An empirical study of the equilibrium term structure of interest rates in Australia

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AN EMPIRICAL STUDY OF THE EQUILIBRIUM TERM STRUCTURE OF INTEREST RATES IN AUSTRALIA

TOAN MY PHAM

SUBMITTED IN FULFILMENT OF THE

REQUIREMENT FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

15 NOVEMBER 1997

CERTIFICATION

I, Toan My Pham, declare that this thesis ("An Empirical Study of the Equilibrium Term Structure of Interest Rates in Australia") does not contain work or material which I have previously submitted for any other university degree or similar award.

12 November 1997

CERTIFICATE OF ORIGINALITY

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of a university or other institute of higher learning, except where due acknowledgement is made in the text.

I also declare that the intellectual content of this thesis is the product of my own work, even though I may have received assistance from others on style, presentation and language expression.

(Signed)

ABSTRACT

This study develops an integrated framework to empirically test a class of equilibrium models of the term structure of interest rates in the Australian context: (i) Cox, Ingersoll and Ross (1985), (ii) Vasicek (1977); and (iii) a generalised version of Cox, Ingersoll and Ross (1985), first introduced in this study.

The research involves an examination of the following specific issues:

(i) to what extent the empirical implications of these models are supported by the Australian bond market; and

(ii) which model performs best in terms of goodness of fit and predictiveness.

To examine these issues, the study develops an integrated estimation methodology capable of incorporating zero coupon and coupon paying bonds, which was used in two distinct phases of the research design:(i) A new technique based upon Chebyshev polynomials is designed to generate zero-coupon term structures from a limited number of coupon paying bonds; (ii) The second the stage is concerned with estimating, testing, and ranking the following three models: CIR (1985), generalised CIR, and Vasicek (1977), using all available zero-coupon and coupon-paying Australian bonds from 1985 to 1992.

The main results of the study may be summarised as follows:

(i) The newly introduced Chebyshev polynomial based curve fitting

technique performs best relatively to the differential equation based Nelson-Siegel model.

(ii) While the instantaneous spot rate in equilibrium models is the predominant and driving factor in bond pricing it consistently underestimates its two closest observed rates, the cash rate and 13-week Treasury Note rate. However, the underestimation is removed when model rates of 13-week maturity are compared with 13-week Treasury Note rates, indicating that the biasedness is probably caused by the lack of an observed instantaneous spot rate.

(iii) There is mild support for the models' implication of parameter stability.

(iv) On the basis of performance criteria the generalised CIR model, first introduced in this study, is the best performer although it is also the most computationally difficult.

(v) The tested models display considerable multicollinearity, a characteristic consistently recognised in this literature.

In conclusion, the study found empirical support for equilibrium models of the term structure of interest rates in Australia.

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Finally and most importantly, I dedicate this work to my family. To my beloved wife, Roseline, and precious children, Cyril and Marilee, I bow and say humbly: " A thousand apologies for many a late night." I hope that it would not be "a thousand apologies that come a lifetime too late!"[1]

[1] Rephrased from the title of chapter 24 of Chang J. *Wild Swan*. Flamingo (1992).

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CHAPTER 1

OBJECTIVES AND SCOPE OF THE STUDY

The term structure of interest rates, commonly defined as the relationship between times to maturities and yields on default free bonds, is a topic that attracts intensive research in finance and economics. The history of this research dates back to the end of the 19th century[1] and was based upon some ad-hoc characterisations of the notion of expectation in financial markets while current research views bonds as financial assets and concentrates on developing: (i) conditions for the pricing of these bonds that are consistent with equilibrium in financial markets (partial equilibrium) or the whole economy (general equilibrium); or (ii) conditions for the existing term structure to evolve over time such as to preclude arbitrage opportunities. These theories of the term structure are known as equilibrium and preference free respectively and they form the paradigm of modern[2] term structure research in the last twenty years. The models that feature most prominently in this literature include equilibrium models by Vasicek (1977), Cox, Ingersoll and Ross (1985), and the preference free model by Heath, Jarrow and Morton (1992).

1.1 MOTIVATION, RESEARCH ISSUES AND SCOPE OF INVESTIGATION

As with any scientific endeavour, theoretical developments entail empirical verification. In contrast to the testing of traditional theories of the term structure, tests of modern term structure models are limited in number and these have been

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[1] See Irving Fisher (1896).

[2] The term 'modern' is used here to differentiate these models from the so-called traditional theories based upon some ad-hoc characterisation of expectations.

carried out with mostly US data[3]. The choice between equilibrium theories and preference theories largely depends upon the focus of interest. If pricing derivatives on the assumption that the existing term structure is in equilibrium is of primary interest, then testing preference free theories will be a natural step. While the assumption may be questionable, the empirical verification of the no-arbitrage condition to ensure the ensuing term structures to be consistent with the initial term structure would constitute a separate study in its own right. However, our primary interest in this thesis lies in the term structure that can be both related to economic factors and consistent with either the financial markets or the economy in equilibrium.

A review[4] of the empirical evidence of equilibrium theories reveals that: (i) there are substantially different and conflicting results across models; (ii) there is evidence of significant parameter instability, inconsistent with the specification of equilibrium theories; and (iii) assessing the validity of model performance becomes a difficult task since there is a serious lack of a comprehensive comparative study of competing models. These problems provide the principal motivation for undertaking an empirical investigation of some of the most prominent equilibrium models in the existing literature: Vasicek (1977), Cox, Ingersoll and Ross (1985b) - hereafter CIR - and an extended version of the CIR model, first introduced in this study.

One of the functions of a theory of the term structure, including equilibrium

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[3] To our knowledge, there are three empirical studies of modern term structure models on Australian data. These include a published paper by Chiarella, Mackenzie and Pham (1992) and two unpublished working papers by Chiarella, Lo and Pham (1990), and Hathaway (1988).

[4] See Chapter 2, Section 2.4.4.

theories, is to estimate the unobserved term structure using existing information. In this respect, the theory performs a function similar to that of the popular practice of fitting the yield curve using existing coupon bond prices. This curve fitting is based upon the no-arbitrage condition in an efficient bond market, namely the price of a coupon bond must equal the conditional expectation of the present value of its coupons and terminal face value. While the theoretical underpinning of curve fitting is insufficient to explain what determines the shape and level of the term structure, it is adequate for the purpose of interpolating unobserved interest rates from observed interest rates. Interpolation[5] is also one of the functions of theories of term structure. In this respect, curve fitting may be viewed as a naive model which shares a similar purpose and which is easier to estimate. That, perhaps, is where the analogy ends because while term structure theories explain and predict how the current and future term structures are determined by economic variables, curve fitting treats the current term structure as a 'black box' which has to be uncovered at each point in time using ex-post bond prices. By definition a curve-fitted term structure tracks ex-post data better than an economic based model. This property is exploited to generate term structures (from a limited number of bonds) to act as benchmarks to judge the estimated equilibrium models. Thus, in the absence of sufficient observed zero coupon bonds[6] a generated curve can be considered an 'almost surely observed' term structure which may be used to assess an equilibrium

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[5] While a term structure is estimated from a limited number of observed maturities, it is anticipated that interest rates corresponding to unobserved maturities may be inferred from this curve.

[6] In Australia there are only three zero coupon bonds, namely, 5-week, 13-week and 26-week Treasury Notes.

model in so far as goodness of fit and model predictiveness are concerned.

Given the objectives outlined above, the study develops an integrated estimation methodology - capable of incorporating all zero coupon and coupon paying bonds - that enables the empirical work to be carried out in two distinct phases:

(i) A new curve fitting technique based upon Chebyshev polynomials is introduced to utilise all observed bond prices from 1985 to 1992 to construct daily term structures;

(ii) Estimation and analysis of equilibrium models by Vasicek (1977), Cox, Ingersoll, and Ross (1985b), and a generalised version of Cox, Ingersoll, and Ross (1985b) are carried out with a view to ranking these models on the basis of two criteria: (a) empirical support for the implications of the models; and (b) goodness of fit and predictive contents.

1.2 CONTRIBUTIONS OF THIS STUDY

In view of the substantially different and often conflicting evidence found in existing studies of equilibrium models and the serious lack of comparisons across competing models especially in the Australian context, a major contribution of this thesis is to provide a remedy in this regard.

Specifically this study makes several contributions in two broad categories: (1) contributions to the empirical literature, especially Australian, of equilibrium term structure of interest rates; and (2) contributions to the estimation methodology.

(1) Contributions to the literature

(a) It offers the first comparative analysis of the empirical verification of equilibrium models in the Australian context.

(b) The original Cox, Ingersoll and Ross (1985b) model is generalised and its numerical estimation presents an alternative to the closed form solution to stochastic differential equations. In many respects the numerical method is substantially more flexible in that it is less restricted by assumed forms of the interest rate process and volatility function.

(c) It provides a theoretical basis for pricing interest rate derivative securities in Australia in place of ad-hoc yield curve fitting.

(2) Contributions to the estimation methodology

(a) It introduces a curve fitting technique based upon Chebyshev polynomials which effectively removes two commonly encountered econometric difficulties in term structure estimation, namely, maturity dependent errors and multicollinearity caused by the mismatch of coupon payment dates.

(b) The bond pricing functions of equilibrium models are both highly nonlinear and multiplicative in parameters. Existing methods of estimation assumes additive disturbances which are less appropriate for bond prices. Hence, a multiplicative type of error is introduced. This specification is both consistent with the nature of the pricing functions and is significantly more accurate than additive errors.

(c) It introduces a minimisation procedure based upon nonlinear regression that incorporates zero-coupon and coupon paying bonds, hence significantly increases the degrees of freedom and accuracy of estimation.

1.3 OUTLINE OF SUBSEQUENT CHAPTERS

Chapter 2 reviews the literature of the term structure of interest rates. It begins by providing a careful analysis of traditional expectations theories and then proceeds to examine modern bond pricing theories (equilibrium theories and preference free theories). Finally an appraisal of the three main strands of theories of the term structure (traditional, equilibrium and preference free) is given. Throughout the review, the theoretical underpinning of the models, the empirical methodology applied in testing these models and the evidence therefrom are critically analysed and presented. In particular, two common methods of estimation, generalised method of moments (GMM) and nonlinear least squares, are compared and contrasted.

Chapter 3 presents the empirical methodology which is used in this thesis and a description of the data set. The first part of this chapter summarises the method of estimation used in this study, nonlinear regression, which numerically solves for the vector of the parameters of tested models by minimising the sum of squares of the deviations of the observed bond prices from their theoretical (model generated) prices. The hypotheses of each tested model and the criteria for model selection are then outlined. The second part presents the methodology of yield curve fitting and introduces a new numerical method based upon Chebyshev polynomials. The chapter concludes with a description of the data set supplied by the Reserve Bank of Australia.

Chapter 4 presents the empirical results of the two statistical models, Nelson-Siegel (1987) and Chebyshev polynomials. This is followed by an analysis and discussion of the results, which ultimately leads to the choice of the Chebyshev polynomial based model to be used as benchmark to judge the performance of equilibrium models.

Chapter 5 presents and discusses the empirical results of the general equilibrium Cox, Ingersoll and Ross (1985) model, and then examines the statistical properties of its parameters with a view to (i) determining the extent to which well the model conforms to its theoretical prescriptions in the Australian empirical context; and (ii) determining its performance against the benchmarks generated by the Chebyshev polynomial based model. This estimation and investigative methodology is again applied to a generalised version of the Cox, Ingersoll and Ross model, first introduced in this study, and the Vasicek (1977) model. The empirical results are presented in Chapter 6 and Chapter 7 respectively. The overall results of Chapters 4, 5, 6 and 7 are placed in perspective in Chapter 8 where a comparison of the models is made. Then Chapter 9 summarises the objectives, issues of investigation, contributions of the study, and suggests areas of future research.

CHAPTER 2

REVIEW OF THE LITERATURE

Theories have been advanced to explain the shapes and levels of the term structure of interest rates over time. The history of the study of this subject has been long and voluminous. These theories, however, can be categorised into two major strands paralleling their historical evolutions: traditional and 'modern'. The first group consists of theories known as theories of expectations, liquidity preference and preferred habitat. The 'modern' group grew out of the development of continuos time finance by Black-Scholes and Merton era in the early 1970's when techniques of forming arbitrage portfolios and the application of stochastic differential equations to derive equilibrium asset pricing relationships were extended to default free bonds.

The major aim of this chapter is to review the literature relevant to the objectives of this thesis. The chapter is organised as follows: Section 1 reviews basic concepts and interest rate mathematics while Section 2 contains a brief description of the traditional theories. Section 3 then concentrates on the modern theories and Section 4 concludes the chapter.

2.1 PRELIMINARIES

The objective of this section is to develop some common definitions of terms and notation to facilitate the ensuing discussion. To simplify the notation we shall construct a reference framework under certainty using the default free zero coupon bond as the building block. This can be achieved without loss of generality and can be easily adapted to coupon paying bonds, each of which can be viewed as a portfolio of zero coupon bonds.

By definition a discount bond pays \$1 at maturity and no coupon; hence we have:

$$P(T,T) = 1 \tag{1}$$

The yield to maturity, $y(t, T)$, is the continuously compounded rate of return that the discount bond earns between t and T. Thus,

$$P(t,T) \exp\left[(T - t)\, y(t, T)\right] = 1 \tag{2}$$

or

$$y(t, T) = -\frac{\ln P(t, T)}{(T - t)} \tag{3}$$

The instantaneous spot rate of interest, r, is the yield to maturity of a bond maturing instantly:

$$r(t) = y(t, t) \tag{4}$$

The forward rate of return, f(t,T), is the return implied in current prices for some future time. Thus, f(t,T) is the return on an investment P(t, T) for an instant after T:

$$f(t, T) = -\frac{\partial P(t,T)/\partial T}{P(t, T)} \tag{5}$$

Under certainty all securities must earn the same rate of return, $t \forall T$:

$$-\frac{\partial P(t, T)}{\partial T} \cdot \frac{1}{P(t, T)} = r(t) \tag{6}$$

or

$$\frac{\partial P(t,\ T)}{\partial T} + r(t)\ P(t,\ T) = 0 \tag{7}$$

The unique solution to (7) under the boundary conditions, $P(T,\ T) = 1$, $P(t,T) \geq 0$, is

$$P(t,\ T) = \exp\left[-\int_t^T r(s)ds\right] \tag{8}$$

Then substituting (8) into (3) yields:

$$y(t,\ T) = \frac{1}{(T-t)} \int_t^T r(s)ds \tag{9}$$

It can be seen that the instantaneous forward rate, instantaneous spot rate, and instantaneous yield to maturity are equal:

$$f(t,\ t) = r(t) = y(t,\ t) \tag{10}$$

The term structure of interest rates is then defined as the yield to maturity on default free bonds as a function of their time to maturity.

2.2 TRADITIONAL THEORIES

The common thread underlying the traditional theories is that expectations play a central role in determining the term structure. Over time it has been proposed that other factors may also be at work. The preference for liquidity, which is assumed to grow out of the imbalance between investors wanting to borrow long and lenders lending short, gives rise to a possible liquidity premium[1] for long-term bonds. Further, bond markets may be segmented by preferences for chosen maturities such that each class of investors choose to stay within its 'preferred habitat'.

2.2.1 Expectations Theory

In a riskless economy the term structure of interest rates entails the following no-arbitrage equilibrium relationships[2]: (i) the instantaneous forward rate for date T equals the instantaneous spot rate for time T;

$$f(t,T) = r_T \tag{11}$$

(ii) the return of holding a long term bond at time t which matures at time T equals the return on rolling over a series of short term bonds from time t to time T;

$$\frac{1}{P(t,T)} = (1+r_t)(1+r_{t+1})...(1+r_{T-1}) \tag{12}$$

and (iii) the realised return on any bond at any time equals 1 plus the prevailing riskless rate

$$\frac{P(t+1,T)}{P(t,T)} = 1+r_t \tag{13}$$

In an uncertain economy these relationships need to be modified. It is then natural to think of the market's expectations about future interest rates as a way to characterise the term structure under uncertainty. This idea was initially suggested by Fisher (1896), subsequently refined by Hicks (1939) and Lutz (1940), and has

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[1] Cox, Ingersoll and Ross (1981) interpret this as a risk premium.

[2] See Ingersoll (1987, pp.388-9).

come to be known as the expectations theory of interest rates. Over time various concrete formulations of the theory have emerged but two of the most basic and identifiable forms have been expressed as follows: (i) the expected one-period rate of return on holding an n-period bond equals the one period riskless spot interest rate; and (ii) forward rates implied in the current term structure equal expected future spot rates. Cox, Ingersoll and Ross (1981) (CIR hereafter) study the theoretical validity of the expectations theory in the context of arbitrage equilibrium and find these various forms to be inconsistent. To prevent arbitrage opportunities, equilibrium is characterised by an equality among expected holding period returns of all default-free bonds over all holding periods. For example, the return on any series of investments over the period t_0 to t_n must have the same expectation:

$$E\left[\frac{P(t_1,T_1)P(t_2,T_2)...P(t_n,T_n)}{P(t_0,T_1)P(t_1,T_2)...P(t_{n-1},T_n)}\right] = \phi(t_n,t_0) \tag{14}$$

CIR use Jensen's inequality[3] to show that equation (14) generally leads to an internal contradiction. This contradiction, however, no longer exists if the expected holding returns are equal for only one specific holding period. It is then natural to specify the next shortest period as this specific period. Equilibrium is then characterised by

$$E_t\left[\frac{dP(Y,t,T)}{P(Y,t,T)}\right] = r_t\, dt \tag{15}$$

This version of the expectations theory is named the local expectations hypothesis and is the only specification consistent with arbitrage equilibrium.

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[3] See Cox, Ingersoll and Ross (1981,pp.769-799). An explanation of Jensen's inequality can be found in Ingersoll (1987, p. 16).

Equation (15) contains no risk premium as if risk neutrality were assumed. CIR (1981), however, demonstrate that this is not necessarily the case and that there are two other cases where a default-free bond would not require a premium: (i) locally certain consumption in a single good pure exchange economy; and (ii) state independent Bernoulli-logarithmic utility of consumption in a production-exchange economy. Both these cases seem to contradict observed human behaviour.

In the literature, the version of the expectations theory characterised by zero term premium has come to be known as the pure expectation theory or PET and is usually attributed to Fisher (1930) or Lutz (1940). It is based upon risk neutrality or special cases of risk preferences which may not reflect actual investment behaviour. Despite the lack of realism, the idea of expectations and recently the popular view that expectations are rational, hold a great deal of appeal in applied research. In fact, applied researchers have rarely taken the zero-risk premium prediction seriously[4] and when it is decisively rejected by evidence, alternative theoretical justification for the expectations theory is sought.

Campbell (1986) provides the theoretical justification for those versions of the expectations theory which are studied in the empirical literature. He constructs a general equilibrium example to show that both the holding period term premium and the forward premium are constant instead of being zero as in the CIR economy. The key difference is that the instantaneous variances of the underlying sources of uncertainty are constant through time rather than proportional to a state variable as prescribed by CIR (1981). He then shows that the versions of the

[4] Shiller and McCulloch (1987, p.27)

expectations theory which have been formulated in the empirical literature are not necessarily incompatible with each other and with arbitrage pricing equilibrium. These non-zero term premium versions constitute what is called the expectations theory, in contrast to the zero term premium pure expectation theory.

The empirical literature is extremely voluminous and has a long history. A detailed survey of this literature, which is presented in Appendix A (section AI), leads to some identifiable generalisations. The crucial factor is how to model expectations. Various methods have been used, including the perfect foresight model, the error learning model, the rational expectations theory, and surveys of interest-rate expectations. The pure expectations theory is decisively rejected. Moreover, the rational expectations theory has replaced the error-learning model as the dominant paradigm and it has spawned a wealth of research. Evidence produced within the rational expectations framework, however, has been largely inconsistent with its predictions. Furthermore the empirical evidence indicates that: (i) forward rates are not good forecasts of expected future spot rates, especially at the short end of maturity spectrum; (ii) the term premia are positive and tend to increase, though not monotonically, with maturity; and (iii) the term premia are positively related to the level of interest rates.

2.2.2 Liquidity Preference Theory

In the liquidity preference theory investors demand a risk premium for holding long term bonds as risk is perceived to increase with maturity. Hence, Hicks (1939) suggests that forward rates should exceed subsequent spot rates with the difference, known as the liquidity premium increasing with maturity. The reason is that borrowers want to finance long to match the long term nature of their investments but lenders want to lend short because of liquidity considerations and the perceived risk of long- term securities. A weakness of this theory is that the persistence of the one-sided spectrum of maturities and the consequential positive premium[5] are hard to explain in actively arbitraged markets. The empirical evidence, (see Appendix A (section AII)), is inconclusive. This could be attributed to the theory being refuted by observed data or the problem of modelling expectations[6].

2.2.3 Preferred Habitat Theory

The preferred habitat theory originated with Culbertson (1957) who argued that markets are segmented and hence arbitrage across markets is limited. Hedging against risk, which is the primary force determining the term structure is primarily concerned with matching the maturity of the liability side of a portfolio of investments with that of its assets. Lenders and borrowers who want to avoid capital and income risk prefer instruments with maturities matching their investment horizons and are reluctant to leave their 'preferred habitat'.

Modigliani and Sutch (1966, 1967) integrated the notion of market segmentation with rational expectations to allow for negative or positive liquidity premia which should not be systematically tied to maturity. They claim that their theory blends pure expectation, liquidity preference and market segmentation. This theory is characterised by two distinctive features: (i) the term premia could be positive or negative reflecting the excess demand for, or supply of, loans

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[5] This also implies that securities of different maturities are not perfect substitutes.

[6] See Appendix A (section AI) for a discussion of alternative models of expectations.

corresponding to the maturity characterising the habitat; and (ii) the expected current return on an n-period bond is equal to the one period rate plus the expected capital gain. However, the expected capital gain is assumed to be proportional to the expected fall in the long interest rate. Hence the spread between the short and long rate should primarily be determined by the long rate. Drawing on Keynes (1936) and De Leeuw (1965), Modigliani and Sutch contend that expectations of long rates are determined by two factors: (i) long rates tend to regress to a normal level (approximated by some average of the long rates in the past); and (ii) recent trends in interest rates. These two influences are then combined in an Almon lag structure to model expectations. The long rate is then rewritten as a combination of the lag structure, representing expectations, and other variables representing term premia. Theoretically, a main criticism of this model is that the separation of term premia and expectations is arbitrary and implausible. The empirical evidence, (see Appendix A, section AIII), supporting it is modest.

2.2.4 Concluding remarks on traditional theories

Theoretically, a major weakness of the traditional theories is that they are not developed as a result of the utility maximisation process subject to either market clearing constraints or less stringently to a no-arbitrage condition. Rather the theories are put forward as a series of propositions that interest rates are determined by expectations. Although it is possible to find some form of market equilibrium consistent with some of these propositions, the lack of a integrated theoretical foundation[7] provides little guidance to empirical testing. Thus, the

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[7] Shiller and McCulloch (1987) refer to traditional theories as "heuristic theoretical models" (p.60).

empirical literature contains mixed and often conflicting evidence. The Australian evidence, which is presented in Appendix A (section IV), is no exception[8]. In sum, the empirical literature has generated more questions[9] than answers.

The new class of theories, which will be reviewed in the next section, contain all the elements of the traditional theories, namely rational expectations and risk preferences. Yet the issue of equilibrium under uncertainty is explicitly accounted for by introducing uncertainty into bond price dynamics via the incorporation of one or more Wiener processes. In this sense, the theory of the term structure constitutes a fully integrated component of the theory of dynamic asset pricing[10].

2.3. MODERN BOND PRICING THEORIES

In this section we review partial equilibrium (in the bond market) and general equilibrium (in the economy at large) bond pricing models which grew out of the Merton-Black-Scholes era in the early 1970s[11].

A default-free bond is considered an asset in the representative consumer's portfolio of investments. He/she is assumed to maximise the sum of his/her expected utilities of consumption over time. A necessary condition for a maximum

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[8] See Appendix A (section AIV) for a detailed survey of Australian contributions to the testing of the traditional theories of the term structure.

[9] Some of these questions are : (i) why do the term premia vary?; and (ii) what is the correct way to describe the relationship between the term premia and other economic variables? More unresolved issues are listed in Shiller and McCulloch (1987, pp.60-61).

[10] The unifying characteristic of this theory is a valuation equation which states that the price of a claim is equal to the product of the conditional mathematical expectation of its future payoff and the marginal rate of substitution of current and future consumption of the representative investor.

[11] Specifically, the pricing of a risky asset is governed by a partial differential equation which is the end result of imposing a no-arbitrage condition on a riskless portfolio of assets.

is the marginal utility of consuming the proceeds at time t from selling the bond is equal to the marginal disutility of the expected loss of future consumption at time $t+1$. This equality of the marginal rate of substitution of utilities must hold for all assets in the portfolio and at all times[12]. Uncertainty is introduced by one or more Wiener processes and Ito calculus is used to relate the driving force of the term structure to the price of a default-free bond. By varying the term to maturity the arbitrage-free bond price will determine the entire term structure.

2.3.1 Partial Equilibrium Models

2.3.1.1 Introduction

In this framework it is assumed that the term structure is driven by a state variable, usually specified to be the spot interest rate[13], resulting in a one-factor term structure model. As the equilibrium condition used is that of no-arbitrage, these models are called partial equilibrium. Although more factors can be incorporated we shall start with a single-factor model to facilitate the exposition.

2.3.1.2 Theory

The stochastic process of the spot interest rate is specified to be a Wiener process:

$$dr = \alpha(r,t)dt + \sigma(r,t)dz \qquad (16)$$

where $\alpha(r,t)$ and $\sigma(r,t)$ are called the drift rate and volatility rate of the process

[12] See Samuelson and Merton (1969), Merton (1971), Rubinstein (1976), Lucas (1978), and LeRoy (1982).

[13] A further subclassification of partial equilibrium models is on whether they use bond price, $P(t, T)$, or forward rate, $f(t, T)$, as the driving forces for the interest rate dependence. For example, Ball and Torous (1983), and Heath, Jarrow and Morton (1992) start from an assumed process for $P(t, T)$ and $f(t, T)$ respectively. See also Appendix B.

respectively, and $dz(t)$ are the increments of a Wiener process $z(t)$ such that $dz(t) \sim N(0,\sqrt{dt})$. If the price of a default-free bond, $P(r, t, T)$, is assumed to be a function of the spot interest, time and time to maturity, and by Ito's lemma it must also follows the same type of process[14]:

$$\frac{dP(t, T)}{P(t, T)} = \mu\,(r, t, T)dt + s\,(r, t, T)\,dz \tag{17}$$

where

$$\mu(r,t,T) = \frac{\left[\frac{\partial P}{\partial t} + \alpha\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}\right]}{P} \tag{18}$$

$$s(r,t,T) = \frac{\sigma\frac{\partial P}{\partial r}}{P} \tag{19}$$

We now form a portfolio of such bonds, V, with weights w_1 and w_2 corresponding to maturity dates T_1, T_2 respectively and such that $w_2 = 1 - w_1$. The return on this portfolio is:

$$\frac{dV}{V} = \left[w_1\mu(r,t,T_1) + w_2\mu(r,t,T_2)\right] dt + \left[w_1 s(r,t,T_1) + w_2 s(r,t,T_2)\right] dz \tag{20}$$

which can be made non-stochastic by intertemporally choosing the weights such that:

$$w_1 s\,(r,t,T_1) + w_2 s\,(r,t,T_2) = 0 \tag{21}$$

The stochastic process describing the dynamic evolution of the portfolio V then becomes:

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[14] An exposition of Ito's calculus can be found in Hull (1993).

$$\frac{dV}{V} = \left[w_1\mu(r,t,T_1) + w_2\mu(r,t,T_2)\right]dt \tag{22}$$

It is clear that at any point in time the portfolio is riskless from time t to $t + dt$ and hence it must earn the risk free rate over that period, that is:

$$\frac{dV}{V} = rdt \tag{23}$$

From (20), (21) and (22) we then obtain the no-arbitrage condition between two bonds of different maturity:

$$\frac{\mu(r,t,T_1)-r}{s(r,t,T_1)} = \frac{\mu(r,t,T_2)-r}{s(r,t,T_2)} \tag{24}$$

Hence $(\mu - r)/s$ is independent of maturity and the no-arbitrage condition can be written as

$$\frac{\mu-r}{s} = \lambda \tag{25}$$

The left hand side of (25) is the ratio of the excess return over the variance rate of a bond of maturity T_i. This ratio is commonly denoted by λ and is known as the market price of interest rate risk. It has to be satisfied by bonds of all maturities for there to be no arbitrage in the bond market.

Substituting the definitions of $\mu(r,t,T)$ and $s(r,t,T)$ of equations (18) and (19) into equation (25) yields the following partial differential equation for the pricing of the pure discount bond:

$$\frac{1}{2}\sigma^2(r,t,T)\frac{\partial^2 P}{\partial r^2} + \left[\alpha(r,t,T)-\lambda(r,t)\sigma(r,t,T)\right]\frac{\partial P}{\partial r} + \frac{\partial P}{\partial t} - rP = 0 \tag{26}$$

In order to solve this equation the following additional conditions are required: (i) $0 \leq t \leq T$; $|r| < \infty$; (ii) $P(r,t,t) = 1$; and (iii) $\lim_{r \to \infty} P(r, t, T) = 0$. The first condition means that the spot interest rate must be bounded. The second condition means that the price of an instantly maturing pure discount bond is 1 while the third condition means that the price of a pure discount bond as $r \to \infty$ approaches zero.

The above general framework underlies a number of single-factor and two-factor models where the factors may be the instantaneous spot rate and the rate of inflation (Richard (1978)) or the instantaneous spot rate and the yield on a console bond which pays coupons continuously (Brennan and Schwartz (1979)). The major models in this group are now reviewed.

2.3.1.2(a) One Factor Models

A summary of one-factor models is provided in Table 1. These models assume that the term structure is driven by a single factor such as the spot interest rate, r. Thus Vasicek (1977) assumes that $\alpha\ (r,t) = \kappa(\theta - r)$, $\lambda(r,t) = \lambda$ and $\sigma\ (r,t) = \sigma$, where κ may be interpreted as the adjustment speed of the spot rate as it reverts to the long-run equilibrium rate, θ. This process characterises the fluctuations of r about θ. The assumption of a constant market price of risk is implied by the continuous time CAPM with logarithmic utility of consumption functions[15]. Thus, the resulting process of the spot rate is :

$$dr = \kappa(\theta - r)dt + \sigma dz \tag{27}$$

while the solution to the stochastic differential equation for bond prices implied by the no-arbitrage condition is:

[15] See Dothan (1977), pp.61-2.

$$P(t,s,r) = \exp\left[\frac{1}{\kappa}(1-e^{-\kappa(T-t)})(R(\infty)-r)-(T-t)R(\infty)-\frac{\sigma^2}{4\kappa^3}(1-e^{-\kappa(T-t)})^2\right] \tag{28}$$

where R $_{vasi}(\infty)$[16] is the yield on a very long bond as T → ∞ :

$$R_{vasi}(\infty) = \theta-\lambda-\frac{1}{2}\left[\frac{\sigma^2}{\kappa^2}\right] \tag{29}$$

Like Vasicek (1977), Dothan (1977) also assumes a constant market price of risk, $\lambda(r,t) = \lambda$. However, he assumes a proportional process for the short rate:

$$dr = \sigma r dz \tag{30}$$

The resultant bond price $P(r,t,T)$ solution is a decreasing convex function of the spot rate, r, time to maturity, $\tau = T-t$, and an increasing concave function of the variance rate of the spot rate, σ^2.

The only difference between Dothan (1977) and Vasicek (1977) is in the specification of the dynamics of the spot interest rate. Both assumptions of the dynamics are plausible but a disadvantage of the Vasicek model is that interest rates can become negative. Dothan (1977) overcomes this problem by assuming that the stochastic term has a standard deviation proportional to r, implying a log-normal distribution for r. It should be noted, however, that the Vasicek model may be a reasonable approximation to reality as the probability of negative interest rates

[16] Note that λ in the Vasicek model is defined to be positive (see Vasicek (1977, equation (14)). To make it consistent with λ in the CIR model, a negative sign is placed in front of λ in equation (29).

TABLE 1

Partial Equilibrium (One Factor) Models

Model	State variable	Stochastic Process	Solution	Remarks
Vasicek (1977)	Spot rate	dr = $\kappa(\theta$-r)dt + σdz	$P(t,s,r) = \exp\left[\frac{1}{\kappa}(1-e^{-\kappa(T-t)})(R(\infty)-r)-(T-t)R(\infty)-\frac{\sigma^2}{4\kappa^3}(1-e^{-\kappa(T-t)})^2\right]$	Mean reverting interest rates Possible negative interest rates
Dothan (1977)	Spot rate	dr = σr(t)dz	Discount bond price* is decreasing convex function of r and τ (=T-t) and increasing concave function of σ^2	Prices of discount bonds of long maturities do not approach zero

Note: * The formula is long and complex. See Dothan (1977, pp.62-65) is small. In any case the issue of superiority of one model over the other has to be settled empirically[17].

2.3.1.2(b) Multi-factor Models

Table 2 provides a summary of multifactor models. Apart from the belief that one factor is sufficient to determine the term structure, the choice of one factor is essentially one of modelling convenience. In this section we shall review a number of two-factor models. Richards (1978) developed a two-factor model based upon the real rate of interest, R, and the current rate of inflation, π. The dynamics of these factors are modelled by:

$$dR = -a(R-R^*)dt + \sigma_R R^{1/2} dz_R \tag{31}$$

$$d\pi = -c(\pi-\pi^*)dt + \sigma_\pi \pi^{1/2} dz_\pi \tag{32}$$

where a and c are the speeds of adjustment of R and π to their long-run equilibria R^* and π^* respectively; $\sigma_R R^{1/2}$ and $\sigma_\pi \pi^{1/2}$ are the volatilities of the changes in the real rate of interest and the rate of inflation respectively.

The market prices of risk of R and π are assumed to be increasing functions of R and π respectively:

[17] Only the Vasicek model has been tested empirically. See Munnik and Schotman (1994).

TABLE 2

Partilal Equilibrium (Multifactor) Models

Model	State Variables	Stochastic Processes	Solution	Remarks
Richards (1978)	Real rate (R) Rate of inflation (π)	$dR(t) = -a(R-R^*)dt + \sigma(R,t)R^{1/2}dz(R,t)$ $d\pi(t) = -c(\pi-\pi^*)dt + \sigma(\pi,t)\pi^{1/2}dz(\pi,t)$	Discount bond price is a function of R, π, market prices of interest and inflation risks	Market prices of interest and inflation risks are increasing functions of R and π Bond prices are decreasing and convex in both R and π
Brennan & Schwartz (1979)	Spot rate Long rate	$dr = [a_1+b_1(l-r)]dt + r\sigma_1 dz_1$ $dl = l(a_2+b_2+c_2 l)dt + l\sigma_2 dz_2$	No bond price closed form solution. Instead, bond price is obtained by numerically solving the partial differential equation	Only one market price of risk (of spot rate) is needed by assuming the long bond to be the console bond, a traded asset
Langetieg (1980)	Multi stochastic factors	$dx_i = \alpha_i(x,t) + \sigma^i(x,t)dz_i$	$P(t,T) = E_t exp(A(t))$	Spot rate is represented as linear combination of economic factors that follow a joint elastic random walk process. Numerical solution available in the general case Closed form solutions where distributions are those specified by Dothan (1977), Vasicek (1977) and Dothan (1977)

$$\lambda_R^* \ (R,\pi,t) \ = \ \lambda_R R^{1/2} \tag{33}$$

$$\lambda_\pi^* \ (R,\pi,t) \ = \ \lambda_\pi \pi^{1/2} \tag{34}$$

Given these assumptions and applying to this two factor world the condition of no riskless arbitrage between bonds of different maturities, Richards derives a formula which relates the price of discount bonds to the real rate of interest, the anticipated rate of inflation, and the market prices of interest and inflation risks. A weakness of this model is that it introduces two utility dependent parameters, namely λ_R^* and λ_π^* , into the valuation formula, which complicates considerably the empirical estimation of the model.

Brennan and Schwartz (1979,1980,1982) assume that bond prices are driven by the spot interest rate, r, and long interest rate, ℓ, and that r and ℓ are modelled by a joint stochastic process:

$$dr \ = \ \beta_1(r,l,t)dt \ + \ \eta_1(r,l,t)dz_1 \tag{35}$$

$$dl \ = \ \beta_2(r,l,t)dt \ + \ \eta_2(r,l,t)dz_2 \tag{36}$$

where dz_1 and dz_2 are Wiener processes with $E(dz_1) = E(dz_2) = 0$ and $E[dz_1 dz_2] = \rho dt$ where ρ is the correlation between dz_1 and dz_2 . Furthermore $\beta_1(.)$ and $\beta_2(.)$ are the expected instantaneous rates of change in the spot and long-term rates of interest respectively, $\eta_1^2(.)$ and $\eta_2^2(.)$ are the instantaneous variance rates of the changes in r and ℓ, and ρ is the instantaneous correlation between the unanticipated changes in r and ℓ. The model as specified by equations (35) and (36) relates changes in each interest rate to changes in the other interest rate and to its own level. A specific form of equations (35) and (36) is chosen for the purpose of estimation:

$$dr = [a_1 + b_1(\ell - r)]dt + r\sigma_1 dz_1 \tag{37}$$

$$d\ell = \ell(a_2 + b_2 + c_2\ell)dt + \ell\sigma_2 dz_2 \tag{38}$$

where a_1, b_1, a_2, b_2, c_2, σ_1 and σ_2 are constants. This formulation is based upon the following assumptions: (i) the instantaneous standard deviation of each interest rate is proportional to its level; (ii) the drift of the spot interest rate reflects the tendency of the spot rate to regress to the long interest rate; and (iii) the market price of risk of the long interest rate is a linear function of r and ℓ. The resultant partial differential equation includes two market prices of risk, λ_1 and λ_2, corresponding to the spot rate and long rate respectively. Assuming that ℓ is the rate of a console bond, which is a traded asset, then λ_2 can be expressed in terms of σ_2, b_2, r and ℓ and thus does not appear in the bond pricing equation:

$$\begin{gathered}\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\eta_1^2 + \frac{\partial^2 P}{\partial r \partial l}\rho\eta_1\eta_2 + \frac{1}{2}\frac{\partial^2 P}{\partial l^2}\eta_2^2 + \frac{\partial P}{\partial r}(\beta_1 - \lambda_1\eta_1) + \frac{\partial P}{\partial l}(\eta_2^2/l + l^2 - rl) \\ - \frac{\partial P}{\partial t} - Pr = 0\end{gathered} \tag{39}$$

However, while the model does not have a closed form solution, it can be estimated numerically.

Schaefer and Schwartz (1984) build upon the earlier work of Brennan and Schwartz (1979,1980,1982) by specifying the console rate and the spread between the console rate and the short rate as the state variables. Employing an empirically supported observation that these two state variables are orthogonal they simplify the bond pricing partial differential equation. However, this still does not yield a closed form solution. Instead, an approximate solution is found.

Langetieg (1980) develops a model of the term structure characterised by multi stochastic factors with joint elastic random walk (or Ornstein-Uhlenbeck) processes and where the instantaneous riskless rate of interest is a linear combination of these factors. A general solution is obtained but an explicit and practical solution is only possible for special distributions, namely those stochastic processes of the underlying state variables specified by Vasicek (1977), Dothan (1977), Cox, Ingersol and Ross (1985), and Richards (1977). Theoretically this model encompasses an arbitrary number of economic factors that are related to the term structure but practically only a single-factor or two-factor version can be implemented.

Multifactor modelling is an attempt to overcome the sparsity of information contained in one factor models and the difficulty of correctly identifying such a factor. Theoretically, it presents a prima facie stronger case as a richer set of information is being used. The cost, however, is mathematical and computational complexity.

2.3.1.3 Empirical evidence

In this section we review the empirical tests of the Vasicek one-factor and Brennan-Schwartz two-factor models, the only two partial equilibrium models that have been tested to date. The overall objective of these tests is to ascertain how well the models fit the data. The Vasicek model yields a model bond price formula, equation (28), which can be applied directly to cross-sections of observed bond prices. Alternatively, a time series regression can be implemented, using a discrete analogue to equation (27)[18]. The data in the latter case would consist of a time series of an observed proxy for the instantaneous spot interest rate.

Table 3 provides a summary of the empirical evidence of partial equilibrium models. Using daily observations of the one-month Amsterdam InterBank Offered

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[18] See Sanders and Unal (1988)

Rate for the 1985-1991 period and actively traded Dutch Government Bonds for 1989-1990, Munnik and Schotman (1994) estimated equations (27) and (28) of the Vasicek model. They find that: (i) the model provides a good fit for bond prices (the average pricing error is 0.18 guilder per 100 guilder par bond); (ii) the mean reversion parameter, κ, and the volatility parameter, σ^2, behave erratically; (iii) in many cross sections σ^2 falls to zero and a lower bound for σ_2 is set at 10^{-5} which was attained in more than half of the sample times. The effect of the lower bound on the fit of the model was checked and found to be negligible[19] ; and (iv) the model is overspecified[20] so that without significantly affecting the goodness of fit of the model we can set one of the structural parameters (κ, θ, σ^2) at some "reasonable" value and optimise over the others. A weakness of this study is that it does not provide out-of-sample verification of the estimated model. Thus, the issue of model accuracy and parameter stability remains unexplored.

The testing of the Brennan-Schwartz model involves three steps: (i) linear regression is applied to the stochastic processes of the spot rate and long rate to yield estimates of the coefficients of the processes; (ii) these estimates are then used in the partial differential equation to yield $P(r,l,\lambda,t,T)$ which is the model price of a pure discount bond maturing in τ periods, $\tau = T\text{-}t$, when the two interest rates are r and ℓ, and λ is the appropriate market price of risk parameter; and (iii) λ is determined by minimising the sum of squared differences between observed bond prices and model prices.

Brennan and Schwartz (1979) numerically solve equation (39) and then compare predicted bond prices with observed bond prices and estimated yields from

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[19] For example a lower bound of 10^{-3} could have been set without significantly reducing the goodness of fit of the model, indicating that the likelihood function is very flat.

[20] A symptom of overparameterisation is the near singularity of the Hessian.

TABLE 3

Partial Equilibrium Models: Empirical Evidence

Study	Tested Model	Methodology	Data	Findings	Remarks
Munnik & Schotman (1994)	Vasicek	Cross section	Daily observations of Amsterdam Interbank Offered Rate (1985-91) and Dutch Government Bonds (1989-1990)	Good fit for bond prices κ (mean reversion parameter) and σ^2 behave erratically σ^2 is usually very small and have negligible impact on the fit of the model Model is overspecified	Does not provide out of sample validation
Brennan and Schwartz (1979)	Brennan & Schwartz	Numerically solving bond price partial differential equation (37) Linear regression of actual bond prices (or yields) on predicted bond prices	Monthly observations of 30 day Canadian Bankers' Acceptances and average yields on Canadian bonds with maturities in excess of 10 years (1964-1976) Quarterly prices of 101 Canadian bonds with maturities of less than 10 years (1964-1979)	Slope coefficient of 0.93 for bond prices and 0.79 for yields to maturity Overall, results are weak	Further work needed on specification and estimation of interest rate processes and the market price of risk

TABLE 3 (Continued)

Partial Equilibrium Models: Empirical Evidence

Study	Tested Model	Methodology	Data	Findings	Remarks
Brennan & Schwartz (1980)	Brennan & Schwartz	Numerically solving bond price partial differential equation Factor analysis of bond price errors Focussing on predicting bond returns conditional on the factors recovered from bond price errors	Monthly observations of 30 day Canadian Bankers' Acceptances and average yields on Canadian bonds with maturities in excess of 25 years (1964-1979) Quarterly prices of 126 Canadian bonds (1964-1979)	A third factor is present in addition to the long and short rates Three factors predict bond returns with lesser errors than two factor	Does not provide a benchmark with which to compare the estimated model of the term structure. Comparison is only indirect via the prediction of bond returns
Brennan & Schwartz (1982)	Brennan & Schwartz	Replicates Brennan & Schwartz (1980)	Monthly data in CRSP (US) Government Bond File (1958-1979)	A third factor is found to affect bond prices Strong relation between bond price errors and subsequent bond returns	Results indicate either market inefficiency, model misspecification, or data errors
Dietrich-Campbell & Schwartz (1986)	Brennan and Schwartz model is used to price American call and put options bonds and bills	Replicates Brennan & Schwartz (1980) to estimate the term structure	CRSP Government Bond File (1970-1982) Daily observations of call and put options on US Government Bonds and Treasury Bills (1982-1983)	Increased accuracy of pricing options relative to the Black & Scholes (1973) model	Focus of study is on pricing options hence it does not report how well the Brennan & Schwartz predicted bond prices

predicted bond prices with actual yields. Hence, they use a linear regression of actual values on predicted values to test the hypothesis of unbiased prediction which requires a zero intercept and unity slope. As proxies for the spot interest rate and the long-term interest rate they use yields on 30-day Canadian Bankers' Acceptances and average yields to maturity on Government of Canada bonds with maturities in excess of 10 years. Both series are monthly data and extend from January 1964 to December 1976. Quarterly prices of 101 Government of Canada bonds from January 1964 to January 1977 are used to estimate the market price of risk, λ_1. Brennan and Schwartz (1979) found a slope coefficient of 0.93 for bond prices and 0.79 for yields to maturity, while the intercepts are significantly different from zero. Overall, the evidence can be described as weak, at best. In fact, Brennan and Schwartz (1979) concede that there are other factors that may affect the term structure, and that their work should be seen as a " first step". In addition, further work is required within their model to improve on the specification and estimation of the interest rate processes and the market price of risk. The latter, being utility dependent, will prove to be a major hurdle in estimating any bond pricing equation.

Brennan and Schwartz (1980) extend their 1979 paper with a more detailed empirical analysis of the intertemporal stability and predictive ability of the assumed stochastic process for interest rates and the statistical analysis of the estimated market price of risk due to the uncertainty of the short rate. In addition, the focus is on predicting bond returns on the basis of the factors recovered from bond pricing errors. The data include a sample of 126 Government of Canada bonds from January 1964 to April 1979 and yields on 30-day Canadian Bankers' Acceptances and average yields to maturity on Government of Canada bonds with maturities in excess of 25 years. The quarterly bond prices and monthly yields are used to estimate the market price of risk and the stochastic processes of the interest rates respectively. The pricing errors (model prices less observed prices) are factor analysed and the results suggest the presence of a third factor in addition to the long and short rates[21]. Using a model incorporating these three factors to predict bond returns leads to smaller errors (measured by root mean squared errors, RMSE) than models incorporating only the long and short rates. The market price of risk parameter, λ_1, is found to be much smaller than the value reported in their 1979 paper, the discrepancy being attributed to a different sample size of bonds and a different method of estimation. Nevertheless, in both papers this parameter does not have a substantial impact on the estimated model. A major weakness of the Brennan and Schwartz (1980) study is that it does not provide a benchmark with which to compare the estimated model of the term structure. An indirect test, namely the contribution of the model to uncovering the factors, is not sufficient to demonstrate how well the proposed model captures the forces that determine the shape of the term structure. Besides, it is well known that factor analysis can only produce 'blind' factors. Further interpretation is needed to identify the economic content of the uncovered factors.

Brennan and Schwartz (1982) replicate the Brennan and Schwartz (1980) study on US government bond data from 1964 to 1979. Again, a third factor appears to be present in explaining bond prices. They also found a strong relation between bond price errors and subsequent bond returns, and suggests that this could be due to either market inefficiency, inadequacy of the model, or data errors.

Dietrich-Campbell and Schwartz (1986) use the two-factor Brennan-

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[21] It should be noted that factor analysis assumes constant coefficients and hence is not appropriate for an intertemporal model,

Schwartz model to price American call and put options on US government bonds and Treasury Bills from November 21, 1982 through October 31, 1983. As the study focuses on options the authors do not report how well the Brennan-Schwartz model performs in predicting bond prices. However it does contribute to increased accuracy of predicted option prices relative to the Black-Scholes model (1973). The performance of the term structure model depends upon that of the option pricing models.

2.3.1.4 Concluding remarks on partial equilibrium models

There are many similarities between the Black-Scholes option pricing model and the partial equilibrium models of the term structure of interest rates. The former assumes that the economy (which includes the stock price dynamics and the market price of risk) has reached equilibrium. An option is a derivative security whose value depends only on the stock price dynamics and time. To preclude arbitrage profits the option has to be priced in a prescribed manner relative to the stock price dynamics and the market price of risk. In this sense the Black-Scholes model is a partial equilibrium model. The relationship between a partial equilibrium term structure model and its state variable(s) bears close resemblance to that between an option and its underlying asset. Thus, to price the bond in this arbitrage framework it is necessary to assume that the economy has reached equilibrium and this gives rise to the equilibrium state variable(s). That, perhaps, is where the analogy ends. To price a pure discount bond where the state variable is, for example, an interest rate is much more difficult than pricing an option written on an underlying stock. As the stock is a traded asset, and hence its <u>price</u> is observed, it can be combined with the option to create a riskless portfolio. In contrast the specified interest rate is <u>not</u> a market traded security, and hence cannot be used in forming portfolios. The Black-Scholes riskless portfolio also eliminates the market price of risk of the stock in the final partial differential equation. The market price of risk of the interest rate cannot be similarly eliminated. As a result, to price the bond it is necessary to specify both the state variable(s) and the market price(s) of risk of the state variable(s), one dimension more than is required by the Black-Scholes technology.

The preceding survey of the empirical testing of partial equilibrium models leads to the following observations:(i) because the bond pricing equation, in both the closed form or the partial differential equation form, is highly non-linear, computation is extremely difficult while the market price of risk and the speed of adjustment are highly unstable. Yet they do not seem to have a substantial impact on estimated bond prices; (ii) most studies find that the model being tested produces small within-sample bond price errors; and (iii) though a third factor has been found to determine the term structure, the first factor appears to explain the largest percentage of the total variance of the residual errors[22]

To sum up, partial equilibrium models are based upon a number of assumptions: (i) the economy in equilibrium; (ii) state variables modelled by Markov processes; (iii) bond price is a function of state variables. Then the use of the no-arbitrage argument leads to a bond pricing partial differential equation which may or may not have a closed form solution. Empirically, there are problems in regard to the choice of the state variable(s) to be included in the model and the functional form of the market price(s) of risk. These problems are mostly resolved by expediency rather than economic rationale. Further, there is no guarantee that a dynamics chosen for a state variable will be consistent with an economy in

[22] Factor analysis by Brennan and Schwartz (1980) reports the first factor accounts for 83.5% of the total variance of the bond pricing errors.

equilibrium.

2.3.2 General Equilibrium Models

2.3.2.1 Introduction

Table 4 provides a summary of general equilibrium models. In this class of models, the term structure of interest rates is an integral part of the equilibrium theory of dynamic asset pricing. Some distinguishing features of this theory are: (i) the real (production) and financial (exchange) markets are endogenised; (ii) all participants are rational and utility maximising; (iii) individuals have time-additive state-independent utility functions exhibiting linear risk tolerance[23], and hence there exists a representative agent for the constructed economy; and (iv) uncertainty is introduced by modelling state variables as diffusion processes.

2.3.2.2 Theory

The genesis of general equilibrium models of the term structure lies in Cox, Ingersoll and Ross (1985a, 1985b). While CIR (1985a) lays the general equilibrium foundations for contingent claim valuations, CIR (1985b) applies the general theory developed in CIR (1985a) to the pricing of default free bonds, thus bringing the term structure of interest rates into the unified general equilibrium theory of dynamic asset pricing.

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[23] Specifically these utility functions include power functions and negative exponential functions. For further details see Huang and Litzenberger (1988, Ch.5).

TABLE 4

General Equilibrium Models

Model	State Variable(s)	Stochastic Process(es)	Solution	Remarks
CIR(1985b)	Spot rate	$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}\,dz$	$P(r,t,T) = A(t,T)e^{-B(r,T)r}$	Shapes of yield curve: humped or monotone. Yield to maturity → constant as $T \to \infty$
Longstaff (1989)	Spot rate	$dr = \kappa(\theta - \sqrt{r})dt + \sigma\sqrt{r}\,dz$	P(r,t,T)= A(t,T)exp(B(t,T)r+C(t,T)	Yield to maturity → constant as $T \to \infty$ Shapes of yield curve: hump, trough, monotone

TABLE 4 (Continued)

General Equilibrium Models

Longstaff & Schwartz (1991)	Spot rate and Volatility of Spot rate	$dr = (\alpha\gamma+\beta\eta-\frac{\beta\delta-\alpha\xi}{\beta-\alpha}\tau-\frac{\xi-\delta}{\beta-\alpha}V)dt$ $+\alpha\sqrt{\frac{\beta\tau-V}{\alpha(\beta-\alpha)}}dZ_2 + \beta\sqrt{\frac{V-\alpha\tau}{\beta(\beta-\alpha)}}dZ_3$ $dV = \left(\alpha^2\gamma+\beta^2\eta-\frac{\alpha\beta(\delta-\xi)}{\beta-\alpha}\tau-\frac{\beta\xi-\alpha\delta}{\beta-\alpha}V\right)dt$ $+ \alpha^2\sqrt{\frac{\beta r-V}{\alpha(\beta-\alpha)}}dZ_2 + \beta^2\sqrt{\frac{V-\alpha r}{\beta(\beta-\alpha}}dZ_3$	$F(r,V,\tau) =$ $A^{2\gamma}(\tau)B^{2\eta}(\tau)\exp(\kappa+C(\tau)r+D(\tau)V)$	Two factors, hence a richer set of information
Beaglehole & Tenney (1992)		$dr = (\theta-\kappa r)dt + \sigma dz$	*	Reverts to Longstaff (1991) model for $\kappa = 0$

Note: * A long formula (see Beaglehole and Tenny (1992, pp.351-352)

The CIR general equilibrium valuation model, CIR (1985a), is based upon the following assumptions: (i) there is a single good which can be allocated to consumption or investment; (ii) the production possibilities consist of a set of linear activities, and production processes are characterised by constant returns to scale; (iii) there are markets for a variety of contingent claims to the amounts of goods where these claims are securities whose returns are functions of the individual's wealth, W, and the state of technology (Y); and (iv) individuals have identical preferences and allocate wealth (W) among the set of production processes, contingent claims, and the amount to be borrowed or lent at the instantaneous risk free rate, r.

The representative consumer is assumed to maximise a lifetime objective function of the following form:

$$E\int_{t}^{t'} U[C(s),Y(s),s]ds \tag{40}$$

where t' is the terminal date and $C(s)$ is the consumption flow at time s. Equilibrium at the individual level is achieved by solving the lifetime utility function for the optimal consumption, C^*, the optimal proportion (of wealth) invested in the production processes, a^*, and the optimal proportion invested in the contingent claims, b^*. The expected returns on the production processes, α, the expected returns on the contingent claims, denoted β, and the riskfree interest rate, r, are given at the individual level. Equilibrium at the economy level determines α, β, and r, the total production plan, and the total consumption plan, and requires that the net supply of contingent claims and riskless lending be zero.

In the above general equilibrium setting CIR (1985a) show that the price of any contingent claim , F, must satisfy the partial differential equation:

$$
\begin{aligned}
&\frac{1}{2}(var\ W)F_{ww} + \sum_{i=1}^{k}(cov\ W,Y_i)F_{Y_iY_j} \\
&+ \left[r(W,Y,t)W - C^*(W,Y,t)\right] \\
&+ \sum_{i=1}^{k}F_{Y_i}\left[\mu_i - (-\frac{J_{ww}}{J_w})(cov\ W,Y_i) - \sum_{j=1}^{k}(-\frac{J_{WY_j}}{J_W})(cov\ Y_i,Y_j)\right] \\
&+ F_t - r(W,Y,t)F + \delta(W,Y,t) = 0
\end{aligned}
\tag{41}
$$

where $r(W,Y,t)$ is the equilibrium instantaneous interest rate, $J(W,Y,t)$ the indirect utility function, μ_i the expected change in the i^{th} state of technology, and δ(W,Y,t) the payout flow received by the security. The subscripts to F and J denote partial derivatives.

Under suitably chosen assumptions CIR(1985b) show that equation (41) can be reduced to one that describes the dynamics of a pure discount bond, $P(r,t,T)$ under the following restrictions. The utility function $U(C,s)$ is of the constant relative risk aversion class (CRRA), namely $U(C,s) = exp(-\rho s)(C^\gamma - 1)/\gamma$, where ρ is the time discount parameter and γ describes the level of risk aversion for the representative agent. For this class of utility functions the price of bonds does not depend upon wealth.

If the single state variable (instantaneous interest rate) is of the square root process type and the utility function is logarithmic then equilibrium entails the following relationships:

$$
dr = \kappa(\theta - r)dt + \sigma\sqrt{r}\,dz \tag{42}
$$

$$
\lambda^* = \frac{\lambda}{\sigma}\sqrt{r} \tag{43}
$$

where κ, θ, σ^2 and λ are constants and λ^* is the market price of risk of the instantaneous interest rate. For positive κ, and θ this process is the continuous time version of an autoregressive process of order 1 which tends to regress elastically to its central location, θ. κ is the speed of adjustment towards θ. The speed of adjustment κ toward θ is such that κ or the time to maturity of the bond tends to infinity then the conditional expected interest rate tends to θ and the conditional variance of the interest rate tends to zero. In this framework θ can be interpreted as the long run equilibrium of the instantaneous interest rate.

The above assumptions enable us to write down a specialised form of the partial differential equation (41) for the price of a pure discount bond:

$$\frac{1}{2}\sigma^2 r P_{rr} + \left[\kappa\theta-(\kappa+\lambda)r\right]P_r + P_t - rP = 0 \tag{44}$$

This partial differential equation, (44), has a closed form solution:

$$P(r,t,T) = A(t,T)e^{-B(t,T)r} \tag{45}$$

where

$$A(t,T) \equiv \left[\frac{2\gamma\exp[(\kappa+\lambda+\gamma)(T-t)/2]}{(\gamma+\kappa+\lambda)(\exp(\gamma(T-t))-1)+2\gamma}\right]$$

$$B(t,T) \equiv \frac{2(\exp(\gamma(T-t))-1)}{(\gamma+\kappa+\lambda)(\exp(\gamma(T-t)-1)+2\gamma} \tag{46}$$

$$\gamma \equiv ((\kappa+\lambda)^2+2\sigma^2)^{1/2}$$

Equations (45) and (46) constitute what has become known as the Cox, Ingersoll and Ross (hereafter CIR) model of the term structure of interest rates. The model has the following properties:

(i) the bond price is a decreasing convex function of θ;

(ii) the bond price is an increasing concave (decreasing convex) function of κ if the interest rate is greater (smaller) than θ;

(iii) the bond price is an increasing concave function of the market price of risk, λ;

(iv) the bond price is an increasing concave function of the interest rate variance, σ^2;

(v) interest rates can reach zero if $\sigma^2 > 2\kappa\theta$ but negative interest rates are precluded;

(vi) the term premium, defined as the arithmetic expected rate of return less rate of interest ($= \lambda r P_r / P$) of a given maturity discount bond, is of uniform sign for all states of the world;

(vii) the term premium is of uniform sign for all maturities of discount bonds;

(viii) the yield curve[24] can only have three shapes[25]: uniformly rising if $r < R_{cir}(\infty)$, declining if $r > \kappa\theta/(\kappa+\lambda)$, and humped for intermediate values of r;

(ix) expected interest rates with respect to time are monotonically converging to θ given that $\kappa > 0$;

(x) the yields on bonds of different maturities are perfectly correlated as they are all related to the single source of risk in the model, namely the state of technology; and

(xi) as T approaches infinity the yield to maturity approaches a constant, and hence is independent of the current spot interest rate:

---

[24] The yield to maturity, $R(r,t,T)$, is defined by $\exp[-(T-t)R(r,t,T)] \equiv P(r,t,T)$.

[25] This means the CIR term structure precludes cyclical behaviour.

$$R_{cir}(\infty) = \frac{2\kappa\theta}{\gamma+\kappa+\lambda} \tag{47}$$

Although the CIR model is derived in terms of real interest rate, it is equally derivable in terms of nominal interest rates by means of the no-arbitrage argument (see Brown and Dybvig, (1986)) and assuming that inflation follows the same dynamics as the spot interest rate, namely the square root process. The nominal version is more empirically tractable than the real version as nominal prices are available whilst real prices generally need further transformation via an inflation index which can conceivably distort the observable data.

