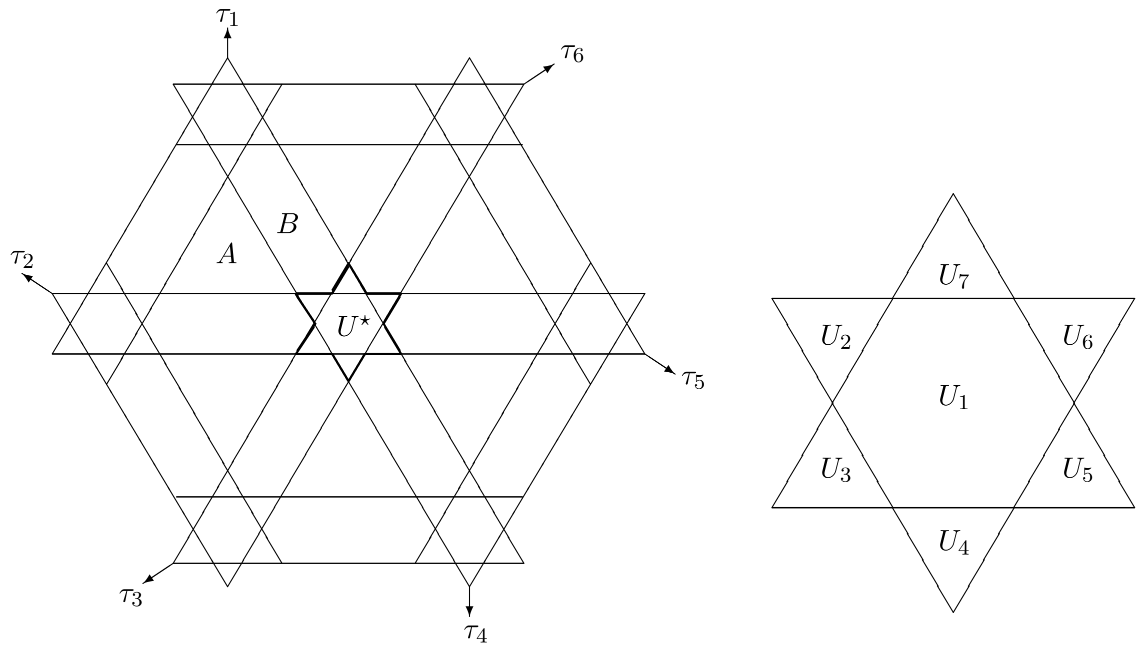

Figure 6.3: τ_i: equilateral triangles, $i = 1, \ldots, 6$.

