# Solutions to differential equations via fixed point approaches: new mathematical foundations and applications

**Author:**
Almuthaybiri, Saleh

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# Solutions to differential equations via fixed point approaches: New mathematical foundations and applications

Saleh Salhan G. Almuthaybiri

Supervised by Professor Christopher C. Tisdell

School of Mathematics and Statistics
Faculty of Science
The University of New South Wales
Sydney, Australia

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A thesis in fulfillment of the requirements
for the degree of Doctor of Philosophy

# Thesis/Dissertation Sheet

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**Abstract 350 words maximum:**

The central aim of this thesis is to construct a fuller and firmer mathematical foundation for the solutions to various classes of nonlinear differential equations than is currently available in the literature. This includes boundary value problems (BVPs) that involve ordinary differential equations, and initial value problems (IVPs) for fractional differential equations.

In particular, we establish new conditions that guarantee the existence, uniqueness and approximation of solutions to second-order BVPs, third-order BVPs, and fourth-order BVPs for ordinary differential equations. The results enable us, in turn, to shed new light on problems from applied mathematics, engineering and physics, such as: the Emden and Thomas-Fermi equations; the bending of elastic beams through an application of our general theories; and laminar flow in channels with porous walls.

We also ensure the existence, uniqueness and approximation of solutions to some IVPs for fractional differential equations. An understanding of the existence, uniqueness and approximation of solutions to these problems is fundamental from both pure and applied points of view.

Our methods involve an analysis of nonlinear operators through fixed-point theory in new and interesting ways. Part of the novelty involves generating new conditions under which these operators are contractive, invariant and/or establishing new *a priori* bounds on potential solutions. As such, we draw on: Banach fixed-point theorem, Schauder fixed-point theorem, Rus's contraction mapping theorem, and a continuation theorem due to A. Granas and its constructive version known as continuation method for contractive maps.

The ideas in this thesis break new ground at the intersection of pure and applied mathematics. Thus, this work will be of interest to those who are researching the theoretical aspects of differential equations, and those who are interested in better understanding their applications.

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# Acknowledgements

I believe in that it is Allah (God) Who guides and helps us to gain knowledge by means of many ways such as hearing, looking, thinking, reading, and writing: Allah says "Read! In the Name of your Lord Who has created (all that exist). Created man, out of a (mere) clot of congealed blood. Read! And your Lord is the Most Generous. He Who taught (the writing) by the pen. Taught man that which he knew not", Quran, Chapter 96, Al-Alaq (The Clot), verse 1-5; and He also says "And Allah has brought you forth from the wombs of your mothers when you knew nothing, and He gave you hearing and sight and intelligence and affections: that you may give thanks (to Allah)", Quran, Chapter 16, An-Aahl (The Bee), verse 78.

Therefore, I will have always primarily and foremost been grateful and thankful to Allah for giving me so much including the opportunity, ability, strength, and knowledge, which have made this research be completed; all Praise be to Allah. Then, since my journey of study until this research is completed has been more than 8 years, there are many people in my life who really have helped me complete this research and so they truly deserve to be thanked and appreciated.

- My journey of study began in 2013 when I obtained a position (after finishing my undergraduate degree) at Qassim University (QU), Saudi Arabia, as a teaching assistant in their Mathematics Department. After a year of teaching at the department, I was awarded a full scholarship for international study and I chose to begin my studies in the United States where I first met the English language requirements and then pursed my master of science degree in applied mathematics from the University of Dayton (UD) in 2017. After this, I decided to relocating to The University of New South Wales (UNSW) in Australia to pursue a PhD. Therefore, I would like to take this opportunity to thank both QU and the Saudi Arabian Cultural Mission (SACM) in both Australia and the United States for having provided full financial and administrative supports during my such a long journey. This has really made this research possible at UNSW. In return, it is my hope to use my educational experiences at UNSW and UD to benefit not only myself, and the mathematical field, but also more importantly QU and the people of my country, Saudi Arabia.

- My parents and family: I've always been thankful and grateful to them for their never ending love, care, and ongoing support and encouragement. Special thanks to my dad whom I had asked before accepting my position at QU and said to him "Dad, would you support me and let me accept this position, which will requires me to study abroad for at least 7 years", then he said "O son go ahead and accept it and come back to me whenever you are satisfied with your study". Without his permission and support I would have not achieved this stage. In addition very special thanks to my mother: at the beginning she did not really want me to take this position but she was told how it works and then understood it, so I would really thank her for her truly never ending love, care, and ongoing prayer for me, and I believe in that her prayer have helped me complete this research. In return, I ask Allah to make me the most kind to my mother and bad, and not overbearing or miserable. Again and always, many thanks to my family-my wife and daughter- for their never ending love, care, and ongoing support and for being with me at every stage that I have gone through during my journey. Therefore, I am truly dedicating this achievement to them all.

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Since "A PhD is a stepping stone into a research career" ( Mullins and Kiley, 2002 p.386), I would ask Allah, "O my Lord! advance me in knowledge", Quran, Chapter 20, Taha, verse 114. Also, I would like to ask Him, last but not least, to give me all the success in my new life career in a way that in one day I shall make a history in the field of mathematics.

**Saleh S. Almuthaybiri**

June 03, 2021

# Contents

# Chapter 1

# Introduction

## 1.1 Motivations

A core purpose of differential equations is to describe, model and predict a range of phenomena. Equations such as ordinary differential equations (ODEs) where the order is an integer; and fractional differential equations (FDEs) where the order is not necessarily an integer, continue to form important tools that are employed by the mathematical, scientific and engineering communities in the above ways. ODEs, in particular, have served as a powerful and essential tool to describe and analyze physical problems for more than 300 years. For example, ODEs have been applied to generate a better understanding of electronic structure of atoms, astrophysics and electron density theory see [15, 38, 43, 41, 42, 44, 46, 48, 49, 51, 50, 53, 59, 70, 83, 92, 102, 110, 119, 129, 166, 167, 162, 163, 164, 165, 191, 196, 213, 214, 259] and the references therein. ODEs can also be connected with models from the deflection of a curved beam having a constant or varying cross-section, three layer beams, electromagnetic waves, gravity-driven flows, and laminar flows see [45, 81, 112, 221, 226, 242] and the references therein; oscillation theory see [72, 252]; and elastic beam deflections for details see [10, 31, 34, 77, 84, 88, 91, 97, 99, 101, 104, 116, 118, 120, 124, 130, 154, 155, 157, 159, 178, 185, 183, 184, 209, 222, 277, 282, 284, 285, 286, 289, 290, 288, 295].

FDEs can also serve as a powerful tool to describe natural phenomena. However, despite that FDEs are as old as the ODEs [142, 228], there has been considerable interest in the researching of FDEs in just recent decades due to their development of the connection with certain scientific models, for example, from viscoelasticity see [5, 71, 139, 235] and the references therein, and quantum mechanics see [65, 75, 127, 140, 223, 224, 239, 241, 276] and the references therein.

Additionally, FDEs can also be connected with models from Brownian motion see [12, 13, 32, 33, 54, 86, 103, 131, 168, 169, 182, 186, 188, 189, 238, 292, 302] and the references therein. The reader is also referred to [125, 140, 215, 236, 302] for more references.

The important applications of ODEs and FDEs has motivated many researchers to explore questions involving solutions to these problems. These questions include:

(**1**) When does a differential equation have a solution?

(**2**) When is there only one solution?

(**3**) How can we approximate or construct this solution?

Part of the significance of this kind of knowledge comes from theoretical and practical perspectives, as "knowing an equation has a unique solution is important from both a modeling and theoretical point of view" [263, p.794].

In fact, this has been supported by the late Louis Nirenberg's comments in his Abel Prize lecture of 2015 (Some Remarks on Mathematics):

"I've also worked on the theory of the (differential) equations themselves. Do solutions exist? In general, you cannot write down, specifically, a solution. Sometimes you can use computers to compute very good approximations to solutions, but sometimes, somebody comes up with a mathematical model of some problem - some (partial) differential equation - and it turns out that it does not have solutions at all. There are equations that don't have solutions. So, part of the problem is, given some model, are there solutions? Are the solutions regular? Are they unique? What properties can you show for the solutions - maybe some kind of symmetry or monotonicity, or things like that? These are things that you want to investigate".

To address the aforementioned questions, a significant number of important theorems in fixed point theory (FPT) has been since the twentieth century developed in order to analyze the differential equations (brief details about FPT are given in Section 1.4). Such ideas have played vital role in deepening our understanding of various features of, for example, such aforementioned models. Therefore, this thesis is devoted to examining the existence, uniqueness and approximation of solutions to various classes of nonlinear differential equations. This includes boundary value problems (BVPs) that involve ordinary differential equations, and initial value problems (IVPs) for fractional differential equations, where the majority of these problems can be linked with multiple applications. Our methods involve an analysis of these problems through FPT.

Before introducing my main results, let me first provide a brief history of fractional differential equations, some basic concepts, a brief introduction of fixed-point methods, and outline of this thesis. Then, at the end of this Chapter, I shall provide a direct references of the published and submitted results that arise from this thesis.

## 1.2 Brief history of some differential equations

Differential equations were first introduced in the 17th century by Isaac Newton and Gottfried Wilhelm Leibniz, emerging from their calculus research. Three forms of the first order differential equations that were found in unpublished rough notes made by Newton are

$$\frac{dy}{dx} = f(x),$$

$$\frac{dy}{dx} = f(y),$$

$$\frac{dy}{dx} = f(x, y).$$

The above equations include only ordinary derivatives of or more dependent variables, which are currently referred to as ODEs and it is a modern term. Another form is

$$x\frac{\partial u}{\partial x} + y\frac{\partial y}{\partial y} = u,$$

and it includes partial derivatives of dependent variables, which are referred to as partial differential equations (PDEs) and it is also a modern term. I remark that both of these terms, ODEs and PDEs, were called 'fluxions' in Newtons' original work. Although Newton was the first to determine the fundamental results of calculus, Leibniz, working independently, was the first to publish them in 1684. Leibniz was aware of the necessity of having a great mathematical notations and so he was in charge of the modern notations for the derivative $\frac{dy}{dx}$ and for the derivative of order $n \in \mathbb{N}_0 := \{0, 1, 2, 3, \ldots\}$ that is $\frac{d^n y}{dx^n}$ .

In 1693, that is about two decades following Newton and Leibniz's groundbreaking work on differential equations, they officially published their results, see [142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 152, 197, 198, 199, 200, 201] and the references therein. This is when the larger mathematics community became aware of the subject and since then the fields of ODEs and PDEs have enjoyed significant developments, particularly the development of its extension forms. This includes boundary-value problems (BVPs), which first emerged in the 18th century.

BVPs are now one of the important areas of the fields of ODEs and PDEs. Despite the substantial amount of research into BVPs that already exists, it continues to be an important area of research for both applied mathematics and especially for physics scholars to their connections with very important scientific models. Therefore, BVPs, that involve ODEs, form the basis of most of my thesis, particularly in Chapter 2, Chapter 3, Chapter 4, Chapter 5, and Chapter 6 where will see important examples of scientific applications that are modelled by BVPs.

Moreover, FDEs have been in recent decades considered to be an important area due to their importance and development of the connection with certain scientific applications. The idea of a differential equation of noninteger order, for example, a differential equation of order $1/2$ in particular, is what Marquis de L'Hospital, in 1695, discussed with Gottfried Wilhelm Leibniz [142] (also see [228]) and so the fractional of calculus is believed to have emerged from such idea. In particular, L'Hospital asked Leibniz about the meaning of Leibniz's notation $\frac{d^n y}{dx^n}$ ("What if $n$ be $1/2$"), and the response of Leibniz to L'Hospital , dated 30 September, 1695, was the following

"... This is an apparent paradox from which, one day, useful consequences will be drawn. ..."

Following this discussion, the subject of fractional of calculus was mentioned by many great mathematicians. For example, in 1730 Euler, Leonhard [80, 228] wrote

"When $n$ is a positive integer, and if $p$ should be a function of $x$, the ratio $d^n p$ to $dx^n$ can always be expressed algebraically, so that if $n = 2$ and $p = x^3$, then $d^2(x^3)$ to $d(x^2)$ is $6x$ to $1$. Now it is asked what kind of ratio can then be made if $n$ be a fraction. The difficulty in this case can easily be understood. For if $n$ is a positive integer $d^n$ can be found by continued differentiation. Such a way, however, is not evident if $n$ is a fraction. But yet with the help of interpolation which I have already explained in this dissertation, one may be able to expedite the matter."

In 1772, J. L. Lagrange [135, 228] developed the law of exponents for differential operators of integer orders and stated the following

$$\frac{d^m}{dx^m} \cdot \frac{d^n}{dx^n} y = \frac{d^{m+n}}{dx^{m+n}} y.$$

Note that the "dot" is not a multiplication and is omitted in modern notation. Lagrange's result may be considered to be an indirect contribution to the subject of fractional of calculus, since after the development of the theory of fractional calculus, the analogous rule holds for $m$ and $n$ being arbitrary with some restrictions imposed on $y$, see for example [68].

In 1812, P. S. Laplace defined a fractional derivative by means of an integral. Seven years later, a formula for $n$th order derivatives of $y = x^m$, where $m$ is a positive integer, was developed by S. F. Lacroix [134, pp.409-410]. His formula is given by

$$D^\alpha y := \frac{d^\alpha}{dx^\alpha}(x^m) = \frac{m!}{(m-\alpha)!} x^{m-\alpha}, \quad m \geq \alpha,$$

where $(\cdot)!$ is the factorial function with (as usual) $0! = 1$. It is a well-known result that the Gamma function $\Gamma(\cdot)$ (a full definition is given in 1.1 a little later) is a generalization of factorial function. Thus, by replacing the factorial function by the Gamma function, S. F. Lacroix obtained the following formula

$$D^\alpha y = \frac{\Gamma(m+1)}{\Gamma(m-\alpha+1)} x^{m-\alpha}.$$

Lacroix then gave an example, which provides the correct answer to the problem raised on the discussion between L'Hospital and Leibniz, that is for $y = x$, $m = 1$ and $\alpha = 1/2$, he obtained

$$\frac{d^{1/2}x}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = 2\sqrt{x/\pi}. \tag{1.1}$$

Surprisingly, the Lacroix's result is the same as that yielded by well-known of present-day definition of a fractional derivative known as Riemann–Liouville's definition (a full definition is given in Section 1.3.2). In 1822, it was Joseph B. J. Fourier [87, 228] who made the next contribution to the subject of fractional of calculus by giving a generalization of notation for differentiation of arbitrary function.

As we can see from above discussion, the derivatives of arbitrary order was mentioned by Leibniz, Euler, Laplace, Lacroix, and Fourier. However, the first application of the fractional operation was done by Niels Henrik Abel [3] in 1823. He applied the fractional of calculus in the solution of an integral equation, that arises in physical problem, which has been described as "elegant."

All of this was a motivation for Joseph Liouville to study the fractional of calculus and made the first major breakthrough into the subject of fractional of calculus when he published several publications in rapid succession during the third decade of 19th century [170, 171, 172, 173, 174, 175, 176]. Since then, the subject of fractional of calculus has enjoyed significant developments thus far and when we look back to this period, the famous book of Oldham and Spanier [204], published in 1971, may naturally spring to mind since it was the first work devoted specifically for the subject of fractional of calculus. Following this work, there has been a vast amount of literature on the subject, for example see [60, 68, 106, 125, 194, 215, 236, 293] and the references therein. Today, there are some famous international journals devoted mostly to the field of fractional of calculus. Therefore, IVPs for fractional differential equations will be a core dimension of this thesis, particularly in Chapter 7 and Chapter 8. I refer the reader to the significant references cited above and in these chapters for the developments in the field of IVPs for fractional differential equations.

## 1.3 Some basic concepts

In this Section I give some basic concepts and preliminaries that I will use throughout the course of the thesis. In particular, I shall discuss some basic concepts of the fractional calculus including some special functions, definitions and basic properties of the fractional calculus. I will then briefly introduce and construct some new classical and weighted Banach spaces.

Let me first start by introducing briefly some special functions.

### 1.3.1 Some special functions

Historically, special functions such as exponential function, factorial function, Gamma function and their extensions functions, have a strong connection with solutions to many differential equations including fractional differential equations. Thus I shall briefly introduce two special functions: (1) Gamma function, which is the extension of the factorial function, (2) the Mittag–Leffler function, which is the extension of the exponential function; see [14, 26, 79, 89, 125, 141, 215, 236] . These functions will be essentially important in my work, particularly in Chapters 7 and 8, in which they will be used.

**Gamma Function:**

The Gamma function is such an important function in various areas such as definite integration, hypergeometric series, Riemann zeta function, number theory as well as in the fractional calculus, and so I give the following definition of the Gamma function.

**Definition 1.1** (See [236, 125, 215] ). *The function* $\Gamma : (0, \infty) \to \mathbb{R} := (-\infty, \infty)$, *defined by*

$$\Gamma(x) = \int_0^\infty s^{x-1} e^{-s} ds, \quad x > 0, \tag{1.2}$$

*is called the Euler's Gamma function (known as the Euler integral of the second kind).*

One can easily calculate $\Gamma(1)$, that is $\Gamma(1) = 1$.

The following formula can be obtained from (1.2) by integration by parts

$$\Gamma(x+1) = x\Gamma(x), \quad x > 0. \tag{1.3}$$

(1.3) is an important functional equation, and so for $n \in \mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$, functional equation (1.3) becomes

$$\Gamma(n+1) = n!, \tag{1.4}$$

with (as usual) $0! = 1$.

Now I introduce my second function: the Mittag–Leffler Function.

**Mittag–Leffler Function:**

In the fractional calculus the Mittag–Leffler function plays a similar role to that of the exponential function in classical calculus, see for example [125, 215, 236].

**Definition 1.2** (See [125, 215, 236] )**.** *Let $\mu > 0$ and $\nu > 0$ be real numbers. Then a two-parameter Mittag-Leffler function $E_{\mu,\nu} : \mathbb{R} \to \mathbb{R}$ is defined by the series expansion given by*

$$E_{\mu,\nu}(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\mu k + \nu)}. \tag{1.5}$$

From above definition of the two-parameter Mittag-Leffler function (1.5), it follows that if $\nu = 1$, then we have

$$E_{\mu,1}(x) := E_{\mu}(x). \tag{1.6}$$

The above function (1.6) will be my particular interest for my work in Chapters 7 and 8. Moreover, if $\mu = \nu = 1$ then (1.5) becomes an exponential function that is

$$E_{1,1}(x) := e^x. \tag{1.7}$$

I refer the reader to [14, 26, 79, 89, 125, 141, 215, 236] for more details regarding the above special functions and others special functions.

### 1.3.2 Some definitions and basic properties of the fractional derivatives/Integral

I now introduce some notations, definitions and basic properties that are related to the fractional calculus. In particular, I shall introduce two popular definitions of fractional of calculus, which are of particular relevance to my work. These definitions are known as the Riemann–Liouville fractional integrals and fractional derivatives and Caputo fractional integrals and fractional derivatives. But I first need to introduce some function spaces.

**Definition 1.3** (See [68]). *Let $[a,b]$ $(0 \le a < b < \infty)$ be a finite interval of $\mathbb{R}_+ := [0,\infty)$, $k \in \mathbb{N}_0 := \{0,1,2,3,\ldots\}$, $m \in \mathbb{N} := \{1,2,3,\ldots\}$ and $1 \le q$.*

$$\begin{aligned}
L_p([a,b]) &:= \left\{ y : [a,b] \to \mathbb{R}; y \text{ is measurable on } [a,b] \text{ and } \int_a^b |y(x)|^p \, dx < \infty \right\}, \\
L_\infty([a,b]) &:= \left\{ y : [a,b] \to \mathbb{R}; y \text{ is measurable and essentially bounded on } [a,b] \right\}, \\
C^k([a,b]) &:= \left\{ y : [a,b] \to \mathbb{R}; y \text{ has a continuous k-th derivative} \right\}, \\
C^0([a,b]) &:= C[a,b], \\
AC^m([a,b]) &:= \left\{ y : [a,b] \to \mathbb{R}; y \text{ and } D^{m-1}y(x) \in AC[a,b] \ (D = \frac{d}{dx}) \right\}.
\end{aligned}$$

Above, for $1 \le q \le \infty$, $L_p([a,b])$ is the usual Lebesgue space. I denote by $C^k([a,b])$ the set of real-valued functions $y$ that are defined on $[a,b]$ and are $k-$times continuously differentiable therein, also by $AC^m[a,b]$ I denote the set of real-valued functions $y$ that are defined on $[a,b]$ and have continuous derivatives up to order $m-1$ therein. I assume that the reader is familiar with the above concepts of function spaces from functional analysis and I refer the reader to [52, 66, 229, 294].

I now state the following definition of the Riemann–Liouville fractional integrals.

**Definition 1.4** (See [125, 215, 236]). *Let $\alpha > 0$. The $\alpha$-th Riemann–Liouville fractional integral operator $I_a^\alpha$of a function, $y \in L_1[a,b]$, is defined for a.e $x \in [a,b]$ by*

$$I_a^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-s)^{\alpha-1} y(s) ds, \quad x > a. \tag{1.8}$$

*Here $\Gamma(\alpha)$ is the Gamma function (1.2). When $\alpha = m \in \mathbb{N} := \{1,2,3,\ldots\}$, the definition (1.8) coincides with the $m$th integral of the form*

$$I_a^m y(x) = \frac{1}{(m-1)!} \int_a^x (x-s)^{m-1} y(s) ds, \quad (m \in \mathbb{N}). \tag{1.9}$$

*For $\alpha = 0$, define $I_a^0$ to be the identity operator.*

I state the following definition of the Riemann–Liouville fractional derivatives.

**Definition 1.5** (See [125, 215, 236]). *Let $m \in \mathbb{N}$ and assume $1 - m < \alpha < m$. The Riemann–Liouville fractional differential operator $D_a^\alpha$ of order $\alpha$ is defined when $D^{m-1}(I^{m-\alpha}y) \in AC([a,b])$, that is $I^{m-\alpha}y \in AC^{m-1}([a,b])$, by*

$$D_a^\alpha y(x) := \left(\frac{d}{dx}\right)^m (I_a^{m-\alpha}y)(x) = \frac{1}{\Gamma(m-\alpha)}\left(\frac{d}{dx}\right)^m \int_a^x (x-s)^{m-\alpha-1}y(s)ds, \quad x > a. \tag{1.10}$$

*In particular, when $\alpha = m \in \mathbb{N}_0$, then*

$$D_a^0 y(x) := y(x),$$

*and*

$$D_a^m y(x) := y^{(m)}(x),$$

*where $y^{(m)}(x)$ denotes the classical $m$th order derivative. Moreover, if $0 < \alpha < 1$, then (1.10) takes the following form*

$$D_a^\alpha y(x) := \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-s)^{-\alpha}y(s)ds, \quad x > a. \tag{1.11}$$

The following Lemma shows how we can verify the the Riemann–Liouville fractional integration and differentiation operators of a power function such as $(x-a)^{\beta-1}$. These will yield a power function of the same form. For their proofs I refer to [125, 215].

**Lemma 1.1** (See [125]). *If $\alpha \geq 0$ and $\beta > 0$, then*

$$(I_a^\alpha (s-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}, \tag{1.12}$$

*and*

$$(D_a^\alpha (s-a)^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}. \tag{1.13}$$

*In particular, if $\beta = 1$ and $\alpha \geq 0$, then the Riemann–Liouville fractional derivatives of a constant are, in general, not equal zero:*

$$(D_a^\alpha 1)(x) = \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad 0 < \alpha < 1. \tag{1.14}$$

**Remark 1.1.** *If $a = 0$, $\alpha = 1/2$, and $\beta = 2$. The (1.13) becomes*

$$(D_0^{1/2}s)(x) := (D^{1/2}s)(x) = \frac{\Gamma(2)}{\Gamma(3/2)}x^{1/2} = 2\sqrt{x/\pi},$$

*that yields as the result of Lacroix (1.1).*

I now state the definitions of the Caputo fractional derivatives.

**Definition 1.6** (See [125, 215, 236]). *Let*

$$T_{m-1}y(x) := \sum_{k=0}^{m-1} y^{(i)}(a)\frac{x^i}{i!}$$

*be Taylor polynomial of degree* $m-1$ *such that* $y^{(i)}(a)$, *exists for* $i = 0, 1, \ldots, m-1$. *If* $I^{m-\alpha}y \in AC^{m-1}([a,b])$ *and* $T_{m-1}y$ *exists, then the Caputo fractional derivative of order* $\alpha$ *is defined by*

$${}^cD_a^\alpha y(x) = D^\alpha(y(x) - T_{m-1}y(x)).$$

*In particular, when* $0 < \alpha < 1$, *we have*

$${}^cD_a^\alpha y(x) = D_a^\alpha(y(x) - y(a)).$$

*Moreover, if* $y \in AC^{m-1}([a,b])$ *then Caputo fractional derivative of order* $\alpha$ *is defined by*

$${}^cD_a^\alpha y(x) = I^{m-\alpha}D_a^m y(x). \tag{1.15}$$

*Again for* $\alpha = 0$, *define* ${}^cD_a^0$ *to be the identity operator.*

I remark that when I explicitly choose $a = 0$ on the interval $[a,b]$ I shall use the symbols $I_a^\alpha := I^\alpha$, $D_a^\alpha := D^\alpha$, and ${}^cD_a := {}^cD$.

Also, throughout my work, I may chose to work on a specific interval for example in Chapter 2 my analysis shall involve the interval when $a = 0$, that is $[0,b]$, so it will be mentioned on each Chapter.

I state the following properties for the fractional integral.

**Remark 1.2** (See [125, 236]). *The following properties hold:*
**(i)** $I^\alpha I^\beta y(x) = I^{\alpha+\beta}y(x)$, *and* $D^\alpha I^\beta y(x) = y(x)$, $\alpha > 0$, $\beta > 0$, *for a.e.* $x \in [0,1]$ *where* $y \in L^1(0,1)$;
**(ii)** $I^\alpha D^\alpha y(x) = y(x)$, $0 < \alpha < 1$, $y \in C([0,1])$ *and* $D^\alpha y \in C(0,1) \cap L^1(0,1)$;
**(iii)** $I^\alpha : C([0,1]) \to C([0,1])$, $\alpha > 0$.

For more details of the Riemann–Liouville fractional integrals and fractional derivatives and Caputo fractional integrals and fractional derivatives, I refer the reader to [125, 215, 236].

### 1.3.3 Construction of Banach spaces

I now briefly introduce and construct various Banach spaces including weighted Banach spaces. My analysis throughout my work will be set within the environment of complete, normed linear, and metric spaces, known as Banach spaces. This will be important for my results and so let me first give some notation and related definitions. I assume that the reader is familiar with the usual concepts of Banach spaces from functional analysis and I again refer the reader to [52, 66, 229, 294].

The following definition sheds light on what I mean by a metric space.

**Definition 1.7.** *Let $Y$ be a nonempty set. A metric for $Y$ is a function $\varrho : Y \times Y \to \mathbb{R}$ such that*

**(a)** $\varrho(y,z) \geq 0$ *for all* $y, z \in Y$;

**(b)** $\varrho(y,z) = 0 \Longleftrightarrow y = z$;

**(c)** $\varrho(y,z) = \varrho(z,y)$ *for all* $y, z \in Y$, *(Symmetry);*

**(d)** $\varrho(y,z) \leq \varrho(y,u) + \varrho(u,z)$ *for all* $y, z, u \in Y$ *(Triangle Inequality).*

*If $\varrho$ is a metric for $Y$, then the ordered pair $(Y,\varrho)$ is called a metric space (I may sometimes refer to the metric space $(Y,\varrho)$ by $Y$).*

The completeness of a metric space is usually one of the most important assumptions in many fixed point theorems, and so I introduce the idea of the completeness of the metric space but first I give the definition of a Cauchy sequence.

**Definition 1.8.** *Let $\{y_n\}_{n=1}^{\infty}$ be a sequence in a metric space $(Y,\varrho)$. Then $\{y_n\}_{n=1}^{\infty}$ is called a Cauchy sequence if for any $\epsilon > 0$ there exists an $N \in \mathbb{N}_+$ such that $\varrho(y_n, y_l) < \epsilon$, for all $n, l \geq N$.*

**Definition 1.9.** *We say that the metric space $(Y,\varrho)$ is complete if every Cauchy sequence of points in $Y$ converges to a point in $Y$.*

Another way to determine the completeness of a metric space based on another metric space is to use the idea of equivalence of metrics, and so the following definition gives an idea what I mean by the equivalence between two metrics.

**Definition 1.10.** *Let $Y$ be a nonempty set and let $\varrho$ and $\tau$ be two metrics on $Y$. Then the metric $\tau$ is equivalent to the metric $\varrho$ if there exists $m, m_1 > 0$ such that*

$$m\tau(y,z) \leq \varrho(y,z) \leq m_1\tau(y,z), \text{ for all } y, z \in Y.$$

**Lemma 1.2.** *Let $\varrho$ and $\tau$ be two equivalent metrics on $Y$. Then the ordered pair $(Y,\varrho)$ is complete if and only if the ordered pair $(Y,\tau)$ is complete.*

I am now ready to construct my Banach spaces. Consider the space of continuous functions

$C([a,b])$ coupled with a suitable metric, either

$$d_0(y,z) := \max_{x\in[a,b]} |y(x) - z(x)|; \tag{1.16}$$

or

$$\delta_p(y,z) := \left( \int_a^b |y(x) - z(x)|^p \, dx \right)^{1/p}, \quad p > 1. \tag{1.17}$$

Also consider $C([a,b])$ coupled with a suitable norm, either

$$\|y\|_0 := \max_{x\in[a,b]} |y(x)|; \tag{1.18}$$

or

$$\|y\|_p := \left( \int_a^b |y(x)|^p \, dx \right)^{1/p}, \quad p > 1. \tag{1.19}$$

It is a well-known result that each of the pairs $(C([a,b]), d_0)$ and $(C([a,b]), \|y\|_0)$ form complete metric spaces. Also each of the pairs $(C([a,b]), \delta_p)$ and $(C([a,b]), \|y\|_p)$ form metric spaces, but it is not complete.

Now consider the set of real-valued functions that are defined on $[a,b]$ and are $k-$times continuously differentiable therein. Denote this space by $C^k([a,b])$. For functions $y, z \in C^k([a,b])$ and appropriate nonnegative constants $L_i$ and $W_i$ to be determined in the statements or proofs of my main results, I construct the following metrics from $d_0$ and $\delta_p$:

$$d^*(y,z) := \sum_{i=0}^{k} \max_{x\in[a,b]} |y^{(i)}(x) - z^{(i)}(x)|; \tag{1.20}$$

$$\delta(y,z) := \sum_{i=0}^{k} \left[ L_i \left( \int_a^b |y^{(i)}(x) - z^{(i)}(x)|^p \, dx \right)^{1/p} \right], \quad p > 1. \tag{1.21}$$

$$d(y,z) := \max_{i\in\{0,1,2,\ldots,k\}} \left\{ W_i \max_{x\in[a,b]} |y^{(i)}(x) - z^{(i)}(x)| \right\}. \tag{1.22}$$

Again, it is known that each of the pairs $(C^k([a,b]), d^*)$ and $(C^k([a,b]), d)$ form a complete metric space. The pair $(C^k([a,b]), \delta)$ also forms a metric space, however they are not complete.

I now state the following important relationships between my above metrics on $C^2([a,b])$, which I will draw on in the proofs of my main results.

**Theorem 1.1.** *For $y, z \in C^2([a,b])$ we have*

$$\delta(y,z) \le (b-a)^{1/p} \max_{i\in\{0,1,2\}} \{L_i\} \, d^*(y,z); \tag{1.23}$$

*and*

$$\delta(y,z) \le (b-a)^{1/p}\left(\sum_{i=0}^{2}\frac{L_i}{W_i}\right)d(y,z). \tag{1.24}$$

*Proof.* It is a well-known result that

$$\delta_p(y,z) \le (b-a)^{1/p} d_0(y,z), \quad \text{for all } y,z \in C([a,b]), \tag{1.25}$$

and so repeatedly applying (1.25) we have

$$\begin{aligned}
\delta(y,z) &= L_0\delta_p(y,z) + L_1\delta_p(y',z') + L_2\delta_p(y'',z'') \\
&\le (b-a)^{1/p}\left(L_0 d_0(y,z) + L_1 d_0(y',z') + L_2 d_0(y'',z'')\right) \\
&\le (b-a)^{1/p} \max_{i\in\{0,1,2,3\}}\{L_i\}\left(d_0(y,z) + d_0(y',z') + d_0(y'',z'')\right) \\
&= (b-a)^{1/p} \max_{i\in\{0,1,2\}}\{L_i\}\ d^*(y,z).
\end{aligned} \tag{1.26}$$

Thus we have obtained (1.23).

Finally, I show that the inequality (1.24) holds. From (1.26), for $y,z \in C^3([a,b])$, we have

$$\begin{aligned}
\delta(y,z) &\le (b-a)^{1/p}\left(L_0 d_0(y,z) + L_1 d_0(y',z') + L_2 d_0(y'',z'')\right) \\
&\le (b-a)^{1/p}\left(\sum_{i=0}^{2}\frac{L_i}{W_i}\right)d(y,z).
\end{aligned}$$

Thus we have obtained (1.24). □

The following Theorem is a generalization of Theorem 1.1

**Theorem 1.2.** *Let $k \ge 0$ be an integer, for $y,z \in C^k([a,b])$ we have*

$$\delta(y,z) \le (b-a)^{1/p} \max_{i\in\{0,1,2,\ldots,k\}}\{L_i\}\ d^*(y,z); \tag{1.27}$$

*and*

$$\delta(y,z) \le (b-a)^{1/p}\left(\sum_{i=0}^{k}\frac{L_i}{W_i}\right)d(y,z). \tag{1.28}$$

*Proof.* The proof follows similar lines of argument as that of Theorem 1.1 and so it is omitted. □

I am now ready to construct so-called weighted Banach spaces. I shall do so by using Biekecki's metric/or norm, which is known as Biekecki's method [47] of weighted metric/or norm that dates back to the mid of the last century. This method has become an interesting technique to obtain existence and uniqueness results for a very wide classes of differential, integral, and many other functional equations. Since then Biekecki's method has captivated the scientific attention of research communities in both pure and applied mathematics and I refer the reader to see [18, 78, 123, 132, 187, 255, 261] and the references therein. Therefore, I first introduce a definition of measuring distance in weighted metric spaces, which I will be concerned within the context of the my work in Chapter 7.

**Definition 1.11.** *Let $\kappa_0 > 0$ be a constant, $[a,b] := [0,1]$, and $\alpha > 0$. Define the space of continuous functions $C([0,1])$ coupled with a suitable metric, either*

$$d_{\kappa_0}(y,z) := \max_{x\in[0,1]} \frac{|y(x) - z(x)|}{E_\alpha(\kappa_0 x^\alpha)}; \tag{1.29}$$

*or*

$$d_0(y,z) := \max_{x\in[0,1]} |y(x) - z(x)|. \tag{1.30}$$

The above definition of $d_{\kappa_0}$ is a generalization of Bielecki's metric [261, p.309], [74, pp.153–155], [247, p.44]. This Bielecki's metric type involves the Mittag–Leffler function.

I now list the following an important properties of $d_{\kappa_0}$, which will play an important role in my work in Chapter 7. And for their proofs I refer to [261, Lemma 6.3].

**Lemma 1.3** (See [261, Lemma 6.3])**.** *If $\kappa_0 > 0$ is a constant and $\alpha > 0$, then:*

**(i)** *$d_{\kappa_0}$ is a metric;*

**(ii)** *$d_{\kappa_0}$ is equivalent to $d_0$;*

**(iii)** *$(C([0,1]), d_{\kappa_0})$ is a complete metric space.*

Again the following definition is a new definition of measuring distance in a normed space, which is a generalization of Bielecki's norm type.

**Definition 1.12.** *Let $\kappa > 0$, $0 < \rho_1 < 1$ and $\alpha > \rho_1$ be constants. Let $I_0 := [0,1]$ and define the space $X = \{u : u \in C(I_0)$ and $D^{\rho_1}u \in C(I_0)\}$ coupled with a suitable norm, either*

$$\|u\|_{X_\kappa} = \max_{x\in I_0} \frac{|u(x)|}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})} + \max_{x\in I_0} \frac{|D^{\rho_1}u(x)|}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}, \tag{1.31}$$

*or*

$$\|u\|_X = \max_{x\in I_0} |u(x)| + \max_{x\in I_0} |D^{\rho_1}u(x)|. \tag{1.32}$$

Motivated by the importance of Biekecki's method [47] of weighted norm and [253, Lemma 3.2] and [254, Lemma 3.1], I now list the following an important properties of $\|u\|_{X_\kappa}$, which will play an important role in my work in Chapter 8. The proof of the following Lemma is similar to the proof of [261, Lemma 6.3].

**Lemma 1.4.** *Let $\kappa > 0$ and $\alpha > \rho_1$ with $0 < \rho_1 < 1$ be constants, then:*

**(i)** $\|u\|_{X_\kappa}$ *is norm;*

**(ii)** $\|u\|_{X_\kappa}$ *is norm and is equivalent to* $\|u\|_X$*;*

**(iii)** $(X, \|\cdot\|_{X_\kappa})$ *is a Banach space.*

*Proof.* **(i)** Let $\kappa > 0$ and $\alpha > \rho_1$ with $0 < \rho_1 < 1$ be constants. Then we have $E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1}) > 0$ for all $x \in I_0$ and $E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})$ is continuous on $[0,1]$. The three properties of a norm are now easily verified.

**(ii)** We need to show that there exist two positive consents $m$ and $m_1$ such that

$$m\|u\|_X \leq \|u\|_{X_\kappa} \leq m_1\|u\|_X.$$

To see this, since $\kappa > 0$ and $\alpha > \rho_1$, $E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})$ is continuous and strictly increasing on $[0,1]$ we have

$$\frac{1}{E_{\alpha-\rho_1}(\kappa)} \leq \frac{1}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})} \leq 1, \quad \text{for all } x \in I_0,$$

and so

$$\frac{1}{E_{\alpha-\rho_1}(\kappa)}\|u\|_X \leq \|u\|_{X_\kappa} \leq \|u\|_X, \quad \text{for all } u \in X. \tag{1.33}$$

Thus, (1.33) ensures that our norms are equivalent with $m = \frac{1}{E_{\alpha-\rho_1}(\kappa)}$ and $m_1 = 1$.

**(iii)** follows from **(ii)** and $\|u\|_X$ being a Banach space (see [253, Lemma 3.2], also see [254, Lemma 3.1]): If $\{u_n\}$ is a Cauchy sequence in $\|u\|_X$ then **(ii)** ensures that $\{u_n\}$ is a Cauchy sequence in $\|u\|_{X_\kappa}$. Thus, $\|u\|_{X_\kappa}$ is a Banach space. □

For further purposes I will consider the following Banach space

$$Y = \left\{v : v \in C(I_0) \text{ and } D^{\rho_1}v \in C(I_0)\right\}$$

coupled with a suitable norm, either

$$\|v\|_{Y_\kappa} = \max_{x\in I_0} \frac{|v(x)|}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})} + \max_{x\in I_0} \frac{|D^{\rho_1}v(x)|}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}, \tag{1.34}$$

or

$$\|v\|_Y = \max_{x\in I_0} |v(x)| + \max_{x\in I_0} |D^{\rho_1} v(x)|. \tag{1.35}$$

For $(u,v) \in X \times Y$, let $\|(u,v)\|_{X_\kappa \times Y_\kappa} = \|u\|_{X_\kappa} + \|v\|_{Y_\kappa}$.

Clearly,

$$(X \times Y, \|(u,v)\|_{X_\kappa \times Y_\kappa}) \tag{1.36}$$

is a Banach space.

## 1.4 Fixed point methods

For more than 100 years, fixed point theory has enjoyed significant developments and one of the motivations for developing fixed point theory lies in its central aim, which is to deepen our understanding of when certain classes of equations admit solutions. In particular, fixed point theory establishes conditions under which certain classes of operators will have existence, uniqueness and approximation of fixed points. The famous fixed point theorems such as Banach [35] published in 1922 and Schauder [240] published in 1930 may naturally spring to mind due to their importance and significance, to name just a few. Following this a number of important advancements in fixed point theory have occurred. This includes fixed point theorem due to Rus [231] published in 1970 and theorem known as continuation method for contractive maps (which I shall abbreviate henceforth to CMCM) due to Granas [95] published in 1994. Another important theorem is a constructive version of CMCM due to Precup [217, Theorem 2.2]. Therefore, the aim of this Chapter is to present these theorems, which I will employ to prove my results in this thesis.

I shall first introduce some well-known definitions from functional analysis, for this matter I refer the reader to [52, 66, 229, 294].

**Definition 1.13.** *Let $Y$ be a set. A self map on $Y$ is a mapping from $Y$ to itself: $T : Y \to Y$.*

The following definition sheds light on what I mean by a fixed point of an operator.

**Definition 1.14.** *Let the operator $T : Y \to Y$ be a self map. A $y \in Y$ is said to be a fixed point of $T$ if $Ty = y$.*

I now present the following fixed point theorem known as Schauder's fixed point theorem [240] (also see [294, Theorem 2.A, p.56]). This theorem only asserts the existence of a fixed point without its uniqueness.