Longstaff's model (1989) is cast in the CIR framework except that the stochastic process of the short rate is specified to be non-linear in the drift term. This process has come to be known as "double square root" process:

$$dr = \kappa(\theta-\sqrt{r})dt + \sigma\sqrt{r}\,dz \tag{48}$$

The non-linearity assumption is made to overcome a weakness in the CIR model, which only allows for essentially three types of yield curves, monotonically increasing, monotonically decreasing, and humped. All the other assumptions underlying the CIR model are retained by Longstaff (1989). As a result the partial differential equation governing the price of a pure discount bond is:

$$\frac{\sigma^2}{2}\frac{\partial^2 P}{\partial r^2} + \left[\frac{\sigma^2}{4}-\kappa\sqrt{r}-2\lambda r\right]\frac{\partial P}{\partial r}-rP+\frac{\partial P}{\partial t} = 0 \tag{49}$$

Define $\tau = T - t$ as the maturity of the bond then the closed form solution to (49) is:

$$P(r,\tau) = A(\tau)\exp\left(B(\tau)r+C(\tau)\sqrt{(r)}\right) \tag{50}$$

where

$$
\begin{aligned}
A(\tau) &= \left[\frac{1-c_0}{1-c_0\exp(\gamma\tau)}\right]^{\frac{1}{2}} \exp\left[c_1+c_2\tau+\frac{c_3+c_4e^{\gamma\tau/2}}{1-c_0e^{\gamma\tau}}\right] \\
B(\tau) &= \frac{2\lambda-\gamma}{\sigma^2}+\frac{2\gamma}{\sigma^2(1-c_0e^{\gamma\tau})} \\
C(\tau) &= \frac{2\kappa(2\lambda+\gamma)(1-e^{\gamma\tau/2})^2}{\gamma\sigma^2(1-c_0e^{\gamma\tau})} \\
\gamma &= \sqrt{(4\lambda^2+2\sigma^2)} \\
c_0 &= (2\lambda+\gamma)/(2\lambda-\gamma) \\
c_1 &= \frac{-\kappa^2}{\gamma^3\sigma^2}(4\lambda+\gamma)(2\lambda-\gamma) \\
c_2 &= \frac{2\lambda+\gamma}{4}-\frac{\kappa^2}{\gamma^2} \\
c_3 &= \frac{4\kappa^2}{\gamma^3\sigma^2}(2\lambda^2-\sigma^2) \\
c_4 &= \frac{-8\lambda\kappa^2}{\gamma^3\sigma^2}(2\lambda+\gamma)
\end{aligned} \tag{51}
$$

Longstaff's model (1989) has the following properties:

(i) the double square root process has conditional mean and conditional variance :

$$
\begin{aligned}
E[r_t|r] &= r + \sigma^2 t/4 \\
Var[r_t|r] &= \sigma^2(rt+\sigma^2t^2/8)
\end{aligned} \tag{52}
$$

(ii) as $t \to \infty$ the steady state distribution of the interest rate is

$$
\frac{2\kappa}{\sigma^2\sqrt{r}}\exp\left(\frac{-4\kappa\sqrt{r}}{\sigma^2}\right) \tag{53}
$$

with mean equal $\sigma^4/8\kappa^2$, and variance equal $5\sigma^8/64\kappa^4$; and

(iii) the yield to maturity as $\tau \to \infty$ is

$$R_{long}(\infty) = \kappa^2/\gamma^2 + (\gamma - 2\lambda)/4>0 \tag{54}$$

which is independent of the current interest rate.

Longstaff(1992) shows that when the condition that r is inaccessible to zero, that is when $\sigma^2 < 2\kappa\theta$, the equilibrium bond price is unique and the behaviour of $P(r,\tau)$ as $r \to 0$ is implicitly specified by the fundamental valuation equation (49). However, when r is accessible to zero there are many possible equilibrium solutions to the fundamental valuation equation. He provides three examples: (i) an absorbing equilibrium where the short term interest rate is absorbed at zero if it reaches zero; (ii) an unrestricted equilibrium where the interest rate process returns immediately to positive values if it reaches zero; and (iii) an empirical equilibrium where boundary conditions on the bond price at $r = 0$ can be directly imposed. For example, a boundary condition must imply non-negative forward rates for there to be no arbitrage opportunities.

Longstaff's model (1989) has been criticised by Beaglehole and Tenney (1992) who argue that equation (50) is not the solution to the problem being posed. This error is due to the failure to properly account for a boundary condition. Specifically, the formula does not satisfy the condition that the derivative of the bond price with respect to r must approach zero as r approaches zero. Instead, when $r \to 0^+$ then $\partial P/\partial r \to \infty$. This implies that the expected return of the discount bond at $r = 0$ is not necessarily the same as the limiting return as $r \to 0^+$. Equation (50), however, is correct in a model economy in which the variable r is not reflected (i.e. absorbed) at $r = 0$[26]. Beaglehole and Tenney (1992) show that the Longstaff formula is a special case of their model whose state variable, r, is modelled by an Ornstein-Uhlenbeck process:

$$dr = (\theta - \kappa r)dt + \sigma dz \tag{55}$$

where θ, κ and σ are positive parameters.

Both the CIR and Longstaff models are driven by a single state variable which may not be sufficient to capture the driving force(s) of the term structure. Thus, Longstaff and Schwartz (1992) have proposed a two-factor model within the CIR framework which includes the short-term interest rate and the volatility of the short term interest rate. Following CIR (1985a) it is assumed that there is one physical good which is produced by a single constant-returns-to-scale technology. The returns on physical investments, denoted dQ/Q, are assumed to be driven by two economic factors (state variables), X and Y . While X only drives expected returns, Y drives both expected returns and production volatility. The rationale for this specification is that expected returns and production volatility are not necessarily perfectly correlated. The assumed process is:

$$\frac{dQ}{Q} = (\mu X + \theta Y)dt + \sigma\sqrt{Y}dz_1 \tag{56}$$

where μ, θ, and σ are positive constants. The two state variables are specified as follows:

---

[26] Beagonhole and Tenney also note that the Longstaff economy is characterised by low and nonstationary interest rates. Hence t-statistics tend to be unreliable.

$$dX = (a-bX)dt + c\sqrt{X}dz_2$$
$$dY = (d-eY)dt + f\sqrt{Y}dz_3 \tag{57}$$

where $a,\ b,\ c,\ d,\ e,\ f > 0$ and z_2 is uncorrelated with z_1 and z_3 as the factor X is specified to only drive expected returns and not production volatility. Given this framework, Longstaff and Schwartz obtain a pricing formula for the discount bond which is a function of three variables r, V, and $\tau = T\text{-}t$, and six parameters.

The strength of the Longstaff and Schwartz (1992) model is that it combines the general equilibrium feature with a richer set of information. This can be best seen by comparing its features with those of partial equilibrium models.

According to CIR (1985b, p.397-398, section 5), pricing a bond requires specifying : (i) a state variable which drives the bond price; (ii) the stochastic process of the variable; and (iii) the exact form of the factor risk premium. In the CIR general equilibrium framework all these are endogenously determined whilst in the arbitrage derivation there are some problems. For example, denote $\Pi(r,t,T)$ as the excess expected return on a bond of maturity date T. Then an arbitrage driven equilibrium requires $\Pi(r,t,T)$ to be of the form:

$$\Pi(r,t,T) = \psi(r,t,T)\frac{\partial P(r,t,T)}{\partial r} \tag{58}$$

One of the problems with this no-arbitrage equilibrium condition is that there is no guarantee that there is some underlying general equilibrium that is consistent with both (i) and (ii) above[27]. Further, to be consistent with general equilibrium the choice of the functional form of ψ is restricted. For example, CIR (1985b) provide an example where $\psi(r,t) = \psi_0 + \lambda r$. This linear form of the risk

[27] According to Harrison and Kreps (1979) this problem may be solved by viewing models that admit no arbitrage as preference free.

premium is reasonable but the CIR model, in fact, guarantees arbitrage profits (see CIR (1985b, p.398)). Thus, the advantages of general equilibrium modelling, and hence disadvantages of partial equilibrium, include: (i) the general equilibrium model can be completely specified to be internally consistent; and (ii) it allows predicting the changes in the term structure (CIR (1985b, p.398)) consistent with the changes in the general equilibrium of the economy.

2.3.2.3 Empirical Methodology and Evidence

Table 5 provides a summary of the empirical evidence of the general equilibrium models. The nominal CIR model can be derived by the no-arbitrage argument (see Brown and Dybvig (1986)). The justification for this approach lies in the desirable properties of the interest rate process, namely no negative nominal interest rates. Gibbons and Ramaswamy (1993, p.7) opine that testing the CIR model with nominal prices would show how robust it is with respect to a misspecification of the interest rate process. Moreover CIR (1985b, p.405) note that "the interest equation and the fundamental valuation equation have exactly the same form when all variables are expressed in nominal terms as when all variables are expressed in real terms" . This means the pricing formula is as applicable to nominal as well as real prices of bonds.

Empirical testing of general equilibrium models have taken two routes: cross-sectional regression and generalised methods of moments (hereafter GMM). The first was pioneered by Brown and Dybvig (1986) and the latter by Gibbons and Ramaswamy (1993). If the error term is assumed to be normally distributed then cross-sectional regression is equivalent to maximum likelihood estimation and hence its statistical inference is stronger than that of non-parametric GMM. The GMM approach, on the other hand, does not require that the error term distribution be specified. However, large sample theory is employed to generate confidence regions for parameter estimates. Furthermore the GMM estimators and their standard errors are consistent even in the presence of heteroskedasticity, serial correlation and correlation across maturities[28].

[28] See Longstaff and Schwartz (1992), p.1276.

TABLE 5

General Equilibrium Models: Empirical Evidence

Study	Tested Model	Methodology	Data	Findings	Remarks Strengths/Weaknesses
Brown & Dybvig(1986)	CIR	Cross section	Monthly prices U.S. T-bills, Notes and bonds Dec 1952 - Dec 1983	implied volatility highly unstable CIR model overestimates spot rate, r Model misspecified	Weakness associated with nonlinear estimation
Gibbons & Ramaswamy (1993)	CIR	GMM			
Chiarela, Lo & Pham (1989)	CIR	Cross section	Monthly prices Australian Treasury bonds January 1978 - December 1987		
Longstaff (1989)	Longstaff (Double square root)	GMM	Monthly(check) yields US Treasury bills 1964-1968	Longstaff model outperforms CIR model Actual pricing more complex than can be explained by single state models	
Brown & Schaefer(1994)	CIR	Cross section	Daily prices British Government Index-Linked Securities Mar 1981 - Dec 1989	Long term yield is stable Implied volatility consistent with time series estimate Spot rate (r), speed of adjustment (κ), market price of risk (λ), volatility , and long term equilibrium spot rate highly correlated and unstable over time	● 'Real' (indexed) data ● $\kappa+\lambda$, $\kappa\theta$, and σ^2 kept constant over quarters and years

2.3.2.3(a) Cross sectional regression

The observed price of a zero-coupon bond i, defined by $P_i\ (r,\ t,\ T)$, is written as a sum of the price given by the model under consideration, $\hat{P}_i\ (r,\ t,\ T)$, and an error term, ϵ_t :

$$P_i\ (r,\ t,\ T) = \hat{P}_i\ (r,\ t,\ T) + \epsilon_t \tag{59}$$

If the error term is assumed to be independent and identically distributed, then maximum likelihood estimates of the parameters of the model can be obtained from a cross-section of bonds of varying maturities and which are traded at a given point in time. This method, which places no intertemporal restrictions on the parameters of the model, is similar to the cross-sectional estimation of the Black-Scholes implied standard deviation, ISD. The ISD methodology is now well accepted and volatility change well recognised. Thus, the justification for Brown-Dybvig's methodology, discussed below, rests on the same ground.

Brown and Dybvig (1986) rearrange the CIR bond pricing formula as follows:

$$P(r,t,T) = A(t,T)exp(-B(t,T)) \tag{60}$$

where for $\tau = T\text{-}t$

$$A(t,T) \equiv \left[\frac{\phi_1 \exp(\phi_2 \tau)}{\phi_2[\exp(\phi_1 \tau)-1]+\phi_1}\right]^{\phi_3}$$

$$B(t,T) \equiv \frac{\exp(\phi_1 \tau)-1}{\phi_2[\exp(\phi_1 \tau)-1]+\phi_1} \tag{61}$$

$$\phi_1 \equiv \left[(\kappa+\lambda)^2+2\sigma^2\right]^{1/2}$$

$$\phi_2 \equiv (\kappa+\lambda+\phi_1)/2$$

$$\phi_3 \equiv 2\kappa\theta/\sigma^2$$

From these parameters, the long term yield $R_{cir}(\infty)$ and the volatility of the short term rate, σ^2, are expressed as:

$$R_{cir}(\infty) = (\phi_1 - \phi_2)\phi_3$$

$$\sigma^2 = 2[\phi_1\phi_2 - \phi_2^2] \tag{62}$$

The estimation methodology involves applying nonlinear least squares procedures to a cross section of prices of U.S. Treasury issue at a point in time to obtain maximum likelihood estimates of r, ϕ_1, ϕ_2, and ϕ_3 . The model has one variable, r, and four parameters, κ, λ, θ, and σ, but κ, θ and λ cannot be separately identified[29]. Consequently, this methodology only estimates r, ϕ_1, ϕ_2, ϕ_3 and hence $R(r,t,\infty)$ and σ^2. While the parameters provide a snapshot of a single term structure at a given point in time, a time series can be obtained by repeating the cross sectional regression over time. The main findings of Brown and Dybvig (1986) are: (i) the CIR variance, σ^2, is much more volatile than that estimated from a weekly time series of Treasury Bills with 13 weeks to maturity; (ii) the CIR model systematically overestimates the short rate, r, relative to the Treasury Bill rate with at most 14 days to maturity; and (iii) analysis of the residuals indicates

[29] Inspection of equations (60) and (61) confirms that it is overidentified.

the model is misspecified in the context of the data[30].

A major weakness of this methodology is an inherent weakness associated with non-linear statistical models, that is, the estimates of the parameters are generally biased[31]. Further, as only asymptotic properties of the estimates are known, hypothesis testing is only valid asymptotically.

Brown and Schaefer (1994) test the CIR model, using the Brown-Dybvig methodology, on British Government Index-Linked bonds. Estimation is implemented in two ways: unconstrained and constrained. The former places no constraints on the parameters in weekly cross section estimates of the model whilst the latter requires parameters, other than the short rate, to be constant across quarters and years. The main findings are: (i) the estimated long term zero coupon yield, R_L, is stable and the implied volatility of the spot rate is consistent with its time series estimate counterpart; (ii) estimates of other parameters, namely r, $\kappa+\lambda$, $\kappa\theta$, and σ^2, are often highly correlated and highly unstable over time; and (iii) the mean pricing error for most bonds is less than £0.2 per approximately £100 price. While the pricing errors are small relative to bond prices the instability of parameter estimates is a symptom of multicollinearity (see Maddala (1992, Ch.7). However, the strength of the Brown and Schaefer (1994) study lie in : (i) the data set which is, in a sense, 'real' so that the modelling of inflation is no longer necessary; and (ii) the constraining of $\kappa+\lambda$, $\kappa\theta$, and σ^2 to be constant which is both consistent with model specification and an attempt to reduce the convergence difficulty in highly non-linear estimation.

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[30] Using the cross-sectional methodology of Brown and Dybvig, Munnik and Schotman (1992) test the CIR model on Dutch daily Government bond prices from 1989 to 1990. They report a good fit for bond prices. The average pricing error is 0.18 guilder (par bonds are normalised to 100 guilder). Further the mean reversion parameter, κ, and the volatility parameter σ^2 behave erratically.

[31] See Judge et al (1985, p. 209).

Barone, Cuoco, and Zautzik (1991) apply the Brown-Dybvig and Brown-Schaefer approach to a sample of Italian Treasury bonds traded on the secondary market from December 30, 1983 to December 31, 1990. The parameter estimates are ϕ_1, ϕ_2, ϕ_3, r, $R_{cir}(\infty)$, $\sigma r^{1/2}$, θ, and κ. The last two, θ and κ, are estimated separately by assuming a zero risk premium, $\lambda=0$[32]. However, Barone, Cuoco, and Zautzik (1991) differ from previous studies in that : (i) daily data are used; and (ii) the error term is assumed to be proportional to the derivative of bond price with respect to yield to maturity[33]. They find that: (i) the mean value of the distribution of the differentials between actual and theoretical bond prices is not significantly different from zero; (ii) the spot rate moves closely with the yield of three-month Treasury bills; and (iii) the implied volatilities, $\sigma r^{1/2}$, are reasonably close to the standard deviations calculated from the time series of the instantaneous rate. The other parameters, $R_{cir}(\infty)$, θ, and κ, become more stable over time, reflecting a larger number of bonds being used in the estimation.

2.3.2.3(b) Generalised method of moments (GMM)

A recent approach in model specification testing is based on the idea that correct specification requires that random quantities, which are functions of the error terms, should have zero expectations (or moments) conditional upon some information set (see Newey (1985), Tauchen (1986), Hansen (1982), Hansen and Singleton (1982)). These specification tests are known as conditional moments tests.

Historically the method of moments is one of the oldest methods of estimation. Using the law of large numbers the moment of any distribution can be

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[32] Assuming zero risk premium is incorrect because the objective of the CIR study is to model the term structure of interest rates under uncertainty.

[33] This implies that the errors in the prices of short-term bonds are smaller than those in long-term bonds.

estimated by the corresponding sample moment of independent drawings from the distribution. The generalisation of this method is based upon two facts: (i) both conditional and unconditional moments can be used; and (ii) moments may depend upon unknown parameters and conditional moment tests can be used not only for model specification but also in estimating parameters. Thus, setting the population moments (which are functions of parameters) equal to their sample counterparts and then solving the resulting system of moment equations would yield point estimates of the parameters[34]. Applying this method to estimating the parameters of bond pricing models involves deriving the theoretical unconditional moments of yields and then setting them equal to their sample moments.

The requirement that the theoretical unconditional moments be zero can be written as:

$$E(g_{\tau i}(P,\tau,\underline{\beta})) = 0 \qquad i = 1,\ldots,k \tag{63}$$

where $g_{\tau i}(P,\tau,\underline{\beta})$ is the yield of a bond P of maturity τ and $\underline{\beta}$ is the k-vector of parameters of the model under consideration.

Substituting the sample moments for their theoretical moments gives:

$$\frac{1}{n}\sum_{\tau=1}^{n} g_{\tau i}(P,\tau,\underline{\beta}) = 0 \qquad i=1,\ldots,k \tag{64}$$

GMM seeks a vector $\hat{\underline{\beta}}$, called the GMM estimator of $\underline{\beta}$,(in the sample of data) that makes the LHS of equation (64) as close as possible to zero. The advantage and popularity of the GMM estimation is that it simultaneously corrects the problems of heteroskedasticity, serial correlation, simultaneity bias ($E(X^*\epsilon) \neq 0$), and measurement bias. Furthermore, it is distribution free. The

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[34] Clearly we need as many moment equations as there are parameters.

disadvantage is that being not based upon the normal distribution, GMM statistical tests are weak.

Gibbons and Ramaswamy (1993) were the first to apply the GMM methodology to test the CIR model. Their test focused on the holding period real return) relative, defined as $1/P(r,\tau)$, where $P(r,\tau)$ is the CIR price of a discount bond in real terms. From this definition the first conditional moment can be written as:

$$E_t\big(1/P(r,\tau)\big) = A(\tau)^{-1} e^{B(\tau)r} \tag{65}$$

By iterated expectations the conditional moment can then be written as unconditional moment:

$$f_1(\tau,\beta) \equiv E(1/P(\tau,r)) - A(\tau)^{-1} E\left[e^{B(\tau)r}\right] = 0 \tag{66}$$

Similarly, higher moments are derived and equated to their corresponding empirical moments to yield the vector of parameter estimates of the CIR model, namely $\hat{\underline{\beta}}\ (\kappa, \sigma^2, \lambda, \theta)$.

The data used by Gibbons and Ramaswamy (1993) include monthly observations of U.S. Treasury bill prices and the Consumer Price Index. From these two sources the holding period real return relatives were constructed for each month and for maturities of 30 and 90 days. All the parameter estimates, κ, σ^2, λ, and θ, are found to be more than two standard errors away from zero while λ was found to be negative and consistent with a positive risk premium as predicted by the CIR theory[35].

There are several problems in applying the GMM in tests of the CIR model:

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[35] The sample period was further divided into two subsamples (1/64-9/79, 10/79-12/83). The estimates of the parameters change in magnitude but remain statistically significant.

(i) the problem of overlapping observations[36] becomes worse with coupon paying bonds as the maturity of the bonds increase and this places a heavy burden on the algorithm that searches for the parameter estimates[37]; and (ii) at the end of each month it was usually not possible to find Bills of exactly 30, 90, 180, and 345 days to maturity so that relative yields were interpolated from bills maturing in the surrounding dates.

To sum up, the data arrangement required by the GMM methodology is not observable and some form of data construction is needed. Longstaff (1989) estimates the parameters of both his double square root model and the CIR square root model using the generalised method of moments. The data used to estimate κ, θ, σ^2 and λ, for the square root model and κ, σ^2, and λ, for double square root model consist of average yields to maturity of two-, three-, four-, and five-month U.S. Treasury bills over the 1964-1986 period. The parameter estimates enable the computation of the theoretical yields implied by both models for longer term U.S. Treasury bills. Longstaff (1989) then compares these theoretical yields with observed U.S. Treasury bills with maturities of six to twelve months over the same period. The comparison shows that the Longstaff model outperforms the CIR model in capturing the level and variation of the term structure over this spectrum. However, in both models the biases, defined as the difference between model yields and actual yields, show that actual pricing is more complex than can be accounted

[36] Coupon payments of various bonds do not fall on the same dates so that observations of coupon payments are not time-aligned. This gives rise to the problem of multi-collinearity.

[37] In fact, even with such short term maturities the authors pointed out that 'the data in the series involve adjacent observations that have overlapping intervals over which the returns are computed" (Gibbons and Ramaswamy (1986, p.16-17)). Initially real return relatives were calculated for maturities of 30, 90, 180 and 345 days but the consideration of overlapping observations led them to finally use only bills of 30 and 90 days to maturity. The exclusion of coupon paying bonds at longer maturities mean that a richer set of information is not being utilised.

for by single state variable models.

Using the GMM methodology Longstaff and Schwartz (1992) test their two-factor and CIR one-factor models on U.S. Treasury bill yields with maturities ranging from three months to five years over the June 1964 - December 1989 period. While the Longstaff and Schwartz model cannot be rejected by the data the CIR model is rejected at the 10 percent level[38].

2.3.2.4 Concluding remarks on general equilibrium models

One of the significant contributions of the general equilibrium approach to term structure research is that it integrates this area into the equilibrium theory of dynamic asset pricing. The significance should be seen in the general framework of the purpose of scientific inquiry, namely, to explain (financial) phenomena. In this perspective, the general equilibrium approach is well rooted in the foundation of financial economics and its received concepts of market clearing and utility maximisation. In sum, the general equilibrium models are based on economic theory.

At the practical level a major disadvantage of the general equilibrium approach is the difficulty in treating the market price of risk. Either it has to be inferred from market prices or be specified a priori. If the first route is taken then the term structure has to be inverted, and this is not easy computationally because the bond pricing formulae are highly nonlinear and the spot rate and the bond price process parameters are not independent of the market price of risk. Alternatively, specifying the market price of risk as a function of the state variable(s) necessarily involves some measure of arbitrariness and could lead to models inconsistent with general equilibrium that admit arbitrage opportunities. Another empirical problem

[38] The *p*-values for the Longstaff and Schwartz and CIR models are 0.652 and 0.064 respectively.

is that statistical tests in non-linear estimation are only asymptotically correct because only asymptotic properties of the parameter estimates are generally available (Judge et al (1985, p.209)).

Turning to the relative merits of cross sectional regression and GMM as alternative empirical methodologies in tests of general equilibrium models, it is not possible to establish the superiority of one methodology to the other. While the requirements of GMM are less stringent than those of cross sectional regression, GMM tests are less powerful as they focus on over-identifying the tested model[39]. Furthermore the null hypothesis of the GMM approach is tested against an unspecified alternative so that if it is rejected, little is known about what goes wrong. The choice between the two methodologies is largely dictated by the emphasis on the short-term or entire spectrum of the term structure[40]. The data required by GMM are usually not available in a readily observed form, and have to be constructed to be of approximately equally spaced intervals, such as months or years.

On balance, the evidence shows that the CIR model and its variants provide a reasonably good fit on within-sample data. Furthermore, the spot rate appears to be the most important factor[41], while other parameters of the model, κ, θ, λ, have much less impact and are possibly negligible. A major weakness of the studies is

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[39] Over-identifying is necessary to test restrictions of the model, thereby making the estimators more consistent but statistical tests become less powerful.

[40] Cross sectional tests make use of bond prices with the full range of maturities and frequencies while GMM tests focus on the data set constructed by Fama (1984) and maintained by CRSP. The CRSP data set consists of U.S. Treasury bill and Treasury Bond yields. These yields are based on the average of bid and ask prices for these securities with maturities ranging from one month to eleven months, then from one year to 5 years.

[41] The t statistics are always significant.

the lack of out-of-sample validation.

2.3.3 Preference free models

In addition to explaining the term structure of interest rates, models of the term structure have been used extensively in pricing derivatives where one of the inputs is the existing term structure. Whether the current term structure is in equilibrium is debatable but there certainly is a strong interest and practice in using the information in the current term structure to price derivative assets. The class of models which concentrates on the current term structure has come to be known as preference free models. While Appendix B contains a detailed survey of the theory and empirical evidence this section provides a general discussion of these models based on Heath, Jarrow and Morton (1992), hereafter HJM, undoubtedly the most important piece of work in preference free modelling of the term structure.

We begin by highlighting the difference between equilibrium and preference free models. Equilibrium models are usually based upon a specified stochastic process for the short rate and a specified form for the market price of risk from which the yield curve or term structure is deduced. The parameters of the models are chosen to reflect market data as closely as possible. Preference free models are analogous to the Black-Scholes methodology of pricing options where equity is determined exogenously. Within the preference free framework, information implied by the initial term structure (including traders' preferences) is taken as exogenous. No-arbitrage conditions are then imposed to determine how the initial term structure evolves. It is important to note that in the Black-Scholes framework no risk preference is needed to price options. Similarly, no risk preference is needed to price interest rate derivatives because the risk preference is already incorporated into the observed term structure. However, to take the analogy a step further, risk preferences are clearly needed if one wishes to price equity. Similarly, risk preferences are also needed to price bonds. Therefore regardless of how the underlying asset is priced, via an equilibrium based model such as Cox Ingersoll and Ross (1985) or a preference free model such as or Heath, Jarrow and Morton (1992), a risk preference is clearly implicit. Thus, Jamshidian (1990) argues that "it is more fruitful to regard the preference-free approach as the evaluation of contingent-claims prices in terms of the yield curve. Stated in this way, it is irrelevant whether the "given" term structure is generated from an equilibrium model or is obtained from the market with an essentially arbitrary shape". Furthermore as no-arbitrage is a necessary condition of the equilibrium approach, it is a subset of the preference free approach.

HJM take the initial forward rate structure as given and then specify how it would evolve over time to preclude arbitrage opportunities. This is achieved by recognising a link between the drift and volatility of the specified forward rate process[42]:

$$\alpha(t,T) = -\sum_{i=1}^{n} \sigma_i(t,T) \left[\gamma_i(t) - \int_t^T \sigma_i(t,v)dv \right] \tag{67}$$

where $\alpha(t,T)$, $\sigma(t,T)$ and γ are the drift, volatility of the instantaneous forward rate and the market price of risk respectively. The subscript i refers to the random factor i, $i = 1,\dots n$.

A constraint of the HJM model is that " the bond price process, spot rate process, and the market price of risks cannot be chosen independently" (HJM (1992, p.88-89). Furthermore, to characterise the term structure relative to an

[42] This is the no-arbitrage condition.

earlier date requires knowledge of the entire path the spot rate followed in reaching the present value. In addition to the path being non observable, the HJM spot rate process is in general non-Markov (Hull (1993, pp.400-401), which makes the HJM model very slow computationally. In short, the theoretical generality of the HJM model is a major strength but additional structure needs to be imposed at the practical level.

The ultimate objective of the preference free approach is to price derivatives <u>given the existing term structure of interest rates</u>. As relative pricing does not depend upon knowledge of the utility foundation of the underlying asset, modelling the term structure is then reduced to developing no-arbitrage conditions for the current term structure to evolve over time. These conditions impose little restriction, and hence, the approach is far more general than the equilibrium approach at the theoretical level. The generality, however, implies that a preference free model is incapable of being implemented unless additional restrictions are imposed on the volatility function of the state variable(s)[43], the forward rate, spot rate, or bond price. The issue of specification criteria of the volatility structure remains unsettled and, hence an active area of research.

Empirically, the testing of preference free models is still very limited in number and mostly relies on GMM, a non-parametric procedure. Hence, only weak statistical results are obtained. A related problem[44] is that all the extant empirical studies have used only short-term T-Bills[45], and thus ignore the long-term

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[43] See Appendix B for choices of state variables in preference free modelling.

[44] The data requirement of the GMM methodology is not observable and some form of data construction is needed.

[45] The CRSP bond file consists of cross-sections of monthly bond prices of approximately one- to twelve-month and one- to five-year maturity. The use of cross sections of bonds of unequal time to maturity would induce a problem of serial correlation.

spectrum of the term structure.

2.3.4 Australian evidence of modern bond pricing

Empirical testing of modern bond pricing in the Australian financial markets is confined to three studies[46]: Chiarella, Pham and Mackenzie (1990), Chiarella, Lo and Pham (1989) and Bhar (1993). Chiarella, Pham and Mackenzie (1990) provide a test of the Brennan-Schwartz two- factor model. Specifically, it evaluates the assumed discrete version of the joint interest rate process for the spot and long rates, equations (37 and (38), by estimating it under varying market conditions in Australia from September 1964 to February 1987. In the absence of data on individual bonds the bond pricing valuation, equation (39), was applied to a Government bond index and the pricing errors, the predicted index less observed index, were found to be biased upwards. As Chiarella, Pham and Mackenzie (1990) did not perform a regression of predicted values on actual values it is not possible to compare their results with the US and Canadian studies of Brennan and Schwartz (1979, 1980, 1982) and Dietrich-Campbell and Schwartz (1986). A serious weakness of the Chiarella, Mackenzie and Pham (1989) study is that the estimation is performed on one bond, namely the index, and hence the market price of risk cannot be estimated directly from a cross-section of bonds. A cross-sectional approach would reveal a more reliable indicator of risk preferences than a single bond. Chiarella, Lo and Pham (1989) were the first Australian study to test the CIR model by applying the Brown-Dybvig method of cross-sectional estimation to a set of monthly Treasury bond data from January 1978 to December 1987. The spot rate, r, the long rate, R_L and the variance of the spot rate process implied by the

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[46] An unpublished Australian study by Hathaway (1988) tests a mean reverting interest dynamics where the long-term equilibrium spot rate is an exponentially weighted average of the past spot rates (Ingersoll(1987),pp.407-409). As this study does not test any term structure model per se it is outside the scope of investigation of this thesis.

model, $\sigma r^{1/2}$, are compared to their close counterparts estimated from historical series. Generally the results are mixed for the entire period. The model-implied spot rate is found to be an unbiased estimator of the yield on 90 day Treasury Notes and 90 day bank accepted bills[47]. On the other hand, while the long rate consistently underestimates the yield on 15 year Treasury bonds, their trends move in close correspondence. Furthermore, the implied variance is not a good predictor of its time-series estimate from the observed Treasury note and bank bill yields. However, the Chiarella, Lo and Pham (1989) study suffers from a serious weakness in that the coupon aspect of the bonds is not properly accounted for[48] so that the model is largely tested on a wrongly transformed set of observations.

Bhar (1993) replicates the Ho and Lee's (1990) study on a sample of 90-day Bank Bill futures options traded on the Sydney Futures Exchange (SFE) over a period of five weeks. Consistent with the findings of Ho and Lee (1986), Bhar (1993) found : (i) the Ho and Lee model perform better than the Black (1976) model in terms of mean squared residuals; and (ii) it exhibits no bias with respect to moneyness and time-to-maturity. The absence of bias is an improvement on many prior studies on pricing biases of the Black (1976) and Black-Scholes (1973) models. The finding that the Ho and Lee (1990) model is superior to the Black (1976) model tends to confirm the conjecture that incorporating stochastic interest rates into option pricing models leads to better fit to observations[49].

All the Australian studies are preliminary and the first two, Chiarella, Pham

[47] The spot rate implied by the CIR model is compared with two observed proxies, 90-day Bank bills for the pre-1980 period and 91-day Treasury notes for the post-1980 period.

[48] A coupon bond is equivalent to a portfolio of zero-coupon bonds where a coupon payment is considered the face value of such bonds. Chiarella, Lo and Pham (1989) wrongly equate the price of a coupon bond with the present value of its face value, thus ignoring the effect of coupon payments.

[49] See also Harrison, Pham and Sim (1992).

and Mackenzie (1990) and Chiarella, Lo and Pham (1989) contain weaknesses. Specifically, these include: (i) failure to make use of bonds of all maturities; (ii) observed coupon bonds are not properly transformed into zero-coupon bonds; and (iii) absence of zero-coupon bonds precluding the verification of long-term theoretical interest rates. Hence, further research is necessary to address these issues.

2.4 SUMMARY AND CONCLUDING APPRAISAL

This section summarises the salient features of the three main developments in the study of the term structure (traditional theories, equilibrium theories and preference free theories) and discusses their weaknesses and strengths. The section concludes with justifications for the thesis.

2.4.1 Traditional theories

The key theoretical feature of the traditional theories is that expectations play a central role in determining the level and shape of the term structure of interest rates. The crucial factor then is how to model expectations. The current accepted view is that expectations are rational. Although these theories are consistent with the utility maximising behaviour of a representative agent, the utility foundation remains restricted to risk neutrality which, in turn, implies zero risk premia.

Empirically, the zero risk premia hypothesis has been rejected decisively and the empirical literature is concerned with uncovering the determinants of the risk premium and confirming that the risk premium changes with time. However, there is a lack of consensus on why the risk premium changes over time. In sum, studying the term structure in the traditional framework represents a snapshot of an important problem. Although it may not provide a totally satisfactory answer, it certainly has enriched our understanding of many aspects of the problem. The effort is, thus, not wasted.

2.4.2 Equilibrium theories

The equilibrium theories include models based upon general and partial equilibrium theories, where the common feature is that bond prices of all maturities satisfy a partial differential equation. Partial equilibrium models ensure no-arbitrage conditions in the bond market while general equilibrium models require bond prices to be consistent with the general equilibrium economy as well. In the latter framework bond pricing is considered an integrated part of asset pricing. A distinct advantage of equilibrium models is that it follows the traditional methodology of developing theories in finance of seeking to <u>explain</u> the relationship between bonds of different maturities and how bonds are priced in such a way as to be consistent with the <u>assumed</u> behaviour of a representative utility maximising investor. However, a disadvantage is the possibility that the assumed behaviour and utility functions fail to reflect actual behaviour and preferences. This issue can only be settled empirically.

The empirical literature, however, remains limited despite the theoretical developments. The evidence surveyed in this chapter indicates some limited support for the general usefulness of these models. In terms of implementation the partial equilibrium models are more flexible while the general equilibrium models require a well chosen utility function and the market price of risk to yield a formula. This requirement may be so restrictive and arbitrary that the chosen measures do not lend themselves easily to economic interpretation.

2.4.3 Preference free models

Unlike the equilibrium (partial or general) theories which seek to price discount bonds[50] given investors' preferences, preference free models are developed for the ultimate purpose of pricing interest rate derivatives. Hence, preference free models focus on the current term structure and set the condition for the evolution of the subsequent term structures such that arbitrage is precluded. The advantage of this framework is its generality and hence it includes equilibrium models as special cases while the disadvantage is the difficulty of implementation unless additional structure is imposed. Further, the HJM spot rate process which characterises the initial term structure is, in general, non-Markov and this would significantly increase the level of computation.

2.4.4 Justification for an empirical inquiry of equilibrium theories in the Australian context

Our survey indicates that, relative to the traditional theories, empirical verification of the modern theories remains sparse, especially in Australian research. Theoretically the modern theories are consistent with the general theory of asset pricing, and hence belong to the main stream of finance. However, it is not possible to demonstrate a clear-cut superiority of one class of theories over another[51]. It would be more appropriate to say that these theories represent different ways of investigating a problem that continues to attract the attention of finance researchers. Each has its own strengths, weaknesses and <u>purpose of inquiry</u>. In a nutshell, the traditional theories focus on how the term structure is affected by expectations, however defined; the equilibrium theories seek to model

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[50] Hence the term structure because of the one to one relation between the price of the discount bond and its underlying spot rate.

[51] There has been a consistent effort to 'rehabilitate' the traditional theories. See Campbell (1986), Froot (1989), McCulloch (1993).

the term structure in the context of economic theory, and the preference theories seek to price interest rate derivatives assuming the current term structure is in equilibrium. Current term structure research, however, is undoubtedly dominated by the modern theories - equilibrium and preference free theories. While these two strands of modern theories pose distinct and justifiable research questions, our focus lies in the equilibrium theories, namely the purpose of our inquiry is to ascertain whether the observed term structure is consistent with these theories. If a positive answer to this question emerges from this study, then it will be a natural step to embark on a test of preference free theories and then the pricing of derivatives. Thus, the scope of a study that includes both these theories would go beyond the normal limit of a thesis project. Consequently, in this thesis we concentrate on the equilibrium theories while leaving the preference theories to future research.

The literature survey in this chapter suggests several reasons for undertaking an empirical examination of equilibrium models: (i) equilibrium models of the term structure remain an important and integrated part of the existing interest rate literature; and (ii) Australian studies of the equilibrium theories are both limited in number and deficient in methodology (see section 2.3.4); hence the validation of these theories in Australian bond markets justifies further research. The present study addresses itself to the difficulties that confront previous Australian studies. Specifically, the innovations include: (i) a new method of term structure fitting based upon the Chebyshev polynomials is used to overcome the limited range of zero-coupon yields which hinders the verification of theoretical rates; (ii) bonds of all maturities are used to incorporate as much information as possible into the estimation; (iii) the empirical method introduces an alternative specification of the error term, the multiplicative error, which is both consistent with the models and which reduces the non-linearity of the tested models. Finally, two of the three models tested, Vasicek (1977), CIR (1985), represent the most influential studies in this literature while the third model, a generalisation of the CIR model, has not been proposed and tested previously. Thus, the range of the tested models represents the most comprehensive empirical study of equilibrium models in Australia. The empirical methodology to implement this inquiry is the subject matter of the next chapter.

CHAPTER 3

EMPIRICAL METHODOLOGY AND DATA

The objective of this chapter is to present the empirical methodology and the data used to estimate and test the bond pricing models. Chapter 2 discusses in detail the two empirical methods which have been used to test modern bond pricing models: nonlinear regression and generalised method of moments (GMM)[1]. Nonlinear regression is adopted for this thesis for three main reasons: (i) it makes use of the entire spectrum of bond maturity; (ii) if the errors are found to be normally distributed then nonlinear regression is equivalent to maximum likelihood estimation; hence the power of its statistical tests tends to be stronger than that of GMM; and (iii) limitations of Australian data makes it impossible to overidentify the system of empirical moments[2], an essential feature of GMM estimation.

As the focus of this thesis is the testing of partial and general equilibrium models of the term structure of interest rates the following models are chosen: (i) Vasicek's (1977) partial equilibrium model; (ii) CIR's (1985) general equilibrium models; and (iii) a generalised CIR model. Vasicek's (1977) partial equilibrium model is chosen over the models proposed by Dothan (1977), Richards (1977) and Brennan and Schwartz (1977) because, without a drift term, Dothan's (1977) interest rate process is not as general as Vasicek's while the Richards (1977) and Brennan and Schwartz (1977) two factor models face serious empirical problems[3]

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[1] See Chapter 2.

[2] The CIR model, for example, has four parameters and hence needs more than four observed zero coupon prices while only three such prices are observed.

[3] See Chiarella, Pham and Mackenzie (1992) and Chapter 2 for details.

in the Australian context. For example, Richards (1977) and Brennan and Schwartz (1977, 1982) introduce two market prices of risk, the inflation rate and the long rate respectively, which are not traded securities and hence incapable of being measured. CIR (1985b) is the pioneering equilibrium model while Longstaff (1989) model is flawed[4].

While the relative performance of a model is usually determined by a comparison of the predicted values from the model with observations, no zero-coupon bond term structures exist beyond the 180 day maturities. Hence it is necessary to generate the term structure of interest rates from observed coupon paying bonds to provide a benchmark of comparison. In this respect a curve fitting exercise was implemented using the Nelson and Siegel (1989) model and Chebychev polynomials.

The chapter is organised as follows. The general framework of estimation using cross-sectional regression is introduced in Section 1 while a more detailed description involving each of the tested models is given in Section 2. Section 3 describes the curve fitting using the Nelson and Siegel model and Chebychev polynomials. Criteria for model selection is discussed in Section 4. Then the data is described in Section 5 and Section 6 concludes the chapter.

3.1 CROSS SECTION REGRESSION METHOD

In general the cross section regression method involves minimising the sum of squared errors of the sample of zero-coupon bond prices where error is defined as the difference between the theoretical (model-generated) zero-coupon price and

[4] See Chapter 2.

the observed zero-coupon bond price. While the models of the term structure of interest rates are built on zero-coupon bonds, most observed bonds are coupon-paying. As a coupon paying bond denoted i, $i = 1,\ldots,m$, can be viewed as a portfolio of zero-coupon bonds denoted j, $j = 1,\ldots,n$, to estimate the parameters of the theoretical zero-coupon bond model by means of the observed coupon paying bonds involves nesting the coupon payments (treated as zero coupon bonds) within the observed coupon paying bonds. To illustrate the minimisation process we use the following notation:

P_i = observed coupon paying bond, $i = 1,\ldots,m$;

$\hat{P}_i(\underline{a}, \underline{x}, \tau_n)$ = model price for the i^{th} coupon paying bond, $i = 1,\ldots,m$;

$P_z(\underline{a}, \underline{x}, \tau_j)$ = zero-coupon bond price at coupon paying time $j = 1,\ldots,n$;

$\tau_j = t_j - t_0$ = maturity of payment j;

τ_n = maturity of coupon paying bond i;

τ_s = time to settlement from today (t_0);

m = number of coupon paying bonds;

n = number of coupons of each coupon paying bond;

$C(\tau_j)$ = coupon payment for maturity τ_j ;

$\underline{x}$ = vector of variables of the model price function;

$\underline{a}$ = vector of the parameters of the model being tested;

ϵ_i = error of the i^{th} coupon paying bond;

The model price of the i^{th} coupon paying bond at time 0 (today) is then given by

$$\hat{P}_i(\underline{a}, \underline{x}, \tau_n) = \left[\sum_{j=1}^{n} C(\tau_j).P_z(\underline{a}, \underline{x}, \tau_j)\right].\frac{1}{P_z(\underline{a}, \underline{x}, \tau_s)} \tag{1}$$

As the bond data used in this study are quoted at time 0 (today) for settlement dates[5], the function of the term $\frac{1}{P_z(\underline{a}, \underline{x}, \tau_s)}$ is to obtain the future bond price at a settlement date by multiplying the bond price at time *0* with the future value of a zero discount bond at settlement time, *s*.

To estimate a model of the term structure it is necessary to solve for the vector of the parameters of the model, $\underline{a}$, by minimising the sum of squared errors associated with *m* coupon paying bonds:

$$\underset{\underline{a}}{Min} \sum_{i=1}^{m} w_i \left[P_i - \hat{P}_i(\underline{a}, \underline{x}, \tau_n) \right]^2 = \underset{\underline{a}}{Min} \sum_{i=1}^{m} w_i \, \epsilon_i^2 \tag{2}$$

where w_i is the weight of coupon bond *i, i = 1,..., m*.

Heteroskedasticity is a common problem in cross-section data and is well recognised in the econometric literature but unless its form is known 'there is no 'best' way to test for and model heteroskedasticity' (Judge, et al (1985), p.454). Thus, despite there being many methods to correct for this problem, they tend to be ad-hoc (Gujarati (1988), p.338, 341) and usually the choice is based on estimation convenience (Judge et al (1985, p.455). White's general heteroskedasticity test was conducted on unweighted nonlinear regression estimates of the term structure and it confirms the presence of heteroskedasticity with respect to bond maturity. To

[5] Settlement dates vary according to bond maturities: following the purchase date settlement is effected the next day for bonds under 5 years of maturity, and 7 days for bonds of 5 years and over .

correct for heteroskedasticity nonlinear regression was weighted by duration, a measure of the average life of a bond. As duration gives greater weight to those cash payments which have larger present values it is a better indication of risk than maturity[6].

The minimisation problem given by (2) is expressed in terms of bond prices and is the standard formulation in the existing literature for estimating modern bond price models. An alternative minimisation program to (2) can be built around the logarithmic form of observed prices and model prices:

$$\underset{\underline{a}}{Min}\sum_{i=1}^{m} w_i \left[v_i \right]^2 \tag{3}$$

where $v_i = \ln P_i - \ln \hat{P}_i = \ln \left[\dfrac{P_i}{\hat{P}_i(\underline{a}, \underline{x}, \tau_n)} \right]$.

The use of the logarithmic form can be justified in the following way. The minimisation can be performed in terms of either zero-coupon bond yields or zero-coupon bond prices. These are referred to as yield norm and price norm respectively. While the price norm has been used by other authors, including Brown and Dybvig (1986), the yield norm has not been used in previous research. Yet there is no a priori reason for the superiority of one norm over the other. In fact, being in percentage form, the yields are a more sensitive set of data than the prices (in dollars), and there is evidence that the yield norm would be, on average,

[6] Empirically Brennan and Schwartz (1983) show that duration is a good measure of bond return variability while theoretically Schaefer and Schwartz (1987) propose to model bond return variability by duration.

more accurate than the price norm (Diament (1993)). Further, If υ_i is normally distributed then bond price is lognormal, a desirable property which entails non-negative prices.

The price of a zero coupon bond can be written in continuous compounding as:

$$P_z = C\, e^{(-y\,\tau)} \tag{4}$$

where y is the yield, τ is the maturity of the bond, and C is a coupon payment which is both the only payment and final face value. Alternatively, rearranging (4) and taking the logarithm gives:

$$-\ln(\, P_z \,/\, C\,) = y\,\tau \tag{5}$$

The yield, y, in (5) is the quantity to be estimated with an error term η:

$$-\ \ln(\, P_z \,/\, C\,) = (\, y + \eta\,)\,\tau \tag{6}$$

An n-coupon paying bond, denoted P_i , can be written as a portfolio of n zero coupon bonds and to avoid clustering of notations we assume, without loss of generality, that the bond is valued at the first coupon payment date,

$$P_i = \sum_{j=1}^{n} e^{[-y\,-\,\eta\,]\cdot\tau_j}\, C\,(\tau_j\,) \tag{7}$$

so that (7) may be approximated by

$$P_i = \left[\, \sum_{j=1}^{n} C\,(\tau_j\,)\, e^{(-y\,\tau_j\,)} \right] e^{-\bar{\eta}\,\bar{\tau}} \tag{8}$$

The term $\bar{\eta}\,\bar{\tau}$ in (8) may be viewed as weighted errors where the weights are the durations of zero-coupon bonds. Taking the logarithm of (8) gives:

$$\ln P_i = \ln \hat{P}_i + \bar{\eta}\,\bar{\tau} \tag{9}$$

While the LHS of equation (9) may be regarded as an observation of $ln\ P_i$ the RHS consists of its theoretical model, $\ln \hat{P}_i$, and an error term. Hence equation (9) is a regression with $\ln \hat{P}$ as the logarithm of the estimated model price and $\bar{\eta}\,\bar{\tau}$ as the error term. Equation (9) then may be viewed as an empirical version of equation (7).

An alternative justification for the yield norm lies in the specification of the form of the error. The functions that underlie the models to be estimated have the following common form:

$$P\ (r,\ t,\ T) = A(t,\ T)\ e^{-B\,(t,\ T)\ r} \tag{10}$$

While the conventional approach is to add an <u>additive</u> error term to (10), the nonlinear nature of (10) may justify a multiplicative error. While the variables, $A(t,T)$ and $B(t,T)$, enter multiplicatively, implying that each depends upon the levels of the other variables, the additivity of the error means that each is independent of all the others. The multiplicative nature of the models described by (10) may then justify specifying an error term to be proportional to the levels of the variables and hence a multiplicative error would be appropriate. The additive error form of (10),

$$P\ (r,\ t,\ T) = A(t,\ T)\ e^{-B(t,\ T)\ r} + \epsilon_t \tag{11}$$

may be converted to a multiplicative error by re-writing (10) as:

$$P\ (r,\ t,\ T) = A(t,\ T)\ e^{(-B(t,\ T)\ r)\ (1\ +\ u_t)} \tag{12}$$

where u_t is additive and i.i.d. so that u_t enters (12) multiplicatively. If the fit is good then u_t would be small and $(1 + u_t)$ is approximately equal to $exp(u_t)$. Hence, (12) is re-written as:

$$P\ (r,\ t,\ T) = A(t,\ T)\ e^{(-B(t,\ T)\ r)}\ e^{u_t} \tag{13}$$

Taking the logarithm of (13) yields:

$$\ln P\ (r,\ t,\ T) = \ln A(t,\ T) - B(t,\ T)\ r + u_t \tag{14}$$

Equation (14) may be interpreted as the empirical version of equation (10). To sum up, the logarithmic norm is proposed because: (i) it has not been used in previous research; (ii) it is consistent with the multiplicative nature of model (11); (iii) the nonlinearity of the models to be estimated can be reduced by the logarithmic form; and (iv) if the error term, u_t, is normally distributed then bond price is lognormal and hence is always constrained to be non-negative. In addition to being consistent with observations the non-negativity of bond prices also facilitates the search process in the minimisation program.

3.2 MODELS TO BE TESTED

This section describes the tests of individual models within the general framework of the minimisation program specified by (2) and (3). The overall aim is to estimate the vector of parameter estimates, $\underline{a}$,and then establish how well the models describe the term structure. While the models to be tested imply that their parameters are constant, the issue of the duration of the time period remains an empirical question. Hence, the estimation process is implemented over various time intervals and under various specifications of the behaviour of model parameters: (i) daily estimation where all the individual parameters are allowed to vary freely, (ii) quarterly estimation where only the short rate is allowed to vary from day to day while all the remaining parameters are held fixed;and (iii) semi-annual estimation where all the parameters are held fixed while the short rate varies from day to day. This scheme of time division was also adopted[7] by Brown and Schaeffer (1993).

As argued in section 3.1 both the logarithmic norm and price norm are used in the estimation. Once it can be established that one norm is superior[8] to the other then results from the superior norm will be reported.

3.2.1 Vasicek (1977) Model

3.2.1.1 Structure of the model

Vasicek (1977) model is characterised by: (i) the stochastic process of the spot rate; (ii) the bond pricing equation; and (iii) the long term yield (R(∞)) which are given by equations (15), (16) and (17) respectively.

$$dr = \kappa(\theta - r)dt + \sigma dz \tag{15}$$

$$P(t,s,r) = \exp\left[\frac{1}{\kappa}(1-e^{-\kappa(T-t)})(R(\infty)-r)-(T-t)R(\infty)-\frac{\sigma^2}{4\kappa^3}(1-e^{-\kappa(T-t)})^2\right] \tag{16}$$

$$R_{vasi}(\infty) = \theta - \frac{\lambda\sigma}{\kappa} - \frac{1}{2}\left[\frac{\sigma^2}{\kappa^2}\right] \tag{17}$$

The vector of the parameter estimates of the Vasicek model, $\underline{a}_{vasi}$, consists of r,

[7] Unaware of Brown and Schaeffer (1993), the author used the same idea to test the stability of model parameters in 1992.

[8] Criteria for model selection are explained in Section 3.5

κ, σ and $(\theta - \lambda)$[9] where r is the instantaneous short rate, κ is the speed of adjustment of the short rate to its equilibrium θ, σ is the volatility of the change in the short rate, and $-\lambda$ is the market price of risk. Hence, the parameter vector to be estimated (by both the logarithmic and price norm) is re-written as:

$$\underline{a}_{vasi} = \begin{bmatrix} r \\ \kappa \\ \sigma \\ \theta - \lambda \end{bmatrix} \tag{18}$$

3.2.1.2 Hypotheses

An overall empirical implication is that the parameters are constant over time while individual parameters may be either negative, or positive or both. For example, the market price of risk, $-\lambda$, is negative while the spot rate may be either positive or negative owing to the assumed form of the volatility function. The implied signs of Vasicek parameters are summarised in Table 3.1.

[9] As noted in Chapter 2, the parameter λ in the Vasicek model is defined to be positive (see Vasicek (1977, equation (14)). To make it consistent with λ in the CIR model a negative sign is placed before Vasicek λ.

TABLE 3.1

Empirical implications of Vasicek model

Parameter	Sign	Remarks
r	+/-	Assumed form of volatility function allows for negative real rate of interest
κ	+	Mean reversion requires positive speed of adjustment
σ	+	Risk factor
$\theta - \lambda$	+/-	Long run spot rate, θ, is positive while market price of risk, $-\lambda$, is negative
$R_{vasi}(\infty)$	+/-	Value of long rate depends upon relative values of all parameters

TABLE 3.2

Empirical implications of CIR model

Parameter	Sign	Remarks
r	+	Assumed form of volatility function precludes negative spot rate
σ	+	Risk factor
$\kappa\theta$	+	Speed of adjustment and long term spot rate are both positive
$\kappa + \lambda$	+/-	κ is positive, λ is negative
$R_{cir}(\infty)$	+	Long term rate includes κ, θ, σ, λ

3.2.2 CIR Model

3.2.2.1 Structure of the model

The CIR model is characterised by: (i) the stochastic interest rate process; (ii) the bond pricing equation; and (iii) the long term yield which are given by equation (19), (20), and (22) respectively.

$$dr = \kappa(\theta - r)dt + \sigma r^{1/2}dz \tag{19}$$

where κ is the speed of adjustment of r to its equilibrium level θ and $\sigma r^{1/2}$ is the volatility of the change in r;

$$P(r,t,T) = A(t,T)e^{-B(t,T)r} \tag{20}$$

where

$$A(t,T) \equiv \left[\frac{2\gamma\exp[(\kappa+\lambda+\gamma)(T-t)/2]}{(\gamma+\kappa+\lambda)(\exp(\gamma(T-t))-1)+2\gamma}\right]$$

$$B(t,T) \equiv \frac{2(\exp(\gamma(T-t))-1)}{(\gamma+\kappa+\lambda)(\exp(\gamma(T-t)-1)+2\gamma} \tag{21}$$

$$\gamma \equiv ((\kappa+\lambda)^2+2\sigma^2)^{1/2}$$

and

$$R_{cir}(\infty) = \frac{2\kappa\theta}{\gamma+\kappa+\lambda} \tag{22}$$

As (20) is overidentified κ cannot be estimated separately from θ and λ. Hence the vector of parameter estimates consists of r, σ, $\kappa\theta$ and $\kappa + \lambda$:

$$\underline{a}_{cir} = \begin{bmatrix} r \\ \sigma \\ \kappa\theta \\ \kappa+\lambda \end{bmatrix} \tag{23}$$

3.2.2.2 Hypotheses

An implication of the CIR model is that r, σ and $\kappa\theta$ are positive. While the market price of risk, λ, is theoretically negative the sign of $\kappa + \lambda$ cannot be predicted without the knowledge of the relative sizes of κ and λ. Further, all the parameters are assumed to be constant over time. These hypotheses are summarised in Table 3.2.

3.2.3 Generalised CIR Model

3.2.3.1 Structure of the model

The closed form solution to the CIR bond pricing equation results from the assumption that the change in the instantaneous rate of interest follows the square root process. The rationale of this assumption is not based on economics but technical convenience[10]. A generalisation of the interest rate process would lead to the model performing better (in the mean squared error sense). Thus, it is proposed that the stochastic interest rate process and the market price of risk be of the form:

$$dr = \kappa(\theta - r)dt + \sigma r^{\beta} dz \tag{24}$$

and

$$\lambda^{*} = \lambda r^{\gamma}/\sigma \tag{25}$$

where both β and γ are positive. If β and γ are not statistically different from 0.5 then it is evidence that the CIR model is well specified with respect to the data

[10] CIR (1985b) use this particular process because a closed form solution to the differential equation is already in existence (see Feller (1951)).

used in the estimation. As (24) and (25) depart from the CIR specification the bond price can no longer be given by (20). Instead it has to be recovered from the numerical solution of a partial differential equation. In the general estimation framework of this study, the vector of parameter estimates becomes:

$$\underline{a}_{gcir} = \begin{bmatrix} r \\ \kappa \\ \theta \\ \sigma \\ \beta \\ \lambda \\ \gamma \end{bmatrix} \tag{26}$$

where the subscripts denote partial derivatives.

For the case of the single state variable, r, the CIR fundamental valuation equation[11] is given by:

$$P_r(\mu - \lambda^* \sigma^*) + P_t + (1/2)(\sigma^*)^2 P_{rr} - rP = 0 \tag{27}$$

where $\mu = \kappa(\theta - \mathrm{r})$ and $\lambda^* = \lambda \mathrm{r}^{\gamma}$. Then the fundamental equation becomes:

$$P_r[\kappa(\theta - r) - \lambda r^{\gamma} r^{\beta}] + P_t + (1/2)(\sigma r^{\beta})^2 P_{rr} - rP = 0 \tag{28}$$

The process of estimating $\underline{a}_{gcir}$ consists of three stages: (i) the first and second order partial derivatives are estimated; (ii) the derivative estimates are substituted into the partial differential equation to solve for an array of bond prices; and (iii) the array of bond prices is used in the minimisation program (2) or (3) to yield the vector of parameter estimates, $\underline{a}_{gcir}$. The method of explicit finite difference[12] is used to estimate $\underline{a}_{gcir}$. While implicit finite difference is usually considered more stable, the scheme of difference adopted in this study is based

[11] See Cox, Ingersoll and Ross (1985, p. 393).

[12] See Brennan and Schwartz (1978) and Courtadon (1991) in Figlewski, Silber and Subrahmanyam, eds., (1991).

upon Chiarella (1991, pp. 189-192) and the conditions suggested by Hill and Dewynne (1987) to overcome the problem of instability and slow convergence[13].

The essential steps are explained in detail below:

(i) Estimating the partial derivatives P_r , P_{rr} and P_t

We construct a lattice of (a) a time dimension ranging from time $t = 0$ to the maturity of a bond, $t = T$; and (b) an interest rate dimension ranging from $r_{min} = 0\%$ to a maximum, say $r_{max} = 200\%$.

Taylor's expansion of the bond price $P(r, t)$ at the point (r, t) is given by:

$$P(r+h,t) = P(r,t)+h\frac{\partial P}{\partial r}+\frac{1}{2}h^2\frac{\partial^2 P}{\partial r^2}+\frac{1}{6}h^3\frac{\partial^3 P}{\partial r^3}+O(h^4) \tag{29}$$

$$P(r-h,t) = P(r,t)-h\frac{\partial P}{\partial r}+\frac{1}{2}h^2\frac{\partial^2 P}{\partial r^2}-\frac{1}{6}h^3\frac{\partial^3 P}{\partial r^3}+O(h^4) \tag{30}$$

where $h = \Delta r$. Adding (29) to (30) yields a finite difference approximation of the second derivative, P_{rr} :

$$\frac{\partial^2 P}{\partial r^2} = \frac{P(r+h,t)-2P(r,t)+P(r-h,t)}{h^2}+O(h^4) \tag{31}$$

which can be approximated, in finite difference form, by:

$$\frac{\partial^2 P}{\partial r^2}\bigg|_{ij} = \frac{P_{i+1,j} - 2P_{ij} + P_{i-1,j}}{h^2} \tag{32}$$

where P_{ij} is the finite difference approximation of $P(r,t)$ at the point $r = r_{min} + ih$

[13] In a commercial environment the issue of computational cost would be more important than in an academic environment. We faced no computational constraint at the University of New South Wales.

and $t = s + jk$.

