Mobility analyses of spatial linkages : studies in instantaneous and full-cycle mobility of linkages

Author:
Baker, Jo'n Eddie

Publication Date:
1975

DOI:
https://doi.org/10.26190/unsworks/4663

Form 1
WAIVER

THE UNIVERSITY OF NEW SOUTH WALES

DECLARATION RELATING TO DISPOSITION OF THESIS

This is to certify that I, J E BAKER, being a candidate for the degree of Doctor of Philosophy, am fully aware of the policy of the University relating to the retention and use of higher degree theses, namely that the University retains the copies of any thesis submitted for examination, "and is free to allow the thesis to be consulted or borrowed. Subject to the provisions of the Copyright Act (1968) the University may issue the thesis in whole or in part, in photostat or microfilm or other copying medium."

In the light of these provisions I grant the University Librarian permission to publish, or to authorise the publication of my thesis, in whole or in part, as he deems fit.

I also authorize the publication by University Microfilms of a 600 word abstract in Dissertation Abstracts International (D.A.I.).

Signatu

Witness

Date

Studies in instantaneous and full-cycle mobility of linkages:

MOBILITY ANALYSES OF SPATIAL LINKAGES

Jo'n Eddie Baker, M.Sc., B.E., M.Eng.Sc.

October 1975

A dissertation submitted to the School of Mechanical and Industrial Engineering in The University of New South Wales towards the degree of Doctor of Philosophy.

I dedicate this work to Mum, whose considerable sacrifices, constant encouragement and unshakable confidence in my ability were largely responsible for its having been attempted.

I was indeed fortunate to have as supervisor Dr. Ken Waldron, and as active correspondent Prof. Kenneth Hunt. Both men are in the vanguard of kinematicians, the former as algebraicist, the latter as geometrician, and I gained much from their insight and knowledge in the field of spatial linkage analysis.

I also thank Prof. Noel Svensson, who bore the onerous duty of supervising my work after Dr. Waldron's resignation from the university.

Abstract

This thesis may be summarised as follows.

1

A resumé is presented of the fundamentals of linkage closure algebra and screw system theory. These techniques are the chief tools used in the subsequent analyses. Other important phenomena and terms are defined.

2

Screw system theory is employed to develop a general method for determining limiting configurations of linkages, in respect of both joints and links. The method is applied to several individual chains. The significance of the reciprocal screw system in limiting configurations is also investigated.

3

Linkage joint motion limitation is considered by means of a method which makes use of the fact that a loop in such a limiting configuration has transitory part-chain mobility.

4

Three matters, each of later significance, are considered. The five-bar hybrid linkages obtained by combining Delassus three- and four-bar loops are listed. The special kinematic nature of coaxial screws in certain circumstances is investigated. An important theorem on the mobility of loops containing screw joints is established.

5

That group of thirteen four-bar linkages, each of mobility one and connectivity sum four, was isolated by Delassus more

than fifty years ago. For the reasons given, a fresh analysis is presented, culminating in the same results, but with greater detail.

6

To complete our survey of four-bar overconstrained linkages, it had been necessary to consider those containing screw joints and possessing connectivity sum in excess of four. An attempt is made here, using linkage closure algebra, with almost complete success.

7

Although some particular mobile five-bar loops were known, no systematic procedure to isolate all of them had been mounted. Several categories are fully investigated here, using linkage closure algebra, and an indication is given of the steps required to complete the analysis.

Opening Remarks

The conceptual gulf between planar and spatial linkages, it must be conceded, is enormous. Certainly, designers of practical mechanisms appear to see spatial movement as a series of superimposed planar motions, and therefore obtainable only by combining variations of the handful of planar four-bars. Little true spatial hardware has been developed, despite the availability of those linkages associated with the names of Bennett, Sarrus, Bricard, Myard, Delassus and Goldberg.

Part of the reason for this limited view is undoubtedly the fact that planar mechanisms have been known and used since antiquity. Application preceded theory, and latter-day synthesis has refined knowledge and expertise, but extended them by very little. Of course, manual graphical synthesis of planar mechanisms is well-developed, but it is out of the question for spatial ones. Three-dimensional linkage kinematics, let alone kinetics, is unknown territory to most engineering graduates, and there is no heritage of traditional spatial mechanisms for the designer to build upon. Moving from the plane to space is not simply a shift from two- to three-dimensional trigonometry, however considerable that alone may be. Also involved is an expansion from two to six kinds of kinematic joints, even if we limit ourselves to lower pairs. In addition, where a screw joint is involved, the equations of kinematics will include transcendental relationships.

It will be incumbent upon the design-minded spatial linkage kinematician to demonstrate any superiority of three-dimensional mechanisms over their composite planar counterparts. In the meantime, those of us who are so inclined have it as our considerable task to first complete the analysis of spatial loops. Then, hopefully, we shall be well-prepared to begin synthesis. It is the still formidable area of analysis to which this thesis is addressed. I have directed my efforts towards unsolved problems, in the main, rather than the deeper geometrical principles, and have largely succeeded in that aim.

I see the following work as one of consolidation and extension of past research. Its 'originality' lies not in basic truths, which remain elusive, but in meeting kinematic challenges which others have been unable or unwilling to pursue. An example is the inclusion of screw joints in the analysis; past workers have avoided the algebraic difficulties they entail. I have depended heavily on previous researchers but, in so doing, have tied up some loose ends and filled in some gaps.

The quantity of algebraic material to follow makes the thesis bulkier than I should wish. For this reason partly, the text of the thesis has been made pithy rather than discursive. More detailed statements on similar aspects of linkage analysis can be found, for example, in references [30,45]. One realises the arduous nature of working through a mass of algebraic exposition, but any further reduction in the following presentation would make it impenetrable for the reader.

Chapter 1 of this work consists principally of basic material, well-known or readily available to researchers in the field. This information is essential for the understanding of the subsequent analyses. The remainder of the thesis is original with me, except section 2.1 which is largely due to K. J. Waldron. Parts of chapters 2 and 5 have been published, in references [3,2,5] respectively.

1
FUNDAMENTAL CONSIDERATIONS

Linkage algebra

A linkage is an assemblage of rigid bodies interconnected by kinematic joints. The joints may be any of an available multitude, but we shall restrict consideration to the most commonly used types, the so-called lower pairs in which, ideally, contact between the jointed bodies is entirely surface-to-surface. There are precisely [46] six lower pairs:

> The screw or helical joint H, the revolute or hinge R and the prismatic or sliding joint P all have one degree of freedom. The revolute can be regarded as a screw joint with zero pitch and, theoretically, we may also think of the prismatic joint as a screw of infinite pitch.
>
> The cylindric joint C has two degrees of freedom, permitting translation along its axis as well as independent rotation about it. The three prior joints can each be considered as degeneracies of this one; this particular property, in fact, will be used to great advantage in much of the following analyses.
>
> The spherical or global joint S has three degrees of freedom, permitting independent rotations about three axes.

The plane joint F, allowing completely free relative planar movement, also has three degrees of freedom.

If a rigid body were completely free to move in space, it would have six degrees of freedom - in a plane, it would have three degrees of freedom - in a straight line only, it would possess one degree of freedom. Let us denote by F the number of degrees of freedom relevant for a rigid body in a given context. Jointing a rigid body to a frame introduces constraints which reduce its number of degrees of freedom; the size of the reduction depends on the nature of the joint, and we have listed above the resulting numbers of degrees of freedom for the six lower pairs. To avoid confusion in terminology, the number of degrees of freedom of a joint j is called its connectivity, denoted by f_j.

Now, a linkage as a whole also possesses degrees of freedom - the number applicable we call its mobility M. The mobility of a linkage is the number of free variables which must be fixed to determine the values of all other such variables. These quantities are normally either joint rotation or translation, for example, input crank angle. Most practical mechanisms have mobility one, and theoretical interest also is mainly centred on linkages with mobility of unity.

In the general case, there is a relationship among linkage mobility, joint connectivities and the appropriate number of degrees of freedom associated with the realm of operation of the linkage. This relationship, which can be simply expressed, is the celebrated Grübler or Kutzbach criterion, namely

$$M = F(L-1) - \sum_{j=1}^{J} (F-f_j). \tag{i}$$

L is the number of members (links) in the linkage and J the number of joints. The quantity (L-1) is due to the necessity of choosing a frame of reference, or reference member, which may be any of the links comprising the linkage; the quantity $(F-f_j)$ is the number of constraints imposed by joint j.

It can be otherwise shown, for example in reference [45], for a linkage consisting of a single closed loop of jointed members that

$$M = \sum_{j=1}^{J} f_j - f_{1,J+1} . \tag{ii}$$

Here $f_{1,J+1}$, the "connectivity of the joint linking members 1 and J+1", is in fact some geometrical property of the linkage as a whole. It is, alternatively, a property of the "closing" of the loop. Parenthetically, for such a loop, the number of members is equal to the number of joints. Let us therefore put L=J in equation (i), and compare the result with (ii). We have

$$M = F(J-1) - FJ + \sum_{j=1}^{J} f_j$$

$$= \sum_{j=1}^{J} f_j - F. \tag{iii}$$

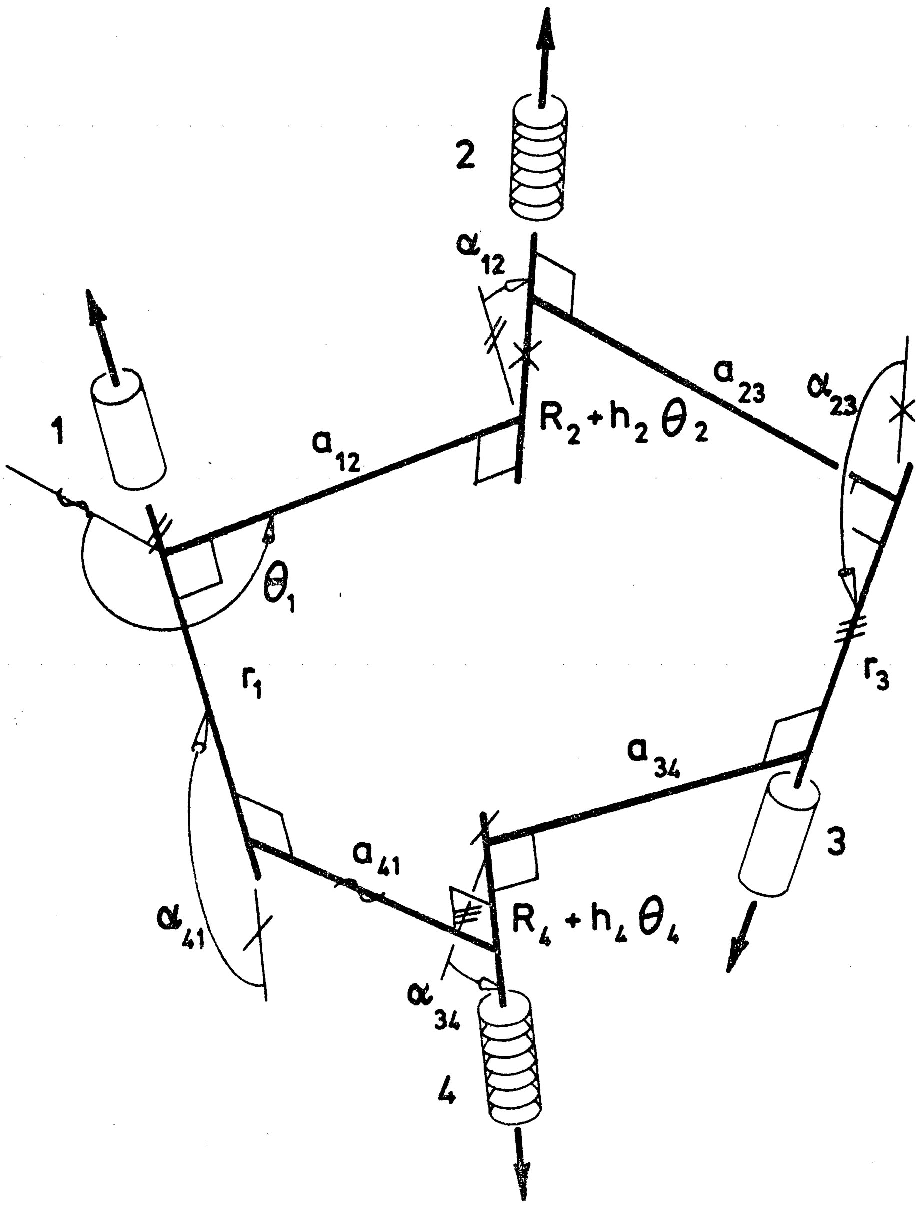
Thus, for a single closed loop,

$$f_{1,J+1} = F.$$

We shall be concerned in this work only with linkages of the single closed loop form.

An example of a linkage obeying equation (iii) is the R-R-R-P- slider-crank mechanism in the plane. For this linkage, F=3 and f_j=1, for all j; the mobility is one. Also following (iii), for mobility unity in space, a single closed loop all joints of which are turning pairs must have seven members. Other examples of spatial linkages with expected mobility one are C-C-C-R-, S-S-R- and S-C-C-. In short, for mobility unity in a single closed loop spatial linkage, the connectivity sum of the joints present, according to (iii), must be seven. It was discovered around the turn of the century, however, that there exist linkages which disobey the Grübler criterion. These, the so-called overconstrained linkages, have special geometrical properties which grant them mobility beyond that allowed by (iii) or (i). One of the simplest and certainly the most famous of these is the Bennett linkage [7] which possesses only four revolutes, their axes skew in space, and has mobility of one. The special relationships among lengths and twists of members in this linkage are responsible for its remarkable, unexpected mobility.

While these overconstrained linkages have mobility at variance with equation (iii), with the terms as defined, they could still be regarded as in accord with equation (ii), due to the flexible definition of $f_{1,J+1}$. Unfortunately, for most cases, there is no known way of pre-determining the value of $f_{1,J+1}$ for a given linkage. The tendency among researchers, then, is to accept the Grübler criterion, modifying it where possible, and to regard overconstrained



2
α_{12}
1
a_{12}
$R_2 + h_2\,\theta_2$
a_{23}
α_{23}
θ_1
r_1
r_3
a_{34}
3
a_{41}
$R_4 + h_4\,\theta_4$
α_{41}
α_{34}
4

Fig. 1.1

linkages as exceptions due to their singular geometrical properties.

It is theoretically possible, nevertheless, to analyse every closed linkage by attempting to solve the "closure equations". Many workers [1,2,4,5,14,35,36,45,47,48] have done just this and with particular reference to overconstrained linkages. The results obtained in this way are not so much parametric solutions, as existence criteria for the linkages under examination, that is dimensional conditions demanded of link lengths and twists in order for a linkage to be feasible. While early workers like Bennett and Bricard obtained relatively isolated linkages largely through inspiration, the more systematic technique of 'solving' a set of governing equations usually leads to families of solutions. Many of these sets of results have been found to include the Bennett linkage as a special case.

It is possible to represent the closure equations in a number of ways. The form and notation adopted here is basically that used by Waldron [45, 47], but altered slightly for universality. With reference to Fig. 1.1 (which actually applies specifically to the C-H-C-H- chain), we define the following terms.

$a_{i\ i+1}$ the constant length of the common perpendicular between the axes of successive joints i and i+1

r_i the variable distance, measured along joint axis i, between successive common perpendiculars; this symbol applies particularly to prismatic and cylindric joints

R_i a quantity similar to r_i, but constant, applying especially to revolutes and screws; for a revolute, it is the so-called fixed offset

h_i the pitch of screw pair i

$\alpha_{i\ i+1}$ the angle of twist between the directions of successive joint axes i and i+1

θ_i the joint angle between the two common perpendiculars relating to joint i

It should be recognised that, where a single closed loop has L members, joint L+1 coincides with joint 1.

To render the algebra of the thesis a little less overpowering in appearance, we shall abbreviate 'cos', 'sin' and 'tan' by c, s and t respectively - for example,

$$c\theta_1 \equiv \cos\theta_1.$$

The symbols ρ,σ,τ will each stand for ± 1. k,l,m,n, will stand for the integers $(-\infty,\infty)$, and M,N will refer to small sets of integers, as indicated. C and K will be constants, as defined.

It can be shown [45] that, if we represent linkage joint positions by column vectors and displacements by 3×3 matrices, specifically putting

$$\underline{s}_i = \begin{bmatrix} a_{i\ i+1} \\ 0 \\ r_i \end{bmatrix} \quad \underline{\underline{U}}_i = \begin{bmatrix} c\theta_i & -s\theta_i & 0 \\ s\theta_i & c\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underline{\underline{V}}_{i\ i+1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\alpha_{i\ i+1} & -s\alpha_{i\ i+1} \\ 0 & s\alpha_{i\ i+1} & c\alpha_{i\ i+1} \end{bmatrix},$$

we can simulate closure of the linkage by means of the relations

$$\prod_{i=1}^{J} \underline{\underline{U}}_i \, \underline{\underline{V}}_{i\ i+1} = \underline{\underline{I}} \tag{iv}$$

and

$$\underline{\underline{U}}_1 \, \underline{s}_1 + \sum_{j=2}^{J} \left(\prod_{i=1}^{j-1} \underline{\underline{U}}_i \, \underline{\underline{V}}_{i\ i+1} \right) \underline{\underline{U}}_j \, \underline{s}_j = \underline{0}. \tag{v}$$

(iv) clearly represents nine scalar equations, and

(v) three scalar equations.

It can be demonstrated that no more than three of the scalar equations included in (iv) are independent; thus (iv) and (v) together yield no more than six independent equations. In the general case of a constrained (mobility = 1) spatial linkage, precisely six equations covered by (iv) and (v) are independent. For a given overconstrained linkage, an appropriate procedure to test mobility is to determine under what conditions we might have the suitably fewer number of independent equations - our guide will be the relevant value of $f_{1,J+1}$. For example, a C-H-C-H- linkage has connectivity sum 6; for this loop to possess mobility 1, satisfaction of (ii) demands that $f_{1,J+1}$ equals 5. Thus, a C-H-C-H- linkage must have only five independent closure equations. By the same reasoning, for say a R-R-R-R- linkage (connectivity sum 4), mobility of unity would require that only three closure equations were independent.

An equivalent expression of the condition for a linkage to have mobility one is that the twelve closure equations have a single infinity of solutions. That is, any variable must be expressible as a unique function of any other variable. Since there are generally six independent equations among (iv) and (v), if the linkage under examination manifests less than seven variables, the implication is that these six equations are actually dependent. In practice, we successively eliminate variables until an equation containing only a single variable is obtained. This equation must be an identity in the variable if the original set of six equations is to be dependent [45,47].

In order to eliminate multiple solutions in this form of analysis, we make the following restrictions on the values of the constant quantities $a_{i\ i+1}, \alpha_{i\ i+1}$.

$$\left.\begin{array}{ll} 0 \leqslant a_{i\ i+1}, & \text{all } i \\ 0 \leqslant \alpha_{i\ i+1} < \pi, & \text{all } i \neq I \\ 0 \leqslant \alpha_{I\ I+1} < 2\pi, & \text{some } I \end{array}\right\} \qquad (1.1)$$

where the choice of I, one value only, is arbitrary.

We shall often find it convenient to replace a joint of connectivity greater than 1 by an equivalent chain of connectivity 1 joints. Thus, a cylindric joint might be replaced by a coaxial combination of two screws (of different pitch) or screw and revolute, or by a combination of screw and slider or revolute and slider with parallel axes. A spherical joint will usually be replaced by the equivalent combination of three concurrent revolutes, where we might choose the orientations of the axes to suit our purposes. We might replace a planar joint by any of the combinations -R-R-R-, -R-R-P-, -R-P-R-, -R-P-P-, -P-R-P-, where the prisms are parallel to the plane of the joint and the revolutes perpendicular to it. We should then generally locate the individual joint axes as convenient.

A dominant characteristic of mobile overconstrained linkages is the property of parallel or concurrent joint axes. It will be seen in part II of this work how both properties, particularly parallelism, can aid a systematic analysis of the mobility of linkages.

Some definitions

We shall, at all times, be interested only in those linkages which are "proper" and have mobility one. A linkage is proper if none of its joints is replaceable by an alternative joint of lower connectivity while mobility is retained. An example of an improper linkage is that produced by replacing a slider in the P-P- two-bar by a cylindric joint. The cylindric pair would function as a slider, and the mobility of the linkage would remain unchanged. The cylindric joint would be locked in rotation - it would exhibit a "passive degree of freedom". Improperness always implies redundant joint connectivity, not excess mobility.

On occasion, we shall find a linkage with "part-chain mobility". This is a wide term which includes improperness, locked joints and greater mobility as special cases. The epithet applies when less than the full complement of joints present is required for mobility of a given loop. For example, a planar four-revolute linkage in which two revolutes are coaxial has part-chain mobility within those two joints.

There is a difference between a 'derivative' and a 'degeneracy', and we shall use both terms. A derivative will be a linkage obtained from a mobile 'parent' loop (of higher connectivity sum) by reducing certain joint freedoms and imposing additional dimensional constraints. For example, the Bennett linkage (connectivity sum four) may be derived from a mobile C-H-C-H- chain (connectivity sum six). A degeneracy will be that linkage virtually produced when geometrical constraints are imposed on another loop of higher connectivity sum in an attempt to make it mobile. It is inappropriate to exemplify the term here, but cases will appear in part II of the thesis.

Several times in part II, we shall make use of Bennett's [8] spherical indicatrix. This device is of great value in detecting joint rotation capability in many linkages. In section 4.3, the proof of an important theorem will be based on it. Several workers [4,14,15,25,35,36,45,47,48] have utilised the device since Bennett first [9] applied it. The spherical indicatrix of a spatial linkage is produced by suppressing translation in joints. Thus, prismatic joints are locked, and cylindric and screw joints become turning pairs. The spherical indicatrix provides a representation, then, of the rotational joint motions of a given linkage. A spatial four-bar, for example, has as its spherical indicatrix a spherical quadrilateral. The kinematics of the indicatrix will be governed by the rotational closure equations (iv) alone. Just as the translational closure equations may be obtained from the algebraic dual of the rotational ones, a spatial linkage may be produced from the physical dual of its spherical indicatrix.