**Theorem 1.3** (See **Schauder, 1930, [240]**). *Let $\Omega$ be a nonempty, closed, bounded and convex subset of the Banach space $Y$. If $T : \Omega \to \Omega$ is a compact map then there is at least one $y \in \Omega$ such that $Ty = y$.*

The compactness idea is also one of the most important assumptions in many fixed point theorems and so the following definition shows when the map $T$ is defined to be *compact*.

**Definition 1.15.** *Let $Y$ be a Banach space and let $T : Y \to Y$. The map $T$ is defined to be* compact *if: $T$ is continuous; and $T$ maps bounded sets into relatively compact sets.*

The Arzelá-Ascoli theorem [28] gives the necessary and sufficient conditions that confirm the set $\Omega$ to be relatively compact in $C([a, b])$. Before stating the The Arzelá-Ascoli theorem I need to give two important definitions: the definition of equicontinuous and uniformly bounded sets.

**Definition 1.16.** *A set $\Omega$ is said to be equicontinuous if, for every $\epsilon > 0$, there exists some $\delta > 0$ such that,*

$$|g(x_1) - g(x_2)| < \epsilon,$$

*for all $g \in \Omega$ and all $x_1, x_2 \in [a, b]$ with $|x_1 - x_2| < \delta$.*

**Definition 1.17.** *A set $\Omega$ is said to be uniformly bounded if there exists a constant $M > 0$ such that,*

$$\|g\| := \max_{x \in [a,b]} |g(x)| < M,$$

*for every $g \in \Omega$.*

I am now ready to state the Arzelá-Ascoli theorem.

**Theorem 1.4** (See **Arzelá-Ascoli,1896, [28]**). *Let $\Omega \subseteq C([a, b])$ for some $a < b$, and assume the sets to be coupled with the norm $\|g\|$. Then $\Omega$ is relatively compact in $C[a, b]$ if $\Omega$ is equicontinuous and uniformly bounded.*

Another very well-known and powerful fixed point theorem concerning fixed points of operators is Banach fixed point theorem [35]. It asserts both the existence, uniqueness and approximation of a fixed point.

**Theorem 1.5** (See **Banach, 1922, [35]**). *Let $(Y, \varrho)$ be a complete metric space and let $T : Y \to Y$. If $T$ is contractive in the sense that there exists a positive constant $\sigma < 1$ with*

$$\varrho(Ty, Tz) \le \sigma \varrho(y, z), \quad \text{for all } y, z \in Y \tag{1.37}$$

*then: $T$ has a unique fixed–point $u$, that is, $Tu = u$ for a unique $u \in Y$; and $T^m y \to u$ for each $y \in Y$, where $T^0 y := y$ and $T^{m+1} y := T(T^m y)$.*

**Remark 1.3.** *It is well–known [96, p.10] that by beginning at an arbitrary $y \in Y$, Banach's theorem provides the following estimate on the "error" between the $m$th iteration $T^m y$ and the fixed point $u$, namely*

$$\varrho(T^m y, u) \leq \frac{\sigma^m}{1-\sigma} \varrho(y, Ty). \tag{1.38}$$

It is a well-known result that for a given $T : Y \to Y$, we can sometimes have a case where $T$ is not contractive on the whole of the set $Y$. However, it is possible that $T$ is instead contractive only on a subset (say an open ball) of $Y$. Therefore, Banach fixed point theorem (Theorem 1.5) has a very useful local version for the balls, and it is presented as the following Corollary

**Corollary 1.1.** *Let $(Y, \varrho)$ be a complete metric space containing an open ball with radius $r > 0$ and center $y_0$. That is, there exists*

$$\mathcal{B}_r := \{y \in Y : \varrho(y, y_0) < r\} \subseteq Y.$$

*Let $T : \mathcal{B}_r \to Y$ be a contractive map with constant $\sigma < 1$, that is,*

$$\varrho(Ty, Tz) \leq \sigma \varrho(y, z), \quad \textit{for all } y, z \in \mathcal{B}_r. \tag{1.39}$$

*If*

$$\varrho(Ty, y_0) < (1-\sigma) r, \tag{1.40}$$

*then: $T$ has a unique fixed–point in $\mathcal{B}_r$.*

I shall use the following terminology when I generalize (1.37) and/or (1.40).

**Definition 1.18.** *An operator $T : \mathcal{B}_r \subseteq Y \to Y$ on a metric space $(Y, \varrho)$ is said to be Lipschitz continuous on $\in \mathcal{B}_r$ if and only if (1.39) holds for all $y, z \in \mathcal{B}_r$ with fixed $\sigma$, $0 \leq \sigma < \infty$. If this holds for $\sigma = 1$, $T$ is called nonexpansive; and obviously if this hold for fixed $\sigma$, $0 \leq \sigma < 1$, $T$ is called $\sigma$–contractive.*

The following fixed point theorem due to Rus [231] will also form a core part of my approach to obtaining my results in Chapters 3, 4, and 5.

**Theorem 1.6** (See **Rus,1970, [231]**). *Let $Y$ be a nonempty set and let $\varrho$ and $\tau$ be two metrics on $Y$ such that $(Y, \varrho)$ forms a complete metric space. If the mapping $T : Y \to Y$ is continuous with respect to $\varrho$ on $Y$ and:*

$$\varrho(Ty, Tz) \le c\tau(y, z), \text{ for some } c > 0 \text{ and all } y, z \in Y; \tag{1.41}$$

$$\tau(Ty, Tz) \le \alpha\tau(y, z), \text{ for some } 0 < \alpha < 1 \text{ and all } y, z \in Y; \tag{1.42}$$

*then there is a unique $y \in Y$ such that $Ty = y$.*

When compared with more well known techniques from fixed point theory, such as Theorem 1.5, there has been limited research involving applications of Theorem 1.6 to explore the questions of existence and uniqueness of solutions to ordinary differential equations. It is surprising that Theorem 1.6 has not received more attention from researchers, given that it was published more than 40 years ago. Perhaps its sheltered state has more to do with the human tendency to favor approaches that are more well known. That is, humankind tends to continue to tread along methodological paths that are more well traveled without exploring alternative perspectives and techniques. However, developing alternative perspectives are important as they can open up new ways of thinking and working [262, 266].

Theorem 1.6 differs from Theorem 1.5. For example, Theorem 1.6 involves two metrics which may not necessarily be equivalent. In addition, the underlying space in Theorem 1.6 is assumed to be complete with respect to the first of these metrics, but not necessarily complete with respect to the second metric. The operator is assumed to be contractive with respect to the second metric. As we will discover, it is these very properties that have the potential to advance recent (or longer-standing) results on existence and uniqueness of solutions to differential equations: particularly, boundary value problems. Thus, I take the position that Theorem 1.6 forms an important, underappreciated and untapped tool that has the potential to open up new lines of inquiry and thus is most worthy of attention. For more recent applications of Rus' fixed point theorem, see [18, 20, 21, 22, 251].

In more recent years, a number of important advancements in fixed point theory have occurred. This includes the following Theorem known as continuation method for contractive maps (which I shall abbreviate henceforth to CMCM) due to Granas, see [95].

**Theorem 1.7** (See **Granas, 1994, [95]**). *Let $(Y, \varrho)$ be a complete metric space, let $\overline{U} \subset Y$ be a closed set, and let $H : \overline{U} \times [0, 1] \to Y$. Assume:*

**(i)** *there exists a constant* $\sigma \in [0,1)$ *with*

$$\varrho(H_\lambda y, H_\lambda z) \le \sigma \varrho(y,z), \quad \textit{for all} \;\; y,z \in \overline{U} \;\; \textit{and all} \;\; \lambda \in [0,1];$$

**(ii)** $H_\lambda(y) \ne y$, *for all* $y \in \partial U$ *and all* $\lambda \in [0,1]$*;*

**(iii)** *there exists a constant* $L > 0$ *such that*

$$\varrho(H_{\lambda_1} y, H_{\lambda_2} y) \le L|\lambda_1 - \lambda_2|, \quad \textit{for all} \;\; \lambda_1, \lambda_2 \in [0,1] \;\; \textit{and} \;\; y \in \overline{U}.$$

*If* $H_0$ *has a fixed point in* $U$*, then for every* $\lambda \in [0,1]$*,* $H_\lambda$ *also has a fixed point.*

Another important theorem is a constructive version of CMCM due to Precup, see [217, Theorem 2.2] and [217, Corollary 2.5] (also see [207, Theorem 2.3], and [207, Theorem 2.4]). Thus, I state the following Theorem without proof, see [217, Theorem 2.2] and [217, Corollary 2.5] (also see, [207, Theorem 2.4]), which is a special case of Precup's constructive extension of CMCM. For its proof I refer to [95, pp.376–377] (also see, [219, Theorem 2.3]).

**Theorem 1.8.** *Let* $(Y,\varrho)$ *be a complete metric space, let* $U \subset Y$ *be open, and let* $H : \overline{U} \times [0,1] \to Y$*. Assume that the following conditions are satisfied:*

**(a1)** *there is a* $\sigma \in [0,1)$ *such that*

$$\varrho(H_\lambda y, H_\lambda z) \le \sigma \varrho(y,z), \quad \textit{whenever} \;\; y,z \in \overline{U} \;\; \textit{and all} \;\; \lambda \in [0,1];$$

**(a2)** $H_\lambda(y) \ne y$, *for all* $y \in \partial U$ *and all* $\lambda \in [0,1]$*;*

**(a3)** $H_\lambda(y)$ *is continuous in* $\lambda$*, uniformly for* $y \in \overline{U}$*.*

*In addition, suppose that there exists a nonempty set* $U_1 \subset \overline{U}$ *with* $H_0(U_1) \subset U_1$*. Then, for each* $\lambda \in [0,1]$*, there exists a unique fixed point* $y(\lambda)$ *of* $H_\lambda$*. Moreover,* $y(\lambda)$ *depends continuously on* $\lambda$ *and there exists* $0 < r \le \infty$*, integers* $m, n_1, n_2, \dots, n_{m-1}$*, and numbers* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_{m-1} < \lambda_m = 1$ *such that for any* $y_0 \in y$ *satisfying* $\varrho(y_0, y(0)) \le r$*, the sequences* $(y_{j,k})_{k\ge 0}$*,* $j = 1,2,\dots,m$*,*

$$\begin{aligned} y_{1,0} &:= y_0 \\ y_{j,k+1} &:= H_{\lambda_1}(y_{j,k}), \quad k = 0,1,\dots \\ y_{j+1,0} &:= y_{j,n_j}, \quad j = 1,2,\dots,m-1 \end{aligned}$$

*are well defined and satisfy*

$$\varrho(y_{j,k}, y(\lambda_j)) \le \frac{\sigma^k}{1-\sigma} \varrho(y_{j,0}, H_{\lambda_j}(y_{j,0})), \quad (k \in \mathbb{N}). \tag{1.43}$$

This modern theory of CMCM due to Granas and Precup is less well known than the more established fixed point theories such as Schauder's and Local Banch's fixed–point theorems. Nevertheless, the ideas of CMCM have enjoyed interesting applications in the research study of a range of ordinary and partial differential equations, shedding light on questions such as existence, uniqueness, location and approximation of "global" solutions of these general types of problems see, [8, 9, 27, 96, 126, 206, 207, 208, 216, 217, 218] for more details about advancements of CMCM over traditional theory and known results. Such an analysis of these general types of problems through CMCM is badly needed, to illuminate my comprehension of "global" solutions of these general types of problems, as the following discussion illustrates. Let $f(x,y)$ represent the right hand side of an ordinary or partial differential equation and let $[0,1]$ be the entire $x$-domain of definition of the function $f(x,y)$. By *global* solutions to this differential equation, I mean that there exists a solution $y = y(x)$ to the problem, with $y(x)$ defined on the whole $x$-interval $[0,1]$. A key challenge regarding *global* to this differential equation is as follows. If $f(x,u)$ satisfies a global Lipschitz condition in $u$ on the entire infinite strip (say, on $[0,1]\times\mathbb{R}$), then the problem under consideration has a unique global solution which can be obtained by means of the classical Banach contraction principle. However, if $f$ only satisfies a *local* Lipschitz condition (say, on $[0,1]\times[-2,-2]$), then by Banach's *local* theorem we can only prove the existence and uniqueness of a *local* solution (that is, a solution defined only on a subinterval of $[0,1]$). The subinterval restriction stems from the invariance condition on the associated operator imposed in the local version of Banach's fixed point theorem, see [207, pp.18–23].

On the other hand, we can see that the condition $[(\mathbf{a2})]$ of Theorem 1.8, known as Lipschitz continuous or Lipschitz condition is both classical and well known within the context of Banach's fixed point theorems, however, note that this condition $[(\mathbf{a2})]$ holds only *locally* instead of, say for example, on an infinite strip $[0,1]\times\mathbb{R}$. Consequently, any application of Banach's classical fixed point theorems in this setting must be restricted to a local version. This then leads to existence and uniqueness of only a locally-defined solution on a mere subinterval $I_{S_1}\subset[0,1]$, rather than yielding existence and uniqueness of a global solution defined on the whole interval $[0,1]$. The restriction is a result of the invariance condition of the local version of Banach's fixed point theorem which demands the operator $T$ satisfies $T(\bar{B}_{S_1})\subset\bar{B}_{S_1}$, where $\bar{B}_{S_1}$ is a closed ball in $C(I_{S_1};\mathbb{R})$.

As we will discover that Theorem 1.8 advances knowledge in a way that a localized version of Banach's fixed point theorem cannot by establishing *global* existence and uniqueness of solutions and showing that the invariance condition outlined above can be avoided. This is one of the main advancements of Theorem 1.8 over traditional theory and known results. Thus, Theorem 1.8 will be of particular relevance to my work in Chapter 7 where I form new existence theory for *global* solutions to initial value problems for fractional differential equations (IVP) since the area of fractional differential equations has remained sheltered from an analysis involving CMCM and its constructive extension.

## 1.5 Outline of this thesis

This thesis is organized as follows:

The first set of new results is formulated in Chapter 2 where I construct a firm mathematical foundation for the second-order boundary value problem (second-order BVP) associated with a generalized Emden equation that embraces Thomas–Fermi-like theories. The second-order BVP for the relativistic and nonrelativistic Thomas–Fermi equations are included as special cases. Herein I prove that each of these second-order BVPs that are subjected to two-point boundary conditions admit a unique solution. my methods involve an analysis of the problems through arguments that apply differential inequalities and Schauder's fixed point theorem [294, Theorem 2.A, p.56]. The new results guarantee the existence of a unique solution, ensuring the generalized Emden equation that embraces Thomas–Fermi theory sits on a firm mathematical foundation. The problems, methods and ideas in this Chapter provide a logical starting point for navigating the latter chapters of this thesis, where the analysis moves to more complicated problems such as those: of higher order, with more complex boundary conditions or problems with fractional derivatives. The contents of this Chapter has been published in The Journal of Engineering Mathematics [19].

In Chapter 3, the second result is presented regarding the existence, uniqueness and approximation of solutions to third-order ordinary differential equations that are subjected to two- and three-point boundary conditions (third-order BVP). The differential equation under consideration in this Chapter features a scalar-valued, nonlinear right-hand side that does not depend on derivatives. my results are obtained in the following ways. Firstly, I provide sharp and sharpened estimates for integrals regarding various Green's functions that are associated with the third-order BVP. Secondly, I apply these sharper estimates to problems in conjunction with Banach's fixed point theorem [35] to establish my first novel result of this Chapter for the existence and uniqueness of solutions to the third-order BVP. Thirdly, I sharpen my first result of this Chapter by showing that a larger class of third-order BVP admit a unique solution. I achieve this by drawing on fixed-point theory in an interesting and alternative way via an application of Rus's contraction mapping theorem [231]. The idea of this is to utilize two metrics on a metric space, where one pair is complete. my both results of this Chapter are given within a global (unbounded) context, and improve the recent results of Smirnov [248]. This is achieved by showing that my both results enable a larger class of boundary value problems admit a unique solution than results of Smirnov. Finally, I illustrate the essence of the advancements of this Chapter over existing literature via the discussion of examples. The contents of this Chapter has been published in Mathematical Modelling and Analysis [22].

In Chapter 4, the third-order BVP that features a scalar-valued, a fully nonlinear right-hand side that depends on each of the lower-order derivatives is considered. The goal of this is to establish a more complete and wider-ranging theory than is the results obtained in Chapter 3 regarding the existence, uniqueness and approximation of solutions to the third-order BVP. To develop this, my strategy involves an analysis of the problem under consideration, and its associated operator equations, within closed and bounded sets. By means of the methods of Chapter 3 I am able to form a fuller theory regarding the existence, uniqueness and approximation of solutions to the third-order BVP, that is applicable to a wider range of problems than the results obtained in Chapter 3, and I discuss finally several examples to illustrate the nature of these advancements. The contents of this Chapter has been published in Differential Equations and Applications (DEA) [20].

In Chapter 5, I prove the existence and uniqueness of solutions to two-point boundary value problems involving fourth-order, ordinary differential equations (fourth-order BVP). The differential equation under consideration in this Chapter features a scalar-valued, a fully nonlinear right-hand side that depends on each of the lower-order derivatives. Such problems have interesting applications to modelling the deflections of beams. I sharpen traditional results and approaches such as Banach's fixed point theorem in bounded and unbounded setting by showing that a larger class of problems admit a unique solution. I achieve this by drawing Rus's contraction mapping theorem. My theoretical results are applied to the area of elastic beam deflections when the beam is subjected to a loading force and the ends of the beam are either: both clamped; or one end is clamped and the other end is free. Existence and uniqueness of solutions to the models are guaranteed for certain classes of linear and nonlinear loading forces. The contents of this Chapter has been published in Open Mathematics journal [21].

In Chapter 6 my aim is to develop a more complete theory regarding solutions to the problem of laminar flow in channels with porous walls, where we have been motivated by the general theory established in Chapter 5. In particular, my aim is to introduce contraction mapping ideas in what appears to be a first time synthesis and application to the problem of laminar flow in channels with porous walls that is modelled by a fourth order boundary value problem (BVP). My strategy involves establishing new *a priori* bounds on solutions and draws on Banach's fixed point theorem. This enables a deeper understanding of the problem by strategically addressing the questions of existence, uniqueness and approximation of solutions under one integrated framework, rather than applying somewhat disjointed approaches. Through this strategy, I advance current knowledge by extending the range of values of the Reynolds number under which the problem will admit a unique solution; and I furnish a sequence of functions whose limit converges to this solution, enabling an iterative approximation to any theoretical degree of accuracy. The contents of this Chapter has been submitted to Partial Differential Equations and Applications (PDEA).

In Chapter 7, my aim is to form new existence theory for *global* solutions to initial value problems for fractional differential equations (IVP). Traditional approaches to existence, uniqueness and approximation of global solutions for initial value problems involving fractional differential equations have been unwieldy or intractable due to the limitations of previously used methods. This includes certain invariance conditions of the underlying local fixed point strategies. Herein I draw on an alternative tactics, applying the more modern ideas of continuation methods for contractive maps to IVP. In doing so, I shed new light on the situation, producing these new perspectives through a range of novel theorems that involve sufficient conditions under which global existence, uniqueness, approximation and location of solutions are ensured. The contents of this Chapter has been published in Analysis: International mathematical journal of analysis and its applications [18].

In Chapter 8, I form a new uniqueness result for a class of initial value problems involving a coupled system of nonlinear Riemann–Liouville fractional differential equations. The main tools involve the Banach contraction principle and the introduction of a new definition of measuring distance in an appropriate normed space. My new results improve some work of Sun et al. 2012 [254]. An example is given at the end of this Chapter to illustrate my result. The contents of this Chapter has been published in Communications on Applied Nonlinear Analysis journal [23].

Chapter 9 contains my discussion and conclusion where I explore the underlying meaning of my all results and the possible avenues for further research and developments. This includes the formulation of some open questions that arise from the present thesis to stimulate further research into the areas.

Before I start introducing my main results, let me provide a direct references of the publications and the submitted results that arise from each Chapter of this thesis.

## 1.6 Published and submitted results from this thesis

**Published results from Chapter 2:**

The contents of Chapter 2 has been published. See the following reference

[1] Saleh S. Almuthaybiri and Christopher C. Tisdell. Establishing existence and uniqueness of solutions to the boundary value problem involving a generalized Emden equation, embracing Thomas-Fermi-like theories. *J. Engrg. Math.*, 124:1–10, 2020.

**Published results from Chapter 3:**

The contents of Chapter 3 has been published. See the following reference

[2] Saleh S. Almuthaybiri and Christopher C. Tisdell. Sharper existence and uniqueness results for solutions to third-order boundary value problems. *Math. Model. Anal.*, 25(3):409–420, 2020.

**Published results from Chapter 4:**

The contents of Chapter 4 has been published. See the following reference

[3] Saleh S. Almuthaybiri and Christopher C. Tisdell. Existence and uniqueness of solutions to third–order boundary value problems: Analysis in closed and bounded sets. *Differ. Equ. Appl.*, 12(12):291–312, 2020.

**Published results from Chapter 5:**

The contents of Chapter 5 has been published. See the following reference

[4] Saleh S. Almuthaybiri and Christopher C. Tisdell. Sharper existence and uniqueness results for solutions to fourth-order boundary value problems and elastic beam analysis. *Open Math.*, 18(1):1006–1024, 2020.

**Submitted results from Chapter 6:**

The contents of Chapter 6 has been submitted to the following journal

[5] Saleh S. Almuthaybiri and Christopher C. Tisdell. Laminar Flow in Channels with Porous Walls: Advancing the Existence, Uniqueness and Approximation of Solutions via Fixed Point Approaches. *Differ. Equ. Appl.*, has been submitted: July 2021.

**Published results from Chapter 7:**

The contents of Chapter 7 has been published. See the following reference

[6] Saleh S. Almuthaybiri and Christopher C. Tisdell. Global existence theory for fractional differential equations: New advances via continuation methods for contractive maps. *Analysis (Berlin)*, 39(4):117–128, 2019.

**Published results from Chapter 8:**

The contents of Chapter 8 has been published. See the following reference

[7] Saleh S. Almuthaybiri and Christopher C. Tisdell. Uniqueness of solutions for a coupled system of nonlinear fractional differential equations via weighted norms. *Comm. Appl. Nonlinear Anal.*, 28(1):65–76, 2021.

# Chapter 2

# Second-order BVPs of Emden equations type

## 2.1 Introduction

In this Chapter I examine the existence and uniqueness of solutions to boundary value problems that feature second-order, ordinary differential equations and two point boundary conditions. The problems, methods and ideas in this Chapter provide a logical starting point for navigating the latter chapters of this thesis, where the analysis moves to more complicated problems such as those: of higher order, with more complex boundary conditions or problems with fractional derivatives. The second-order BVPs involves the following differential equation:

$$x^{2\alpha-1}y'' = \left[xy + \lambda y^2\right]^{\alpha}; \tag{2.1}$$

subjected to the following (Dirichlet) boundary conditions

$$y(0) = 1, \quad y(b) = 0, \quad b > 0. \tag{2.2}$$

Above $\alpha$ and $\lambda$ are constants. The general form (2.1) is known as Emden boundary value problem which can be linked with multiple models that are of physical interest and I briefly discuss some special cases to help motivate and contextualize my study.

The case $\alpha = 3/2$ and $\lambda = 0$ in (2.1) leads to the differential equation

$$y'' = \frac{y^{3/2}}{x^{1/2}}, \tag{2.3}$$

which is known as the classic, dimensionless Thomas–Fermi equation [83, 259]. For an almost encyclopedic account of the literature on (2.3), see [213, Sec 1.2]. Equation (2.3) arises, for example, in the study of the electronic structure of atoms via an account of the electron density $\rho(\mathbf{r})$ for nonrelativistic atomic ions in a magnetic field with strength $B = 0$. The electron density $\rho(\mathbf{r})$ is defined as the number of electrons per unit volume at position $\mathbf{r}$ in the atomic charge cloud [191, p.1]. The potential energy $V(\mathbf{r})$ is related to $\rho(\mathbf{r})$ via Poisson's equation of electrostatics [191, p.26]. Assuming spherical symmetry, a function $y$ is termed as a "screening function" and is defined via the relationship

$$y := \frac{r}{Ze^2}(\mu - V)$$

where: $\mu$ is the (constant) chemical potential; $r$ is the distance from the nucleus; and $Z$ is the atomic number of the atom. The independent variable $x$ in (2.3) is dimensionless and defined via $x = r/b_0$, where $b_0 > 0$ is a constant that is in terms of the Bohr radius and the atomic number $Z$ [191, p.26]. Poisson's equation then leads to the ordinary differential equation (2.3).

The case $\alpha = 1/2$ and $\lambda = 0$ in (2.1) produces the differential equation

$$y'' = [xy]^{1/2}, \tag{2.4}$$

essentially due to Kadomstev [119]. In this situation, I make similar assumptions to the above case for nonrelativistic atomic ions, but now the atoms are classed as heavy and the magnetic field has a very large strength $B$. Once again, Poisson's equation is employed to obtain (2.4), see [107, pp. 2301–2301] and [246, p.546].

The case $\alpha = 3/2$ and $\lambda > 0$ in (2.1) generates the differential equation

$$x^2 y'' = \left[xy + \lambda y^2\right]^{3/2} \tag{2.5}$$

which can be rearranged to form

$$y'' = \frac{y^{3/2}}{x^{1/2}}\left[1 + \lambda\frac{y}{x}\right]^{3/2}$$

and aligns with the equation of Vallarta and Rosen [271], see [191, p.169]. The presence of the positive $\lambda$, which depends on constants including $Z$ and the fine structure constant, is connected with the incorporation of special relativity into the Thomas–Fermi model in the case where the field strength $B = 0$. Spherical symmetry is preserved and, once again, Poisson's equation leads to our ordinary differential equation (2.5). Note that $\lambda$ scales with $c^{-2}$ and hence we recover (2.3) as $c \to \infty$, see [179, 210].

In the instance when $\alpha = 1/2$ and $\lambda > 0$ in (2.1), we have the differential equation

$$y'' = \left[xy + \lambda y^2\right]^{1/2} \tag{2.6}$$

which was formulated by Hill, Grout and March [109]. Once again, the inclusion of the positive $\lambda$ is to embed considerations of special relativity [191, p.171] where the magnetic field has a very high strength $B$. Spherical symmetry and Poisson's equation lead to (2.6).

The first boundary condition in (2.2) arises from the physical condition at the nucleus, namely

$$V(\mathbf{r}) \to -\frac{Ze}{r}, \quad \text{as } r \to 0.$$

The second boundary condition in (2.2) indicates the constraint of a (large, but) finite-sized nucleus. The significance of this form has been acknowledged in [192, p.9] and [108, p.4821] as necessary for solving (2.1). In the unbounded interval case, the problem leads to a divergence of $\int \rho \, d\mathbf{r}$ and thus the electron density cannot be normalized. The second boundary condition in (2.2) provides us with a means around this challenge.

In addition to the above links with the electron density theory of atoms, the differential equation (2.1) can be connected with models from astrophysics. For example, if $\lambda = 0$, then the differential equation (2.1) becomes

$$x^{2\alpha-1}y'' = [xy]^{\alpha}$$

which, under the change of variables $\phi_2 = xy$ becomes the equation

$$\frac{1}{x^2}\left(x^2\phi_2'\right)' = \phi_2^{\alpha}.$$

Apart from a sign, this is Emden's famous equation

$$\frac{1}{x^2}\left(x^2\theta'\right)' = -\theta^{\alpha}$$

which arises in the study of the gravitational equilibrium of a mass of a gas [191, p.43]. The dimensionless $\theta$ is related to the density; $x$ is a dimensionless radius; and $\alpha$ is known as the polytropic index, entering the effective equation of state via the power law form $P = K\rho^{1+1/\alpha}$, where $P$ and $\rho$ are the pressure and density, and $K$ is a proportionality constant.

As we can see, the evolution of the Thomas–Fermi theory [83, 259] of atoms has captivated the scientific attention of research communities in applied mathematics and physics for in excess of eighty years and so in addition to the above references, I refer the reader to see [15, 38, 43, 41, 42, 44, 46, 48, 49, 51, 50, 53, 59, 70, 92, 102, 110, 119, 129, 166, 167, 162, 163, 164, 165, 196, 214] and the references therein.

Thus the study of (2.1) is connected with a number of problems of physical interest and is well worth my attention regarding applications and so let me discuss the state of play regarding my current understanding of the above problems regarding theoretical viewpoints to the above problems.

In [107] Grout and March furnished the nonlinear problem (2.4) and explored solutions via numerical approaches through a power series expansion and integration via an Adams-variable step technique. Questions related to the global existence and uniqueness of solutions to (2.4), (2.2) remained open until recently when Tisdell and Holzer [268] illustrated that the problem does possess a unique solution on the interval $[0, b]$. This was achieved via an application of fixed-point strategies.

The relativistic form (2.6) was explored by Hill, Grout and March [109]. Once again, the problem under consideration was analyzed via numerical methods via an Adams-variable step technique. However, questions of global existence and uniqueness of solutions were not discussed and remain open.

The relativistic equation (2.6) then further evolved in [192, 193] to become the singular problem (2.1). Part of the motivation for March and Nieto [193] to investigate the form (2.1) was to produce results that could "embrace equations arising in the simplest self-consistent density functional theory - namely the Thomas–Fermi statistical method" [193, L341]. March and Nieto employed a power series expansion approach for solutions to (2.1). However, questions regarding global existence and uniqueness of solutions were not discussed and remain open.

In March's monograph he acknowledges that "at the time of writing, pure electron density theory is not at a fully quantitative stage" [191, p.1]. As we can also see from my previous discussion, pure electron density theory does not appear to be at a fully qualitative stage either.

Therefore, the question of existence and uniqueness of solutions for my problems forms a fundamental and important area of investigation regarding whether the above mathematical models for physical phenomena are well-posed. So in this Chapter my purpose is to address the aforementioned open questions and gaps by establishing a firm mathematical foundation for the nonlinear forms (2.6) and (2.1) where each is subjected to (2.2). In particular, I am concerned with the "well-posedness" of these problems [263]. My methods involve an analysis of the problems through arguments that apply differential inequalities and Schauder's fixed point theorem [294, Theorem 2.A, p.56] (Theorem 1.3) .

This Chapter is organized as follows.

In Section 2.2 I briefly introduce the notation and definitions that are necessary for navigating this Chapter and then formulate my main results, where I prove the problems under consideration are well-posed. Inspired by the approaches in [268] for the nonrelativistic problem (2.4), (2.2), my main strategy involves an analysis of the problem through arguments that apply differential inequalities and Schauder's fixed point theorem (Theorem 1.3) to the generalized Emden problem (2.1), (2.2). Well-posedness for the relativistic problem (2.6), (2.2) then follows as a special case.

By addressing the open questions raised earlier, this Chapter not only advances our understanding of well-posedness for the Emden equation (2.1), but as a special case, it also deepens our understanding of the relativistic situation (2.6). The new mathematical results presented herein will be of importance as the field of Thomas–Fermi theory continues to develop.

## 2.2 Existence results via Schauder's fixed–point theorem

My analysis will be set within a complete, normed linear space, known as a Banach space. For my analysis, I therefore choose the interval to be $[0, b]$ and consider the space of continuous functions $C([0, b])$ coupled the normed $\|y\|_0$ defined in (1.18) with $a = 0$, that is

$$\|y\|_0 := \max_{x \in [0,b]} |y(x)|, \quad \text{for all } y \in C([0, b]).$$

It is a well-known result that the pair $(C([0, b]), \|\cdot\|_0)$ forms a complete metric space, which I will draw on when I prove my main results.

The following definition sheds light on what I mean by a solution to my problem (2.1), (2.2).

**Definition 2.1.** *For each fixed $\alpha > 0$ and each fixed $\lambda \geq 0$, we say $y = y(x)$ is a* solution to (2.1), (2.2)*, if a function $y : [0, b] \to \mathbb{R}$ such that $y'$ is continuous on $[0, b]$; and $y''$ is continuous on $(0, b]$; and $y$ satisfies (2.1) on $(0, b]$ and $y$ satisfies (2.2).*

I thus denote this solution space by $C^1([0, b]) \cap C^2((0, b])$. In particular, I would expect solutions to be nonnegative and decreasing on $[0, b]$ and concave up, which aligns with the intuition from our physical models.

Let me now formulate my main results. Firstly, I establish new findings regarding the uniqueness of solutions, that is, ensuring that there is at most one solution to my problems. Secondly, I then advance current knowledge by proving the existence of solutions. Combining the two sets of results ensures my problems are well-posed.

### 2.2.1 First main result: Uniqueness of solutions

I begin by proving the uniqueness of solutions to the generalized Emden boundary value problem. By uniqueness, I really mean the term "nonmultiplicity". My results of this subsection alone do not ensure the existence of a unique solution. Rather, under the assumption of the existence of a solution, my results guarantee that it must be the only solution to the problem.

**Theorem 2.1.** *For every $\alpha > 0$ and each $\lambda \geq 0$, there is, at most, one solution to the generalized Emden boundary value problem* (2.1)*,* (2.2).

*Proof.* My approach involves an indirect proof. Assume that the boundary value problem (2.1), (2.2) has, at least, two nonidentical solutions. Let $u$ and $v$ denote any two such nonidentical solutions, that is $u \not\equiv v$ on $[0, b]$. In particular, there must a point $x_0 \in [0, b]$ such that $u(x_0) \neq v(x_0)$. I show that this leads to a contradiction by discussing two cases.

**Case 1:** $u(x_0) > v(x_0)$.

Let

$$r(x) := u(x) - v(x), \quad \text{for all } x \in [0, b]. \tag{2.7}$$

From the defined properties of solutions $u$ and $v$ we see that $r$ is continuous on $[0, b]$. Thus $r$ must achieve its maximum (and minimum) values on $[0, b]$.

Without loss of generality, let $x_0 \in [0, b]$ be such that

$$r(x_0) = \max_{x \in [0,b]} r(x) > 0. \tag{2.8}$$

If $x_0 = 0$, then the first constraint in (2.2) ensures $r(0) = 0$ and so (2.8) cannot hold for $x_0 = 0$. Similarly, the second constraint in (2.2) ensures that (2.8) cannot hold for $x_0 = b$.

We note that due to the properties of solutions $u$ and $v$, our $r$ has a continuous derivative on $[0, b]$ and a continuous second derivative on $(0, b]$. Thus, if $x_0 \in (0, b)$ then the maximum principle [220, p.1] applies, ensuring $r'(x_0) = 0$ and $r''(x_0) \leq 0$. From (2.7) and (2.1) we also have

$$x_0^{2\alpha-1}(u - v)''(x_0) = \left[x_0 u(x_0) + \lambda(u(x_0))^2\right]^\alpha - \left[x_0 v(x_0) + \lambda(v(x_0))^2\right]^\alpha > 0$$

where the last inequality holds due to $u(x_0) > v(x_0)$ and we have exploited the monotonicity of the right hand side of (2.1) in $y$. Thus, we reach a contradiction regarding $r''(x_0)$ and so (2.8) cannot hold for any $x_0 \in (0, b)$. Thus, the case $u(x_0) > v(x_0)$ leads to a contradiction.

**Case 2:** $v(x_0) > u(x_0)$.

Repeating the above argument of Case 1, it can be shown that we are led to a contradiction. This involves applying the maximum principle to $-r$ on $[0, b]$. For brevity, I omit the details of this repetition.

We have reached contradictions in each of our cases. We thus conclude that the problem cannot have more than one solution, so that there is, at most, one solution to (2.1), (2.2). □

The following interpretation of Theorem 2.1 provides some additional insight into its practical value regarding the uniqueness of solutions.

**Lemma 2.1.** *If, for each $\alpha > 0$ and each $\lambda \geq 0$, the generalized Emden boundary value problem (2.1), (2.2) has a solution then it must be the only one.*

Given the relationship between (2.1) and (2.6) we can now formulate the following Lemmas for the uniqueness of solutions to the relativistic Thomas–Fermi boundary value problem as a corollary of Theorem 2.1

**Lemma 2.2.** *For every $\lambda \geq 0$ there is, at most, one solution of the relativistic Thomas–Fermi boundary value problem (2.6), (2.2).*

*Proof.* Drawing on Theorem 2.1 with $\alpha = 1/2$ we thus see that the conclusion on nonmultiplicity of solutions applies to the relativistic Thomas–Fermi boundary value problem (2.6), (2.2). □

**Lemma 2.3.** *If the relativistic Thomas–Fermi boundary value problem (2.6), (2.2) has a solution then it must be the only one.*

Let me compare my new results of this Section with the known literature.

**Lemma 2.4.** *Theorem 2.1 and Remark 2.2 form extensions of [268, Theorem 2] for the nonrelativistic problem (2.4), (2.2). In fact, [268, Theorem 2] follows as a special case of Theorem 2.1 ($\alpha = 1/2$, $\lambda = 0$) and Remark 2.2 ($\lambda = 0$).*

### 2.2.2 Second main result: Existence of a unique solution

Let me now turn my attention to questions of the *existence* of a unique solution. I will combine the results of the preceding subsection together with an application of Schauder's fixed point theorem (Theorem 1.3) .

The next new result guarantees that for each fixed $\alpha \in (0,1)$ and each fixed $\lambda \geq 0$ the generalized Emden boundary value problem (2.1), (2.2) has a unique solution.

**Theorem 2.2.** *For each fixed $\alpha \in (0,1)$ and $\lambda \geq 0$, the generalized Emden boundary value problem (2.1), (2.2) has a unique solution $y \in C^1([0,b]) \cap C^2((0,b])$ such that*

$$0 \leq y(x) \leq 1 - \frac{x}{b}, \quad \textit{for all } x \in [0,b]. \tag{2.9}$$

*Proof.* My proof is summarized as follows. For each fixed $\alpha \in (0,1)$ and $\lambda \geq 0$, the idea is to modify the right-hand-side of (2.1) to form a modified function $h$ that is defined on the infinite strip $[0,b] \times \mathbb{R}$ and is continuous and uniformly bounded therein. At least one solution to the modified boundary value problem is then guaranteed to exist by Schauder's fixed point theorem. The solutions to the modified problem are then shown to be solutions to (2.1), (2.2). Finally, an application of Theorem 2.1 shows that this solution must be unique.

For each fixed $\alpha \in (0,1)$ and $\lambda \geq 0$, consider the modified differential equation

$$x^{2\alpha-1} y'' = h(x,y), \qquad x \in (0,b], \tag{2.10}$$

subject to (2.2), where

$$h(x,z) := \begin{cases} \left[ x\left(1 - \dfrac{x}{b}\right) + \lambda\left(1 - \dfrac{x}{b}\right)^2 \right]^\alpha + \dfrac{z - \left(1 - \dfrac{x}{b}\right)}{z + \dfrac{x}{b}}, & \text{for } z \geq 1 - \dfrac{x}{b}; \\ \left[ xz + \lambda z^2 \right]^\alpha, & \text{for } 0 \leq z \leq 1 - \dfrac{x}{b}; \\ \dfrac{z}{1-z}, & \text{for } z \leq 0. \end{cases}$$

Note that $h$ is continuous and uniformly bounded on $[0,b] \times \mathbb{R}$. Let this bound be denoted by $M > 0$. Showing the existence of a unique solution to the modified Emden boundary value problem (2.10), (2.2) is equivalent to showing the existence of a $y \in C([0,b])$ that satisfies the following the integral equation

$$y(x) := 1 - \frac{x}{b} + \frac{x}{b} \int_0^b \int_s^b \frac{1}{p^{2\alpha-1}} h(p, y(p)) \, dp \, ds - \int_0^x \int_s^b \frac{1}{p^{2\alpha-1}} h(p, y(p)) \, dp \, ds, \quad x \in [0,b]. \tag{2.11}$$

The proof of this equivalence is found in [205].

I draw on the Banach space $(C([0,b]), \|\cdot\|_0)$ defined early in this Section. Define an operator $V : C([0,b]) \to C([0,b])$ by

$$[Vy](x) := 1 - \frac{x}{b} + \frac{x}{b} \int_0^b \int_s^b \frac{1}{p^{2\alpha-1}} h(p, y(p)) \, dp \, ds - \int_0^x \int_s^b \frac{1}{p^{2\alpha-1}} h(p, y(p)) \, dp \, ds, \quad x \in [0,b]. \tag{2.12}$$

For each $\lambda \geq 0$ and $\alpha \in (0,1)$ our operator $V$ is well-defined.

The continuity of $h$ ensures that for all $y \in C([0,b])$ we have $Vy \in C([0,b])$. In addition, comparing (2.11) with (2.12) we see that $y$ is a solution to (2.10), subject to (2.2), if and only if $Vy = y$. Hence, I seek an application of the Schauder fixed point theorem to show that the modified boundary value problem (2.10), (2.2) has at least one solution.

Define the closed ball $\Omega_1 \subset C([0,b])$ by

$$\Omega_1 := \left\{ y \in C([0,b]) : \|y\|_0 = \max_{x\in[0,b]} |y(x)| \leq 2 + \frac{Mb^{3-2\alpha}}{1-\alpha} \right\}. \tag{2.13}$$

Thus, $\Omega_1$ is a nonempty, closed, bounded and convex subset of $C([0,b])$.

