Developments in the theory of partitions

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DEVELOPMENTS IN THE THEORY OF PARTITIONS

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To my dearest friend, Terry.

I would like to thank a number of people: Terry, my wife, for her support and encouragement over the past five years; Jeremy and Renate, my parents, whose faith in my ability to achieve this goal never wavered; "Nicky" Nikov, Simon Prokhovnik and George Szekeres for their help and encouragement; Richard Askey, Leonard Carlitz and Paul Erdős for their constructive advice; Loy Yeo for help with computing, and Helen Langley for her neat (and speedy!) typing.

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Contents

Abstract

This thesis comprises, in the main, results obtained recently by the author in the theory of partitions and closely related areas.

We begin with the study of a very general continued fraction, and obtain explicit formulae for the convergents. These formulae are transformed in various ways, so that, in the limit, most of the classical results involving continued fractions, due to such people as Ramanujan, Gordon and Carlitz, are obtained. Moreover, our approach obviates the need to quote such product-sum formulae as are given by Slater.

The polynomials which arise in this study are of great interest, both from the point of view of analysis and of combinatorics. Their analytic interest stems from the fact that they form an orthogonal family, not previously studied, and generalise the extended q-Hermite polynomials. A start is made on an investigation of this aspect, but there remain some open questions.

On the other hand, many combinatorial applications of the polynomials have been found. We are led to discover a new polynomial identity which implies the celebrated Rogers-Ramanujan identities. We give a new proof Sylvester's remarkable refinement of Euler's partition theorem using the polynomials, and the proof gives rise to a new polynomial identity which implies Lebesgue's identity. Our final combinatorial application is to prove two identities of Slater. There is, undoubtedly, much that deserves to be learnt about these polynomials.

We then turn to an investigation of several problems in the theory of partitions. We give proofs, again via polynomials, of celebrated identities due to Euler and Jacobi, and use these identities to give an elementary proof of a very beautiful result of Ramanujan. Elementary proofs are given of some partition relations of Kolberg, and it is shown that there are an infinite family of such relations.

Not much is known concerning the parity of $p(n)$, the number of partitions of n. We define $r(n)$, an arithmetic function which grows far less quickly than $p(n)$, give a recurrence for $r(n)$, and show how $p(n)$ and $r(n)$ are related modulo 4. These results contain, as a corollary, the relations modulo 2 for $p(n)$ given by MacMahon.

We close with the proof of a result given by Ramanujan in his recently discovered "lost" notebook, concerning an unusual continued fraction. This result contains earlier results of Eisenstein and Andrews, amongst others. Once again our approach is via the convergents, and explicit formulae are found. This gives rise to another family of polynomials which may have further interesting properties.

Chapter 1 Introduction

The theory of partitions is probably the oldest branch of additive number theory. It is primarily concerned with decompositions of integers into sums of integers. The subject originated with L. Euler, and its fitful development is marked by the major contributions of a small number of men, among them J.J. Sylvester, P.A. MacMahon, B. Gordon and G.E. Andrews. This development has been closely linked to the study of transformations of series. For example, Euler considered $p(n)$, the total number of partitions of n, and showed that*

$$\begin{aligned}\sum_{n\geq 0} p(n)q^n &= \prod_{n\geq 1} \frac{1}{(1-q^n)} \\ &= 1 + \sum_{n\geq 1} \frac{q^n}{(1-q)(1-q^2)\dots(1-q^n)} \\ &= 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(1-q)^2(1-q^2)^2\dots(1-q^n)^2} \\ &= 1/\{1 + \sum_{n\geq 1} (-1)^n (q^{\frac{1}{2}(3n^2-n)} + q^{\frac{1}{2}(3n^2+n)})\} .\end{aligned}$$

Notable contributors to the study of such series include C.F. Gauss, E. Heine, C.G.J. Jacobi, L.J. Rogers, S. Ramanujan, G.N. Watson and W.N. Bailey. It is fair to say that these areas are as active today as they have ever been.

As the study of partitions and related series advanced, Ramanujan and others were able to apply their knowledge to results on continued fractions. The oldest and most famous theorem in this regard is the Rogers-Ramanujan continued fraction

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* Throughout the thesis, $|q| < 1$. All results are then true as stated, except at those points where a denominator may vanish.

$$1 + \frac{q}{1+} \; \frac{q^2}{1+} \; \frac{q^3}{1+} \; \cdots = \prod_{n\geq 0} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})} \quad .$$

The first half of this thesis is devoted to reversing the historical relationship between such continued fractions and the theory of partitions and series. Indeed by considering the convergents to the continued fraction

$$1 + a + b + \frac{cq-a}{1+a+bq+} \; \frac{cq^2-a}{1+a+bq^2+} \; \cdots$$

we are led in Chapter 2 to an extensive study of a new family $P_n(a,b,c,q)$ of polynomials with diverse applications. For example we show that*

$$\sum_{r\geq 0} q^{r^2} \begin{bmatrix} n-r \\ r \end{bmatrix} = \sum_{r\geq 0} (-1)^r q^{\frac{1}{2}r(5r-1)} \begin{bmatrix} n-2r \\ r \end{bmatrix} H_{n-3r}(q^r)$$
$$- \sum_{r\geq 0} (-1)^r q^{\frac{1}{2}r(5r+3)} \begin{bmatrix} n-1-2r \\ r \end{bmatrix} H_{n-1-3r}(q^r) \; . \tag{1.1}$$

This result yields the first of the celebrated Rogers-Ramanujan identities on letting $n \to \infty$. Indeed, numerous such transformations are found, so that the representations of almost all the classical infinite continued fractions as quotients of theta-series become transparent from the convergents. Among the new continued fractions that we discover utilising our approach is

$$1 + \frac{q}{1-q+q^2+} \; \frac{q}{1-q+q^4+} \cdots = \prod_{n\geq 0} \frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})} \; . \tag{1.2}$$

The polynomials $P_n(a,b,c,q)$ generalise the previously

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* All special notation is explained in the appendix.

studied, and well understood, extended q-Hermite polynomials. In Chapter 3 we analyse this aspect of these polynomials. In particular, some connection coefficient theorems are given.

We next utilise these polynomials in Chapter 4 to present a new proof of Sylvester's remarkable partition theorem "The number of partitions of n into odd parts, s of which are distinct, is equal to the number of partitions of n into distinct parts with s sequences of consecutive parts."

While all previous analytic proofs have relied on V.A. Lebesgue's identity

$$\sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}(1+c)\ldots(1+cq^{n-1})}{(1-q)\ldots(1-q^{n})} = \prod_{n\geq 0}\left(\frac{1+cq^{2n+1}}{1-q^{2n+1}}\right) ,$$

we are able to treat the problem purely with our polynomials, and find the new identity

$$\sum_{r,s\geq 0} q^{\binom{r+s+1}{2}+\binom{r}{2}} c^{r} \begin{bmatrix} r+s \\ r \end{bmatrix}\begin{bmatrix} n-r \\ r+s \end{bmatrix} =$$

$$= \sum_{r\geq 0} c^{r} q^{r^{2}} (-q^{r+1})_{n-2r} \begin{bmatrix} n-r \\ r \end{bmatrix} , \qquad (1.3)$$

itself a special case of a general polynomial identity obtained in Chapter 2. Lebesgue's identity follows from our result as $n \to \infty$.

In the same chapter is presented another new proof of Sylvester's theorem which shows clearly how Sylvester's theorem is equivalent to Lebesgue's identity.

Sylvester's partition theorem is a refinement of Euler's partition theorem "The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts". Andrews has given a generalisation of Euler's theorem; his theorem and three similar results are presented. One of these is

" $\sum_{\pi\in\Pi_n} \binom{s(\pi)}{r}$ is equal to the number of partitions of $n-r^2$ with no even part greater than $2r$". (1.4)

(Here Π_n is the set of partitions of n into distinct parts, and $s(\pi)$ is the number of sequences of consecutive parts in π.) Euler's theorem is the case $r=0$.

The study in Chapter 4 of partitions with distinct parts and exactly k sequences leads naturally to the study of partitions with distinct parts where now k bounds the length of sequences. We discover the generating function for such partitions with all parts less than n. In the case $k=2$, we are led to the polynomial identity

$$\sum_{r\geq 0} c^r q^{r^2} \left[{n-r \atop r}\right] =$$

$$= \sum_{r\geq 0} c^r q^{\binom{r+1}{2}} \sum_{2s\leq r} (-1)^s q^{\binom{s}{2}} \left[{n-r \atop s}\right]_{(q^2)} \left[{n-2s-1 \atop r-2s}\right] \qquad (1.5)$$

This identity in turn yields a combinatorial interpretation of two identities of L.J. Slater, namely

$$\prod_{r\geq 1} (1+q^r) \sum_{r\geq 0} \frac{(-1)^r q^{3r^2}}{(q^2;q^2)_r (-q;q)_{2r}} = 1/\prod_{r\geq 0} (1-q^{5r+1})(1-q^{5r+4})$$

and

$$\prod_{r\geq 1}(1+q^r)\sum_{r\geq 0}\frac{(-1)^r q^{3r^2-2r}}{(q^2;q^2)_r(-q;q)_{2r}} = 1/\prod_{r\geq 0}(1-q^{5r+2})(1-q^{5r+3}) .$$

Thus, the study of the polynomials in Chapters 2-5 has led to various new results in the theory of partitions. These polynomials should, in the future, lead to further interesting work. As evidence for this claim, we recall that one of I.J. Schur's most ingenious proofs of the Rogers-Ramanujan identities was effectively based on the polynomial identity

$$\sum_{r\geq 0} q^{r^2}\begin{bmatrix} n-r \\ r \end{bmatrix} = \sum_{\lambda=-\infty}^{\infty}(-1)^{\lambda} q^{\frac{1}{2}(5\lambda^2+\lambda)}\begin{bmatrix} n \\ [\frac{1}{2}(n-5\lambda)] \end{bmatrix} .$$

An investigation of the partition-theoretic implications of this identity led Andrews to his extensive study of sieves in partitions. No such explanation of (1.1) is known; however, as we come to understand more about the $P_n(a,b,c,q)$ we may well find the answer to this and the other open questions mentioned in Chapters 2-5.

The remainder of the thesis presents several topics closely related to the theme already developed.

G.H. Hardy nominated as the most beautiful identity given by Ramanujan

$$\sum_{n\geq 0} p(5n+4)q^n = 5\prod_{n\geq 1}\frac{(1-q^{5n})^5}{(1-q^n)^6} .$$

In Chapter 7 we give a truly elementary proof of this result, as well as elementary proofs of the identities of O.Kolberg,

$$P_0P_4 + P_1P_3 - 2P_2^2 = 0 ,$$

$$P_0P_2 + P_3P_4 - 2P_1^2 = 0 ,$$

and

$$3P_1P_2 - 2P_0P_3 - P_4^2 = 0 ,$$

where $$P_i = \sum_{n\equiv i \bmod 5} p(n)q^n .$$

Further, we prove that identities of the same sort hold not only for the modulus 5, but for any modulus not a power of 2. Thus in particular for the modulus 3,

if $P_i = \sum_{n\equiv i \bmod 3} p(n)q^n$, then

$$(P_0^2-P_1P_2)(P_2^2-P_0P_1)^2 + (P_2^2-P_0P_1)(P_1^2-P_0P_2)^2 + (P_1^2-P_0P_2)(P_0^2-P_1P_2)^2 = 0 . \tag{1.5}$$

All that are needed to prove these results are the identities

$$\prod_{n\geq 1} (1-q^n) = 1 + \sum_{n\geq 1} (-1)^n (q^{\frac{1}{2}(3n^2-n)} + q^{\frac{1}{2}(3n^2+n)})$$

and

$$\prod_{n\geq 1} (1-q^n)^3 = \sum_{n\geq 0} (-1)^n (2n+1) q^{\frac{1}{2}(n^2+n)}$$

due respectively to Euler and Jacobi. The first of these follows from the polynomial identity

$$\prod_{r=1}^{i} (1+a^{-1}q^{2r-1}) \prod_{r=1}^{j} (1+aq^{2r-1}) = \sum_{r=-i}^{j} a^r q^{r^2} \begin{bmatrix} i+j \\ i+r \end{bmatrix}_{(q^2)} \tag{1.6}$$

while the second follows from the polynomial identity

$$\prod_{r=1}^{n} (1-q^r)^2 = \sum_{r=0}^{n} (-1)^r (2r+1) q^{\frac{1}{2}(r^2+r)} \begin{bmatrix} 2n+1 \\ n-r \end{bmatrix} , \tag{1.7}$$

both proved in Chapter 6.

Not much is known concerning the behaviour of $p(n)$ modulo 2. If we define $r(n)$ to be the number of representations of n as a sum

$$n = \Delta(n_1) + 4\Delta(n_2) + 16\Delta(n_3) + \dots$$

where $\Delta(n)$ is the triangular number $\frac{1}{2}(n^2+n)$, then $r(n)$ grows far more slowly than $p(n)$, and satisfies the simple recurrence relation

$$r(4n) = r(n) + \sum_{k\geq 1} r(n-(8k^2-k)) + \sum_{k\geq 1} r(n-(8k^2+k)),$$

$$r(4n+1) = r(n) + \sum_{k\geq 1} r(n-(8k^2-3k)) + \sum_{k\geq 1} r(n-(8k^2+3k)),$$

$$r(4n+3) = r(n) + \sum_{k\geq 1} r(n-(8k^2-5k)) + \sum_{k\geq 1} r(n-(8k^2+5k)),$$

$$r(4n+6) = r(n) + \sum_{k\geq 1} r(n-(8k^2-7k)) + \sum_{k\geq 1} r(n-(8k^2+7k)), \tag{1.8}$$

and as we show in Chapter 8,

$$p(n) \equiv r(n) + 2 \sum_{k\in S} r(n-2k^2) \bmod 4 \tag{1.9}$$

where $S = \{1,3,4,5,7,9,11,12,\dots\}$ is the set of numbers k with $t(k)$ even, where $t(k) = t$ is defined by $2^t | k$, $2^{t+1} \nmid k$.

This result contains as a corollary MacMahon's congruences modulo 2 for $p(n)$, namely

$$p(4n) \equiv p(n) + \sum_{k\geq 1} p(n-(8k^2-k)) + \sum_{k\geq 1} p(n-(8k^2+k))$$

and so on.

It is well known that the identity

$$\sum_{n\geq 0} \frac{q^{\frac{1}{2}(n^2+n)}}{(q)_n} = \prod_{n\geq 0} \frac{1}{(1-q^{2n+1})}$$

is equivalent to Euler's partition theorem (mentioned earlier), while the Rogers-Ramanujan identities

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_n} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}$$

and

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_n} = \prod_{n\geq 0} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}$$

are equivalent to partition theorems, the first to

"The number of partitions of n into parts which differ by at least 2 is equal to the number of partitions of n into parts which are congruent to 1 or 4 modulo 5".

Several of the limiting cases of results we treat in Chapter 2 are amenable to partition-theoretic interpretation. We consider

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q)_{2n}} = \prod_{\substack{n>0 \\ n\equiv\pm1,\pm3,\pm4,\pm5,\pm7,\pm9 \bmod 20}} (1-q^n)^{-1} ,$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q)_{2n+1}} = \prod_{\substack{n>0 \\ n\equiv\pm1,\pm2,\pm5,\pm6,\pm8,\pm9 \bmod 20}} (1-q^n)^{-1} ,$$

$$\sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q)_{2n+1}} = \prod_{\substack{n>0 \\ n\equiv\pm1,\pm4,\pm6,\pm7 \bmod 16}} (1-q^n)^{-1} ,$$

and

$$\sum_{n\geq 0} \frac{q^{2n^2}}{(q)_{2n}} = \prod_{\substack{n>0 \\ n\equiv\pm2,\pm3,\pm4,\pm5 \bmod 16}} (1-q^n)^{-1} ,$$

all of which appear in Slater's compendium of such identities. We show that, for example, the fourth yields

"The number of partitions of n,

$$n = a_1 + a_2 + a_3 + a_4 + a_5 + \dots$$

with

$$a_1 \geq 2,\ a_2 - a_1 \geq 0,\ a_3 - a_2 \geq 2,\ a_4 - a_3 \geq 0,\ a_5 - a_4 \geq 2,\ \dots$$

is equal to the number of partitions of n into parts congruent to $2, 3, 4, 5, 11, 12, 13$ or 14 modulo 16." (1.10)

Straightforward proofs of Slater's identities also are included. We remark that Gordon, W. Connor, and Andrews and Askey have, independently, given partition-theoretic treatments of the first two of the four identities given above.

In 1976, Andrews discovered a manuscript of Ramanujan which he has called "the 'lost' notebook". This manuscript, probably written in the last year of Ramanujan's life, contains about six hundred identities, of which Andrews has to date proved more than half. One of these, with which Andrews had some difficulty, concerns the unusual continued fraction

$$F(a,b,\lambda,q) = 1 + \frac{aq+\lambda q}{1+}\ \frac{bq+\lambda q^2}{1+}\ \frac{aq^2+\lambda q^3}{1+}\ \frac{bq^2+\lambda q^4}{1+}\ \cdots .$$

As in the case of the continued fraction of Chapter 2, our approach is via the convergents, for which an explicit expression is obtained. Ramanujan's result then follows easily. Some particular cases of this result are

$$1+\frac{q+q^2}{1+}\ \frac{q^2+q^4}{1+}\ \frac{q^3+q^6}{1+}\ \cdots\ =\prod_{n\geq 0}\frac{(1-q^{6n+3})^2}{(1-q^{6n+1})(1-q^{6n+5})}$$

and

$$1+\frac{q}{1+}\ \frac{q+q^2}{1+}\ \frac{q^3}{1+}\ \frac{q^2+q^4}{1+}\ \frac{q^5}{1+}\ \cdots\ =\prod_{n\geq 0}\frac{(1-q^{4n+2})^2}{(1-q^{4n+1})(1-q^{4n+3})}\quad .$$

Again we have a new family of polynomials, which may prove as fruitful as the $P_n(a,b,c,q)$.

Chapter 2 A Continued Fraction

§1 We consider the continued fraction

$$F(a,b,c,q) = 1+a+b+\frac{cq-a}{1+a+bq+}\;\frac{cq^2-a}{1+a+bq^2+}\;\cdots\cdots \tag{2.1.1}$$

As we shall see, many of the continued fractions which have been found by earlier writers to be expressible as products can be obtained from (2.1.1) by specialisation of the parameters.

Indeed we show that (if $|a| < 1$)

$$F(a,b,c,q) =$$

$$= \sum_{r\geq 0} \frac{q^{\binom{r}{2}} b^r (-cq/b)_r}{(q)_r (a)_{r+1}} \Bigg/ \sum_{r\geq 0} \frac{q^{\binom{r+1}{2}} b^r (-cq/b)_r}{(q)_r (a)_{r+1}} \tag{2.1.2}$$

$$= \frac{1 + \sum\limits_{r\geq 1} \frac{(1-cq^{2r})}{(1-cq^r)} \frac{(c/a)_r}{(aq)_r} \frac{(-cq/b)_r}{(-b)_r} \frac{(cq)_r}{(c)_r} q^{3\binom{r}{2}+r} a^r b^r}{\sum\limits_{r\geq 0} (1-cq^{2r+1}) \frac{(cq/a)_r}{(aq)_r} \frac{(-cq/b)_r}{(-b)_{r+1}} \frac{(cq)_r}{(q)_r} q^{3\binom{r}{2}+2r} a^r b^r} . \tag{2.1.3}$$

There are many special cases of (2.1.2) and (2.1.3) in which the series appearing on the right-hand-side can, via Jacobi's triple product theorem (see Chapter 6), or some other device, be expressed as products. In order to demonstrate the power of (2.1.2) and (2.1.3), we give a number of these special cases, most, but not all, of which have previously appeared in the literature, before we turn to proofs. Note that (2.1.2) has appeared in Hirschhorn (1974a). A result of the same type appears in Andrews (1968) [Theorem 6].

If in (2.1.3) we set $a = b = 0,\ c = 1,$ we obtain

$$1 + \frac{q}{1+}\ \frac{q^2}{1+}\ \frac{q^3}{1+}\ \cdots =$$

$$= \frac{1 + \sum_{r\geq 1} (-1)^r q^{\frac{1}{2}r(5r-1)}(1+q^r)}{\sum_{r\geq 0} (-1)^r q^{\frac{1}{2}r(5r+3)}(1-q^{2r+1})}$$

$$= \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty}{(q\ ;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty}$$

$$= (q^2;q^5)_\infty (q^3;q^5)_\infty / (q;q^5)_\infty (q^4;q^5)_\infty, \qquad (2.1.4)$$

a result due independently to L.J. Rogers (1894) p.328 and S. Ramanujan (1919b).

If in (2.1.3) we set q^2 for q, $a = 0,\ b = q,\ c = 1,$ we obtain

$$1 + q + \frac{q^2}{1+q^3+}\ \frac{q^4}{1+q^5+}\ \cdots =$$

$$= \frac{1 + \sum_{r\geq 1} (-1)^r q^{4r^2-r}(1+q^{2r})}{\sum_{r\geq 0} (-1)^r q^{4r^2+3r}(1-q^{2r+1})}$$

$$= \frac{(q^3;q^8)_\infty (q^5;q^8)_\infty (q^8;q^8)_\infty}{(q\ ;q^8)_\infty (q^7;q^8)_\infty (q^8;q^8)_\infty}$$

$$= (q^3;q^8)_\infty (q^5;q^8)_\infty \ / \ (q;q^8)_\infty (q^7;q^8)_\infty\ , \qquad (2.1.5)$$

a result due to B. Gordon (1965).

If in (2.1.2) we set $a = 0$, and subtract b, we obtain

$$1 + \frac{cq}{1+bq+}\ \frac{cq^2}{1+bq^2+}\ \cdots =$$

$$= \sum_{r\geq 0} \frac{q^{\binom{r+1}{2}} b^r (-c/b)_r}{(q)_r} \Bigg/ \sum_{r\geq 0} \frac{q^{\binom{r+1}{2}} b^r (-cq/b)_r}{(q)_r} \qquad (2.1.6)$$

a result of Carlitz (1965),

$$= (-bq)_\infty \sum_{t\geq 0} \frac{c^t q^{t^2}}{(q)_t\,(-bq)_t} \Big/ (-bq)_\infty \sum_{t\geq 0} \frac{c^t q^{t^2+t}}{(q)_t(-bq)_t}$$

$$= \sum_{t\geq 0} \frac{c^t q^{t^2}}{(q)_t(-bq)_t} \Big/ \sum_{t\geq 0} \frac{c^t q^{t^2+t}}{(q)_t(-bq)_t} , \tag{2.1.7}$$

a result stated by Ramanujan (Notebooks Vol. II p. 196). If we now set $b = 1$, we obtain

$$1 + \frac{cq}{1+q+}\,\frac{cq^2}{1+q^2+}\cdots = \sum_{t\geq 0} \frac{c^t q^{t^2}}{(q^2;q^2)_t} \Big/ \sum_{t\geq 0} \frac{c^t q^{t^2+t}}{(q^2;q^2)_t}$$

$$= (-cq;q^2)_\infty/(-cq^2;q^2)_\infty , \tag{2.1.8}$$

also a result of Carlitz (1965).

The special case $c = 1/q$ was given earlier by Gordon (1965).

If in (2.1.2) we subtract a, set $c = 0$ and $b = a$, we obtain

$$1+a-\frac{a}{1+a+aq-}\,\frac{a}{1+a+aq^2-}\cdots =$$

$$= \sum_{r\geq 0} \frac{q^{\binom{r}{2}} a^r}{(q)_r(a)_r} \Big/ \sum_{r\geq 0} \frac{q^{\binom{r+1}{2}} a^r}{(q)_r(a)_{r+1}} . \tag{2.1.9}$$

Now, it is not hard to show (see §9.3) that

$$(a)_\infty \sum_{r\geq 0} \frac{q^{\binom{r}{2}} a^r}{(q)_r(a)_r} = \sum_{r\geq 0} \frac{q^{2r^2-r} a^{2r}}{(q^2;q^2)_r} \tag{2.1.10}$$

and thence (put aq for q) that

$$(a)_\infty \sum_{r\geq 0} \frac{q^{\binom{r+1}{2}} a^r}{(q)_r(a)_{r+1}} = \sum_{r\geq 0} \frac{q^{2r^2+r} a^{2r}}{(q^2;q^2)_r} . \tag{2.1.11}$$

It follows from (2.1.9), (2.1.10), (2.1.11) that

$$1+a-\frac{a}{1+a+aq-}\,\frac{a}{1+a+aq^2-}\ \cdots\ =$$

$$= \sum_{r\geq 0}\frac{q^{2r^2-r}a^{2r}}{(q^2;q^2)_r}\Bigg/\sum_{r\geq 0}\frac{q^{2r^2+r}a^{2r}}{(q^2;q^2)_r}\ . \tag{2.1.12}$$

If we now set q^2 for q, $a=q$, and use the Rogers-Ramanujan identities (see Appendix §8) we find that

$$1+q-\frac{q}{1+q+q^3-}\,\frac{q}{1+q+q^5-}\ \cdots\ =$$

$$= \sum_{r\geq 0}\frac{q^{4r^2}}{(q^4;q^4)_r}\Bigg/\sum_{r\geq 0}\frac{q^{4r^2+4r}}{(q^4;q^4)_r}$$

$$= (q^8;q^{20})_\infty(q^{12};q^{20})_\infty/(q^4;q^{20})_\infty(q^{16};q^{20})_\infty\ , \tag{2.1.13}$$

Gordon (1965).

As a final example, if in (2.1.2) we subtract a, then subtract b and set $c=0$, we obtain

$$1-\frac{a}{1+a+bq-}\,\frac{a}{1+a+bq^2-}\ \cdots\ =$$

$$= \sum_{r\geq 0}\frac{q^{\binom{r+1}{2}}b^r}{(q)_r(a)_r}\Bigg/\sum_{r\geq 0}\frac{q^{\binom{r+1}{2}}b^r}{(q)_r(a)_{r+1}}\ . \tag{2.1.14}$$

If we now set q^2 for q, $-a$ for a, and then $b=a/q$, we find

$$1+\frac{a}{1-a+aq+}\,\frac{a}{1-a+aq^3+}\ \cdots\ =$$

$$= \sum_{r\geq 0}\frac{q^{r^2}a^r}{(q^2;q^2)_r(-a;q^2)_r}\Bigg/\sum_{r\geq 0}\frac{q^{r^2}a^r}{(q^2;q^2)_r(-a;q^2)_{r+1}}\ . \tag{2.1.15}$$

Now (see §9.3),

$$(-a;q^2)_\infty \sum_{r\geq 0} \frac{q^{r^2}a^r}{(q^2;q^2)_r(-a;q^2)_r} = \sum_{r\geq 0} \frac{q^{r^2-r}a^r}{(q)_r} \tag{2.1.16}$$

and

$$(-a,q^2)_\infty \sum_{r\geq 0} \frac{q^{r^2}a^r}{(q^2;q^2)_r(-a;q^2)_{r+1}} = \sum_{r\geq 0} \frac{q^{r^2}a^r}{(q)_r} \quad . \tag{2.1.17}$$

From (2.1.15), (2.1.16) and (2.1.17) it follows that

$$1 + \frac{a}{1-a+aq+} \; \frac{a}{1-a+aq^3+} \; \cdots =$$

$$= \sum_{r\geq 0} \frac{q^{r^2-r}a^r}{(q)_r} \Bigg/ \sum_{r\geq 0} \frac{q^{r^2}a^r}{(q)_r} \quad . \tag{2.1.18}$$

If we now set $a = q$, and use the Rogers-Ramanujan identities once again, we find that

$$1 + \frac{q}{1-q+q^2+} \; \frac{q}{1-q+q^4+} \; \cdots =$$

$$= (q^2;q^5)_\infty(q^3;q^5)_\infty/(q;q^5)_\infty(q^4;q^5)_\infty \quad . \tag{2.1.19}$$

§2. We prove (2.1.2) as follows: We can write

$$1+a+b+ \frac{cq-a}{1+a+bq+} \; \frac{cq^2-a}{1+a+bq^2+} \; \cdots \; \frac{cq^n-a}{1+a+bq^n} =$$

$$= \frac{P_{n+1}(a,b,c,q)}{Q_{n+1}(a,b,c,q)} \quad , \tag{2.2.1}$$

where

$$P_0 = 1, \quad P_1 = 1+a+b \; ,$$

$$Q_0 = 0, \quad Q_1 = 1 \; ,$$

and, for $n \geq 1$, (2.2.2)

$$P_{n+1} = (1+a+bq^n)P_n+(cq^n-a)P_{n-1} \; ,$$

$$Q_{n+1} = (1+a+bq^n)Q_n+(cq^n-a)Q_{n-1} \quad .$$

It follows from (2.2.2) that

$$Q_{n+1}(a,b,c,q) = P_n(a,bq,cq,q) \quad . \tag{2.2.3}$$

However, for the purposes of this chapter, we find that it is convenient to distinguish the P_n and Q_n, and not use (2.2.3).

We set

$$P(z) = \sum_{n\geq 0} P_n z^n , \quad Q(z) = \sum_{n\geq 0} Q_n z^n . \tag{2.2.4}$$

It follows from (2.2.2) that

$$\begin{aligned}(1-z)(1-az)P(z) - z(b+cqz)P(qz) &= 1 , \\ (1-z)(1-az)Q(z) - z(b+cqz)Q(qz) &= z .\end{aligned} \tag{2.2.5}$$

We can write (2.2.5)

$$\begin{aligned}P(z) &= \frac{1}{(1-z)(1-az)} + \frac{zb(1+cqz/b)}{(1-z)(1-az)} P(qz) , \\ Q(z) &= \frac{z}{(1-z)(1-az)} + \frac{zb(1+cqz/b)}{(1-z)(1-az)} Q(qz) .\end{aligned} \tag{2.2.6}$$

It follows by iteration of (2.2.6) that

$$\begin{aligned}P(z) &= \sum_{n\geq 0} P_n z^n = \sum_{r\geq 0} \frac{q^{\binom{r}{2}} z^r b^r (-cqz/b)_r}{(z)_{r+1}(az)_{r+1}} , \\ Q(z) &= \sum_{n\geq 0} Q_n z^n = \sum_{r\geq 0} \frac{q^{\binom{r+1}{2}} z^{r+1} b^r (-cqz/b)_r}{(z)_{r+1}(az)_{r+1}} .\end{aligned} \tag{2.2.7}$$

We now appeal to Abel's lemma. If $\lim_{n\to\infty} P_n$ exists (and we shall see that it does if $|a| < 1$),

$$\begin{aligned}P_\infty = \lim_{n\to\infty} P_n &= \lim_{z\to 1-} (1-z)P(z) \\ &= \sum_{r\geq 0} \frac{q^{\binom{r}{2}} b^r (-cq/b)_r}{(q)_r (a)_{r+1}}\end{aligned} \tag{2.2.8a}$$

and similarly

$$Q_\infty = \lim_{n\to\infty} Q_n = \sum_{r\ge 0} \frac{q^{\binom{r+1}{2}} b^r (-cq/b)_r}{(q)_r (a)_{r+1}} . \tag{2.2.8b}$$

If we let $n \to \infty$ in (2.2.1) and use (2.2.8), we obtain (2.1.2).

