

Pricing and hedging of VIX derivatives in modified stochastic models

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# Pricing and hedging of VIX derivatives in modified stochastic models

Jiefei Yu

A thesis in fulfilment of the requirements for the degree of

Doctor of Philosophy



School of Mathematics and Statistics

Faculty of Science

The University of New South Wales

20th August 2022

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After Cboe launched the VIX and its corresponding derivatives, investors have maintained a high level of interest in this unique index. Unlike indices that focus on asset prices, the VIX can reflect investors' subjective expectations of the market by incorporating varying volatility into the calculation. However, in the past literature, the prices of the VIX derivatives have often been just obtained from models with fixed parameters [1] or considered only the long-term volatility of the asset price [2]. Also, we find sparse discussions on the compatibility of the VIX valuation process with modifications of time series stochastic models, such as the regime switching and the subordinator method. At last, we note little literature specifically mentions hedging VIX derivatives and the relevant strategy-obtaining process. To address these issues, we explore the pricing and hedging of VIX derivatives in this thesis, using the VIX European call option as an example. In the pricing process covered in Chapters 3-4, we incorporate the regime switching factor into the continuous 4/2 model and discretize the model based on Heston-Nandi's idea in combination with various modifications, to improve the model's capture of various volatility and changes in the market environment. After comparing the results obtained by the saddlepoint method, we find that those modifications significantly improved the quality of the model, increasing the accuracy of the pricing results and allowing the model to adapt to a more general market environment. In Chapter 5, we hedge the VIX options based on the GARCH framework using a local quadratic hedging approach. After taking advantage of the GARCH model, we optimize the method of obtaining option hedging strategies by reducing the weights of stochastic simulations and reducing the number of simulations required while enhancing the model accuracy.

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# Abstract

After Cboe launched the VIX and its corresponding derivatives, investors have maintained a high level of interest in this unique index. Unlike indices that focus on asset prices, the VIX can reflect investors' subjective expectations of the market by incorporating varying volatility into the calculation. However, in the past literature, the prices of the VIX derivatives have often been just obtained from models with fixed parameters [1] or considered only the long-term volatility of the asset price [2]. Also, we find sparse discussions on the compatibility of the VIX valuation process with modifications of time series stochastic models, such as the regime switching and the subordinator method. At last, we note little literature specifically mentions hedging VIX derivatives and the relevant strategy-obtaining process. To address these issues, we explore the pricing and hedging of VIX derivatives in this thesis, using the VIX European call option as an example. In the pricing process covered in Chapters 3-4, we incorporate the regime switching factor into the continuous 4/2 model and discretize the model based on Heston-Nandi's idea in combination with various modifications, to improve the model's capture of various volatility and changes in the market environment. After comparing the results obtained by the saddlepoint method, we find that those modifications significantly improved the quality of the model, increasing the accuracy of the pricing results and allowing the model to adapt to a more general market environment. In Chapter 5, we hedge the VIX options based on the GARCH framework using a local quadratic hedging approach. After taking advantage of the GARCH model, we optimize the method of obtaining option hedging strategies by reducing the weights of stochastic simulations and reducing the number of simulations required while enhancing the model accuracy.

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# Chapter 1

# Introduction

When people talk about a security or market index, people tend to analyze it from two aspects, which are the price curve and the volatility. Due to the presence of volatility, different assets are assigned different potential/apparent risks. This makes investors not always chase high-yielding assets, but turn to portfolio investments that contain both risky and risk-free assets. Volatility is a statistical measure of the spread of asset gains and losses, also described as "risk" When the volatility of an asset is high, the change in the price of the investments can be dramatic. This means that investors holding such risky assets are likely to suffer significant losses in a short period. In contrast, less volatile assets are also relatively less risky. The magnitude of volatility is determined by various market factors, such as asset type, market period, unexpected events, etc.

In financial research and model construction, volatility is often measured as a function of the variance associated with asset prices or their returns, such as daily return, beta index or Volatility Index as detailed in this thesis. We can divide them into two categories, historical volatility and implied volatility. As a retrospective indicator, historical volatility measures volatility by assessing price changes over time. Historical volatility will be greater over this period if there is some apparent or potential economic event that caused a dramatic change in the capital markets. By contrast, when the effects of this event dissipate, historical volatility will return to its original level. The advantage of this indicator is

#### CHAPTER 1. INTRODUCTION

that its value is relatively simple to calculate, and its modelling techniques are relatively sophisticated. But it also has an obvious drawback: it does not tell investors what to do before future volatility arrives and this makes the indicator difficult to apply.

Consequently, another indicator, implied volatility, has been introduced as a popular tool for those investors who will keep a close eye on the future. As a forward indicator, implied volatility offers investors a way to predict the future volatility of assets. Because implied volatility represents expectations of potential future volatility, it tends to be generalised from the price of the options in the market, rather than looking back at past asset prices. Among the various alternatives for implied volatility, the Volatility Index introduced by Cboe is one of the measures that has received the most attention in recent studies.

After being introduced by Cboe Global Market in 1993, the Volatility Index (VIX) has proven its practical value with its prediction of the stock market and relevant derivatives. As a real-time market index, VIX expresses the expectation of traders for the next 30 days. The VIX tends to rise when a large price swing happens in the market and stays at a low and stable value when the market eases upward in a prolonged bull market (Lin 2017) [3]. Initially, VIX only considered the options on the S&P 100 index and then was modified by Cboe by considering the S&P 500 index (Carr and Wu 2007) [4]. The current VIX is an index of options prices across all strikes in a given time interval, constructed by maturities with average weights that vary deterministically over time. In March 2004, Cboe introduced VIX futures and then launched the VIX option in February 2006. These derivatives became popular quickly with remarkable trading volume growth over time as they offer the forecast traders a mechanism to invest volatility of the S&P 500 directly without considering the price changes, dividends, and interest of the underlying assets. This development takes researchers' motivation to construct models to reproduce the dynamic of VIX and predict its further volatility. This thesis focuses on the pricing and hedging of VIX call options over different maturity time with varying strikes.

Generally, the VIX is simulated in two methods by previous papers. The first method mentioned by Zhang and Zhu (2006) [5] expresses the value of VIX as a square root of the realised variance of S&P index(SPX). This method taking advantage of the relationship of the SPX and the VIX is implemented on those models which are constructed with the joint dynamics of SPX and its instantaneous daily variance over time. The second method considers the instantaneous daily variance of the return only. It calculates VIX as the annualised arithmetic average of the expected daily variance under the risk-neutral measure (Wang,2016) [1]. This thesis prefers the first approximation to model construction because it distinguishes the CIR-type from the flip CIR-type of stochastic volatility in different models.

While the prices of the underlying assets and their returns are difficult to predict directly, their corresponding volatilities are highly predictable(Bollerslev, et al. 1994) [6]. Hence, it is natural to implement this phenomenon to price and hedge relevant derivatives by considering correlations of the financial indexes and their volatility. As an efficient alternative solution to simulate heteroscedasticity over time, stochastic volatility has been regarded as a popular concept in mathematical financial market analysis after the assumption of constant volatility is broken. Some tractable stochastic volatility models were developed in Hull and White (1987) [7] and Heston(1993) [8] have been proven accurate models in real-world market performance.

Although the success of the Heston model is supported by the literature for its reproducing Characteristics and clear financial meaning parameters (Grasselli 2017) [9], some realworld data calibration revealed some limitations. To remedy this shortcoming of this model, an inverse CIR model structure is developed with the power of variance being 3/2(Heston 1997) [10]. Empirical research has revealed that this improvement provides the model with the ability to capture short-term high fluctuations. For example, Drimus (2011) [11] compared the predictive performance of the original Heston model and the 3/2 model on the short-term variance. As a result, the 3/2 model offers a sharp skew for terms when the instantaneous variance grows remarkably while the Heston model flattens the skew. Nonetheless, the Heston model still provides a relatively good fit for moderate market fluctuations and remains a benchmark for financial predictable models in the stochastic volatility field. To make a general model that fits all conditions, Grasselli (2017) [9] offered the 4/2 model by combining the two kinds of variance processes of the Heston model and the 3/2 model. In this way, the process of variance comes from the overlap of the CIR process and flipped CIR process with weights calibrated from market data.

One of the aims of this thesis is to improve the 4/2 model by adding the regime-switching factor and offer its closed-form solution to the value of corresponding options whose prices are expressed as tail expectations and approached by the saddlepoint method. The motivation for this improvement is that the original model mentioned above is assuming that the parameters are constant throughout business time. However, this assumption is not easily satisfied because fluctuations in financial markets naturally affect its parameters. This fact gives us a reason to raise an alternative model structure, the regime-switching framework. The regime-switching models reflect the market trend effect by varying parameters changed following market statements such as boom and bust caused by the trade of investors or other economic activities. An obvious disadvantage of this approach is that it is difficult to observe the underlying rules of parameter variation. One possible solution is to divide the interval in the financial index, defining that the market is in a specific statement when the value of the financial index is in a specific interval. Lee (2007) [12] conducted an empirical experiment to test the advantages of regime-switching models over traditional models for prediction.

Although there is growing literature on the application of regime-switching models to different financial problems, this technique is not often used in the context of pricing VIX options [13]. As a well-known approximation method, the saddlepoint method is used in pricing options under stochastic volatility with regime-switching models. Following this method, the option prices can be obtained directly by solving the first order saddlepoint differential equation and taking it into the relevant saddlepoint approximation. After being introduced by Daniels (1954) [14], the saddlepoint approximation is widely used since most financial derivatives have non-standard density functions that make the calculation of the expectation of the underlying asset very difficult. Consequently, this issue can be solved perfectly with the saddlepoint method. Based on this idea from the literature, we plan to gain the regime-switching 4/2 model by incorporating some fixed parameters into the

regime switching types and then approach the closed-form solution of the price of the VIX call option by the alternative saddlepoint method with the modified cumulative probability function and the regime-switching matrix.

On the other hand, those financial stochastic models based on Heston's idea represent volatility through current spot market parameters when the model considers the variance to be instantaneous. This requirement raises the problem that some parameters are difficult to obtain, such as the variance of volatility. Discrete-time Generalised Autoregressive Conditional Heteroskedasticity model(GARCH) is a practical candidate for dealing with continuous-time dilemmas because it does not require any current spot parameters. As a standard application to this class, Heston and Nandi (1993) [8] provided a closed-form GARCH option model to describe the iterative process of log returns and their volatility. Because the instantaneous variance is not explicit to traders, the GARCH model does not imply an immediate conflict. Instead, this model assumes that the log returns are derived from the risk-free rate and that the value of the conditional variance is determined by its historical performance. The moment generating function (MGF) is also available since a closed-form is provided. Therefore, we can use the saddlepoint method to solve the tail expectations of option prices by extending the MGF of the conditional variance to the MGF of the VIX.

Another disadvantage of continuous-time models based on the Heston type is that these models do not accurately describe trading periods and volumes. They assume that market participants operate uniformly and evenly. That is not true because the trade frequency expressed as the volume, often varies during the trade time. This fact leads to a regular period in processes corresponding to underlying assets. However, traditional continuoustime models constructed by real time and standard Brownian motions cannot describe those periods, which typically follow the non-Gaussian distribution. To deal with the problem of non-normality, we draw on Shirvani's approach of introducing a dependent model in the process of modelling for asset returns and used the subordinator to generate commercial times instead of real times [15].

After Clark (1973) [16] used it in financial modelling to describe continuous-time time

#### CHAPTER 1. INTRODUCTION

variation in the cotton futures market, the subordination process, also called random time change, is widely used in finance. This approach, while maintaining the original process X(T), assumes that the time T driving this process obeys some time-varying distribution T(t). This approach allows the original model to adapt to more market situations and capture more market characteristics, such as the heavy-tailed phenomenon. Aguilar and Kirkby (2021) [17] considered time subordination to stochastic processes in several market models to test whether the introduction of these subordinator captures complex phenomena such as volatility aggregation or long memory. Janczura and WyAlomanska(2011) [18] analysed three models of different diffusion distributions, presented the similarities and differences between the models and pointed out their main characteristics. Buchmann et al. derive an overall model by subordinating multivariate Brownian motion to a subordinator model. The excellent characteristics of the multivariate class allow such models to be applied to option pricing based on PIDEs or tree methods. Finally, they performed pricing using European and American options on two assets.

This thesis introduces the 4/2 GARCH model with its subordination modification based on the traditional Heston Nandi model. This model combines the advantages of a discrete model framework that is more practical to implement in real-world fundamental markets and the adaptability of the 4/2 model to the volatility of different magnitudes and periods. Moreover, because of the utilisation of subordination, this model can periodically vary its time parameters based on a specific distribution to simulate different trading volumes and trading periodic phenomena.

After introducing the valuation of financial derivatives on the VIX, it was natural to wonder if we could hedge these derivatives. A hedging is a strategic portfolio whose intention is to reduce the risk of adverse price volatility of an underlying asset. For this purpose, the value of such a portfolio tends to move in the negative correlations of the hedged asset in order to be used to offset losses arising from volatility in the prices or returns of the hedged asset. Therefore, unlike constructing a portfolio for profit, when investors decide to build a hedged asset portfolio, they pay for insurance to prevent the hedged asset from suffering a great financial loss from an unexpected event. While this action does not usually reduce the probability of this contingency occurring, its impact will be kept within acceptable limits by this portfolio after the worst case happens.

We can demonstrate the construction and effectiveness of a hedged portfolio with a simple example. An investor is preparing to order a turkey for Thanksgiving to celebrate the holiday. As can be expected, the price of turkeys will rise dramatically on Thanksgiving Day due to the high demand. Therefore, if the investor buys the turkey on Thanksgiving Day, he will pay extra money due to market volatility.

Being a visionary, this investor reserves the turkeys in advance at a relatively reasonable price by paying a deposit. In this way, he gains the right to buy the turkey on Thanksgiving Day at that price. With this move, the upper bound of the investor's Thanksgiving budget has been fixed in advance, regardless of how the cost of the turkey changes in the market. Of course, rights, unlike obligations, can be waived in unfavourable circumstances. If, on Thanksgiving Day, the turkey drops significantly below the predetermined price due to over-stocking at the mall, etc. Then the investor also has the option to forfeit the previously acquired rights and instead purchase the turkey at the current market price.

This example illustrates the process of hedging and reveals the critical issue of hedging: whether the losses avoided by the hedge cover the cost of the hedge. Whether the right to hedge is executed at maturity time or not, the cost of hedging is unavoidable. Therefore, after estimating the value of the asset, the evaluation of the hedging cost will be the key task.

The cost of hedging varies depending on the hedging method. For example, if delta hedging is implemented, the investor needs to hold the same number of derivatives corresponding to the hedged asset, such as options and futures, in order to achieve delta-neutral and offset the full risk caused by its unexpected volatility. Despite the simplicity and reliability of this hedging method, it is not easy to consistently achieve a delta-neutral statement. Traders must constantly trade the derivatives in hand to make the hedging strategy real time and effective. Trading too frequently, especially with expensive derivatives such as options, raises the cost of hedging and makes hedging meaningless. If the asset volatility

#### CHAPTER 1. INTRODUCTION

does not come as expected, the unnecessary hedging cost will be a significant loss.

To compensate for this drawback, the quadratic hedging method is chosen to hedge the VIX options in this thesis. Quadratic hedging produces a martingale measure with optimal variance. It makes the initial capital position of the hedge equal to the expectation of the discounted option cash flows taken under this measure(Secomandi 2022) [19]. The last part of this thesis discusses the hedging work of the corresponding options by minimising the quadratic risk. From past research, little literature focuses on the hedging of VIX derivatives under the GARCH framework, while this framework has been widely used for option pricing. To fill this gap, recent studies have estimated the parameters with information from both the underlying asset and its conditional volatility under risk-neutral measures. In this thesis, we investigate the hedging of VIX options with the discrete-time Heston–Nandi GARCH model and discuss the simplification allows us to obtain a more accurate hedging strategy with fewer Monte Carlo experiments than the traditional quadratic hedging approach.

# Chapter 2

# Literature and background

In this section, we focus on some of the models that will be used in this thesis and their corresponding hedging methods. These methods and models will be used in combination in the following sections to address the valuation and hedging of VIX derivatives.

# 2.1 Continuous-time stochastic model

## 2.1.1 Heston model

We begin our model construct with a basic continuous-time stochastic volatility model proposed by Heston (1993) [8]. This model was presented to correct the bias caused by the assumption of normality in the traditional Black-Scholes model. In fact, in the assumptions of the conventional model, stock returns are often considered to be normally distributed and the mean and the variance are constant to its long-term stable value. However, this assumption often does not hold in real markets. Motivated by this fact, Heston suggested a closed-form solution, the Heston model, in which conditional volatility is correlated with the spot stock prices to capture the bias that arises from such correlation. Because it can accurately capture the dynamics of spot prices and deviations in the distribution of spot returns, the Heston model has the feature of correcting for deviations caused by option pricing. By this method, one can characterise the option models with the dynamics of spot price and conditional volatility.

Although there are some assumptions to be satisfied, we still use the risk-neutral measure for all option pricing and hedging work. This is because compared to historical measures, risk-neutral measures take into account price drift caused by risk aversion and are more suitable for calculations of strong volatility assets such as derivatives. Since European options are priced and hedged under EMM, we skip the measure transformation in this chapter and directly introduce the Heston model in risk-neutral form. Given a risk-neutral probability space  $(\Omega, \mathcal{F}, \mathbb{Q})$  with an information filtration  $\mathcal{F}_t$  where  $t \in (0, T)$ . Heston introduced the following model

$$dS_t = rS_t dt + \sqrt{V_t S_t} dZ_1$$

$$dV_t = k(\theta - V_t) dt + \sigma \sqrt{V_t} dZ_2$$
(2.1.1)

In the model, the conditional volatility is also called Cox-Ingersoll-Ross (CIR) process that was first developed by Cox, Ingersoll, and Ross (1985) [20] for imparting interest rate to stochastic modelling. With this process, the volatility possesses a mean-reverting structure as k represents the mean-reversion speed and  $\theta$  represents the long-term volatility.

As a result, Heston presented a stochastic model for volatility-varying assets. Compared with the traditional Black-Scholes method, this model is versatile enough to describe stock options, bond options, and currency options (Heston 1993) [8] and capture most bias during the option pricing. The Correlation between conditional volatility and the stock price is necessary to capture the biases. However, the empirical test of Mikhailov and Noegel (2003) [21] revealed that the Heston model may not capture the biased fully, especially for short maturities. The diffusion power of the volatility  $\frac{1}{2}$  performed inaccurately in some literature. It is because the Heston model is difficult to capture short-term high volatility dynamics (Jones 2003) [22]. These considerations motivated one to develop a more advanced model that would compensate for this disadvantage.

## 2.1.2 3/2 model

Therefore, we introduced a 3/2 stochastic volatility model to solve the problem in the Heston model mentioned in the previous subsection. This model inherits the mean-reverting structure but raises the power of  $v_t$  on the second stochastic differential equations to make the process of  $V_t$  non-affine. The 3/2 model was first extended to the application of this model to options and swaps. According to Lewis's study, the diffusion power of  $V_t$  is nearly 1.3 instead of 0.5, supporting the application of the 3/2 model. The study of Carr and Sun (2007) [4] showed some works on the model effectiveness of different powers of  $V_t$  and obtained the conclusion that the numerical results of the non-affine structure are more accurate than those of affine structure implemented in the CIR process.

Given a general probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with an information filtration  $\mathcal{F}_t$  where  $t \in (0, T)$ . The 3/2 model is built by multiplying an extra  $V_t$  to the conditional volatility process dynamics of the Heston model as follows

$$dS_t = rS_t dt + \sqrt{V_t} S_t dZ_1$$

$$dV_t = V_t [k(\theta - V_t) dt + \sigma \sqrt{V_t} dZ_2]$$
(2.1.2)

To make the conditional volatility of this model the same structure as that of the Heston model, we let  $\hat{V}_t = \frac{1}{V_t}$ ,  $\hat{k} = k\theta$  and  $\hat{\sigma} = -\sigma$ . With this change, the model becomes

$$dS_t = rS_t dt + \frac{1}{\sqrt{\hat{V}_t}} S_t dZ_1$$

$$d\hat{V}_t = \hat{k}(\theta - \hat{V}_t) dt + \hat{\sigma} \sqrt{\hat{V}_t} dZ_2$$
(2.1.3)

Because of the power increment, the 3/2 model offers extreme paths to present short-term instantaneous volatility. Drimus (2011) [11] showed that when the instantaneous variance increases, the short-term biases turn to be flat in the Heston model, but the short-term biases turn to be sharp in the 3/2 model. This numerical result means that the Heston model incorrectly outlines the variance smiles in a market with the high-pressure market.