Subtracting (30) from (29) yields a finite difference approximation of P_r:

$$\frac{\partial P}{\partial r} = \frac{P(r+h,t)-P(r-h,t)}{2h} + O(h^2) \tag{33}$$

or in finite difference form:

$$\frac{\partial P}{\partial r}\bigg|_{ij} = \frac{P_{i+1,\,j} - P_{i-1,\,j}}{2h} \tag{34}$$

The time difference approximation can be obtained by expanding $P(r,\ t\text{-}k)$ around $P(r,\ t)$ where $k = \Delta t$:

$$P(r,\ t-k) = P(r,\ t) - k\frac{\partial P}{\partial t} + O(k^2) \tag{35}$$

which implies:

$$\frac{\partial P}{\partial t} = \frac{P(r,\ t) - P(r,\ t-k)}{k} + O(|k|) \tag{36}$$

or in finite difference form:

$$\frac{\partial P}{\partial t}\bigg|_{ij} = \frac{P_{i,\,j} - P_{i,\,j-1}}{k} \tag{37}$$

(ii) The approximations for the partial derivatives are substituted in the fundamental equation (28) to yield:

$$\frac{P_{i+1,j}-P_{i-1,j}}{2h}\left[\kappa(\theta-r)-\lambda r^{\gamma}r^{\beta}\right] + \frac{P_{i,j}-P_{i,j-1}}{k} + \tag{38}$$
$$\frac{1}{2}(\sigma\ r^{\beta})^2\ \frac{P_{i+1,j}-2P_{i\,j}+P_{i-1,j}}{h^2}\ -rP_{i\,j} = 0$$

Rearranging the terms which are functions of $P_{i,\,j-1}$, $P_{i-1,\,j}$, P_{ij} and $P_{i+1,\,j}$ we have the following system of equations:

$$P_{i,\,j-1} = aP_{i-1,\,j} + bP_{i\,j} + cP_{i+1,\,j} \tag{39}$$

where

$$a = k\left(\frac{B}{h^2}-\frac{A}{2h}\right)P_{i-1,\,j} \tag{40}$$

$$b = \left(1-\frac{2kB}{h^2}\right) \tag{41}$$

$$c = k\left(\frac{A}{2h}+\frac{B}{h^2}\right) \tag{42}$$

$$A = \kappa(\theta-r)-\lambda r^{\gamma}r^{\beta} \tag{43}$$

$$B = \frac{1}{2}(\sigma r^{\beta})^2 \tag{44}$$

or in matrix form[14]:

$$\bar{P}_{j-1} = \bar{E}\,\bar{P}_j \tag{45}$$

where

$$\bar{P}_{j-1} = \begin{bmatrix} P_{1,j-1} \\ P_{2,j-1} \\ \cdot \\ \cdot \\ \cdot \\ P_{N-2,j-1} \\ P_{N-1,j-1} \end{bmatrix}$$

$$\bar{E} = \begin{bmatrix} b\ c \\ a\ b\ c \\ \cdots \\ \cdots \\ \cdots \\ a\ b\ c \\ a\ b \end{bmatrix} \tag{46}$$

$$\bar{P}_j = \begin{bmatrix} P_{1j} \\ P_{2j} \\ \cdot \\ \cdot \\ \cdot \\ P_{N-2,j} \\ P_{N-1,j} \end{bmatrix}$$

(iii) The system of N-1 equations (45) is solved for a bond price, for example an observed price of a 13-week Treasury Note, by stepping back from t = T to t = 0

[14] At the boundaries where r = 0 and r = maximum, only the forward difference equation is used to approximate P_r, ie: $\frac{\partial P}{\partial r} = \frac{P_{i+1,j} - P_{ij}}{h}$. This explains why the top row and bottom row of $\bar{E}$ have only two terms.

using P(T,T) = 1. Denoting this bond price as $\hat{P}$, and repeating the process for an array of observed bond prices, then substitute these solved prices into the minimisation program (2) and (3) to solve for the vector of parameters, $\underline{a}_{gcir}$.

3.2.3.2 Hypotheses

The market price of risk, λ, is necessarily negative while all the other parameters are positive. Furthermore, all the parameters are specified to be constant over time. A summary of the parameters and their empirical implications are given in Table 3.3. The generalised CIR model converges to the CIR model if the two parameters, γ and β, are not statistically different from 0.5.

$$\begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \end{pmatrix}$$

TABLE 3.3

Empirical implications of generalised CIR model

Parameter	Sign	Remarks
r	+	Assumed form of volatility function precludes negative interest rate
κ	+	Mean-reverting speed of adjustment
θ	+	Long run equilibrium spot rate
σ	+	Risk factor
β	+	Volatility is positively proportional to spot rate (see (24))
λ	-	Market price of risk
γ	+	Market price of risk is proportional to spot rate (see (25))

3.3 YIELD CURVE FITTING

While equilibrium theories provide the theoretical underpinning of the term structure, there exists a strand of literature known as yield curve fitting which is primarily concerned with measuring the term structure from observed coupon bonds. The theory of yield curve fitting is largely based upon the notion that bond price is equal to the expected value of its coupons and terminal face value in an arbitrage-free market:

$$P_i = E_t \left[\sum_{t=1}^{n} \frac{C_t}{1 + r_t} + \frac{FV}{(1 + r_n)^n} \right] \tag{47}$$

where P_i is observed bond price, C_i is coupon payment, r_t and r_n are discount rates for bonds maturing at time t and n respectively, and FV is bond face value.

While equation (47) does not possess the strong theoretical foundation underlying equilibrium theories its usefulness has long been recognised where there is a need to extrapolate or interpolate discount rates from existing maturities. Equation (47) may be called a naive model of the term structure. One of the criteria of judging the adequacy of a theory of the term structure is how well it approximates existing bond prices. In this respect the equilibrium models and the naive model share a common objective of fitting existing bond prices as accurately as possible. Hence their performance can be compared using the goodness-of-fit criterion. In essence, yield curve fitting is a mathematical approximation exercise; and hence, in an ex-post sense, it is expected to perform better than equilibrium theories. A hypothesis then can be postulated:

H1: Fitted term structures perform better than equilibrium models by the goodness of fit criteria.

An implication of this hypothesis is that, if accepted, the naive model is preferred where the primary interest lies in the maximum accuracy of approximating the unobserved term structure in order to price fixed income securities or interest rate derivatives. The naive model, however, has to be fitted on a daily basis as its theoretical underpinning does not allow for forecasting over time. Thus, for the practical purpose of the day-to-day needs, the naive model may be viewed as an alternative to equilibrium models. Toward the objective of testing the above hypothesis we begin by reviewing the general framework of yield curve fitting in the existing literature in Section 3.4.1, then in Section 3.4.2 a new methodology is proposed which overcomes some of the weaknesses found in the existing literature.

3.3.1 Survey of yield curve fitting

The early studies of yield curve fitting typically consisted of free hand smoothing and some form of regression of observed bond yields on their corresponding maturities[15] (see Durand (1942), Bradley and Crane (1973), Echols and Elliot (1976), Cohen, Kramer, and Waugh (1966)). While yield curves are useful they are not term structures unless all the expected future rates are equal. McCulloch (1971, 1975) was first to suggest a methodology which involves the estimation of the discount function (the present value of $1 repayable at time t) to

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[15] For example, Cohen, Kramer, and Waugh (1969) specify the yield to maturity, ($y(\tau)$, as a function of maturity, τ, maturity squared, and the square of the logarithm of maturity:

$$y(\tau) = a + b\tau + c\,(\log\tau)^2$$

yield the term structure[16]. The operations of this estimation methodology are best seen in the following exposition.

The price of a coupon paying bond can be written as the sum of the present values of its coupon payments and its face value:

$$P_i = \sum_{t=1}^{n_i} C_i \, D(t) + 100 \, D(n_i) \tag{48}$$

where P_i is the price of bond i, C_i is the coupon payment of bond i, $D(t)$ is the discount factor (present value of \$1) at time t and n_i is bond i's remaining term to maturity.

To illustrate the solution to the discount function $D(t)$ we assume there are three bonds maturing at dates *1, 2* and *3* so that equation (48) can be written out as:

$$\begin{bmatrix} C_1+100 & 0 & 0 \\ C_2 & C_2+100 & 0 \\ C_3 & C_3 & C_3+100 \end{bmatrix} \begin{bmatrix} D(1) \\ D(2) \\ D(3) \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \tag{49}$$

The discount function can then be calculated as long as there is at least a bond maturing for each payment date to ensure the non-singularity of the matrix of payments. The estimation of the discount function, $D(t)$, in equation (49) is usually implemented by cross-sectionally regressing the bond prices on the RHS of equation (49) against the coupons on the LHS. In practice, bonds do not mature at the same dates and the coupon payment dates vary from one bond to another (the

[16] In discrete time the discount function (present value of \$1 payable at time t) is $(1 + r_t)^{-t}$ where r_t is the rate of interest appropriate for discounting \$1 at time t. Obviously estimated discount functions at different t's yield an estimated term structure.

problem of mismatch of payment dates) so that there are more payment dates than bonds. As cross-section regression can only provide point estimates of D(t), some form of interpolation is needed to smooth the discontinuities and to interpolate outside the sample maturities[17]. The technical basis of this curve fitting exercise is Weiertrass' Approximation Theorem[18] which states that there is a class of functions that may approximate any continuous function over a specified interval with an arbitrarily small degree of error. Examples of these functions are polynomials and spline functions (see McCulloch (1971), Vasicek and Fong (1982), Shea (1984, 1985)). For example, McCulloch (1971, 1975) assumes that coupons are paid continuously and then chooses a cubic spline[19] in t:

$$D(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + b_1 d_1 (t-t_1)^3 + \ldots + b_k d_k (t-t_k)^3 \tag{50}$$

where $t_1, t_2, \ldots, t_k$ are chosen knot points, $d_k = 0$ if $t < t_j$, $d_k = 1$ if $t > t_j$, a_0, a_1, a_3 and $b_1, \ldots, b_k$ are parameters to be estimated.

Thus, the estimation of the continuous discount function, $D(t)$, is usually implemented by cross-sectional regression of (49) with the restriction on $D(t)$

[17] The problem of discontinuities of the discount function artificially causes the zero-coupon bond rates (derived from the estimated discount factors) to change dramatically from one period to another. The smoothness of the term structure is necessary if the term structure is to be used for maturities other than those included in the sample. Consequently, a common solution to the problems of discontinuities and smoothing is to fit a smooth curve through the estimated discount factors.

[18] See Phillips, G.M. and P.J. Taylor (1973).

[19] Exponential spline has been suggested by Vasicek and Fong (1982) and Chambers, Carlton and Waldman (1984) but it was criticised by Shea (1984, 1985) as not performing any better than ordinary spline techniques. Mcullock (1971) uses quadratic spline which results in 'knuckles' in the forward rate curve, which is corrected in McCulloch (1975) by requiring the discount function to be twice continuously differentiable, hence the cubic spline is chosen.

specified by (50).

The voluminous literature on fitting various polynomials to a set of observed yields or bond prices has not reached a consensus on the best functional form and the degree of smoothing. Langetieg and Smoot (1989) tested twenty two functional forms on a sample of US Treasury bill, Bond, and Note prices from July 1973 to June 1981. They recommend a simple linear methodology for reasonably accurate measurements and a nonlinear methodology for greater accuracy. As nonlinear estimation is more accurate it is adopted for our purpose of constructing the 'observed' term structure. In particular, two nonlinear methods are implemented: (i) modified Nelson and Siegel's (1987) method and a <u>new</u> method based on Chebychev polynomials. Although in general there is no economic rationale for curve fitting, Nelson and Siegel's (1987) model is based upon technical considerations which are also consistent with observed properties of the term structure. The new method is proposed to overcome the weaknesses found in Nelson and Siegel's (1987) model.

Nelson and Siegel's (1987) method is well known for its 'parsimonious'[20] characterising of the yield curve in that it is capable of generating a range of shapes of the yield curve with a relatively small number of parameters. They assume that the instantaneous forward rate, *f(t)*, is a solution to a second order differential equation with equal roots:

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[20] Parsimony is a desirable characteristic of modelling. Obviously the goodness of fit can be increased by increasing the number of explanatory variables (see Judge et al, (1985), p.862). Other things being equal, a model with a smaller number of variables is preferred.

$$f(t,T) = \beta_0 + \beta_1 \exp[-(T-t)/k] + \beta_2\big((T-t)/k . \exp[-(T-t)/k]\big) \tag{51}$$

where β_0, β_1, β_2, and k are constants. Then the yield to maturity, $y(t, T)$, is the average of the forward rates:

$$y(t,T) = \frac{1}{T-t} \int_0^{T-t} f(x)dx \tag{52}$$

The resulting function from the integration in (52) is

$$y(t,T) = \beta_0 + (\beta_1 + \beta_2)[1 - \exp(-(T-t)/k)]/[(T-t)/k] \tag{53}$$

which is also linear in coefficients, given k.

To apply regression, equation (53) is rewritten in the form:

$$y(T-t) = a + b\big[1 - \exp(-(T-t)/k\big]/\big[(T-t)/k\big] + c\exp\big[-(T-t)/k\big] \tag{54}$$

Recognising the relationship between (zero coupon) bond price and yield to maturity, $P_z = exp[-y(T\text{-}t).(T\text{-}t)]$, then the coefficients a, b and c are estimated by nonlinear least squares[21] with the minimisation programs (2) and (3) over a grid of values of k. These coefficients then determine the term structure of interest rates[22]. It should be noted that Nelson and Siegel (1987, p.478) use linear least squares to estimate equation (54) directly on (zero-coupon) Treasury bills data. As such data is usually limited to very short term maturities, coupon paying bonds with longer maturities are excluded from their estimation. In our case, equation

[21] Note that in this case P is the zero-coupon bond price whose yield to maturity y(t, T) is also its spot rate. Hence, an array of y(t, T) over (t, T) is the term structure.

[22] In the minimisation program given by (2) and (3) each coupon payment is considered a zero coupon bond. Hence the yield to maturity, $y(T\text{-}t)$, is also the spot rate of a zero coupon bond of maturity $(T\text{-}t)$.

(54) is incorporated into (1) and (2) which is then estimated by nonlinear least square on coupon paying bonds data; hence, making it possible to use a much larger observed data set.

A major theoretical weakness of Nelson and Siegel's (1987) model is the arbitrary choice of a second order differential equation with equal roots as this forms a very small set of differential equations that may describe the term structure. While our nonlinear regression method improves upon Nelson-Siegel's ordinary least squares a new method is proposed in the next section to overcome some common econometric problems in term structure fitting.

3.3.2 Chebychev polynomial based method

In this section we propose a new method of fitting the term structure based upon Chebychev polynomials. While the technical properties of Chebychev polynomials as an approximation of functions are well known, they have not previously been applied to term structure estimation. Moreover, we estimate the inverse of the discount function, the so called cumulator, which is broadly consistent with existing theories of the term structure. Apart from having a theoretical base, the proposed method overcomes most, if not all, of the estimation difficulties of the traditional methods.

To simplify the exposition we assume no taxes. Unlike current practice, we estimate the inverse of the discount function[23] at time t, denoted $(1 + Y_t^*)$. The function Y_t^* is related to the discount function, $D(t)$, and the annualised yield to maturity (compounded semi-annually), denoted y_t, by the following relationships:

[23] The use of the inverse of the discount function is largely for programming convenience.

$$Y_t^* = \frac{1}{1+D(t)} \tag{55}$$

$$\frac{1}{\left[1+\frac{y_t}{2}\right]^{2t}} = \frac{1}{1+Y_t^*} \tag{56}$$

$$y_t = 2\left[(1+Y_t^*)^{1/2t}-1\right] \tag{57}$$

As the discount function is the present value of $1, its inverse, $(1 + Y_t^*)$, is the value at time t of $1 invested at time *0* for $t > 0$ so that $Y_t^* > 0$ for positive interest rates. Y_t^* is named the *interest cumulator* because it adds up the interest amounts over the period from time *0* to time *t*.

The interest cumulator has the following desirable properties: (i) Because an investment for a zero time earns a zero return, hence $Y_0^* = 0$; (ii) As investing for a longer period earns more returns for positive nominal interest rates, hence $\frac{dY_t^*}{dt} > 0$; and (iii) While the discount function is monotonically decreasing, the interest cumulator is monotonically increasing.

To estimate the cumulator and hence the zero coupon term structure by means of (55) to (57), we use the framework of the minimisation program (2) and (3) while incorporating Chebychev polynomials into the model price of the coupon paying bond, $\hat{P}_i\ (\underline{a}, \underline{x}, \tau_n)$. Chebychev polynomials which have been used extensively for curve fitting in engineering and science, possess two particularly desirable properties which improve the econometrics of yield curve fitting: (i) the error of a Chebychev polynomial approximation is uniformly distributed over the specified range unlike many other polynomials where the error can vary wildly over some specified range; and (ii) the Chebychev polynomials form an orthogonal set. These two properties effectively remove two commonly encountered problems in term structure estimation: (i) As bond price and yield data errors are dependent upon maturity (the longer the maturity, the larger the error), the even distribution property neutralises this source of errors; (ii) Being fixed, coupons cause multicollinearity which is rectified by the orthogonality property.

As the time range of the Chebychev variable, x, is (-1, 1), bond maturity, $\tau = T - t$, may be converted to this range by the following relationship:

$$x = \frac{\tau}{\frac{\tau_{\max} - \tau_{\min}}{2}} - 1 \tag{58}$$

where

$\tau_{\max}$ = longest maturity at time t

$\tau_{\min}$ = shortest maturity at time t

The Chebychev polynomials are given by the recurrence relation:

$$T_0 = 1$$

$$T_1(x) = x$$

$$T_j(x) = 2xT_{j-1}(x) - T_{j-2}(x) \qquad j = 2, 3, 4, \ldots \tag{59}$$

then the parameter estimates of the cumulator are the solution to the minimisation problem:

$$\underset{\underline{a}}{\text{Min}} \sum_{i=1}^{m} w_i \left[P_i - \hat{P}_i (\underline{a}, \underline{x}, \tau_n) \right]^2 \tag{60}$$

$$\hat{P}_i = \sum_{j=1}^{n} \frac{C}{1+Y_j^*} + \frac{100}{1+Y_n^*} \qquad (61)$$

$$Y_t^* = a_0 + a_1 T_1(x) + \ldots + a_k T_k(x) \qquad (62)$$

where P_i is the observed price of bond i, C is the coupon payment, m is the number of coupon paying bonds, $\underline{a} = \{a_0 , a_1 , \ldots a_k\}$ is the vector of the Chebychev parameters, and n is the number of coupon payments in each coupon paying bond such that $Y_0^* = 0$, $\frac{dY_t^*}{dt} > 0$ for $t > 0$.

The monotonically increasing property of the interest cumulator is ensured by the use of the Chebychev series constrained within the range [-1, 1]. To monitor the behaviour of the Chebychev series in tracking the cumulator, our minimisation program verifies that each step of the estimated Chebychev series is always monotonically increasing.

Our method has a number of advantages over both Nelson and Siegel's (1987) method and the traditional splining and regression method: (i) it partly overcomes the problem of arbitrary degree of smoothing in that it does not require specifying a functional form and a degree of differentiability; (ii) by using nonlinear least squares it circumvents the problem of the number of payments exceeding the number of bonds[24]; (iii) unlike McCulloch (1971, 1975) we do not need to assume that coupons are paid continuously, (iv) the minimisation program given by (2) and (3) makes use of the entire range of available data; and (v) the

[24] Linear least squares require inversion of the matrix of payments (see equation (49). If the number of payments exceed the number of bonds, the coupon payment matrix is singular and non invertible.

interest cumulator function is well behaved with respect to time.

While the Chebychev polynomial method has a number of practical advantages its merit has to be settled empirically by a comparison with the Nelson-Siegel model.

3.4 MODEL SELECTION

The testing of a number of term structure models raises a natural question: what is the best model ? It should be noted that it is not appropriate to compare a curve-fitted model with a theoretical and economic model. Thus, model ranking in this thesis will be carried out in two distinct stages: (i) the Nelson-Siegel model and the Chebyshev polynomial are compared to determine the best fitted model which then serves as the 'observed' term structure; and (ii) the equilibrium models are then compared with this 'observed' term structure.

A variety of measures are used to measure model comparative performance. A common set of statistics, known as information criteria, are calculated for each of the models while in the analysis of individual models, other additional model selection criteria will be used. These revolve around the idea of minimum mean bond pricing errors where error is defined as observed price *less* model generated bond price. Where observed interest rates and model generated interest rates are available the quantity of mean rate errors is also calculated. Furthermore, model comparison also distinguishes between within-sample errors and out-of-sample (or predictive) errors. While model selection is based upon standard econometric tools, these are slightly re-defined to reflect the particular requirements of our tests or to exploit unique features of the data set.

Although model selection criteria are largely developed for linear models

Akaike and Schwartz information criteria are commonly used in nonlinear models[25]. The general thrust of the information criteria is based upon the notion that the adequacy of approximation is measured by the distance of the 'model of reality' and the true distribution of the random variable of interest (see Judge et al (1985), pp. 869-875). The formulae of the Akaike and Schwartz information criteria are developed on the basis of the notion of <u>minimising</u> this distance (see SHAZAM (1993), p. 13):

Akaike information criterion (log AIC) $= \ln\hat{\sigma}^2 + \dfrac{2k}{N}$

Schwartz criterion (log SC) $= \ln\hat{\sigma}^2 + \dfrac{k \ln N}{N}$

where $\hat{\sigma}^2 = \dfrac{\sum_{i=1}^{m} (\, residual\,)^2}{N}$ and N is the number of observations.

A common and popular measure of the goodness of fit of a model is adjusted R^2 while its predictive power is measured by the Theil's inequality and the components of the inequality: error due to bias, error due to variation and error due to covariation (see Theil (1961, 1966)). The formulae of these measures are given by:

Theil's inequality: $U = \sqrt{\dfrac{\sum (P_i - \hat{P}_i)}{\sum \hat{P}_i}}$

Bias proportion: $U^M = \dfrac{(\bar{P}-\bar{\hat{P}})^2}{\frac{1}{N}\sum (P_i - \hat{P}_i)^2}$

Variance proportion: $U^S = \dfrac{(S_P - S_{\hat{P}})}{\frac{1}{N}\sum (P_i - \hat{P}_i)^2}$

[25] See Maddala, G.(1992), pp.500-501.

Covariance proportion: $U^C = \dfrac{2(1-r)s_P\, s_{\hat{P}}}{\frac{1}{N}\sum (P_i-\hat{P}_i)^2}$

where P_i is the observed series, $\hat{P}_i$ is the predicted series, s_P and $s_{\hat{P}}$ are the variances of the observed and predicted series respectively, $\overline{P}$ and $\overline{\hat{P}}$ are the means of the observed and predicted series respectively, and N is the number of observations. A perfect prediction implies the following set of values:

$U = 0$ and for $U > 0$, the ideal distribution is $U^M = U^S = 0$, and $U^C = 1$

3.5 DESCRIPTION OF THE DATA SET

The daily data used for the estimation of the models in this thesis consists of: (i) 13-week and 26-week Treasury Note rates; (ii) Commonwealth Government bonds; and (iii) 5-week Treasury Note rates. While the first two categories cover the period from 2 January 1985 to 30 December 1992, the last category was first available on November 13, 1991 and hence only covers the period from November 1991 to December 1992.

The bond data are in the form of press releases issued by the Reserve Bank of Australia at 5.00 pm from Monday to Friday. These releases consist of bonds traded on the secondary markets by 18 dealers designated as Reporting Bond Dealers. These bond dealers are active in the market for Commonwealth Government bonds and report their transactions each day by 3.00 pm to the Reserve Bank. Further, under the arrangement set up in 1985 the Bank dealt only with these dealers[26] in buying and selling Government bonds of more than one

[26] The practice of dealing only with the Reporting Bond Dealers was discontinued in November 1991. Instead, the Bank deals with any member of the Reserve Bank Information and Transfer System (RITS) whose current membership exceeds 50.

year to maturity. Thus, the press releases represent the trading of the most active bonds each day. Given the thinness of the secondary bond market in Australia and the need to have a reasonable sample size for estimation, those days with less than four traded bonds are excluded. A list of the Reporting Bond Dealers and the distribution of the bond data are given in Tables 3.3 and 3.4 respectively.

To obtain proxies for the short term theoretical rates we use the 11 AM cash rate and 13 week Treasury Note rate[27] which, perhaps, are their closest observed counterparts. Further, the cash rate is not used in the estimation, a comparison between the 11 AM cash rate and the theoretical rate shows the extrapolative power of the estimated model.

3.6 CONCLUSION

The objective of this chapter is to describe the empirical methodology and the data used in the estimation, testing and ranking of models of the term structure of interest rates. While hypothesis testing verifies the consistency of model restrictions and implications with the data, model selection is based upon Akaike and Schwartz information criteria, the goodness of fit statistics and the relative predictive powers of the tested models. The lack of observed zero-coupon bonds outside the range of 5-week, 13-week and 26-week maturities provides the motivation to fit the term structure by means of two numerical methods, the Nelson-Siegel method and a new method based on Chebyshev polynomials. The term structure generated by the better of these two models will be used as a benchmark to judge equilibrium models' performance.

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[27] 5-week Treasury Notes were introduced on November 13, 1991 while other observed rates were available for the period 02/01/1985 - 30/12/1992.

Table 3.3

Reporting Bond Dealers (15 October 1991)

Australia and New Zealand Banking Group Ltd
Australian Gilt Securities
Bain Capital Markets Ltd
Bankers Trust Australia Ltd
Citibank Ltd
Commonwealth Bank of Australia
County NatWest Australia Capital Markets Ltd
Fay, Richwhite Securities Ltd
FBA (Discount) Ltd
Merrill Lynch International (Australia) Ltd
National Australia Bank Ltd
National Mutual Life Association of Australasia Ltd
Potter Warburg Discount Ltd
Rothschild Australia Securities Ltd
SBC Dominguez Barry Ltd
Schroders Australia Ltd
JB Were & Son
Westpac Banking Corporation

Table 3.4

Distribution of bond data
(1985-1992)

Maturity (Years)	Number of bonds	Percentage
Maturity < 3	13740	30.21
3 < Maturity < 5	7302	16.06
5 < Maturity < 10	18105	39.81
10 < Maturity < 15	5566	12.24
15 < Maturity	764	1.68

Note: It should be noted that on any given day in the sample a cash rate, a 13-week Treasury Note rate, and a 26-week Treasury Note rate are available from 2 January 1985 to 30 December 1992. Furthermore, since November 1991 a 5-week Treasury Note rate is also available. Hence, these rates are not included in this distribution.

CHAPTER 4

CURVE FITTING MODELS: EMPIRICAL RESULTS

In this chapter we present the empirical results of fitting the term structure of interest rates by means of the Nelson-Siegel (1987) and the Chebyshev polynomial models. The objective of this exercise is twofold: (i) to determine the better of these two statistical models; and (ii) to use the better model to provide (zero-coupon) interest rates where no such observed rates are available. Toward this end, the chapter is organised as follows. Sections 4.1 and 4.2 briefly recapitulate the estimation procedures for both statistical models. Sections 4.3 and 4.4 present the empirical results of the estimation of the Nelson-Siegel model and Chebyshev model respectively. This is followed by a comparison of the two models in section 4.5. A sample of the fitted term structures on a given day are presented in Section 4.6, then Section 4.7 concludes the chapter.

4.1 ESTIMATION PROCEDURES

As the estimation procedure is explained in detail in Chapter 3 only the general structure[1] is given here to capture the particular properties of these two models. In a nutshell the estimation problem is to estimate the parameter vector, $\underline{a}$, of the theoretical[2] bond pricee, $\hat{P}_i\left(\underline{a},\ (T-t)\right)$, on a set of observed coupon paying bonds, P_i , i =1, 2, 3, ...,m , by minimising the sum of weighted residuals between the observed and theoretical prices:

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[1] Full details are given in Chapter 3.

[2] The term theoretical bond price in this chapter denotes an assumed functional form for the bond price. It is not 'theoretical' in the sense of CIR or Vasicek price.

$$\underset{\underline{a}}{Min} \sum_{i=1}^{m} w_i \left[P_i - \hat{P}_i \right]^2 = \underset{\underline{a}}{Min} \sum_{i=1}^{m} w_i \, \epsilon_i^2 \tag{1}$$

where $\epsilon_i = P_i - \hat{P}_i$ and w_i is the weight of coupon paying bond i, $i = 1,\ldots, m$.

As each theoretical coupon paying bond consists of n coupon payments, C_j , $j = 1, 2, 3, \ldots, n$, it can be viewed as a portfolio of n theoretical zero coupon bonds $P_{z,j}$, $j = 1, 2, 3, \ldots, n$,

$$\hat{P}_i = \sum_{j=1}^{n} C_j \, \hat{P}_{z,j} \left(\underline{a}, (T-t)\right) \tag{2}$$

Thus, substituting equation (2) into equation (1) and implementing the minimisation procedure yield the estimates of the parameter vector, $\underline{a}$, of the theoretical zero coupon bond price, $P_{z,j}$, $j = 1, 2, 3, \ldots, n$.

Alternatively, a <u>multiplicative</u> residual may be defined as $v_i = \ln P_i - \ln \hat{P}_i$ and hence an alternative procedure may be implemented to minimise the sum of the multiplicative residuals:

$$\underset{\underline{a}}{Min} \sum_{i=1}^{m} w_i \left[v_i \right]^2 \tag{3}$$

Minimisation procedures (1) and (3) are referred to as the price norm and logarithmic norm respectively. These general procedures will be used to estimate the vector of parameters of Nelson-Siegel and Chebyshev models.

4.1.1 Nelson-Siegel model

The empirical version of the Nelson-Siegel model is given by

$$y(t,\ T) = a + \frac{b\left[\,1 - \exp(-(T - t)\ /\ k)\,\right]}{(T-t)/k} + c\ \exp(-(T-t)/k) \tag{4}$$

where $y(t,\ T)$, being the interest rate of a zero coupon paying bond is related to its price by

$$P_{z,\,j}\left(\underline{a}_{ns},\ (T-t)\right) = \exp\ \left((T-t)\ y(T,t)\right) \tag{5}$$

Thus, substituting (2) and (5) into (1) and (3) and implementing the minimisation process yield the vector of the parameter estimates of the Nelson-Siegel model:

$$\underline{a}_{ns} = [a,b,c,k]$$

4.1.2 Chebyshev polynomial based model

The Chebyshev polynomial relations are given by:

$$\begin{aligned} T_0 &= 1 \\ T_1\ (x) &= x \\ T_j\ (x) &= 2xT_{j-1}(x) - T_{j-2}(x)\ ;\ j = 2,\ 3,\ 4,\ \ldots \end{aligned} \tag{6}$$

while bond prices are related to the interest cumulator[3] by the following relations:

$$\hat{P}_i = \sum_{j=1}^{n} \frac{C_i}{1+Y_j^*} + \frac{100}{1+Y_n^*} \tag{7}$$

$$Y_t^* = a_0 + a_1T_1(x) + \ldots + a_kT_k(x) \tag{8}$$

where P_i is the observed price of bond i, C_i is the coupon payment, x is $(T-t)/B$, B is the arbitrarily chosen upper bound on all observed bond maturities, $\underline{a}_{ch} = \{a_0\ ,\ a_1\ ,\ \ldots\ a_k\}$ is the vector of the Chebyshev parameters, and n is the number of coupon payments in each coupon paying bond such that

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[3] See Chapter 3, section 3.3.

$$Y_0^* = 0 \ , \ \frac{dY_t^*}{dt} > 0 \quad \text{for } t > 0.$$

Substituting equations (6), (7) and (8) into (1) and (2) and then implementing the minimisation program yields an estimated vector of parameter of the Chebyshev model:

$$\underline{a}_{ch} = [a_0, a_1, a_2, a_3]$$

4.2 LOGARITHMIC NORM AND PRICE NORM

Both the price norm (1) and the logarithmic norm (3) estimation procedures are implemented on each day of the data set from January 2, 1985 to December 30, 1992. While the Chebyshev method is estimated with a range of coefficients from 2 to 7, and the degree of accuracy in terms of the sum of squared residuals increases with the number of coefficients, it was decided to use results based upon 4 coefficients to provide a common basis of comparison with the 4-parameter Nelson and Siegel model. For each daily cross sectional regression, two information criteria (Akaike and Schwartz) are calculated. On the basis of these criteria, namely the better model possessing the lower Akaike and/or Schwartz statistic, the logarithmic norm outperforms the price norm and hence only the results from the former are reported (see Table 4.1).

Table 4.1

Distribution of information criteria of logarithmic and price norm
Daily cross-sections

	Akaike Criterion		Schwartz Criterion	
Nelson-Siegel	Log Norm	Price Norm	Log Norm	Price Norm
Mean	-4.4722	-2.4148	-3.2876	-2.2302
Standard deviation	0.1617	0.146	0.1617	0.146
Chebyshev				
Mean	-5.8978	-3.3829	-5.7237	-3.3432
Standard deviation	0.148	0.1528	0.148	0.1528

Note:
The information criteria are based on minimising the residual sum of squares; hence the smaller the criteria the better the model.

4.3 EMPIRICAL RESULTS OF NELSON AND SIEGEL MODEL

This section presents the results from the estimation of the Nelson-Siegel model in three main areas: (i) parameter estimates; (ii) diagnostic statistics of the estimation process in terms of the distribution and heteroskedasticity of the residuals; and (iii) predictive[4] powers of Nelson-Siegel interest rates.

4.3.1 Distribution of Nelson-Siegel parameter estimates

The distribution of the Nelson-Siegel parameters are presented in Table 4.2 while their time series are graphed in Figs. 4.1 - 4.4. It is important to distinguish between the time series significance of these parameters and their cross sectional significance. The former shows their variability over time and in the absence of an economic based dynamic theory, parameters are likely to behave randomly. This seems reasonable as it implies the inability of forecasting future term structures. The cross-sectional significance of a parameter on the other hand reveals the extent to which it contributes to a term structure at a given point in time. Thus, its significance is evidence of a well specified function.

The time series behaviour of parameters, a, b, and c reveals that while they are not stable as implied by the model it is reasonable to conclude that they vibrate within a band characterised by their respective means (see Table 4.2 and Figs. 4.1-4.4). In other words, while the hypothesis of parameter constancy is rejected, it can be argued that these parameters wander around some fixed value and the fluctuations may be regarded as stochastic errors. The minimisation procedure is

[4] Predictive in the Theil's sense (see Chapter 3), that is, a comparison of observed rates at time t with rates generated (by the model) at the same time t. It does not imply prediction in the sense of generating rates at time $t + n$ by means of parameters estimated for time t. This is because the Nelson-Siegel and Chebyshev models are curve fitting techniques and hence do not have predictive contents based upon economic theory.

Table 4.2

Distribution of the parameters of the Nelson-Siegel model

	a	b	c	k
Mean	0.106876	0.034076	-0.01376	3.835002
Variance	0.003285	0.020231	0.007102	12.49471
Standard deviation	0.057315	0.142236	0.084274	3.534786
Skewness	-8.81735	7.891165	-3.71887	0.596292
Kurtosis	154.9733	132.7028	32.58109	-1.38758
Minimum	-1.1816	-0.41289	-0.99499	0.5
Maximum	0.273746	3.110633	0.307398	9
Range	1.455345	3.523521	1.302385	10.92982
Coefficient of variation	0.53628	4.174057	-6.12529	0.921717
LCL for Mean	0.104351	0.02781	-0.01747	3.679289
UCL for Mean	0.109401	0.040342	-0.01005	3.990715
LCL for Variance	0.00309	0.019028	0.00668	11.75168
UCL for Variance	0.0035	0.021553	0.007566	13.31086

Notes:

LCM ≡ lower confidence limit at 95 %

UCL ≡ upper confidence limit at 95 %

Fig. 4.1

Nelson-Siegel parameter *a* (1985-92)

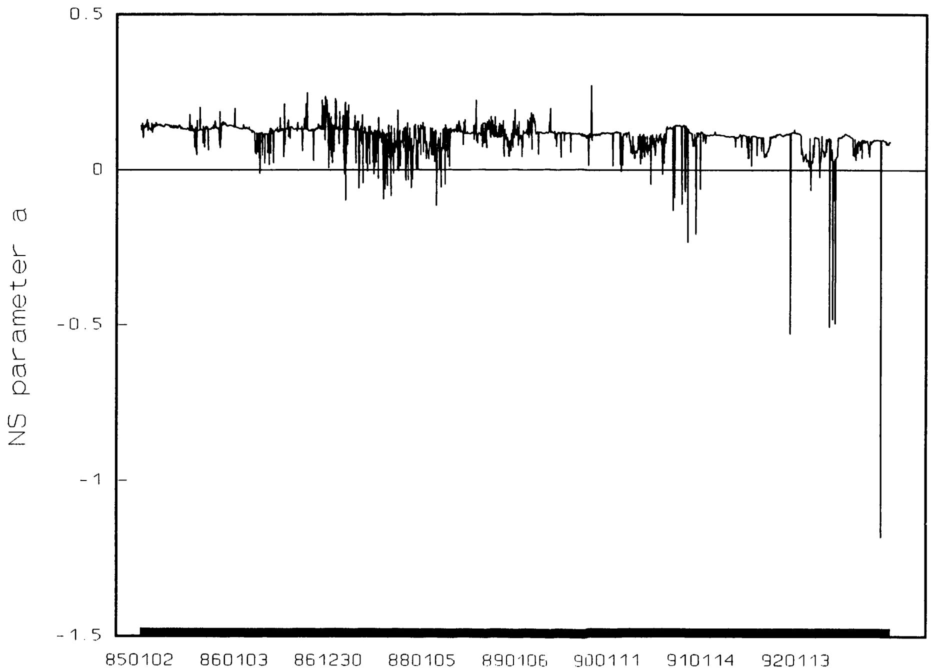

Fig. 4.2

Nelson-Siegel parameter *b* (1985-92)

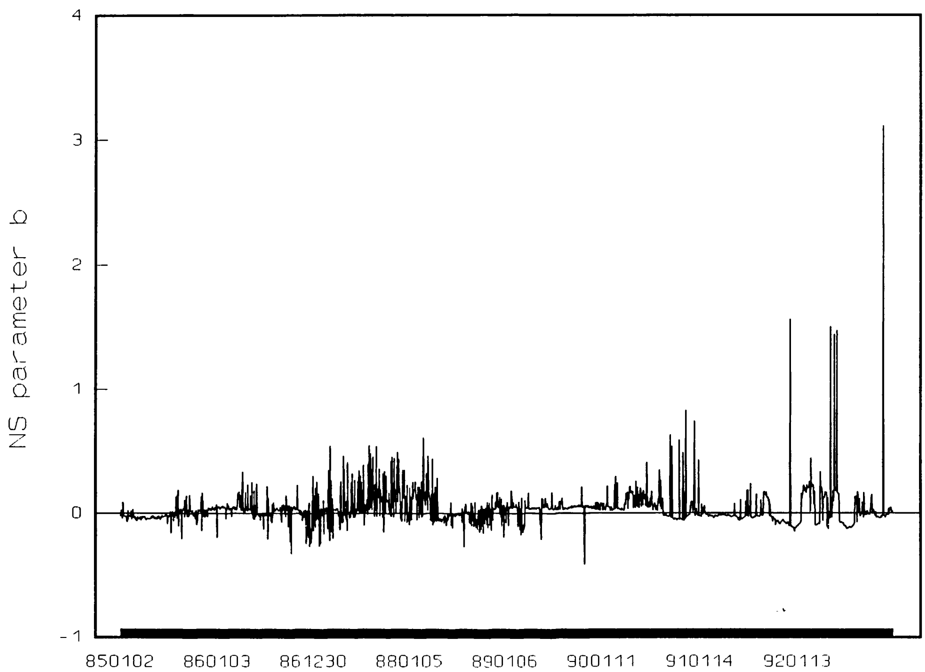

Fig. 4.3

Nelson-Siegel parameter *c* (1985-92)

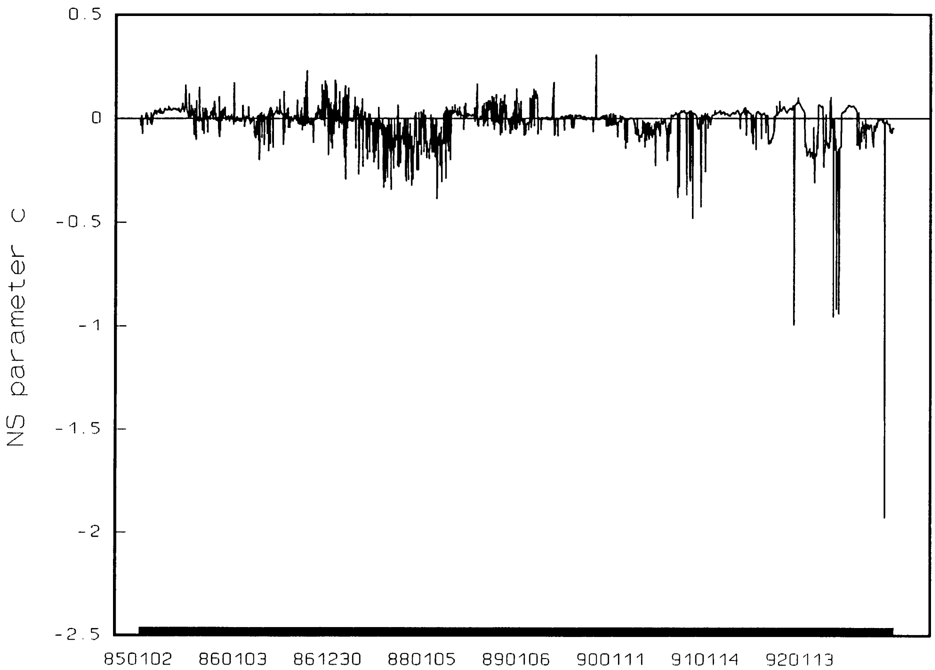

Fig. 4.4

Nelson-Siegel parameter k (1985-92)

performed over a grid of values of k to produce the overall best fit for an optimal combination of a, b, c, and k. Fig. 4.4 shows the time series of optimal values of k, that is, those values that yield the smallest sum of squared errors. As there is no a-priori economic reason for its dynamic behaviour, k fluctuates randomly. Nelson-Siegel (1987) were first to test their model and report the range of k of (10, 365) while in this study it is (2,9) with steps equal to 0.5. It is not possible to infer from this discrepancy as the basis of their estimation is linear and yield while ours is nonlinear and price. Furthermore, our procedure is capable of incorporating coupon paying bonds which increase the data set substantially.

4.3.2 Distribution of t-statistics of Nelson-Siegel parameter estimates

On each daily cross section the four parameters, a, b, c, k, and their corresponding statistics are calculated. While the Wald[5] statistics (which tests the hypothesis that all the four parameters are simultaneously zero) are all significant at 1% level of significance each individual parameter does not always share the same level of significance. Hence, Table 4.3 gives a distribution of these t-statistics and the proportions of the sample in which they are significant. Equation (4) implies that as bond maturity, $(T-t)$, gets large, $y(t, T)$ tends to a while it tends to $a + c$ as bond maturity gets small. Thus, a and $a + c$ are interpreted as having the long term and short term influence on the term structure respectively[6]. It is clear that these parameters (a, b, c) fluctuate erratically and are not constant as implied by the model (see Table 4.2, Fig. 4.1 and Fig. 4.3). It should be noted that a

[5]As the Wald statistic is significant for each and every daily cross section, it is not tabulated.

[6] In this respect the Nelson-Siegel (1987) model shares the same prediction as the CIR (1985b) model.

dominates c as the mean values of a and c are 0.106876 and -0.01376 respectively (see Table 4.2) and the negativity of c implies that the long term influence, $a + c$, is less than the short term influence. Thus the term structure is expected to hump down as maturity increases. Furthermore, parameter a is significant for most of the cross sections (see Table 4.3). Overall, parameter a is the dominant force in Nelson-Siegel term structures. In this respect it is similar to the dominant role of the instantaneous spot rate in CIR, and generalised CIR, and Vasicek models[7].

4.3.3 Diagnostic statistics

This section presents the diagnostic statistics of the estimation with respect to the distribution and heteroskedasticity of the residuals (see Table 4.4). These statistics show whether the minimisation procedure meets some conventional properties of estimation. On the whole the residuals appear normally distributed for 55 per cent and 70 per cent of the sample at the 1 per cent and 5 per cent significance level respectively. This result is particularly important because it shows the extent to which our nonlinear regression estimation is equivalent to maximum likelihood, and hence parameter estimates share the desirable asymptotic properties of the maximum likelihood estimator. In a similar vein the problem of heteroskedasticity is largely solved by using bond durations as weights in minimisation program (1) or (3). The results are particularly significant with 83 per cent and 89 percent of the sample significant at the 1 per cent and 5 per cent respectively. It is well known that heteroskedasticity leads to inefficient estimation

[7] Reported in Chapters 5, 6, 7 respectively.

and biased variances of the estimates[8], thus invalidating the tests of significance. The general absence of heteroskedasticity reinforces the Wald[9] tests and t-tests on the model parameter estimates.

4.3.4 Analysis of the predictiveness of Nelson-Siegel zero coupon rates

This section reports the goodness of fit of Nelson-Siegel short-term rates (1 day, 5 week, 13-week, 26-week) vis-a-vis the series of observed Treasury Notes rates of corresponding maturities (see Table 4.5). It should be noted that these observed series are not of equal length as 5-week Treasury Notes were only issued toward November 1991 and the one day rate is the 11 AM hour cash rate. The goodness of fit is exceptionally good in terms of R^2 and various mean error measures. The Theil's statistics are of mixed quality in that the inequality coefficient is close to zero but the breakdown into the bias, variation and covariation is far from the ideal which requires zero bias and variation and unitary covariation. The inequality, being close to zero, means that there is no systematic bias and the model is well specified while the bias, being relatively large, implies that the observed series fluctuated considerably more than the predicted series. The covariation, despite being different from unity, is less worrisome because it is expected that the predicted series is never perfect.

---

[8] See Gallant (1987, Chapters 1 and 2).

[9] The hypothesis (under the Wald test) is that the four parameters are simultaneously equal to zero. The Wald statistics (though not reported in the chapter) is always significant for each and every daily cross section.

Table 4.3

Distribution of t-statistics of Nelson-Siegel model

Parameter	Significance level	Sample proportion
a	1%	0.85
	5%	0.86
	10%	0.87
b	1%	0.36
	5%	0.39
	10%	0.42
c	1%	0.29
	5%	0.33
	10%	0.37
k	1%	0.28
	5%	0.32
	10%	0.36

Table 4.4

Diagnostic tests of Nelson-Siegel residuals

1985-1992

NS Model	Significance level	Sample Proportion
Normality Test	1%	0.55
	5%	0.15
Heteroskedasticity	1%	0.83
	5%	0.06

Table 4.5

Predictiveness of Nelson-Siegel short-term rates

	1 Day	5 Week	13 Week	26 Week
Correlation coefficient	0.9889	0.87509	0.99411	0.9933
Correlation coefficient squared	0.9781	0.76578	0.98826	0.98664
Root mean squared error	7.3343E-03	5.6358E-03	5.6169E-03	4.5023E-03
Mean absolute error	5.5369E-03	2.5622E-03	4.1013E-03	3.369E-03
Mean error	5.1945E-03	2.5018E-03	3.9037E-03	2.178E-03
Regression coefficient of actual on predicted	1.01318	0.82956	1.05076	1.04166
Theil's inequality* coefficient (1966)	5.4167E-02	8.5457E-02	4.2449E-02	3.4881E-02
Theil's inequality* coefficient (1961)	2.7617E-02	4.3516E-02	2.157E-02	1.761E-02
Fraction of error due to bias	5.016E-01	1.9706E-01	4.8302E-01	2.3407E-01
Fraction of error due to different variation	1.2872E-02	9.072E-03	1.07E-01	1.1054E-01
Fraction of error due to difference covariation	4.8553E-01	7.9387E-01	4.0999E-01	6.5539E-01

Note: * While this inequality is defined differently in Theil (1961), equation (2.20), p.32) and Theil (1966, equation (4.1), p. 28) it conveys a similar concept.

4.4 EMPIRICAL RESULTS OF CHEBYSHEV MODEL

Results from the estimation of the Chebyshev model are presented to highlight three areas: (i) parameter estimates; (ii) diagnostic statistics of the estimation process in terms of the distribution and heteroskedasticity of residuals; and (iii) relative predictiveness[10] of the Chebyshev interest rates with reference to the observed rates.

4.4.1 Distribution of Chebyshev parameters

Table 4.6 reports the distributions of the Chebyshev coefficients while their time series are graphed in Figs. 4.5 - 4.8. Parameter a_0 is most important in terms of: (i) absolute size with mean equal to 2.09815 (see Table 4.6); and (ii) total significance in cross sectional estimation (see Table 4.7). It is not possible, however, to identify the economic content of this parameter.

4.4.2 Distribution of *t*-statistics of Chebyshev model

The Wald[11] statistics and parameter a_0 are significant at 1% level of significance for all cross sections while parameters a_1, a_2 and a_3 are highly significant for most of the cross sections (see Table 4.7). Overall the Chebyshev parameters are significant for larger proportions of the sample than those of the Nelson-Siegel model; for example, parameter a_0 is significant for every cross section (see Table 4.7) in contrast to only 87 per cent for the Nelson-Siegel model (see Table 4.3).

---

[10] In the Theil's sense (see footnote 4).

[11] The Wald statistics which test the hypothesis that all parameters are simultaneously zero are consistently significant at 1% for each and every daily cross section.

Table 4.6

Distribution of the Chebyshev parameters

	a_0	a_1	a_2	a_3
Mean	2.098185	0.407106	0.026526	-0.01063
Variance	1.834643	0.152631	0.006824	0.000701
Standard deviation	1.35449	0.39068	0.082606	0.026471
Skewness	1.57591	2.076778	0.13918	-2.9958
Kurtosis	2.091356	4.713533	4.778179	17.59844
Minimum	0.424727	-0.26032	-0.39953	-0.34924
Maximum	7.300791	2.276414	0.48322	0.071586
Range	6.876064	2.536735	0.882746	0.420827
Coefficient of variation	0.645553	0.959651	3.114131	-2.48976
LCL for Mean	2.038152	0.389791	0.022865	-0.01181
UCL for Mean	2.158217	0.424422	0.030188	-0.00946
LCL for Variance	1.724905	0.143501	0.006416	0.000659
UCL for Variance	1.95525	0.162665	0.007272	0.000747

Note: LCL ≡ lower confidence limit at 95 %
UCL ≡ upper confidence limit at 95 %

Fig. 4.5
Chebyshev parameter a_0 (1985-92)

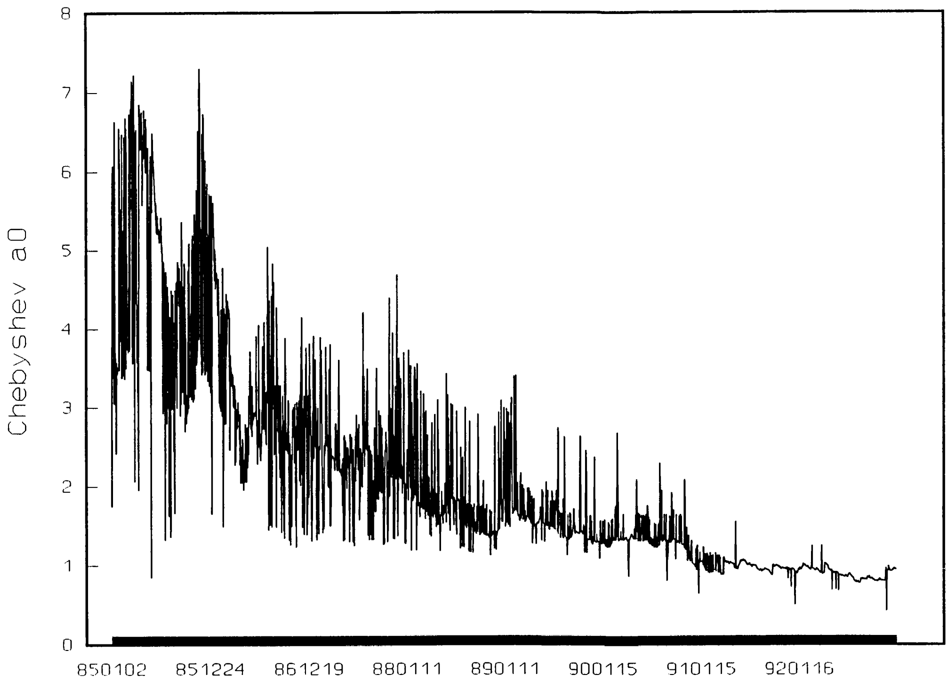

Fig. 4.6
Chebyshev parameter a_1 (1985-92)

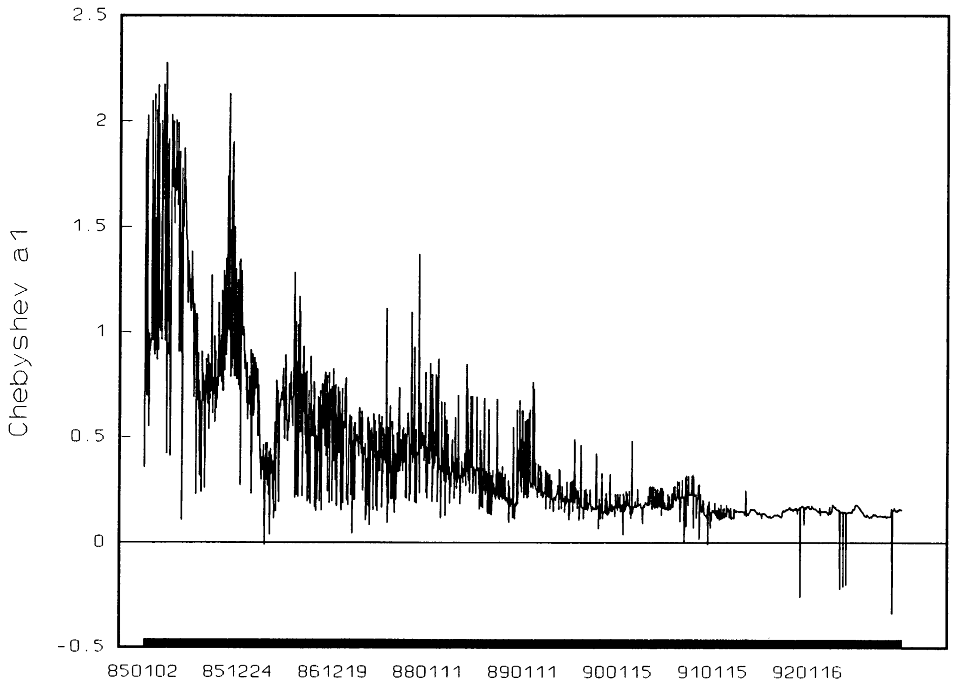

Fig. 4.7
Chebyshev parameter a_2 (1985-92)

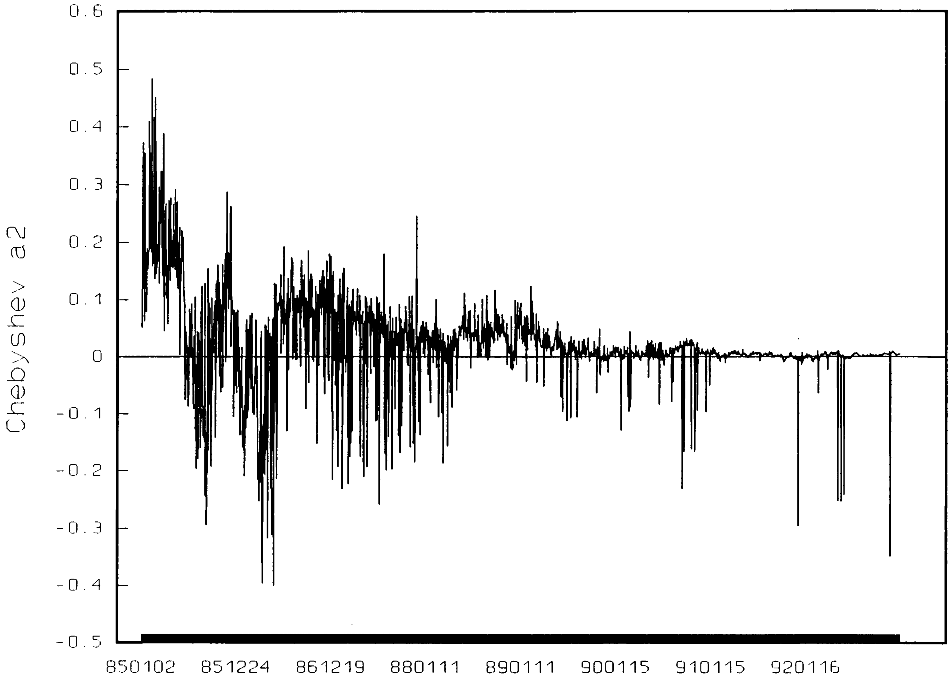

Fig. 4.8

Chebyshev parameter a_3 (1985-92)

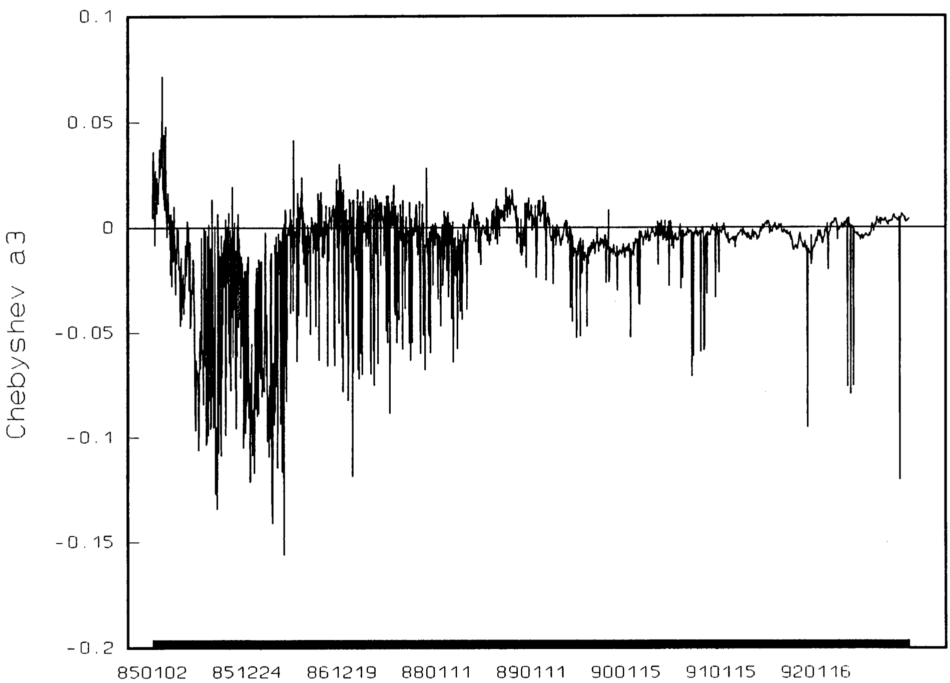

Table 4.7

Distribution of t-statistics of Chebyshev parameters

Parameter	Significance level	Sample proportion
a_0	1%	1.00
	5%	1.00
	10%	1.00
a_1	1%	0.96
	5%	0.97
	10%	0.98
a_2	1%	0.32
	5%	0.39
	10%	0.46
a_3	1%	0.36
	5%	0.43
	10%	0.50

4.4.3 Diagnostic statistics

In this section we present the diagnostic statistics of the Chebyshev estimation with respect to the distribution and heteroskedasticity of the residuals. The statistics in Table 4.8 confirm the normal distribution and homoskedasticity of the residuals for a very high proportion of the cross sections. The results mean that the nonlinear regression of the Chebyshev model is asymptotically maximum likelihood and hence Chebyshev estimates share the properties of maximum likelihood estimator.

4.4.4 Analysis of the predictiveness of Chebyshev zero coupon bond rates

The performance of the Chebyshev rates and their observed rates of corresponding maturities is analysed in this section. While the overall goodness of fit is comparable with that of the Nelson-Siegel model, of particular significance is the fit of the 5 week Treasury Note rates which show the Chebyshev model outperforming the Nelson-Siegel model with an R^2 of 91 per cent (see Table 4.9, third row, third column) in contrast to 76.58 per cent (see Table 4.5, third row, third column).