We can obtain explicit expressions for P_n and Q_n from (2.2.7), and then pass to the limit. Thus

$$\sum_{n\ge 0} P_n z^n =$$

$$= \sum_{r\ge 0} \frac{q^{\binom{r}{2}} z^r b^r (-cqz/b)_r}{(z)_{r+1} (az)_{r+1}}$$

$$= \sum_{r\ge 0} q^{\binom{r}{2}} z^r b^r \sum_{s=0}^{r} q^{\binom{s+1}{2}} (cz/b)^s \begin{bmatrix} r \\ s \end{bmatrix} \sum_{t\ge 0} z^t \begin{bmatrix} r+t \\ r \end{bmatrix} \times$$

$$\times \sum_{u\ge 0} (az)^u \begin{bmatrix} r+u \\ r \end{bmatrix} ,$$

and, putting $r = s+v$, this becomes

$$= \sum_{s,t,u,v\ge 0} q^{\binom{s+v}{2}+\binom{s+1}{2}} a^u b^v c^s \times$$

$$\times \begin{bmatrix} s+v \\ s \end{bmatrix} \begin{bmatrix} s+t+v \\ t \end{bmatrix} \begin{bmatrix} s+u+v \\ u \end{bmatrix} z^{2s+t+u+v} . \tag{2.2.9a}$$

It follows that

$$P_n = \sum_{2s+t+u+v=n} q^{\binom{s+v}{2}+\binom{s+1}{2}} a^u b^v c^s \times$$

$$\times \begin{bmatrix} s+v \\ s \end{bmatrix} \begin{bmatrix} s+t+v \\ t \end{bmatrix} \begin{bmatrix} s+u+v \\ u \end{bmatrix}$$

$$= \sum_{s,u,v\ge 0} q^{\binom{s+v}{2}+\binom{s+1}{2}} a^u b^v c^s \times$$

$$\times \begin{bmatrix} s+v \\ s \end{bmatrix} \begin{bmatrix} s+u+v \\ u \end{bmatrix} \begin{bmatrix} n-s-u \\ s+v \end{bmatrix} , \tag{2.2.10a}$$

and similarly,

$$Q_n = \sum_{s,u,v\geq 0} q^{\binom{s+v+1}{2}+\binom{s+1}{2}} a^u b^v c^s \times$$

$$\times \begin{bmatrix} s+v \\ s \end{bmatrix} \begin{bmatrix} s+u+v \\ u \end{bmatrix} \begin{bmatrix} n-1-s-u \\ s+v \end{bmatrix} . \qquad (2.2.10b)$$

Letting $n \to \infty$ in (2.2.10a), we find that if $|a| < 1$,

$$P_\infty = \sum_{s,u,v\geq 0} q^{\binom{s+v}{2}+\binom{s+1}{2}} a^u b^v c^s \times$$

$$\times \begin{bmatrix} s+v \\ s \end{bmatrix} \begin{bmatrix} s+u+v \\ u \end{bmatrix} . \frac{1}{(q)_{s+v}}$$

$$= \sum_{s,v\geq 0} q^{\binom{s+v}{2}+\binom{s+1}{2}} b^v c^s \begin{bmatrix} s+v \\ s \end{bmatrix} \frac{1}{(q)_{s+v}} \times$$

$$\times \sum_{u\geq 0} a^u \begin{bmatrix} s+u+v \\ u \end{bmatrix}$$

$$= \sum_{s,v\geq 0} q^{\binom{s+v}{2}+\binom{s+1}{2}} b^v c^s \begin{bmatrix} s+v \\ s \end{bmatrix} \frac{1}{(q)_{s+v}} \frac{1}{(a)_{s+v+1}}$$

and, putting $s+v=r$, this becomes

$$= \sum_{r\geq 0} q^{\binom{r}{2}} b^r \frac{1}{(q)_r} \frac{1}{(a)_{r+1}} \sum_{s=0}^{r} q^{\binom{s+1}{2}} (c/b)^s \begin{bmatrix} r \\ s \end{bmatrix}$$

$$= \sum_{r\geq 0} \frac{q^{\binom{r}{2}} b^r (-c/b)_r}{(q)_r (a)_{r+1}} ,$$

which is (2.2.8a).

(2.2.8b) follows similarly from (2.2.10b).

§3. Having proved (2.1.2) we proceed to prove (2.1.3).

Watson's theorem (Watson (1929a), see Appendix §8) is

$$
{}_4\phi_3\left[\begin{matrix}\frac{Aq}{BC},\ D,\ E,\ q^{-N} & ;\ q;\ q\\ \frac{Aq}{B},\ \frac{Aq}{C},\ \frac{DEq^{-N}}{A} & \end{matrix}\right] =
$$

$$
= \frac{(Aq/D)_N(Aq/E)_N}{(Aq)_N\ (Aq/DE)_N} \times {}_8\phi_7\left[\begin{matrix}A, q\sqrt{A},\ -q\sqrt{A},\ B,\ C,\ D,\ E,\ q^{-N};\ q;\ \frac{A^2q^2}{BCDEq^{-N}}\\ \sqrt{A},\ -\sqrt{A},\ \frac{Aq}{B},\ \frac{Aq}{C},\ \frac{Aq}{D},\ \frac{Aq}{E},\ \frac{Aq}{q^{-N}}\end{matrix}\right]. \tag{2.3.1}
$$

Letting $B,D,N \to \infty$ in (2.3.1), we obtain

$$
\sum_{r\geq 0} \frac{q^{\binom{r}{2}}(-\frac{Aq}{E})^r\ (E)_r}{(q)_r\left(\frac{Aq}{C}\right)_r} =
$$

$$
= \frac{(Aq/E)_\infty}{(Aq)_\infty} \sum_{r\geq 0} \frac{(1-Aq^{2r})}{(1-A)} \frac{(C)_r}{\left(\frac{Aq}{C}\right)_r} \frac{(E)_r}{\left(\frac{Aq}{E}\right)_r} \frac{(A)_r}{(q)_r} q^{3\binom{r}{2}+2r} \left(\frac{-A^2}{CE}\right)^r . \tag{2.3.2}
$$

If we now set $A=c$, $C=c/a$, $E=-cq/b$, we obtain

$$
P_\infty = \frac{1}{(1-a)} \sum_{r\geq 0} \frac{q^{\binom{r}{2}} b^r(-cq/b)_r}{(q)_r(aq)_r}
$$

$$
= \frac{1}{(1-a)} \frac{(-b)_\infty}{(cq)_\infty} \sum_{r\geq 0} \frac{(1-cq^{2r})}{(1-c)} \frac{(c/a)_r}{(aq)_r} \frac{(-cq/b)_r}{(-b)_r} \frac{(c)_r}{(q)_r} q^{3\binom{r}{2}+r} a^r b^r
$$

$$
= \frac{1}{(1-a)} \frac{(-b)_\infty}{(cq)_\infty} \{1 + \sum_{r\geq 1} \frac{(1-cq^{2r})}{(1-cq^r)} \frac{(c/a)_r}{(aq)_r} \frac{(-cq/b)_r}{(-b)_r} \frac{(cq)_r}{(q)_r} q^{3\binom{r}{2}+r} a^r b^r\}. \tag{2.3.3a}
$$

If, on the other hand, in (2.3.2) we set $A=cq$, $C=cq/a$, $E= -cq/b$, we obtain

$$
Q_\infty = \frac{1}{(1-a)} \sum_{r\geq 0} \frac{q^{\binom{r+1}{2}} b^r(-cq/b)_r}{(q)_r(aq)_r}
$$

$$
= \frac{1}{(1-a)} \frac{(-bq)_\infty}{(cq^2)_\infty} \sum_{r\geq 0} \frac{(1-cq^{2r+1})}{(1-cq)} \frac{(cq/a)_r}{(aq)_r} \frac{(-cq/b)_r}{(-bq)_r} \frac{(cq)_r}{(q)_r} q^{3\binom{r}{2}+2r} a^r b^r
$$

$$= \frac{1}{(1-a)} \frac{(-b)_\infty}{(cq)_\infty} \sum_{r\geq 0} (1-cq^{2r+1}) \frac{(cq/a)_r}{(aq)_r} \frac{(-cq/b)_r}{(-b)_{r+1}} \frac{(cq)_r}{(q)_r} q^{3\binom{r}{2}+2r} a^r b^r . \tag{2.3.3b}$$

It follows from (2.3.3) that

$$F(a,b,c,q) = P_\infty/Q_\infty$$

$$= \frac{1 + \sum_{r\geq 1} \frac{(1-cq^{2r})}{(1-cq^r)} \frac{(c/a)_r}{(aq)_r} \frac{(-cq/b)_r}{(-b)_r} \frac{(cq)_r}{(q)_r} q^{3\binom{r}{2}+r} a^r b^r}{\sum_{r\geq 0} (1-cq^{2r+1}) \frac{(cq/a)_r}{(aq)_r} \frac{(-cq/b)_r}{(-b)_{r+1}} \frac{(cq)_r}{(q)_r} q^{3\binom{r}{2}+2r} a^r b^r},$$

which is (2.1.3).

§4 It is possible to transform the P_n, Q_n, again via Watson's theorem (2.3.1) in such a way that on letting $n \to \infty$ we obtain (2.3.3) and thus (2.1.3) directly.

We have (2.2.7)

$$\sum_{n\geq 0} P_n z^n = \sum_{r\geq 0} \frac{q^{\binom{r}{2}} z^r b^r (-cqz/b)_r}{(z)_{r+1} (az)_{r+1}}$$

$$= \sum_{r\geq 0} \frac{q^{\binom{r}{2}} z^r b^r (-cqz/b)_r}{(az)_{r+1}} \sum_{s\geq 0} z^s \begin{bmatrix} r+s \\ r \end{bmatrix}$$

$$= \sum_{r,s\geq 0} \frac{q^{\binom{r}{2}} b^r (-cqz/b)_r}{(az)_{r+1}} \begin{bmatrix} r+s \\ r \end{bmatrix} z^{r+s} ,$$

which, putting $r+s = t$, becomes

$$= \sum_{t\geq 0} z^t \sum_{r=0}^{t} \frac{q^{\binom{r}{2}} b^r (-cqz/b)_r}{(az)_{r+1}} \begin{bmatrix} t \\ r \end{bmatrix}$$

$$= \sum_{t\geq 0} z^t \sum_{r=0}^{t} \frac{q^{\binom{r}{2}} b^r (-cqz/b)_r (1-q^t)\dots(1-q^{t-r+1})}{(az)_{r+1} \qquad (q)_r}$$

$$= \sum_{t\geq 0} z^t \sum_{r=0}^{t} \frac{(q^{-t})_r (-cqz/b)_r}{(q)_r (az)_{r+1}} (-bq^t)^r$$

$$= \sum_{t\geq 0} z^t \cdot \frac{1}{1-az} \; {}_2\phi_1 \left(\begin{matrix} q^{-t}, -cqz/b; q; -bq^t \\ aqz \end{matrix} \right) \tag{2.4.1a}$$

Now, if we let $B,D \to \infty$ in Watson's theorem (2.3.1) we obtain

$${}_2\phi_1 \left(\begin{matrix} q^{-N}, \; E \;\; ; \; q; \; \frac{Aq^{N+1}}{E} \\ \frac{Aq}{C} \end{matrix} \right) =$$

$$= \frac{(Aq/E)_N}{(Aq)_N} \sum_{r\geq 0} \frac{(1-Aq^{2r})}{(1-A)} \frac{(C)_r}{(\frac{Aq}{C})_r} \frac{(E)_r}{(\frac{Aq}{E})_r} \frac{(A)_r}{(q)_r} \frac{(q^{-N})_r}{(Aq^{N+1})_r} q^{2\binom{r}{2}} \left(\frac{A^2 q^{2+N}}{CE} \right)^r . \tag{2.4.2}$$

If we now set $A=cz$, $C=c/a$, $E=-cqz/b$, $N=t$, we obtain

$${}_2\phi_1 \left(\begin{matrix} q^{-t}, \; -cqz/b; \; q; \; -bq^t \\ aqz \end{matrix} \right) =$$

$$= \frac{(-b)_t}{(cqz)_t} \sum_{r\geq 0} \frac{(1-cq^{2r}z)}{(1-cz)} \frac{(c/a)_r}{(aqz)_r} \frac{(-cqz/b)_r}{(-b)_r} \frac{(cz)_r}{(q)_r} \frac{(q^{-t})_r}{(cq^{t+1}z)_r} \times$$

$$\times \; q^{2\binom{r}{2}+r} \; (-abq^t z)^r$$

$$= (-b)_t \sum_{r\geq 0} (1-cq^{2r}z) \frac{(c/a)_r}{(aqz)_r} \frac{(-cqz/b)_r}{(-b)_r} \times \frac{(cz)_r}{(cqz)_t (1-cz)(cq^{t+1}z)_r} \times$$

$$\times \frac{q^{-\binom{r}{2}} (-1)^r q^{tr} (q^{-t})_r}{(q)_r} \times q^{3\binom{r}{2}+r} a^r b^r z^r$$

$$= (-b)_t \sum_{r\geq 0} (1-cq^{2r}z) \frac{(c/a)_r}{(aqz)_r} \frac{(-cqz/b)_r}{(-b)_r} \frac{1}{(cq^r z)_{t+1}} q^{3\binom{r}{2}+r} \begin{bmatrix} t \\ r \end{bmatrix} a^r b^r z^r . \tag{2.4.3a}$$

It follows from (2.4.1a) and (2.4.3a) that

$$\sum_{n\geq 0} P_n z^n =$$

$$= \sum_{t\geq 0} z^t(-b)_t \sum_{r\geq 0}(1-cq^{2r}z)\frac{(c/a)_r}{(az)_{r+1}}\,\frac{(-cqz/b)_r}{(-b)_r}\,\frac{1}{(cq^r z)_{t+1}}\,q^{3\binom{r}{2}+r}\begin{bmatrix}t\\ r\end{bmatrix}a^r b^r z^r$$

$$= \sum_{r,t\geq 0}(1-cq^{2r}z)\frac{(c/a)_r}{(az)_{r+1}}\,\frac{(-cqz/b)_r}{(cq^r z)_{t+1}}\,\frac{(-b)_t}{(-b)_r}\,q^{3\binom{r}{2}+r}\begin{bmatrix}t\\ r\end{bmatrix}a^r b^r z^{r+t}$$

which, on putting $t = r+s$, becomes

$$= \sum_{r,s\geq 0}(1-cq^{2r}z)\frac{(c/a)_r}{(az)_{r+1}}\,\frac{(-cqz/b)_r}{(cq^r z)_{r+s+1}}\,(-bq^r)_s\,q^{3\binom{r}{2}+r}\begin{bmatrix}r+s\\ s\end{bmatrix}a^r b^r z^{2r+s}\,. \qquad (2.4.4a).$$

In the same way, we can show that

$$\sum_{n\geq 0} Q_n z^n =$$

$$= \sum_{r,s\geq 0}(1-cq^{2r+1}z)\,\frac{(cq/a)_r}{(az)_{r+1}}\,\frac{(-cqz/b)_r}{(cq^{r+1}z)_{r+s+1}}\,(-bq^{r+1})_s\,q^{3\binom{r}{2}+2r}\begin{bmatrix}r+s\\ s\end{bmatrix}$$

$$\times\ a^r b^r z^{2r+s+1} \qquad (2.4.4b)$$

We have $\sum_{n\geq 0} P_n z^n =$

$$= \sum_{r,s\geq 0}(1-cq^{2r}z)(c/a)_r(-bq^r)_s\,q^{3\binom{r}{2}+r}\begin{bmatrix}r+s\\ r\end{bmatrix}a^r b^r z^{2r+s}\times\frac{(-cqz/b)_r}{(az)_{r+1}(cq^r z)_{r+s+1}}$$

$$= \sum_{r,s\geq 0}(1-cq^{2r}z)(c/a)_r(-bq^r)_s\,q^{3\binom{r}{2}+r}\begin{bmatrix}r+s\\ r\end{bmatrix}a^r b^r z^{2r+s}\times$$

$$\times\sum_{t=0}^{r} q^{\binom{t+1}{2}}\begin{bmatrix}r\\ t\end{bmatrix}(cz/b)^t \sum_{u\geq 0}\begin{bmatrix}r+u\\ r\end{bmatrix}(az)^u \sum_{v\geq 0}\begin{bmatrix}r+s+v\\ v\end{bmatrix}(cq^r z)^v\ ,$$

which, on putting $r = t+w$, becomes

$$= \sum_{s,t,u,v,w\geq 0} (1-cq^{2t+2w}z)(c/a)_{t+w}(-bq^{t+w})_s \times$$

$$\times q^{3\binom{t+w}{2}+(t+w)+\binom{t+1}{2}+(t+w)v} \times$$

$$\times \begin{bmatrix} s+t+w \\ s \end{bmatrix} \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} s+t+v+w \\ v \end{bmatrix} \times$$

$$\times a^{t+u+w} b^w c^{t+v} z^{s+3t+u+v+2w}, \tag{2.4.5a}$$

and similarly,

$$\sum_{n\geq 0} Q_n z^n =$$

$$= \sum_{s,t,u,v,w\geq 0} (1-cq^{2t+2w+1}z)(cq/a)_{t+w}(-bq^{t+w+1})_s \times$$

$$\times q^{3\binom{t+w}{2}+2(t+w)+\binom{t+1}{2}+(t+w+1)v} \times$$

$$\times \begin{bmatrix} s+t+w \\ s \end{bmatrix} \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} s+t+v+w \\ v \end{bmatrix} \times$$

$$\times a^{t+u+w} b^w c^{t+v} z^{s+3t+u+v+2w+1} . \tag{2.4.5b}$$

It follows from (2.4.5a) that

$$P_n = \sum_{s+3t+u+v+2w=n} (c/a)_{t+w}(-bq^{t+w})_s \times$$

$$\times q^{3\binom{t+w}{2}+(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^w c^{t+v} \times$$

$$\times \begin{bmatrix} s+t+w \\ s \end{bmatrix} \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} s+t+v+w \\ v \end{bmatrix}$$

$$- c \sum_{s+3t+u+v+2w=n-1} (c/a)_{t+w}(-bq^{t+w})_s \times$$

$$\times q^{3\binom{t+w}{2}+3(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^w c^{t+v} \times$$

$$\times \begin{bmatrix} s+t+w \\ s \end{bmatrix} \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} s+t+v+w \\ v \end{bmatrix}$$

$$= \sum_{t,u,v,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_{n-3t-u-v-2w} \times$$

$$\times\ q^{3\binom{t+w}{2}+(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^{w} c^{t+v} \times$$

$$\times \begin{bmatrix} n-2t-u-v-w \\ t+w \end{bmatrix} \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} n-2t-u-w \\ v \end{bmatrix}$$

$$- c \sum_{t,u,v,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_{n-1-3t-u-v-2w} \times$$

$$\times\ q^{3\binom{t+w}{2}+3(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^{w} c^{t+v} \times$$

$$\times \begin{bmatrix} n-1-2t-u-v-w \\ t+w \end{bmatrix} \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} n-1-2t-u-w \\ v \end{bmatrix}$$

$$= \sum_{t,u,v,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_{n-3t-u-v-2w} \times$$

$$\times\ q^{3\binom{t+w}{2}+(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^{w} c^{t+v} \times$$

$$\times \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} t+v+w \\ v \end{bmatrix} \begin{bmatrix} n-2t-u-w \\ t+v+w \end{bmatrix}$$

$$- c \sum_{t,u,v,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_{n-1-3t-u-v-2w} \times$$

$$\times\ q^{3\binom{t+w}{2}+3(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^{w} c^{t+v} \times$$

$$\times \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} t+v+w \\ v \end{bmatrix} \begin{bmatrix} n-1-2t-u-w \\ t+v+w \end{bmatrix}, \tag{2.4.6a}$$

and similarly,

$$Q_n = \sum_{t,u,v,w\geq 0} (cq/a)_{t+w}(-bq^{t+w+1})_{n-1-3t-u-v-2w} \times$$

$$\times\ q^{3\binom{t+w}{2}+2(t+w)+\binom{t+1}{2}+(t+w+1)v} a^{t+u+w} b^{w} c^{t+v} \times$$

$$\times \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} t+v+w \\ v \end{bmatrix} \begin{bmatrix} n-1-2t-u-w \\ t+v+w \end{bmatrix}$$

$$- \; cq \sum_{t,u,v,w\geq 0} (cq/a)_{t+w}(-bq^{t+w+1})_{n-2-3t-u-v-2w} \;\times$$

$$\times \; q^{3\binom{t+w}{2}+4(t+w)+\binom{t+1}{2}+(t+w+1)v} a^{t+u+w} b^{w} c^{t+v} \;\times$$

$$\times \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} t+v+w \\ v \end{bmatrix} \begin{bmatrix} n-2-2t-u-w \\ t+v+w \end{bmatrix} \; . \qquad (2.4.6b)$$

If in (2.4.6a) we let $n \to \infty$, we obtain

$$P_\infty = \sum_{t,u,v,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_\infty \;\times$$

$$\times \; q^{3\binom{t+w}{2}+(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^{w} c^{t+v} \;\times$$

$$\times \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} t+v+w \\ v \end{bmatrix} \cdot \frac{1}{(q)_{t+v+w}}$$

$$- \; c \sum_{t,u,v,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_\infty \;\times$$

$$\times \; q^{3\binom{t+w}{2}+3(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^{w} c^{t+v} \;\times$$

$$\times \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} t+v+w \\ v \end{bmatrix} \cdot \frac{1}{(q)_{t+v+w}}$$

$$= \sum_{t,u,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_\infty \;\times$$

$$\times \; q^{3\binom{t+w}{2}+(t+w)+\binom{t+1}{2}} a^{t+u+w} b^{w} c^{t} \;\times$$

$$\times \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \cdot \frac{1}{(q)_{t+w}} \cdot \frac{1}{(cq^{t+w})_\infty}$$

$$- c \sum_{t,u,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_\infty \times$$

$$\times\ q^{3\binom{t+w}{2}+(t+w)+\binom{t+1}{2}} a^{t+u+w} b^w c^t \times$$

$$\times \left[{t+w \atop t}\right]\left[{t+u+w \atop u}\right] \cdot \frac{1}{(q)_{t+w}} \cdot \frac{1}{(cq^{t+w})_\infty}$$

$$= \sum_{t,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_\infty \times$$

$$\times\ q^{3\binom{t+w}{2}+(t+w)+\binom{t+1}{2}} a^{t+w} b^w c^t \times$$

$$\times \left[{t+w \atop t}\right] \frac{1}{(q)_{t+w}} \cdot \frac{1}{(cq^{t+w})_\infty} \cdot \frac{1}{(a)_{t+w+1}}$$

$$- c \sum_{t,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_\infty \times$$

$$\times\ q^{3\binom{t+w}{2}+3(t+w)+\binom{t+1}{2}} a^{t+w} b^w c^t \times$$

$$\times \left[{t+w \atop t}\right] \frac{1}{(q)_{t+w}} \cdot \frac{1}{(cq^{t+w})_\infty} \cdot \frac{1}{(a)_{t+w+1}} ,$$

which, putting $t+w = r$, becomes

$$= \sum_{r\geq 0} (c/a)_r(-bq^r)_\infty q^{3\binom{r}{2}+r} a^r b^r \frac{1}{(q)_r} \cdot \frac{1}{(cq^r)_\infty} \cdot \frac{1}{(a)_{r+1}} \times$$

$$\times \sum_{t=0}^{r} q^{\binom{t+1}{2}} (c/b)^t$$

$$- c \sum_{r\geq 0} (c/a)_r(-bq^r)_\infty\, q^{3\binom{r}{2}+3r} a^r b^r \cdot \frac{1}{(q)_r} \cdot \frac{1}{(cq^r)_\infty} \cdot \frac{1}{(a)_{r+1}} \times$$

$$\times \sum_{t=0}^{r} q^{\binom{t+1}{2}} (c/b)^t$$

$$= \sum_{r\geq 0} (1-cq^{2r})(c/a)_r(-bq^r)_\infty\, q^{3\binom{r}{2}+r} a^r b^r (-cq/b)_r \times \frac{1}{(q)_r\,(cq^r)_\infty (a)_{r+1}}$$

$$= \frac{(-b)_\infty}{(cq)_\infty} \sum_{r\geq 0} \frac{(1-cq^{2r})}{(1-cq^r)} \frac{(c/a)_r}{(a)_{r+1}} \frac{(-cq/b)_r}{(-b)_r} \frac{(cq)_r}{(q)_r} q^{3\binom{r}{2}+r} a^r b^r$$

$$= \frac{1}{1-a} \frac{(-b)_\infty}{(cq)_\infty} \left\{ 1+ \sum_{r\geq 1} \frac{(1-cq^{2r})}{(1-cq^r)} \frac{(c/a)_r}{(aq)_r} \frac{(-cq/b)_r}{(-b)_r} \frac{(cq)_r}{(q)_r} q^{3\binom{r}{2}+r} a^r b^r \right\},$$

which is (2.3.3a).

Similarly, if in (2.4.6b) we let $n \to \infty$, we obtain (2.3.3b). We omit the details.

§5. We have obtained two quite different expression for P_n, namely those in (2.2.10a) and in (2.4.6a). Equating them yields a new polynomial identity which involves three parameters and which implies the Rogers-Ramanujan identities.

Thus

$$\sum_{s,u,v\geq 0} q^{\binom{s+v}{2}+\binom{s+1}{2}} a^u b^v c^s \times$$

$$\times \begin{bmatrix} s+v \\ s \end{bmatrix} \begin{bmatrix} s+u+v \\ u \end{bmatrix} \begin{bmatrix} n-s-u \\ s+v \end{bmatrix}$$

$$= \sum_{t,u,v,w\geq 0} (c/a)_{t+w} (-bq^{t+w})_{n-3t-u-v-2w} \times$$

$$\times\, q^{3\binom{t+w}{2}+(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^w c^{t+v} \times$$

$$\times \begin{bmatrix} t+w \\ t \end{bmatrix} \begin{bmatrix} t+u+w \\ u \end{bmatrix} \begin{bmatrix} t+v+w \\ v \end{bmatrix} \begin{bmatrix} n-2t-u-w \\ t+v+w \end{bmatrix}$$

$$- \; c \sum_{t,u,v,w\geq 0} (c/a)_{t+w}(-bq^{t+w})_{n-1-3t-u-v-2w} \times$$

$$\times \; q^{3\binom{t+w}{2}+3(t+w)+\binom{t+1}{2}+(t+w)v} a^{t+u+w} b^{w} c^{t+v} \times$$

$$\times \begin{bmatrix} t+w \\ t \end{bmatrix}\begin{bmatrix} t+u+w \\ u \end{bmatrix}\begin{bmatrix} t+v+w \\ v \end{bmatrix}\begin{bmatrix} n-1-2t-u-w \\ t+v+w \end{bmatrix} \quad . \tag{2.5.1}$$

This simplifies considerably in the case $b = 0$. Thus

$$\sum_{s,u\geq 0} q^{s^2} a^u c^s \begin{bmatrix} s+u \\ u \end{bmatrix}\begin{bmatrix} n-s-u \\ s \end{bmatrix} =$$

$$= \sum_{t,u,v\geq 0} (c/a)_t q^{2t^2+tv} a^{t+u} c^{t+v} \begin{bmatrix} t+u \\ u \end{bmatrix}\begin{bmatrix} t+v \\ v \end{bmatrix}\begin{bmatrix} n-2t-u \\ t+v \end{bmatrix}$$

$$-c \sum_{t,u,v\geq 0} (c/a)_t q^{2t^2+2t+tv} a^{t+u} c^{t+v} \begin{bmatrix} t+u \\ u \end{bmatrix}\begin{bmatrix} t+v \\ v \end{bmatrix}\begin{bmatrix} n-1-2t-u \\ t+v \end{bmatrix} \quad . \tag{2.5.2}$$

Since there is essentially only one appearance of n on each side of (2.5.2), it should be possible to prove (2.5.2) by induction.

Further, (2.5.2) can be written

$$\sum_{s,u\geq 0} q^{s^2} a^u c^s \begin{bmatrix} s+u \\ s \end{bmatrix}\begin{bmatrix} n-s-u \\ s \end{bmatrix} =$$

$$= \sum_{t,u\geq 0} (c/a)_t q^{2t^2} a^{t+u} c^t \begin{bmatrix} t+u \\ u \end{bmatrix}\begin{bmatrix} n-2t-u \\ t \end{bmatrix} \sum_{v\geq 0} c^v q^{tv} \begin{bmatrix} n-3t-u \\ v \end{bmatrix}$$

$$- \; c \sum_{t,u\geq 0} (c/a)_t q^{2t^2+2t} a^{t+u} c^t \begin{bmatrix} t+u \\ u \end{bmatrix}\begin{bmatrix} n-1-2t-u \\ t \end{bmatrix} \sum_{v\geq 0} c^v q^{tv} \begin{bmatrix} n-1-3t-u \\ v \end{bmatrix}$$

$$= \sum_{t,u\ge 0} (c/a)_t q^{2t^2} a^{t+u} c^t \begin{bmatrix} t+u \\ u \end{bmatrix} \begin{bmatrix} n-2t-u \\ t \end{bmatrix} H_{n-3t-u}(cq^t) \quad *$$

$$-c \sum_{t,u\ge 0} (c/a)_t q^{2t^2+2t} a^{t+u} c^t \begin{bmatrix} t+u \\ u \end{bmatrix} \begin{bmatrix} n-1-2t-u \\ t \end{bmatrix} H_{n-1-3t-u}(cq^t) \ . \qquad (2.5.3)$$

In particular, if $a=0$,

$$\sum_{s\ge 0} q^{s^2} c^s \begin{bmatrix} n-s \\ s \end{bmatrix} =$$

$$= \sum_{t\ge 0} (-1)^t q^{\frac{1}{2}t(5t-1)} c^{2t} \begin{bmatrix} n-2t \\ t \end{bmatrix} H_{n-3t}(cq^t)$$

$$-c \sum_{t\ge 0} (-1)^t q^{\frac{1}{2}t(5t+3)} c^{2t} \begin{bmatrix} n-1-2t \\ t \end{bmatrix} H_{n-1-3t}(cq^t) \ . \qquad (2.5.4)$$

The Rogers-Ramanujan identities follow on letting $n \to \infty$ and setting $c=1$, $c=q$. Thus, letting $n \to \infty$ in (2.5.4), we obtain

$$\sum_{s\ge 0} \frac{q^{s^2} c^s}{(q)_s} = \sum_{t\ge 0} (-1)^t q^{\frac{1}{2}t(5t-1)} c^{2t} \frac{1}{(q)_t} \cdot \frac{1}{(cq^t)_\infty}$$

$$- c \sum_{t\ge 0} (-1)^t q^{\frac{1}{2}t(5t+3)} c^{2t} \frac{1}{(q)_t} \cdot \frac{1}{(cq^t)_\infty}$$

$$= \sum_{t\ge 0} (-1)^t (1-cq^{2t}) q^{\frac{1}{2}t(5t-1)} c^{2t} \frac{1}{(q)_t} \cdot \frac{1}{(cq^t)_\infty}$$

$$= \frac{1}{(c)_\infty} \sum_{t\ge 0} (-1)^t (1-cq^{2t}) q^{\frac{1}{2}t(5t-1)} c^{2t} \frac{(c)_t}{(q)_t} \qquad (2.5.5)$$

---

* Here $H_n(x) = \sum_{r=0}^{n} x^r \begin{bmatrix} n \\ r \end{bmatrix}$ (see Appendix §6).

$$= \frac{1}{(cq)_\infty} \sum_{t\geq 0} (-1)^t \frac{(1-cq^{2t})}{(1-cq^t)} q^{\frac{1}{2}t(5t-1)} c^{2t} \frac{(cq)_t}{(q)_t}$$

$$= \frac{1}{(cq)_\infty} \left\{1+\sum_{t\geq 1} (-1)^t \frac{(1-cq^{2t})}{(1-cq^t)} q^{\frac{1}{2}t(5t-1)} c^{2t} \frac{(cq)_t}{(q)_t}\right\} . \tag{2.5.6}$$

If in (2.5.6) we set $c=1$, and in (2.5.5) we set $c=q$, we obtain

$$\sum_{s\geq 0} \frac{q^{s^2}}{(q)_s} = \frac{1}{(q)_\infty} \left\{1 + \sum_{t\geq 1} (-1)^t (1+q^t) q^{\frac{1}{2}t(5t-1)}\right\}$$

$$= \frac{1}{(q)_\infty} (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty$$

$$= 1/(q;q^5)_\infty (q^4;q^5)_\infty , \tag{2.5.7}$$

and

$$\sum_{s\geq 0} \frac{q^{s^2+s}}{(q)_\infty} = \frac{1}{(q)_\infty} \sum_{t\geq 0} (-1)^t (1-q^{2t+1}) q^{\frac{1}{2}t(5t+3)}$$

$$= \frac{1}{(q)_\infty} (q;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty$$

$$= 1/(q^2;q^5)_\infty (q^3;q^5)_\infty , \tag{2.5.8}$$

which are the Rogers-Ramanujan identities. Incidentally, if we set $c=1$ and let $q \to 1$ in (2.5.4) we obtain the following formula for the Fibonacci numbers:

$$F_n = \sum_{s\geq 0} \binom{n-s}{s}$$

$$= \sum_{t\geq 0} (-1)^t \binom{n-2t}{t} 2^{n-3t} - \sum_{t\geq 0} (-1)^t \binom{n-1-2t}{t} 2^{n-1-3t} \tag{2.5.9}$$

which follows easily from

$$\sum_{n\geq 0} F_n x^n = \frac{1}{1-x-x^2} = \frac{1}{1-2x+x^3} - \frac{x}{1-2x+x^3} . \tag{2.5.10}$$

§6. If we compare coefficients of c^r on both sides of (2.5.4), we obtain

$$q^{r^2}\begin{bmatrix}n-r\\r\end{bmatrix} =$$

$$= \sum_{2t+v=r} (-1)^t q^{\frac{1}{2}t(5t-1)+vt}\begin{bmatrix}n-2t\\t\end{bmatrix}\begin{bmatrix}n-3t\\v\end{bmatrix}$$

$$- \sum_{2t+v=r-1} (-1)^t q^{\frac{1}{2}t(5t+3)+tv}\begin{bmatrix}n-1-2t\\t\end{bmatrix}\begin{bmatrix}n-1-3t\\v\end{bmatrix}$$

$$= \sum_{t\geq 0} (-1)^t q^{\frac{1}{2}t(5t-1)+t(r-2t)}\begin{bmatrix}n-2t\\t\end{bmatrix}\begin{bmatrix}n-3t\\r-2t\end{bmatrix}$$

$$- \sum_{t\geq 0} (-1)^t q^{\frac{1}{2}t(5t+3)+t(r-1-2t)}\begin{bmatrix}n-1-2t\\t\end{bmatrix}\begin{bmatrix}n-1-3t\\r-1-2t\end{bmatrix}$$

$$= \sum_{t\geq 0} (-1)^t q^{\binom{t}{2}} q^{rt}\begin{bmatrix}n-2t\\t\end{bmatrix}\begin{bmatrix}n-3t\\r-2t\end{bmatrix}$$

$$- \sum_{t\geq 0} (-1)^t q^{\binom{t+1}{2}} q^{rt}\begin{bmatrix}n-1-2t\\t\end{bmatrix}\begin{bmatrix}n-1-3t\\r-1-2t\end{bmatrix}$$

$$= \sum_{t\geq 0} (-1)^t q^{\binom{t}{2}} q^{rt}\begin{bmatrix}n-r\\t\end{bmatrix}\begin{bmatrix}n-2t\\n-r\end{bmatrix}$$

$$- \sum_{t\geq 0} (-1)^t q^{\binom{t+1}{2}} q^{rt}\begin{bmatrix}n-r\\t\end{bmatrix}\begin{bmatrix}n-1-2t\\n-r\end{bmatrix} \quad . \tag{2.6.1}$$

Putting $n+r$ for n, we obtain the identity

$$q^{r^2}\begin{bmatrix}n\\r\end{bmatrix} = \sum_{t\geq 0} (-1)^t q^{\binom{t}{2}} q^{rt}\begin{bmatrix}n\\t\end{bmatrix}\begin{bmatrix}n+r-2t\\n\end{bmatrix}$$

$$- \sum_{t\geq 0} (-1)^t q^{\binom{t+1}{2}} q^{rt}\begin{bmatrix}n\\t\end{bmatrix}\begin{bmatrix}n+r-1-2t\\n\end{bmatrix} \quad . \tag{2.6.2}$$

This identity is also deducible fairly directly from Watson's theorem. Let $B,C,D,E \to \infty$ in (2.3.1).

$$\sum_{r\geq 0} q^{r^2} \begin{bmatrix} N \\ r \end{bmatrix} A^r = \sum_{t\geq 0} (1-Aq^{2t})(-1)^t q^{\frac{1}{2}t(5t-1)} \frac{1}{(Aq^t)_{N+1}} \begin{bmatrix} N \\ t \end{bmatrix} A^{2t}$$

$$= \sum_{t,v\geq 0} (1-Aq^{2t})(-1)^t q^{\frac{1}{2}t(5t-1)+tv} \begin{bmatrix} N \\ t \end{bmatrix} \begin{bmatrix} N+v \\ N \end{bmatrix} A^{2t+v} \quad . \qquad (2.6.3)$$

Now compare coefficients of A^r on both sides of (2.6.3), and obtain (2.6.2).