Apart from the special situations dealt with in chapter 4, any joint of a spatial linkage will have rotational freedom if and only if the corresponding joint in the spherical indicatrix does. If a joint in the indicatrix is locked, then, the corresponding joint in the spatial loop will be locked in rotation, regardless of what translation is possible in its joints. Because the indicatrix contains only turning pairs, it will usually have a lower connectivity sum than its spatial counterpart; this reduction in the effective number of joint variables can be of great assistance in analysis.

The most important general use of the spherical indicatrix is its application to infer either parallelism or locking of certain joint axes. For example, the indicatrix of a C-H-P-P- four-bar is a R-R- spherical two-bar. But spherical two- and three-bar loops are not mobile unless the joints are coaxial. If the turning pairs of the indicatrix are coaxial, the cylindric and screw joints of the original four-bar must be parallel. So we conclude that a fully mobile solution for the C-H-P-P- chain must have its cylindric and screw joint axes parallel.

Spatial linkage analysis has reached the stage where it has become essential to introduce additional symbols, either as shorthand or as aids to ready recognition of important features. We shall use the symbol J for a joint which is either arbitrary or undetermined as, for example, in the chain P-P-P-P-J-, which has part-chain mobility. In section 4.1 and thereafter, we shall employ the symbols $^\frown$ and = to indicate special joint relationships. The first will imply parallelism, as in the mobile P$^\frown$P- and H$^\frown$H$^\frown$H$^\frown$H-; the second will refer to certain pairs of coaxial joints, as in the Delassus linkage H=H$^\frown$H=H-. Details concerning the use of these latter symbols will be given in section 4.2.

Screw system theory

Any discrete finite change in spatial position of a rigid body may be described by a screwing motion about a unique axis with a certain associated screw pitch. The pitch relates linearly the translation of the body parallel to the axis with its rotation about the axis. By taking into account the time interval involved in very small spatial position changes, we develop the notion of the instantaneous angular velocity $\underset{\sim}{\omega}$ of the screwing body about its spatial axis. The axis is referred to as the instantaneous screw axis (ISA) of the body at that instant of its gross motion. The pitch h associated with the instantaneous motion is regarded as a property of the ISA. By choosing an origin, we can locate the ISA by the direction cosines of $\underset{\sim}{\omega}$ and the distance $\underset{\sim}{\rho}$ connecting the origin with the axis. The perpendicular distance between the origin and the ISA is designated $\underset{\sim}{\rho}_P$, so that

$$\underset{\sim}{\rho} = \underset{\sim}{\rho}_P + c\underset{\sim}{\omega}, \qquad \text{(vi)}$$

where c is a parameter. We can easily show that, if $\underset{\sim}{\mu}$ is the velocity of that point in the moving body instantaneously at the origin,

$$\underset{\sim}{\mu} = h\underset{\sim}{\omega} + \underset{\sim}{\rho}\times\underset{\sim}{\omega}. \qquad \text{(vii)}$$

It is convenient for us to specify an ISA by its screw motor $\underset{\sim}{S} = (\underset{\sim}{\omega},\underset{\sim}{\mu})$. These six independent co-ordinates allow us to determine the velocity of any point in the body. If $\underset{\sim}{\omega}$ is non-zero, we define base vectors in the directions of $\underset{\sim}{\omega}$ and $\underset{\sim}{\mu}$ by the equations

$$\left.\begin{aligned} \hat{\underset{\sim}{\omega}} &= \frac{\underset{\sim}{\omega}}{\omega} \\ \hat{\underset{\sim}{\mu}} &= \frac{\underset{\sim}{\mu}}{\omega} \end{aligned}\right\} . \tag{viii}$$

If the body is undergoing pure translation, $\underset{\sim}{\omega}$ is zero, and then we define

$$\hat{\underset{\sim}{\mu}} = \frac{\underset{\sim}{\mu}}{\mu} . \tag{ix}$$

If follows from elementary vector algebra that

$$\left.\begin{aligned} h &= \hat{\underset{\sim}{\omega}}.\hat{\underset{\sim}{\mu}} \\ \text{and} \qquad \underset{\sim}{\rho}_p &= \hat{\underset{\sim}{\omega}}\times\hat{\underset{\sim}{\mu}} \end{aligned}\right\} . \tag{1.2}$$

The ISA itself is determined by five parameters included in $\hat{\underset{\sim}{\omega}}$, $\underset{\sim}{\rho}_p$ and h. The magnitude ω is a sixth, independent quantity.

The general theory of screws was formulated by Ball [6] and given a vectorial interpretation by Everett [23]. The theory's application to linkages was due largely to Hunt [27-30] and Waldron [41-46,49]. The significance of the theory for linkage analysis stems from the representation of the joint between two links as an ISA; the consequent rigid body screwing represents the motion of one link with respect to the other. For an open chain of links, the motion of the last relative to the first is then determined by superimposing all the individual relative motions. If the joints 1,2,...,J produce motions governed by

$$\begin{aligned} \underset{\sim}{S}_1 &= (\underset{\sim}{\omega}_1, \underset{\sim}{\mu}_1) \\ \underset{\sim}{S}_2 &= (\underset{\sim}{\omega}_2, \underset{\sim}{\mu}_2) \\ &\vdots \\ \underset{\sim}{S}_J &= (\underset{\sim}{\omega}_J, \underset{\sim}{\mu}_J) , \end{aligned}$$

the resultant relative motion will be given by

$$\underset{\sim}{S} = \sum_{j=1}^{J} \underset{\sim}{S}_j = (\underset{\sim}{\omega}_1+\underset{\sim}{\omega}_2+\ldots+\underset{\sim}{\omega}_J, \underset{\sim}{\mu}_1+\underset{\sim}{\mu}_2+\ldots+\underset{\sim}{\mu}_J).$$

This composite motor is called a screw system. If we represent each ISA by the vector

$$\underset{\sim}{\$}_j = (\hat{\underset{\sim}{\omega}}_j, \hat{\underset{\sim}{\mu}}_j),$$

the screw system can be written as

$$\underset{\sim}{S} = \sum_{j=1}^{J} \omega_j \underset{\sim}{\$}_j, \tag{1.3}$$

a linear combination of the component vectors. The number of linearly independent component vectors is called the order of the screw system, and cannot be greater than six.

The connectivity of a linkage lower pair is, in general, the order of its screw system. This result is easily seen by replacing the joint with a chain of connectivity one joints. It is possible, however, in certain configuration of a linkage, for the order of the screw system of a joint, or series of joints, to be instantaneously lower than the connectivity sum. Thus, the term instantaneous mobility is used in this context, and the matter will be of basic significance in part I of this thesis.

The screw system of a closed loop is obtained as for an open chain, but with the first and last (1 and J+1) members placed together in a closed configuration. The screw system of the joints of the closed loop has been called by Waldron the equivalent screw system (ESS), and its order is identical with the quantity $f_{1,J+1}$ of equation (ii). We may therefore express the mobility of a closed loop by

$$M = \sum_{j=1}^{J} f_j - (\text{order of ESS}). \tag{x}$$

The linkage will have instantaneous mobility of one if, at the instant being considered, the order of the ESS is one less than the connectivity sum of the loop. For full-cycle mobility of 1, the order of the ESS must be one less than the connectivity sum for almost all possible configurations of the linkage. There is, however, the likelihood of exceptional configurations, such as a limit position, when the order of the ESS is even lower.

For a screw system of high order, it is sometimes easier to make use of its reciprocal screw system. Two screws, given by

$$\underset{\sim}{\$}_1 = (\hat{\underset{\sim}{\omega}}_1, \hat{\underset{\sim}{\mu}}_1)$$

and

$$\underset{\sim}{\$}_2 = (\hat{\underset{\sim}{\omega}}_2, \hat{\underset{\sim}{\mu}}_2),$$

are said to be reciprocal if and only if

$$\hat{\underset{\sim}{\omega}}_1 \cdot \hat{\underset{\sim}{\mu}}_2 + \hat{\underset{\sim}{\mu}}_1 \cdot \hat{\underset{\sim}{\omega}}_2 = 0. \tag{1.4}$$

Given a screw system, the reciprocal system is that consisting of all the screws which are reciprocal to every screw in the given system. If two systems are reciprocal, the sum of their orders equals six. Hence, instead of working with a given system of order five, one might be able to obtain the required results more readily by using the reciprocal one-system.

I

INSTANTANEOUS MOBILITY

2

LIMIT POSITIONS VIA SCREW SYSTEM THEORY

Introduction

The occurrence of limit positions is a familiar phenomenon in both planar and spatial linkages of mobility one. A limit position occurs whenever motion about a joint momentarily ceases and then reverses. The limit positions of a linkage define its range of motion; thus, their locations are of interest to the designer. Alternatively, the absence of a position of motion limitation for a particular joint implies that continuous motion about that joint is possible. Such a joint may conveniently be used to drive the linkage.

Limit position analyses for a number of particular spatial linkages are available in the literature. Some workers [26,31,37] have approached the problem of locating limit positions by using the method of generated surfaces, or related methods. This approach is limited in its potential, since only relatively simple linkage geometries can be so treated.

Another technique, which also yields additional useful information, is that adopted by Duffy and others [16-22,24]. The first step is to successively eliminate variables from the linkage closure equations until only two selected variables remain, in such a way that we can write an explicit input-output equation. This equation is obviously itself of value. By requiring that the first derivative of the output variable be zero while that of the input is not, we are able to write down an equation in the input variable alone. In symbols,

$$\theta_o = f(\theta_i) \tag{i}$$

$$\therefore \quad 0 = \dot{\theta}_o = f'(\theta_i) \times \dot{\theta}_i$$

$$\therefore \quad g(\theta_i) = f'(\theta_i) = 0. \tag{ii}$$

Equation (ii) can be solved, either algebraically or numerically, for the input variable θ_i. Substitution of this value into equation (i) gives the corresponding value of θ_o. Back substitution into the closure equations then yields the values of the remaining joint variables. The chief drawback of this procedure is the elimination process needed to reach an equation of the form (i). This is always difficult and, for more general linkage geometries, may be too complicated to be practicable. In view of the complexity of the elimination process, and the fact that relatively minor changes in the linkage geometry can completely alter its character, it is virtually necessary to approach each variation of linkage geometry as a separate problem. For example, establishing the input-output relationship for the R-R-C-R-C- chain is a different problem from determining it for the R-C-R-R-C- loop.

Another possible approach is to differentiate the closure equations with respect to time, set the derivative of the joint variable which is at a limit position to zero, and eliminate the remaining derivatives. The result is an algebraic equation between the joint variables. It may be solved simultaneously with the closure equations to obtain the values of all joint variables at the limit position.

This approach has, in principle, a big advantage, in that algebraic elimination can be replaced by numerical solution of a system of non-linear algebraic equations. However, the equation derived from the elimination of the derivatives is usually enormously complicated. Also, the solution of such systems of non-linear equations numerically is not easy. The related algebraic method presented in the following section has the advantage that the additional required equation is usually much simpler than that given by the approach just described.

2.1 Limit positions and the linkage screw system

A joint with connectivity one has, in each of its positions, a single unique screw axis. A joint with connectivity $\nu \geq 1$ has a screw system of order ν, each member of the system being a possible instantaneous screw axis for motion about the joint. Such a joint can, at any instant, be replaced by a chain with ν connectivity one joints and ν-1 additional links. In particular, cylindric, spherical and planar joints can be replaced by chains of connectivity one joints which are their kinematic equivalents in all their positions. We shall consider such substitutions to have been effected, so that we need deal only with linkages all joints of which have connectivity one.

A single-loop linkage can only have mobility one if the linkage screw system has order one less than the connectivity sum of the joints (See chapter 1.). At a limit position, one joint is instantaneously locked while the remainder of the linkage continues to move. Thus, the linkage obtained by locking the joint at which the limit position occurs is instantaneously mobile. In terms of screw systems, the screw system defined by the remaining joints of the linkage must decrease in order by at least one when a specified joint is at a limit position.

This result may either be used directly, in geometric form, or may be converted to algebraic form by means of motor notation. When used geometrically, success is largely dependent on geometrical peculiarities. A general approach is not feasible. Nevertheless, the technique is attractive in some cases.

Using motor notation, we recall from chapter 1 that $\underset{\sim}{\rho}$ is the position vector of any point on the ISA with respect to the origin of co-ordinates. The location of the origin is arbitrary, but it can often be chosen to advantage, dependent on the geometry of the particular linkage under examination. In a single closed loop linkage with N connectivity one joints, the following equations must be satisfied by the elements of the motors $(\underset{\sim}{\omega}_i, \underset{\sim}{\mu}_i)$.

$$\left.\begin{aligned} \sum_{i=1}^{N} \underset{\sim}{\omega}_i &= \underset{\sim}{0} \\ \sum_{i=1}^{N} \underset{\sim}{\mu}_i &= \underset{\sim}{0} \end{aligned}\right\} \tag{2.1.1}$$

These equations resemble the displacement closure equations and are, in principle, obtainable from them by manipulation of their time derivatives. They state that there is no net velocity of the linkage as a whole. The summations refer to the individual velocities due to each axis totalled around the loop.

Since $\underset{\sim}{\omega}_i$ and $\underset{\sim}{\mu}_i$ each have three components, we may rewrite the vector equations (2.1.1) above as six, normally independent, scalar equations in seven or less variables, $\omega_1, \omega_2, \ldots, \omega_M$, $\mu_{M+1}, \ldots, \mu_N$, where $N\text{-}M \leqslant 3$. In an overconstrained linkage, where $N<7$, the set of six equations will not be independent. In any case, the number of independent equations will be again one less than the number of variables.

We used in the Introduction the fact that, when a joint variable attains a limit position, its first derived value is zero. But the screw motor incorporates the variable's first derivative. By putting ω or μ, whichever is appropriate, equal to zero for a particular joint axis while allowing others to vary, we express motion limitation for that joint. We are thus able to extract from equations (2.1.1) an algebraic condition, usually in determinantal form, to be satisfied at a limiting position. This condition, together with the linkage closure equations (or sometimes without them), enables us to determine the linkage configurations for the limit positions of the joint being considered.

In the next section, we present several linkages which have been analysed using this method. It is necessary to consider a wide selection of chains in order to observe the variations in specific lines of attack and nature of results obtained. In determining the instantaneous screw axes, we shall make constant use of the quantities $\underline{s}_i$, $\underline{\underline{U}}_i$ and $\underline{\underline{V}}_{i\ i+1}$ as defined in chapter 1.

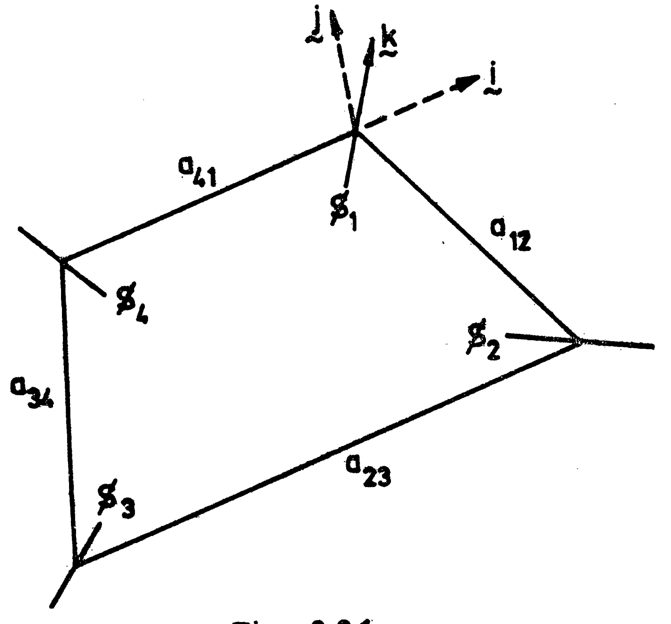


Fig. 2.2.1

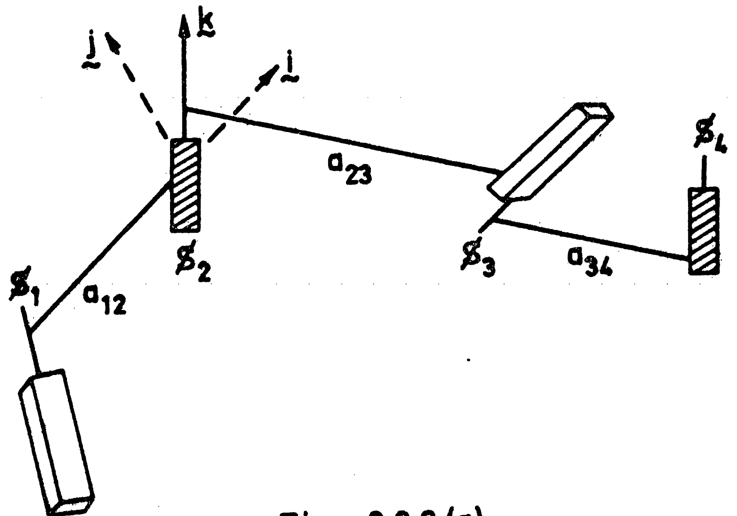


Fig. 2.2.2 (a)

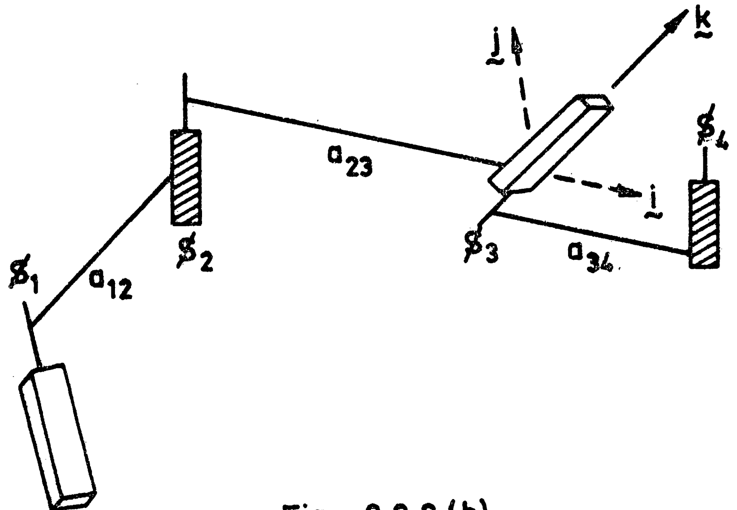


Fig. 2.2.2 (b)

2.2 Examples

The Bennett linkage

Refer to Fig. 2.2.1.

$$R_1 = R_2 = R_3 = R_4 = 0 \qquad a_{12} = a_{34} \qquad a_{23} = a_{41}$$

$$\alpha_{12} = \alpha_{34} \quad \alpha_{23} = \alpha_{41} \qquad a_{12}s\alpha_{23} = a_{23}s\alpha_{12}$$

For this linkage, we find that

$$\underset{\sim}{\omega}_1 = \omega_1\underset{\sim}{k} \qquad \underset{\sim}{\mu}_1 = \underset{\sim}{0}$$

$$\underset{\sim}{\omega}_4 = \omega_4(s\alpha_{23}\underset{\sim}{j}+c\alpha_{23}\underset{\sim}{k}) \qquad \underset{\sim}{\rho}_4 = -a_{23}\underset{\sim}{i}$$

$$\therefore \quad \underset{\sim}{\mu}_4 = a_{23}\omega_4(c\alpha_{23}\underset{\sim}{j}-s\alpha_{23}\underset{\sim}{k})$$

$$\underset{\sim}{\omega}_2 = \omega_2(s\theta_1 s\alpha_{12}\underset{\sim}{i}-c\theta_1 s\alpha_{12}\underset{\sim}{j}+c\alpha_{12}\underset{\sim}{k}) \quad \underset{\sim}{\rho}_2 = a_{12}(c\theta_1\underset{\sim}{i}+s\theta_1\underset{\sim}{j})$$

$$\therefore \quad \underset{\sim}{\mu}_2 = a_{12}\omega_2(c\alpha_{12}s\theta_1\underset{\sim}{i}-c\alpha_{12}c\theta_1\underset{\sim}{j}-s\alpha_{12}\underset{\sim}{k}).$$

We set $\underset{\sim}{\omega}_3 = \underset{\sim}{0}$.

$$\therefore \quad \underset{\sim}{\mu}_3 = \underset{\sim}{0}$$

Thus, from equations (2.1.1), we obtain the scalar results below.

$$\omega_2 s\theta_1 s\alpha_{12} = 0 \tag{i}$$

$$\omega_4 s\alpha_{23} - \omega_2 s\alpha_{12}c\theta_1 = 0 \tag{ii}$$

$$\omega_1 + \omega_2 c\alpha_{12} + \omega_4 c\alpha_{23} = 0 \tag{iii}$$

$$a_{12}\omega_2 c\alpha_{12}s\theta_1 = 0 \tag{iv}$$

$$a_{23}\omega_4 c\alpha_{23} - a_{12}\omega_2 c\alpha_{12}c\theta_1 = 0 \tag{v}$$

$$-a_{12}\omega_2 s\alpha_{12} - a_{23}\omega_4 s\alpha_{23} = 0 \tag{vi}$$

Assuming $c\alpha_{12}\neq 0$, eliminating $\omega_2 c\theta_1$ between equations (ii) and (v) yields

$$\omega_4 a_{12} s\alpha_{23}(c\alpha_{12}-c\alpha_{23}) = 0.$$

Thus, since $c\alpha_{12}\neq c\alpha_{23}$ for this linkage, we conclude that $\omega_4=0$.
Note that, if $c\alpha_{12}=0$, equation (v) implies that $\omega_4=0$.
From (vi), $\omega_2=0$.
From (iii), $\omega_1=0$.
We conclude that the Bennett linkage has no limit positions.