Working from (2.12), we see that the continuity of $h$ and the bound $M$ ensure that for all $y \in C([0,b])$ and each fixed $0 < \alpha < 1$ and $\lambda \geq 0$ we have

$$\begin{aligned} \|Vy\|_0 &= \max_{x\in[0,b]} |[Vy](x)| \\ &\leq 1 + \frac{Mb^{3-2\alpha}}{1-\alpha} \\ &< 2 + \frac{Mb^{3-2\alpha}}{1-\alpha}. \end{aligned}$$

Thus we see that $V : \Omega_1 \to \Omega_1$.

It can be shown that $V$ is completely continuous on $C([0,b])$ (see [205]) and so forms a compact map. Thus, for every fixed $\alpha \in (0,1)$ and $\lambda \geq 0$, the Schauder fixed point theorem implies there exists at least one fixed point, $y \in \Omega_1$, that is, $Vy = y$. These $y$ are not only in $C([0,b])$, but are actually in $C^1([0,b]) \cap C^2((0,b])$ due to the continuity of $f$ (and $h$) which follows from (2.11). Thus, the modified Emden boundary value problem (2.10), (2.2) has at least one solution $y \in C^1([0,b]) \cap C^2((0,b])$.

I now show that solutions $y$ to (2.10), (2.2) must satisfy (2.9) and so they must be solutions to the unmodified Emden boundary value problem (2.1), (2.2).

I first show that

$$y(x) \leq 1 - \frac{x}{b}, \quad \text{for all } x \in [0,b]. \tag{2.14}$$

I establish (2.14) via an indirect proof. Assume (2.14) does not hold. Let

$$q(x) := y(x) - \left(1 - \frac{x}{b}\right)$$

and let $x_1 \in [0, b]$ be the point such that

$$q(x_1) := \max_{x \in [0,b]} q(x) > 0.$$

The boundary conditions (2.2) ensure $x_1 \in (0, b)$. Thus, the maximum principle gives $q'(x_1) = 0$ and $q''(x_1) \leq 0$. But, for each $0 < \alpha < 1$ and $\lambda \geq 0$, we have

$$\begin{aligned} x_1^{2\alpha-1} q''(x_1) &= x_1^{2\alpha-1} y''(x_1) \\ &= h(x_1, y(x_1)) \\ &= \left[ x_1 \left(1 - \frac{x_1}{b}\right) + \lambda \left(1 - \frac{x_1}{b}\right)^2 \right]^\alpha + \frac{y(x_1) - \left(1 - \frac{x_1}{b}\right)}{y(x_1) + \frac{x_1}{b}} > 0. \end{aligned}$$

Thus, we reach a contradiction.

The case showing $y \geq 0$ on $[0, b]$ is shown similarly to the above working (just apply the maximum principle to $-y$) and so is omitted for brevity.

We have thus established that all solutions to the modified Emden problem must satisfy (2.9). Hence, these solutions are also solutions to the unmodified Emden problem (2.1), (2.2). That is, the boundary value problem has at least one solution.

Combining this with the conclusion of Theorem 2.1, we see that (2.1), (2.2) has at least one solution and at most one solution. We conclude that the solution is unique. That is, we conclude that the generalized Emden boundary value problem is well-posed. □

The following Corollary concerning well-posedness of the relativistic Thomas–Fermi problem.

**Corollary 2.1.** *For every $\lambda \geq 0$ there exists a unique solution of the relativistic Thomas–Fermi boundary value problem* (2.6)*,* (2.2)*.*

*Proof.* Drawing on Theorem 2.2 with $\alpha = 1/2$ we thus see that the conclusion on existence and uniqueness of solutions applies to the relativistic Thomas–Fermi boundary value problem (2.6), (2.2). □

Let me compare the two new results of this Section with the known literature.

**Remark 2.1.** *Theorem 2.2 and Corollary 2.1 form extensions of [268, Theorem 3] for the nonrelativistic problem* (2.4)*,* (2.2)*. In fact, [268, Theorem 3] follows as a special case of Theorem 2.2 ($\alpha = 1/2$, $\lambda = 0$) and Corollary 2.1 ($\lambda = 0$).*

# Chapter 3

# Third-order BVPs: Analysis in unbounded domain

## 3.1 Introduction

In this Chapter I consider a higher order ordinary differential equation than considered in the previous Chapter. In particular I consider the following third-order ordinary differential equation:

$$y''' + f(x, y) = 0, \quad x \in [a, b]. \tag{3.1}$$

where $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuous and (3.1) is subject to the following three-point boundary conditions:

$$y(a) = 0, \; y'(a) = 0, \; y(b) = ky(\eta); \tag{3.2}$$

where $a < \eta < b$ and $k \in \mathbb{R}$. Observe that if $k = 0$ then (3.2) collectively becomes two-point conditions.

The study of the third-order equations has gained considerable attention for more than 50 years due to their importance and connection with many number of physical and technological processes such as the deflection of a curved beam having a constant or varying cross-section, three layer beams and electromagnetic waves, gravity-driven flows or laminar flows for example see [45, 81, 112, 221, 226, 242] and the references therein. Since “knowing an equation has a unique solution is important from both a modeling and theoretical point of view" [263, p.794], a range of authors have investigated the existence, uniqueness and approximation of solutions to third-order ordinary differential equations that are subjected to appropriate boundary conditions including three-point boundary conditions, which was first formulated by Sansone [237] in 1948. They have pursued a spectrum of approaches to the existence and/or uniqueness of solution to third-order BVPs. This includes methods such as: Schauder fixed point theorem [249, 250]; Leray–Schauder degree [90]; Leray–Schauder continuation theorem [7]; lower and upper solutions [56, 225, 232, 234]; monotone positive solutions [301]; nonconjugate boundary conditions and Lypapunov functions [82]; positive solutions to singular problems [105]; and oscillation theory [72, 252]. The reader is also referred to [6, 25, 37, 98, 160, 161, 190, 203, 212, 233] for some additional developments in the field of third-order BVPs and their applications.

In addition to this, an author named Smirnov recently [248] considered the BVP (3.1) ,(3.2) and he skillfully developed a theory regarding existence and uniqueness of solutions for the BVP of the form (3.1), (3.2) via use of Banach's fixed point theorem (Theorem 1.5) in a complete metric space and also established interesting properties of the associated Green's function.

Motivated by the above discuss and the importance of studying third-order BVPs, the purpose of this Chapter is to examining the existence, uniqueness and approximation of solutions to the BVP of the form (3.1) ,(3.2). In particular, I am interesting in sharpening Smirnov's existence and uniqueness results for the BVP (3.1), (3.2). This is achieved in three directions and in complementary ways.

Firstly, I provide in Section 3.2 sharp and sharpened estimates than these provided by Smirnov for integrals regarding various Green's functions.

Secondly, these sharper estimates are applied to problems in Section 3.3 via Theorem 1.5. Even though in this step I use Theorem 1.5, which was used by Smirnov, my result improves those of Smirnov by illustrating that a larger class of these kinds of problems admit a unique solution.

Since applying the Rus fixed point theorem (Theorem 1.6) appears to occupy a unique position within the literature as a strategy to ensure existence and uniqueness of solutions to third-order BVPs, I thirdly in Section 3.4 apply Theorem 1.6 in a metric space. The result in this step shall form an advancement over applications of Banach's fixed point theorem. This is achieved through the use of two metrics and Theorem 1.6. As we will discover, this enables a greater class of problems to be better understood regarding existence and uniqueness of solutions, which includes sharpening the Lipschitz constants. I also devote discussion to fully illustrate the nature of the advancements made via use of remarks and examples in Section 3.5.

## 3.2 Estimates of integrals of Green's functions

In this Section I establish improved inequalities for integrals involving various Green's function that are associated with the BVP (3.1), (3.2).

I first give a definition on what I mean by a solution to (3.1), (3.2)

**Definition 3.1.** *We say $y = y(x)$ is a* solution to (3.1), (3.2) *, if a function $y : [a, b] \to \mathbb{R}$ such that $y$ is a three-times continuously differentiable function (that is, $y \in C^3([a, b])$) that satisfies the BVP* (3.1)*,* (3.2).

The BVP (3.1), (3.2) can be recast as an equivalent integral equation [248, pp. 173–174]

$$y(x) = \int_a^b g(x, s) f(s, y(s))\, ds, \quad x \in [a, b], \tag{3.3}$$

where

$$g(x, s) := R(x, s) + \frac{k(x-a)^2}{(b-a)^2 - k(\eta - a)^2} R(\eta, s) \tag{3.4}$$

and $R$ is given explicitly by

$$R(x, s) = \frac{1}{2} \begin{cases} \frac{(x-a)^2(b-s)^2}{(b-a)^2} - (x-s)^2, & \text{for } a \le s \le x \le b, \\ \\ \frac{(x-a)^2(b-s)^2}{(b-a)^2}, & \text{for } a \le x \le s \le b. \end{cases} \tag{3.5}$$

The following result establishes the nonnegativity of the function $R$ and will be useful in developing my estimates on the integrals of $R$ and $|g|$.

**Theorem 3.1.** *The function $R(x, s)$ in* (3.5) *satisfies $R \ge 0$ on $[a, b] \times [a, b]$.*

*Proof.* From (3.5) we can see that the case showing $R(x, s) \ge 0$ on the region $a \le x \le s \le b$ is obvious due the the squared form of the function therein.

The remaining situation to show $R(x, s) \ge 0$ on the region $a \le s \le x \le b$ involves some algebraic manipulation in the following manner. From (3.5), if we apply the formula for the difference of two squares, then we see that for $a \le s \le x \le b$ we have

$$\frac{(x-a)^2(b-s)^2}{2(b-a)^2} - \frac{(x-s)^2}{2} = \frac{1}{2}\left[\frac{(x-a)(b-s)}{b-a} + (x-s)\right]\left[\frac{(x-a)(b-s)}{b-a} - (x-s)\right] \ge 0.$$

The nonnegativity follows from the fact that each expression contained in the square brackets of the above product is nonnegative and so the product in question is nonnegative. For example, in the first square bracket we have a sum of two nonnegative terms; while in the second square bracket we may equivalently write the terms in as

$$\frac{(s-a)(b-x)}{b-a} \geq 0.$$

□

Let me utilize Theorem 3.1 to form the following sharp and sharper estimates on the integrals of various Green's functions for the BVP (3.1), (3.2). The estimates will be of a helpful form for my analysis in Sections 3.3 and 3.4 as well as the next Chapter. The estimates are also of independent interest.

**Theorem 3.2.** *The function $R(x,s)$ in (3.5) satisfies*

$$\int_a^b R(x,s)\ ds \leq \frac{2}{81}(b-a)^3, \quad \textit{for all } x \in [a,b]. \tag{3.6}$$

*Inequality (3.6) is sharp in the sense that it is the best inequality possible.*

*Proof.* For all $x \in [a,b]$ we have

$$\begin{aligned}
\int_a^b R(x,s)\ ds &= \int_a^x R(x,s)\ ds + \int_x^b R(x,s)\ ds \\
&= \int_a^x \frac{(x-a)^2(b-s)^2}{2(b-a)^2} - \frac{(x-s)^2}{2}\ ds + \int_x^b \frac{(x-a)^2(b-s)^2}{2(b-a)^2}\ ds \\
&= \frac{(x-a)^2}{6(b-a)^2}\left[-(b-x)^3 + (b-a)^3\right] - \frac{(x-a)^3}{6} + \frac{(x-a)^2(b-x)^3}{6(b-a)^2} \\
&= \frac{(x-a)^2(b-x)}{6}.
\end{aligned}$$

Now, if we define

$$r(x) := \int_a^b R(x,s)\ ds$$

then an application of basic calculus reveals that

$$\begin{aligned}
\max_{x\in[a,b]} r(x) &= \max_{x\in[a,b]} \frac{(x-a)^2(b-x)}{6} \\
&= \frac{2}{81}(b-a)^3.
\end{aligned}$$

In particular, the maximum of $r$ on $[a,b]$ is attained when

$$x = a + \frac{2}{3}(b-a)$$

and this illustrates that the inequality (3.6) is sharp. □

**Remark 3.1.** *Smirnov [248, p.175] forms the estimate*

$$\int_a^b |R(x,s)|\, ds \le \frac{(b-a)^3}{3}, \quad \textit{for all } x \in [a,b]. \tag{3.7}$$

*If we compare* (3.7) *with my sharper estimate* (3.6) *then it is easy to see that Theorem 3.2 extends [248, Proposition 3].*

An analogue of Theorem 3.2 for $g$ now follows.

**Theorem 3.3.** *The function* $g(x,s)$ *in* (3.4) *satisfies*

$$\int_a^b |g(x,s)|\, ds \le (b-a)^3 \left[\frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right], \textit{ for all } x \in [a,b]. \tag{3.8}$$

*where we have assumed* $k(\eta-a)^2 \ne (b-a)^2$ *with* $a < \eta < b$.

*Proof.* Consider

$$\begin{aligned}
\int_a^b R(\eta,s)\, ds &= \int_a^\eta \frac{(\eta-a)^2(b-s)^2}{2(b-a)^2} - \frac{(\eta-s)^2}{2}\, ds + \int_\eta^b \frac{(\eta-a)^2(b-s)^2}{2(b-a)^2}\, ds \\
&= \int_a^b \frac{(\eta-a)^2(b-s)^2}{2(b-a)^2}\, ds - \int_a^\eta \frac{(\eta-s)^2}{2}\, ds \\
&= \frac{(\eta-a)^2(b-a)}{6} - \frac{(\eta-a)^3}{6} \\
&= \frac{1}{6}(\eta-a)^2(b-\eta) \\
&\le \frac{1}{3}(b-a)^3. 
\end{aligned} \tag{3.9}$$

So similarly to the proof of Theorem 3.2, for $x \in [a,b]$, we have

$$\begin{aligned}
\int_a^b |g(x,s)|\, ds &= \int_a^b \left| R(x,s) + \frac{k(x-a)^2}{(b-a)^2 - k(\eta-a)^2} R(\eta,s) \right| ds \\
&\le \int_a^b |R(x,s)| + \left| \frac{k(x-a)^2}{(b-a)^2 - k(\eta-a)^2} \right| |R(\eta,s)|\, ds \\
&= \frac{1}{6}(x-a)^2(b-x) + \frac{|k|(x-a)^2}{|(b-a)^2 - k(\eta-a)^2|} \frac{1}{6}(\eta-a)^2(b-\eta) \\
&\le \frac{2}{81}(b-a)^3 + \frac{|k|(b-a)^2}{|(b-a)^2 - k(\eta-a)^2|} \frac{1}{3}(b-a)^3 \\
&= (b-a)^3 \left[\frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right].
\end{aligned}$$

Above, we employed the fact that $R \ge 0$ and (3.6). Thus we have established (3.8). □

**Remark 3.2.** *Smirnov [248, p.176] forms the estimate*

$$\int_a^b |g(x,s)|\, ds \le \frac{(b-a)^3}{3} \left[1 + \frac{|k|(b-a)^2}{|(b-a)^2 - k(\eta-a)^2|}\right], \textit{ for all } x \in [a,b]. \tag{3.10}$$

*If we compare* (3.10) *with my sharp estimate* (3.8) *then it is easy to see that Theorem 3.3 extends [248, Proposition 4.].*

## 3.3 Existence results via Banach fixed point theorem

In this Section I establish my first novel result for the existence and uniqueness of solutions to the BVP (3.1), (3.2) via Banach's fixed point theorem (Theorem 1.5) where I use the results of Section 3.2.

**Theorem 3.4.** *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous, let $f(x,0) \neq 0$ for all $x \in [a,b]$ and let $L$ be a nonnegative constant such that*

$$|f(x,u_0) - f(x,v_0)| \leq L|u_0 - v_0|, \text{ for all } (x,u_0),(x,v_0) \in [a,b] \times \mathbb{R}. \tag{3.11}$$

*If $k(\eta - a)^2 \neq (b-a)^2$ with $a < \eta < b$ and*

$$L(b-a)^3\left[\frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right] < 1, \tag{3.12}$$

*then the BVP* (3.1), (3.2) *has a unique (nontrivial) solution in $C^3([a,b])$.*

*Proof.* Consider the operator $T : C([a,b]) \to C([a,b])$ defined by

$$(Ty)(x) := \int_a^b g(x,s) f(s,y(s))\, ds, \quad x \in [a,b].$$

In view of (3.3) we wish to show that there exists a unique $y \in C([a,b])$ such that

$$Ty = y.$$

Every such solution will also lie in $C^3([a,b])$ as can be directly shown by differentiating (3.3) and confirming the continuity.

To establish the existence and uniqueness to $Ty = y$, we show that the conditions of Theorem 1.5 hold.

We consider the space of continuous functions $Y := C([a,b])$ coupled with the metric $d_0$ defined in (1.16), so the pair $(Y,\varrho) := (C([a,b]), d_0)$ forms a complete metric space.

For $y, z \in C([a,b])$ and $x \in [a,b]$, consider

$$\begin{aligned} |(Ty)(x) - (Tz)(x)| &\leq \int_a^b |g(x,s)|\, |f(s,y(s)) - f(s,z(s)|\, ds \\ &\leq \int_a^b |g(x,s)|\, L\, |y(s) - z(s)|\, ds \end{aligned}$$

$$\begin{aligned} &\le L d_0(y,z) \int_a^b |g(x,s)|\, ds \\ &\le L(b-a)^3 \left[ \frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2 - k(\eta-a)^2|} \right] d_0(y,z), \end{aligned} \tag{3.13}$$

where we have applied (3.8).

Taking the maximum of both sides of the inequality (3.13) over $[a,b]$ we thus have for all $y, z \in C([a,b])$

$$d_0(Ty,Tz) \le L(b-a)^3 \left[ \frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2 - k(\eta-a)^2|} \right] d_0(y,z),$$

and in light of (3.12) we see that $T$ satisfies all of the conditions of Theorem 1.5. Thus, the operator $T$ has a unique fixed point in $C([a,b])$. This solution is also in $C^3([a,b])$ and we have equivalently shown that the BVP (3.1), (3.2) has a unique (nontrivial) solution. □

**Remark 3.3.** *Smirnov's result [248, Theorem 1] assumes*

$$L\frac{(b-a)^3}{3}\left[1 + \frac{|k|(b-a)^2}{|k(\eta-a)^2 - (b-a)^2|}\right] < 1. \tag{3.14}$$

*If we compare (3.14) with my (3.12) then we can see that (3.12) forms a less restrictive condition.*

The following results are a consequence of Theorem 1.5 holding for the operator $T$ therein, see [294, Theorem 1.A]. I will use it to form the following results that involve approximations to the unique solution $y$ of the BVP (3.1), (3.2).

**Remark 3.4.** *Let the conditions of Theorem 3.4 hold. If we recursively define a sequence of approximations $y_n = y_n(x)$ on $[a,b]$ via*

$$y_0 := 0, \quad y_{n+1}(x) := \int_a^b g(x,s)\, f(s, y_n(s))\, ds, \quad n = 0,1,2,\dots$$

*then:*

- *the sequence $y_n$ converges to the solution $y$ of (3.1), (3.2) with respect to the $d_0$ metric and the rate of convergence is given by*

$$d_0(y_{n+1},y) \le L(b-a)^3 \left[ \frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2 - k(\eta-a)^2|} \right] d_0(y_n,y);$$

- *for each $n$, an* a priori *estimate on the error is*

$$d_0(y_n,y) \le \frac{\left( L(b-a)^3 \left[ \frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2-k(\eta-a)^2|} \right] \right)^n}{1 - L(b-a)^3 \left[ \frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2-k(\eta-a)^2|} \right]} d_0(y_1,0);$$

- *for each $n$, an* a posteriori *estimate on the error is*

$$d_0(y_{n+1}, y) \leq \frac{L(b-a)^3\left[\frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right]}{1 - L(b-a)^3\left[\frac{2}{81} + \frac{|k|(b-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right]} d_0(y_{n+1}, y_n).$$

## 3.4 Existence results via Rus fixed point theorem

In this Section I state and prove my second novel result on existence and uniqueness of solutions to (3.1), (3.2) where I employ two metrics under Rus's theorem (Theorem 1.6).

**Theorem 3.5.** *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous, let $f(x,0) \neq 0$ for all $x \in [a,b]$ and let $L$ be a nonnegative constant such that*

$$|f(x,u_0) - f(x,v_0)| \leq L|u_0 - v_0|, \textit{ for all } (x,u_0), (x,v_0) \in [a,b] \times \mathbb{R}. \tag{3.15}$$

*If $k(\eta - a)^2 \neq (b-a)^2$ with $a < \eta < b$ and there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with*

$$L\left(\int_a^b \left(\int_a^b |g(x,s)|^q \, ds\right)^{p/q} dt\right)^{1/p} < 1, \tag{3.16}$$

*then the BVP (3.1), (3.2) has a unique (nontrivial) solution in $C^3([a,b])$.*

*Proof.* Consider the operator $T : C([a,b]) \to C([a,b])$ defined by

$$(Ty)(x) := \int_a^b g(x,s) f(s, y(s)) \, ds, \quad x \in [a,b].$$

In light of (3.3) we want to show that there exists a unique $y \in C([a,b])$ such that

$$Ty = y.$$

Such a solution will also lie in $C^3([a,b])$ as can be directly shown by differentiating (3.3) and confirming the continuity.

To establish the existence and uniqueness to $Ty = y$, we show that the conditions of Theorem 1.6 hold.

Consider the pair $(Y, \varrho) := (C([a,b]), d_0)$ which forms a complete metric space. In addition, consider the metric $\delta_p = \tau$ on $Y$, where $p > 1$, where $\delta_p$ is defined in (1.17).

For $y, z \in C([a,b])$ and $x \in [a,b]$, consider

$$|(Ty)(x) - (Tz)(x)| \leq \int_a^b |g(x,s)| \, |f(s,y(s)) - f(s,z(s)| \, ds$$

$$
\begin{aligned}
&\leq \int_a^b |g(x,s)|\; L\; |y(s)-z(s)|\; ds \\
&\leq \left(\int_a^b |g(x,s)|^q\; ds\right)^{1/q} L \left(\int_a^b |y(s)-z(s)|^p\; ds\right)^{1/p} \qquad (3.17)\\
&\leq L \max_{x\in[a,b]} \left(\int_a^b |g(x,s)|^q\; ds\right)^{1/q} \delta_p(y,z).
\end{aligned}
$$

Above, we have used (3.15) and Hölder's inequality [113, 227] to obtain (3.17). Thus, defining

$$c := L \max_{x\in[a,b]} \left(\int_a^b |g(x,s)|^q\; ds\right)^{1/q}$$

we see that

$$d_0(Ty,Tz) \leq c\delta_p(y,z), \quad \text{for some } c>0 \text{ and all } y,z \in C([a,b]) \tag{3.18}$$

and so the inequality (1.41) of Theorem 1.6 holds.

Now, for all $y, z \in C([a,b])$ we may apply (1.25) to (3.18) to obtain

$$d_0(Ty,Tz) \leq c\delta_p(y,z) \leq c(b-a)^{1/p} d_0(y,z).$$

Thus, given any $\varepsilon > 0$ we can choose $\Delta = \varepsilon/c(b-a)^{1/p}$ so that $d_0(Ty,Tz) < \varepsilon$ whenever $d_0(y,z) < \Delta$. Hence $T$ is continuous on $C([a,b])$ with respect to the $d_0$ metric.

Finally, we show that $T$ is contractive on $C([a,b])$ with respect to the $\delta_p$ metric, that is, the inequality (1.42) in Theorem 1.6 holds. From (3.17), for each $y, z \in C([a,b])$ consider

$$\left(\int_a^b |(Ty)(x)-(Tz)(x)|^p\; dt\right)^{1/p} \leq L\left(\int_a^b \left(\int_a^b |g(x,s)|^q\; ds\right)^{p/q} dt\right)^{1/p} \delta_p(y,z),$$

and so we obtain

$$\delta_p(Ty,Tz) \leq L\left(\int_a^b \left(\int_a^b |g(x,s)|^q\; ds\right)^{p/q} dt\right)^{1/p} \delta_p(y,z).$$

From my assumption (3.16), we thus have

$$\delta_p(Ty,Tz) \leq \alpha\delta_p(y,z),$$

for some $\alpha < 1$ and all $y, z \in C([a,b])$.

Thus, Theorem 1.6 is applicable and the operator $T$ has a unique fixed point in $C([a,b])$. This solution is also in $C^3([a,b])$ and we have equivalently shown that the BVP (3.1), (3.2) has a unique (nontrivial) solution. □

For the choices $p = 2$ and $q = 2$ my Theorem 3.5 becomes the following new result.

**Theorem 3.6.** *Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be continuous, let $f(x,0) \neq 0$ for all $x \in [a,b]$ and let $L$ be a nonnegative constant such that*

$$|f(x,u_0) - f(x,v_0)| \leq L|u_0 - v_0|, \text{ for all } (x,u_0),(x,v_0) \in [a,b] \times \mathbb{R}. \tag{3.19}$$

*If $k(\eta - a)^2 \neq (b-a)^2$ with $a < \eta < b$ and*

$$L\left(\int_a^b \left(\int_a^b \left|g(x,s)\right|^2 \, ds\right) dx\right)^{1/2} < 1, \tag{3.20}$$

*then the BVP (3.1), (3.2) has a unique (nontrivial) solution in $C^3([a,b])$.*

**Remark 3.5.** *The left hand side of the condition (3.20) does not appear to be particularly pleasant to calculate by hand. Indeed, even when I employed Maple to evaluate the left hand side of (3.20) I produced a very complicated expression that took up nearly an entire page (even after attempts at "simplification"). Thus, I have elected not to expressly include this bound for general intervals $[a,b]$ but I have the Maple code [17] for those who are interested. However, as we will see below, I can discuss some special cases that shed some light on the situation.*

## 3.5 Examples, comparisons and remarks

Let me discuss the nature of the advancement of my theorems through exemplification, comparisons and remarks.

**Remark 3.6.** *In the case $[a,b] = [0,1]$, Smirnov's result [248, Theorem 1] for (3.1), (3.2) assumes (3.15) holds for some constant $L$ such that*

$$\frac{L}{3}\left[1 + \frac{|k|}{|1 - k\eta^2|}\right] < 1. \tag{3.21}$$

*Observe the limit on the size of the Lipschitz constant $L$ governed by (3.21). Given $k$ and $\eta$, for sufficiently small $L$, the inequality (3.21) will hold.*

*Let me illustrate how condition (3.21) is sharpened through my Theorem 3.6 by discussing an example. Consider $k = 1$, $\eta = 1/2$. In this situation, Smirnov's condition (3.21) becomes*

$$L < 9/7. \tag{3.22}$$

*Whereas the left hand side of (3.20) can be evaluated (for example, using Maple [17]) with my particular values of $k$, $a$, $b$ and $\eta$, which leads to*

$$\int_0^1 g(x,s)^2 \, ds = \frac{4}{27}x^7 - \frac{5}{18}x^6 + \frac{2}{15}x^5 - \frac{4}{45}x^4(x-1)^5$$

*and so*

$$\int_0^1 \int_0^1 g(x,s)^2 \, ds \, dt = \frac{16}{14175}. \tag{3.23}$$

*Thus, in this special case,* (3.20) *takes the form*

$$L\frac{4\sqrt{7}}{315} < 1. \tag{3.24}$$

*Condition* (3.24) *will be satisfied, for example, if*

$$L \leq 29.$$

*For an $f$ such as*

$$f(x,y) := 20\sin y + (x+1)^2$$

*the smallest constant $L$ that can be chosen so that $f$ satisfies* (3.15) *on* $[0,1] \times \mathbb{R}$ *is* $L = 20$. *The value* $L = 20$ *does not satisfy Smirnov's condition* (3.22)*, but it does satisfy* (3.24).

*Thus we can see that Theorem 3.5 and Theorem 3.6 apply to a wider class of problems than [248, Theorem 1].*

**Remark 3.7.** *If we let* $k = 0$ *in* (3.2) *then we observe that we have classical two-point boundary conditions. In this case, the result of Theorem 3.5 leads to existence and uniqueness for the corresponding two-point problem with* (3.16) *becoming*

$$L\left(\int_a^b \left(\int_a^b \left|R(x,s)\right|^q \, ds\right)^{p/q} dt\right)^{1/p} < 1, \tag{3.25}$$

*where $R$ is defined in* (3.5). *In this situation, with* $p = 2$ *and* $q = 2$*, the left hand side of* (3.25) *can be computed (by using Maple [17], for example) to obtain the equivalent condition*

$$L\frac{\sqrt{266}}{840}(b-a)^3 < 1. \tag{3.26}$$

*This can then be compared with Smirnov's condition [248, Theorem 1] (with* $k = 0$*), namely*

$$L\frac{(b-a)^3}{3} < 1, \tag{3.27}$$

*and with the condition* (3.12) *(with* $k = 0$*), namely*

$$L\frac{2}{81}(b-a)^3 < 1. \tag{3.28}$$

*Observe the restriction on the length of the interval and/or the Lipschitz constant in* (3.27). *Clearly, the inequality* (3.26) *is sharper than* (3.27). *Thus, for this special case, we can see that Theorem 3.5 applies to a wider class of problems than [248, Theorem 1]. Furthermore, we can see that my* (3.26) *is sharper than my* (3.28).

**Remark 3.8.** *Theorem 3.5 is sharper than Theorem 3.4. However, as we have noted, the left-hand side of* (3.16) *may not be so straightforward to calculate in general situations. On the other hand, the left-hand side of* (3.12) *may be much easier to calculate. Thus, Theorem 3.4 still has advantages, despite its limitations when compared with Theorem 3.5.*

**Remark 3.9.** *Finally, I observe how my work not only confirms the importance of* $L$ *and* $(b - a)$ *as influencing factors in the existence and uniqueness of solutions to BVPs, but it also illustrates how the consideration of: the sign of Green's functions; estimates on Green's functions; and choice of metrics can play an important role.*

# Chapter 4

# Third-order BVPs: Analysis in bounded domain

## 4.1 Introduction

The third-order BVP (3.1) can be made more challenging, that is considering $f$ to be fully nonlinear as the following

$$y''' + f(x, y, y', y'') = 0, \quad x \in [a, b], \tag{4.1}$$

where $f : \Omega_3 \subset [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is assumed to be continuous and (4.1) is subject to the same three-point boundary conditions considered on the previous Chapter, namely (3.2).

The goal of this Chapter is to establish a more complete and wider-ranging theory than is the results obtained in the previous Chapter. In particular, I am interested in proving the existence, uniqueness and approximation of solutions to (4.1), (3.2). This is motivated by the obtained results in the previous Chapter, which was published recently [22] and by the result of Smirnov [248]. In these results, the BVP (3.1), (3.2) was analyzed and sufficient conditions were established under which the BVP (3.1), (3.2) admitted a unique (nontrivial) solution that could be approximated by Picard iterants.

Two fundamental assumptions in the obtained results in the previous Chapter [22] and in [248] were: $f : [a, b] \times \mathbb{R} \to \mathbb{R}$, that is, $f$ was defined on the whole "infinite strip" $[a, b] \times \mathbb{R}$; and $f$ satisfied a Lipschitz condition on the entire set $[a, b] \times \mathbb{R}$, that is, there was a constant $L > 0$ such that

$$|f(x, u_0) - f(x, v_0)| \le L|u_0 - v_0|, \text{ for all } (x, u_0), (x, v_0) \in [a, b] \times \mathbb{R}. \tag{4.2}$$

Furthermore, one can see that the $f$ in (3.1) is of a form that does not depend on derivatives of the solution $y$.

The results in the previous Chapter ([22]) and in [248] form important and interesting contributions to knowledge, however, a complete qualitative theory for the existence and uniqueness of solutions to (4.1), (3.2) is yet to be achieved, as the following examples illustrate.

**Example 4.1.** *Consider the BVP*

$$y''' + x + 2 + y^2 = 0; \tag{4.3}$$

$$y(0) = 0,\ y'(0) = 0, \quad y(1) = y(1/2). \tag{4.4}$$

*Here, my $f$ in (4.3) is well-defined on $[0,1] \times \mathbb{R}$, but it does not satisfy the Lipschitz condition (4.2) therein. Thus, the results in [22, 248] do not apply to this example.*

**Example 4.2.** *Consider*

$$y''' + x + 1 + \frac{y}{5} + \frac{(y')^3}{3000} = 0, \tag{4.5}$$

*subject to (4.4). The results in [22, 248] do not apply to this example because the $f$ in (4.5) is of a more general form than that in (3.1) due to its dependency on $y'$.*

**Example 4.3.** *Consider*

$$y''' + \frac{1}{2-y} = 0, \tag{4.6}$$

*subject to (4.4). The results in [22, 248] do not apply to this example because the $f$ in (4.6) is not well defined on the whole of the strip $[0,1] \times \mathbb{R}$.*

Sufficiently motivated by some of the gaps that have been identified through the above discussion, the aim of this Chapter is to advance the current state of knowledge on (4.1), (3.2) in a way that addresses the aforementioned challenges. My strategy involves undertaking an analysis: within closed and bounded sets of $[a,b] \times \mathbb{R}$; and in closed balls within infinite dimensional space. In doing so, I am able to form a fuller theory and a deeper understanding of the qualitative properties of the solutions to (4.1), (3.2). In particular, I develop a set of results that is applicable to a wider range of problems than the obtained results in previous Chapter ([22]) and the results obtained in [248]

This Chapter is organized as follows. In Section 4.2 I build on some of the ideas in [22, 248] by establishing new estimates on the integrals of derivatives of various Green's functions. This includes "sharp" estimates. These estimates are then applied to (4.1), (3.2) in Section 4.3

and Section 4.4 via Banach fixed point theorem and Rus fixed point theorem respectively to ensure the existence and uniqueness of solutions under sufficient conditions. In addition, I establish some constructive results regarding the approximation of solutions through the use of Picard iterations. Finally, I illustrate the essence of the advancements of my work over existing literature via the discussion of examples in Section 4.5.

## 4.2 Estimates of integrals of Green's functions

In this Section I establish various inequalities for integrals that involve a range of Green's functions and their derivatives that are connected with the BVP (4.1), (3.2). While these results are of interest in their own right, I will draw on them when I form my existence, uniqueness and approximation theorems for solutions to (4.1), (3.2).

I first give a definition on what I mean by a solution to (4.1), (3.2).

**Definition 4.1.** *We say $y = y(x)$ is a solution to* (4.1)*,* (3.2)*, if a function $y : [a, b] \to \mathbb{R}$ such that $y$ has a third-order derivative that is continuous on $[a, b]$ (which I denote by $y \in C^3([a, b])$); and $y$ satisfies: $(x, y(x), y'(x), y''(x)) \in \Omega_3$ for all $x \in [a, b]$; and* (4.1) *on $[a, b]$; and the boundary conditions* (3.2).

By employing the procedure in [248, pp.173-174], it can be shown that the BVP (4.1), (3.2) can be equivalently reformulated as the integral equation

$$y(x) = \int_a^b g(x, s) f(s, y(s), y'(s), y''(s))\, ds, \quad x \in [a, b], \tag{4.7}$$

where $g$ and $R$ are given explicitly by (3.4) and (3.5) .

I now establish the following new estimate involving $R_x = \partial R/\partial x$ that complements Theorem 3.1 and Theorem 3.2.

**Theorem 4.1.** *The function $R(x, s)$ in* (3.5) *satisfies*

$$\int_a^b |R_x(x, s)|\, ds \le \frac{5}{6}(b - a)^2, \quad \textit{for all } x \in [a, b]. \tag{4.8}$$

*Proof.* For all $x \in [a, b]$ we have

$$\begin{aligned}
\int_a^b |R_x(x, s)|\, ds &= \int_a^x |R_x(x, s)|\, ds + \int_x^b |R_x(x, s)|\, ds \\
&= \int_a^x \left| \frac{(x - a)(b - s)^2}{(b - a)^2} - (x - s) \right| ds + \int_x^b \frac{(x - a)(b - s)^2}{(b - a)^2}\, ds \\
&\le \int_a^x \frac{(x - a)(b - s)^2}{(b - a)^2} + (x - s)\, ds + \int_x^b \frac{(x - a)(b - s)^2}{(b - a)^2}\, ds
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^x (x-s)\, ds + \int_a^b \frac{(x-a)(b-s)^2}{(b-a)^2}\, ds \\
&= \frac{1}{2}(x-a)^2 + \frac{1}{3}(x-a)(b-a) \\
&\le \frac{5}{6}(b-a)^2.
\end{aligned}
$$

Thus we have obtained (4.8). □

Similarly, we have the following complementary estimate involving $R_{xx} = \partial^2 R/\partial x^2$.

**Theorem 4.2.** *The function $R(x,s)$ in (3.5) satisfies*

$$
\int_a^b |R_{xx}(x,s)|\, ds \le \frac{2}{3}(b-a), \quad \textit{for all } x \in [a,b]. \tag{4.9}
$$

*Inequality (4.9) is sharp in the sense that it is the best inequality possible.*

*Proof.* For all $x \in [a,b]$ we have

$$
\begin{aligned}
\int_a^b |R_{xx}(x,s)|\, ds &= \int_a^x |R_{xx}(x,s)|\, ds + \int_x^b |R_{xx}(x,s)|\, ds \\
&= \int_a^x \left| \frac{(b-s)^2}{(b-a)^2} - 1 \right| ds + \int_x^b \frac{(b-s)^2}{(b-a)^2}\, ds \\
&= \int_a^x 1 - \frac{(b-s)^2}{(b-a)^2}\, ds + \int_x^b \frac{(b-s)^2}{(b-a)^2}\, ds \\
&= (x-a) + \frac{(b-x)^3 - (b-a)^3}{3(b-a)^2} + \frac{(b-x)^3}{3(b-a)^2} \\
&= (x-a) + \frac{2(b-x)^3}{3(b-a)^2} - \frac{1}{3}(b-a).
\end{aligned}
$$

In particular, if we apply basic calculus to the above cubic function then we see that it achieves its maximum value on $[a,b]$ at $x = b$, with the maximum value being $2(b-a)/3$. Thus we have established (4.9) and illustrated that the bound is sharp. □

Through a more careful analysis of the ideas in Section 3.2 ([22]) I may sharpen Theorem 3.3.

**Theorem 4.3.** *For all $x \in [a,b]$ the function $g(x,s)$ in (3.4) satisfies*

$$
\int_a^b |g(x,s)|\, ds \le (b-a)^3 \left[ \frac{2}{81} + \frac{|k|(\eta-a)^2}{6|(b-a)^2 - k(\eta-a)^2|} \right], \tag{4.10}
$$

*where we have assumed $k(\eta-a)^2 \ne (b-a)^2$ with $a < \eta < b$.*

*Proof.* From the proof of Theorem 3.3 we have

$$
\int_a^b R(\eta,s)\, ds = \frac{1}{6}(\eta-a)^2(b-\eta). \tag{4.11}
$$

Thus we have

$$\begin{aligned}
\int_a^b |g(x,s)|\,ds &= \int_a^b \left| R(x,s) + \frac{k(x-a)^2}{(b-a)^2 - k(\eta-a)^2} R(\eta,s) \right| \,ds \\
&\leq \int_a^b |R(x,s)| + \left| \frac{k(x-a)^2}{(b-a)^2 - k(\eta-a)^2} \right| |R(\eta,s)|\,ds \\
&= \frac{1}{6}(x-a)^2(b-x) + \frac{|k|(x-a)^2}{|(b-a)^2 - k(\eta-a)^2|}\frac{1}{6}(\eta-a)^2(b-\eta) \\
&\leq \frac{2}{81}(b-a)^3 + \frac{|k|(b-a)^2}{|(b-a)^2 - k(\eta-a)^2|}\frac{1}{6}(\eta-a)^2(b-a) \\
&= (b-a)^3 \left[ \frac{2}{81} + \frac{|k|(\eta-a)^2}{6|(b-a)^2 - k(\eta-a)^2|} \right].
\end{aligned}$$

□

**Remark 4.1.** *In addition to the sharpening of previous estimates, part of the significance in establishing* (4.10) *is seen in its increased dependency on* $\eta$ *when compared with* (3.8). *This dependency acknowledges and incorporates the very nature of the three point conditions that are embedded within my problem to a higher degree than that of* (3.8).

Let me now establish an analogue of Theorem 4.1 for $g_x = \partial g/\partial x$.

**Theorem 4.4.** *For all* $x \in [a,b]$*, the function* $g(x,s)$ *in* (3.4) *satisfies*

$$\int_a^b |g_x(x,s)|\,ds \leq (b-a)^2 \left[ \frac{5}{6} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|} \right], \tag{4.12}$$

*where we have assumed* $k(\eta-a)^2 \neq (b-a)^2$ *with* $a < \eta < b$.

*Proof.* For $x \in [a,b]$, we have

$$\begin{aligned}
\int_a^b |g_x(x,s)|\,ds &= \int_a^b \left| R_x(x,s) + \frac{2k(x-a)}{(b-a)^2 - k(\eta-a)^2} R(\eta,s) \right| \,ds \\
&\leq \int_a^b |R_x(x,s)| + \left| \frac{2k(x-a)}{(b-a)^2 - k(\eta-a)^2} \right| R(\eta,s)\,ds \\
&= \int_a^b |R_x(x,s)|\,ds + \frac{2|k|(x-a)}{|(b-a)^2 - k(\eta-a)^2|} \int_a^b R(\eta,s)\,ds \\
&\leq \frac{5}{6}(b-a)^2 + \frac{2|k|(x-a)}{|(b-a)^2 - k(\eta-a)^2|}\frac{1}{6}(\eta-a)^2(b-\eta) \\
&\leq \frac{5}{6}(b-a)^2 + \frac{|k|(b-a)\left[(\eta-a)^2(b-a)\right]}{3|(b-a)^2 - k(\eta-a)^2|} \\
&= (b-a)^2 \left[ \frac{5}{6} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|} \right].
\end{aligned}$$

Above, we employed (4.8) and (4.11). Thus we have established (4.12). □

Similarly, we can establish the following analogue of Theorem 4.2 for $g_{xx} = \partial^2 g/\partial x^2$.