The identity (2.6.2) is a q-generalisation of the binomial coefficient identity

$$\binom{n}{r} = \sum_{t\geq 0} (-1)^t \binom{n}{t} \binom{n+r-2t}{n} - \sum_{t\geq 0} (-1)^t \binom{n}{t} \binom{n+r-1-2t}{n} \qquad (2.6.4)$$

which follows easily from

$$(1+x)^n = \frac{(1-x^2)^n}{(1-x)^{n+1}} - x\,\frac{(1-x^2)^n}{(1-x)^{n+1}} \quad . \qquad (2.6.5)$$

It would be nice to have a combinatorial proof of (2.6.2), for this would give us significant insight into the Rogers-Ramanujan identities. Indeed a first step would be a combinatorial proof of (2.6.4). We note with regard to (2.6.4) that

$$\sum_{t\geq 0} (-1)^t \binom{n}{t} \binom{n+r-2t}{n} - \sum_{t\geq 0} (-1)^t \binom{n}{t} \binom{n+r-1-2t}{n}$$

$$= \sum_{t\geq 0} (-1)^t \binom{n}{t} \binom{n+r-1-2t}{n-1}$$

$$= \frac{(n+r-1)!}{(n-1)!r!} \sum_{t\geq 0} \frac{(-n)_t}{t!} \frac{r(r-1)\ldots(r-2t+1)}{(n+r-1)\ldots(n+r-2t)}$$

$$= \frac{(n+r-1)!}{(n-1)!r!}\; {}_3F_2 \left[\begin{matrix} -n, & -\frac{r}{2}, & \frac{-r+1}{2} \\ & \frac{-n-r+1}{2}, & \frac{-n-r+2}{2} \end{matrix} ; 1 \right]$$

$$= \frac{(n+r-1)!}{(n-1)!r!} \frac{(n-1)!n!}{(n+r-1)!(n-r)!}$$

$= \binom{n}{r}$ by the Pfaff-Saalschutz summation (see App'x §5). Thus (2.6.4) is a special case of a non-trivial hypergeometric identity.

Chapter 3 A Family of Orthogonal Polynomials

§1. We study the polynomials $P_n(a,b,c,q)$ defined by

$$P_0 = 1, \quad P_1 = 1+a+b \ , \tag{3.1.1a}$$

and for $n \geq 1$,

$$P_{n+1} = (1+a+bq^n)P_n + (cq^n-a)P_{n-1} \ . \tag{3.1.1b}$$

These polynomials arise as the numerators of the convergents to the continued fraction

$$1+a+b+\frac{cq-a}{1+a+bq+}\ \frac{cq^2-a}{1+a+bq^2+}\ \cdots$$

considered in Chapter 2.

We can write (3.1.1b) as

$$bP_n = q^{-n}\, P_{n+1} - q^{-n}(1+a)P_n + q^{-n}(a-cq^n)P_{n-1} \ . \tag{3.1.2}$$

By a theorem of Favard (1935) we deduce that the P_n form an orthogonal family of polynomials in the variable b. As we shall see, the P_n are, in the special case $c=a$, closely related to the extended q-hermite polynomials. However, these polynomials do not appear to have been previously studied in the present generality.

In §2 we shall state a formula for the P_n, simpler than those found in Chapter 2. It shows that the P_n reduce to the extended q-Hermite polynomials when $c=a$, and at the same time solves the "connection coefficient problem" between the fully general P_n and the simpler polynomials.

Proofs are given in §§3,4.

§2. We shall show that

$$P_n(a,b,c,q) =$$

$$= \sum_{\ell,m\geq 0} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m a^\ell (-bq^m/a)_\ell \begin{bmatrix} n-m \\ m \end{bmatrix} \begin{bmatrix} n-2m \\ \ell \end{bmatrix} \quad . \tag{3.2.1}$$

In particular, if $c=a$,

$$P_n(a,b,a,q) = \sum_{\ell\geq 0} a^\ell (-b/a)_\ell \begin{bmatrix} n \\ \ell \end{bmatrix} \tag{3.2.2}$$

$$= \sum_{\ell\geq 0} (a+b)\ldots.(a+bq^{\ell-1}) \frac{(1-q^n)\ldots.(1-q^{n-\ell+1})}{(1-q)\ \ldots.\ (1-q^\ell)} \quad .$$

It follows that

$$P_n(a,b,a,q^{-1}) = \sum_{\ell\geq 0} (a+b)\ldots.(a+b/q^{\ell-1}) \frac{(1-q^{-n})\ldots(1-q^{-n+\ell-1})}{(1-q^{-1})\ldots\ (1-q^{-\ell})}$$

$$= \sum_{\ell\geq 0} q^{-\binom{\ell}{2}} (b+a)\ldots.(b+aq^{\ell-1})(q^{-n})_\ell \frac{(-1)^\ell q^{\binom{\ell+1}{2}}}{(q)_\ell}$$

$$= \sum_{\ell\geq 0} \frac{(-a/b)_\ell (q^{-n})_\ell}{(q)_\ell} (-bq)^\ell$$

$$= (-a/q)^n h_n(b;-q,-q/a|q) \ , \tag{3.2.3}$$

where the h_n are the extended q-hermite polynomials defined by

$$h_n(x;a,b|q) = b^n \sum_{\ell\geq 0} \frac{(q/bx)_\ell (q^{-n})_\ell}{(q)_\ell} (ax)^\ell \quad . \tag{3.2.4}$$

The h_n have been studied extensively by Andrews and Askey (1980?) and are well understood. We mention just two properties of these polynomials.

$$\sum_{n\geq 0} \frac{(-1)^n q^{\binom{n}{2}} h_n(x;a,b|q)z^n}{(q)_n} = \frac{(az)_\infty (bz)_\infty}{(abxz/q)_\infty} \tag{3.2.5}$$

and

$$\mathfrak{S}_{b^{-1}}^{a^{-1}} h_n(x;a,b|q)h_m(x;a,b|q)(xa)_\infty (xb)_\infty d(q,x)$$

$$= \begin{cases} 0 & \text{if} \quad m \neq n \\ (-1)^n \dfrac{(ab)^n}{a-b} q^{1-\binom{n}{2}} (q)_\infty (q)_n (a/b)_\infty (b/a)_\infty & \text{if} \quad m=n \end{cases} \tag{3.2.6}$$

The formula corresponding to (3.2.5) for the $P_n(a,b,a,q)$ is

$$\sum_{n\geq 0} \frac{P_n(a,b,a,q)}{(q)_n} z^n = \frac{(-bz)_\infty}{(az)_\infty (z)_\infty} . \tag{3.2.7}$$

For, by (3.2.2),

$$\sum_{n\geq 0} \frac{P_n(a,b,a,q)}{(q)_n} z^n = \sum_{n\geq 0} \frac{z^n}{(q)_n} \sum_{\ell\geq 0} a^\ell (-b/a)_\ell \begin{bmatrix} n \\ \ell \end{bmatrix}$$

$$= \sum_{\ell\geq 0} a^\ell (-b/a)_\ell \sum_{n\geq \ell} \frac{z^n}{(q)_n} \begin{bmatrix} n \\ \ell \end{bmatrix}$$

$$= \sum_{\ell\geq 0} a^\ell \frac{(-b/a)_\ell}{(q)_\ell} z^\ell \sum_{n\geq 0} \frac{z^n}{(q)_n}$$

$$= \frac{1}{(z)_\infty} \sum_{\ell\geq 0} a^\ell \frac{(-b/a)_\ell}{(q)_\ell} z^\ell$$

$$= \frac{1}{(z)_\infty} \frac{(-bz)_\infty}{(az)_\infty} .$$

The problem of finding the formula corresponding to (3.2.6) for the $P_n(a,b,a,q)$ is as yet unsolved.

We have from (3.2.1) that

$P_n(a,b,c,q) \quad =$

$$= \sum_{m\geq 0} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m \left[{n-m \atop m}\right] \sum_{\ell\geq 0} a^\ell (-bq^m/a)_\ell \left[{n-2m \atop \ell}\right]$$

$$= \sum_{m\geq 0} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m \left[{n-m \atop m}\right] P_{n-2m}(a,bq^m,a,q) \qquad (3.2.8)$$

which shows how the $P_n(a,b,c,q)$ are related to the $P_n(a,b,a,q)$.

Before proving (3.2.1) we note that Andrews (unpublished) has proved a somewhat different formula for the P_n, namely

$$P_n(a,b,c,q) = \sum_{\ell,m\geq 0} c^\ell (-b/c)_\ell a^m (c/a)_m \left[{\ell+m \atop \ell}\right]\left[{n-m \atop \ell}\right] \qquad (3.2.9)$$

which again generalises (3.2.2).

A proof of (3.2.9) along the lines of the proof in §3 of (3.2.1) can easily be given.

§3. (3.2.1) was discovered empirically by the author in attempting to generalise (3.2.2), which had been found earlier by Andrews. Our first proof is purely a verification. Thus,

$$\sum_{\ell,m\geq 0} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m a^\ell (-bq^m/a)_\ell \left[{n-m \atop m}\right]\left[{n-2m \atop \ell}\right]$$

$$= \sum_{r,s,t,u\geq 0} (-1)^{r+s} q^{\binom{r+s+1}{2}} (-1)^s a^r c^s q^{\binom{s}{2}} \left[{r+s \atop s}\right] a^t (-bq^{r+s})^u q^{\binom{u}{2}} \left[{t+u \atop u}\right]$$

$$\times \left[{n-r-s \atop r+s}\right]\left[{n-2r-2s \atop t+u}\right]$$

$$= \sum_{r,s,t,u\geq 0} (-1)^r a^{r+t} b^u c^s q^{\binom{r+s+1}{2}+\binom{s}{2}+(r+s)u+\binom{u}{2}} \times$$

$$\times \begin{bmatrix} r+s \\ s \end{bmatrix} \begin{bmatrix} t+u \\ u \end{bmatrix} \begin{bmatrix} n-r-s \\ r+s \end{bmatrix} \begin{bmatrix} n-2r-2s \\ t+u \end{bmatrix}$$

$$= \sum_{s,u,v\geq 0} a^v b^u c^s q^{\binom{s+u}{2}+\binom{s+1}{2}} \begin{bmatrix} s+u \\ s \end{bmatrix} \begin{bmatrix} s+u+v \\ v \end{bmatrix} \times$$

$$\times \sum_{r\geq 0} (-1)^r q^{\binom{r+1}{2}+(s+u)r} \begin{bmatrix} v \\ r \end{bmatrix} \begin{bmatrix} n-s-r \\ s+u+v \end{bmatrix}$$

$$= \sum_{s,u,v\geq 0} a^v b^u c^s q^{\binom{s+u}{2}+\binom{s+1}{2}} \begin{bmatrix} s+u \\ s \end{bmatrix} \begin{bmatrix} s+u+v \\ v \end{bmatrix} \begin{bmatrix} n-s-v \\ s+u \end{bmatrix}$$

$$= P_n(a,b,c,q) \ .$$

Here we have used the identity

$$\sum_{r\geq 0} (-1)^r q^{\binom{r+1}{2}+(s+u)r} \begin{bmatrix} v \\ r \end{bmatrix} \begin{bmatrix} n-s-r \\ s+u+v \end{bmatrix} = \begin{bmatrix} n-s-v \\ s+u \end{bmatrix} \tag{3.3.1}$$

which follows from

$$\frac{(xq^{s+u+1})_v}{(x)_{s+u+v+1}} = \frac{1}{(x)_{s+u+1}} \tag{3.3.2}$$

on comparing coefficients of $x^{n-2s-u-v}$.

§4. Another proof of (3.2.1), making use of the generating function of the P_n, can be given.

$$\sum_{n\geq 0} P_n(a,b,c,q) z^n = \sum_{r\geq 0} \frac{q^{\binom{r}{2}} z^r b^r (-cqz/b)_r}{(z)_{r+1}(az)_{r+1}} \tag{3.4.1}$$

$$= \sum_{r\geq 0} \frac{q^{\binom{r}{2}} z^r b^r (-cqz/b)_r}{(z)_{r+1}} \sum_{s\geq 0} \begin{bmatrix} r+s \\ r \end{bmatrix} (az)^s$$

$$= \sum_{r,s\geq 0} \frac{q^{\binom{r}{2}} b^r a^s (-cqz/b)_r}{(z)_{r+1}} \begin{bmatrix} r+s \\ r \end{bmatrix} z^{r+s}$$

and, putting $r+s=t$, this becomes

$$= \sum_{t\geq 0} a^t z^t \sum_{r=0}^{t} q^{\binom{r}{2}} \frac{(-cqz/b)_r}{(z)_{r+1}} \begin{bmatrix} t \\ r \end{bmatrix} (b/a)^r$$

$$= \sum_{t\geq 0} a^t z^t \sum_{r=0}^{t} \frac{(-cqz/b)_r (q^{-t})_r}{(z)_{r+1} (q)_r} (-bq^t/a)^r$$

$$= \sum_{t\geq 0} \frac{a^t z^t}{1-z} \; {}_2\phi_1 \left(\begin{matrix} -cqz/b, & q^{-t} \\ & qz \end{matrix} ; q; -bq^t/a \right)$$

which, using the second iterate of Heine's transformation (see App'x §3), becomes

$$= \sum_{t\geq 0} \frac{a^t z^t}{1-z} \frac{(-b/a)_t}{(qz)_t} \; {}_2\phi_1 \left(\begin{matrix} c/a, & q^{-t} \\ & -b/a \end{matrix} ; q; q^{t+1} z \right)$$

$$= \sum_{t\geq 0} \frac{a^t z^t}{(z)_{t+1}} (-b/a)_t \sum_{m=0}^{t} \frac{(c/a)_m (q^{-t})_m}{(-b/a)_m (q)_m} (q^{t+1} z)^m$$

$$= \sum_{t\geq 0} \frac{a^t z^t}{(z)_{t+1}} \sum_{m=0}^{t} (-1)^m q^{\binom{m+1}{2}} (c/a)_m (-bq^m/a)_{t-m} \begin{bmatrix} t \\ m \end{bmatrix} z^m$$

which, on putting $t = \ell+m$, becomes

$$= \sum_{\ell,m\geq 0} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m a^\ell (-bq^m/a)_\ell \begin{bmatrix} \ell+m \\ \ell \end{bmatrix} \frac{z^{\ell+2m}}{(z)_{\ell+m+1}}$$

$$= \sum_{\ell,m\geq 0} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m a^\ell (-bq^m/a)_\ell \begin{bmatrix} \ell+m \\ \ell \end{bmatrix} z^{\ell+2m} \times$$

$$\times \sum_{p\geq 0} \begin{bmatrix} \ell+m+p \\ \ell+m \end{bmatrix} z^p$$

$$= \sum_{\ell,m,p\geq 0} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m a^\ell (-bq^m/a)_\ell \begin{bmatrix} \ell+m \\ \ell \end{bmatrix} \begin{bmatrix} \ell+m+p \\ \ell+m \end{bmatrix} z^{\ell+2m+p} . \qquad (3.4.2)$$

It follows that $P_n(a,b,c,q) =$

$$= \sum_{\ell+2m+p=n} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m a^\ell (-bq^m/a)_\ell \begin{bmatrix} \ell+m \\ \ell \end{bmatrix} \begin{bmatrix} \ell+m+p \\ \ell+m \end{bmatrix}$$

$$= \sum_{\ell,m\geq 0} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m a^\ell (-bq^m/a)_\ell \begin{bmatrix} \ell+m \\ \ell \end{bmatrix} \begin{bmatrix} n-m \\ \ell+m \end{bmatrix}$$

$$= \sum_{\ell,m\geq 0} (-1)^m q^{\binom{m+1}{2}} a^m (c/a)_m a^\ell (-bq^m/a)_\ell \begin{bmatrix} n-m \\ m \end{bmatrix} \begin{bmatrix} n-2m \\ \ell \end{bmatrix}$$

which is (3.2.1), as required.

§5. We have

$$\sum_{n\geq 0} P_n(a,b,c,q) z^n = \sum_{r\geq 0} \frac{q^{\binom{r}{2}} z^r b^r (-cqz/b)_r}{(z)_{r+1} (az)_{r+1}} .$$

Putting $\frac{1}{a}$ for a, $\frac{b}{a}$ for b, $\frac{c}{a^2}$ for c, az for z,

we obtain

$$\sum_{n\geq 0} a^n P_n\left(\frac{1}{a}, \frac{b}{a}, \frac{c}{a^2}, q\right) z^n = \sum_{r\geq 0} \frac{q^{\binom{r}{2}} z^r b^r (-cqz/b)_r}{(z)_{r+1} (az)_{r+1}}$$

$$= \sum_{n\geq 0} P_n(a,b,c,q) z^n . \qquad (3.5.1)$$

It follows that

$$P_n(a,b,c,q) \;=\; a^n P_n\left(\frac{1}{a}, \frac{b}{a}, \frac{c}{a^2}, q\right) \quad . \tag{3.5.2}$$

In particular, if $c=a$,

$$P_n(a,b,a,q) \;=\; a^n P_n\left(\frac{1}{a}, \frac{b}{a}, \frac{1}{a}, q\right) \quad , \tag{3.5.3}$$

which also follows from (3.2.7).

(3.5.2), together with (2.2.10a), (2.4.6a), (3.2.1) or (3.2.9), yields yet more formulae for $P_n(a,b,c,q)$.

Thus, for instance, (3.5.2) together with (3.2.1) yields

$$P_n(a,b,c,q) \;=$$

$$= a^n \sum_{\ell,m\geq 0} (-1)^m q^{\binom{m+1}{2}} a^{-m}(c/a)_m a^{-\ell}(-bq^m)_\ell \left[{n-m \atop m}\right]\left[{n-2m \atop \ell}\right]$$

$$= \sum_{\ell,m\geq 0} a^{n-2m-\ell} q^{\binom{m+1}{2}} (c-a)\ldots(cq^{m-1}-a)(-bq^m)_\ell \left[{n-m \atop m}\right]\left[{n-2m \atop \ell}\right] \; . \tag{3.5.4}$$

In particular if we set $c=a$, we obtain

$$P_n(a,b,a,q) \;=\; \sum_{\ell\geq 0} a^{n-\ell}(-b)_\ell \left[{n \atop \ell}\right] \; , \tag{3.5.5}$$

which should be compared with (3.2.2).

Comparison of (2.2.10) with (3.5.4) yields the identity

$$\sum_{s,v\geq 0} q^{\binom{s+v}{2}+\binom{s+1}{2}} a^u b^v c^s \left[{s+v \atop s}\right]\left[{s+u+v \atop u}\right]\left[{n-s-u \atop s+v}\right]$$

$$= \sum_{\ell,m\geq 0} a^{n-2m-\ell} q^{\binom{m+1}{2}} (c-a)\ldots(cq^{m-1}-a)(-bq^m)_\ell \left[{n-m \atop m}\right]\left[{n-2m \atop \ell}\right] \tag{3.5.6}$$

which, as we shall see in Chapter 4, has a special case of considerable significance.

Chapter 4. Sylvester's Partition Theorem

§1. What must be one of the first partition theorems is Euler's

Theorem 4.1.1.

The number of partitions of n into odd parts is equal to the number of partitions of n into distinct parts.

Sylvester (1884-6) stated and proved the following startling refinement of Euler's theorem, namely

Theorem 4.1.2

The number, $A(n,s)$, of partitions of n into odd parts, s of which are distinct, is equal to the number, $B(n,s)$, of partitions of n into distinct parts, with s sequences of consecutive parts.

Andrews (1966) gave a proof of Theorem 4.1.2 based on generating functions, later simplified by Hirschhorn (1974b). In §2 we see that the polynomials which arise in this proof are intimately related to the polynomials studied in Chapters 2 and 3. Indeed, using an explicit expression obtained earlier for these polynomials we obtain a polynomial identity which in the limit yields Theorem 4.1.2.

In doing so, we are led to consider the special case $a=0$, $b=q$ of identity (3.5.6), which is

$$\sum_{s,v\geq 0} q^{\binom{s+v+1}{2}+\binom{s}{2}} c^s \begin{bmatrix} s+v \\ s \end{bmatrix} \begin{bmatrix} n-s \\ s+v \end{bmatrix} =$$

$$= \sum_{m\geq 0} c^m q^{m^2} (-q^{m+1})_{n-2m} \begin{bmatrix} n-m \\ m \end{bmatrix} . \tag{4.1.3}$$

This new identity, which is easily proved directly, yields, as $n \to \infty$

$$\sum_{r\geq 0} \frac{q^{\binom{r+1}{2}}(-c)_r}{(q)_r} = (-q)_\infty(-cq;q^2)_\infty ,$$

$$= \frac{(-cq;q^2)_\infty}{(q;q^2)_\infty} , \qquad (4.1.4)$$

an identity attributed to V.A. Lebesgue (1840), and a special case of the q-analogue of Kummer's identity (see App'x §4). Lebesgue's identity is essentially equivalent to Sylvester's theorem, as we shall see in §3.

In his 1966 paper, Andrews also gives the following generalisation of Euler's theorem.

<u>Theorem 4.1.5</u>

$$G(n,r) = \sum_{\pi\in\Pi(n)} \binom{g(\pi)}{r}$$ is equal to the number of partitions of n with r distinct even parts and all other parts odd. Here $\Pi(n)$ is the set of partitions of n into distinct parts, and $g(\pi)$ is the number of "gaps" in the partition π, related to the number of sequences $s(\pi)$ by

$$g(\pi) = s(\pi)$$

unless the smallest part in π is 1, when

$$g(\pi) = s(\pi) - 1 .$$

In §4 I present a simplified version of Andrews' proof of Theorem 4.1.5 and prove the following further three new generalisations of Theorem 4.1.1.

Theorem 4.1.6

$G(n,r)$ is equal to the number of partitions of $n-r^2-r$ with no even part greater than $2r$.

Theorem 4.1.7

$S(n,r) = \sum_{\pi\in\Pi(n)} \binom{s(\pi)}{r}$ is equal to the number of partitions of $n+r$ with r distinct even parts and all other parts odd.

Theorem 4.1.8

$S(n,r)$ is equal to the number of partitions of $n-r^2$ with no even part greater than $2r$.

Theorem 4.1.1 is the case $r=0$ of each of Theorems 4.1.5-4.1.8.

§2. In order to prove Theorem 4.1.2, let $B(n,s,\ell)$ denote the number of partitions of n into distinct parts with s sequences of consecutive parts, and no part greater than ℓ.

(4.2.1)

Then, as Andrews shows,

$$\begin{aligned} B(n,s,\ell) &= \\ &= B(n,s,\ell-1) + (B(n-\ell,s,\ell-1) - B(n-\ell,s,\ell-2)) \\ &\qquad + B(n-\ell,s-1,\ell-2) \quad . \end{aligned} \tag{4.2.2}$$

Now let

$$B_\ell(a,q) = \sum_{n,s} B(n,s,\ell)a^s q^n \quad . \tag{4.2.3}$$

Then

$$B_0(a,q) = 1, \quad B_1(a,q) = 1+aq \quad , \tag{4.2.4a}$$

and it follows from (4.2.2) that

$$B_\ell(a,q) =$$

$$= B_{\ell-1}(a,q)+q^\ell(B_{\ell-1}(a,q)-B_{\ell-2}(a,q))+aq^\ell B_{\ell-2}(a,q)$$

$$= (1+q^\ell)B_{\ell-1}(a,q)+(a-1)q^\ell B_{\ell-2}(a,q) \quad . \tag{4.2.4b}$$

It follows from (4.2.4) that

$$B_\ell(a,q) = P_\ell(0,q,(a-1)q,q)+(a-1)qP_{\ell-1}(0,q^2,(a-1)q^2,q) \ , \tag{4.2.5}$$

where the P_n are the polynomials studied in Chapter 3. For, by virtue of (3.1.1), the right-hand-side of (4.2.5) satisfies (4.2.4), which defines the $B_\ell(a,q)$ uniquely.

From (3.5.4), we have

$$P_\ell(0,q,(a-1)q,q) =$$

$$= \sum_{m\geq 0} (a-1)^m q^{m^2+m}(-q^{m+1})_{\ell-2m} \left[\begin{matrix}\ell-m\\ m\end{matrix}\right] \quad , \tag{4.2.6}$$

and

$$P_{\ell-1}(0,q^2,(a-1)q^2,q) =$$

$$= \sum_{m\geq 0} (a-1)^m q^{m^2+2m}(-q^{m+2})_{\ell-1-2m}\left[\begin{matrix}\ell-1-m\\ m\end{matrix}\right] \quad . \tag{4.2.7}$$

It follows from (4.2.5)-(4.2.7) that

$$B_\ell(a,q) = \sum_{m\geq 0} (a-1)^m q^{m^2}(-q^{m+1})_{\ell-2m} \quad \times$$

$$\times \quad \left\{q^m\left[\begin{matrix}\ell-m\\ m\end{matrix}\right] + (1+q^{\ell-m+1})\left[\begin{matrix}\ell-m\\ m-1\end{matrix}\right]\right\}$$

$$= \sum_{m\geq 0} (a-1)^m q^{m^2}(-q^{m+1})_{\ell-2m} \ \frac{(q)_{\ell-m}}{(q)_m(q)_{\ell-2m+1}} \ (1-q^{\ell+1})$$

$$= (1-q^{\ell+1}) \sum_{m\geq 0} \frac{(a-1)^m q^{m^2}}{(q^2;q^2)_m} \ \frac{(q^2;q^2)_{\ell-m}}{(q)_{\ell-2m+1}} \quad . \tag{4.2.8}$$

Letting $\ell \to \infty$ in (4.2.8), we obtain

$$\sum_{n,s} B(n,s)a^s q^n = \frac{1}{(q;q^2)_\infty} \sum_{m\geq 0} \frac{(a-1)^m q^{m^2}}{(q^2;q^2)_m}$$

$$= \frac{((1-a)q;q^2)_\infty}{(q;q^2)_\infty}$$

$$= \prod_{r\geq 0} \left(\frac{1+(a-1)q^{2r+1}}{1-q^{2r+1}} \right)$$

$$= \prod_{r\geq 0} \left(1 + \frac{aq^{2r+1}}{1-q^{2r+1}} \right)$$

$$= \sum_{n,s} A(n,s)a^s q^n \quad , \qquad (4.2.9)$$

which is Sylvester's Theorem.

§3. I now give an alternative proof of Sylvester's Theorem. It is, in my opinion, as simple and as straightforward as any of the proofs in the literature, including that of Ramamani and Venkatachaliengar (1972), reproduced in Andrews (1976), and shows clearly the equivalence of Sylvester's Theorem and Lebesgue's identity (4.1.4).