## 2.1.3 4/2 model

The long-term market performance of the original Heston model has proven to be a useful tool for the S&P 500 and its derivative indices such as the VIX. However, empirical research also revealed its shortcomings in capturing short-term volatility because the power of the conditional variance is  $\frac{1}{2}$ , which leads to the error of downward-sloping volatility of the variance smile(Drimus 2011) [11]. Since the volatility term is flipped to the denominator, this process mimics the fast reverts process that occurs when volatility goes up and down. The motivation to maintain good performance over short and long business periods led us to consider combining the original Heston model with the 3/2 model. Thanks to Grasselli (2017) [9], who provided an effective model to implement this idea, called the "4/2" model.

$$dS_t = rS_t dt + (a\sqrt{V_t} + \frac{b}{\sqrt{V_t}})S_t dZ_1$$
  

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dZ_2$$
(2.1.4)

## 2.2 Discrete motivation and GARCH model

After addressing the inability of the Heston model to capture short-term high volatility, we look at another problem with this model: the unobservable continuous parameters. Because of that, continuous-time stochastic volatility models are usually hard to implement and test with real-world market data. Although the literature assumes that those parameters are known and often gives an experimental one, it is not possible to extract the volatility variable precisely from the discrete observations of the spot asset price from the total conditional variance change in continuous-time stochastic volatility models(Heston 1993) [8]. At the same time, the calculation of the model's implied volatility would become very tedious: the current value of implied volatility can only be obtained by calculating a large amount of other option price information. Therefore, one realistic idea of model modification is to overcome the difficulty in obtaining the parameters of the continuoustime stochastic model but keeping the variance variation of time in the Heston model. One candidate is the GARCH model. Option pricing under a GARCH model is a popular method in modern financial analysis. In 1995, Duan introduced a solid theoretical foundation based on the locally risk-neutral valuation relationship to price options under GARCH models (Duan 1995) [23]. Heston and Nandi introduced a closed-form option pricing formula for a spot asset whose variance and its asset itself follow a GARCH(p,q) process. It provides option formulas for stochastic volatility models that are easier to calculate than previous approaches, while the model can be valued based entirely on observable historical data. In keeping with the idea of Heston's continuous-time stochastic volatility model, the correlation between volatility and returns in this model is similar to that of the Heston model, but with reduced observation requirements. Instead of requiring investors to keep an eye on current market conditions to update returns and volatility in real time, this model allows investors to price options by using historical data. Here we apply its simplest but most useful branch, GARCH(1, 1) model to assume the log asset price and conditional variance in the risk-neutral measurement as follows:

$$R_{t} = r_{t} + \lambda h_{t} + \sqrt{h_{t} z_{t}^{*}}$$

$$h_{t} = \omega + \beta h_{t-1} + \alpha (z_{t-1}^{*} - \delta^{*} \sqrt{h_{t-1}})^{2}$$
(2.2.1)

where

$$z^*(t) = z(t) + (\lambda + \frac{1}{2})\sqrt{h(t)}$$
$$\delta^* = \delta + \lambda + \frac{1}{2}$$

In the first SDE of (2.2.1),  $R_t$  is the log return of the underlying assets' spot price such as  $R_t = \log \frac{S_t}{S_{t-1}}$ . Because this model is built under a risk-neutral measure,  $r_t$  represents the risk-neutral interest rate at time t. The second assumption of the GARCH(1, 1) model is considered  $z^*(t)$  to be standard normally distributed. The second SDE of (2.2.1) introduces  $h_t$  as the conditional variance of the log return between the time interval bounded by t and t - 1. After the measure modification, the GARCH model gains the ability to yield a closed-form solution for European options prices and capture the kurtosis and fat tail characteristics of returns distribution in the real world market. Furthermore, the autoregressive structure of the model describes the leverage effect and persistence of the conditional variance.

Following Heston's similar approach, we transformed the continuous-time Heston model into a discrete version, the Heston-Nandi model. We could derive a discrete version of the 4/2 model based on the continuous-time 4/2 model and GARCH(1,1) model. we assume the new model follows two statements,

1. The underlying asset price logarithm return and the conditional variance h(t) follow the autoregressive process,

$$R_{t} = r_{t} + \lambda h_{t} - \frac{1}{2}h_{t} + (a\sqrt{h_{t}} + b\frac{1}{\sqrt{h_{t}}})z_{t}$$

$$h_{t} = \omega + \beta h_{t-1} + \alpha(z_{t-1} - \delta\sqrt{h_{t-1}})^{2}$$
(2.2.2)

where  $r_t$  is the risk-neutral interest rate for time intervals and  $z_t$  is a standard normal distribution.  $h_t$  is the conditional variance of the logarithm return for time intervals and is known from the information set at time t. All those three variables are updated by discrete innovation steps and kept constant in the time interval.

2. The value of a call option with a one-time interval fits the Black–Scholes-Rubinstein formula with spot price following the conditional log-normal distribution.

# 2.3 Subordinator and regime switching

## 2.3.1 Subordinator

Whether continuous or discrete models are implemented, the innovation dynamics of the log return and conditional variances are always assumed to be standard Brownian diffusion processes. However, Rachev rejected this hypothesis in 2005 and noted that asset returns exhibit asymmetries and fat tails(Rachev et al. 2005) [24] that one cannot model the time series spot value(in both continuous-time and discrete-time model) by using traditional standard Brownian diffusion with the real world time like data. Based on this finding, it seems questionable to simply define the innovation part of returns and variance iteration as standard Brownian motions.

A classic solution to this problem that arises in the model analysis is to introduce a subordinated process to describe the so-called "business time" that corresponds to the update of financial information and trade behaviours. The subordinated process in finance defines a random time change like Y(t) = X(T(t)) assuming two processes T and X are independent of each other. This change was first used by Clark (1973) [16] to replace Brownian motion to capture skews and fat-tail during pricing model construction and Carr implemented this modification to the options field in 2003(Carr et al. 2003). [25]. Following this modification, it is evident that finding a suitable subordinator according to the relevant processes is crucial work. In this thesis, we analyze the Brownian diffusion with the Lévy subordinator such as the gamma subordinator.

### 2.3.2 Lévy subordinator

A stochastic process G(t) is said to be a homogeneous Lévy process with G(0) = 0 and with independent increments over time intervals that do not overlap. Moreover, the increment only depends on the difference in time but is independent of the initial spot time. This property is sometimes referred to as the stationary increment. A Lévy process G(t) with non-decreasing trajectories is called a Lévy subordinator which is regarded as a business time used to record the upcoming information and trade behaviours. If we apply a typical Lévy subordinator, gamma subordinator, to the traditional Heston Nandi model, the model becomes

$$R_{G} = r_{G} + \lambda h_{G} - \frac{1}{2}h_{G} + \sqrt{h_{G}}z_{G}$$

$$h_{G} = \omega + \beta h_{G-1} + \alpha(z_{G-1} - \delta\sqrt{h_{G-1}})^{2}$$
(2.3.1)

where G(t) follows the gamma process with its Lévy density

$$G(t;\alpha,\beta) = \frac{\alpha e^{-\beta t}}{t} \mathbf{1}_{t>0}$$

## 2.4 Regime switching

Another questionable assumption of the traditional stochastic model is that the stochastic process of the underlying asset remains constant throughout the business time. This means that the parameters used in the process of estimating the prices of assets and their relationship between the log returns and their conditional variances remain unchanged But this assumption does not correspond to the actual market situation. For example, the regime dependence in the VIX index could be expressed as a continuous-time Markov chain. Using the historical VIX value, Guo and Wohar (2006) [26] provided the statistical evidence of two structural breaks, demonstrating that market volatility varies over time.

A possible method to endogenize this time variation into the stochastic process is the Markov regime switching model implemented by Hamilton (1990) [27] to express the U.S. economic cycle characterised by a periodic transition from boom to bust and vice versa. This model is popular in stochastic model construction because it can capture extra market trends by switching the models' crucial parameter values between different market statements.

The reason for the switching is comprehensive which may be the result of different volumes of trades, emotional volatility of the traders, an external event significant enough to cause a market jump, or other financial cases. Following the empirical study, Elliott et al. (2007) [28] offered a price approximation of the volatility derivatives under a Markov regime switching model modified from the original Heston model. Vo (2009) [29] discussed the effect of applying the regime switching factors in the crude oil market and figured out that regime switching improvement of original models performed better in curving the actual market volatility. In this thesis, this model is used to implement the parameter switches between regimes with different market statements. Specifically, this model modifies the original 4/2 model by improving the conditional variance of the underlying assets by a market statement coefficient decided by a two-state Markov chain whose states represent different statements of the market, named the Markov regime switching 4/2 model.

## 2.4.1 4/2 Regime switching model

Following Elliott et al. (2007) [28], we adopt the regime switching modification on the original stochastic volatility model and transfer it into a continuous-time Markov-modulated stochastic volatility model.

Firstly, we assume the market dynamics follows a continuous-time finite-state observable Markov chain  $M_t$  with state-space  $S = \{s_1, s_2, s_3...s_N\}$ . Each element of the statespace represents a kind of market condition like "boom" or "depression". Accordingly, the dynamics of a Markov chain is expressed by the N-by-N matrix A with each element  $a_{ij}$ , where  $i, j \in (1, 2, ..., N)$  under the historical probability measure  $\mathbb{P}$ . In this way,  $a_{ij}$ describes the intensity of transition from the state i to state j, satisfying  $a_{ij} \ge 0$  if  $i \neq j$  and  $\Sigma_{i=1}^{N} a_{ij} = 0$  (Elliott and Lian 2013) [30]. Furthermore, Elliott has shown that the Markov chain has a semi-martingale representation:

$$M_T = M_0 + \int_0^T A' \cdot M_t dt + D_T$$
 (2.4.1)

where  $D_T$  is a  $\mathbb{R}^N$  martingale with respecting to the filtration generated by  $M_T$  under the historical probability measure  $\mathbb{P}$  and A' is the transpose of A. In this session, we only assume that the long-term variance  $\theta$  switches according to the state of the indicator  $M_t$ for simplification, which means

$$\theta_t = <\vec{\theta}, M_t > \tag{2.4.2}$$

where  $\vec{\theta}$  is the state space of long-term variance and  $\langle , \rangle$  denotes the scalar product in  $\mathbb{R}^N$ . In this way, the traditional 4/2 model turns to

$$dS_t = rS_t dt + (a\sqrt{V_t} + \frac{b}{\sqrt{V_t}})SdZ_1$$
  

$$dV_t = k(\theta_t - V_t)dt + \sigma\sqrt{V_t}dZ_2$$
(2.4.3)

## 2.4.2 Saddlepoint method and its approach to VIX derivatives

From the above definition, the price of the European VIX call option can be expressed as a right-hand tail expectation under a risk neutral measure like  $E_{\mathbb{Q}}[VIX - K]^+$ . Then our subsequent work is to find an analytic solution to approaching this expectation, if possible. Various approximation methods in the literature have been adopted to find this type of tail expectation such as the Edge-worth expansion and the method of steepest descents. This section introduces the background of the saddlepoint approximation that is commonly recognised improved from the steepest descent method and applies it to approach the tail expectation. Although this method may sound like it was invented for statistical tasks, it does exist in many areas of scientific research. When CGF is known, one can obtain the closed-form of the tail probabilities of the distributions and their expectation. By this method Lugannani and Rice (1980) [31] offered a closed-form of the tail probabilities under the Gaussian assumption. Wood et al. (1993) [32] improved the saddlepoint approximation with a non-normal distribution and obtained a better approximation when the Gaussian assumption is not satisfied. Rogers and Zane (1999) [33] used it to research the underlying asset price by solving its cumulative distribution function and then applied the result to calculate the price of a European option. Glasserman and Kim (2009) [34] applied this method to approach the analytic approximation of the cumulative distribution functions of an affine jump-diffusion process.

The core of this method is constructed by approximating the conditional expectation (tail expectation):

$$C(Y,t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}[(Y-K)^+]$$
(2.4.4)

where

$$\mathbb{E}_t^{\mathbb{Q}}[(Y-K)^+] = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{k(z)-zK}}{z^2} dz \qquad r \in (0,\alpha_+) \quad \alpha_+ > 0$$

here Y is the underlying asset which we are interested in, T and K are the delivery time and strike price, k(z) is the cumulant generating function (CGF) of Y. There are several methods to solve this integral like the Lugannani-Rice formula, the frozen method local quadratic method and so on. In this thesis, we prefer the alternative saddlepoint method for the complication of the VIX call option closed-form solution which we will offer in this section.

# 2.5 The closed-form solution of the $VIX_T^2$ and its call option

Unlike other options, the VIX option, which originated in 2006, uses the Cboe Volatility Index as its underlying asset. If we follow the original method, we need to solve the equation

$$C(VIX_T, t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}[(VIX_T - K)^+]$$
(2.5.1)

That means we need to find the CGF of VIX itself. However, the VIX is so complex that we can hardly find a closed-form solution of its CGF directly. We noticed that there is always a linear relationship between  $VIX_T^2$  and  $V_T$  in both the 3/2 model and the traditional Heston model. Rationally, we guess that there may exist a similar affine structure that expresses  $VIX_T^2$  by a linear form of  $V_T$ . The approach of  $VIX_T^2$  by implementing the Ito formula on the underlying assets and solving its expectation till time t is expressed as follows:

$$VIX^{2} = \frac{1}{\tau} \lim_{N \to \infty} \Sigma_{i=1}^{N} \mathbb{E}_{T}^{\mathbb{Q}} [(\log \frac{S_{T_{i}}}{S_{T_{i}-1}})^{2}]$$
(2.5.2)

where  $\tau = \frac{30}{365}$ . For the first part of this integral, we consider the process  $e^{kT}V_T$  which satisfies the following SDE:

$$de^{kT}V_T = e^{kT}(k\theta dT + \sigma\sqrt{V_T}dZ_T^2)$$
(2.5.3)

From that, we obtain:

$$\mathbb{E}_{T}^{\mathbb{Q}}[V_{s}] = e^{k(T-s)}V_{t} + \theta(1 - e^{k(T-s)})$$
(2.5.4)

Thus, we could express (2.13) as: set the initial time t as 0 and solve the second part of the integral in (2.13) and the approximate affine structure of  $VIX^2$  will be shown as

$$VIX_{T}^{2} \approx \frac{a^{2}(1-e^{-k\tau})}{k\tau}V_{T} + a^{2}\theta(1-(\frac{(1-e^{-k\tau})}{k\tau})) + 2ab + \frac{1}{\tau}b^{2}\int_{T}^{T+\tau} \mathbb{E}_{T}^{\mathbb{Q}}[\frac{1}{V_{s}}]ds$$

$$(2.5.5)$$

As the closed-form expression of its CGF is known, we could implement the saddlepoint method to approximate the tail expectation. By following these steps, the closed-form of option pricing models of the VIX with initial time 0, delivery time T and strike K is presented as:

$$C(VIX_T, T) = e^{-rT} \mathbb{E}_0^{\mathbb{Q}}[(VIX_T - K)^+]$$
  
=  $e^{-rT} \mathbb{E}_0^{\mathbb{Q}}[(\sqrt{VIX_T^2} - K)^+]$  (2.5.6)

To obtain the results of the option price, we should discuss how to obtain the tail expectation with a known cumulant generating function (CGF) first, then present the method of approaching that CGF. By the definition of the tail expectation,

$$E[(\sqrt{X} - K)^{+}] = \int_{K^{2}}^{\infty} \sqrt{X} p(X) dX \qquad (2.5.7)$$
where p(X) is the probability density function of X. The analytical closed-form expression of this tail expectation was offered by Kowk and Zheng (2014) [2], as:

$$\mathbb{E}_{0}^{\mathbb{Q}}[(\sqrt{VIX_{T}^{2}}-K)^{+}] = \frac{1}{4\sqrt{\pi i}} \int_{r-i\infty}^{r+i\infty} \frac{e^{k(\hat{z})+g(\hat{z})}}{\hat{z}^{3/2}} d\hat{z} \qquad r \in (0,\alpha_{+})$$
(2.5.8)

where k(z) is the CGF of the  $VIX_T^2$  and analytic in some open vertical strip  $\{z \in C : \alpha_- < Rez < \alpha_+\}$  with  $\alpha_- < 0$  and  $\alpha_+ > 0$ ,  $g(z) = log(1 - erf(\sqrt{z}K))$ .  $\hat{z}$  is the positive root of the alternative saddlepoint equation

$$k'(\hat{z}) + g'(\hat{z}) - \frac{3}{2\hat{z}} = 0.$$
(2.5.9)

### 2.6 VIX derivatives quadratic hedging

In the last decade, discrete model-based stock pricing models have become popular such as the GARCH model mentioned above. This motivated one to consider the hedging model of the relevant pricing model. The core of hedging is to replicate the terminal payoff of the underlying assets by introducing a portfolio consisting of risky and risk-free assets with suitable strategies. A standard example is hedging a VIX call option with the expiration date T, strike price K, and its payoff function  $H = (VIX_T - K)^+$ . Following this example, the strategies that hedge the contingent claims of the option dynamically during the whole trade time is introduced by a  $\mathcal{F}_t$  process. As investors are trying to reduce the sensitivity of their assets to market volatility, the total gain of a good hedging strategy should be close to H as possible. Moreover, as no further information is offered at t = 0, the relevant strategies do not depend on any cash flows other than initial costs in the option hedging process.

Another point that we should note about the discrete hedging of the GARCH model is that achieving a full hedging statement during the whole trade time is impossible as hedging

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strategies will not change till the next discrete time interval arrived. Naturally, instead of finding a hedging portfolio that perfectly replicates the payoff function, we would like to introduce a hedging strategy that can minimise a particular measure of the difference between contingent claims and the value of the hedging portfolio. One popular measure of this aspect is quadratic hedging risk. Several approaches for minimising the quadratic hedging measure have been introduced in the literature, Laurent (1999) [35] outlined the results and developments in the area of hedging contingent claims in incomplete markets. Schweizer (1995) [36] identified the optimal strategy with the smallest variance of net loss  $H - G_t(\theta)$ . Currently, there are two main quadratic hedging methods used for quadratic hedging risk measures and their corresponding optimal strategies. One of them is called "global quadratic hedging" and suggests minimising the total risk. Its strategy is selffinancing so its cumulative cost process is constant. However, following the existence and the uniqueness of a total risk-minimizing strategy discussed by Schweizer (1995) [36], in such a condition where bounded mean-variance trade off of asset prices is not available, the total risk-minimizing strategy may not exist. The other is to control the local incremental cost of risk hedging, called "local quadratic hedging". This strategy no longer needs selffinancing condition but is mean-self-financing. Thus, the cumulative cost process is a martingale in historical measure  $\mathbb{P}$ .

Considering the inevitable set of difficulties that arise when applying continuous-time stochastic models above, this thesis uses the discrete-time frame as the basis for pricing and hedging. In this frame, stock price information is updated at fixed time points  $t \in 1, 2, 3...$ with the same intervals. We then follow the definition of portfolios strategy mentioned by Schweizer and Martinis (1998) [37] as  $\varphi_t = (\theta_t, \eta_t)$  where  $\theta_t$  represent the number of risky assets held at time t and  $\eta_t$  is the number of the risk-free asset(bond) held at time t. This frame also assumes that traders cannot modify their stock strategies in parallel with stock price changes, i.e., that strategy changes have a lag, even though the lag can be infinitely close to zero. That means the traders keeps their strategies unchanged during (t - 1, t]and the number of stocks  $\theta_t$  is  $\mathcal{F}_{t-1}$ -measurable when  $t \geq 1$ . On the other hand, under the local quadratic hedging strategy we choose in this thesis, bonds are always modified in parallel with the stock to automatically balance the total price of the hedged portfolios. Therefore, the number of bonds  $\eta_t$  is  $\mathcal{F}_t$  and held unchanged during [t, t+1).

Because no trade happens before t = 1, we define  $\phi_0 = (0, V_0)$  where  $V_0$  is the initial value of the hedging portfolio. Following these initial conditions, the value of the portfolio is defined as

$$V_t = \theta_t X_t + \eta_t \tag{2.6.1}$$

and its discrete cumulative gain from initial time 1 up to time t is

$$G_t(\theta_1, \theta_2 \dots \theta_t) = \Sigma_{u=1}^T \theta_u \Delta X_u \tag{2.6.2}$$

where  $\Delta X_u = X_u - X_{u-1}$ .

At last, the discrete cost process will be defined as

$$C_t = V_t - G_t$$
  
=  $V_t - \sum_{u=1}^t \theta_u \Delta X_u.$  (2.6.3)

Following the above definitions, we can then define quadratic hedging in an incomplete market environment that is raised by discrete trade restrictions from the real world. In fact, in an incomplete market environment, regardless of what hedging method we use and how much it costs, we can never fully hedge the potential risk in future. Therefore, there are two main options for hedging: 1, to achieve the minimum exposure at an appropriate hedging cost such as the quadratic hedging mentioned in this thesis. 2, to achieve the minimum hedging cost in a reasonable exposure such as quantile hedging. In 1995, Schweizer introduced self-financial hedging to deal with the risk control problem in the incomplete market of the real world measure(Schweizer 1995) [36]. A self-financing hedging strategy is a hedging strategy that always maintains  $V_t = V_0 + G_t$  during the total business time. According to the definition of cost process, this hedging strategy always keeps the cost process constant, i.e.  $C_t = V_0$ . Under the premise that the cost process is unchanged, the goal of this strategy is to minimise the quadratic loss between contingent claims  $H_T$  and hedging portfolio  $V_T$  at maturity time T. This loss is represented as a hedging risk  $R_t$ expressed as

$$R_t = E[(H_T - V_T)^2 | \mathcal{F}_t], \qquad (2.6.4)$$

and the hedging proposed is

$$\underset{(V_0,\theta)\in R\times\Theta}{\operatorname{arg\,min}} \quad R_t \tag{2.6.5}$$

where  $\Theta$  represents the set of all feasible self-financing trading strategies at time t. Since this hedging is performed to deal with the quadratic criterion, known as the global quadratic hedging error, this hedging is called global quadratic hedging (Heath et al. 2001) [38].

Local quadratic hedging is another quadratic hedging mentioned by Schweizer in 1988 [39] first which discards the self-financing rules. It no longer requires the hedging strategy cost process to be constant but imposing  $V_t = H_t$  by modifying the risk-free asset  $\eta_t$  at time t. Because of this, its hedging purpose is to minimise the squared increment in hedging cost caused by strategy modification like

$$\underset{(V_t,\theta)\in R\times\Theta}{\operatorname{arg\,min}} \quad E[(C_{t+1}-C_t)^2|\mathcal{F}_t]. \tag{2.6.6}$$

Although some literature says that local is inferior to global because it minimises the risk arising from all future time increments and not only the next one (Augustyniak, 2017) [40], the local model is more suitable for hedging in the GARCH when we consider some characteristics of this model: given t statement variables, the GARCH looks forward and simulates future processes of stocks return and volatility but ignore the initial state variables such as  $S_0$  and  $h_0$ . The parameters of GARCH are also only used to estimate future processes and cannot be used for backward projection of previous steps. This makes it impossible to use the GARCH model only to obtain information before t from

the current statement when it is not quite observable. Compared with the global one, the local hedging focuses on the stepwise increments from the current time to the next onetime spot and is independent with of all information before t. Furthermore, The stepwise iteration property of the GARCH model coincides with the stepwise minimisation property of local hedging.