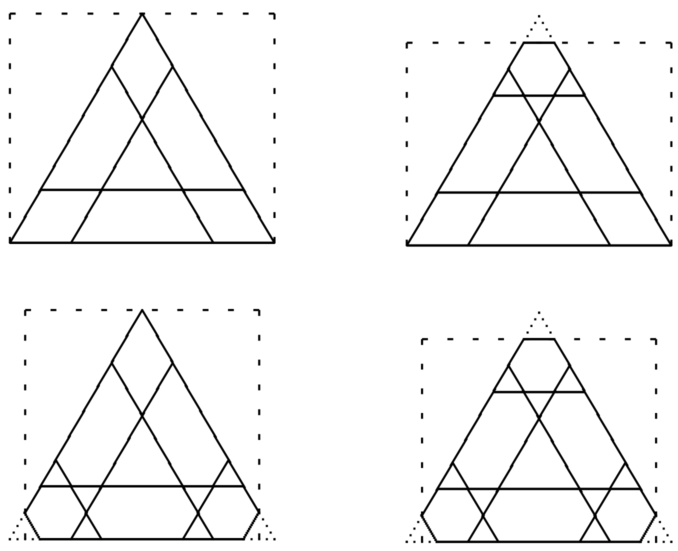

Figure 6.4: Supports T_i and rectangles W_i

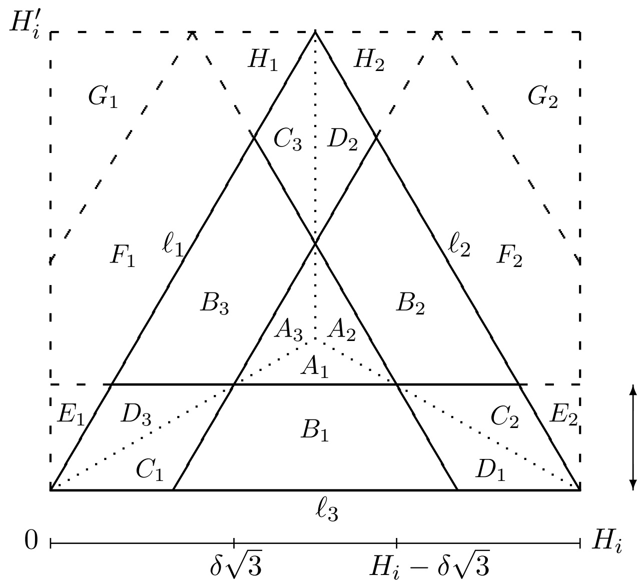

Figure 6.5: T_i and W_i

We note that even though (6.6.1) and (6.6.2) are proved in [20] for $\beta = \beta'$, the results still hold for $\beta \neq \beta'$.

Recall that for each subset $\Omega \subset \mathbb{S}^2$, $R(\Omega)$ is the image of Ω under the inverse of the radial projection $\mathcal{R}_\Omega$; see (5.3.1) and (5.3.2). It was shown in [54, Lemma 3.1] that

$$\|v\|_{L_2(\Omega)} \simeq \|\bar{v}_0\|_{L_2(R(\Omega))} \quad \text{and} \quad \|v\|_{H^1(\Omega)} \simeq \|\bar{v}_0\|_{H^1(R(\Omega))}$$

for any v belongs to $L_2(\Omega)$ and $H^1(\Omega)$, respectively. Using Theorem 2.11, we obtain

$$\|v\|_{H^{1/2}(\Omega)} \simeq \|\bar{v}_0\|_{H^{1/2}(R(\Omega))}, \quad v \in H^{1/2}(\Omega). \tag{6.6.3}$$

From (6.2.3),(6.2.4) and (6.2.5) and repeating the argument used in the proof of [54, Lemma 3.1], we obtain that for any $v \in H^{1/2}(\Omega)$, there hold

$$\begin{aligned} |||v|||_{H^{1/2}(\Omega)} &\simeq |||\bar{v}_0|||_{H^{1/2}(R(\Omega))} \\ |||v|||_{\widetilde{H}^{1/2}(\Omega)} &\simeq |||\bar{v}_0|||_{\widetilde{H}^{1/2}(R(\Omega))} \\ |v|_{H^{1/2}(\Omega)} &\simeq |\bar{v}_0|_{H^{1/2}(R(\Omega))}, \end{aligned} \tag{6.6.4}$$

where the constants are independent of the size of Ω.

Proof for Lemma 6.8: Recall that we need to prove

$$\sum_{i=1}^{J} |||\theta_i v|||^2_{\widetilde{H}^{1/2}(T_i)} \preceq \sum_{i=1}^{J} \left(1 + \log \frac{H_i}{\delta}\right)^2 |||v|||^2_{H^{1/2}(W_i)},$$

where for any $i = 1, \dots, J$, T_i is the support of the partition function θ_i and W_i is the rectangle which contains T_i and shares a common edge with T_i (see Figure 6.4). We will prove

$$|||\theta_i v|||^2_{\widetilde{H}^{1/2}(T_i)} \preceq \left(1 + \log \frac{H_i}{\delta}\right)^2 |||v|||^2_{H^{1/2}(W_i)} \tag{6.6.5}$$

for T_i being of the first shape in Figure 6.4. The cases when T_i is of other shapes can be proved in the same manner.

Equivalences (6.6.4) and (6.6.3) allow us to prove instead of (6.6.5) the following inequality

$$|||\overline{\theta_i v_0}|||^2_{\widetilde{H}^{1/2}(R(T_i))} \preceq \left(1 + \log \frac{H_i}{\delta}\right)^2 |||\overline{v}_0|||^2_{\widetilde{H}^{1/2}(R(W_i))}. \tag{6.6.6}$$

For notational convenience, in this proof we write T_i, W_i, θ_i, and v instead of $R(T_i)$, $R(W_i)$, $(\bar{\theta}_i)_0$, and $\bar{v}_0$, namely we think of T_i and W_i as planar regions, and θ_i and v as two variables functions. Here H_i and δ are the size of T_i and the size of the overlap.