For comparison, let us now carry out a limit position analysis by direct differentiation of the closure equations. From section 5.7, we can write down the following relevant closure equations.

$$\theta_2 + \theta_4 = 2k\pi \tag{i}$$

$$\theta_1 + \theta_3 = 2l\pi \tag{ii}$$

$$(c\theta_1 s\theta_2 + s\theta_1 c\theta_2 c\alpha_{12})s\alpha_{23} = -(s\theta_2 + s\theta_1 c\alpha_{23})s\alpha_{12} \tag{iii}$$

$$(s\theta_1 s\theta_2 - c\theta_1 c\theta_2 c\alpha_{12})s\alpha_{23} = (c\theta_1 + c\theta_2)s\alpha_{12}c\alpha_{23} + c\alpha_{12}s\alpha_{23} \tag{iv}$$

$$(c\theta_1 c\theta_2 - s\theta_1 s\theta_2 c\alpha_{12})s\alpha_{23} = -(c\theta_1 + c\theta_2)s\alpha_{12} - s\alpha_{23} \tag{v}$$

$$(s\theta_1 c\theta_2 + c\theta_1 s\theta_2 c\alpha_{12})s\alpha_{23} = -(s\theta_1 + s\theta_2 c\alpha_{23})s\alpha_{12} \tag{vi}$$

Differentiating (ii) with respect to time and setting $\dot{\theta}_3$ equal to zero results in

$$\dot{\theta}_1 = 0.$$

From (i), in the same way, if $\dot{\theta}_2$ or $\dot{\theta}_4$ is zero, so is the other. We must assume, therefore, for a limiting position of joint 3, that neither is zero.

Differentiation of (iii) implies

$$(c\theta_1 c\theta_2 - s\theta_1 s\theta_2 c\alpha_{12})\dot{\theta}_2 s\alpha_{23} = - c\theta_2 \dot{\theta}_2 s\alpha_{12}.$$

Since $\dot{\theta}_2$ is not zero, by using (v), we conclude that

$$c\theta_1 = - \frac{s\alpha_{23}}{s\alpha_{12}} . \tag{a}$$

Taking the time-derivative of (vi) leads to

$$(-s\theta_1 s\theta_2 + c\theta_1 c\theta_2 c\alpha_{12})s\alpha_{23} = - c\theta_2 c\alpha_{23} s\alpha_{12},$$

since $\dot{\theta}_2$ is not zero. By comparison with equation (iv), we see that

$$c\theta_1 s\alpha_{12} c\alpha_{23} = - c\alpha_{12} s\alpha_{23}.$$

Substituting for $c\theta_1$ from (a),

$$c\alpha_{23} = c\alpha_{12}.$$

But this relationship cannot hold for the Bennett linkage.

We therefore conclude that $\dot{\theta}_2 = \dot{\theta}_4 = 0$; that is, the linkage has no limit positions.

As suggested in the Introduction, this latter approach is considerably more devious than that based on screw system theory, despite the fact that we were dealing with essentially the same entities; for example,

$$\dot{\theta}_3 \equiv \omega_3.$$

Techniques which are basically elimination procedures, because of the large number of variables present, must either be devious or produce greatly complex relationships. Screw system theory has the advantage of being more naturally connected with joint motion rather than displacement.

The Delassus linkage, P-H-P-H-

Refer to Figs. 2.2.2.

$$c\theta_1 = c\theta_3 = \sigma$$

$$\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{41} = \frac{\pi}{2} \qquad R_2 = \sigma R_4 + 2m\pi h$$

We shall demonstrate here that a judicious choice of origin can simplify the algebra.

With reference to Fig. 2.2.2(a), to determine the limit positions of screw joint 4, we locate the origin on joint axis 2. We can write down the following results for the motor components.

$$\underset{\sim}{\omega}_2 = \omega_2 \underset{\sim}{k} \qquad \underset{\sim}{\mu}_2 = h\omega_2 \underset{\sim}{k}$$

$$\underset{\sim}{\omega}_1 = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_1 = \mu_1 \underset{\sim}{j}$$

$$\underset{\sim}{\omega}_3 = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_3 = \mu_3(s\theta_2 \underset{\sim}{i} - c\theta_2 \underset{\sim}{j})$$

We set $\underset{\sim}{\omega}_4 = \underset{\sim}{0}$.

$$\therefore \quad \underset{\sim}{\mu}_4 = \underset{\sim}{0}$$

Using equations (2.1.1), we conclude that

$$\underset{\sim}{\omega}_2 = \underset{\sim}{0}$$

and, consequently,

$$s\theta_2 = 0.$$

Setting $c\theta_2 = \tau$, we have that

$$\mu_1 = \tau\mu_3.$$

That is, joints 2 and 4 reach their limit positions simultaneously, at which time the linkage instantaneously functions as a mobile two-slider.

Referring now to Fig. 2.2.2(b), in order to determine the limit positions of joint 1, we locate the origin on joint 3. The following results are found for the motor components.

$$\underset{\sim}{\omega}_3 = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_3 = \mu_3\underset{\sim}{k}$$

$$\underset{\sim}{\omega}_2 = \omega_2\underset{\sim}{j} \qquad \underset{\sim}{\rho}_2 = -a_{23}\underset{\sim}{i} - (R_2 + h\theta_2)\underset{\sim}{j}$$

$$\therefore \quad \underset{\sim}{\mu}_2 = h\omega_2\underset{\sim}{j} - a_{23}\omega_2\underset{\sim}{k}$$

$$\underset{\sim}{\omega}_4 = \omega_4(-\sigma\underset{\sim}{j}) \qquad \underset{\sim}{\rho}_4 = r_3\underset{\sim}{k} + a_{34}\sigma\underset{\sim}{i}$$

$$\therefore \quad \underset{\sim}{\mu}_4 = r_3\sigma\omega_4\underset{\sim}{i} - h\sigma\omega_4\underset{\sim}{j} - a_{34}\omega_4\underset{\sim}{k}$$

$\underset{\sim}{\omega}_1 = \underset{\sim}{0}$ and we set $\underset{\sim}{\mu}_1 = \underset{\sim}{0}$.

By equations (2.1.1), then,

$$\omega_4 = \sigma\omega_2.$$

Hence, since we cannot allow $\omega_2 = 0$,

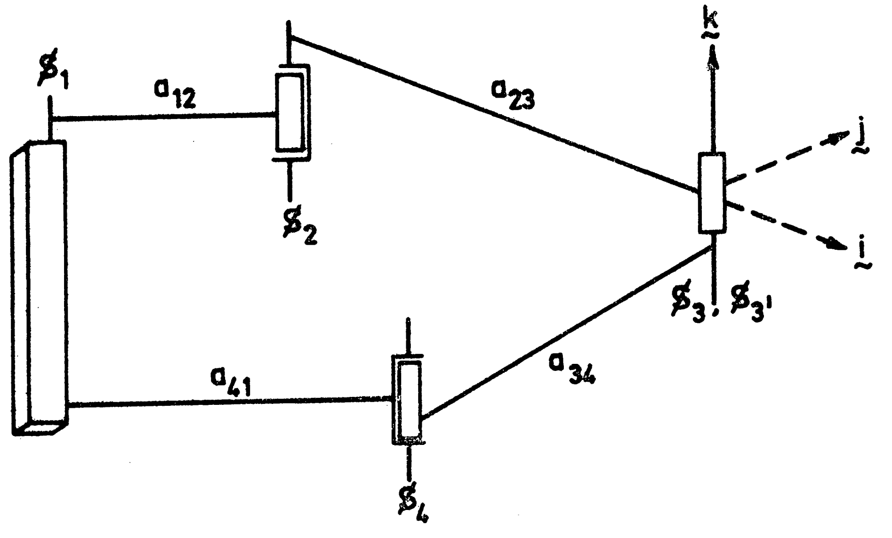


Fig. 2.2.3

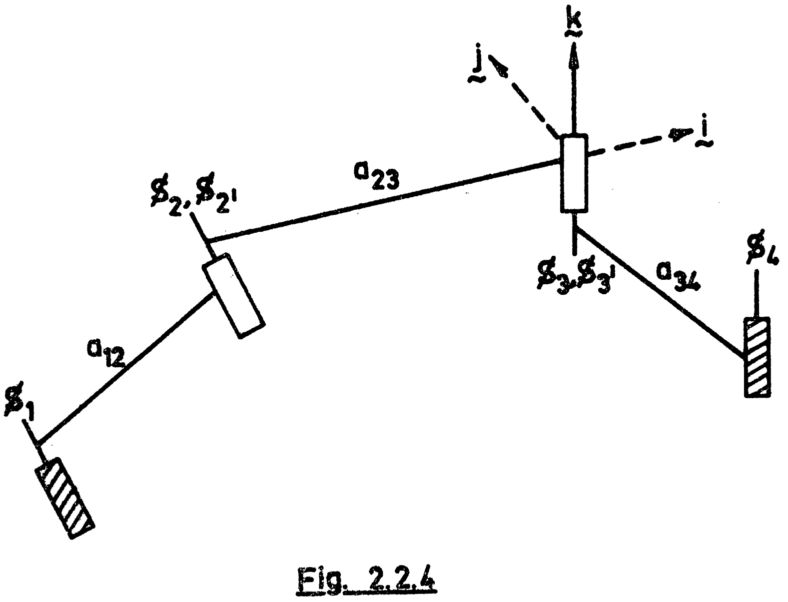


Fig. 2.2.4

$$r_3 = 0$$

and $$\mu_3 = \omega_2(a_{23}+\sigma a_{34}).$$

We may use the linkage closure equations to determine the values of the other joint variables for this instant.

The P-R-C-R- linkage

Refer to Fig. 2.2.3.

$$\theta_1 = \pi \qquad \alpha_{23} = \alpha_{34} = 0 \qquad \alpha_{41} = \alpha_{12} \neq 0$$

The linkage is improper for $\alpha_{12} = \frac{\pi}{2}$.
Here we use a fairly general approach to demonstrate how we can select a joint variable at random for motion limitation analysis.
The closure equations for this linkage are as follows.

$$\theta_2 + \theta_3 + \theta_4 = (2k+1)\pi \tag{a}$$

$$-a_{41} - a_{34}c\theta_4 + a_{23}c\theta_2 + a_{12} = 0 \tag{b}$$

$$a_{34}s\theta_4 + a_{23}s\theta_2 + r_1 s\alpha_{12} = 0 \tag{c}$$

$$R_4 + r_3 + R_2 + r_1 c\alpha_{12} = 0 \tag{d}$$

We may now write down the velocity expression for each joint:

$\underset{\sim}{\omega}_3 = \omega_3\underset{\sim}{k} \qquad \underset{\sim}{\mu}_3 = \underset{\sim}{0}$

$\underset{\sim}{\omega}_{3'} = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_{3'} = \mu_{3'}\underset{\sim}{k}$

$\underset{\sim}{\omega}_2 = \omega_2\underset{\sim}{k} \qquad \underset{\sim}{\rho}_2 = -a_{23}\underset{\sim}{i} - R_2\underset{\sim}{k}$

$\therefore \quad \underset{\sim}{\mu}_2 = a_{23}\omega_2\underset{\sim}{j}$

$\underset{\sim}{\omega}_4 = \omega_4\underset{\sim}{k} \qquad \underset{\sim}{\rho}_4 = a_{34}c\theta_3\underset{\sim}{i} + a_{34}s\theta_3\underset{\sim}{j} + r_3\underset{\sim}{k}$

$\therefore \qquad \underset{\sim}{\mu}_4 = a_{34}\omega_4(s\theta_3\underset{\sim}{i} - c\theta_3\underset{\sim}{j})$

$\underset{\sim}{\omega}_1 = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_1 = (s\theta_2 s\alpha_{12}\underset{\sim}{i} + c\theta_2 s\alpha_{12}\underset{\sim}{j} + c\alpha_{12}\underset{\sim}{k})$

From equations (2.1.1), we can write down the scalar equations below.

$$\omega_2 + \omega_3 + \omega_4 = 0 \qquad \text{(i)}$$

$$s\theta_2 s\alpha_{12}\mu_1 + a_{34}s\theta_3\omega_4 = 0 \qquad \text{(ii)}$$

$$c\theta_2 s\alpha_{12}\mu_1 + a_{23}\omega_2 - a_{34}c\theta_3\omega_4 = 0 \qquad \text{(iii)}$$

$$c\alpha_{12}\mu_1 + \mu_{3'} = 0 \qquad \text{(iv)}$$

We may now consider each joint freedom in turn.

$\mu_1 = 0$:

From (iv), $\mu_{3'} = 0$.

From (ii), $s\theta_3 = 0$.

We may obtain the values of all other variables now from equations (a)-(d).

$\omega_2 = 0$:

For a non-trivial solution of equations (i)-(iv), we require that

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & a_{34}s\theta_3 & s\theta_2 s\alpha_{12} & 0 \\ 0 & -a_{34}c\theta_3 & c\theta_2 s\alpha_{12} & 0 \\ 0 & 0 & c\alpha_{12} & 1 \end{vmatrix} = 0.$$

That is, $s(\theta_2+\theta_3) = 0.$

Thus, from (a), $s\theta_4 = 0.$

All other variables may be determined from equations (b)-(d).

$\omega_3 = 0$:

For a non-trivial solution of equations (i)-(iv), we require that

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & a_{34}s\theta_3 & s\theta_2 s\alpha_{12} & 0 \\ a_{23} & -a_{34}c\theta_3 & c\theta_2 s\alpha_{12} & 0 \\ 0 & 0 & c\alpha_{12} & 1 \end{vmatrix} = 0.$$

That is, $a_{34}(s\theta_3 c\theta_2 + c\theta_3 s\theta_2) + a_{23}s\theta_2 = 0$,

whence $a_{34}s\theta_4 + a_{23}s\theta_2 = 0$,

from equation (a).

Thus, from equation (c), $r_1 = 0$.

As before, all other variable values may now be determined.

$\mu_{3'} = 0$:

From (iv), $\mu_1 = 0$.

$\omega_4 = 0$:

From (ii), $s\theta_2 = 0$.

All other variable values may now be established from equations (a)-(d).

A H-C-C-H- linkage

Refer to Fig. 2.2.4.

We shall consider that known H-C-C-H- solution, for which there are two pairs of parallel joint axes.

$$\alpha_{12} = \alpha_{34} = 0 \qquad \alpha_{23} = \alpha_{41} \neq 0$$

This linkage has a connectivity sum of six and, as will be seen below, solution of the limit position equations for one joint variable is no longer a trivial matter.

The closure equations for this linkage are as follows.

$$\theta_1 + \theta_2 = (2k+1)\pi \tag{a}$$

$$\theta_3 + \theta_4 = (2l+1)\pi \tag{b}$$

$$-a_{41}+a_{34}c\theta_3+a_{23}+a_{12}c\theta_2 = 0 \tag{c}$$

$$a_{34}s\theta_3+r_2s\alpha_{23}-a_{12}s\theta_2c\alpha_{23}+(R_1+h_1\theta_1)s\alpha_{23} = 0 \tag{d}$$

$$(R_4+h_4\theta_4)+r_3+r_2c\alpha_{23}+a_{12}s\theta_2s\alpha_{23}+(R_1+h_1\theta_1)c\alpha_{23} = 0 \tag{e}$$

The following results may be obtained for the motor components.

$\underset{\sim}{\omega}_3 = \omega_3\underset{\sim}{k}$ $\qquad \underset{\sim}{\mu}_3 = \underset{\sim}{0}$

$\underset{\sim}{\omega}_{3'} = \underset{\sim}{0}$ $\qquad \underset{\sim}{\mu}_{3'} = \mu_{3'}\underset{\sim}{k}$

$\underset{\sim}{\omega}_4 = \omega_4\underset{\sim}{k}$ $\qquad \underset{\sim}{\rho}_4 = a_{34}c\theta_3\underset{\sim}{i}+a_{34}s\theta_3\underset{\sim}{j}+r_3\underset{\sim}{k}$

$$\therefore \underset{\sim}{\mu}_4 = \omega_4(a_{34}s\theta_3\underset{\sim}{i}-a_{34}c\theta_3\underset{\sim}{j}+h_4\underset{\sim}{k})$$

$\underset{\sim}{\omega}_2 = \omega_2(s\alpha_{23}\underset{\sim}{j}+c\alpha_{23}\underset{\sim}{k})$ $\qquad \underset{\sim}{\rho}_2 = -a_{23}\underset{\sim}{i}-r_2s\alpha_{23}\underset{\sim}{j}-r_2c\alpha_{23}\underset{\sim}{k}$

$$\therefore \underset{\sim}{\mu}_2 = a_{23}\omega_2(c\alpha_{23}\underset{\sim}{j}-s\alpha_{23}\underset{\sim}{k})$$

$\underset{\sim}{\omega}_{2'} = \underset{\sim}{0}$ $\qquad \underset{\sim}{\mu}_{2'} = \mu_{2'}(s\alpha_{23}\underset{\sim}{j}+c\alpha_{23}\underset{\sim}{k})$

$\underset{\sim}{\omega}_1 = \omega_1(s\alpha_{23}\underset{\sim}{j}+c\alpha_{23}\underset{\sim}{k})$

$$\underset{\sim}{\rho}_1 = -(a_{12}c\theta_2+a_{23})\underset{\sim}{i} + (a_{12}s\theta_2c\alpha_{23}-[R_1+h_1\theta_1]s\alpha_{23} - r_2s\alpha_{23})\underset{\sim}{j}$$
$$- (a_{12}s\theta_2s\alpha_{23}+ [R_1+h_1\theta_1]c\alpha_{23} + r_2c\alpha_{23})\underset{\sim}{k}$$

$$\therefore \underset{\sim}{\mu}_1 = \omega_1\{a_{12}s\theta_2\underset{\sim}{i}+(h_1s\alpha_{23}+a_{12}c\theta_2c\alpha_{23}+a_{23}c\alpha_{23})\underset{\sim}{j}$$
$$+(h_1c\alpha_{23}-a_{12}c\theta_2s\alpha_{23}-a_{23}s\alpha_{23})\underset{\sim}{k}\}$$

By means of equations (2.1.1), we obtain the required relationships below, after some simplification.

$$\omega_1 + \omega_2 = 0 \tag{i}$$

$$\omega_3 + \omega_4 = 0 \tag{ii}$$

$$\omega_1a_{12}s\theta_2 + \omega_4a_{34}s\theta_3 = 0 \tag{iii}$$

$$\omega_1(h_1s\alpha_{23}+a_{12}c\theta_2c\alpha_{23}) + \mu_{2'}s\alpha_{23}-\omega_4a_{34}c\theta_3 = 0 \tag{iv}$$

$$\omega_1(h_1c\alpha_{23}-a_{12}c\theta_2s\alpha_{23}) + \mu_{2'}c\alpha_{23}+\mu_{3'}+\omega_4h_4 = 0 \tag{v}$$

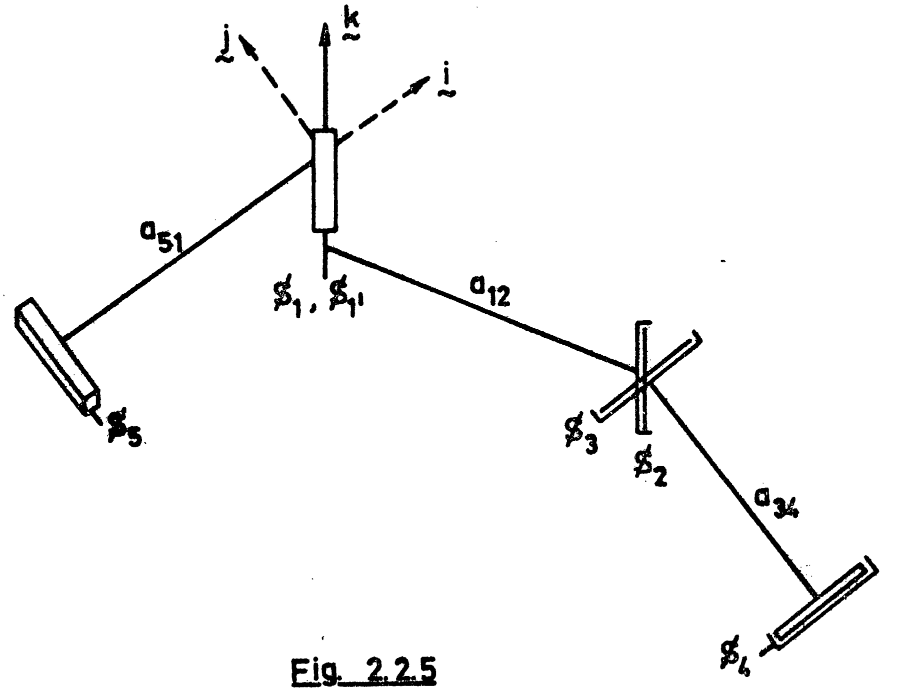


Fig. 2.2.5

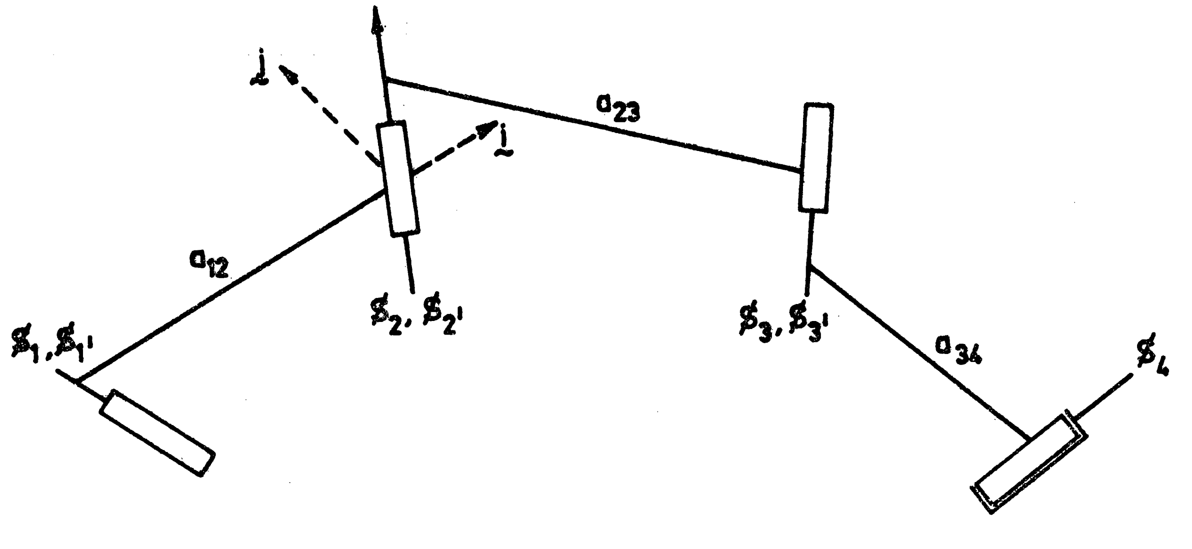


Fig. 2.2.6

By symmetry, there are only three joint variables to consider; in fact, they collapse into two.