Chapter 3

# Pricing VIX call option in a 4/2model with a regime switching approximation by the saddlepoint method

# 3.1 Introduction

Many esteemed scholars have done plenty of research on the "change" over a long period. Just like the change of tactics is driven by the opponent, market fluctuation is full of uncertainty. Unlike others who focus on the underlying assets, some traders with unique views on future market volatility or risk intend to make their judgements and predictions with financial derivatives. To catch the change and measure the volatility, a good market index is necessary for those special traders. The Volatility Index (VIX) has been considered the world's premier barometer of equity market volatility since it was introduced by the Chicago Board Options Exchange in 1993. Compared with the other Indexes displaying the objective statements of the market, the VIX Index reflects investors' opinions and attitudes towards future market volatility subjectively, which means it could be regarded as the market's "fear gauge".

Considering the excellent performance of the VIX, the Chicago Board Options Exchange launched the Volatility Index (VIX) future plan in March 2004 and VIX options in February 2006 as its derivatives to markets. Compared with constructing relative portfolios following the weights calculated, directly investing in VIX derivatives allows the investors to spend less and have a better risk-control ability. Unexpectedly, the trade volume of those derivatives increased remarkably in the last decades. As a result, increasing papers have been published to improve the accuracy of the VIX derivatives pricing approximation. Zhang and Zhu (2006) [5] produced a closed-form approximation of VIX futures under the Heston's stochastic volatility framework(Heston 1993) [8]. Sepp (2008) [41] developed an analytical methodology for pricing and hedging options by employing the generalised Fourier transform for swaps on the realised volatility and variance and options on these swaps. Lian and Zhu (2013) [42] presented an exact solution for VIX option prices under a stochastic volatility model with simultaneous jumps in asset prices and volatility. Zhu and Lian (2011) [43] provided a closed-form exact solution for the pricing of discrete-sampled variance swaps and volatility swaps.

In 2015, Lin et al. developed a 4/2 stochastic volatility plus jumps model and obtained a closed-form solution for the joint Fourier-Laplace transform so that the VIX derivatives can be priced. Combing the classic Heston model with the 3/2 model, one could get a new stochastic volatility model called the "4/2 model" as the instantaneous volatility part structured as  $a\sqrt{V_t} + \frac{b}{\sqrt{V_t}}$  with given constants a and b. Then, they utilised the model to make a parameter consistently framework VIX derivatives and gained the approximated prices. To figure out the merits of the 4/2 model, they tested their model with the Heston model and the 3/2 model and compared their different performances in practice. The result illustrated that the 4/2 model indeed has an "overall performance" in pricing VIX derivatives and fits more general market status.

However, parameters should be inconsistent and changeable during the whole monitored time. To ensure the accuracy of the approximation, we should consider the volatility of

parameters with the volatility of target derivatives themselves. One of the useful methods to show the parameter volatility is assuming that the parameters switch from different regimes as time pass. The regime switching model is the stock parameters switching among a finite number of states depending on the time. The market regimes could reflect the state of the underlying economy, the general mood of investors in the market or other economic factors (Zhang and Chan 2016) [44]. However, little work has been done about pricing the VIX derivatives in the context of regime switching models, while Vo (2009) [29] found reliable evidence of regime switching in the market, which supposed that the regime switching stochastic volatility model may result in a more accurate pricing approximation as it reflects more information on the market. To fill this gap, it is meaningful to modify the consistent 4/2 model by adding the regime switching factors.

This thesis aims to take the regime switching factor into the 4/2 model and launch a suitable modified closed-form solution of the VIX option pricing approximation which transfers the value of derivative prices into tail expectations. To make the whole evaluation process more operational, we prefer the saddlepoint method which was suggested by Daniels (1954) [14]. The structure and steps of this chapter are planned as follows: First, we build a consistent 4/2 model and contribute the closed-form solution for the call option price. This part will be mentioned in section 2. Second, we introduce the saddlepoint method and apply it to solve the tail expectation which represents the option price in section 3. Then, we change the consistent parameters into the regime switching ones, resulting in a 4/2 regime switching model and work out a comprehensive closed-form solution of option prices and offer their approximated method. These works will be done in section 4. In section 5 and 6 we will make a numerical test and discuss the conclusion of this method.

## 3.2 The 4/2 model and its martingale judgement

In this section, we introduce the 4/2 model which includes both CIR and flipped CIR components in its instantaneous variance. From previous work, we find that the Heston

model takes the instantaneous variance as a mean-reverting squared Bessel process and performs well on the long-term stable market situation, while the subsequent the 3/2model which is the inverse of a CIR process capturing the short-term market volatility better. Naturally, one will consider if we could produce a model which contains components reflecting both the short- and long-term volatility. Thus, as a combination of the classic Heston model and the 3/2 model, the 4/2 model assumes that the instantaneous variance is linearly related to  $\sqrt{V_t}$  and  $\frac{1}{\sqrt{V_t}}$ , where  $V_t$  expresses the instantaneous variance of the model based on CIR component at time t. Considering that the 4/2 model is based on two classic models which could both be solved by saddlepoint methods, it is meaningful to apply saddlepoint methods to this model during the solution processing. In this chapter, we focus on taking the saddlepoint method to calculate the call option price of the 4/2model. For simplification, we apply the 4/2 model without jump to describe the variance process  $V_t$  of underlying asset price  $S_t$  as follows:

$$dS_t = rS_t dt + (a\sqrt{V_t} + \frac{b}{\sqrt{V_t}})S_t dZ_1$$
  

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dZ_2$$
(3.2.1)

where r is determined as the risk-free interest rate,  $\theta$  is the long-term variance, k is the mean-reverting rate,  $\sigma$  is the volatility of variance,  $dZ_1$  and  $dZ_2$  are correlated Wiener processes with constant correlation parameter  $\rho$  in the risk-neutral measure  $\mathbb{Q}$ . In fact, by taking a or b zero, we could recover (3.2.1) into the 3/2 model and Heston model respectively. Correspondingly, (3.2.1) involves both the volatility processes of the Heston model and the 3/2 model. Specially, when we take a = 0 and  $\nu_t = 1/V_t$  and the second equation of (3.2.1) becomes:

$$d\nu_t = \nu_t k_\nu (\theta_\nu - \nu_t) dt + \sigma_\nu {\nu_t}^{3/2} dZ_2$$
(3.2.2)

where

$$k_{\nu} = k\theta - \sigma^2, \sigma_{\nu} = \frac{k}{(k\theta - \sigma^2)}, \sigma_{\nu} = -\sigma.$$

In that way, (3.2.2) simulates the volatility process of the 3/2 model and composes the whole 3/2 model when we take a = 0. On the other hand, it is obvious that the 4/2 model

could be transferred into the Heston model when we just let b = 0. Thus, we could say, the volatility of the 4/2 model is driven by both the Heston and the 3/2 components.

Similar to other stochastic volatility models, the 4/2 model also needs to be implemented in a given risk-neutral measure to ensure no arbitrage during the trades. However, we should handle it carefully because the risk-neutral measure  $\mathbb{Q}$  may not exist under the 3/2 model framework (Lewis 1999) [45]. Lewis defined that the risk-neutral measure is not applicable when the discounted asset price  $\tilde{S}_t = S_t e^{-rt}$  is a strict local martingale in the measure  $\mathbb{Q}$ . Thus, to determine if the risk-neutral measure exists, we could check whether  $\tilde{S}_t$  is a true martingale (not a local martingale). Generally, we could make the Feller non-explosion test for  $V_t$  for both the risk-neutral probability measure and historical probability measure to see whether the local martingale conditions are met. Lin and Li (2017) [3] proved that the discounted stock price  $\tilde{S}_t$  is a true martingale, if and only if

$$2k\theta \ge \sigma^2 \ge 2k\theta + 2\rho\sigma b \tag{3.2.3}$$

*Proof*: From Grasselli (2017) [9], we transfer (3.2.1) from risk-neutral measure to historical measure by a measure change :

$$d\hat{Z}_2 = dZ_2 - \rho(a\sqrt{V_t} + \frac{b}{\sqrt{V_t}})dt$$
 (3.2.4)

where  $\hat{Z}_2$  presents a standard Brownian motion under the historical measure. With this change,  $V_t$  in (3.2.1) could be expressed as:

$$dV_t = [k(\theta - V_t) - \rho\sigma(aV_t + b)]dt + \sigma\sqrt{V_t}dZ_2.$$
(3.2.5)

Grasselli mentioned, the discounted asset price is a local martingale if (3.2.5) meets the Feller condition as

$$2k\theta + 2\rho\sigma b \ge \sigma^2 \tag{3.2.6}$$

and the Feller condition is satisfied with the risk-neutral measure when

$$2k\theta \ge \sigma^2 \tag{3.2.7}$$

Therefore, the discounted asset price  $\tilde{S}_t$  is a true martingale when the  $\sigma^2$  is valued just in the range pointed out above.

# 3.3 The closed-form solution of VIX call option and its approach

An option is a right to purchase or sell an asset or derivatives at a specific price in a specific period. As the call option and put option follow put-call parity and have similar solution processes, we will just talk call option which represents the buyer part of the option in the remainder of this chapter. Without loss of the generality, we consider the call option price C(Y, t) as a conditional expectation (tail expectation):

$$C(Y,t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}[(Y-K)^+]$$
(3.3.1)

where

$$\mathbb{E}_t^{\mathbb{Q}}[(Y-K)^+] = \frac{1}{2\pi i} \int_{r-i\infty}^{r+i\infty} \frac{e^{k(z)-zK}}{z^2} dz \qquad r \in (0,\alpha_+) \quad \alpha_+ > 0]$$

and Y is the underlying asset which we are interested in, T and K are the delivery time and strike price, k(z) is the cumulant generating function (CGF) of Y. There are several methods to solve this integral like the Lugannani-Rice formula, the frozen method local

quadratic method and so on. In this chapter, we prefer the alternative saddlepoint method for the complication of the VIX call option closed-form solution which we will offer in this section.

## **3.3.1** The closed-form solution of the $VIX_T^2$ and its call option

Unlike other options, the VIX option, which originated in 2006, uses the Cboe Volatility Index as its underlying asset. If we follow the original method, we need to solve the equation

$$C(VIX_T, t) = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}}[(VIX_T - K)^+]$$
(3.3.2)

That means we need to find the CGF of VIX itself. However, the VIX is so complex that we can hardly find a closed-form solution of its CGF directly. We noticed that there is always a linear relationship between  $VIX_T^2$  and  $V_T$  in both the 3/2 model and the traditional Heston model. Rationally, we guess that there may exist a similar affine structure that expresses  $VIX_T^2$  by a linear form of  $V_T$ . At last, the approach of  $VIX_T^2$ by implementing the Ito formula on the underlying assets and solving its expectation till time t is expressed as follows:

$$VIX^{2} = \frac{1}{\tau} \lim_{N \to \infty} \Sigma_{i=1}^{N} \mathbb{E}_{T}^{\mathbb{Q}} [(\log \frac{S_{T_{i}}}{S_{T_{i}-1}})^{2}]$$
  

$$\approx \frac{1}{\tau} \lim_{N \to \infty} \Sigma_{i=1}^{N} (a^{2} \mathbb{E}_{t}^{\mathbb{Q}} [V_{T_{i-1}}] + b^{2} \mathbb{E}_{T}^{\mathbb{Q}} [\frac{1}{V_{T_{i-1}}}] + 2ab) \Delta T_{i} \qquad (3.3.3)$$
  

$$= \int_{T}^{T+\tau} (a^{2} \mathbb{E}_{T}^{\mathbb{Q}} [V_{s}] + b^{2} \mathbb{E}_{T}^{\mathbb{Q}} [\frac{1}{V_{s}}] + 2ab) ds$$

where  $\tau = \frac{30}{365}$ . For the first part of this integral, we consider the process  $e^{kT}V_T$  which satisfies the following SDE:

$$de^{kT}V_T = e^{kT}(k\theta dT + \sigma \sqrt{V_T} dZ_T^2).$$
(3.3.4)

From that, we obtain:

$$\mathbb{E}_T^{\mathbb{Q}}[V_s] = e^{k(T-s)}V_t + \theta(1 - e^{k(T-s)}).$$
(3.3.5)

Thus, we could express (3.3.3) as: set the initial time t as 0 and solve the second part of the integral in (3.3.3) and the approximate affine structure of  $VIX^2$  will be shown as

$$VIX_T^2 \approx \frac{a^2(1-e^{-k\tau})}{k\tau}V_T + a^2\theta(1-(\frac{(1-e^{-k\tau})}{k\tau})) + 2ab + \frac{1}{\tau}b^2\int_T^{T+\tau}\mathbb{E}_T^{\mathbb{Q}}[\frac{1}{V_s}].$$
 (3.3.6)

Thus, the crucial work of figuring out the approximated expression of  $VIX^2$  is to find the solution of  $\int_T^{T+\tau} \mathbb{E}_T^{\mathbb{Q}}[\frac{1}{V_s}] ds$ . By using proposition 8.1. of Chan and Platen's work (2015) [46] and taking parameter p to be 0 and keeping  $\beta_1 > 0$ , we have

$$\mathbb{E}[\int_{0}^{T} \frac{1}{V_{s}} ds] = -\frac{d}{d\mu} \frac{1}{2^{v} V_{0}^{m}} e^{-\frac{b_{1} V_{0}}{\gamma(e^{b_{1} T} - 1)} + b_{1} m T} (\frac{b_{1} e^{b_{1} T}}{(e^{b_{1} T} - 1)\gamma}) \times (\frac{b_{1}^{2} V_{0}}{\gamma^{2} sinh^{2}(\frac{b_{1} T}{2})})^{v/2} \frac{\Gamma(\beta_{1})}{\Gamma(1+v)} {}_{1}F_{1}(\beta_{1}, 1+v, \frac{b_{1} V_{0}}{\gamma(e^{b_{1} T} - 1)})$$
(3.3.7)

where

$$dV_T = (a_1 - b_1 V_T)dt + \sqrt{2\gamma V_T}dW_t$$
  
$$\beta_1 = 1 + m + v/2, m = \frac{1}{2}(a_1/\gamma - 1), v = \frac{1}{\gamma}\sqrt{(a_1 - \gamma)^2 + 4\mu\gamma}$$

Without loss of generality, we could regard the integral  $\int_T^{T+\tau} \mathbb{E}_T^{\mathbb{Q}}[\frac{1}{V_s}] ds$  as the minus result of  $\int_0^{T+\tau} \mathbb{E}_T^{\mathbb{Q}}[\frac{1}{V_s}] ds$  and  $\int_0^T \mathbb{E}_T^{\mathbb{Q}}[\frac{1}{V_s}] ds$  and imply the formula above separately, expressing the closed-form of  $VIX^2$  by (3.3.6) and (3.3.7). As the closed-form expression of its CGF is known, we could implement the saddlepoint method to approximate the tail expectation. By following these steps, the closed-form of option pricing models of the VIX with initial time 0, delivery time T and strike K is presented as:

$$C(VIX_T, T) = e^{-rT} \mathbb{E}_0^{\mathbb{Q}}[(VIX_T - K)^+]$$
  
=  $e^{-rT} \mathbb{E}_0^{\mathbb{Q}}[(\sqrt{VIX_T^2} - K)^+]$  (3.3.8)

To obtain the results of the option price, we should discuss how to obtain the tail expectation form  $E[(\sqrt{X} - K)^+]$  with a known cumulant generating function (CGF) first, then present the method of approaching that CGF. By the definition of the tail expectation,

$$E[(\sqrt{X} - K)^{+}] = \int_{K^{2}}^{\infty} \sqrt{X} p(X) dX$$
 (3.3.9)

where p(X) is the probability density function of X. If we have known the CGF of X, we could obtain the analytical closed-form expression of this tail expectation according to Kowk and Zheng's work in 2014 [2] like :

$$\mathbb{E}_{0}^{\mathbb{Q}}[(\sqrt{VIX_{T}^{2}}-K)^{+}] = \frac{1}{4\sqrt{\pi i}} \int_{r-i\infty}^{r+i\infty} \frac{e^{k(\hat{z})+g(\hat{z})}}{\hat{z}^{3/2}} d\hat{z} \qquad r \in (0,\alpha_{+})$$
(3.3.10)

where k(z) is the CGF of the  $VIX_T^2$  and analytic in some open vertical strip that owns the range  $\{z \in C : \alpha_- < Rez < \alpha_+\}$  with  $\alpha_- < 0$ ,  $\alpha_+ > 0$  and  $g(z) = log(1 - erf(\sqrt{z}K))$ .  $\hat{z}$  is the positive root of the alternative saddlepoint equation

$$k'(\hat{z}) + g'(\hat{z}) - \frac{3}{2\hat{z}} = 0$$
(3.3.11)

## **3.3.2** The approach of $VIX_T^2$ CGF

Since we have set up the closed-form solution of the VIX call option and its approximation, it is feasible to figure out the closed-form of the CGF of  $VIX_T^2$ . Considering that  $VIX_T^2$ could be expressed as an affine structure of  $V_T$ , we should gain the closed-form CGF of  $V_T$  first and figure out the CGF of  $VIX_T^2$  by the linear transform. Thanks to Chan and Platen who offered the affine structure of the governing dynamics equation of  $V_t$  [46], we are able to obtain the CGF closed-form of  $V_T$  in the analytic form:

$$E_0^{\mathbb{Q}}[e^{zV_T}] = B(z;T)V_0 + \gamma(z;T), Rez < \alpha_+, \quad \alpha_+ > 0$$
(3.3.12)

where

$$B(z;T) = \frac{2kz}{\sigma^2(1-e^{kT})z + 2ke^{kT}}$$
$$\gamma(z;T) = -\frac{2k\theta}{\sigma^2}\log(1+\frac{\sigma^2 z}{2k}(e^{-kT}-1))$$

where  $\alpha_+$  is determined by requiring the arguments of the above logarithm terms to be greater than zero. Considering the affine structure of  $VIX_T^2$  and variance process  $V_T$ , we are able to derive the CGF of  $VIX^2$  as k(z)

$$k(z) = \log \mathbb{E}_0^{\mathbb{Q}} (e^{zVIX_T^2})$$
  
=  $\beta z + E_0^{\mathbb{Q}} [e^{\alpha zV_T}]$   
=  $\beta z + B(\alpha z; T)V_0 + \gamma(\alpha z; T), \qquad Rez < \alpha_+$   
(3.3.13)

where

$$\begin{split} \alpha &\approx \frac{a^2(1-e^{-k\tau})}{k\tau} \\ \beta &\approx a^2\theta(1-(\frac{(1-e^{-k\tau})}{k\tau})) + 2ab + \frac{b^2}{\tau} \mathbb{E}_0^{\mathbb{Q}}[\int_0^{T+\tau} \frac{1}{V_s} ds] - \frac{b^2}{\tau} \mathbb{E}_0^{\mathbb{Q}}[\int_0^T \frac{1}{V_s} ds] \end{split}$$

### 3.4 Regime switching

As we mentioned before, we need to consider the short- and long-term situations of the market and make the model more general. However, we find that the Heston model, the 3/2 model and the 4/2 model all assume that the parameters are consistent during the whole monitored time as common models implicitly ignore long-term or wider economic factors that may influence the volatility dynamics (Elliott and Lian 2013) [30]. To fill this gap, we attempt to modify the traditional model by adding the regime switching factor. Combining stochastic volatility models with regime switching allows us to take the short-term price dynamics effect into the whole pricing approximated process. Several empirical

works have shown that the regime switching factor contributes obviously to the market dynamics and fits the "real-world" market better. Following Elliott et al. (2007) [28], we adopt the regime switching modification on the original stochastic volatility model and transfer it into a continuous-time Markov-modulated stochastic volatility model.