Table 4.8

Diagnostic tests of Chebyshev residuals

NS Model	Significance level	Sample Proportion
Normality Test (Jarque-Berra)	1%	0.66
	5%	0.80
Heteroskedasticity (White)	1%	0.94
	5%	0.97

Table 4.9

Predictiveness of the Chebyshev model

	1 day	5 week	13 week	26 w
Correlation coefficient	0.9876	0.95395	0.99789	0.99634
Correlation coefficient squared	0.9753	0.91001	0.99578	0.99268
Root mean squared error	7.2E-03	3.5E-03	4.0E-03	3.2055E-03
Mean absolute error	5.68E-03	2.4E-03	3.2E-03	2.4954E-03
Mean error	4.6E-03	1.9E-03	2.9E-03	1.0197E-03
Regression coefficient of actual on predicted	1.01133	0.93657	1.0508	1.04054
Theil's inequality coefficient (1966)*	5.32E-02	5.35E-02	3.01E-02	2.48E-02
Theil's inequality coefficient (1961)*	2.71E-02	2.71E-02	1.52E-02	1.24E-02
Fraction of error due to bias	4.1903E-01	2.8399E-01	5.2018E-01	1.0199E-01
Fraction of error due to different variation	1.2912E-02	2.6173E-03	1.8585E-01	1.8398E-01
Fraction of error due to difference covariation	5.6806E-01	7.1339E-01	2.9396E-01	7.1492E-01

Note: * see Note, Table 4.5

4.5 COMPARATIVE ANALYSIS OF CHEBYSHEV AND NELSON-SIEGEL MODELS

Results of a comparative analysis of Nelson-Siegel and Chebyshev models to determine the better model are reported in this section. This assessment is based upon the Akaike and Schwartz information criteria[12] and mean pricing errors[13] which show that the Chebyshev model outperforms the Nelson and Siegel model (see Tables 4.10 and 4.11). For example, the Chebyshev model displays smaller mean pricing errors in 1985, 1987-91, and over the entire period, 1985-92 (see Table 4.11). In addition, the diagnostic statistics (Tables 4.4 and 4.8) and the goodness of fit statistics (Tables 4.5 and 4.9) also indicate that the Chebyshev model is better than the Nelson-Siegel model in that both the means of the former's Akaike and Schwartz criteria are smaller and the former outperforms the latter in 69 percent of the cross-sections(see Table 4.10). Consequently, for the purpose of constructing the term structure of interest rates by means of observed coupon paying bonds, the Chebyshev model is the preferred model to provide those unobserved long term (zero-coupon) rates to act as benchmarks for the equilibrium model rates of corresponding maturities.

4.6 A SAMPLE OF CHEBYSHEV AND NELSON-SIEGEL TERM STRUCTURES

In this section a snap shot of the term structures generated by the Nelson-Siegel and Chebyshev models is provided (see Fig. 4.9, Fig. 4.10, and Fig. 4.11).

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[12]The smaller are these statistics the better the model.

[13] Pricing error is defined as observed bond price *less* price fitted by either Chebyshev polynomials or Nelson-Siegel model.

Table 4.10

Distribution of information criteria

Overall performance under logarithmic norm		
	Mean Akaike criterion	Mean Schwartz criterion
Nelson-Siegel	-4.4722 (0.1617)	-3.2876 (0.1617)
Chebyshev	-5.8978 (0.148)	-5.7237 (0.148)
Distribution of cross-sectional performance		Percentage of daily cross sections in sample set
Nelson-Siegel outperforms Chebyshev		31
Chebyshev outperforms Nelson-Siegel		69

Note: As the Akaike and Schwartz criteria are based upon minimising the residual sum of squares the smaller these statistics the better the model. Furthermore, they differ by a constant, hence their standard errors (in parentheses) are equal.

Table 4.11

Comparative analysis of mean pricing errors per $100 bond

	Chebyshev	Nelson-Siegel
1985*	0.43024	0.44127
1986	0.47108	0.46988
1987*	0.44065	0.46214
1988*	0.35833	0.37728
1989*	0.31263	0.36469
1990*	0.19081	0.22526
1991*	0.09859	0.13471
1992	0.14335	0.11805
1985-92*	0.305711	0.32416

Note: *denotes the year(s) in which the Chebyshev model displays smaller mean pricing errors than the Nelson-Siegel model.

Several observations may be made. Firstly, the Chebyshev curve (see Fig. 4.10) is much more flexible in tracking the variations of observations while the Nelson-Siegel shape (see Fig. 4.9), being controlled by its underlying particular differential equation, is more restricted. This is particularly accentuated for maturities beyond the maximum observed maturity[14]. Secondly, in the range of observed maturities, the difference between the two models is smaller than that outside them (see Fig. 4.11). For example, the maximum difference (see Fig. 4.11) is in the order of 30 basis points.

4.7 CONCLUSION

This chapter documents the empirical evidence of constructing the term structure of interest rates by means of two statistical models: the Nelson-Siegel and Chebyshev polynomials models. The evidence indicates that the specification of a multiplicative residual in the estimation model led to improvement in accuracy. Furthermore, the nonlinear regression estimation used in this chapter is asymptotically maximum likelihood so that all the tests enjoy the strength of a maximum likelihood estimator. Lastly, while the goodness of fit of both these models is high in absolute terms, the Chebyshev model has an edge over the Nelson-Siegel model and hence will be used as a benchmark in the testing of the equilibrium models.

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[14]Approximately 9.5 years on 20 December 1990.

Fig. 4.9

Nelson-Siegel term structure (20/12/90)

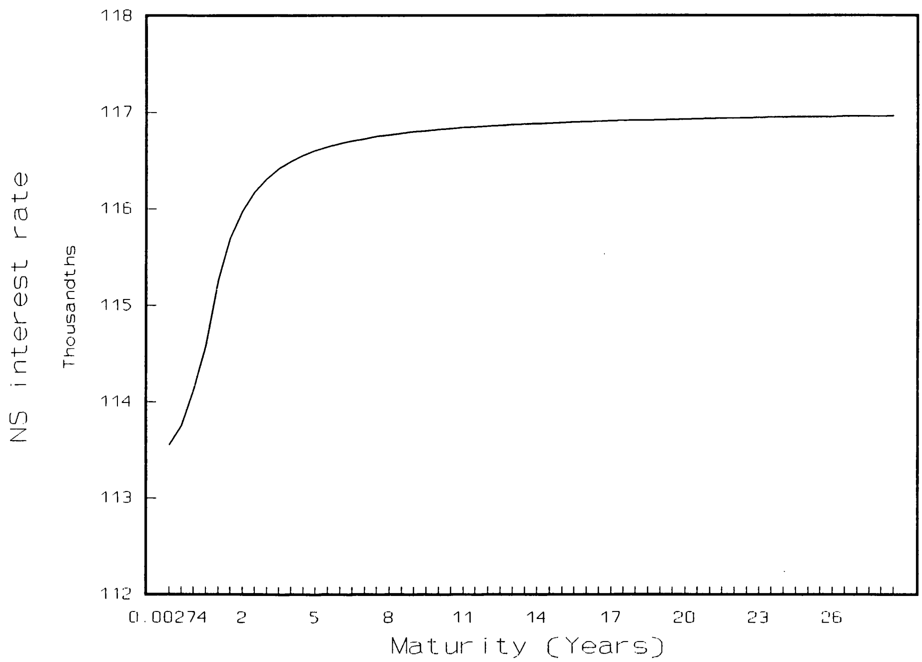


Fig. 4.10

Chebyshev term structure (20/12/90)

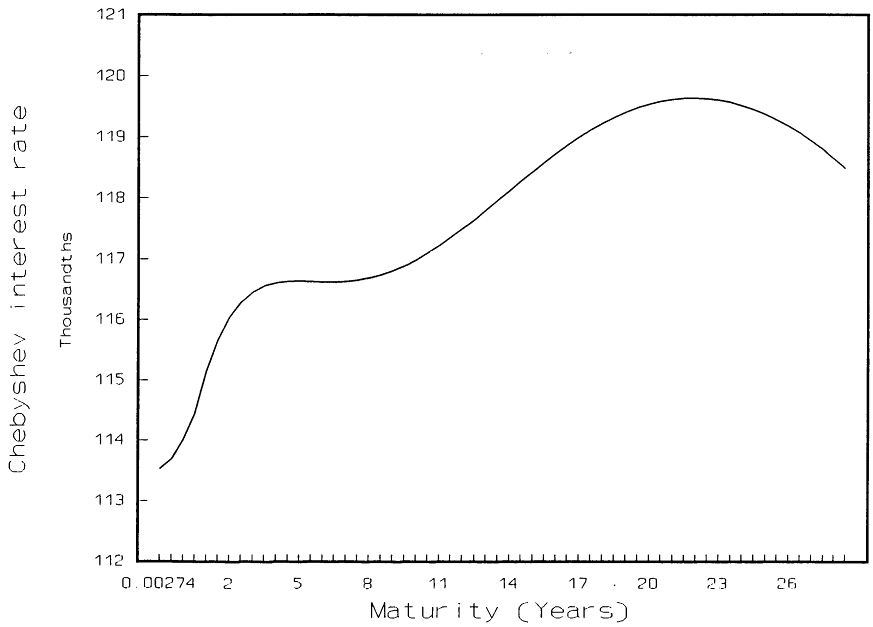
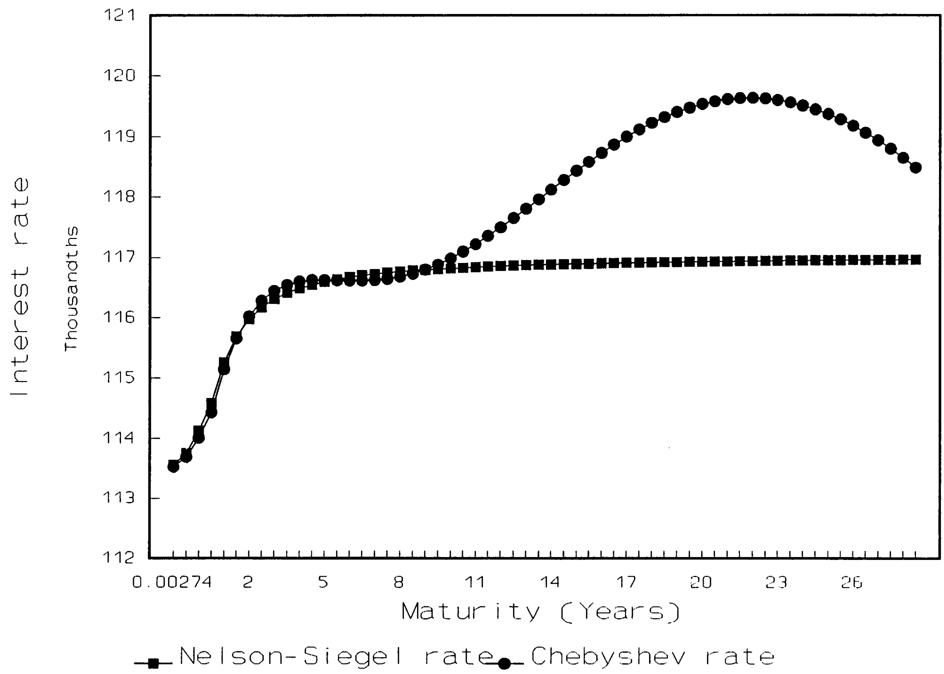


Fig. 4.11

Chebyshev and Nelson-Siegel term structures (20/12/90)



CHAPTER 5

CIR MODEL: EMPIRICAL RESULTS

The objective of this chapter is to present and discuss the results of the empirical estimation of the CIR model undertaken in this study. The two major issues we shall address ourself to are: (i) empirical support for parameter stability as implied by the model; and (ii) goodness of fit and predictiveness of the model. Toward this end, various statistical aspects of parameter estimates are explored together with several investigations undertaken to deepen our understanding of the behaviour of the model in the Australian context. The chapter is organised as follows. Sections 5.1-5.5 are concerned with the results and goodness of fit of estimates of the parameters, r, $\sigma\sqrt{r}$, $\kappa\theta$, $\kappa+\theta$, and the long term yield, $R_{cir}(\infty)$. Section 5.6 then provides a snapshot of cross-sectional results in the form of graphs of a typical CIR term structure and its deviations from observations. Finally, to conclude the chapter, a summary of general results about the model as a whole is given in Section 5.7

Model estimation is implemented using unconstrained and constrained nonlinear regression where the error functions are expressed in terms of bond prices and the logarithm of bond prices[1]. Unconstrained daily regression allows all the model parameters, including the spot rate, to vary freely while constrained regression keeps the parameters fixed, except the spot rate, over each quarterly and semi-annual interval. In Chapter 3 it is argued that the logarithmic norm, based upon the multiplicative error, is more accurate. Comparing the information

[1] For justification for the two types of estimation see Chapter 3.

criterion statistics[2] of <u>each</u> and <u>every</u> cross-section regression confirms the superiority of the logarithmic norm[3]. The results in distributional form are presented in Table 5.1. As the logarithmic norm is first proposed in this study, no evidence in previous research is available for comparison or contrast. The evidence, however, is consistent with the a priori arguments for the appropriateness of the logarithmic norm[4]. Consequently, the results reported in this chapter are based upon the logarithmic norm. Furthermore the homoskedasticity and normality tests as shown in Table 5.2 imply that the nonlinear regression procedure adopted is equivalent to maximum likelihood[5] for the majority of the cross sections. Hence, it possesses all the desirable properties of maximum likelihood estimators (see Green (1993, pp. 305-307)). The null hypothesis of the White test is homoskedasticity and the test statistic is asymptotically distributed as χ^2 with $k\text{-}1$ degrees of freedom where k is the number of regressors (see Green(1993, pp. 392-393)) while the null hypothesis of the Jarque-Berra test is normality and the test statistic has a χ^2 distribution with two degrees of freedom (see Kmenta (1986, pp. 265-267)). These tests are applied to all cross sections[6] and the proportions of days

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[2] See Judge et al (1985, p. 242 and Chapter 21) and Maddala (1992, p. 500-501) who argue that Aikake's information criterion (AIC) is commonly used (at least in nonlinear models). Schwartz criterion is developed within the Baysian framework. Information criteria are based upon minimising the residual sum of squares; thus, among competing regression models the one with the minimum criteria statistic is preferred.

[3] For the case of the superiority of the logarithmic norm see Chapter 3.

[4] See Chapter 3.

[5] See Green (1993, pp. 305-307).

[6] While parameters σ, κ, θ, and λ are constrained to be constant in each quarterly and semi-annual cross section, the spot rate, r, is allowed to vary from day to day. Hence the White and Jarque-Berra tests are also applied from day to day for quarterly and semi-annual estimations.

where they are significant are shown in Table 5.2. For example, using daily estimation where all the parameters, including the spot rate r, vary from day to day, the normality and homoskedasticity hypotheses cannot be rejected for 65.5% and 85.16% of the daily cross sections respectively at a 1% level of significance (see row 3 and 5 and column 3 of Table 5.2).

Table 5.1

Distribution of information criteria of logarithmic and price norm
Daily cross-sections

	Akaike Criterion		Schwartz Criterion	
	Log Norm	Price Norm	Log Norm	Price Norm
Mean	-2.5788	-1.4955	-2.3942	-1.3109
Standard deviation	0.1721	0.1662	0.1721	0.1662

Note:
The information criteria are based on minimising the residual sum of squares; hence the smaller the criteria the better the model.

Table 5.2

Normality and heteroskedasticity tests of regression residuals

		Daily	Quarterly	Semi-annual
	Significance level	Proportion of cross-sections	Proportion of cross-sections	Proportion of cross-sections
Normality Test (Jarque-Berra)	1%	0.6550	0.8576	0.8121
	5%	0.7854	0.8822	0.8872
Heteroskedasticity (White)	1%	0.8516	0.7579	0.7804
	5%	0.9245	0.8475	0.8680

Note:
The entry 0.8576(column 4, row 3) means that at 1% level of significance the regression residual is normally distributed for 85.76% of the total quarterly cross sections (the spot rate is allowed to vary from day to day while the other parameters, σ, $\kappa\theta$ and $\kappa+\lambda$, are kept fixed over each quarter). Similarly the entry 0.7579 (column 4, row 5) means that at 1% level of significance the regression residual is homoskedastic for 75.79% of the total quarterly cross sections. Other entries are interpreted accordingly.

5.1 ESTIMATES OF SPOT RATE

5.1.1 Distributional statistics

Spot rate estimates are presented in Tables 5.3(a) and 5.3(b) while Figs. 5.1(a) and 5.1(b) provide a visual contrast of the differences between the spot rate and its two closest observed counterparts in terms of maturity, the cash rate[7] and the 13-week Treasury Note (TN) rate. It is worth noting that the estimated CIR spot rate tends to underestimate these observed rates (by approximately 1.6% - 1.8%) as evidenced by the predominantly positive differences between observed rates and CIR rates (see Figs. 5.1(a) and 5.1(b)). This difference is then regressed on the level of the observed cash rate or Treasury Note rate , which indicates that the underestimation is positively related to the level of the observed rates (see Table 5.3(d)).

While the CIR spot rate tends to track the 13-week TN rate slightly better than the cash rate as the mean difference between the TN rate and the spot rate is smaller than the mean difference between the cash rate and the spot rate (see Table 5.3(c)), the high *t*-values indicate the CIR spot rate significantly underestimates both these observed rates. Apparently the CIR rate tracks the 13-week TN rate better, probably due to the fact that the 13-week TN rate is less influenced by the Reserve Bank's money market operations than the cash rate. Furthermore, estimates of the spot rate are not significantly affected by the method of estimation, constrained or unconstrained. Thus in Table 5.3(b) the distributional statistics across daily, quarterly and semi-annual intervals are essentially similar. This is

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[7] The cash rate is the overnight funds rate among financial institutions and is influenced by monetary policy considerations.

further supported by the *t-values* which cannot reject the hypothesis that the means of the spot rate under the three methods of estimation are equal.

Lastly, among the CIR parameter estimates the spot rate is always significant[8] at each and every (daily, quarterly, semi-annual) cross section. Thus, consistent with the specification of the model, it is the driving force of the CIR term structure.

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[8] The *t*-statistic underlying the hypothesis that the spot rate is 0.0 always exceed 5.00; and hence they are not tabulated.

Table 5.3(a)

Daily estimates of CIR spot rate

	Mean	Std	Min	Max
1985	0.13406	0.012461	0.11796	0.18498
1986	0.14011	0.015686	0.11080	0.17635
1987	0.12512	0.014464	0.09986	0.16386
1988	0.11729	0.012908	0.09563	0.14438
1989	0.16290	0.009202	0.13970	0.17754
1990	0.13385	0.014014	0.10580	0.17047
1991	0.09636	0.012419	0.06735	0.11474
1992	0.05979	0.008355	0.04949	0.09503
1985-92	0.12148	0.031649	0.049489	0.18498

Table 5.3(b)

Distribution of CIR spot rate
Daily, quarterly and semi-annual estimates
(1985-1992)

	t-value	Daily	Quarterly	Semi-annual
Mean		1.2148E-01	1.2532E-01	1.2092E-01
Std		3.1649E-02	3.3906E-02	2.9663E-02
Minimum		4.9489E-02	4.5641E-02	4.5415E-02
Maximum		1.8498E-01	2.1162E-01	1.8444E-01
H_0 *(d, q)*	0.85			
H_0 *(d, sa)*	0.46			
H_0 *(q, sa)*	0.52			

Notes:

(a) H_0 *(d, q)* is the null hypothesis that the mean of daily estimates of CIR spot rate is equal to the mean of its quarterly estimates;

(b) H_0 *(d, sa)* is the null hypothesis that the mean of daily estimates of CIR spot rate is equal to the mean of its semi-annual estimates;

(c) H_0 *(q, sa)* is the null hypothesis that the mean of quarterly estimates of CIR spot rate is equal to the mean of its semi-annual estimates.

Table 5.3(c)

Differences between observed TN rates and CIR spot rate (1985-1992)

	13-week TN rate *less* CIR spot rate	Cash rate *less* CIR spot rate
Mean	0.016162	0.018171
Standard deviation	3.12E-04	3.75E-04
t-value[a]	51.74*	48.31*
t-value[b]	4.10*	

Notes:

[a] The null hypothesis is that the mean of (observed rate *less* theoretical rate) is 0.0.
[b] The null hypothesis is that the mean of the series (13-week TN rate *less* r) is equal to the mean of the series (cash rate *less* r).
* Significant at 1%.

Table 5.3(d)

Regression of cash rate, 13-week Treasury Note rate and CIR spot rate (1985-1992)

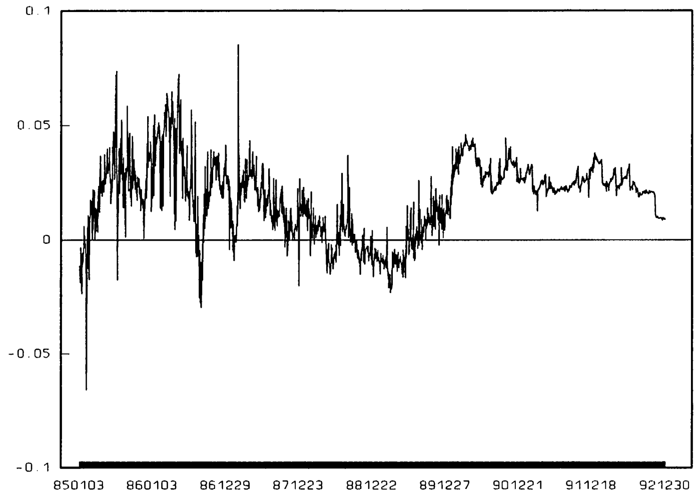
Dependent variable	Independent variable	Intercept	Reg Coef	$\bar{R}^2$	DW
Diff_cash[a]	Level of cash rate	0.6748E-02	0.0472	0.0126	2.00
t-value		20.30*	3.78*		
Diff_tn[b]	Level of TN rate	0.1726E-04	0.0864	0.3807	2.30
t-value		0.067	9.28*		

Notes:
* significant at 1%
[a]Diff_cash ≡ cash rate *less* CIR spot rate
[b]Diff_tn ≡ 13-week Treasury Note rate *less* CIR spot rate

Fig. 5.1(a)

Difference between cash rate and CIR spot rate

Rate difference

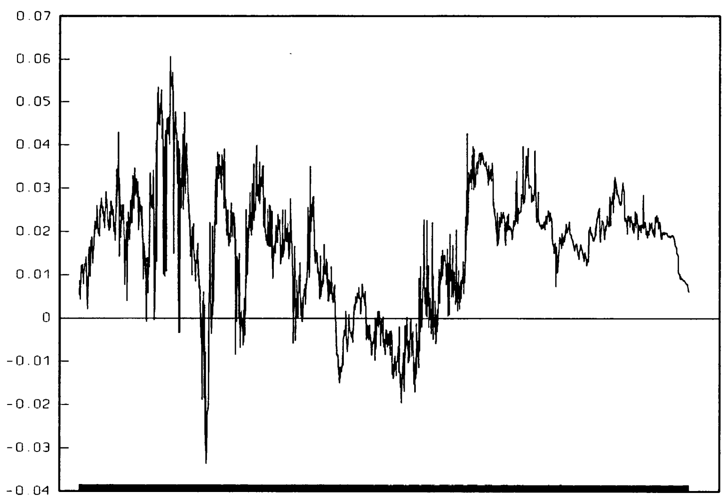


Note: Rate difference = cash rate *less* CIR spot rate

Fig. 5.1(b)

Difference between 13-week TN rate and CIR spot rate

Rate difference



Note:

Rate difference = 13-week TN rate *less* CIR spot rate

5.1.2 A test of the unbiasedness of 13-week CIR interest rates

Section 5.1.1 shows that the CIR spot rate significantly underestimates the cash rate and 13-week Treasury Note rate. This result is probably due to the fact that the comparison is not strictly valid as the spot rate and the observed rates are not of similar maturity. To address this mismatch of maturity we use 13-week Treasury Note rates to explore whether the 13-week CIR rates are unbiased estimates of these observed rates. The investigation is implemented by regressing the CIR rates on the observed rates and then testing the hypothesis that the intercept and the gradient of the regression equation are equal to 0.0 and 1.0 respectively:

$$r_{obs} = \alpha + \beta r_{est} + \epsilon \tag{1}$$

where r_{obs} is the observed 13-week Treasury Note rate and r_{est} is the estimated 13-week CIR rate. The null hypothesis is $\alpha = 0.0$ and $\beta = 1.0$.

In particular, the unbiasedness of the 13-week CIR interest rates[9] is assessed by: (i) regressing observed 13-week Treasury Note rates on day t against 13-week CIR interest rates based upon parameter estimates on day t; and (ii) regressing observed 13-week Treasury Note rates on day t + *one month* against 13-week CIR interest rates calculated for day t + *one month* using parameter values estimated on day t. The first regression is equivalent to within-sample validation as it uses the information embedded in the sample of observations on day t while the second regression is equivalent to out-of-sample validation, in a temporal sense, as it uses the information embedded on day t to predict CIR interest rates on day t +

[9] Assessment of goodness of fit of the CIR model for the entire term structure is provided in Section 5.5.

one month. Results from the regression are presented in Tables 5.4(a) and 5.4(b) where on the basis of the *F*-statistics the null hypothesis that α - 0.0 and $\beta = 1.0$ is accepted. These results indicate that 13-week CIR interest rates are unbiased estimators of 13-week Treasury Note rates. This finding is a significant improvement over the finding in section 5.1.1 where the CIR instantaneous spot rate significantly underestimates the 13-week Treasury Note rate. The unbiasedness of the 13-week CIR rate may be attributed to: (i) the CIR rate being of similar maturity as the Treasury Note rate; and (ii) 13-week Treasury Notes are a popular market-driven short-term zero-coupon instruments at least within the sample period (1985-1992).

Table 5.4(a)

Tests of unbiasedness of daily 13-week CIR rates of interest (1985-1992)

Observed rate (Dependent variable)	Estimated CIR rate (Independent variable)	α	β	F-value	$\bar{R}^2$
13-week TN	13-week	0.97E-12 (0.10E-03)	0.9932 (0.40E-02)	1.40	0.9717

Notes:

(a) CIR rates are calculated for day t, using the parameters estimated on day t.

(b) The null of the F test is $\alpha = 0.0$ and $\beta = 1.0$.

(c) **: Significant at 0.01 level.

(d) Standard errors are in parentheses.

Table 5.4(b)

Tests of unbiasedness of daily 13-week predicted CIR rates of interest (1985-1992)

Observed rate (Dependent variable)	Estimated CIR rate (Independent variable)	α	β	F-value	$\bar{R}^2$
13-week TN	13-week	0.24E-11 (0.14E-03)	0.9994 (0.57E-02)	0.42E-02	0.9354

Notes

(a) Predicted CIR rates are calculated for day t + *one month*, using the parameters estimated on day t.
(b) The null of the F test is $\alpha = 0.0$ and $\beta = 1.0$.
(c) **: Significant at 0.01 level.
(d) Standard errors are in parentheses.

5.1.3 Comparative analysis of the performance (predictiveness) of the CIR model

While the data cross-sections include bonds that vary from day to day with respect to coupon rate and maturity they all contain 13-week and 26-week Treasury Notes. This unique feature is exploited to examine how the CIR model performs relatively to a naive model of the term structure which assumes that interest rates remain constant[10] over time. This is a direct test of the predictiveness of two CIR bond prices with respect to their observed prices. Essentially the question we seek to answer is " Will the CIR model estimated on day t and then used to predict the prices of 13-week and 26-week Treasury Notes on day $t + n$ months outperform the naive model ?". The value of n is chosen to be *one month* and *three months*. While the choice is arbitrary, prediction in this time frame should give a reasonable indication of the robustness of the CIR model in view of the daily fluctuations in interest rates in the market place.

The criterion used to measure performance is the pricing error defined as the difference between the observed price and the estimated price.

For the naive model we have:

$$PE_{naive} = P_{obs,t+n} - P_{obs,t} \tag{2}$$

where PE_{naive} is the pricing error under the naive model, $P_{obs,t}$ the observed price at time t, $P_{obs,t+n}$ the observed price at time $t+n$, and n is either 1 or 3 months. In other words, on day t $P_{obs,t}$ is assumed to prevail at time $t+n$ (under a constant interest regime) and is then compared with $P_{obs,t+n}$ which actually prevails at

[10] This is a 'do nothing and cost-free' model. Any constructed model should generally beat it to justify costs of prediction.

time $t+n$.

For the CIR model we have

$$PE_{cir} = P_{obs,t+n} - P_{cir,t,t+n} \tag{3}$$

where PE_{cir} is the pricing error under the CIR model and $P_{cir,t,t+n}$ is the CIR price predicted at time t to prevail at time $t+n$.

The results presented in Tables 5.5(a) and 5.5(b) indicate that over the entire sample period :(i) the mean errors of the CIR model are significantly smaller than those of the naive model. For example, the naive model and CIR model generate, on average, errors of $0.21 and $0.10 per $100 Treasury Note respectively (see Table 5.5(a), third row); and (ii) on the basis of the *F*-statistics[11] the null hypothesis that the mean squared errors of the naive model are smaller than those of the CIR model is decisively rejected. Thus, overall the CIR model outperforms the naive model.

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[11] It should be noted that in these two tables the ratio of two *MSEs* is *F*-distributed with degrees of freedom equal to the number of observations used in the calculation of the *MSEs* (see Johnson and Kotz (1970, Chapter 26).

Table 5.5(a)

Comparative mean dollar pricing errors of $100 13-week Treasury Notes

	One-month prediction		Three-month prediction	
	Naive model	CIR model	Naive model	CIR model
	\$0.21	-\$0.10	\$0.39	-0.14
t-value of $\overline{PE}$	29.48	-9.59	33.32	-15.24
Hypotheses	$H_0 : MSE_{naive} < MSE_{cir}$ $H_1 : MSE_{naive} \geq MSE_{cir}$		$H_0 : MSE_{naive} < MSE_{cir}$ $H_1 : MSE_{naive} \geq MSE_{cir}$	
F-value	4.41*		7.76*	

Note: * Significant at 1%.

Table 5.5(b)

Comparative mean dollar pricing errors of $100 26-week Treasury Notes

	One-month prediction		Three-month prediction	
	Naive model	CIR model	Naive model	CIR model
	\$0.35	-\$0.12	\$0.30	-\$0.16
t-value of $\overline{PE}$	28.97	-8.30	34.31	-20.27
Hypotheses	$H_0 = MSE_{naive} < MSE_{cir}$ $H_1 : MSE_{naive} \geq MSE_{cir}$		$H_0 : MSE_{naive} < MSE_{cir}$ $H_1 : MSE_{naive} \geq MSE_{cir}$	
F-value	15.12*		4.59*	

Note: * Significant at 1%.

5.1.4 Comparison with previous studies

In this section estimates of the spot rate are compared with those reported by previous studies, in particular Brown and Dybvig (1986), Chiarella, Lo and Pham (1989), Barone et al (1991) and Munik and Schotman (1994) (see Table 5.5(c)). Our finding that the spot rate underestimates the cash rate is inconsistent with Brown and Dybvig (1986) who used monthly U.S. Treasury Bill data to estimate parameters of the CIR. They found the CIR model systematically overestimating the implied short rate of return. However, their 'observed' rate is the <u>mean</u> yield on US Treasury Bills with at most 14 days to maturity while our observed rate is the single <u>daily</u> cash rate. The cash rate is a more appropriate proxy for the instantaneous spot rate in terms of maturity. Furthermore the cash rate is observed while the mean yield is simply an average of observed yields. Munnik and Schotman (1994) reported that the CIR spot rate is very close to the observed one month Amsterdam Interbank Offered Rate although no statistical evidence is provided. Similarly Barone et al (1991) found the estimated spot rate highly correlated with three-month Treasury Bills in Italy. The issue of overestimation or underestimation, however, is not investigated by Barone et al and Munnik and Schotman (1994). Chiarella, Lo and Pham (1989) found that the CIR spot rate <u>significantly underestimates</u> the Australian 13-week Treasury Note rate for three years (1978, 1979, 1984) out of the ten-year period (Jan/1977-Dec/1987). For the three years which overlap with the sample set of this study (1985, 1986, 1987) Chiarella, Lo and Pham (1989) found overestimation which is inconsistent with our results. This inconsistency is most likely caused by the flaw in Chiarella, Lo and

Pham's estimation procedure[12] which was noted in Chapter 2 of this thesis.

In sum, the mixed results of the CIR spot rate over-estimating or under-estimating an observed risk-free rate reported by this study and others may be attributed to the following factors: (i) different data sets; and (ii) choice of an (observed) proxy for the spot rate. In terms of the latter factor our choice of the cash rate is perhaps more appropriate owing to its shorter maturity. It is interesting to note that where the mismatch of maturity is rectified, the 13-week CIR rate is an unbiased estimator of the 13-week Treasury Note rate. As the issue of unbiasedness has not been considered in previous studies no comparison with previous research is possible.

[12] These models are reviewed in detail in Chapter 2.

Table 5.5(c)

Comparison of estimates of CIR spot rates with observed rates

Study	Market	Observed rate	Results
Brown & Dibvig (1986)	USA (1977-83) Monthly data	Mean yield on T-Bills with at most 14 days to maturity	CIR *r* overestimates observed rate
Munnik & Schotman (1994)	Netherlands (1989-90) Monthly data	Amsterdam Interbank Offered Rate	CIR *r* is close to observed rate
Baron et al (1991)	Italy (1983-90) Daily data	1 month, 3 month interbank rate and 3 month T-Bill rate	CIR *r* is highly correlated with observed rates
Chiarella et al (1989)	Australia (1978-87) Monthly data	13-week T-Note rate	CIR *r* overestimates observed rate
This chapter	Australia (1985-92) Daily data	Overnight cash rate	CIR *r* underestimates observed rate

5.2 ESTIMATES OF VOLATILITY

In this section estimates of the volatility of the CIR stochastic interest rate, $\sigma\sqrt{(r)}$ are presented and discussed. As r is always significant and positive, attention will be concentrated on the parameter of volatility, namely σ.

5.2.1 Distributional statistics and graphs

Basic statistics of estimates of σ are presented in Tables 5.6(a), 5.6(b), 5.6(c) and 5.6(d) while Figs. 5.2(a), 5.2(b), 5.2(c), and 5.2(d) graph the time series of these estimates. While daily estimates of σ appear smaller than quarterly and semi-annual estimates, three pairwise tests of the null hypothesis that their means are equal indicate that it cannot be rejected at 1% level of significance (see Table 5.6(c)). Despite this, it is problematic to conclude that σ is stable over time, considering some significant outliers of σ displayed in Figures 5.2(a-d) which are due to: (i) the highly nonlinear nature of the CIR pricing formula[13] which allows for substantial variations among its variables while the functional value remains stable under the minimisation process; and (ii) multicollinearity among the explanatory variables[14]. Multicollinearity is indeed the case as shown by Table 5.6(e) where the correlations among the parameters are documented. The spot rate, however, is least correlated with the remaining parameters (see Table 5.6(e), second column) and hence is least unstable (see Section 5.1). Parameter instability in the CIR model has also been noted by Brown and Schaefer (1994) and Singh

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[13] Apparently the nonlinearity of the CIR formula and the multicollinearity have been a major source of instability in parameter estimates. Brown and Dybvig (1986) refer to zero and negative estimates of σ which they excluded from their analysis (see Brown and Dybvig (1986, Table I, note *a*, p. 622)). See also Singh (1995) and Chan et al (1992).

[14] See Johnston (1984, p.240).

(1995), which is inconsistent with the model's specification of constant parameters.

For each daily, quarterly and semi-annual cross-section a t test is conducted to assess the significance of σ and consequently the proportions of the cross-sections in which σ is significant are reported in Table 5.6(d). For example, at 1% level of significance 5.98% of the series of daily estimates of σ are statistically different from zero. By constraining σ to be fixed over quarterly and semi-annual estimation the degree of freedom in estimation is substantially increased[15] and hence it is more likely to be significant than under daily estimation. For example, 62.5% and 75% of the quarterly and semi-annual estimates of σ are significant compared to 5.98% of the daily estimate series (see Table 5.6(d), third row). It is important to distinguish between the significance of cross-sectional σ and time-series σ. The former shows whether it is a significant factor in the pricing of a bond at a given time point while the latter shows how volatile the volatility factor is over time. Thus, the significance of time-series σ does not impact on the pricing of bonds.

Generally while the CIR model is silent on the time frame, its volatility parameter is more significant over longer intervals of estimation.

[15] With daily estimation the number of parameters to estimate is four (r, σ, $\kappa\theta$, $\kappa+\lambda$) for a sample of, say, 20 bonds while with quarterly estimation (66 trading days) the number of parameters consists of 66 spot rates and three parameters for a sample of approximately 1320 bonds (= 66 x 20).

Table 5.6(a)

Time series of daily estimates of σ[16]

	Mean	Std	Min	Max
1985	0.05220	0.14362	1.0E-08	1.06329
1986	0.13356	0.31177	1.0E-08	1.07633
1987	0.06107	0.07912	1.0E-08	1.04764
1988	0.03459	0.04312	1.0E-08	0.18008
1989	0.04283	0.19337	2.0E-08	1.01761
1990	0.05551	0.13484	6.7E-04	0.96867
1991	0.01830	0.05247	0.08114	0.16553
1992	0.12803	0.14106	0.10336	0.39471
1985-1992	0.06546	0.16434	0.01033	1.07633

[16] It is important to distinguish between the significance of cross-sectional σ and time-series σ. The former shows whether it is a significant factor in the pricing of a bond at a given time point while the latter shows how volatile the volatility factor is over time. Thus, the significance of time-series σ does not impact on the pricing of bonds.

Table 5.6(b)

Time series of daily estimates of CIR volatility, $\sigma \sqrt{r}$ volatility)

	Mean	Std	Min	Max
1985	0.0180	0.0541	3.50E-09	0.4510
1986	0.0184	0.0540	3.49E-09	0.4500
1987	0.0556	0.1309	3.49E-09	0.4510
1988	0.0207	0.0305	3.64E-09	0.4240
1989	0.0117	0.0140	3.24E-09	0.0660
1990	0.0128	0.0680	3.80E-09	0.4140
1991	0.0251	0.0670	3.70E-09	0.3950
1992	0.0063	0.0160	0.0304	0.0519
1985-1992	0.0224	0.0644	0.0304	0.4516

Table 5.6(c)

Distribution of estimates of σ
(1985-92)

		σ		
	t-value	DAILY	QUARTERLY	SEMIANNUAL
Mean		6.5469E-02	1.2688E-01	1.2244E-01
Std		1.6434E-01	2.3541E-01	2.4052E-01
Minimum		1.0336E-01	6.4598E-03	0.0
Maximum		1.0763	9.8731E-01	9.9484E-01
H_0 (d, q)	1.47			
H_0 (d, sa)	0.98			
H_0 (q, sa)	0.03			

Notes:

(a) H_0 (d, q) is the null hypothesis that the mean of daily estimates of σ is equal to the mean of the quarterly estimates;

(b) H_0 (d, sa) is the null hypothesis that the mean of daily estimates of σ is equal to the mean of the semi-annual estimates;

(c) H_0 (q, sa) is the null hypothesis that the mean of quarterly estimates of σ is equal to the mean of the semi-annual estimates.

Table 5.6(d)

Proportions of estimates of σ being significant over the sample period

	Daily estimates	Quarterly estimates	Semi-annual estimates
Significance level	Sample Proportion	Sample Proportion	Sample proportion
1%	0.0598	0.6250	0.7500
5%	0.1322	0.6250	0.7500
10%	0.2173	0.6875	0.7500

Note: The entry 0.6250 (third row, third column) means that at 1% level of significance 62.5 % of the 32 quarterly estimates of σ are different from zero. Other entries are interpreted similarly.

Table 5.6(e)

Correlation matrix of r, σ, $\kappa+\lambda$ and $\kappa\theta$

	r	σ	$\kappa\theta$	$\kappa+\lambda$
r	1.000000			
σ	0.018264	1.0000		
$\kappa\theta$	0.290040	0.67657	1.0000	
$\kappa+\lambda$	0.341350	0.56363	0.98547	1.00000

Fig. 5.2(a)

Daily estimates of σ

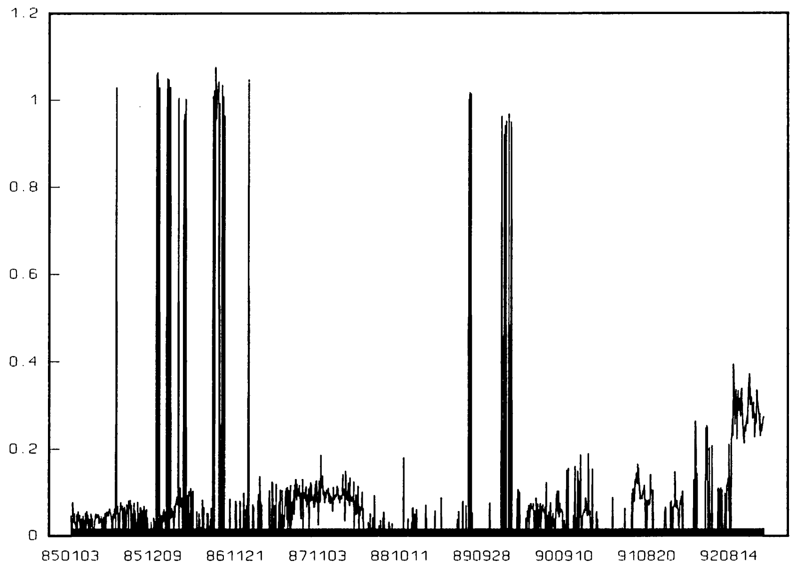


Figure 5.2(b)

Time series distribution of daily estimates of $\sigma\sqrt{r}$

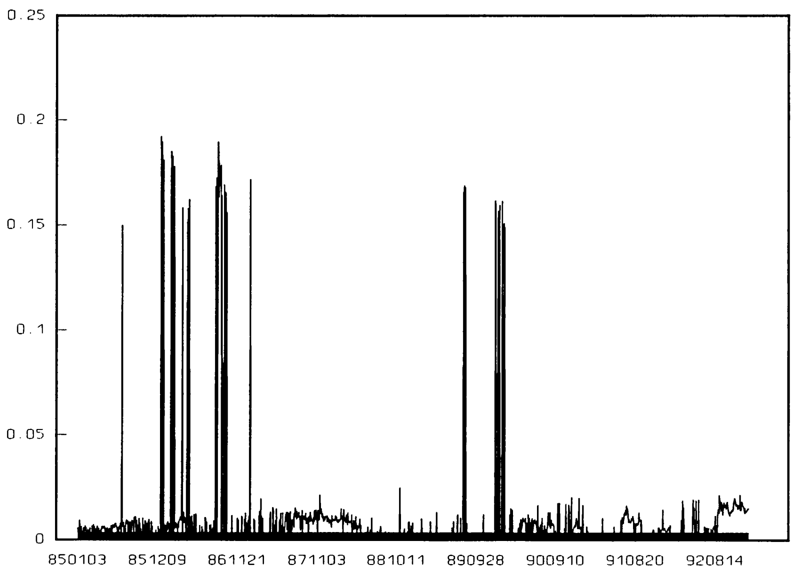


Fig. 5.2(c)

Quarterly estimates of CIR σ

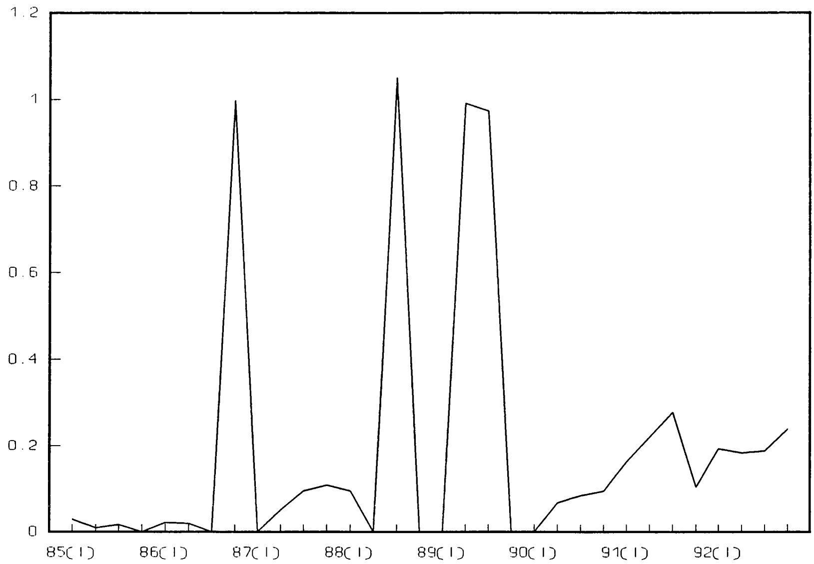
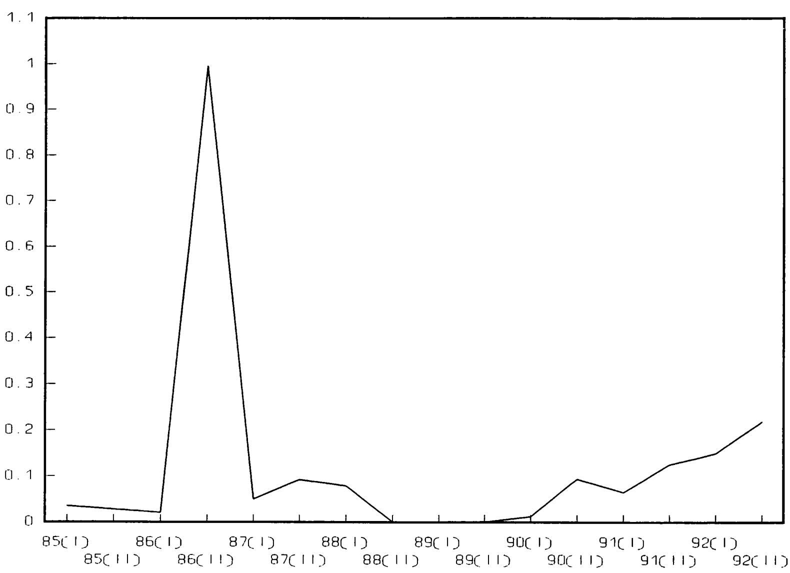


Fig. 5.2(d)

Semi-annual estimates of σ



5.2.2 CIR volatility and time series volatility of estimates of 90-day TN rates

To investigate further the CIR spot rate volatility we examine its relationship with the 13-week Treasury Note series[17]. We choose 13-week Treasury Notes because of their complete series (1985-1992) and relatively short maturity. This exercise serves two purposes: (i) if the CIR volatility bears a close relationship with Treasury Note volatility then its evolution may be explained in terms of the volatility of Treasury Notes; and (ii) to the extent that 13-week Treasury Notes may be considered proxies for CIR spot rates then it is a natural step to examine their respective volatilities.

Following Brown and Dybvig (1986) the volatility of changes in the Treasury Note rate is measured by their monthly standard deviations while the daily estimates of CIR volatility are averaged over the corresponding month[18]. Results are presented and graphed in Table 5.7(a) and Figs. 5.3(a) and 5.3(b) respectively. While on average there is no significant difference between the means of the two volatility estimates on the basis of the *t*-statistic in Table 5.7(a), the evolutions of the two series display little resemblance (see Fig. 5.3(a) and Fig. 5.3(b)). Thus, to formally investigate whether the CIR volatility is an unbiased predictor of the Treasury Note volatility the following OLS regression is estimated:

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[17] Other series (5-week Treasury Notes, cash rate, 26-week Treasury Notes) are also used. Results are largely similar.

[18] A moving monthly window approach was tried whereby a subsequent window is formed by discarding the first observation and adding an additional observation in the series. However, this approach introduces serial correlation for the regression and hence was not adopted.

$$\log \hat{s}_t = \alpha + \beta \log \hat{s}_{cir} + \epsilon \tag{4}$$

where $\log \hat{s}_t$ and $\log \hat{s}_{cir}$ are the logarithms of the volatilities of the TN and CIR interest rates respectively. The logarithmic form is used to circumvent the problem that the dependent variable, namely standard deviation, cannot take negative values. The null hypothesis is that α and β are equal to 0.0 and 1.0 respectively. On the basis of the F test, the results of the OLS estimation in Table 5.7(b) decisively reject the hypothesis that CIR volatility is an unbiased estimator of Treasury Note volatility.

Table 5.7(a)

Distribution of monthly average of CIR volatility ($\sigma\sqrt{r}$) and monthly average standard deviation of change in 13-week TN rate

	t-value	Monthly average of $\sigma\sqrt{r}$ ($\hat{s}_{cir}$)	Monthly average of standard deviation of change in 13-week TN rate ($\hat{s}_t$)
Mean		0.0230	0.0226
Std		0.0354	0.0190
Min		7.26E-06	2.10E-03
Max		0.2754	0.1207
H_0	0.1026		

Note:
H_0 is the null that the mean of monthly averages of CIR volatility is equal to the mean of monthly averages of standard deviation of change in 13-week TN rate.

Table 5.7(b)

Regression of the volatility of the change in TN rate on volatility of CIR

Dependent variable	Independent variable	α	β	F-value	$\overline{R}^2$
$\log \hat{s}_t$	$\log \hat{s}_{cir}$	-4.09 (0.23)	0.1344E-02 (0.036)	485.75	0.0138

Notes:
(a) Standard errors are in parentheses;
(b) The null of the *F*-test is that $\alpha = 0.0$ and $\beta = 1.0$.

Figure 5.3(a)

Monthly CIR Volatility

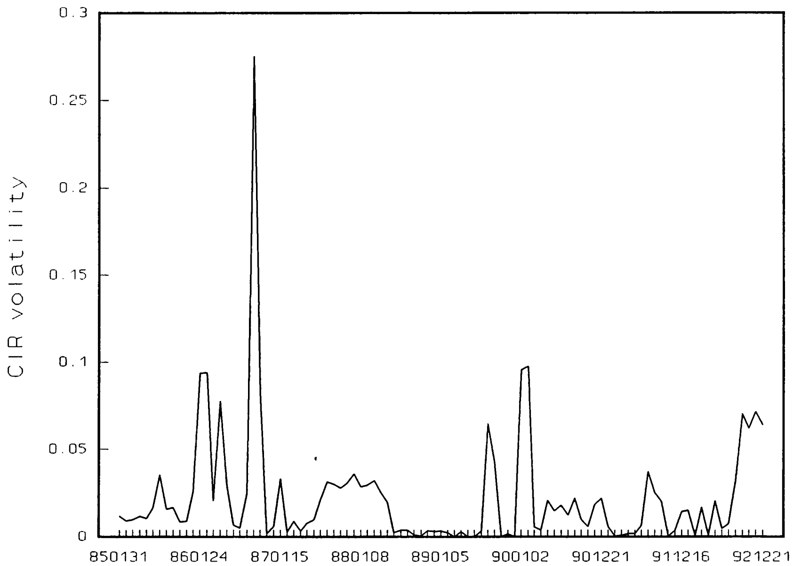
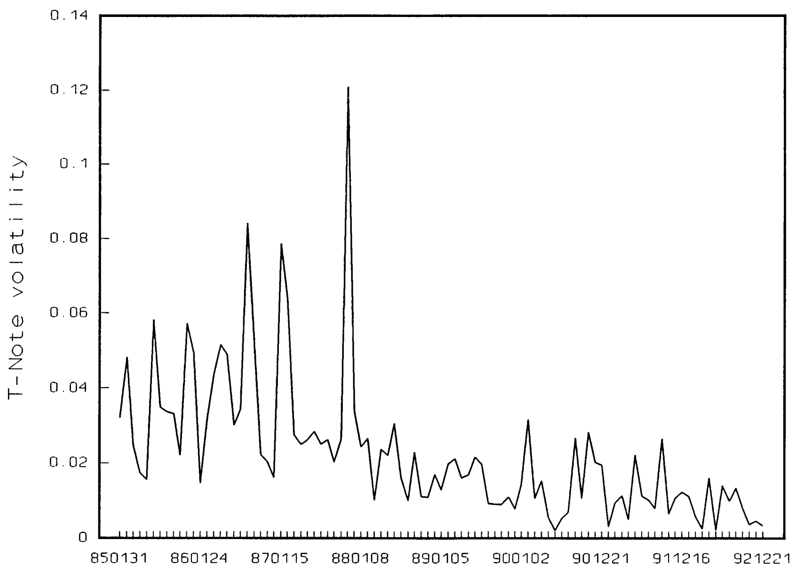


Figure 5.3(b)

Monthly Treasury Note Volatility



5.2.3 Comparison with previous studies

The results of CIR volatility estimates and time series volatility estimates in this study and previous studies are presented in Table 5.7(c). It is remarkable that despite differences in data, sample periods and markets, volatility estimates in these studies are remarkably close. Moreover, with the exception of Brown and Dybvig (1986), the hypothesis that the CIR volatility is an unbiased estimator[19] of the time series estimates is not supported. The inconsistency may be attributed to (i) Brown and Dybvig's OLS estimation which uses the standard deviation of time series estimates as dependent variable[20]; (ii) their ad-hoc procedure of averaging daily estimates of CIR volatility to calculate monthly estimate of volatility; and (iii) the two series of estimates do not share a common maturity with CIR short rate being instantaneous while Treasury-Note rate being of 13-week duration.

It is significant to note that while the means of daily, quarterly and semi-annual estimates of σ are not significantly different as shown by the t-values in Table 5.6(c), quarterly and semi-annual estimates of σ are more likely to be significant than daily estimates (see Table 5.6(d)). This is due to the increase in the degrees of freedom of quarterly and semi-annual estimation (see Section 5.2).

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[19] They report an intercept not significantly different from zero and a slope not significantly different from unity and an R^2 of 0.61.

[20] OLS assumes that the dependent variable is normally distributed, hence capable of taking negative values while the standard deviation is non-negative.

Table 5.7(c)

Comparison of CIR volatility and time series estimates

Study	Market	Mea n $\sigma\sqrt{r}$ (%)	Mean Std (%)	Is CIR volatility unbiased predictor of time series estimates ?
Brown & Dibvig (1986)	USA (1977-83) Monthly data	1.95[a]	2.26[a]	Yes
Brown & Schaefer (1994)	UK (1984-89) Daily data	2.65[b]	2.33[b]	No
Baron et al (1991)	Italy (1983-90) Daily data	2.05[c]	2.797[c]	Not reported
Chiarella et al (1989)	Australia (1978-87) Monthly data	2.95[d]	3.16[d]	No
This chapter	Australia (1985-92) Daily data	2.30	3.54	No

Sources:
[a] Calculated from Brown & Dibvig (1986, Table I);
[b] Brown & Schaefer (1994, p. 28);
[c] Barone et al (1991, Tables 3 and 5);
[d] Calculated from Chiarella et al (1989, Tables 1 and 3).

5.3 ESTIMATES OF THE LONG RATE, $R_{cir}(\infty)$

Estimates of the CIR long-term yield are presented and discussed in this section.

5.3.1 Distributional statistics and graphs

Distributional statistics of estimates of the long-term yield, $R_{cir}(\infty)$, are presented in Tables 5.8(a) and 5.8(b) while the time series of the daily, quarterly and semi-annual estimates are graphed in Figs. 5.4(a), 5.4(b) and 5.4(c). The means of the estimates across the three modes of estimation show a long term yield in the region of 7% to 9 % (see Table 5.8(b), second row). In fact the mean in each year and over the whole sample period (1985-92) is not significantly different from 0.075[21]. Given the fluctuations of parameter estimates it is remarkable that the long term yield remains within a fairly tight band. The relatively high standard errors and hence low *t*-statistics of $R_{cir}(\infty)$ (see Tables 5.8(a) and 5.8(b)) suggest that the parameter estimates, σ, $\kappa+\lambda$, $\kappa+\theta$ on which $R_{cir}(\infty)$ depends, are mutually correlated[22]. This is indeed the case from Table 5.6(e). Further, as shown by their higher standard errors (see Table 5.8(b)), daily estimates fluctuate substantially more than quarterly and semi-annual estimates. This may be attributed to the larger degree of freedom in quarterly and semi-annual estimation[23]. Except for the means of daily and quarterly estimates, the *t*-statistics in Table 5.8(b) indicate that there is no significant difference in the means of daily and semi-annual

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[21] The *t*-statistic for this null hypothesis is 0.06.

[22] See Gujarati (1988), pp. 292-293 or Johnston (1984), p.240.

[23] As explained in Section 5.2, the degrees of freedom increase substantially when four parameters (r, σ, $\kappa\theta$, $\kappa+\lambda$) are to be estimated from a sample of bonds pooled over three months or six months.

estimates; and the means of quarterly and semi-annual estimates. While these findings suggest that estimates of the long rate are sensitive to time interval, and hence cannot be a constant, evidence seems to suggest that they vary within a band of 7%-9% where the fluctuations may be considered stochastic errors.

Table 5.8(a)

CIR long-term yield, $R_{cir}(\infty)$

Year	Mean	Std	Min	Max
1985	0.056338	0.086101	6.8E-07	0.68889
1986	0.083598	0.062342	1.2E-07	0.31656
1987	0.068680	0.061202	3.0E-08	0.46254
1988	0.069805	0.075848	0.0	0.89806
1989	0.106550	0.025904	3.3E-06	0.12641
1990	0.063262	0.072713	1.0E-08	0.45227
1991	0.089594	0.068503	7.5E-06	0.70372
1992	0.094120	0.031904	3.3E-04	0.14491
1985-92	0.078911	0.065720	0.0	0.89806

Table 5.8(b)

Distribution of CIR long-term yield, $R_{cir}(\infty)$

	t-value	DAILY	QUARTERLY	SEMIANNUAL
Mean		7.891E-02	8.932E-02	6.860E-02
Std		6.572E-02	1.354E-02	4.431E-02
Min		0.0	4.304E-02	1.030E-04
Max		8.980E-01	1.147E-01	1.299E-01
$H_0(d,q)$	3.69^{**}			
$H_0(q,sa)$	1.83^{*}			
$H_0(d,sa)$	0.92			

Notes:
(a) $H_0(d,q)$ is the null hypothesis that the mean of the daily estimates is equal to the mean of the quarterly estimates;
(b) $H_0(q,sa)$ is the null hypothesis that the mean of the quarterly estimates is equal to the mean of the semi-annual estimates;
(c) $H_0(d,sa)$ is the null hypothesis that the mean of the daily estimates is equal to the mean of the semi-annual estimates;
(d) *: significant at 5%; **: significant at 1%

Figure 5.4(a)

Daily estimates of CIR long-term yield, $R_{cir}(\infty)$

Daily long-term yield

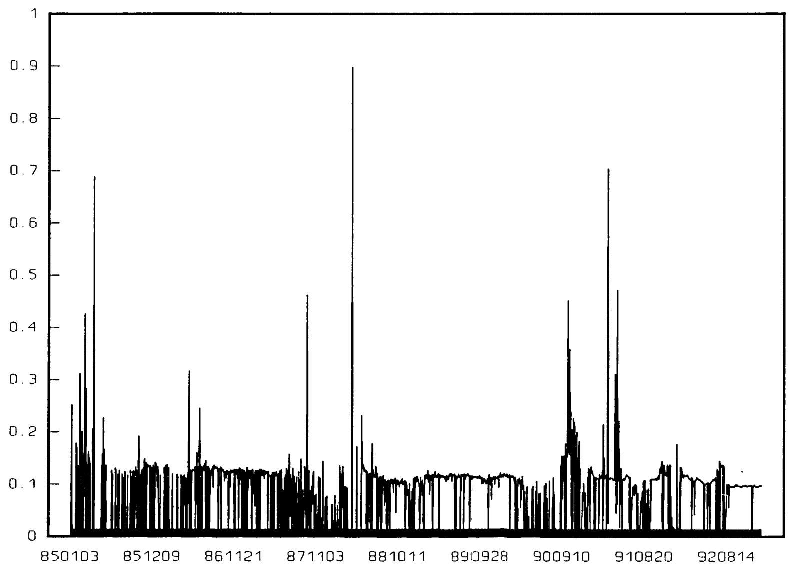

Fig. 5.4(b)

Quarterly estimates of $R_{cir}(\infty)$

Quarterly long-term yield

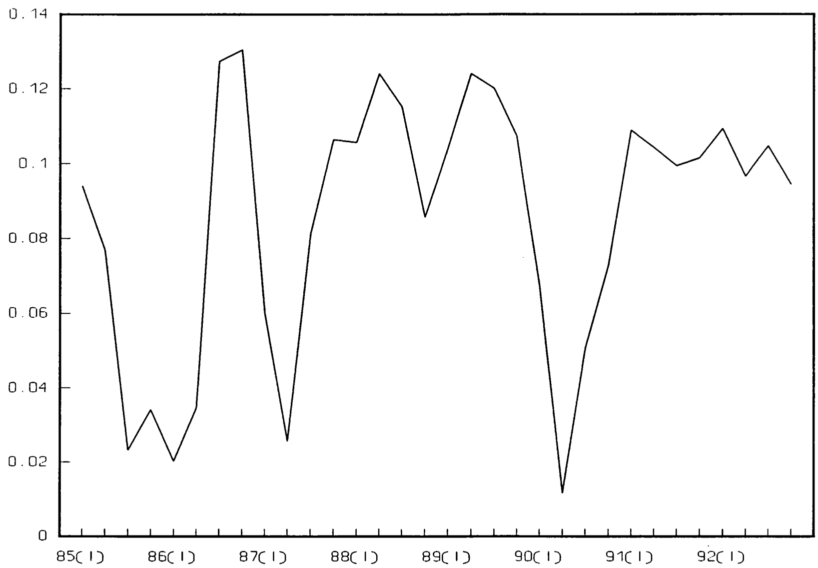

Fig. 5.4(c)

Semi-annual estimates of $R_{cir}(\infty)$

Semi-annual long-term yield

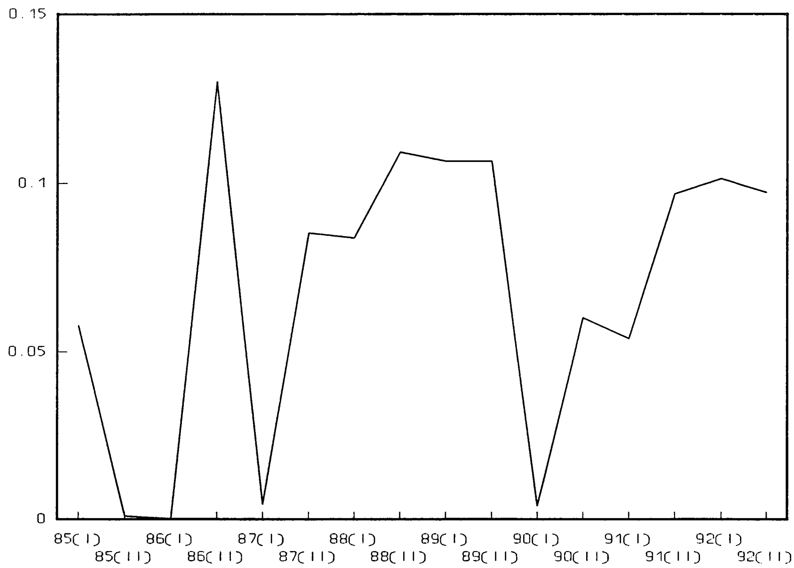

Table 5.8(c)

Comparison of estimates of CIR long term yield, $R_{cir}(\infty)$

Study	Market	Mean $R(\infty)$ (%)	Stability
Brown & Dibvig (1986)	USA (1977-83) Monthly data	10.31[a]	Not reported
Brown & Schaefer (1994)	UK (1984-89) Daily data	2.50[b]	Yes
Barone et al (1991)	Italy (1983-90) Daily data	11.56[c]	Not reported
Chiarella et al (1989)	Australia (1978-87) Monthly data	12.04[d]	Not reported
This study	Australia (1985-92) Daily data	7.89	Yes

Sources:

[a] Brown & Dibvig (1986, Table I);
[b] Brown & Schaefer (1994, Table 6, p. 23);
[c] Barone et al (1991, Table 3);
[d] Chiarella et al (1989, Table 2).