**Theorem 4.5.** *For all $x \in [a,b]$, the function $g(x,s)$ in (3.4) satisfies*

$$\int_a^b |g_{xx}(x,s)|\, ds \le (b-a)\left[\frac{2}{3} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right], \tag{4.13}$$

*where we have assumed $k(\eta-a)^2 \neq (b-a)^2$ with $a < \eta < b$.*

*Proof.* For $x \in [a,b]$, we have

$$\begin{aligned}
\int_a^b |g_{xx}(x,s)|\, ds &= \int_a^b \left| R_{xx}(x,s) + \frac{2k}{(b-a)^2 - k(\eta-a)^2} R(\eta,s) \right| ds \\
&\le \int_a^b |R_{xx}(x,s)| + \left| \frac{2k}{(b-a)^2 - k(\eta-a)^2} \right| R(\eta,s)\, ds \\
&= \int_a^b |R_{xx}(x,s)|\, ds + \frac{2|k|}{|(b-a)^2 - k(\eta-a)^2|} \int_a^b R(\eta,s)\, ds \\
&\le \frac{2}{3}(b-a) + \frac{2|k|}{|(b-a)^2 - k(\eta-a)^2|} \frac{1}{6}\left[(\eta-a)^2(b-a)\right] \\
&= (b-a)\left[\frac{2}{3} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right].
\end{aligned}$$

Above, we employed (4.9) and (4.11). Thus we have established (4.13). □

## 4.3 Existence results via Banach fixed point theorem

In this Section I establish my first novel results for the existence, uniqueness and approximation of solutions to the BVP (4.1), (3.2) via Banach's fixed point theorem within closed and bounded sets of $[a,b] \times \mathbb{R}$; and in closed balls within infinite dimensional space. My approach involves applications of: the metric $d$ defined in (1.22); the bounds formed in Section 4.2; and through Banach fixed point theorem (Theorem 1.5). This shall be applicable to a wider range of problems than the work obtained on Section 3.3 ([22]) and the result of Smirnov [248].

To avoid the repeated use of long and complicated expressions, I define the following constants to simplify my application of the bounds that I established in Section 4.2. The following notation will be used in the statement and proof of my main results:

$$\begin{aligned}
\omega_0 &:= (b-a)^3\left[\frac{2}{81} + \frac{|k|(\eta-a)^2}{6|(b-a)^2 - k(\eta-a)^2|}\right]; \\
\omega_1 &:= (b-a)^2\left[\frac{5}{6} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right]; \\
\omega_2 &:= (b-a)\left[\frac{2}{3} + \frac{|k|(\eta-a)^2}{3|(b-a)^2 - k(\eta-a)^2|}\right];
\end{aligned} \tag{4.14}$$

where we assume that $(b-a)^2 \neq k(\eta-a)^2$.

The following Theorem is my first novel result of this Section.

**Theorem 4.6.** *Let $f : B \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the "block"*

$$B := \left\{(x,u,v,w) \in \mathbb{R}^4 : x \in [a,b],\ |u| \le R,\ |v| \le \frac{\omega_1}{\omega_0}R,\ |w| \le \frac{\omega_2}{\omega_0}R\right\},$$

*where $R > 0$ is a constant and each $\omega_i$ is defined in (4.14). Let $f(x,0,0,0) \neq 0$ for all $x \in [a,b]$ and assume $M\omega_0 \le R$. For $i = 0,1,2$, let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x,u_0,u_1,u_2) - f(x,v_0,v_1,v_2)| \le \sum_{i=0}^{2} L_i |u_i - v_i|,$$

$$\text{for all } (x,u_0,u_1,u_2),(x,v_0,v_1,v_2) \in B. \tag{4.15}$$

*If $k(\eta - a)^2 \neq (b-a)^2$ with $a < \eta < b$ and*

$$L_0\omega_0 + L_1\omega_1 + L_2\omega_2 < 1, \tag{4.16}$$

*then the BVP (4.1), (3.2) has a unique (nontrivial) solution in $C^3([a,b])$ such that*

$$(x, y(x), y'(x), y''(x)) \in B \text{ for all } x \in [a,b].$$

*Proof.* Consider the pair $(Y,\varrho) := (C^2([a,b]), d)$, where the constants $W_i$ in our $d$ in (1.22) are chosen to form

$$d(y,z) := \max\left\{d_0(y,z),\ \frac{\omega_0}{\omega_1}d_0(y',z'),\ \frac{\omega_0}{\omega_2}d_0(y'',z'')\right\}$$

(that is, $W_0 = 1$, $W_1 = \omega_0/\omega_1$ and $W_2 = \omega_0/\omega_2$). Our pair forms a complete metric space. Now, for the constant $R > 0$ in the definition of $B$, consider the following ball $\mathcal{B}_R \subset C^2([a,b])$ defined via

$$\mathcal{B}_R := \{y \in C^2([a,b]) : d(y,0) \le R\}.$$

Since $\mathcal{B}_R$ is a closed subspace of $C^2([a,b])$, the pair $(\mathcal{B}_R, d)$ forms a complete metric space.

Consider the operator $\mathcal{T} : \mathcal{B}_R \to C^2([a,b])$ defined by

$$(\mathcal{T}y)(x) := \int_a^b g(x,s) f(s, y(s), y'(s), y''(s))\, ds, \quad x \in [a,b].$$

In view of (4.7) we wish to show that there exists a unique $y \in \mathcal{B}_R$ such that

$$\mathcal{T}y = y.$$

Every such solution will also lie in $C^3([a,b])$ as can be directly shown by differentiating (4.7) and confirming the continuity.

To establish the existence and uniqueness to $\mathcal{T}y = y$, we show that the conditions of Theorem 1.5 hold with $Y = \mathcal{B}_R$.

Let me show $\mathcal{T} : \mathcal{B}_R \to \mathcal{B}_R$. For $y \in \mathcal{B}_R$ and $x \in [a,b]$, consider

$$\begin{aligned}
|(\mathcal{T}y)(x)| &\leq \int_a^b |g(x,s)|\,|f(s,y(s),y'(s),y''(s))|\,ds \\
&\leq M \int_a^b |g(x,s)|\,ds \\
&\leq M\omega_0
\end{aligned}$$

where we have applied Theorem 4.3. Thus we have $d_0(\mathcal{T}y,0) \leq M\omega_0$.

Similarly,

$$\begin{aligned}
|(\mathcal{T}y)'(x)| &\leq \int_a^b |g_x(x,s)|\,|f(s,y(s),y'(s),y''(s))|\,ds \\
&\leq M \int_a^b |g_x(x,s)|\,ds \\
&\leq M\omega_1
\end{aligned}$$

where we have applied Theorem 4.4. Thus $\omega_0 d_0((\mathcal{T}y)',0)/\omega_1 \leq M\omega_0$.

In addition, via similar arguments, we obtain

$$|(\mathcal{T}y)''(x)| \leq M\omega_2$$

by drawing on Theorem 4.5, so that $\omega_0 d_0((\mathcal{T}y)'',0)/\omega_2 \leq M\omega_0$.

Thus, for all $y \in \mathcal{B}_R$ we have

$$\begin{aligned}
d(\mathcal{T}y,0) &= \max\left\{d_0(\mathcal{T}y,0),\ \frac{\omega_0}{\omega_1}d_0((\mathcal{T}y)',0),\ \frac{\omega_0}{\omega_2}d_0((\mathcal{T}y)'',0)\right\} \\
&\leq \max\{M\omega_0, M\omega_0, M\omega_0\} \\
&= M\omega_0 \\
&\leq R
\end{aligned}$$

where the final inequality holds by assumption. Thus, for all $y \in \mathcal{B}_R$ we have $\mathcal{T}y \in \mathcal{B}_R$ so that $\mathcal{T} : \mathcal{B}_R \to \mathcal{B}_R$.

Let me now show that $\mathcal{T}$ is contractive on $\mathcal{B}_R$ with respect to $d$. For $y, z \in \mathcal{B}_R$ and $x \in [a,b]$, consider

$$|(\mathcal{T}y)(x) - (\mathcal{T}z)(x)| \leq \int_a^b |g(x,s)|\,|f(s,y(s),y'(s),y''(s)) - f(s,z(s),z'(s),z''(s))|\,ds$$

$$
\begin{aligned}
&\leq \int_a^b |g(x,s)| \left( \sum_{i=0}^{2} L_i \, |y^{(i)}(s) - z^{(i)}(s)| \right) ds \\
&\leq \omega_0 \left( L_0 d_0(y,z) + L_1 d_0(y',z') + L_2 d_0(y'',z'') \right) \\
&\leq \omega_0 \left( L_0 d(y,z) + L_1 \frac{\omega_1}{\omega_0} d(y,z) + L_2 \frac{\omega_2}{\omega_0} d(y,z) \right) \\
&= (L_0\omega_0 + L_1\omega_1 + \omega_2 L_2) d(y,z)
\end{aligned}
$$

where we have applied (4.10) and (4.15).

Similarly, we can show

$$
\begin{aligned}
|(\mathcal{T}y)'(x) - (\mathcal{T}z)'(x)| &\leq \omega_1 \left( L_0 + L_1 \frac{\omega_1}{\omega_0} + L_2 \frac{\omega_2}{\omega_0} \right) d(y,z); \\
|(\mathcal{T}y)''(x) - (\mathcal{T}z)''(x)| &\leq \omega_2 \left( L_0 + L_1 \frac{\omega_1}{\omega_0} + L_2 \frac{\omega_2}{\omega_0} \right) d(y,z).
\end{aligned}
$$

Thus, for all $y, z \in \mathcal{B}_R$ we have

$$
\begin{aligned}
d(\mathcal{T}y, \mathcal{T}z) &= \max \left\{ d_0(\mathcal{T}y, \mathcal{T}z),\ \frac{\omega_0}{\omega_1} d_0((\mathcal{T}y)', (\mathcal{T}z)'),\ \frac{\omega_0}{\omega_2} d_0((\mathcal{T}y)'', (\mathcal{T}z)'') \right\} \\
&\leq (L_0\omega_0 + L_1\omega_1 + \omega_2 L_2) d(y,z).
\end{aligned}
$$

Due to our assumption (4.16) we see that $\mathcal{T}$ is a contractive map on $\mathcal{B}_R$. Thus all of the conditions of Theorem 1.5 hold with $Y = \mathcal{B}_R$. We conclude that the operator $\mathcal{T}$ has a unique fixed point in $\mathcal{B}_R \subset C^2([a,b])$. This solution is also in $C^3([a,b])$ and we have equivalently shown that the BVP (4.1), (3.2) has a unique solution.

We note that our solution cannot be the zero function, as our assumption $f(x,0,0,0) \neq 0$ excludes this possibility. □

As we can see from the proof of Theorem 4.6, the assumption $M\omega_0 \leq R$ is applied to ensure the "invariance" of $\mathcal{T}$, namely $\mathcal{T} : \mathcal{B}_R \to \mathcal{B}_R$. Let us explore this idea further with the following variations on the theme of Theorem 4.6 where I modify the aforementioned condition.

**Theorem 4.7.** *Let $f : B_1 \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the "block"*

$$
B_1 := \left\{ (x,u,v,w) \in \mathbb{R}^4 : x \in [a,b],\ |u| \leq \frac{\omega_0}{\omega_1} R,\ |v| \leq R,\ |w| \leq \frac{\omega_2}{\omega_1} R \right\},
$$

*where $R > 0$ is a constant and each $\omega_i$ is defined in (4.14). Let $f(x,0,0,0) \neq 0$ for all $x \in [a,b]$ and assume $M\omega_1 \leq R$. For $i = 0,1,2$, let $L_i$ be nonnegative constants (not all zero) such that*

$$
|f(x,u_0,u_1,u_2) - f(x,v_0,v_1,v_2)| \leq \sum_{i=0}^{2} L_i |u_i - v_i|,
$$

$$\textit{for all } (x, u_0, u_1, u_2), (x, v_0, v_1, v_2) \in B_1. \tag{4.17}$$

*If* $k(\eta - a)^2 \neq (b-a)^2$ *with* $a < \eta < b$ *and*

$$L_0\omega_0 + L_1\omega_1 + L_2\omega_2 < 1, \tag{4.18}$$

*then the BVP* (4.1), (3.2) *has a unique (nontrivial) solution in* $C^3([a,b])$ *such that*

$$(x, y(x), y'(x), y''(x)) \in B_1 \textit{ for all } x \in [a,b].$$

*Proof.* The proof follows similar ideas to that of the proof of Theorem 4.6 and so is only summarized.

Consider the pair $(Y, \varrho) := (C^2([a,b]), d)$ where now the constants $W_i$ in our $d$ in (1.22) are chosen to form

$$d(y,z) := \max\left\{\frac{\omega_1}{\omega_0} d_0(y,z),\ d_0(y',z'),\ \frac{\omega_1}{\omega_2} d_0(y'',z'')\right\}$$

(that is, $W_0 = \omega_1/\omega_0$, $W_1 = 1$ and $W_2 = \omega_1/\omega_2$). For the constant $R > 0$ in the definition of $B_1$, consider the following ball $\mathcal{B}_{1R} \subset C^2([a,b])$ defined via

$$\mathcal{B}_{1R} := \{y \in C^2([a,b]) : d(y,0) \le R\}.$$

Since $\mathcal{B}_{1R}$ is a closed subspace of $C^2([a,b])$, the pair $(\mathcal{B}_{1R}, d)$ forms a complete metric space.

Following the same type of arguments as in the proof of Theorem 4.6, it can be shown that the condition $M\omega_1 \le R$ ensures $\mathcal{T} : \mathcal{B}_{1R} \to \mathcal{B}_{1R}$. Furthermore, (4.17) and (4.18) guarantee that $\mathcal{T}$ is contractive on $\mathcal{B}_{1R}$.

The existence and uniqueness now follows from Theorem 1.5. □

Similarly, we have the following result.

**Theorem 4.8.** *Let* $f : B_2 \to \mathbb{R}$ *be continuous and uniformly bounded by* $M > 0$ *on the "block"*

$$B_2 := \left\{(x,u,v,w) \in \mathbb{R}^4 : x \in [a,b],\ |u| \le \frac{\omega_0}{\omega_2}R,\ |v| \le \frac{\omega_1}{\omega_2}R,\ |w| \le R\right\},$$

*where* $R > 0$ *is a constant and each* $\omega_i$ *is defined in* (4.14). *Let* $f(x,0,0,0) \neq 0$ *for all* $x \in [a,b]$ *and assume* $M\omega_2 \le R$. *For* $i = 0,1,2$, *let* $L_i$ *be nonnegative constants (not all zero) such that*

$$|f(x,u_0,u_1,u_2) - f(x,v_0,v_1,v_2)| \le \sum_{i=0}^{2} L_i |u_i - v_i|,$$

$$\textit{for all } (x, u_0, u_1, u_2), (x, v_0, v_1, v_2) \in B_2. \tag{4.19}$$

*If* $k(\eta - a)^2 \neq (b-a)^2$ *with* $a < \eta < b$ *and*

$$L_0\omega_0 + L_1\omega_1 + L_2\omega_2 < 1, \tag{4.20}$$

*then the BVP* (4.1)*,* (3.2) *has a unique (nontrivial) solution in* $C^3([a,b])$ *such that*

$$(x, y(x), y'(x), y''(x)) \in C \text{ for all } x \in [a,b].$$

*Proof.* Once again, the proof follows similar ideas to that of the proof of Theorem 4.6 and so I provide just an outline of the ideas.

Consider the pair $(Y, \varrho) := (C^2([a,b]), d)$ where now the constants $W_i$ in our $d$ in (1.22) are chosen to form

$$d(y,z) := \max\left\{\frac{\omega_2}{\omega_0} d_0(y,z),\ \frac{\omega_2}{\omega_1} d_0(y',z'),\ d_0(y'',z'')\right\}$$

(that is, $W_0 = \omega_2/\omega_0$, $W_1 = \omega_2/\omega_1$ and $W_2 = 1$). For the constant $R > 0$ in the definition of $B_2$, consider the following ball $\mathcal{B}_{2R} \subset C^2([a,b])$ defined via

$$\mathcal{B}_{2R} := \{y \in C^2([a,b]) : d(y,0) \leq R\}.$$

Since $\mathcal{B}_{2R}$ is a closed subspace of $C^2([a,b])$, the pair $(\mathcal{B}_{2R}, d)$ forms a complete metric space.

Following the same type of arguments as in the proof of Theorem 4.6, it can be shown that the condition $M\omega_2 \leq R$ ensures $\mathcal{T} : \mathcal{B}_{2R} \to \mathcal{B}_{2R}$. Furthermore, (4.19) and (4.20) guarantee that $\mathcal{T}$ is contractive on $\mathcal{B}_{2R}$.

The existence and uniqueness now follows from Theorem 1.5. □

**Remark 4.2.** *As flagged earlier, part of the significance in including Theorem 4.7 and Theorem 4.8 in addition to Theorem 4.6 involves exploring variations on the theme of the invariance condition* $M\omega_i \leq R$*. We see from their statements and proofs therein that we can modify the invariance condition in each of the theorems at the expense of "modifying" the block on which we consider* $f$ *and the associated metric.*

Picard iterations form an important structure for successively approximating solutions [266, 294]. I can now form the following results that involve approximations to the unique solution $y$ of the BVP (4.1), (3.2). They are a consequence of Theorem 1.5 holding for the operator $\mathcal{T}$ therein, see [294, Theorem 1.A].

**Remark 4.3.** *Let the conditions of Theorem 4.6, Theorem 4.7 or Theorem 4.8 hold. If we recursively define a sequence of approximations $y_n = y_n(x)$ on $[a,b]$ via*

$$y_0 := 0, \quad y_{n+1}(x) := \int_a^b g(x,s)\ f(s, y_n(s), y_n'(s), y_n''(s))\ ds, \quad n = 1, 2, \cdots$$

*then, for each of the corresponding metrics defined in the proofs of Theorem 4.6, Theorem 4.7 and Theorem 4.8:*

- *the sequence $y_n$ converges to the solution $y$ of (3.1), (3.2) with respect to the $d$ metric and the rate of convergence is given by*

$$d(y_{n+1}, y) \le (L_0\omega_0 + L_1\omega_1 + \omega_2 L_2) d(y_n, y);$$

- *for each $n$, an* a priori *estimate on the error is*

$$d(y_n, y) \le \frac{(L_0\omega_0 + L_1\omega_1 + \omega_2 L_2)^n}{1 - (L_0\omega_0 + L_1\omega_1 + \omega_2 L_2)} d(y_1, 0);$$

- *for each $n$, an* a posteriori *estimate on the error is*

$$d(y_{n+1}, y) \le \frac{(L_0\omega_0 + L_1\omega_1 + \omega_2 L_2)}{1 - (L_0\omega_0 + L_1\omega_1 + \omega_2 L_2)} d(y_{n+1}, y_n).$$

**Remark 4.4.** *We can see that Theorem 4.6 and its variations involve the same condition (4.16). Here, it would seem that "all roads lead to Rome", as no matter which other sets or variations of (1.22) we employ, we keep returning to the same inequality (4.16).*

## 4.4 Existence results via Rus fixed point theorem

Now in this Section I am ready to establish my second novel results for the existence, uniqueness and approximation of solutions to the BVP (4.1), (3.2) via Rus' fixed point theorem within closed and bounded sets of $[a,b] \times \mathbb{R}$; and in closed balls within infinite dimensional space. My approach involves applications of: the two metrics that is $\delta$ defined in (1.21) and $d$ defined in (1.22) and; the bounds formed in Section 4.2; and through Rus fixed point theorem (Theorem 1.6). This shall be applicable to a wider range of problems than the work obtained on Section 3.4 ([22]) and the result of Smirnov [248].

To avoid the repeated use of complicated expressions, I define the following constants to simplify certain notation. Let $p > 1$ and $q > 1$ be constants such that $1/p + 1/q = 1$. Define

$$\nu_0 := \max_{x \in [a,b]} \left[ \left( \int_a^b |g(x,s)|^q\ ds \right)^{1/q} \right];$$

$$\nu_1 := \max_{x\in[a,b]} \left[ \left( \int_a^b |g_x(x,s)|^q \, ds \right)^{1/q} \right];$$
$$\nu_2 := \max_{x\in[a,b]} \left[ \left( \int_a^b |g_{xx}(x,s)|^q \, ds \right)^{1/q} \right]; \tag{4.21}$$

and

$$\varpi_0 := \left( \int_a^b \left( \int_a^b |g(x,s)|^q \, ds \right)^{p/q} dx \right)^{1/p};$$
$$\varpi_1 := \left( \int_a^b \left( \int_a^b |g_x(x,s)|^q \, ds \right)^{p/q} dx \right)^{1/p};$$
$$\varpi_2 := \left( \int_a^b \left( \int_a^b |g_{xx}(x,s)|^q \, ds \right)^{p/q} dx \right)^{1/p}. \tag{4.22}$$

In the proof of my results in this Section, I will draw on Theorem 1.2, in particular the relationship between the two metrics $\delta$ and $d$ given by (1.24).

The following Theorem is my first novel result of this Section.

**Theorem 4.9.** *Let $f : B \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the "block" $B$ defined in Theorem 4.6. Let $f(x,0,0,0) \neq 0$ for all $x \in [a,b]$ and assume $M\omega_0 \le R$. For $i = 0,1,2,$ let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x,u_0,u_1,u_2) - f(x,v_0,v_1,v_2)| \le \sum_{i=0}^{2} L_i |u_i - v_i|,$$
$$\text{for all } (x,u_0,u_1,u_2), (x,v_0,v_1,v_2) \in B. \tag{4.23}$$

*If $k(\eta-a)^2 \neq (b-a)^2$ with $a < \eta < b$ and there are constants $p > 1$ and $q > 1$ with $1/p + 1/q = 1$ such that*

$$L_0\varpi_0 + L_1\varpi_1 + L_2\varpi_2 < 1, \tag{4.24}$$

*where each of the $\varpi_i$ are defined in (4.22), then the BVP (4.1), (3.2) has a unique (nontrivial) solution in $C^3([a,b])$ such that $(x, y(x), y'(x), y''(x)) \in B$ for all $x \in [a,b]$.*

*Proof.* Define $\mathcal{B}_R$, $d$ and $\mathcal{T}$ as in the proof of Theorem 4.6. We want to show that there exists a unique $y \in \mathcal{B}_R$ such that

$$\mathcal{T}y = y.$$

Such a solution will also lie in $C^3([a,b])$ as can be directly shown by differentiating (4.7) and confirming the continuity.

To establish the existence and uniqueness to $\mathcal{T}y = y$, we show that the conditions of Theorem 1.6 hold.

The pair $(Y, \varrho) := (\mathcal{B}_R, d)$ forms a complete metric space. Following the same type of arguments as in the proof of Theorem 4.6, it can be shown that the condition $M\omega_0 \le R$ ensures $\mathcal{T} : \mathcal{B}_R \to \mathcal{B}_R$.

In addition, consider the metric $\delta = \tau$ in (1.21) on $\mathcal{B}_R = Y$, where $p > 1$ and the $L_i$ come from (4.23).

For $y, z \in \mathcal{B}_R$ and $x \in [a, b]$, consider

$$\begin{aligned}
|(\mathcal{T}y)(x) - (\mathcal{T}z)(x)| &\le \int_a^b |g(x,s)|\, |f(s, y(s), y'(s), y''(s)) - f(s, z(s), z'(s), z''(s))|\, ds \\
&\le \int_a^b |g(x,s)| \left( \sum_{i=0}^{2} L_i\, |y^{(i)}(s) - z^{(i)}(s)| \right) ds \\
&\le \left( \int_a^b |g(x,s)|^q\, ds \right)^{1/q} \left( \sum_{i=0}^{2} L_i \left( \int_a^b |y(s) - z(s)|^p\, ds \right)^{1/p} \right) \qquad (4.25) \\
&= \nu_0 \delta(y, z).
\end{aligned}$$

Above, we have used (4.23) and Hölder's inequality [113, 227] to obtain (4.25). Similar calculations lead us to

$$\begin{aligned}
|(\mathcal{T}y)'(x) - (\mathcal{T}z)'(x)| &\le \nu_1 \delta(y, z) \\
|(\mathcal{T}y)'(x) - (\mathcal{T}z)'(x)| &\le \nu_2 \delta(y, z).
\end{aligned}$$

Combining the above inequalities, we obtain

$$d(\mathcal{T}y, \mathcal{T}z) \le c\delta(y, z), \quad \text{for some } c > 0 \text{ and all } y, z \in \mathcal{B}_R, \qquad (4.26)$$

where,

$$c := \max\left\{ \nu_0, \frac{\omega_0}{\omega_1}\nu_1, \frac{\omega_0}{\omega_2}\nu_2 \right\}.$$

Thus, the inequality (1.41) of Theorem 1.6 holds.

Furthermore, $\mathcal{T}$ is continuous on $\mathcal{B}_R$ with respect to the $d$ metric as can be shown from the following arguments. For all $y, z \in \mathcal{B}_R$ we may apply (1.24) from Theorem 1.1 to (4.26) to obtain

$$\begin{aligned}
d(\mathcal{T}y, \mathcal{T}z) &\le c\delta(y, z) \\
&\le c(b-a)^{1/p} \left( L_0 + L_1 \frac{\omega_1}{\omega_0} + L_2 \frac{\omega_2}{\omega_0} \right) d(y, z).
\end{aligned}$$

Thus, given any $\varepsilon > 0$ we can choose

$$\Delta = \frac{\varepsilon}{c(b-a)^{1/p}\left(L_0 + L_1\frac{\omega_1}{\omega_0} + L_2\frac{\omega_2}{\omega_0}\right)}$$

so that $d(\mathcal{T}y, \mathcal{T}z) < \varepsilon$ whenever $d(y,z) < \Delta$. Hence $\mathcal{T}$ is continuous on $\mathcal{B}_R$ with respect to the $d$ metric.

Finally, we show that $\mathcal{T}$ is contractive on $\mathcal{B}_R$ with respect to the $\delta$ metric, that is, the inequality (1.42) in Theorem 1.6 holds. From (4.25) and the associated discussion, for each $y, z \in \mathcal{B}_R$ and $x \in [a,b]$ we have

$$\left(\int_a^b |(\mathcal{T}y)(x) - (\mathcal{T}z)(x)|^p \, dx\right)^{1/p} \le \varpi_0\delta(y,z);$$
$$\left(\int_a^b |(\mathcal{T}y)'(x) - (\mathcal{T}z)'(x)|^p \, dx\right)^{1/p} \le \varpi_1\delta(y,z);$$
$$\left(\int_a^b |(\mathcal{T}y)''(x) - (\mathcal{T}z)''(x)|^p \, dx\right)^{1/p} \le \varpi_2\delta(y,z);$$

and so we obtain

$$\delta(\mathcal{T}y, \mathcal{T}z) \le (L_0\varpi_0 + L_1\varpi_1 + L_2\varpi_2)\,\delta(y,z).$$

From our assumption (4.24), we thus have

$$\delta(\mathcal{T}y, \mathcal{T}z) \le \alpha\delta(y,z),$$

for some $\alpha < 1$ and all $y, z \in \mathcal{B}_R$.

Thus, Theorem 1.6 is applicable and the operator $\mathcal{T}$ has a unique fixed point in $\mathcal{B}_R$. This solution is also in $C^3([a,b])$ and we have equivalently shown that the BVP (4.1), (3.2) has a unique solution.

We note that my solution cannot be the zero function, as my assumption $f(x,0,0,0) \neq 0$ excludes this possibility. □

Similarly, we have the following two results which I state without proof due to concerns of brevity and repetition. The proofs follow similar lines to that of the proof of Theorem 4.9.

**Theorem 4.10.** *Let $f : B_1 \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the "block" $B_1$ defined in Theorem 4.7. Let $f(x,0,0,0) \neq 0$ for all $x \in [a,b]$ and assume $M\omega_1 \le R$. For $i = 0,1,2$, let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x,u_0,u_1,u_2) - f(x,v_0,v_1,v_2)| \le \sum_{i=0}^{2} L_i|u_i - v_i|,$$

$$\text{for all } (x,u_0,u_1,u_2),(x,v_0,v_1,v_2) \in B_1. \tag{4.27}$$

*If $k(\eta-a)^2 \neq (b-a)^2$ with $a<\eta<b$ and there are constants $p>1$ and $q>1$ with $1/p+1/q=1$ such that*

$$L_0\varpi_0 + L_1\varpi_1 + L_2\varpi_2 < 1, \tag{4.28}$$

*where each of the $\varpi_i$ are defined in (4.22), then the BVP (4.1), (3.2) has a unique (nontrivial) solution in $C^3([a,b])$ such that $(x,y(x),y'(x),y''(x)) \in B_1$ for all $x\in[a,b]$.*

**Theorem 4.11.** *Let $f: B_2 \to \mathbb{R}$ be continuous and uniformly bounded by $M>0$ on the "block" $B_2$ defined in Theorem 4.8. Let $f(x,0,0,0)\neq 0$ for all $x\in[a,b]$ and assume $M\omega_2 \le R$. For $i=0,1,2$, let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x,u_0,u_1,u_2)-f(x,v_0,v_1,v_2)| \le \sum_{i=0}^{2} L_i|u_i-v_i|,$$

$$\text{for all } (x,u_0,u_1,u_2),(x,v_0,v_1,v_2) \in B_2. \tag{4.29}$$

*If $k(\eta-a)^2 \neq (b-a)^2$ with $a<\eta<b$ and there are constants $p>1$ and $q>1$ with $1/p+1/q=1$ such that*

$$L_0\varpi_0 + L_1\varpi_1 + L_2\varpi_2 < 1, \tag{4.30}$$

*where each of the $\varpi_i$ are defined in (4.22), then the BVP (4.1), (3.2) has a unique (nontrivial) solution in $C^3([a,b])$ such that $(x,y(x),y'(x),y''(x)) \in C$ for all $x\in[a,b]$.*

**Remark 4.5.** *Similarly to Remark 4.4, my Theorem 4.9 and its variations involve the same condition (4.24).*

**Remark 4.6.** *Let $m := d_0(f(\cdot,0,0,0),0)$. Each of the invariance conditions $M\omega_i \le R$ can be replaced with*

$$m\omega_i \le (1-(L_0\omega_0 + L_1\omega_1 + \omega_2 L_2))R$$

*in my existence theorems herein and their conclusions will still hold. To see this, for example, we show that $\mathcal{T}:\mathcal{B}_R\to\mathcal{B}_R$. For all $y\in\mathcal{B}_R$ we have*

$$\begin{aligned} d(\mathcal{T}y,0) &\le d(\mathcal{T}y,\mathcal{T}0)+d(\mathcal{T}0,0)\\ &\le (L_0\omega_0+L_1\omega_1+\omega_2L_2)d(y,0)+m\omega_0\\ &\le (L_0\omega_0+L_1\omega_1+\omega_2L_2)R + (1-(L_0\omega_0+L_1\omega_1+\omega_2L_2))R\\ &= R. \end{aligned}$$

*Thus we see that under this condition we ensure that $\mathcal{T}:\mathcal{B}_R\to\mathcal{B}_R$. The other cases may be shown in similar ways.*

## 4.5 Examples, comparisons and remarks

Let me discuss the nature of the advancement of my new results by revisiting my intractable examples originally posed in Section 4.1. I show how my new results can be applied.

**Example 4.4.** *The BVP (4.3), (4.4) has a unique solution such that* $|y(x)| \le 1$ *for all* $x \in [a, b]$.

*Proof.* We show that the conditions of Theorem 4.6 are satisfied. Choose $R = 1$ and consider my

$$f(x, y) := x + 2 + y^2$$

restricted to the accompanying rectangle

$$B := \{(x, u) \in \mathbb{R}^2 : x \in [0, 1], |u| \le 1\}. \tag{4.31}$$

Observe that $|f| \le 4 =: M$ on $B$. Furthermore we can obtain $\omega_0 = 13/162$. Thus we have $M\omega_0 = 52/162 \le 1 =: R$. In addition, $|\partial f/\partial y| = |2x| \le 2$ on $B$ and thus we may choose $L_0 = 2$ to ensure (4.15) holds on $B$ (with the other $L_i$ being zero). Finally, we note that $L_0\omega_0 = 26/162 < 1$. Thus, we see that all of the conditions of Theorem 4.6 are satisfied and its conclusion holds for this example. □

**Example 4.5.** *The BVP (4.5), (4.4) has a unique solution such that* $|y(x)| \le 1$ *and* $|y'(x)| \le 153/13$ *for all* $x \in [a, b]$.

*Proof.* We show that the conditions of Theorem 4.6 are satisfied for the rectangle $B$ defined via $R = 1$, namely

$$B := \{(x, u, v) \in \mathbb{R}^3 : x \in [0, 1], |u| \le 1, |v| \le \omega_1/\omega_0\}. \tag{4.32}$$

As in the previous example, $\omega_0 = 13/162$ and now $\omega_1 = 17/18$ with $\omega_1/\omega_0 = 153/13 < 12$. Observe that $|f| < 2 + 1/5 + (12)^3/3000 < 3 =: M$ on $B$. We have $M\omega_0 = 39/162 \le 1$. In addition, on $B$ we have: $|\partial f/\partial y| = 1/5$; and $|\partial f/\partial y'| = |(y')^2/1000| < 1/5$ and thus we may choose $L_0 = 1/5$ and $L_1 = 1/5$ so that (4.15) holds on $B$. Finally, we note that $L_0\omega_0 + L_1\omega_1 < 1$. Thus, we see that all of the conditions of Theorem 4.6 are satisfied and its conclusion holds for this example. □

**Example 4.6.** *The BVP (4.6), (4.4) has a unique solution such that* $|y(x)| \le 1$ *for all* $x \in [a, b]$.

*Proof.* We show that the conditions of Theorem 4.6 are satisfied for the rectangle $B$ defined in (4.31) where $R = 1$. Observe that $|f| \le 1 =: M$ on $B$. As before, $\omega_0 = 13/162$. Thus we have $M\omega_0 = 13/162 \le 1$. In addition, $|\partial f/\partial y| = |1/(2 - y)^2| \le 2$ on $B$ and thus we may choose $L_0 = 2$

so that (4.15) holds on $B$. Finally, we note that $L_0\omega_0 = 26/162 < 1$. Thus, we see that all of the conditions of Theorem 4.6 are satisfied and its conclusion holds for this example. □

Let me discuss on example involving the conditions of Theorem 4.9.

**Remark 4.7.** *In the case: $[a,b] = [0,1]$; $\eta = 1/2$; $k = 1$; $p = 2 = q$; the left hand side of (4.24) is evaluated on the previous Chapter namely (3.23) (see [22]) to obtain*

$$\int_0^1 \int_0^1 g(x,s)^2 \, ds \, dx = \frac{16}{14175}.$$

*Thus, (4.24) takes the form*

$$L_0\varpi_0 = L_0\frac{4\sqrt{7}}{315} < 1 \tag{4.33}$$

*which will be satisfied, for example, if*

$$L_0 \leq 29.$$

*The condition (4.16) takes the form*

$$L_0\omega_0 = L_0\frac{13}{162} < 1. \tag{4.34}$$

*For an $f$ such as*

$$f(x,y) := 13y^2 + (x+1)^2$$

*the assumptions of the obtained results in the previous Chapter ([22]) and in [248] are not satisfied because this $f$ is not Lipschitz on the strip $[0,1]\times\mathbb{R}$. In addition, note that for $R = 1$ and $L_0 = 26$ the condition (4.16) in its form (4.34) does not hold and so Theorem 4.6 does not apply in this case. On the other hand, my $f$ does satisfy (4.33) on the ball $B$ with $R = 1$ with $L_0 = 26$. Thus we see that Theorem 4.9 is sharper than Theorem 4.6.*

We note that Theorem 4.6 and its variations do not rule out the existence of additional solutions to my problem whose graphs are not completely contained in the sets under consideration. For instance, in Example 4.4 we restricted our attention to a subset of the domain of $f$, rather than working with its maximal domain of $[0,1]\times\mathbb{R}$. Other solutions may exist whose graphs are not completely contained in our $B$.

**Remark 4.8.** *We note my new estimates in Section 4.2 and those used in the discussion of my examples are a mixture of sharp and rough estimates. However, the rough estimates are simple and reasonably easy to calculate. The significance of rough inequalities such as (4.12) and (4.13) has been promoted by mathematicians such as Nirenberg and Friedrichs, "who often stressed the* *applicability of rough inequalities to various problems" [202, p.483]. In this spirit, we note that* (4.11) *may be further estimated to form*

$$\begin{aligned}\int_a^b R(\eta, s)\ ds &\le \frac{1}{6}(\eta - a)^2(b-a)\\ &\le \frac{1}{6}(b-a)^3.\end{aligned}$$

*This rougher estimate can also be applied in a similar fashion to the ideas and methods herein.*

It can be the case that certain problems do not satisfy the assumptions of fixed point theory, but the operators therein actually will admit a fixed point. Thus, it is important that we keep developing alternative perspectives in mathematics because they can open up new ways of thinking and working [262, p. 1292], [264], [265], [266, Sec. 3], [267]. This includes a need to think beyond the current limitations of fixed point theory.

# Chapter 5

# Fourth-order BVPs

## 5.1 Introduction

In this Chapter I consider fourth-order ordinary differential equation

$$y^{(iv)} = f(x, y, y', y'', y'''), \quad x \in [0,1], \tag{5.1}$$

where $f : \Omega_4 \subseteq [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ is assumed to be continuous and (5.1) is subject to either of the two-point boundary conditions:

$$y(0) = d_0,\ y'(0) = d_2,\ y(1) = d_1,\ y'(1) = d_3; \tag{5.2}$$

$$y(0) = d_0,\ y'(0) = d_2,\ y''(1) = d_4,\ y'''(1) = d_5; \tag{5.3}$$

and $d_0, d_1, d_2, d_3, d_4, d_5$ are given constants in $\mathbb{R}$.

A natural motivation for the investigation of fourth-order boundary value problems (BVPs) arises in the analysis of elastic beam deflections. Consider a beam occupying the interval [0,1] with $x$ denoting the position along the beam. The beam is subjected to certain forces and if $y = y(x)$ represents the resultant deflection of the beam at position $x$, then the equation of motion leads to the differential equation (5.1). In simplified situations the problem is subjected to either of the boundary conditions:

$$y(0) = 0,\ y'(0) = 0,\ y(1) = 0,\ y'(1) = 0; \tag{5.4}$$

$$y(0) = 0,\ y'(0) = 0,\ y''(1) = 0,\ y'''(1) = 0. \tag{5.5}$$

The boundary conditions (5.4) may be interpreted in a physical sense as the beam having clamped ends at $x = 0$ and at $x = 1$, while (5.5) may be interpreted in a material sense as the beam having a clamped end at $x = 0$ and a free end at $x = 1$.

Fourth-order BVPs and their application to elastic beam deflections have been studied by many researchers. Indeed, entire monographs have appeared regarding the field such as [10, 101]. The reader is also referred to [31, 34, 77, 84, 88, 91, 97, 99, 104, 116, 118, 120, 124, 130, 154, 155, 157, 159, 178, 185, 183, 184, 209, 222, 277, 282, 284, 285, 286, 289, 290, 288, 295] for some additional developments in the field of fourth-order BVPs and applications to beam analysis. However, let me situate my work of this Chapter within the field of research by discussing its differences and connections with recent and noteworthy publications in the area. I analyze four dimensions: the type of problem under consideration; the assumptions imposed; the methods employed; and the nature of the results obtained.

Observe that the differential equation under consideration (5.1) in this Chapter features a scalar-valued, fully nonlinear right-hand side that depends on each of the lower-order derivatives. This is in contrast to such works as [57, 158, 280, 298] where either: $f$ does not depend in a nonlinear way on each of the derivatives $y$, $y'$, $y''$ and $y'''$; or a system of equations were considered. The two sets of boundary conditions (5.2) and (5.3) that I consider herein differ in form from those considered by [117, 122, 158, 195, 280, 283, 298]. Thus, the problem under consideration in this Chapter is distinct from the above works.

On the other hand, the fully nonlinear problem (5.1), (5.5) was analyzed in [63, 64, 156]. The assumptions on $f$ in [63, 64] are of a local nature, that is, the domain of $f$ is restricted to closed and bounded sets. While these types of assumptions are quite wide-ranging, the very nature of *localized* assumptions means that only limited, localized information about solutions can be necessarily obtained. For instance, in the context of localized assumptions, nothing can be concluded about existence and uniqueness of solutions that may lie outside of the closed and bounded set that is under consideration. This is especially important as multiple solutions to fourth-order BVPs have been shown to exist [34]. However, herein I provide an analysis in both global and local settings. In doing so, I obtain more complete and balanced knowledge in my conclusions regarding the existence and uniqueness of solutions.

In [156] the approach involved obtaining the existence of solutions via the application of fixed point index theory in cones. In contrast, my methods herein involve a suitable application of the Rus fixed point theorem [231] via two metrics. Thus, my assumptions and methods herein are different from the aforementioned works.