First, we assume the market dynamics follows a continuous-time finite-state observable Markov chain  $M_t$  with state-space  $S = \{s_1, s_2, s_3...s_N\}$ . Each element of the statespace represents a kind of market condition like "boom" or "depression". Accordingly, the dynamics of a Markov chain is expressed by the N-by-N matrix A with each element  $a_{ij}$ , where  $i, j \in (1, 2, ...N)$  under the historical probability measure  $\mathbb{P}$ . In this way,  $a_{ij}$ describes the intensity of transition from the state *i* to state *j*, satisfying  $a_{ij} \ge 0$  if  $i \neq j$ and  $\sum_{i=1}^{N} a_{ij} = 0$  (Elliott and Lian 2013). Furthermore, Elliott has shown that the Markov chain has a semi-martingale representation:

$$M_T = M_0 + \int_0^T A' \cdot M_t dt + D_T$$
 (3.4.1)

where  $D_T$  is a  $\mathbb{R}^N$  martingale with respecting to the filtration generated by  $M_T$  under the historical probability measure  $\mathbb{P}$  and A' is the transpose of A. In this section, we only assume that the long-term variance  $\theta$  switches according to the state of the indicator  $M_t$ for simplification, which means

$$\theta_t = <\vec{\theta}, M_t > \tag{3.4.2}$$

where  $\vec{\theta}$  is the state space of long-term variance and  $\langle , \rangle$  denotes the scalar product in  $\mathbb{R}^N$ . In this way, the traditional 4/2 model turns to

$$dS_t = rS_t dt + (a\sqrt{V_t} + \frac{b}{\sqrt{V_t}})SdZ_1$$
  

$$dV_t = k(\theta_t - V_t)dt + \sigma\sqrt{V_t}dZ_2.$$
(3.4.3)

As the model changed, we need to figure out a new CGF, which fit the regime switching conditions. Following the previous work applied to the Heston model offered by Zhang and Chan (2016) [44], we let  $\mathcal{F}_0^v$  and  $\mathcal{F}_0^M$  be the natural filtration decided by the Brownian motion factor and the regime switching factor till time t. If the information about the regime switching factor presented by the Markov chain till time T is known, the CGF with a regime switching factor of the variance process  $CGFV_T(z; T, V_0|\mathcal{F}_T^M)$  should be

$$CGFV_T(z;T,V_0|\mathcal{F}_T^M) = B(z;T)V_0 + \Gamma(z;T), Rez < \alpha_+$$
(3.4.4)

where

$$B(z;T) = \frac{2kz}{\sigma^2(1-e^kT)z + 2ke^kT}$$
  

$$\Gamma(z;T) = \int_0^T \langle k\vec{\theta}B(z;t), M_t \rangle dt$$

However, under the condition that the evaluated time t = 0, the information from the evaluated time to the delivery time T is unknown to us. That means, instead of  $\mathcal{F}_T^M$ , the information of regime switching conditions we could collect belongs to the natural filtration  $\mathcal{F}_0^M$ . Thus, the CGF of  $V_T$  till time t is

$$CGFV_{T}(z;T,V_{0}|\mathcal{F}_{0}^{M}) = \mathbb{E}^{\mathbb{Q}}[CGFV_{T}(z;T,V_{0}|\mathcal{F}_{T}^{M})|\mathcal{F}_{0}^{M} \vee \mathcal{F}_{0}^{v}]$$

$$= B(z;T)V_{0} + \mathbb{E}^{\mathbb{Q}}[\int_{0}^{T} \langle k\vec{\theta}B(z;t), M_{t} \rangle dt|\mathcal{F}_{0}^{M} \vee \mathcal{F}_{0}^{v}].$$
(3.4.5)

From (3.4.5), we could find that the crucial point of solving the CGF of  $V_T$  is solving the core calculation  $E^{\mathbb{Q}}[\int_0^T \langle k\vec{\theta}B(z;t), M_t \rangle dt | \mathcal{F}_0^M]$ . From Elliott and Lian (2013) [30], the core calculation part could be expressed as

$$\mathbb{E}^{\mathbb{Q}}[\int_{0}^{T} < k\vec{\theta}B(z;t), M_{t} > dt | F_{0}^{M} \lor \mathcal{F}_{0}^{v}] = <\Phi(z,T)M_{0}, I>$$
(3.4.6)

where

$$\Phi(z,T) = A'T + diag\{k\vec{\theta}\frac{2}{\sigma^2}log[1 + \frac{\sigma^2 z}{2k}(e^{-kT} - 1)]\},$$

I = (1, 1...1) in  $\mathbb{R}^N$  and A' is the transpose of A.

After obtaining the closed-form of  $V_t$ 's CGF, we naturally consider the CGF of VIX with a regime switching till the time T. Just like (3.3.13), the CGF of VIX with a regime switching till T could be expressed as an affine structure of  $V_T$  CGF.

$$k(z) = log \mathbb{E}_{0}^{\mathbb{Q}}(e^{zVIX^{2}})$$

$$= \beta(\theta)z + logCGFV_{T}(\alpha z; T, V_{0})$$

$$= \beta(\theta)z + B(\alpha z; T)V_{0} + \langle \Phi(\alpha z, T)M_{0}, I \rangle, \quad Rez < \alpha_{+}$$
(3.4.7)

From (3.4.7),  $\beta$  contains the regime switching parameter  $\theta$  so that we could refer to the method of gaining the closed-form expression of  $V_t$ 's CGF and assume that the information till time T is known first. As the regime switching factor is only included in the CGF of VIX, the price of the call option of the regime switching model could be expressed as

$$E_0^{\mathbb{Q}}[(\sqrt{VIX^2} - K)^+ | \mathcal{F}_T^M \lor \mathcal{F}_T^v] = \mathbb{E}^{\mathbb{Q}}[\mathbb{E}_T^{\mathbb{Q}}[(\sqrt{VIX^2} - K)^+] | \mathcal{F}_0^M \lor \mathcal{F}_0^v]$$

$$= \frac{1}{4\sqrt{\pi}i} \int_{r-i\infty}^{r+i\infty} \frac{e^{\mathbb{E}^{\mathbb{Q}}[k(\hat{z})|\mathcal{F}_0^M \lor \mathcal{F}_0^v] + g(\hat{z})}}{\hat{z}^{3/2}} d\hat{z}$$
(3.4.8)

where

 $r \in (0, \alpha_+)$ 

We need to calculate  $\mathbb{E}^{\mathbb{Q}}[k(\hat{z})|\mathcal{F}_0^M \vee \mathcal{F}_0^v]$  if we want to approximate the option price shown in (3.4.8) similarly as evaluating the CGF of  $V_t$ , we need to figure out the expectation under the natural filtration  $\mathcal{F}_0^M$ . Thus, the CGF of VIX with a regime switching factor till time T evaluated at time 0,  $k_{rs}(\hat{z})$  is

$$k_{rs}(\hat{z}) = E^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[k(\hat{z})|\mathcal{F}_{T}^{M}]|\mathcal{F}_{0}^{M} \vee \mathcal{F}_{0}^{v}]$$
  
$$= \mathbb{E}^{\mathbb{Q}}[\int_{0}^{T} \langle k(\hat{z};t,\vec{\theta},VIX), M_{t} \rangle dt|\mathcal{F}_{0}^{M} \vee \mathcal{F}_{0}^{v}]$$
  
$$= \langle \Phi_{vix}(\hat{z};T,\vec{\theta},VIX)M_{0}, I \rangle$$
  
(3.4.9)

where

$$\Phi_{vix}(\hat{z};T,\vec{\theta},VIX) = A'T + diag\{k(\hat{z};T,\vec{\theta},VIX)\}$$

At last, similarly to the approach method for consistent parameters VIX model offered by Kowk and Zhang (2017) [2], the first order approximation of the call option price under the 4/2 regime switching model could be approximated as:

$$\left[\mathbb{E}_{0}^{\mathbb{Q}}(VIX-K)^{+}\right] \approx \frac{\sqrt{2}e^{k_{rs}(\hat{z})+g(\hat{z})-\frac{3}{2}log(\hat{z})}}{4\sqrt{\pi[k_{rs}'(\hat{z})+g''(\hat{z})+\frac{3}{2\hat{z}^{2}}]}}$$
(3.4.10)

where  $\hat{z}$  is the positive real root of the alternative saddlepoint equation (3.3.11).

Also, the higher order form of this approximation can be obtained by performing the Taylor expanding of  $e^{k_{rs}(z)+g(z)-\frac{3}{2}log(z)}$  up to the corresponding order near the estimated saddlepoint  $\hat{z}$ . As an example, the second-order form of this approximation obtained by the fourth-order Taylor expansion on the above exponential is presented as follows:

$$[\mathbb{E}_{0}^{\mathbb{Q}}(VIX - K)^{+}] \approx \frac{\sqrt{2}e^{k_{rs}(\hat{z}) + g(\hat{z}) - \frac{3}{2}log(\hat{z})}}{4\sqrt{\pi[k_{rs}''(\hat{z}) + g''(\hat{z}) + \frac{3}{2\hat{z}^{2}}]}} \times \left(1 + \frac{1}{8}\frac{k_{rs}'''(\hat{z}) + g'''(\hat{z}) + \frac{9}{\hat{z}^{4}}}{[k_{rs}''(\hat{z}) + g''(\hat{z}) + \frac{3}{2\hat{z}^{2}}]^{2}} - \frac{5}{24}\frac{[k_{rs}''(\hat{z}) + g''(\hat{z}) - \frac{3}{\hat{z}^{2}}]^{2}}{[k_{rs}''(\hat{z}) + g''(\hat{z}) + \frac{3}{2\hat{z}^{2}}]^{3}}\right).$$

$$(3.4.11)$$

There is no doubt that this form of approximation becomes more accurate as the order increases. We note, however, that even the second order form of the approximation already requires the fourth-order derivative of the saddlepoint equation,  $k_{rs}^{\prime\prime\prime\prime}(z)$ . The potential computational complexity of the fourth-order derivatives can be very high, given that we are estimating in the framework of the regime switching model. After balancing the model accuracy and computational complexity, we acknowledge that the higher order forms of the approximation are able to increase the accuracy of the option pricing model, but do not perform the corresponding numerical test in the next section.

### 3.5 Numerical test

In this section, we present numerical examples for VIX call option prices with different delivery time and strikes by using the saddlepoint approximation method. In each combination of time and strike, we test the approximation under both the regime switching

parameter	value
V0	$0.87^{2}$
k	3.46
р	0.82
σ	0.14
a	1.5
b	0.01
r	0.08
$\vec{ heta}$	[0.009, 0.004]
$M_0$	[1,0]
А	$\begin{pmatrix} 3 & -3 \\ -5 & 5 \end{pmatrix}$

Table 3.1

framework and the non-switching one. Then, we figure out the accuracy of the regime switching model by comparing the numerical result with Monte-Carlo simulation ones. Although the regime switching model can accommodate a variety of shifts in market states, to keep the test straightforward, we assume that the market in which this option is issued has only two states, one defined as a boom and the other defined as a bust. The numerical test in this section assumes this market that starts in a boom state and may switch to a depressed state in the whole business time. In this case, due to price depression caused by the market state switch, the final European VIX call option price in this market is lower than the corresponding option price that would have remained in the boom state. In addition, we choose a delivery time of 0.7 to 0.9 so that the long-term volatility included in the 4/2 model can be reflected and distinguished from the short-term volatility. All other parameters needed in the calculation process are given in Table 3.1, the calculation results are shown in Table 3.2 and the accuracy issues are demonstrated in Table 3.3.

Obviously, it is difficult to find a market that fits our assumptions in the financial data. For the lack of market information, this section uses Monte Carlo simulations with 10000

#### 3.5. NUMERICAL TEST

-							
Т	0.7		0.8		0.9		
strike	switch	constant	switch	constant	switch	constant	
0.07	0.1807	0.1956	0.1828	0.20025	0.1845	0.2049	
0.11	0.1387	0.1532	0.1409	0.15785	0.1421	0.1625	
0.15	0.09709	0.1111	0.09921	0.11575	0.1000	0.1204	
0.19	0.05646	0.06908	0.05854	0.07372	0.05789	0.07836	

Table	3.2

Т	0.7			0.8			0.9		
strike	switch	MC	error	switch	MC	error	switch	MC	error
0.07	0.1807	0.1802	0.277	0.1828	0.1811	0.939	0.1845	0.1818	1.485
0.11	0.1387	0.1403	1.140	0.1409	0.1412	0.212	0.1426	0.1423	0.211
0.15	0.09709	0.1005	3.393	0.09921	0.1018	2.544	0.1009	0.1024	1.465
0.19	0.05646	0.0606	6.832	0.05854	0.062	5.581	0.06016	0.0626	3.898
0.23	0.0212	0.0207	2.415	0.02284	0.0224	1.964	0.02406	0.0227	5.991

#### Table 3.3

repetitions as true values for verification and the results were listed as "MC" in the table. During the Monte Carlo test, we implemented a traditional Heston simulation method and modified the original model by adding a parameter  $\epsilon$  which is valued at 1 or 0. The frequencies of 1 and 0 are produced by the Monte Carlo matrix A to simulate the regime switching statements. In this way, we could express the regime switching Monte Carlo model for the value of VIX as

$$VIX_{n} = \frac{a^{2}(1-e^{-k\tau})}{k\tau}V_{n} - \left(\frac{b^{2}(1-e^{-k\tau})}{k\tau} - 2b^{2}\right)\frac{1}{V_{n}} - b^{2}\theta\left(1 - \frac{(1-e^{-k\tau})}{k\tau}\right)\frac{1}{V_{n}^{2}} + a^{2}\theta\left(1 - \frac{(1-e^{-k\tau})}{k\tau}\right) + 2ab$$
(3.5.1)

z is a standard normal random variable,  $\delta$  is the step length of time in the test, the value of  $\theta_{simu}$  is picked from  $\theta$  state-space and decided by the simulated results of  $\epsilon$  in each step and the value of  $V_n$  could be obtained by the Milstein method (Platen 2010) [47] which is used to approximate numerical solution of a stochastic differential equation.

From table 2, we figure out that the prices of the call option calculated from the regime switching 4/2 stochastic volatility model are obviously different from their non-switching counterparts in each time-strike combination. This numerical result corresponds to our

expectation that the price of the option in a state switching market is lower than option prices in a market that remains unchanged. A more likely reason for this is that boom markets have greater trade volume and trade frequency than a state switching market, so asset prices in this market will be given higher volatility, causing investors to expect a greater rise in the VIX, which in turn enhances the price of the corresponding call option. The results also show that when delivery time T gets close to 1(one whole year), the price of the VIX option also increases, as longer business hours bring greater uncertainty and higher VIX values.

From table 3, we show the accuracy of the regime switching model by the error%. We find that the relationship between error and delivery time was not significant. However, in every delivery time, an increase in strikes always leads to an increase in error%. One possible reason for this is that when the strikes gradually increase, the value of the price is so low that a very small inaccuracy can lead to a large error%.

### 3.6 Conclusion

We have extended the 4/2 model in a regime switching framework and approached first order numerical results. In order to capture both short and long-term price dynamics effects of the market volatility, we adopt the regime switching method to reflect the short dynamics factor and apply the 4/2 model to describe the long-term effects. To ensure that the discounted stock price is a true harness under the risk-neutral measure, we examine the Feller condition to ensure this. Subsequently, we introduce an alternative saddlepoint method by which the price of VIX options expressed by a tail expectation could be priced. Then, we take a numerical test and point out that the 4/2 model with regime switching provides a different approximated value from the original model and show its accuracy by comparing the approached results with simulation values from the Monte-Carlo method. Summarising all findings, the introduction of the regime switching factor leads to a significant improvement in VIX option pricing approximations, suggesting that regime switching helps in accuracy issues and describes the market trend "more truly". Finally, several works remained in this chapter, and we expect to solve them in the future. For example, we have not yet found a way to simplify the calculation of the higher order closed-form of pricing or to numerically approach the higher order derivatives of the saddlepoint equation, preventing us from effectively improving the accuracy of the model by increasing the order of the approximation. Furthermore, a non-Gaussian saddlepoint method should be offered as none can ensure whether the VIX follows the normal distribution. These topics deserve more attention and will be aspects and subjects of future work. CHAPTER 4. GENERALISATION OF THE GARCH MODEL AND ITS APPLICATION TO THE PRICING OF VIX CALL OPTIONS

Chapter 4

# Generalisation of the GARCH model and its application to the pricing of VIX call options

## 4.1 Introduction

As time goes by, the modern financial market becomes more and more mature and diversified. Not only the assets themselves but also the derivatives and volatility could be priced by those advanced traders who believe they have special predictability on the trends of individual stocks, portfolios or even the total market. As a real time index of traders' expectation of market trends, Volatility Index(VIX), which has been calculated by Chicago Board Options Exchange since 1993, expresses the traders' subjective feelings about risk when making financial decisions. However, the definition and criteria for the VIX were not fixed from the beginning. For example, in 1993, the Cboe used the S&P100 rather than the S&P500 to obtain the value of VIX. The calculation methods of VIX vary but are usually based on the volatility of the underlying asset or derivative. After the VIX was recognised and widely used by many investors, Cboe launched VIX futures and VIX options in 2004 and 2006 respectively.

One of the common methods to calculate the VIX derivatives is the Heston method which is provided by Heston in 1993. He purposed that the volatility of the relevant assets is continuous over time and follows the Cox–Ingersoll–Ross process(CIR). Following this assumption, the volatility is related to the current spot prices which are instantaneous. Much of the literature demonstrates the superiority of this model but also signals some shortcomings of this model in the empirical tests. For example, some parameters like  $\xi$ , describing the variance of instantaneous volatility  $V_t$ , is hard to gain accurately as we cannot just separate the variance factor of the total change of the volatility as the time trend and jumps also contribute to the complete change of  $V_t$ . This model flaw is not only present in the Heston model but is prevalent in all continuous-time financial models. On the other hand, the way of the Heston model that describes the volatility does not capture the changing curve well when it runs into short-term and dramatic fluctuations led by some sudden market events. In this condition, the skew of the Heston model goes flat, but the real-world skew of volatility should steepen as the  $V_t$  increases. At last, the original Heston model assumes the diffusion terms of asset price and its volatility follow standard Gaussian distribution. But the empirical work has shown that curve of densities of return and volatility are both fat tails compared with the Gaussian distribution. Therefore, the main objective of this chapter is to propose modified stochastic models that can solve the above problem by combining various modelling techniques.

Regarding the first problem, we can introduce some popular discrete stochastic models, inherit their modelling ideas and bring some modifications to extend them. Discrete-time Generalised Autoregressive Conditional Heteroskedasticity models(GARCH) are good candidates for this part. Heston and Nandi offered a GARCH type model which describes the evolution of log return and its volatility. Instead of implying the instantaneous variance which is continuous but not so practical to traders, the GARCH model assumes that the logarithm return is derived by the increment over time and the change of conditional variance decided by its historic paths and spot innovation term. Furthermore, Heston offered the risk-neutral modification of this model and the closed-form of the moment generation function(MGF) of the conditional variance. This makes it possible to price the option of VIX by transferring it to a tail expectation and applying the saddlepoint which will be mentioned in section 5.

After considering the shortcomings of the continuous-time stochastic model, we will proceed to the problem of the Heston model's inability to correctly capture short-term fluctuations. To overcome the wrong skew exhibition and some other limitations, Heston (1997) [10] presented the 3/2 model and assume the  $V_t$  follows the flipped CIR process because it could return to the long variance faster when the magnitude of volatility is high. These characteristics coincide with empirical research and real market performance. Bakshi et al. (2006) [48] ran a numerical test to simulate the dynamics of a Constant Elasticity of Variance volatility and compare it with both the Heston model and the 3/2model. From this work, the coefficient of the diffusive power is about 1.3, closer to the  $3/2 \mod(1.5)$  than the Heston  $\mod(0.5)$ . But we still agree that the Heston  $\mod(0.5)$ performs well in stress-free and long-period conditions. To make a general model that fits all conditions, Grasselli (2017) [9] offered the 4/2 model by combining the two kinds of the  $V_t$  processes of the Heston model and the 3/2 model. In this way, the process of  $V_t$  comes from the overlap of the CIR process and flipped CIR process with weights calibrated from market data. In Section 5, this chapter will combine the 4/2 model constructed by Grasselli and Heston's discretization method for the 1/2 model to construct the 4/2 GARCH model.

The last issue planned to be discussed is the varying business time. The empirical studies indicate that the probability distribution of the innovation term may not follow Gaussian much and own fat tails. This is because there are some assets and derivatives that are affected by market cycle and trading volume, and their innovation terms exhibit different characteristics from the standard normal distribution. This fact violates the Gaussian assumption of the Black-Scholes family model, impairing the predicted ability of the original model by unexpected drifting. To address the effects and capture the fat tail of the return, we could implement a random time subordinator to take place of the constant time in the model. The structure of the chapter is as follows. Section 2 will present a GARCH framework of the Heston-Nandi model with its VIX option structure. We will introduce GARCH the 4/2 model and its moment generating function(MGF) for the saddlepoint method in the third section. Section 4 introduced the subordinators and the saddlepoint method is implemented in section 5. The last two sections contain the empirical test with a typical subordinator and briefly summarise this chapter.

### 4.2 Pricing VIX derivatives under GARCH framework

In a statistical field, heteroskedasticity tends to indicate that the volatility of the residuals does not follow an invariant constant, but varies with time. It allows us to reduce forecast errors and improve the accuracy of future forecasts by taking into account the residuals from the previous value. As a natural generalisation of the ARCH (Autoregressive Conditional Heteroskedasticity) process, GARCH and its modified models are widely used to analyze time series data concerning exchange rates, stock prices, etc. Introduced by Bollerslev (1986) [6], the GARCH model assumed that the volatility of the residuals follows an ARMA (autoregressive moving average) process with parameters p and qwhich control the contribution rate of historic conditional variance to leverage terms and innovation terms of current conditional variance. Then, Heston developed a closed-form option pricing formula for the spot assets following a specific GARCH(p, q) process with the log return correlated with its conditional variance(Heston 2000) [49]. In this chapter, we implement its simplest but most useful branch, GARCH(1,1) model. This model assumes that both the leverage terms and innovation terms of the conditional variance in the present moment are only related to the value of its previous step.

#### 4.2.1 VIX description in GARCH(1, 1) model

Since the GARCH models are based on the discrete framework, it is necessary to define the discrete time first. In this chapter, we assume that the stock price information is updated

at fixed time points  $t \in 1, 2, 3...$  with the same intervals. This means that all stochastic processes develop in a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with an information filtration  $\mathcal{F}_t$ . When the business time reaches the moment t, we assume that all traders have access to all public information at the same time and make rational decisions based on this information.