Let $B^*(n,s,p,u)$ be the number of partitions of n into p distinct parts, with s sequences of consecutive parts, and u 1's $(u = 0$ or $1)$. (4.3.1)

Subtracting 1 from every part shows that

$$B^*(n,s,p,0) = B^*(n-p,s,p,0) + B^*(n-p,s,p,1)$$

and

$$B^*(n,s,p,1) = B^*(n-p,s-1,p-1,0) + B^*(n-p,s,p-1,1) \ . \qquad (4.3.2)$$

Now let

$$B^*_{p,u}(a,q) \;=\; \sum_{n,s} B(n,s,p,u)a^s q^n \quad . \tag{4.3.3}$$

It follows from (4.3.2) that

$$B^*_{p,0}(a,q) \;=\; q^p B^*_{p,0}(a,q) + q^p B^*_{p,1}(a,q) \;,$$

and
$$B^*_{p,1}(a,q) \;=\; aq^p B^*_{p-1,0}(a,q) + q^p B^*_{p-1,1}(a,q) \;. \tag{4.3.4}$$

From (4.3.4) it follows that

$$B^*_{p,0}(a,q) \;=\; \frac{q^p}{1-q^p}\; B^*_{p,1}(a,q)$$

and that

$$B^*_{p,1}(a,q) = q^p \left(1 + \frac{aq^{p-1}}{1-q^{p-1}}\right) \; B^*_{p-1,1}(a,q) \;. \tag{4.3.5}$$

A trivial induction yields, since $B^*_{1,1}(a,q) = aq$,

$$B^*_{p,0}(a,q) \;=\; \frac{aq^{\frac{1}{2}(p^2+3p)}}{1-q^p} \left(1 + \frac{aq}{1-q}\right) \cdots \left(1 + \frac{aq^{p-1}}{1-q^{p-1}}\right) ,$$

$$B^*_{p,1}(a,q) \;=\; aq^{\frac{1}{2}(p^2+p)} \left(1 + \frac{aq}{1-q}\right) \cdots \left(1 + \frac{aq^{p-1}}{1-q^{p-1}}\right) . \tag{4.3.6}$$

Let $\quad B^*(n,s,p) = B^*(n,s,p,0) + B^*(n,s,p,1) \quad .$ (4.3.7)

Then

$$\sum_{n,s} B^*(n,s,p)a^s q^n \;=$$

$$= \; B^*_{p,0}(a,q) + B^*_{p,1}(a,q)$$

$$= \; \frac{aq^{\frac{1}{2}(p^2+p)}}{1-q^p} \left(1 + \frac{aq}{1-q}\right) \cdots \left(1 + \frac{aq^{p-1}}{1-q^{p-1}}\right) \{q^p + (1-q^p)\}$$

$$= \; \frac{aq^{\frac{1}{2}(p^2+p)}}{1-q^p} \left(1 + \frac{aq}{1-q}\right) \cdots \left(1 + \frac{aq^{p-1}}{1-q^{p-1}}\right)$$

$$= q^{\binom{p+1}{2}} \frac{(1-(1-a))(1-(1-a)q) \;\ldots\; (1-(1-a)q^{p-1})}{(1-q^p) \quad (1-q) \quad \ldots \quad (1-q^{p-1})}$$

$$= \frac{q^{\binom{p+1}{2}} (1-a)_p}{(q)_p} \quad . \tag{4.3.8}$$

Summing on p, we obtain

$$\sum_{n,s} B(n,s) a^s q^n = 1 + \sum_{p\geq 1} \frac{q^{\binom{p+1}{2}} (1-a)_p}{(q)_p}$$

$$= \sum_{p\geq 0} \frac{q^{\binom{p+1}{2}} (1-a)_p}{(q)_p} \quad . \tag{4.3.9}$$

On the other hand,

$$\sum_{n,s} A(n,s) a^s q^n = \prod_{r\geq 0} \left(1 + \frac{aq^{2r+1}}{1-q^{2r+1}} \right)$$

$$= \frac{((1-a)q;q^2)_\infty}{(q;q^2)_\infty} \quad . \tag{4.3.10}$$

Thus Sylvester's Theorem is equivalent to

$$\sum_{p\geq 0} \frac{q^{\binom{p+1}{2}} (1-a)_p}{(q)_p} = \frac{((1-a)q;q^2)_\infty}{(q;q^2)_\infty} \quad , \tag{4.3.11}$$

which is Lebesgue's identity (4.1.4).

§4. We prove Theorems 4.1.5-4.1.8 as follows.

Let
$$G(n,r,k) = \sum_{\pi\in\Pi(n,k)} \binom{g(\pi)}{r} \quad , \tag{4.4.1}$$

where $\Pi(n,k)$ is the set of partitions of n into precisely k distinct parts.

Andrews (1966)(equation 3.1) showed that

$$G(n,r,k) = G(n-k,r,k) + G(n-k,r,k-1) + G(n-2k,r-1,k-1). \qquad (4.4.2)$$

If we let

$$G_k(a,q) = \sum_{n,r} G(n,r,k)a^r q^n , \qquad (4.4.3)$$

then

$$G_k(a,q) = q^k G_k(a,q) + q^k G_{k-1}(a,q) + aq^{2k}G_{k-1}(a,q), \qquad (4.4.4)$$

or,

$$G_k(a,q) = \frac{q^k(1+aq^k)}{1-q^k} G_{k-1}(a,q) . \qquad (4.4.5)$$

A trivial induction, together with $G_0(a,q) = 1$, yields

$$G_k(a,q) = \frac{q^{\binom{k+1}{2}}(-aq)_k}{(q)_k} . \qquad (4.4.6)$$

That is,

$$\sum_{n,r} G(n,r,k)a^r q^n = \frac{q^{\binom{k+1}{2}}(-aq)_k}{(q)_k} . \qquad (4.4.7)$$

Summing on k yields

$$\sum_{n,r} G(n,r)a^r q^n = \sum_{k\geq 0} \frac{q^{\binom{k+1}{2}}(-aq)_k}{(q)_k}$$

$$= \frac{(-aq^2;q^2)_\infty}{(q;q^2)_\infty}$$

$$= \prod_{r\geq 0} \left(\frac{1+aq^{2r+2}}{1-q^{2r+1}}\right) , \qquad (4.4.8)$$

from which Theorem 4.1.5 follows.

We have

$$\sum_{n,r} G(n,r)a^r q^n = \prod_{r\geq 0} \left(\frac{1+aq^{2r+2}}{1-q^{2r+1}} \right)$$

$$= \prod_{r\geq 0} \frac{1}{1-q^{2r+1}} \sum_{r\geq 0} \frac{a^r q^{r^2+r}}{(1-q^2)(1-q^4)\ \ldots\ (1-q^{2r})}$$

$$= \sum_{r\geq 0} \frac{a^r q^{r^2+r}}{(1-q)(1-q^2)\ \ldots\ (1-q^{2r})(1-q^{2r+1})(1-q^{2r+3})\ \ldots} \,, \tag{4.4.9}$$

from which Theorem 4.1.6 follows.

$$\text{Let} \quad S(n,r,k) = \sum_{\pi\in\Pi(n,k)} \binom{s(\pi)}{r} \ . \tag{4.4.10}$$

Using Andrews' device, it can be shown that

$$S(n,r,k) = S(n-k,r,k) + S(n-k,r,k-1) + S(n-2k+1,r-1,k-1). \tag{4.4.11}$$

Letting

$$S_k(a,q) = \sum_{n,r} S(n,r,k)a^r q^n \ , \tag{4.4.12}$$

it follows as before that

$$S_k(a,q) = \frac{q^{\binom{k+1}{2}}(-a)_k}{(q)_k} \ , \tag{4.4.13}$$

or,

$$\sum_{n,r} S(n,r,k)a^r q^n = \frac{q^{\binom{k+1}{2}}(-a)_k}{(q)_k} \ . \tag{4.4.14}$$

It follows from (4.4.14) and (4.4.7) that

$$S(n+r,r,k) = G(n,r,k) . \tag{4.4.15}$$

Summing on k yields

$$S(n+r,r) = G(n,r) . \tag{4.4.16}$$

Theorems 4.1.7 and 4.1.8 follow from (4.4.16) taken together with Theorems 4.1.5 and 4.1.6.

Chapter 5. Further Combinatorial Aspects of the Polynomials of Chapters 2,3

§1. It is well-known that the polynomials which arise in the study of the continued fraction

$$1 + \frac{cq}{1+}\ \frac{cq^2}{1+}\ \cdots , \tag{5.1.1}$$

namely

$$P_n(0,0,c,q) = \sum_{2j\le n} c^j q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix} \tag{5.1.2}$$

are capable of a combinatorial interpretation. Thus, $P_n(0,0,c,q)$ is the generating function for partitions into parts differing by at least 2, with the parts all $< n$, and where the power of c counts the number of parts (see Hirschhorn (1972), MacMahon (1916) Art. 286).

In attempting to find a more general family of polynomials with a combinatorial interpretation, George Andrews suggested the possibility of finding the generating function for partitions into distinct parts with all sequences of length less than k, and with all parts $< n$. This generating function is, as we shall see in §2,

$$\sum_{j\ge 0} c^j q^{\binom{j+1}{2}} \sum_{k\ell\le j} (-1)^\ell q^{k\binom{\ell}{2}} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_{(q^k)} \begin{bmatrix} n-k\ell-1 \\ j-k\ell \end{bmatrix} , \tag{5.1.3}$$

where again the power of c counts the number of parts.

In the case $k=2$, the expression in (5.1.3) is equal to that in (5.1.2), by virtue of the fact that they generate the same set of partitions. Equating the two expressions yields, as we shall see in §3, the following identity of Slater (1951), [19], namely

$$\prod_{j\geq 1} (1+q^j) \sum_{j\geq 0} \frac{(-1)^j q^{3j^2}}{(q^2;q^2)_j(-q;q)_{2j}} = 1/\prod_{j\geq 0} (1-q^{5j+1})(1-q^{5j+4}) \ . \tag{5.1.4}$$

We also obtain the companion identity

$$\prod_{j\geq 1} (1+q^j) \sum_{j\geq 0} \frac{(-1)^j q^{3j^2+2j}}{(q^2;q^2)_j(-q;q)_{2j+1}} = 1/\prod_{j\geq 0} (1-q^{5j+2})(1-q^{5j+3}) \ , \tag{5.1.5}$$

which, we show, is equivalent to another identity of Slater (loc.cit.), [15], namely

$$\prod_{j\geq 1} (1+q^j) \sum_{j\geq 0} \frac{(-1)^j q^{3j^2-2j}}{(q^2;q^2)_j(-q;q)_{2j}} = 1/\prod_{j\geq 0} (1-q^{5j+2})(1-q^{5j+3}) \ . \tag{5.1.6}$$

§2. Let $T_k(n,j;q)$ denote the generating function for partitions with no part occurring more than $k-1$ times, with all parts $\leq n$, and with precisely j parts. (5.2.1)

Then

$$\sum_{j\geq 0} T_k(n,j;q)c^j =$$

$$= (1+cq+\ldots+c^{k-1}q^{k-1})(1+cq^2+\ldots+c^{k-1}q^{2(k-1)}) \times \ldots$$

$$\ldots \times (1+cq^n+\ldots+c^{k-1}q^{n(k-1)}) \ .$$

$$= \left(\frac{1-c^kq^k}{1-cq}\right)\left(\frac{1-c^kq^{2k}}{1-cq^2}\right)\cdots\left(\frac{1-c^kq^{nk}}{1-cq^n}\right)$$

$$= \prod_{j=1}^{n} (1-c^k(q^k)^j) \times \prod_{j=1}^{n} (1-cq^j)^{-1}$$

$$= \frac{(-(cq)^k;q^k)_n}{(-cq;q)_n}$$

$$= \sum_{\ell\ge 0} (-c^k q^k)^\ell (q^k)^{\binom{\ell}{2}} \begin{bmatrix} n \\ \ell \end{bmatrix}_{(q^k)} \cdot \sum_{m\ge 0} (cq)^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix}$$

$$= \sum_{\ell, m\ge 0} c^{k\ell+m} (-1)^\ell q^{k\binom{\ell+1}{2}+m} \begin{bmatrix} n \\ \ell \end{bmatrix}_{(q^k)} \begin{bmatrix} n+m-1 \\ m \end{bmatrix} . \tag{5.2.2}$$

It follows that

$$T_k(n,j;q) =$$

$$= \sum_{k\ell+m=j} (-1)^\ell q^{k\binom{\ell+1}{2}+m} \begin{bmatrix} n \\ \ell \end{bmatrix}_{(q^k)} \begin{bmatrix} n+m-1 \\ m \end{bmatrix}$$

$$= \sum_{k\ell\le j} (-1)^\ell q^{k\binom{\ell+1}{2}+j-k\ell} \begin{bmatrix} n \\ \ell \end{bmatrix}_{(q^k)} \begin{bmatrix} n+j-k\ell-1 \\ j-k\ell \end{bmatrix}$$

$$= \sum_{k\ell\le j} (-1)^\ell q^{k\binom{\ell}{2}+j} \begin{bmatrix} n \\ \ell \end{bmatrix}_{(q^k)} \begin{bmatrix} n+j-k\ell-1 \\ j-k\ell \end{bmatrix} . \tag{5.2.3}$$

Now let $S_k(n,j;q)$ denote the generating function for partitions into distinct parts with all sequences of length less than k, with all parts $< n$, and with j parts. Subtracting $0,1,\dots,j-1$ respectively from the j parts shows that

$$S_k(n,j;q) = q^{\binom{j}{2}} T_k(n-j,j;q) \tag{5.2.4}$$

$$= q^{\binom{j}{2}} \sum_{k\ell\le j} (-1)^\ell q^{k\binom{\ell}{2}+j} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_{(q^k)} \begin{bmatrix} n-k\ell-1 \\ j-k\ell \end{bmatrix}$$

$$= q^{\binom{j+1}{2}} \sum_{k\ell\le j} (-1)^\ell q^{k\binom{\ell}{2}} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_{(q^k)} \begin{bmatrix} n-k\ell-1 \\ j-k\ell \end{bmatrix} . \tag{5.2.5}$$

It follows that

$$\sum_{j\ge0} S_k(n,j;q)\, c^j =$$

$$= \sum_{j\ge0} c^j q^{\binom{j+1}{2}} \sum_{k\ell\le j} (-1)^\ell q^{k\binom{\ell}{2}} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_{(q^k)} \begin{bmatrix} n-k\ell-1 \\ j-k\ell \end{bmatrix} \quad ,$$

which is (5.1.3).

§3. We have, by virtue of what was said in §1, that

$$\sum_{j\ge0} c^j q^{j^2} \begin{bmatrix} n-j \\ j \end{bmatrix} =$$

$$= \sum_{j\ge0} c^j q^{\binom{j+1}{2}} \sum_{2\ell\le j} (-1)^\ell q^{2\binom{\ell}{2}} \begin{bmatrix} n-j \\ \ell \end{bmatrix}_{(q^2)} \begin{bmatrix} n-2\ell-1 \\ j-2\ell \end{bmatrix} \quad . \qquad (5.3.1)$$

Letting $n \to \infty$, we obtain

$$\sum_{j\ge0} \frac{c^j q^{j^2}}{(q)_j} = \sum_{j\ge0} c^j q^{\binom{j+1}{2}} \sum_{2\ell\le j} (-1)^\ell q^{2\binom{\ell}{2}} \frac{1}{(q^2;q^2)_\ell} \cdot \frac{1}{(q)_{j-2\ell}}$$

$$= \sum_{\ell\ge0} \frac{(-1)^\ell q^{2\binom{\ell}{2}}}{(q^2;q^2)_\ell} \sum_{j\ge2\ell} \frac{c^j q^{\binom{j+1}{2}}}{(q)_{j-2\ell}}$$

$$= \sum_{\ell\ge0} \frac{(-1)^\ell q^{2\binom{\ell}{2}}}{(q^2;q^2)_\ell} \cdot \sum_{j\ge0} \frac{c^{j+2\ell} q^{\binom{j+2\ell+1}{2}}}{(q)_j}$$

$$= \sum_{\ell\ge0} \frac{c^{2\ell}(-1)^\ell q^{2\binom{\ell}{2}+\binom{2\ell+1}{2}}}{(q^2;q^2)_\ell} \sum_{j\ge0} \frac{c^j a^{\binom{j}{2}} (q^{2\ell+1})^j}{(q)_j}$$

$$= \sum_{\ell\ge0} \frac{c^{2\ell}(-1)^\ell q^{3\ell^2}}{(q^2;q^2)_\ell} (1+cq^{2\ell+1})(1+cq^{2\ell+2}) \;\ldots .$$

$$= \prod_{j\geq 1} (1+cq^j) \sum_{\ell\geq 0} \frac{c^{2\ell}(-1)^\ell q^{3\ell^2}}{(q^2;q^2)_\ell} \frac{1}{(1+cq)(1+cq^2)\dots(1+cq^{2\ell})}$$

$$= \prod_{j\geq 1} (1+cq^j) \sum_{\ell\geq 0} \frac{c^{2\ell}(-1)^\ell q^{3\ell^2}}{(q^2;q^2)_\ell(-cq;q)_{2\ell}} \quad . \tag{5.3.2}$$

If in (5.3.2) we set $c=1$, and use the first Rogers-Ramanujan identity, we obtain

$$\prod_{j\geq 1} (1+q^j) \sum_{j\geq 0} \frac{(-1)^j q^{3j^2}}{(q^2;q^2)_j(-q;q)_{2j}} = 1/\prod_{j\geq 0} (1-q^{5j+1})(1-q^{5j+4}) ,$$

which is (5.1.4), while if we set $c=q$ and use the second Rogers-Ramanujan identity, we obtain

$$\prod_{j\geq 1} (1+q^j) \sum_{j\geq 0} \frac{(-1)^j q^{3j^2+2j}}{(q^2;q^2)_j(-q;q)_{2j+1}} = 1/\prod_{j\geq 0} (1-q^{5j+2})(1-q^{5j+3}) ,$$

which is (5.1.5).

Now,

$$\sum_{j\geq 0} \frac{(-1)^j q^{3j^2+2j}}{(q^2;q^2)_j(-q;q)_{2j+1}} =$$

$$= \sum_{j\geq 0} \frac{(-1)^j q^{3j^2+2j}}{(q^2;q^2)_j(-q;q)_{2j}} \left(1 - \frac{q^{2j+1}}{1+q^{2j+1}}\right)$$

$$= \sum_{j\geq 0} \frac{(-1)^j q^{3j^2+2j}}{(q^2;q^2)_j(-q;q)_{2j}} + \sum_{j\geq 0} \frac{(-1)^{j+1} q^{3j^2+4j+1}}{(q^2;q^2)_j(-q;q)_{2j+1}}$$

$$= 1 + \sum_{j\geq 1} \frac{(-1)^j q^{3j^2-2j}}{(q^2;q^2)_{j-1}(-q;q)_{2j-1}} + \sum_{j\geq 1} \frac{(-1)^j q^{3j^2+2j}}{(q^2;q^2)_j(-q;q)_{2j}}$$

$$= 1 + \sum_{j\geq 1} \frac{(-1)^j q^{3j^2-2j}}{(q^2;q^2)_j(-q;q)_{2j}} \{(1-q^{2j})(1+q^{2j})+ q^{4j}\}$$

$$= 1 + \sum_{j\geq 1} \frac{(-1)^j q^{3j^2-2j}}{(q^2;q^2)_j(-q;q)_{2j}}$$

$$= \sum_{j\geq 0} \frac{(-1)^j q^{3j^2-2j}}{(q^2;q^2)_j(-q;q)_{2j}} \quad . \qquad (5.3.3)$$

(5.1.6) follows from (5.1.5) together with (5.3.3).

Chapter 6. Jacobi's Triple-Product Identity

§1. The identity referred to in the title of this chapter is Jacobi's celebrated identity (Gesammelte Werke, Vol.1, pp.232-4)

$$\prod_{r\geq 1} \left(1+a^{-1}q^{2r-1}\right)\left(1+aq^{2r-1}\right)\left(1-q^{2r}\right) = \sum_{r=-\infty}^{\infty} a^r q^{r^2} . \tag{6.1.1}$$

This identity plays a role of great importance in the theory of partitions and related areas, and in particular in those areas touched upon in this thesis.

Thus, if we set $q^{k/2}$ for q, $-q^{\ell/2}$ for a, we obtain

$$\prod_{r\geq 1} \left(1-q^{kr-\frac{1}{2}k-\frac{1}{2}\ell}\right)\left(1-q^{kr-\frac{1}{2}k+\frac{1}{2}\ell}\right)\left(1-q^{kr}\right) =$$

$$= \sum_{r=-\infty}^{\infty} (-1)^r q^{\frac{1}{2}kr^2+\frac{1}{2}\ell r}$$

$$= 1 + \sum_{r\geq 1} (-1)^r \left(q^{\frac{1}{2}kr^2-\frac{1}{2}\ell r} + q^{\frac{1}{2}kr^2+\frac{1}{2}\ell r}\right) \tag{6.1.2}$$

while if we set $q^{k/2}$ for q, $q^{\ell/2}$ for a, we obtain

$$\prod_{r\geq 1} \left(1+q^{kr-\frac{1}{2}k-\frac{1}{2}\ell}\right)\left(1+q^{kr-\frac{1}{2}k+\frac{1}{2}\ell}\right)\left(1-q^{kr}\right) =$$

$$= \sum_{r=-\infty}^{\infty} q^{\frac{1}{2}kr^2+\frac{1}{2}\ell r}$$

$$= 1 + \sum_{r\geq 1} \left(q^{\frac{1}{2}kr^2-\frac{1}{2}\ell r} + q^{\frac{1}{2}kr^2+\ell r}\right) . \tag{6.1.3}$$

In particular, setting $k=3$, $\ell=1$ in (6.1.2) yields Euler's identity (Opera Omnia Series Prima, Vol. VIII, p.334)

$$\prod_{r\geq 1}(1-q^r) = 1+\sum_{r\geq 1}(-1)^r\left(q^{3r^2/2-\frac{1}{2}r}+q^{3r^2/2+\frac{1}{2}r}\right) . \qquad (6.1.4)$$

We have had occasion in Chapter 2 to use other special cases of (6.1.2) and (6.1.3) without comment, and will do so again in Chapters 7, 8, 10.

A further consequence of (6.1.1), obtained by setting $q^{\frac{1}{2}}$ for q, $-aq^{\frac{1}{2}}$ for a, dividing by $1-\frac{1}{a}$, then letting $a \to 1$ (the limiting process requires some justification) is Jacobi's identity (Gesammelte Werke, Vol. 1, pp236-7)

$$\prod_{r\geq 1}(1-q^r)^3 = \sum_{r\geq 0}(-1)^r(2r+1)q^{\frac{1}{2}(r^2+r)} . \qquad (6.1.5)$$

We shall make good use of (6.1.4) and (6.1.5) in Chapter 7.

Jacobi's identity (6.1.1) can be derived from an identity due to Ramanujan (Notebooks, Vol.II, p.196) recently given easy proofs by Andrews and Askey (1978) and Ismail (1977) . A proof of Ramanujan's identity, and the derivation from it of (6.1.1) are given in the Appendix, §7.

Alternatively, one can obtain (6.1.1) by simply letting $i,j \to \infty$ in the identity

$$\prod_{r=1}^{i}\left(1+a^{-1}q^{2r-1}\right)\prod_{r=1}^{j}\left(1+aq^{2r-1}\right) = \sum_{r=-i}^{j} a^r q^{r^2}\left[{i+j \atop i+r}\right]' , \qquad (6.1.6)$$

where $\left[{i \atop j}\right]'$ is obtained from $\left[{i \atop j}\right]$ by replacing q by q^2. The identity (6.1.6), ascribed by Hirschhorn (1976) to MacMahon, is actually much older; R. Askey has traced it back as far as Schweins (1819). Indeed, (6.1.6) can be written

$$q^{i^2}a^{-i}\prod_{r=1}^{i+j}\left(1+(aq^{-2i})q^{2r+1}\right) = q^{i^2}a^{-i}\sum_{r=0}^{i+j}\left(aq^{-2i}\right)^r q^{r^2}\left[{i+j \atop r}\right], \tag{6.1.7}$$

and so is seen to be equivalent to the q-binomial theorem (see Appendix §2). In §2 we present the proof by induction of (6.1.6) given by Hirschhorn (loc.cit.). This proof is a verification; it is neater than the related proof given by Grosswald (1966) in which (6.1.6) is "generated".

We can obtain a finite version of (6.1.5) from (6.1.6) as follows. Set $i=n+1$, $j=n$, set $q^{\frac{1}{2}}$ for q, $-aq^{\frac{1}{2}}$ for a, divide by $1-\frac{1}{a}$, and let $a \to 1$. We then have

$$\prod_{r=1}^{n}(1-q^r)^2 = \sum_{r=0}^{n}(-1)^r(2r+1)q^{\frac{1}{2}(r^2+r)}\left[{2n+1 \atop n-1}\right]. \tag{6.1.8}$$

Identity (6.1.8), first observed by Hirschhorn (1977) is particularly simple, and yields (6.1.5) on letting $n \to \infty$. Because it works so nicely, I give, in §3, a direct proof by induction of (6.1.8).

Finally, we note that (6.1.4) can be written

$$q\prod_{r\geq 1}(1-q^{24r}) = q + \sum_{r\geq 1}(-1)^r\left(q^{(6r-1)^2}+q^{(6r+1)^2}\right) \tag{6.1.9}$$

while (6.1.5) can be written

$$q\prod_{r\geq 1}(1-q^{8r})^3 = \sum_{r\geq 0}(-1)^r(2r+1)q^{(2r+1)^2}. \tag{6.1.10}$$

It follows easily from (6.1.9) and (6.1.10) that if $\alpha = \frac{1}{2}$ or $\frac{3}{2}$, and we write

$$q \prod_{r\geq 1} \left(1-q^{(12/\alpha)r}\right)^{2\alpha} = \sum_{r\geq 0} c_{\alpha,r} q^r , \tag{6.1.11}$$

then the $c_{\alpha,r}$ are multiplicative. That is

$$c_{\alpha,rs} = c_{\alpha,r} c_{\alpha,s} \quad \text{if} \quad (r,s) = 1 . \tag{6.1.12}$$

Ramanujan (1916) stated and Mordell (1917) proved that more generally if α is any divisor of 12, or if $\alpha = \frac{1}{2}$ or $\frac{3}{2}$, and if the $c_{\alpha,r}$ are defined by (6.1.11), then (6.1.12) holds.

§2. In order to prove (6.1.6), let

$$S_{i,j} = \sum_{r=-i}^{j} a^r q^{r^2} \begin{bmatrix} i+j \\ i+r \end{bmatrix}' . \tag{6.2.1}$$

Then, if $i > 0$,

$$\begin{aligned} S_{i,j} &= \sum_{r=-i}^{j} a^r q^{r^2} \left\{ \begin{bmatrix} i+j-1 \\ i+r-1 \end{bmatrix}' + q^{2i+2r} \begin{bmatrix} i+j-1 \\ i+r \end{bmatrix}' \right\} \\ &= \sum_{r=-i+1}^{j} a^r q^{r^2} \begin{bmatrix} i+j-1 \\ i+r-1 \end{bmatrix}' + \frac{q^{2i-1}}{a} \sum_{r=-i}^{j-1} a^{r+1} q^{r^2+2r+1} \begin{bmatrix} i+j-1 \\ i+r \end{bmatrix}' \\ &= \sum_{r=-i+1}^{j} a^r q^{r^2} \begin{bmatrix} i+j-1 \\ i+r-1 \end{bmatrix}' + \frac{q^{2i-1}}{a} \sum_{r=-i+1}^{j} a^r q^{r^2} \begin{bmatrix} i+j-1 \\ i+r-1 \end{bmatrix}' \\ &= \left(1 + \frac{q^{2i-1}}{a}\right) S_{i-1,j} , \end{aligned} \tag{6.2.2}$$

while if $j > 0$,

$$S_{i,j} = \sum_{r=-i}^{j} a^r q^{r^2} \left\{ \begin{bmatrix} i+j-1 \\ i+r \end{bmatrix}' + q^{2j-2r} \begin{bmatrix} i+j-1 \\ i+r-1 \end{bmatrix}' \right\}$$

$$= \sum_{r=-i}^{j-1} a^r q^{r^2} \begin{bmatrix} i+j-1 \\ i+r \end{bmatrix}' + aq^{2j-1} \sum_{r=-i+1}^{j} a^{r-1} q^{r^2-2r+1} \begin{bmatrix} i+j-1 \\ i+r-1 \end{bmatrix}'$$

$$= \sum_{r=-i}^{j-1} a^r q^{r^2} \begin{bmatrix} i+j-1 \\ i+r \end{bmatrix}' + aq^{2j-1} \sum_{r=-i}^{j-1} a^r q^{r^2} \begin{bmatrix} i+j-1 \\ i+r \end{bmatrix}'$$

$$= (1+aq^{2j-1}) S_{i,j-1} \; . \tag{6.2.3}$$

Since $S_{0,0} = 1$, it follows by induction that for $i,j \geq 0$,

$$S_{i,j} = \prod_{r=1}^{i} \left(1+a^{-1}q^{2r-1}\right) \prod_{r=1}^{j} \left(1+aq^{2r-1}\right) \quad , \tag{6.2.4}$$

which is (6.1.6).

§3. In order to prove (6.1.8), let

$$S_n = \sum_{r=0}^{n} (-1)^r (2r+1) q^{\frac{1}{2}(r^2+r)} \begin{bmatrix} 2n+1 \\ n-r \end{bmatrix} \quad . \tag{6.3.1}$$

Then for $n > 0$,

$$S_n = \sum_{r=0}^{n} (-1)^r (2r+1) q^{\frac{1}{2}(r^2+r)} \left\{ \begin{bmatrix} 2n \\ n-r-1 \end{bmatrix} + q^{n-r} \begin{bmatrix} 2n \\ n-r \end{bmatrix} \right\}$$

$$= \sum_{r=0}^{n-1} (-1)^r (2r+1) q^{\frac{1}{2}(r^2+r)} \begin{bmatrix} 2n \\ n-r-1 \end{bmatrix}$$

$$+ q^n \sum_{r=0}^{n} (-1)^r (2r+1) q^{\frac{1}{2}(r^2-r)} \begin{bmatrix} 2n \\ n-r \end{bmatrix}$$

$$= \sum_{r=0}^{n-1} (-1)^r (2r+1) q^{\frac{1}{2}(r^2+r)} \left\{ \begin{bmatrix} 2n-1 \\ n-r-1 \end{bmatrix} + q^{n+r+1} \begin{bmatrix} 2n-1 \\ n-r-2 \end{bmatrix} \right\}$$

$$+ q^n \sum_{r=0}^{n} (-1)^r (2r+1) q^{\frac{1}{2}(r^2-r)} \left\{ \begin{bmatrix} 2n-1 \\ n-r \end{bmatrix} + q^{n+r} \begin{bmatrix} 2n-1 \\ n-r-1 \end{bmatrix} \right\}$$

$$= \sum_{r=0}^{n-1} (-1)^r (2r+1) q^{\frac{1}{2}(r^2+r)} \begin{bmatrix} 2n-1 \\ n-r-1 \end{bmatrix}$$

$$+ q^n \sum_{r=0}^{n-2} (-1)^r (2r+1) q^{\frac{1}{2}(r^2+3r+2)} \begin{bmatrix} 2n-1 \\ n-r-2 \end{bmatrix}$$

$$+ q^n \sum_{r=0}^{n} (-1)^r (2r+1) q^{\frac{1}{2}(r^2-r)} \begin{bmatrix} 2n-1 \\ n-r \end{bmatrix}$$

$$+ q^{2n} \sum_{r=0}^{n-1} (-1)^r (2r+1) q^{\frac{1}{2}(r^2+r)} \begin{bmatrix} 2n-1 \\ n-r-1 \end{bmatrix}$$

$$= (1+q^{2n}) S_{n-1}$$

$$+ q^n \sum_{r=1}^{n-1} (-1)^{r-1} (2r-1) q^{\frac{1}{2}(r^2+r)} \begin{bmatrix} 2n-1 \\ n-r-1 \end{bmatrix}$$

$$+ q^n \Big\{ \begin{bmatrix} 2n-1 \\ n \end{bmatrix} - 3 \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} + \sum_{r=1}^{n-1} (-1)^{r+1} (2r+3) q^{\frac{1}{2}(r^2+r)} \begin{bmatrix} 2n-1 \\ n-r-1 \end{bmatrix} \Big\}$$

$$= (1+q^{2n}) S_{n-1}$$

$$- q^n \Big\{ 2 \begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} + \sum_{r=1}^{n-1} (-1)^r \{(2r-1)+(2r+3)\} q^{\frac{1}{2}(r^2+r)} \begin{bmatrix} 2n-1 \\ n-r-1 \end{bmatrix} \Big\}$$

$$= (1+q^{2n}) S_{n-1}$$

$$- 2q^n \sum_{r=0}^{n-1} (-1)^r (2r+1) q^{\frac{1}{2}(r^2+r)} \begin{bmatrix} 2n-1 \\ n-r-1 \end{bmatrix}$$

$$= (1-2q^n+q^{2n}) S_{n-1}$$

$$= (1-q^n)^2 S_{n-1} \quad . \tag{6.3.2}$$

Since $S_0=1$, it follows by induction that for $n \geq 0$,

$$S_n = \prod_{r=1}^{n} (1-q^r)^2 \ , \tag{6.3.3}$$

which is (6.1.8).