Indeed, a range of authors have pursued a spectrum of approaches to the existence and/or uniqueness of solution to fourth-order BVPs. This includes methods such as: Schauder fixed point theorem [158, 195] and topological degree [181]; monotone iteration [298]; reduction to second order systems [63]; lower and upper solutions [280]; fixed point theorems in cones [57, 117, 156, 296, 298]; fixed points of general $\alpha$-concave operators; and fixed point theorems in partially ordered metric spaces [58]. Thus, I can see that my approach of applying the Rus fixed point theorem appears to occupy a unique position within the literature as a strategy to ensure existence and uniqueness of solutions to fourth-order BVPs.

Moreover, in some of above cited works, the fourth-order BVP under consideration is reducible to a larger system of second-order BVPs and thus the underlying analysis for the fourth-order problem is closely linked with that of second-order problems. Herein I make no such reduction, preferring to work directly on the original form of the problems. As flagged in [97, p.108], no reduction of order (to become a second-order problem) is available for the BVP (5.1), (5.4) due to the nature of the boundary conditions. This realization may partially explain Yao's position [288, p.237] regarding reasons for the slow progress of research into (5.1), (5.2) and (5.1), (5.4). Thus, new methods and perspectives are needed [262, 266] to advance the associated existence and uniqueness theory and its application to beam deflection analysis.

Sufficiently motivated by the above discussion and the importance of investigating the existence and uniqueness theory and its application to beam deflection analysis, the purpose of this Chapter is to address the aforementioned challenges by examining the existence and uniqueness of solutions to the BVPs (5.1), (5.2) and (5.1), (5.3). My results of this Chapter form an advancement over traditional approaches such as applications of Banach's fixed point theorem. This is achieved through the use of two metrics and the Rus fixed point theorem which utilizes two metrics on a metric space. As we will discover, this enables a greater class of problems to be better understood regarding existence and uniqueness of solutions. This includes sharpening the Lipschitz constants involved within a global (unbounded) context and within closed and bounded domains.

This Chapter is organized as follows.

In Section 5.2 I present the existence and uniqueness results for solutions to BVPs (5.1), (5.2) within both a global (unbounded) context and within closed and bounded domains via applications of Rus's contraction mapping theorem through two metrics. The approach used in Section 5.2 is then used to obtain existence and uniqueness results for solutions to BVPs (5.1), (5.3) in Section 5.3. This follows by Section 5.4 where my existence and uniqueness results obtained in Sections 5.2 and 5.3 are applied to the area of elastic beam deflections when the beam is subjected to a loading force and the ends of the beam are either: fully clamped ends (5.4); or clamped/free ends (5.5). Existence and uniqueness of solutions to the models are guaranteed under linear and nonlinear loading forces — that is, the models are identified to be well-posed.

## 5.2 Existence results via Rus' fixed point theorem: Fully clamped ends

In this Section I establish my novel results for the existence and uniqueness of solutions to the BVP (5.1), (5.2) via applications of Rus's contraction mapping theorem. The first result of this Section is given within a global (unbounded) context and then I give my second result within closed and bounded sets of $[0,1]\times\mathbb{R}$; and in closed balls within infinite dimensional space.

Before stating my results, I briefly introduce the notation, definitions and some preliminaries results that are necessary for navigating this Section and then formulate my main results.

The following definition gives what I mean by a solution to (5.1), (5.2).

**Definition 5.1.** *By a solution to* (5.1), (5.2) *we mean a function* $y:[0,1]\to\mathbb{R}$ *such that* $y$ *is four times differentiable, with a continuous fourth-order derivative on* $[0,1]$, *which we denote by* $y\in C^4([0,1])$, *and our* $y$ *satisfies both* (5.1) *and* (5.2).

In order to construct an appropriate operator $N$ and corresponding fixed-point problem for (5.1), (5.2) I note that the BVP (5.1), (5.2) is equivalent to the integral equation [288, p.238], [184]

$$y(x) = \int_0^1 G(x,s) f(s,y(s),y'(s),y''(s),y'''(s))\, ds + \phi_4(x), \quad x\in[0,1]. \tag{5.6}$$

Above, $G(x,s)$ is the Green's function for the BVP

$$y^{(iv)} = 0,\ x\in[0,1],$$
$$y(0)=0,\ y'(0)=0,\ y(1)=0,\ y'(1)=0,$$

and is given explicitly by

$$0\le G(x,s) = \frac{1}{6}\begin{cases} s^2(1-x)^2[(x-s)+2(1-s)x], & \text{for } 0\le s\le x\le 1,\\ \\ x^2(1-s)^2[(s-x)+2(1-x)s], & \text{for } 0\le x\le s\le 1;\end{cases} \tag{5.7}$$

and $\phi_4$ is the unique solution to the BVP

$$y^{(iv)} = 0,\ x\in[0,1],$$

$$y(0) = d_0,\ y'(0) = d_2,\ y(1) = d_1,\ y'(1) = d_3,$$

which is given explicitly by

$$\phi_4(x) = d_0(2x^3 - 3x^2 + 1) + d_1(-2x^3 + 3x^2) + d_2(x^3 - 2x^2 + x) + d_3(x^3 - x^2). \tag{5.8}$$

To avoid the repeated use of complicated expressions, I define the following constants to simplify certain notation.

For $i = 0, 1, 2, 3$, define positive constants $\epsilon_i$ via

$$\epsilon_i \geq \max_{x\in[0,1]} \int_0^1 \left| \frac{\partial^i}{\partial x^i} G(x,s) \right| \, ds. \tag{5.9}$$

Such choices are always possible due to the smoothness of $G$.

Let $p > 1$ and $q > 1$ be constants such that $1/p + 1/q = 1$. For $i = 0, 1, 2, 3$, define

$$c_i := \max_{x\in[0,1]} \left[ \left( \int_0^1 \left| \frac{\partial^i}{\partial x^i} G(x,s) \right|^q \, ds \right)^{1/q} \right] \tag{5.10}$$

and

$$\gamma_i := \left( \int_0^1 \left( \int_0^1 \left| \frac{\partial^i}{\partial x^i} G(x,s) \right|^q \, ds \right)^{p/q} \, dx \right)^{1/p}. \tag{5.11}$$

In this Chapter, I shall draw on a special case of Theorem 1.2 and my analysis shall involves the interval when $a = 0$ and $b = 1$. So I state the following special case of Theorem 1.2 that is when $k := 3$.

**Theorem 5.1.** *For $y, z \in C^3([0,1])$ we have*

$$\delta(y,z) \leq \max_{i\in\{0,1,2,3\}} \{L_i\} \, d^*(y,z) \tag{5.12}$$

*and*

$$\delta(y,z) \leq \left( \sum_{i=0}^{3} \frac{L_i}{W_i} \right) d(y,z). \tag{5.13}$$

### Analysis in unbounded domain:

I am now in a position to state and prove my first result for the existence and uniqueness of solutions to (5.1), (5.2) within a global (unbounded) context. In this part my approach involves applications of the metrics $d^*$ defined in (1.20) and $\delta$ defined in (1.21).

**Theorem 5.2.** *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and let $L_i$ be nonnegative constants for $i = 0,1,2,3$ (not all zero) such that*

$$|f(x,u_0,u_1,u_2,u_3) - f(x,v_0,v_1,v_2,v_3)| \le \sum_{i=0}^{3} L_i|u_i - v_i|$$
$$\text{for all } (x,u_0,u_1,u_2,u_3),(x,v_0,v_1,v_2,v_3) \in [0,1] \times \mathbb{R}^4. \tag{5.14}$$

*If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\gamma_i$ defined in (5.11) such that*

$$\sum_{i=0}^{3} L_i\gamma_i < 1, \tag{5.15}$$

*then the BVP (5.1), (5.2) has a unique solution in $C^4([0,1])$.*

*Proof.* Based on the form (5.6), we define the operator $N : C^3([0,1]) \to C^3([0,1])$ by

$$(Ny)(x) := \int_0^1 G(x,s)f(s,y(s),y'(s),y''(s),y'''(s))\, ds + \phi_4(x), \quad x \in [0,1].$$

We wish to show that there exists a unique $y \in C^3([0,1])$ such that

$$Ny = y$$

which is equivalent to proving the BVP (5.1), (5.2) has a unique solution. (Any solutions lying in $C^3([0,1])$ will also lie in $C^4([0,1])$ as repeatedly differentiating (5.6) will show.)

To prove that our $N$ has a unique fixed point, we show that the assumptions of Theorem 5.2 ensure that the conditions of Theorem 1.6 hold.

Consider the pair $(Y,\varrho) := (C^3([0,1]), d^*)$ to form a complete metric space and consider the metric $\delta = \tau$ on $Y$ where $p > 1$.

For $y, z \in C^3([0,1])$ and $x \in [0,1]$, consider

$$\begin{aligned}
&|(Ny)(x) - (Nz)(x)| \\
&\le \int_0^1 |G(x,s)|\, |f(s,y(s),y'(s),y''(s),y'''(s)) - f(s,z(s),z'(s),z''(s),z'''(s))|\, ds \\
&\le \int_0^1 |G(x,s)| \sum_{i=0}^{3} \left( L_i|y^{(i)}(s) - z^{(i)}(s)| \right)\, ds \\
&\le \left( \int_0^1 |G(x,s)|^q\, ds \right)^{1/q} \left[ \sum_{i=0}^{3} L_i \left( \int_0^1 |y^{(i)}(x) - z^{(i)}(x)|^p\, dx \right)^{1/p} \right] \\
&\le c_0\delta(y,z),
\end{aligned} \tag{5.16}$$

where we have invoked our assumption (5.14) and Hölder's inequality for $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ and $c_0$ is defined in (5.10).

By repeating the above approach on derivatives we can obtain

$$\max_{x\in[0,1]} |(Ny)^{(i)}(x) - (Nz)^{(i)}(x)| \le c_i\delta(y,z)$$

for $i = 0,1,2,3$. Thus, defining

$$c := \sum_{i=0}^{3} c_i$$

we see that

$$d^*(Ny,Nz) \le c\delta(y,z), \quad \text{for some } c > 0 \text{ and all } y,z \in C^3([0,1]).$$

and so the inequality (1.41) of Theorem 1.6 holds.

Now, from (5.12), for all $y,z \in C^3([0,1])$ we have

$$d^*(Ny,Nz) \le c\delta(y,z) \le c \max_{i\in\{0,1,2,3\}} \{L_i\}\ d^*(y,z).$$

Thus, given any $\varepsilon > 0$ we can choose $\Delta = \varepsilon/(c\max_{i\in\{0,1,2,3\}}\{L_i\})$ so that $d^*(Ny,Nz) < \varepsilon$ whenever $d^*(y,z) < \Delta$. Hence $N$ is continuous on $C^3([0,1])$ with respect to the $d^*$ metric.

Finally, we show that $N$ is contractive on $C^3([0,1])$ with respect to the $\delta$ metric. From (5.16), for each $y,z \in C^3([0,1])$ and $i = 0,1,2,3$, we have

$$L_i\left(\int_0^1 |(Ny)^{(i)}(x) - (Nz)^{(i)}(x)|^p\ dx\right)^{1/p} \le L_i\gamma_i\delta(y,z)$$

where the $\gamma_i$ are defined in (5.11). Summing both sides of the previous inequality over $i$ we obtain

$$\delta(Ny,Nz) \le \left(\sum_{i=0}^{3} L_i\gamma_i\right)\delta(y,z)$$

for all $y,z \in C^3([0,1])$. From our assumption (5.15), we have ensured

$$\delta(Ny,Nz) \le \alpha\delta(y,z)$$

for some $\alpha < 1$ and all $y,z \in C^3([0,1])$.

Thus, Theorem 1.6 is applicable and yields the existence of a unique fixed point to $N$ that lies in $C^3([0,1])$. This solution is also in $C^4([0,1])$ as can be verified by differentiating the integral equation (5.6). Thus we have equivalently shown that the BVP (5.1), (5.2) has a unique solution. □

The Lipschitz condition (5.14) will be satisfied if, for example, our

$$f(x,u_0,u_1,u_2,u_3)$$

has partial derivatives $\partial f/\partial u_i$ that are uniformly bounded and continuous on $[0,1]\times\mathbb{R}^4$ for each corresponding $i = 0,1,2,3$. In this case, each bound $M_i$ can form $L_i$ for $i = 0,1,2,3$.

For example, for an $f$ such as

$$f(x,u_0,u_1,u_2,u_3) = [\sin(u_0) + \cos(u_1) + \sin(u_2) + \cos(u_3)]/10$$

we could choose $M_i = L_i = 1/10$.

### Analysis in bounded domain:

Although the Lipschitz condition (5.14) imposed in Theorem 5.2 is more difficult to be satisfied on the unbounded domain $[0,1]\times\mathbb{R}^4$ when compared with a closed and bounded subset of this region, the conclusion of Theorem 5.2 provides robust, global information about the solutions to the problem under consideration. In comparison, the very nature of *localized* assumptions means that only limited, localized information about solutions can be necessarily obtained. For instance, in the context of localized assumptions, nothing can be concluded about existence and uniqueness of solutions that may lie outside of the closed and bounded set that is under consideration. This is especially important as multiple solutions to fourth-order BVPs have been shown to exist [34].

Motivated by the above discussion, in this part I provide some balance by examining questions of existence and uniqueness of solutions to (5.1), (5.2) within subsets of $[0,1]\times\mathbb{R}^4$. Essentially, I impose less restrictive conditions on $f$ in exchange for obtaining less information regarding the solutions. My approach in this part involves applications of the metrics $d$ defined in (1.22) and $\delta$ defined in (1.21).

**Theorem 5.3.** *Let $f : B_3 \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set*

$$B_3 := \Big\{(x,u,v,w,z) \in \mathbb{R}^5 : x \in [0,1],\ |u - \phi_4(x)| \le R,$$
$$|v - \phi_4'(x)| \le \frac{\epsilon_1}{\epsilon_0}R,\ |w - \phi_4''(x)| \le \frac{\epsilon_2}{\epsilon_0}R,\ |z - \phi_4'''(x)| \le \frac{\epsilon_3}{\epsilon_0}R\Big\},$$

*where $R > 0$, $\phi_4$ is defined in (5.8) and the $\epsilon_i$ are defined in (5.9). Assume $M\epsilon_0 \le R$. For $i = 0,1,2,3$, let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x,u_0,u_1,u_2,u_3) - f(x,v_0,v_1,v_2,u_3)| \le \sum_{i=0}^{3} L_i|u_i - v_i|,$$
$$\text{for all } (x,u_0,u_1,u_2,u_3),(x,v_0,v_1,v_2,v_3) \in B_3. \tag{5.17}$$

*If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\gamma_i$ defined in (5.11) and*

$$\sum_{i=0}^{3} L_i\gamma_i < 1, \tag{5.18}$$

*then the BVP* (5.1)*,* (5.2) *has a unique solution in* $C^3([0,1])$ *such that*

$$(x, y(x), y'(x), y''(x), y'''(x)) \in B_3, \quad \textit{for all } x \in [0,1].$$

*Proof.* Consider the pair $(Y, \varrho) := (C^3([0,1]), d)$ where the constants $W_i$ in our $d$ in (1.22) are chosen such that $W_0 = 1$, $W_1 = \epsilon_0/\epsilon_1$, $W_2 = \epsilon_0/\epsilon_2$ and $W_3 = \epsilon_0/\epsilon_3$. Our pair forms a complete metric space. Now, for the constant $R > 0$ and function $\phi_4$ in the definition of $B_3$, consider the following set $\mathcal{B}_{3R} \subset C^3([0,1])$ defined via

$$\mathcal{B}_{3R} := \{y \in C^3([0,1]) : d(y, \phi_4) \le R\}.$$

Since $\mathcal{B}_{3R}$ is a closed subspace of $C^3([0,1])$, the pair $(\mathcal{B}_{3R}, d)$ forms a complete metric space.

Consider the operator $N : \mathcal{B}_{3R} \to C^3([0,1])$ defined as in the proof of Theorem 5.2.

To establish the existence and uniqueness to $Nx = x$, we show that the conditions of Theorem 1.6 hold with $Y = \mathcal{B}_{3R}$.

Let me show $N : \mathcal{B}_{3R} \to \mathcal{B}_{3R}$. For $y \in \mathcal{B}_{3R}$ and $x \in [0,1]$, consider

$$\begin{aligned}
|(Ny)(x) - \phi_4(x)| &\le \int_0^1 |G(x,s)|\,|f(s, y(s), y'(s), y''(s), y'''(s))|\,ds \\
&\le M \int_0^1 |G(x,s)|\,ds \\
&\le M\epsilon_0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|(Ny)'(x) - \phi_4'(x)| &\le \int_0^1 \left|\frac{\partial}{\partial x} G(x,s)\right|\,|f(s, y(s), y'(s), y''(s), y'''(s))|\,ds \\
&\le M \int_0^1 \left|\frac{\partial}{\partial x} G(x,s)\right|\,ds \\
&\le M\epsilon_1.
\end{aligned}$$

Thus $\epsilon_0 |(Ny)'(x) - \phi_4'(x)|/\epsilon_1 \le M\epsilon_0$.

In addition, via similar arguments, we obtain

$$|(Ny)''(x) - \phi_4''(x)| \le M\epsilon_2, \quad |(Ny)'''(x) - \phi_4'''(x)| \le M\epsilon_3;$$

so that $\epsilon_0 |(Ny)''(x) - \phi_4''(x)|/\epsilon_2 \le M\epsilon_0$ and $\epsilon_0 |(Ny)'''(x) - \phi_4'''(x)|/\epsilon_3 \le M\epsilon_0$.

Thus, for all $y \in \mathcal{B}_{3R}$ we have

$$d(Ny, \phi_4) \le \max\{M\epsilon_0, M\epsilon_0, M\epsilon_0, M\epsilon_0\}$$

$$= M\epsilon_0$$
$$\leq R$$

where the final inequality holds by assumption. Thus, for all $y \in \mathcal{B}_{3R}$ we have $Ny \in \mathcal{B}_{3R}$ so that $N : \mathcal{B}_{3R} \to \mathcal{B}_{3R}$.

Similarly to the proof of Theorem 5.2, consider the metric $\delta = \tau$ on $Y$ where $p > 1$ and we have, for $i = 0, 1, 2, 3$, and all $x \in [0, 1]$, that

$$|(Ny)^{(i)}(x) - (Nz)^{(i)}(x)| \leq c_i \delta(y, z)$$

via applications of Hölder's inequality.

Combining the above inequalities we obtain,

$$d(Ny, Nz) \leq c\delta(y, z), \quad \text{for some } c > 0 \text{ and all } y, z \in \mathcal{B}_{3R}, \tag{5.19}$$

where,

$$c := \max \left\{ c_0, \frac{\epsilon_0}{\epsilon_1} c_1, \frac{\epsilon_0}{\epsilon_2} c_2, \frac{\epsilon_0}{\epsilon_3} c_3 \right\}.$$

Thus, the first inequality of Theorem 1.6 holds.

Furthermore, $N$ is continuous on $\mathcal{B}_{3R}$ with respect to the $d$ metric as can be shown from the following arguments. For all $y, z \in \mathcal{B}_{3R}$ we may apply (5.13) to (5.19) to obtain

$$\begin{aligned} d(Ny, Nz) &\leq c\delta(y, z) \\ &\leq c \left( L_0 + L_1 \frac{\epsilon_1}{\epsilon_0} + L_2 \frac{\epsilon_2}{\epsilon_0} + L_3 \frac{\epsilon_3}{\epsilon_0} \right) d(x, y). \end{aligned}$$

Thus, given any $\varepsilon > 0$ we can choose

$$\Delta = \frac{\varepsilon}{\left( L_0 + L_1 \frac{\epsilon_1}{\epsilon_0} + L_2 \frac{\epsilon_2}{\epsilon_0} + L_3 \frac{\epsilon_3}{\epsilon_0} \right)}$$

so that $d(Ny, Nz) < \varepsilon$ whenever $d(x, y) < \Delta$. Hence $N$ is continuous on $\mathcal{B}_{3R}$ with respect to the $d$ metric.

Finally, the contraction condition of $N$ on $\mathcal{B}_{3R}$ follows from essentially the same arguments as those used in the proof of Theorem 5.2 and so we do not repeat it here.

Thus, we conclude that all the conditions of Theorem 1.6 hold for $N$ on $\mathcal{B}_{3R}$ and so the unique fixed point of $N$ in $\mathcal{B}_{3R}$ is guaranteed to exist. □

The following three new results are variations on the theme of Theorem 5.3.

**Theorem 5.4.** *Let $f : B_4 \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set*

$$B_4 := \Big\{(x,u,v,w,z) \in \mathbb{R}^5 : x \in [0,1],\ |u - \phi_4(x)| \le \frac{\epsilon_0}{\epsilon_1}R,$$
$$|v - \phi_4'(x)| \le R,\ |w - \phi_4''(x)| \le \frac{\epsilon_2}{\epsilon_1}R,\ |z - \phi_4'''(x)| \le \frac{\epsilon_3}{\epsilon_1}R\Big\},$$

*where $R > 0$, $\phi_4$ is defined in (5.8) and the $\epsilon_i$ are defined in (5.9). Assume $M\epsilon_1 \le R$. For $i = 0,1,2,3$, let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x,u_0,u_1,u_2,u_3) - f(x,v_0,v_1,v_2,u_3)| \le \sum_{i=0}^{3} L_i|u_i - v_i|,$$
$$\text{for all } (x,u_0,u_1,u_2,u_3),(x,v_0,v_1,v_2,v_3) \in B_4. \tag{5.20}$$

*If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\gamma_i$ defined in (5.11) such that*

$$\sum_{i=0}^{3} L_i\gamma_i < 1, \tag{5.21}$$

*then the BVP (5.1), (5.2) has a unique solution in $C^3([0,1])$ such that*

$$(x, y(x), y'(x), y''(x), y'''(x)) \in B_4, \quad \text{for all } x \in [0,1].$$

*Proof.* The proof follows similar lines of argument as that of Theorem 5.3 and so I just sketch the basic idea.

Consider the pair $(Y,\varrho) := (C^3([0,1]), d)$ where the constants $W_i$ in our $d$ in (1.22) are chosen such that $W_0 = \epsilon_1/\epsilon_0$, $W_1 = 1$, $W_2 = \epsilon_1/\epsilon_2$ and $W_3 = \epsilon_1/\epsilon_3$. Our pair forms a complete metric space. Now, for the constant $R > 0$ and function $\phi_4$ in the definition of $B_4$, consider the following set $\mathcal{B}_{4R} \subset C^3([0,1])$ defined via

$$\mathcal{B}_{4R} := \{y \in C^3([0,1]) : d(y,\phi_4) \le R\}.$$

Since $\mathcal{B}_{4R}$ is a closed subspace of $C^3([0,1])$, the pair $(\mathcal{B}_{4R}, d)$ forms a complete metric space.

Following similar steps as that of Theorem 5.3, the condition $M\epsilon_1 \le R$ ensures $N : \mathcal{B}_{4R} \to \mathcal{B}_{4R}$. Furthermore, (5.20) and (5.21) guarantee that $N$ is contractive on $\mathcal{B}_{4R}$ with respect to $\delta$. □

**Theorem 5.5.** *Let $f : B_5 \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set*

$$B_5 := \Big\{(x,u,v,w,z) \in \mathbb{R}^5 : x \in [0,1],\ |u - \phi_4(x)| \le \frac{\epsilon_0}{\epsilon_2}R,$$

$$|v - \phi_4'(x)| \le \frac{\epsilon_1}{\epsilon_2} R, \ |w - \phi_4''(x)| \le R, \ |z - \phi_4'''(x)| \le \frac{\epsilon_3}{\epsilon_2} R \Big\},$$

*where $R > 0$, $\phi_4$ is defined in (5.8) and the $\epsilon_i$ are defined in (5.9). Assume $M\epsilon_2 \le R$. For $i = 0, 1, 2, 3$, let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, u_3)| \le \sum_{i=0}^{3} L_i |u_i - v_i|,$$

$$\text{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in B_5. \tag{5.22}$$

*If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\gamma_i$ defined in (5.11) such that*

$$\sum_{i=0}^{3} L_i \gamma_i < 1, \tag{5.23}$$

*then the BVP (5.1), (5.2) has a unique solution in $C^3([0, 1])$ such that*

$$(x, y(x), y'(x), y''(x), y'''(x)) \in B_5, \quad \text{for all } x \in [0, 1].$$

*Proof.* The proof follows similar lines of argument as that of Theorem 5.3 and so I just sketch the basic idea.

Consider the pair $(Y, \varrho) := (C^3([0, 1]), d)$ where the constants $W_i$ in our $d$ in (1.22) are chosen such that $W_0 = \epsilon_2/\epsilon_0$, $W_1 = \epsilon_2/\epsilon_1$, $W_2 = 1$ and $W_3 = \epsilon_2/\epsilon_3$. Our pair forms a complete metric space. Now, for the constant $R > 0$ and function $\phi_4$ in the definition of $B_5$, consider the following set $\mathcal{B}_{5R} \subset C^3([0, 1])$ defined via

$$\mathcal{B}_{5R} := \{y \in C^3([0, 1]) : d(y, \phi_4) \le R\}.$$

Since $\mathcal{B}_{5R}$ is a closed subspace of $C^3([0, 1])$, the pair $(\mathcal{B}_{5R}, d)$ forms a complete metric space.

Following similar steps as that of Theorem 5.3, the condition $M\epsilon_2 \le R$ ensures $N : \mathcal{B}_{5R} \to \mathcal{B}_{5R}$. Furthermore, (5.22) and (5.23) guarantee that $N$ is contractive on $\mathcal{B}_{5R}$ with respect to $\delta$. □

**Theorem 5.6.** *Let $f : B_6 \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set*

$$B_6 := \Big\{(x, u, v, w, z) \in \mathbb{R}^5 : x \in [0, 1], \ |u - \phi_4(x)| \le \frac{\epsilon_0}{\epsilon_3} R,$$

$$|v - \phi_4'(x)| \le \frac{\epsilon_1}{\epsilon_3} R, \ |w - \phi_4''(x)| \le \frac{\epsilon_2}{\epsilon_3} R, \ |z - \phi_4'''(x)| \le R \Big\},$$

*where $R > 0$, $\phi_4$ is defined in (5.8) and the $\epsilon_i$ are defined in (5.9). Assume $M\epsilon_3 \le R$. For $i = 0, 1, 2, 3$, let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, u_3)| \le \sum_{i=0}^{3} L_i |u_i - v_i|,$$

$$\textit{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in B_6. \tag{5.24}$$

*If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\gamma_i$ defined in (5.11) such that*

$$\sum_{i=0}^{3} L_i \gamma_i < 1, \tag{5.25}$$

*then the BVP (5.1), (5.2) has a unique solution in $C^3([0,1])$ such that*

$$(x, y(x), y'(x), y''(x), y'''(x)) \in B_6, \quad \textit{for all } x \in [0,1].$$

*Proof.* Omitted due to brevity. □

## 5.3 Existence results via Rus' fixed point theorem: Clamped/free ends

In this Section I establish my novel results for the existence and uniqueness of solutions to the BVP (5.1), (5.3) via applications of Rus's contraction mapping theorem. My approach in this Section is identical to the approach of the previous Section i.e., I first give my result within a global (unbounded) context and then I give my second result within closed and bounded sets of $[0,1] \times \mathbb{R}$; and in closed balls within infinite dimensional space.

Let me briefly first introduce the notation, and definitions that are necessary for navigating this Section and then formulate my main results of this Section.

The following definition gives what I mean by a solution to (5.1), (5.3).

**Definition 5.2.** *By a solution to (5.1), (5.3) we mean a function $y : [0,1] \to \mathbb{R}$ such that $y$ is four times differentiable, with a continuous fourth-order derivative on $[0,1]$, which we denote by $y \in C^4([0,1])$, and our $y$ satisfies both (5.1) and (5.3).*

The BVP (5.1), (5.3) is equivalent to the integral equation [288, p.238], [184]

$$y(x) = \int_0^1 \mathcal{G}(x,s) f(s, y(s), y'(s), y''(s), y'''(s)) \, ds + \psi_4(x), \quad x \in [0,1], \tag{5.26}$$

where $\mathcal{G}(x,s)$ is the Green's function [285, p.2] for the following BVP

$$y^{(iv)} = 0, \;\; x \in [0,1],$$

$$y(0) = 0, \;\; y'(0) = 0, \;\; y''(1) = 0, \;\; y'''(1) = 0,$$

and is given explicitly by

$$0 \le \mathcal{G}(x,s) := \frac{1}{6}\begin{cases} s^2(3x-s), & \text{for } 0 \le s \le x \le 1; \\ \\ x^2(3s-x), & \text{for } 0 \le x \le s \le 1, \end{cases} \tag{5.27}$$

and $\psi_4$ is the unique solution to the BVP

$$y^{(iv)} = 0, \;\; x \in [0,1],$$
$$y(0) = d_0, \;\; y'(0) = d_2, \;\; y''(1) = d_4, \;\; y'''(1) = d_5,$$

which is given explicitly by

$$\psi_4(x) = d_5x^3/6 + (d_4 - d_5)x^2/2 + d_2x + d_0. \tag{5.28}$$

Once again, to avoid the repeated use of complicated expressions, I define the following constants to simplify certain notation.

For $i = 0, 1, 2, 3$, define positive constants $\theta_i$ via

$$\theta_i \ge \max_{x\in[0,1]} \int_0^1 \left| \frac{\partial^i}{\partial x^i}\mathcal{G}(x,s) \right| \, ds. \tag{5.29}$$

Such choices are always possible due to the smoothness of $\mathcal{G}$.

Let $p > 1$ and $q > 1$ be constants such that $1/p + 1/q = 1$. For $i = 0, 1, 2, 3$, define

$$e_i := \max_{x\in[0,1]} \left[ \left( \int_0^1 \left| \frac{\partial^i}{\partial x^i}\mathcal{G}(x,s) \right|^q \, ds \right)^{1/q} \right] \tag{5.30}$$

and

$$\lambda_i := \left( \int_0^1 \left( \int_0^1 \left| \frac{\partial^i}{\partial x^i}\mathcal{G}(x,s) \right|^q \, ds \right)^{p/q} \, dx \right)^{1/p}. \tag{5.31}$$

### 5.3.1 Analysis in unbounded domain

I am now in a position to state and prove my first result for the existence and uniqueness of solutions to (5.1), (5.3) within a global (unbounded) context. In this part my approach involves applications of the metrics $d^*$ defined in (1.20) and $\delta$ defined in (1.21).

**Theorem 5.7.** *Let $f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ be continuous and let $L_i$ be nonnegative constants for $i = 0, 1, 2, 3$ (not all zero) such that*

$$|f(x,u_0,u_1,u_2,u_3) - f(x,v_0,v_1,v_2,v_3)| \le \sum_{i=0}^{3} L_i|u_i - v_i|$$

$$\textit{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in [0,1] \times \mathbb{R}^4. \tag{5.32}$$

*If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\lambda_i$ defined in (5.31) such that*

$$\sum_{i=0}^{3} L_i \lambda_i < 1, \tag{5.33}$$

*then the BVP (5.1), (5.3) has a unique solution in $C^4([0,1])$.*

*Proof.* Consider the operator $\mathcal{N} : C^3([0,1]) \to C^3([0,1])$ constructed from the form (5.26), namely

$$(\mathcal{N}y)(x) := \int_0^1 \mathcal{G}(x,s) f(s, y(s), y'(s), y''(s), y'''(s)) \, ds + \psi_4(x), \quad x \in [0,1].$$

We want to show that there exists a unique $y \in C^3([0,1])$ such that

$$\mathcal{N}y = y.$$

To prove this, we shall show that the conditions of Theorem 1.6 hold. Consider the pair $(Y, \varrho) = (C^3([0,1]), d^*)$ which forms a complete metric space. In addition, consider the metric $\delta = \tau$ on $Y$ where $p > 1$.

For $y, z \in C^3([0,1])$ and $x \in [0,1]$, consider

$$\begin{aligned}
&|(\mathcal{N}y)(x) - (\mathcal{N}z)(x)| \\
&\le \int_0^1 |\mathcal{G}(x,s)| \, |f(s, y(s), y'(s), y''(s), y'''(s)) - f(s, z(s), z'(s), z''(s), z'''(s))| \, ds \\
&\le \int_0^1 |\mathcal{G}(x,s)| \sum_{i=0}^{3} \left( L_i |y^{(i)}(s) - z^{(i)}(s)| \right) \, ds \\
&\le \left( \int_0^1 |\mathcal{G}(x,s)|^q \, ds \right)^{1/q} \left[ \sum_{i=0}^{3} L_i \left( \int_0^1 |y^{(i)}(s) - z^{(i)}(s)|^p \, ds \right)^{1/p} \right] \\
&\le \max_{x \in [0,1]} \left( \int_0^1 |\mathcal{G}(x,s)|^q \, ds \right)^{1/q} \delta(y,z). \\
&= e_0 \delta(y,z)
\end{aligned}$$

where $e_0$ is defined in (5.30). Repeating the above argument to derivatives of the operator $\mathcal{N}$ yields

$$|(\mathcal{N}y)^{(i)}(x) - (\mathcal{N}z)^{(i)}(x)| \le e_i \delta(y,z)$$

for $i = 0, 1, 2, 3$. Thus, defining

$$c := \sum_{i=0}^{3} e_i$$

we see that

$$d^*(\mathcal{N}y,\mathcal{N}z) \le c\delta(y,z), \quad \text{for some } c > 0 \text{ and all } y,z \in Y.$$

and so the inequality (1.41) of Theorem 1.6 holds.

Now, for all $y,z \in C^3([0,1])$ consider

$$d^*(\mathcal{N}y,\mathcal{N}z) \le c\delta(y,z) \le c \max_{i\in\{0,1,2,3\}}\{L_i\}\, d^*(y,z).$$

Thus, given any $\varepsilon > 0$ we can choose $\Delta = \varepsilon/(c\max_{i\in\{0,1,2,3\}}\{L_i\})$ so that $d^*(\mathcal{N}y,\mathcal{N}z) < \varepsilon$ whenever $d^*(y,z) < \Delta$. Hence $\mathcal{N}$ is continuous on $C^3([0,1])$ with respect to the $d^*$ metric.

Finally, we show that $\mathcal{N}$ is contractive on $C^3([0,1])$ with respect to the $\delta$ metric. For each $y,z \in C^3([0,1])$ and $i = 0,1,2,3$ we have

$$L_i\left(\int_0^1 |(\mathcal{N}y)^{(i)}(x) - (\mathcal{N}z)^{(i)}(x)|^p\, dx\right)^{1/p} \le L_i\left(\int_0^1\left(\int_0^1\left|\frac{\partial^i}{\partial x^i}\mathcal{G}((x,s)\right|^q ds\right)^{p/q} dx\right)^{1/p} \delta(y,z)$$

and so summing the previous inequality over $i$ we obtain

$$\delta(\mathcal{N}y,\mathcal{N}z) \le \left(\sum_{i=0}^{3} L_i\lambda_i\right)\delta(y,z).$$

Thus, by (5.33), we have

$$\delta(\mathcal{N}y,\mathcal{N}z) \le \alpha_1\delta(y,z)$$

for some $\alpha_1 < 1$ and all $y,z \in C^3([0,1])$.

Thus, Theorem 1.6 is applicable and the operator $\mathcal{N}$ has a unique fixed point in $C^3([0,1])$. This solution is also in $C^4([0,1])$ as differentiating the integral equation (5.26) shows. My conclusion is equivalent to showing the BVP (5.1), (5.3) has a unique solution. □

### 5.3.2 Analysis in bounded domain

Now let me now explore in this part the existence and uniqueness of solutions to (5.1), (5.3) on subsets of $[0,1]\times\mathbb{R}^4$.

**Theorem 5.8.** *Let $f : B_7 \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set*

$$B_7 := \Big\{(x,u,v,w,z) \in \mathbb{R}^5 : x \in [0,1],\ |u - \psi_4(x)| \le R,$$
$$|v - \psi_4'(x)| \le \frac{\theta_1}{\theta_0}R,\ |w - \psi_4''(x)| \le \frac{\theta_2}{\theta_0}R,\ |z - \psi_4'''(x)| \le \frac{\theta_3}{\theta_0}R\Big\},$$

*where* $R > 0$, $\psi_4$ *is defined in* (5.28) *and the* $\theta_i$ *are defined in* (5.29). *Assume* $M\theta_0 \le R$. *For* $i = 0, 1, 2, 3$, *let* $L_i$ *be nonnegative constants (not all zero) such that*

$$|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, u_3)| \le \sum_{i=0}^{3} L_i |u_i - v_i|,$$
$$\text{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in B_7. \tag{5.34}$$

*If there are constants* $p > 1$ *and* $q > 1$ *such that* $1/p + 1/q = 1$ *with* $\lambda_i$ *defined in* (5.31) *such that*

$$\sum_{i=0}^{3} L_i \lambda_i < 1, \tag{5.35}$$

*then the BVP* (5.1), (5.3) *has a unique (nontrivial) solution in* $C^3([0,1])$ *such that*

$$(x, y(x), y'(x), y''(x), y'''(x)) \in B_7, \quad \text{for all } x \in [0,1].$$

*Proof.* The proof is very similar to that of the proof of Theorem 5.3 by making the appropriate modifications (e.g., $\mathcal{G}$ instead of $G$, etc.). Thus I omit the proof for brevity. □

Similarly, we have the following three results which I state without proof due to brevity and concerns of repetition.

**Theorem 5.9.** *Let* $f : B_8 \to \mathbb{R}$ *be continuous and uniformly bounded by* $M > 0$ *on the set*

$$B_8 := \Big\{ (x, u, v, w, z) \in \mathbb{R}^5 : x \in [0,1],\ |u - \psi_4(x)| \le \frac{\theta_0}{\theta_1} R,$$
$$|v - \psi_4'(x)| \le R,\ |w - \psi_4''(x)| \le \frac{\theta_2}{\theta_1} R,\ |z - \psi_4'''(x)| \le \frac{\theta_3}{\theta_1} R \Big\},$$

*where* $R > 0$, $\psi_4$ *is defined in* (5.28) *and the* $\theta_i$ *are defined in* (5.29). *Assume* $M\theta_1 \le R$. *For* $i = 0, 1, 2, 3$, *let* $L_i$ *be nonnegative constants (not all zero) such that*

$$|f(x, u_0, u_1, u_2, u_3) - f(x, v_0, v_1, v_2, u_3)| \le \sum_{i=0}^{3} L_i |u_i - v_i|,$$
$$\text{for all } (x, u_0, u_1, u_2, u_3), (x, v_0, v_1, v_2, v_3) \in B_8. \tag{5.36}$$

*If there are constants* $p > 1$ *and* $q > 1$ *such that* $1/p + 1/q = 1$ *with* $\gamma_i$ *defined in* (5.11) *such that*

$$\sum_{i=0}^{3} L_i \lambda_i < 1, \tag{5.37}$$

*then the BVP* (5.1), (5.3) *has a unique (nontrivial) solution in* $C^3([0,1])$ *such that*

$$(x, y(x), y'(x), y''(x), y'''(x)) \in B_8, \quad \text{for all } x \in [0,1].$$

**Theorem 5.10.** *Let $f : B_9 \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set*

$$B_9 := \Big\{ (x,u,v,w,z) \in \mathbb{R}^5 : x \in [0,1],\ |u - \psi_4(x)| \leq \frac{\theta_0}{\theta_2} R,$$
$$|v - \psi_4'(x)| \leq \frac{\theta_1}{\theta_2} R,\ |w - \psi_4''(x)| \leq R,\ |z - \psi_4'''(x)| \leq \frac{\theta_3}{\theta_2} R \Big\},$$

*where $R > 0$, $\psi_4$ is defined in (5.28) and the $\theta_i$ are defined in (5.29). Assume $M\theta_2 \leq R$. For $i = 0,1,2,3$, let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x,u_0,u_1,u_2,u_3) - f(x,v_0,v_1,v_2,u_3)| \leq \sum_{i=0}^{3} L_i |u_i - v_i|,$$
$$\text{for all } (x,u_0,u_1,u_2,u_3), (x,v_0,v_1,v_2,v_3) \in B_9. \tag{5.38}$$

*If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\gamma_i$ defined in (5.11) such that*

$$\sum_{i=0}^{3} L_i \lambda_i < 1, \tag{5.39}$$

*then the BVP (5.1), (5.3) has a unique (nontrivial) solution in $C^3([0,1])$ such that*

$$(x, y(x), y'(x), y''(x), y'''(x)) \in B_9, \quad \text{for all } x \in [0,1].$$

**Theorem 5.11.** *Let $f : \varsigma \to \mathbb{R}$ be continuous and uniformly bounded by $M > 0$ on the set*

$$\varsigma := \Big\{ (x,u,v,w,z) \in \mathbb{R}^5 : x \in [0,1],\ |u - \psi_4(x)| \leq \frac{\theta_0}{\theta_3} R,$$
$$|v - \psi_4'(x)| \leq \frac{\theta_1}{\theta_3} R,\ |w - \psi_4''(x)| \leq \frac{\theta_2}{\theta_3} R,\ |z - \psi_4'''(x)| \leq R \Big\},$$

*where $R > 0$, $\psi_4$ is defined in (5.28) and the $\theta_i$ are defined in (5.29). Assume $M\theta_3 \leq R$. For $i = 0,1,2,3$, let $L_i$ be nonnegative constants (not all zero) such that*

$$|f(x,u_0,u_1,u_2,u_3) - f(x,v_0,v_1,v_2,u_3)| \leq \sum_{i=0}^{3} L_i |u_i - v_i|,$$
$$\text{for all } (x,u_0,u_1,u_2,u_3), (x,v_0,v_1,v_2,v_3) \in \varsigma. \tag{5.40}$$

*If there are constants $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$ with $\lambda_i$ defined in (5.31) such that*

$$\sum_{i=0}^{3} L_i \lambda_i < 1, \tag{5.41}$$

*then the BVP (5.1), (5.3) has a unique (nontrivial) solution in $C^3([0,1])$ such that*

$$(x, y(x), y'(x), y''(x), y'''(x)) \in \varsigma \quad \text{for all } x \in [0,1].$$

## 5.4 Application to beam deflections

This Section analyses fourth-order simplified BVPs arising from the deflection of elastic beams subject to linear and nonlinear loading forces. Through this discussion I illustrate the nature of the advancement of this Chapter when compared with traditional approaches.