In GARCH (1,1) model, the log return  $R_t$  follows the following process in the physical measurement:

$$R_{t} = r_{t} + \lambda h_{t} + \sqrt{h_{t}} z_{t}$$

$$h_{t} = \omega + \beta h_{t-1} + \alpha (z_{t-1} - \delta \sqrt{h_{t-1}})^{2}.$$
(4.2.1)

In this first equation,  $R_t$  is the log return of spot price  $S_t$ ,  $r_t$  is the risk-free interest rate,  $\lambda$  is a coefficient for the risk premium,  $z_t$  is a standard normal distribution which drives the price volatility caused by stochastic factor. This equation relates the logistic regression to its conditional variance, arguing that the change of the log return is a joint result of fluctuations in the interest rate and conditional variance. In the second equation,  $h_t$  could be regarded as an instantaneous conditional variance at time t and formed as a function of previous information which could be observed in the explicit window. Two positive parameters  $\alpha$  and  $\beta$  represent the drift and leverage coefficient which are constants in this assumption. In addition, we note that the innovation term of  $h_t$  has a negative shock, which is to express the negative correlation between return and variance, which is consistent with Black's findings in 1976 [50].

Before moving on to assets and relative derivatives pricing work, we need to transfer the governing equations into risk-neutral measure  $\mathbb{Q}$  because the purpose of this chapter is to use the model to value the option. Duan (1995) [23] and Heston (1997) [10] provided ways to transform the risk-neutral form of the GARCH model respectively as:

$$z^{*}(t) = z(t) + (\lambda + \frac{1}{2})\sqrt{h(t)}$$
  

$$\delta^{*} = \delta + \lambda + \frac{1}{2}.$$
(4.2.2)

In this way, the risk-neutral GARCH (1,1) model turns out to be

$$R_{t} = r_{t} - \frac{1}{2}h_{t} + \sqrt{h_{t}}z_{t}^{*}$$

$$h_{t} = \omega + \beta h_{t-1} + \alpha (z_{t-1}^{*} - \delta^{*}\sqrt{h_{t-1}})^{2}.$$
(4.2.3)

#### 4.2.2 Definitions of VIX under GARCH model

Starting from 1989, a series of approximations on price volatility derivatives and hedging volatility risk has been gradually proposed. A popular method of approximating the VIX options constructs the value of  $VIX^2$  by an affine structure of associated volatility. In this way, the expression  $VIX^2$  is  $VIX^2 = av_t + b$  with a and b depending on the parameter values of applied financial models. As a description of market varying, volatility can often be expressed by a variety of commercial indexes or derivative prices so the definition of a and b in this form, too. Therefore, the definition of VIX also varies with not only the reference index and derivatives but also the structure of VIX itself. For example, Hao and Zhang (2013) [51] applied a practical  $VIX^2$  form which associates its value with the expected daily variance introduced by Cboe. This form expresses the VIX as the annualised average of expected daily variance under a risk-neutral measure. It first defines an implied volatility  $V_t(n)$  as:

$$V_t(n) = \frac{1}{n} \sum_{i=1}^n E_{\mathbb{Q}}[h_{T+i} | \mathcal{F}_t].$$
(4.2.4)

In this form, n tends to be taken as 22 (trading days of a month), as the VIX is an indicator of volatility for the next coming month. Then, the value of  $VIX^2$  is described as an annualised variance transformed from  $V_t(n)$  that:

$$VIX_t^2 = 252 \times V_t(22). \tag{4.2.5}$$

According to Wang's (2011) [52] proof, if the conditional variance  $h_t$  follows the Heston-Nandi model, then  $V_t(n)$  can be further expressed as:

$$V_t(n) = (1 - \Gamma(n))\bar{h} + \Gamma(n)h_{t+1}$$
(4.2.6)

where

$$\bar{h} = \frac{w + \alpha}{1 - \beta - \alpha \delta^2}$$

and

$$\Gamma(n) = \frac{1 - (\beta - \alpha \delta^2)^n}{n(1 - \beta - \alpha \delta^2)}$$

$$VIX_t^2 = \frac{1}{n} E_{\mathbb{Q}}[Var_t^{t+n} | \mathcal{F}_s]$$
(4.2.7)

where 0 < s < t and the discrete realised variance is defined as :

$$Var_t^{t+n} = \sum_{i=1}^n (R_{T+i})^2.$$

After defining the  $VIX^2$ , we move on to the derivatives of the VIX, namely the VIX options. In this chapter, we use the VIX European call option as an example to show the method of obtaining the closed-form of the derivative prices. To see this, we consider a call option on VIX, with the terminal payoff  $(VIX_T - K)^+$ . This option form indicates that this option will expire at time T with an expiration value max $[VIX_T - K, 0]$ . Given the above conditions, this pricing formula of a European VIX call option, C(t,T) with strike K can be written as:

$$C(t,T) = e^{-r(T-t)} E_{\mathbb{Q}}[(VIX_T - K, 0)^+ | \mathcal{F}_t].$$
(4.2.8)

The next work is to expand the realised variance into an affine form of  $h_t$  and obtain the value of this tail expectation by the saddlepoint method. Considering that the purpose of this chapter is to compute the option price in the 4/2 GARCH model, these two parts will be carried out after the introduction of the 4/2 GARCH model.

# 4.3 Expression of VIX option under 4/2 GARCH framework

The long-term market performance of the original Heston model has proved that it is a utility tool for S&P 500 and its derivative index such as VIX. But the empirical research

also revealed its shortage of capturing the short-term volatility as the power of the conditional variance is  $\frac{1}{2}$ , which will lead to downward-sloping volatility of variance smiles in mistake(Drimus 2011) [11]. Furthermore, Bakshi (2006) [48] simulated the dynamics of a Constant Elasticity of Variance (CEV) volatility and approximated the power of variance term, resulting in 1.3 nearly. Both of these two simulations suggest that we need to find an advanced way to express the variance part in governing equations. To solve this block, Heston (1997) [10] offered an inverse CIR volatility process named the "3/2 model". As the volatility term is flipped to the denominator, this process mimics the fast reverting phenomenon when the volatility fluctuates widely. The motivation of keeping good performance for both short- and long-term business periods makes us consider a combination of the original Heston model and the 3/2 model. Thanks to Grasselli (2017) [9], who offered an effective model to realise this idea named the "4/2" model.

In this section, we present the 4/2 model in a discrete framework and consider it with the GARCH model, based on the idea of Grasselli (2017) [9] who presented the continuous-time 4/2 model. This model combines the diffusion terms of the Heston-Nandi model and its flip type just like the 3/2 model. The derivation of this model starts from the continuous-time 4/2 model and is obtained by a discrete process. In the discrete model, we note that not only the parameters of the diffusion term change (from  $\sqrt{v_t}$  to  $a\sqrt{v_t} + b/\sqrt{v_t}$ ), but also the parameters of its leverage term change and are also related to a, b. This model, which inherits the advantages of the Heston-Nandi model, makes the power of the diffusion term adjustable, allowing the model to be adapted to the more generalised underlying asset and its derivatives.

### 4.3.1 Structure of 4/2 GARCH model

The original continuous-time 4/2 model describes the following two processes

$$dS_t = rS_t dt + (a\sqrt{V_t} + \frac{b}{\sqrt{V_t}})S_t dZ_1$$

$$dV_t = k(\theta - V_t)dt + \sigma\sqrt{V_t}dZ_2.$$
(4.3.1)

From those equations, we find that the diffusion term of this model combines the Heston model type and the 3/2 model. Parameters a and b are implemented to make the volatility of variance fit the real market situation in both the short- and long-term. After the modification of the flip CIR factor, this model owns a similar variance process with the Heston model but keeps the flip  $\sqrt{V_t}$  coming from the 3/2 model to capture the short-term volatility correctly.

Following a similar way Heston transfers the continuous-time Heston model into a discrete version, the Heston-Nandi model. We can derive a discrete version of the 4/2 model through Heston's process of discretizing the continuous-time Heston model to a Heston-Nandi model. With the distribution of  $S_t$  in 4/2 model, the expression of logarithm of  $S_t$  follows:

$$dlogS_t = (r - \frac{1}{2}(a^2v_t + \frac{b^2}{v_t} + 2ab))dt + (a\sqrt{v_t} + b\frac{1}{\sqrt{v_t}})dZ_t^1.$$
(4.3.2)

The discrete form of the above equation is written as :

$$R_t = \log \frac{S_{t+1}}{S_t} \approx (r - \frac{1}{2}(a^2v_t + \frac{b^2}{v_t} + 2ab))\Delta t + (a\sqrt{v_t} + b\frac{1}{\sqrt{v_t}})\Delta Z_t^1.$$
(4.3.3)

As the discrete form of  $R_t$  is obtained in the 4/2 model, we assume the log return  $R_t$  and its conditional variance  $h_t$  in the GARCH 4/2 model follow discrete stochastic processes like:

$$R_{t} = r_{t} - \frac{1}{2} (a^{2}h_{t} + \frac{b^{2}}{h_{t}} + 2ab\Delta t) + (a\sqrt{h_{t}} + b\frac{1}{\sqrt{h_{t}}})z_{t}^{*}$$

$$h_{t} = \omega + \beta h_{t-1} + \alpha (z_{t-1}^{*} - \delta^{*}\sqrt{h_{t-1}})^{2}.$$
(4.3.4)

Because this model is for option pricing, we present the risk-neutral version of the GARCH 4/2 model and keep the meaning of the symbols in the Heston-Nandi model. Similar to

the process described by the Heston-Nandi model, variables in this model are updated by discrete time steps and kept constant in each time interval.

To have the same properties, the 4/2 model also inherits the normality assumption of the Heston-Nandi model. This assumption claims that the value of a call option with a one-time interval fits the Black–Scholes-Rubinstein formula with spot price following the conditional log-normal distribution. We keep the second assumption of the Heston-Nandi model (2000) [49]. It reveals that even the governing model is discrete with steps, but it has a continuous-time stochastic process in each time interval. This assumption ensures that we could discretize the 4/2 model by a similar method to Heston's.

Because the 4/2 model contains the volatility term of the 3/2 model, we need to consider the Heston trap that the volatility may reach zero and become negative during the autoaggressive process. Normally, one could limit the volatility by the Feller condition. But the Feller condition is only available for the CIR process and hard to meet in the real market. Hence, we implement another method that reflects the value of volatility when it passes through zero by taking its absolute value. Following this idea, the new process of conditional model  $h_t$  turns out to be:

$$h_t = |\omega + \beta h_{t-1} + \alpha (z_{t-1} - \delta \sqrt{h_{t-1}})^2|.$$
(4.3.5)

#### 4.3.2 Cumulant generating function of 4/2 GARCH model

In this chapter, we introduce an affine structure to express  $VIX^2$  as a conditional variance  $h_t$ . This structure is introduced in this section and is used in the construction of the CGF for  $VIX^2$  for application in the saddlepoint method for solving the tail expectation. Before presenting the affine structure, we discuss the CGF of  $h_t$  first.

According to the solution of Wang (2017) [1], the semi-closed-form for moment-generating function of the forward conditional variance at m time intervals from now  $h_{t+m}$  is expressed as an exponential structure as following:

$$E_{\mathbb{O}}[e^{zh_{t+m}}|\mathcal{F}_t] = f(z, m, h_t) = e^{C(z,m) + H(z,m)h_t}$$
(4.3.6)

where,

$$\begin{split} C(z,m) &= C(z,m-1) - \frac{1}{2} log(1 - 2\alpha H(z,m-1)) + \omega H(z,m-1) \\ H(z,m) &= \beta H(z,m-1) + \frac{\alpha \delta^* H(z,m-1)}{1 - 2\alpha H(z,m-1)} \end{split}$$

with the initial condition as:

$$C(z,m) = 0$$
$$H(z,m) = z.$$

Then, the CGF of  $h_t$ ,  $K_h(z)$  is expressed as the power part of this exponential structure that:

$$K_h(z) = C(z,m) + H(z,m)h_t.$$
 (4.3.7)

In addition, this form offers a method to calculate the risk-neutral conditional expectation of  $h_{t+m}$  under the filtration  $\mathcal{F}_t$  like :

$$E_{\mathbb{Q}}[h_{t+m}|\mathcal{F}_t] = E'_{\mathbb{Q}}[e^{zh_{t+m}}|\mathcal{F}_t]|_{z=0}.$$
(4.3.8)

After we get the CGF of  $h_t$ , we move on to the CGF of  $VIX^2$  under the 4/2 GARCH model by associating the CGF of  $h_t$  with the relations of  $h_t$  and the VIX. From (4.3.3), we figure out the square of log return as :

$$(R_{t+1})^{2} = (r_{t} + \lambda h_{t} - \frac{1}{2}(a^{2}h_{t} + \frac{b^{2}}{h_{t}} + 2ab\Delta t) + (a\sqrt{h_{t}} + b\frac{1}{\sqrt{h_{t}}})z_{t})^{2}$$

$$\approx (a^{2}h_{t} + \frac{b^{2}}{h_{t}} + 2ab\Delta t)z^{2}.$$
(4.3.9)

Here z is the standard normal distribution and the higher order (> 1) of  $\Delta t$  is ignored. This approximation loses some of its precision, but greatly reduces the calculation difficulty and

avoids estimating the value of  $h_t^2$  and its reciprocal. Once the square of log return is figured out, the value of  $VIX_T^2$  can be expressed as an expectation of a function relevant with  $h_t$ that:

$$VIX_{T}^{2} = \frac{1}{n} \sum_{i=1}^{n} E_{\mathbb{Q}}[(R_{T+i})^{2} | \mathcal{F}_{t}]$$

$$\approx \frac{1}{n} \sum_{i=1}^{n} (a^{2} \mathbb{E}_{t}^{\mathbb{Q}}[h_{T+i-1}] + b^{2} \mathbb{E}_{t}^{\mathbb{Q}}[\frac{1}{h_{T+i-1}}] + 2ab\Delta t).$$
(4.3.10)

Using this method, we transform the value of  $VIX_T^2$  into the affine structure of the expectation of  $h_t$  and its reciprocal. However, it is more to approach the expectation of  $\frac{1}{h_t}$ , as the process and distribution that  $\frac{1}{h_t}$  follows is unknown. Therefore, we use Taylor expansions to transform the expectation of  $\frac{1}{h_t}$  into a linear form of  $E[h_t]$  as :

$$E_s^{\mathbb{Q}}[\frac{1}{h_t}] \approx \frac{2}{h_s} - \frac{1}{h_s^2} E_s^{\mathbb{Q}}[h_t]$$
(4.3.11)

where s < t presents the last time for updating explicit market information.

Combining the two transformations above, we can value  $VIX_T^2$  as:

$$VIX_{T}^{2} \approx \frac{1}{n} \sum_{i=1}^{n} \left( a^{2} \mathbb{E}_{t}^{\mathbb{Q}}[h_{T+i-1}] + b^{2} \mathbb{E}_{t}^{\mathbb{Q}}[\frac{1}{h_{T+i-1}}] + 2ab\Delta t \right)$$
  
$$\approx \frac{1}{n} \sum_{i=1}^{n} \left( \left( a^{2} - \frac{b^{2}}{h_{t}^{2}} \right) \mathbb{E}_{t}^{\mathbb{Q}}[h_{T+i-1}] + \frac{2b^{2}}{h_{t}} + 2ab\Delta t \right)$$
  
$$= \left( a^{2} - \frac{b^{2}}{h_{t}^{2}} \right) V_{T}(n) + \frac{2b^{2}}{h_{t}} + 2ab\Delta t$$
(4.3.12)

where  $V_T(n)$  is defined in (4.2.6). Then we are able to derive the CGF of  $VIX^2$ , k(z) as follows:

$$k(z) = log \mathbb{E}_0^{\mathbb{Q}} (e^{zVIX_T^2})$$
$$= \beta z + K_h(\alpha z)$$
(4.3.13)

$$=\beta z + C(\alpha z; T-t) + H(\alpha z; T-t)h_t$$

where

$$\begin{aligned} \alpha &\approx (a^2 - \frac{b^2}{h_t^2}) \Gamma(n) \\ \beta &\approx \frac{2b^2}{h_t} + 2ab\Delta t + (a^2 - \frac{b^2}{h_t^2})(1 - \Gamma(n)\bar{h}) \end{aligned}$$

and  $\Gamma$  follows the definition of (4.2.6).

### 4.4 Model modification by business time

In addition to adjusting the powers of the innovation terms, we can also start with the distribution of the innovation terms themselves and modify the business time by subordinators. Carr and Wu (2003) [25] mentioned that in addition to stochastic volatility, market returns are also influenced by a number of other factors, such as the non-normality of non-volatility and the negative correlation between returns and their volatility. They further observe that the operations in market trading are not a uniform pass-through process; instead, the number and frequency of trades may fluctuate over time. In a GARCH model, we can reflect this market characteristic by varying the iteration rate of the model. High-volume and high-frequency trades will speed up the iteration rate and shorten the time interval of each iteration. Low-volume trades slow down the iteration rate of the model and lengthen the corresponding time interval. This method of changing the time rate to suit the market conditions is called the stochastic clock or business time.

#### 4.4.1 Introduction of subordinator

After Clark (1973) [16] in the first application of continuous-time stochastic to financial modelling to estimate futures prices, several common subordinated processes that were applied to fit periods with different characteristics have been introduced in literature like  $\alpha - stable$ , compound Poisson and gamma. Finding a suitable subordinator allows us to draw relatively accurate conclusions about the processes described by the time series models. Before selecting a subordinator, we briefly review the main properties of the
distribution corresponding to the subordinated process. We first consider the general definition of subordinated processes that are followed by subordinators. We know that both the Heston model and the GARCH model use standard Brownian motion for their diffusion term  $Z_t$ . When this process is modified as a subordinated process, the subordinated Brownian motion is represented as:

$$Z_G = Z(G(t)) \tag{4.4.1}$$

where Z(t) represents the standard Brownian motion and G(t) is the subordinator that follows a given distribution. Through this subordinated process, the real time and business time are linked: G(t) changes the rate of time and thus affects the rate of evolution of the innovation term, even if the distribution of the term remains unchanged (still in standard Brownian motion).

In fact, a subordinated process is a non-negative Lévy process. This process has three characteristics: Lévy process L(t) is a stochastic process that satisfies the following requirement, 1), L(0) = 0; 2), L is an independent and stationary increment process. 3), Lis continuous on its domain. This integral operator, also known as the Lévy measure, has different forms depending on the distribution of business time.

In a probability space  $\mathbb{P}$ , the description of the subordinated process comes from the following aspects: distribution function (density function) f(x), Lévy measure v(dx), characteristic function  $\Psi(z,t) = E[e^{izx}]$  and Lévy exponent  $\psi = \log \Psi(z,1)$ . By using this process to replace the original Brownian motion in the equation, the improved stochastic differential equation is able to describe more general and complex volatility characteristics.

#### 4.4.2 Construction of the 4/2 GARCH model with subordinator

To transform a process that follows real time into a process that follows business time, we first choose a subordinator G = G(t), t > 0 and then define G(0) = 0 with its trajectories only taking non-negative values. In the practical market, the G is regarded as the business time, flowing differently with the real time, t. As a stochastic process, the innovation rate of the G is decided by the information gained between the real time intervals. Usually, the time-modified function g(G) is gained from the time-relative function f(t) of the original process by an integral as:

$$g(G) = \int_0^t f(s)p(s)ds$$
 (4.4.2)

where p(s) is the rule of G such as the closed-form of a stochastic process or the probability density function of well-known distributions.

Thus, based on the above notation and definitions, we introduce the 4/2 GARCH model which could be modified as:

$$R_{G} = r_{G} + \lambda h_{G} - \frac{1}{2}h_{G} + (a\sqrt{h_{G}} + b\frac{1}{\sqrt{h_{G}}})z_{G}$$

$$h_{G} = \omega + \beta h_{G-1} + \alpha(z_{G-1} - \delta\sqrt{h_{G-1}})^{2}.$$
(4.4.3)

Also, when the real-time based option price C is obtained, we can express the option price based on the business time as:

$$C_G(t,T) = \int_t^T C(h_G,G)p(G)dG.$$
 (4.4.4)

In the empirical experiment section, we prefer to use a technically popular modification, gamma subordinator as a general example to show how the choice of subordinator affects the pricing of underlying assets or derivatives.

# 4.5 Solution of the VIX options pricing

As the closed-form expression of its CGF is known, we could implement the saddlepoint method to approximate the tail expectation. By following these steps, the closed-form of option pricing models of the VIX with initial time 0, delivery time T and strike K is presented as:

$$C(VIX_T, T) = e^{-rT} \mathbb{E}_0^{\mathbb{Q}}[(VIX_T - K)^+]$$

$$= e^{-rT} \mathbb{E}_0^{\mathbb{Q}}[(\sqrt{VIX_T^2} - K)^+].$$
(4.5.1)

To obtain the results of the option price, we should discuss how to obtain the tail expectation form  $E[(\sqrt{X} - K)^+]$  with a known cumulant generating function (CGF) first, then present the method of approaching that CGF. By the definition of the tail expectation:

$$E[(\sqrt{X} - K)^{+}] = \int_{K^{2}}^{\infty} [\sqrt{X}p(X)dX]$$
(4.5.2)

where p(X) is the probability density function of X. Now the CGF of  $VIX^2$  is available only if the CGF of volatility is known.

# 4.5.1 The approach of $VIX_T^2$ CGF

In this section, we will obtain the CGF of VIX based on equations (4.3.6) and (4.3.15), after which one could approximate the price of option continually by the tail expectation  $C_t = e^{-r\Delta t}E[(S_t - K)^+]$  like characteristic method or saddlepoint method. Thanks to Heston (1993) [8], we have had a closed -form moment generation function(MGF) of the conditional variance  $h_{t+n}$  under the Heston-Nandi model framework when p = q = 1 like:

$$e^{k(z)} = E_t^{\mathbb{Q}}[e^{zh_{t+n}}] = e^{A(t;n,z) + B(t;n,z)h_t}$$
(4.5.3)

where,

$$A(t;n,z) = A(t;n-1,z) + B(t;n,z)h_t\omega + zr - \frac{1}{2}log(1 - 2\alpha B(t;n-1,z))$$
$$B(t;n,z)) = \frac{\alpha\delta^* B(t;n-1,z))}{1 - 2\alpha B(t;n-1,z))} + \beta B(t;n-1,z)).$$

Because we only consider the GARCH(1,1) model in this chapter, the initial conditions follow:

A(T; 0, z) = B(T; 0, z) = 0.