Chapter 7. Some Relations involving the Partition Function

§1. Let $p(n)$ denote the number of partitions of n.

Then
$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} (1-q^n)^{-1} . \tag{7.1.1}$$

Ramanujan (1919a) conjectured, on the basis of the evidence provided by a table of values of $p(n)$ that

$$p(5n+4) \equiv 0 \bmod 5, \tag{7.1.2}$$

$$p(7n+5) \equiv 0 \bmod 7, \tag{7.1.3}$$

$$p(11n+6) \equiv 0 \bmod 11, \tag{7.1.4}$$

and more generally that

if $\Delta = 5^a 7^b 11^c$ and $24\lambda \equiv 1 \bmod \Delta$, then

$$p(n\Delta + \lambda) \equiv 0 \bmod \Delta . \tag{7.1.5}$$

He succeeded (Ramanujan (1919a),(1921)) in proving this conjecture for certain choices of (a,b,c); in particular, he proved (7.1.2)-(7.1.4), but (7.1.5) was seen by S.Chowla to be false as it stands. The following theorem is the result of the work of G.N.Watson (1938) and A.O.L. Atkin (1967).

Theorem 7.1.6

If $\Delta = 5^a 7^b 11^c$, and $24\lambda \equiv 1 \bmod \Delta$, then

$$p(n\Delta + \lambda) \equiv 0 \bmod 5^a 7^{[\frac{1}{2}(b+2)]} 11^c .$$

In §2 I give an elementary proof of (7.1.2), based on a technique for "splitting" the generating function of the $p(n)$. A similar proof can be given for (7.1.3), but I have not succeeded in proving (7.1.4) by the same method.

Ramanujan (1919a) proved the identity

$$\sum_{n\geq 0} p(5n+4)q^n = 5\{\prod_{r\geq 1}(1-q^{5r})^5/\prod_{r\geq 1}(1-q^r)^6\} \quad . \qquad (7.1.7)$$

G.H. Hardy (1927) pp.xxxiv-v, says "It would be difficult to find more beautiful formulae than the Rogers-Ramanujan identities but if I had to select one formula from all Ramanujan's work, I would agree with Major MacMahon in selecting [the above identity]."
In §2 I give an elementary proof of (7.1.7). The proofs I give in §2 can be shown to be related to those of O.Kolberg (1957).

Kolberg (loc.cit) has further shown that if

$$P_i = \sum_{n\equiv i \bmod 5} p(n)q^n , \qquad (7.1.8)$$

then

$$P_0P_4 + P_1P_3 - 2P_2^2 = 0 , \qquad (7.1.9)$$

$$P_0P_2 + P_3P_4 - 2P_1^2 = 0 , \qquad (7.1.10)$$

$$3P_1P_2 - 2P_0P_3 - P_4^2 = 0 . \qquad (7.1.11)$$

In §3 I prove (7.1.9)-(7.1.11), and in §4 I further show that similar polynomial identities hold for any modulus not a power of 2. In particular, we will see in §5 that if

$$P_i = \sum_{n\equiv i \bmod 3} p(n)q^n \qquad (7.1.12)$$

then

$$(P_0^2-P_1P_2)(P_2^2-P_0P_1)^2 + (P_2^2-P_0P_1)(P_1^2-P_0P_2)^2 + (P_1^2-P_0P_2)(P_0^2-P_1P_2)^2 = 0 .$$
$$(7.1.13)$$

In §6 I employ the same techniques to prove that $F_{5n+4} \equiv 0 \bmod 5$, where the F_n are the Fibonacci numbers.

§2. In order to prove (7.1.2), write

$$\prod_{n\geq 1}(1-q^n) = \phi(q) \ , \tag{7.2.1}$$

and suppose $\omega^5=1, \ \omega \neq 1$. (7.2.2)

Then $\sum_{n\geq 0} p(n)q^n = 1/\phi(q)$

$$= \phi(\omega q)\phi(\omega^2 q)\phi(\omega^3 q)\phi(\omega^4 q)/\phi(q)\phi(\omega q)\phi(\omega^2 q)\phi(\omega^3 q)\phi(\omega^4 q). \tag{7.2.3}$$

Consider the denominator of (7.2.3),

$$\Phi = \phi(q)\phi(\omega q)\phi(\omega^2 q)\phi(\omega^3 q)\phi(\omega^4 q) =$$

$$= \prod_{r\geq 1} (1-q^r)(1-\omega^r q^r)(1-\omega^{2r}q^r)(1-\omega^{3r}q^r)(1-\omega^{4r}q^r)$$

$$= \prod_{r\equiv 0(5)} (1-q^r)^5 \prod_{r\not\equiv 0(5)} (1-q^{5r})$$

$$= \prod_{r\geq 1} (1-q^{5r})^5 \prod_{r\not\equiv 0(5)} (1-q^{5r})$$

$$= \prod_{r\geq 1} (1-q^{5r})^6 \Big/ \prod_{r\equiv 0(5)} (1-q^{5r})$$

$$= \prod_{r\geq 1} (1-q^{5r})^6 \Big/ \prod_{r\geq 1} (1-q^{25r})$$

$$= (\phi(q^5))^6/\phi(q^{25}) \ . \tag{7.2.4}$$

We now consider the numerator of (7.2.3). We have (6.1.4) that

$$\phi(q) = 1 + \sum_{r\geq 1} (-1)^r\left(q^{3r^2/2-\frac{1}{2}r} + q^{3r^2/2+\frac{1}{2}r}\right)$$

$$= \phi_0(q)+\phi_1(q)+\phi_2(q) \ , \tag{7.2.5}$$

where $\phi_i(q)$ contains all terms of $\phi(q)$ of which the exponent is congruent to i mod 5. (Note that $\phi_3(q)=\phi_4(q)= 0$ since $3r^2/2 \pm \frac{1}{2}r \not\equiv 3,4 \bmod 5$.)

It follows from (7.2.5) that

$$\begin{aligned}
\phi(\omega q) &= \phi_0 + \omega\phi_1 + \omega^2\phi_2 \;, \\
\phi(\omega^2 q) &= \phi_0 + \omega^2\phi_1 + \omega^4\phi_2 \;, \\
\phi(\omega^3 q) &= \phi_0 + \omega^3\phi_1 + \omega\phi_2 \;, \\
\phi(\omega^4 q) &= \phi_0 + \omega^4\phi_1 + \omega^3\phi_2 \;,
\end{aligned} \tag{7.2.6}$$

and thence that the numerator

$$\begin{aligned}
&\phi(\omega q)\phi(\omega^2 q)\phi(\omega^2 q)\phi(\omega^4 q) = \\
&= (\phi_0^4 - \phi_1^3\phi_2 + 2\phi_0\phi_1\phi_2^2) + (-\phi_0^3\phi_1 - \phi_0\phi_2^3 + \phi_1^2\phi_2^2) \\
&\quad + (-\phi_0^3\phi_2 - \phi_1\phi_2^3 + \phi_0^2\phi_1^2) + (\phi_2^4 - \phi_0\phi_1^3 + 2\phi_0^2\phi_1\phi_2) \\
&\quad + (\phi_1^4 - 3\phi_0\phi_1^2\phi_2 + \phi_0^2\phi_2^2) \;.
\end{aligned} \tag{7.2.7}$$

It follows from (7.2.3), (7.2.4) and (7.2.7) that

$$\begin{aligned}
\sum_{n\geq 0} p(5n)q^{5n} &= (\phi_0^4 - \phi_1^3\phi_2 + 2\phi_0\phi_1\phi_2^2)/\Phi \;, \\
\sum_{n\geq 0} p(5n+1)q^{5n+1} &= (-\phi_0^3\phi_1 - \phi_0\phi_2^3 + \phi_1^2\phi_2^2)/\Phi \;, \\
\sum_{n\geq 0} p(5n+2)q^{5n+2} &= (-\phi_0^3\phi_2 - \phi_1\phi_2^3 + \phi_0^2\phi_1^2)/\Phi \;, \\
\sum_{n\geq 0} p(5n+3)q^{5n+3} &= (\phi_2^4 - \phi_0\phi_1^3 + 2\phi_0^2\phi_1\phi_2)/\Phi \;, \\
\sum_{n\geq 0} p(5n+4)q^{5n+4} &= (\phi_1^4 - 3\phi_0\phi_1^2\phi_2 + \phi_0^2\phi_2^2)/\Phi \;.
\end{aligned} \tag{7.2.8}$$

We further have (6.1.5) that

$$(\phi(q))^3 = \sum_{r\geq 0} (-1)^r(2r+1)q^{\frac{1}{2}(r^2+r)} \;.$$

That is,

$$(\phi_0+\phi_1+\phi_2)^3 = (\phi_0^3+3\phi_1\phi_2^2) + (\phi_2^3+3\phi_0^2\phi_1)$$

$$+ (3\phi_0\phi_1^2 + 3\phi_0^2\phi_2) + (\phi_1^3+6\phi_0\phi_1\phi_2)$$

$$+ (3\phi_0\phi_2^2 + 3\phi_1^2\phi_2) .$$

$$= \sum_{r\geq 0} (-1)^r(2r+1)q^{\frac{1}{2}(r^2+r)} \qquad (7.2.9)$$

Since $\frac{1}{2}(r^2+r) \not\equiv 2,4 \bmod 5$, it follows from (7.2.9) that

$$3\phi_0\phi_1^2 + 3\phi_0^2\phi_2 = 0 , \qquad (7.2.10)$$

and

$$3\phi_0\phi_2^2 + 3\phi_1^2\phi_2 = 0 . \qquad (7.2.11)$$

From both (7.2.10) and (7.2.11) it follows that

$$\phi_0\phi_2 = -\phi_1^2 \qquad (7.2.12)$$

It follows from (7.2.8) and (7.2.12) that

$$\sum_{n\geq 0} p(5n+4)q^{5n+4} = 5\phi_1^4/\Phi . \qquad (7.2.13)$$

(7.1.2) follows directly from (7.2.13).

Further, we have

$$\phi_1 = \sum_{3r^2/2-\frac{1}{2}r\equiv 1(5)} (-1)^r q^{3r^2/2-\frac{1}{2}r} + \sum_{3r^2/2+\frac{1}{2}r\equiv 1(5)} (-1)^r q^{3r^2/2+\frac{1}{2}r}$$

$$= \sum_{s\geq 0} (-1)^{5s+1} q^{\frac{1}{2}(75s^2+25s+2)} + \sum_{s\geq 1} (-1)^{5s-1} q^{\frac{1}{2}(75s^2-25s+2)}$$

$$= -q\{1 + \sum_{s\geq 1} (-1)^s (q^{\frac{1}{2}(75s^2-25s)} + q^{\frac{1}{2}(75s^2+25s)})\}$$

$$= -q \prod_{r\geq 1} (1-q^{75r-50})(1-q^{75r-25})(1-q^{75r})$$

$$= -q \prod_{r\geq 1} (1-q^{25r})$$

$$= -q\, \phi(q^{25}) . \qquad (7.2.14)$$

It follows from (7.2.4), (7.2.13) and (7.2.14) that

$$\sum_{n\geq 0} p(5n+4)q^{5n+4} = 5q^4(\phi(q^{25}))^5/(\phi(q^5))^6, \qquad (7.2.15)$$

or

$$\sum_{n\geq 0} p(5n+4)q^n = 5(\phi(q^5))^5/(\phi(q))^6 , \qquad (7.2.16)$$

which is (7.1.7).

§3. We have shown (7.2.8) that

$$P_i = \psi_i . \phi(q^{25})/(\phi(q^5))^6 , \qquad (7.3.1)$$

where

$$\psi_0 = \phi_0^4 - \phi_1^3\phi_2 + 2\phi_0\phi_1\phi_2^2 ,$$

$$\psi_1 = - \phi_0^3\phi_1 - \phi_0\phi_2^3 + \phi_1^2\phi_2^2 ,$$

$$\psi_2 = - \phi_0^3\phi_2 - \phi_1\phi_2^3 + \phi_0^2\phi_1^2 ,$$

$$\psi_3 = \phi_2^4 - \phi_0\phi_1^3 + 2\phi_0^2\phi_1\phi_2 ,$$

$$\psi_4 = \phi_1^4 - 3\phi_0\phi_1^2\phi_2 + \phi_0^2\phi_2^2 , \qquad (7.3.2)$$

and where (7.2.12)

$$\phi_0\phi_2 = - \phi_1^2 .$$

It follows that

$$\psi_0 = \phi_0^4 - 3\phi_1^3\phi_2 ,$$

$$\psi_1 = - \phi_0^3\phi_1 + 2\phi_1^2\phi_2^2 ,$$

$$\psi_2 = 2\phi_0^2\phi_1^2 - \phi_1\phi_2^3 ,$$

$$\psi_3 = - 3\phi_0\phi_1^3 + \phi_2^4 ,$$

$$\psi_4 = 5\phi_1^4 ,$$

$$\psi_0\psi_4 = 5\phi_0^4\phi_1^4 - 15\phi_1^7\phi_2 ,$$

$$\psi_1\psi_3 = 3\phi_0^4\phi_1^4 + 7\phi_1^7\phi_2 + 2\phi_1^2\phi_2^6 ,$$

$$\psi_2^2 = 4\phi_0^4\phi_1^4 - 4\phi_1^7\phi_2 + \phi_1^2\phi_2^6 ,$$

$$\psi_0\psi_2 = 2\phi_0^6\phi_1^2 + 7\phi_0\phi_1^7 + 3\phi_1^4\phi_2^4 ,$$

$$\psi_3\psi_4 = - 15\phi_0\phi_1^7 + 5\phi_1^4\phi_2^4 ,$$

$$\psi_1^2 = \phi_0^6\phi_1^2 - 4\phi_0\phi_1^7 + 4\phi_1^4\phi_2^4 ,$$

$$\psi_1\psi_2 = - 2\phi_0^5\phi_1^3 + 3\phi_1^8 - 2\phi_1^3\phi_2^5 ,$$

$$\psi_0\psi_3 = - 3\phi_0^5\phi_1^3 - 8\phi_1^8 - 3\phi_1^3\phi_2^5 ,$$

$$\psi_4^2 = 25\phi_1^8 , \tag{7.3.3}$$

and so

$$\psi_0\psi_4 + \psi_1\psi_3 - 2\psi_2^2 = 0 , \tag{7.3.4}$$

$$\psi_0\psi_2 + \psi_3\psi_4 - 2\psi_1^2 = 0 , \tag{7.3.5}$$

$$3\psi_1\psi_2 - 2\psi_0\psi_3 - \psi_4^2 = 0 . \tag{7.3.6}$$

(7.1.9) - (7.1.11) follow from (7.3.4) - (7.3.6) together with (7.3.1).

§4. For $m \geq 1$, $i = 0,1,\ldots,m-1$, let

$$P_i = \sum_{n \equiv i \bmod m} p(n)q^n \tag{7.4.1}$$

We prove the following as yet unpublished results:

Theorem 7.4.2.

If m is not of the form $2^\alpha 3^\beta$, then there is at least one non-trivial polynomial in $P_0,\ldots,P_{m-1}$ homogeneous of degree $m-1$, which, considered as a series in q, is identically zero.

Theorem 7.4.3.

If m is not a power of 2, then there is at least one non-trivial polynomial in $P_0,\ldots,P_{m-1}$, homogeneous of degree $3(m-1)$, which, considered as a series in q, is identically zero.

Proof of Theorem 7.4.2.

Suppose m is not of the form $2^{\alpha}3^{\beta}$. Then there is a prime p, $p \neq 2,3$, with $p|m$. Since $p \equiv \pm 1 \bmod 6$, $(p,24)=1$. As j runs through a complete set of residues mod p, so does $24j+1$. So, for some j the congruence $x^2 \equiv 24j+1 \bmod p$ has no solution. For these j, the congruence $(6r\pm1)^2 \equiv 24j+1$ has no solution, so the congruence $\frac{3}{2}r^2 \pm \frac{1}{2}r \equiv j \bmod p$ has no solution, from which it follows that the congruence $\frac{3}{2}r^2 \pm \frac{1}{2}r \equiv j \bmod m$ has no solution.

Now write

$$\phi = \phi(q) = \prod_{r\geq 1} (1-q^r) = 1 + \sum_{r\geq 1} (-1)^r (q^{3r^2/2-\frac{1}{2}r} + q^{3r^2/2+\frac{1}{2}r})$$

$$= \phi_0 + \phi_1 + \ldots + \phi_m \qquad (7.4.4)$$

where, as in §2, ϕ_i contains all terms of ϕ of which the exponent is congruent to $i \bmod m$.

Then for those j for which the congruence $3r^2/2 \pm \frac{1}{2}r \equiv j \bmod m$ has no solution,

$$\phi_j = 0 \quad . \qquad (7.4.5)$$

Now, writing $P = P(q) = \sum_{n\geq 0} p(n)q^n$, and $\omega = e^{2\pi i/m}$

we have

$$\phi = 1/P$$

$$= P(\omega q)P(\omega^2 q)\dots P(\omega^{m-1}q)/P(q)P(\omega q)\dots P(\omega^{m-1}q). \tag{7.4.6}$$

The denominator of (7.4.6),

$$\mathcal{P}(q) = P(q)P(\omega q)\dots P(\omega^{m-1}q) \tag{7.4.7}$$

is a series in q^m, for

$$\mathcal{P}(\omega q) = P(\omega q)P(\omega^2 q)\dots P(\omega^{m-1}q)P(q) = \mathcal{P}(q), \tag{7.4.8}$$

so if we write

$$\mathcal{P}(q) = \sum_{r\geq 0} a_r q^r , \tag{7.4.9}$$

then

$$\omega^r a_r = a_r , \tag{7.4.10}$$

so for $r \not\equiv 0 \bmod m$, $a_r = 0$. (7.4.11)

Further, we have

$$P = P_0 + P_1 + \dots + P_{m-1} , \tag{7.4.12}$$

so

$$P(\omega q) = P_0 + \omega P_1 + \dots + \omega^{m-1}P_{m-1} ,$$
$$P(\omega^2 q) = P_0 + \omega^2 P_1 + \dots + \omega^{m-2}P_{m-1} .$$
$$\vdots$$
$$P(\omega^{m-1}q) = P_0 + \omega^{m-1}P_1 + \dots + \omega P_{m-1} . \tag{7.4.13}$$

Thus the numerator of (7.4.6) is

$P(\omega q)P(\omega^2 q) \dots P(\omega^{m-1}q) =$

$$= (P_0+\omega P_1 + \dots + \omega^{m-1}P_{m-1})\times\dots\times(P_0+\omega^{m-1}P_1+\dots+\omega P_{m-1})$$

$$= \sum_{\alpha_0+\dots+\alpha_{m-1}=m-1} c(\alpha_0,\dots,\alpha_{m-1})P_0^{\alpha_0} \dots P_{m-1}^{\alpha_{m-1}} \tag{7.4.14}$$

It follows from (7.4.4), (7.4.6), (7.4.7) and (7.4.14) that

$$\phi_0+\phi_1+\ldots+\phi_{m-1} = \sum_{\alpha_0+\ldots+\alpha_{m-1}=m-1} c(\alpha_0,\ldots,\alpha_{m-1})P_0^{\alpha_0}\ldots P_{m-1}^{\alpha_{m-1}}/P \quad (7.4.15)$$

whence

$$\phi_i = \sum_{\substack{\alpha_0+\ldots+\alpha_{m-1}=m-1 \\ \alpha_1+2\alpha_2+\ldots+(m-1)\alpha_{m-1}\equiv i \bmod m}} c(\alpha_0,\ldots,\alpha_{m-1})P_0^{\alpha_0}\ldots P_{m-1}^{\alpha_{m-1}}/P \ . \quad (7.4.16)$$

Thus, for each j for which $\phi_j = 0$, we obtain a polynomial in $P_0,\ldots,P_{m-1}$, homogeneous of degree $m-1$, which, considered as a series in q, is identically zero, viz.

$$\sum_{\substack{\alpha_0+\ldots+\alpha_{m-1}=m-1 \\ \alpha_1+2\alpha_2+\ldots+(m-1)\alpha_{m-1}\equiv j \bmod m}} c(\alpha_0,\ldots,\alpha_{m-1})P_0^{\alpha_0}\ldots P_{m-1}^{\alpha_{m-1}} = 0 \ . \quad (7.4.17)$$

It is easy to check that the coefficient of $P_0^{m-2}P_j^1$ is

$$c(m-2,0,\ldots,0,1,0,\ldots 0) = \omega^j + \omega^{2j} + \ldots + \omega^{(m-1)j} \neq 0 \ , \quad (7.4.18)$$

so the polynomial is non-trivial.

Proof of Theorem 7.4.3.

Suppose m is not a power of 2. Then there is a prime p, $p\neq 2$, with $p|m$. Since $p \equiv 1 \bmod 2$, $(p,8) = 1$. As j runs through a complete set of residues mod p, so does $8j+1$. So for some j, the congruence $x^2 \equiv 8j+1 \bmod p$ has no solution. For these j, the congruence $(2r+1)^2 \equiv 8j+1 \bmod p$ has no solution, so the congruence $\frac{1}{2}r^2+\frac{1}{2}r \equiv j \bmod p$ has no solution, from which it follows that the congruence $\frac{1}{2}r^2+\frac{1}{2}r \equiv j \bmod m$ has no solution.

Now write

$$\psi = \phi^3 = \sum_{r\geq 0} (-1)^r(2r+1)q^{\frac{1}{2}r^2+\frac{1}{2}r}$$

$$= \psi_0 + \psi_1 + \ldots + \psi_{m-1} , \tag{7.4.19}$$

where ψ_i contains all those terms of ψ in which the exponent is congruent to $i \bmod m$.

For those j for which the congruence $\frac{1}{2}r^2+\frac{1}{2}r \equiv j \bmod m$ has no solutions, we have

$$\psi_j = 0 . \tag{7.4.20}$$

Now,

$$\psi_0 + \ldots + \psi_{m-1} = (\phi_0 + \ldots + \phi_{m-1})^3$$

$$= \sum_{\beta_0+\ldots+\beta_{m-1}=3} d(\beta_0,\ldots,\beta_{m-1})\phi_0^{\beta_0}\ldots\phi_{m-1}^{\beta_{m-1}} \tag{7.4.21}$$

(each coefficient $d(\beta_0,\ldots,\beta_{m-1})$ is 1,3, or 6), from which it follows that

$$\psi_i = \sum_{\substack{\beta_0+\ldots+\beta_{m-1}=3 \\ \beta_1+2\beta_2+\ldots+(m-1)\beta_{m-1} \equiv i \bmod m}} d(\beta_0,\ldots,\beta_{m-1})\phi_0^{\beta_0}\ldots\phi_{m-1}^{\beta_{m-1}} . \tag{7.4.22}$$

Thus, for those j for which the congruence $\frac{1}{2}r^2+\frac{1}{2}r \equiv j \bmod m$ has no solution, we have

$$\sum_{\substack{\beta_0+\ldots+\beta_{m-1}=3 \\ \beta_1+2\beta_2+\ldots+(m-1)\beta_{m-1} \equiv j \bmod m}} d(\beta_0,\ldots,\beta_{m-1})\phi_0^{\beta_0}\ldots\phi_{m-1}^{\beta_{m-1}} = 0. \tag{7.4.23}$$

If in (7.4.23) we substitute (7.4.16) and multiply throughout by P^3, we obtain a polynomial in $P_0,\ldots,P_{m-1}$, homogeneous of degree $3(m-1)$, which, considered as a series in q, is identically zero.

Further, the coefficient $d(2,0,\ldots,0,1,0,\ldots,0)$ of $\phi_0^2\phi_j$ in ψ_j is 3, while the coefficient $c(m-1,0,\ldots,0)$ of P_0^{m-1} in ϕ_0 is 1, and the coefficient $c(m-2,0,\ldots,0,1,0,\ldots 0)$ of $P_0^{m-2}P_j$ in ϕ_j is, as noted in (7.4.18), non-zero, so the coefficient of $P_0^{3m-4}P_j$ in the above polynomial is non-zero, so the polynomial is non-trivial.

§4. Suppose $m=3$.

The congruence $\tfrac{1}{2}r^2+\tfrac{1}{2}r \equiv 2 \bmod 3$ has no solutions, so

$$\psi_2 = 0 \ . \tag{7.5.1}$$

Now,

$$\begin{aligned}\psi_0+\psi_1+\psi_2 &= (\phi_0+\phi_1+\phi_2)^3 \\ &= (\phi_0^3+\phi_1^3+\phi_2^3+6\phi_0\phi_1\phi_2) \\ &\quad + (3\phi_0^2\phi_1+3\phi_1^2\phi_2+3\phi_2^2\phi_0) \\ &\quad + (3\phi_0\phi_1^2+3\phi_1\phi_2^2+3\phi_2\phi_0^2) \ ,\end{aligned} \tag{7.5.2}$$

so

$$\begin{aligned}\psi_0 &= \phi_0^3+\phi_1^3+\phi_2^3+6\phi_0\phi_1\phi_2 \ , \\ \psi_1 &= 3\phi_0^2\phi_1+3\phi_1^2\phi_2+3\phi_2^2\phi_0 \ , \\ \psi_2 &= 3\phi_0\phi_1^2+3\phi_1\phi_2^2+3\phi_2\phi_0^2 \ .\end{aligned} \tag{7.5.3}$$

It follows from (7.5.1) and (7.5.3) that

$$3\phi_0\phi_1^2+3\phi_1\phi_2^2+3\phi_2\phi_0^2 = 0 \ . \tag{7.5.4}$$

Further,

$$\phi_0+\phi_1+\phi_2 = (P_0+\omega P_1+\omega^2 P_2)(P_0+\omega^2 P_1+\omega P_2)/P$$

$$= [(P_0^2-P_1P_2) + (P_2^2-P_0P_1) + (P_1^2-P_0P_2)]/P \,, \tag{7.5.5}$$

so

$$\phi_0 = (P_0^2-P_1P_2)/P$$

$$\phi_1 = (P_2^2-P_0P_1)/P$$

$$\phi_2 = (P_1^2-P_0P_2)/P \quad . \tag{7.5.6}$$

It follows from (7.5.4) and (7.5.6) that

$$3(P_0^2-P_1P_2)(P_2^2-P_0P_1)^2 + 3(P_2^2-P_0P_1)(P_1^2-P_0P_2)^2 + 3(P_1^2-P_0P_2)(P_0^2-P_1P_2)^2 = 0 \,. \tag{7.5.7}$$

(Note that the coefficient of $P_0^5P_2$ is $3 \times 1^2 \times -1 = -3 \neq 0$, so the polynomial is non-trivial.)

Dividing by 3, we obtain

$$(P_0^2-P_1P_2)(P_2^2-P_0P_1)^2 + (P_2^2-P_0P_1)(P_1^2-P_0P_2)^2 + (P_1^2-P_0P_2)(P_0^2-P_1P_2)^2 = 0 \,, \tag{7.5.8}$$

which is (7.1.13).

As a second example of the foregoing, let $m=5$. The congruences $3r^2/2 \pm \frac{1}{2}r \equiv 3, 4 \bmod 5$ have no solutions so

$$\phi_3 = 0 \tag{7.5.9}$$

and

$$\phi_4 = 0 \,. \tag{7.5.10}$$

Now, $\phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 =$

$$= (P_0+\omega P_1+\omega^2P_2+\omega^3P_3+\omega^4P_4)(P_0+\omega^2P_1+\omega^4P_2+\omega P_3+\omega^3P_4)$$

$$\times (P_0+\omega^3P_1+\omega P_2+\omega^4P_3+\omega^2P_4)(P_0+\omega^4P_1+\omega^3P_2+\omega^2P_3+\omega P_4)/P \qquad (7.5.11)$$

from which it follows that

$$\phi_0 = (P_0^4 - 3P_0^2P_1P_4 - 3P_0^2P_2P_3 + 2P_0P_1^2P_3 + 2P_0P_1P_2^2 + 2P_0P_2P_4^2$$
$$+ 2P_0P_3^2P_4 - P_1^3P_2 + P_1^2P_4^2 - P_1P_2P_3P_4 - P_1P_3^3 - P_2^3P_4$$
$$+ P_2^2P_3^2 - P_3P_4^3)/P$$

$$\phi_1 = (-P_0^3P_1 + 2P_0^2P_2P_4 + 2P_0P_1^2P_4 - P_0P_1P_2P_3 - P_0P_2^3$$
$$- 3P_0P_3P_4^2 - P_1^3P_3 + P_1^2P_2^2 - 3P_1P_2P_4^2 + 2P_1P_3^2P_4$$
$$+ 2P_2^2P_3P_4 - P_2P_3^3 + P_4^4)/P$$

$$\phi_2 = (-P_0^3P_2 + P_0^2P_1^2 + 2P_0^2P_3P_4 - P_0P_1P_2P_4 - 3P_0P_1P_3^2 + 2P_0P_2^2P_3$$
$$- P_0P_4^3 - P_1^3P_4 + 2P_1^2P_2P_3 - P_1P_2^3 + 2P_1P_3P_4^2 + P_2^2P_4^2$$
$$-3P_2P_3^2P_4 + P_3^4)/P$$

$$\phi_3 = (-P_0^3P_3 + 2P_0^2P_1P_2 + P_0^2P_4^2 - P_0P_1^3 - P_0P_1P_3P_4 - 3P_0P_2^2P_4$$
$$+ 2P_0P_2P_3^2 + 2P_1^2P_2P_4 + P_1^2P_3^2 - 3P_1P_2^2P_3 - P_1P_4^3$$
$$+ P_2^4 + 2P_2P_3P_4^2 - P_3^3P_4)/P$$

and

$$\phi_4 = (-P_0^3P_4 + 2P_0^2P_1P_3 + P_0^2P_2^2 - 3P_0P_1^2P_2 + 2P_0P_1P_4^2 - P_0P_2P_3P_4$$
$$- P_0P_3^3 + P_1^4 - 3P_1^2P_3P_4 + 2P_1P_2^2P_4 + 2P_1P_2P_3^2 - P_2^3P_3$$
$$- P_2P_4^3 + P_3^2P_4^2)/P \;. \qquad (7.5.12)$$

Thus (7.5.9) becomes

$$- P_0^3P_3 + 2P_0^2P_1P_2 + P_0^2P_4^2 - P_0P_1^3 - P_0P_1P_3P_4 - 3P_0P_2^2P_4$$

$$+ 2P_0P_2P_3^2 + 2P_1^2P_2P_4 + P_1^2P_3^2 - 3P_1P_2^2P_3 - P_1P_4^3 + P_2^4$$

$$+ 2P_2P_3P_4^2 - P_3^3P_4 = 0 \tag{7.5.13}$$

while (7.5.10) becomes

$$- P_0^3P_4 + 2P_0^2P_1P_3 + P_0^2P_2^2 - 3P_0P_1^2P_2 + 2P_0P_1P_4^2 - P_0P_2P_3P_4$$

$$- P_0P_3^3 + P_1^4 - 3P_1^2P_3P_4 + 2P_1P_2^2P_4 + 2P_1P_2P^2{}_3 - P_2^3P_3$$

$$- P_2P_4^3 + P_3^2P_4^2 = 0 \quad . \tag{7.5.14}$$

6. The technique developed in §2 can be applied far more widely than has so far been indicated. Thus, for example, we can use it to prove that

$$F_{5n+4} \equiv 0 \bmod 5 \; . \tag{7.6.1}$$

We start with

$$F = F(q) = \sum_{n\geq 0} F_n q^n = 1/(1-q-q^2) = 1/\phi \quad . \tag{7.6.2}$$

Here

$$\phi = \phi_0 + \phi_1 + \phi_2 \quad , \tag{7.6.3}$$

where

$$\phi_0 = 1, \ \phi_1 = -q, \ \phi_2 = -q^2 \quad . \tag{7.6.4}$$

(Notice that $\phi_3 = \phi_4 = 0$ and $\phi_0\phi_2 = -\phi_1^2$, just as in the case of the partition function.)