$\omega_1 = 0$:

We see that, if either ω_1 or ω_2 is zero, so too is the other. From (iii),

$$s\theta_3 = 0.$$

Corresponding values for the other joint variables may be now determined from the closure equations.

$\mu_{2'} = 0$:

From (iii) and (iv), for a non-trivial solution, we require that

$$a_{34}s\theta_3(h_1 s\alpha_{23}+a_{12}c\theta_2 c\alpha_{23}) + a_{12}s\theta_2 a_{34}c\theta_3 = 0.$$

Either θ_2 or θ_3 may be eliminated between this equation and (c) to obtain a quartic in $c\theta_3$ or $c\theta_2$ respectively. After the quartic is solved, the other variable values may be obtained by means of the closure equations.

The P-C-S-R- linkage

Refer to Fig. 2.2.5.

We consider the limit positions of the revolute. The actual connectivity sum of this linkage is not seven, but six [45].

$$\alpha_{12} = \alpha_{34} = 0 \qquad a_{23} = r_2 = r_3 = 0$$

We may write the following expressions for linkage velocities.

$$\underset{\sim}{\omega}_1 = \omega_1\underset{\sim}{k} \qquad \underset{\sim}{\mu}_1 = \underset{\sim}{0}$$

$$\underset{\sim}{\omega}_{1'} = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_{1'} = \mu_{1'}\underset{\sim}{k}$$

$$\underset{\sim}{\omega}_5 = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_5 = \mu_5(c\alpha_{51}\underset{\sim}{k}+s\alpha_{51}\underset{\sim}{j})$$

$$\underset{\sim}{\omega}_2 = \omega_2\underset{\sim}{k} \qquad \underset{\sim}{\rho}_2 = a_{12}(c\theta_1\underset{\sim}{i}+s\theta_1\underset{\sim}{j}) + r_1\underset{\sim}{k}$$

$$\therefore \quad \underset{\sim}{\mu}_2 = a_{12}\omega_2(s\theta_1\underset{\sim}{i}-c\theta_1\underset{\sim}{j})$$

$$\underset{\sim}{\omega}_3 = \omega_3([c\theta_1 s\theta_2 s\alpha_{23}+s\theta_1 c\theta_2 s\alpha_{23}]\underset{\sim}{i} + [s\theta_1 s\theta_2 s\alpha_{23}-c\theta_1 c\theta_2 s\alpha_{23}]\underset{\sim}{j}+c\alpha_{23}\underset{\sim}{k})$$

$$\underset{\sim}{\rho}_3 = a_{12}(c\theta_1\underset{\sim}{i}+s\theta_1\underset{\sim}{j}) + r_1\underset{\sim}{k}$$

$$\therefore \underset{\sim}{\mu}_3 = \omega_3[\{a_{12}c\alpha_{23}s\theta_1+r_1 s\alpha_{23}c(\theta_1+\theta_2)\}\underset{\sim}{i}$$

$$-\{a_{12}c\alpha_{23}c\theta_1-r_1 s\alpha_{23}s(\theta_1+\theta_2)\}\underset{\sim}{j}-a_{12}s\alpha_{23}c\theta_2 k\}]$$

We set $\underset{\sim}{\omega}_4 = \underset{\sim}{0}$.

$\therefore \quad \underset{\sim}{\mu}_4 = \underset{\sim}{0}$.

By equations (2.1.1), we now write down the six scalar velocity closure equations:

$$\omega_3 s\alpha_{23}s(\theta_1+\theta_2) = 0 \tag{i}$$

$$\omega_3 s\alpha_{23}c(\theta_1+\theta_2) = 0 \tag{ii}$$

$$\omega_1+\omega_2+\omega_3 c\alpha_{23} = 0 \tag{iii}$$

$$a_{12}\omega_2 s\theta_1 = 0 \tag{iv}$$

$$\mu_5 s\alpha_{51}-a_{12}\omega_2 c\theta_1 = 0 \tag{v}$$

$$\mu_5 c\alpha_{51}+\mu_{1'} = 0 \tag{vi}$$

From equation (iv), $s\theta_1 = 0$.

There are also the subsidiary results relating velocities, such as

$$\omega_3 = 0,$$

and
$$\omega_1 + \omega_2 = 0.$$

The closure equations for this linkage, which are obtained from the general five-bar equations, may now be used to determine the values of all other joint variables.

The C-C-C-R- linkage

Refer to Fig. 2.2.6.

This is the only constrained linkage we consider in this section. As for the last linkage treated, we shall exemplify the procedure by examining the conditions for motion limitation of the revolute only.

The linkage velocities may be expressed as follows.

$$\underset{\sim}{\omega}_2 = \omega_2 \underset{\sim}{k} \qquad \underset{\sim}{\mu}_2 = \underset{\sim}{0}$$

$$\underset{\sim}{\omega}_{2'} = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_{2'} = \mu_{2'} \underset{\sim}{k}$$

$$\underset{\sim}{\omega}_1 = \omega_1(s\alpha_{12}\underset{\sim}{j}+c\alpha_{12}\underset{\sim}{k}) \qquad \underset{\sim}{\mu}_1 = a_{12}\omega_1(c\alpha_{12}\underset{\sim}{j}-s\alpha_{12}\underset{\sim}{k})$$

$$\underset{\sim}{\omega}_{1'} = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_{1'} = \mu_{1'}(s\alpha_{12}\underset{\sim}{j}+c\alpha_{12}\underset{\sim}{k})$$

$$\underset{\sim}{\omega}_3 = \omega_3(s\theta_2 s\alpha_{23}\underset{\sim}{i}-c\theta_2 s\alpha_{23}\underset{\sim}{j}+c\alpha_{23}\underset{\sim}{k})$$

$$\underset{\sim}{\mu}_3 = \omega_3([a_{23}c\alpha_{23}s\theta_2+r_2 s\alpha_{23}c\theta_2]\underset{\sim}{i}$$

$$- [a_{23}c\alpha_{23}c\theta_2-r_2 s\alpha_{23}s\theta_2]\underset{\sim}{j}-a_{23}s\alpha_{23}\underset{\sim}{k})$$

$$\underset{\sim}{\omega}_{3'} = \underset{\sim}{0}$$

$$\underset{\sim}{\mu}_{3'} = \mu_{3'}(s\theta_2 s\alpha_{23}\underset{\sim}{i}-c\theta_2 s\alpha_{23}\underset{\sim}{j}+c\alpha_{23}\underset{\sim}{k})$$

We set $\underset{\sim}{\omega}_4 = \underset{\sim}{0}$.

$$\therefore \quad \underset{\sim}{\mu}_4 = \underset{\sim}{0}$$

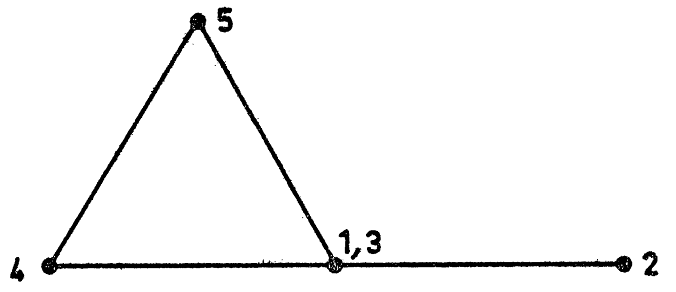


Fig. 2.2.7(a)

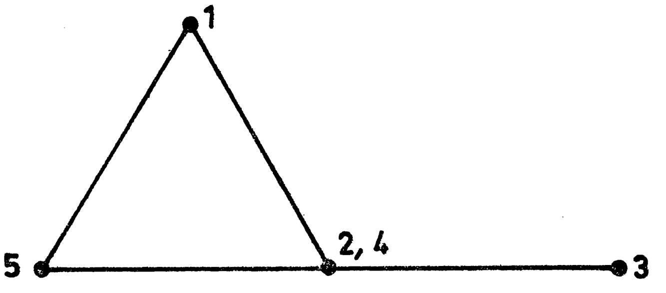


Fig. 2.2.7(b)

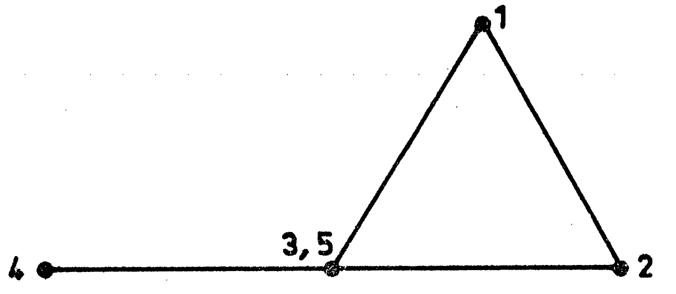


Fig. 2.2.7(c)

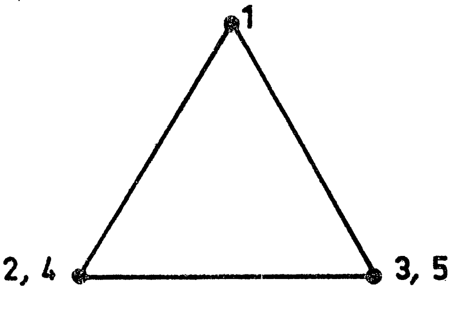


Fig. 2.2.7(d)

Then, by (2.1.1), the following results must hold.

$$\omega_3 s\theta_2 s\alpha_{23} = 0 \quad \text{(i)}$$

$$\omega_1 s\alpha_{12} - \omega_3 c\theta_2 s\alpha_{23} = 0 \quad \text{(ii)}$$

$$\omega_1 c\alpha_{12} + \omega_2 + \omega_3 c\alpha_{23} = 0 \quad \text{(iii)}$$

$$\mu_{3'} s\theta_2 s\alpha_{23} + \omega_3 (a_{23} c\alpha_{23} s\theta_2 + r_2 s\alpha_{23} c\theta_2) = 0 \quad \text{(iv)}$$

$$a_{12}\omega_1 c\alpha_{12} + \mu_{1'} s\alpha_{12} - \mu_{3'} c\theta_2 s\alpha_{23} - \omega_3 (a_{23} c\alpha_{23} c\theta_2 - r_2 s\alpha_{23} s\theta_2) = 0 \quad \text{(v)}$$

$$-a_{12}\omega_1 s\alpha_{12} + \mu_{1'} c\alpha_{12} + \mu_{2'} + \mu_{3'} c\alpha_{23} - \omega_3 a_{23} s\alpha_{23} = 0 \quad \text{(vi)}$$

Clearly, from (i),

$$s\theta_2 = 0.$$

The corresponding values of the other joint variables may now be determined by using the four-bar closure equations.

The 5H- parallel-screw linkage

Refer to Figs. 2.2.7.

For this case,

$$\underset{\sim}{\omega}_2 = \omega_2 \underset{\sim}{k}; \quad \underset{\sim}{\omega}_3 = \omega_3 \underset{\sim}{k}; \quad \underset{\sim}{\omega}_4 = \omega_4 \underset{\sim}{k}; \quad \underset{\sim}{\omega}_5 = \omega_5 \underset{\sim}{k}.$$

We set $\underset{\sim}{\omega}_1 = \underset{\sim}{0}$,

so that $\underset{\sim}{\mu}_1 = \underset{\sim}{0}$.

Locating our reference frame at joint 4, we find the following results.

$\underset{\sim}{\mu}_4 = h_4 \omega_4 \underset{\sim}{k}$

$\underset{\sim}{\rho}_3 = - a_{34} \underset{\sim}{i} - (R_3 + h_3 \theta_3) \underset{\sim}{k}$

$\underset{\sim}{\mu}_3 = h_3 \omega_3 \underset{\sim}{k} + a_{34} \omega_3 \underset{\sim}{j}$

$$\underset{\sim}{\rho}_2 = - \begin{pmatrix} a_{23}c\theta_3 + a_{34} \\ -a_{23}s\theta_3 \\ (R_2+h_2\theta_2) + (R_3+h_3\theta_3) \end{pmatrix}$$

$$\underset{\sim}{\mu}_2 = h_2\omega_2\underset{\sim}{k} + \omega_2\{a_{23}s\theta_3\underset{\sim}{i} + (a_{23}c\theta_3+a_{34})\underset{\sim}{j}\}$$

$$\underset{\sim}{\rho}_5 = a_{45}c\theta_4\underset{\sim}{i} + a_{45}s\theta_4\underset{\sim}{j} + (R_4+h_4\theta_4)\underset{\sim}{k}$$

$$\underset{\sim}{\mu}_5 = h_5\omega_5\underset{\sim}{k} + \omega_5\{a_{45}s\theta_4\underset{\sim}{i}-a_{45}c\theta_4\underset{\sim}{j}\}$$

From equations (2.1.1) then,

$$\omega_2 + \omega_3 + \omega_4 + \omega_5 = 0 \qquad \text{(i)}$$

$$a_{23}s\theta_3\omega_2 + a_{45}s\theta_4\omega_5 = 0 \qquad \text{(ii)}$$

$$(a_{23}c\theta_3+a_{34})\omega_2 + a_{34}\omega_3 - a_{45}c\theta_4\omega_5 = 0 \qquad \text{(iii)}$$

$$h_2\omega_2 + h_3\omega_3 + h_4\omega_4 + h_5\omega_5 = 0 \qquad \text{(iv)}$$

For a non-trivial solution of equations (i)-(iv) we must have that

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_{23}s\theta_3 & 0 & 0 & a_{45}s\theta_4 \\ a_{23}c\theta_3+a_{34} & a_{34} & 0 & -a_{45}c\theta_4 \\ h_2 & h_3 & h_4 & h_5 \end{vmatrix} = 0.$$

That is,

$$a_{23}a_{45}(h_4-h_3)s(\theta_3+\theta_4) = a_{23}a_{34}(h_5-h_4)s\theta_3+a_{34}a_{45}(h_3-h_2)s\theta_4. \quad \text{(a)}$$

Now, for this linkage, the closure equations may be written as follows.

$$a_{51}c(\theta_4+\theta_5) + a_{45}c\theta_4 + a_{34} + a_{23}c\theta_3 + a_{12}c(\theta_2+\theta_3) = 0 \qquad (b)$$

$$a_{51}s(\theta_4+\theta_5) + a_{45}s\theta_4 \qquad - a_{23}s\theta_3 - a_{12}s(\theta_2+\theta_3) = 0 \qquad (c)$$

$$\theta_1 + \theta_2 + \theta_3 + \theta_4 + \theta_5 = 2\pi \qquad (d)$$

$$R_1 + R_2 + R_3 + R_4 + R_5 + h_1\theta_1 + h_2\theta_2 + h_3\theta_3 + h_4\theta_4 + h_5\theta_5 = 0 \quad (e)$$

We assume that $h_1 \neq h_5$ and eliminate θ_1 between (d) and (e) to obtain

$$R_1 + R_2 + R_3 + R_4 + R_5 + 2\pi h_1 + (h_2-h_1)\theta_2 + (h_3-h_1)\theta_3 + (h_4-h_1)\theta_4$$
$$= (h_1-h_5)\theta_5,$$

whence

$$\theta_5 = C + K_2\theta_2 + K_3\theta_3 + K_4\theta_4$$

and $\quad \theta_4 + \theta_5 = C + K_2\theta_2 + K_3\theta_3 + (K_4+1)\theta_4. \qquad (f)$

For completely general parameter values, substitution of (f) into (b) and (c) will result in transcendental equations. But if K_2, K_3, K_4 are rational numbers, the equations will not be transcendental. As an example, we choose

$C = 0$

$K_2 = 0$, so that $h_2 = h_1$

$K_3 = -1$, so that $h_5 = h_3$

$K_4 = -2$, so that $h_4 = 2h_3 - h_1$

$a_{12} = a_{23} = a_{34} = a_{45} = a_{51} \neq 0.$

Then $\theta_4 + \theta_5 = -(\theta_3+\theta_4)$,

and equations (a), (b), (c) become

$$s(\theta_3+\theta_4) = s\theta_4 - s\theta_3 \tag{α}$$

$$c(\theta_3+\theta_4) + c\theta_4 + 1 + c\theta_3 + c(\theta_2+\theta_3) = 0 \tag{β}$$

$$-s(\theta_3+\theta_4) + s\theta_4 \quad -s\theta_3 - s(\theta_2+\theta_3) = 0. \tag{γ}$$

The choice of the value of C in fact governs the 'start angle' of a screw joint. Different values for C, as we shall see, yield totally different limiting configurations.

From equations (α) and (γ) we have the immediate result that

$$s(\theta_2+\theta_3) = 0. \tag{δ}$$

Now, equation (α) may be re-expressed as

$$\begin{aligned} s\theta_3(1+c\theta_4) &= s\theta_4(1-c\theta_3) \\ \pm\sqrt{(1-c\theta_3)(1+c\theta_3)}(1+c\theta_4) &= \sqrt{(1-c\theta_4)(1+c\theta_4)}(1-c\theta_3), \end{aligned}$$

for which there are the three possible solutions

$$\left.\begin{aligned} c\theta_3 &= 1 \\ c\theta_4 &= -1 \\ (1+c\theta_3)(1+c\theta_4) &= (1-c\theta_3)(1-c\theta_4) \end{aligned}\right\} .$$

The possibilities resulting from the latter two of these are easily seen to be either contained in the first or to involve us in contradictions when substitutions into equations (β) and (γ) are carried out.

We may conclude then that $c\theta_3 = 1$.

So $\theta_3 = 0$ and, from (δ), $s\theta_2 = 0$.
Hence,

$$c\theta_2 = \pm 1.$$

Substitution in (β) yields

$$2c\theta_4 + 2 = \mp 1,$$

whence $\quad c\theta_4 = -1 \mp .5$.

We conclude that $\quad c\theta_2 = -1$

and $\quad c\theta_4 = -.5$.

Hence, $\quad \theta_2 = \pi$ and $\theta_4 = \pm \frac{2\pi}{3}$.

From (d) and (e),

$$\theta_5 = \mp \frac{4\pi}{3} \quad \text{and} \quad \theta_1 = \frac{5\pi}{3} \quad \text{or} \quad \frac{\pi}{3}.$$

Either of these solutions is represented by the limiting configuration shown in Fig. 2.2.7(a). At first glance, it seems as if we have obtained the configuration for a stationary joint 5 rather than joint 1. In fact, <u>both</u> joints are here stationary; it happens that, for the pitch relationships chosen, joints 2, 3 and 4 are so arranged in Fig. 2.2.7(a) to result in their belonging to the same <u>second</u> order screw system. This fact allows the two remaining joints to be instantaneously at rest, although the linkage is mobile.

It is perhaps surprising that the only limit configuration obtained for stationary joint 1 was not produced by the other four joints being in line. Let us therefore look briefly at the consequences of assuming one of these three configurations as representing motion limitation for joint 1. The possibilities are shown in Figs. 2.2.7(b)-(d). We assume the same values for K_2, K_3, K_4 as before, with $h_1 \neq h_5$, and the same relationships among the $a_{i\ i+1}$.

In Fig. 2.2.7(b)

$$\theta_1 = \frac{2\pi}{3};\quad \theta_2 = -\frac{\pi}{3};\quad \theta_3 = \pi;\quad \theta_4 = 0;\quad \theta_5 = \frac{2\pi}{3}.$$

Since, from (f) above,

$$\theta_5 = C - \theta_3 - 2\theta_4,$$

we conclude that a satisfactory value of C is $\frac{5\pi}{3}$.

In Fig. 2.2.7(c),

$$\theta_1 = \frac{2\pi}{3};\quad \theta_2 = \frac{2\pi}{3};\quad \theta_3 = 0;\quad \theta_4 = \pi;\quad \theta_5 = \frac{\pi}{3}.$$

Again, since

$$\theta_5 = C - \theta_3 - 2\theta_4,$$

we find that $C = \frac{5\pi}{3}$ also satisfies in this case.

Note that this configuration also indicates a limiting position for joint 5.

In Fig. 2.2.7(d),

$$\theta_1 = -\frac{2\pi}{3};\quad \theta_2 = -\frac{2\pi}{3};\quad \theta_3 = \pi;\quad \theta_4 = \pi;\quad \theta_5 = -\frac{2\pi}{3}.$$

We find that $C = \frac{7\pi}{3}$ is a satisfactory value.

We have presented in this section and the preceding one, as an alternative to current methods of limit position analysis, a new technique based on instantaneous screw theory. This approach is more direct, easier to apply and wider-ranging than its counterparts, as well as being inherently capable of yielding useful information about relationships between the velocities of different joints in a linkage. It is true, as with all aspects of spatial linkage analysis, that, as the number of links increases, so does the difficulty of solving a problem completely. On the other hand, in every case without exception, this technique will yield an algebraic condition among joint variables alone which is to be satisfied for a joint limit position. Such a condition can be interpreted, without further algebra, as a geometrical constraint on the linkage configuration concomitant with that joint's stationary position.

Some of the examples treated here are linkages with screw joints. It is believed that this is the first time a limit position analysis has been presented for such loops. Although they are some of the simplest linkages incorporating screw joints, the method is, in principle, capable of handling more general geometries. It is true, however, that the resulting equations will frequently be transcendental.

In the examples given here, closed form solutions are available. This is not an essential feature of the method. The augmented set of closure equations can be solved numerically to give limit positions for linkages of given dimensions and any degree of generality.

2.3 Use of the reciprocal screw

The technique developed in section 2.1 and applied widely in section 2.2 is capable of producing, for every linkage, that extra relationship needed to determine a limit position, if it exists, for any joint variable. In this section, we shall present an alternative formulation, based on the same screw system theory, for the analysis of limiting configurations of mobile linkages. This new development has the extra advantage of locating for us the reciprocal screws of the linkage's limit position ESS and determining the pitches of those screws.

It became clear, through the number of examples used in 2.2, that, as the connectivity sum of a linkage increased, so did the difficulty of finding its limit position configurations For the method below, however, the reverse is the case. Linkages of higher connectivity sum have fewer reciprocal screws and their determination becomes, therefore, a smaller algebraic task than for chains of relatively low connectivity sum.