### 5.4.1 Nonlinear loading force

If the we consider a loading force on the beam given by $f(x,y)$, which may be nonlinear, then we obtain the following fourth-order differential equation

$$y^{(iv)} = f(x,y), \quad x \in [0,1], \tag{5.42}$$

where $y = y(x)$ represents the resultant deflection of the beam at position $x$. This form is similar to that of [34].

In this case, the Lipschitz condition of my theorems reduces to

$$|f(x,u_0) - f(x,v_0) \le L_0|u_0 - v_0| \tag{5.43}$$

on $[0,1] \times \mathbb{R}$ or on suitable subsets.

A standard approach to the problem (5.42) with either: fully clamped ends (5.4); or clamped/free ends (5.5), could employ Banach's contraction mapping theorem within the space of continuous functions $C([0,1])$ coupled with the maximum metric

$$d_0(y,z) := \max_{x\in[0,1]} |y(x) - z(x)|, \quad \text{for all } y, z \in C([0,1]).$$

To obtain a contraction for the operator $N$ with respect to our $d$ as per Banach's contraction mapping theorem, the standard condition for clamped ends takes the form

$$L_0 \max_{x\in[0,1]} \int_0^1 |G(x,s)|\, ds = L_0/384 < 1 \tag{5.44}$$

see [58, Theorem 3.1 Condition c] or [64].

For clamped/free ends, to obtain a contraction for the operator $\mathcal{N}$ with respect to our $\varrho$ as per Banach's contraction mapping theorem, the condition

$$L_0 \max_{x\in[0,1]} \int_0^1 |\mathcal{G}(x,s)|\, ds = L_0/8 < 1 \tag{5.45}$$

is involved, see for example, [63, p.58] or [24, Remark 1].

In comparison, the simplified version of Theorem 5.2 (or related theorems) with $p = q = 2$ involves (5.15) taking the form

$$L_0\left(\int_0^1\left(\int_0^1 |G(x,s)|^2\,ds\right)\,dx\right)^{1/2} = L_0\left(71/17463600\right)^{1/2} < 1. \tag{5.46}$$

The left-hand side of (5.46) can be verified by direct integration, or through use of a suitable computing package, via

$$\begin{aligned}\int_0^1 |G(x,s)|^2\,ds &= \left[\int_0^x\left(\frac{s^2(1-x)^2[(x-s)+2(1-s)x]}{6}\right)^2\,ds\right]\\ &\quad+\int_x^1\left(\frac{x^2(1-s)^2[(s-x)+2(1-x)s]}{6}\right)^2\,ds\\ &=\left[(x-1)^4x^7(20x^2-50x+33)/1260\right]-x^4(20x^2+10x+3)(x-1)^7/1260\end{aligned}$$

and another integration yields

$$\begin{aligned}&\int_0^1\left(\int_0^1 |G(x,s)|^2\,ds\right)\,dx\\ &=\int_0^1 (x-1)^4x^7(20x^2-50x+33)/1260-x^4(20x^2+10x+3)(x-1)^7/1260\,dx\\ &=71/17463600.\end{aligned}$$

One can observe that my condition (5.46) is sharper than condition (5.44) and this illustrates one aspect of how the new results of this Chapter represent an advancement over traditional approaches, and how they are applicable to a wider class of problems.

The simplified version of Theorem 5.7 (or related theorems) with $p = q = 2$ involves (5.33) taking the form

$$L\left(\int_0^1\left(\int_0^1 |\mathcal{G}(x,s)|^2\,ds\right)\,dx\right)^{1/2} = L\left(11/1680\right)^{1/2} < 1. \tag{5.47}$$

The left-hand side of (5.47) can be verified via the following steps.

$$\begin{aligned}\int_0^1 |\mathcal{G}(x,s)|^2\,ds &= \left[\int_0^x\left(\frac{s^2(3x-s)}{6}\right)^2\,ds\right]+\int_x^1\left(\frac{x^2(3s-x)}{6}\right)^2\,ds\\ &=\left[11x^7/420\right]+x^4/12-x^5/12+x^6/36-x^7/36\end{aligned}$$

and another integration yields

$$\begin{aligned}\int_0^1\left(\int_0^1 |\mathcal{G}(x,s)|^2\,ds\right)\,dx &= \int_0^1 11x^7/420+x^4/12-x^5/12+x^6/36-x^7/36\,dx\\ &=11/1680.\end{aligned}$$

Again, observe that my condition (5.47) is sharper than condition (5.45) and this illustrates another element of my advancement.

Let me further demonstrate the more aspects of my sharpened assumptions above through the discussion of some concrete cases.

If we consider the special case of $f$ in (5.42) in the form

$$f(y) := 10 \sin y + 1 \tag{5.48}$$

then we see that the smallest Lipschitz constant that we can calculate so that (5.43) holds on $[0,1] \times \mathbb{R}$ is $L_0 = 10$, which is formulated from the bound on $|f'(y)|$. Such an $L_0$ satisfies (5.47) but not (5.45). Thus, our $f$ satisfies the conditions of Theorem 5.7 (with appropriate boundary conditions) and we can obtain information in unbounded sets regarding the existence, uniqueness and approximation of solutions.

On the other hand, if we consider $f$ on a closed and bounded domain $[-R, R]$, for any constant $R > 0$, then we still cannot get (5.45) to hold for the above choice of $L_0 = 10$. Hence, the results of [63] do not apply to the $f$ in this example.

Furthermore, we can rerun the same argument as above for the example

$$f(y) = 385 \sin y + 1$$

to show that (5.46) holds but (5.44) does not. Hence, the results of [64] do not apply to this $f$ with $L_0 = 385$.

Furthermore, we consider the special case

$$f(x,y) := x + 1 + 8y^2, \tag{5.49}$$

subject to clamped/free ends (5.5).

Choose $R = 1/2$ to form the set $F$ in the statement of Theorem 5.8. Here, choose $\theta_0 = 1/8$ (with the remaining $\theta_i$ not coming in to play) and a bound $M$ on $f$ over $F$ can be chosen to be $M = 4$.

In addition, we see $\partial f/\partial y$ is continuous and bounded on $F$ by $8$. Thus, we can choose our Lipschitz constant to be $L_0 = 8$. (In the notation of Theorem 5.8 we have $L_0 = 8$ and the remaining $L_i$ are zero.)

Choosing $p = q = 2$, our $\lambda_0$ is contained in (5.47) (with the other $\lambda_i$ not coming in to play). Thus, we have

$$L_0\lambda_0 = 8\left(11/1680\right)^{1/2} < 1.$$

Hence all of the conditions of Theorem 5.8 are satisfied and its conclusion may be applied to our problem.

On the other hand, if we try to verify or apply the conditions in [63] to our problem then we run into an impossibility. For all $x \in [0,1]$ and $|y| \le M/8$, where $M > 0$, the assumption in [63] becomes

$$|f(x,y)| \le 2 + 8(M/8)^2 \le M.$$

which has only the solution $M = 4$. However, (5.45) takes the form

$$L_0/8 = 8/8 < 1,$$

which is clearly impossible. Thus the results in [63] do not apply to this example.

The above examples and discussion illustrates how the new results of this Chapter represent an advancement over traditional approaches, and how they are applicable to a wider class of problems.

### 5.4.2 Linear loading force

If the loading force on the beam is linear and given by $f(x,y) = h(x)y + j(x)$, then fourth-order ordinary differential equation

$$y^{(iv)} = h(x)y + j(x), \quad x \in [0,1], \tag{5.50}$$

is obtained and we can form the following corollaries.

**Corollary 5.1.** *Let $h$ and $j$ be continuous. If $|h(x)| < \left(17463600/71\right)^{1/2}$ for all $x \in [0,1]$, then the elastic beam deflection BVP* (5.50)*,* (5.4) *with linear loading force has a unique solution in* $C^4([0,1])$.

*Proof.* This is a special case of Theorem 5.2 with $p = q = 2$ and $f(x,y) = h(x)y + j(x)$. □

**Corollary 5.2.** *Let $h$ and $j$ be continuous. If $|h(x)| < \left(1680/11\right)^{1/2}$ for all $x \in [0,1]$, then the elastic beam deflection BVP* (5.50)*,* (5.5) *with linear loading force has a unique solution in* $C^4([0,1])$.

*Proof.* This is a special case of Theorem 5.7 with $p = q = 2$. □

# Chapter 6

# Fourth-order BVPs: An application to laminar flows in channels with porous walls

## 6.1 Introduction to the problem of laminar flow

Laminar flows in channels with porous walls have attracted the attention of applied mathematicians and engineers since the 1940s. This is partly due to their connection with a diverse range of physical problems that are of significant interest. For example, in aeronautics the method of transpiration cooling has gained attention:

> "In this method, the surfaces to be protected against the influence of a hot fluid stream are manufactured from a porous material and a cold fluid is ejected through the wall to form a protective layer along the surface. Certain areas on the skin of high-velocity aircraft may be provided with these surfaces as protection against the influence of aerodynamic heating. Porous surfaces with suction also are used on airfoils and bodies of aircraft to delay separation or transition to turbulence; in these cases, the flow along the surface is of a boundary-layer type." [73, pp.1-2]

In addition, channel flows are seen in plants [111] and animals [133], where vascular systems distribute energy to where it is needed, and enable distal parts of the organism to communicate [114]. Furthermore, channels play a significant role in the transportation of liquids or gases and energy from sites of production to the consumer or industry [114], and the protection of channel walls via transpiration cooling is of primary interest in nuclear applications [73].

Thus, the purpose of this Chapter is to develop a more complete theory regarding solutions to the problem of laminar flow in channels with porous walls. In particular, my aim is to introduce contraction mapping ideas in what appears to be a first time synthesis and application to the problem of laminar flow in channels with porous walls that is modelled by nonlinear, fourth-order differential equation (BVP)

$$y^{(iv)} + \mathcal{R}(y'y'' - yy''') = 0, \quad x \in [0,1], \tag{6.1}$$

where $y = y(x)$, $\mathcal{R}$ is a Reynolds number and (6.1) is subject to the two-point boundary conditions:

$$y(0) = 0,\ y''(0) = 0,\ y'(1) = 0,\ y(1) = 1. \tag{6.2}$$

There are at least three significant points of distinction between my current work in this Chapter and the existing literature. They include: the mathematical form of the problem under consideration; the types of methods employed; and the nature of the results obtained. I discuss them below.

In the literature relating to laminar flow within in channels with porous walls (and its variations) [45, 73, 100, 112, 114, 180, 221, 226, 243, 244, 257, 258, 272, 281], the majority of scholars have exclusively considered and analyzed the problem as an equivalent third-order BVP

$$y''' + \mathcal{R}[(y')^2 - yy''] = K$$

which was coupled with the three point conditions

$$y(0) = y'(1) = y''(0) = 0$$

where the constant of integration $K$ is to be determined from the remaining boundary condition $y(1) = 1$. There is a minority of authors who have analyzed the problem as equivalent third and fourth order BVPs (and, even fifth order, on occasion), however the attention on the third-order problem mostly dominates the scientific discussion therein. Thus we can see that a focus on the equivalent fourth order BVP in the extent literature has not been prevalent. This may have been due to the authors therein favouring lower order problems perhaps due to a perception that its form is more agreeable to work with and seeing its potential to open up interesting avenues. The continued focus on the third-order form of the BVP seen in the literature may also be partly due to human nature and the act of conditioning — I tend to see and continue to work with the mathematical forms that I have been conditioned and accustomed to.

In contrast, herein I take the position that the fourth order BVP (6.1), (6.2) presents a natural form to work with. For example, the form enables a complete integration between the differential equation and the boundary conditions, synthesizing the data from the problem as an integral equation. This is in contrast to third order approaches where there are constants of integration in the equation and a fourth "hanging" boundary condition to consider. In addition, I in the previous Chapter [21] have advanced the mathematical theory regarding solutions to fourth order BVPs in directions that potentially can shine new light on (6.1), (6.2) and so I feel that this presents a timely opportunity to directly work with the form of the fourth order BVP (6.1), (6.2).

Extent mathematical methods regarding laminar flow in channels with porous walls can be broadly grouped into: perturbation techniques; asymptotic approaches; numerical and initial value methods; and fixed point techniques with differential inequalities. The above approaches have enabled a deeper understanding of (6.1), (6.2) through: a development of series solutions [45, 221, 257, 258, 281]; fostering the existence and uniqueness of solutions [112, 180, 243, 272]; and furnishing multiple solutions [100, 112, 226] for various values of $\mathcal{R}$. In particular, the dominant approach for the existence of solutions via fixed point theory has involved topological ideas, such as the Leray–Schauder degree theory. This has been subsequently coupled with uniqueness (or nonmultiplicity) concepts involving differential inequalities and then separate approximation methods are drawn on to gain additional insight. In comparison, in this Chapter I introduce contraction mapping ideas in what appears to be a first time synthesis and application to the problem of laminar flow in channels with porous walls. There are several advantages in this synthesis. Firstly, a contractive mapping approach forms an integrated strategy towards existence, uniqueness and approximation of solutions by its very nature. Secondly, this synthesis does not depend on whether $\mathcal{R}$ is positive or negative (unlike some previous approaches that concentrate on either suction or injection). Together, my synthesis offers a more integrated approach than previously developed strategies regarding the existence, uniqueness and numerical aspects of solutions.

Most importantly, my employment of contractive mappings enables an extension of previous results. While the case $\mathcal{R} < 0$ has been shown to possess a unique solution, the case $\mathcal{R} > 0$ is far more open, with the best range for the existence and uniqueness set in [272] at

$$0 < \mathcal{R} < \frac{-(72\sqrt{3}+1) + \sqrt{(72\sqrt{3}+1)^2 + 12\sqrt{3}(72\sqrt{3}-24)}}{48(3\sqrt{3}-1)} \approx 4.005014 \times 10^{-2}.$$

I extend this range herein by at least an order of magnitude.

My results complement the recent and growing body of knowledge regarding the theory and applications of Navier–Stokes equations [211, 245], laminar flow [29, 260, 270, 300] and swirling flow [299] by establishing a firm mathematical foundation for the problem (6.1), (6.2).

My Chapter is organized as follows. In Section 6.2 I briefly derive the problem (6.1), (6.2) with aims of completeness and context for my work, and to enable a comparison between the form of my equations and those that have been previously analyzed. Furthermore, I construct an integral equation that is equivalent to (6.1), (6.2) that will form the basis of my contractive mapping approach. In Section 6.3 I establish new bounds on integrals of various Green's functions associated with (6.1), (6.2). Some of the estimates therein are sharp and they prove to be useful when developing my main existence, uniqueness and approximation results in Section 6.4. Therein I establish the main results drawing on an approach involving contractive mappings and fixed point theory.

## 6.2 Formulation of the problem of laminar flow

In this Section I briefly derive the equations of interest, drawing on the ideas and exposition of Berman [45] and Robinson [226]. Further details may be found therein and in [221, 256, 257, 258, 281].

Consider a channel with a rectangular cross section. One side of the cross section that represents the distance between the porous walls is much smaller than the other, and this constraint enables an analysis of the problem as an instance of two-dimensional flow.

Furthermore, consider the steady, incompressible, laminar flow where the fluid is subject to either injection or suction with constant velocity $\mathcal{V}$ through the walls. I assume that both channel walls have equal permeability.

We choose a coordinate system so that its origin is placed at the centre of the channel. Let $\xi$ and $\chi$ denote the co-ordinate axes that are, respectively, parallel and perpendicular to the channel walls, and let $u = u(\xi, \chi)$ and $v = v(\xi, \chi)$ denote the velocity components in the $\xi$ and $\chi$ directions, respectively. Let the width of the channel (ie, the distance between the walls) be $2h$ and let the channel have length $L$.

Let $p = p(\xi, \chi)$ denote the pressure that we assume is a sufficiently smooth function. Let $\rho$ denote the density of the fluid and let $\nu$ denote the constant kinematic viscosity of the fluid. Under the assumed conditions and choice of axes, we introduce the dimensionless variable

$$x = \frac{\chi}{h}$$

and then the Navier–Stokes equations can be expressed as

$$u\frac{\partial u}{\partial \xi} + \frac{v}{h}\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial \xi} + \nu\left(\frac{\partial^2 u}{\partial \xi^2} + \frac{1}{h^2}\frac{\partial^2 u}{\partial x^2}\right)$$
$$u\frac{\partial v}{\partial \xi} + \frac{v}{h}\frac{\partial v}{\partial x} = -\frac{1}{h\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v}{\partial \xi^2} + \frac{1}{h^2}\frac{\partial^2 v}{\partial x^2}\right).$$

The continuity equation takes the form

$$\frac{\partial u}{\partial \xi} + \frac{1}{h}\frac{\partial v}{\partial x} = 0$$

and the associated boundary conditions are

$$u(\xi, \pm 1) = 0, \;\; v(\xi, 0) = 0$$
$$v(\xi, \pm 1) = \mathcal{V}, \;\; \frac{\partial u}{\partial x}(\xi, 0) = 0.$$

For a two-dimensional incompressible flow, a stream function $\psi$ exists such that

$$u(\xi, x) = \frac{1}{h}\frac{\partial \psi}{\partial x} \tag{6.3}$$
$$v(\xi, x) = -\frac{\partial \psi}{\partial \xi} \tag{6.4}$$

with the continuity equation being satisfied.

Due to a symmetrical flow about the plane lying midway between the channel walls, we will analyze the solution over half of the channel, i.e., from the midplane to one wall.

For constant wall velocity $\mathcal{V}$, Berman [45] cleverly observed that the equations of motion and the boundary conditions could be satisfied under an assumption that the velocity component $v$ is independent of $\xi$ and he skillfully introduced a stream function, $\psi$, of the form

$$\psi(\xi, x) := [h\bar{u}(0) - \mathcal{V}\xi]y(x)$$

where $y$ is a suitably smooth function of the distance parameter $x$ and $y$ is to be determined later. In addition, $\bar{u}(0)$ is an arbitrary velocity at $\xi = 0$ that will be managed away in due course.

From (6.3) and (6.4) we can derive the velocity components

$$u(\xi, x) = \left[\bar{u}(0) - \mathcal{V}\frac{\xi}{h}\right]y'(x) \tag{6.5}$$

$$v(\xi, x) = v(x) = \mathcal{V}y(x). \tag{6.6}$$

For constant wall velocity $\mathcal{V}$, the $\chi$ component of velocity $v$ becomes a function of $x$ only. If (6.5) and (6.6) are substituted into the equations of motion then we obtain

$$-\frac{1}{\rho}\frac{\partial p}{\partial \xi} = \left[\bar{u}(0) - \mathcal{V}\frac{\xi}{h}\right]\left[-\frac{\mathcal{V}}{h}[(y')^2 - yy''] - \frac{\nu}{h^2}y'''\right] \tag{6.7}$$

$$-\frac{1}{h\rho}\frac{\partial p}{\partial x} = \frac{v}{h}\frac{dv}{d\eta} - \frac{\nu}{h^2}\frac{d^2v}{dx^2}. \tag{6.8}$$

The right-hand side of (6.8) is seen to be a function of $x$ only and so differentiation with respect to $\xi$ yields

$$\frac{\partial^2 p}{\partial \xi \partial x} = 0.$$

If we now differentiate (6.7) with respect to $x$ then we obtain

$$\left[\bar{u}(0) - \mathcal{V}\frac{\xi}{h}\right]\frac{d}{dx}\left[\frac{\mathcal{V}}{h}[(y')^2 - yy''] + \frac{\nu}{h^2}y'''\right] = \frac{\partial^2 p}{\partial x \partial \xi}$$

and employing the symmetry of mixed partial derivatives of $p$ we thus obtain

$$\left[\bar{u}(0) - \mathcal{V}\frac{\xi}{h}\right]\frac{d}{dx}\left[\frac{\mathcal{V}}{h}[(y')^2 - yy''] + \frac{\nu}{h^2}y'''\right] = 0.$$

If the above equation is to hold for all $\xi$ then we must have

$$\begin{aligned} 0 &= \frac{d}{dx}\left[\frac{\mathcal{V}}{h}[(y')^2 - yy''] + \frac{\nu}{h^2}y'''\right] \\ &= y^{(iv)} + \mathcal{R}[y'y'' - yy'''] \end{aligned}$$

where

$$\mathcal{R} := \frac{\mathcal{V}h}{\nu}$$

is a Reynolds number and we have thus derived (6.1).

The boundary conditions on the function $y$ and its derivatives are obtained from (6.5) and (6.6) to produce (6.2). Note that we have $\mathcal{R} > 0$ for suction at both walls and $\mathcal{R} < 0$ for injection at both walls.

Now let me first give a definition on what I mean by a solution to (6.1), (6.2).

**Definition 6.1.** *We say $y = y(x)$ is a solution to (6.1), (6.2) , if a function $y : [0,1] \to \mathbb{R}$ such that $y$ is four times differentiable, with a continuous fourth-order derivative on $[0,1]$, which we denote by $y \in C^4([0,1])$, and our $y$ satisfies both (6.1) and (6.2) for some value of $\mathcal{R}$.*

Let me now establish an equivalency between the BVP (6.1), (6.2) and an integral equation. The integral equation will be critical in Section 6.3 to develop my main results.

**Theorem 6.1.** *The BVP* (6.1), (6.2) *is equivalent to the integral equation*

$$y(x) = \int_0^1 \mathcal{M}(x,s)\mathcal{R}\left(y'(s)y''(s) - y(s)y'''(s)\right)\,ds + \phi(x), \quad x \in [0,1]. \tag{6.9}$$

*Above:* $\mathcal{M}(x,s)$ *is a Green's function given explicitly by*

$$\mathcal{M}(x,s) := \frac{1}{12}\begin{cases} s(1-x)^2[(s^2-3)x+2s^2], & \text{for } 0 \le s \le x \le 1, \\ \\ x(1-s)^2[(x^2-3)s+2x^2], & \text{for } 0 \le x \le s \le 1; \end{cases} \tag{6.10}$$

*and* $\phi$ *is given by*

$$\phi(x) = \frac{1}{2}(3x - x^3). \tag{6.11}$$

*Proof.* It is sufficient to construct $y$ from the form

$$y(x) = \phi_1(x) + \phi(x)$$

where $\phi$ is the solution to

$$\phi^{(iv)} = 0; \;\; \phi(0) = 0, \; \phi''(0) = 0, \; \phi(1) = 1, \; \phi'(1) = 0;$$

and $\phi_1$ is the solution to

$$\phi_1^{(iv)} + \mathcal{R}(\phi_1'\phi_1'' - \phi_1\phi_1''') = 0; \; \phi_1(0) = 0, \; \phi_1''(0) = 0, \; \phi_1(1) = 0, \; \phi_1'(1) = 0.$$

Direct integration and determination of the associated constants shows that

$$\phi(x) = \frac{1}{2}(3x - x^3).$$

Integrate both sides of the differential equation for $\phi_1$ from $s = 0$ to $s = x$ four times to obtain

$$\phi_1(x) = -\frac{1}{6}\int_0^x (x-s)^3\mathcal{R}\left(\phi_1'(s)\phi_1''(s) - \phi_1(s)\phi_1'''(s)\right)\,ds + A\eta^3 + B\eta^2 + Cx + D. \tag{6.12}$$

and we determine the constants of integration $A, B, C, D$ from the homogeneous boundary conditions for $\phi_1$. Our left-hand conditions $\phi_1(0) = 0$ and $\phi_1''(0) = 0$ ensure $D = 0$ and $B = 0$, respectively. In addition, employing the right-hand conditions, we obtain

$$\phi_1(1) = 0 = -\frac{1}{6}\int_0^1 (1-s)^3\mathcal{R}\left(\phi_1'(s)\phi_1''(s) - \phi_1(s)\phi_1'''(s)\right)\,ds + A + C$$

$$\phi_1'(1) = 0 = -\frac{1}{2}\int_0^1 (1-s)^2 \mathcal{R}\left(\phi_1'(s)\phi_1''(s) - \phi_1(s)\phi_1'''(s)\right)\, ds + 3A + C.$$

Solving the above system of equations for $A$ and $C$ we obtain

$$\begin{aligned}
A &= \frac{1}{12}\left[\int_0^1 \left[3(1-s)^2 - (1-s)^3\right]\mathcal{R}\left(\phi_1'(s)\phi_1''(s) - \phi_1(s)\phi_1'''(s)\right)\, ds\right] \\
&= \frac{1}{12}\left[\int_0^x (1-s)^2(s+2)\mathcal{R}\left(\phi_1'(s)\phi_1''(s) - \phi_1(s)\phi_1'''(s)\right)\, ds\right. \\
&\quad \left. + \int_x^1 (1-s)^2(s+2)\mathcal{R}\left(\phi_1'(s)\phi_1''(s) - \phi_1(s)\phi_1'''(s)\right)\, ds\right], \\
C &= \frac{1}{6}\int_0^1 (1-s)^3\mathcal{R}\left(\phi_1'(s)\phi_1''(s) - \phi_1(s)\phi_1'''(s)\right)\, ds - A \\
&= \frac{1}{12}\left[\int_0^x (1-s)^2(-3s)\mathcal{R}\left(\phi_1'(s)\phi_1''(s) - \phi_1(s)\phi_1'''(s)\right)\, ds\right. \\
&\quad \left. + \int_x^1 (1-s)^2(-3s)\mathcal{R}\left(\phi_1'(s)\phi_1''(s) - \phi_1(s)\phi_1'''(s)\right)\, ds\right].
\end{aligned}$$

Substituting these expressions into (6.12) and applying some algebraic manipulation finally leads us to the form (6.9).

Direct differentiation of our $y$ with the aforementioned values of $A$ and $C$ lead us to the differential equation (6.1). Substitution of appropriate values of $x$ into (6.1) and its derivatives reveals that the boundary conditions (6.2) also hold. □

## 6.3 Estimates of integrals of Green's functions

In this Section I establish some new bounds involving the integral of the Green's function in (6.10) and its derivatives. The results will be applied in Section 6.4 to form my main existence, uniqueness and approximation results. In addition, the bounds are of independent mathematical interest as they have the potential to be helpful outside the scope of the present Chapter, for example, in topological approaches to BVPs.

My first result establishes the nonpositivity of $\mathcal{M}$ and a new, sharp bound on the integral of $|\mathcal{M}|$.

**Theorem 6.2.** *The Green's function $\mathcal{M}$ in (6.10) satisfies $\mathcal{M} \le 0$ on $[0,1] \times [0,1]$ and*

$$\int_0^1 |\mathcal{M}(x,s)|\, ds \le \frac{39 + 55\sqrt{33}}{65536} < \frac{3}{500} =: \iota_0, \text{ for all } x \in [0,1]. \tag{6.13}$$

*Our estimate is sharp in the sense it is the best result possible.*

*Proof.* For $0 \le s \le x \le 1$ we have

$$(s^2 - 3)x + 2s^2 = s^2(x+2) - 3x \le x^2(x+2) - 3x \le 0$$

and so

$$s(1-x)^2[(s^2-3)x+2s^2] \le 0$$

therein. Similarly, for $0 \le x \le s \le 1$ we have

$$(x^2-3)s+2x^2 \le s^2(s+2)-3s \le 0$$

and the nonpositivity of $\mathcal{M}$ thus also holds on this region.

Combining the above two cases we obtain $\mathcal{M} \le 0$ on $[0,1]\times[0,1]$.

For all $x \in [0,1]$ consider

$$\begin{aligned}
\int_0^1 |\mathcal{M}(x,s)|\, ds &= -\int_0^1 \mathcal{M}(x,s)\, ds \\
&= -\frac{1}{12}\left[\int_0^x s(1-x)^2[(s^2-3)x+2s^2]\, ds + \int_x^1 x(1-s)^2[(x^2-3)s+2x^2]\, ds\right] \\
&= \frac{x^4}{24} - \frac{x^3}{16} + \frac{x}{48} \\
&= \frac{1}{48}x(2x+1)(x-1)^2.
\end{aligned}$$

If we apply calculus to the above quartic function then we see that it achieves its maximum value on $[0,1]$ at

$$x^* = \frac{1+\sqrt{33}}{16}$$

which may be substituted into the above quartic function to obtain

$$\begin{aligned}
\max_{x\in[0,1]} \int_0^1 |\mathcal{M}(x,s)|\, ds &= \int_0^1 |\mathcal{M}(x^*,s)|\, ds \\
&= \frac{39+55\sqrt{33}}{65536} < \frac{3}{500}.
\end{aligned}$$

□

My second result complements Theorem 6.2 by generating a new bound on the integral of $|\partial\mathcal{M}/\partial x|$.

**Theorem 6.3.** *The Green's function $\mathcal{M}$ in (6.10) satisfies*

$$\int_0^1 \left|\frac{\partial}{\partial x}\mathcal{M}(x,s)\right|\, ds < \frac{1}{25} =: \iota_1, \text{ for all } x \in [0,1]. \tag{6.14}$$

*Proof.* For all $x \in [0,1]$ consider

$$\int_0^1 \left|\frac{\partial}{\partial x}\mathcal{M}(x,s)\right|\, ds = \int_0^x \left|\frac{(x-1)s((s^2-3)x+s^2+1)}{4}\right|\, ds + \int_x^1 \left|\frac{(1-s)^2((x^2-1)s+2x^2)}{4}\right|\, ds$$

$$
\begin{aligned}
&\le \frac{1}{4}\left[\int_0^x (1-x)s(-(s^2-3)x+s^2+1)\,ds\right] \\
&+ \frac{1}{4}\left[\int_x^1 (1-s)^2(-(x^2-1)s+2x^2)\,ds\right] \\
&= -\frac{7}{36}x^5 + \frac{1}{3}x^4 - \frac{1}{3}x^3 + \frac{25}{144}x^2 + \frac{1}{48} \\
&= \frac{1}{144}(1-x)(28x^4-20x^3+28x^2+3x+3).
\end{aligned}
$$

Now, if we apply calculus to the above quintic function then we see that it achieves its maximum value on $[0,1]$ at

$$x^* = \frac{(13378+70\sqrt{94137})^{2/3} + 32(13378+70\sqrt{94137})^{1/3} - 656}{70(13378+70\sqrt{94137})^{1/3}}$$

which may be substituted into the above quintic function to obtain

$$
\begin{aligned}
\max_{x\in[0,1]} \int_0^1 \left|\frac{\partial}{\partial x}\mathcal{M}(x,s)\right| ds &\le \frac{(-9228485\sqrt{94137}+14747147607)(13378+70\sqrt{94137})^{1/3}}{19373188800000} \\
&+ \frac{(-2111935\sqrt{94137}+317136861)(13378+70\sqrt{94137})^{2/3}}{19373188800000} \\
&+ \frac{13592477}{360150000} \\
&< \frac{1}{25}.
\end{aligned}
$$

□

My third result constructs a new bound on the integral of $|\partial^2\mathcal{M}/\partial x^2|$.

**Theorem 6.4.** *The Green's function* $\mathcal{M}$ *in* (6.10) *satisfies*

$$\int_0^1 \left|\frac{\partial^2}{\partial x^2}\mathcal{M}(x,s)\right| ds \le \frac{9}{8} =: \iota_2, \text{ for all } x \in [0,1]. \tag{6.15}$$

*Proof.* For all $x \in [0,1]$ consider

$$
\begin{aligned}
\int_0^1 \left|\frac{\partial^2}{\partial x^2}\mathcal{M}(x,s)\right| ds &= \int_0^x \left|\frac{s((s^2-3)x+2)}{2}\right| ds + \int_x^1 \left|\frac{x(s^3-3s+2)}{2}\right| ds \\
&\le \int_0^x \frac{s(-(s^2-3)x+2)}{2}\,ds + \int_x^1 \frac{x(s^3-3s+2)}{2}\,ds \\
&= -\frac{1}{4}\left(x^4-6x^2+2x-\frac{3}{2}\right)x.
\end{aligned}
$$

The above quintic function is strictly increasing on $[0,1]$ and thus must achieve its maximum value on $[0,1]$ at $x^* = 1$ which gives

$$\max_{x\in[0,1]} \int_0^1 \left|\frac{\partial^2}{\partial x^2}\mathcal{M}(x,s)\right| ds \le \frac{9}{8}.$$

□

My final result constructs a new, sharp bound on the integral of $|\partial^3\mathcal{M}/\partial x^3|$.

**Theorem 6.5.** *The Green's function* $\mathcal{M}$ *in* (6.10) *satisfies*

$$\int_0^1 \left| \frac{\partial^3}{\partial x^3}\mathcal{M}(x,s) \right| ds \le \frac{5}{8} =: \iota_3, \text{ for all } x \in [0,1]. \tag{6.16}$$

*Our estimate is sharp in the sense it is the best result possible.*

*Proof.* For all $x \in [0,1]$ consider

$$\begin{aligned} \int_0^1 \left| \frac{\partial^3}{\partial x^3}\mathcal{M}(x,s) \right| ds &= \int_0^x \left| \frac{s(s^2-3)}{2} \right| ds + \int_x^1 \left| \frac{(1-s)^2(s+2)}{2} \right| ds \\ &= -\frac{1}{2}\int_0^x s(s^2-3)\, ds + \frac{1}{2}\int_x^1 (1-s)^2(s+2)\, ds \\ &= -\frac{1}{4}x^4 + \frac{3}{2}x^2 + \frac{3}{8} - x. \end{aligned}$$

The above function is increasing on $[0,1]$ and so must achieve its maximum value on $[0,1]$ at $x^* = 1$. Thus, we have

$$\max_{x\in[0,1]} \int_0^1 \left| \frac{\partial^3}{\partial x^3}\mathcal{M}(x,s) \right| ds = \left[ \int_0^1 \left| \frac{\partial^3}{\partial x^3}\mathcal{M}(x,s) \right| ds \right]_{x=1} = \frac{5}{8}$$

as claimed. □

## 6.4 Existence results via Banach fixed point theorem

In this Section I formulate my main results regarding existence, uniqueness and approximation of solutions via fixed point methods under contraction mappings.

But let me first give some lemmas and notations, which I will use on my main results.

My analysis will be set within a complete, normed linear space, known as a Banach space, and so I consider the pair $(Y, \varrho) := (C^3([a,b]), d)$, where the constants $W_i$ in my $d$ in (1.22) are chosen to form

$$d(y,z) := \max\left\{ d_0(y,z),\ \frac{3}{20}d_0(y',z'),\ \frac{2}{375}d_0(y'',z''),\ \frac{3}{625}d_0(y''',z''') \right\}, \tag{6.17}$$

that is

$$W_0 = 1,\ W_1 = \frac{\iota_0}{\iota_1} = \frac{3}{20},\ W_2 = \frac{\iota_0}{\iota_2} = \frac{2}{375},\ W_3 = \frac{\iota_0}{\iota_3} = \frac{6}{625}, \tag{6.18}$$

where constants $\iota_i$ are defined in (6.13), (6.14), (6.15), (6.16). My pair forms a complete metric space

Now let $R > 0$ be a constant and let $\phi$ be defined in (6.11). My analysis will involve the following set

$$\begin{aligned}\Upsilon := \Big\{&(x,u,v,w,z) \in \mathbb{R}^5 : x \in [0,1],\ |u - \phi(x)| \le R,\\ &|v - \phi'(x)| \le \frac{20}{3}R,\ |w - \phi''(x)| \le \frac{375}{2}R,\ |z - \phi'''(x)| \le \frac{625}{6}R\Big\}.\end{aligned} \tag{6.19}$$

We note that our $\phi$ in (6.11) satisfies the following inequalities on $[0,1]$:

$$|\phi| \le 1,\ |\phi'| \le 3/2,\ |\phi''| \le 3,\ |\phi'''| \le 3. \tag{6.20}$$

The following result establishes a critically important bound on parts of (6.1) and will be used in the proof of my main results. In particular, this bound will be of importance in establishing an invariance condition for a mapping between two balls.

**Lemma 6.1.** *Let*

$$h(x,u,v,w,z) := \mathcal{R}(vw - uz). \tag{6.21}$$

*Then $h$ is bounded on $\Upsilon$ by*

$$M := |\mathcal{R}|\left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]. \tag{6.22}$$

*Proof.* For $(x,u,v,w,z) \in \Upsilon$ consider

$$\begin{aligned}|h(u,v,w,z)| &= |\mathcal{R}(vw - uz)|\\ &\le |\mathcal{R}|\big(|v|\,|w| + |u|\,|z|\big)\\ &= |\mathcal{R}|\big[|(v - \phi'(x) + \phi'(x)|)(|w - \phi''(x) + \phi''(x)|)\\ &+ (|u - \phi(x) + \phi(x)|)(|z - \phi'''(x) + \phi'''(x)|)\big]\\ &\le |\mathcal{R}|\big[|(v - \phi'(x)| + |\phi'(x)|)(|w - \phi''(x)| + |\phi''(x)|)\\ &+ (|u - \phi(x)| + |\phi(x)|)(|z - \phi'''(x)| + |\phi'''(x)|)\big]\\ &\le |\mathcal{R}|\left[\left(\frac{20}{3}R + \frac{3}{2}\right)\left(\frac{375}{2}R + 3\right) + (R+1)\left(\frac{625}{6}R + 3\right)\right]\\ &= |\mathcal{R}|\left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right].\end{aligned}$$

Above, we have repeatedly applied the triangle inequality and used the form of $\Upsilon$ and (6.20). $\square$

Unfortunately, the function $h$ is not globally Lipschitz in the sense of (6.23) on the whole of $[0,1] \times \mathbb{R}^4$. A global Lipschitz state is a desirable condition in the theory and application of differential equations. However, by strategically restricting our attention to the subset $\Upsilon$, the following result ensures that our $h$ will be Lipschitz therein.

**Lemma 6.2.** *Let*

$$h(x,u,v,w,z) := \mathcal{R}(vw - uz).$$

*For any fixed $R > 0$ our $h$ is Lipschitz on $\Upsilon$ in the sense that there are nonnegative constants $L_i$ (not all zero) such that*

$$|h(x,u_0,u_1,u_2,u_3) - h(x,v_0,v_1,v_2,v_3)| \le \sum_{i=0}^{3} L_i |u_i - v_i|$$
$$\text{for all } (x,u_0,u_1,u_2,u_3),(x,v_0,v_1,v_2,v_3) \in \Upsilon. \tag{6.23}$$

*Proof.* It is sufficient to show that $h$ has bounded partial deriatives on $\Upsilon$. As we will see, these bounds can then act as the Lipschitz constants $L_i$.

For all $(x,u,v,w,z) \in \Upsilon$ consider

$$\begin{aligned}
\left|\frac{\partial h}{\partial u}\right| &= |-\mathcal{R}z| \\
&= |\mathcal{R}||z - \phi'''(x) + \phi'''(x)| \\
&\le |\mathcal{R}|[|z - \phi'''(x)| + |\phi'''(x)|] \\
&\le |\mathcal{R}|\left[\frac{625}{6}R + 3\right] =: L_0. \qquad (6.24)
\end{aligned}$$

Also, we can also obtain the following inequalities on $\Upsilon$ via similar arguments

$$\left|\frac{\partial h}{\partial v}\right| \le |\mathcal{R}|\left[\frac{375}{2}R + 3\right] =: L_1 \tag{6.25}$$
$$\left|\frac{\partial h}{\partial w}\right| \le |\mathcal{R}|\left[\frac{20}{3}R + \frac{3}{2}\right] =: L_2 \tag{6.26}$$
$$\left|\frac{\partial h}{\partial z}\right| \le |\mathcal{R}|\left[R + 1\right] =: L_3. \tag{6.27}$$

By the fundamental theorem of calculus we have

$$\begin{aligned}
h(u_0,u_1,u_2,u_3) - h(v_0,v_1,v_2,v_3) &= \int_{v_0}^{u_0} \frac{\partial h}{\partial s}(s,v_1,v_2,v_3)\, ds + \int_{v_1}^{u_1} \frac{\partial h}{\partial t}(u_0,t,v_2,v_3)\, dt \\
&\quad + \int_{v_2}^{u_2} \frac{\partial h}{\partial q}(u_0,u_1,q,v_3)\, dq + \int_{v_3}^{u_3} \frac{\partial h}{\partial p}(u_0,u_1,u_2,p)\, dp
\end{aligned}$$

and since all partial derivatives are bounded on $\Upsilon$ we thus have

$$|h(u_0,u_1,u_2,u_3) - h(v_0,v_1,v_2,v_3)| \le \sum_{i=0}^{3}\left|\int_{v_i}^{u_i} L_i\, dv\right| = \sum_{i=0}^{3} L_i|u_i - v_i|.$$

□

### Main Results: Contraction mapping approach

I am now in a position to synthesize my previous results to form my main results. In this Section I establish my novel result for the existence and uniqueness of solutions to the BVP (6.1), (6.2) via Banach's fixed point theorem (Theorem 1.5). It involves sufficient conditions under which a mapping will admit a unique fixed point, and generates a sequence that converges to this fixed point.