It is noted that $W_i = [0, H_i] \times [0, H_i']$ where $H_i' = \sqrt{3}/2 H_i$. Recall that

$$|||\theta_i v|||^2_{\widetilde{H}^{1/2}(T_i)} = |\theta_i v|^2_{H^{1/2}(T_i)} + \int_{T_i} \frac{[\theta_i v(\boldsymbol{x})]^2}{\operatorname{dist}(\boldsymbol{x}, \partial T_i)} \, d\boldsymbol{x}, \tag{6.6.7}$$

where

$$|\theta_i v|^2_{H^{1/2}(T_i)} = \int_{T_i} \int_{T_i} \frac{|\theta_i v(\boldsymbol{x}) - \theta_i v(\boldsymbol{x}')|^2}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, d\boldsymbol{x} \, d\boldsymbol{x}'. \tag{6.6.8}$$

We first estimate the second term in the right hand side of (6.6.7), which is split into a sum of integrals over the triangles $T_i^\ell := A_\ell \cup B_\ell \cup C_\ell \cup D_\ell$ (see Figure 6.5), for $\ell = 1, 2, 3$, as follows:

$$\int_{T_i} \frac{[\theta_i v(\boldsymbol{x})]^2}{\operatorname{dist}(\boldsymbol{x}, \partial T_i)} \, d\boldsymbol{x} = \sum_{\ell=1}^{3} \int_{T_i^\ell} \frac{[\theta_i v(\boldsymbol{x})]^2}{\operatorname{dist}(\boldsymbol{x}, \partial T_i)} \, d\boldsymbol{x}.$$

We only need to estimate the integral over T_i^1, the other two can be bounded similarly. Recall that

$$\theta_i(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \boldsymbol{x} \in A_1, \\ \delta^{-1} \operatorname{dist}(\boldsymbol{x}, \partial T_i), & \text{if } \boldsymbol{x} \in B_1, \\ \delta^{-2} \operatorname{dist}(\boldsymbol{x}, \partial T_i) \operatorname{dist}(\boldsymbol{x}, \ell_1), & \text{if } \boldsymbol{x} \in C_1, \\ \delta^{-2} \operatorname{dist}(\boldsymbol{x}, \partial T_i) \operatorname{dist}(\boldsymbol{x}, \ell_2), & \text{if } \boldsymbol{x} \in D_1. \end{cases}$$

Since $\delta^{-1} \operatorname{dist}(\boldsymbol{x}, \ell_k) \le 1$ for $k = 1, 2$ and $\boldsymbol{x}$ respectively belongs to C_1 and D_1, there holds

$$\theta_i(\boldsymbol{x}) \le \delta^{-1} \operatorname{dist}(\boldsymbol{x}, \partial T_i) \quad \forall \boldsymbol{x} \in B_1 \cup C_1 \cup D_1.$$

Hence

$$\begin{aligned} \int_{T_i^1} \frac{[\theta_i v(\boldsymbol{x})]^2}{\operatorname{dist}(\boldsymbol{x}, \partial T_i)} \, d\boldsymbol{x} &= \int_{A_1} \frac{[\theta_i v(\boldsymbol{x})]^2}{\operatorname{dist}(\boldsymbol{x}, \partial T_i)} \, d\boldsymbol{x} + \int_{B_1 \cup C_1 \cup D_1} \frac{[\theta_i v(\boldsymbol{x})]^2}{\operatorname{dist}(\boldsymbol{x}, \partial T_i)} \, d\boldsymbol{x} \\ &\preceq \int_{A_1} \frac{|v(\boldsymbol{x})|^2}{y} \, d\boldsymbol{x} + \frac{1}{\delta} \int_{B_1 \cup C_1 \cup D_1} |v(\boldsymbol{x})|^2 \, d\boldsymbol{x} \\ &\le \int_0^{H_i} \left(\int_\delta^{H_i'} \frac{|v(x,y)|^2}{y} \, dy \right) dx + \frac{1}{\delta} \int_0^{H_i} \int_0^\delta |v(x,y)|^2 \, dy \, dx. \end{aligned}$$