5.3.2 Comparison with previous studies

While our numerical estimates of the long term yield differ from previous studies (see Table 5.8(c)) it is not possible to judge whether ours are more or less reasonable for the following reasons: (i) the long term yield being the internal rate of return of a discount bond of infinite maturity is not observed anywhere; and (ii) institutional differences exist among markets. It is, however, significant that the daily long-term yield in this study displays remarkable stability (see Fig. 5.5(a)) which is in agreement with Brown and Schaefer (1994). No other previous studies have explored this issue.

5.4 ESTIMATES OF $\kappa\theta$ AND $\kappa+\lambda$

Estimates of the speed of adjustment, κ, the equilibrium spot rate, θ, and the market price of risk, λ, are presented and discussed in this section. As it is not possible to estimate these separately, results are presented in the form of $\kappa\theta$ and $\kappa+\lambda$.

5.4.1 Distributional statistics, *t*-statistics and graphs

Distributional statistics, *t*-statistics and graphs of $\kappa\theta$ and $\kappa+\lambda$ are presented in Tables 5.9(a)-5.9(d) and Figs. 5.5(a)-5.5(f) respectively. The CIR model assumes that these quantities are constant while their signs[24] are positive for $\kappa\theta$ and positive/negative for $\kappa+\lambda$. At first sight the evidence seems to indicate that they vary according to the interval over which the estimation is made. In particular, quarterly and semi-annual estimates display more extreme observations than daily

[24] See Chapter 3.

estimates (see Figs. 5.5(a)-5.5(f)). As the total numbers of quarterly and semi-annual estimates are 32 and 16 respectively, elimination of outliers means a large number of observations are discarded and hence this is not attempted. However, allowing for estimation errors it can be argued that the means of daily estimates of $\kappa\theta$ in Table 5.9(a) and $\kappa+\lambda$ in Table 5.9(b) are not significantly different[25] from 0.02 in all years (see Table 5.9(a), second column). Similarly we can find other values[26] from which quarterly and semi-annual estimates do not significantly deviate. Thus while estimates of $\kappa\theta$ and $\kappa+\lambda$ are sensitive to the sample period (daily, quarterly, semi-annual) they converge to some constants with stochastic errors. Furthermore, the signs of the estimates are consistent with the prescriptions of the model, namely $\kappa\theta$ is invariably positive (see Table 5.9(a)) while $\kappa+\lambda$ is, on average, positive but also negative in some cross sections (see Table 5.9(b) and Fig.5.5(d)). Hence, overall the estimates can be said to be in broad agreement with the model. The results also contain some extreme estimates which are due to the nonlinear nature of the CIR formula. Furthermore, the predominant role of the spot rate[27] in the formula allows for substantial variations of other parameters with little impact on bond price. On each cross-section (daily, quarterly, semi-annual) the statistical significance of $\kappa\theta$ and $\kappa+\lambda$ is tested and the proportions of significant

[25] For example, the null hypothesis that the population mean of $\kappa\theta$ is 0.02 while the best estimator of the population mean is 0.023288 and the standard deviation is 0.051199 (see Table 5.9(a), last row), then the t-value for this null hypothesis is (0.023288 - 0.02)/0.051199 = 0.06. Similarly the t-statistic for $\kappa + \lambda$ is 0.38.

[26] The t-values for the null hypotheses that the means of the quarterly and semi-annual estimates are 0.10 and 0.017 are 1.91 and 0.047 respectively.

[27] Estimates of the spot rate are always positive and closely track some observed rates such as the cash rate and 13-week Treasury Note rate (see section 5.1).

estimates out of the total cross-sections at various levels of significance are reported in Table 5.9(d). For example, at 1% level of significance, $\kappa\theta$ is significant for 24.11 % of the whole number of cross-sections under daily estimation. The proportions of significant cross-sections under quarterly and semi-annual estimation are similarly interpreted. Again the low t-statistics indicate that these parameters are highly correlated[28], which is indeed the case (see Table 5.6(e)). The implication is that while these quantities are important from a theoretical point of view, in practical terms they make little impact on bond price in comparison with the spot rate which plays a dominant role.

In summary, the CIR model assumes that $\kappa\theta$ and $\kappa+\lambda$ are constant and independent of time. While literally this assumption cannot be supported, the evidence[29] (see Table 5.9(c)), on balance, suggests that, under the pairwise t-tests, (i) the means of $\kappa+\lambda$ are equal under the three methods of estimation; and (ii) the means of $\kappa\theta$ are equal under daily and semi-annual estimations only. Overall, these parameters are in <u>broad</u> agreement with model specifications.

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[28] See Gujarati (1988), pp. 292-293.

[29] In the statistical sense that they differ from a certain constant with random errors.

Table 5.9(a)

Daily estimates of $\kappa\theta$

Year	Mean	Std	Min	Max
1985	0.013443	0.065231	2.00000E-08	0.85933
1986	0.048772	0.10673	1.00000E-08	0.83930
1987	0.011239	0.026897	0.00000	0.36399
1988	0.0083500	0.010532	0.00000	0.046412
1989	0.049593	0.033352	2.00000E-07	0.17931
1990	0.013934	0.031582	0.00000	0.18537
1991	0.016250	0.015341	3.40000E-07	0.053836
1992	0.024820	0.012981	0.000010880	0.053320
1985-92	0.023288	0.051199	0.0	0.085933

Table 5.9(b)

Daily estimates of $\kappa+\lambda$

Year	Mean	Std	Min	Max
1985	0.087311	0.48629	-0.040645	4.01108
1986	0.34866	0.76458	-0.070601	3.92082
1987	0.058310	0.20828	-0.116950	2.54957
1988	0.056166	0.11498	-0.124170	0.41071
1989	0.41865	0.23305	0.010477	1.20373
1990	0.10597	0.24310	-0.126050	1.26975
1991	0.12450	0.15550	-0.093239	0.49334
1992	0.14612	0.17617	-0.162780	0.49417
1985-92	0.16827	0.38406	-0.16278	4.01108

Table 5.9(c)

Distribution of estimates of $\kappa\theta$ and $\kappa+\lambda$
(1985-92)

$\kappa\theta$				
	t-value	DAILY	QUARTERLY	SEMIANNUAL
Mean		2.3288E-02	1.779327E-01	1.920465E-02
Std		5.1199E-02	4.06009E-02	4.281162E-02
Minimum		0.0	0.0	0.0
Maximum		8.5933E-01	2.0282318E-01	1.7558785E-01
$H_0(d,q)$	21.27^{**}			
$H_0(d,sa)$	0.38			
$H_0(q,sa)$	12.32^{**}			
$\kappa+\lambda$				
	t-value	DAILY	QUARTERLY	SEMI-ANNUAL
Mean		1.6827E-01	9.767318E-02	1.1238595E-01
Std		3.8406E-01	2.4048834E-01	2.5114796E-01
Maximum		4.01108	1.45084023	9.8514925E-01
Minimum		-1.6278E-01	0.0	0.0
$H^0(d,q)$	1.62			
$H^0(d,sa)$	0.88			
$H^0(q,sa)$	0.19			

Notes:
$H_0(d,q)$ is the null hypothesis that the mean of the daily estimates is equal to the mean of the quarterly estimates;
$H_0(q,sa)$ is the null hypothesis that the mean of the quarterly estimates is equal to the mean of the semi-annual estimates;
$H_0(d,sa)$ is the null hypothesis that the mean of the daily estimates is equal to the mean of the semi-annual estimates;
*: significant at 5%
**: significant at 1%

Table 5.9(d)

Proportions of significant $\kappa\theta$ and $\kappa+\lambda$

		DAILY	QUARTERLY	SEMI-ANNUAL
	Significance level	Sample Proportion	Sample Proportion	Sample proportion
$\kappa\theta$	1%	0.2411	0.6250	0.6875
	5%	0.5167	0.6562	0.7500
	10%	0.8318	0.6875	0.8125
$\kappa+\lambda$	1%	0.2734	0.5625	0.4375
	5%	0.5796	0.5938	0.3750
	10%	0.9186	0.6250	0.5000

Note:
The entry 0.6250 (third row, fourth column) means that at 1% level of significance 62.5 % of the 32 quarterly estimates of $\kappa\theta$ are different from zero. Other entries are interpreted similarly.

Table 5.9(e)

Comparison of estimates of $\kappa\theta$ and $\kappa+\lambda$

	Market	$\kappa\theta$	$\kappa+\lambda$
Brown & Schaefer (1994)	UK (1984-89) Daily data	1.99E-03	-5.7E-02
This chapter	Australia (1985-92) Daily data	2.3288E-02	1.6827E-01

Fig. 5.5(a)

Daily estimates of $\kappa\theta$

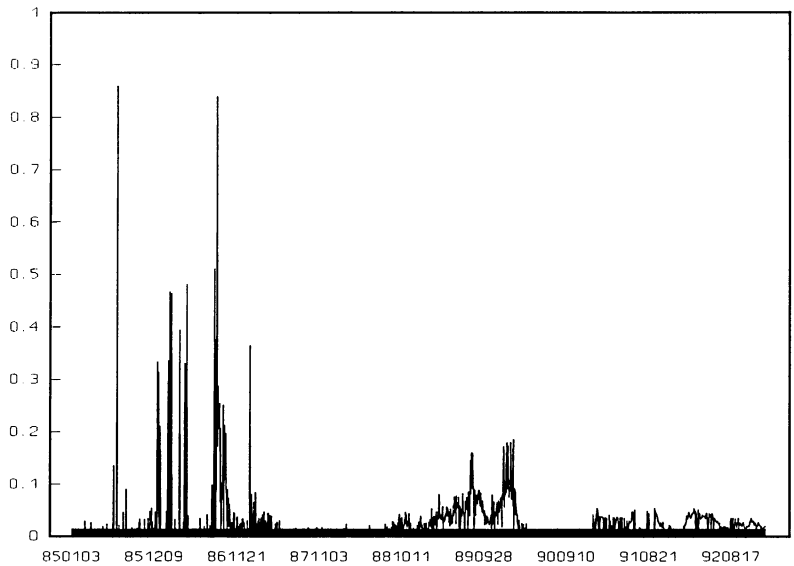


Fig. 5.5(b)

Quarterly estimates of $\kappa\theta$

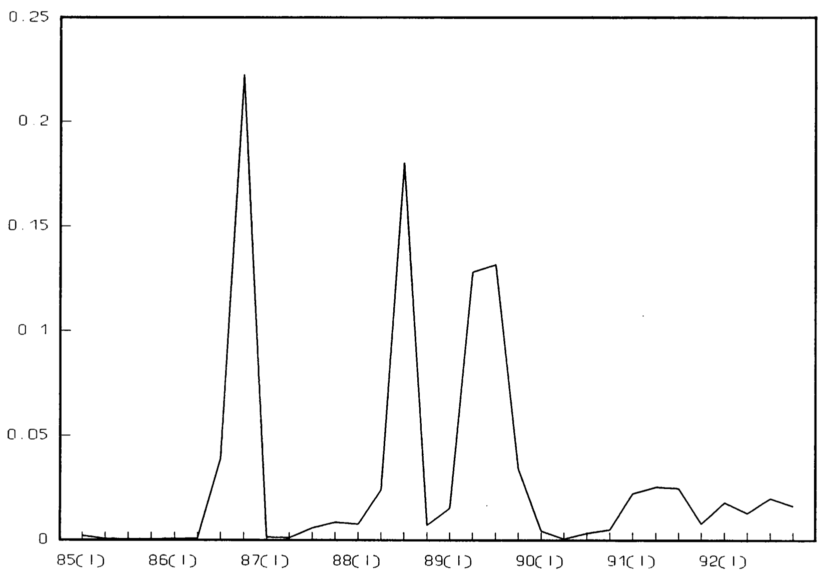


Fig. 5.5(c)

Semi-annual estimates of $\kappa\theta$

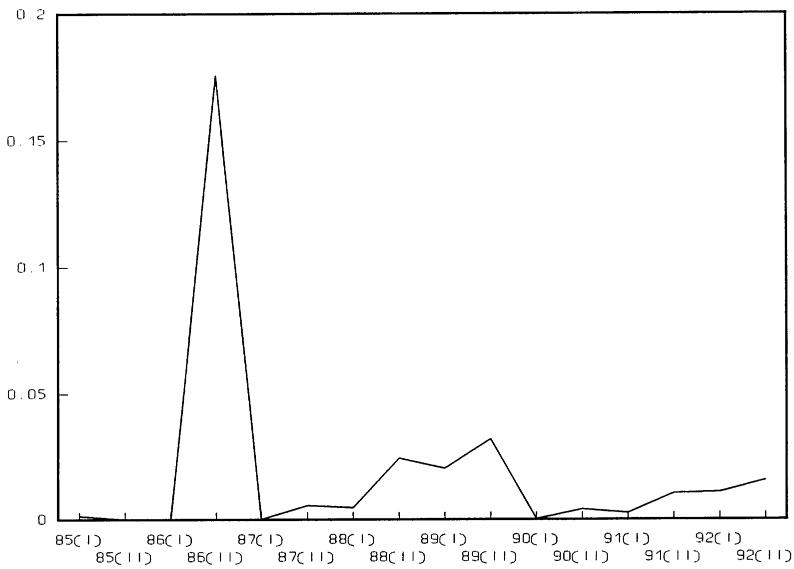


Figure 5.5(d)

Daily estimates of $\kappa+\lambda$

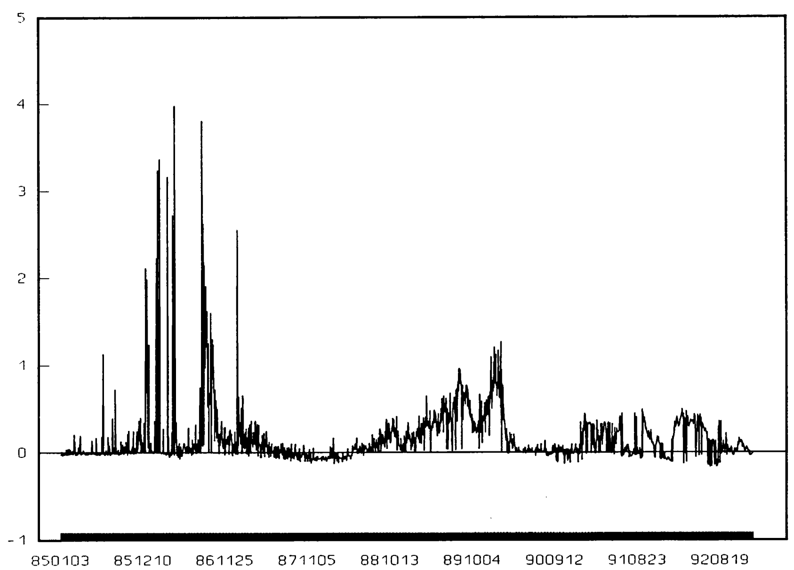


Fig. 5.5(e)

Quarterly $\kappa+\lambda$

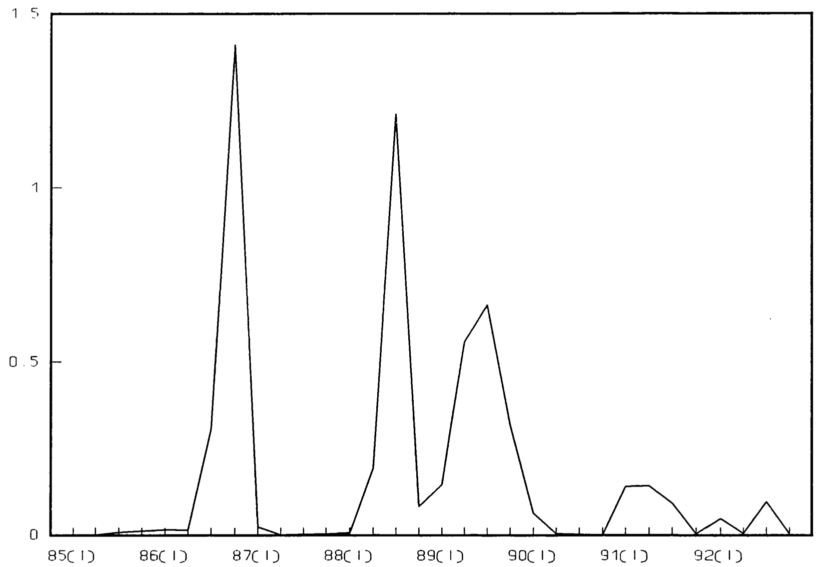
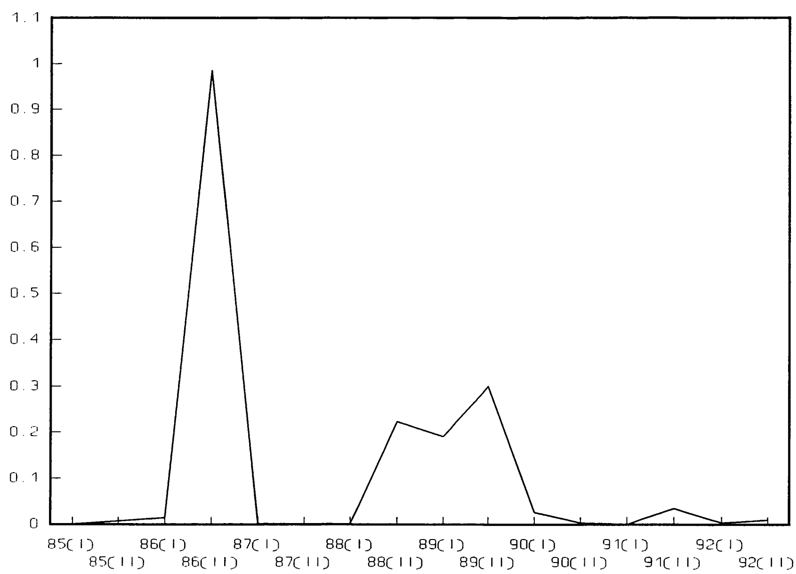


Fig. 5.5(f)

Semi-annual estimates of $\kappa+\lambda$



5.4.2 Comparison with previous studies

Brown and Schaefer (1994) is the only study that reports estimates of $\kappa\theta$ and $\kappa+\lambda$ (see Table 5.9(e)). While our estimates of $\kappa+\lambda$ are larger than Brown and Schaefer's (1994), on balance, ours are predominantly positive. As κ and λ are incapable of being estimated separately any further comparison is not meaningful. Turning to $\kappa\theta$, our positive estimates are consistent with both Brown and Schaefer's (1994) and the CIR's assumption that κ and θ are positive but the order is nearly eleven times that of Brown and Schaefer's (1994). Apart from the fact that these parameters reflect local market features, a possible explanation for the difference is that their parameters are in real terms while ours are in nominal terms[30]. For example, the average of their real spot rate over the sample period (1981-1989) is 3.16 per cent for the U.K. bond market while our nominal spot rate (1985-1992) is 12.15 per cent[31]. Furthermore, the instability of their parameter estimates as observed by Brown and Schaefer (1994) suggests that any comparison across markets should be interpreted with caution especially when they cannot be estimated separately.

[30] There are two versions of the CIR model (see Cox, Ingersoll, Ross (1985b)). The justification for estimating the nominal version is given in Chapter 2.

[31] While the difference is substantial it should be pointed out the CIR model allows for the current spot rate to be either above or under its long run equilibrium, θ; hence the higher spot rate in Australia (even after allowing for inflation) does not suggest that κ is higher in Australia.

5.5 EXTENSION OF THE GOODNESS OF FIT AND PREDICTIVENESS OF THE CIR MODEL

5.5.1 Measurement

Section 5.1.3 examines the predictiveness of the CIR model by means of the pricing errors associated with 13-week Treasury Note rates. In this section the analysis is extended to the whole spectrum of the term structure in terms of two measures of the performance of the CIR model: price mean error and rate mean squared error[32]. Essentially these are designed to compare the predictions of the CIR model with observations. The observations include coupon bond prices and three zero-coupon Treasury Notes (5-week, 13-week and 26-week). The estimated CIR model is capable of generating coupon-bond prices as well as zero-coupon interest rates. To exploit this dual capacity for the purpose of comparison we need observed quantities in terms of prices and rates. While the former quantity is in the form of observed coupon bond prices the latter quantity has to be generated from the Chebyshev model as zero-coupon term structure is not observed except for the few short-term rates (5-week, 13-week and 26-week Treasury Note rates). Thus, the price error highlights the difference between the observed coupon bond price and the predicted CIR coupon bond price while the rate mean squared error concentrates on the difference between the zero-coupon CIR term structure and the zero-coupon Chebyshev term structure. It should be noted that these two measures are not directly transformable as the price error contains coupons while the rate error is related to zero-coupon rates.

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[32] The justification for using slightly different quantities will be given in this section.

The price mean error is defined as

$$PME_{t+j} = \frac{1}{n}\sum_{i}^{n} (P_{i,obs,t+j} - P_{i,cir,t,t+j}) \tag{5}$$

where $P_{i,obs,t+j}$ is the observed bond price i at time $t + j$ where j is either *one month, three months* or *six months*, $P_{i,cir,t,t+j}$ is the forecasted CIR price at time t[33] to prevail at time $t + j$, PME_{t+j} is the price mean error at time $t + j$, and n is the number of bonds. The justification for using this form of price error is twofold: (i) conceptually, the pricing error is the average dollar error of pricing an approximately $100 bond; and (ii) the pricing error can be assumed to be normally distributed[34] so that statistical tests of significance can be carried out[35]. Specifically three price errors are calculated to correspond to three forecasting periods: (i) for daily estimation we use the parameter values estimated on day t to calculate CIR predicted bond prices for day t + *one month* and then we compare these with observed bond prices on day t + *one month*; (ii) for quarterly estimation we use the parameters estimated for quarter j to calculate predicted CIR bond prices for each day in quarter $j + 1$; and (iii) for semiannual estimation the parameters estimated in half year k are used to calculate predicted CIR bond prices for each day of half-year $k + 1$. Thus daily, quarterly and semi-annual errors are strictly not comparable because while daily errors involve predicting prices in the

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[33] It should be noted that the parameters of the CIR model are estimated at time t and then used to predict bond prices at time $t+j$. Hence the errors between observed bond prices at time $t+j$ and predicted bond prices at $t+j$ are not regression errors.

[34] Note that these errors are not regression errors and hence are not constrained to sum to zero.

[35] The assumption of normality is reasonable in view of a large number of daily cross sectional regressions over 8 years from 1985 to 1992.

next month, the quarterly and semi-annual errors result from predicting prices in the next three months and six months respectively. Obviously prediction in six months is more likely to result in greater errors than prediction in one month and three months.

The rate mean square error (*MSE*) is defined as

$$Rate\ MSE_t = \frac{1}{n}\sum_i^n (r_{cheby,t} - r_{cir,t})^2$$

where $r_{cheby,t}$, $r_{cir,t}$ and n are the Chebyshev, CIR rates of interest and the number of observed bonds respectively. The justification for using this form of rate error is twofold: (i) it is a common measure of goodness of fit; and (ii) the ratio of two *MSEs* is *F*-distributed[36] so that significance tests can be carried out.

Conceptually, the rate *MSE* on day *t* is an indication of the average error of estimating the zero-coupon rates for coupon bonds observed on day *t*. Specifically, two rate *MSEs* are measured: (i) <u>within-sample</u> rate *MSE* where both interest rates, CIR and Chebyshev, are calculated up to and including the maximum observed maturity on day *t*; and (ii) <u>out-of-sample rate *MSE*</u> where both interest rates are calculated, on day *t*, for maturities 10 years beyond the <u>maximum observed</u> maturity. As the average maturity of observed term structure is approximately 20 years, the extension of 10 years would provide a maturity of 30 years, a time frame long enough to cover most long term asset pricing considerations. While the first measure, within-sample *MSE*, is a common measure of goodness of fit, the second measure, out-of-sample rate *MSE*, extends the goodness of fit of the CIR term

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[36] The degrees of freedom of this distribution are the numbers of observations used in the calculation of the two MSEs respectively (see Johnson and Kotz (1970, Chapter 26)).

structure (with reference to the Chebyshev term structure) in the long term spectrum where no observations are used in its construction. Unlike price errors, rate MSEs, being calculated for day t using the parameters estimated for the *same day*, do not have predictive content across time because the Chebyshev term structure has no theoretical basis that predicts intertemporal parameter stability[37].

Estimates of pricing and rate errors are reported in Tables 5.10(a) and 5.10(b). In terms of minimising pricing errors the CIR model performs best when parameter values estimated on a daily basis are used (see Table 5.10(a)). For example, the CIR model underestimates the observed prices of \$100 bonds by \$1.29, \$1.83, and \$3.24 in predicting their prices in *one month, three months* and *six months* respectively by using the daily, quarterly and semi-annual parameter values estimated for day t. Further, the pairwise t-tests indicate that these predictive price mean errors (over one month, three months and six months) are significantly different while the F-test shows that out-of-sample MSEs are significantly larger than within-sample MSEs. These results are consistent with expectation.

Turning to rate errors, these are smaller within the maturity of the term structure based upon observed prices than those outside this maturity. In terms of root mean square errors, these are 1.6733E-03 and 7.5813E-03 or approximately 16 and 75 basis points[38]. Thus, with reference to a zero-coupon term structure constructed from observed coupon bond prices, the CIR term structure deviates

[37] There is no arbitrage argument underlying Chebyshev curve fitting, so its evolution over time is unknown. This essentially precludes pooling data across time to fit the term structure by means of Chebyshev polynomials. Therefore only daily Chebyshev term structures are estimated.

[38] 16 and 75 basis points are the squared roots of 2.8E-06 and 5.7476E-05 (see Table 5.10(b)).

from this 'Chebyshev polynomial constructed' term structure by an average error of 16 basis points. In sum, this result confirms the expectation that within sample errors are smaller than out-of-sample errors.

5.5.2 Comparison with previous research

The price mean errors and rate MSEs reported in the preceding section measure the predictiveness and goodness of fit of the CIR model respectively. Furthermore, the method of calculating these errors has not been attempted in previous research. Hence, no comparison is possible, though the results in Table 5.10(a) and 5.10(b) confirm that: (i) out-of-sample errors are larger than within-sample errors; and (ii) pricing errors are smaller under daily estimation. While pricing errors are also reported by Munnik and Schotman (1994) and Brown and Schaefer (1994) these errors are not compatible with ours. This is because their errors result from the objective of least square regression, namely the minimisation of the sum of square errors. Thus, their errors by design, tend to zero. In order to provide a comparison of similar quantities, our mean pricing errors (from regression) are reported in Table 5.10(c) together with the results of these studies. It can be seen that our regression pricing errors are consistent with previous research (see Table 5.8(c)) and that they are not statistically different from zero, the latter satisfying the requirement of least square estimation.

Table 5.10(a)

CIR pricing errors per $100 bond
(1985-1992)

	Daily estimation	Quarterly estimation	Semi-annual estimation
	One-month prediction	Three-month prediction	Six-month prediction
Price Mean Error	-$1.29	-$1.83	-$3.24
t-value	-2.24	-3.86	-2.45
Hypothesis	$H_0(d,q)$	$H_0(q,sa)$	$H_0(d,sa)$
t-value	8.56	4.75	6.56

Notes:
(a) $H_0(d,q)$ is the null that the mean of daily pricing errors is equal to the mean of quarterly pricing errors;
(b) $H_0(q,sa)$ is the null that the mean of quarterly pricing errors is equal to the mean of semi-annual pricing errors;
(c) $H_0(d,sa)$ is the null that the mean of daily pricing errors is equal to the mean of semi-annual pricing errors.

Table 5.10(b)

Daily within-sample and out-of-sample rate MSEs
(1985-1992)

	Within sample rate MSE	Out-of-sample rate MSE
Rate MSE	2.8E-06	5.7476E-05
Null hypothesis	$\frac{Out\text{-}of\text{-}sample\ rate\ MSE}{Within\text{-}sample\ rate\ MSE} = 1$	
F-value	46.80*	

Notes:
(a) * Significant at 0.01% (F-value for 120 degrees of freedom is 1.00 while our degrees of freedom are over 1900.
(b) Only daily MSEs are calculated because the Chebyshev polynomials are estimated daily only.

Table 5.10(c)

Comparison of regression pricing errors

	Market	Regression pricing error (Standard error)
Brown & Schaefer (1994)	UK Daily data (1984-89)	£0.20 (0.40)
Munik & Schotman (1994)	Dutch Daily data (1989-90)	0.17 guilder (0.04)
This chapter	Australian Daily (1985-92)	-$0.25 (0.22)

Notes:

(a) Mean regression errors are calculated by comparing observed prices on day t with CIR prices estimated for day t using the observed prices on day t. The objective of least square regression is to minimise the sum of these errors. In other words, the sum of errors tends to zero asymptotically.

(b) While the errors are in different currencies they result from fitting bonds of 100 units of respective currencies. Hence the errors can be interpreted as percentage of errors and are thus free from the exchange rate problem.

5.6 A SAMPLE OF CIR, CHEBYSHEV TERM STRUCTURES AND OBSERVED AND FITTED BOND PRICES

In this section a sample of daily CIR, Chebyshev term structures, CIR-fitted and observed bond prices are presented. While in the preceding sections only distributional results are reported, the graphs in this section provide a snapshot of the ultimate aim, namely an estimated term structure. These graphs are chosen on the basis of their parameters being close to their means. For the purpose of comparison, each of the graphs describes some aspect of the term structure on the same day. Thus, Fig. 5.6(a) shows the sample of observed bonds and their CIR-estimated prices on 20/12/1990. As these two prices are visually indistinct, their pricing errors, defined as the difference between observed prices and estimated prices, are graphed in Fig. 5.6(b). The CIR term structure in Fig. 5.6(c) is constructed from the parameters estimated from the observed bond prices in Fig. 5.6(a). Similarly, the Chebyshev and CIR term structures are graphed in Fig. 5.6(d) while Fig. 5.6(e) highlights their differences with respect to maturity.

Generally speaking several conclusions can be drawn from these graphs: (i) the goodness of fit of the CIR model is exceptionally good with bond price errors in the maximum order of $0.30 per $100 bond or 0.3 per cent error (see Figs 5.6(a) and 5.6(b)); (ii) while the maximum maturity of an observed bond on this day is 3495 days or more than 9.5 years (see Fig. 5.6(a)), the CIR and Chebyshev term structures extend well over this maturity; (iii) this extrapolation beyond observed bonds explains why the out-of-sample (beyond 9.5 years) rate errors are larger than within-sample rate errors though the maximum error corresponding to approximately 25 year maturity is under 0.4 per cent (see Fig. 5.6(e)); and (iv) the

CIR term structure has the expected inverted U[39] shape while the Chebyshev term structure, being the result of a curve-fitting technique, is more flexible (see Fig. 5.6(d)). These conclusions are representative over the whole sample period as the mean regression errors is in the order of 25 cents per $100 bond (see Table 5.10(c), last row)[40].

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[39] This is one of the admissible shapes of the CIR model. For further details see Chapter 2.

[40] In addition, *adjusted R-squared* for each estimation is well over 99 per cent.

Fig. 5.6(a)

Observed and CIR-fitted dollar bond prices (20/12/1990)

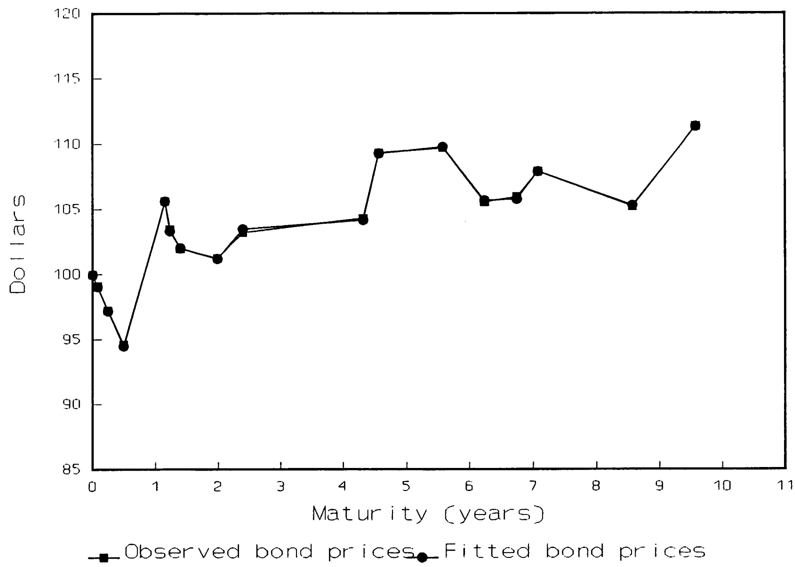
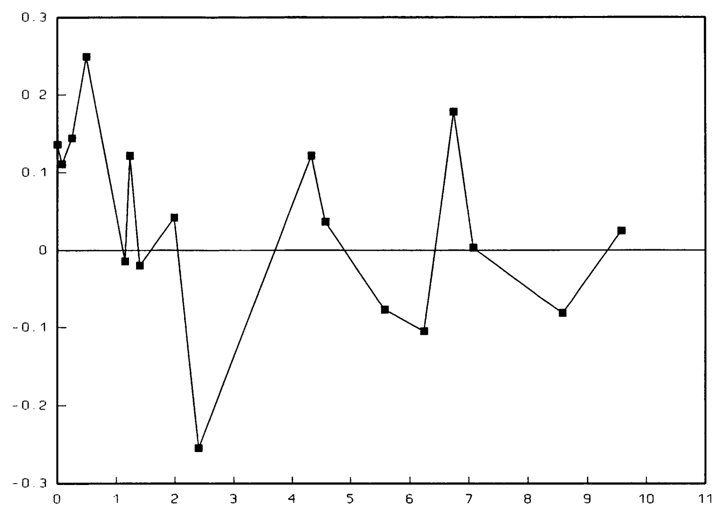


Fig. 5.6(b)

Dollar bond price errors (20/12/1990)



Maturity (Years)

Note: Bond price error is the difference between observed bond price and CIR-fitted bond price. The error is the dollar error per \$100 bond.

Fig. 5.6(c)

CIR Term Structure - Daily estimates - (20/12/90)

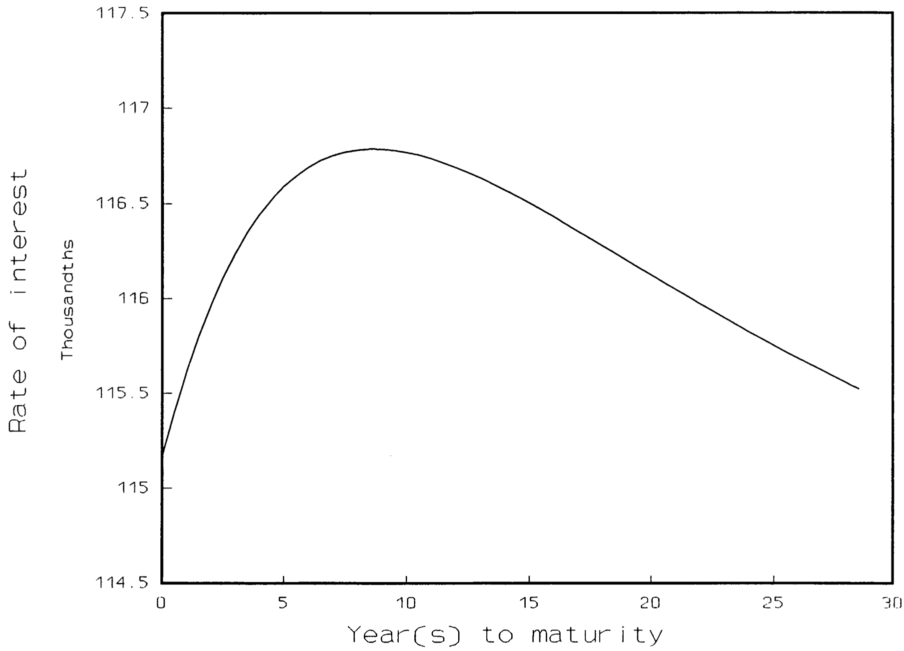


Fig. 5.6(d)

Chebyshev and CIR term structures - Daily estimates (20/12/1990)

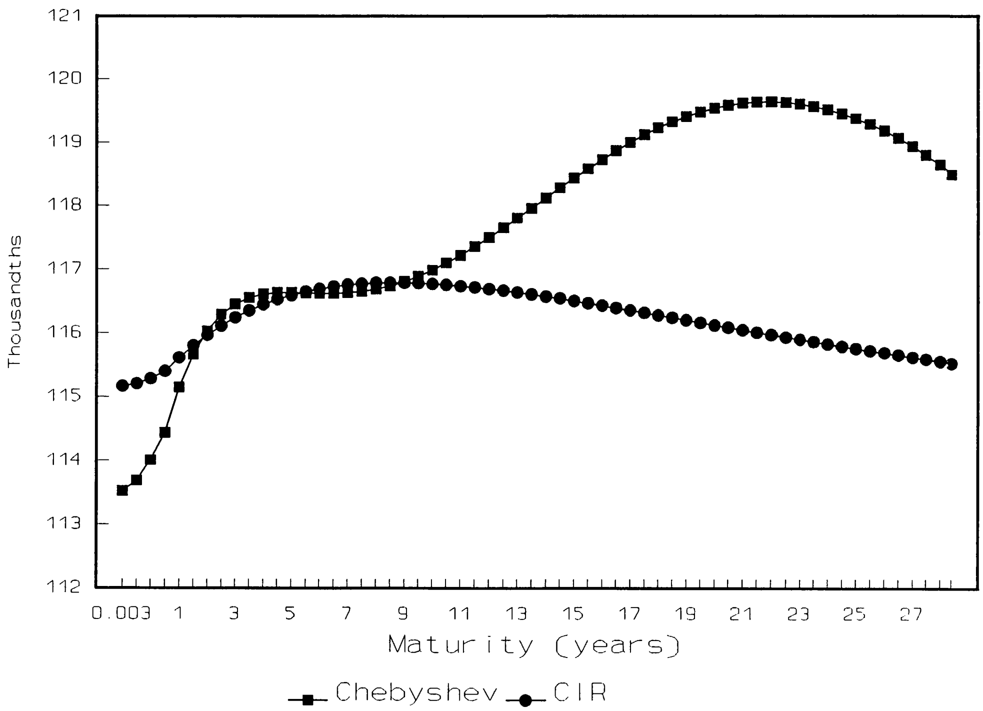
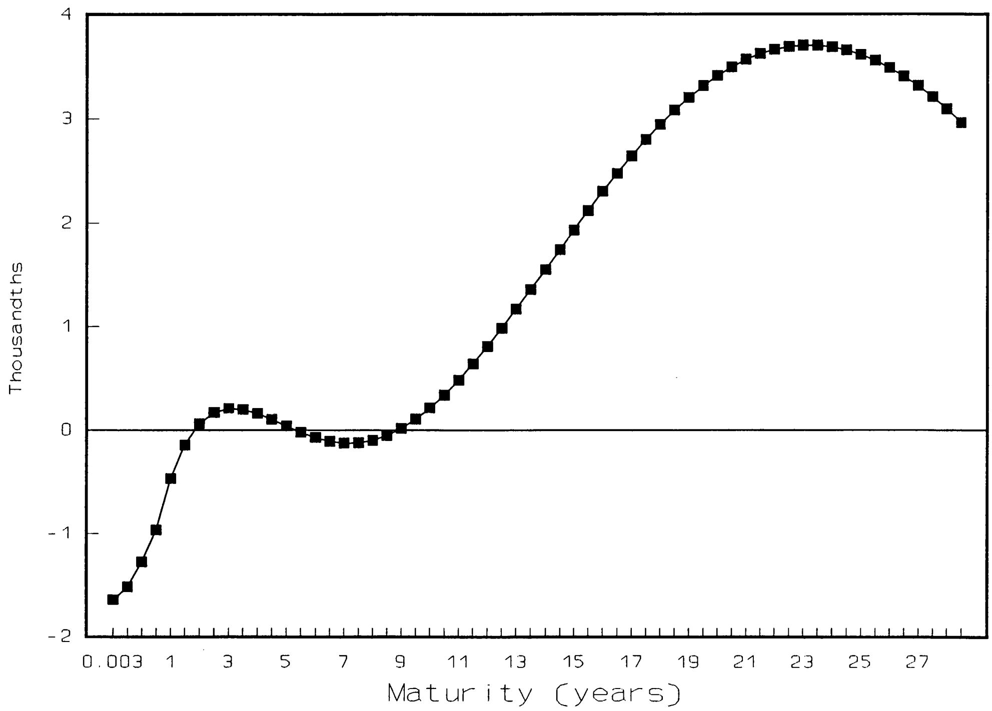


Fig. 5.6(e)

Rate difference between Chebyshev and CIR term structures (20/12/1990)



Notes:

(a) A thousandth on the vertical axis is equal to 10 basis point or 0.1%;

(b) Rate difference = Chebyshev rate of interest *less* CIR interest rate.

5.7 CONCLUSION

The overall objective of this chapter is to estimate the CIR model using daily bond price data from 1985 to 1992. Essentially we seek answers to two issues: (i) parameter stability as implied by the model; and (ii) model's goodness of fit and predictive powers. Towards this end, estimation is implemented by means of unconstrained and constrained nonlinear regression where the error function is expressed in terms of bond prices and the logarithm of bond prices. While various results are reported, the following findings are the more significant:

Firstly, in terms of estimation methodology the logarithmic norm, first proposed in this study, outperforms the price norm, the standard used in current research.

Secondly, unconstrained (daily) regression is the most accurate method of estimation in contrast to constrained (quarterly and semi-annual) regression.

Thirdly, while the spot rate remains a dominant factor in the pricing of Australian bonds during the sample period (1985-1992), it significantly underestimates its two closest observed proxies, the cash rate and 13-week treasury Note rate. However, this underestimation is removed when a CIR generated rate of equal maturity is compared to the observed 13-week treasury Note rate; hence indicating that either overestimation or underestimation is probably due to the lack of an observed instantaneous spot rate.

Fourthly, an implication of the CIR model, namely constant long term yield, $\kappa\theta$, and $\kappa+\theta$ is broadly supported while the stability of the volatility parameter, σ , remains problematic. Overall, the requirement of constant parameters can be compared to a similar assumption underlying the Black-Scholes model of option pricing, namely constant volatility. While this assumption has been decisively rejected empirically, it has not diminished the usefulness of the Black-Scholes model. In this perfective the same defence can be made with respect to the Cox-Ingersoll-Ross model.

Fifthly, the model's goodness of fit using existing information is consistent with previous research while its predictive power beyond current time and observed bond maturity, first measured in this study, are consistent with expectations.

On balance our findings are broadly in agreement with previous research, in particular Brown and Dybvig (1986), Barone et al (1991), Munnik and Schotman (1994) and Brown and Schaefer (1992). Finally, while the CIR model cannot outperform the curve-fitting Chebyshev technique its deviation from the latter is remarkably small. On the basis of daily estimation, it could just substitute for a curve-fitting technique, yet retaining the qualities of a general equilibrium model.

CHAPTER 6

GENERALISED CIR MODEL: EMPIRICAL RESULTS

In this chapter results of the estimation of the generalised CIR model are presented and discussed. The two major issues we shall address ourself to are: (i) empirical support for parameter stability as implied by the model; and (ii) goodness of fit and predictiveness of the model. Toward this end, various statistical aspects of parameter estimates are explored together with several investigations undertaken to deepen our understanding of the bahaviour of the model in the Australian context. While the general framework of Chapter 5 is retained, this chapter concentrates on highlighting the main empirical findings pertaining to the generalised model. In order to avoid repetition relevant references to Chapter 5 will be made where it is necessary.

The generalised CIR model[1] is the solution to the following partial differential equation:

$$\frac{\partial P}{\partial r}\left[\kappa(\theta-r)-\lambda r^{\gamma}r^{\beta}\right]+\frac{\partial P}{\partial t}+\frac{1}{2}(\sigma r^{\beta})^{2}\frac{\partial^{2}P}{\partial r^{2}}-rP = 0 \tag{1}$$

where the stochastic interest rate process, r, and the market price of risk, λ^{*}, are of the form:

$$dr = \kappa(\theta-r)dt+\sigma r^{\beta}dz \tag{2}$$

$$\lambda^{*} = \frac{\lambda r^{\gamma}}{\sigma} \tag{3}$$

The parameters of the model, $\underline{a}_{gcir}$, are estimated by the method of explicit

[1] See Chapter 3 for a complete derivation of this model.

difference[2]:

$$\underline{a}_{gcir} = \begin{bmatrix} r \\ \kappa \\ \theta \\ \sigma \\ \beta \\ \lambda \\ \gamma \end{bmatrix} \tag{4}$$

where r is the instantaneous interest rate, κ is the speed of adjustment of r to its equilibrium θ, and σ, β, γ are positive constants. If β and γ are not statistically different from 0.5 then the generalised CIR model reverts to the CIR model. While both the price and logarithmic norms of estimation are implemented the latter is found to be superior[3] (see Table 6.1). For example, the mean of the Aikake criterion for the logarithmic norm is -3.6767 while that of the price norm is -2.4696 (see Table 6.1, third row, second and third column). Hence only results from the logarithmic norm are reported. Furthermore the normality and heteroskedasticity tests of regression residuals (see Table 6.2) imply that the nonlinear regression procedure used in the estimation is, for the majority of the cross sections, equivalent to maximum likelihood estimation. For example, at 1% level of significance, 71.18% of the daily cross sectional regression errors are normally distributed while 77.15% are homoskedastic (see Table 6.2, third row, third column). It should be noted that while analytical results for linear regression

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[2] See Chapter 3 for full details of the method of explicit difference.

[3] See Judge et al (1985, p. 242 and Chapter 21), and Maddala (1992, p. 500-501) who argue that Aikake's information criterion (AIC) is commonly used (at least in nonlinear models). Schwartz criterion is developed within the Baysian framework. Information criteria are based upon minimising the residual sum of squares; thus, among competing regression models the one with the minimum criteria statistic is preferred. See also Chapter 5.

estimators (concerning unbiasedness, consistency, and efficiency) are available, these are only valid in nonlinear regression in an asymptotic sense[4].

The structure of this chapter essentially follows that of Chapter 5. Thus estimates of the spot rate and volatility are presented and discussed in Sections 6.1 and 6.2 respectively while Section 6.3 is concerned with the remaining parameters, namely κ, θ, λ, and γ. This is followed by an analysis of the goodness of fit and predictiveness of the model in Section 6.4. Then a sample of term structures generated by this model and the Chebyshev polynomial based technique are provided in Section 6.5. Finally, the chapter concludes in Section 6.6.

[4] See Judge et al (1985) and Green (1993).

Table 6.1

Distribution of information criteria of logarithmic and price norm
Daily cross-sections (1985-1992)

	Akaike Criterion		Schwartz Criterion	
	Log Norm	Price Norm	Log Norm	Price Norm
Mean	-3.6767	-2.4696	-3.4933	-2.2862
Standard deviation	0.1724	0.1487	0.1724	0.1487

Note:
The information criteria are based on minimising the residual sum of squares; hence the smaller the criteria the better the model.

Table 6.2
Normality and heteroskedasticity tests of regression residuals

		Daily	Quarterly	Semi-annual
	Significance level	Proportion of cross-sections	Proportion of cross-sections	Proportion of cross-sections
Normality Test (Jarque-Berra)	1%	0.7118	0.8249	0.8411
	5%	0.7994	0.9231	0.9190
Heteroskedasticity (White)	1%	0.7715	0.6210	0.6898
	5%	0.8688	0.7707	0.8209

Note:
The entry 0.8249(column 4, row 3) means that at 1% level of significance the regression residual is normally distributed for 82.49% of the total quarterly cross sections (the spot rate is allowed to vary from day to day while the other parameters, σ, $\kappa\theta$ and $\kappa+\lambda$, are kept fixed over each quarter). Similarly the entry 0.6210 (column 4, row 5) means that at 1% level of significance the regression residual is homoskedastic for 62.10% of the total quarterly cross sections. Other entries are interpreted accordingly.

6.1 ESTIMATES OF SPOT RATE r

6.1.1 Distributional statistics

Estimates of the generalised spot rates are presented in Tables 6.3(a) and 6.3(b) while the differences between these estimates and their closest observed rates, the cash rate and 13-week Treasury Note rate are graphed in Figs. 6.1(a) and 6.1(b). From these tables and graphs two observations can be made: (i) the generalised CIR spot rate is significantly less than both the cash rate and the 13-week Treasury Note rate (see Table 6.3(c)) as shown by the significant differences between the spot rate on the one hand, and the cash rate and 13-week Treasury Note rate on the other hand. Moreover this underestimation is somewhat related to the level of interest rates[5] (see Table 6.3(d) as shown by the relatively low R^2 values ; and (ii) except for the significant difference between the mean of daily estimates and the mean of quarterly estimates, the estimates of the spot rate under the three methods are essentially similar (see Table 6.3(b)). In addition, regardless of the interval over which the spot rate is estimated it is always positive and significant for each and every cross section[6]. In this respect the spot rate remains the predominant force in the pricing of bonds, which is consistent with the specification that it drives the single factor model.

6.1.2 An unbiasedness test of 13-week generalised CIR rate

The underestimation of the cash rate and the 13-week Treasury Note rate by

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[5] To ascertain whether the underestimation is related to the level of interest rates, a regression is performed (see Chapter 5 and Table 6.3(d)).

[6] Further, because the spot rate is always significant the Wald statistic to test the hypothesis that all the parameters, including the spot rate, is always significant. Results are not tabulated.

the spot rate (see Section 6.1.1) may be attributed to the mismatch of maturity. To address this issue we use a common maturity of 13 weeks for generalised CIR rates and Treasury Note rates. This maturity is chosen owing to the availability of a full series of daily 13-week Treasury Note rates (1985-1992). The test of unbiasedness is carried out by regressing the Treasury Note rates against the generalised CIR rates and then testing the hypothesis that the intercept and the gradient of the regression equation are equal to 0.0 and 1.0 respectively:

$$r_{obs} = \alpha + \beta r_{est} + \epsilon \tag{5}$$

where r_{obs} is the observed 13-week Treasury Note rate and r_{est} is the estimated 13-week generalised CIR rate. The null hypothesis is $\alpha = 0.0$ and $\beta = 1.0$.

Results of this test are reported in Tables 6.4(a) and 6.4(b) where the *F*-statistics indicate that while 13-week generalised CIR rates are unbiased estimators of 13-week Treasury Note rates, <u>predicted</u> 13-week generalised CIR rates are not unbiased estimators of 13-week Treasury Note rates. This finding is not unexpected as predicted rates are calculated for day t + *one month*, using parameters estimated for day t. Thus, when CIR interest rates of the same maturity are compared <u>within sample</u>, the underestimation observed in section 6.1.1 no longer exists. Hence, it appears that the mismatch of maturity[7] may be responsible for the underestimation of observed rates by the generalised CIR spot rate.

---

[7] Mismatch of maturity refers to the fact that <u>instantaneous</u> CIR rates are compared with <u>daily</u> observed cash rates and 13-week Treasury rates.

Table 6.3(a)

Daily estimates of generalised CIR spot rate

	Mean	Std	Min	Max
1985	0.12158	0.008427	0.10905	0.14298
1986	0.12399	0.008627	0.10564	0.14082
1987	0.11412	0.011536	0.094465	0.13613
1988	0.10686	0.010425	0.090729	0.12678
1989	0.13812	0.004997	0.12516	0.15100
1990	0.12133	0.009233	0.099709	0.13675
1991	0.087936	0.011013	0.063404	0.10509
1992	0.056381	0.006484	0.042858	0.06816
1985-92	0.10872	0.025840	0.042858	0.15100

Table 6.3(b)

Distribution of generalised CIR spot rate
Daily, quarterly and semi-annual estimates
(1985-1992)

	t-value	Daily	Quarterly	Semi-annual
Mean		1.0872E-01	1.3124E-01	1.2104E-01
Std		2.5840E-02	1.113E-02	2.9388E-02
Minimum		4.2858E-02	1.1248E-02	4.7526E-02
Maximum		1.5100E-01	1.8696E-01	1.8037E-01
H_0 *(d, q)*	-2.41*			
H_0 *(d, sa)*	-1.61			
H_0 *(q, sa)*	0.04			

Notes:
(a) H_0 *(d, q)* is the null hypothesis that the mean of daily estimates of generalised CIR spot rate is equal to the mean of its quarterly estimates;
(b) H_0 *(d, sa)* is the null hypothesis that the mean of daily estimates of generalised CIR spot rate is equal to the mean of its semi-annual estimates;
(c) H_0 *(q, sa)* is the null hypothesis that the mean of quarterly estimates of generalised CIR spot rate is equal to the mean of its semi-annual estimates.
(d) *: significant at 1% level.

Table 6.3(c)

Differences between observed TN rates and generalised CIR spot rate (1985-1992)

	90-day TN rate - r	*Cash rate - r*
Mean	0.015563	0.018171
Standard deviation	3.086E-04	3.75E-04
t^a value	50.42*	48.31*
t^b value	8.44*	

Notes:

[a] The null hypothesis is that the mean of (observed rate *less* theoretical rate) is 0.0.
[b] The null hypothesis is that the mean of the series (13-week TN rate *less* r) is equal to the mean of the series (cash rate *less* r).
* Significant at 1%.

Table 6.3(d)

Regression of spot rate and cash rate and 13-week Treasury Note rate and Generalised CIR spot rates (1985-1992)

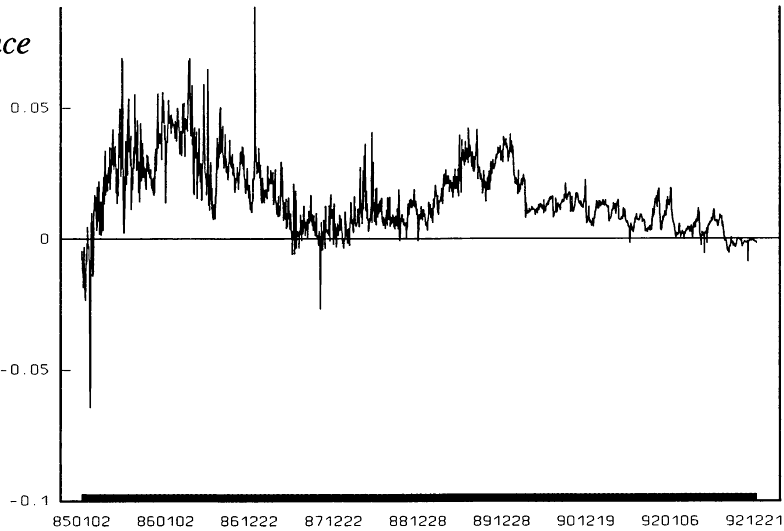
Dependent variable	Independent variable	Intercept	Reg Coef	$\overline{R}^2$	DW
Diff_cash[a]	Level of cash rate	-1.027E-02	0.1772	0.1791	2.12
t-value		0.7988	20.59*		
Diff_tn[b]	Level of TN rate	-1.691E-02	0.2411	0.3708	1.95
t-value		-1.4411	33.84*		

Notes:
* significant at 1%
[a]Diff_cash ≡ cash rate *less* Generalised CIR spot rate
[b]Diff_tn ≡ 13-week Treasury Note rate *less* Genralised CIR spot rate

Fig. 6.1(a)

Difference between cash rate and generalised CIR spot rate

Rate difference

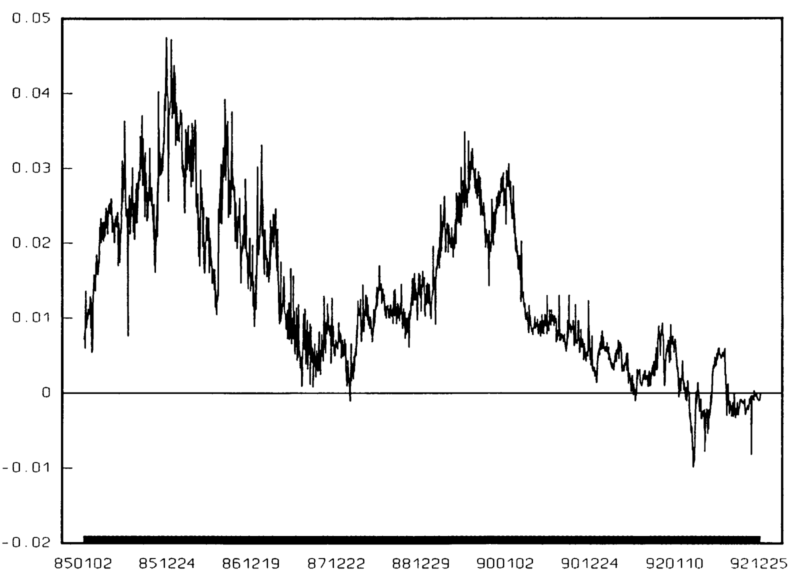


Note: Rate difference = cash rate *less* generalised CIR spot rate

Fig. 6.1(b)

Difference between 13-week TN rate and generalised CIR spot rate

Rate difference



Note: Rate difference = 13-week TN rate *less* generalised CIR spot rate

6.1.2 A test of the unbiasedness of 13-week CIR interest rate[8]

In this section we explore whether 13-week CIR interest rates are unbiased estimators of 13-week Treasury Note rates. The motivation for this inquiry is to address the issue of mismatch of maturities in Section 6.1.1 where the instantaneous generalised spot rate is compared with the cash rate and 13-week Treasury Note rate. This investigation is implemented by regressing the generalised CIR rates on the observed rates and then testing the hypothesis that the intercept and the gradient of the regression equation are equal to 0.0 and 1.0 respectively:

$$r_{obs} = \alpha + \beta r_{est} + \epsilon \tag{5}$$

where r_{obs} is the observed 13-week Treasury Note rate and r_{est} is the estimated 13-week genralised CIR rate. The null hypothesis is $\alpha = 0.0$ and $\beta = 1.0$.