# 4.5.2 Saddlepoint methods to VIX option

In this section, we discuss this work using the saddlepoint method in solving for option prices. The basic principle of the saddlepoint method is to use the CGF to find the approximate result of the tail probability and further obtain an approximation of the option valuation. In effect, we calculate the option price by inverting the Fourier formula to approximate the principal contribution of the integral by choosing a "perfect" profile (Rogers and Zane 1999) [33]. In applying the saddlepoint method, we can often find a closed-form solution based on the parameters of the stochastic process. This allows us to quickly change the calculation results by transforming the parameters without the need for additional numerical simulations in the presence of changing market conditions.

One of the more popular saddlepoint methods in the literature is the generalised Lugannini-Rice approximation proposed by Wood et al. (1993) [32], which can accommodate a wide variety of model diffusion terms simultaneously, whether they are standard normal or not. However, we note that this method is cumbersome and not directly applicable to the  $P(\sqrt{X} > K)$  case. This difficulty drives us to find a more straightforward way to obtain a closed-form solution for this tail probability, even though it may have more restrictions.

Since we have made the same assumption of normality as the Heston-Nandi model in the construction of the 4/2 model, we can directly follow the idea of Kowk and zheng (2014) [2] and represent the alternative saddlepoint method that:

$$\mathbb{E}_{t}^{\mathbb{Q}}[(\sqrt{VIX_{T}^{2}}-K)^{+}] = \frac{1}{4\sqrt{\pi i}} \int_{r-i\infty}^{r+i\infty} \frac{e^{k(\hat{z})+g(\hat{z})}}{\hat{z}^{3/2}} d\hat{z} \qquad r \in (0,\alpha_{+})$$
(4.5.4)

where  $g(z) = log(1 - erf(\sqrt{z}K))$  and  $\hat{z}$  is the positive root of the alternative saddlepoint

equation:

$$k'(\hat{z}) + g'(\hat{z}) - \frac{3}{2\hat{z}} = 0.$$
(4.5.5)

# **4.6** Back to GARCH (p,q)

#### 4.6.1 Existence of long-term memory in conditional variance

In previous sections, we discussed some modifications and extensions of the GARCH(1,1) type model. Previous studies have also focused on GARCH (1, 1) models, while less research has been done on GARCH(p, q) models and even less work has been done on valuation and hedging with GARCH (p, q) models. Barone (2008) [53] proposed a new method for options pricing based on GARCH models with filtered historical innovation but in the iteration of the conditional variance  $h_t$  only the information from the previous step  $h_{t-1}$ , i.e., the GARCH(1,1) model, is implemented. Hsieh (2006) [54] scrutinised and tested the Heston-Nandi model [49] and the Duan's NGARCH model [23] that has a conditional variance iteration process like  $h_t = \beta_0 + \beta_1 h_t + (z_t - \gamma)^2 \beta_2 h_t$  to determine whether there is a cost to restricting model in the affine family.

There are two possible reasons for the choice of GARCH(1,1): 1, the current conditional variance  $h_t$  is less influenced by the conditional variance of the previous 2 steps than  $h_{t-1}$ ; 2, the GARCH(p, q) model greatly increases the complexity of the model, and the gain does not match the increase in its computational cost in computational accuracy. In addition to these two reasons, we note that no literature systematically describes how to select or model the values of p and q, and there is little literature that treats p and q as random values rather than constants. However, by studying the persistence of the conditional variance of the U.S. stock return index, Crato (1994) [55] found evidence of the existence of long-term memory in its conditional side. In his tests of long-term memory in the conditional variances,  $h_t$  of GARCH (p, q) can be expressed as an infinite-sum form

as:

$$h_t = \omega + \sum_{i=1}^{\infty} \delta_i X_{t-i}^2 \tag{4.6.1}$$

where  $X_t$  is the spot time observations that  $h_t = E(X_t^2 | \mathcal{F}_{t-1})$ . Also, he mentioned that these findings suggest that later researchers should make the model flexible enough to experience the effects of long-term memory and dependence in the modelling process. Based on this motivation, it is suggested that in the process of pricing and hedging under the GARCH model framework, we should properly consider and test whether the variance has long-term memory and model for this result.

# 4.6.2 Selection of p and q in GARCH(p, q) model

As mentioned before, in the original GARCH(p,q) model, p and q are determined as constants from the beginning by other conditions. However, as the information is updated at each time node t, the long-term memorability of its model may change. At this point, we should try to adjust the model so that the backward iterative process of the model can change with whether the conditional variance depends on the earlier steps' values. It is meaningful to make p and q from constants to some random variables that follow some processes.

In this section, we use the non-stationary Poisson process as an example and assume that p and q fit different parameters of the non-stationary Poisson process. Unlike the Poisson process which has a fixed  $\lambda$  and a uniform mean, the non-stationary Poisson process owns an unfixed intensity  $\lambda(t)$ . If p and q follow these processes, the leverage terms and the innovation terms of the conditional variance actually become compound non-stationary Poisson processes. This modification turns the GARCH(p,q) model in historic measure to be:

$$R_{t} = r_{t} + \lambda h_{t} + \sqrt{h_{t} z_{t}}$$

$$h_{t} = \omega + \sum_{i=1}^{p(t)} \beta_{i} h_{t-i} + \sum_{i=1}^{q(t)} \alpha_{i} (z_{t-i} - \delta_{i} \sqrt{h_{t-i}})^{2}$$
(4.6.2)

where p(t) and q(t) are Poisson distributed with intensity  $\lambda_p(t)$  and  $\lambda_q(t)$ . Following this, the means and probability mass functions of these two variables are

$$m_{i}(t) = \int_{0}^{t} \lambda_{i}(s) ds$$

$$f_{i}(k) = \frac{e^{m_{i}(t)}m_{i}(t)^{k}}{k!} \qquad k \ge 0; \quad i = \{p, q\}.$$
(4.6.3)

At this point, we can control the distribution of p and q by determining the  $\lambda_i(t)$ . We will use a hyperbolic cosine function used by Hong (1995) [56] in his study of earthquake occurrence in subsequent empirical experiments and simulate the Increasing Mean-Occurrence statement, corresponding to the financial case where the long-term memory dependence of the conditional variance becomes more robust as the number of iterations increases.

### 4.6.3 Direct generation of long-term memory parameters

Next, we discuss a more generalised model where not only p and q, but even  $\alpha_i$ ,  $\beta_i$ and  $\delta_i$ , change with time. In fact, switching different parameters according to different market conditions are widely accepted means of model improvement. A proven solution born from this idea is the regime switching GARCH model. Previous studies found that variance may produce structural changes in the course of market volatility forecasting. Different parameter structures will allow the variance to persist over different ranges until the following change in parameter structure arrives. However, the traditional GARCH model does not capture this property of the variance because it always assumes that the conditional variance is based on iterative changes in the same parameter structure. This allows GARCH models that use high positions as starting points for forecasting to be excessive, and vice versa. Based on this reality, a better idea for model improvement is the introduction of Markov regime switching factors into traditional GARCH models.

Without loss of generality, we use the GARCH model that switches between two regimes as an example. This is not only because it is the simplest form of the regime switching model, but also because the states of the market tend to be dichotomous: high and

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low, frequent and episodic, boom and bust. In each regime, we apply a separate set of parameters framework for the iterative process of variance and decide at each discrete point in time whether to make a regime switching or not (typically this decision is modelled by a state transition matrix.) The effectiveness of this improvement was reported by Klaassen (2002) [57] as he generated an additional source of volatility persistence and thus improves the flexibility in describing the volatility persistence of shocks. Based on the regime switching GARCH(1, 1) model proposed by him, this section integrates the GARCH(p, q) model with the regime switching factor and proposes parameter vector structure switching based on the state transition matrix.

Before introducing the regime switching GARCH model, we need to first define the relevant regimes and paths that the volatility follows. Also, we need to decide how the regimes are switched between each other, i.e., the expression of the Markov transition matrix. As we mentioned previously, we assume that this model switches between two regimes only. To express this process, we introduce two regimes,  $r_h$  and  $r_l$  and supposed that the conditional variance follows two different distributions and parameters in two different regimes. Next, we define the path  $p_t$  as a series of regimes that generate over time like:

$$p_t = (r_1, r_2 \dots r_s \dots r_t), \qquad r_s \in \{r_h, r_l\}.$$
(4.6.4)

Thus, generating a new regime statement for the next time step based on the previous regime has only two possible outcomes: keeping the previous regime or switching to another regime. Therefore, we define the probability that a process in regime  $r_h$  remains in this regime in the next step as  $p_h$  and specify  $p_l$  with a similar definition. Based on the above definitions, we assume that the regime switching is based on the following Markov state transition matrix:

$$M = \begin{bmatrix} p_h & 1 - p_h \\ 1 - p_l & p_l \end{bmatrix}.$$

We can now discuss the method of constructing the regime switching GARCH model

under different information observations. We first discuss the best case, in which regime information is recorded from the beginning. This means that we can obtain the exact probability of the regime at moment t, based on historical path information, i.e.,  $P(r_t) = P(r_t|p_{t-1})$ . In this case, the conditional variance process of traditional GARCH(p,q) will be developed as:

$$h_t = \omega^{r_t} + \sum_{i=1}^p \beta_i^{r_t} h_{t-i} + \sum_{i=1}^q \alpha_i^{r_t} (z_{t-i} - \delta_i^{r_t} \sqrt{h_{t-i}})^2.$$
(4.6.5)

From the equation we can see that under the regime  $r_t$ , we apply the parameter structure  $\{\omega^{r_t}, \beta_i^{r_t}, \alpha_i^{r_t}, \delta_i^{r_t}\}$  which is determined by empirical experiment before modelling.

However, in modelling based on historical data, it is difficult to distinguish whether the generation of historical volatility is due to different regimes or determined by other reasons. Therefore, we cannot distinguish the contribution of different factors to the historical volatility from the market data. Based on this fact, Gray (1996) [58] improved this model so that it no longer relies on the full path, but on the time information  $\mathcal{F}_{t-1}$ . In his model, instead of trying to obtain full information about the volatility path, Gray made the conditional variance process from an iteration of past values to an iteration of past volatility based on the last step information to approach the current volatility iteration process. The conditional variance process based on this improved GARCH(p, q) model is:

$$h_{t} = \omega^{r_{t}} + \sum_{i=1}^{p} \beta_{i}^{r_{t}} E[h_{t-i} | \mathcal{F}_{\sqcup -\infty}] + \sum_{i=1}^{q} \alpha_{i}^{r_{t}} (z_{t-i} - \delta_{i}^{r_{t}} \sqrt{E[h_{t-i} | \mathcal{F}_{\sqcup -\infty}]})^{2}.$$
(4.6.6)

Finally, Klaassen [57] showed that the current regime can also contribute to the estimation of the expectation of the past variance. Following this line of thought, we put information about the current regime into the expectation condition as well and describe the regime switching GARCH(p, q) model as follows:

$$h_{t} = \omega^{r_{t}} + \sum_{i=1}^{p} \beta_{i}^{r_{t}} E[h_{t-i} | \mathcal{F}_{\sqcup -\infty}, r_{t}] + \sum_{i=1}^{q} \alpha_{i}^{r_{t}} (z_{t-i} - \delta_{i}^{r_{t}} \sqrt{E[h_{t-i} | \mathcal{F}_{\sqcup -\infty}, r_{t}]})^{2}.$$
(4.6.7)

# 4.7 Numerical experiment

In this chapter, we conduct a series of mathematical experiments to show the effect of different model modifications on the pricing results. The thing we want to convey is that with model modification, we can adapt the original discrete GARCH model to a broader market environment, account for more market factors, more accurately describe the distribution of prices and volatility, and obtain more optimal conclusions. We first consider some adjustments to the GARCH (1,1) model and then consider the GARCH (p, q) model cases. In conducting the Empirical experiment, we first give the parameters of the original model, the GARCH (1, 1) model. All model modifications are made on these model parameters. In addition, in the subsequent modelling process, we provide only the corresponding modified parameters and keep the other parameters consistent, which allows us to compare the impact of different models on option pricing. For ease of understanding, we do not give the parameters in the form of tables or lists but put them directly into the model to allow the evolution of the model to be observed more intuitively.

Because the object of our pricing work is VIX European options, we specify the parameters directly under EMM as the price of the option needs to be evaluated under this measure as a benchmark to make the results meaningful. The initial adjusted stock price is set to be  $S_0 = 100$ , and the initial conditional volatility is set to be  $h_0 = 2 \times 10^{-4}$ . Then we implement a risk-neutral type of GARCH model with parameters used as follows:

$$R_{t} = 7.69 \times 10^{-5} - \frac{1}{2}h_{t} + \sqrt{h_{t}}z_{t}^{*}$$

$$h_{t} = 2.31 \times 10^{-6} + 0.682h_{t-1} + 2.85 \times 10^{-6}(z_{t-1}^{*} - 365.25\sqrt{h_{t-1}})^{2}$$
(4.7.1)

where  $z_t^* = z_t + 2.52\sqrt{h_t}$  and long stable volatility  $\bar{h} = 2.85 \times 10^{-4}$ . The pricing claim is set as a European VIX call option such as  $H_t = E_{\mathbb{Q}}[(VIX_T - K)^+ | \mathcal{F}_t]$  with a strike range from 0.04 to 0.2, and maturity of T = 22 (business days of a month). This is because after one month of iterations through this model, the conditional volatility will be very close to the long-term volatility. This makes the latter option prices perform similarly to one-month values. Moreover, to facilitate our comparison of the option pricing results of the original Heston-Nandi model with the model containing the subordinator, we assume that the diffusion term  $z^*$  of the original GARCH model follows the standard Brownian motion.

# 4.7.1 The comparison of the 4/2 model and the Heston-Nandi model

One of the improvements to the GARCH (1, 1) model in this chapter is to adapt it from a fixed diffusion term to a flexible one containing  $h_t$  and its flip. Therefore, in the option pricing process of the 4/2 model, we vary only a and b, but the rest of the parameters are the same as in the Heston-Nandi model. The pricing results are as follows:

VIX option pricing with varying volatility characteristics									
K	0.0400	0.0600	0.0800	0.1000	0.1200	0.1400	0.1600	0.1800	0.2000
Heston	0.2590	0.2354	0.2180	0.1981	0.1772	0.1580	0.1363	0.1159	0.0980
a = 1.5 b = .0005	0.5405	0.5193	0.4999	0.4806	0.4615	0.4397	0.4192	0.4005	0.3811
a=1 b=.0005	0.3342	0.3201	0.3030	0.2841	0.2620	0.2379	0.2160	0.1977	0.1777
a = 1.5 b = .001	0.5146	0.4835	0.4681	0.4525	0.4214	0.4281	0.3922	0.3912	0.3374
a=1 b=.001	0.2942	0.2728	0.2495	0.2356	0.2158	0.1921	0.1749	0.1532	0.1315

# Table 4.1

Table 4.1 shows the performance of option pricing under different model parameter constructions. We can see that the pricing results through the Heston-Nandi model are smaller than the pricing results through the 4/2 GARCH model for the same time steps and strikes. It is because short-term volatility is considered in the pricing process through the 4/2 GARCH model. This suggests that the 4/2 model captures more of the volatility characteristics than the Heston model. Also, we observe the impact of changes in long-term volatility and short-term volatility weights on the VIX value. For example, with a fixed long-term volatility weight a, a larger short-term volatility weight b leads to a decrease in the value of the VIX. One possible explanation is that short-term volatility has less impact on overall future volatility relative to long-term volatility, so a more significant proportion of short-term volatility instead represents a more stable future market in investors' views. This also explains why a significant weighting of long-term volatility would significantly increase the value of the VIX and raise the price of VIX call options. Overall, table 4.1 reflects the superior performance of the 4/2 model compared to the Heston model, while also providing guidance on the impact of short- and long-term volatility on the final VIX through the change in a and b. We show this more visually in the following figure:



Figure 4.1: option pricing performance in varying model parameters

# 4.7.2 The Comparison of real time and the gamma subordinator

In the previous section, we mentioned that we would use the gamma subordinator as an example to illustrate the impact of business time on the valuation results, which is different from the real time. Here we give the distribution of the gamma subordinator as  $G \sim \gamma(k = 0.5, \eta = 1)$ . We then show the results of the pricing option in both the real time and the business time frameworks in the following table:

V	VIX opt	ion prici	ng in th	ie frame	work of	differen	t diffusi	on	
K	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
real time	0.259	0.235	0.218	0.198	0.177	0.158	0.136	0.116	0.098
gamma sub	0.183	0.167	0.151	0.135	0.119	0.103	0.087	0.072	0.056

Table 4.	.2
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From Table 4.2, we can find that after the gamma subordinator is applied to the GARCH 4/2 model, the prices of its options are smaller than those of the GARCH 4/2 model

with Brownian motion as its diffusion term under different strikes. This shows that the diffusion terms that follow different distributions can greatly affect the pricing results, confirming once again that even under the same parameters, different business time rates caused by market factors such as trading periods and trading volume can greatly affect the final pricing results of the underlying asset and derivatives. Similarly, we provide a figure to show their performance,



Figure 4.2: option pricing performance in different time systems

# 4.7.3 The Performance of GARCH(p, q) with p and q following nonstationary Poisson processes

Finally, we show the effect of p and q as stochastic variables on option pricing results by randomising p and q through non-stationary Poisson processes. This process allows us to vary the Poisson distribution followed by p and q to simulate the change of volatility memory factor in time. Here, we assume that the long-term memory of volatility becomes progressively more substantial over time. This means that at the beginning time, p and q are more likely to randomise to smaller values, and as iterative steps increase, the probability of p and q taking more significant values increases. After fixing K = 0.1, the option pricing results of this model over time compared with GARCH (1, 1) are exhibited

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in table 4.3.

VIX option pricing in stochastic p&q GARCH model vs GARCH(1,1)										
steps	4	6	8	10	12	14	16	18	20	22
GARCH(p, q)	0.188	0.205	0.223	0.242	0.263	0.284	0.308	0.333	0.359	0.387
GARCH(1,1)	0.171	0.174	0.175	0.178	0.18	0.183	0.185	0.188	0.191	0.195

Table -	4.3	3
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As the table above demonstrates, because the incremental effect of the conditional variance more than the last step on the current value is taken into account, the GARCH(p, q) model driven by the non-stationary Poisson process has a larger conditional variance than that of the original GARCH(1,1) model, and thus a higher VIX value. We further find that as the effect of the long-term memory grows with time, the stochastic simulated results for p and q gradually increase and the option pricing results change from linear to convex. This process stimulates the iterative process for those volatilities whose long-term memory effect grows with time. The difference between this process and GARCH (1, 1) in option pricing can be observed in the figure below.



Figure 4.3: VIX Option prices under in in stochastic p&q GARCH model vs GARCH(1,1)

# 4.8 Conclusion

In this thesis, we mention the problem of pricing VIX options under various changes to the model: for the GARCH(1,1) model, we introduce the 4/2 model modification and considers different business times (subordinator); for the GARCH(p, q) model, we randomise p and q that make them follow non-stationary Poisson processes and discuss the regime switching technique, too. In these modifications, we perform empirical experiments on 4/2GARCH models, subordinator GARCH models and stochastic (p, q) GARCH models by applying the saddlepoint method to approximate VIX European call option prices. Previous work shows that despite the widespread implementation of VIX and its derivative, there are few relevant methods for pricing its options, let alone generalised model modifications. In the future, the market environment will naturally become more complicated with the usage of more advanced financial products. This makes it difficult for traditional GARCH models to be applied to future markets. Therefore, these chapters outline a series of improvements and extensions to the GARCH model that will enable future investors to utilise these improvements and extensions in their financial models, thereby improving the accuracy of their pricing tools or adapting them to more complicated financial environments.

CHAPTER 5. A GARCH-BASED METHOD FOR OBTAINING LOCAL QUADRATIC HEDGING STRATEGIES FOR VIX CALL OPTIONS

Chapter 5

# A GARCH-based method for Obtaining local quadratic hedging strategies for VIX Call options

# 5.1 Introduction

In the financial market, the risk of certain financial derivatives is often high relative to the underlying asset. It makes it particularly important to study the hedging aspects of these financial derivatives. In this chapter, we will give an example of one of these representative financial derivatives, VIX options, for hedging. We know that the VIX has been an index-based instrument reflecting market volatility and investors' subjective mindset since its introduction by Cboe in 1993. It predicts future volatility sharpness mainly by way of measuring potential implied volatility. A high VIX value signals the onset of dramatic market volatility and vice versa. Unlike those investors who invest in the underlying asset, investors who invest in VIX options are more likely to profit from high volatility (call options). Such profits are necessarily accompanied by greater risk, so hedging VIX options and developing related strategies is a meaningful financial modelling practice.

Several methods have been implemented for pricing the VIX option and its other derivatives to obtain the numerical simulation results or analytic results outside of hedging. Wang and Daigler (2011) [52] investigated the pricing performance of VIX option models by testing three representative VIX option models. Lin and Chang (2009) [59] derived a VIX option model that considers the price processes of SPX according to the literature, showing that state-dependent jumps in prices and volatility both play crucial roles in VIX options pricing. Sepp(2008) [41] discussed the positive volatility skew observed in the implied volatility of VIX options and capture this feature by a jump-diffusion model. By introducing a jump, Baldeaux et al. [60] generated an implied volatility model that fits short-term index options perfectly, demonstrating the advantage of a purely diffusive 3/2 model in capturing implied volatility in VIX options. Simon's study (2017) [61] examined VIX options trading strategies based on historical market data about VIX futures. Their study indicated that an increase in the volatility of the underlying VIX futures contract will cause the implied volatility of VIX options to trend upward, allowing long VIX options strategies to benefit greatly.