It was indicated in chapter 1 that the number of linearly independent screws reciprocal to a screw system is equal to six less the order of the system. A linkage of mobility 1 and connectivity sum N has an ESS of order N-1. The order of its reciprocal screw system is therefore 7-N. Thus, a constrained linkage (connectivity sum 7) has no reciprocal screws during its gross motion. A Bennett linkage has three linearly independent reciprocal screws. When a joint variable reaches a limiting value, the order of the linkage's ESS decreases by one for mobility in the remainder of the loop. At such a

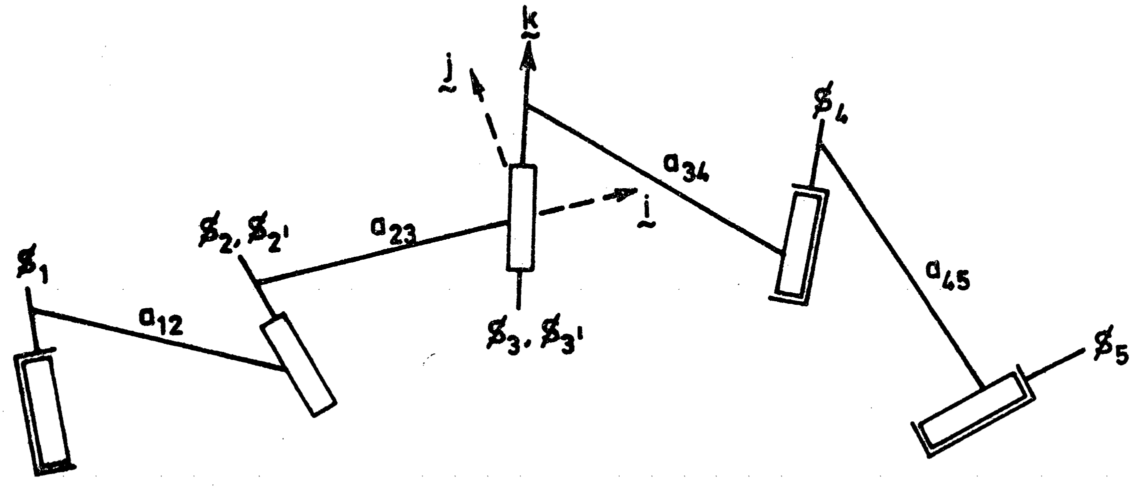


Fig. 2.3.1

limiting configuration, then, the order of the reciprocal screw system increases by one. For a single joint variable limitation in a constrained linkage, therefore, there will be one reciprocal screw.

To locate the reciprocal screws associated with linkage mobility, instantaneous or full-cycle, we need only recall equation (1.4). Having found the linkage velocities, as we did in the examples of section 2.2, we may apply this equation to each of them in turn, thereby determining the reciprocal screw system. The pitch and location of a reciprocal screw in the system may be then found by application of equations (1.2).

We shall demonstrate the use of the procedure for full-cycle mobility in section 6.6. For our present purposes, it should be sufficient to apply the method to a linkage we did not treat in the last section, the constrained five-bar loop, R-C-C-R-R- (Fig. 2.3.1). We shall attempt to find the limiting configurations for hinge joint 5, knowing that the reciprocal screw system of the remaining joints for any such configuration consists of a single, unique screw. The linkage velocities are as follows.

$$\underset{\sim}{\omega}_3 = \omega_3 \underset{\sim}{k} \qquad \underset{\sim}{\mu}_3 = \underset{\sim}{0}$$

$$\underset{\sim}{\omega}_{3'} = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_{3'} = \mu_{3'} \underset{\sim}{k}$$

$$\underset{\sim}{\omega}_2 = \omega_2 (s\alpha_{23} \underset{\sim}{j} + c\alpha_{23} \underset{\sim}{k}) \qquad \underset{\sim}{\mu}_2 = a_{23}\omega_2 (c\alpha_{23} \underset{\sim}{j} - s\alpha_{23} \underset{\sim}{k})$$

$$\underset{\sim}{\omega}_{2'} = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_{2'} = \mu_{2'} (s\alpha_{23} \underset{\sim}{j} + c\alpha_{23} \underset{\sim}{k})$$

$$\underset{\sim}{\omega}_1 = \omega_1 \{s\theta_2 s\alpha_{12} \underset{\sim}{i} + (c\theta_2 s\alpha_{12} c\alpha_{23} + c\alpha_{12} s\alpha_{23}) \underset{\sim}{j} + (-c\theta_2 s\alpha_{12} s\alpha_{23} + c\alpha_{12} c\alpha_{23}) \underset{\sim}{k}\}$$

$$\underset{\sim}{\rho}_1 = - \begin{pmatrix} a_{12}c\theta_2 + R_1 s\alpha_{12} s\theta_2 + a_{23} \\ -a_{12}s\theta_2 c\alpha_{23} + R_1 s\alpha_{12} c\alpha_{23} c\theta_2 + R_1 c\alpha_{12} s\alpha_{23} + r_2 s\alpha_{23} \\ a_{12} s\theta_2 s\alpha_{23} - R_1 s\alpha_{12} s\alpha_{23} c\theta_2 + R_1 c\alpha_{12} c\alpha_{23} + r_2 c\alpha_{23} \end{pmatrix}$$

$$\therefore \underset{\sim}{\mu}_1 = \omega_1\{(a_{12}c\alpha_{12}s\theta_2+r_2s\alpha_{12}c\theta_2)\underset{\sim}{i}$$
$$-(a_{12}s\alpha_{12}s\alpha_{23}+r_2s\alpha_{12}c\alpha_{23}s\theta_2+a_{23}s\alpha_{12}s\alpha_{23}c\theta_2$$
$$-a_{12}c\alpha_{12}c\alpha_{23}c\theta_2-a_{23}c\alpha_{12}c\alpha_{23})\underset{\sim}{j}$$
$$-(a_{12}s\alpha_{12}c\alpha_{23}-r_2s\alpha_{12}s\alpha_{23}s\theta_2+a_{23}s\alpha_{12}c\alpha_{23}c\theta_2$$
$$+a_{12}c\alpha_{12}s\alpha_{23}c\theta_2+a_{23}c\alpha_{12}s\alpha_{23})\underset{\sim}{k}\}$$

$$\underset{\sim}{\omega}_4 = \omega_4(s\theta_3s\alpha_{34}\underset{\sim}{i}-c\theta_3s\alpha_{34}\underset{\sim}{j}+c\alpha_{34}\underset{\sim}{k})$$
$$\underset{\sim}{\mu}_4 = \omega_4\{(a_{34}c\alpha_{34}s\theta_3+r_3s\alpha_{34}c\theta_3)\underset{\sim}{i}$$
$$-(a_{34}c\alpha_{34}c\theta_3-r_3s\alpha_{34}s\theta_3)\underset{\sim}{j}-a_{34}s\alpha_{34}\underset{\sim}{k}\}$$

We set $\underset{\sim}{\omega}_5 = \underset{\sim}{0}$.

$$\therefore \underset{\sim}{\mu}_5 = \underset{\sim}{0}$$

Let us represent the screw reciprocal to $\$_1$-$\$_4$ by the motor

$$(\underset{\sim}{\Omega},\underset{\sim}{M}) = (\alpha,\beta,\gamma,\delta,\varepsilon,\zeta).$$

We now apply equation (1.4) to the reciprocal screw and each of the non-zero linkage velocities in turn.

$$\begin{array}{lrl}
3: & \omega_3\zeta = 0; & \therefore\ \zeta = 0 \\
3': & \mu_{3'}\gamma = 0; & \therefore\ \gamma = 0 \\
2': & \mu_{2'}s\alpha_{23}\beta = 0; & \therefore\ \beta = 0 \\
2: & \omega_2s\alpha_{23}\varepsilon = 0; & \therefore\ \varepsilon = 0 \\
1: & \multicolumn{2}{l}{\omega_1(a_{12}c\alpha_{12}s\theta_2+r_2s\alpha_{12}c\theta_2)\alpha + \omega_1s\alpha_{12}s\theta_2\delta = 0} \\
4: & \multicolumn{2}{l}{\omega_4(a_{34}c\alpha_{34}s\theta_3+r_3s\alpha_{34}c\theta_3)\alpha + \omega_4s\alpha_{34}s\theta_3\delta = 0}
\end{array}$$

We see that, since α is the only non-zero component of $\underset{\sim}{\Omega}$, the reciprocal screw is parallel to $\underset{\sim}{i}$. Further, from the second of equations (1.2), $\underset{\sim}{\rho}_p=\underset{\sim}{0}$. We conclude that the reciprocal screw lies along the common normal from $\$_2$ to $\$_3$. We may write its ISA vector as

$$\underset{\sim}{\$}_r = (1,0,0,\tfrac{\delta}{\alpha},0,0).$$

From the first of (1.2), we find that the pitch of the reciprocal screw may be given as

$$h_r = \frac{\delta}{\alpha}.$$

The last two relations obtained by the application of (1.4) above may now be used to yield the results

$$-\frac{\delta}{\alpha} = \frac{a_{12}c\alpha_{12}s\theta_2+r_2s\alpha_{12}c\theta_2}{s\alpha_{12}s\theta_2} = \frac{a_{34}c\alpha_{34}s\theta_3+r_3s\alpha_{34}c\theta_3}{s\alpha_{34}s\theta_3}.$$

Hence, the extra algebraic constraint to be satisfied for this linkage when θ_5 takes a limiting value is concisely expressed as

$$\frac{a_{12}}{t\alpha_{12}}+\frac{r_2}{t\theta_2} = \frac{a_{34}}{t\alpha_{34}}+\frac{r_3}{t\theta_3}. \qquad \text{(i)}$$

The variables r_2 and r_3 may be eliminated from the solution procedure by making use of the translational closure equation (7.10), given in the Introduction to chapter 7. We may rearrange this equation and the one obtained from it by advancing the indices by 3, producing the two following results.

$$\begin{aligned} r_2s\theta_3 = -\frac{1}{s\alpha_{23}}\{&a_{51}(c\theta_4c\theta_5-s\theta_4s\theta_5c\alpha_{45})+R_5s\theta_4s\alpha_{45}+a_{45}c\theta_4+a_{34}+a_{23}c\theta_3\\ &+a_{12}(c\theta_2c\theta_3-s\theta_2s\theta_3c\alpha_{23})+R_1(s\theta_2c\theta_3s\alpha_{12}+c\theta_2s\theta_3s\alpha_{12}c\alpha_{23}\\ &+s\theta_3c\alpha_{12}s\alpha_{23})\} \end{aligned}$$

$$\begin{aligned} r_3s\theta_2 = -\frac{1}{s\alpha_{23}}\{&a_{34}(c\theta_2c\theta_3-s\theta_2s\theta_3c\alpha_{23})+a_{23}c\theta_2+a_{12}+a_{51}c\theta_1+R_5s\theta_1s\alpha_{51}\\ &+a_{45}(c\theta_5c\theta_1-s\theta_5s\theta_1c\alpha_{51})+R_4(s\theta_5c\theta_1s\alpha_{45}+c\theta_5s\theta_1s\alpha_{45}c\alpha_{51}\\ &+s\theta_1c\alpha_{45}s\alpha_{51})\} \end{aligned}$$

Substituting for r_2 and r_3 from these two equations into (i), we find the following relation.

$$
\begin{aligned}
& s\theta_2 s\theta_3 s\alpha_{23}(a_{12}c\alpha_{12}s\alpha_{34}-a_{34}s\alpha_{12}c\alpha_{34}) \\
&\quad = s\alpha_{12}s\alpha_{34}\{a_{51}(c\theta_2 c\theta_4 c\theta_5-c\theta_2 s\theta_4 s\theta_5 c\alpha_{45}-c\theta_1 c\theta_3) \\
&\qquad +a_{45}(c\theta_2 c\theta_4-c\theta_1 c\theta_3 c\theta_5+s\theta_1 c\theta_3 s\theta_5 c\alpha_{51}) \\
&\qquad +a_{34}s\theta_3(c\theta_2 s\theta_3+s\theta_2 c\theta_3 c\alpha_{23})-a_{12}s\theta_2(s\theta_2 c\theta_3+c\theta_2 s\theta_3 c\alpha_{23}) \\
&\qquad +R_5(c\theta_2 s\theta_4 s\alpha_{45}-s\theta_1 c\theta_3 s\alpha_{51}) \\
&\qquad +R_1 c\theta_2(s\theta_2 c\theta_3 s\alpha_{12}+c\theta_2 s\theta_3 s\alpha_{12}c\alpha_{23}+s\theta_3 c\alpha_{12}s\alpha_{23}) \\
&\qquad -R_4 c\theta_3(c\theta_1 s\theta_5 s\alpha_{45}+s\theta_1 c\theta_5 s\alpha_{45}c\alpha_{51}+s\theta_1 c\alpha_{45}s\alpha_{51})\} \quad \text{(ii)}
\end{aligned}
$$

A second equation containing only joint angles is obtained by advancing the indices in equation (7.10) by 4:

$$
\begin{aligned}
& a_{45}(c\theta_3 c\theta_4-s\theta_3 s\theta_4 c\alpha_{34})+R_4 s\theta_3 s\alpha_{34}+a_{34}c\theta_3+a_{23}+a_{12}c\theta_2+R_1 s\theta_2 s\alpha_{12} \\
& +a_{51}(c\theta_1 c\theta_2-s\theta_1 s\theta_2 c\alpha_{12})+R_5(s\theta_1 c\theta_2 s\alpha_{51}+c\theta_1 s\theta_2 s\alpha_{51}c\alpha_{12}+s\theta_2 c\alpha_{51}s\alpha_{12}) \\
& = 0 \qquad \text{(iii)}
\end{aligned}
$$

Equations (ii) and (iii) may now be solved, in principle, simultaneously with three independent rotational five-bar closure equations, to yield the values of θ_1-θ_5 for the limit positions of joint 5. Certainly, for given linkage dimensions, solutions may be determined iteratively.

2.4 On stationary points of a linkage

In the preceding sections, whenever we set $\underset{\sim}{\omega}_j=\underset{\sim}{0}$ for joint j, we understood that rotation about that joint would instantaneously cease. That is, there would be no relative motion between the two links containing the joint. The practical significance of this fact lay in taking one of those links as permanently fixed. Then the other would be instantaneously fixed, and the joint variable θ_j would represent either an input or output angle. The analysis therefore indicated absolute motion limitation only if one of the two adjacent links was fixed to a frame of reference. In this context, we offered a screw system technique as a viable alternative (albeit a more profound one) to those already available for investigating the stationary points of input-output relationships.

Screw system theory has wider application, however, because it is of the essence in rigid body motion, constrained or unconstrained. We have already mentioned some of the other facets of its latent power in the uses given it above. Consider now the matter of stationary values for any point in a linkage. We may write down a position vector $\underset{\sim}{\rho}_P$, referred to our fixed frame, which locates a given point P on a link. If we denote the velocity of P by $\underset{\sim}{\mu}_P$, the point is stationary in space when

$$\dot{\underset{\sim}{\rho}}_P = \underset{\sim}{\mu}_P = \underset{\sim}{0}.$$

This result can be interpreted as a determinantal condition on linkage joint velocities. As in section 2.2, the condition yields algebraic constraints which, in principle, allow us to determine the linkage configurations for which the chosen point is motionless.

Hence we may, by this technique, find the stationary configurations for all points on the couplers of any linkage. To illustrate the method, we shall take a sample from the linkages already analysed for joint limitation positions.

The Bennett linkage

Refer to Fig. 2.2.1.

Let us regard the link joining axes 4 and 1 as fixed, so that the coupler is the link between joints 2 and 3. Consider any point P on the coupler distant pa_{23} $(0<p<1)$ from joint axis 2. The location of P is given by

$$\underset{\sim}{\rho}_P = \begin{pmatrix} a_{12}c\theta_1+pa_{23}c\theta_1c\theta_2-pa_{23}c\alpha_{12}s\theta_1s\theta_2 \\ a_{12}s\theta_1+pa_{23}s\theta_1c\theta_2+pa_{23}c\alpha_{12}c\theta_1s\theta_2 \\ pa_{23}s\alpha_{12}s\theta_2 \end{pmatrix}.$$

By differentiation, we obtain the following result for the velocity of P.

$$\underset{\sim}{\mu}_P = \begin{pmatrix} (-a_{12}s\theta_1-pa_{23}s\theta_1c\theta_2-pa_{23}c\alpha_{12}c\theta_1s\theta_2)\omega_1 \\ \qquad +(-pa_{23}c\theta_1s\theta_2-pa_{23}c\alpha_{12}s\theta_1c\theta_2)\omega_2 \\ (a_{12}c\theta_1+pa_{23}c\theta_1c\theta_2-pa_{23}c\alpha_{12}s\theta_1s\theta_2)\omega_1 \\ \qquad +(-pa_{23}s\theta_1s\theta_2+pa_{23}c\alpha_{12}c\theta_1c\theta_2)\omega_2 \\ pa_{23}s\alpha_{12}c\theta_2\omega_2 \end{pmatrix}$$

The stationary positions of P are indicated by setting $\underset{\sim}{\mu}_P=\underset{\sim}{0}$. Since, by the results of section 2.2, we know that $\omega_j\neq0$ for all j, we may conclude here that

$$c\theta_2 = 0 \qquad s\theta_2 = \sigma,$$

and so

$$\begin{vmatrix} -a_{12}s\theta_1 - pa_{23}c\alpha_{12}\sigma c\theta_1 & -pa_{23}\sigma c\theta_1 \\ a_{12}c\theta_1 - pa_{23}c\alpha_{12}\sigma s\theta_1 & -pa_{23}\sigma s\theta_1 \end{vmatrix} = 0.$$

The latter constraint implies

$$pa_{12}a_{23} = 0,$$

which cannot hold.

We therefore find that there is no point on the Bennett linkage which becomes stationary.

The P-R-C-R- linkage

Refer to Fig. 2.2.3.

We fix the link joining axes 2 and 3 and regard as coupler that joining axes 4 and 1. The point P to be considered will be distant pa_{41} from revolute axis 4. Then its position vector will be as follows.

$$\underset{\sim}{\rho}_P = \begin{pmatrix} a_{34}c\theta_3 + pa_{41}c\theta_3 c\theta_4 - pa_{41}s\theta_3 s\theta_4 \\ a_{34}s\theta_3 + pa_{41}s\theta_3 c\theta_4 + pa_{41}c\theta_3 s\theta_4 \\ r_3 + R_4 \end{pmatrix}$$

$$\therefore \quad \underset{\sim}{\mu}_P = \begin{pmatrix} (-a_{34}s\theta_3 - pa_{41}s\theta_3c\theta_4 - pa_{41}c\theta_3s\theta_4)\omega_3 \\ \qquad + (-pa_{41}c\theta_3s\theta_4 - pa_{41}s\theta_3c\theta_4)\omega_4 \\ (a_{34}c\theta_3 + pa_{41}c\theta_3c\theta_4 - pa_{41}s\theta_3s\theta_4)\omega_3 \\ \qquad + (-pa_{41}s\theta_3s\theta_4 + pa_{41}c\theta_3c\theta_4)\omega_4 \\ \mu_{3'} \end{pmatrix}$$

At stationary positions of P, $\underset{\sim}{\mu}_P = \underset{\sim}{0}$. Then

$$\mu_{3'} = 0.$$

From the results of section 2.2, then,

$$\mu_1 = 0 \qquad \text{and} \qquad s\theta_3 = 0 \quad c\theta_3 = \sigma.$$

Hence, from the expression for $\underset{\sim}{\mu}_P$ above,

$$\begin{vmatrix} -pa_{41}\sigma s\theta_4 & -pa_{41}\sigma s\theta_4 \\ a_{34}\sigma + pa_{41}\sigma c\theta_4 & pa_{41}\sigma c\theta_4 \end{vmatrix} = 0,$$

in order that ω_3, ω_4 are non-zero. Then

$$s\theta_4 = 0 \qquad c\theta_4 = \tau.$$

So we find that the coupler itself is stationary when $s\theta_3 = s\theta_4 = 0$. At this time, there is no translational motion in the prismatic and cylindric joints, but rotation continues about joints 2,3,4. The accompanying values of the other joint variables may be found from the linkage's closure equations.

The R-C-C-R-R- linkage

Refer to Fig. 2.3.1.

Here, we shall permanently fix the link between joints 2 and 3 and consider the coupler joining axes 4 and 5. The point P will be distant pa_{45} from joint 4. The position vector of P will be given by

$$\underset{\sim}{\rho}_P = \begin{pmatrix} a_{34}c\theta_3+pa_{45}c\theta_3c\theta_4-pa_{45}c\alpha_{34}s\theta_3s\theta_4+R_4s\alpha_{34}s\theta_3 \\ a_{34}s\theta_3+pa_{45}s\theta_3c\theta_4+pa_{45}c\alpha_{34}c\theta_3s\theta_4-R_4s\alpha_{34}c\theta_3 \\ r_3+pa_{45}s\alpha_{34}s\theta_4+R_4c\alpha_{34} \end{pmatrix}.$$

$$\therefore \underset{\sim}{\mu}_P = \begin{pmatrix} (-a_{34}s\theta_3-pa_{45}s\theta_3c\theta_4-pa_{45}c\alpha_{34}c\theta_3s\theta_4+R_4s\alpha_{34}c\theta_3)\omega_3 \\ +(-pa_{45}c\theta_3s\theta_4-pa_{45}c\alpha_{34}s\theta_3c\theta_4)\omega_4 \\ (a_{34}c\theta_3+pa_{45}c\theta_3c\theta_4-pa_{45}c\alpha_{34}s\theta_3s\theta_4+R_4s\alpha_{34}s\theta_3)\omega_3 \\ +(-pa_{45}s\theta_3s\theta_4+pa_{45}c\alpha_{34}c\theta_3c\theta_4)\omega_4 \\ \mu_3,+pa_{45}s\alpha_{34}c\theta_4\omega_4 \end{pmatrix}$$

In order for a non-trivial solution for ω_3, ω_4, μ_3, to be available, we must have the following relationship.

$$(-s\theta_3s\theta_4+c\alpha_{34}c\theta_3c\theta_4)(-a_{34}s\theta_3-pa_{45}s\theta_3c\theta_4-pa_{45}c\alpha_{34}c\theta_3s\theta_4 +R_4s\alpha_{34}c\theta_3)$$

$$=(-c\theta_3s\theta_4-c\alpha_{34}s\theta_3c\theta_4)(a_{34}c\theta_3+pa_{45}c\theta_3c\theta_4-pa_{45}c\alpha_{34}s\theta_3s\theta_4 +R_4s\alpha_{34}s\theta_3)$$

Simplifying, we require that

$$pa_{45}s^2\alpha_{34}s\theta_4c\theta_4+a_{34}s\theta_4+R_4s\alpha_{34}c\alpha_{34}c\theta_4 = 0.$$

This equation can be solved for θ_4, and substitution into the linkage's closure equations will permit the determination of the other joint variable values at the stationary positions of P.