As before, if $\omega^5 = 1, \omega \neq 1$,

$$F = \phi(\omega q)\phi(\omega^2 q) \; \ldots \; \phi(\omega^4 q)/\phi(q)\phi(\omega q) \; \ldots \; \phi(\omega^4 q) \; . \tag{7.6.5}$$

The denominator of (7.6.5),

$$\Phi = \phi(q)\phi(\omega q)\ \dots\ \phi(\omega^4 q)$$

$$= (1-q-q^2)(1-\omega q-\omega^2 q^2)(1-\omega^2 q-\omega^4 q^2)(1-\omega^3 q-\omega q^2)(1-\omega^4 q-\omega^3 q^2)$$

is a power series in q^5, and is easily computed to be

$$= 1 - 11q^5 - q^{10}\ . \qquad (7.6.6)$$

The numerator of (7.6.5) is, similarly,

$$\phi(\omega q)\phi(\omega^2 q)\phi(\omega^3 q)\phi(\omega^4 q) =$$

$$= (1-\omega q-\omega^2 q^2)(1-\omega^2 q-\omega^4 q^2)(1-\omega^3 q-\omega q^2)(1-\omega^4 q-\omega^3 q^2)$$

$$= 1 + q + 2q^2 + 3q^3 + 5q^4 - 3q^5 + 2q^6 - q^7 + q^8\ . \qquad (7.6.7)$$

It follows that

$$\sum_{n\geq 0} F_{5n}\, q^{5n} = (1-3q^5)/(1-11q^5-q^{10})\ ,$$

$$\sum_{n\geq 0} F_{5n+1} q^{5n+1} = (q+2q^6)/(1-11q^5-q^{10})\ ,$$

$$\sum_{n\geq 0} F_{5n+2} q^{5n+2} = (2q^2-q^7)/(1-11q^5-q^{10})\ ,$$

$$\sum_{n\geq 0} F_{5n+3} q^{5n+3} = (3q^3+q^8)/(1-11q^5-q^{10})\ ,$$

$$\sum_{n\geq 0} F_{5n+4} q^{5n+4} = 5q^4/(1-11q^5-q^{10})\ . \qquad (7.6.8)$$

In particular, it follows from (7.6.8) that

$$\sum_{n\geq 0} F_{5n+4} q^n = 5/(1-11q-q^2)\ , \qquad (7.6.9)$$

and that

$$F_{5n+4} \equiv 0 \mod 5. \qquad (7.6.10)$$

Chapter 8. The Parity of the Partition Function

§1. Whereas something is known of the arithmetic behaviour of $p(n)$ modulo 5,7 and 11 for example, (see Chapter 7), very little is known concerning the parity of $p(n)$.

Kolberg (1959) proved (see §2)

Theorem 8.1.1: $p(n)$ is even infinitely often and odd infinitely often.

In order to obtain a better picture, it is desirable to have a table of values. Since $p(n)$ grows very quickly, it is convenient to make use of the following result of P.A. MacMahon (1921), namely

Theorem 8.1.2 Modulo 2,

$$p(4n) \equiv p(n) + \sum_{k\geq 1} p(n-(8k^2-k)) + \sum_{k\geq 1} p(n-(8k^2+k))$$

$$p(4n+1) \equiv p(n) + \sum_{k\geq 1} p(n-(8k^2-3k)) + \sum_{k\geq 1} p(n-(8k^2+3k))$$

$$p(4n+3) \equiv p(n) + \sum_{k\geq 1} p(n-(8k^2-5k)) + \sum_{k\geq 1} p(n-(8k^2+5k))$$

$$p(4n+6) \equiv p(n) + \sum_{k\geq 1} p(n-(8k^2-7k)) + \sum_{k\geq 1} p(n-(8k^2+7k)) .$$

Indeed, J.R. Parker and D. Shanks (1967) computed a table for $n \leq 2 \times 10^6$ and found that $p(n)$ seeems to be distributed randomly modulo 2.

We prove the following results, to appear in Hirschhorn (1980?), which improve on Theorem 8.1.2. Here $p^*(n)$ denotes the number of partitions of n into distinct odd parts, $r(n)$ denotes the number of solutions with $n_i \geq 0$ of the equation

$$n = \Delta(n_1) + 4\Delta(n_2) + 16\Delta(n_3) + \ldots$$

where $\Delta(n) = \frac{1}{2}n(n+1)$, and $t(n) = t$ is defined by $2^t | n$, $2^{t+1} \nmid n$.

Theorem 8.1.3

$$p(n) \equiv p^*(n) \equiv r(n) \bmod 2$$

Theorem 8.1.4

$$p(n) \geq p^*(n) \geq r(n)$$

Theorem 8.1.5

$$r(0) = 1, \quad r(2) = 0, \quad \text{and for} \quad n \geq 0 ,$$

$$r(4n) = r(n) + \sum_{k\geq 1} r(n-(8k^2-k)) + \sum_{k\geq 1} r(n-(8k^2+k)),$$

$$r(4n+1) = r(n) + \sum_{k\geq 1} r(n-(8k^2-3k)) + \sum_{k\geq 1} r(n-(8k^2+3k)) ,$$

$$r(4n+3) = r(n) + \sum_{k\geq 1} r(n-(8k^2-5k)) + \sum_{k\geq 1} r(n-(8k^2+5k)) ,$$

$$r(4n+6) = r(n) + \sum_{k\geq 1} r(n-(8k^2-7k)) + \sum_{k\geq 1} r(n-(8k^2+7k)) .$$

Theorem 8.1.6

$$p(n) \equiv p^*(n) + 2 \sum_{k\geq 1} p^*(n-2k^2) \bmod 4$$

Theorem 8.1.7

$$p(n) \equiv r(n) + 2 \sum_{t(k)\text{ even}} r(n-2k^2) \bmod 4$$

Theorem 8.1.8

$$p^*(n) \equiv r(n) + 2 \sum_{t(k)\text{ even}} r(n-8k^2) \bmod 4 .$$

Clearly, Theorem 8.1.2 is a corollary of Theorems 8.1.3 and 8.1.5, while Theorem 8.1.3 is a corollary of Theorems 8.1.7 and 8.1.8.

Further, Theorem 8.1.4 greatly understates the facts. Indeed, $p(n)$ is far greater than $r(n)$. The first few values of $p(n)$, $p^*(n)$ and $r(n)$ are given in the following table:

n	0	1	2	3	4	5	6	7	8	9	10	11	12
p(n)	1	1	2	3	5	7	11	15	22	30	42	56	77
$p^*(n)$	1	1	0	1	1	1	1	1	2	2	2	2	3
r(n)	1	1	0	1	1	1	1	1	0	0	2	0	1

§2. We have

$$\sum_{n\geq 0} p(n)q^n = 1/\prod_{n\geq 1} (1-q^n) . \tag{8.2.1}$$

Therefore

$$\prod_{n\geq 1} (1-q^n) \sum_{n\geq 0} p(n)q^n = 1 . \tag{8.2.2}$$

Since (6.1.3)

$$\prod_{n\geq 1} (1-q^n) = 1 + \sum_{n\geq 1} (-1)^n(q^{\frac{1}{2}(3n^2-n)} + q^{\frac{1}{2}(3n^2+n)}) , \tag{8.2.3}$$

it follows that

$$p(n) = \sum_{k\geq 1} p(n-\tfrac{1}{2}(3k^2-k)) + \sum_{k\geq 1} p(n-\tfrac{1}{2}(3k^2+k)) \tag{8.2.4}$$

Suppose $p(n)$ is even only finitely often.
Then $p(n)$ is odd for $n \geq m$ (say) But if $n = \frac{1}{2}(3m^2+m)+m$
we have

$$\begin{aligned} p(n) &= p(n-1) + p(n-2) -- ++ \ldots. \\ &\quad - (-1)^m(p(n-\tfrac{1}{2}(3m^2-m)) + p(n-\tfrac{1}{2}(3m^2+m))) \\ &= p(n-1) + p(n-2) -- ++ \ldots.. \\ &\quad - (-1)^m(p(2m) + p(m)) \end{aligned}$$

which is even, a contradiction.

On the other hand, suppose $p(n)$ is odd only finitely often. Then $p(n)$ is even for $n \geq m$ (say). But if $n = \frac{1}{2}(3m^2+m)$,

$$p(n) = p(n-1) + p(n-2) -- ++ \ldots.$$

$$+ (-1)^m((p(n-\tfrac{1}{2}(3m^2-m)) + p(n-\tfrac{1}{2}(3m^2+m))$$

$$= p(n-1) + p(n-2) -- ++ \ldots$$

$$+ (-1)^m(p(m) + p(0)),$$

which is odd, again a contradiction.

Thus Theorem 8.1.1 is proved.

§3. We have

$$\sum_{n\geq 0} p(n)q^n = 1/\prod_{n\geq 1} (1 - q^n)$$

$$= \prod_{n\geq 1} (1 + q^n)/\prod_{n\geq 1} (1 - q^{2n})$$

$$= \prod_{n\geq 1} (1+q^{2n-1}) \cdot \prod_{n\geq 1} \left(\frac{1 + q^{2n}}{1 - q^{2n}}\right)$$

$$= \sum_{n\geq 0} p^*(n)\, q^n \cdot \prod_{n\geq 1} \left(\frac{1 + q^{2n}}{1 - q^{2n}}\right) \tag{8.3.1}$$

$$\equiv \sum_{n\geq 0} p^*(n)q^n \mod 2 \tag{8.3.2}$$

since

$$\frac{1 + q^{2n}}{1 - q^{2n}} = 1 + \frac{2q^{2n}}{1 - q^{2n}} \equiv 1 \mod 2 . \tag{8.3.3}$$

It follows from (8.3.2) that

$$p(n) \equiv p^*(n) \mod 2. \tag{8.3.4}$$

Further,

$$\sum_{n\geq 0} p^*(n)q^n = \prod_{n\geq 1} (1+q^{2n-1})$$

$$= \prod_{n\geq 1} (1+q^{4n-3})(1+q^{4n-1})$$

which by (6.1.3)

$$= \{1 + \sum_{n\geq 1} (q^{2n^2-n}+q^{2n^2+n})\} / \prod_{n\geq 1} (1-q^{4n})$$

$$= \sum_{n\geq 0} q^{\Delta(n)} / \prod_{n\geq 1} (1-q^{4n})$$

$$= \sum_{n\geq 0} q^{\Delta(n)} \sum_{n\geq 0} p(n)\, q^{4n} \tag{8.3.5}$$

which by (8.3.4)

$$\equiv \sum_{n\geq 0} q^{\Delta(n)} \sum_{n\geq 0} p^*(n)q^{4n} \mod 2. \tag{8.3.6}$$

It follows by iteration of (8.3.6) that, mod 2,

$$\sum_{n\geq 0} p^*(n)q^n \equiv \sum_{n\geq 0} q^{\Delta(n)} \sum_{n\geq 0} q^{4\Delta(n)} \sum_{n\geq 0} q^{16\Delta(n)} \; \dots\dots$$

$$= \sum_{n\geq 0} r(n)q^n , \tag{8.3.7}$$

so

$$p^*(n) \equiv r(n) \mod 2 . \tag{8.3.8}$$

(8.3.4) and (8.3.8) constitute Theorem 8.1.3.

Clearly $p(n) \geq p^*(n)$. (8.3.9)

Let us write

$$\sum_{n\geq 0} a_n q^n \geq \sum_{n\geq 0} b_n q^n$$

if

$$a_n \geq b_n \quad \text{for every} \quad n \; . \tag{8.3.10}$$

We have by (8.3.5) and (8.3.9) that

$$\begin{aligned}\sum_{n\geq 0} p^*(n)q^n &= \sum_{n\geq 0} q^{\Delta(n)} \sum_{n\geq 0} p(n)q^{4n} \\ &\geq \sum_{n\geq 0} q^{\Delta(n)} \sum_{n\geq 0} p^*(n)q^{4n} \; . \end{aligned} \tag{8.3.11}$$

It follows by iteration of (8.3.11) that

$$\sum_{n\geq 0} p^*(n)q^n \geq \sum_{n\geq 0} r(n)q^n \tag{8.3.12}$$

or

$$p^*(n) \geq r(n) \; . \tag{8.3.13}$$

(8.3.9) and (8.3.13) constitute Theorem 8.1.4.

We have

$$\begin{aligned}\sum_{n\geq 0} r(n)q^n &= \sum_{n\geq 0} q^{\Delta(n)} \sum_{n\geq 0} q^{4\Delta(n)} \sum_{n\geq 0} q^{16\Delta(n)} \cdots \\ &= \sum_{n\geq 0} q^{\Delta(n)} \sum_{n\geq 0} r(n)q^{4n} \; . \end{aligned} \tag{8.3.14}$$

If we now substitute

$$\begin{aligned}\sum_{n\geq 0} q^{\Delta(n)} &= \Big(\sum_{n\geq 0} q^{\Delta(8n)} + q^{\Delta(8n+7)}\Big) + \Big(\sum_{n\geq 0} q^{\Delta(8n+1)} + q^{\Delta(8n+6)} \Big) \\ &\quad + \Big(\sum_{n\geq 0} q^{\Delta(8n+2)} + q^{\Delta(8n+5)}\Big) + \Big(\sum_{n\geq 0} q^{\Delta(8n+3)} + q^{\Delta(8n+4)} \Big)\end{aligned}$$

$$= \{1+\sum_{n\geq 1}(q^{32n^2-4n}+q^{32n^2+4n})\} + q\{1+\sum_{n\geq 1}(q^{32n^2-12n}+q^{32n^2+12n})\}$$

$$+q^3\{1+\sum_{n\geq 1}(q^{32n^2-20n}+q^{32n^2+20n})\}+q^6\{1+\sum_{n\geq 1}(q^{32n^2-28n}+q^{32n^2+28n})\} \tag{8.3.15}$$

into (8.3.14), and compare coefficients, we obtain Theorem 8.1.5.

We have (8.3.1)

$$\sum_{n\geq 0} p(n)q^n = \sum_{n\geq 0} p^*(n)q^n \cdot \prod_{n\geq 1}\left(\frac{1+q^{2n}}{1-q^{2n}}\right)$$

$$= \sum_{n\geq 0} p^*(n)q^n / \prod_{n\geq 1}\left(\frac{1-q^{2n}}{1+q^{2n}}\right)$$

$$= \sum_{n\geq 0} p^*(n)q^n / \prod_{n\geq 1}(1-q^{2n})(1-q^{4n-2})$$

$$= \sum_{n\geq 0} p^*(n)q^n / \prod_{n\geq 1}(1-q^{4n-2})^2(1-q^{4n})$$

which, by (6.1.2)

$$= \sum_{n\geq 0} p^*(n)q^n / \{1+2\sum_{n\geq 1}(-1)^n q^{2n^2}\}$$

$$\equiv \sum_{n\geq 0} p^*(n)q^n \times \{1+2\sum_{n\geq 1} q^{2n^2}\} \bmod 4 \tag{8.3.16}$$

since

$$\{1+2\sum_{n\geq 1} q^{2n^2}\}\{1+2\sum_{n\geq 1}(-1)^n q^{2n^2}\} \equiv 1+4\sum_{n\geq 1} q^{8n^2} \equiv 1 \bmod 4 . \tag{8.3.17}$$

Theorem 8.1.6 follows from (8.3.16).

Modulo 4, we have by (8.3.16) and (8.3.5),

$$\sum_{n\geq 0} p(n)q^n \equiv \{1 + 2 \sum_{n\geq 1} q^{2n^2}\} \sum_{n\geq 0} p^*(n)q^n \mod 4$$

$$= \{1 + 2 \sum_{n\geq 1} q^{2n^2}\} \sum_{n\geq 0} q^{\Delta(n)} \sum_{n\geq 0} p(n)q^{4n} \quad . \tag{8.3.18}$$

It follows by iteration of (8.3.18) that, mod 4,

$$\sum_{n\geq 0} p(n)q^n \equiv \{1 + 2 \sum_{n\geq 1} q^{2n^2}\}\{1 + 2 \sum_{n\geq 1} q^{8n^2}\}\ldots \times \sum_{n\geq 0} r(n)q^n$$

$$\equiv \{1 + 2 \left(\sum_{n\geq 1} q^{2n^2} + \sum_{n\geq 1} q^{8n^2} + \ldots\right)\} \times \sum_{n\geq 0} r(n)q^n$$

$$= \{1 + 2 \sum_{n\geq 1} (t(n)+1)q^{2n^2}\} \times \sum_{n\geq 0} r(n)q^n$$

$$\equiv \{1 + 2 \sum_{t(n)\text{ even}} q^{2n^2}\} \sum_{n\geq 0} r(n)q^n \tag{8.3.19}$$

from which Theorem 8.1.7 follows.

Finally, we have from (8.3.5) and (8.3.19) that, mod 4,

$$\sum_{n\geq 0} p^*(n)q^n = \sum_{n\geq 0} q^{\Delta(n)} \sum_{n\geq 0} p(n)q^{4n}$$

$$\equiv \sum_{n\geq 0} q^{\Delta(n)} \{1 + 2 \sum_{t(n)\text{ even}} q^{8n^2}\} \sum_{n\geq 0} r(n)q^{4n}$$

$$= \{1 + 2 \sum_{t(n)\text{ even}} q^{8n^2}\} \sum_{n\geq 0} r(n)q^n , \tag{8.3.20}$$

from which Theorem 8.1.8 follows.

Chapter 9. Some Partition Theorems of the Rogers-Ramanujan Type

§1. It is well known that the identity

$$\sum_{r\geq 0} \frac{q^{\frac{1}{2}(r^2+r)}}{(q)_r} = 1/(q;q^2)_\infty \tag{9.1.1}$$

is equivalent to Euler's partition theorem

Theorem 9.1.2

The number of partitions of n into distinct parts is equal to the number of partitions of n into odd parts, while the Rogers-Ramanujan identities

$$\sum_{r\geq 0} \frac{q^{r^2}}{(q)_r} = 1/(q;q^5)_\infty(q^4;q^5)_\infty \tag{9.1.3}$$

and

$$\sum_{r\geq 0} \frac{q^{r^2+r}}{(q)_r} = 1/(q^2;q^5)_\infty(q^3;q^5)_\infty \tag{9.1.4}$$

are, as first realised by MacMahon (1916)Art.276, equivalent, respectively, to the partition theorems

Theorem 9.1.5

The number of partitions of n into parts which differ by 2 is equal to the number of partitions of n into parts congruent to 1 or 4 mod 5,

and

Theorem 9.1.6

The number of partitions of n into parts which differ by 2, but with no 1's, is equal to the number of partitions of n into parts congruent to 2 or 3 mod 5.

L.J. Slater (1951) gave a list of 130 identities involving q-series and infinite products, some of which other than those already considered can be interpreted as results about partitions. Thus for example, the four identities, Slater (loc.cit.) [79]=[98], [94], [38]=[86] and [39]=[83], proved in §§3,4,

$$\sum_{r\geq 0} \frac{q^{r^2}}{(q)_{2r}} = 1/\{(q;q^{20})_\infty(q^3;q^{20})_\infty(q^4;q^{20})_\infty(q^5;q^{20})_\infty(q^7;q^{20})_\infty(q^9;q^{20})_\infty \cdot(q^{11};q^{20})_\infty(q^{13};q^{20})_\infty(q^{15};q^{20})_\infty(q^{16};q^{20})_\infty(q^{17};q^{20})_\infty(q^{19};q^{20})_\infty\} \tag{9.1.7}$$

$$\sum_{r\geq 0} \frac{q^{r^2+r}}{(q)_{2r+1}} = 1/\{(q;q^{20})_\infty(q^2;q^{20})_\infty(q^5;q^{20})_\infty(q^6;q^{20})_\infty(q^8;q^{20})_\infty(q^9;q^{20})_\infty \cdot(q^{11};q^{20})_\infty(q^{12};q^{20})_\infty(q^{14};q^{20})_\infty(q^{15};q^{20})_\infty(q^{18};q^{20})_\infty(q^{19};q^{20})_\infty\} \tag{9.1.8}$$

$$\sum_{r\geq 0} \frac{q^{2r^2+2r}}{(q)_{2r+1}} = 1/\{(q;q^{16})_\infty(q^4;q^{16})_\infty(q^6;q^{16})_\infty(q^7;q^{16})_\infty \cdot (q^9;q^{16})_\infty(q^{10};q^{16})_\infty(q^{12};q^{16})_\infty(q^{15};q^{16})_\infty\} \tag{9.1.9}$$

and

$$\sum_{r\geq 0} \frac{q^{2r^2}}{(q)_{2r}} = 1/\{(q^2;q^{16})_\infty(q^3;q^{16})_\infty(q^4;q^{16})_\infty(q^5;q^{16})_\infty \cdot (q^{11};q^{16})_\infty(q^{12};q^{16})_\infty(q^{13};q^{16})_\infty(q^{14};q^{16})_\infty\} \tag{9.1.10}$$

yield, respectively, as we prove in §2,

Theorem 9.1.11.

The number of partitions of n,

$$n = a_1 + a_2 + a_3 + \dots .$$

with

$$a_1 > a_2 \geq a_3 > a_4 \geq a_5 > \dots .$$

is equal to the number of partitions of n into parts congruent to 1, 3, 4, 5, 7, 9, 11, 13, 15, 16, 17 or 19 mod 20, Theorem 9.1.12

The number of partitions of n,

$$n = a_1 + a_2 + a_3 + \ldots.$$

with

$$a_1 \geq a_2 > a_3 \geq a_4 > a_5 \geq \ldots.$$

is equal to the number of partitions of n into parts congruent to 1, 2, 5, 6, 8, 9, 11, 12, 14, 15, 18 or 19 mod 20,

Theorem 9.1.13

The number of partitions of n,

$$n = a_1 + a_2 + a_3 + \ldots.$$

with

$$a_2 - a_1 \geq 2, \quad a_3 - a_2 \geq 0, \quad a_4 - a_3 \geq 2, \quad a_5 - a_4 \geq 0, \ldots.$$

is equal to the number of partitions of n into parts congruent to 1, 4, 6, 7, 9, 10, 12 or 15 mod 16, and

Theorem 9.1.14

The number of partitions of n,

$$n = a_1 + a_2 + a_3 + \ldots.$$

with

$$a_1 \geq 2, \ a_2 - a_1 \geq 0, \ a_3 - a_2 \geq 2, \ a_4 - a_3 \geq 0, \ a_5 - a_4 \geq 2, \ldots.$$

is equal to the number of partitions of n into parts congruent to 2, 3, 4, 5, 11, 12, 13 or 14 mod 16.

Whereas Theorems 9.1.11 and 9.1.12 have appeared previously in the literature (Gordon (1965), W. Connor (1975)), Theorems 9.1.13 and 9.1.14 are new. All four are to appear in Hirschhorn (1979?a).

Andrews has pointed out that, as we show in §5, the left hand side of (9.1.9) enumerates the number of partitions of $2n+1$ into distinct odd parts, while the left-hand side of (9.1.10) enumerates the number of partitions of $2n$ into distinct odd parts, giving further partition theorems.

§2. We start by proving Theorem 9.1.11. Let $p_1(n)$ denote the number of partitions of n,

$$n = a_1 + a_2 + a_3 + \dots.$$

with $$a_1 > a_2 \geq a_3 > a_4 \geq a_5 > \dots\quad .$$

Any such partition of n has for some $r \geq 1$, $2r-1$ or $2r$ parts.

Suppose $n = a_1 + a_2 + a_3 + \dots + a_{2r-1}$

with $a_1 > a_2 \geq a_3 > \dots \geq a_{2r-1}$.

Then

$$a_{2r-1} \geq 1,\ a_{2r-2} \geq 1,\ a_{2r-3} \geq 2,\ \dots,\ a_2 \geq r-1,\ a_1 \geq r\ .$$

If we subtract

1 from a_{2r-1}, 1 from a_{2r-2}, 2 from a_{2r-3}, ... r from a_1,

there remains a partition of $n-r^2$ into at most $2r-1$ parts, and this process is reversible, giving a one-to-one correspondence between the two sets of partitions.

Similarly, if $n = a_1 + a_2 + a_3 + \dots + a_{2r}$

with $a_1 > a_2 \geq a_3 > \dots > a_{2r}$,

then $a_{2r} \geq 1,\ a_{2r-1} \geq 2,\ a_{2r-2} \geq 2,\ \dots,\ a_2 \geq r,\ a_1 \geq r+1,$

and if we subtract

1 from a_{2r}, 2 from a_{2r-1}, 2 from a_{2r-2}, ..., r+1 from a_1 ,

the remains a partition of $n-(r^2+2r)$ into at most $2r$ parts, and again the process is reversible, giving a one-to one correspondence.

Thus

$$1+\sum_{n\geq 1} p_1(n)q^n = 1+\sum_{r\geq 1}\frac{q^{r^2}}{(q)_{2r-1}}+\sum_{r\geq 1}\frac{q^{r^2+2r}}{(q)_{2r}}$$

$$= 1+\sum_{r\geq 1}\left(\frac{q^{r^2}}{(q)_{2r-1}}+\frac{q^{r^2+2r}}{(q)_{2r}}\right)$$

$$= \sum_{r\geq 0}\frac{q^{r^2}}{(q)_{2r}}\ , \qquad (9.2.1)$$

which, by (9.1.7), yields Theorem 9.1.11.

The proofs of Theorems 9.1.12-9.1.14 are similar.

Thus, if $p_2(n)$ denotes the number of partitions of n,

$$n = a_1 + a_2 + a_3 + \ldots .$$

with $$a_1 \geq a_2 > a_3 \geq a_4 > a_5 \geq \ \ldots .$$

then

$$1+\sum_{n\geq 1} p_2(n)q^n = 1+\sum_{r\geq 1}\frac{q^{r^2+r-1}}{(q)_{2r-1}}+\sum_{r\geq 1}\frac{q^{r^2+r}}{(q)_{2r}}$$

$$= \left(1+\frac{q}{(q)_1}\right)+\sum_{r\geq 1}\left(\frac{q^{r^2+r}}{(q)_{2r}}+\frac{q^{r^2+3r+1}}{(q)_{2r+1}}\right)$$

$$= \sum_{r\geq 0}\frac{q^{r^2+r}}{(q)_{2r+1}}\ , \qquad (9.2.2)$$

which, by (9.1.8), yields Theorem 9.1.12.

If $p_3(n)$ denotes the number of partitions of n ,

$$n = a_1 + a_2 + a_3 + \ldots .$$

with

$$a_2 - a_1 \geq 2,\ a_3 - a_2 \geq 0,\ a_4 - a_3 \geq 2,\ a_5 - a_4 \geq 0,\ \ldots\ ,$$

then

$$\begin{aligned} 1 + \sum_{n\geq 1} p_3(n)q^n &= 1 + \sum_{r\geq 1} \frac{q^{2r^2-1}}{(q)_{2r-1}} + \sum_{r\geq 1} \frac{q^{2r^2+2r}}{(q)_{2r}} \\ &= \left(1 + \frac{q}{(q)_1}\right) + \sum_{r\geq 1} \left(\frac{q^{2r^2+2r}}{(q)_{2r}} + \frac{q^{2r^2+4r+1}}{(q)_{2r+1}}\right) \\ &= \sum_{r\geq 0} \frac{q^{2r^2+2r}}{(q)_{2r+1}}\ , \end{aligned} \tag{9.2.3}$$

which, by (9.1.9), yields Theorem 9.1.13.

Finally, if $p_4(n)$ denotes the number of partitions of n,

$$n = a_1 + a_2 + a_3 + \ldots .$$

with

$$a_1 \geq 2,\ a_2 - a_1 \geq 0,\ a_3 - a_2 \geq 2,\ a_4 - a_3 \geq 0,\ a_5 - a_4 \geq 2\ ,\ \ldots\ ,$$

then

$$\begin{aligned} 1 + \sum_{n\geq 1} p_4(n)q^n &= 1 + \sum_{r\geq 1} \frac{q^{2r^2}}{(q)_{2r-1}} + \sum_{r\geq 1} \frac{q^{2r^2+2r}}{(q)_{2r}} \\ &= 1 + \sum_{r\geq 1} \left(\frac{q^{2r^2}}{(q)_{2r-1}} + \frac{q^{2r^2+2r}}{(q)_{2r}}\right) \\ &= \sum_{r\geq 0} \frac{q^{2r^2}}{(q)_{2r}}\ , \end{aligned} \tag{9.2.4}$$

which, by (9.1.10), yields Theorem 9.1.14.

§3. In order to prove (9.1.7) we first establish the identity

$$(a)_\infty \sum_{r\ge 0} \frac{q^{\frac12(r^2-r)}a^r}{(q)_r(a)_r} = \sum_{r\ge 0} \frac{q^{2r^2-r}a^{2r}}{(q^2;q^2)_r} . \qquad (9.3.1)$$

Thus,

$$(a)_\infty \sum_{r\ge 0} \frac{q^{\frac12(r^2-r)}a^r}{(q)_r(a)_r} = \sum_{r\ge 0} \frac{q^{\binom{r}{2}}a^r(aq^r)_\infty}{(q)_r}$$

$$= \sum_{r\ge 0} \frac{q^{\binom{r}{2}}a^r}{(q)_r} \sum_{s\ge 0} \frac{(-1)^s(aq^r)^s q^{\binom{s}{2}}}{(q)_s}$$

$$= \sum_{r,s\ge 0} \frac{a^{r+s}(-1)^s q^{\binom{r+s}{2}}}{(q)_r(q)_s}$$

$$= \sum_{t\ge 0} \frac{a^t q^{\binom{t}{2}}}{(q)_t} \sum_{s\ge 0}^{t} (-1)^s \begin{bmatrix} t \\ s \end{bmatrix}$$

$$= \sum_{t \text{ even}} \frac{a^t q^{\binom{t}{2}}}{(q)_t} \cdot \frac{(q)_t}{(q^2;q^2)_{t/2}} \qquad \text{(see App'x §6)}$$

$$= \sum_{r\ge 0} \frac{a^{2r} q^{\frac12(2r)(2r-1)}}{(q^2;q^2)_r} ,$$

which is (9.3.1).

If in (9.3.1) we set q^2 for q, $a=q$ and use the first Rogers-Ramanujan identity, we obtain

$$(q;q^2)_\infty \sum_{r\ge 0} \frac{q^{r^2}}{(q)_{2r}} = \sum_{r\ge 0} \frac{q^{4r^2}}{(q^4;q^4)_r}$$

$$= 1/(q^4;q^{20})_\infty(q^{16};q^{20})_\infty , \qquad (9.3.2)$$

so

$$\sum_{r\geq 0} \frac{q^{r^2}}{(q)_{2r}} = 1/(q;q^2)_\infty (q^4;q^{20})_\infty (q^{16};q^{20})_\infty$$

$$= 1/\{(q;q^{20})_\infty (q^3;q^{20})_\infty (q^4;q^{20})_\infty (q^5;q^{20})_\infty (q^7;q^{20})_\infty (q^9;q^{20})_\infty$$
$$(q^{11};q^{20})_\infty (q^{13};q^{20})_\infty (q^{15};q^{20})_\infty (q^{16};q^{20})_\infty (q^{17};q^{20})_\infty (q^{19};q^{20})_\infty\} ,$$

which is (9.1.7).