It follows from Lemma 6.12 and $H_i' \simeq H_i$ that

$$\int_{T_i^1} \frac{[\theta_i v(\boldsymbol{x})]^2}{\operatorname{dist}(\boldsymbol{x}, \partial T_i)} \, d\boldsymbol{x} \preceq \left(1 + \log \frac{H_i}{\delta}\right)^2 |||v|||^2_{H^{1/2}(W_i)}. \tag{6.6.9}$$

We now need to use (6.2.7) to estimate the double integral in (6.6.8). This motivates us to transform the integral over the triangle T_i into the integral over the rectangle W_i. To do so, we first introduce an extension $\widetilde{\theta}_i$ of θ_i over the rectangle W_i as follows (see Figure 6.5):

$$\widetilde{\theta}_i(\boldsymbol{x}) := \begin{cases} \theta_i(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in T_i, \\ \delta^{-1} \operatorname{dist}(\boldsymbol{x}, \ell_k), & \text{if } \boldsymbol{x} \in F_k, \quad k = 1, 2, \\ \delta^{-2} \operatorname{dist}(\boldsymbol{x}, \ell_k) \operatorname{dist}(\boldsymbol{x}, \ell_3), & \text{if } \boldsymbol{x} \in E_k, \quad k = 1, 2, \\ \delta^{-2} \operatorname{dist}(\boldsymbol{x}, \ell_1) \operatorname{dist}(\boldsymbol{x}, \ell_2), & \text{if } \boldsymbol{x} \in H_k, \quad k = 1, 2, \\ 1, & \text{if } \boldsymbol{x} \in G_k, \quad k = 1, 2. \end{cases} \tag{6.6.10}$$

The extension $\widetilde{\theta}_i$ is defined in order to preserve the continuity and symmetry of θ_i across the edges ℓ_1 and ℓ_2 of T_i. Since $T_i \subset W_i$ and $\widetilde{\theta}_i = \theta_i$ on T_i, there holds

$$\int_{T_i}\int_{T_i} \frac{|\theta_i v(\boldsymbol{x}) - \theta_i v(\boldsymbol{x}')|^2}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, d\boldsymbol{x} \, d\boldsymbol{x}' \le \int_{W_i}\int_{W_i} \frac{|\widetilde{\theta}_i v(\boldsymbol{x}) - \widetilde{\theta}_i v(\boldsymbol{x}')|^2}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, d\boldsymbol{x} \, d\boldsymbol{x}'.$$

Noting (6.2.7), we have

$$\begin{aligned} \int_{W_i}\int_{W_i} \frac{|\widetilde{\theta}_i v(\boldsymbol{x}) - \widetilde{\theta}_i v(\boldsymbol{x}')|^2}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, d\boldsymbol{x} \, d\boldsymbol{x}' &\simeq \int_I\int_I \frac{\|\widetilde{\theta}_i v(x, \cdot) - \widetilde{\theta}_i v(x', \cdot)\|^2_{L_2(I')}}{|x - x'|^2} \, dx \, dx' \\ &\quad + \int_{I'}\int_{I'} \frac{\|\widetilde{\theta}_i v(\cdot, y) - \widetilde{\theta}_i v(\cdot, y')\|^2_{L_2(I)}}{|y - y'|^2} \, dy \, dy' \\ &=: \mathcal{I}_1 + \mathcal{I}_2, \end{aligned} \tag{6.6.11}$$

where $I = [0, H_i]$, $I' = [0, H_i']$, $\boldsymbol{x} = (x, y)$ and $\boldsymbol{x}' = (x', y')$. We will show that $\mathcal{I}_1$ is bounded by

$$\left(1 + \log \frac{H_i}{\delta}\right)^2 |||v|||^2_{H^{1/2}(W_i)}.$$

The term $\mathcal{I}_2$ can be estimated in the same manner. By using the triangular inequality, and noting that $\widetilde{\theta}_i \le 1$, we have

$$\mathcal{I}_1 \le \int_I\int_I \frac{\|[\widetilde{\theta}_i(x, \cdot) - \widetilde{\theta}_i(x', \cdot)]v(x, \cdot)\|^2_{L_2(I')}}{|x - x'|^2} \, dx \, dx' + \int_I\int_I \frac{\|v(x, \cdot) - v(x', \cdot)\|^2_{L_2(I')}}{|x - x'|^2} \, dx \, dx'. \tag{6.6.12}$$