**Theorem 6.6.** *If there is an $R > 0$ and $\mathcal{R}$ such that*

$$|\mathcal{R}|\left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right] \leq R \tag{6.28}$$

$$|\mathcal{R}|\left[\frac{65}{4}R + \frac{4901}{2000}\right] < 1 \tag{6.29}$$

*then the BVP* (6.1)*,* (6.2) *admits a unique solution $y$ with*

$$(x, y(x), y'(x), y''(x), y'''(x)) \in \Upsilon, \quad \textit{for all } x \in [0,1].$$

*Proof.* To avoid the repeated use of complicated constants and expressions we will draw on the notation defined earlier in this Chapter. Let the constants $\iota_i$ be defined in (6.13), (6.14), (6.15), (6.16). Let the function $h$ be defined in (6.21). Let the constants $L_i$ be defined in (6.24), (6.25), (6.26), (6.27) and let $M$ be defined in (6.22).

Choose $R > 0$ to form $\Upsilon$ where $R$ and $\mathcal{R}$ satisfy (6.28) and (6.29).

Based on the form (6.9), we define the operator $\mathcal{W} : C^3([0,1]) \to C^3([0,1])$ by

$$(\mathcal{W}y)(x) := \int_0^1 \mathcal{M}(x,s)\mathcal{R}(y'(s)y''(s) - y(s)y'''(s))\, ds + \phi(x), \quad x \in [0,1].$$

Consider the pair $(C^3([0,1]), d)$ where $d$ is defined in (6.17). Our pair forms a complete metric space.

Now, for the constant $R > 0$ and function $\phi$ in the definition of $\Upsilon$, consider the following set $\Upsilon_R \subset C^3([0,1])$

$$\Upsilon_R := \{y \in C^3([0,1]) : d(y,\phi) \leq R\}.$$

Since $\Upsilon_R$ is a closed subspace of $C^3([0,1])$, the pair $(\Upsilon_R, d)$ forms a complete metric space.

Consider the operator $\mathcal{W} : \Upsilon_R \to C^3([0,1])$ where we have restricted its domain. We wish to show that there exists a unique $y \in \mathcal{B}_R$ such that

$$\mathcal{W}y = y$$

which is equivalent to proving the BVP (6.1), (6.2) has a unique solution in $\Upsilon_R$. (Any solutions lying in $C^3([0,1])$ will also lie in $C^4([0,1])$ as repeatedly differentiating (6.9) will show.)

To prove that our $\mathcal{W}$ has a unique fixed point in $\Upsilon_R$, we show that the assumptions of Theorem 1.5 hold with $Y = \Upsilon_R$.

Let me show the invariance condition $\mathcal{W} : \Upsilon_R \to \Upsilon_R$ holds. For $y \in \Upsilon_R$ and $x \in [0,1]$, consider

$$\begin{aligned}|(\mathcal{W}y)(x) - \phi(x)| &\leq \int_0^1 |\mathcal{M}(x,s)|\, |\mathcal{R}(y'(s)y''(s) - y(s)y'''(s))|\, ds \\ &\leq M \int_0^1 |\mathcal{M}(x,s)|\, ds \\ &\leq M\iota_0 \\ &= |\mathcal{R}| \left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]\frac{3}{500}.\end{aligned}$$

Similarly,

$$\begin{aligned}|(\mathcal{W}y)'(x) - \phi'(x)| &\leq \int_0^1 \left|\frac{\partial}{\partial x}\mathcal{M}(x,s)\right|\, |\mathcal{R}(y'(s)y''(s) - y(s)y'''(s)))|\, ds \\ &\leq M \int_0^1 \left|\frac{\partial}{\partial x}\mathcal{M}(x,s)\right|\, ds \\ &\leq M\iota_1 \\ &= |\mathcal{R}| \left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]\frac{1}{25}.\end{aligned}$$

Thus

$$\begin{aligned}\frac{\iota_0}{\iota_1}|(\mathcal{W}y)'(x) - \phi'(x)| &\leq M\iota_0 \\ &= |\mathcal{R}| \left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]\frac{3}{500}.\end{aligned}$$

In addition, via similar arguments, we can obtain

$$\begin{aligned}|(\mathcal{W}y)''(x) - \phi''(x)| &\leq M\iota_2, \\ &= |\mathcal{R}| \left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]\frac{9}{8}; \\ |(\mathcal{W}y)'''(x) - \phi'''(x)| &\leq M\iota_3 \\ &= |\mathcal{R}| \left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]\frac{5}{8};\end{aligned}$$

so that

$$\frac{\iota_0}{\iota_2}|(\mathcal{W}y)''(x) - \phi''(x)| \leq M\iota_0 = |\mathcal{R}| \left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]\frac{3}{500};$$

$$\frac{\iota_0}{\iota_3}|(\mathcal{W}y)'''(x) - \phi'''(x)| \leq M\iota_0 = |\mathcal{R}|\left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]\frac{3}{500}.$$

Thus, for all $y \in \Upsilon_R$ we have

$$\begin{aligned} d(\mathcal{W}y, \phi) &\leq \max\{M\iota_0, M\iota_0, M\iota_0, M\iota_0\} \\ &= M\iota_0 \\ &= |\mathcal{R}|\left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]\frac{3}{500} \\ &\leq R \end{aligned}$$

by assumption (6.28). Hence, for all $y \in \Upsilon_R$ we have $\mathcal{W}y \in \Upsilon_R$ so that $\mathcal{W} : \Upsilon_R \to \Upsilon_R$.

Let me now show that $\mathcal{W}$ is contractive on $\Upsilon_R$ with respect to $d$. For $y, z \in \Upsilon_R$ and $x \in [0,1]$, consider

$$\begin{aligned} &|(\mathcal{W}y)(x) - (\mathcal{W}z)(x)| \\ &\leq \int_0^1 |\mathcal{M}(x,s)|\,|h(y(s), y'(s), y''(s), y'''(s)) - h(z(s), z'(s), z''(s), z'''(s))|\,ds \\ &\leq \int_a^b |\mathcal{M}(x,s)| \left(\sum_{i=0}^{3} L_i\,|y^{(i)}(s) - z^{(i)}(s)|\right) ds \\ &\leq \iota_0\left(L_0 d_0(y,z) + L_1 d_0(y',z') + L_2 d_0(y'',z'') + L_3 d_0(y''',z''')\right) \\ &\leq \iota_0\left(L_0 d(y,z) + L_1\frac{\iota_1}{\iota_0}d(y,z) + L_2\frac{\iota_2}{\iota_0}d(y,z) + L_3\frac{\iota_3}{\iota_0}d(y,z)\right) \\ &= (L_0\iota_0 + L_1\iota_1 + L_2\iota_2 + L_3\iota_3)d(y,z) \\ &= |\mathcal{R}|\left[\frac{65}{4}R + \frac{4901}{2000}\right]d(y,z) \end{aligned}$$

where we have applied Lemma 6.2.

Similarly, we can show

$$\begin{aligned} |(\mathcal{W}y)'(x) - (\mathcal{W}z)'(x)| &\leq \iota_1\left(L_0 + L_1\frac{\iota_1}{\iota_0} + L_2\frac{\iota_2}{\iota_0} + L_3\frac{\iota_3}{\iota_0}\right)d(y,z); \\ |(\mathcal{W}y)''(x) - (\mathcal{W}z)''(x)| &\leq \iota_2\left(L_0 + L_1\frac{\iota_1}{\iota_0} + L_2\frac{\iota_2}{\iota_0} + L_3\frac{\iota_3}{\iota_0}\right)d(y,z); \\ |(\mathcal{W}y)'''(x) - (\mathcal{W}z)'''(x)| &\leq \iota_3\left(L_0 + L_1\frac{\iota_1}{\iota_0} + L_2\frac{\iota_2}{\iota_0} + L_3\frac{\iota_3}{\iota_0}\right)d(y,z). \end{aligned}$$

Thus, for all $y, z \in \Upsilon_R$ we have

$$d(\mathcal{W}y, \mathcal{W}z)$$

$$
\begin{aligned}
&= \max\left\{ d_0(\mathcal{W}y, \mathcal{W}z),\ \frac{\iota_0}{\iota_1} d_0((\mathcal{W}y)', (\mathcal{W}z)'),\ \frac{\iota_0}{\iota_2} d_0((\mathcal{W}y)'', (\mathcal{W}z)''), \frac{\iota_3}{\iota_0} d_0((\mathcal{W}y)''', (\mathcal{W}z)''') \right\} \\
&\leq (L_0\iota_0 + L_1\iota_1 + L_2\iota_2 + L_3\iota_3) d(y, z) \\
&= |\mathcal{R}| \left[ \frac{65}{4} R + \frac{4901}{2000} \right] d(y, z).
\end{aligned}
$$

Due to my assumption (6.29) we see that $\mathcal{W}$ is a contractive map on $\Upsilon_R$.

Hence all of the conditions of Theorem 1.5 hold with $Y = \Upsilon_R$. Theorem 1.5 is applicable and yields the existence of a unique fixed point to $\mathcal{W}$ that lies in $\mathcal{B}_R \subset C^3([0,1])$. This solution is also in $C^4([0,1])$ as can be verified by differentiating the integral equation (6.9). Thus we have equivalently shown that the BVP (6.1), (6.2) has a unique solution. $\square$

The question remains: when do the constraints (6.28) and (6.29) hold? The following result addresses this question by choosing a $R > 0$ that maximizes $\mathcal{R}$.

**Theorem 6.7.** *For all*

$$
|\mathcal{R}| < \frac{2000\sqrt{65}}{19500 + 4901\sqrt{65}} \approx 0.2732360884,
$$

*the BVP* (6.1), (6.2) *has a unique solution lying in* $\Upsilon$ *with*

$$
R = 3\frac{\sqrt{65}}{325} \approx 0.07442084075.
$$

*Proof.* Note that (6.28) and (6.29) are equivalent to

$$
|\mathcal{R}| \leq \frac{R}{\left[\frac{8125}{6}R^2 + \frac{4901}{12}R + \frac{15}{2}\right]\frac{3}{500}} \tag{6.30}
$$

$$
|\mathcal{R}| < \frac{1}{\left[\frac{65}{4}R + \frac{4901}{2000}\right]}. \tag{6.31}
$$

The two curves of the functions of $R$ that make up the right-hand sides of the inequalities (6.30) and (6.31) intersect at

$$
R = 3\frac{\sqrt{65}}{325} \approx 0.07442084075.
$$

The value of these functions at their point of intersection is

$$
\frac{2000\sqrt{65}}{19500 + 4901\sqrt{65}} \approx 0.2732360884 \tag{6.32}
$$

and so, for values of $\mathcal{R}$ strictly less than (6.32), both of our inequalities (6.28) and (6.29) will hold. Thus, for these values of $R$ and $\mathcal{R}$ the conclusion of Theorem 6.6 holds. $\square$

**Remark 6.1.** *The range*

$$|\mathcal{R}| < \frac{2000\sqrt{65}}{19500 + 4901\sqrt{65}} \approx 2.732360884 \times 10^{-1}$$

*in Theorem 6.7 improves the result in [272] for $\mathcal{R} > 0$, which established the existence of a unique solution for*

$$0 < \mathcal{R} < \frac{-(72\sqrt{3}+1) + \sqrt{(72\sqrt{3}+1)^2 + 12\sqrt{3}(72\sqrt{3}-24)}}{48(3\sqrt{3}-1)} \approx 4.005014 \times 10^{-2}.$$

*We observe that our upper limit for $\mathcal{R}$ is at least an order of magnitude higher than the result in [272].*

**Remark 6.2.** *Due to the rather small value of $R$ in Theorem 6.7, the result can be interpreted as establishing the existence of a solution that uniquely lies within a thin strip, where the graph of function $\phi$ lies at the centre, and*

$$\left| y(x) - \frac{1}{2}(3x - x^3) \right| \leq 3\frac{\sqrt{65}}{325}, \quad \text{for all } x \in [0,1].$$

*Part of the signficance with the small value of $R$ can be related to the location of our solution. For small $R$ we know that our solution cannot deviate "too much" from the known function $\phi$.*

**Remark 6.3.** *Note that the conclusions of Theorem 6.6 and Theorem 6.7 say nothing about what might happen outside of the set $\Upsilon$. Additional solutions may exist whose graphs are not completely contained in $\Upsilon$.*

Let me now pivot our attention to examine the approximation of solutions to (6.1), (6.2). The following results involve Picard iterants [266, Sec. 2] that will form approximations to the unique solution $y$ of the BVP (6.1), (6.2). The following approximation results are a consequence of Theorem 1.5 holding for the operator $\mathcal{W}$ therein, see [294, Theorem 1.A].

**Remark 6.4.** *Let the conditions of Theorem 6.7 hold. If we recursively define a sequence of approximations $y_n = y_n(x)$ on $[0,1]$ via*

$$y_0(x) := \phi(x) = \frac{1}{2}(3x - x^3)$$
$$y_{n+1}(x) := \int_0^1 \mathcal{M}(x,s)\mathcal{R}(y_n'(s)y_n''(s) - y_n(s)y_n'''(s))\, ds + y_0(x), \quad n = 0, 1, 2, \ldots$$

*then:*

- *the sequence $y_n$ converges to the solution $y$ of (6.1), (6.2) with respect to the $d$ metric and the rate of convergence is given by*

$$d(y_{n+1}, y) \leq (L_0\iota_0 + L_1\iota_1 + L_2\iota_2 + L_3\iota_3)d(y_n, y)$$

$$= |\mathcal{R}| \left[ \frac{65}{4} R + \frac{4901}{2000} \right] d(y_n, y)$$

- *for each $n$, an* a priori *estimate on the error is*

$$\begin{aligned} d(y_n, y) &\leq \frac{(L_0\iota_0 + L_1\iota_1 + L_2\iota_2 + L_3\iota_3)^n}{1 - (L_0\iota_0 + L_1\iota_1 + \iota_2 L_2 + \iota_3 L_3)} d(y_1, \phi) \\ &= \frac{\left[ |\mathcal{R}| \left[ \frac{65}{4} R + \frac{4901}{2000} \right] \right]^n}{1 - |\mathcal{R}| \left[ \frac{65}{4} R + \frac{4901}{2000} \right]} d(y_1, \phi) \end{aligned}$$

- *for each $n$, an* a posteriori *estimate on the error is*

$$\begin{aligned} d(y_{n+1}, y) &\leq \frac{(L_0\iota_0 + L_1\iota_1 + L_2\iota_2 + L_3\iota_3)}{1 - (L_0\iota_0 + L_1\iota_1 + L_2\iota_2 + L_3\iota_3)} d(y_{n+1}, y_n) \\ &= \frac{|\mathcal{R}| \left[ \frac{65}{4} R + \frac{4901}{2000} \right]}{1 - |\mathcal{R}| \left[ \frac{65}{4} R + \frac{4901}{2000} \right]} d(y_1, \phi). \end{aligned}$$

**Remark 6.5.** *If we begin with $y_0$, then we can compute*

$$\begin{aligned} y_1(x) &= -\frac{x}{280} \left( \mathcal{R}x^6 + (-3\mathcal{R} + 140)x^2 + 2\mathcal{R} - 420 \right), \\ y_2(x) &= -\frac{x}{8736000} \left[ \left( x^{14} - \frac{300}{77} x^{13} - \frac{78}{11} x^{10} + \frac{390}{7} x^9 + \frac{65}{21} x^8 \right. \right. \\ &\quad \left. - \frac{780}{7} x^7 + \frac{1053}{49} x^6 - \frac{234}{7} x^4 + \frac{296027}{1617} x^2 - \frac{58496}{539} \right) \mathcal{R}^3 \\ &\quad + \left( \frac{3640}{11} x^{10} - 2600x^9 - 650x^8 + 23400x^7 - \frac{14040}{7} x^6 + 8580x^4 - \frac{6190600}{77} x^2 + \frac{4107350}{77} \right) \mathcal{R}^2 \\ &\quad \left. + 46800(x^2 - 5)(-1 + x)^2 (x + 1)^2 \mathcal{R} + 4368000x^2 - 13104000 \right]. \end{aligned}$$

*One of the advantages in my method of approximation over Terrill's [256] is that there I have no constants of integration that need to be calculated and re-calculated with every step of the process.*

# Chapter 7

# Caputo fractional IVPs

## 7.1 Introduction

In this Chapter, I consider the following fractional initial value problem (IVP) of a Caputo type of arbitrary order $\alpha > 0$,

$$D^{\alpha}\left(y - T_{\lceil\alpha\rceil-1}y\right) = f(x,y), \quad x \in [0,1], \quad \alpha > 0, \tag{7.1}$$

$$y(0) = A_0,\ y'(0) = A_1, \dots,\ y^{(\lceil\alpha\rceil-1)}(0) = A_{\lceil\alpha\rceil-1}, \tag{7.2}$$

where, $\lceil\alpha\rceil$ is the ceiling value of $\alpha$; $D^{\alpha}$ represents the Riemann–Liouville fractional differentiation operator of arbitrary order $\alpha > 0$; $T_{\lceil\alpha\rceil-1}y$ is the Taylor polynomial of degree $\lceil\alpha\rceil - 1$ of $y = y(x)$; $A_i$ are constants; and $f : S_2 \to \mathbb{R}$ is assumed to be continuous with $S_2$ being a finite strip in $\mathbb{R}^2$ defined by

$$S_2 = \left\{(x,u) \in \mathbb{R}^2 : 0 \le x \le 1 \text{ and } \left|u - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i}{i!}x^i\right| \le R\right\}, \quad R > 0.$$

In particular, when $0 < \alpha < 1$ the above IVP becomes

$$D^{\alpha}\left(y - y(0)\right) = f(x,y), \quad 0 < \alpha < 1, \tag{7.3}$$

$$y(0) = A_0, \tag{7.4}$$

where, $A_0$ is a constant and $f : S_1 \to \mathbb{R}$ is assumed to be continuous with $S_1$ being a rectangle in $\mathbb{R}^2$ defined by

$$S_1 := \{(x,u) \in \mathbb{R}^2 : 0 \le x \le 1, \text{ and } |u - A_0| \le R\}, \quad R > 0.$$

It is a well-known result that the left–hand side of (7.1) is the Caputo derivative of $y$ of order $\alpha > 0$, that is, ${}^cD^\alpha y(x) = D^\alpha(y(x) - T_{\lceil\alpha\rceil-1}y(x))$ and so I may sometimes use this notation (see Section 1.3 for more details).

The field of fractional differential equations has gained more considerable attention in recent decades due to its importance and development of the connection with scientific applications, for example see [60, 68, 106, 125, 194, 215, 236, 293]. In particular, there is a vast amount of interesting literature regarding initial and boundary value problems for fractional differential equations where a range of authors have investigated the existence, uniqueness and approximation of solutions to these problems. They have pursued a spectrum of traditional approaches to the existence and/or uniqueness of solution to these problems. This includes methods such as: Banach theorem, Schauder theorem, Schaefer theorem, Leray–Schauder degree, Leray–Schauder nonlinear alternative and Leray–Schauder continuation theorem, for example see [11, 16, 39, 55, 69, 115, 136, 137, 138, 230, 261, 269]; lower and upper solutions, for example see [40, 115, 177, 275, 297]; Picard's existence and uniqueness theorem, Peano's existence theorem, extendibility of solutions to the right, maximal intervals of existence, a Kamke type convergence theorem, and the continuous dependence of solutions on parameters, for example see [36, 76, 278, 303]; and Gronwall inequality, for example see [2, 4, 85, 121, 273, 274, 279, 287, 291]. The reader is also referred to [1, 125, 140, 153, 215, 236, 302] for some additional developments in the field of fractional differential equations and their applications.

However, despite much important work being carried out, significant gaps still exist in our state of understanding of these problems. For example, the area of fractional differential equations, particularly initial value problems for fractional differential equations, has remained sheltered from an analysis involving CMCM (Theorem 1.8) and its constructive extension. In particular, as we will discover, CMCM has the potential to open up new directions and understanding regarding the *global* existence, uniqueness and approximation of solutions to initial value problems for fractional differential equations when compared with more traditional approaches. Such ideas of global existence, uniqueness and approximation of solutions would form the bedrock to underpin advanced studies in the area, especially with respect to applications, and thus appear to be of significant interest, see [261], for example.

Therefore, the aim of this Chapter is to formulate new theorems whose proofs involve an application of the constructive version of CMCM. In particular, I show that, under a local Lipschitz condition and *a priori* bounds on solutions to a certain family of problems, the problems (7.1), (7.2) and (7.3), (7.4) will admit a unique, global solution through an application of a constructive version of CMCM.

This Chapter is organized as follows:

In Section 7.2 I consider the problem (7.3), (7.4) where I establish new global existence theory in the case when $f$ satisfies a *local* Lipschitz condition, in conjunction with *a priori* bounds on solutions to a family of problems. This is achieved by the application of CMCM. This Section provides a logical starting point for navigating Section 7.3, where the analysis moves to more complicated problems such as higher order. In particular, in Section 7.3 I consider the problem (7.1), (7.2) where I extend the ideas from Section 7.2 from the case $0 < \alpha < 1$ to arbitrary $\alpha > 0$ through similar arguments. I devote Section 7.4 to furnishing an example where I illustrate how to apply my new ideas to a concrete problem and finish my work with Section **??** where I discuss benefits, limitations and potential ways forward for future work.

## 7.2 Global existence results via continuation theorems: $0 < \alpha < 1$

In this Section I begin my study by considering the problem (7.3), (7.4) and I shall show that, under a local Lipschitz condition and *a priori* bounds on solutions to a certain family of problems, the problem (7.3), (7.4) will admit a unique, global solution. I ensure this through an application of a constructive version of CMCM (Theorem 1.8).

Let me first state the following well-known Lemma, which provides insight into the equivalency between: fractional differential and fractional integral forms.

**Lemma 7.1.** *If $f : S_1 \to \mathbb{R}$ is continuous then the initial value problem (7.3), (7.4) is equivalent to the integral equation*

$$y(x) = A_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, y(s))\, ds, \quad x \in [0,1]. \tag{7.5}$$

The following definition sheds light on what I mean by a *global* solution to my problem.

**Definition 7.1.** *We say $y = y(x)$ is a global solution to the fractional initial value problem (7.3), (7.4) (ie, on $[0,1]$) if: $D^\alpha \left(y - y(0)\right)$ is well-defined on $[0,1]$; $y = y(x)$ satisfies: (7.3) for all $x \in [0,1]$; and (7.4); with $|y(x) - A_0| \leq R$ for all $x \in [0,1]$.*

The following is my first novel result for global solutions to (7.3), (7.4). As we can see, the main sufficient assumptions involve a local Lipschitz condition on $f$ and the obtention of *a priori* bounds on a related family of fractional differential equations.

**Theorem 7.1.** *Let $f : S_1 \to \mathbb{R}$. If the following conditions are satisfied:*

**(H1)** *$f$ is continuous in $S_1$;*

**(H2)** *there exists a constant $L > 0$ such that*

$$|f(x, u_0) - f(x, v_0)| \leq L|u_0 - v_0|, \textit{for all } (x, u_0), (x, v_0) \in S_1; \tag{7.6}$$

**(H3)** *for all solutions $y_\lambda$ to the family of problems*

$$\begin{cases} D^\alpha(y - y(0)) = \lambda f(x, y), & 0 \leq x \leq 1,\ 0 < \alpha < 1,\ 0 \leq \lambda \leq 1, \\ y(0) = A_0, \end{cases} \tag{7.7}$$

*one has $|y_\lambda(x) - A_0| < R$, for all $x \in [0, 1]$ and all $\lambda \in [0, 1]$;*

*then (7.3), (7.4) has a unique global solution $y = y(x)$ with $|y(x) - A_0| < R$ for all $x \in [0, 1]$.*

*Proof.* Consider the complete metric space $(C([0, 1]), d_{\kappa_0})$ where $\kappa_0 > 0$ is to be precisely defined a little later and let $U \subset C([0, 1])$ be an open set with

$$U := \{y \in C([0, 1]) : |y(x) - A_0| < R, \quad \text{for all } x \in [0, 1]\}.$$

Define $H_\lambda(y)$ by

$$[H_\lambda(y)](x) := A_0 + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} f(s, y(s))\, ds, \quad x \in [0, 1], \quad \lambda \in [0, 1]. \tag{7.8}$$

Here

$$H_\lambda : [0, 1] \times \overline{U} \to C([0, 1]).$$

We wish to show that all of the conditions of Theorem 1.8 hold. This will then ensure that for each $\lambda \in [0, 1]$ the operator $H_\lambda$ has a unique fixed point in $U$. For the special case $\lambda = 1$ this is equivalent to ensuring the original problem (7.3), (7.4) has a unique solution.

We do this in four stages.

First we will show that if (H1) holds then $H_\lambda(y)$ is continuous in $\lambda$, uniformly for $y \in \overline{U}$. By this, we mean that given any $\varepsilon > 0$, we can choose a $\delta = \delta(\varepsilon)$ such that

$$d_{\kappa_0}\left(H_{\lambda_1}(y), H_{\lambda_2}(y)\right) < \varepsilon, \quad \text{whenever } |\lambda_1 - \lambda_2| < \delta.$$

Since $f$ is continuous on the compact rectangle $S_1$, there exists a constant $M > 0$ such that

$$|f(x,y)| \le M, \quad \text{for all} \quad (x,y) \in S_1.$$

Let $\lambda_1, \lambda_2 \in [0,1]$ and for $y \in \overline{U}$ consider

$$\begin{aligned}
d_{\kappa_0}\left(H_{\lambda_1}(y), H_{\lambda_2}(y)\right) &:= \max_{x\in[0,1]}\left[\frac{|[H_{\lambda_1}(y)](x) - [H_{\lambda_2}(y)](x)|}{E_\alpha(\kappa_0 x^\alpha)}\right] \\
&\le \max_{x\in[0,1]}\left[\frac{1}{E_\alpha(\kappa_0 x^\alpha)}\frac{|\lambda_1-\lambda_2|}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|f(s,y(s))|\,ds\right] \\
&\le \max_{x\in[0,1]}\left[\frac{1}{E_\alpha(\kappa_0 x^\alpha)}\frac{|\lambda_1-\lambda_2|}{\Gamma(\alpha)}\int_0^x M(x-s)^{\alpha-1}\,ds\right] \\
&= M|\lambda_1-\lambda_2|\max_{x\in[0,1]}\left[\frac{1}{E_\alpha(\kappa_0 x^\alpha)}\frac{x^\alpha}{\Gamma(\alpha+1)}\right] \\
&\le \frac{M}{\Gamma(\alpha+1)}|\lambda_1-\lambda_2| \\
&< \varepsilon
\end{aligned}$$

for $|\lambda_1 - \lambda_2| < \delta := \varepsilon\Gamma(\alpha+1)/M$. Thus $H_\lambda(y)$ is continuous in $\lambda$, uniformly for $y \in \overline{U}$.

Second, we will show that if (H2) is satisfied then $H_\lambda(y)$ is a contractive map in $y$ on $\overline{U}$ with respect to the metric $d_{\kappa_0}$. By this I mean that there is a constant $\sigma < 1$ such that

$$d_{\kappa_0}(H_\lambda(y), H_\lambda(z)) \le \sigma d_{\kappa_0}(y,z), \text{ for all } \lambda \in [0,1],\ (y,z) \in \overline{U}.$$

Let $L > 0$ be the constant defined in (7.6) and let $\kappa_0 := L\gamma$ where $\gamma > 1$ is an arbitrary constant.

For all $y$, $z \in \overline{U}$ and $\lambda \in [0,1]$ consider

$$\begin{aligned}
d_{\kappa_0}(H_\lambda(y), H_\lambda(z)) &:= \max_{x\in[0,1]}\frac{|[H_\lambda(y)](x) - [H_\lambda(z)](x)|}{E_\alpha(\kappa_0 x^\alpha)} \\
&\le \max_{x\in[0,1]}\left[\frac{1}{E_\alpha(\kappa_0 x^\alpha)}\frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|f(s,y(s)) - f(s,z(s))|\,ds\right] \\
&\le \max_{x\in[0,1]}\left[\frac{1}{E_\alpha(\kappa_0 x^\alpha)}\frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}L|y(s)-z(s)|\,ds\right] \\
&= L\max_{x\in[0,1]}\left[\frac{1}{E_\alpha(\kappa_0 x^\alpha)}\frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}E_\alpha(\kappa_0 s^\alpha)\frac{|y(s)-z(s)|}{E_\alpha(\kappa_0 s^\alpha)}\,ds\right] \\
&\le Ld_{\kappa_0}(y,z)\max_{x\in[0,1]}\left[\frac{1}{E_\alpha(\kappa_0 x^\alpha)}\frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}E_\alpha(\kappa_0 s^\alpha)\,ds\right] \\
&\le Ld_{\kappa_0}(y,z)\max_{x\in[0,1]}\left[\frac{1}{E_\alpha(\kappa_0 x^\alpha)}\left(\frac{E_\alpha(\kappa_0 x^\alpha)-1}{\kappa_0}\right)\right] \\
&= \frac{d_{\kappa_0}(y,z)}{\gamma}\max_{x\in[0,1]}\left[1 - \frac{1}{E_\alpha(\kappa_0 x^\alpha)}\right] \\
&= \frac{d_{\kappa_0}(y,z)}{\gamma}\left[1 - \frac{1}{E_\alpha(\kappa_0)}\right]
\end{aligned}$$

$$\leq \frac{d_{\kappa_0}(y,z)}{\gamma}$$

where we have used (7.6) and $\kappa_0 = L\gamma$ above. Since $\gamma > 1$ we see that $H_\lambda(y)$ is a contraction map in $y$ on $\overline{U}$ with respect to $d_{\kappa_0}$ and has a contraction constant $\sigma = 1/\gamma < 1$.

Thirdly, we show that if (H3) is satisfied then $H_\lambda(y) \neq y$ for all $y \in \partial U$ and $\lambda \in [0,1]$. To see this, assume for the sake of contradiction that there exists $\lambda = \lambda_1 \in [0,1]$, $x = x_1 \in [0,1]$ and a solution $y$ with

$$y(x_1) = H_{\lambda_1}y(x_1) = A_0 + \frac{\lambda_1}{\Gamma(\alpha)} \int_0^{x_1} (x_1 - s)^{\alpha-1} f(s, y(s))\, ds. \tag{7.9}$$

Due to Lemma 7.1 we observe that (7.9) is equivalent to the situation where there is at least one $x_1 \in [0,1]$ and one $\lambda_1 \in [0,1]$ such that $|y_{\lambda_1}(x_1) - A_0| = R$ for solutions to the family of problems (7.7). However, this contradicts my assumption $|y_\lambda(x) - A_0| < R$ for all $x \in [0,1]$ and for all $\lambda \in [0,1]$. We obtain a contradiction.

Finally, we define the set $U_1 \subset \overline{U}$

$$U_1 := \{y \in C([0,1]) : |y(x) - A_0| < R/2, \text{ for all } x \in [0,1]\}.$$

We observe that for all $y \in U_1$ we have $H_0(y) \equiv A_0 \in U_1$ and so $H_0(U_1) \subset U_1$.

Collectively, my four steps have shown that all of the conditions of Theorem 1.8 are satisfied and thus for each $\lambda \in [0,1]$ we conclude that $H_\lambda$ has a unique fixed point in $U$. This includes the special case $\lambda = 1$, which is equivalent to proving that the fractional initial value problem (7.3), (7.4) possesses a unique solution. □

One may wonder on how I may ensure the *a priori* bound condition (H3) in Theorem 7.1. In this regard I am motivated by [269, Theorem 4.1] to obtain the following novel result that will ensure (H3) of Theorem 7.1 is satisfied.

**Theorem 7.2.** *Let $f : S_1 \to \mathbb{R}$. If the following conditions are satisfied:*

($\hat{\mathbf{H}}$**1**) *$f$ is continuous in $S_1$;*

($\hat{\mathbf{H}}$**2**) *there exists a constant $L > 0$ such that*

$$|f(x,u_0) - f(x,v_0)| \leq L|u_0 - v_0|, \textit{for all } (x,u_0),(x,v_0) \in S_1; \tag{7.10}$$

($\hat{\mathbf{H}}$**3**) *there exist nonnegative constants $N_1$, $N_2$ such that*

$$|f(x,y)| \leq -2N_1 y f(x,y) + N_2, \quad \textit{for all } (x,y) \in S_1; \tag{7.11}$$

$$N_1 A_0^2 + \frac{N_2}{\Gamma(\alpha+1)} < R;$$

*then* (7.3)*,* (7.4) *has a unique solution* $y = y(x)$ *with* $|y(x) - A_0| < R$ *for all* $x \in [0,1]$.

*Proof.* We show that the condition (H3) of Theorem 7.1 is satisfied. To see this, let $y_\lambda$ be any family of solutions to (7.7). Then similarly to [269, Theorem 4.1], all solutions of (7.7) satisfy the *a priori* bound

$$|y_\lambda(x) - A_0| \leq N_1 A_0^2 + \frac{N_2}{\Gamma(\alpha+1)}, \quad \text{for all } x \in [0,1] \text{ and } \lambda \in [0,1]. \tag{7.12}$$

Set

$$B = N_1 A_0^2 + \frac{N_2}{\Gamma(\alpha+1)},$$

then we have $|y_\lambda(x) - A_0| \leq B < R$ for all $x \in [0,1]$ and for all $\lambda \in [0,1]$, as I claim. □

Let me now turn to the question of construction (or approximation) of these global solutions that are ensured by my previous theorems. I draw on the constructive elements of CMCM Theorem 1.8 to form the following result.

**Theorem 7.3.** *Let* $f : S_1 \to \mathbb{R}$. *If all conditions of Theorem 7.2 are satisfied, then* (7.3)*,* (7.4) *has a unique solution* $y = y(x)$ *with* $|y(x) - A_0| < R$ *for all* $x \in [0,1]$*, Moreover, if a sequence of functions* $\{y_{\lambda_1,k}\}_{k\geq 0}$*,* $\lambda_1 := 1$*, is defined inductively by choosing any* $y_{1,0} \in C([0,1])$ *such that* $|y_{1,0} - A_0| \leq R$ *for all* $x \in [0,1]$ *and setting*

$$y_{1,k+1}(x) := H_1(y_{1,k}) := A_0 + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, y_{1,k}(s))\, ds, \quad x \in [0,1],\ k = 0,1,2,\ldots \tag{7.13}$$

*then the sequence* $\{y_{1,k}\}_{k\geq 0}$ *converges uniformly on* $[0,1]$ *to the unique solution* $y \in U$ *of* (7.3)*,* (7.4) *with respect to* $d_{\kappa_0}$.

*Proof.* By Theorem 7.1, the fractional initial value proplem (7.3), (7.4) has a unique solution $y = y(x)$ with $|y(x) - A_0| < R$ for all $x \in [0,1]$. The proof of showing that the sequence $\{y_{1,k}\}_{k\geq 0}$ converges uniformly on $[0,1]$ to the unique solution $y \in U$ of (7.3), (7.4) with respect to $d_{\kappa_0}$ is ensured by Theorem 1.8, see also ([217, Corollary 2.5] and [207, Theorem 2.4]); thus I only give the outline.

Set,

$$d = d_{\kappa_0},\ Y = C([0,1]),\ r = R,\ j = m = 1,\ \lambda = \lambda_1 = 1,$$

and then apply similar principles as in the proof of [207, Theorem 2.3]. □

**Remark 7.1.** *The Lipschitz condition (7.6) is both classical and well known within the context of Banach's fixed point theorems, however, note that (7.6) holds only* locally *instead of, say, on an infinite strip* $[0,1] \times \mathbb{R}$. *Consequently, any application of Banach's classical fixed point theorems in this setting must be restricted to a local version. This then leads to existence and uniqueness of only a locally-defined solution on a mere subinterval* $I_{S_1} \subset [0,1]$*, rather than yielding existence and uniqueness of a global solution defined on the whole interval* $[0,1]$. *The restriction is a result of the invariance condition of the local version of Banach's fixed point theorem which demands the operator* $T$ *satisfies* $T(\bar{B}_{S_1}(A_0) \subset \bar{B}_{S_1}(A_0)$*, where* $\bar{B}_{S_1}(A_0)$ *is a closed ball in* $C(I_{S_1};\mathbb{R})$.

*Thus, Theorem 7.1 advances knowledge in a way that a localized version of Banach's fixed point theorem cannot by establishing* global *existence and uniqueness of solutions. This is one of my main advancements over traditional theory and known results.*

## 7.3 Global existence results: Generalization to case $\alpha > 0$

In this Section I establish my second novel result for global solutions to the problem (7.1), (7.2) via an application of a constructive version of CMCM (Theorem 1.8). My approach in this Section is identical to the approach of the previous Section i.e., I briefly illustrate a generalization of the results obtained in previous Section from the case $0 < \alpha < 1$ to arbitrary $\alpha > 0$.

Let me first start with stating the relationship between the more general problem (7.1), (7.2) and its integral form is contained in the following Lemma.

**Lemma 7.2.** *If* $f : S_2 \to \mathbb{R}$ *is continuous then the initial value problem (7.1), (7.2) is equivalent to the integral equation*

$$y(x) = \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i}{i!} x^i + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, y(s))\, ds, \quad x \in [0,1]. \tag{7.14}$$

Let me now define what I mean by a global solution to my problem (7.1), (7.2).

**Definition 7.2.** *We say* $y = y(x)$ *is a global solution to the fractional initial value problem (7.1), (7.2) on* $[0,1]$ *if:* ${}^cD^\alpha y(x)$ *is well-defined on* $[0,1]$*;* $y = y(x)$ *satisfies: (7.1) for all* $x \in [0,1]$*; and (7.2); with* $\left|y(x) - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i}{i!} x^i\right| \leq R$ *for all* $x \in [0,1]$.

The following Theorem is a generalization of Theorem 7.1 from the case $0 < \alpha < 1$ to arbitrary $\alpha > 0$.

**Theorem 7.4.** *Let $f : S_2 \to \mathbb{R}$. If the following conditions are satisfied:*

**(P1)** *$f$ is continuous in $S_2$;*

**(P2)** *there exists a constant $L > 0$ such that*

$$|f(x,u_0) - f(x,v_0)| \le L|u_0 - v_0|, \text{for all } (x,u_0),(x,v_0) \in S_2; \tag{7.15}$$

**(P3)** *for any solution $y_\lambda$ to the family of problems*

$$\begin{cases} {}^cD^\alpha y = \lambda f(x,y), \quad 0 \le x \le 1,\ 0 \le \lambda \le 1,\ \alpha > 0, \\ y(0) = A_0,\ y'(0) = A_1, \ldots,\ y^{(\lceil \alpha \rceil - 1)}(0) = A_{n-1}, \end{cases} \tag{7.16}$$

*one has $|y_\lambda(x) - \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{A_i x^i}{i!}| < R$, for all $x \in [0,1]$ and all $\lambda \in [0,1]$;*

*then (7.1), (7.2) has a unique global solution $y = y(x)$ with $|y(x) - \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{A_i}{i!} x^i| < R$ for all $x \in [0,1]$.*

*Proof.* The proof is very similar to the proof of Theorem 7.1; therefore, I only give the outline. Consider the complete metric space $(C([0,1]), d_{\kappa_0})$ where $\kappa_0$ is defined as in Theorem 7.1 and let $\mathcal{S}_2 \subset C([0,1])$ be an open set with

$$\mathcal{S}_2 := \left\{ y \in C([0,1]) : \left| y(x) - \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{A_i x^i}{i!} \right| < R, \quad \text{for all } x \in [0,1] \right\}.$$

Then apply similar principles as in the proof of Theorem 7.1 to the operator $H_\lambda$ that is defined by

$$[H_\lambda(y)](x) := \sum_{i=0}^{\lceil \alpha \rceil - 1} \frac{A_i x^i}{i!} + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha - 1} f(s, y(s))\, ds, \quad x \in [0,1], \quad \lambda \in [0,1]. \tag{7.17}$$

□

Again, let me provide some concreteness regarding condition (P3). If we draw on [261, Theorem 7.1] then we can obtain sufficient conditions under which the *a priori* bounds to the associated family will be ensured. I then have the following new result.

**Theorem 7.5.** *Let $f : S_2 \to \mathbb{R}$. If the following conditions are satisfied:*

**($\hat{\mathbf{P}}$1)** *$f$ is continuous in $S_2$;*

($\hat{\mathbf{P}}\mathbf{2}$) *there exists a constant* $L > 0$ *such that*

$$|f(x,u_0) - f(x,v_0)| \leq L|u_0 - v_0|, \textit{for all } (x,u_0),(x,v_0) \in S_2; \tag{7.18}$$

($\hat{\mathbf{P}}\mathbf{3}$) *there exists a continuous function* $h : [0,1] \to [0,\infty)$ *and a continuous function* $g : S_2 \to [0,\infty)$ *such that*

$$|f(x,y)| \leq h(x) g\left(\left|u - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i x^i}{i!}\right|\right), \textit{ for all } (x,u) \in S_2 \tag{7.19}$$

$$g\left(\left|u - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i x^i}{i!}\right|\right) \leq g(R), \textit{ for all } (x,u) \in S_2,$$

$$\frac{R}{g(R)} > \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} h(s)\, ds, \textit{ for all } x \in [0,1], \tag{7.20}$$

*then* (7.1), (7.2) *has a unique global solution* $y = y(x)$ *with* $|y(x) - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i x^i}{i!}| < R$ *for all* $x \in [0,1]$.