In order to prove (9.1.8) we first establish the identity

$$(-a;q^2)_\infty \sum_{r\geq 0} \frac{q^{r^2} a^r}{(q^2;q^2)_r (-a;q^2)_{r+1}} = \sum_{r\geq 0} \frac{q^{r^2} a^r}{(q)_r} . \quad (9.3.3)$$

Thus,

$$(-a;q^2)_\infty \sum_{r\geq 0} \frac{q^{r^2} a^r}{(q^2;q^2)_r (-a;q^2)_{r+1}} =$$

$$= \sum_{r\geq 0} \frac{q^{r^2} a^r}{(q^2;q^2)_r} (-aq^{2r+2};q^2)_\infty$$

$$= \sum_{r\geq 0} \frac{q^{r^2} a^r}{(q^2;q^2)_r} \sum_{s\geq 0} \frac{(aq^{2r})^s q^{s^2+s}}{(q^2;q^2)_s}$$

$$= \sum_{r,s\geq 0} \frac{a^{r+s} q^s q^{(r+s)^2}}{(q^2;q^2)_r (q^2;q^2)_s}$$

$$= \sum_{t\geq 0} \frac{a^t q^{t^2}}{(q^2;q^2)_t} \sum_{s=0}^{t} q^s \begin{bmatrix} t \\ s \end{bmatrix}_{(q^2)}$$

$$= \sum_{t\geq 0} \frac{a^t q^{t^2}}{(q^2;q^2)_t} \cdot \frac{(q^2;q^2)_t}{(q)_t} \quad \text{(see App'x §6)}$$

$$= \sum_{t\geq 0} \frac{a^t q^{t^2}}{(q)_t} \quad ,$$

which is (9.3.3).

If in (9.3.3) we set $a=q$, and use the second Rogers-Ramanujan identity, we obtain

$$(-q;q^2)_\infty \sum_{r\geq 0} \frac{q^{r^2+r}}{(q^2;q^2)_r(-q;q^2)_{r+1}} = \sum_{r\geq 0} \frac{q^{r^2+r}}{(q)_r}$$

$$= 1/(q^2;q^5)_\infty(q^3;q^5)_\infty \quad , \tag{9.3.4}$$

so

$$\sum_{r\geq 0} \frac{q^{r^2+r}}{(q^2;q^2)_r(-q;q^2)_{r+1}} = 1/(-q;q^2)_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty$$

$$= 1/\{(-q;q^{10})_\infty(-q^3;q^{10})_\infty(-q^5;q^{10})_\infty(-q^7;q^{10})_\infty(-q^9;q^{10})_\infty \cdot$$
$$\cdot (q^2;q^{10})_\infty(q^7;q^{10})_\infty(q^3;q^{10})_\infty(q^8;q^{10})_\infty\}$$

$$= 1/\{(-q;q^{10})_\infty(q^2;q^{10})_\infty(-q^5;q^{10})_\infty(q^8;q^{10})_\infty(-q^9;q^{10})_\infty \cdot$$
$$\cdot (q^6;q^{20})_\infty(q^{14};q^{20})_\infty\}$$

$$= 1/\{(-q;q^{20})_\infty(q^2;q^{20})_\infty(-q^5;q^{20})_\infty(q^6;q^{20})_\infty(q^8;q^{20})_\infty(-q^9;q^{20})_\infty \cdot$$
$$\cdot (-q^{11};q^{20})_\infty(q^{12};q^{20})_\infty(q^{14};q^{20})_\infty(-q^{15};q^{20})_\infty(q^{18};q^{20})_\infty(-q^{19};q^{20})_\infty\}$$

$$\tag{9.3.5}$$

If in (9.3.5) we put $-q$ for q, we obtain

$$\sum_{r\geq 0} \frac{q^{r^2+r}}{(q)_{2r+1}} =$$

$$= 1/\{(q;q^{20})_\infty (q^2;q^{20})_\infty (q^5;q^{20})_\infty (q^6;q^{20})_\infty (q^8;q^{20})_\infty (q^9;q^{20})_\infty \,.$$

$$. (q^{11},q^{20})_\infty (q^{12};q^{20})_\infty (q^{15};q^{20})_\infty (q^{16};q^{20})_\infty (q^{18};q^{20})_\infty (q^{19};q^{20})_\infty\} ,$$

which is (9.1.8) .

§4. We now prove (9.1.9) and (9.1.10)

$$\sum_{r\geq 0} \frac{q^{2r^2+2r}}{(q)_{2r+1}} = \sum_{r\geq 0} \frac{q^{\frac12(r^2-1)}}{(q)_r} \cdot \left\{ \frac{1-(-1)^r}{2} \right\}$$

$$= q^{-\frac12} \cdot \frac{1}{2} \left\{ \sum_{r\geq 0} \frac{q^{\frac12(r^2-r)}(q^{\frac12})^r}{(q)_r} - \sum_{r\geq 0} (-1)^r \frac{q^{\frac12(r^2-r)}(q^{\frac12})^r}{(q)_r} \right\}$$

$$= q^{-\frac12} \cdot \frac{1}{2} \left\{ \prod_{r\geq 0} (1+q^{r+\frac12}) - \prod_{r\geq 0} (1-q^{r+\frac12}) \right\}$$

$$= q^{-\frac12} \cdot \frac{1}{2} \left\{ \prod_{r\geq 1} (1+q^{2r-\frac32})(1+q^{2r-\frac12}) - \prod_{r\geq 0} (1-q^{2r-\frac32})(1-q^{2r-\frac12}) \right\}$$

$$= q^{-\frac12} \frac{1}{\prod_{r\geq 1}(1-q^{2r})} \cdot \frac{1}{2} \left\{ \sum_{r=-\infty}^{\infty} q^{r^2+\frac12 r} - \sum_{r=-\infty}^{\infty} (-1)^r q^{r^2+\frac12 r} \right\}$$

by (6.1.3) and (6.1.2)

$$= q^{-\frac12} \frac{1}{(q^2;q^2)_\infty} \sum_{r=-\infty}^{\infty} q^{r^2+\frac12 r} \left\{ \frac{1-(-1)^r}{2} \right\}$$

$$= q^{-\frac12} \frac{1}{(q^2;q^2)} \sum_{r=-\infty}^{\infty} q^{(2r-1)^2+\frac12(2r-1)}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{r=-\infty}^{\infty} q^{4r^2-3r}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{r=-\infty}^{\infty} q^{4r^2+3r}$$

$$= \frac{1}{(q^2;q^2)_\infty} \prod_{r\geq 1} (1+q^{8r-7})(1+q^{8r-1})(1-q^{8r})$$

by (6.1.3)

$$= \frac{1}{(q^2;q^2)_\infty} (-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty$$

$$= \frac{1}{(q^2;q^2)_\infty} \frac{(q^2;q^{16})_\infty}{(q;q^8)_\infty} \frac{(q^{14};q^{16})_\infty}{(q^7;q^8)_\infty} (q^8;q^8)_\infty$$

$$= \{(q^2;q^{16})_\infty(q^8;q^{16})_\infty(q^{14};q^{16})_\infty(q^{16};q^{16})_\infty\}/$$

$$/\{(q;q^{16})_\infty(q^2;q^{16})_\infty(q^4;q^{16})_\infty(q^6;q^{16})_\infty(q^7;q^{16})_\infty(q^8;q^{16})_\infty$$

$$.(q^9;q^{16})_\infty(q^{10};q^{16})_\infty(q^{12};q^{16})_\infty(q^{14};q^{16})_\infty(q^{15};q^{16})_\infty(q^{16};q^{16})_\infty\}$$

$$= 1/\{(q;q^{16})_\infty(q^4;q^{16})_\infty(q^6;q^{16})_\infty(q^7;q^{16})_\infty\ .$$

$$.\ (q^9;q^{16})_\infty(q^{10};q^{16})_\infty(q^{12};q^{16})_\infty(q^{15};q^{16})_\infty\}$$

which is (9.1.9), while

$$\sum_{r\geq 0} \frac{q^{2r^2}}{(q)_{2r}} = \sum_{r\geq 0} \frac{q^{\frac{1}{2}r^2}}{(q)_r} \left\{\frac{1+(-1)^r}{2}\right\}$$

$$= \frac{1}{2} \left\{ \sum_{r\geq 0} \frac{q^{\frac{1}{2}(r^2-r)}(q^{\frac{1}{2}})^r}{(q)_r} + \sum_{r\geq 0} (-1)^r \frac{q^{\frac{1}{2}(r^2-r)}(q^{\frac{1}{2}})^r}{(q)_r} \right\}$$

$$= \frac{1}{2}\left\{ \prod_{r\geq 0} (1+q^{r+\frac{1}{2}}) + \prod_{r\geq 0} (1-q^{r+\frac{1}{2}})\right\}$$

$$= \frac{1}{2} \left\{ \prod_{r\geq 1} (1+q^{2r-\frac{3}{2}})(1+q^{2r-\frac{1}{2}}) + \prod_{r\geq 1} (1-q^{2r-\frac{3}{2}})(1-q^{2r-\frac{1}{2}})\right\}$$

$$= \frac{1}{\prod_{r\geq 1}(1-q^{2r})} \cdot \frac{1}{2} \left\{ \sum_{r=-\infty}^{\infty} q^{r^2+\frac{1}{2}r} + \sum_{r=-\infty}^{\infty} (-1)^r q^{r^2+\frac{1}{2}r} \right\}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{r=-\infty}^{\infty} q^{r^2+\frac{1}{2}r} \left\{ \frac{1+(-1)^r}{2} \right\}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{r=-\infty}^{\infty} q^{4r^2+r}$$

$$= \frac{1}{(q^2;q^2)_\infty} \cdot \prod_{r\geq 1} (1+q^{8r-5})(1+q^{8r-3})(1-q^{8r})$$

$$= \frac{1}{(q^2;q^2)_\infty} (-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty$$

$$= \frac{1}{(q^2;q^2)_\infty} \frac{(q^6;q^{16})_\infty}{(q^3;q^8)_\infty} \frac{(q^{10};q^{16})_\infty}{(q^5;q^8)_\infty} (q^8;q^8)_\infty$$

$$= \{(q^6;q^{16})_\infty(q^8;q^{16})_\infty(q^{10};q^{16})_\infty(q^{16};q^{16})_\infty\}/$$

$$/\{(q^2;q^{16})_\infty(q^3;q^{16})_\infty(q^4;q^{16})_\infty(q^5;q^{16})_\infty(q^6;q^{16})_\infty(q^8;q^{16})_\infty \ .$$

$$.\ (q^{10};q^{16})_\infty(q^{11};q^{16})_\infty(q^{12};q^{16})_\infty(q^{13};q^{16})_\infty(q^{14};q^{16})_\infty(q^{16};q^{16})_\infty\}$$

$$= 1/\{(q^2;q^{16})_\infty(q^3;q^{16})_\infty(q^4;q^{16})_\infty(q^5;q^{16})_\infty \ .$$

$$.\ (q^{11};q^{16})_\infty(q^{12};q^{16})_\infty(q^{13};q^{16})_\infty(q^{14};q^{16})_\infty\}$$

which is (9.1.10).

§5. We now justify the remarks made at the end of §1.

A partition of $2n+1$ into distinct odd parts must for some $r \geq 0$ contain $2r+1$ parts. If we subtract $1,3,5,\ldots,4r+1$ respectively from these parts, there remains a partition of $(2n+1) - (4r^2+4r+1)$ into at most $2r+1$ even parts. Thus, if $p^*(n)$ denotes the number of partitions of $2n+1$ into distinct odd parts, we have

$$\sum_{n\geq 0} p^*(n)q^{2n+1} = \sum_{r\geq 0} \frac{q^{4r^2+4r+1}}{(q^2;q^2)_{2r+1}}, \tag{9.5.1}$$

from which it follows that

$$\sum_{n\geq 0} p^*(n)q^{n} = \sum_{r\geq 0} \frac{q^{2r^2+2r}}{(q)_{2r+1}}, \tag{9.5.2}$$

as asserted.

Similarly, if $p^{**}(n)$ denotes the number of partitions of $2n$ into distinct odd parts, then

$$\sum_{n\geq 0} p^{**}(n)q^{2n} = \sum_{r\geq 0} \frac{q^{4r^2}}{(q^2;q^2)_{2r}}, \tag{9.5.3}$$

or,

$$\sum_{n\geq 0} p^{**}(n)q^{n} = \sum_{r\geq 0} \frac{q^{2r^2}}{(q)_{2r}}, \tag{9.5.4}$$

again as asserted.

Chapter 10. A Continued Fraction of Ramanujan

§1. In 1976, G.E. Andrews discovered a manuscript of Ramanujan (1920?) containing more than six hundred identities. (For the interesting details of this discovery, see Andrews (1979).) One of these identities concerns the continued fraction

$$F(a,b,\lambda,q) = 1 + \frac{aq+\lambda q}{1+}\ \frac{bq+\lambda q^2}{1+}\ \frac{aq^2+\lambda q^3}{1+}\ \frac{bq^2+\lambda q^4}{1+}\ \cdots . \tag{10.1.1}$$

Ramanujan states without proof that

$$F(a,b,\lambda,q) = G(a,b,\lambda)/G(aq,b,\lambda q) \tag{10.1.2}$$

where

$$G(a,b,\lambda) = \sum_{n\geq 0} \frac{q^{\frac{1}{2}(n^2+n)}(a+\lambda)\ldots(a+\lambda q^{n-1})}{(1-q)\ldots(1-q^n)(1+bq)\ldots(1+bq^n)} . \tag{10.1.3}$$

Andrews (loc.cit.) proves (10.1.2) directly, though with some difficulty. In §2 we give a proof via the convergents to $F(a,b,\lambda,q)$.

Applying Watson's theorem (2.3.1) to the numerator and denominator of (10.1.2) yields

$$F(a,b,\lambda,q) =$$

$$\frac{1 + \sum_{r\geq 1} \frac{(1-\lambda q^{2r})}{(1-\lambda q^r)} \frac{(-\lambda/b)_r}{(-bq)_r} \frac{(-\lambda/a)_r}{(-aq)_r} \frac{(\lambda q)_r}{(q)_r} q^{\frac{1}{2}(3r^2+r)}(-ab)^r}{\sum_{r\geq 0} (1-\lambda q^{2r+1}) \frac{(-\lambda q/b)_r}{(-bq)_r} \frac{(-\lambda/a)_r}{(-aq)_{r+1}} \frac{(\lambda q)_r}{(q)_r} q^{\frac{1}{2}(3r^2+3r)}(-ab)^r} . \tag{10.1.4}$$

(10.1.4) contains as corollaries several elegant continued fractions, all given by Ramanujan (1920?), some of which have appeared previously in the literature. Thus,

$$1+\frac{q}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+}\;\cdots = \prod_{n\geq 0}\frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})} \tag{10.1.5}$$

(Rogers (1894) p.328, Ramanujan (1919b)),

$$1+\frac{q}{1-}\;\frac{q-q^2}{1+}\;\frac{q^3}{1-}\;\frac{q^2-q^4}{1+}\;\frac{q^5}{1-}\;\cdots = 1/\sum_{n\geq 0}(-1)^n q^{\frac{1}{2}(n^2+n)} \tag{10.1.6}$$

(Eisenstein (1844)) ,

$$1+\frac{q+q^2}{1+}\;\frac{q^2+q^4}{1+}\;\frac{q^3+q^6}{1+}\;\cdots = \prod_{n\geq 0}\frac{(1-q^{6n+3})^2}{(1-q^{6n+1})(1-q^{6n+5})} \tag{10.1.7}$$

(Watson (1929b) p.236, Gordon (1965) p.742, Andrews (1968)),

$$1+\frac{q}{1+}\;\frac{q+q^2}{1+}\;\frac{q^3}{1+}\;\frac{q^2+q^4}{1+}\;\frac{q^5}{1+}\cdots = \prod_{n\geq 0}\frac{(1-q^{4n+2})^2}{(1-q^{4n+1})(1-q^{4n+3})} \tag{10.1.8}$$

(Ramanujan (1920?)) ,

$$1+\frac{q+q^2}{1+}\;\frac{q^4}{1+}\;\frac{q^3+q^6}{1+}\;\frac{q^8}{1+}\;\cdots = \prod_{n\geq 0}\frac{(1-q^{8n+3})(1-q^{8n+5})}{(1-q^{8n+1})(1-q^{8n+7})} \tag{10.1.9}$$

(Ramanujan (1920?)) ,

and

$$1-\frac{q-q^2}{1-}\;\frac{q^2-q^4}{1-}\;\frac{q^3-q^6}{1-}\;\cdots = 1/\sum_{n\geq 0}(-1)^n q^{3n^2+2n}(1+q^{2n+1}) \tag{10.1.10}$$

(Ramanujan (1920?)).

§2. Our main result, proved in §3, is

$$1+\frac{aq+\lambda q}{1+}\;\frac{bq+\lambda q^2}{1+}\;\cdots\;\frac{aq^n+\lambda q^{2n-1}}{1} = \frac{P_{2n-1}(a,b,\lambda)}{P_{2n-2}(b,aq,\lambda q)},$$

$$1+\frac{aq+\lambda q}{1+}\;\frac{bq+\lambda q^2}{1+}\;\cdots\;\frac{bq^n+\lambda q^{2n}}{1} = \frac{P_{2n}(a,b,\lambda)}{P_{2n-1}(b,aq,\lambda q)} \tag{10.2.1}$$

where

$$P_n(a,b,\lambda) = \sum a^s b^t \lambda^u q^{\Delta(s)+\Delta(t)+st+su+tu+u^2} \times$$

$$\times \begin{bmatrix} n+1-s-t-u \\ u \end{bmatrix} \begin{bmatrix} [(n+1)/2]-t-u \\ s \end{bmatrix} \begin{bmatrix} [n/2]-s-u \\ t \end{bmatrix} , \tag{10.2.2}$$

the sum is taken over all $s,t,u \geq 0$ such that $s+t+u \leq [(n+1)/2]$, $\Delta(n) = \frac{1}{2}(n^2+n)$, and where for our present purposes, $\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 1$.

Letting $n \to \infty$ in (10.2.1) and (10.2.2), we obtain

$$F(a,b,\lambda,q) = \frac{P(a,b,\lambda)}{P(b,aq,\lambda q)} , \tag{10.2.3}$$

where

$$P(a,b,\lambda) = \sum_{s,t,u\geq 0} a^s b^t \lambda^u \frac{q^{\Delta(s)+\Delta(t)+st+su+tu+u^2}}{(q)_s (q)_t (q)_u} . \tag{10.2.4}$$

Now, it is obvious from (10.2.4) that

$$P(a,b,\lambda) = P(b,a,\lambda) . \tag{10.2.5}$$

Also

$$P(a,b,\lambda) = \prod_{n\geq 1} (1+bq^n) . G(a,b,\lambda) , \tag{10.2.6}$$

where $G(a,b,\lambda)$ is given by (10.1.3).

For,

$$P(a,b,\lambda) = \sum_{s,t,u\geq 0} a^s b^t \lambda^u \frac{q^{\Delta(s)+\Delta(t)+st+su+tu+u^2}}{(q)_s (q)_t (q)_u}$$

$$= \sum_{s,u\geq 0} a^s \lambda^u \frac{q^{\Delta(s)+su+u^2}}{(q)_s (q)_u} \sum_{t\geq 0} \frac{q^{\Delta(t)} (bq^{s+u})^t}{(q)_t}$$

$$= \sum_{s,u\geq 0} a^s \lambda^u \frac{q^{\Delta(s)+su+u^2}}{(q)_s (q)_u} (-bq^{s+u+1})_\infty$$

$$= (-bq)_\infty \sum_{s,u\geq 0} \frac{a^s\lambda^u q^{\Delta(s)+su+u^2}}{(q)_s(q)_u(-bq)_{s+u}}$$

$$= (-bq)_\infty \sum_{n\geq 0} \frac{q^{\Delta(n)}}{(q)_n(-bq)_n} \sum_{s+u=n} a^s\lambda^u q^{\Delta(u-1)}\begin{bmatrix} n \\ u \end{bmatrix}$$

$$= (-bq)_\infty \sum_{n\geq 0} \frac{q^{\Delta(n)}\ a^n(-\lambda/a)_n}{(q)_n(-bq)_n}$$

$$= \prod_{n\geq 1} (1+bq^n)\ G(a,b,\lambda)\ .$$

From (10.2.3), (10.2.5) and (10.2.6) it follows that

$$F(a,b,\lambda,q) = \frac{P(a,b,\lambda)}{P(aq,b,\lambda q)} = \frac{G(a,b,\lambda)}{G(aq,b,\lambda q)}\ ,$$

which is (10.1.2)

§3. We establish (10.2.1) by showing that if $P_n(a,b,\lambda)$ is defined by (10.2.2) then

$$P_0 = 1,\quad P_1 = 1+aq+\lambda q\ , \tag{10.3.1}$$

and

$$P_n(a,b,\lambda) = P_{n-1}(b,aq,\lambda q) + (aq+\lambda q)\ P_{n-2}(aq,bq,\lambda q^2)\ . \tag{10.3.2}$$

We can write (10.3.2)

$$\frac{P_n(a,b,\lambda)}{P_{n-1}(b,aq,\lambda q)} = 1 + \frac{aq+\lambda q}{\left(\dfrac{P_{n-1}(b,aq,\lambda q)}{P_{n-2}(aq,bq,\lambda q^2)}\right)}\ . \tag{10.3.3}$$

(10.2.1) follows by iteration of (10.3.3), together with (10.3.1).

In order to prove (10.3.1), write

$$P_n(a,b,\lambda) = \sum a^s b^t \lambda^u q^{f(s,t,u)} c_n(s,t,u) \tag{10.3.4}$$

where

$$f(s,t,u) \;=\; \Delta(s)+\Delta(t)+st+su+tu+u^2 \tag{10.3.5}$$

and

$$c_n(s,t,u) \;=\; \genfrac{[}{]}{0pt}{}{n+1-s-t-u}{u}\genfrac{[}{]}{0pt}{}{[(n+1)/2]-t-u}{s}\genfrac{[}{]}{0pt}{}{[n/2]-s-u}{t} \;. \tag{10.3.6}$$

It is trivial to show that

$$f(t,s,u) \;=\; f(s,t,u)\;,$$

$$s+t+u+f(s-1,t,u) = f(s,t,u)\;,$$

and

$$s+t+2u-1+f(s,t,u-1) = f(s,t,u)\;. \tag{10.3.7}$$

Also

$$c_{n-2}(s,t,u-1)+q^u(c_{n-2}(s-1,t,u)+q^s c_{n-1}(t,s,u)) =$$

$$= \; c_n(s,t,u)\;. \tag{10.3.8}$$

For,

$$c_{n-2}(s-1,t,u)+q^s c_{n-1}(t,s,u) \;=$$

$$= \genfrac{[}{]}{0pt}{}{n-s-t-u}{u}\genfrac{[}{]}{0pt}{}{[(n-1)/2]-t-u}{s-1}\genfrac{[}{]}{0pt}{}{[n/2]-s-u}{t} \;+$$

$$+\; q^s\genfrac{[}{]}{0pt}{}{n-t-s-u}{u}\genfrac{[}{]}{0pt}{}{[n/2]-s-u}{t}\genfrac{[}{]}{0pt}{}{[(n-1)/2]-t-u}{s}$$

$$= \genfrac{[}{]}{0pt}{}{n-s-t-u}{u}\genfrac{[}{]}{0pt}{}{[n/2]-s-u}{t}\left\{\genfrac{[}{]}{0pt}{}{[(n-1)/2]-t-u}{s-1}+ q^s\genfrac{[}{]}{0pt}{}{[(n-1)/2]-t-u}{s}\right\}$$

$$= \genfrac{[}{]}{0pt}{}{n-s-t-u}{u}\genfrac{[}{]}{0pt}{}{[n/2]-s-u}{t}\genfrac{[}{]}{0pt}{}{[(n+1)/2]-t-u}{s} \;,$$

and so

$$c_{n-2}(s,t,u-1)+q^u(c_{n-2}(s-1,t,u)+q^s c_{n-1}(t,s,u)) \;=$$

$$= \begin{bmatrix} n-s-t-u \\ u-1 \end{bmatrix} \begin{bmatrix} [(n+1)/2]-t-u \\ s \end{bmatrix} \begin{bmatrix} [n/2]-s-u \\ t \end{bmatrix}$$

$$+ q^u \begin{bmatrix} n-s-t-u \\ u \end{bmatrix} \begin{bmatrix} [n/2]-s-u \\ t \end{bmatrix} \begin{bmatrix} [(n+1)/2]-t-u \\ s \end{bmatrix}$$

$$= \begin{bmatrix} [(n+1)/2]-t-u \\ s \end{bmatrix} \begin{bmatrix} [n/2]-s-u \\ t \end{bmatrix} \left\{ \begin{bmatrix} n-s-t-u \\ u-1 \end{bmatrix} + q^u \begin{bmatrix} n-s-t-u \\ u \end{bmatrix} \right\}$$

$$= \begin{bmatrix} n+1-s-t-u \\ u \end{bmatrix} \begin{bmatrix} [(n+1)/2]-t-u \\ s \end{bmatrix} \begin{bmatrix} [n/2]-s-u \\ t \end{bmatrix}$$

$$= c_n(s,t,u) \ .$$

It follows from (10.3.4), (10.3.7) and (10.3.8) that

$$P_{n-1}(b,aq,\lambda q)+(aq+\lambda q)P_{n-2}(aq,bq,\lambda q^2) =$$

$$= \sum b^s a^t \lambda^u q^{t+u+f(s,t,u)} c_{n-1}(s,t,u)$$

$$+ a \sum a^s b^t \lambda^u q^{s+t+2u+1+f(s,t,u)} c_{n-2}(s,t,u)$$

$$+ \lambda \sum a^s b^t \lambda^u q^{s+t+2u+1+f(s,t,u)} c_{n-2}(s,t,u)$$

$$= \sum a^s b^t \lambda^u q^{s+u+f(t,s,u)} c_{n-1}(t,s,u)$$

$$+ \sum a^s b^t \lambda^u q^{s+t+2u+f(s-1,t,u)} c_{n-2}(s-1,t,u)$$

$$+ \sum a^s b^t \lambda^u q^{s+t+2u-1+f(s,t,u-1)} c_{n-2}(s,t,u-1)$$

$$= \sum a^s b^t \lambda^u q^{f(s,t,u)} \quad \times$$

$$\times \quad \{q^{s+u} c_{n-1}(t,s,u)+q^u c_{n-2}(s-1,t,u)+c_{n-2}(s,t,u-1)\}$$

$$= \sum a^s b^t \lambda^u q^{f(s,t,u)} \times$$

$$\times \{c_{n-2}(s,t,u-1)+q^u(c_{n-2}(s-1,t,u)+q^s c_{n-1}(t,s,u))\}$$

$$= \sum a^s b^t \lambda^u q^{f(s,t,u)} c_n(s,t,u)$$

$$= \quad P_n(a,b,\lambda) \quad ,$$

which is (10.3.2), as required.

Appendix

In this Appendix, we establish a number of fundamental results in the theory of basic hypergeometric series. Most of these results are quoted at some point in the body of the thesis, but their proofs do not properly belong there. They are collected here rather than cited because they are important to the thesis and are not easily accessible.

§1. Notation

For $n \geq 1$, let

$$(a)_n = (a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}) , \tag{A.1.1a}$$

and in particular,

$$(q)_n = (q;q)_n = (1-q)(1-q^2)\dots(1-q^n) . \tag{A.1.1b}$$

Then

$$(a)_n = \frac{(a)_\infty}{(aq^n)_\infty} , \tag{A.1.2}$$

and this is used to define $(a)_n$ for $n \leq 0$, for a not a power of q.

For $n \geq r \geq 0$, let

$$\left[{n \atop r}\right] = \frac{(q)_n}{(q)_r(q)_{n-r}} , \tag{A.1.3a}$$

and more generally,

$$\left[{n \atop r}\right]_{(q^k)} = \frac{(q^k;q^k)_n}{(q^k;q^k)_r(q^k;q^k)_{n-r}} . \tag{A.1.3b}$$

For $r,s \geq 0$, let

$$ {}_r\phi_s\begin{pmatrix}\alpha_1,\ldots,\alpha_r;q;x\\ \beta_1,\ldots,\beta_s\end{pmatrix} = $$

$$ = \sum_{n\geq 0} \frac{(\alpha_1)_n\cdots(\alpha_r)_n}{(q)_n(\beta_1)_n\cdots(\beta_s)_n} x^n \quad . \tag{A.1.4} $$

(In general, $|x| < 1$ for convergence.)

§2. The q-binomial theorem.

The q-binomial theorem is

$$ {}_1\phi_0(a;q;x) = \sum_{n\geq 0} \frac{(a)_n}{(q)_n} x^n = \frac{(ax)_\infty}{(x)_\infty} \quad . \tag{A.2.1} $$

Proof:

$$ {}_1\phi_0(a;q;x) - {}_1\phi_0(a;q;xq) = $$

$$ = \sum_{n\geq 0} \frac{(a)_n}{(q)_n} x^n(1-q^n) $$

$$ = \sum_{n\geq 1} \frac{(a)_n}{(q)_{n-1}} x^n $$

$$ = \sum_{n\geq 0} \frac{(a)_{n+1}}{(q)_n} x^{n+1} $$

$$ = \sum_{n\geq 0} \frac{(a)_n}{(q)_n} x^n(1-aq^n) $$

$$ = x\,{}_1\phi_0(a;q;x) - ax\,{}_1\phi_0(a;q;xq) \quad . \tag{A.2.2} $$

It follows that

$$ (1-x)\,{}_1\phi_0(a;q;x) = (1-ax)\,{}_1\phi_0(a;q;xq) \quad , \tag{A.2.3} $$

or

$$ {}_1\phi_0(a;q;x) = \frac{(1-ax)}{(1-x)}\,{}_1\phi_0(a;x;xq) \quad . \tag{A.2.4} $$

It follows by iteration that for $n \geq 1$

$$ {}_1\phi_0(a;q;x) = \frac{(ax)_n}{(x)_n}\ {}_1\phi_0(a;q;xq^n) \ . \qquad \text{(A.2.5)}$$

Letting $n \to \infty$, we obtain

$$ {}_1\phi_0(a.q.x) = \frac{(ax)_\infty}{(x)_\infty} \ ,$$

which is (A.2.1) .

In particular, if in (A.2.1) we set $a=q^{n+1}$, we obtain

$$\begin{aligned} \frac{1}{(x)_{n+1}} &= \frac{(xq^{n+1})_\infty}{(x)_\infty} \\ &= {}_1\phi_0(q^{n+1};q;x) \\ &= \sum_{r\geq 0} \frac{(q^{n+1})_r}{(q)_r} x^r \\ &= \sum_{r\geq 0} \frac{(q)_{n+r}}{(q)_n (q)_r} x^r \\ &= \sum_{r\geq 0} \begin{bmatrix} n+r \\ n \end{bmatrix} x^r \ , \qquad \text{(A.2.6)} \end{aligned}$$

and letting $n \to \infty$ we obtain

$$ \frac{1}{(x)_\infty} = \sum_{r\geq 0} \frac{x^r}{(q)_r} \ , \qquad \text{(A.2.6a)}$$

while if in (A.2.1) we put $-xq^n$ for x, q^{-n} for a , we obtain

$$ (-x)_n = \frac{(-x)_\infty}{(-xq^n)_\infty} $$

$$= {}_1\phi_0(q^{-n};q;-xq^n)$$

$$= \sum_{r\geq 0} \frac{(q^{-n})_r}{(q)_r} (-xq^n)^r$$

$$= \sum_{r\geq 0} \frac{(1-q^{-n})\dots(1-q^{-n+r-1})}{(q)_r} (-1)^r q^{nr} x^r$$

$$= \sum_{r\geq 0} \frac{(1-q^n)\dots(q^{r-1}-q^n)}{(q)_r} x^r$$

$$= \sum_{r\geq 0} \frac{q^{\binom{r}{2}}(1-q^n)\dots(1-q^{n-r+1})}{(q)_r} x^r$$

$$= \sum_{r\geq 0} q^{\binom{r}{2}} \left[{n \atop r}\right] x^r \quad , \tag{A.2.7}$$

and letting $n \to \infty$,

$$(-x)_\infty = \sum_{r\geq 0} \frac{q^{\binom{r}{2}} x^r}{(q)_r} \quad . \tag{A.2.7a}$$

§3. Heine's transformation, and its 2^{nd} and 3^{rd} iterates.