It follows from (6.2.7) that the last integral in (6.6.12) is bounded by $|||v|||^2_{H^{1/2}(W_i)}$. We still need to estimate

$$A_{I,I}(\widetilde{\theta}_i v) := \int_I\int_I \frac{\|[\widetilde{\theta}_i(x, \cdot) - \widetilde{\theta}_i(x', \cdot)]v(x, \cdot)\|^2_{L_2(I')}}{|x - x'|^2} \, dx \, dx'.$$

We denote $I_1 := [0, \delta\sqrt{3}]$, $I_2 := [\delta\sqrt{3}, H_i - \delta\sqrt{3}]$, and $I_3 := [H_i - \delta\sqrt{3}, H_i]$. In order to show that $A_{I,I}(\widetilde{\theta}_i v)$ is bounded by $|v|^2_{H^{1/2}(W_i)}$, we will prove that $A_{I_k,I_\ell}(\widetilde{\theta}_i v)$ are bounded by $|||v|||^2_{H^{1/2}(W_i)}$, for $k, \ell = 1, 2, 3$. By the symmetry of $\widetilde{\theta}_i$, it is sufficient to prove the estimation for $A_{I_k,I_\ell}(\widetilde{\theta}_i v)$ for $(k, \ell) \in \{(1, 1), (1, 2), (1, 3), (2, 2)\}$.

Let $x, x' \in I_1$. Elementary calculation (though tedious) reveals

$$\max_{y\in I'} |\widetilde{\theta}_i(x, y) - \widetilde{\theta}_i(x', y)| \preceq \delta^{-1}|x - x'|.$$

Thus

$$A_{I_1,I_1}(\widetilde{\theta}_i v) \preceq \frac{1}{\delta}\int_{I_1} \|v(x, \cdot)\|^2_{L_2(I')}\, dx = \frac{1}{\delta}\|v(\cdot, \cdot)\|^2_{L_2(I_1\times I')}.$$

Noting that the size of I_1 is defined to be proportional to δ, and by using (6.6.2), we deduce

$$A_{I_1,I_1}(\widetilde{\theta}_i v) \preceq \left(1 + \log \frac{H_i}{\delta}\right)^2 |||v|||^2_{H^{1/2}(W_i)}. \tag{6.6.13}$$

We next estimate $A_{I_1,I_2\cup I_3}(\widetilde{\theta}_i v)$ by estimating the two integral $A_{I_1,[\delta\sqrt{3},\, 2\delta\sqrt{3}]}(\widetilde{\theta}_i v)$ and $A_{I_1,[2\delta\sqrt{3},\, H_i]}(\widetilde{\theta}_i v)$. Repeating the argument used in estimating $A_{I_1,I_1}(\widetilde{\theta}_i v)$, we obtain

$$A_{I_1,[\delta\sqrt{3},\ 2\delta\sqrt{3}]}(\widetilde{\theta}_i v) \preceq \left(1 + \log \frac{H_i}{\delta}\right)^2 |||v|||^2_{H^{1/2}(W_i)}. \tag{6.6.14}$$

We note here that

$$\max_{y\in I'}[\widetilde{\theta}_i(x, y) - \widetilde{\theta}_i(x', y)]^2 \le 4, \quad \forall x, x' \in I, \tag{6.6.15}$$

and

$$|x - 2\delta\sqrt{3}| \ge \delta\sqrt{3} \quad \forall x \in I_1. \tag{6.6.16}$$