*Proof.* We show that if ($\hat{P}3$) is satisfied then $H_\lambda(y) \neq y$ for all $y \in \partial\mathcal{S}_2$ and $\lambda \in [0,1]$. To see this, assume for the sake of contradiction that there exists at least one $\lambda \in [0,1]$ and $x \in [0,1]$, and a solution $y$ with

$$y(x) = H_\lambda y(x) = \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i x^i}{i!} + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s,y(s))\, ds. \tag{7.21}$$

So,

$$\begin{aligned}
R &= \left|H_\lambda y(x) - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i x^i}{i!}\right| \\
&= \left|\frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s,y(s))\, ds\right| \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} h(s) g\left(\left|y(s) - \sum_{i=0}^{\lceil\alpha\rceil-1} A_i s^i/i!\right|\right) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} h(s) g\left(\left|y(s) - \sum_{i=0}^{\lceil\alpha\rceil-1} A_i s^i/i!\right|\right) ds \\
&\leq \frac{g(R)}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} h(s)\, ds
\end{aligned}$$

where we have used (7.19), but this contradicts my assumption (7.20). Hence, for all $x \in [0,1]$ and for all $\lambda \in [0,1]$, we have $|y_\lambda(x) - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i}{i!} x^i| < R$. □

If the initial conditions in (7.2) take the simplified form

$$y(0) = A_0, \; y'(0) = 0 = y''(0) = \cdots = y^{(\lceil\alpha\rceil - 1)}(0), \tag{7.22}$$

then the condition $(\hat{P}3)$ can be relaxed in the following way.

**Corollary 7.1.** *Suppose the condition $(\hat{P}3)$ of Theorem 7.6 is replaced by the following condition:*

$(\hat{\mathbf{P}}\mathbf{4})$ *there exists a continuous function $h : [0,1] \to \mathbb{R}$ and a nondecreasing continuous function $g : [0,R] \to [0,\infty)$ such that*

$$|f(x,y)| \le h(x) g\left(|y - A_0|\right), \quad \textit{for all } x \in [0,1], \; \left|y - A_0\right| \le R, \tag{7.23}$$

*and*

$$\frac{R}{g(R)} > \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} h(s)\, ds, \quad \alpha > 0. \tag{7.24}$$

*Then the result of Theorem 7.6 holds for the special initial conditions (7.22).*

**Remark 7.2.** *The result of Theorem 7.2 holds if the condition $(\hat{H}3)$ is replaced by the condition of Corollary 7.1, for the case $0 < \alpha < 1$.*

Let me now turn to construction (or approximation) of solutions for my problem.

**Theorem 7.6.** *Let $f : S_2 \to \mathbb{R}$. If all conditions of Theorem 7.4 are satisfied, then (7.1), (7.2) has a unique global solution $y = y(x)$ with $\left|y(x) - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i x^i}{i!}\right| < R$ for all $x \in [0,1]$. Moreover, a sequence of functions $\{y_{\lambda_1,k}\}_{k\ge0}$, $\lambda_1 := 1$, is defined inductively by choosing any $y_{1,0} \in C([0,1])$ such that $\left|y_{1,0} - \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i x^i}{i!}\right| \le R$ for all $x \in [0,1]$ and setting*

$$\begin{aligned} y_{1,k+1}(x) &:= H_1(y_{1,k}) \\ &:= \sum_{i=0}^{\lceil\alpha\rceil-1} \frac{A_i x^i}{i!} + \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, y_{1,k}(s))\, ds, \quad x \in [0,1], \; k = 0,1,2,\ldots \end{aligned} \tag{7.25}$$

*then the sequence $\{y_{1,k}\}_{k\ge0}$ converges uniformly on $[0,1]$ to the unique global solution $y \in S_2$ of (7.1), (7.2) with respect to $d_{\kappa_0}$.*

*Proof.* The proof is virtually identical to the proof of Theorem 7.3; therefore, it is omitted. □

**Remark 7.3.** *In the view of (1.43), the approach in the proof of Theorem 7.3 and Theorem 7.6 can be used to evaluate the rate of convergence of iterates. If $y$, $y_{1,0} \in C([0,1])$ and $\kappa_0 := L\gamma$ with*

$\gamma > 1$ *then* (1.43) *yields*

$$d_{\kappa_0}(H_1^k(y_{1,0}), y(1)) \le \frac{\gamma^{-k}}{1-\gamma^{-1}} d_{\kappa_0}(y_{1,0}, H_1(y_{1,0})), \qquad k = 1, 2, \ldots$$

*and so*

$$\|H_1^k(y_{1,0}) - y(1)\|_0 \le E_\alpha(L\gamma x^\alpha) \frac{\gamma^{-k}}{1-\gamma^{-1}} \|y_{1,0} - H_1(y_{1,0})\|_0, \qquad k = 1, 2, \ldots \tag{7.26}$$

*where* $\|\cdot\|_0$ *is the norm induced by the max–metric* (1.16)*. The choice* $\gamma := k/L$ *yields a nice evaluation of the rate of convergence in* (7.26)*, namely*

$$\|H_1^k(y_{1,0}) - y(1)\|_0 \le E_\alpha(kx^\alpha) \left(\frac{L}{k}\right)^k \frac{k}{k-L} \|y_{1,0} - H_1(y_{1,0})\|_0, \qquad k = 1, 2, \ldots.$$

## 7.4 An example

Let me discuss an example, aiming to illustrate how to apply my new results to concrete problems.

**Example 7.1.** *Consider the following fractional initial value problem (FIVP)*

$$\begin{cases} D^{1.1}[y - y(0) - ty'(0)] = x^2 e^{y/2}, \quad 0 \le x \le 1, \\ y(0) = 0, \; y'(0) = 0. \end{cases} \tag{7.27}$$

*Then FIVP* (7.27) *has a unique solution* $y = y(x)$ *with* $|y(x)| < 3/2$*, for all* $x \in [0,1]$.

*Proof.* In this example we have a special case of (7.1), (7.2) with:
$f(x,u) = x^2 e^{u/2}$; $A_0 = 0, A_1(0) = 0$; and $\alpha = 1.1$. We define the filled rectangle $S_1$ by

$$S_1 := \left\{(x,u): \; x \in [0,1], \; |u| \le 3/2\right\}.$$

Note that for all $(x,u) \in S_1$ we have

$$\begin{aligned} |f(x,u)| &= |x^2 e^{u/2}| \\ &\le |x^2| e^{|u/2|} \end{aligned}$$

and so (7.23) holds with: $h(x) = x^2$; and $g(|u|) = e^{|u/2|}$. Now for $R = 3/2$ and in view of (7.24) and its context, we see that

$$\frac{1}{\Gamma(1.1)} \int_0^x (x-s)^{0.1} s^2 \, ds = \frac{\Gamma(3)}{\Gamma(4.1)} x^{3.1} \le \frac{\Gamma(3)}{\Gamma(4.1)} = \frac{2}{\Gamma(\alpha+3)} \approx 0.2936.$$

Also,

$$\frac{R}{g(R)} = \frac{\hat{R}}{e^{\hat{R}/2}} = \frac{3/2}{e^{3/4}} \approx 0.7085.$$

Thus (7.24) holds and by Corollary 7.1 we have that the FIVP (7.27) has a unique global solution $y = y(x)$ with $|y(x)| < 3/2$ for all $x \in [0,1]$. □

# Chapter 8

# Coupled systems of nonlinear Riemann–Liouville fractional differential equations

## 8.1 Introduction

The central aim of this Chapter is to develop a new theory concerning existence and uniqueness of solutions to a special case of the following initial value problem for a coupled system of multi-term nonlinear fractional differential equations (SIVP):

$$D^{\alpha}u(x) = f(x, v(x), D^{\beta_1}v(x), \dots, D^{\beta_N}v(x)), \quad D^{\alpha-i}u(0) = 0, \quad i = 1, 2, \dots, n_1; \tag{8.1}$$

$$D^{\sigma}v(x) = g(x, u(x), D^{\rho_1}u(x), \dots, D^{\rho_N}u(x)), \quad D^{\sigma-j}v(0) = 0, \quad j = 1, 2, \dots, n_2; \tag{8.2}$$

where: $x \in (0,1]$; $\alpha > \beta_1 > \beta_2 > \dots > \beta_N > 0$; $\sigma > \rho_1 > \rho_2 > \dots > \rho_N > 0$; $n_1 = [\alpha] + 1$, $n_2 = [\sigma] + 1$ for $\alpha, \sigma \notin \mathbb{N}$, and $n_1 = \alpha$, $n_2 = \sigma$ for $\alpha, \sigma \in \mathbb{N}$; $\beta_{q_0}, \rho_{q_0} < 1$ for any $q_0 \in \{1, 2, \dots, N\}$; $D$ represents the standard Riemann–Liouville fractional differentiation operator; and $f, g : [0,1] \times \mathbb{R}^{N+1} \to \mathbb{R}$ are given functions.

In 2012, Sun et al. [254] considered the existence and uniqueness of solutions to SIVP (8.1), (8.2) and under sufficient conditions imposed in [254], the existence and uniqueness of the solutions to SIVP (8.1), (8.2) were obtained by the means of Schauder fixed point theorem(Theorem 1.3) and Banach contraction principle (Theorem 1.5), respectively. In the uniqueness results obtained by Sun et al. [254], which is my central aim to improve in this Chapter, the nonlinear

term $f$ and $g$ are assumed to be continuous functions and satisfy the following conditions:
[(**H1**)] there exist nonnegative functions $\eta_0, \eta_1, \ldots, \eta_N \in L^1[0,1]$ and $h_0, h_1, \ldots, h_N \in L^1[0,1]$ such that

$$
\begin{aligned}
|f(x, y_0, y_1, \ldots, y_N) - f(x, z_0, z_1, \ldots, z_N)| &\leq \eta_0(x)|y_0 - z_0| + \eta_1(x)|y_1 - z_1| \\
&\quad + \ldots + \eta_N(x)|y_N - z_N|, \\
|g(x, y_0, y_1, \ldots, y_N) - g(x, z_0, z_1, \ldots, z_N)| &\leq h_0(x)|y_0 - z_0| + h_1(x)|y_1 - z_1| \\
&\quad + \ldots + h_N(x)|y_N - z_N|,
\end{aligned}
$$

for all $x \in [0,1]$, $y_i, z_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, N$, and the functions $f, g$ satisfy $f(0, 0, \ldots, 0) = 0$ and $g(0, 0, \ldots, 0) = 0$.
[(**H2**)] Assume that $\xi = \max\{B_0, B_1, \ldots, B_N, H_0, H_1, \ldots, H_N\} < 1$, where

$$
B_i = \max_{x \in [0,1]} \left| \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} \eta_i(s) ds + \sum_{j=1}^{N} \int_0^x \frac{(x-s)^{\alpha-\rho_j-1}}{\Gamma(\alpha-\rho_j)} \eta_i(s) ds \right|, \quad i = 0, 1, 2, \ldots, N,
$$

$$
H_i = \max_{x \in [0,1]} \left| \int_0^x \frac{(x-s)^{\sigma-1}}{\Gamma(\sigma)} h_i(s) ds + \sum_{j=1}^{N} \int_0^x \frac{(x-s)^{\sigma-\beta_j-1}}{\Gamma(\sigma-\beta_j)} h_i(s) ds \right|, \quad i = 0, 1, 2, \ldots, N.
$$

However, the condition [(**H2**)] turns out to be a strong assumption as the following example illustrates.

**Example 8.1.** *Consider the following coupled system of fractional initial value problem (FIVP)*

$$
\begin{cases}
D^{2.2}u(x) = x + 2xv(x) + \frac{10|D^{0.2}v(x)|}{1+|D^{0.2}v(x)|}, & D^{2.2-i}u(0) = 0, \quad i = 1, 2, 3; \\
D^{2.2}v(x) = x^2 + \frac{12|u(x)|}{1+|u(x)|} + xD^{0.2}u(x), & D^{2.2-j}v(0) = 0, \quad j = 1, 2, 3;
\end{cases}
\tag{8.3}
$$

*where* $x \in (0,1]$.

To see that [(**H2**)] does not hold, let

$$
f(x, y_0, y_1) = x + 2xy_0 + \frac{10|y_1|}{1+|y_1|}, \quad \text{and} \quad g(x, y_0, y_1) = x^2 + \frac{12|y_0|}{1+|y_0|} + xy_1.
$$

Then for $x \in (0,1]$, $y_0, y_1, z_0, z_1 \in \mathbb{R}$ we get

$$
\begin{aligned}
|f(x, y_0, y_1) - f(x, z_0, z_1)| &\leq 2|y_0 - z_0| + \left| \frac{10|y_1|}{1+|y_1|} - \frac{10|z_1|}{1+|z_1|} \right| \\
&= 2|y_0 - z_0| + \left| \frac{10(|y_1| - |z_1|)}{(1+|y_1|)(1+|z_1|)} \right| \\
&\leq 2|y_0 - z_0| + 10|y_1 - z_1|.
\end{aligned}
\tag{8.4}
$$

Similarly we have

$$|g(x,y_0,y_1)-g(x,z_0,z_1)| \leq 12|y_0-z_0|+|y_1-z_1|. \tag{8.5}$$

It is obvious that [(**H2**)] does not hold since $\xi > 1$ with: $\alpha=\sigma=2.2$, $\beta_1=\rho_1=0.2$, and the functions defined in condition [(**H1**)] are replaced by nonnegative constants, namely: $\eta_0 = 2, \eta_1 = 10$ and $h_0 = 12, h_1 = 1$.

Motivated by the aforementioned gap, my objective is to develop a theory of existence and uniqueness of solutions to the following initial value problem involving a coupled system of nonlinear fractional differential equations

$$D^{\alpha}u(x) = f(x,v(x),D^{\rho_1}v(x)), \quad D^{\alpha-i}u(0)=0; \quad i=1,2,\ldots,n_1, \tag{8.6}$$

$$D^{\alpha}v(x) = g(x,u(x),D^{\rho_1}u(x)), \quad D^{\alpha-i}v(0)=0; \quad i=1,2,\ldots,n_1, \tag{8.7}$$

where: $x\in(0,1]$; $\alpha>\rho_1>0$; $n_1=[\alpha]+1$ for $\alpha\notin\mathbb{N}$, and $n_1=\alpha$ for $\alpha\in\mathbb{N}$; $\rho_1<1$; $D$ represents the standerd Riemann–Liouville fractional differentiation operator; and $f,g:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ are given functions. As can be seen above that the coupled system (8.6), (8.7) is a special case of (8.1), (8.2) with $\alpha=\sigma$, and $\beta_1=\rho_1$.

I am now ready to state and prove my main result, but let me first provides some lemmas that are associated with my work.

**Lemma 8.1** (See [128, Lemma 2.9]). *The following integration formula is valid for any* $\kappa,\mu>0:$

$$I^{\mu}E_{\mu}(\kappa x^{\mu}) = \frac{1}{\kappa}(E_{\mu}(\kappa x^{\mu})-1), \quad \textit{for all } x\in[0,1]. \tag{8.8}$$

*Let* $\mu>\nu>0$. *Then there exists a constant* $M_{\mu,\nu}$ *(dependent on* $\mu$ *and* $\nu$*) such that the following inequality is valid for any* $\kappa>0$*:*

$$\sup_{x\in[0,1]}\frac{I^{\mu}E_{\mu-\nu}(\kappa x^{\mu-\nu})}{E_{\mu-\nu}(\kappa x^{\mu-\nu})} \leq \frac{M_{\mu,\nu}}{\kappa}, \quad \textit{for all } x\in[0,1]. \tag{8.9}$$

For the purpose of the forthcoming analysis, in Lemma 8.1 I fix the constant $\kappa$ via ,

$$\kappa := \gamma(P_1+P_2), \tag{8.10}$$

where: $\gamma>1$, $P_1 := \max\{L_0,L_1\}\left(M_{\alpha,\rho_1}+1\right)$, $P_2 := \max\{K_0,K_1\}\left(M_{\alpha,\rho_1}+1\right)$, $M_{\alpha,\rho_1}$ is the constant in (8.9) with $\mu=\alpha$ and $\nu=\rho_1$, and $L_1,L_2,K_0$, and $K_1$ are nonnegative constants to be defined in Theorem 8.1.

**Remark 8.1.** *The inequality* (8.8) *remains valid for my choice of the constant* $\kappa$ *given by* (8.10) *since* $\kappa > 0$ *and* $\alpha > \rho_1$.

Consider the following coupled system of integral equations

$$u(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} f(s, v(s), D^{\rho_1} v(s))\, ds, \tag{8.11}$$

$$v(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s, u(s), D^{\rho_1} u(x))\, ds. \tag{8.12}$$

My analysis will involve an equivalent fractional integral equations, so I introduce the following Lemma which states the equivalency between fractional differential and integral forms.

**Lemma 8.2** (See [254, Lemma 3.2])**.** *Suppose that two functions* $f, g : I_0 \times \mathbb{R}^2 \to \mathbb{R}$ *are continuous. Then* $(u, v) \in X \times Y$ *is a solution of* (8.6), (8.7) *if and only if* $(u, v) \in X \times Y$ *is a solution of coupled system of integral equations* (8.11), (8.12).

## 8.2 Existence result via Banach fixed point theorem

In this Section I state and prove my main result: Theorem 8.1.

**Theorem 8.1.** *Let* $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ *be continuous functions. Suppose that the following condition is satisfied*

**(H3)** *there exist nonnegative constants* $L_0, L_1$ *and* $K_0, K_1$ *such that*

$$\begin{aligned} |f(x, y_0, y_1) - f(x, z_0, z_1)| &\leq L_0 |y_0 - z_0| + L_1 |y_1 - z_1|, \\ |g(x, y_0, y_1) - g(x, z_0, z_1)| &\leq K_0 |y_0 - z_0| + K_1 |y_1 - z_1|, \end{aligned} \tag{8.13}$$

*where* $x \in [0,1]$, $y_i, z_i \in \mathbb{R}$, $i = 0, 1$, *and the functions* $f, g$ *satisfy* $f(0,0,0) = 0$ *and* $g(0,0,0) = 0$. *Then the coupled system* (8.6), (8.7) *has a unique solution.*

*Proof.* Consider the Banach space $(X \times Y, \|(u,v)\|_{X_\kappa \times Y_\kappa})$ defined in (1.36) and let the operator $\mathcal{H} : X \times Y \to X \times Y$ be defined by

$$\begin{aligned} \mathcal{H}(u,v)(x) &= \left( \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, v(s), D^{\rho_1} v(s))\, ds, \int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} g(s, u(s), D^{\rho_1} u(x))\, ds \right) \\ &= (\mathcal{H}_1 v(x), \mathcal{H}_2 u(x)). \end{aligned} \tag{8.14}$$

By Lemma 8.2, showing the existence of fixed–points of operator $\mathcal{H}$ is equivalent to showing the existence of solutions to the coupled system (8.6), (8.7). Thus, we want to prove that there exists a unique $(u, v)$ such that $\mathcal{H}(v, u) = (u, v)$. To do this, we first show that $\mathcal{H} : X \times Y \to X \times Y$. Then we show that our operator $\mathcal{H}$ is a contractive map with contraction constant $\theta < 1$ under the norm $\|\cdot\|_{X_\kappa \times Y_\kappa}$, and Banach contraction principle will then apply. For convenience we let $I_0 := [0, 1]$ and set $\psi(x) := E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})$ where $\kappa > 0$ is given by (8.10) and set $a(t) := |v_2(x) - v_1(x)|$ and $b(t) := |D^{\rho_1} v_2(x) - D^{\rho_1} v_1(x)|$.

We now show that $\mathcal{H} : X \times Y \to X \times Y$. For any $(u, v) \in X \times Y$, we have

$$\begin{aligned}
\|\mathcal{H}_1 v\|_{X_\kappa} &= \max_{x\in I_0}\left[\frac{|\mathcal{H}_1 v|}{\psi(x)}\right] + \max_{x\in I_0}\left[\frac{|D^{\rho_1}\mathcal{H}_1 v|}{\psi(x)}\right]\\
&= \max_{x\in I_0}\left[\frac{1}{\psi(x)}\left|\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, v(s), D^{\rho_1}v(s))ds\right|\right]\\
&+ \max_{x\in I_0}\left[\frac{1}{\psi(x)}\left|\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)} f(s, v(s), D^{\rho_1}v(s))ds\right|\right]\\
&\le \max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\left(L_0|v(s)| + L_1|D^{\rho_1}v(s)|\right)ds\right]\\
&+ \max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\left(L_0|v(s)| + L_1|D^{\rho_1}v(s)|\right)ds\right]\\
&\le \max_{x\in I_0}\left[\frac{\max\{L_0, L_1\}}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\left(|v(s)| + |D^{\rho_1}v(s)|\right)ds\right]\\
&+ \max_{x\in I_0}\left[\frac{\max\{L_0, L_1\}}{\psi(x))}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\left(|v(s)| + |D^{\rho_1}v(s)|\right)ds\right]\\
&= \max_{x\in I_0}\left[\frac{\max\{L_0, L_1\}}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\psi(s)\left(\frac{|v(s)|}{\psi(s)} + \frac{|D^{\rho_1}v(s)|}{\psi(s)}\right)ds\right]\\
&+ \max_{x\in I_0}\left[\frac{\max\{L_0, L_1\}}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\psi(s)\left(\frac{|v(s)|}{\psi(s)} + \frac{|D^{\rho_1}v(s)|}{\psi(s)}\right)ds\right]\\
&\le \|v\|_{Y_\kappa}\max\{L_0, L_1\}\max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\psi(s)ds\right]\\
&+ \|v\|_{Y_\kappa}\max\{L_0, L_1\}\max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\psi(s)ds\right]\\
&= \|v\|_{Y_\kappa}\max\{L_0, L_1\}\left[\max_{x\in I_0}\frac{I^\alpha E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})} + \max_{x\in I_0}\frac{I^{\alpha-\rho_1}E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}\right];
\end{aligned}$$

above we applied Lemma 8.1, (8.9) and (8.8) with $\mu = \alpha$, and $\nu = \rho_1$. Hence, we see that

$$\begin{aligned}
\|\mathcal{H}_1 v\|_{X_\kappa} &\le \|v\|_{Y_\kappa}\max\{L_0, L_1\}\left[\frac{M_{\alpha,\rho_1}}{\kappa} + \max_{x\in I_0}\left(\frac{1}{\kappa}\left(1 - \frac{1}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}\right)\right)\right]\\
&\le \|v\|_{Y_\kappa}\max\{L_0, L_1\}\left[\frac{M_{\alpha,\rho_1}}{\kappa} + \frac{1}{\kappa}\right] = \frac{P_1}{\kappa}\|v\|_{Y_\kappa}.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
\|\mathcal{H}_2 u\|_{Y_\kappa} &= \max_{x\in I_0}\left[\frac{|\mathcal{H}_2 u|}{\psi(x)}\right] + \max_{x\in I_0}\left[\frac{|D^{\rho_1}\mathcal{H}_2 u|}{\psi(x)}\right]\\
&= \max_{x\in I_0}\left[\frac{1}{\psi(x)}\left|\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)} g(s,u(s),D^{\rho_1}u(s))ds\right|\right]\\
&+ \max_{x\in I_0}\left[\frac{1}{\psi(x)}\left|\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)} g(s,u(s),D^{\rho_1}u(s))ds\right|\right]\\
&\le \max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\left(K_0|u(s)| + K_1|D^{\rho_1}u(s)|\right)ds\right]\\
&+ \max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\left(K_0|u(s)| + K_1|D^{\rho_1}u(s)|\right)ds\right]\\
&\le \max_{x\in I_0}\left[\frac{\max\{K_0,K_1\}}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\left(|u(s)| + |D^{\rho_1}u(s)|\right)ds\right]\\
&+ \max_{x\in I_0}\left[\frac{\max\{K_0,K_1\}}{\psi(x))}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\left(|u(s)| + |D^{\rho_1}u(s)|\right)ds\right]\\
&= \max_{x\in I_0}\left[\frac{\max\{K_0,K_1\}}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\psi(s)\left(\frac{|u(s)|}{\psi(s)} + \frac{|D^{\rho_1}u(s)|}{\psi(s)}\right)ds\right]\\
&+ \max_{x\in I_0}\left[\frac{\max\{K_0,K_1\}}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\psi(s)\left(\frac{|u(s)|}{\psi(s)} + \frac{|D^{\rho_1}u(s)|}{\psi(s)}\right)ds\right]\\
&\le \|u\|_{X_\kappa}\max\{K_0,K_1\}\max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\psi(s)ds\right]\\
&+ \|u\|_{X_\kappa}\max\{K_0,K_1\}\max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\psi(s)ds\right]\\
&= \|u\|_{X_\kappa}\max\{K_0,K_1\}\left[\max_{x\in I_0}\frac{I^\alpha E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})} + \max_{x\in I_0}\frac{I^{\alpha-\rho_1}E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}\right];
\end{aligned}
$$

above we apply Lemma 8.1, (8.9) and (8.8) with $\mu = \alpha$, and $\nu = \rho_1$. Hence, we have that

$$
\begin{aligned}
\|\mathcal{H}_2 u\|_{Y_\kappa} &\le \|u\|_{X\kappa}\max\{K_0,K_1\}\left[\frac{M_{\alpha,\rho_1}}{\kappa} + \max_{x\in I_0}\left(\frac{1}{\kappa}\left(1 - \frac{1}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}\right)\right)\right]\\
&\le \|u\|_{X_\kappa}\max\{K_0,K_1\}\left[\frac{M_{\alpha,\rho_1}}{\kappa} + \frac{1}{\kappa}\right] = \frac{P_2}{\kappa}\|u\|_{X_\kappa}.
\end{aligned}
$$

It follows that

$$
\|\mathcal{H}(u,v)\|_{X_\kappa\times Y_\kappa} \le \frac{P_1}{\kappa}\|v\|_{Y\kappa} + \frac{P_2}{\kappa}\|u\|_{X_\kappa} \le \frac{(P_1+P_2)}{\kappa}\|(u,v)\|_{X_\kappa\times Y_\kappa} = \frac{1}{\gamma}\|(u,v)\|_{X_\kappa\times Y_\kappa}.
$$

where we have used the constant $\kappa$ given by (8.10). As $\gamma > 1$ we see that $\mathcal{H} : X\times Y \to X\times Y$.

We now show that $\mathcal{H}$ is contractive with respect to $\|\cdot\|_{X\kappa\times Y\kappa}$. For any $(u_1,v_1),(u_2,v_2)\in X\times Y$, we have

$$
\begin{aligned}
&\|\mathcal{H}_1 v_2 - \mathcal{H}_1 v_1\|_{X\kappa}\\
&= \max_{x\in I_0}\left[\frac{|\mathcal{H}_1 v_2(x) - \mathcal{H}_1 v_1(x)|}{\psi(x)}\right] + \max_{x\in I_0}\left[\frac{|D^{\rho_1}\mathcal{H}_1 v_2(x) - D^{\rho_1}\mathcal{H}_1 v_1(x)|}{\psi(x)}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \max_{x\in I_0}\left[\frac{1}{\psi(x)}\left|\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\big(f(s,v_2(s),D^{\rho_1}v_2(s)) - f(s,v_1(s),D^{\rho_1}v_1(s))\big)\,ds\right|\right]\\
&+ \max_{x\in I_0}\left[\frac{1}{\psi(x)}\left|\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\big(f(s,v_2(s),D^{\rho_1}v_2(s)) - f(s,v_1(s),D^{\rho_1}v_1(s))\big)\,ds\right|\right]\\
&\le \max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\big(L_0a(s)+L_1b(s)\big)\,ds\right]\\
&+ \max_{x\in I_0}\left[\frac{1}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\big(L_0a(s)+L_1b(s)\big)\,ds\right]\\
&\le \max_{x\in I_0}\left[\frac{\max\{L_0,L_1\}}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\big(a(s)+b(s)\big)\,ds\right]\\
&+ \max_{x\in I_0}\left[\frac{\max\{L_0,L_1\}}{\psi(x))}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\big(a(s)+b(s)\big)\,ds\right]\\
&= \max_{x\in I_0}\left[\frac{\max\{L_0,L_1\}}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-1}}{\Gamma(\alpha)}\psi(s)\left(\frac{a(s)}{\psi(s)}+\frac{b(s)}{\psi(s)}\right)ds\right]\\
&+ \max_{x\in I_0}\left[\frac{\max\{L_0,L_1\}}{\psi(x)}\int_0^x \frac{(x-s)^{\alpha-\rho_1-1}}{\Gamma(\alpha-\rho_1)}\psi(s)\left(\frac{a(s)}{\psi(s)}+\frac{b(s)}{\psi(s)}\right)ds\right]\\
&\le \|v_2-v_1\|_{Y_\kappa}\max\{L_0,L_1\}\max_{x\in I_0}\left[\frac{1}{\psi(x)}\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\psi(s)ds\right]\\
&+ \|v_2-v_1\|_{Y_\kappa}\max\{L_0,L_1\}\max_{x\in I_0}\left[\frac{1}{\psi(x)}\frac{1}{\Gamma(\alpha-\rho_1)}\int_0^x (x-s)^{\alpha-\rho_1-1}\psi(s)ds\right]\\
&= \|v_2-v_1\|_{Y_\kappa}\max\{L_0,L_1\}\left[\max_{x\in I_0}\frac{I^\alpha E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}+\max_{x\in I_0}\frac{I^{\alpha-\rho_1}E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}\right];
\end{aligned}
$$

above we applied Lemma 8.1 (8.9) and (8.8) with $\mu=\alpha$, and $\nu=\rho_1$. We see that

$$
\begin{aligned}
\|\mathcal{H}_1v_2-\mathcal{H}_1v_1\|_{X_\kappa} &\le \|v_2-v_1\|_{Y_\kappa}\max\{L_0,L_1\}\left[\frac{M_{\alpha,\rho_1}}{\kappa}+\max_{x\in I_0}\left(\frac{1}{\kappa}\left(1-\frac{1}{E_{\alpha-\rho_1}(\kappa x^{\alpha-\rho_1})}\right)\right)\right]\\
&\le \|v_2-v_1\|_{Y_\kappa}\max\{L_0,L_1\}\left[\frac{M_{\alpha,\rho_1}}{\kappa}+\frac{1}{\kappa}\right]\\
&= \frac{P_1}{\kappa}\|v_2-v_1\|_{Y_\kappa}.
\end{aligned}
$$

Similarly, we can arrive at

$$
\|\mathcal{H}_2u_2-\mathcal{H}_2u_1\|_{Y_\kappa} \le \frac{P_2}{\kappa}\|u_2-u_1\|_{X_\kappa}.
$$

Hence, we have

$$
\begin{aligned}
\|\mathcal{H}(u_2,v_2)-\mathcal{H}(u_1,v_1)\|_{X_\kappa\times Y_\kappa} &\le \frac{(P_1+P_2)}{\kappa}\|(u_2-u_1,v_2-v_1)\|_{X_\kappa\times Y_\kappa}\\
&= \frac{1}{\gamma}\|(u_2-u_1,v_2-v_1)\|_{X_\kappa\times Y_\kappa},
\end{aligned}
$$

where we have used the constant $\kappa$ given by (8.10). As $\gamma>1$ we see that $\mathcal{H}$ is a contractive map with contraction constant $\theta := 1/\gamma$ and the Banach contraction principle (Theorem 1.5) applies that the operator $\mathcal{H}$ has a unique fixed point that is a solution of coupled system (8.6), (8.7). The proof is completed. □

**Remark 8.2.** *The condition* $[(\mathbf{H2})]$ *is not contained in Theorem 8.1. Moreover, If the norm* $\|\cdot\|_{X\times Y}$ *had been used in the proof of Theorem 8.1 rather than the norm* $\|\cdot\|_{X_\kappa\times Y_\kappa}$*, then we would have needed an additional assumption in Theorem 8.1, namely condition* $[(\mathbf{H2})]$.

**Remark 8.3.** *The inequalities* (8.4) *and* (8.5) *show that the FIVP* (8.3) *satisfies the condition* $(\mathbf{H3})$ *of Theorem 8.1 with* $L_0 := 2$, $L_1 := 10$, $K_0 := 12$, *and* $K_1 := 1$. *This shows the existence of unique solution to FIVP* (8.3).

*On the other hand, the condition* $(\mathbf{H2})$*, which was imposed by [254], does not hold. This shows my result improves the result obtained in [254] by illustrating that a larger class of these kinds of problems admit a unique solution.*

# Chapter 9

# Conclusion

In this Chapter I briefly discuss the results obtained in this thesis and also identify some potential open problems for further research.

In Chapter 2, I constructed a firm mathematical foundation for the second-order boundary value problem (second-order BVP) associated with a generalized Emden equation that embraces Thomas–Fermi-like theories. My new results in Chapter 2 guaranteed the existence of a unique solution, ensuring the generalized Emden equation that embraces Thomas–Fermi theory sits on a firm mathematical foundation.

The significance of these results can be interpreted as making progress towards a fuller qualitative theory of pure electron density theory. My existence and uniqueness results for the generalized Emden problem address open questions and gaps by establishing a firm mathematical foundation for the nonlinear forms (2.6) and (2.1) where each is subjected to (2.2). In a broader sense, my work complements recent literature on the theory and applications of boundary value problems, such as [30, 61, 93, 94]. However, there are some limitations of my work and so let us discuss them and identify some potential avenues for exploration.

One shortcoming of my new results is that they are nonconstructive. While I have developed new knowledge that my problems do admit a unique solution, I do not have a specific way of calculating or approximating the solution embedded within the techniques. This is a consequence of my choice to employ the Schauder fixed point theorem, which is nonconstructive.

The Banach contraction mapping theorem [294] is more constructive than Schauder's. It relies on iterations, and an open challenge lies in showing that the operator $V$ is contractive on a suitable set and the convergence of the Picard iterations. At the time of writing, I have explored the standard Picard iterations [266] for (2.4), (2.2)

$$y_{n+1} = V(y_n), \quad n = 0, 1, 2, \ldots$$

where $V$ is defined in (2.12) and an initial starting approximation could be $y_0 := 1 - x/b$. I can show that each of the subsequences $y_{2n}$ and $y_{2n-1}$ are monotonic, uniformly bounded and equicontinuous on $[0, b]$. Thus, each of these subsequences must converge uniformly to a continuous function on $[0, b]$. However, I have been unable to prove that they converge to the *same* limit. This remains an open question.

Regarding the location of solutions regarding my problems, we can see that each of my theorems bound the unique solution to each of my problems so that it lies within a triangular wedge, described by (2.9). If we can potentially shrink the shape or size of the region of existence by establishing sharper inequalities then we may be able to obtain more precise estimates and a better understanding of the location of the unique solution. At this stage, if and how this can be achieved is an open question.

Despite the generality in the form of the Emden problem (2.1) and the new qualitative theory herein, there is a range of other problems that are yet to be explored in this direction. For example, my work has been concerned with the equations associated atoms in free space. It appears that there are open qualitative questions regarding the problem of an atom in a plasma and the associated boundary value problems, see [191, Ch.11].

In Chapter 3 and Chapter 4, I established a more complete and wider-ranging theory than is currently available in the literature regarding the existence, uniqueness and approximation of solutions to third-order BVPs that are subjected to two- and three-point boundary conditions. My strategy involved an analysis of the problem under consideration, and its associated operator equations, within both unbounded sets and closed and bounded sets. For each set, I first applied Banach's fixed point theorem [35] to establish the existence and uniqueness of solutions to the third-order BVPs. I then sharpened this results by showing that a larger class of third-order BVPs admit a unique solution. I achieved this by drawing on fixed-point theory in an interesting and alternative way via an application of Rus's contraction mapping theorem [231] (Theorem 1.6).

In Chapter 5, I proved the existence and uniqueness of solutions to two-point boundary value problems involving fourth-order, ordinary differential equations (fourth-order BVP). In particular, I sharpened traditional results and approaches such as Banach's fixed point theorem in bounded and unbounded setting by showing that a larger class of problems admit a unique solution. I achieved this by drawing on Theorem 1.6. My theoretical results were applied to the area of elastic beam deflections when the beam is subjected to a loading force and the ends of the beam are either: both clamped; or one end is clamped and the other end is free. Existence and uniqueness of solutions to the models were also guaranteed for certain classes of linear and nonlinear loading forces.

Let me identify some limitations of Chapters 3, 4 and 5 as well as some open problems for further research.

I remark that there are many potential ways forward regarding the ideas of the these chapters, in both pure and applied forms. The application of Theorem 1.6 to all kinds of problems still appears to be underutilized and so there are opportunities to improve my understanding of solutions to a range of differential, integral and difference equations via Theorem 1.6. For example, important work into the area of nano scaled beams has been carried out in [62, 67] where non local and higher order theories have been used to capture effect sizes with nonlocal forms of boundary conditions. It is unknown if Theorem 1.6 can be used in these kinds of problems and this remains an open question.

However, the results achieved by drawing on Theorem 1.6 were sharper than the results achieved by drawing on Theorem 1.5. However, as we have noted, some of assumptions of the theorems that were achieved by drawing on Theorem 1.6 may not be so straightforward to calculate in general situations, see for example Remark 3.5. On the other hand, some of assumptions of the theorems that were achieved by drawing on Theorem 1.5 may be much easier to calculate. Thus, the theorems that were achieved by drawing on Theorem 1.5 still has advantages, despite its limitations when compared with the results achieved by drawing on Theorem 1.6.

In Chapter 6 I provided a more complete theory of existence, uniqueness and approximation of solutions to the BVP from laminar flow in channels with porous walls. This idea, where I advanced the current state of play via a contractive mapping approach and extended the range of Reynolds number ($\mathcal{R}$) under which a unique solution exists, appeared to be a first time synthesis and application to the problem of laminar flow in channels with porous walls.

In Chapter 6, there are some potential open problems for further research and so let me briefly identify them as well as some of the limitations of these results.

Two of my estimates in Section 6.3 are sharp, while the remaining two appear to be of a rougher nature. Is it possible to sharpen the bounds in Section 6.3? This would have the potential to further extend the range of $\mathcal{R}$ under which (6.1), (6.2) would admit a unique solution. Moreover, is it possible to sharpen the conditions (6.28) and (6.29), perhaps via the consideration of alternative metrics or sets? Once again, this would potentially enable an extension of the range of $\mathcal{R}$ that would ensure uniqueness of solutions.

Unfortunately, the results in Chapter 6 were analyzed under the assumptions on the function $h$ being of a local nature, that is, the domain of $h$ is restricted to closed and bounded sets $\Upsilon$. While these types of assumptions are quite wide-ranging, the very nature of *localized* assumptions means that only limited, localized information about solutions can be necessarily obtained. For instance, in the context of localized assumptions, nothing can be concluded about existence and uniqueness of solutions that may lie outside of the subset $\Upsilon$. However, the results did not provide an analysis in global settings. This is especially important as multiple solutions to fourth-order BVPs (6.1), (6.2) have been shown to exist [226, 112, 100].

Hence I welcome research into all of this.

Traditional approaches to existence, uniqueness and approximation of global solutions for initial value problems involving fractional differential equations have been unwieldy or intractable due to the limitations of previously used methods. This includes certain invariance conditions of the underlying local fixed point strategies. Therefore, in Chapter 7, I particularly chose to draw on Precup's constructive version of CMCM (Theorem 1.8) to prove new global existence, uniqueness, approximation and location of solutions to initial value problems for fractional differential equations (IVP). I did so because Theorem 1.8 advanced knowledge in a way that traditional approaches such as a localized version of Banach's fixed point theorem cannot by establishing *global* existence and uniqueness of solutions and showing that the invariance condition can be avoided. It was also because I was interested in provided a "holistic" analysis of my problems: existence, uniqueness *and* approximation / computation of global solutions. For those who may be only interested in questions regarding existence and uniqueness, Granas's original nonconstructive version of CMCM would also suffice (and deliver novel results) as the assumptions regarding these are essentially the same as Precup's version.

My theorems delivered significant insight into the nature of global solutions by ensuring: existence, uniqueness, location and approximation. However, there is also a price to pay for such insight. In this "economy" there is a trade-off between the scale of assumptions in my theorems and the size of the conclusions that they deliver. My results are no exception to this trade-off.

If we wish to have deeper insight, then we must make more assumptions about my problem. However, as I have shown in Chapter 7, my ideas do enjoy applicability to problems.

I also remark that my main theorems in Chapter 7 can be extended to the case of systems of equations where $f$ would be vector-valued. Finally, I am excited about the opportunities for new lines of inquiry that my present work opens up. This includes opportunities to advance knowledge on global solutions to various initial value problems from the literature and I welcome research into the area.

In Chapter 8, I formed a new uniqueness result for a class of initial value problems involving a coupled system of nonlinear Riemann–Liouville fractional differential equations. The main tools involved the Banach contraction principle and the introduction of a new definition of measuring distance in an appropriate normed space. My new results improve some work of Sun et al. 2012 [254]. However it is not possible for us at this stage to apply my method to the coupled system (8.1), (8.2) due to the complications that arise in the fractional orders of the weighted function and fractional integral. This remains an open question, and so I welcome research into this.

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