3
LIMIT POSITIONS VIA CONNECTIVITY SUM REDUCTION

3.1 The ploy

We know, from our investigations into gross mobility criteria for overconstrained linkages, that it is always some kind of geometric specialty which is responsible for mobility in a linkage which has connectivity sum less than seven. The geometric constraints are in the form of relationships among the parameters of the linkage, that is the $a_{i\ i+1}$, $\alpha_{i\ i+1}$, R_i and h_i. The dimensional conditions so imposed provide the existence criteria for a linkage of a certain full-cycle mobility. There is another kind of geometric specialty, however, namely the actual configuration of the linkage, the spatial locations and orientations of the joints. For a linkage of gross mobility one, the configuration at any time ultimately depends on the value of one joint variable. Generally, in a given linkage, there are configurations which produce a transient mobility greater than the full-cycle mobility. In particular, such a situation characterises the joint limit positions of the linkage.

At such a limit position, at least one linkage variable will be instantaneously fixed while the others continue to change. In effect, the original linkage is momentarily functioning as another mobile linkage of smaller connectivity sum and, often, fewer links. One of the difficulties in algebraic linkage analysis is that equations become considerably more formidable as the number of variables increases. In view of the previous statement, we might consider the possibility of actually reducing the number of variables involved in a limit position analysis. Suppose that we were to lock the joint which is to exhibit a stationary position, and to replace the given linkage by another which is identical apart from the absence of the joint freedom being examined. We might then try to determine the geometric (or algebraic) constraints on the remaining joint variables which permit instantaneous mobility of the replacement linkage. It would possibly be necessary to re-interpret any results in terms of the original linkage variables and parameters. The great significance of the suggested technique is that it enables us to treat equations of lesser complexity than we should otherwise face. It is basically a labour-saving device, and is obviously closely related to the technique outlined in the second-last paragraph of the Introduction to chapter 2.

Let us now formalise the procedure. Given a linkage and a particular joint freedom to consider, we shall fix the appropriate variable and thereby alter the character of the loop in that region. The linkage will have a lower connectivity sum and we might have to renumber the joints. In doing so, we shall use roman capitals to distinguish them from the original numbering system. Where the number of links has been decreased, the new closure equations will be of lower order than the original ones. Also, near the joint being considered, the new variables and parameters will have a different meaning from their counterparts in the original linkage. At more than one joint distant from that being examined, however, variables and parameters will be unchanged in value. Having written down the closure equations for the replacement chain, we can, by direct differentiation, find the determinantal condition under which it will be instantaneously mobile. This condition, possibly together with the closure equations, allows us to solve, in principle, the limit position problem.

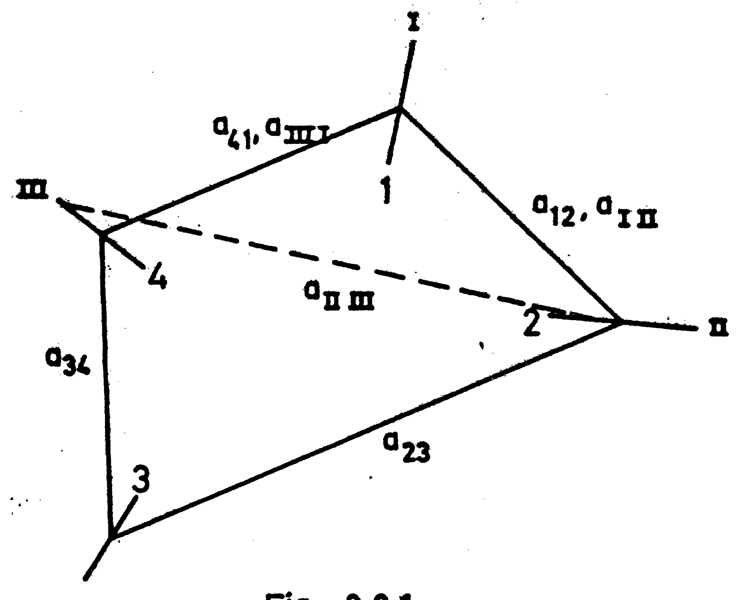


Fig. 3.2.1

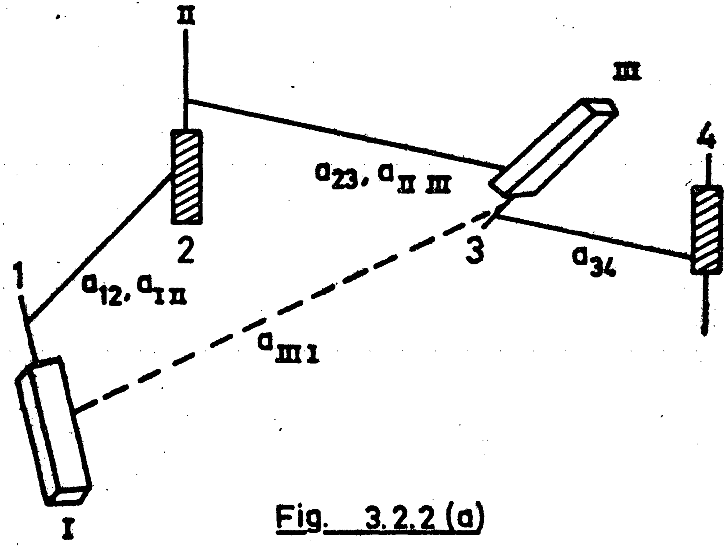


Fig. 3.2.2 (a)

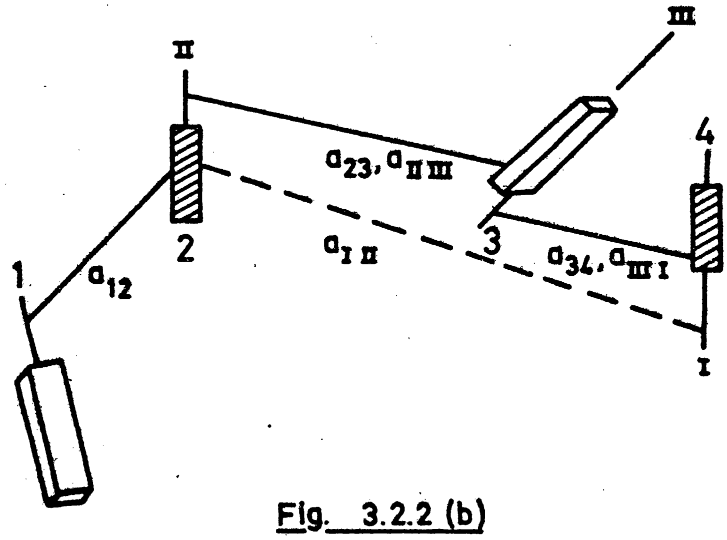


Fig. 3.2.2 (b)

3.2 Examples

We shall illustrate the procedure by reference to some of the examples used in section 2.2.

The Bennett linkage

Refer to Fig. 3.2.1.

To consider the limit positions of joint 3, we fix that joint and introduce a new link connecting axes 2 and 4. We now have a three-bar loop and must renumber the joints, as shown. We investigate the possible instantaneous mobility of this three-bar. We can use the dimensional constraints

$$R_I = 0 \qquad a_{III\ I}s\alpha_{I\ II} = a_{I\ II}s\alpha_{III\ I}.$$

Two appropriate closure equations are as follows.

$$-c\theta_I s\alpha_{III\ I}s\alpha_{I\ II} + c\alpha_{III\ I}c\alpha_{I\ II} = c\alpha_{II\ III} \tag{i}$$

$$a_{I\ II}s\theta_I s\alpha_{III\ I} + R_{II}c\alpha_{II\ III} + R_{III} = 0 \tag{ii}$$

Differentiation of (i) and (ii) yields the results

$$s\theta_I\dot{\theta}_I = c\theta_I\dot{\theta}_I = 0,$$

from which we may conclude

$$\dot{\theta}_1 = \dot{\theta}_I = 0.$$

It is then clear that the other two joints are also locked unless $\alpha_{12}=\alpha_{23}$; but this case has locked joints anyway. Hence, we again find that the Bennett linkage has no joint limit positions.

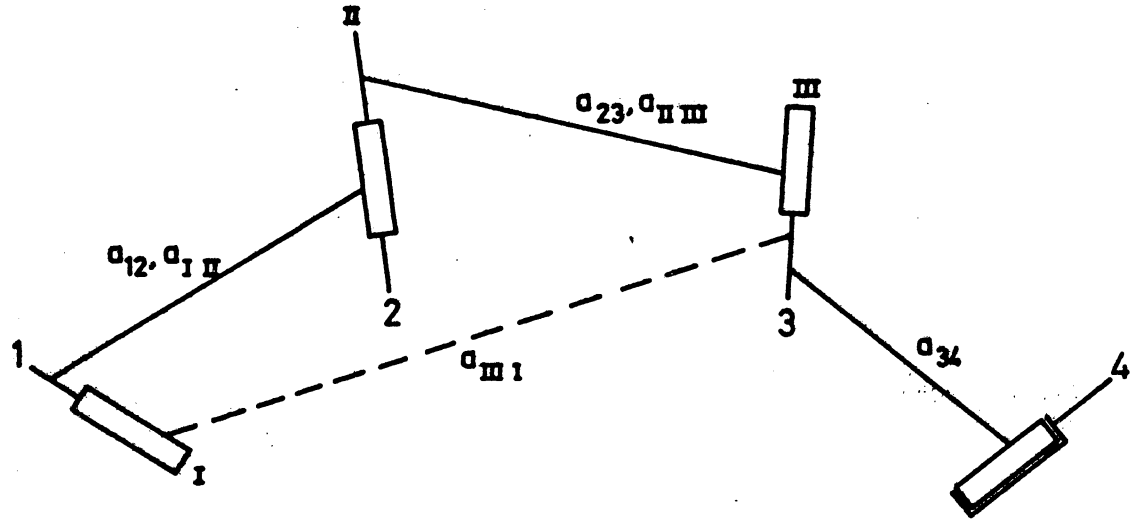


Fig. 3.2.3

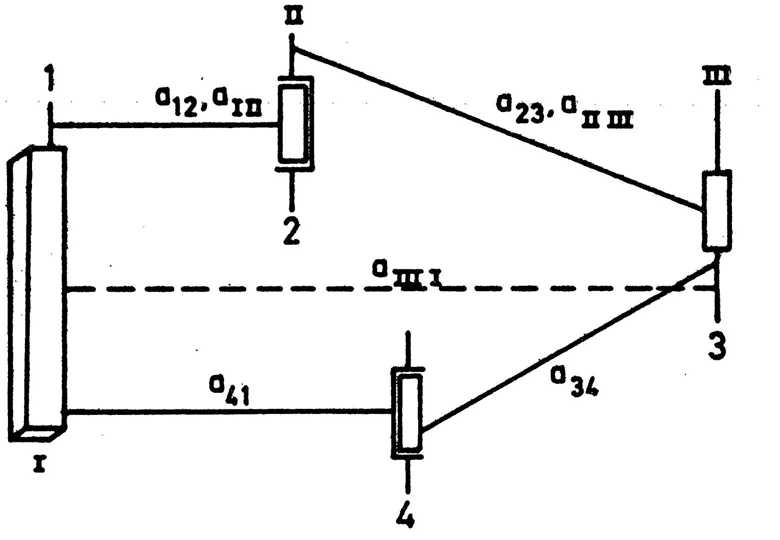


Fig. 3.2.4

The P-H-P-H- and C-C-C-R- linkages

Refer to Figs. 3.2.2 and 3.2.3.

We may consider the limit positions of joint 4 for both of these linkages simultaneously. In the same manner as for the Bennett linkage, we find that the following closure equation applies to the three-bar replacement loop.

$$-c\theta_{II} s\alpha_{I\ II} s\alpha_{II\ III} + c\alpha_{I\ II} c\alpha_{II\ III} = c\alpha_{III\ I} \tag{i}$$

(Of course, for the P-H-P-H- chain, the equation can be simplified due to the dimensional constraints obtaining. Such simplification serves no purpose here, however.) Differentiation of (i) results in

$$s\theta_{II}\dot{\theta}_{II} = 0.$$

Hence, in order for joint 2 (II) to remain mobile, we must have

$$s\theta_2 = 0.$$

The values of the remaining joint variables for either linkage may be found by means of the closure equations.

To seek out the limit positions of a slider in the P-H-P-H- linkage, we fix joint 1 and introduce a link connecting joints 4 and 2, as illustrated in Fig. 3.2.2(b). We may use the constraints

$$\alpha_{II\ III} = \alpha_{III\ I} = \frac{\pi}{2} \qquad c\alpha_{I\ II} = \sigma \qquad c\theta_{III} = \tau.$$

The four following closure equations are relevant.

$$a_{II\ III}c\theta_{II} + a_{I\ II} + a_{III\ I}c\theta_{I} + r_{III}s\theta_{I} = 0 \qquad \text{(i)}$$

$$\sigma a_{II\ III}s\theta_{II} \qquad - a_{III\ I}s\theta_{I} + r_{III}c\theta_{I} = 0 \qquad \text{(ii)}$$

$$s\theta_{II} \qquad - s\theta_{I} \qquad = 0 \qquad \text{(iii)}$$

$$c\theta_{I}s\theta_{II} + \sigma s\theta_{I}c\theta_{II} = 0 \qquad \text{(iv)}$$

By differentiation of the first three equations, we obtain the following results.

$$(-a_{II\ III}s\theta_{II})\dot{\theta}_{II} + (-a_{III\ I}s\theta_{I} + r_{III}c\theta_{I})\dot{\theta}_{I} + s\theta_{I}\dot{r}_{III} = 0$$

$$(\sigma a_{II\ III}c\theta_{II})\dot{\theta}_{II} + (-a_{III\ I}c\theta_{I} - r_{III}s\theta_{I})\dot{\theta}_{I} + c\theta_{I}\dot{r}_{III} = 0$$

$$c\theta_{II}\ \dot{\theta}_{II} \qquad - c\theta_{I}\ \dot{\theta}_{I} \qquad = 0$$

A non-trivial solution of these equations requires that

$$c\theta_{II}\{c\theta_{I}(-a_{III\ I}s\theta_{I} + r_{III}c\theta_{I}) + s\theta_{I}(a_{III\ I}c\theta_{I} + r_{III}s\theta_{I})\}$$
$$+ c\theta_{I}\ \{c\theta_{I}(-a_{II\ III}s\theta_{II}) - s\theta_{I}(\sigma a_{II\ III}c\theta_{II})\} = 0.$$

Hence,

$$r_{III}c\theta_{II} = a_{II\ III}c\theta_{I}\{c\theta_{I}s\theta_{II} + \sigma s\theta_{I}c\theta_{II}\} = 0,$$

by equation (iv) above.
Therefore, either r_{III} or $c\theta_{II}$ is zero. If $c\theta_{II}=0$, by (iv) $c\theta_{I}=0$. Then, by (ii), $a_{II\ III}=a_{III\ I}$; but such a constraint does not generally hold. We conclude that

$$r_3 = r_{III} = 0.$$

The linkage closure equations may be employed to determine the values of the other joint variables.

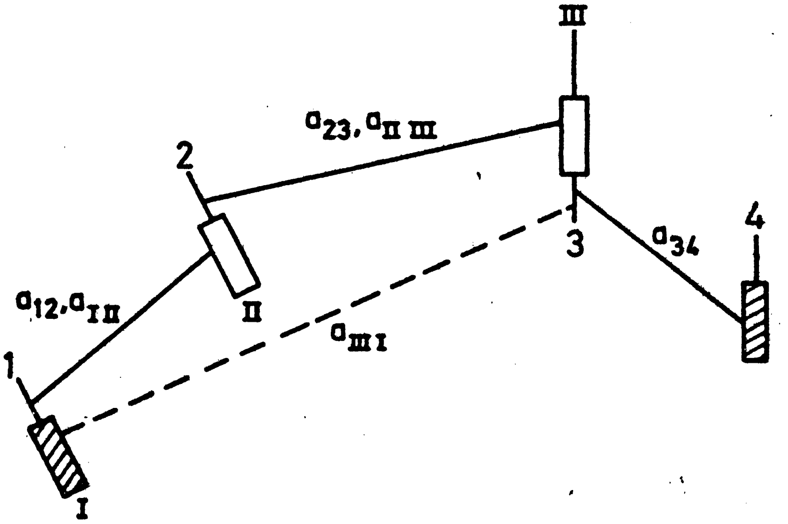


Fig. 3.2.5

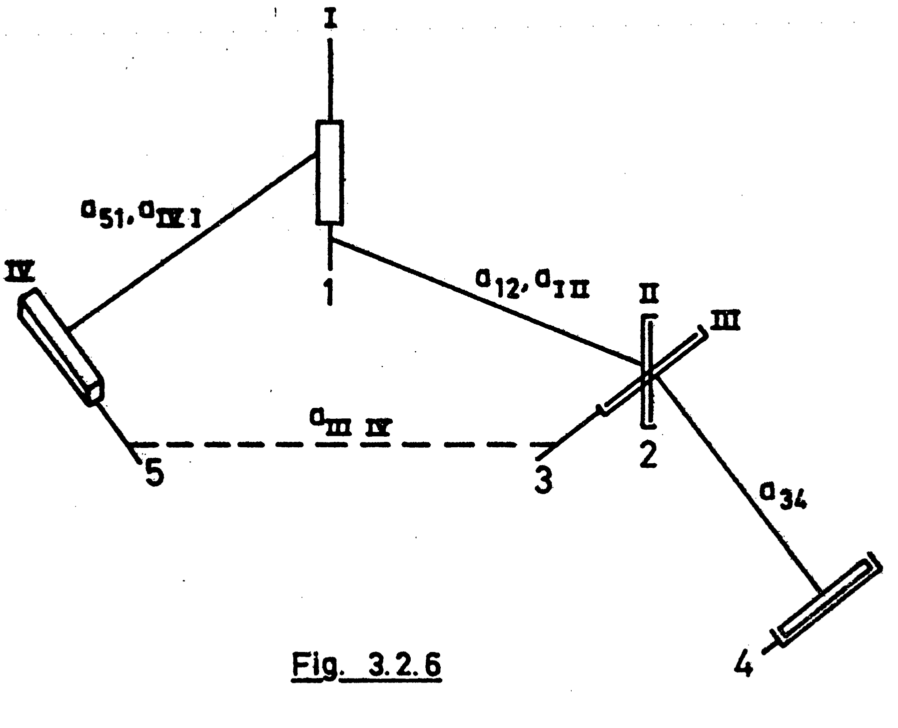


Fig. 3.2.6

The P-R-C-R- and H-C-C-H- linkages

Refer to Figs. 3.2.4 and 3.2.5.

Let us consider the limit positions of joint 4 of the P-R-C-R- linkage by locking the joint and providing a link which connects joints 3 and 1. We may use the dimensional condition

$$\alpha_{II\ III} = 0.$$

Relevant closure equations for the replacement three-bar are as follows.

$$s\theta_{I} = 0 \qquad \text{(i)}$$

$$a_{I\ II}c\theta_{I} + a_{III\ I} + a_{II\ III}c\theta_{III} = 0 \qquad \text{(ii)}$$

$$s\theta_{III} + c\theta_{I}s\theta_{II} = 0 \qquad \text{(iii)}$$

Differentiation of (ii) yields

$$a_{I\ II}s\theta_{I}\dot{\theta}_{I} + a_{II\ III}s\theta_{III}\dot{\theta}_{III} = 0.$$

Hence, by (i), if $\dot{\theta}_{III}$ is not to be zero,

$$s\theta_{III} = 0.$$

Then, by (iii),

$$s\theta_{2} = s\theta_{II} = 0.$$

The accompanying values of the other variables may be found from the closure equations.

Since, for the H-C-C-H- linkage, locking joint 4 and connecting joints 3 and 1 allows us to use the constraint

$$\alpha_{I\ II} = 0,$$

the analysis follows the same lines as for the P-R-C-R- loop above. We may conclude, therefore, that

$$s\theta_2 = 0.$$

The P-C-S-R- linkage

Refer to Fig. 3.2.6.

We consider the limit positions of the revolute by locking it and connecting joints 3 and 5. We may assume the following dimensional conditions.

$$\alpha_{I\ II} = R_{II} = a_{II\ III} = 0$$

Let us use the two following relevant four-bar closure equations for the replacement linkage.

$$c\alpha_{II\ III} = -c\theta_{IV}s\alpha_{III\ IV}s\alpha_{IV\ I} + c\alpha_{III\ IV}c\alpha_{IV\ I} \quad \text{(i)}$$

$$a_{I\ II}c\theta_{I} + a_{IV\ I} + a_{III\ IV}c\theta_{IV} + R_{III}s\theta_{IV}s\alpha_{III\ IV} = 0 \quad \text{(ii)}$$

Equation (i) implies, as is clear physically, that θ_{IV} is a constant, so that $\dot{\theta}_{IV}=0$. Differentiation of (ii) then yields

$$s\theta_I\dot{\theta}_I = 0,$$

whence, for joint I to be mobile,

$$s\theta_1 = s\theta_I = 0.$$

Once again, the other variables may be evaluated by means of the closure equations.

II

FULL-CYCLE MOBILITY

4

SOME PRELIMINARY RESULTS

4.1 Delassus hybrid five-bars

After reviewing the work of Bennett, Myard and Goldberg in producing new mobile overconstrained linkages by combining mobile loops of smaller connectivity sum, Waldron [43,45] examined whole classes of such hybrid linkages, paying particular attention to those with six links. The six-bar hybrids which he tabulated were produced by aligning the eighty-four different suitable pairs available from the thirteen Delassus four-bars in such a way that a common joint from each member linkage was removed and the remainder of each loop fused with the other. The removed joints would coincide with the fixed relative ISA of the two links which connected the two original loops.