All of the above literature discusses and approaches the VIX option within a continuoustime model. In these models, the price and the current volatility of the underlying stock are expressed as stochastic differential equations, which contain several unobserved parameters. The Heston model, for example, requires that the variance of the variance be provided in the process of conditional variance. But the stochastic variation of the wave caused by the variance is difficult to isolate from the volatility variation alone, which makes the simulation of this parameter tricky. In addition, in real markets, trades take time to complete. This makes it impossible for trading behaviour to be instantaneous. Therefore, discrete models tend to be more practical in real markets. Heston and Nandi (2000) [49] modified the original Heston model(Heston 1993) [8] by discretizing it according to the generalised autoregressive conditional heteroskedasticity (GARCH) model introduced by Bollerslev in 1986 [6], named the Heston-Nandi model. By relating the spot price of a stock to itself and to the historical path of volatility, this model describes the development

# CHAPTER 5. A GARCH-BASED METHOD FOR OBTAINING LOCAL QUADRATIC HEDGING STRATEGIES FOR VIX CALL OPTIONS

of cost as a function of the minimising of volatility and the correlation between volatility and spot prices.

Even though much work has been done in the literature on the pricing of the VIX option and its related trading strategies, very little has been written about hedging this derivative. Considering that the VIX option is also a type of derivative, hedging is a natural thing to do. However, we find that most of the previous literature has focused on pricing studies of VIX options, with little mention of hedging such derivatives. To fill this gap, the motivation of this chapter is to hedge VIX options based on other financial derivatives to fill the gap in this area of research.

Since there has been relatively little previous research on hedging VIX options, we first need to find a suitable hedging method for this derivative. Several methods are proposed to hedge the number of potential payoffs caused by the option exercise. Different hedging methods are applied to assets and derivatives that follow different processes and distributions. Since the underlying derivative in this chapter is the VIX option and is assumed to follow the GARCH(1,1) model, this chapter applies quadratic hedging as a hedging method for this option. Quadratic hedging is a method of controlling risk by minimising hedging errors over time. There are two types of strategies depending on whether they are self-financing: global quadratic hedging and local quadratic hedging. Considering a stochastic process X and its call option payoff function  $H = (X_T - K)^+$ , the principal goal is finding a self-financial strategy minimising  $E[(H-G_T-V_0)^2]$ . As a practical approximation, quadratic hedging is implemented as an approximate replication of option cash flows based on self-trading policies (Follmer and Sondermann 1986) [62]. Even if typically used for European options, for which the option exercise policy is given, a simple adaptation of the model of Seconandi and Yang (2021) [63] made it possible to apply quadratic hedging to options for which an exercise policy needs to be determined. In particular, Seconandi and Yang (2022) [19] extended the conditional quadratic hedging proposal of Seconandi for assets with a single cash flow on a fixed date, such as European options, to assets distinguished by streams of cash flows, of which American options are a particular case. As it is mentioned, the model implemented in this chapter assumes that the volatility before pricing spot time does not affect the estimated results and that the volatility and stock price are autoregressive. These conditions make the original global quadratic hedging fit model inadequate. As an alternative hedging method, the local risk-minimization quadratic was introduced by Schweizer first (Schweizer 1988) [39]. Compared with global quadratic hedging, local risk-minimization hedging doesn't require self-financing conditions but controls the value of  $\eta_t$  to ensure  $V_t = H_t$  during the business time t. Therefore, this hedging strategy is produced by a series of steps of local optimisations whose goals are to minimise the incremental cost in each time interval.

Based on the above, the objective of this chapter is to hedge European VIX call options under the assumption that the log return and its conditional variance follow the Heston-Nandi model [49]. Finally, we improve the simulation process to obtain the hedging strategy. For the denominator part of the hedging strategy, we give the approximate semi-closed-form directly based on the characteristics of the Heston Nandi model. For the numerator part of the strategy, we use the transition probability matrix to enhance the accuracy of the numerical simulation. Finally, we solve for the semi-closed-form chunks of the strategy using the saddle-point method.

The rest of the chapter is structured as follows, in section 2 we introduce some preliminaries and definitions; in section 3 we provide the Decomposition of the hedging strategy; in section 4 we present the semi-closed-form of some of the decomposed chunks according to the GARCH(1, 1) model; in section 5 the remaining chunks will be simulated according to a transition probability matrix. Numerical experiments and conclusions will be presented in sections six and seven.

# 5.2 Preliminaries of quadratic hedging

### 5.2.1 Notations and definition

The purpose of this chapter is to find a suitable quadratic hedging strategy  $\Phi_t = (\theta_t, \eta_t)$  to deal with the European VIX call option. Before talking about the hedging method, we will define a series of notations and basic research frameworks.

No matter whether the models are built in continuous or discrete type, they are commonly set on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with an information filtration  $\mathcal{F}_t$  where  $t \in (0, T)$ . A hedging portfolio contains a risky asset  $S_t^r$  and a risk-free asset  $B_t^r$  traded in an arbitragefree market that is not affected by trading frictions, Typically,  $S_t^r$  represents the stopping price of a specific stock that follows a stochastic process mentioned above and  $B^r_t$  represent the spot price of a fixed income financial products like bond follows a deterministic process. As the prices of those financial products only grow over time with constant interest rate r, the process to  $B_t^r$  is expressed as  $B_t^r = exp(rt)$ . To avoid repeated discounting operations, we use the price of a risk-free asset as a discounted numéraire. Hence, we define the adjusted price of one unit risk-free asset  $B_t = B_t^r/B_t^r = 1$  and  $S_t = S_t^r/B_t^r$  to eliminate the inherent increment of prices caused by time. The measure  $\mathbb{P}$  represents the real-world (historical) probability measure and  $\mathbb{Q}$  represents the equivalent martingale measure that  $\mathbb{Q} \approx \mathbb{P}$  with  $S_t$  being a local martingale in this measure. Without other definitions, all subsequent work is done in this measure. At last, we define the value of a T-maturity contingent claim we want to hedge and the European VIX option,  $C(h_t, t)$ , where  $h_t$ represents the conditional volatility at time t.

Considering a series of unavoidable difficulties arising from the application of continuoustime models for real transactions, this chapter uses the discrete-time framework as the basis for pricing and hedging. In this framework, stock price information is updated at fixed time points  $t \in 1, 2, 3...$  with the same intervals. We then follow the definition of portfolios strategy mentioned by Schweizer and Martinis (1999) [64] as  $\varphi_t = (\theta_t, \eta_t)$  where  $\theta_t$  represents the number of the risky asset (stock) held at time t and  $\eta_t$  is the number of the risk-free asset (bond) held at time t. This framework also assumes that traders cannot modify their stock positions in parallel with stock price changes, i.e., that strategy changes have a lag, even though the lag can be infinitely close to zero. That means the traders keep their strategies unchanged during (t-1,t] and the number of stocks  $\theta_t$  is  $\mathcal{F}_{t-1}$ -measurable when  $t \geq 1$ . On the other hand, under the local quadratic hedging strategy we choose in this chapter, bonds are always modified in parallel with the stock to automatically balance the total price of the hedged portfolios. Therefore, the number of bonds  $\eta_t$  is  $\mathcal{F}_t$  and held unchanged during [t, t + 1).

Because no trade happens before t = 1, we define  $\phi_0 = (0, V_0)$  where  $V_0$  is the initial value of the hedging portfolio. Following these initial conditions, the value of the portfolio is defined as

$$V_t = \theta_t X_t + \eta_t \tag{5.2.1}$$

and its discrete cumulative gain from initial time 1 up to time t is

$$G_t(\theta_1, \theta_2 \dots \theta_t) = \Sigma_{u=1}^T \theta_u \Delta X_u \tag{5.2.2}$$

where  $\Delta X_u = X_u - X_{u-1}$ .

At last, the discrete cost process will be defined as

$$C_t = V_t - G_t$$
  
=  $V_t - \sum_{u=1}^t \theta_u \Delta X_u.$  (5.2.3)

# 5.2.2 Selecting a hedging strategy for the GARCH model

Following these definitions, we can then define the local quadratic hedging into an incomplete market environment that is raised by discrete trades restricted from the real world. The local quadratic hedging is not a perfect replication model as its inevitable risk  $R_t$ is always beyond zero. In fact, in an incomplete market environment, no matter what hedging method we use and how much it costs, we can never fully hedge the potential risk in future. Therefore, there are two main options for hedging: 1, to achieve the minimum exposure at a reasonable hedging cost such as the quadratic hedging mentioned in this chapter. 2, to achieve the minimum hedging cost in an exposure such as the quantile hedging, Schweizer (1995) [36] introduced self-financing hedging to deal with the risk control problem in the incomplete market of the real world measure. The goal of this strategy is to minimise the quadratic error between contingent claims  $H_T$  and hedging portfolio  $V_T$ at maturity time T. Then the hedging risk  $R_t$  is expressed as

$$R_t = E[(H_T - V_T)^2 | \mathcal{F}_t], \qquad (5.2.4)$$

and the objective of hedging is

$$\underset{(V_0,\theta)\in R\times\Theta}{\operatorname{arg\,min}} \quad R_t \tag{5.2.5}$$

where  $\Theta$  represents the set of all feasible self-financing trading strategies at time t. Since this hedging is performed to deal with the quadratic criterion, known as the global quadratic hedging error, this hedging is called global quadratic hedging (Schweizer, 2001) [38].

Local quadratic hedging is another quadratic hedging mentioned by Schweizer in 1988 first which discards the self-financing rules. It no longer requires the hedging strategy cost process to be constant but imposing  $V_t = H_t$  by modifying the risk-free asset  $\eta_t$  at time t. Because of this, its hedging purpose is to minimise the squared increment in hedging cost caused by strategy modification like

$$\underset{(V_t,\theta)\in R\times\Theta}{\operatorname{arg\,min}} \quad E[(C_{t+1}-C_t)^2|\mathcal{F}_t] \tag{5.2.6}$$

Although some literature says that the local model is inferior to the global one because it minimises the risk arising from all future time increments and not only the next one (Augustyniak 2016) [40], the local model is more suitable for hedging in the GARCH when we consider some characteristics of this model: given t statement variables, the GARCH just looks forward and simulates future processes of stocks return and volatility but ignore the initial state variables such as  $S_0$  and  $h_0$ . The parameters of GARCH are also only used to estimate future processes and cannot be used for backward projection of previous processes. This makes it impossible to use the GARCH model only to obtain information before the moment t from the current statement when it is not quite observable. Compared with the global one, local hedging focus on the stepwise increments from the current time to the next one-time spot and is independent of all information before t. The stepwise iteration property of the GARCH model also coincides with the stepwise minimisation property of local hedging.

# 5.3 Decomposition of local hedging approach

After the choice of hedging, the next step is to figure out its solution form. In the previous literature, the analytical solution is difficult to obtain because of the complexity of the hedging approach. However, in the GARCH model, this problem has been solved to some extent. As a motivation for this section, we will propose an analytical solution for local hedging based on conditional expectations and the transition matrix.

Following the definition above, local quadratic hedging develops strategies based on  $V_t = H_t$  by modifying the risk-free asset  $\eta_t$  at time t and its hedging purpose is to minimise the squared increment in hedging cost. Because the contingent claims to be hedged is a VIX option, the hedging is set in an EMM measure  $\mathbb{Q}$ . Then what we need to do is minimise the expectation of the quadratic increment of the cost process  $E_{\mathbb{Q}}[(C_t - C_{t-1})^2 | \mathcal{F}_{t-1}]$  by determining  $\theta_t$  and  $\eta_{t-1}$  based on the information till t - 1. Following the decomposition of Schweizer (1988) [39] to the expectation of quadratic increment, we note that

$$E[(C_t - C_{t-1})^2 | \mathcal{F}_{t-1}] = E[(V_t - V_{t-1} - \theta_t \Delta X_t)^2 | \mathcal{F}_{t-1}]$$
  
=  $E[(V_t - V_{t-1} - \theta_t \Delta X_t) | \mathcal{F}_{t-1}]^2 + Var[(V_t - V_{t-1} - \theta_t \Delta X_t) | \mathcal{F}_{t-1}]$   
=  $(E[(V_t - V_{t-1} - \theta_t \Delta X_t) | \mathcal{F}_{t-1}])^2 + Var[(V_t - \theta_t \Delta X_t) | \mathcal{F}_{t-1}].$   
(5.3.1)

By the previous definition, the number of risk-free unit assets  $\eta_{t-1}$  is relevant only with the current values of stock prices and the underlying contingent claim. So it cannot affect the value of the second part of (5.3.1). This fact makes the determination of  $\eta_{t-1}$  relatively simple: we only need to approach the first half of the formula to the minimum. By this idea,  $\eta_{t-1}$  is modified to make  $E[(V_t - V_{t-1} - \theta_t \Delta X_t)|\mathcal{F}_{t-1}] = 0$  so it is determined as

$$\eta_{t-1} = E[V_t - \Delta \theta_t X_{t-1} - \theta_t X_t | \mathcal{F}_{t-1}]$$
(5.3.2)

After (5.3.2) holds, the value of the expectation of quadratic increment is determined only by the second part of (5.3.1). The second part could be minimised if and only if

$$Cov(V_t - \theta_t \Delta X_t, \Delta X_t) = 0.$$
(5.3.3)

By using the Doob decomposition, the number of stocks that an investor should hold after adopting this hedging strategy can be obtained in a backward iterative form like

$$\theta_t = \frac{E[V_t \Delta X_t | \mathcal{F}_{t-1}]}{Var[\Delta X_t | \mathcal{F}_{t-1}]}$$
(5.3.4)

We note that the denominator of (5.3.4) is the variance of the stock price increment. In the GARCH model, it can be further expressed as

$$Var[\Delta X_t | \mathcal{F}_{t-1}] = Var[X_t - X_{t-1} | \mathcal{F}_{t-1}] = Var[X_t | \mathcal{F}_{t-1}]$$
(5.3.5)

here we discuss the conditional volatility of  $X_t$  which is not commonly mentioned in the GARCH type model as the previous literature usually cares about the conditional volatility of log return,  $h_t$ . However, we can still approach (5.3.5) the GARCH model processes.

The numerator of (5.3.4) is the conditional expectation of the product of the stock price increment and the value of the underlying contingent claim. According to the definitions and iterated rule, the expectation can be rewritten as follows

$$E[V_{t}\Delta X_{t}|\mathcal{F}_{t-1}] = E[E[H|F_{t}](X_{t} - X_{t-1})\mathcal{F}_{t-1}]$$

$$= E[E[H(X_{t} - X_{t-1})|F_{t}]\mathcal{F}_{t-1}]$$

$$= E[H(X_{t} - X_{t-1})|\mathcal{F}_{t-1}]$$

$$= E[HX_{t}|\mathcal{F}_{t-1}] - E[H|\mathcal{F}_{t-1}]X_{t-1}$$
(5.3.6)

The second half of (5.3.6) is a linear transformation of the underlying contingent claim under  $\mathcal{F}_{t-1}$  as the value of  $X_{t-1}$  is known at this filtration. But the first part is much more complicated than it is described as the analytical closed-form of the product of two unobserved stochastic processes. It is not relatively approachable. In the previous literature, this expectation was often simulated by the Monte Carlo simulation. However, because the prices of stock prices are path-dependent, obtaining accurate results typically requires a particularly large number of experiments. Alternatively, we implement an approximate analytical solution based on the transition probability matrix to solve this problem.

# 5.4 Heston-Nandi model and its application in pricing option

This section reviews a closed-form of a specific GARCH model introduced by Heston and obtains the conditional expectation of stock volatility and the tail expectation of the VIX option by saddlepoint method.

# 5.4.1 Definition of Heston-Nandi model

We consider the model under the physical measure  $\mathbb{P}$  with its probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . Following a specific GARCH(p, q) model introduced by Heston and Nandi (2000) [49], we decide the log-return of the stock spot price by these two equations.

$$R_{t} = r_{t} + \lambda h_{t} + \sqrt{h_{t}} z_{t}$$

$$h_{t} = \omega + \sum_{i=1}^{p} \beta_{i} h_{t-i} + \sum_{i=1}^{q} \alpha_{i} (z_{t-i} - \gamma_{i} \sqrt{h_{t-i}})^{2}$$
(5.4.1)

where  $r_t$  is the risk-free compounded interest rate at the time t defined in the last section.  $z_t$  is a standard normal distribution that presents the innovation of the conditional variance  $h_t$ . This chapter assumes the underlying asset follows the GARCH(p,q) model of the first order case that p = q = 1 and the risk-free interest is a constant r. Furthermore, this

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model will also be used to simplify the Monte-Carlo simulation of the quadratic hedging process.

Commonly, the VIX option is pricing under a risk-neutral measure  $\mathbb{Q}$ . Under the modification of Heston, the risk-neutral version of GARCH(1, 1) is presented as

$$R_{t} = r - \frac{1}{2}h_{t} + \sqrt{h_{t}}z_{t}^{*}$$

$$h_{t} = \omega + \beta h_{t-1} + \alpha (z_{t-1}^{*} - \gamma_{1}^{*}\sqrt{h_{t-1}})^{2}$$
(5.4.2)

where,

$$z_{t}^{*} = z_{t} + (\lambda + \frac{1}{2})\sqrt{h_{t}}$$
  

$$\gamma_{1}^{*} = \gamma_{1} + \lambda + \frac{1}{2}$$
(5.4.3)

To ensure the innovation factor  $z_t^*$  keeps normally distributed, the price of the VIX call option in one period (t - 1, t) is often assumed to follow the Black–Scholes-Rubinstein formula.

Once the processes of the GARCH model are decided, we can now figure out the approximation of (5.3.5). By definitions, we note that  $R(t) = log(X_t) - log(X_{t-1})$  and the conditional variance of R(t) is expressed as  $h_t$ . Thus, the conditional variance of  $log(X_t)$  under filtration  $\mathcal{F}_t$  is obtained as

$$Var[log(X_t)|\mathcal{F}_{t-1}] = Var[R(t)] = h_t.$$
 (5.4.4)

Here the conditional variance is a varying random variable, which is exactly what the heteroskedasticity model shows. However, in order to get a deterministic value when formulating the strategy, we need to fix the denominator component of the number of stocks,  $\theta_t$ , as a constant. One of the approaches is using the conditional expectation of  $h_t$  as a fixed approximation to the conditional variance of  $R_t$ . Therefore, the variance of log return under the filtration  $\mathcal{F}_{t-1}$  becomes

$$E[h_t|\mathcal{F}_{t-1}] \approx Var[R_t|\mathcal{F}_{t-1}] = Var[log(X_t)|\mathcal{F}_{t-1}].$$
(5.4.5)

Next, we perform a first-order Taylor expansion on  $log(X_t)$  under the same filtration to obtain

$$log(X_t) \approx log(E[X_t | \mathcal{F}_{t-1}]) + \frac{X_t - E[X_t | \mathcal{F}_{t-1}]}{E[X_t | \mathcal{F}_{t-1}]}.$$
(5.4.6)

If we find the expectation for both sides of the above equation, we have

$$E[log(X_t)|\mathcal{F}_{t-1}] \approx log(E[X_t|\mathcal{F}_{t-1}]).$$
(5.4.7)

By the process of  $R_t$ , we note that the conditional expectation of  $log(X_t)$  can be expressed as

$$E[log(X_t)|\mathcal{F}_{t-1}] \approx log(X_{t-1}) + r + \lambda E[h_t|\mathcal{F}_{t-1}].$$
(5.4.8)

Here we omit the expectation of the diffusion part because it has a negligible impact on the final result and can significantly reduce the computational complexity. Similarly, we find that the variance between the two sides satisfies

$$Var[log(X_t)|\mathcal{F}_{t-1}] \approx \frac{Var[X_t|\mathcal{F}_{t-1}]}{E[X_t|\mathcal{F}_{t-1}]^2}.$$
(5.4.9)

Combining these results, we obtain the expression of the conditional variance of  $X_t$  under  $\mathcal{F}_{t-1}$  as

$$Var[X_t|\mathcal{F}_{t-1}] \approx E[h_t|\mathcal{F}_{t-1}] \times E[X_t|\mathcal{F}_{t-1}]^2$$
  

$$\approx E[h_t|\mathcal{F}_{t-1}] \times e^{2E[log(X_t)|\mathcal{F}_{t-1}]}$$
  

$$\approx E[h_t|\mathcal{F}_{t-1}] \times e^{2(log(X_{t-1})+r+\lambda E[h_t|\mathcal{F}_{t-1}])}.$$
(5.4.10)

So far, we have given the approximate closed-form of the denominator part of the hedging strategy  $\theta$  based on the GARCH(1, 1) model provided by Heston-Nandi. About the numerator part of the strategy, we note that it consists of an expectation of the product of a discount price and an underlying derivative with its expectation. Regarding the target derivative expectation, we can find an approximation by the saddle point method. As for the first half of the numerator, we need to find another technique.