Results are presented in Table 6.4(a) and Table 6.4(b) where the *F*-statistics indicate that while 13-week generalised CIR rates are unbiased estimators of 13-week Treasury Note rates on the same day (see Table 6.4(a)), they are not unbiased estimators of 13-week Treasury Note rates <u>one month ahead</u> (see Table 6.4(b). The latter result is not unexpected as it requires generalised CIR rates to be <u>predictors</u> for <u>future</u> observed rates. The former result, however, addresses the issue that the underestimation reported in Section 6.1.1 is due to the mismatch of maturities.

[8] For full details of the basis of this test see Chapter 5, Section 5.1.2.

Table 6.4(a)

Test of unbiasedness of 13-week generalised CIR rates of interest

(1985-1992)

Observed rate (Dependent variable)	Estimated CIR rate (Independent variable)	α	β	F-value	$\bar{R}^2$
13-week TN rate	13-week	0.129E-10 (0.149E-03)	1.247 (1.759E-01)	0.58	0.9364

Notes:

(a) CIR rates are calculated for day t, using the parameters estimated on day t;

(b) The null of the F test is $\alpha = 0.0$ and $\beta = 1.0$;

(c) Standard errors are in parentheses.

(d) TN = Treasury Note

Table 6.4(b)

Test of unbiasedness of daily predicted 13-week generalised CIR rates of interest

(1985-1992)

Observed rate (Dependent variable)	Estimated CIR rate (Independent variable)	α	β	F-value	$\bar{R}^2$
13-week TN rate	13-week	0.316E-11 (0.197E-03)	1.4276 (0.102)	248.3*	0.8887

Notes

(a) Predicted CIR rates are calculated for day t + *one month*, using the parameters estimated on day t;

(b) The null of the F test is $\alpha = 0.0$ and $\beta = 1.0$;

(c) Standard errors are in parentheses;

(d) *: significant at 1% level;

(e) TN = Treasury Note.

6.1.3 Comparative analysis of the performance (predictiveness) of the generalised CIR model

In this section the performance of the generalised CIR model is assessed with reference to a naive model which assumes that interest rates remain constant over time[9]. The criterion used for this purpose is the pricing error defined as observed price *less* estimated price. Two pricing errors are calculated: (i) the error associated with the naive model; and (ii) the error associated with the generalised CIR model. To gain a perspective on the relative performance over maturities, each of these errors is estimated for two time points ahead of *day t*: one month and three months. The two observed securities chosen for this investigation are 13-week and 26-week Treasury Notes which are zero-coupon bonds.

For the naive model we have:

$$PE_{naive,t+n} = P_{obs,t+n} - P_{obs,t} \tag{6}$$

where $PE_{naive,t+n}$ is the pricing error under the naive model, $P_{obs,t}$ is the observed price at time t (assumed to remain constant over time), $P_{obs,t+n}$ is the observed price at time $t+n$, and n is either 1 or 3 months.

For the generalised CIR model we have:

$$PE_{gcir,t+n} = P_{obs,t+n} - P_{gcir,t,t+n} \tag{7}$$

where $PE_{gcir,t+n}$ is the pricing error under the generalised CIR model and $P_{gcir,t,t+n}$ is the generalised CIR price predicted at time t to prevail at time $t+n$.

---

[9] See Chapter 5 for further details.

Results of the comparative errors are presented in Tables 6.5(a) and 6.5(b) where the errors under the generalised CIR are significantly less than the errors[10] under the naive model for two maturities, 13-week and 26-week, for which there are observed zero-coupon bond prices. For example, the dollar error of the generalised CIR model is \$0.16 for predicting the price of a 13-week Treasury Note in one month's time while the error associated with the naive model is \$0.21 (see Table 6.5(a), third row, second and third columns). Furthermore, the *F*-statistics show that the smaller errors are significant at the 1% level (see Tables 6.5(a)-6.5(b)). Results with respect to 26-week Treasury Notes (see Table 6.5(b)) are similarly interpreted.

[10] The ratio of two *MSE*s is *F*-distributed with degrees of freedom equal to the numbers of observations used in the calculation of the MSEs (see Chapter 5, section 5.1.3).

Table 6.5(a)

Comparative mean dollar pricing errors of \$100 13-week Treasury Notes

	One-month prediction		Three-month prediction	
	Naive model	Generalised CIR model	Naive model	Generalised CIR model
	\$0.21	-\$0.16	\$0.39	-0.29
t-value of $\overline{PE}$	29.48	-61.62	33.32	-42.07
Hypotheses	$H_0 : MSE_{naive} < MSE_{gcir}$ $H_1 : MSE_{naive} \geq MSE_{gcir}$		$H_0 : MSE_{naive} < MSE_{gcir}$ $H_1 : MSE_{naive} \geq MSE_{gencir}$	
F^**-value*	1.72^{**}		1.81^{**}	

Notes:

* Ratio of two *MSEs* is F-distributed (see Johnson and Kotz(1970, Chapter 26));

** Daily data were used in calculating *MSEs*; hence the degrees of freedom exceed 1900. Note that with 120 degrees of freedom $F_{0.01,120,120} = 1.53$.

Table 6.5(b)

Comparative mean dollar pricing errors of $100 26-week Treasury Notes

	One-month prediction		Three-month prediction	
	Naive model	Generalised CIR model	Naive model	Generalised CIR model
	$0.35	-$0.09	$0.30	-$0.14
t-value of $\overline{PE}$	28.97	-10.37	34.31	-24.47
Hypotheses	H_0 : $MSE_{naive} < MSE_{gcir}$ H_1 : $MSE_{naive} \geq MSE_{gcir}$		H_0 : $MSE_{naive} < MSE_{gcir}$ H_1 : $MSE_{naive} \geq MSE_{gcir}$	
F^**-value*	15.12^{**}		4.59^{**}	

Notes:

* Ratio of two *MSEs* is F-distributed (see Johnson (1970, Chapter 26);

** Daily data were used in calculating *MSEs*; hence the degrees of freedom exceed 1900. Note that with 120 degrees of freedom $F_{0.01,120,120} = 1.53$.

6.1.4 Comparison with previous studies

Strictly speaking, this comparison is not valid because the generalised CIR model is first estimated in this study. However, the results in Table 6.5(c) are reproduced to provide a perspective on estimates of the instantaneous spot rate across <u>different</u> models. Thus, estimates of the generalised CIR spot rates significantly underestimates both the observed cash rate and 13-week T-Note rate. This finding is consistent with our estimates of the CIR spot rate (see Chapter 5) but inconsistent with Brown and Dybvig (1986) and Chiarella et al (1989). A difficulty in resolving this issue of overestimation or underestimation lies in the lack of an <u>instantaneous</u> observed rate. Thus the comparison is strictly not valid owing to the mis-match of maturity. Nevertheless the observed rates used in this study are closer to the instantaneous spot rate, in terms of maturities, than those used by previous studies. Furthermore when the mis-match of maturities is resolved by comparing 13-week generalised CIR rates and 13-week Treasury Note rates the former are <u>unbiased</u> estimators of the former (see Table 6.4(a)). This exercise of comparing model generated rates and observed rates of identical maturities is first attempted in this study. Hence comparison with previous research is limited.

Table 6.5(c)

Comparison of estimates of CIR spot rates with observed rates

Study	Market	Observed rate	Results
Brown & Dibvig (1986)	USA (1977-83) Monthly data	Mean yield on T-Bills with at most 14 days to maturity	CIR *r* overestimates observed rate
Munnik & Schotman (1994)	Netherlands (1989-90) Monthly data	Amsterdam Interbank Offered Rate	CIR *r* is close to observed rate
Baron et al (1991)	Italy (1983-90) Daily data	1 month, 3 month interbank rate and 3 month T-Bill rate	CIR *r* is highly correlated with observed rates
Chiarella et al (1989)	Australia (1978-87) Monthly data	13-week T-Note rate	CIR *r* overestimates observed rate
Chapter 5: CIR model	Australia (1985-92) Daily data	Overnight cash rate 13-week T-Note rate	CIR *r* underestimates observed rates
This chapter: Generalised CIR model	Australia (1985-92) Daily data	Overnight cash rate 13-week T-Note rate	Generalised CIR *r* underestimates observed rates

6.2 ESTIMATES OF VOLATILITY

Estimates of the volatility of the generalised CIR stochastic interest rate are presented and discussed in this section.

6.2.1 Distributional statistics and graphs

Basic estimates of the generalised CIR volatility and their distribution are presented in Tables 6.6(a)-6.6(e) while Figures 6.4(a)-6.4(d) graph the time series of these estimates. Several observations can be made: (i) daily estimates of σ are significantly larger and more volatile than quarterly and semi-annual estimates (see Table 6.6(c)); (ii) estimates of σ become more significant as more data are pooled over quarters and half years (see Table 6.6(d)); and (iii) over the whole sample period (1985-1992) generalised CIR volatility, σr^{β} , is approximately one half of CIR volatility, $\sigma\sqrt{r}$, (see Table 6.6(e)).

Table 6.6(a)

Daily estimates of σ

	Mean	Std	Min	Max
1985	0.020265	0.012825	4.7E-05	0.05098
1986	0.010597	0.012801	9.89E-06	0.04829
1987	0.039277	0.033382	2.88E-06	0.18919
1988	0.027710	0.028285	2.4E-05	0.11803
1989	0.012622	0.013763	6.4E-05	0.10775
1990	0.029883	0.019400	1.35E-04	0.08947
1991	0.057964	0.022140	8.4E-03	0.11479
1992	0.10844	0.031365	2.06E-02	0.15900
1985-92	0.03844	0.038192	2.88E-06	0.18919

Note: The volatility of the generalised CIR model is σr^{β} where σ is a constant.
It is important to distinguish between the significance of cross-sectional σ and time-series σ. The former shows whether it is a significant factor in the pricing of a bond at a given time point while the latter shows how volatile the volatility factor is over time. Thus, the significance of time-series σ does not impact on the pricing of bonds.

Table 6.6(b)

Daily estimates of generalised CIR volatility, σr^{β}

	Mean	Std	Min	Max
1985	0.007015	4.37E-03	1.69E-05	0.01692
1986	0.003661	4.36E-03	3.60E-06	0.01613
1987	0.012792	1.08E-02	1.01E-06	0.06086
1988	0.008713	8.67E-03	8.48E-06	0.03625
1989	0.004716	5.18E-03	2.40E-05	0.04092
1990	0.010340	6.70E-03	4.84E-05	0.03087
1991	0.016922	6.49E-03	2.58E-03	0.034106
1992	0.025887	7.89E-03	4.41E-03	0.040361
1985-1992	0.011269	9.89E-03	1.01E-06	0.060857

Table 6.6(c)

Distribution of estimates of σ
(1985-92)

		σ		
	t-value	DAILY	QUARTERLY	SEMIANNUAL
Mean		0.03844	0.015055	1.2479E-02
Std		3.8192E-02	9.0229E-03	4.81E-03
Minimum		2.88395E-06	9.3629E-03	2.192E-03
Maximum		1.8919E-01	5.2959E-02	1.6517E-02
$H_0(d, q)$	50.33			
$H_0(d, sa)$	39.51			
$H_0(q, sa)$	37.71			

Note: H_0 *(d, q)* is the null hypothesis that the mean of daily estimates of σ is equal to the mean of its quarterly estimates;

H_0 *(d, sa)* is the null hypothesis that the mean of daily estimates of σ is equal to the mean of its semi-annual estimates;

H_0 *(q, sa)* is the null hypothesis that the mean of quarterly estimates of σ is equal to the mean of its semi-annual estimates.

Table 6.6(d)

Significance proportions of t-statistics of generalised CIR σ parameter

	Daily Estimates	Quarterly Estimates	Semi-annual Estimates
Significance level	Sample Proportion	Sample Proportion	Sample proportion
1%	0.0539	0.4375	0.5625
5%	0.1637	0.4375	0.5625
10%	0.1707	0.4375	0.5625

Table 6.6(e)

Comparison of estimates of generalised CIR volatility, σr^{β} ,
and CIR volatility, $\sigma\sqrt{r}$.
(1985-1992)

	Mean	Standard deviation	Min	Max
CIR volatility*, $\sigma\sqrt{r}$	0.0224	0.0644	0.0304	0.4516
Generalised CIR volatility, σr^{β}	0.01127	9.89E-03	1.01E-06	0.06086

Note: * see Table 5.6(b), Chapter 5.

Fig. 6.2(a)

Time series distribution of daily estimates of σ

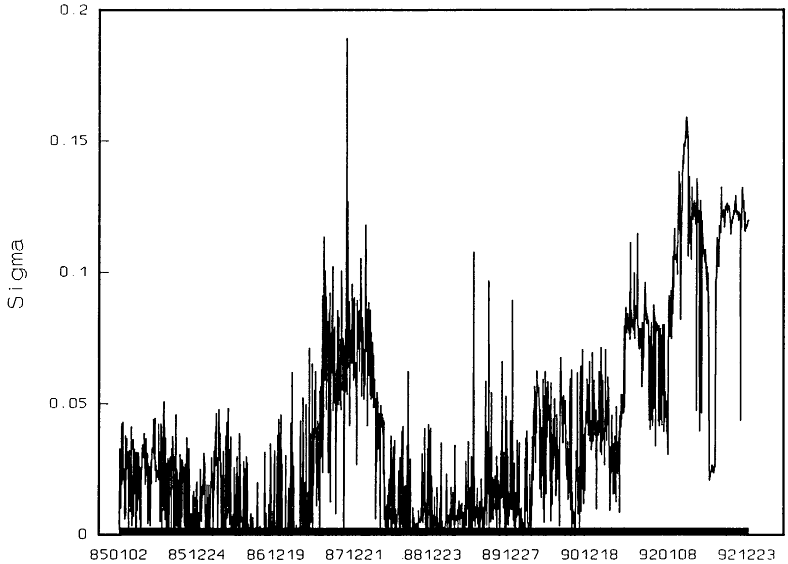


Figure 6.2(b)

Time series distribution of daily estimates of σr^{β}

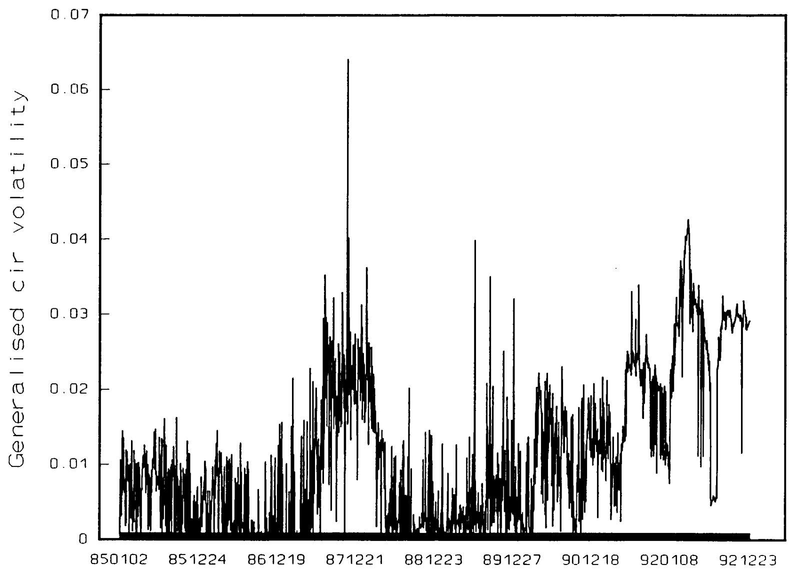


Fig. 6.2(c)

Quarterly estimates of generalised CIR σ

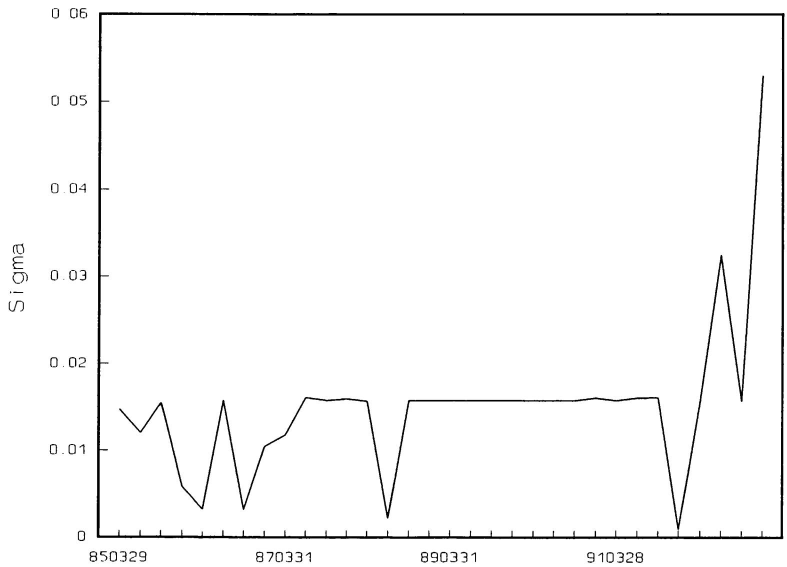
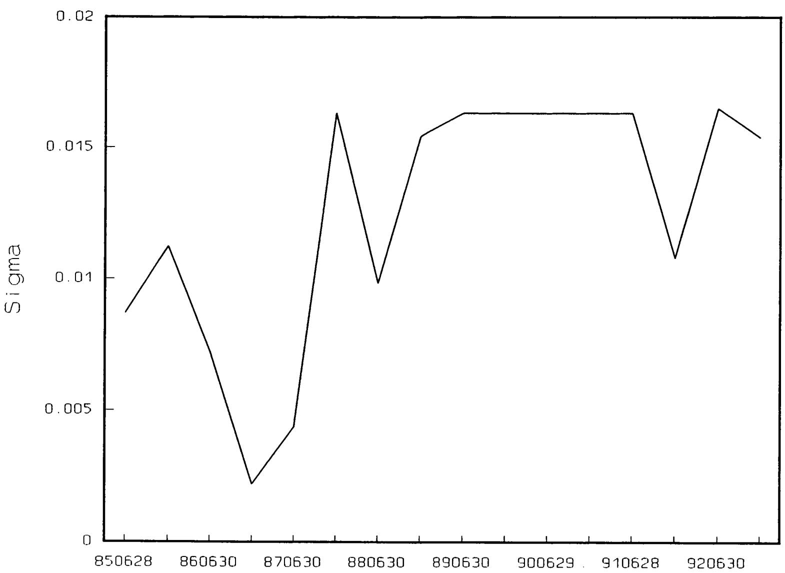


Fig. 6.2(d)

Semi-annual estimates of σ



6.2.2 Generalised CIR volatility and time series volatility of 90-day TN rates

In this section we investigate the relationship between generalised CIR volatility and volatility of 13-week Treasury Note rates[11]. If such a relationship exists then generalised CIR volatility can be used as a predictor for Treasury Note rate volatility. Thus, monthly estimates of Treasury Note rate changes are regressed against daily estimates of generalised CIR volatility averaged over the corresponding months:

$$\log \hat{s}_t = \alpha + \beta \log \hat{s}_{gcir} + \epsilon \tag{8}$$

where $\log \hat{s}_t$ and $\log \hat{s}_{gcir}$ are the logarithms of the volatilities of the TN and generalised CIR interest rates. The null hypothesis is $\alpha = 0.0$ and $\beta = 1.00$. Distributional statistics and regression results are presented in Tables 6.7(a) and 6.7(b) while the time series of generalised CIR volatility and Treasury Note volatility are graphed in Figs. 6.3(a) and 6.3(b).

On the basis of this evidence, generalised CIR rates are much less volatile than TN rates (see Table 6.7(a)and Figs. 6(a)-6(b)) and the hypothesis that generalised CIR volatility is an unbiased estimator of TN rate volatility is rejected (see Table 6.7(b) by the F-test.

---

[11] See Brown and Dybvig (1986) and Chapter 5, section 5.2.2 for justification and elaboration.

Table 6.7(a)

Distribution of monthly average of generalised CIR volatility (σr^{β}) and monthly average of standard deviation of change in 13-week TN rate

	t-value	Monthly average of σr^{β}	Monthly average of std of change in 13-week TN rate
Mean		0.011269	0.0226
Std		0.009890	0.019
Min		1.018E-06	2.1E-03
Max		0.060857	0.1207
H_0	5.25*		

Note:
H_0 is the null that the mean of monthly averages of generalised CIR volatility is equal to the mean of monthly averages of standard deviation of change in 13-week TN rate.
* significant at 1%.

Table 6.7(b)

Regression of the volatility of the change in TN rate on volatility of generalised CIR

Dependent variable	Independent variable	Intercept	Reg. Coeff	*F*-value	$\overline{R}^2$
$\hat{s}_t$	$\hat{s}_{gcir}$	0.0234 (0.33E-02)	-0.416 (0.2257)	59.58	0.1605

Notes:
(a) standard errors are in parentheses;
(b) The null of the *F*-test is that $\alpha = 0.0$ and $\beta = 1.0$.

6.2.3 Comparison with previous studies

A comparison with previous studies on the issue of theoretical CIR volatility and TN volatility is presented in Table 6.7(c). The order of our result (1.13%) is about one half or one third of those reported by Brown and Dybvig (1986), Brown and Schaefer (1994), Chiarella et al (1989) and Chapter 5 where the original CIR model was tested. This inconsistency was considered in Chapter 5 and a number of reasons were offered (see section 5.2.3). In particular reference to this chapter, this low value is more likely to be caused by the form of the volatility function, σr^{β} , where σ in the present model is about one half of σ in the CIR model (see Table 6.6(e)) and β is greater than 0.5 (see section 6.3). A natural question is which form of the volatility function is best. As a specified volatility function bears on the overall performance and goodness of fit of the model, the answer has to be settled in favour of the volatility function of the model that performs best. This issue will be considered in Chapter 8 where all the tested models are assessed in a comparative perspective.

In addition, quarterly and semi-annual estimates of σ are more likely to be significant than daily estimates (see Table 6.6(d)). This is due to the increase in the degrees of freedom as data are pooled (see Chapter 5).

Figure 6.3(a)

Monthly generalised CIR Volatility

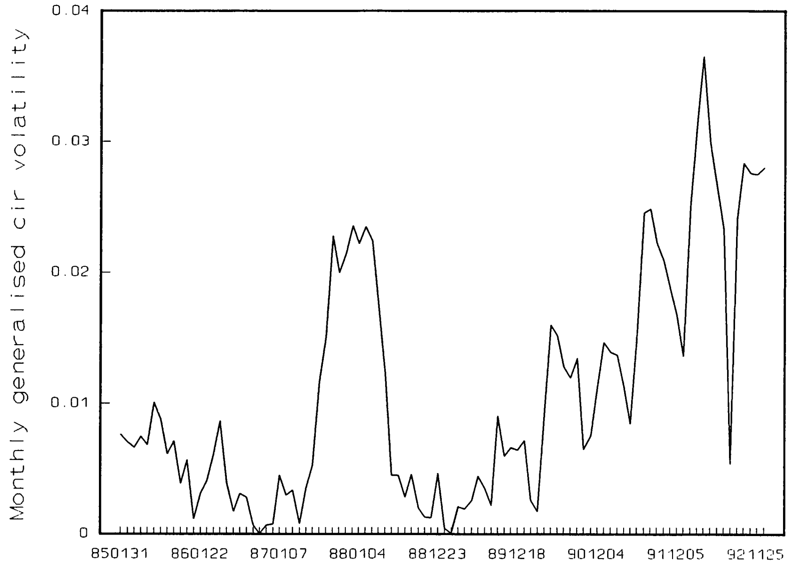


Figure 6.3(b)

Monthly Treasury Note Volatility

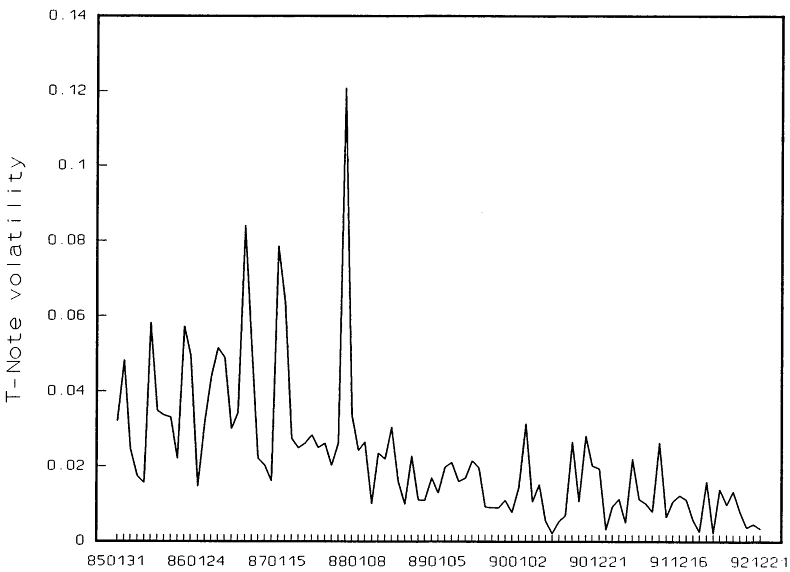


Table 6.7(c)

Comparison of Generalised CIR volatility* and time series estimates

Study	Market	Mean (%)	Mean Std (%)	Is CIR volatility unbiased predictor of time series estimates ?
Brown & Dibvig (1986)	USA (1977-83) Monthly data	1.95[a]	2.26[a]	Yes
Brown & Schaefer (1994)	UK (1984-89) Daily data	2.65[b]	2.33[b]	No
Baron et al (1991)	Italy (1983-90) Daily data	2.05[c]	2.797[c]	Not reported
Chiarella et al (1989)	Australia (1978-87) Monthly data	2.95[d]	3.16[d]	No
Chapter 5: CIR model	Australia (1985-92) Daily data	2.30[e]	3.54[e]	No
This chapter: Generalised CIR model	Australia (1985-92) Daily data	1.13[f]	0.99[f]	No

Sources:
* Note that volatility of generalised CIR model is σr^{β} while volatility of CIR model is $\sigma\sqrt{r}$;
a Calculated from Brown & Dibvig (1986, Table I);
[b] Brown & Schaefer (1994, p. 28);
[c] Barone et al (1991, Tables 3 and 5);
[d] Calculated from Chiarella et al (1989, Tables 1 and 3);
[e] Chapter 5 (Table 5.7(c);
[f] Table 6.7(a).

6.3 ESTIMATES OF κ, θ, β, λ, γ

The high correlation and instability of the variables of equilibrium models of the term structure are well recognised in the empirical literature[12] in this area of research. A consequence of the correlation problem is the high standard errors of parameter estimates (or low t-statistics)[13] while the instability problem throws doubt on the assumption of constant parameters underlying these models. Evidence of collinearity and instability of the CIR variables is presented and discussed in Chapter 5. These problems are exacerbated in the generalised CIR model by the presence of two additional parameters, β and γ. Hence only the cross-sectional *t*-statistics of θ, β, and γ are significant and reported[14] (see Table 6.8(d)).

Summary statistics of daily, quarterly, and semi-annual estimates of κ, θ, β, λ, γ are presented in Tables 6.8(a)-6.8(c) while the time series of these estimates are graphed in Fig. 6.4(a)-6.4(r). These results indicate that:

Firstly, β and γ are significantly different[15] from 0.5[16] (see Tables 6.8(a)-6.8(c) and Fig. 6.4(d), Fig. 6.4(j), Fig. 6.4(p), Fig. 6.4(f), Fig. 6.4(l), and Fig. 6.4(r)). This result implies that the generalised CIR model is distinct from the CIR model with respect to this sampling data.

Secondly, the speed of adjustment, κ and the market price of risk, λ, being positive and negative respectively during the sampling period, hence they are

---

[12] See Longstaff (1989,), Longstaff and Schwartz (1992, p.1278), Chan et al (1992), and Chapter 5.

[13] See Johnstone (1984, p.240).

[14] Note the difference between cross sectional and time series significance of parameter estimates. The former shows whether they are significant factors in the pricing of a bond at a given point in time while the latter shows how volatile they are over time. Thus, the significance of time series parameter estimates do not impact on the pricing of bonds.

[15] The *t*-statistics for the null hypotheses that $\beta = 0.5$ and $\gamma = 0.5$ are 34.73 and 794.18 respectively while the *t*-statistics for the cases of quarterly and semi-annual estimates of β and γ are even much higher.

[16] Under the CIR model β and γ are 0.5.

consistent with model specification (see Tables 6.8(b)-6.8(c) and Figs. 6.4(g), 6.4(k), 6.4(m), 6.4(q));

Thirdly, across all the three modes of estimation, the long run equilibrium spot rate, θ, vibrates around 10 percent (see Tables 6.8(a)-6.8(c), Figs. 6.4(b), 6.4(h), 6.4(n)). In fact, in daily estimation, θ appears to follow an oscillating process (see Fig. 6.4(b)).

Fourthly, all the remaining parameters are not stable or constant as specified by the model. In any event, both the CIR and generalised models are overparameterised, resulting in serious collinearity and instability of parameter estimates.

Lastly, comparison with previous research is limited as this generalised model is first estimated in this study.

Table 6.8(a)

Summary statistics of daily parameter estimates

	Mean	Std	Min	Max
κ	0.034846	0.034871	0.000051	0.461247
θ	0.118155	0.026615	0.02018	0.214978
β	0.516026	0.020532	0.000003	0.672209
λ	-0.03176	0.026848	-0.1669	-3.3E-06
γ	0.535279	0.029991	0.480407	0.710986

Table 6.8(b)

Summary statistics of quarterly parameter estimates

	Mean	Std	Min	Max
κ	0.028244	0.026489	0.012886	0.120818
θ	0.09696	0.014034	0.053193	0.145552
σ	0.015055	0.009023	0.000936	0.052959
β	0.298261	0.007053	0.27783	0.311365
λ	-0.00552	0.003087	-0.01135	-0.00006
γ	0.017137	0.007544	0.000724	0.021805

Table 6.8(c)

Summary statistics of semi-annual parameter estimates

	Mean	Std	Min	Max
κ	0.04094	0.031761	0.012803	0.094845
θ	0.096515	0.010411	0.063221	0.105306
σ	0.012479	0.00481	0.002192	0.016517
β	0.296348	0.009718	0.265138	0.301679
λ	-0.00493	0.003196	-0.01206	-0.00031
γ	0.020847	0.013162	0.002937	0.064514

Table 6.8(d)

Significance proportions of t-statistics of generalised CIR parameters

Daily estimates

	Significance level	Sample Proportion	Sample Proportion	Sample proportion
θ	1%	0.2381	0.2500	0.3125
	5%	0.5013	0.5312	0.5625
	10%	0.8080	0.8750	0.9375
β	1%	0.4987	0.5000	0.5625
	5%	0.5827	0.6250	0.6250
	10%	0.8371	0.8750	0.8750
γ	1%	0.5099	0.5325	0.5625
	5%	0.5791	0.6875	0.6250
	10%	0.8642	0.9062	0.9375

Fig. 6.4(a)

Daily estimates of κ (1985-1992)

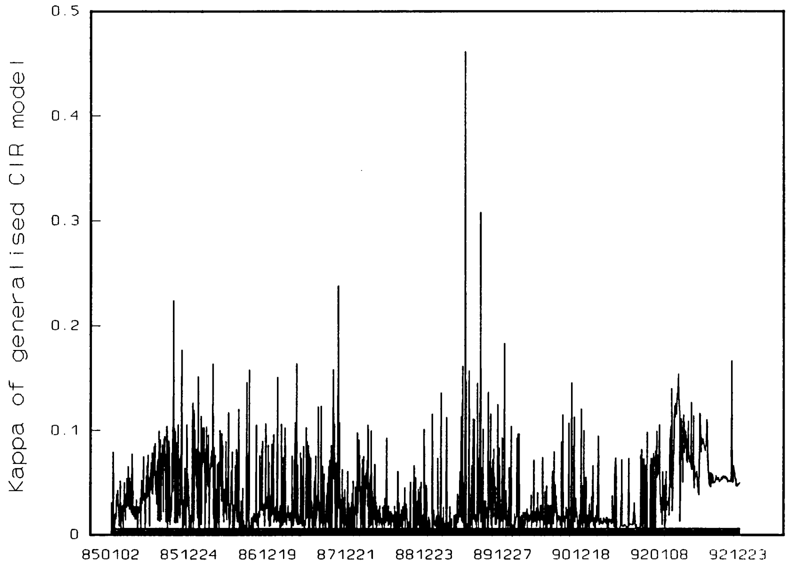


Fig. 6.4(b)

Daily estimates of θ (1985-1992)

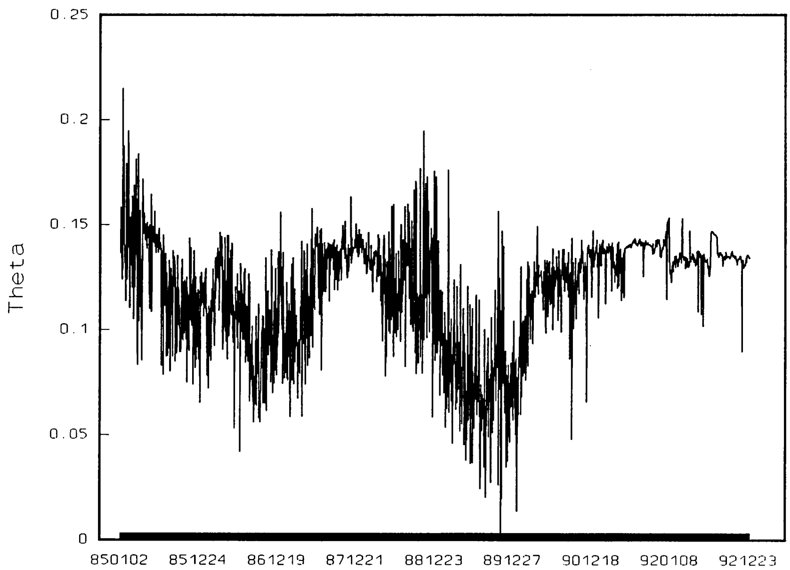


Fig. 6.4(c)

Daily estimates of σ (1985-1992)

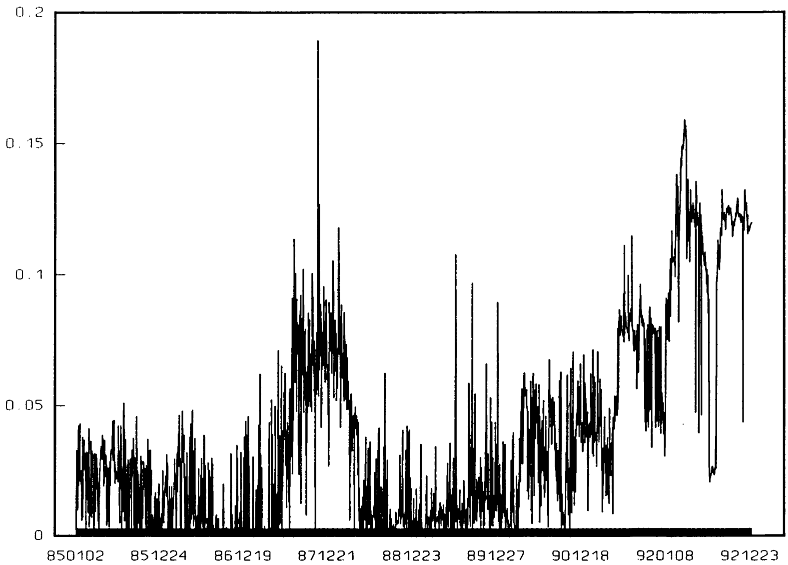


Fig. 6.4(d)

Daily estimates of β (1985-1992)

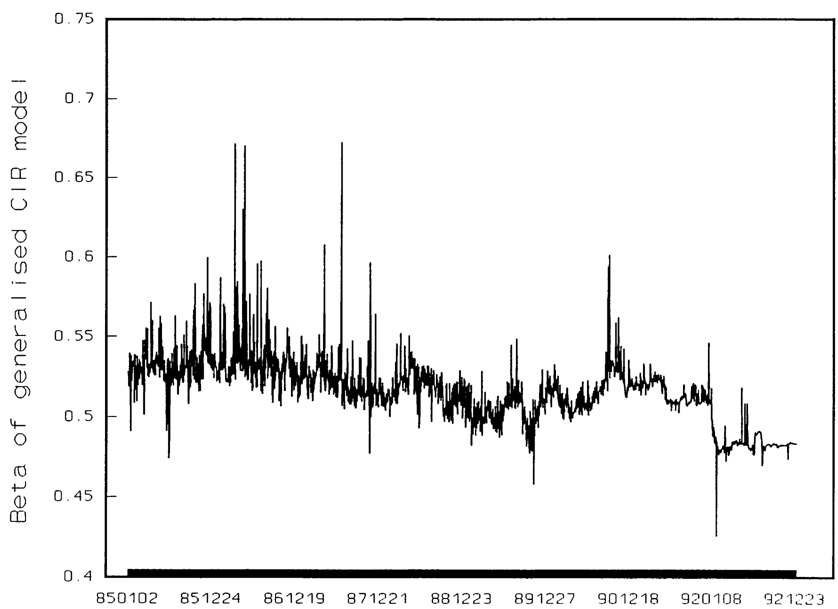


Fig. 6.4(e)

Daily estimates of λ (1985-1992)

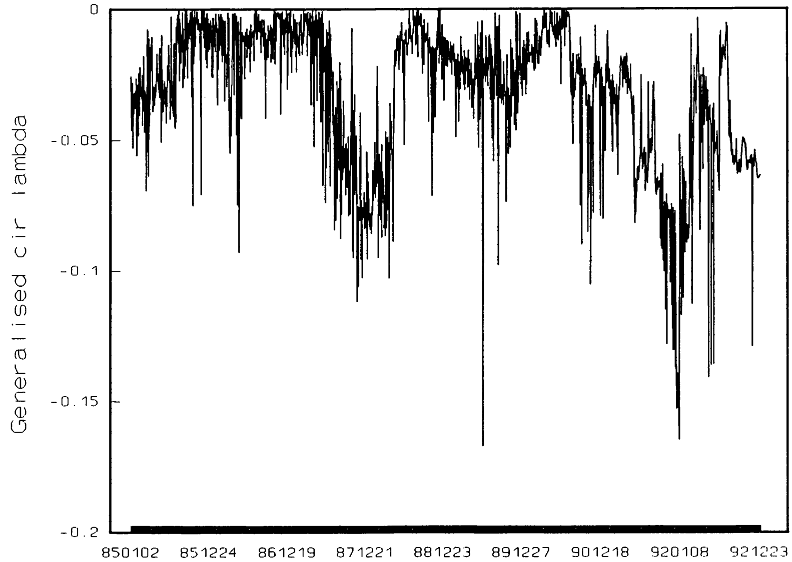


Fig. 6.4(f)

Daily estimates of γ (1985-1992)

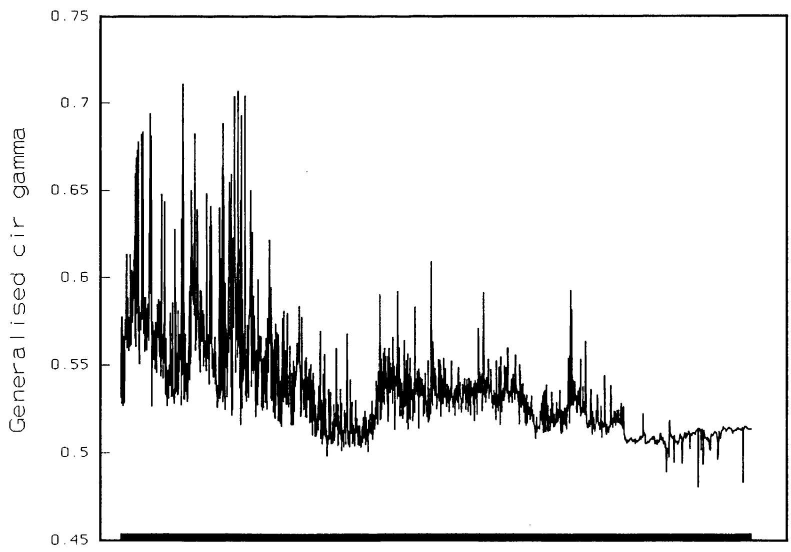


Fig. 6.4(g)

Quarterly estimates of κ (1985-1992)

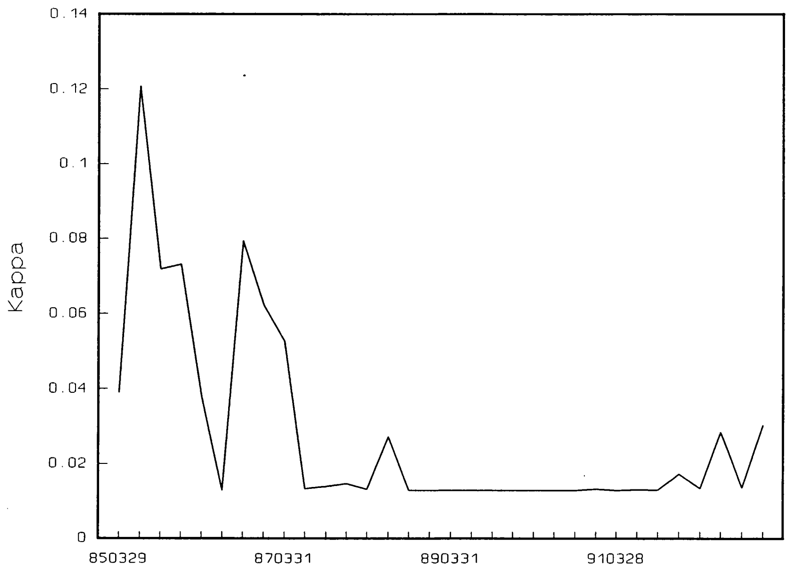


Fig. 6.4(h)

Quarterly estimates of θ (1985-1992)

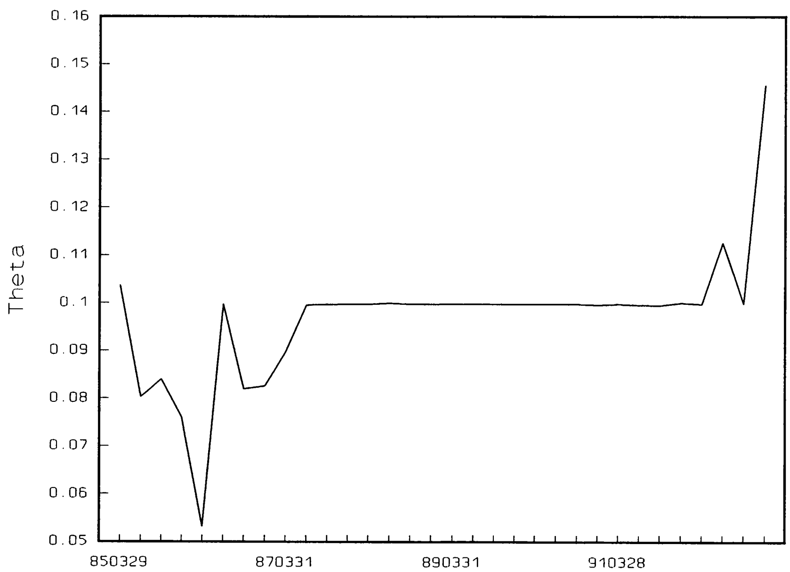


Fig. 6.4(i)

Quarterly estimates of σ (1985-1992)

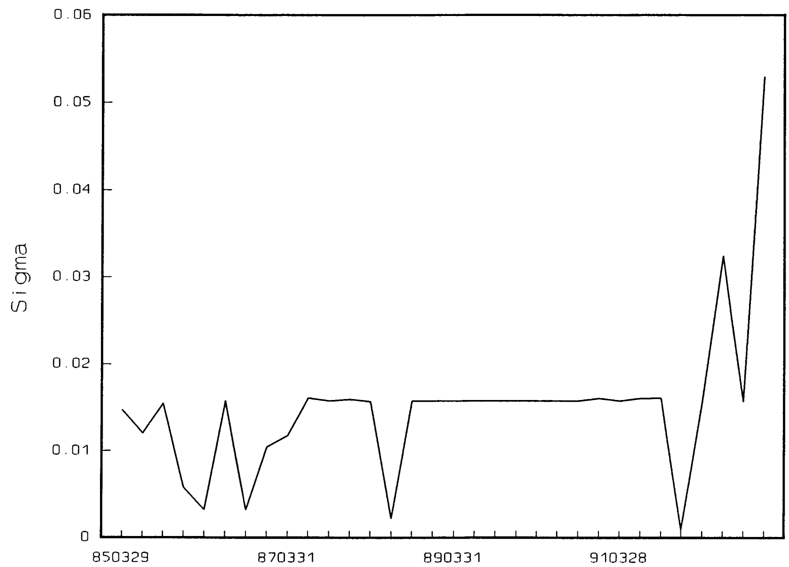


Fig. 6.4(j)

Quarterly estimates of β (1985-1992)

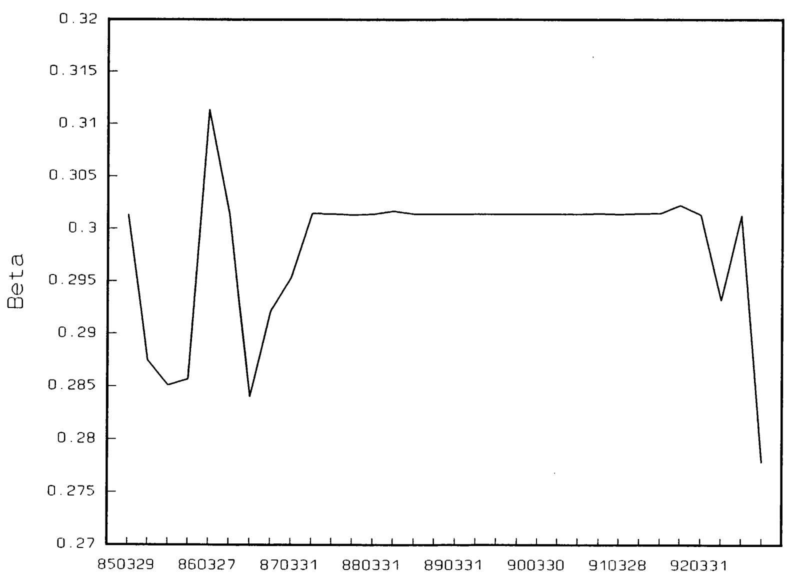


Fig. 6.4(k)

Quarterly estimates of λ (1985-1992)

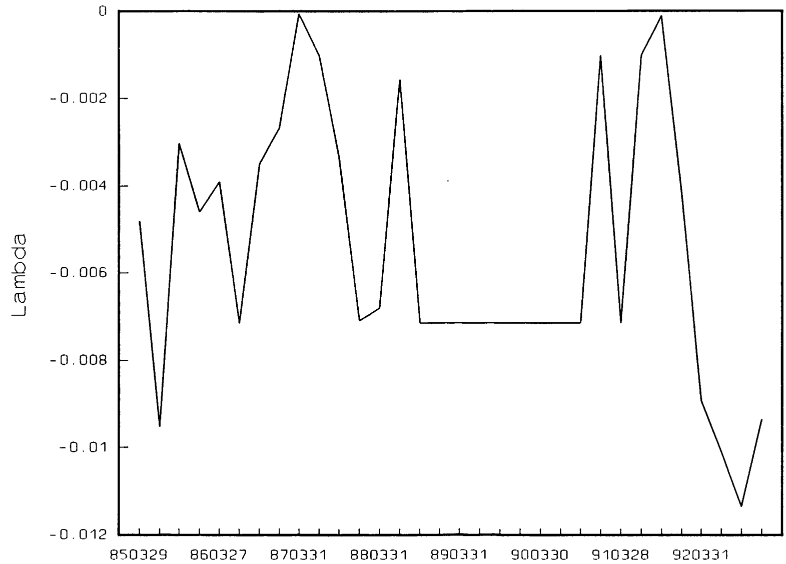


Fig. 6.4(l)

Quarterly estimates of γ (1985-1992)

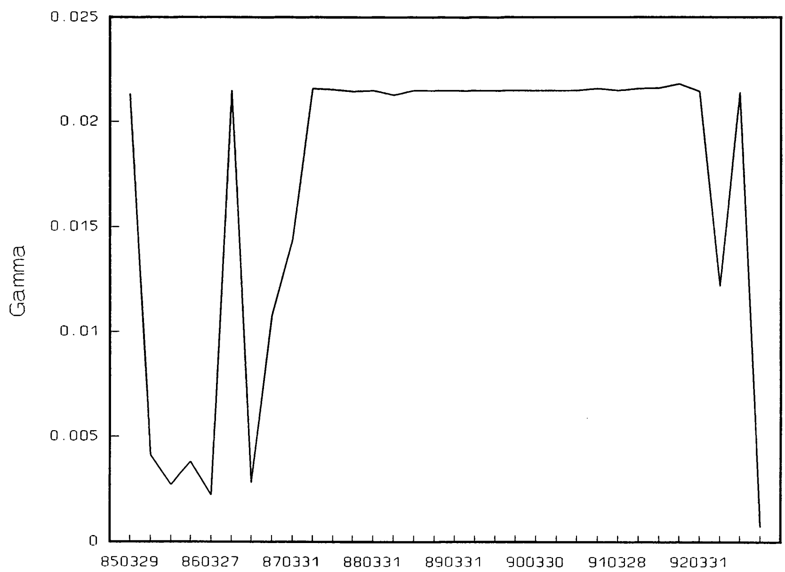


Fig. 6.4(m)

Semi-annual estimates of κ (1985-1992)

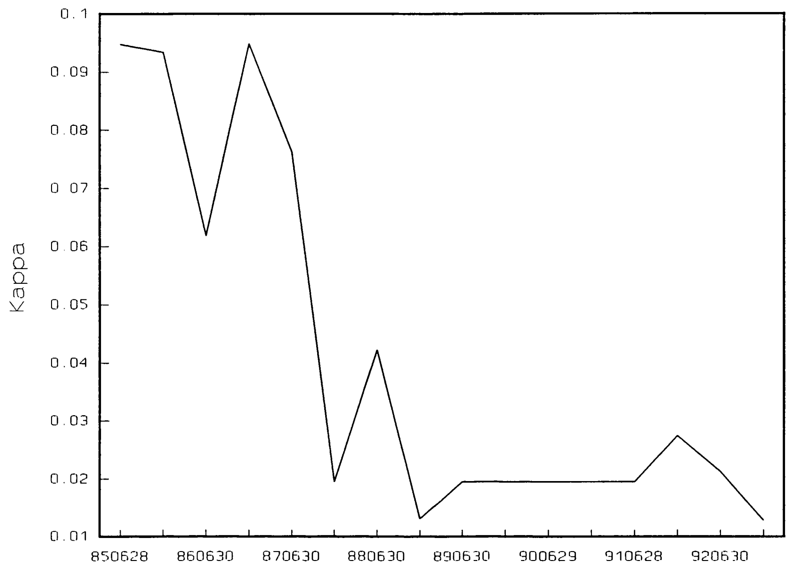


Fig. 6.4(n)

Semi-annual estimates of θ (1985-1992)

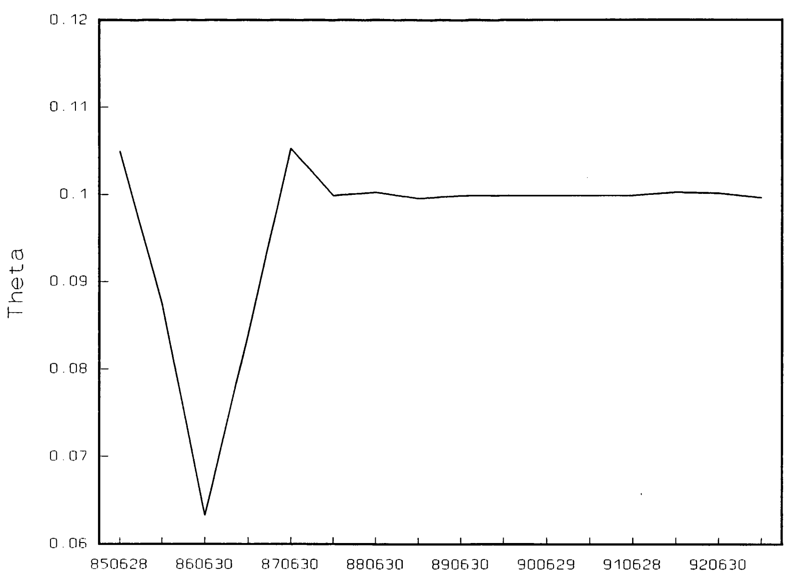


Fig. 6.4(o)

Semi-annual estimates of σ (1985-1992)

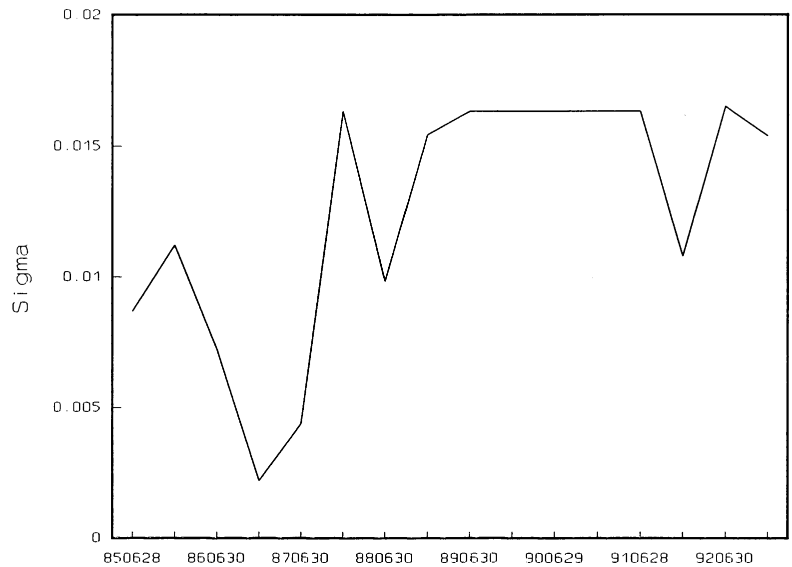


Fig. 6.4(p)

Semi-annual estimates of β (1985-1992)

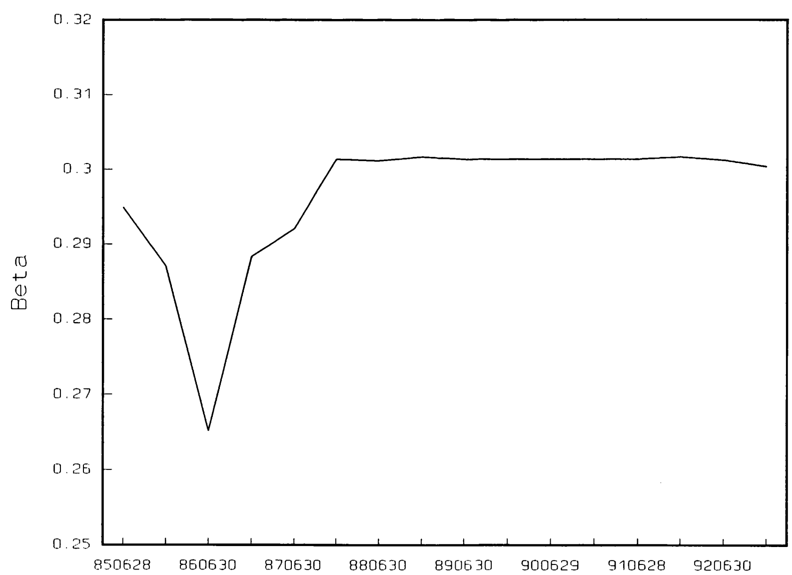


Fig. 6.4(q)

Semi-annual estimates of λ (1985-1992)

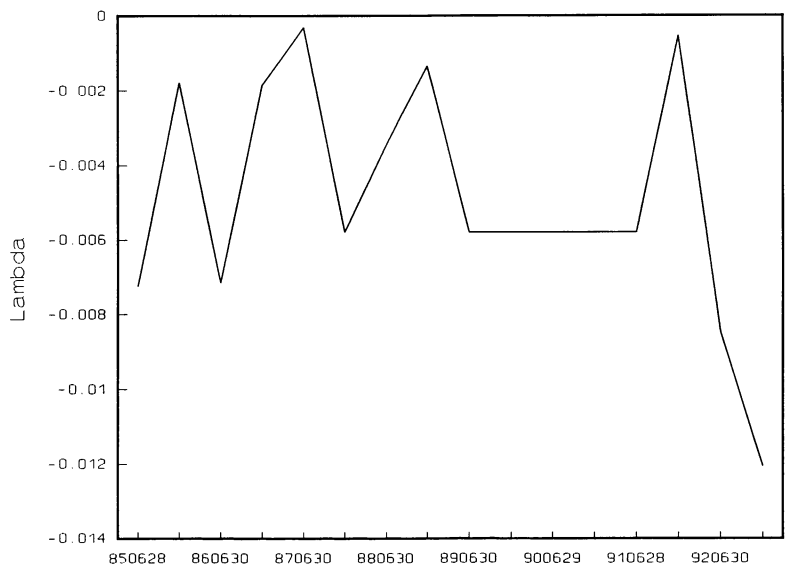
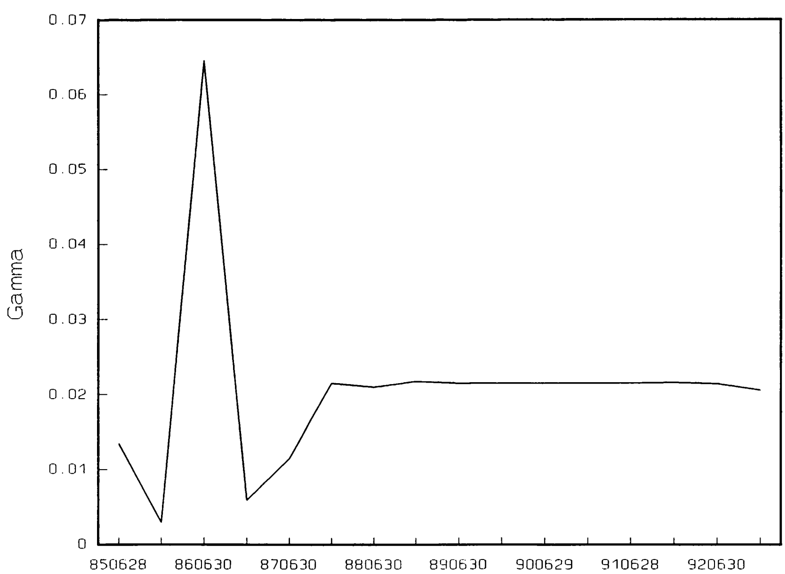


Fig. 6.4(r)

Semi-annually estimates of γ (1985-1992)



6.4 EXTENSION OF THE GOODNESS OF FIT AND PREDICTIVENESS OF THE GENERALISED CIR MODEL[17]

6.4.1 Measurement

In this section we extend the measurement of the goodness of fit and predictiveness of the generalised CIR model to the whole spectrum of the term structure of interest rate by means of the price mean error and the rate mean square error. The former highlights the difference between the observed coupon bond price and the predicted generalised coupon bond price while the latter shows the difference between the zero-coupon generalised CIR term structure and the zero-coupon Chebychev term structure[18]. These measures are calculated using daily, quarterly and semi-annual estimates to ascertain which method of estimation is more accurate.

Results are presented in Tables 6.9(a) and 6.9(b) which confirm the expectation that daily estimation and within sample errors outperform longer-term estimation and outside sample errors respectively. In particular, the mean price errors are \$0.43, \$0.70, and \$0.73 in predicting bond prices one month, three months, and six months ahead of a given day (see Table 6.9(a)). As bonds are of \$100 denominations these errors can be interpreted as percentage errors. Furthermore, these errors are significantly different from each other, except the case of daily and six-month errors (see Table 6.9(a), last row). This rather paradoxical finding is the result of large variances of errors associated with the six-month prediction procedure.

---

[17] See Chapter 5, section 5.5 for a detailed discussion of the rationale of the measures of goodness of fit and predictiveness.

[18] The price mean error is restricted to the longest maturity of an observed coupon bond on a given day while the rate mean square error is extended to any maturity. Thus the former highlights the errors of the model relative to observed bond prices while the latter shows the errors relative to the Chebychev term structure. It should be recalled that the Chebychev term structure is a curve-fitted term structure.

The mean rate errors are in the order of 11 and 75 basis points[19]for within-sample and out-of-sample[20] sample term structures respectively. These figures are consistent with the expectation that within sample errors are smaller than out-of-sample errors.

6.4.2 Comparison with previous research

The errors reported in the preceding section have not been calculated in previous research. Hence, comparison is limited. The errors reported in the empirical literature of the term structure are regression errors which are, by design, forced to converge to zero. In order to provide a comparison consistent with this literature our regression errors are reported together with those of other studies in Table 6.9(c). Overall they are of the same order as those of previous studies.