Although Waldron mentioned the five-bars which could be constructed in the same way from pairs of Delassus three- and four-bars, he regarded the resulting loops as being "not of much interest". These five-bars are of relevance in this work, however, especially in the remainder of this chapter and in chapter 7. Hence, in Table 4.1.1 below, the five-bar Delassus hybrids are listed, with reference to the loops whence they were constructed. The three columns of the table are assigned to the Delassus three-bars as indicated, and the thirteen rows to the Delassus four-bars, numbered as in chapter 5. The results presented in the table should not be regarded as original here, and the reader is referred to the

Table 4.1.1

	H=H=H=	H=H⁀P-		P-P-P-
d.1	~ parallel-screw	~ parallel-screw	-	-
d.2	~ parallel-screw	~ parallel-screw	H⁀H⁀H-H=H- coaxial pair normal to others	~ parallel-screw
d.3	~ parallel-screw	~ parallel-screw	H⁀H-H=H-P- as prec.	~ parallel-screw
d.4	~ parallel-screw	~ parallel-screw	H$\overline{\text{-H=H-}}$H-P- as prec.	~ parallel-screw
d.5	R-R-R-H=H- spherical	R-R-R-H=P- spherical	-	-
d.6	-	-	P-P-P-H=H-	part-chain mobility
d.7	part-chain mobility	part-chain mobility	-	-
d.8	part-chain mobility	part-chain mobility	H=H-H=H-P- parallel planes	~ parallel-screw
d.9	~ parallel-screw	~ parallel-screw	-	-
d.10	~ parallel-screw	~ parallel-screw	-	-
d.11	~ parallel-screw	~ parallel-screw	-	-
d.12	~ parallel-screw	~ parallel-screw	H$\overline{\text{-P-}}$H-H=H- bilateral symmetry	~ parallel-screw
d.13	R-R-R-H=H- Bennett	R-R-R-H=P- Bennett	-	-

The new notation employed in this table is explained fully in section 4.2.

abovementioned works of Waldron for a full account of the principles involved.

One of the most important features of the five-link hybrid linkages is the -H=H- and, to a lesser extent, the -H=P- joint combinations which are kinematically equivalent to single revolutes or slider. This matter will be discussed at length in the next section. The parallel-screw linkage, which appears so frequently in the table, also figures largely in chapter 7, where detailed references are given for it. Its basic form comprises five parallel screws of arbitrary pitch, but up to three of the screw joints may be replaced by arbitrarily oriented sliders, without affecting the mobility of the linkage.

Table 4.1.1 is necessarily concise and, because it is, lacks some precision. Some additional comment is in order here to amplify those results which are stated possibly too briefly. The P-P-P-H=H- hybrid obtained by combining d.6 with H=H^P- is kinematically equivalent to d.6, the H=H combination acting as a slider; the linkage, however, might be alternatively regarded as a special case of a parallel-screw linkage, where the only two actual screw joints present are coaxial. There are five loops described as having part-chain mobility; some of them will exhibit permanently locked joints, whilst others will possess mobility greater than one. The many parallel-screw linkages which result are all special in some sense; for example, there are cases of coaxial screws, screws of equal pitch, and parallel screw joint and slider.

4.2 On coaxial screws

Consider the 'sequence' of known overconstrained linkages, mobility one, P-P-, H-H-P- and H-H-H-H-. In the first, the two prismatic pairs are parallel; in the second, the screws are coaxial and parallel to the slider; the third consists of two parallel pairs of coaxial screw joints. All three loops have the same overall motion.

We can obtain the second chain from the first by replacing one of the sliders by a pair of coaxial screws. Whilst the screws rotate with respect to each other, there is no net rotation of the combination. Their combined movement consists of translation only, so that together they act as a prismatic joint. Similarly, we may obtain the third from the second by replacing the other slider by another pair of coaxial screws. Within each coaxial combination there is rotation, but the only movement relating the two groups is pure translation parallel to them.

Now regard the 'incomplete sequence' of known over-constrained linkages, mobility one, P-P-P- and H-H-P-P-. In the first, the three prismatic joints lie in parallel planes; in the second, the screws are coaxial and lie in a plane parallel to planes containing the sliders. For the reasons given above, we may complete the sequence with the two linkages, H-H-H-H-P- and H-H-H-H-H-H-. For the first of them, we have two pairs of coaxial screws and a slider, each joint axis being parallel to the same plane. For the second, the axes of the three pairs of coaxial screw joints are all parallel to the same plane. These two linkages have apparently not been previously described in any specific manner.

Let us now consider the general four-slider, and again generate linkages by the same technique. We obtain H-H-P-P-P-, H-H-H-H-P-P-, H-H-P-H-H-P- and H-H-H-H-H-H-P- chains. Once more, we have replaced prismatic joints by coaxial screw combinations which may produce the same net motion. In each case, the new linkage has mobility one. We could not, however, replace all four sliders by coaxial screws. Essentially, the behaviour of coaxial screw pairs as sliders depends on the rest of the loop constraining them to possess no net rotation. The constraint will hold for replacement of up to two prismatic pairs, but breaks down for the third. The seven-bar H-H-H-H-H-H-P- mentioned above will not, in general, be so overconstrained, and will function as a mobility one constrained linkage. Replacement of all four sliding joints by coaxial screws would result in a chain of mobility two. The linkage consisting of four pairs of coaxial screws would be actually equivalent to a C-C-C-C- four-bar.

For later convenience, we shall refer to the eleven relevant chains given above as 'prismatic linkages', since the motion of each joint (regarding a pair of coaxial screws as a single joint in this context) is purely translational. There are no other single-loop linkages which possess this property. The process of replacing -P- by -H-H- is precisely equivalent to Waldron's [43,45] development of hybrid linkages, if one of the primary chains used is the H-H-P- Delassus three-bar. Table 4.1.1 illustrates this fact.

As a consequence of the above discussion, one might well reconsider the present convention for counting joints. Should a coaxial pair of screws equivalent to a slider be regarded as two joints, or one? Should the sequence of linkages, P-P-, H-H-P-, H-H-H-H-, be thought of as variations of a two-bar linkage, where the H-H combination is seen as a special kind of connectivity one joint? We shall henceforth use a new notation, namely H=H, for the coaxial screws combination which acts as a slider, as a concession to the unique role played by it. It will also be of great assistance as a shorthand method of describing relevant loops, especially for a linkage with a large number of joints or more than one coaxial combination. For related reasons, we shall also use the notation $J \frown J$ to indicate that two joints are parallel; in such a case, it may be inferred that the joints act in their normal manner, and that nothing more than parallelism is indicated. Both of these notations were employed to advantage in Table 4.1.1, and will be used to a much greater extent from now on.

Some weight is added to the consideration of the H=H combination as a single joint by comparing the independent closure equations of, for example, the $P \frown P$-, $H{=}H \frown P$-, $H{=}H \frown H{=}H$- sequence of linkages. The following equations are easily obtained by substituting the appropriate dimensional constraints into the two-, three- and four-bar sets of closure equations.

$P \frown P$-: $\quad r_1 + r_2 = 0$.......(i)

with $\quad c\theta_1 = c\theta_2 = -1$

$$\text{H=H}\hat{}\text{P-:} \qquad R_1 + R_2 + h_1\theta_1 + h_2\theta_2 + r_3 = 0 \ldots\ldots\ldots\ldots \text{(i)}$$

$$\theta_1 + \theta_2 = (2k+1)\pi \ldots\ldots \text{(ii)}$$

$$\text{with} \qquad c\theta_3 = -1$$

$$\text{H=H}\hat{}\text{H=H-:} \qquad R_1+R_2+R_3+R_4+h_1\theta_1+h_2\theta_2+h_3\theta_3+h_4\theta_4 = 0 \ldots\ldots\ldots\ldots \text{(i)}$$

$$\theta_1 + \theta_2 = (2k+1)\pi \ldots\ldots \text{(ii)}$$

$$\theta_3 + \theta_4 = (2l+1)\pi \ldots\ldots \text{(iii)}$$

The number of independent closure equations in each case is, as expected, one less than the connectivity sum of the corresponding linkage. For the latter two linkages, however, we can restructure the closure equations and dimensional constraints so that they are directly comparable with those of the first linkage. The alternative formulation is given below.

$$\text{H=H}\hat{}\text{P-:} \qquad ([R_1+h_1\theta_1]+[R_2+h_2\theta_2]) + \mathbf{r}_3 = 0 \ldots\ldots\ldots \text{(i)}$$

$$\text{with} \qquad c(\theta_1+\theta_2) = c\theta_3 = -1$$

$$\text{H=H}\hat{}\text{H=H-:} \qquad ([R_1+h_1\theta_1]+[R_2+h_2\theta_2])+([R_3+h_3\theta_3]+[R_4+h_4\theta_4])=0 \ldots \text{(i)}$$

$$\text{with} \qquad c(\theta_1+\theta_2)=c(\theta_3+\theta_4) = -1$$

Each of the three linkages is now described by one translational closure equation, relating the rectilinear motions of the two sliders or their equivalents. The subsidiary equations are either geometrical constraints, or relations which are internal to the sliding elements.

The prismatic linkages are not the only chains in which the H=H group may be used in place of a prismatic pair. Any known loop which is proper, has mobility one and connectivity sum less than six, with an available prismatic joint, may be operated upon in the same manner. Such linkages as the F-H-P-, C$\widehat{-}$R-P-P- and C$\widehat{-}$R$\widehat{-}$R-P- may be so treated, thus producing an apparently new linkage, but with net motion characteristics the same as those of its progenitor.

It is often convenient to regard a cylindric joint as a pair of coaxial screws. Indeed, Hunt [30] considers the cylindric pair to be fundamentally so composed. In view of the preceding discussion, however, it becomes clear that, whilst we may always make the exact kinematic replacement of a pair of coaxial screws for a cylindric joint, we cannot always perform the reverse operation without introducing a passive degree of freedom. If, for example, we were to replace the H=H combination in H=H$\widehat{-}$P- by a cylindric pair, there would be no rotation of the new joint. All rotation involved in the H=H group is internal to it, and so the cylindric joint would function as a slider.

So far, we have been concerned solely with the equivalence of a pair of coaxial screws and a prismatic joint. But it is also possible for the H=H group to function as a revolute, where now all translational movement is internal to the combination. Examples of this latter equivalence are available in Table 4.1.1. Indeed, there are cases where the combination of parallel screw and prismatic joints acts as a pure turning pair. We shall denote this group by H=P, examples again being evident in Table 4.1.1. Here also, translational motion of the screw and slider is confined to within the combination. As remarked earlier, the net motion of a -H=H- or -H=P- combination is determined by the possible behaviour of the rest of the linkage, and ultimately therefore by the geometrical constraints present.

The fact that the -H=H- or -H=P- combination acts as a connectivity one joint does not imply a relationship with, say, the -S-S- group in the R-S-S-R- chain or the -S- joint in the S-R-P-R- linkage. In the former, the -S-S- combination has connectivity sum five because of the common joint axis; in the latter, the -S- joint has connectivity sum two because only two other turning pairs are present. Both of these examples evince a loss of one potential degree of freedom. The -H=H- or -H=P- combination has suffered no such degeneracy; the two degrees of freedom are employed, but in completely different ways.

4.3 A theorem for linkages with screw joints

In chapters 6 and 7, we shall be concerned with establishing existence criteria for certain overconstrained linkages with screw joints. The transcendentality of the closure equations containing screw joint variables makes them extremely difficult to analyse. The problem can be significantly alleviated, however, in the cases where a simple relationship obtains among the pitches of some of the screw joints present. It would be of great value if, having solved such a special case, there were a means of using the result to solve the general problem. The purpose of this section is to provide that means. Without further comment here, we shall establish a theorem which allows us to proceed from the particular to the general, in this context; the theorem will be cited many times in later pages.

Consider a linkage being tested for mobility. Tie some screw pitches temporarily, for example by setting a pair of pitches equal.

(i) Suppose a certain joint is shown to be locked in rotation. Then that joint will be also locked in the spherical indicatrix. It the screw pitches are now untied, the spherical indicatrix will remain unchanged. Thus, the joint in the original linkage will be likewise locked in rotation.

(ii) Suppose that the linkage is found to be rotationally mobile under certain dimensional conditions. The spherical indicatrix will be mobile. On untying the screw pitches, the spherical indicatrix will remain unchanged. In view of (i), for the original linkage to be potentially mobile in rotation, the previously established constraints must remain.

We exclude from the foregoing consideration the rotational mobility of a screw joint which is part of a -H=H- group acting as a single prismatic pair in a mobile linkage. Rotational movement of either screw in the group is internal to the combination, and the group itself would not rightly appear in the spherical indicatrix.

5

THE DELASSUS LINKAGES

Introduction

Etienne Delassus was probably the pioneer in the study of mobility of spatial linkages. Among his notable achievements was the determination of all four-bar linkages, connectivity sum four and mobility unity. In more recent times, Dimentberg and Yoslovich [15], Waldron [45,47,48] and others [1,2,4,5,9,10,25,32-36,38] have examined for mobility either whole classes of four-bar linkages or particular linkages using the procedure of solution of closure equations.

Delassus's group of overconstrained four-bars, however, remains a classic in the field. It must therefore appear odd that his work was largely either accepted without comment by later workers or ignored. Part of the reason is given by Waldron [45]: Delassus's reference [11] especially is difficult to read, and so application of the techniques developed therein approaches incoherency. It is also worth noting that, whilst within two years Delassus produced eight [12] of the linkages, the remaining few [13] were not published for another two decades. This fact is an indication that the problem was far from simple, even for Delassus.

Because, then, there seemed to be at least a possibility that Delassus's results were erroneous or incomplete, it was considered worthwhile to attempt a rederivation, with the presumably more powerful techniques now available. A new derivation would be of particular significance in that latter-day research which depends to some extent on Delassus's findings; examples are references [1,4,35,36,43,45,48]. A curious feature of Delassus's results was the absence of constraints relating to some of the constants in a linkage construction, particularly the fixed offsets of the joints. For example, it seemed that, in every Delassus linkage, the constant R_i was zero, for all i; this unlikely state of affairs helped to motivate the following investigation.

Whilst the analysis presented in this chapter is original, I have obviously proceeded with the belief that Delassus's findings were substantially correct. Without his prior inspiration, the following work would probably have not been attempted.

In this chapter, since we are interested in four-bar linkages, equations (iv) and (v) of chapter 1 are reducible to

$$\underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{\underline{V}}_{23}\underline{\underline{U}}_3\underline{\underline{V}}_{34}\underline{\underline{U}}_4\underline{\underline{V}}_{41} = \underline{\underline{I}}$$

and $$\underline{\underline{U}}_1\underline{s}_1 + \underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{s}_2 + \underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{\underline{V}}_{23}\underline{\underline{U}}_3\underline{s}_3 + \underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{\underline{V}}_{23}\underline{\underline{U}}_3\underline{\underline{V}}_{34}\underline{\underline{U}}_4\underline{s}_4 = \underline{0}.$$

More conveniently, after Waldron [45,47], we may re-express them as follows, using the fact that the $\underline{\underline{U}}_i$ and $\underline{\underline{V}}_{i\ i+1}$ are orthogonal matrices.

$$\underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{\underline{V}}_{23} = \underline{\underline{V}}_{41}{}^T\underline{\underline{U}}_4{}^T\underline{\underline{V}}_{34}{}^T\underline{\underline{U}}_3{}^T \qquad \text{(i)}$$

and $$\underline{\underline{V}}_{23}{}^T\underline{\underline{U}}_2{}^T\underline{\underline{V}}_{12}{}^T\underline{s}_1 + \underline{\underline{V}}_{23}{}^T\underline{s}_2 + \underline{\underline{U}}_3\underline{s}_3 + \underline{\underline{U}}_3\underline{\underline{V}}_{34}\underline{\underline{U}}_4\underline{s}_4 = \underline{0}. \qquad \text{(ii)}$$

If we now expand (i) and (ii), using the definitions of $\underline{s}_i$, $\underline{\underline{U}}_i$ and $\underline{\underline{V}}_{i\ i+1}$, we obtain the twelve scalar closure equations:

$$c\theta_1 c\theta_2 - s\theta_1 s\theta_2 c\alpha_{12} = c\theta_3 c\theta_4 - s\theta_3 s\theta_4 c\alpha_{34} \tag{5.1}$$

$$-c\theta_1 s\theta_2 c\alpha_{23} - s\theta_1 c\theta_2 c\alpha_{12} c\alpha_{23} + s\theta_1 s\alpha_{12} s\alpha_{23} = s\theta_3 c\theta_4 + c\theta_3 s\theta_4 c\alpha_{34} \tag{5.2}$$

$$c\theta_1 s\theta_2 s\alpha_{23} + s\theta_1 c\theta_2 c\alpha_{12} s\alpha_{23} + s\theta_1 s\alpha_{12} c\alpha_{23} = s\theta_4 s\alpha_{34} \tag{5.3}$$

$$s\theta_1 c\theta_2 + c\theta_1 s\theta_2 c\alpha_{12} = -c\theta_3 s\theta_4 c\alpha_{41} - s\theta_3 c\theta_4 c\alpha_{34} c\alpha_{41} + s\theta_3 s\alpha_{34} s\alpha_{41} \tag{5.4}$$

$$-s\theta_1 s\theta_2 c\alpha_{23} + c\theta_1 c\theta_2 c\alpha_{12} c\alpha_{23} - c\theta_1 s\alpha_{12} s\alpha_{23}$$
$$= - s\theta_3 s\theta_4 c\alpha_{41} + c\theta_3 c\theta_4 c\alpha_{34} c\alpha_{41} - c\theta_3 s\alpha_{34} s\alpha_{41} \tag{5.5}$$

$$s\theta_1 s\theta_2 s\alpha_{23} - c\theta_1 c\theta_2 c\alpha_{12} s\alpha_{23} - c\theta_1 s\alpha_{12} c\alpha_{23} = c\theta_4 s\alpha_{34} c\alpha_{41} + c\alpha_{34} s\alpha_{41} \tag{5.6}$$

$$s\theta_2 s\alpha_{12} = c\theta_3 s\theta_4 s\alpha_{41} + s\theta_3 c\theta_4 c\alpha_{34} s\alpha_{41} + s\theta_3 s\alpha_{34} c\alpha_{41} \tag{5.7}$$

$$c\theta_2 s\alpha_{12} c\alpha_{23} + c\alpha_{12} s\alpha_{23} = s\theta_3 s\theta_4 s\alpha_{41} - c\theta_3 c\theta_4 c\alpha_{34} s\alpha_{41} - c\theta_3 s\alpha_{34} c\alpha_{41} \tag{5.8}$$

$$- c\theta_2 s\alpha_{12} s\alpha_{23} + c\alpha_{12} c\alpha_{23} = - c\theta_4 s\alpha_{34} s\alpha_{41} + c\alpha_{34} c\alpha_{41} \tag{5.9}$$

$$a_{41}(c\theta_3 c\theta_4 - s\theta_3 s\theta_4 c\alpha_{34}) + r_4 s\theta_3 s\alpha_{34} + a_{34} c\theta_3 + a_{23} + a_{12} c\theta_2$$
$$+ r_1 s\theta_2 s\alpha_{12} = 0 \tag{5.10}$$

$$a_{41}(s\theta_3 c\theta_4 + c\theta_3 s\theta_4 c\alpha_{34}) - r_4 c\theta_3 s\alpha_{34} + a_{34} s\theta_3 + r_2 s\alpha_{23}$$
$$- a_{12} s\theta_2 c\alpha_{23} + r_1(c\theta_2 s\alpha_{12} c\alpha_{23} + c\alpha_{12} s\alpha_{23}) = 0 \tag{5.11}$$

$$a_{41} s\theta_4 s\alpha_{34} + r_4 c\alpha_{34} + r_3 + r_2 c\alpha_{23} + a_{12} s\theta_2 s\alpha_{23}$$
$$+ r_1(c\alpha_{12} c\alpha_{23} - c\theta_2 s\alpha_{12} s\alpha_{23}) = 0 \tag{5.12}$$

It should perhaps be noted here that the forms (i), (ii) of the closure equations were arbitrary, although convenient, and a different choice would have produced a set of scalar equations noticeably varied in appearance from that given above. For example, analogously with equation (5.1) above, we might have obtained

$$c\theta_2 c\theta_3 - s\theta_2 s\theta_3 c\alpha_{23} = c\theta_4 c\theta_1 - s\theta_4 s\theta_1 c\alpha_{41}.$$

Similar results can be easily written down for the other relationships by simple cyclic advancing of the indices.

5.1 The strategy

To some extent, Delassus seems to have gone straight to the heart of the problem. That is, he sought out precisely those four-bar linkages which are mobile while possessing only joints of connectivity one. We could also adopt this approach, by testing every eligible loop in the closure equations and attempting to beat them into submission. There would be major drawbacks, such as the large number of linkages to be tried and the even more frightening prospect of checking a multitude of dimensional possibilities for each one. We should also, by such a method, be generally starting again with each new chain, having made no use of results obtained by treating a prior candidate.

We can, instead, reduce the sheer magnitude of the exercise by attacking the problem more systematically. We note that the closure equations can be considerably simplified by imposing certain dimensional constraints. We can, then, almost reverse the technique suggested above by dividing all four-bars into several categories, each one characterised by a particular set of dimensional constraints. We then fit into each category all appropriate linkages with connectivity sum four, inventing a sufficient number of convenient categories to cover all possibilities.

Having done so, we analyse each category in turn. Any set of dimensional conditions will immediately diminish the number of feasible linkages with mobility one and connectivity sum seven. These are quickly determined and become sub-categories. For a reason which will soon be clear, linkages containing spherical or plane joints may be excluded.

It was pointed out in chapter 1 that screws, revolutes and sliders can be regarded as degeneracies of cylindric joints, and that a revolute may be thought of as a screw joint of zero pitch. Now suppose we have established the feasibility of a certain linkage containing a cylindric joint, say joint 2. If we constrain r_2 to be a constant, joint 2 becomes a revolute; if we fix θ_2, the joint becomes a slider; demanding a linear relationship between r_2 and θ_2 makes the joint a screw. The operation of restraining a joint variable in one such way will reduce the number of linkage variables by one. Then, in order for the closure equations to be formed into a new independent set, additional dimensional constraints will usually be demanded.