# 5.4.2 Approach the semi-closed-form of call option price

In this section, we discuss the approach to the conditional expectations of  $h_t$  and the underlying derivative H being hedged to further reduce the need for Monte Carlo simulation for strategy formulation and to improve the stability of the model. We first discuss the conditional expectation of the conditional variance  $h_t$  because of its relatively low complexity. Although there is a range of methods to solve this conditional expectation, given that the cumulant generating function(CGF) of  $h_t$  will also be used later when solving the underlying derivative, the properties of CGF are chosen here to obtain this expectation. We implement a semi-closed-form for moment generating function(MGF) introduced by Zhang (2016) [44] which assumed the MGF of  $h_t$  is related to its historic value and expressed as an exponential structure

$$E[e^{zh_{t+m}}|\mathcal{F}_t] = f(z, m, h_t) = e^{C(z,m) + H(z,m)h_t}$$
(5.4.11)

where

$$C(z,m) = C(z,m-1) - \frac{1}{2}log(1 - 2\alpha H(z,m-1)) + \omega H(z,m-1)$$
  

$$H(z,m) = \beta H(z,m-1) + \frac{\alpha \delta^* H(z,m-1)}{1 - 2\alpha H(z,m-1)}$$
(5.4.12)

with the initial condition as

$$C(z,m) = 0$$

$$H(z,m) = z$$
(5.4.13)

Then, the CGF of  $h_t$ , k(z) is expressed the logarithm of MGF that

$$K_h(z) = C(z,m) + H(z,m)h_t$$
(5.4.14)

According to the properties of MGF, we can approach the conditional expectation of  $h_{t+1}$ under the filtration  $\mathcal{F}_t$  by setting z to 0 after deriving MGF regarding z as

$$E[h_{t+1}|\mathcal{F}_t] = \frac{dE[e^{zh_{t+1}}|\mathcal{F}_t]}{dz}|_{z=0}$$
(5.4.15)

We then discuss the pricing of the underlying derivative, the VIX European call option. From Cboe's definition of a VIX derivative, a VIX call option is constructed as the tail expectation of the difference between the VIX value and a given strike price

$$C(VIX_T, t) = e^{-r(T-t)} \mathbb{E}[(VIX_T - K)^+ | \mathcal{F}_t]$$
(5.4.16)

From this model, one naturally wonders how to predict the value of  $VIX_T$  under  $\mathcal{F}_t$ . In the previous section, since we are not using the original Heston or GARCH model, but apply the modelling approach that relates the value of VIX to the process of  $R_t$  or  $S_t$ . As the original Heston-Nandi model is used as an example in this section, we can consider the approach mentioned by Hao and Zhang (2013) [51] and express the value of  $VIX^2$  with expected daily variance like

$$VIX_t^2 = 252 \times V_t(22) \tag{5.4.17}$$

where 252 is the annual factor and  $V_t(n)$  represents the annualised average of expected daily variance under a risk-neutral measure as

$$V_t(n) = \frac{1}{n} \sum_{i=1}^n E_{\mathbb{Q}}[h_{T+i} | \mathcal{F}_t]$$
(5.4.18)

In this form, we took the average daily expectation of 22 trading days in a month and multiplied it by an annual factor to obtain the value of  $VIX^2$ . According to Wang (2006), this daily variance expectation can be further expressed as  $h_{t+1}$  in affine structure as follows

$$V_t(n) = (1 - \Gamma(n))\bar{h} + \Gamma(n)h_{t+1}$$
(5.4.19)

where  $\bar{h}$  represents the long-term stable volatility as

$$\bar{h} = \frac{w + \alpha}{1 - \beta - \alpha \delta^2}$$

and

$$\Gamma(n) = \frac{1 - (\beta - \alpha \delta^2)^n}{n(1 - \beta - \alpha \delta^2)}.$$

## CHAPTER 5. A GARCH-BASED METHOD FOR OBTAINING LOCAL QUADRATIC HEDGING STRATEGIES FOR VIX CALL OPTIONS

As the semi-closed-form expression of its CGF is known, we further express the option as an approximation of the tail expectation of the square root of  $VIX^2$  as

$$C(VIX_T, T) = e^{-rT} \mathbb{E}_0^{\mathbb{Q}}[(VIX_T - K)^+]$$

$$= e^{-rT} \mathbb{E}_0^{\mathbb{Q}}[(\sqrt{VIX_T^2} - K)^+]$$
(5.4.20)

and then apply the alternative saddlepoint method according to Kowk and Zheng's work in 2014 [2] and obtain the analytical closed-form expression of this tail expectation as

$$\mathbb{E}_{0}^{\mathbb{Q}}[(\sqrt{VIX_{T}^{2}}-K)^{+}] = \frac{1}{4\sqrt{\pi}i} \int_{r-i\infty}^{r+i\infty} \frac{e^{k(\hat{z})+g(\hat{z})}}{\hat{z}^{3/2}} d\hat{z}r \in (0,\alpha_{+})$$
(5.4.21)

where k(z) is the CGF of the  $VIX_T^2$  and  $g(z) = log(1 - erf(\sqrt{z}K))$ ,  $\hat{z}$  is the positive root of the alternative saddlepoint equation

$$k'(\hat{z}) + g'(\hat{z}) - \frac{3}{2\hat{z}} = 0.$$
(5.4.22)

# 5.5 Transition probability matrix simulation for twist part

After gaining the semi-closed-form part of the strategy, we now focus on the simulation part, the expectation of the product of the discount price  $X_t$  and its underlying derivative H. Duan (2001) [65] proposed a Markov Chain method for valuing American options under the GARCH model and prove that it works well for the GARCH option pricing framework as an alternative numerical method. Following this idea, we approach the expectation  $E[HX_t|\mathcal{F}_{t-1}]$  by a transition probability matrix that describes the transition process of the product between different states. To apply the method of transition probability matrix to the simulation, we need to assume that the processes of the discounted price and conditional volatility follows a binary discrete process and transition in finite states. Therefore, we define the discounted price has m states [X(1), X(2), X(3)...X(m)]and the conditional volatility has n different states [h(1), h(2), h(3)...h(n)] in this binary system (X(i), h(j)) where  $i \in 1...m$  and  $j \in 1...n$ . Before further discussing the transition probability matrix, we also need to remove the time trends in this system. We note that even when modelled under the risk-neutral measure,  $X_t$  still tends to grow over time due to the presence of the risk-free rate. This makes the iterative process of  $X_t$  consisting of a combination of state transfer and time trend contributions. Therefore, when simulating this bivariate discrete process, we remove the time trend to generate a new discount price variable  $X_t^*$ 

$$X_t^* = X_t e^{-(r - \frac{1}{2}\bar{h})\Delta t}$$
(5.5.1)

and use it for the state transition simulation. In addition, since the conditional variance  $h_t$  under the risk neutrality measure no longer contains the time trend, we did not make any modifications to it and used a bivariate system  $(X^*(i), h(j))$  for the subsequent state transfer process with similar definitions. After the binary system is settled, we next consider two questions: how each state is defined, and the probability that this bivariate system will enter this state at the next step.

After removing the time trend, the logarithm of  $X_t^*$  is located in intervals centred by  $log X_{t-1}^*$  and settled symmetrically as  $[log X_{t-1}^* - \Delta log X^*, log X_{t-1}^* + \Delta log X^*]$  where  $\Delta log X^*$  is relevant to the expectation of the conditional volatility  $h_t$  at time t-1 and defined as

$$\Delta \log X^* = \sqrt{\frac{E[h_t|\mathcal{F}_{t-1}](m-1)}{2}}.$$
(5.5.2)

To represent m different statements of the discounted price  $log X^*(i)$ , the overall interval should be divided into m cells C(i) with discounted prices locating in the centre

$$log X(i)^{*} = log X_{t-1}^{*} - \Delta log X^{*} + \Delta log X^{*} \frac{2i-2}{m-1}$$

$$C(i) = [log X_{t-1}^{*} - \Delta log X^{*} + \Delta log X^{*} \frac{2i-3}{m-1}, log X_{t-1} - \Delta log X^{*} + 2\Delta log X^{*} \frac{2i-1}{m-1}]$$
(5.5.3)

where

$$i = 2...m - 1$$

$$C(1) = [-\infty, \log X_{t-1}^* - \Delta \log X^* + 2\Delta \log X^* \frac{2i - 1}{m - 1}]$$

$$C(m) = [\log X_{t-1}^* - \Delta \log X^* + 2\Delta \log X^* \frac{2i - 3}{m - 1}, \infty]$$

Similarly, the conditional volatility interval is defined as  $[h_{t-1} - \Delta h, h_{t-1} + \Delta h]$  where the determination of  $\Delta h$  is based on the dispersion of the logarithm of the conditional variance (Duan 2001) [65] such as

$$\Delta h = \log[e^{h_{t-1}} + \sqrt{\frac{n-1}{2}}\sigma_h] - h_{t-1}$$
(5.5.4)

where  $\sigma_h$  is the standard deviation of the conditional volatility  $h_t$ . For the logarithm of the conditional variance, the cells of conditional volatility are defined as

$$h(j) = h_{t-1} - \Delta h + \Delta h \frac{2j-2}{n-1}$$
  

$$D(j) = [h_{t-1} - \Delta h + \Delta h \frac{2j-3}{n-1}, h_{t-1} - \Delta h + 2\Delta h \frac{2j-1}{n-1}]$$
(5.5.5)

where

$$j = 2...n - 1$$

$$D(1) = [-\infty, h_{t-1} - \Delta h + 2\Delta h \frac{2j - 1}{n - 1}]$$

$$D(n) = [h_{t-1} - \Delta h + 2\Delta h \frac{2h - 3}{n - 1}, \infty].$$
(5.5.6)

After defining the states of the two variables in this binary system, we combine these states and assume that the transfer of these state combinations obeys a transfer probability matrix. we define the joint stochastic processes of  $X^*(i)$  and h(j) from t-1 to t as a  $mn \times mn$  transition probability matrix

$$P = \begin{pmatrix} p(11,11) \dots & p(11,ij) \dots & p(11,mn) \\ p(12,11) \dots & p(12,ij) \dots & p(12,mn) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p(ij,11) \dots & p(ij,ij) \dots & p(ij,mn) \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p(mn,11) \dots & p(mn,ij) \dots & p(mn,mn) \end{pmatrix}$$

The element of this matrix p(ij, kl) represents the transition probability matrix from timevolatility statement (i, j) to statement (k, l) during a one-time interval. With the results of the product in each statement, the conditional expectation of the product of the option price H and the underlying asset price  $X_t^*$  is simulated as

$$E[HX_t^*|\mathcal{F}_{t-1}] = \sum_{i=1}^m \sum_{j=1}^n H(X(k), h(l))X(k)p(ij, kl).$$
(5.5.7)

The next step seems to be that we need to find complex probability transfer networks for  $m \times n$ . But we note that in the GARCH(1,1), once  $R_t$  (obtained by  $X_t$  and  $X_{t-1}$ ) and  $h_{t-1}$  are known, the value of  $h_t$  is deterministic. The transfer probabilities provided by Duan [65] also consider only the conditional probability of adjusted discounted price and expressed the transition probability under risk-neutral measure  $\mathbb{Q}$  as

$$p(ij,kl) = Pr\{X_t = X(k) | X_{t-1} = X(i), h_{t-1} = h(j)\}$$
  
=  $Pr\{L_{ij}(k) < Z < L_{ij}(k+1)\}.$  (5.5.8)

Till now, we can simulate the product of adjusted discounted price and the conditional variance by this transition probability matrix and apply it to approach the numerator part of the hedging strategy after returning the time trend to the simulation results.

# 5.6 Numerical experiment

This section introduces two types of the numerical experiment to show the hedging performances of the traditional Monte Carlo simulation and the semi-closed-form approximation to the hedging strategy. All the experiments are conducted under the EMM measure as the price of the option needs to be evaluated under  $\mathbb{Q}$  as a benchmark to make the results meaningful. All the parameters used for experiments are also adjusted for the risk-neutral measure.

Both Monte Carlo simulations and semi-closed-form approximation are applied to 1000 paths of daily stock price and its relevant volatility simulation following the GARCH(1, 1)

model mentioned in section 3. We refer to the parameters estimated by Wang (2017) [1] based on returns/VIX observations from March 2004 to December 2013. After making the necessary adjustments, we set the initial adjusted stock price to be  $S_0 = 1$  and the initial conditional volatility to be  $h_0 = 2 \times 10^{-4}$ . Then we implement a risk-neutral type of GARCH model with parameters used as follows

$$R_{t} = -\frac{1}{2}h_{t} + \sqrt{h_{t}}z_{t}^{*}$$

$$h_{t} = 1.43 \times 10^{-6} + 0.9952h_{t-1} + 1.4468 \times 10^{-6}(z_{t-1}^{*} - 390.7377_{1}\sqrt{h_{t-1}})^{2}$$
(5.6.1)

where  $z_t^* = z_t + 4.283$  and long-term stable volatility  $\bar{h} = 2.9990 \times 10^{-4}$ .

The hedging claim is set as a European VIX call option such as  $H_t = E_{\mathbb{Q}}[(VIX_T - K)^+ | \mathcal{F}_t]$ with strike ranging from 0.03 to 0.13, and maturity of T = 22 (a month). This is because after one month of iterations through this model, the conditional volatility will be very close to the long-term volatility. This makes the latter option prices and hedges perform similarly to one-month values. On the other hand, because of the use of discounted subordinator, the annualised risk-free rate is 0.

To ensure two methods are applied on the same grounds, the hedging experiments are all conducted by the same bonds and stocks when the strike price and maturity of the hedging claim are given. The hedging performance is obtained by the cumulative amount and compared the simulated results in terms of both average and 99% quantile. We first observe the hedging of different strategies against different strikes after half a month (11 days) and then observe the performance of the two hedging strategies over the whole business time with a fixed strike. To avoid small probability events, we performed this numerical experiment one hundred times and then exhibit the average of the results in the following tables.

From the table.1, we observe that the improvement in hedging effectiveness resulting from semi-closed-form local quadratic hedging under  $\mathbb{Q}$  becomes more prominent when the options are in-the-money but perform similarly when their strikes reach the at-the-money interval. At small strikes, the averages of the square of the cost increment to maintain

Average per	form of	cost inc	rement	of 11th	daily he	edging
Κ	0.03	0.05	0.07	0.09	0.11	0.13
OP	0.114	0.103	0.092	0.082	0.071	0.061
MC(10^-4)	37.55	32.84	27.06	24.08	25.22	44.78
CF(10^-4)	3.91	4.04	3.85	3.92	5.54	32.57

Average perform of cost increment for Strike $K=0.07$								
t(business time)	1	4	7	10	13	16	19	22
OP	0.094	0.094	0.095	0.095	0.095	0.097	0.098	0.101
MC(10^-4)	24.84	31.8	31.47	27.78	28.16	36.71	36.98	44.93
CF(10^-4)	3.83	3.43	3.22	3.33	3.97	7.72	14.89	29.73

Table	5	.2
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 $H_t = V_t$  for the strategy developed by the semi-closed-form method is approximately 10.4% of the Monte Carlos'. The advantage of this method becomes smaller as the strike value increases. When the strike reaches 0.11, the mean of squared increment cost simulated by the semi-closed-form method is about 22% of that of the Monte Carlo's. When the strike reaches the money bar, the mean of squared increment costs of the two methods gradually converges. Table 2. reveal the relationship between business time and hedging effectiveness improvement. Similar to the trend of strikes, the hedging performance of the semi-closed-form method has a more significant advantage relative to the Monte Carlo method when the business time t is small. For the same number of iterations, the square cost increment for the semi-closed-form method to reach the local quadratic hedge is only 15% - 20% of that of the Monte Carlo's. This advantage is gradually lost after t increases. The difference in the average square cost increment between the two methods is within 50% when t reaches 22(number of business days of one month).

	99% qu	arter per	form of c	ost incre	ment for	Strike K	=0.07	
Freq	1	4	7	10	13	16	19	22
OP	0.094	0.094	0.095	0.095	0.095	0.097	0.098	0.101
MC(10^-4)	209.49	208.53	168.94	227.41	276.82	642.79	1974.01	3531.86
$CF(10^{-4})$	29.85	18.36	7.59	23.12	61.68	85.56	292.28	584.60

Table 5.3

99% quarte	er perforr	n of cos	t increm	nent of 11	th daily	hedging
K	0.03	0.05	0.07	0.09	0.11	0.13
OP	0.114	0.103	0.092	0.082	0.071	0.061
MC(10-4)	150.04	32.84	27.06	249.06	249.06	312.75
CF(10-4)	14.76	4.04	3.85	65.25	65.25	331.04

Table 5.4	Tal	ble	5.	4
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From Tables 3. and 4., we remark that the 99% quantile hedging cost has a similar profile to the average hedging cost in the fits of the semi-closed-form method and the Monte Carlo method. At times when strike and business time t are relatively small, the semi-closedform method has a clear advantage: over 1000 experiments, the 99% quantile hedging cost induced by the semi-closed-form method is 10% - 15% of that of the Monte Carlo method. As the strike gradually approaches the at-the-money situation, or as the business time tends to 1 month, the hedging cost advantage induced by the different methods diminishes.

# 5.7 Conclusion

This chapter studied the effects of different simulations of local quadratic hedging strategies with GARCH models by scaling its square increment errors and conducting a numerical test to reveal the method's sophistication. The result addressed some crucial properties of the two methods: (i) The average incremental cost of reaching a hedging strategy with the semi-closed-form method is smaller than that of the Monte Carlo method for the same number of simulations. (ii)In the extreme case (99% quantile), the semi-closed-form method demonstrates a similar advantage. (iii)This advantage diminishes at high strike values and near maturity time, but the opposite result does not occur. In summary, we can conclude that the semi-closed-form method to obtain a hedging strategy for VIX call options in the framework of the GARCH model will outperform the Monte Carlo method for the same fitting cost. The advantage of this fitting method mainly comes from the full use of the conditional variance of the GARCH model and the reasonable fitting of the Brownian motion in the innovation factor these are ignored by the traditional Monte
Carlo method. This neglect dramatically increases the number of experiments required to achieve the same fitting accuracy, allowing investors with quadratic hedging based on GARCH models to obtain a more stable and accurate hedging strategy.

In future research, it would be an interesting direction to introduce this simulation method into more hedging under an extended GARCH model and optimise their approximation processes. We note that the GARCH model has many extension branches, such as EGARCG and NGARCH, which change the structure of the model, and subordinator GARCH and regime switching GARCH, which change the meaning of the parameters. Even, in short-term large market shocks, the introduction of 3/2 of the  $\frac{1}{v_t}$  variable in the SDE of log return  $R_t$ . Hence, how the method of this chapter can be applied to these extended models based on the original GARCH model will be a meaningful research topic in future work. Finally, due to the complexity of the semi-closed-form of the CGF, the Semi-closed-form hedging method for each path is extremely computationally intensive, which in turn prevents us from simulating this method in large times. In future research, we expect to improve the efficiency of this hedging method by approximating the option prices more simply while controlling the loss of accuracy.

### Chapter 6

# **Conclusion and Future Directions**

#### 6.1 Solution to problems

This thesis investigates the problem of pricing and hedging VIX derivatives under different characteristics of stochastic models and generalises the traditional model. In general, we have the following conclusions. 1, In the family of continuous-time models, the regime switching method has better compatibility with the more popular 4/2 models nowadays. The integration of these two models makes its results, the regime switching 4/2 methods, both able to control the power of the diffusion term and to adapt to the relevant parameter changes under different periods, making this model more relevant to the financial process described. 2, This thesis explores new discrete financial models by discretizing the 4/2continuous-time model into GARCH type by a technique like the Heston-Nandi model. The 4/2 GARCH model generated by this approximation inherits the power-adjustable property of the traditional 4/2 model in terms of the diffusion factor, but also possesses the ease of implementation of the GARCH model. 3, By numerical integration, this thesis combined some common subordinator and 4/2 GARCH models, making the combined model adaptable to the different distributions of business time generated due to changes in trading volume or trading period. 4, on the properties of the GARCH model, this thesis proposes more specific hedging methods and simulations for VIX derivatives under the GARCH framework. The previous Monte Carlo model does not make full use of the relevant characteristics of the GARCH model in the simulation of the strategy of quadratic hedging. The simulation method of hedging strategies suggested in this thesis utilises the state transfer matrix to obtain better hedging strategies than the traditional Monte Carlo method within the same number of simulations. Corresponding to the abstract and introduction, the main contribution of this thesis is to modify and expand on existing stochastic financial models for specific models. While retaining the advantages of those stochastic models, we exploit the potential of these modifications and extensions in terms of model optimisation. Some meaningful ideas are provided for subsequent researchers and investors to improve the existing financial measurement tools in the future. In addition, the saddlepoint method plays an important role in this thesis as a core method for determining the values of the derivatives.

#### 6.2 Unresolved issues and future research directions

During this thesis research, we left some questions to be solved, while submitting some possible research directions on these questions. 1, The derivatives used in the examples and numerical experiments in this thesis are European VIX call options. But European call options are only one of the simplest derivatives. Therefore, whether the pricing and hedging methods involved in this thesis are feasible for other financial derivatives deserves further research. 2, In addition to the construction method of VIX used in this thesis, there are other methods of obtaining the value of VIX based on forward contracts, etc. 3, Future research could discuss and compare the accuracy of different VIX definitions for pricing and hedging methods and overcome the computational efficiency issues caused by the higher order estimation of VIX options. However, in some financial models, solutions of higher order forms can lead to very complex semi-closed-forms. This often makes the computation time unacceptably long. Therefore, future researchers can consider some better approximation methods to reduce this error. 4, This thesis only discusses the application of local quadratic hedging in VIX derivatives, but this does not mean that

global hedging is not relevant for VIX derivatives. In fact, in most of the literature, global hedging approaches are closer to the expected hedging needs than local hedging. Therefore, a possible research direction is to perform global hedging or more complex hedging of VIX derivatives. 5, Last but not least, to subject the volatility to controlled factors while blocking other influences, we only considered and simulated the model in a laboratory data environment. This means that the performance of our models and their extensions in the real market data environment is unknown, even if they performed well in the laboratory data environment. An important future work is to identify suitable market data sets, apply the financial models and their extensions in this thesis to these data sets, and check their performance in the real market data environment.

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