We then deduce from (6.6.15), (6.6.16) and (6.6.2) that

$$\begin{aligned}
A_{I_1,[2\delta\sqrt{3},\ H_i]}(\widetilde{\theta}_i v) &\preceq \int_{I_1} \|v(x, \cdot)\|^2_{L_2(I')} \int_{[2\delta\sqrt{3},\ H_i]} \frac{1}{(x' - x)^2}\, dx'\, dx \\
&\preceq \int_{I_1} \frac{H_i - 2\delta\sqrt{3}}{(2\delta\sqrt{3} - x)(H_i - x)} \|v(x, \cdot)\|^2_{L_2(I')}\, dx \\
&\preceq \frac{1}{\delta}\|v(\cdot, \cdot)\|^2_{L_2(I_1\times I')} \\
&\preceq \left(1 + \log \frac{H_i}{\delta}\right)^2 |||v|||^2_{H^{1/2}(W_i)}.
\end{aligned} \tag{6.6.17}$$

We note here that the estimation for $A_{I_1,I_1}(\widetilde{\theta}_i v)$ is based on the fact that the size of I_1 is proportional to δ and

$$|\widetilde{\theta}_i(x, y) - \widetilde{\theta}_i(x', y)| \preceq \delta^{-1}|x - x'| \quad \forall x, x' \in I_1, \quad \forall y \in I'. \tag{6.6.18}$$

The proof $A_{I_1,[\delta\sqrt{3},\, H_i]}(\widetilde{\theta}_i v)$ is then obtained by first splitting it into $A_{I_1,[\delta\sqrt{3},\, 2\delta\sqrt{3}]}(\widetilde{\theta}_i v)$ and $A_{I_1,[2\delta\sqrt{3},\, H_i]}(\widetilde{\theta}_i v)$ in which the former is bounded by using similar argument as in the proof for $A_{I_1,I_1}(\widetilde{\theta}_i v)$, requiring the sizes of I_1 and $[\delta\sqrt{3}, 2\delta\sqrt{3}]$ to be proportional to δ and (6.6.18) to hold for $x \in I_1$, $x' \in [\delta\sqrt{3}, 2\delta\sqrt{3}]$. The latter is estimated by first using (6.6.15) to write it in the form containing the integral

$$\int_{[2\delta\sqrt{3},H_i]} \frac{1}{|x - x'|^2}\, dx',$$

		$d=1$	$\omega^2=0.01$	$d=2$	$\omega^2=0.01$	$d=3$	$\omega^2=0.1$
N	h	$\kappa(\boldsymbol{A})$	α	$\kappa(\boldsymbol{A})$	α	$\kappa(\boldsymbol{A})$	α
18	0.7071	12.27		469.36		19109.90	
66	0.3536	20.90	-0.77	736.53	-0.65	25735.72	-0.42
258	0.1768	39.22	-0.91	1422.77	-0.95	50877.82	-0.98
1026	0.0883	75.93	-0.95	2798.35	-0.97	100876.63	-0.99

Table 6.1: Unpreconditioned systems with uniform triangulations; $\kappa(\boldsymbol{A}) = O(h^{\alpha})$.

DoF	h	$\kappa(\boldsymbol{A})$	H	$\kappa(P)$
66	0.3536	20.90	0.7071	5.37
258	0.1768	39.22	0.3536	6.05
			0.7071	6.45
1026	0.0883	75.93	0.1768	6.46
			0.3536	6.68
			0.7071	7.45
4098	0.0442	149.40	0.0883	6.71
			0.1768	6.80
			0.3536	7.90
			0.7071	9.05

Table 6.2: Condition numbers when $d = 1$, $\omega^2 = 0.01$ with uniform triangulations.

which is then proved to be bounded by $c\delta^{-1}$ for some constant $c > 0$. This procedure can be used in estimating $A_{I_2,I_2}(\widetilde{\theta}_i v)$ by first writing

$$A_{I_2,I_2}(\widetilde{\theta}_i v) = \int_{I_2}\int_{I_2}\int_{[0,H_i']} \frac{[\widetilde{\theta}_i(x,y) - \widetilde{\theta}_i(x',y)]^2 v^2(x,y)}{|x-x'|^2}\, dy\, dx\, dx'.$$

The integral is then split into sum of integrals over subregions in which (x, y) and (x', y) can belong to one of the following sets G_1, $F_1 \cup H_1$, $B_3 \cup C_3 \cup D_2 \cup H_2$, $H_1 \cup C_3 \cup D_2 \cup B_2$, $H_2 \cup F_2$, G_2, and $A_1 \cup A_2 \cup A_3$ in Figure 6.5. The integral when (x, y) and (x', y) belong to $A_1 \cup A_2 \cup A_3$ is easily bounded by $|||v|||_{H^{1/2}(W_i)}$ by using (6.2.7), noting that $\widetilde{\theta}_i(x, y) = \widetilde{\theta}_i(x', y) = 1$. The other integrals can be estimated by using similar argument as used in the proof for $A_{I_1,I}(\widetilde{\theta}_i v)$.