We assume for the purposes of this section that

$$|a|,|b| < 1, \quad |c| < |a||b|, \quad |x| < |c|/|a||b| .$$

Heine's transformation is

$${}_2\phi_1 \left[{a, \; b \atop c} ;q;x \right] = \frac{(b)_\infty (ax)_\infty}{(c)_\infty (x)_\infty} \; {}_2\phi_1 \left[{c/b, \; x \atop ax} ;q;b \right]. \tag{A.3.1}$$

Proof:

$${}_2\phi_1 \left[{a, \; b \atop c} ;q;x \right] = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(q)_n (c)_n} x^n$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n\geq 0} \frac{(a)_n (cq^n)_\infty}{(q)_n (bq^n)_\infty} x^n$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n\geq 0} \frac{(a)_n}{(q)_n} x^n \cdot {}_1\phi_0(c/b;q;bq^n) \quad \text{by} \quad (A.2.1)$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{n\geq 0} \frac{(a)_n}{(q)_n} x^n \cdot \sum_{m\geq 0} \frac{(c/b)_m}{(q)_m} (bq^n)^m$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m\geq 0} \frac{(c/b)_m}{(q)_m} b^m \sum_{n\geq 0} \frac{(a)_n}{(q)_n} (xq^m)^n$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m\geq 0} \frac{(c/b)_m}{(q)_m} b^m \cdot {}_1\phi_0(a;q;xq^m)$$

$$= \frac{(b)_\infty}{(c)_\infty} \sum_{m\geq 0} \frac{(c/b)_m (axq^m)_\infty}{(q)_m (xq^m)_\infty} b^m \quad \text{by} \quad (A.2.1)$$

$$= \frac{(b)_\infty (ax)_\infty}{(c)_\infty (x)_\infty} \sum_{m\geq 0} \frac{(c/b)_m (x)_m}{(q)_m (ax)_m} b^m$$

$$= \frac{(b)_\infty (ax)_\infty}{(c)_\infty (x)_\infty} {}_2\phi_1\left(\begin{matrix} c/b, & x \\ & ax \end{matrix};q;b\right) ,$$

as required.

The 2nd iterate of Heine's transformation is

$${}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix};q;x\right] = \frac{(c/b)_\infty (bx)_\infty}{(c)_\infty (x)_\infty} {}_2\phi_1\left[\begin{matrix} abx/c, & b \\ & bx \end{matrix};q;c/b\right] . \qquad (A.3.2)$$

Proof;

$${}_2\phi_1\left[\begin{matrix} a,b \\ c \end{matrix};q;x\right] = \frac{(b)_\infty (ax)_\infty}{(c)_\infty (x)_\infty} {}_2\phi_1\left[\begin{matrix} c/b, & x \\ & ax \end{matrix};q;b\right]$$

$$= \frac{(b)_\infty (ax)_\infty}{(c)_\infty (x)_\infty} {}_2\phi_1\left[\begin{matrix} x,c/b \\ ax \end{matrix};q;b\right]$$

$$= \frac{(b)_\infty (ax)_\infty}{(c)_\infty \ (x)_\infty} \cdot \frac{(c/b)_\infty (bx)_\infty}{(ax)_\infty \ (b)_\infty} \ {}_2\phi_1 \left[\begin{matrix} abx/c, \ b \\ bx \end{matrix};q;c/b\right] \text{ by (A.3.1)}$$

$$= \frac{(c/b)_\infty (bx)_\infty}{(c)_\infty \ (x)_\infty} \ {}_2\phi_1 \left[\begin{matrix} abx/c, \ b \\ bx \end{matrix};q;c/b\right] \quad ,$$

as required.

The 3^{rd} iterate of Heine's transformation is

$${}_2\phi_1 \left[\begin{matrix} a,b \\ c \end{matrix};q;x\right] = \frac{(abx/c)_\infty}{(x)_\infty} \ {}_2\phi_1 \left[\begin{matrix} c/a,c/b \\ c \end{matrix};q;abx/c\right] \ . \qquad (A.3.3)$$

Proof:

$${}_2\phi_1 \left[\begin{matrix} a,b \\ c \end{matrix};q;x\right] = \frac{(c/b)_\infty (bx)_\infty}{(c)_\infty \ (x)_\infty} \ {}_2\phi_1 \left[\begin{matrix} abx/c, \ b \\ bx \end{matrix};q;c/b\right]$$

$$= \frac{(c/b)_\infty (bx)_\infty}{(c)_\infty \ (x)_\infty} \ {}_2\phi_1 \left[\begin{matrix} b,abx/c \\ bx \end{matrix};q;c/b\right]$$

$$= \frac{(c/b)_\infty (bx)_\infty}{(c)_\infty \ (x)_\infty} \cdot \frac{(abx/c)_\infty (c)_\infty}{(bx)_\infty (c/b)_\infty} \ {}_2\phi_1 \left[\begin{matrix} c/a,c/b \\ c \end{matrix};q;abx/c\right]$$

by (A.3.1)

$$= \frac{(abx/c)_\infty}{(x)_\infty} \ {}_2\phi_1 \left[\begin{matrix} c/a,c/b \\ c \end{matrix};q;abx/c\right] \quad ,$$

as required.

§4. The q-analogs of Gauss's theorem and of Kummer's theorem.

The q-analog of Gauss's theorem is

$${}_2\phi_1 \left[\begin{matrix} a,b \\ c \end{matrix};q;c/ab\right] = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty \ (c/ab)_\infty} \quad (|c| < |a||b|) \ . \qquad (A.4.1)$$

Proof:

$$ {}_2\phi_1\left[\begin{matrix}a,b\\c\end{matrix};q;c/ab\right] = \frac{(b)_\infty\ (c/b)_\infty}{(c)_\infty(c/ab)_\infty}\ {}_2\phi_1\left[\begin{matrix}c/b,c/ab\\c/b\end{matrix};q;b\right] \text{ by (A.3.1)} $$

$$ = \frac{(b)_\infty\ (c/b)_\infty}{(c)_\infty(c/ab)_\infty}\ {}_1\phi_0\ (c/ab;q;b) $$

$$ = \frac{(b)_\infty\ (c/b)_\infty}{(c)_\infty(c/ab)_\infty} \cdot \frac{(c/a)_\infty}{(b)_\infty} \qquad \text{by} \quad \text{(A.2.1)} $$

$$ = \frac{(c/q)_\infty(c/b)_\infty}{(c)_\infty\ (c/ab)_\infty}\ , $$

as required. (We have established the result only for $|b| < 1$, but it is true more generally.)

The q-analog of Kummer's theorem is

$$ {}_2\phi_1\left[\begin{matrix}a,\ b\\aq/b\end{matrix};q;-q/b\right] = \frac{(aq;q^2)_\infty(-q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b)_\infty(-q/b)_\infty}\ (|b| > |q|)\ . \tag{A.4.2} $$

Proof:

$$ {}_2\phi_1\left[\begin{matrix}a,\ b\\aq/b\end{matrix};q;-q/b\right] = {}_2\phi_1\left[\begin{matrix}b,\ a\\aq/b\end{matrix};q;-q/b\right] $$

$$ = \frac{(a)_\infty(-q)_\infty}{(aq/b)_\infty(-q/b)_\infty}\ {}_2\phi_1\left[\begin{matrix}q/b,-q/b\\-q\end{matrix};q;a\right] \text{ by (A.3.1)} $$

$$ = \frac{(a)_\infty(-q)_\infty}{(aq/b)_\infty(-q/b)_\infty} \sum_{n\geq 0} \frac{(q/b)_n(-q/b)_n}{(q)_n(-q)_n}\ a^n $$

$$ = \frac{(a)_\infty(-q)_\infty}{(aq/b)_\infty(-q/b)_\infty} \sum_{n\geq 0} \frac{(q^2/b^2;q^2)_n}{(q^2;q^2)_n}\ a^n $$

$$= \frac{(a)_\infty(-q)_\infty}{(aq/b)_\infty(-q/b)_\infty} \; {}_1\phi_0\,(q^2/b^2;q^2;a)$$

$$= \frac{(a)_\infty(-q)_\infty}{(aq/b)_\infty(-q/b)_\infty} \cdot \frac{(aq^2/b^2;q^2)_\infty}{(a;q^2)_\infty} \quad \text{by (A.2.1)}$$

$$= \frac{(aq;q^2)_\infty(-q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b)_\infty(-q/b)_\infty} ,$$

as required. (We have established the result only for $|a| < 1$, but it is true more generally.) (A.4.2) is

$$\sum_{n\geq 0} \frac{(a)_n(b)_n}{(q)_n(aq/b)_n} \left(-\frac{q}{b}\right)^n = \frac{(aq;q^2)_\infty(-q)_\infty(aq^2/b^2;q^2)_\infty}{(aq/b)_\infty(-q/b)_\infty} .$$

Letting $b \to \infty$, we obtain

$$\sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}(a)_n}{(q)_n} = (aq;q^2)_\infty(-q)_\infty$$

$$= \frac{(aq;q^2)_\infty}{(q;\,q^2)_\infty} , \qquad \text{(A.4.3)}$$

an identity due to V.A. Lebesgue (1840).

(A.4.3) is easy to prove directly, thus:

$$\sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}(a)_n}{(q)_n} = \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}}{(q)_n} \sum_{m=0}^{n} (-1)^m q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix} a^m \quad \text{by (A.2.7)}$$

$$= \sum_{m\geq 0} \frac{(-1)^m q^{\binom{m}{2}} a^m}{(q)_m} \sum_{n\geq m} \frac{q^{\binom{n+1}{2}}}{(q)_{n-m}}$$

$$= \sum_{m\geq 0} \frac{(-1)^m q^{\binom{m}{2}} a^m}{(q)_m} \sum_{n\geq 0} \frac{q^{\binom{n+m+1}{2}}}{(q)_n}$$

$$= \sum_{m\geq 0} \frac{(-1)^m q^{m^2} a^m}{(q)_m} \sum_{n\geq 0} \frac{q^{\binom{n}{2}}(q^{m+1})^n}{(q)_n}$$

$$= \sum_{m\geq 0} \frac{(-1)^m q^{m^2} a^m}{(q)_m} (-q^{m+1})_\infty \quad \text{by} \quad (A.2.7a)$$

$$= (-q)_\infty \sum_{m\geq 0} \frac{(-1)^m q^{m^2} a^m}{(q)_m (-q)_m}$$

$$= (-q)_\infty \sum_{m\geq 0} \frac{(-1)^m (q^2)^{\binom{m}{2}} (aq)^m}{(q^2;q^2)_m}$$

$$= (-q)_\infty (aq;q^2)_\infty \quad \text{by} \quad (A.2.7a)$$

$$= \frac{(aq;q^2)_\infty}{(q;\, q^2)_\infty} \quad ,$$

as required.

§5. Saalschutz's theorem and its q-analog.

The q-analog of Saalschutz's theorem is

$$ {}_3\phi_2 \left[\begin{matrix} a, b, q^{-n} \\ c\,,\ ab/cq^{n-1} \end{matrix} ;q;q \right] = \frac{(c/a)_n (c/b)_n}{(c)_n \ (c/ab)_n} \quad . \qquad (A.5.1)$$

Proof:

We have, by (A.3.3) and (A.2.1), for $|x|$ sufficiently small,

$$ {}_2\phi_1 \left[\begin{matrix} a,\ b \\ c \end{matrix} ;q;x \right] = \frac{(abx/c)_\infty}{(x)_\infty} \ {}_2\phi_1 \left[\begin{matrix} c/a, c/b \\ c \end{matrix} ;q;abx/c \right]$$

$$= {}_1\phi_0(ab/c;q;x) \ {}_2\phi_1 \left[\begin{matrix} c/a, c/b; \\ c \end{matrix} q;abx/c \right] \quad ,$$

or,

$$\sum_{n\geq 0} \frac{(a)_n (b)_n}{(q)_n (c)_n} x^n = \sum_{r\geq 0} \frac{(c/a)_r (c/b)_r}{(q)_r (c)_r} (abx/c)^r \sum_{s\geq 0} \frac{(ab/c)_s}{(q)_s} x^s \qquad (A.5.2)$$

Comparing coefficients of x^n yields

$$\frac{(a)_n(b)_n}{(q)_n(c)_n} = \sum_{r+s=n} \frac{(c/a)_r(c/b)_r}{(q)_r(c)_r} \frac{(ab/c)_s}{(q)_s} (ab/c)^r$$

$$= \sum_{r=0}^{n} \frac{(c/a)_r(c/b)_r}{(q)_r(c)_r} \frac{(ab/c)_{n-r}}{(q)_{n-r}} (ab/c)^r , \qquad \text{(A.5.3a)}$$

or,

$$\sum_{r=0} \frac{(c/a)_r(x/b)_r}{(q)_r(c)_r} \frac{(ab/c)_{n-r}}{(ab/c)_n} \frac{(q)_n}{(q)_{n-r}} (ab/c)^r =$$

$$= \frac{(a)_n\ (b)_n}{(c)_n(ab/c)_n} \qquad \text{(A.5.3b)}$$

or,

$$\sum_{r=0}^{n} \frac{(c/a)_r(c/b)_r}{(q)_r(c)_r} \frac{(1-q^{n-r+1})\dots(1-q^n)}{(1-abq^{n-r}/c)\dots(1-abq^{n-1}/c)} (ab/c)^r$$

$$= \frac{(a)_n\ (b)_n}{(c)_n(ab/c)_n} , \qquad \text{(A.5.3c)}$$

or,

$$\sum_{r=0}^{n} \frac{(c/a)_r(c/b)_r}{(q)_r\ (c)_r} \frac{(q^{-n})_r}{(c/abq^{n-1})_r} q^r = \frac{(a)_n(b)_n}{(c)_n(ab/c)_n} . \qquad \text{(A.5.3d)}$$

If we now set $a = C/A$, $b = C/B$, $c = C$, we obtain

$$\sum_{r=0}^{n} \frac{(A)_r(B)_r}{(q)_r(C)_r} \frac{(q^{-n})_r}{(AB/Cq^{n-1})_r} q^r = \frac{(C/A)_n(C/B)_n}{(C)_n(C/AB)_n} ,$$

which is (A.5.1), as required.

If we replace a by q^a , b by q^b , c by q^c and let $q \to 1$,

we obtain

$$\sum_{r=0}^{n} \frac{a(a+1)\ldots(a+r-1)b(b+1)\ldots(b+r-1)(-n)(-n+1)\ldots\ldots(-n+r-1)}{1.\ 2\ \ldots\ r\ \ c(c+1)\ldots(c+r-1)(a+b-c-n+1)\ldots(a+b-c-n+r)}$$

$$= \frac{(c-a)\ldots(c-a+n-1)\ (c-b)\ldots\ldots(c-b+n-1)}{(c)\ \ldots(c+n-1)\ (c-a-b)\ldots.(c-a-b+n-1)}$$

or

$${}_3F_2\left[\begin{matrix} a,\ b,\ -n \\ c,\ a+b-c-n+1 \end{matrix} ;1\right] =$$

$$= \frac{(c-a)\ldots(c-a+n-1)\ (c-b)\ \ldots\ \ (c-b+n-1)}{(c)\ \ldots\ (c+n-1)\ (c-a-b)\ldots(c-a-b+n-1)}$$

$$= \frac{\Gamma(c-a+n)\Gamma(c-b+n)\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)\Gamma(c+n)\Gamma(c-a-b+n)}\ , \tag{A.5.4}$$

which is Saalschutz's theorem.

§6. The polynomials $H_n(x) = \sum_{r=0}^{n} x^r \left[{n \atop r}\right]$.

If $H_n(x) = \sum_{r=0}^{n} x^r \left[{n \atop r}\right]$,

then
$$\sum_{n\geq 0} \frac{H_n(x)}{(q)_n} z^n = \frac{1}{(z)_\infty (xz)_\infty}\ . \tag{A.6.1}$$

Proof:

$$\sum_{n\geq 0} \frac{H_n(x)}{(q)_n} z^n = \sum_{n\geq 0} \frac{z^n}{(q)_n} \sum_{r=0} x^r \left[{n \atop r}\right]$$

$$= \sum_{r\geq 0} \frac{x^r}{(q)_r} \sum_{n\geq r} \frac{z^n}{(q)_{n-r}}$$

$$= \sum_{r\geq 0} \frac{x^r}{(q)_r} \sum_{n\geq 0} \frac{z^{n+r}}{(q)_n}$$

$$\sum_{r\geq 0} \frac{(xz)^r}{(q)_r} \sum_{n\geq 0} \frac{z^n}{(q)_n}$$

$$= \frac{1}{(xz)_\infty} \cdot \frac{1}{(z)_\infty}\ , \quad \text{by (A.2.6a)}$$

as required.

If in (A.6.1) we set $x = -1$, we obtain

$$\sum_{n\geq 0} \frac{H_n(-1)}{(q)_n} z^n = \frac{1}{(z)_\infty(-z)_\infty}$$

$$= \frac{1}{(z^2;q^2)_\infty}$$

$$= \sum_{n\geq 0} \frac{z^{2n}}{(q^2;q^2)_n} \quad \text{by (A.2.6a)} .$$

It follows that

$$H_n(-1) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{(q)_n}{(q^2;q^2)_{n/2}} & \text{if } n \text{ is even.} \end{cases} \qquad \text{(A.6.2)}$$

On the other hand, if in (A.6.1) we set $x = q^{\frac{1}{2}}$, we obtain

$$\sum_{n\geq 0} \frac{H_n(q^{\frac{1}{2}})z^n}{(q)_n} = \frac{1}{(z)_\infty(q^{\frac{1}{2}}z)_\infty}$$

$$= \frac{1}{(z;q^{\frac{1}{2}})_\infty}$$

$$= \sum_{n\geq 0} \frac{z^n}{(q^{\frac{1}{2}};q^{\frac{1}{2}})_n} \quad \text{by (A.2.6a)}$$

It follows that

$$H_n(q^{\frac{1}{2}}) = \frac{(q)_n}{(q^{\frac{1}{2}};q^{\frac{1}{2}})_n} ,$$

or, putting q^2 for q ,

$$\sum_{r=0}^{n} q^r \begin{bmatrix} n \\ r \end{bmatrix}_{(q^2)} = \frac{(q^2;q^2)_n}{(q)_n} \quad . \tag{A.6.3}$$

§7. Ramanujan's ${}_1\psi_1$ summation and Jacobi's triple product identity. Ramanujan's ${}_1\psi_1$ summation is

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(q)_\infty (b/a)_\infty}{(q/a)_\infty (b)_\infty} \cdot \frac{(az)_\infty (q/az)_\infty}{(z)_\infty (b/az)_\infty} \quad . \tag{A.7.1}$$

Proof:

$$\frac{(az)_\infty (q/az)_\infty}{(z)_\infty \ (b/az)_\infty} = \sum_{r\geq 0} \frac{(a)_r}{(q)_r} z^r \ . \ \sum_{s\geq 0} \frac{(q/b)_s}{(q)_s} (b/az)^s \quad \text{by (A.2.1)}$$

$$= \sum_{r\geq 0} \frac{(a)_r}{(q)_r} \frac{(q/b)_r}{(q)_r} (b/a)^r$$

$$+ \sum_{n\geq 1} z^n \sum_{r\geq 0} \frac{(a)_{n+r}}{(q)_{n+r}} \frac{(q/b)_r}{(q)_r} (b/a)^r$$

$$+ \sum_{n\geq 1} z^{-n} \sum_{r\geq 0} \frac{(a)_r}{(q)_r} \frac{(q/b)_{n+r}}{(q)_{n+r}} (b/a)^{n+r}$$

$$= {}_2\phi_1 \left[\begin{matrix} a, \ q/b \\ q \end{matrix} ;q;b/a\right]$$

$$+ \sum_{n\geq 1} z^n \frac{(a)_n}{(q)_n} \ . \ {}_2\phi_1 \left[\begin{matrix} aq^n, \ q/b \\ q^{n+1} \end{matrix} ;q; \ b/a\right]$$

$$+ \sum_{n\geq 1} z^{-n} \frac{(q/b)_n}{(q)_n} (b/a)^n \ . \ {}_2\phi_1 \left[\begin{matrix} a, \ q^{n+1}/b \\ q^{n+1} \end{matrix} ;q;b/a\right]$$

$$= \frac{(q/a)_\infty (b)_\infty}{(q)_\infty (b/a)_\infty}$$

$$+ \sum_{n\geq 1} z^n \frac{(a)_n}{(q)_n} \cdot \frac{(q/a)_\infty (bq^n)_\infty}{(q^{n+1})_\infty (b/a)_\infty}$$

$$+ \sum_{n\geq 1} z^{-n} \frac{(q/b)_n}{(q)_n} (b/a)^n \cdot \frac{(q^{n+1}/a)_\infty (b)_\infty}{(q^{n+1})_\infty (b/a)_\infty} \quad \text{by (A.4.1)}$$

$$= \frac{(q/a)_\infty (b)_\infty}{(q)_\infty (b/a)_\infty}$$

$$+ \frac{(q/a)_\infty (b)_\infty}{(q)_\infty (b/a)_\infty} \sum_{n\geq 1} z^n \frac{(a)_n}{(b)_n}$$

$$+ \frac{(q/a)_\infty (b)_\infty}{(q)_\infty (b/a)_\infty} \sum_{n\geq 1} z^{-n} \frac{b^n (q/b)_n}{a^n (q/a)_n}$$

$$= \frac{(q/a)_\infty (b)_\infty}{(q)_\infty (b/a)_\infty} \left\{ 1 + \sum_{n\geq 1} z^n \frac{(a)_n}{(b)_n} + \sum_{n\geq 1} z^{-n} \cdot \frac{b^n (q/b)_n}{a^n (q/a)_n} \right\}$$

$$= \frac{(q/a)_\infty (b)_\infty}{(q)_\infty (b/a)_\infty} \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n$$

since by (A.1.2),

$$\frac{(a)_{-n}}{(b)_{-n}} = \frac{(a)_\infty}{(aq^{-n})_\infty} \cdot \frac{(bq^{-n})_\infty}{(b)_\infty}$$

$$= \frac{(1-bq^{-n}) \ldots (1-bq^{-1})}{(1-aq^{-n}) \ldots (1-aq^{-1})}$$

$$= \frac{(-1)^n b^n q^{-\binom{n+1}{2}} (1-q/b) \ldots (1-q^n/b)}{(-1)^n a^n q^{-\binom{n+1}{2}} (1-q/a) \ldots (1-q^n/a)}$$

$$= \left.\frac{b^n (q/b)_n}{a^n (q/a)_n}\right\} ,$$

from which (A.7.1) follows. If in (A.7.1) we replace z by z/a, and set $b = 0$, we obtain

$$\frac{(q)_\infty (z)_\infty (q/z)_\infty}{(q/a)_\infty (z/a)_\infty} = \sum_{n=-\infty}^{\infty} z^n (a)_n / a^n \quad . \tag{A.7.2}$$

If we now let $a \to \infty$, we obtain

$$(q)_\infty (z)_\infty (q/z)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} z^n \quad , \tag{A.7.3}$$

and putting q^2 for q, $-qz$ for z, we obtain

$$(-qz;q^2)_\infty (-q/z;q^2)_\infty (q^2;q^2)_\infty = \sum_{n=-\infty}^{\infty} q^{n^2} z^n \quad , \tag{A.7.4}$$

which is Jacobi's triple product identity.

§8. Watson's Theorem and the Rogers-Ramanujan identities.

Watson's theorem is

$${}_8\phi_7 \left(\begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & d, & e, & f & ; q; \frac{a^2q^2}{bcdef} \\ & \sqrt{a}, & -\sqrt{a}, & aq/b, & aq/c, & aq/d, & aq/e, & aq/f & \end{matrix} \right)$$

$$= \frac{(aq)_\infty (aq/de)_\infty (aq/ef)_\infty (aq/df)_\infty}{(aq/d)_\infty (aq/e)_\infty (aq/f)_\infty (aq/def)_\infty} \times$$

$$\times \; {}_4\phi_3 \left[\begin{matrix} aq/bc, & d, & e, & f & ;q;q \\ & aq/b, & aq/c, & def/a & \end{matrix} \right] \tag{A.8.1}$$

provided that d, e, or f is of the form q^{-n}, with n a non-negative integer.

Proof:

The proof is by induction on n. (A.8.1) is trivially true when d, e or f is 1. Assume it is true when d, e or f is q^{-n}, $n \le k-1$. When f has the value q^{-k}, (A.8.1) can be written

$$e^k (aq/d)_k (aq/e)_{k-1} \times$$

$$\times\ (1-aq^k/e)\ {}_8\phi_7 \left[\begin{matrix} a,\ q\sqrt{a},\ -q\sqrt{a},\ b\ ,\ c\ ,\ d\ ,\ e\ ,\ q^{-k};q;\ \dfrac{a^2q^{k+2}}{bcde} \\ \sqrt{a}\ ,\ -\sqrt{a},\ aq/b, aq/c,\ aq/d,\ aq/e,\ aq^{k+1} \end{matrix}\right]$$

$$= e^k (aq)_k (aq/de)_k\ {}_4\phi_3 \left[\begin{matrix} aq/bc,\ d\ ,\ e\ ,\ q^{-k}\ ;q;q \\ aq/b, aq/c, de/aq^k \end{matrix}\right] \qquad \text{(A.8.2)}$$

Now, both sides of (A.8.2) are polynomials in e of degree k. By the induction hypothesis, they are equal if e takes any of the k values $1, q^{-1}, \ldots, q^{-(k-1)}$. Further if $e = aq^k$, the left-hand-side of (A.8.2) becomes

$$(aq^k)^k (aq/d)_k (q^{-(k-1)})_{k-1} \times$$

$$\times\ \frac{(a)_k}{(q)_k} \frac{(1-aq^{2k})}{(1-a)} \frac{(b)_k}{(aq/b)_k} \frac{(c)_k}{(aq/c)_k} \frac{(d)_k}{(aq/d)_k} \frac{(aq^k)_k}{(q^{-(k-1)})_{k-1}} \frac{(q^{-k})_k}{(aq^{k+1})_k} \left(\frac{a^2q^{k+2}}{abcdq^k}\right)^k$$

$$= \frac{(-1)^k a^{2k} q^{\frac{1}{2}(k^2+3k)} (b)_k (c)_k (d)_k (aq)_k}{(bcd)^k (aq/b)_k (aq/c)_k}$$

while the right-hand-side becomes

$$(aq^k)^k(aq)_k(1/dq^{k-1})_k \cdot {}_4\phi_3\left[\begin{matrix}aq/bc, & d, & aq^k, & q^{-k} \\ & aq/b, & aq/c, & d\end{matrix};q;q\right]$$

$$= (aq^k)^k(aq)_k(1/dq^{k-1})_k \cdot {}_3\phi_2\left[\begin{matrix}aq/bc, & aq^k, & q^{-k} \\ & aq/b, & aq/c\end{matrix};q;q\right]$$

$$= (aq^k)^k(aq)_k(1/dq^{k-1})_k \cdot \frac{(c)_k\ (1/bq^{k-1})_k}{(aq/b)_k(c/aq^k)_k} \quad \text{by (A.5.1)}$$

$$= \frac{a^kq^{k^2}(c)_k(aq)_k}{(aq/b)_k} \cdot \frac{(1-1/dq^{k-1})\dots(1-1/d)(1-1/bq^{k-1})\dots(1-1/b)}{(1-c/aq^k)\dots(1-c/aq)}$$

$$= \frac{a^kq^{k^2}(c)_k(aq)_k}{(aq/b)_k} \cdot \frac{(-1)^kd^{-k}q^{-\binom{k}{2}}(d)_k(-1)^kb^{-k}q^{-\binom{k}{2}}(b)_k}{(-1)^kc^ka^{-k}q^{-\binom{k+1}{2}}(aq/c)_k}$$

$$= \frac{(-1)^ka^{2k}q^{\frac{1}{2}(k^2+3k)}(b)_k(c)_k(d)_k(aq)_k}{(bcd)^k(aq/b)_k(aq/c)_k} .$$

Thus the two sides of (A.8.2) are equal for $k+1$ values of e, and so are identical. This proves (A.8.2), and so, by symmetry in d, e, f, (A.8.1) is true when d, e or f is q^{-n}, $n \le k$. This completes the proof of Watson's theorem.

If in (A.8.1) we take $f = q^{-n}$, and let $b,c,d,e \to \infty$, we obtain

$$\sum_{r=0}^{n} \frac{(a)_r}{(q)_r} \frac{(1-aq^{2r})}{(1-a)} q^{4\binom{r}{2}} \frac{(q^{-n})_r}{(aq^{n+1})_r} (a^2q^{n+2})^r =$$

$$= (aq)_n \sum_{r=0}^{n} \frac{(-1)^rq^{\binom{r}{2}}(q^{-n})_r}{(q)_r} (aq^{n+1})^r , \qquad \text{(A.8.3a)}$$

or

$$\sum_{r=0}^{n} (-1)^r q^{5\binom{r}{2}+2r} a^{2r} \frac{(1-aq^{2r})}{(1-a)} \frac{(a)_r}{(aq^{n+1})_r} \begin{bmatrix} n \\ r \end{bmatrix}$$

$$= (aq)_n \sum_{r=0}^{n} q^{2\binom{r}{2}+r} a^r \begin{bmatrix} n \\ r \end{bmatrix} , \qquad \text{(A.8.3b)}$$

or,

$$\sum_{r=0}^{n} q^{r^2} a^r \begin{bmatrix} n \\ r \end{bmatrix} = \frac{1}{(aq)_n} \sum_{r=0}^{n} (-1)^r q^{\frac{1}{2}(5r^2-r)} a^{2r} \frac{(1-aq^{2r})}{(1-a)} \frac{(a)_r}{(aq^{n+1})_r} \begin{bmatrix} n \\ r \end{bmatrix}.$$

(A.8.3c)

Now let $n \to \infty$, and we obtain

$$\sum_{r\geq 0} \frac{q^{r^2} a^r}{(q)_r} = \frac{1}{(aq)_\infty} \sum_{r\geq 0} (-1)^r q^{\frac{1}{2}(5r^2-r)} a^{2r} \frac{(1-aq^{2r})}{(1-a)} \frac{(a)_r}{(q)_r}$$

(A.8.4a)

$$= \frac{1}{(aq)_\infty} \left\{ 1 + \sum_{r\geq 1} (-1)^r q^{\frac{1}{2}(5r-r)} a^{2r} \frac{(1-aq^{2r})}{(1-aq^r)} \frac{(aq)_r}{(q)_r} \right\} .$$

(A.8.4b)

If in (A.8.4b) we set $a=1$, we obtain

$$\sum_{r\geq 0} \frac{q^{r^2}}{(q)_r} = \frac{1}{(q)_\infty} \left\{ 1 + \sum_{r\geq 1} (-1)^r q^{\frac{1}{2}(5r^2-r)} (1+q^r) \right\}$$

$$= \frac{1}{(q)_\infty} \left\{ 1 + \sum_{r\geq 1} (-1)^r \left(q^{\frac{1}{2}(5r^2-r)} + q^{\frac{1}{2}(5r^2+r)} \right) \right\}$$

$$= \frac{1}{(q)_\infty} (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty \quad \text{by (A.7.4)}$$

$$= 1/(q;q^5)_\infty (q^4;q^5)_\infty , \qquad \text{(A.8.5a)}$$

which is the first Rogers-Ramanujan identity, while if in (A.8.4a) we set $a=q$, we obtain

$$\sum_{r\geq 0} \frac{q^{r^2+r}}{(q)_r} = \frac{1}{(q)_\infty} \sum_{r\geq 0} (-1)^r q^{\frac{1}{2}(5r^2+3r)} (1-q^{2r+1})$$

$$= \frac{1}{(q)_\infty} \left\{ 1 + \sum_{r\geq 1} (-1)^r \left(q^{\frac{1}{2}(5r^2-3r)} + q^{\frac{1}{2}(5r^2+3r)}\right)\right\}$$

$$= \frac{1}{(q)_\infty} (q;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty \quad \text{by (A.7.4)}$$

$$= 1/(q^2;q^5)_\infty (q^3;q^5)_\infty \,, \qquad \text{(A.8.5b)}$$

the second Rogers-Ramanujan identity.

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