---

[19] These are the square roots of MSEs in Table 6.9(b). Thus 11 and 75 basis points are the square roots of 1.2908E-06 and 5.7476E-05.

[20] Within sample errors are those calculated using the longest observed maturity of given days while out-of-sample errors while out-of-sample errors extend the longest maturity of an observed bond on any given day by ten years.

Table 6.9 (a)

Generalised CIR pricing errors per $100 bond
(1985-1992)

	Daily estimation	Quarterly estimation	Semi-annual estimation
	One-month prediction	Three-month prediction	Six-month prediction
Price Mean Error	$0.43	$0.70	$0.73
Hypothesis	$H_0(d,q)$	$H_0(q,sa)$	$H_0(d,sa)$
t-value	-4.35**	-2.62**	-0.23

Notes:
(a) $H_0(d,q)$ is the null that the mean of daily pricing errors is equal to the mean of quarterly pricing errors;
(b) $H_0(q,sa)$ is the null that the mean of quarterly pricing errors is equal to the mean of semi-annual pricing errors;
(c) $H_0(d,sa)$ is the null that the mean of daily pricing errors is equal to the mean of semi-annual pricing errors.
** Significant at 1%.

Table 6.9(b)

Daily within-sample and out-of-sample rate MSEs
(1985-1992)

	Within sample rate MSE	Out-of-sample rate MSE
Rate MSE	1.2908E-06	5.7028E-05
Null hypothesis	$\frac{\text{Out-of-sample rate MSE}}{\text{Within-sample rate MSE}} = 1$	
F-value	44.53*	

Notes:
(a) The null hypothesis is that within sample rate MSE is equal to out-of-sample rate MSE
(b) Only daily MSEs are calculated because the Chebyshev polynomials are estimated daily.

Table 6.9(c)

Comparison of regression pricing errors

	Market	Regression pricing error (Standard error)
Brown & Schaefer (1994)	UK Daily data (1984-89)	£0.20 (0.40)
Munik & Schotman (1994)	Dutch Daily data (1989-90)	0.17 guilder (0.04)
Chapter 5: CIR model	Australian Daily (1985-92)	-\$0.25 (0.22)
This chapter: Generalised CIR model	Australian Daily (1985-92)	\$0.22 (0.16)

Notes:
(a)Mean regression errors are calculated by comparing observed prices on day t with CIR prices estimated for day t using the observed prices on day t. Least square regression entails that the sum of these errors tend to zero asymptotically..
(b) While the errors are in different currencies they result from fitting bonds of 100 units of respective currencies. Hence the errors can be interpreted as percentage of prices and are thus free from the exchange rate problem.

6.5 SAMPLES OF GENERALISED CIR, CHEBYCHEV TERM STRUCTURES, OBSERVED AND FITTED BOND PRICES.

In this section we present a sample of daily generalised CIR, Chebychev term structures, generalise CIR-fitted and observed bond prices on 20/12/1990 (see Figs. 6.5(a) - 6.5(f)). Generally speaking, several conclusions can be drawn from these graphs:

(i) Bond price errors are in the maximum order of \$0.30 per \$100 bond (see Figs. 6.5(a)-6.5(b)), which are consistent with the errors measured by the CIR model (see Chapter 5);

(ii) The generalised CIR term structure extending beyond the maximum observed maturity of 9.5 years is humped shape (see Fig. 6.5(c)), hence also consistent with the implication of the generalised CIR model;

(iii) The errors relative to the Chebychev fitted term structure are consistent with the expectation that those outside the maximum observed maturity are larger than those inside (see Figs. 6.5(d)-6.5(e)); and

(iv) Even at the maturity of nearly 30 years the errors are relatively small at approximately 1.5 percent (see Fig. 6.5(e)).

On the whole the results for this given day are similar to those achieved by the CIR model described in Chapter 5.

Fig. 6.5(a)

Observed and generalised CIR-fitted dollar bond prices (20/12/90)

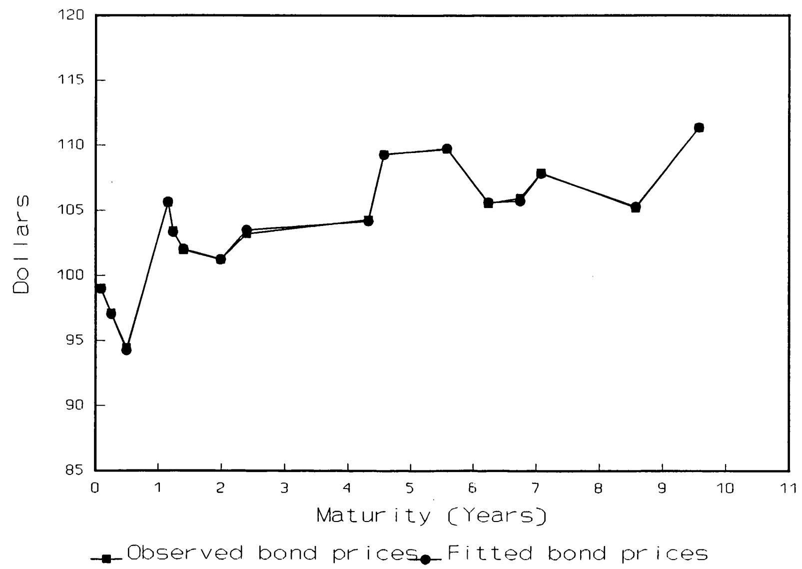


Fig. 6.5(b)

Dollars bond price errors

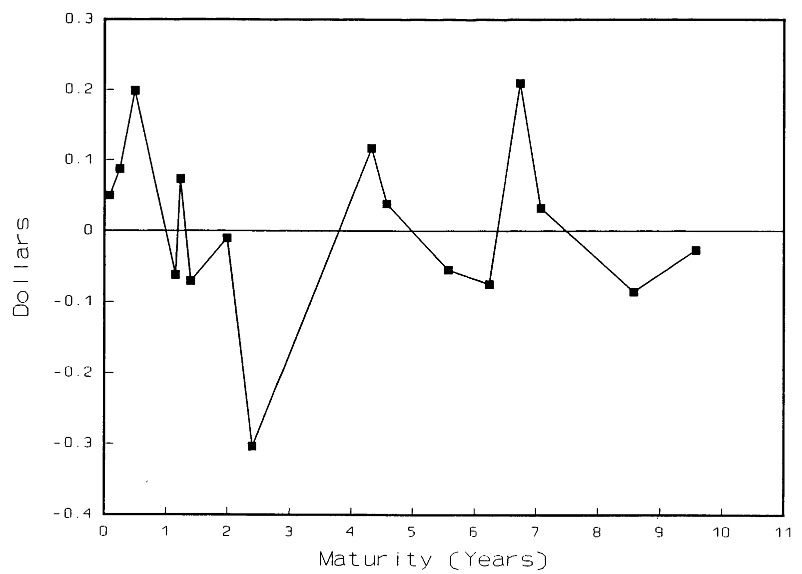


Fig. 6.5(c)

Genralised CIR term structure of interest rates (20/12/1990)

Daily estimates

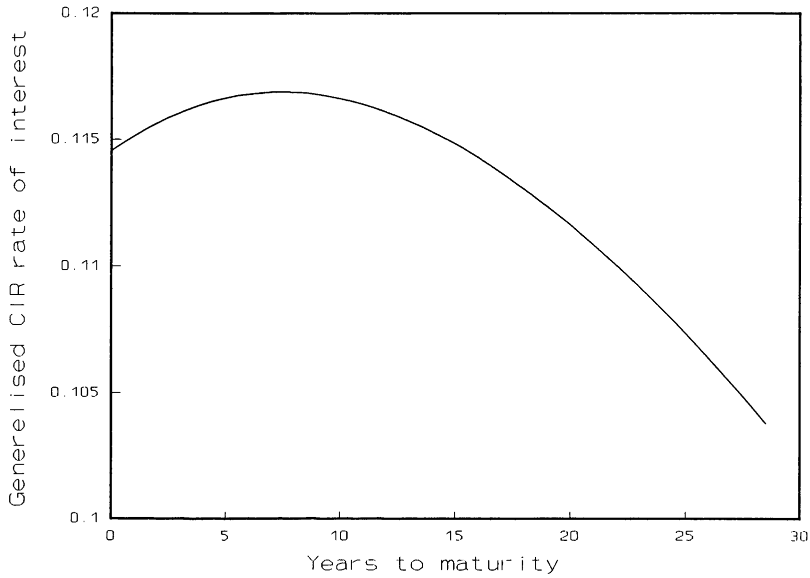


Fig. 6.5(d)

Chebychev and generalised CIR term structures (20/12/1990)

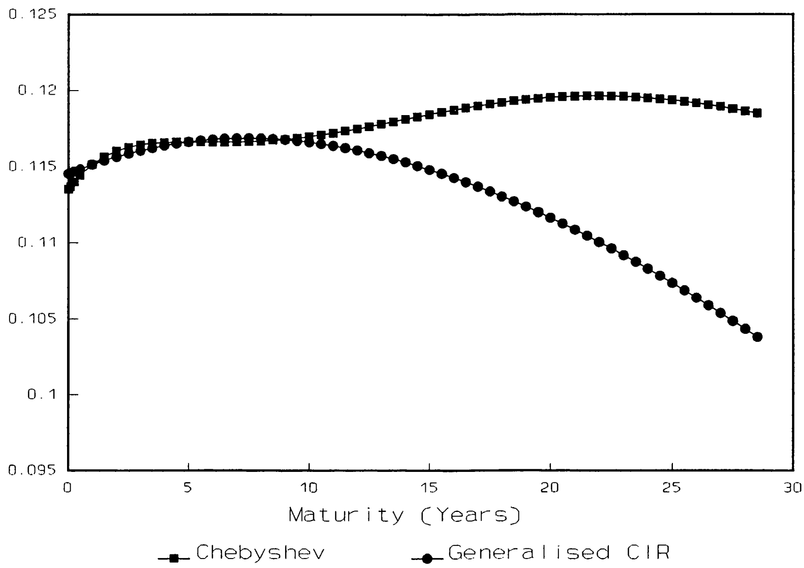
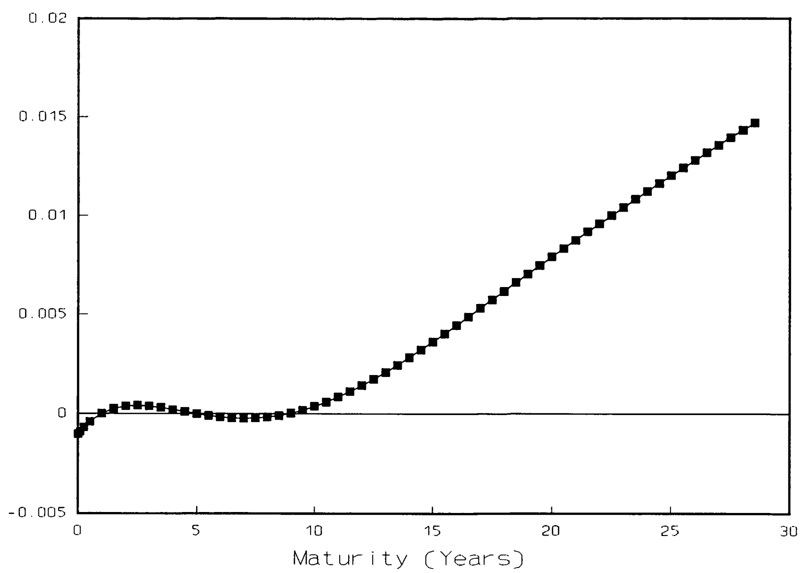


Fig. 6.5(e)

Rate difference between Chebychev and generalised CIR term structures (20/12/1990)



Note: Rate difference = Chebyshev rate *less* generalised CIR rate

6.6 CONCLUSION

The overall objective of this chapter is to estimate the generalised CIR model. A benefit of this exercise, inter alia, is to get separate estimates of those parameters that were not possible under the CIR model, in particular, the speed of adjustment, κ, the equilibrium spot interest rate, θ, and the market price of risk, λ. As in Chapter 5 we address ourself to two major issues: (i) empirical support for parameter stability as implied by the model; and (ii) goodness of fit and predictiveness of the model.

The empirical implementation is effected by means of non-linear regression which minimises the sum of bond price errors. The error is defined as either observed bond price less model price (price norm) or the logarithm of observed bond price less the logarithm of model price (logarithmic norm). Furthermore the nonlinear regression is applied in two modes: (i) unconstrained mode whereby all the model parameters vary from day to day; and (ii) constrained mode whereby only the spot rate, r, varies from day to day while other parameters are kept constant over either each quarter of a year or half of a year.

Several results are obtained:

Firstly, the logarithmic norm, first implemented in this study, is a better method of estimation than the price norm in terms of Akaike and Schwartz criteria;

Secondly, daily estimation performs best in terms of goodness of fit and hence is not unlike a curve fitting technique; yet retaining the qualities of an economic equilibrium model;

Thirdly, the 13-week generalised CIR rates are unbiased estimators of 13-week Treasury Note rates while the model spot rate significantly underestimates both the cash rate and the 13-week Treasury Note rate.

Fourthly, except for the spot rate, the model specifies constant parameters; in this respect the parameters are unstable and vary from one period to another; thus the model is misspecified although in quarterly and semi-annual estimations the instability is decidedly much less than in daily estimation. Collinearity, a symptom of overparameterisation, is even more severe than in the case of the generalised CIR model owing to the addition of two extra parameters.

While daily parameter fluctuations improve the accuracy of estimation, it is evidence of model misspecification. In this regard, it is not unlike the Black-Scholes model whereby increased accuracy by using implied standard deviation is inconsistent with the model's assumption of constant variance.

On balance, while this model has not been estimated in previous research, the overall results are consistent with the CIR model, except that the level of accuracy of estimation of this model is slightly better, in terms of minimum errors. An additional advantage is that it provides measures of <u>separate</u> parameters which are, on the whole, of the right sign as specified by the model.

CHAPTER 7

VASICEK MODEL: EMPIRICAL RESULTS

In this chapter we present the empirical results of estimating the Vasicek model using Australian bond data from 1985 to 1992. The two major issues we shall address ourself to are: (i) empirical support for parameter stability as implied by the model; and (ii) goodness of fit and predictiveness of the model. Toward this end, various statistical aspects of parameter estimates are explored together with several investigations undertaken to deepen our understanding of the bahaviour of the model in the Australian context. The format of this chapter follows that of Chapter 5, hence only those results which are particularly relevant to the Vasicek model are highlighted while references to Chapter 5 are made whenever it is necessary. Thus, estimates of the model are presented, analysed and discussed in sections 7.1-7.5. A snap shot of the Vasicek term structure is provided in section 7.6 while section 7.7 concludes the chapter.

The Vasicek model is given by the following bond pricing equation[1],

$$P(t,s,r) \;=\; \exp\left[\frac{1}{\kappa}(1-e^{-\kappa(T-t)})(R(\infty)-r)-(T-t)R(\infty)-\frac{\sigma^2}{4\kappa^3}(1-e^{-\kappa(T-t)})^2\right] \qquad (1)$$

where the stochastic process of the spot rate is

$$dr \;=\; \kappa(\theta-r)dt \;+\; \sigma dz \qquad (2)$$

and, $R_{vasi}\,(\infty)$, the yield on a very long bond as $T \rightarrow \infty$ is

[1] See Chapter 2 and/or Chapter 3 for details. Also as noted in Chapters 2 and 3, the parameter λ in the Vasicek model is positive, and to make it consistent with the CIR model, a negative sign is placed before Vasicek λ.

$$R_{vasi}(\infty) = \theta - \frac{\sigma\lambda}{\kappa} - \frac{1}{2}\left[\frac{\sigma^2}{\kappa^2}\right] \tag{3}$$

Following the least square minimisation procedure presented in Chapter 3 the vector of parameter estimates are:

$$\underline{a}_{vasi} = \begin{bmatrix} r \\ \kappa \\ \sigma \\ \theta - \lambda \end{bmatrix} \tag{4}$$

As in Chapters 5 and 6 the estimation procedure includes: (i) constrained and unconstrained minimisation[2]; and (ii) price norm and logarithmic norm[3]. The logarithmic norm is found to be superior in terms of the Akaike and Schwartz criteria (see Table 7.1) while the normality and heteroskedasticity tests (see Table 7.2) indicate that nonlinear least square estimation is equivalent to maximum likelihood[4] for the majority of daily, quarterly and semi-annual cross sections. For example, the Akaike statistic[5] for the logarithmic and price norms are -2.1671 and -2.1098 respectively (see Table 7.1, third row, second and third column). Similarly, at 1% level of significance, 73.88% of the daily cross sectional errors are normally distributed.

---

[2] Under constrained minimisation κ, σ, and $\theta - \lambda$ are kept constant over quarterly and semi-annual subperiods while the spot rate is allowed to vary daily. Under unsconstrained minimisation all the four parameters, r, κ, σ, and $\theta - \lambda$ vary from day to day.

[3] The sums of the price errors and the logarithms of the price errors are minimised under the price norm and logarithmic norm respectively.

[4] Hence the nonlinear regression procedure implemented for this chapter has the desirable properties of maximum likelihood estimators (see Green (1993, pp. 305-307)). Furthermore, various properties of linear least square estimation are carried over to nonlinear least squares only in asymptotical sense.

[5] The smaller the Aikake statistic the better the model.

7.1 ESTIMATES OF SPOT RATES

7.1.1 Distributional statistics

Estimates of Vasicek spot rates and their distributional statistics are presented in Tables 7.3(a)-7.3(b). A comparison of these rates with their closest observed counterparts in the Australian financial market, the cash rate and 13-week Treasury Note rate, is provided in Tables 7.3(c)-7.3(d). A graphical presentation of the spot rate estimates and their proxies is given in Figs. 7.1(a)-7.1(c).

Several observations emerge from these tables and graphs. Firstly, while in theory the spot rate may be negative[6] it is, in fact, positive for the entire sample period, (1985-92), (see Table 7.3(a)-7.3(b) and Fig. 7.1)). This is consistent with the conjecture that the case of negative nominal interest rate is rare for practical considerations. Secondly, constrained and unconstrained procedures of estimation (see Table 7.3(b)) reveal that daily and semi-annual rates are insignificantly different while quarterly estimates are significantly different from daily and semi-annual estimates; (iii) the Vasicek spot rate is not significantly different from both the cash rate and the 13-week Treasury Note rate (see Table 7.3(c))[7]. Thirdly, Vasicek spot rates are less than their observed closest proxies, cash rate and 13-week Treasury Note rate (see Table 7.3(c)) as indicated by high *t*-statistics (8.141 and 9.064). The difference between the two series (13-week Treasury Note rate *less* Vasicek spot rate) and (cash rate *less* Vasicek spot rate) is not significant, hence implying that the two observed rates are interchangeable.

---

[6] Under the Vasicek model the specification that volatility is a constant gives rise to possible negative spot rates (see Chapter 2).

[7] Unlike the CIR and Generalised CIR model the insignificant difference implies that there is no need to further investigate the relationship between the difference and the level of either the cash rate or the Treasury Note rate.

7.1.2 A test of the unbiasedness of 13-week Vasicek interest rates[8]

Unlike the comparison of the instantaneous spot rate and 13-week Treasury Note, it is interesting to note that the 13-week Vasicek rate is an unbiased estimator of 13-week Treasury Note rate (see Tables 7.4(a) and 7.4(b)). In this case the maturities of the two rates are exactly matched and the unbiasedness is investigated by means of : (i) regressing observed 13-week Treasury Note rates on day t against 13-week Vasicek interest rates based upon parameter estimates on day t; and (ii) regressing observed 13-week Treasury Note rates on day t + *one month* against 13-week Vasicek interest rates calculated for day t + *one month* using parameter values estimated on day t. It should be noted that the unbiasedness is also carried to <u>predicted</u> rates which are calculated out of the sample from which the model parameters are estimated.

[8] See Chapter 5 for further details on the rationale of this test.

Table 7.1

Distribution of information criteria of logarithmic and price norm
Daily cross-sections

	Akaike Criterion		Schwartz Criterion	
	Log Norm	Price Norm	Log Norm	Price Norm
Mean	-2.1671	-2.1098	-1.8411	-1.7838
Standard deviation	0.2588	0.2452	0.2588	0.2452

Note:
The information criteria are based on minimising the residual sum of squares; hence the smaller the criteria the better the model.

Table 7.2

Normality and heteroskedasticity tests of regression residuals

		Daily	Quarterly	Semi-annual
	Significance level	Proportion of cross-sections	Proportion of cross-sections	Proportion of cross-sections
Normality Test (Jarque-Berra)	1%	0.7388	0.7539	0.7955
	5%	0.8359	0.8656	0.8861
Heteroskedasticity (White)	1%	0.6250	0.4695	0.4886
	5%	0.7881	0.6134	0.6176

Note:
The entry 0.7388(column 4, row 3) means that at 1% level of significance the regression residual is normally distributed for 73.88% of the total quarterly cross sections (the spot rate is allowed to vary from day to day while the other parameters, σ, κ and θ - λ, are kept fixed over each quarter). Similarly the entry 0.4695 (column 4, row 5) means that at 1% level of significance the regression residual is homoskedastic for 46.95% of the total quarterly cross sections. Other entries are interpreted similarly.

Table 7.3(a)

Daily estimates of Vasicek spot rate

	Mean	Std	Min	Max
1985	0.12685	0.01241	0.10061	0.18048
1986	0.13269	0.01504	0.10684	0.17820
1987	0.12604	0.01727	0.08769	0.16214
1988	0.10827	0.01407	0.08362	0.15251
1989	0.16170	0.00772	0.14007	0.17067
1990	0.13278	0.01296	0.10814	0.16610
1991	0.09377	0.01390	0.04479	0.11454
1992	0.06049	0.00892	0.03496	0.07271
1985-1992	0.11812	0.03109	0.03496	0.18048

Table 7.3(b)

Distribution of Vasicek spot rate
Daily, quarterly and semi-annual estimates
(1985-1992)

	t-value	Daily	Quarterly	Semi-annual
Mean		1.1812E-01	5.9493E-02	1.2065E-01
Std		3.1091E-02	8.2584E-03	1.1843E-02
Minimum		3.4958E-02	3.959E-02	3.9462E-02
Maximum		1.8048E-01	7.3759E-02	8.1782E-02
H_0 *(d, q)*	32.29**			
H_0 *(d, sa)*	1.12			
H_0 *(q, sa)*	7.97**			

Notes:

(a) H_0 *(d, q)* is the null hypothesis that the mean of daily estimates of CIR spot rate is equal to the mean of its quarterly estimates;

(b) H_0 *(d, sa)* is the null hypothesis that the mean of daily estimates of CIR spot rate is equal to the mean of its semi-annual estimates;

(c) H_0 *(q, sa)* is the null hypothesis that the mean of quarterly estimates of CIR spot rate is equal to the mean of its semi-annual estimates.

(d) ** : significant at 1%.

Table 7.3(c)

Differences between observed TN rates and Vasicek spot rates (1985-1992)

	13-week TN rate - r	*Cash rate - r*
Mean	0.014278	0.012885
Standard deviation	1.7538E-03	1.4215E-03
t-value[a]	8.14	9.06
t-value[b]	0.5240	

Notes:

[a] The null hypothesis is that the mean of (observed rate *less* theoretical rate) is 0.0.

[b] The null hypothesis is that the mean of the series (13-week TN rate *less* r) is equal to the mean of the series (cash rate *less* r).

Table 7.4(a)

Test of unbiasedness of daily 13-week Vasicek rates of interest (1985-1992)

Observed rate (Dependent variable)	Estimated Vasicek rate (Independent variable)	α	β	F-value	$\bar{R}^2$
13-week TN	13-week	0.34E-09 (1.2E-05)	1.03192 (1.2E-03)	2.50	0.95376

Notes:

(a) Vasicek rates are calculated for day t, using the parameters estimated on day t.

(b) The null of the F test is $\alpha = 0.0$ and $\beta = 1.0$.

(c) Standard errors are in parentheses.

Table 7.4(b)

Test of unbiasedness of daily 13-week predicted Vasicek rates of interest (1985-1992)

Observed rate (Dependent variable)	Estimated Vasicek rate (Independent variable)	α	β	F-value	$\bar{R}^2$
13-week TN	13-week	0.45E-08 (3.1E-05)	1.03176 (3.1E-05)	0.85	0.92661

Notes

(a) Predicted Vasicek rates are calculated for day t + *one month*, using the parameters estimated on day t.

(b) The null of the F test is $\alpha = 0.0$ and $\beta = 1.0$.

(c) Standard errors are in parentheses.

Fig. 7.1 (a)

Daily estimates of Vasicek spot rates

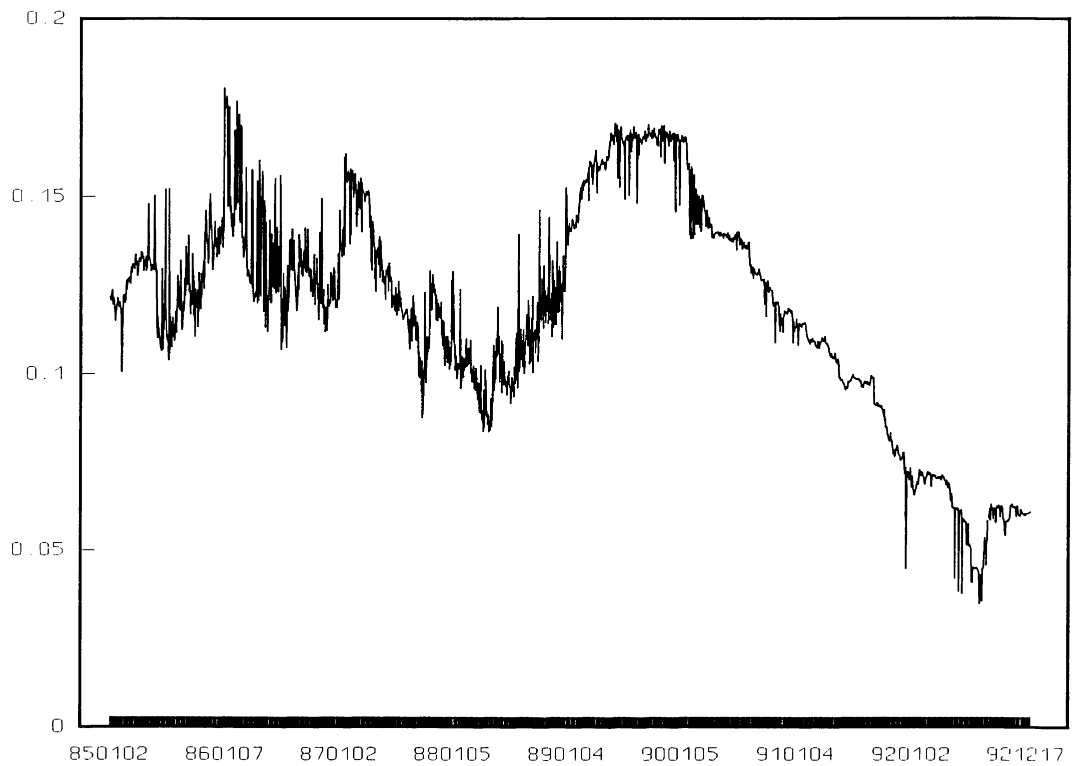
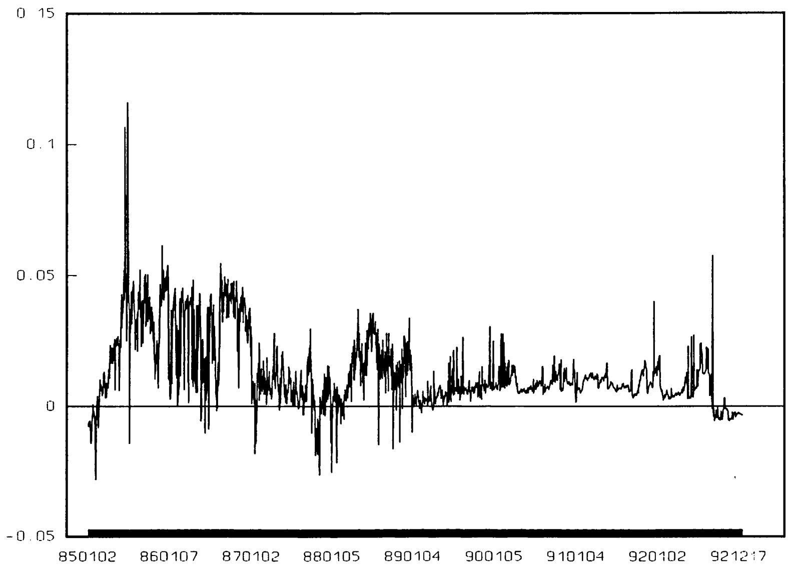


Fig. 7.1 (b)

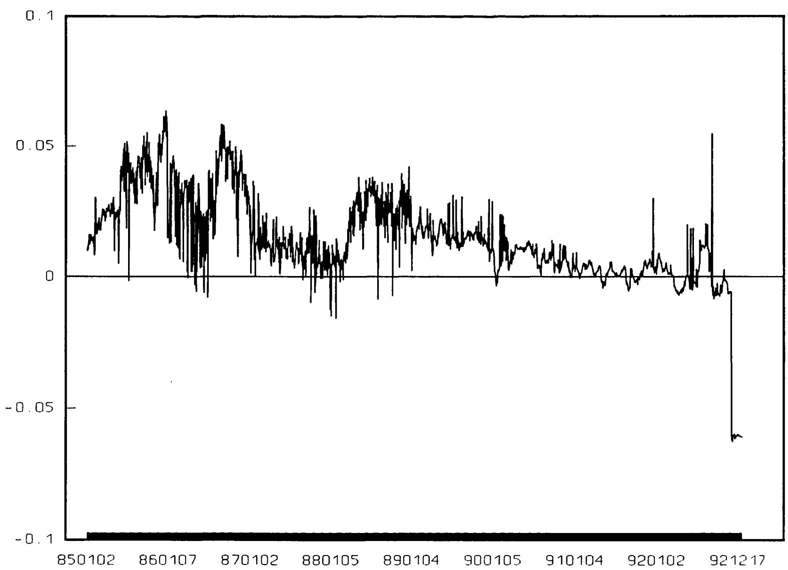
Difference between cash rate and Vasicek spot rate



Note: Difference = Cash rate *less* Vasicek spot rate

Fig. 7.1(c)

Difference between Treasury Note rate and Vasicek spot rate



Note: Difference = Treasury Note rate *less* Vasicek spot rate

7.1.3 Comparative analysis of the performance (predictiveness) of the Vasicek model

Essentially the question we seek to answer is " Will the Vascicek model estimated on day *t* and then used to predict the prices of 13-week and 26-week Treasury Notes on day $t + n$ months, where *n* is either 1 or 3 months, outperform a naive[9] model ?". The justification for this investigation is to exploit a feature of the data set that includes 13-week and 26-week Treasury Note rates on each and every day in the sample (1985-1992). The results are presented in Tables 7.5(a) and 7.5(b) which show that the Vasicek model significantly outperforms the naive model using the MSE criterion[10]. For example, the error associated with the Vasicek model in predicting the 13-week bond price one month in advance is \$0.15 while that of the naive model is \$0.21 (see Table 7.5(a), second row, second and third column). Furthermore, the smaller errors are significant as shown by the *F*-statistics (see Tables 7.5(a) and 7.5(b)).

7.1.4 Comparison with previous research

A comparison of the Vasicek spot rate estimates and those of previous studies is given in Table 7.5(c). This shows the consistency of the results achieved across the three models tested in this study, CIR, generalised CIR, and Vascicek but these results are inconsistent with other studies[11]. Differences in sampling

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[9] A naive model assuming constant interest rates is a 'do nothing and cost nothing' model. Hence on average it should outperform a theoretical model to justify costs. See Chapter 5 for further details.

[10] The link between the *F-statistic* and the ratio of MSEs is explained in Chapter 5. See also Johnson and Kotz (1970, Chapter 26).

[11] See Chapter 5 for a discussion of these issues.

periods and local market conditions may explain the inconsistency with overseas studies (Brown and Dybvig (1986), Munik and Schotman (1994)) while the results in Chiarella et al (1989) are attributed to a flaw[12] in their measurement.

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[12] See Chapter 2.

Table 7.5(a)

Comparative mean dollar pricing errors of $100 13-week Treasury Notes

	One-month prediction		Three-month prediction	
	Naive model	Vasicek model	Naive model	Vasicek model
	\$0.21	-\$0.15	\$0.39	\$0.36
t-value of $\overline{PE}$	29.48	-31.59	33.32	9.30
Hypotheses	$H_0 : MSE_{naive} < MSE_{cir}$ $H_1 : MSE_{naive} \geq MSE_{cir}$		$H_0 : MSE_{naive} < MSE_{cir}$ $H_1 : MSE_{naive} \geq MSE_{cir}$	
F-value	1.96^*		1.74^*	

Note: * Significant at 1% (*F*-value is 1.0 for degrees of freedom exceeding 120).

Table 7.5(b)

Comparative mean dollar pricing errors of \$100 26-week Treasury Notes

	One-month prediction		Three-month prediction	
	Naive model	Vasicek model	Naive model	Vasicek model
	\$0.35	\$0.21	\$0.30	-\$0.26
t-value of $\overline{PE}$	28.97	18.78	34.31	-4.81
Hypotheses	$H_0 = MSE_{naive} < MSE_{cir}$ $H_1 : MSE_{naive} \geq MSE_{cir}$		$H_0 : MSE_{naive} < MSE_{cir}$ $H_1 : MSE_{naive} \geq MSE_{cir}$	
F-value	2.77^{*}		1.33^{*}	

Note: * Significant at 1% (*F*-value is 1.0 for degrees of freedom exceeding 120).

Table 7.5(c)

Comparison of estimates of CIR spot rates with observed rates

Study	Market	Observed rate	Results
Brown & Dibvig (1986)	USA (1977-83) Monthly data	Mean yield on T-Bills with at most 14 days to maturity	CIR *r* overestimates observed rate
Munnik & Schotman (1994)	Netherlands (1989-90) Monthly data	Amsterdam Interbank Offered Rate	CIR *r* is close to observed rate
Baron et al (1991)	Italy (1983-90) Daily data	1 month, 3 month interbank rate and 3 month T-Bill rate	CIR *r* is highly correlated with observed rates
Chiarella et al (1989)	Australia (1978-87) Monthly data	13-week T-Note rate	CIR *r* overestimates observed rate
Chapter 5: CIR model	Australia (1985-92) Daily data	Overnight cash rate 13-week T-Note rate	CIR *r* underestimates observed rates
Chapter 6: Generalised CIR model	Australia (1985-92) Daily data	Overnight cash rate 13-week T-Note rate	Generalised CIR *r* underestimates observed rates
This chapter: Vasicek model	Australia (1985-92) Daily data	Overnight cash rate 13-week T-Note rate	Vasicek *r* underestimates observed rates

7.2 ESTIMATES OF VOLATILITY

In this section estimates of the volatility of the Vasicek stochastic interest rate, σ, are presented and discussed. It should be noted that σ is the Vasicek volatility function while those of the CIR and generalised CIR models are $\sigma\sqrt{r}$ and σr^{β} respectively.

7.2.1 Distributional statistics and graphs

Basic statistics of estimates of σ estimates are presented in Tables 7.6(a), 7.6(b) and 7.6(c) while Figs. 7.2(a), 7.2(b) and 7.2(c) graph the time series of these estimates. Several observations can be made in respect of the evidence. Firstly, daily estimates of Vasicek volatility are, on average, significantly larger than quarterly and semi-annual estimates (see Table 7.6(b)). This implies that one of the assumptions of the model, namely constant volatility, is not supported and the larger daily estimates appear to confirm the conjecture that the specified form of volatility is sensitive to daily bond price changes. Part of the instability of the Vasicek volatility has been found to be caused by the multicollinearity of its variables[13]. This is, indeed, the case where the correlations among the variables are substantial (see Table 7.6(d)). Secondly, for each daily, quarterly and semi-annual cross-section a t test is conducted to assess the significance of σ and consequently the proportions of the cross-sections in which σ is significant are reported in Table 7.6(c). As expected, volatility estimates tend to be more significant under a longer period of estimation as the degree of freedom is substantially increased. Thus, at 10% of significance level, 50% of semi-annual

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[13] See Chapter 5 for more details.

estimates of σ are significant in contrast to 8.37% and 31.25% of daily and quarterly estimates (see Table 7.6(c), last row and last column).

Table 7.6(a)
Time series of daily estimates of σ

1985	0.20883	0.020109	0.219174	2.76824
1986	0.29256	0.013071	0.112599	2.61803
1987	0.12502	0.015253	0.208547	0.24327
1988	0.089689	0.012038	0.146028	0.74984
1989	0.047179	0.89651	0.29616	4.131966
1990	0.016489	0.034256	0.10007	0.14137
1991	0.025251	0.017217	0.000035	0.11152
1992	0.061991	0.020564	0.20793	0.16135
1985-92	0.056726	0.24057	0.000035	4.131966

Table 7.6(b)

Distribution of time series estimates of σ
(1985-92)

		σ		
	t-value	DAILY	QUARTERLY	SEMIANNUAL
Mean		5.6726E-02	1.4459E-02	1.6988E-02
Std		2.4057E-01	1.5942E-02	1.5193E-02
Minimum		3.5E-05	7.1977E-03	4.365E-03
Maximum		4.131966	7.0576E-02	6.6178E-02
H_0 *(d, q)*	6.91			
H_0 *(d, sa)*	5.99			
H_0 *(q, sa)*	-0.53			

Notes:
(a) H_0 *(d, q)* is the null hypothesis that the mean of daily estimates of σ is equal to the mean of the quarterly estimates;
(b) H_0 *(d, sa)* is the null hypothesis that the mean of daily estimates of σ is equal to the mean of the semi-annual estimates;
(c) H_0 *(q, sa)* is the null hypothesis that the mean of quarterly estimates of σ is equal to the mean of the semi-annual estimates.

Table 7.6(c)

Proportions of estimates of σ being significant over the sample period

	Daily estimates	Quarterly estimates	Semi-annual estimates
Significance level	Sample Proportion	Sample Proportion	Sample proportion
1%	0.0282	0.3125	0.25
5%	0.0484	0.3125	0.25
10%	0.0837	0.3125	0.50

Note: The entry 0.3125 (third row, third column) means that at 1% level of significance 62.5 % of the 32 quarterly estimates of σ are different from zero. Other entries are interpreted similarly.

Table 7.6(d)

Correlation matrix of r, κ, σ, and θ - λ

	r	κ	σ	$\theta - \lambda$
r	1.000000			
κ	0.95701	1.0000		
σ	-0.41353	-0.41557	1.0000	
$\theta+\lambda$	-0.1578	-0.015664	0.22548	1.00000

Fig. 7.2 (a)

Daily estimates of Vasicek volatility

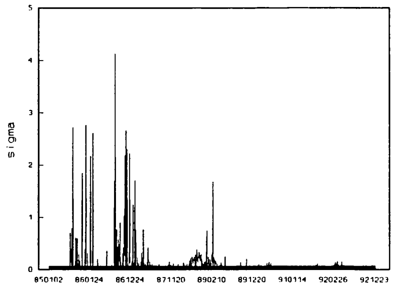


Fig. 7.2 (b)

Quarterly estimates of Vasicek volatility

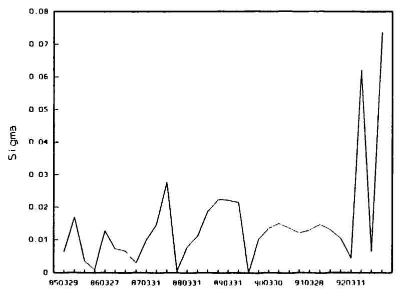
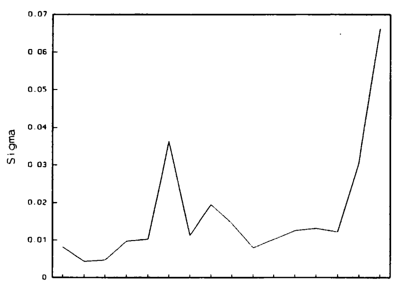


Fig. 7.2 (c)

Semi-annual estimates of Vasicek volatility



7.2.2 Relationship between Vasicek volatility and the time series of volatility[14] of 13-week Treasury Note rate

In this section the volatility of the Vasicek model is further investigated to ascertain if it can act as an unbiased estimator for 13-week Treasury Note volatility. In this respect, the forecast of the volatility of an important short-term rate in Australia would be proof of the usefulness and validity of the Vasicek model. The results of this investigation are presented in Table 7.7(a) and graphed in Figs. 7.3(a)-7.3(b). It should be clear that these two measures are significantly different (see Table 7.7(a)). For example, the Vasicek volatility is nearly 3.5 times as large as its 13-week Treasury Note counterpart. The disparity is further shown in Figs. 7.3(a) and 7.3(b).

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[14] Note that the volatility of the spot rate is σ in the spot rate process, $dr = \kappa(\theta\text{-}r) + \sigma dz$. Hence the corresponding Treasury Note rate volatility is the standard deviation of the series of the <u>differences</u> of daily TN rates.

Table 7.7(a)

Distribution of monthly average of Vasicek volatility (σ) and monthly average standard deviation of change in 13-week TN rate

	t-value	Monthly average of Vasicek σ	Monthly average of std of change in 13-week TN rate ($\hat{s}_t$)
Mean		0.0782	0.0226
Std		0.1072	0.019
Min		1.32E-04	2.1E-03
Max		0.6718	0.1207
H_0	4.86**		

Note:
(a) H_0 is the null that the mean of monthly averages of Vasicek volatility is equal to the mean of monthly averages of standard deviation of change in 13-week TN rate;
(b) Significant at 1%.

Fig. 7.3(a)

Monthly Vasicek volatility

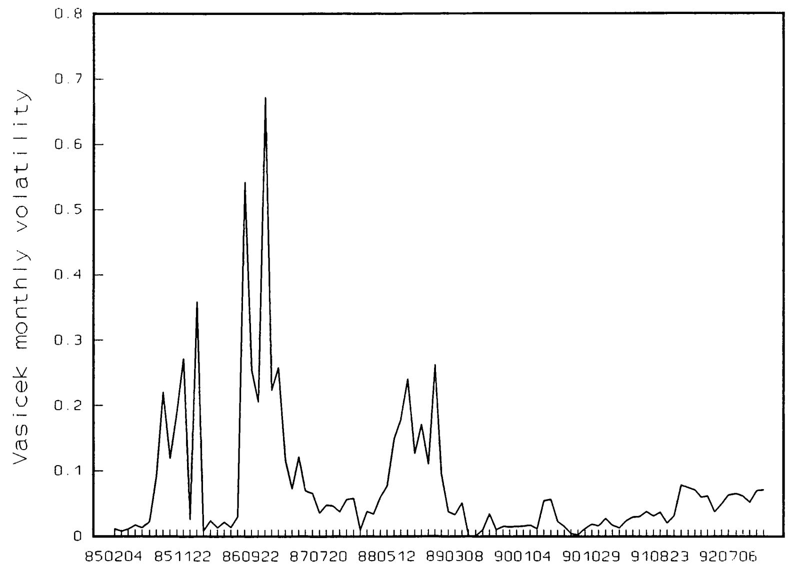
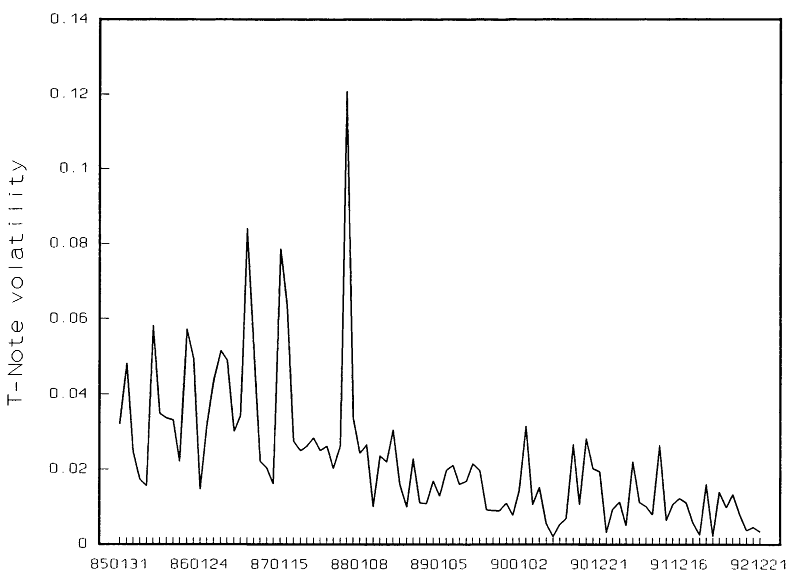


Figure 7.3(b)

Monthly Treasury Note Volatility



7.2.3 Comparison with previous studies

Estimates of the Vasicek volatility together with those of previous studies are presented in Table 7.7(b). While the mean of Vasicek volatility, 5.67% (see Table 7.7(b), last row, third column), appears much larger than those of other models (1.13%-2.95%), it should be noted that its specified form is σ while those of the CIR and generalised CIR models are $\sigma\sqrt{r}$ and σr^{β} respectively. If the value of Vasicek σ is scaled by $\sqrt{r}$ as in the CIR model, it would be of a similar order[15]. Quarterly and semi-annual estimates of σ are not significantly different and both are much smaller than daily estimates (see Table 7.7(b)). In addition, the proportions of significant estimates of σ are higher than daily estimates (see Table 7.7(c)). These results are largely attributed to the increase in the degree of freedom as σ is kept constant over quarterly and semi-annual periods[16].

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[15] With σ = 0.0567 (see Table 7.6(a)) and the mean of the spot rate, 1.1812E-01 (see Table 7.3(a)), this gives 0.0567*1.1812E-01 = 0.019487 or 1.94 percent, which seems consistent with the estimates of other studies (see Table 7.7(b), third column).

[16] See Chapter 5.

Table 7.7(b)

Comparison of CIR, Generalised CIR, and Vasicek volatility* and time series estimates

Study	Market	Mean (%)	Mean Std (%)	Is CIR volatility unbiased predictor of time series estimates ?
Brown & Dybvig (1986)	USA (1977-83) Monthly data	1.95[a]	2.26[a]	Yes
Brown & Schaefer (1994)	UK (1984-89) Daily data	2.65[b]	2.33[b]	No
Baron et al (1991)	Italy (1983-90) Daily data	2.05[c]	2.797[c]	Not reported
Chiarella et al (1989)	Australia (1978-87) Monthly data	2.95[d]	3.16[d]	No
Chapter 5 CIR model	Australia (1985-92) Daily data	2.30[e]	3.54[e]	No
Chapter 6 Generalised CIR model	Australia (1985-92) Daily data	1.13[f]	0.99[f]	No
This chapter Vasicek model	Australia (1985-92) Daily data	5.67[g]	2.41[g]	No

Sources:
* Note that volatility of generalised CIR model is σr^{β} while volatility of CIR model is $\sigma\sqrt{r}$;
a Calculated from Brown & Dibvig (1986, Table I);
[b] Brown & Schaefer (1994, p. 28);
[c] Barone et al (1991, Tables 3 and 5);
[d] Calculated from Chiarella et al (1989, Tables 1 and 3);
[e] Chapter 5 (Table 5.7(c);
[f] Chapter 6 (Table 6.7(a).
[g] This chapter (Table 7.5(b)).

7.3 ESTIMATES OF VASICEK LONG RATE, $R_{vasi}(\infty)$

Estimates of Vasicek long rate are presented in Tables 7.8(a)-7.8(b) and graphed in Figs. 7.4(a)-7.4(c). It is interesting to note that the estimated long rate is remarkably stable around the 11% level across three methods of estimation. For example, the *t-statistics* (see Table 7.8(b)) show that the daily, quarterly and semi-annual mean long rates are not significantly different. Hence, an implication of the model, namely the constancy of the long rate, is supported. Further, the long rate is always positive while in theory its sign can be negative (see eqn (5)):

$$R_{vasi}(\infty) = \theta - \frac{\sigma\lambda}{\kappa} - \frac{1}{2}\left[\frac{\sigma^2}{\kappa^2}\right] \tag{5}$$

The positiveness of the long term rate implies the dominant value of θ relatively to to other parameters, namely, σ, λ, and κ.

Table 7.8(a)

Daily estimates of Vasicek long term rate (1985-1992)

	Mean	Std	Min	Max
1985	0.12355	0.043867	0.001179	0.195512
1986	0.12866	0.026775	0.004169	0.199958
1987	0.11509	0.033673	0.000974	0.193497
1988	0.10466	0.028393	0.002057	0.195175
1989	0.10386	0.020587	0.003713	0.139237
1990	0.11376	0.031352	0.000019	0.193447
1991	0.10409	0.021158	0.016160	0.187085
1992	0.09791	0.014222	0.008383	0.124834
1985-1992	0.11145	0.046244	0.000019	0.199958

Table 7.8(b)

Distribution of Vasicek long term rate
(1985-1992)

	t-value	DAILY	QUARTERLY	SEMIANNUAL
Mean		1.1145E-01	1.11352E-01	1.117732E-01
Std		4.6244E-02	2.7226E-02	1.976E-02
Min		1.9E-05	5.1332E-02	7.1429E-02
Max		1.99958E-01	1.54672E-01	1.38101E-01
$H_0(d,q)$	0.019			
$H_0(q,sa)$	-0.063			
$H_0(d,sa)$	-0.061			

Notes:
(a) $H_0(d,q)$ is the null hypothesis that the mean of the daily estimates is equal to the mean of the quarterly estimates;
(b) $H_0(q,sa)$ is the null hypothesis that the mean of the quarterly estimates is equal to the mean of the semi-annual estimates;
(c) $H_0(d,sa)$ is the null hypothesis that the mean of the daily estimates is equal to the mean of the semi-annual estimates;
(d) *: significant at 5%; **: significant at 1%

Fig. 7.4(a)

Daily Vasicek long term rate

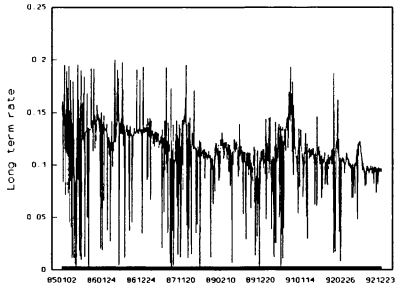
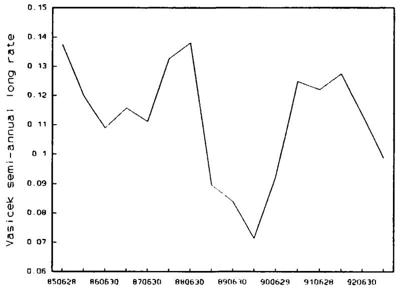


Fig. 7.4(b)

Quarterly Vasicek long term rate

Fig. 7.4(c)

Semi-annual Vasicek long term rate



7.4 ESTIMATES OF κ AND θ - λ[17]

Mean reversion assumed in the Vasicek model requires κ to be a positive constant while a positive θ is consistent with a equilibrium financial market in real[18] terms. Further, the market price of interest rate ris k, - λ, is necessarily negative. Thus, this section is concerned with an examination and analysis of the estimates of these parameters in the Vasicek model. The distributional statistics of κ and θ - λ are summarised in Tables 7.9(a)-7.9(b) while the time series of these estimates are graphed in Figs. 7.5(a)-7.5(c). Several observations may be made:

Firstly, while κ is stable in quarterly and semi-annual constrained estimation with their means in the order of 0.1 to 0.12 respectively, the mean of κ for daily estimates is 0.3449 (see Table 7.9(a)) and is of the same order as those results reported by de Munnik and Schotman (1994, Table 2, p. 1009)[19]. While these estimates do not support the model's assumption of constant parameters over time, it can be argued (see Chapter 5) that these estimates may be interpreted as fluctuations around a specified mean. Thus, for example, we cannot reject the hypothesis that quarterly and semi-annual κ estimates converge to a mean of 0.1 which may be seen as the underlying speed of adjustment while the deviations around this level are stochastic errors.

Turning to the significance of these estimates, it is clear (see Table 7.9(b)) that increasing the length of estimation periods leads to an increase in the

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[17] κ is the speed of adjustment of the spot rate to its long run equilibrium, θ while - λ is the market price of interest risk.

[18] The spot rate in the Vasicek model is real interest rate.

[19] Their estimates of κ are in the range of 0.503-0.596. Note that de Munnik and Schotman (1994) is the only previous study that tests the closed form Vasicek solution.

significance[20] of these parameter estimates. For example, at 1% level, daily estimates are significant for 27.92% of the daily cross sections while the proportions for quarterly and semi-annual estimates are 81.25% and 100% respectively (see Table 7.9(b), third row).

[20] This is consistent with the increase in the degrees of freedom as data are pooled over quarterly and semi-annual periods (see Chapter 5).

Table 7.9(a)
Distribution of estimates of κ and θ - λ
(1985-92)

		κ		
	t-value	DAILY	QUARTERLY	SEMIANNUAL
Mean		3.449E-01	1.09915E-01	1.203E-01
Std		7.041E-01	2.9689E-02	2.9109E-02
Min		0.0	4.0626E-02	7.9665E-02
Max		8.5933	2.15508E-01	1.98163E-01
$H_0(d,q)$	14.05**			
$H_0(d,sa)$	12.86**			
$H_0(q,sa)$	-1.162			
		θ - λ		
		DAILY	QUARTERLY	SEMI-ANNUAL
Mean		1.4548E-01	1.2407E-01	1.23176E-01
Std		4.0019E-02	2.4023E-02	2.2767E-02
Min		0.0	7.0576E-02	7.3175E-02
Max		2.998E-01	1.61363E-01	1.55536E-01
$H_0(d,q)$	4.93			
$H_0(d,sa)$	3.87			
$H_0(q,sa)$	0.13			

Notes:
$H_0(d,q)$ is the null hypothesis that the mean of the daily estimates is equal to the mean of the quarterly estimates;
$H_0(q,sa)$ is the null hypothesis that the mean of the quarterly estimates is equal to the mean of the semi-annual estimates;
$H_0(d,sa)$ is the null hypothesis that the mean of the daily estimates is equal to the mean of the semi-annual estimates;
*: significant at 5%
**: significant at 1%

Table 7.9(b)

Proportions of significant estimates of κ and θ - λ

		DAILY	QUARTERLY	SEMI-ANNUAL
	Significance level	Sample Proportion	Sample Proportion	Sample proportion
κ	1%	0.2729	0.8125	1.0
	5%	0.4678	0.8750	1.0
	10%	0.7309	0.9062	1.0
θ - λ	1%	0.2250	1.0	1.0
	5%	0.2669	1.0	1.0
	10%	0.3088	1.0	1.0

Note:
The entry 0.8125 (third row, fourth column) means that at 1% level of significance 81.25 % of the 32 quarterly estimates of κ are different from zero. Other entries are interpreted similarly.

Fig. 7.5(a)

Daily estimates of κ

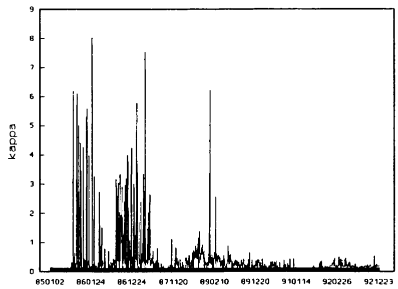


Fig. 7.5(b)

Quarterly estimates of κ

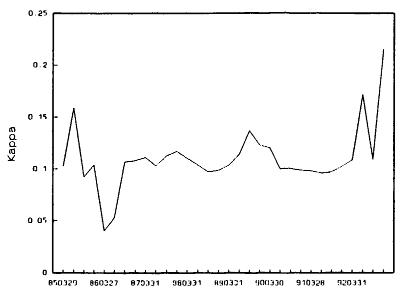


Fig. 7.5(c)

Semi-annual estimates of κ

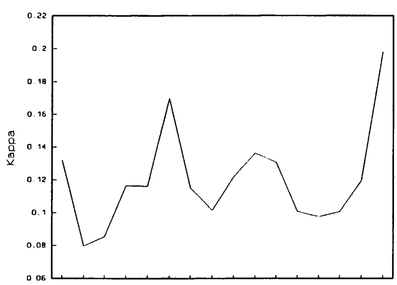


Fig. 7.6(a)

Daily estimates of θ - λ

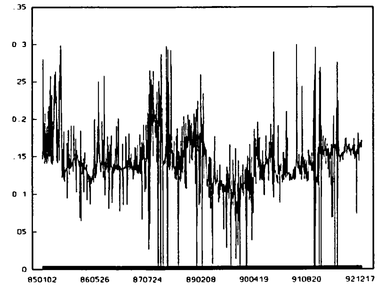


Fig. 7.6(b)

Quarterly estimates of θ - λ

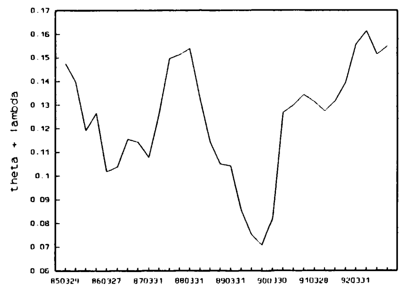
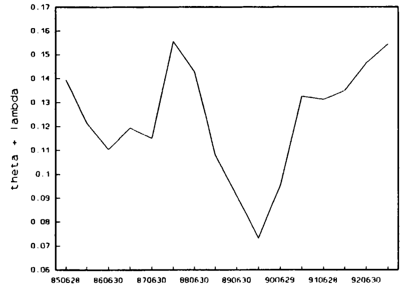


Fig. 7.6(c)

Semi-annual estimates of θ - λ



7.5 EXTENSION OF GOODNESS OF FIT ANALYSIS

In this section the predictiveness of the Vasicek model for the entire spectrum of the term structure of interest rates is examined by means of the pricing errors and the rate mean square errors[21] which make use of results of the Chebyshev curve-fitted term structures. This investigation departs from previous sections in that the performance of the entire term structure over the sampling period (1985-1992) is assessed. As only a few zero-coupon bonds are observed on a given day, term structures have to be constructed from coupon paying bonds. Thus, the methodology of curve fitting introduced in Chapters 3 and 4 is used for this purpose.

Estimates of the errors are reported in Tables 7.10(a)-7.10(c). Two observations may be made. Firstly, all the three errors are not statistically significantly different from each other (see Table 7.10(a)) although the daily errors tend to be smaller on average. For example, the model generates an error of $1.38 in predicting a $100 bond one month in advance, or equivalently a percentage error of 1.38%. The low *t*-statistics (see Table 7.10(a)) associated with daily estimates on the one hand, and quarterly and semiannual estimates on the other, are due to high variances of daily errors over time. As there is no-priori reason for the dynamics of errors, the presence of high variances is indicative of the high risk of prediction by means of daily estimates of the Vasicek and Chebyshev polynomial based models.

Secondly, within-sample rate errors are significantly smaller than out-of-sample rate errors (see Table 7.10(b)) although in absolute terms the errors are

[21]See Chapter 5, Section 5.5 for a justification for the use of these two measures.

relatively small. For example the mean errors in Table 7.10(b) are 2.428E-03 and 1.148E-02[22] or in the range of 24-115 basis points.

The pricing errors and rate MSEs reported in this section have not been attempted in previous research. Table 7.10(c), however, provides the mean of regression errors[23] which can be compared with those of previous studies.

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[22] These are the square roots of 5.9E-06 and 1.32E-04 in Table 7.10(b).

[23] Regression errors are not the same as our MSE in this section. The former is forced by least square regression to converge to zero while the latter does not have this constraint.

Table 7.10(a)

Vasicek pricing errors per $100 bond
(1985-1992)

	Daily estimation	Quarterly estimation	Semi-annual estimation
	One-month prediction	Three-month prediction	Six-month prediction
Price Mean Error	-$1.38	-$5.60	-$5.80
Hypothesis	$H_0(d,q)$	$H_0(q,sa)$	$H_0(d,sa)$
t-value	-1.40	-1.31	-1.40

Notes:
(a) $H_0(d,q)$ is the null that the mean of daily pricing errors is equal to the mean of quarterly pricing errors;
(b) $H_0(q,sa)$ is the null that the mean of quarterly pricing errors is equal to the mean of semi-annual pricing errors;
(c) $H_0(d,sa)$ is the null that the mean of daily pricing errors is equal to the mean of semi-annual pricing errors.