Combining all these we obtain (6.6.5). $\square$

DoF	h	$\kappa(\boldsymbol{A})$	H	$\kappa(P)$
258	0.3536	736.53	0.7071	7.24
1026	0.1768	1422.77	0.3536	6.92
			0.7071	7.06
4098	0.0883	2798.35	0.1768	7.22
			0.3536	6.90
			0.7071	6.59

Table 6.3: Condition numbers when $d = 2$, $\omega^2 = 0.01$ with uniform triangulations.

DoF	h	$\kappa(\boldsymbol{A})$	H	$\kappa(P)$
578	0.3536	25735.72	0.7071	412.96
2306	0.1768	50877.82	0.3536	13.49
			0.7071	24.62
9218	0.0883	100876.63	0.1768	7.94
			0.3536	8.62
			0.7071	8.13

Table 6.4: Condition numbers when $d = 3$, $\omega^2 = 0.1$ with uniform triangulations.

DoF	h	$\kappa(\boldsymbol{A})$	H	$\kappa(P)$
204	0.511	390.4	1.307	19.5
			1.670	19.2
			2.292	26.7
414	0.370	509.9	1.307	17.2
			1.670	17.0
			2.292	27.1
836	0.280	785.2	1.307	13.8
			1.670	14.9
			2.292	28.3
1635	0.184	1048.5	1.307	11.1
			1.670	16.3
			2.292	31.5

Table 6.5: Condition numbers when $d = 1$, $\omega^2 = 0.001$ with MAGSAT satellite data.

DoF	h	$\kappa(\boldsymbol{A})$	H	$\kappa(P)$
810	0.511	1170.9	1.307	23.2
			1.670	11.7
			2.292	5.7
1650	0.370	1616.9	1.307	17.7
			1.670	9.0
			2.292	10.0
3338	0.280	2230.5	1.307	13.2
			1.670	11.1
			2.292	8.1
6534	0.184	3046.7	1.307	13.1
			1.670	9.0
			2.292	6.3

Table 6.6: Condition numbers when $d = 2$, $\omega^2 = 0.01$ with MAGSAT satellite data.

DoF	h	$\kappa(\boldsymbol{A})$	H	$\kappa(P)$
1820	0.511	3322.0	1.307	23.9
			1.670	10.9
			2.292	116.6
3710	0.370	4696.2	1.307	16.5
			1.670	7.7
			2.292	186.9
7508	0.280	6211.0	1.307	12.1
			1.670	12.0
			2.292	167.6

Table 6.7: Condition numbers when $d = 3$, $\omega^2 = 1.2$ with MAGSAT satellite data.

Conclusion

Spaces of spherical radial basis functions and spherical splines are used in the solution of pseudodifferential equations on spheres. Each of the two spaces has both advantages and disadvantages.

The use of spherical radial basis functions results in meshless methods. Good approximation and easy to program in high dimensions are some advantages of spherical radial basis functions. Meanwhile, the use of truncated series to compute singular integrals involving spherical radial basis functions can affect the approximation quality of the method in implementation when solving weakly singular and hypersingular integral equations.

Finite element methods by using spherical splines also apppear to be a very powerful tool to solve pseudodifferential equations. Good approximation, simple forms of the solution as linear combinations of piecewise spherical harmonics, and the ability to control the smoothness of the solution and its derivatives across edges of the triangulations are the most important advantages. Besides, efficient quadrature rules can be used to compute singular integrals involving basis functions of the space of spherical splines. This is another advantage of the use of spherical splines. Meanwhile, mesh generation and refinement costs are some disadvantages. Ill-conditionedness may also arise. However, it can be tackled by using efficient preconditioners as shown in Chapters 5 and 6.

A future study may be coupling of spherical radial basis functions and spherical splines which is not in the scope of this dissertation.

Index

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