Table 7.10(b)

Daily within-sample and out-of-sample rate MSEs
(1985-1992)

	Within sample rate MSE	Out-of-sample rate MSE
Rate MSE	5.9E-06	1.32E-04
Null hypothesis	$\frac{Out\text{-}of\text{-}sample\ rate\ MSE}{Within\text{-}sample\ rate\ MSE} = 1$	
F-value	9.66^*	

Notes:
(a) * Significant at 0.01%;
(b) Only daily MSEs are calculated because the Chebyshev polynomials are estimated daily. (27).

Table 7.10(c)

Comparison of regression[24] pricing errors

	Market	Regression pricing error (Standard error)
Brown & Schaefer (1994)	UK Daily data (1984-89)	£0.20 (0.40)
Munik & Schotman (1994)	Dutch Daily data (1989-90)	0.17 guilder (0.04)
Chapter 5: CIR model	Australian Daily (1985-92)	-$0.25 (0.22)
Chapter 6: Generalised CIR model	Australian Daily (1985-92)	$0.22 (0.16)
This chapter: Vasicek model	Australian Daily (1985-92)	$0.63 (0.19)

Notes:
(a) Mean regression errors are calculated by comparing observed prices on day t with CIR prices estimated for day t using the observed prices on day t. The objective of least square regression is to minimise the sum of these errors.
(b) While the errors are in different currencies they are the errors resulting from fitting bonds of 100 units of respective currencies. Hence the errors can be interpreted as percentage of errors and are, thus, free from the exchange rate problem.

[24] Mean regression errors are calculated by comparing observed prices on day t with CIR prices estimated for day t using the observed prices on day t. Least square regression entails that the sum of these errors tend to zero asymptotically.

7.6 SAMPLES OF VASICEK, CHEBYSHEV TERM STRUCTURES, OBSERVED AND FITTED BOND PRICES

In this section a sample of daily Vasicek and Chebyshev term structures, Vasicek-fitted and observed bond prices on 20 December 1990[25] are presented (see Figs. 7.7(a)-(e)).

Two observations may be made:

Firstly, as observed and fitted bond prices (see Fig. 7.7(a)) are indistinguishable, the differences are magnified in Fig. 7.7(b) which shows the errors are in the range of at most $3 for $100 bonds.

Secondly, the humped shape of the Vasicek term structure (see Fig. 7.7(c)) is consistent with the prescription of the model that does not allow oscillations. The Chebyshev and Vasicek term structures are graphed in Fig. 7.7(d) which reveals: (i) the Vasicek term structure underestimates the Chebyshev fitted curve[26]; and (ii) the shape of the fitted curve is more flexible[27]. To highlight the differences between these two term structures, Fig. 7.7(e) indicates that rate errors increase with those maturities extending beyond 10 years, the maximum observed maturity.

7.7 CONCLUSION

The objective of this chapter is to report on and discuss the empirical results from estimating the Vasicek model in the Australian context. Five parameters are

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[25] The same date is chosen for the CIR, generalised CIR and Vasicek models in order to provide a comparative perspective.

[26] As a curve fitting technique the Chebyshev term structure yields smaller errors and hence is closer to observed term structure (see Chapter 4).

[27] This is as expected as a function of a fitted curve is to track the observations as closely as possible.

estimated, namely the spot rate, r, the volatility, σ, the speed of adjustment, κ, the long rate, $R_{vasi}\ (\infty)$, the equilibrium spot rate and the market price of risk, θ - λ. These estimates are analysed and discussed in the chapter. Several results emerge from this investigation.

Firstly, while the spot rate underestimates its closest proxies, the cash rate and 13-week Treasury Note rate, this is probably due to the mismatch of maturity. In fact, when the mismatch is corrected, 13-week Vasicek interest rates are unbiased estimator of 13-week Treasury Note rates. The underestimation is consistent with the results obtained for the CIR model (Chapter 5) and generalised CIR model (Chapter 6).

Secondly, the volatility factor, σ, the long rate, R_{vasi} (∞), the speed of adjustment, κ, and the equilibrium spot rate, θ, are relatively stable for longer periods of estimation, namely quarterly and semi-annually. Basically this implies that as long as the period of estimation is increased there is a better chance of the parameter stability assumption being met. Overall, in terms of model goodness of fit, daily estimation is exceptionally good though not as accurate as a curve-fitting technique such as Chebyshev polynomials or Nelson and Siegel (1987). The model shares a common problem with the CIR (1985) and generalised CIR (Chapter 6) models, namely they are overparameterised and hence empirically leading to serious collinearrity and unstable parameter estimates. This problem is well recognised in the extant literature and our evidence adds to it.

Finally while there are minor disparities between this study and previous studies, our results are basically consistent with the published empirical research in the area.

Fig. 7.7(a)

Observed and Vasicek-fitted bond prices

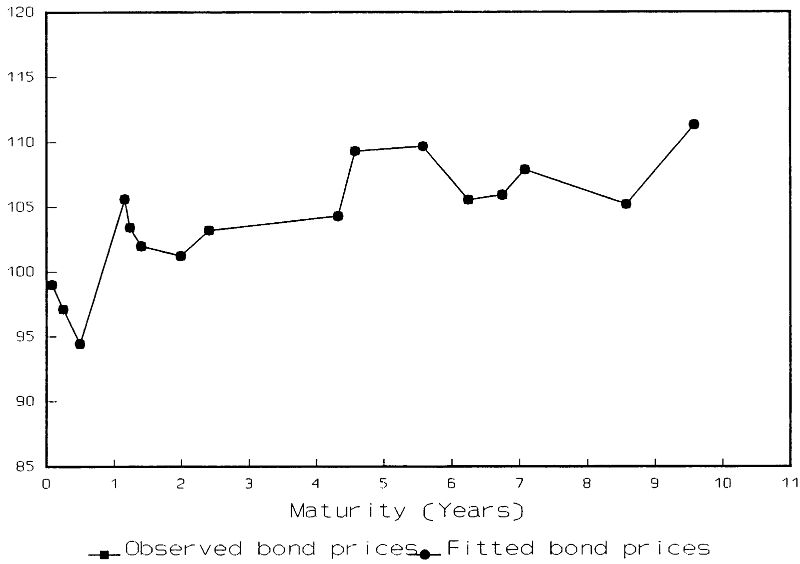
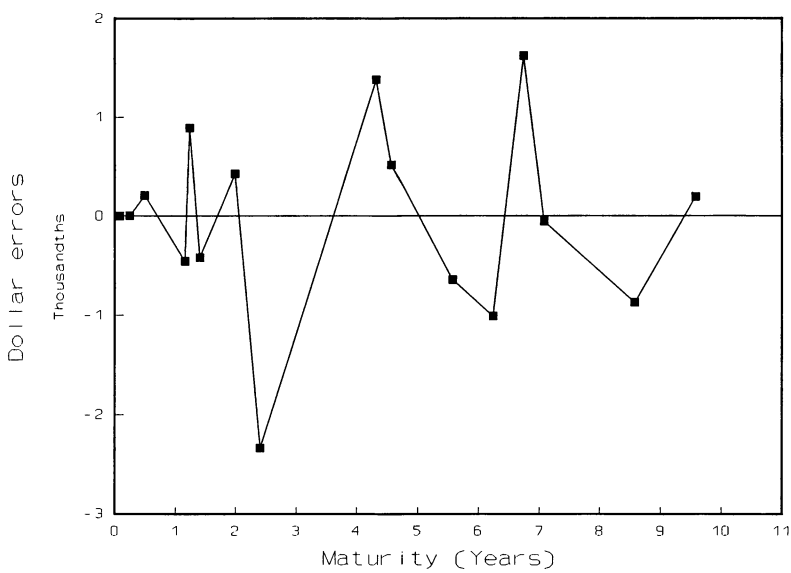


Fig. 7.7(b)

Bond price errors (20/12/90)



Note: Bond price error = Observed bond price *less* fitted bond price

Fig. 7.7(c)

Vasicek Term Structure (20/12/90)

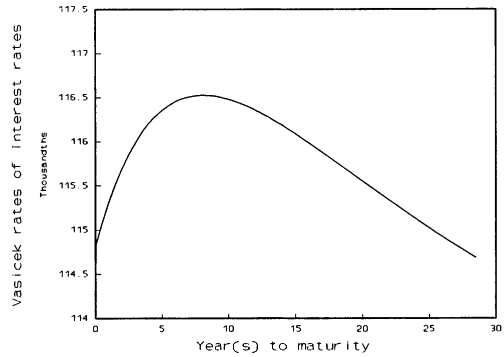


Fig. 7.7(d)

Chebyshev and Vasicek term structures (20/12/90)

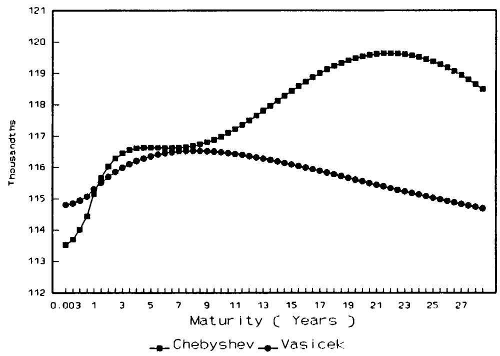
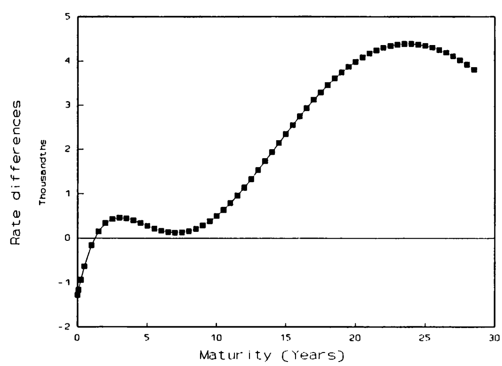


Fig. 7.7(e)

Difference between Chebyshev and Vasicek term structures (20/12/90)



Difference = Chebyshev rate *less* Vasicek rate

CHAPTER 8

A COMPARATIVE PERSPECTIVE OF THE MODELS TESTED IN THIS STUDY

In this study two classes of the term structure of interest rates are tested: curve-fitted and equilibrium models. Within the first class, two models are tested: the Nelson-Siegel model and a Chebyshev polynomial based model. The objective of this exercise is to generate zero-coupon term structures beyond the few observed zero-coupon rates (cash rate, 5-week, 13-week and 13-week rates). The Chebyshev polynomial based technique as applied to bond pricing is found to perform better than the differential equation based Nelson-Siegel model. Thus the term structures generated by this technique are used as 'observed' term structures to assess the performance of economic equilibrium models which include the CIR, generalised CIR, and Vasicek models.

The basic objective of this chapter is to provide a comparative perspective the equilibrium[1] models investigated in this study. Toward this end we reiterate the objectives of this investigation. Essentially we seek answers to the following questions: (i) to what extent do the implications of the models stand up to empirical validation? and (ii) how well do the models fit the data? This is the usual standard of goodness of fit analysis which may be carried within the sample period and/or beyond it.

This chapter is organised as follows. In the first section, the vectors of the models' parameters and their empirical implications are briefly reiterated[2]. This is

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[1] As noted in Chapter 3, curve fitting models are introduced for the purpose of generating term structures against which equilibrium models are assessed. As the basis of curve fitting models are either mathematical or statistical, it would not be valid to compare them with equilibrium models. Hence, this task is not attempted in this study.

[2] See Chapter 3 for full details.

followed by a comparative analysis of the results and a conclusion.

8.1 VECTORS OF MODELS' PARAMETERS AND THEIR EMPIRICAL IMPLICATIONS

The vectors of parameters that are tested are: CIR, generalised CIR and Vasicek.

$$\underline{a}_{cir} = \left[r, \sigma\sqrt{r}, \kappa\theta, \kappa+\lambda\right] \tag{1}$$

$$\underline{a}_{gcir} = \left[r, \sigma, \kappa, \theta, \beta, \lambda, \gamma\right] \tag{2}$$

$$\underline{a}_{vasi} = \left[r, \sigma, \kappa, \theta-\lambda\right] \tag{3}$$

It should be noted that the market price of risk, volatility and long term rate of interest differ among these models (see Table 8.1).

Table 8.1

Market price of risk, volatility and long term rate
CIR, generalised CIR and Vasicek models

	CIR	Generalised CIR	Vasicek
Market price of risk	$\frac{\lambda\sqrt{r}}{\sigma}$	$\frac{\lambda r^{\gamma}}{\sigma}$	$-\lambda$
Volatility	$\sigma\sqrt{r}$	σr^{β}	σ
Long term rate	$2\kappa\theta/(\gamma+\kappa+\lambda)$ where $\gamma=((\kappa+\lambda)^2+2\sigma^2)^{1/2}$	not available	$\theta-\frac{\sigma\lambda}{\kappa}-\frac{1}{2}\left(\frac{\sigma^2}{\kappa^2}\right)$

8.2 TO WHAT EXTENT ARE ASSUMPTIONS ABOUT PARAMETER ESTIMATES SUPPORTED BY EVIDENCE?

An empirical implication of the above models is that the spot rate is a positive[3] varying parameter, the market price of riska negative constant while other parameters are positive constants[4].

A review of the empirical results for these parameters reveals the following:

Firstly, the spot rate is uniformly positive for the three models; this result is particularly meaningful for the Vasicek model which allows for negative interest rates owing to its specified form of volatility. In a practical sense it implies that the Vasicek model is relatively robust to nominal prices. Furthermore, the spot rate, found to be the predominant factor in bond pricing, is consistent with the specification of the models.

Secondly, the market price of interest rate risk cannot be estimated separately for the CIR and Vasicek models while for the generalised CIR this parameter varies more in daily estimation than in quarterly and semi-annual estimation. The evidence, however, supports a market price of risk with stochastic fluctuations around an underlying level.

Thirdly, the long-run equilibrium spot rate, θ, is estimated separately only for the generalised CIR model, and it vibrates around the 10 percent level in all three modes (daily, quarterly, semi-annual) of estimation. In the Vasicek model, a combined quantity θ - λ is estimated. If values of λ are taken from the generalised

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[3] While in the real economy the Vasicek spot rate may be negative owing to the model's assumption of constant volatility, a negative interest rate would be inconsistent with the nominal economy where the time value of money is positive.

[4] See Chapter 3 for full details.

model and added to these estimates of θ - λ, then the resultant θ for the Vasicek model remains remarkably stable around the 11%-12% level.

Fourthly, estimates of σ are more stable over quarterly and semi-annual intervals than daily intervals. Again it can be argued that these estimates fluctuate around a band of values and deviations from this band can be considered stochastic errors.

In sum, several general conclusions may be drawn.

Firstly, the spot rate is fairly similar (see Table 8.2) among the three models and across daily, quarterly[5] and semi-annual estimation. An implication of this result is that the spot rate is the most important factor in these models.

Secondly, there is mild support for the positive constancy of parameters, other than the spot rate, in so far as these are estimated quarterly and semi-annually. Finally, all the three model estimates suffer from the problem of multicollinearity with the generalised CIR model being the worst affected owing to the largest number of parameters that are estimated separately. Thus, on the basis of empirical support for model prescription of parameters, there is not much difference among the models. Then a case may be made for the CIR model to be the preferred model owing to its sounder theoretical foundation[6] on two grounds: (i) it is embedded in a rational utility maximising framework; and (ii) its volatility function precludes negative interest rates.

In the next section the criteria of model performance is applied to assess

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[5] Except for Vasicek quarterly spot rate (see Table 8.2).

[6] Being a general equilibrium model and its specified form of volatility precludes negative interest rates.

these three models.

8.3 COMPARATIVE MODEL PERFORMANCE

While both the price and logarithmic norms are used in the estimation process, the latter is found to be superior on the basis of the Akaike and Schwartz criteria reported in each of the chapters[7] (including the Nelson-Siegel and Chebyshev models). In addition, pricing errors and rate MSEs are also calculated to assess model performance across the entire spectrum of the term structure even beyond the maximum observed bond maturity on each day. This is achieved by means of the Chebyshev curve-fitting technique. These results are summarised in Tables 8.3(a) and Table 8.3(b) which show the generalised CIR model to be the best performer[8]. Thus, in terms of both within-sample and out-of-sample accuracy, the generalised CIR model is the preferred model. Two technical qualifications, however, may be noted: (i) it is more computationally complicated as it involves solving a stochastic differential equation by means of numerical methods; and (ii) computation is considerably slowed down by a relatively flat objective function with seven variables.

8.4 CONCLUSION: WHICH MODEL IS BEST ?

We have assessed these models on the basis of (i) empirical support for their

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[7] Akaike and Schwartz statistics are reproduced here for the logarithmic form of estimation as they are found to be smaller than (or superior to) those of the price norm of estimation.

[8] Note that it is not valid to compare curve-fitting models with equilibrium models. The former, by definition, must always perform better than economic models (see Chapter 3). Hence only the three economic models (CIR, generalised CIR, Vasicek) are assessed for their comparative performance.

theoretical constructs; and (ii) (within-sample) model goodness of fit and (out-of-sample) predictiveness. With respect to the first criterion there is not much difference. With respect to the second criterion, the generalised CIR model outperforms its two competitors. On balance, the generalised CIR model serves two useful purposes: (i) it encompasses the CIR model as a special case; and (ii) its ability to discern all the parameters separately. In the final analysis, it boils down to the trade-off between complicated computation and increased accuracy.

Table 8.2

Mean spot rates

Model	Daily estimation	Quarterly estimation	Semi-annual estimation
CIR	1.2148E-01 (3.1649E-02)	1.2532E-01 (3.3906E-02)	1.2092E-01 (2.9663E-02)
Generalised CIR	1.08872E-01 (2.5840E-02)	1.3124E-01 (1.113E-02)	1.2104E-01 (2.9388E-02)
Vasicek	1.1812E-01 (3.1091E-02)	5.9493E-01 (8.2584E-03)	1.2065E-01 (1.1843E-02)

Note: Standard errors are in parentheses.

Table 8.3(a)

Akaike and Schwartz statistics of tested models

LOGARITHMIC NORM OF ESTIMATION		
	MEAN AKAIKE	MEAN SCHWARTZ
CURVE FITTING MODELS		
NELSON-SIEGEL	-4.4722 (0.1617)	-3.2876 (0.1617)
CHEBYSHEV	-5.8978 (0.148)	-5.7237 (0.148)
EQUILIBRIUM MODELS		
CIR	-2.5788 (0.1721)	-2.3942 (0.1721)
Generalised CIR	-3.6767 (0.1724)	-3.4933 (0.1724)
Vasicek	-2.1671 (0.2588)	-1.8411 (0.2588)

Notes:
(a) Sources: Table 5.1 (Chapter 5), Table 6.1 (Chapter 6), and Table 7.1 (Chapter 7);
(b) A model is judged better by its smaller Akaike and Schwartz statistics (see Judge et al (1985));
(c) Standard errors are in parentheses. As the Akaike and Schwartz criteria differ from each other by a constant their standard errors are equal.

Table 8.3(b)

Mean pricing errors and rate mean square errors

Model	Mean pricing errors (per $100 bond)		
	One month prediction	Three month prediction	Six month prediction
CIR	-$1.29	-$1.83	-$3.24
Generalised CIR	$0.43	$0.70	$0.73
Vasicek	-$1.38	-$5.60	-$5.80
	Rate MSE		
	Within sample	Out-of-sample	
CIR	2.8E-06	5.7476E-05	
Generalised CIR	1.2908E-06	5.7028E-05	
Vasicek	5.9E-06	1.32E-04	

Notes:
(a) Sources: Table 5. (Chapter 5), Table 6. (Chapter 6), Table 7.10(b) (Chapter &);
(b) As the Chebyshev model is chosen to be the better model than Nelson-Siegel. It is then used to generate the term structure against which errors of equilibrium models are calculated. Thus, no errors are calculated for the Nelson-Siegel model.

CHAPTER 9

CONCLUSIONS

9.1 MOTIVATION, OBJECTIVES AND RESEARCH ISSUES

This study develops an integrated framework to empirically test a class of equilibrium models of the term structure of interest rates in the Australian context: (i) Cox Ingersoll and Ross (1985b), (ii) Vasicek (1977); and (iii) a generalised version of Cox, Ingersoll and Ross (1985b).

The research involves an examination of the following specific issues:

(i) to what extent the empirical implications of these models are supported by the Australian bond market; and

(ii) which model performs best in terms of goodness of fit and predictiveness.

To date these issues have not been addressed specifically in the Australian literature on the theory of equilibrium term structure of interest rates. Previous research reveals that: (i) there are substantially different and conflicting results across models; (ii) there is evidence of significant parameter instability, inconsistent with the specification of equilibrium theories; and (iii) assessing the validity of model performance becomes a difficult task since there is a serious lack of a comprehensive comparative study of competing models. Consequently, this study develops an integrated framework in which these and other related issues are further examined and empirically tested.

The research design is implemented within an integrated estimation methodology in two distinct phases: (i) A new technique based upon Chebyshev polynomials is designed to overcome existing weaknesses in the area of curve fitting. The purpose of this exercise is to generate zero-coupon term structures from a limited number of coupon paying bonds; (ii) The second the stage is concerned with estimating the following thee models: CIR (1985b), generalised CIR, and Vasicek (1977) using an integrated framework that utilises all available (zero-coupon and coupon-paying) Australian bonds from 1985 to 1992. Then this is followed by an examination of parameter instability, goodness of fit, model predictiveness an related issues. Finally, a comparative perspective is then provided to enable the models to be ranked.

9.2 RESULTS

The main results of the study may be summarised as follows:

(i) The newly introduced Chebyshev polynomial based curve fitting technique performs best relatively to the differential equation based Nelson-Siegel model.

(ii) While the instantaneous spot rate in equilibrium models is the predominant and driving factor in bond pricing it consistently underestimates its two closest observed rates, the cash rate and 13-week Treasury Note rate. However, the underestimation is removed when model rates of 13-week maturity are compared with 13-week Treasury Note rates, indicating that the biasedness is probably caused by the lack of an observed instantaneous spot rate.

(iii) While constant parameters are implied they are found to be either unstable or fluctuating around a band of values with deviations which can be regarded as stochastic errors. Thus, at best there is mild support for the implication of parameter stability. In this regard the assumption of constant parameters is similar to that of constant volatility and interest rate underlying the Black-Scholes option pricing model. The fact that the Black-Scholes model continues to be the premier model in derivative security pricing in spite of the rejection of this assumption suggests that the next criterion of goodness of fit and predictiveness should be applied.

(iv) On the basis of performance criteria (revealed by both within-sample and out-of-sample statistics) the generalised CIR model, first introduced in this study, is the best performer although it is also the most computationally difficult.

(v) The tested models display considerable multicollinearity, a characteristic consistently recognised in this literature.

(vi) On balance, the equilibrium models may substitute for curve fitting on a daily basis, yet retaining the desirable qualities of economic models.

9.3 CONTRIBUTIONS

This study makes several contributions in two broad categories: (i) contributions to the empirical literature, especially Australian, of equilibrium term structure of interest rates; and (ii) contributions to the estimation methodology.

(i) Contributions to the literature

(a) It offers the first comparative analysis of the empirical verification of equilibrium models in the Australian context.

(b) The extension of the CIR model (1985b) provides an alternative with improved performance and the capacity to estimate separately those parameters that are combined by existing models.

(c) It provides a theoretical basis for pricing interest rate derivative securities in Australia in place of ad-hoc yield curve fitting.

(ii) Contributions to the estimation methodology

(a) It introduces an integrated estimation framework that incorporates zero-coupon and coupon paying bonds, hence significantly increases the degree of freedom, and accuracy of estimation.

(b) It introduces a curve fitting technique based upon Chebyshev polynomials which effectively removes two commonly encountered econometric difficulties in term structure estimation, namely, maturity dependent errors and multicollinearity caused by the mismatch of coupon payment dates.

(c) It introduces an alternative norm of nonlinear regression based upon the logarithm of the error, which is both more accurate and appropriate to the multiplicative nature of bond pricing formulae.

9.4 AREAS OF FUTURE RESEARCH

The present study can be extended in several directions: (i) the term structures generated by various tested models, including curve-fitted models, can be used as input in testing derivative pricing models; (ii) a data base of bond prices constructed in the fashion of Fama-Bliss (which does not exist at this stage) would enable a GMM test of these models in contrast to the cross-sectional nonlinear regression procedure undertaken in this study; and (iii) an empirical test of preference free models of the Heath-Jarrow-Morton type in the Australian context would be in order and would provide a basis for comparison with equilibrium models.

APPENDIX A

EMPIRICAL EVIDENCE[1] OF TRADITIONAL THEORIES OF THE TERM STRUCTURE OF INTEREST RATES

[1] Including Australian evidence (see Section AIV)

AI. EXPECTATIONS THEORIES

The empirical literature is extremely voluminous and has a long history. Broad statements of the expectations theory, being couched in terms of expectations, cannot yield testable propositions without models of expectations. Three models have been proposed: (i) the perfect foresight model; (ii) the error learning model; and (iii) the rational expectations model.

Prior to the 1960s empirical studies concentrated upon the ability of the implied forward rates to predict expected future spot rates. The vehicle to test this proposition was the perfect foresight model which holds that expectations are not only held by the market but are also realised. Macauley (1938) studied the seasonal movement of call money rates and found that they anticipated this seasonal. Kessel (1965) repeated the exercise for the period 1959-61 using a different instrument, twenty seven and fifty five day bills. His results confirmed Macauley's findings that market participants were able to forecast the seasonal in these rates. However, he found forward rates upward-biased forecasters in the case of six-month U.S.Treasury bills for the period from January 1959 to March 1962. Hickman's (1942) comparison of actual spot rates with their predictors, the forward rates implied in the term structure, revealed that the forward rates did not successfully forecast the actual rates. Moreover, Culbertson (1957) found that the Treasury bills and long term Treasury bonds displayed wide differences in holding period yields during 1953. On balance, the major conclusions from these early studies are that forward rates are not good predictors of future spot rates at the short end of maturities and, forward premia are not zero on average.

The mixed results of these early studies are due to their different instruments and periods chosen for analysis. Yet the predictions of the theory are independent of time and maturity. The major flaw in this early literature is the use of a perfect foresight of expectations. The theory posits the role of expectations but is silent on the accuracy of expectations.

Meiselman's (1962) critique of the earlier tests noted that the expectations theory is cast in ex-ante terms while the tests used ex-post data. Consequently the theory can hold while investors might be wrong in forecasting future rates. Hence, one cannot infer anything about whether or not forward rates represent market expectations simply by comparing implied forward rates with the subsequently observed spot rates. Meiselman introduced a mechanical rule known as the error learning model, to model the process of expectation generation. Essentially expectations are revised in light of forecasting errors, defined as the actual one year rate in period t minus the forward rate forecast of a one-year bond beginning at time t as implied in the term structure one year earlier. Using this model of expectation formation and a sample of high grade annual corporate yields from 1901 to 1954 he found evidence consistent with the pure expectation model in the form of a zero term premium.

Meiselman's model has been tested with a variety of data by a number of researchers. Thus Kessel's (1965) test of the error learning model on 28 and 91-day Treasury bills for the periods 1959-62 and 1949-61 yielded positive term premia. Moreover, Buse's (1967) replication of Meiselman's regression model with annual data of British government securities for the period 1933-1963 found intercepts significantly different from zero. Furthermore, Van Horne's (1965) study, using post-war US Treasury bond data, confirmed the presence of positive constants which are interpreted as risk premia.

One source of the mixed results can be attributed to different instruments and periods chosen for analysis. The model, however, suffers from some other more serious problems. For example, a zero intercept is consistent with both zero and non-zero term premium[2]. In addition, Nelson (1972) argues that the error learning rule belongs to a class of linear optimal forecasts (in the minimum mean squared error sense) which represent expectations as linear functions of past data. Hence, it is restrictive because the information set on which expectations are based consists only of the entire history of spot interest rates. Although changes in the spot rate may be useful in revising expectations, it is only one source of information. Further, as the statistical nature of the forecasting error is left unexplored[3],it is not possible to infer whether information has been used in an unbiased and/or consistent manner. It is, however, the first model to explicitly infer expectations from observed data and it is close to the concept of rational expectations which has become an important paradigm in economics and finance.

The growing acceptance of the view that expectations are rational in the early 1970's led to the development of the rational expectations theory of the term structure (see Modigliani and Sutch (1967), Modigliani and Shiller (1973), Shiller (1989)). The simplest form of this theory states that the yield to maturity on n-period bonds equals the weighted average of expected future one-period spot

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[2] Proof is given by Telser (1967), p.108.

[3] Meilseman is not concerned with whether or not expectations are realised. The focus of his model is how expectations are generated.

interest rates plus a constant term premium, π_n, which can be interpreted equivalently as a forward or holding period premium[4] :

$$r(t,T) = \frac{1}{T-t} \{ r(t, t+1) + E_t\, r(t+1, t+2) \ldots E_t\, r(t+n-1, T) \} + \Pi_n \tag{A1}$$

where E_t is the expectations operator conditional on information available at time t, which includes all current and lagged interest rates. Another variant of the rational expectations hypothesis of the term structure is that all term premia are time invariant so that they only depend upon bond maturity and not time.

The rational expectations theory has been tested in at least two ways: (i) by regressing the actual change in m-period spot rates, $r(t+n, t+m+n) - r(t, t+m)$, on the predicted change, $f(t, t+n, t+m+n) - r(t, t+m)$; where the slope coefficient is hypothesised to be 1.00 ; and (ii) checking for excess volatility of long-term interest rates. Shiller and McCulloch (1987) document the results from six major studies[5] across three countries (the U.S., Canada and W.Germany) over a variety of sample periods. The evidence suggests the slope coefficients are well below 1.00 and on some occasions display wrong signs. Further, forward rate forecasts of near-term changes in interest rates are poor but the forecasting power increases

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[4] The holding period term premium is the difference between the conditional expected holding period return and the spot interest rate:

$$\Pi_h = E_t \{ \ln[P(t+1,T)/P(t,T)] \} - r(t,t+1)$$

while the forward term premium, π_f , is the difference between the forward rate and the and the expectation of the corresponding spot rate:

$$\Pi_f = f(t,T) - E_t\,(r_T)$$

[5] These studies include Shiller (1979), Shiller, Campbell and Schoenholtz (1983), Mankiw (1986), Fama (1984), Fama and Bliss (1986), and Shiller (1988).

with the forecast horizon. This seems counter-intuitive.

On the issue of excess volatility Shiller ((1979), (1981), (1988)) and Singleton (1980) reported results which indicate the observed long-term interest rates are far more volatile than can be described by the rational expectation model. Other researchers (Flavin (1983), Marsh and Merton (1986), Kleidon (1988), among others) have disputed tests of excess volatility on the ground of their statistical significance given the generality of these tests. Some of the econometric problems have been overcome by the so-called second-generation tests[6] which also found excess volatility. This issue remains difficult to explain in the context of the representative-consumer, frictionless market model[7].

On balance, empirical tests tend to reject the implications of the pure expectations theory: (i) the term premium is zero; and (ii) the long-term rate is a weighted average of expected future spot rates. The rejection is not a major concern in the empirical literature which concentrates on the testable implications of the expectations theory. In fact, Roll (1970) and McCallum (1975) argue that the CAPM implies a holding premium which is proportional to the conditional covariance of the holding period yield and the market return. Variations of this premium may be caused by changes of this covariance over time and are consistent with the notion that the bond market processes information efficiently. In an equilibrium economy driven by the instantaneous interest rate Longstaff (1990) derived a term premium which depends upon the current level of the risk-free

[6] See Gilles and Leroy (1991).

[7] Gilles and Leroy (1991, p.787).

interest rate. This term premium is time varying, yet consistent with the expectations hypothesis if the time frame for which the expectations hypothesis holds differs from the return measurement period. In particular, if the period over which the hypothesis is supposed to hold is shorter than that over which the bond returns are measured then there is a rich structure of variation of observable term premia. It should be noted that Longstaff (1990) only considers the case of the local expectations hypothesis which states that all bonds have the same expected returns over the next (shortest possible) holding period and these are equal to the instantaneous risk free rate of interest.

There is evidence of time variation in term premia at the short end of the spectrum as well as on long-term bonds (see Shiller (1979), Startz (1982), Shiller, Campbell and Schoenholtz (1983), Fama (1984), Mankiw (1986), Shiller (1988), Campbell (1986), Engel, Lilien, and Robins (1987), Shiller and McCulloch (1987), Froot (1989), and Simon (1989)). As the existence of time varying term premia has been well documented, another related issue of investigation is the relationship between the term premium and some variables other than spot and forward rates. Without guidance from theory, the level and volatility of interest rates have been chosen in the empirical literature.

It is argued that interest rates are mean reverting, thus implying a negative relationship between the level of interest rates and the term premium. This is because if interest rates are high and expected to fall then investors would demand a lower premium. Conversely, if interest rates are low and expected to rise, then investors would demand a higher premium. An alternative view [Kessel (1965), Malkiel (1966), Nelson (1972)] assumes that bonds are near substitutes for money.

When interest rates are low (high) the opportunity cost of holding money (and its substitutes) is low (high). Because of their liquidity short-term bonds are better substitutes for money than long-term bonds. Hence when interest rates rise the opportunity cost of holding long-term bonds is relatively higher than that of holding short-term bonds. In other words, the term premia of long-term bonds rise more than those of short-term bonds. As a result the net effect of an increase in interest rates is an increase in the term premia (long or short) in the forward rates implied in the term structure, and hence a positive relationship between the level of interest rates and the term premium.

In addition to the level of interest rates it is argued that the term premium is positively related to the volatility of interest rates but the consensus is far from being universally accepted[8].

Instead of assuming that expectations are realised (as in perfect foresight model) or rational, another strand of literature uses survey data of market participants' interest expectations. Using quarterly surveys conducted by the Goldsmith-Nagan Bond and Capital Market Letter, Friedman (1979) found the term premium on three-month US Treasury bills to be positive on average, and for a given maturity, to vary positively with the level of interest rates. However, the findings of this study were limited to bonds of short maturities because of data availability. Froot's (1989) study is based upon more recent data of these surveys which include both short-term Treasury bills and 30-year mortgage rates, and it tests a hypothesis implied by the expectations theory and rational expectations: the

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[8] See Van Horne (1990, p.122).

forward premium is an efficient forecast of the future interest change. Froot found that the hypothesis was supported at the long-term maturities but rejected at the short-term maturities. It should be noted that Froot's (1989) results rely on the assumption that the survey data accurately measures the market's expectations[9].

AII. LIQUIDITY PREFERENCE THEORY

Empirical tests of the theory have focused upon demonstrating the existence of the premium and uncovering its determinants. Strong support for the theory comes from Kessel (1965) who found positive premia in Treasury bills and long-term government securities over the 1921-1961 period. Fama (1976) attributes the premium to the uncertainty of inflation as there was a positive relationship between term premia and inflation. This finding is perhaps the first to relate the liquidity premium to the risk of inflation. Evidence from other studies is less encouraging and often accompanied by qualifications. Thus Connard (1966) found positive premia but these diminished rapidly beyond intermediate maturities. Furthermore McCulloch (1975) found positive premia but those for long-term loans were very inaccurate. More recently, Fama (1984) found that expected returns on longer-term bills exceed the returns on one-month bills but these expected returns tend to peak at eight or nine months and do not increase monotonically with maturity.

The level of interest rates has been suggested as another determinant of liquidity premia. Evidence of a positive relationship between liquidity premium and the level of interest rate is found in Kessel (1965), Connard (1966), and Cagan

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[9] There is a strand of literature that investigates the rationality of the market's expectations contained in surveys of financial market participants (see Friedman (1979) and, Chan and Pham (1990)).

(1956) while an inverse relationship is found in Malkiel (1965), Van Horne (1966), Nelson (1972). These conflicting results are due to the use of different types of interest rates and time periods.

AIII. PREFERRED HABITAT THEORY

Empirical tests of the preferred habitat theory are concerned with the effect on the term structure of changes of the relative supplies of securities of various maturities beyond the effect caused by expectations and risk-averse behaviour of lenders. Modigliani and Sutch (1966, p.587) found some support for the influence of national debt structure on the spread of the long and short rates. Their results, however, can only be described as '...at best weak, even in a period in which the national debt was large both in absolute and relative size '. Using a sample of Canadian government bonds with maturities varying between a few months and infinity, McCallum (1975) found a market structure consistent with the preferred habitat theory. His methodology consists of transforming bond prices into expected returns, from which measures of risk are calculated in the form of either a standard deviation of expected returns or systematic risk. Although risk tends to increase with maturity, it levels off after three years. However, other studies have failed to find support for this hypothesis (see Van Horne (1990), footnotes 27 and 28, p.123).

AIV. AUSTRALIAN EVIDENCE

The four Australian contributions to the literature on traditional theories of the term structure of interest rates are all empirical: Dewald (1973), Block (1974), Juttner et al (1975) and Tease (1988). The sample period for the first three papers was characterised by stable interest rates, low turnover in the secondary bond market and extensive financial sector regulation while the focus of research was on the relative importance of expectations and the effects of regulatory activities of the monetary authorities on the term structure of interest rates.

Bloch (1974) tested the error learning model on annual and quarterly Commonwealth government bond yields for the period from March 1954 to September 1968. There were 11 regressions, each with 14 annual observations and 52 regressions, each with 58 quarterly observations. Only four of the annual regressions and fifteen of the quarterly regressions display statistically significant slope coefficients. Compared to Meiselman's results, support for the expectations theory is much weaker. A number of significant constant terms were found, indicating the presence of liquidity premiums, while the liquidity premium was a positive function of term to maturity. In sum, Bloch's results provide limited support for both the pure expectations theory (zero term premium) and expectations theory (increasing term premium).

In a subsequent paper Juttner et al (1975) criticised Bloch's (1974) study on the ground that Meiselman's equation can be transformed into an equation consisting of only spot rates and hence the dependent and independent variables in Meiselman's equation are determined by the difference of two terms containing spot rates. Consequently the results are biased in favour of the maintained hypothesis of a relationship between forward rates and subsequent spot rates.

Juttner et al (1975) apply spectral and cross-spectral analysis to study the interrelatedness between alternative pairs of interest rates and the nature of leads and/ or lags. They found evidence which would appear to lend support to the expectations theory in that rates on different securities are closely related and move together almost simultaneously. However, they attribute this finding to policy interventions by the monetary authorities who set the rates for different maturities by varying buy and offer prices. In other words, the Reserve Bank of Australia generated the rates which are consistent in a statistical sense with Meiselman's model.

Dewald (1973) tested three competing hypotheses on two and ten year Commonwealth Government securities from 1952 through 1966: expectations, preferred habitat, and interest rate policy[10] . He found marginally significant evidence in support of the preferred habitat theory. Interest rate movements were closely controlled at short maturities, but expectations played a more important role at longer maturities.

A more recent paper by Tease (1988) reassesses the role of expectations in Australian term structure research. This study differs from earlier work in at least three respects: (i) it investigates a period relatively free from policy intervention as a result of a number of developments in the financial markets[11]; (ii) Rational expectations are accepted as the proper way to model expectations; and (iii) the author uses a sample of 90 day and 180 day bank bills to test the expectations theory. The joint hypotheses of the expectations theory and a zero (or a constant) risk premium could not be rejected. Thus, unlike empirical studies in the US which fail to support the expectations theory, Australian short term interest rates have behaved in a manner consistent with this theory after the introduction of the tender

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[10] The interest rate policy hypothesis is that the term structure is adjusted by both market forces and policy actions designed to achieve desired economic objectives.

[11] The tender system of issuing Treasury notes and government bonds were introduced in 1979 and 1982 and the Australian dollar was floated in 1983.

system in 1979. However, a weakness of this study is that the data was confined to the short-term maturity spectrum.

APPENDIX B

THEORY AND EVIDENCE OF PREFERENCE FREE MODELLING OF THE TERM STRUCTURE OF INTEREST RATES

The objective of Appendix B is to survey the theoretical developments of the preference free theory and the empirical testing[1] of this theory.

BI. THEORY

Essentially there are three main approaches[2] to constructing preference free models: (i) modelling bond prices ; (ii) modelling forward rates ; and (iii) modelling the short rate. As there is a one to one relationship between the bond price, the forward rate and the instantaneous spot rate, the choice of one variable over the others is a matter of convenience and belief in its driving force rather than economic rationale.

(i) Modelling bond prices

Using the information contained in the current term structure Ho and Lee (1986) were the first to impose the no-arbitrage condition on the evolution of subsequent term structures. They assume that the discount function defined as the equilibrium price of a discount bond, which at any given point in time, can only experience either an upward movement or a downward movement. Furthermore, Ho and Lee (1986) impose two conditions on their model: (i) no riskless arbitrage between bonds of different maturities; and (ii) path independence of the discount function which can depend only on the number of upstate or downstage movements but not on the sequence in which they occur.

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[1] Australian evidence on preference free theory is surveyed in Chapter 2 (section 2.3.4).

[2] This classificatory framework is suggested by Hull and White (1992).

The assumptions of path independence and no riskless arbitrage allow us to define a term structure at any future time given the current term structure. No riskless arbitrage means that it is possible to construct a portfolio of bonds of differing maturities which yields the same return as a one-period discount bond. More specifically, if the upward movement of bond price is large then so is the downward movement so that the weighted average of the movements is the same across all bonds. Path independence requires that the discount function obtained from an upward movement followed by a downward movement be the same as that obtained from a downward movement followed by an upward movement. The assumption of no riskless arbitrage is based on the market efficiency concept while the path independence assumption is to reduce the mathematical complexity of the model.

Central to the Ho and Lee model is the concept of a discount function of a discount bond of maturity *T* which pays \$1 at the end of the *Tth* period. At each time *n* there exists a number of states of the world. A discount function, $P_i^{(n)}(T)$, is defined for each time *n* and state *i* for a discount bond of maturity *T*. This discount function completely describes the term structure of interest rates of the *ith* state at time *n* and the set of discount functions is said to form a binomial lattice if at any given time the discount function can experience only an upward or downward movement. The downward state is always denoted by the upward state less one. For example, $P_{i+1}^{(n)}$ and $P_i^{(n)}$ are the discount functions at time n for the upward and downward state respectively.

If at time n there is no expected change in interest risk then the discount function must be the same for the upward state and the downward state. Moreover it also equals the implied forward function $F_i^{(n)}(T)$:

$$F_i^{(n)}(T) = P_i^{(n+1)}(T) = P_{i+1}^{(n+1)} = \frac{P_i^{(n)}(T+1)}{P_i^{(n)}(1)} \qquad T = 0,1,... \tag{B1}$$

Uncertainty is modelled by the deviation of the discount function from the implied forward function in the following period. Furthermore, this deviation is caused by a shock factor which the authors call the purtabation function. There is a purtabation function for each of the upward state and downward states, denoted $h(T)$ and $h^*(T)$ respectively. The lattice of the term structure movement is completely specified by the set of purtabation functions and the initial discount function $P(T)$.

With the assumptions of no riskless arbitrage and path independence the purtabation functions can be expressed as:

$$\begin{aligned} h(T) &= \frac{1}{\pi+(1-\pi)\delta^T} \\ h^*(T) &= \frac{\delta^T}{\pi+(1-\pi)\delta^T} \end{aligned} \tag{B2}$$

where π is interpreted as the implied binominal probability, in the sense of the 'risk-neutral' probability of Cox, Ross and Rubinstein (1979) model of binomial option pricing, and δ is a perturbation function spread constant such that $h(1) = 1/(\pi + (1 - \pi)\delta)$. Thus π and δ are the parameters of interest volatility.

The Ho and Lee model was initially developed in discrete time. It has been shown to have an equivalent continuous version (Hull and White (1993), Hull (1993)) with spot rate process:

$$dr = [F_t(0,t) + \sigma^2 t]dt + \sigma dz \tag{B3}$$

where $F(0,t)$ is the instantaneous forward rate at time 0 for a bond maturing at time T and the subscript t refers to the derivative with respect to t. The price of the discount bond is:

$$P(t,T) = A(t,T)e^{-r(T-t)} \tag{B4}$$

where

$$\ln A(t,T) = \ln\frac{P(0,T)}{P(0,t)} - (T-t)\frac{\partial \ln P(0,t)}{\partial t} - \frac{1}{2}\sigma^2 t(T-t)^2 \tag{B5}$$

An advantage of the discrete or continuous Ho and Lee model is that it fits the current term structure exactly and it is analytically tractable. Its weaknesses are : (i) it allows for negative[3] interest rates as the interest rate process has no mean reversion ; and (ii) the structure of the volatility function is rather inflexible because the forward rate and the spot rate have the same standard deviation, σ, and the instantaneous standard deviations of all spot and forward rates are the same.

The volatility function of a preference free model needs to be specified carefully to maintain its consistency with the no arbitrage condition and to keep the computation manageable. Thus, Hull and White (1993) develop a theory of bond pricing based upon the assumed process of discount bond price given by

[3] While negative real interest rate is possible, negative nominal interest rate is not.

$$dP(t,T) = r(t)P(t,T)dt + v(t,T)P(t,T)dz \tag{B6}$$

where $v(t,T)$ is the volatility function of bond price differential[4]. The objective of the Hull and White (1993) study is to focus on the volatility function and to identify a necessary and sufficient condition for volatilities to be consistent with a Markov interest rate process where bond prices are lognormally distributed. It is desirable to specify a volatility structure that gives rise to a Markov process for the spot rate because a Markov process is far more tractable than a non-Markov process.

(ii) Modelling forward rates

The most important piece of work in preference free modelling of the term structure is undoubtedly Heath, Jarrow and Morton (1992) (hereafter HJM). It builds on Ho and Lee (1986) by taking the current term structure as given and derives the no arbitrage condition for the evolution of the subsequent term structures. However it expands the Ho and Lee (1986) model by allowing for more than one random variable and by demonstrating that it is possible to use an <u>alternative information set</u>, the forward rate volatility structure, to characterise the term structure. This change of state variable also facilitates the change from discrete time mathematics of Ho and Lee (1986) to continuous time mathematics. Moreover, as they specify the forward rate stochastic structure, rather than the zero coupon bond price stochastic structure, this allows volatilities to change for a fixed value of the bond at maturity. HJM also take the initial forward structure as given and then specify how it would evolve over time to preclude arbitrage opportunities. While the HJM model overcomes the difficulty of specifying the market price of

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[4] As $v(t,T)$ is a function of t and T only, bond prices are lognormal.

risk it introduces new ones. For example, it requires knowledge of the volatility structure for each maturity date whereas in equilibrium theories we need to specify the volatility of the spot rate which is equivalent to only one forward rate. It also requires knowledge of the entire initial forward curve, but this is not observable.

The uncertainty in the HJM economy is characterised by the probability space (Ω,F,Q) where Ω is the state space, F is the σ-algebra representing measurable events and Q is a probability measure. The forward rate is defined as:

$$f(t,T) = -\partial \log P(t,T)/\partial T \tag{B7}$$

so that

$$P(t,T) = \exp\left[-\int_t^T f(t,s)ds\right] \tag{B8}$$

The forward rates satisfy the process:

$$\begin{aligned} f(t,T) - f(0,T) &= \int_0^t \alpha(v,T,\omega)dv \\ &+ \sum_{i=1}^{n} \int_0^t \sigma_i(v,T,\omega)dW_i(v) \quad \textit{for all } 0 \le t \le T \quad ; i=1,\ldots,n \end{aligned} \tag{B9}$$

where $\alpha(v,T,\varpi)$ and $\sigma(v,T,\varpi)$ are the drift and the volatility of the forward rate and ϖ is a member of the state space Ω. This forward rate process consists of n independent Brownian motions denoted by dW_i for $i = 1,\ldots,n$, which determine the stochastic fluctuations of the entire forward rate curve starting from a fixed initial curve $f(0,T)$.

The spot rate process is similar to the forward rate process, except that both the time and maturity arguments vary simultaneously[5]:

$$r(t) = f)t,t) = f(0,t) + \int_0^t \alpha(v,t)dv$$
$$+ \sum_{i=1}^{n} \int_0^t \sigma_i(v,t)dW_i(v) \qquad for\ all\ t \in [0,T] \tag{B10}$$

where n denotes n independent Brownian stochastic processes that characterise the entire forward rate curve. Similarly, the bond price process is described by the stochastic differential equation:

$$dP(t,T) = \left[r(t) + b(t,T)\right]P(t,T)dt$$
$$+ \sum_{i=1}^{n} a_i(t,T)P(t,T)dW_i(t) \tag{B11}$$

where

$$a_i(t,T) \equiv -\int_t^T \sigma_i(t,v)dv \qquad for\ i = 1,\ldots,n \tag{B12}$$

and

$$b(t,T) \equiv -\int_t^T \alpha(t,v)dv + (1/2)\sum_{i=1}^{n} a_i(t,T)^2 \tag{B13}$$

Note that the drift term of the bond price process is expressed in terms of the drift and volatility terms of the forward rate process. The market price of risk is introduced into the analysis by assuming that it exists and is related to the drifts and volatilities of the forward rates in the following manner:

[5] The spot rate is the instantaneous forward rate at t, i.e., $r(t) = f(t,t)$.

$$b(t,T) = \sum_{i=1}^{n} a_i(t,T)(-\gamma_i(t,S_1,\ldots,S_n)) \tag{B14}$$

where $\gamma_i\ (t;S_1,\ldots,S_n)$ is the market price of risk associated with the random factor W_i and the vector of bonds $\{S_1,\ldots,S_n\}$. The left hand side of (B14) is the instantaneous excess expected return on the bond maturing at date T over the risk-free rate while the right hand side is the sum of the market price of risk for factor i times the instantaneous covariance between the T-maturity bond's return and the *i*th random factor for $i = 1,\ldots,n$. If equation (B14) holds then Girsanov's theorem[6] guarantees the existence of an equivalent martingale probability measure which is unique. Its uniqueness leads to the following results (HJM (1992), Proposition 3,pp.86-87): (i) a bond's price depends on the forward rate drifts, the initial forward rate curve, and the forward rate volatilities; (ii) the market price of risk is equal across all bonds of all maturities; and (iii) there is a restriction of the forward rate drifts in order to guarantee the existence of a unique equivalent martingale probability measure, and this is known as the <u>HJM no-arbitrage condition</u>:

$$\alpha(t,T) = -\sum_{i=1}^{n} \sigma_i(t,T)\left(\gamma_i(t) - \int_t^T \sigma_i(t,v)dv\right) \tag{B15}$$

If the volatility structure is chosen to satisfy (B15) then the market price of risk does not enter the bond pricing equation. This is the essence of their argument that " the bond price process, spot rate process, and the market price of risks cannot be chosen independently" (HJM (1992), p.88-89). To characterise the term structure

[6] Girsanov (1960, pp. 285-301).

relative to an earlier date requires knowledge of the entire path the spot rate followed in reaching the present value. In addition to the path being non observable, the HJM spot rate process is in general non-Markov (Hull (1993), pp.400-401), which makes the HJM model very slow computationally. In short, the theoretical generality of the HJM model is a major strength but additional structure need be imposed at the practical level.

(iii) Modelling the spot rate

Hull and White (1990) extend the CIR interest rate process by adding a time dependent term, $a(t)$, to the drift and by making the volatility and the mean reverting rate time dependent so that $\sigma(r,t) = \sigma(t)$ and $\kappa = \kappa(t)$:

$$dr = \left[a(t)+\kappa(t)(\theta-r)\right]dt + \sigma(t)r^{\beta}dz \tag{B16}$$

The stochastic process of equation (B16) incorporates those of the Vasicek model (1977) for $\beta = 0$ and Cox, Ingersoll and Ross (1985b) for $\beta = 0.5$. Hull and White (1990) apply the process specified by (B16) to the partial differential equation of CIR (1985a) which must be satisfied by any contingent claim that depends upon the short rate, r. Hence they derive two bond pricing formulae corresponding to the Vasicek interest process and CIR interest process, which are referenced to as the extended Vasicek and CIR models respectively. The extended Vasicek model is given by:

$$
\begin{aligned}
P(r,t,T) &= A(t,T)\exp(-B(t,T)r) \\
B(t,T) &= \frac{[1-\exp(-\kappa(T-t))]}{\kappa} \\
A(t,T) &= \exp\left[\frac{B(t,T)-T+t)\ (\kappa\phi-\sigma^2/2)}{\kappa^2} - \frac{\sigma^2 B(t,T)^2}{4\kappa}\right] \\
\phi(t) &= \kappa(t)\theta + a(t) - \lambda(t)\sigma(t)
\end{aligned}
\tag{B17}
$$

while the extended CIR model is given by:

$$
\begin{aligned}
P(r,t,T) &= A(t,T)\exp(-B(t,T)r) \\
B(t,T) &= \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\psi)(e^{\gamma(T-t)}-1)+2\gamma} \\
A(t,T) &= \left[\frac{2\gamma e^{(\gamma+\psi)(T-t)/2}}{(\gamma+\psi)(e^{\gamma(T-t)}-1)+2\gamma}\right]^{2\phi/\sigma^2} \\
\phi(t) &= \kappa(t)\theta+a(t) \\
\psi(t) &= \kappa(t)+\lambda(t)\sigma(t)
\end{aligned}
\tag{B18}
$$

Essentially the Hull and White (1990) model is a version of the Ho and Lee (1986) model in continuous time with mean reversion. Thus, it overcomes the non-Markov problem of the HJM model and also allows a richer description of the volatility structure while preserving the analytical tractability of the Ho and Lee model.[7]

As noted previously the choice of an information set to characterise the term structure depends on the belief in what drives the term structure, namely the state variable. Thus, Black, Derman and Toy (1990) concentrate on modelling the spot rate and use the term structure to price bond options and to determine option hedge rations. The Black, Derman and Toy's spot rate is assumed to: (i) drive the prices of all securities; (ii) be lognormally distributed[8] ; and (iii) move through time in a binomial tree (in discrete time). This spot rate process has a continuous time

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[7] see Hull (1993), pp.398-410.

[8] The lognormal assumption is made to preclude negative interest rates. This is particularly desirable from the practitioner's point of view where nominal interest rates are non-negative.

version[9]:

$$dlog(r) = [\theta(t) - \phi(t)\log r]dt + \sigma(t)dz \tag{B19}$$

where r is the local interest rate and $\sigma(t)$ depends on $\phi(t)$. The current structure of long-term rates and their estimated volatilities characterise the current term structure as long as the tree of the spot rate goes out far enough into the future. The spot rate is inferred from the structure of yields of zero-coupon Treasury bonds and their volatilities such that the model's term structure matches today's current market term structure.

Generally assumptions underlying an economic model are guided by little more than common sense or some empirical facts. Thus, relaxing or varying a model's assumptions is considered a valid exercise to determine the robustness of the model. In this context mean reversion is often desired for its perceived conformity to some notion of equilibrium. Hence the process (B19) is slightly modified by Black and Karasinski (1991) :

$$d(logr) = \phi(t)[\log\mu(t) - \log r]dt + \sigma(t)dz \tag{B20}$$

where $\mu(t)$ is the target rate of interest, $\phi(t)$ is the mean reversion factor and σ(t) is the local volatility of the local change in $log(r)$[10]. The modification is aimed at accounting for the adjustment of the spot rate to its target level. The Black, Derman and Toy model overcomes the problem of negative interest rate and the Black and Karasinski model incorporates mean reversion but both are not

[9] The model is initially derived in discrete time.

[10] $\mu(t)$ and $\phi(t)$ are allowed to vary deterministically through time.

analytically tractable[11].

If $\mu(t)$ and $\phi(t)$ are known (inputs) then the Black-Karasinski model will provide estimates of: (i) the current term structure; (ii) the current term structure volatility; and (iii) the cap curve which is the price of an at-the-money differential cap for each maturity)[12]. Alternatively $\mu(t)$, $\phi(t)$, and $\sigma(t)$ can be chosen to match known outputs, namely observed term structure, term structure volatility curve and cap curve.

The HJM's (1992) and Hull and White's (1990) characterisations of the preference free term structure are based on different information sets: the former requiring the forward rate structure while the latter requires the spot rate process. Jeffrey (1992) integrates these two characterisations into a partial differential equation representation and rederives the HJM no arbitrage condition assuming the spot interest rate is a single factor Markov process and the term structure is a function of this rate. Thus, he provides an alternative characterisation to HJM's (1992) and Hull and White's (1990). Further he suggests a class of forward rate volatility structures that are consistent with the preference free representation of the term structure. This class is characterised by the volatility of the forward rate at time t for date T and is proportional to the spot interest rate volatility at time t. Jeffrey demonstrates that if the HJM single-factor model is driven by a Markovian spot rate process, then it is not possible to arbitrarily choose the volatility structure or the functional forms of the initial forward rate structure. This implies that the

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[11] With the advance of computing power the appeal of a closed form solution is not crucial.

[12] A differential cap pays at a rate equal to the difference (if positive) between the short rate and the strike price.

Markovian spot rate based HJM framework does not have the same richness as the general equilibrium models[13]. In short, the generality of the preference free approach is seriously curtailed in terms of practical implementation. Jeffrey's major contribution is that he provides a solution technique in the form of a non linear partial differential equation to characterise the preference free term structure if the spot rate (state variable) is Markovian. Thus, implementability is achieved at the expense of generality[14].

BII. EMPIRICAL EVIDENCE

Empirical tests of preference free models are limited in number despite their popularity among capital market practitioners. As these models are specifically developed to price interest rate derivative securities it is appropriate to use market prices of these securities in empirical tests. This also implies that the models are usually tested with short-term data as traded interest rate options and futures characteristically have a maturity of less than a year. With the exception of one study[15] where maximum likelihood is used , GMM appears to be the preferred technique.

Ho and Lee (1990) develop bond options and bond futures options models based upon Ho and Lee (1986). The futures option model is tested on 90-day

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[13] The choice of the forward volatility structure is restricted if the spot rate is Markovian. Of course, if the spot rate is non-Markovian (by assumption or observation) then term structure research would belong to a different paradigm of investigation, for example non-linear dynamics or chaos.

[14] HJM do not provide a partial differential equation representation of the term structure because they do not preclude the possibility that the forward rate may be non-Markovian. They only provide an intertemporal relation, via the dynamics of the forward rate, among term structures that ensures no arbitrage if one starts at the current term structure.

[15] The exception is Brenner (1989)

Eurodollar futures options from April 10, 1985 to May, 1985. Estimates of the parameters π and δ[16] are significantly different from 1.0 with t-ratios in the order of 10^2 and 10^5 respectively. Compared to the Black (1976) futures option model Ho and Lee (1990) report smaller within-sample estimation errors (sum of the squares of the residuals) and absence of pricing biases with respect to moneyness and time to maturity. While the Black (1976) model assumes a constant risk free rate, the Ho and Lee (1990) incorporates interest movements of Ho and Lee (1986), and hence providing a better fit to the observations of options and futures.

Brenner (1989) tests an N factor HJM model for a class of HJM volatility functions on US Treasury bills using maximum likelihood estimation. He also recasts the CIR (1985b) and Vasicek (1977) models in terms of forward volatilities and then makes an empirical comparison of CIR, Vasicek and two HJM models based upon constant and exponentially decaying volatility. Brenner found: (i) one factor is sufficient in explaining short term T Bills; and (ii) HJM based models fit the data better than CIR and Vasicek models. Despite the results in favour of the HJM framework he also notes that 'the differences in the results could solely be the function of numerical estimation and precision problems'(Brenner(1989),p.167). This is the twin problem of statistical robustness and computation in testing highly-nonlinear models. In particular, weaknesses of this study include: (i) the data are exclusively short term, and thus lose a great deal of information available in longer term bonds; and (ii) the data often violates the normality assumption (Brenner,

[16] As usual the value of an option depends on its own parameters and those of the underlying asset. In this case π and δ are the parameters of the term structure volatility. If $\pi = 1.0$ or $\delta = 1.0$ then there is no interest rate volatility. π and δ are found by minimising the sum of the squares of the residuals where a residual is the difference between the market price and model price of a discount bond of a given maturity.

p.182) and hence the use of the maximum likelihood estimation method for model comparison is not valid.

As noted above, in the HJM framework the entire term structure is characterised by the initial term structure and the volatility structure of forward rates. As the volatility functions cannot be chosen arbitrarily, instead they have to be specified to be consistent with no arbitrage. HJM (1992) suggest a class of three such volatility functions which are discretised and tested by Thurston (1992) on a set of weekly U.S. T-Bill data from 15/2/68 to 30/5/91 using the Generalised Method of Moments[17]. One of these volatility functions is equivalent to that of the continuous time version of Ho and Lee[18] (1986). Overall, Thurston (1992) reports that the Ho and Lee (1986) volatility function best fits the data. However, Thurston's evidence should be judged in the non-parametric GMM estimation framework which yields weaker results than traditional statistical procedures.

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[17] In addition, Thurston (1992) also tests a square root volatility function suggested by Brenner (1989)

[18] Ho and Lee (1986) is a discrete time model.

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