Each time we reduce the connectivity of a joint we are forced to accept more geometrical restrictions on the linkage construction. In so doing, we are constantly eliminating unworkable candidates. After the relevant number of reductions, the endpoint of each sub-category is reached, that is when every joint of the linkage is a slider or a screw (revolute). At this stage, we have isolated the Delassus linkages contained in the group along with the concomitant geometrical constraints. By the principle enunciated in chapter 1, each Delassus linkage will be governed by three independent closure equations in the four remaining joint variables.

As an example, let us briefly consider the category headed by the C-H-C-H- linkage, which has been thoroughly investigated (See section 6.2.). By joint connectivity reductions, this chain will eventually yield the H-H-H-H-, H-H-P-H- and P-H-P-H- derivatives, together with any relevant -R- derivatives, or eliminate them on the way. In addition, during the process, the C-H-H-H- and C-H-P-H- linkages will be tested for mobility. Thus, we reach our goal not directly, but via more general linkages. In so doing, we also reap the benefit of a mobility study for these higher order loops.

While following the general guidelines just put forward, it will be seen in the following sections that the detailed procedure for an arbitrary sub-category cannot be formalised. Special techniques must be applied in many cases; these will be explained as they are required. It is one of the strange and challenging aspects of linkage analysis, especially where screw joints are concerned, that singularities in one form or another continually recur. Mobility, in particular, is heavily dependent on geometric properties. In the recent past, a confusion between full-cycle mobility and transient mobility due to a linkage instantaneously adopting a certain position has led some researchers to incorrect conclusions.

Our classification scheme consists of splitting four-bars into those with parallel adjacent joint axes and those without. For the latter group, we must choose enough convenient categories to include all four-bar chains not covered by the former group. Consider now the following categories.

one pair of adjacent parallel joints
two pairs of adjacent parallel joints
three parallel joints
all joints parallel
general C-H-C-H- derivatives (no two adjacent joints parallel)

On inspection, it is clear that these five categories will include as derivatives all connectivity sum four linkages except those with adjacent sliders and non-parallel adjacent joint axes. Among these exclusions is the P-P-H-H-, no two adjacent joint axes parallel; thus, if we can easily treat the linkage with two adjacent sliders, we shall have also dealt with the case of two adjacent screws (revolutes), which too is not completely covered by the above categories. (Strictly, we should also look at the category headed by the H-H-C-C- linkage, if it exists. It is evident from the results of sections 6.5 and 6.6, however, that, even if there is a proper, mobile H-H-C-C- linkage with no two adjacent joints parallel, its only connectivity sum four derivative will be the spherical linkage, Delassus solution number d.5. We shall show, in those sections, that the H-H-C-C- linkage has the two mobile degeneracies of the spherical four-bar and the Bennett linkage, d.13.)

We can now proceed to analyse these slider chains and leave the five main categories for sections 5.2-5.6. The labels assigned to individual Delassus solutions are the chronologically-based ones chosen by Waldron in reference [45]. There seems to be no real point in altering them to suit the order in which they are uncovered here.

Spatial four-slider

$$\alpha_{12} \neq 0, \qquad \alpha_{23} \neq 0, \qquad \alpha_{34} \neq 0, \qquad \alpha_{41} \neq 0,\pi$$

If any two of the prismatic axes are parallel, the linkage has part-chain mobility, and is based on a P^P- two-bar linkage.

The general P-P-P-P- chain is known to be mobile. In terms of the closure equations, its mobility of unity is easily seen. Since all four joint angles are fixed, there are no rotational equations to be considered. There remain precisely three independent equations, the translational ones (5.10)-(5.12), in four unknowns r_1-r_4.

Thus, mobility unity in the general case is established; special dimensional conditions may produce mobility greater than one. This is the Delassus solution d.6.

Three-slider linkages

$$\alpha_{12} \neq 0, \quad \alpha_{23} \neq 0, \quad \alpha_{34} \neq 0, \quad \alpha_{41} \neq 0,\pi$$

Here we have three joint angles fixed, say θ_1, θ_2, θ_3. From (5.3) for example, since $s\alpha_{34} \neq 0$, θ_4 is also fixed.

The linkage immediately degenerates to the P-P-P-P- chain which has already been treated.

We conclude that there is no four-bar linkage with precisely three sliders and non-parallel joint axes.

Hence there can be no Delassus solution under these conditions.

Linkages with two adjacent sliders

$$\alpha_{12} \neq 0, \quad \alpha_{23} \neq 0, \quad \alpha_{34} \neq 0, \quad \alpha_{41} \neq 0,\pi$$

Let us suppose that joints 1 and 2 are prismatic. Then, by (5.3) for example, since $s\alpha_{34} \neq 0$, θ_4 is also fixed. We now have a three-slider chain which has previously been treated.

Again we have no Delassus solution.

5.2 One pair of adjacent parallel joints

Put

$$\alpha_{34} = 0.$$

$$\alpha_{12} \neq 0, \qquad \alpha_{23} \neq 0, \qquad \alpha_{41} \neq 0, \pi$$

From equation (5.9), joint 2 must be prismatic.

$$c\theta_2 = \frac{1}{s\alpha_{12}s\alpha_{23}}\{c\alpha_{12}c\alpha_{23} - c\alpha_{41}\} \tag{5.2.1}$$

From equations (5.8) and (5.2.1),

$$c(\theta_3+\theta_4) = -\frac{1}{s\alpha_{41}}\{c\alpha_{12}s\alpha_{23} + c\theta_2 s\alpha_{12}c\alpha_{23}\}$$

$$= \frac{1}{s\alpha_{23}s\alpha_{41}}\{c\alpha_{23}c\alpha_{41} - c\alpha_{12}\}. \tag{5.2.2}$$

From equations (5.3) and (5.6), joint 1 is also prismatic. Eliminating $s\theta_1$ and $s\theta_2$ between them,

$$c\theta_2 c\alpha_{12}s\alpha_{23} + s\alpha_{12}c\alpha_{23} = -s\alpha_{41}c\theta_1$$

whence, by use of (5.2.1),

$$c\theta_1 = \frac{1}{s\alpha_{12}s\alpha_{41}}\{c\alpha_{12}c\alpha_{41} - c\alpha_{23}\}. \tag{5.2.3}$$

From equation (5.2.2), if either joint 3 or 4 is a slider, then so is the other. This would give us a special case of the P-P-P-P- linkage with part-chain mobility.

We may assume then that neither of joints 3 and 4 is prismatic. Nor can both joints 3 and 4 be cylindric.

Evidently, (5.2.2) is the only independent rotational closure equation. Together with the translational equations, it provides us with four independent equations. Thus, any linkage with mobility one must have connectivity sum of five or less. In view of the foregoing, the only linkage of mobility unity with connectivity sum five is the P-P-H-C-, a special parallel-screw linkage [27,40,42,45].

. . . .

To seek out mobile linkages of connectivity sum four, we may replace the cylindric joint by a screw only. Our three translational closure equations will be as follows.

$$a_{41}c(\theta_3+\theta_4) + a_{34}c\theta_3 + a_{23} + a_{12}c\theta_2 + r_1 s\theta_2 s\alpha_{12} = 0 \qquad \text{(i)}$$

$$a_{41}s(\theta_3+\theta_4) + a_{34}s\theta_3 + r_2 s\alpha_{23} - a_{12}s\theta_2 c\alpha_{23}$$
$$+ r_1(c\alpha_{23}s\alpha_{12}c\theta_2 + c\alpha_{12}s\alpha_{23}) = 0 \qquad \text{(ii)}$$

$$(R_4+h_4\theta_4) + (R_3+h_3\theta_3) + r_2 c\alpha_{23} + a_{12}s\theta_2 s\alpha_{23}$$
$$+ r_1(c\alpha_{12}c\alpha_{23}-c\theta_2 s\alpha_{12}s\alpha_{23}) = 0 \qquad \text{(iii)}$$

Eliminating r_2 between equations (ii) and (iii),

$$c\alpha_{23}\{a_{41}s(\theta_3+\theta_4) + a_{34}s\theta_3\} + r_1 s\alpha_{12}c\theta_2$$
$$= s\alpha_{23}\{R_4+h_4\theta_4+R_3+h_3\theta_3\} + a_{12}s\theta_2.$$

Eliminating r_1 from this equation by means of (i),

$$c\theta_2\{a_{41}c(\theta_3+\theta_4) + a_{34}c\theta_3 + a_{23} + a_{12}c\theta_2\}$$
$$+ s\alpha_{23}s\theta_2\{R_4+h_4\theta_4+R_3+h_3\theta_3\}+a_{12}s^2\theta_2$$
$$= c\alpha_{23}s\theta_2\{a_{41}s(\theta_3+\theta_4) + a_{34}s\theta_3\}.$$

Using equation (5.2.2), we may rewrite this as

$$s\theta_2\{s\alpha_{23}[R_4+R_3+h_4(-\theta_3+c^{-1}\frac{c\alpha_{23}c\alpha_{41}-c\alpha_{12}}{s\alpha_{23}s\alpha_{41}})+h_3\theta_3] - a_{34}c\alpha_{23}s\theta_3\}$$

$$+ a_{34}c\theta_2c\theta_3+a_{23}c\theta_2+a_{12}+a_{41}\{c\theta_2c(\theta_3+\theta_4)-c\alpha_{23}s\theta_2s(\theta_3+\theta_4)\}=0.$$

$$s\theta_2\{s\alpha_{23}(h_3-h_4)\theta_3-a_{34}c\alpha_{23}(\theta_3 - \frac{\theta_3{}^3}{3!} + \frac{\theta_3{}^5}{5!} - \ldots)\}$$

$$+ a_{34}c\theta_2(1 - \frac{\theta_3{}^2}{2!} + \frac{\theta_3{}^4}{4!} - \ldots) + a_{23}c\theta_2 + a_{12}$$

$$+ s\theta_2s\alpha_{23}\{R_4 + R_3 + h_4c^{-1}\frac{c\alpha_{23}c\alpha_{41}-c\alpha_{12}}{s\alpha_{12}s\alpha_{41}}\}$$

$$+ a_{41}\{c\theta_2c(\theta_3+\theta_4) - c\alpha_{23}s\theta_2s(\theta_3+\theta_4)\} = 0 \qquad \text{(iv)}$$

Remembering that θ_2 and $(\theta_3+\theta_4)$ are constants, we now have an equation in variable θ_3 alone. For the postulated P-P-H-H-linkage to be mobile, we require that this last equation be an identity. Equating coefficients of $\theta_3{}^2$, $\theta_3{}^3$, $\theta_3{}^4$,...., we must have that

$$a_{34} = 0 \qquad \underline{\text{OR}} \qquad c\theta_2 = c\alpha_{23} = 0.$$

$\underline{a_{34} = 0}$

Regarding the coefficient of the remaining term in $\theta_3{}^1$, we have that, since $s\alpha_{23} \neq 0$,

$$h_3 = h_4 \qquad \underline{\text{OR}} \qquad s\theta_2 = 0.$$

Now, from (i), since $s\alpha_{12} \neq 0$, r_1 will be fixed unless $s\theta_2 = 0$. We may then conclude that $s\theta_2 = 0$ and the screw pitches are not necessarily equal. In fact, their equality results in a linkage with part-chain mobility.

Thus, from (5.7),

$$s(\theta_3+\theta_4) = 0.$$

Also, from (5.4),

$$s\theta_1 = 0.$$

Physically, $s\theta_1 = 0$ for example, means that the 4-1 and 1-2 normals are at least parallel. So this condition, with the two similar ones obtained above, implies that the three (Joints 3 and 4 are coaxial because they are parallel and $a_{34} = 0$.) joint axes lie in parallel planes.

The other dimensional condition is the perhaps obvious one, not specified by Delassus, that

$$\left.\begin{aligned} \alpha_{12} + \tau\alpha_{23} + \sigma\alpha_{41} &= 0 \\ a_{12} + \tau a_{23} + \sigma a_{41} &= 0 \end{aligned}\right\},$$

where

$$c\theta_1 = \sigma, \qquad c\theta_2 = \tau.$$

The resulting linkage is the Delassus solution number d.8.

$\underline{c\theta_2 = c\alpha_{23} = 0}$

Consequently, from (5.2.1),

$$c\alpha_{41} = 0.$$

Thus, the prisms are at rightangles to the screws.

From (5.2.3),

$$c\theta_1 = 0.$$

From (5.2.2),

$$c(\theta_3+\theta_4) = -c\alpha_{12},$$

whence $$\theta_3 + \theta_4 = (2k+1)\pi - \sigma\alpha_{12}.$$

Hence, equation (iv) becomes

$$\sigma(h_3 - h_4)\theta_3 + a_{12} + \sigma\{R_4 + R_3 + h_4[(2k+1)\pi - \sigma\alpha_{12}]\} = 0.$$

Evidently,

$$h_3 = h_4 = h, \text{ say},$$

and

$$R_3 + R_4 + \sigma a_{12} + h[(2k+1)\pi - \sigma\alpha_{12}] = 0.$$

This latter dimensional constraint was not given by Delassus. The linkage is Delassus solution number d.3.

5.3 Two pairs of adjacent parallel joints

Put

$$\alpha_{34} = \alpha_{12} = 0$$

$$\alpha_{41} = \alpha_{23} \text{ or } 2\pi-\alpha_{23},$$

where

$$0 < \alpha_{23} < \pi.$$

Hence,

$$c\alpha_{41} = c\alpha_{23} \qquad s\alpha_{41} = \sigma s\alpha_{23}.$$

From equations (5.3) and (5.6), we may write

$$\theta_1 + \theta_2 = 2k\pi + \frac{1+\sigma}{2}\pi. \tag{5.3.1}$$

Similarly, from equations (5.7) and (5.8),

$$\theta_3 + \theta_4 = 2l\pi + \frac{1+\sigma}{2}\pi. \tag{5.3.2}$$

Equations (5.1), (5.2), (5.4) and (5.5) are satisfied by (5.3.1) and (5.3.2). Equation (5.9) is identically satisfied.

Equations (5.10), (5.11) and (5.12) become, respectively,

$$-\sigma a_{41} + a_{34}c\theta_3 + a_{23} + a_{12}c\theta_2 = 0 \tag{5.3.3}$$

$$a_{34}s\theta_3 + r_2 s\alpha_{23} - a_{12}s\theta_2 c\alpha_{23} + r_1 s\alpha_{23} = 0 \tag{5.3.4}$$

$$r_4 + r_3 + r_2 c\alpha_{23} + a_{12}s\theta_2 s\alpha_{23} + r_1 c\alpha_{23} = 0. \tag{5.3.5}$$

Since there are only five independent equations, any proper linkage with mobility 1 must have connectivity sum no greater than six. By (5.3.1) and (5.3.2), if any joint is prismatic, then so too must be its adjacent parallel joint; this would make the linkage part-chain mobile. Likewise, we cannot have two parallel cylindric joints.

For connectivity sum six, then, the only possibilities are C^H-C^H- and H^C-C^H-, which are special parallel-screw linkages [42,45]. Let us now consider the possibility of solutions with connectivity sum five. We have already excluded the chance of a prismatic joint. We shall then try replacing a cylindric joint by a screw.

<u>C^H-C^H-</u> sub-category

We replace cylindric joint 1 by a screw joint.

With the help of equation (5.3.1), equation (5.3.4) may be rewritten as

$$- a_{34}s\theta_3 = - a_{12}s\theta_2 c\alpha_{23} + s\alpha_{23}(h_2-h_1)\theta_2 + s\alpha_{23}(R_2+R_1+[2k + \tfrac{1+\sigma}{2}]\pi h_1).$$

Elimination of θ_3 between this equation and (5.3.3) leads to

$$\begin{aligned} a_{34}{}^2 = {} & a_{12}{}^2c^2\theta_2+(a_{23}-\sigma a_{41})^2+2a_{12}(a_{23}-\sigma a_{41})c\theta_2+K^2s^2\alpha_{23} \\ & + s^2\alpha_{23}(h_2-h_1)^2\theta_2{}^2+a_{12}{}^2s^2\theta_2c^2\alpha_{23}+2Ks^2\alpha_{23}(h_2-h_1)\theta_2 \\ & - 2Ka_{12}s\alpha_{23}c\alpha_{23}s\theta_2-2a_{12}s\alpha_{23}c\alpha_{23}(h_2-h_1)\theta_2s\theta_2, \end{aligned} \qquad \text{(i)}$$

where $\quad K = R_2 + R_1 + [2k + \frac{1+\sigma}{2}]\pi\, h_1.$

We expand $c\theta_2$ and $s\theta_2$ in (i) in terms of powers of θ_2, and then equate coefficients of like powers, since the equation must be an identity in that variable.

$$\theta_2{}^2:\quad 0 = -\,a_{12}{}^2 s^2\alpha_{23} - a_{12}(a_{23} - \sigma a_{41}) - 2a_{12}s\alpha_{23}c\alpha_{23}(h_2 - h_1) + s^2\alpha_{23}(h_2 - h_1)^2$$

$$\theta_2{}^4:\quad 0 = \frac{1}{3}a_{12}{}^2 s^2\alpha_{23} + \frac{1}{12}a_{12}(a_{23} - \sigma a_{41}) + \frac{1}{3}a_{12}s\alpha_{23}c\alpha_{23}(h_2 - h_1)$$

$$\theta_2{}^6:\quad 0 = -\,\frac{2}{45}a_{12}{}^2 s^2\alpha_{23} - \frac{1}{360}a_{12}(a_{23} - \sigma a_{41} - \frac{1}{60}a_{12}s\alpha_{23}c\alpha_{23}(h_2 - h_1)$$

$$\theta_2{}^8:\quad 0 = \frac{1}{315}a_{12}{}^2 s^2\alpha_{23} + \frac{1}{20160}a_{12}(a_{23} - \sigma a_{41}) + \frac{1}{2520}a_{12}s\alpha_{23}c\alpha_{23}(h_2 - h_1)$$

In order for there to be non-trivial solutions of $a_{12}{}^2 s^2\alpha_{23}$, $a_{12}(a_{23} - \sigma a_{41})$, $a_{12}s\alpha_{23}c\alpha_{23}(h_2 - h_1)$ and $s^2\alpha_{23}(h_2 - h_1)^2$, it would be necessary for the following determinant to have the value zero.

$$\begin{vmatrix} -1 & -1 & -2 & 1 \\ \frac{1}{3} & \frac{1}{12} & \frac{1}{3} & 0 \\ -\frac{2}{45} & -\frac{1}{360} & -\frac{1}{60} & 0 \\ \frac{1}{315} & \frac{1}{20160} & \frac{1}{2520} & 0 \end{vmatrix}$$

Since such is not the case, we may conclude that

$$a_{12} = 0 \qquad h_2 = h_1 .$$

Hence, the resultant linkage has part-chain mobility within joints 1 and 2.

There is no need to take the analysis any further for this sub-category. There are no derivatives of the required type.

$\underline{H}{}^\frown\underline{C}$-$\underline{C}{}^\frown\underline{H}$- sub-category

If we replace either cylindric joint by a screw pair, the resulting loop will be precisely the same as the previously attempted $C\hat{-}H-C\hat{-}H-$ derivative. Again, there will be no proper solution with mobility of unity.

. . . .

For the dimensional conditions specified in this category, then, we find that there is no proper linkage with mobility one and connectivity sum less than six. More specifically, there can be no Delassus linkage.

5.4 Three joint axes parallel

Put

$$\alpha_{34} = \alpha_{23} = 0$$

$$\alpha_{41} = \alpha_{12} \quad \text{or} \quad 2\pi - \alpha_{12},$$

where

$$0 < \alpha_{12} < \pi.$$

Hence,

$$c\alpha_{41} = c\alpha_{12} \qquad s\alpha_{41} = \sigma s\alpha_{12}.$$

From equations (5.3) and (5.6), we deduce that joint 1 is prismatic.

$$\theta_1 = \frac{1+\sigma}{2}\pi$$

Equations (5.7) and (5.8) imply that

$$\theta_2 + \theta_3 + \theta_4 = 2k\pi + \frac{1+\sigma}{2}\pi. \qquad (5.4.1)$$

Equation (5.9) is identically satisfied. Equations (5.1), (5.2), (5.4) and (5.5) are satisfied by the above results.

Equations (5.10), (5.11), (5.12) become, respectively,

$$a_{41}c(\theta_3+\theta_4)+a_{34}c\theta_3+a_{23}+a_{12}c\theta_2+r_1s\theta_2s\alpha_{12} = 0 \;.$$

$$a_{41}s(\theta_3+\theta_4)+a_{34}s\theta_3-a_{12}s\theta_2+r_1c\theta_2s\alpha_{12} = 0$$

$$r_4 + r_3 + r_2 + r_1c\alpha_{12} = 0\;. \qquad (5.4.2)$$

We may use (5.4.1) to replace $(\theta_3+\theta_4)$ by θ_2 in the first two of these equations. We may then eliminate $s\theta_2$ and $c\theta_2$ between them to write down the more convenient following equations.

$$-\sigma a_{41} - \sigma a_{34} c\theta_4 + a_{23} c\theta_2 + a_{12} = 0 \qquad (5.4.3)$$

$$\sigma a_{34} s\theta_4 + a_{23} s\theta_2 + r_1 s\alpha_{12} = 0 \qquad (5.4.4)$$

So linkages of connectivity sum five can be mobile. Those with mobility one are the P-H^H^C- and P-H^C^H- chains, which are special parallel-screw linkages [27,40,42,45]. We now test for mobility the connectivity sum four derivatives of these linkages.

<u>P-H^H^C-</u> sub-category

If we were to replace the cylindric joint by a slider, θ_4 would be constant. From (5.4.3), it would then be implied that joint 2 were locked, unless a_{23} were zero. But, by (5.4.4), if a_{23} were zero, joint 1 would be locked. We conclude that there is no relevant P-H^H^P- linkage in this category.

Let us replace the cylindric joint by a screw. Equation (5.4.2) may be now expressed as

$$R_4 + R_3 + R_2 + h_4\theta_4 + h_3\theta_3 + h_2\theta_2 + r_1 c\alpha_{12} = 0. \qquad \text{(i)}$$

This equation, together with (5.4.1), (5.4.3) and (5.4.4), gives us a set of four governing equations in four unknowns. For the linkage to have mobility of one, we must establish the conditions (or sets of conditions) under which only three of the equations are independent.

We note first that we cannot have $a_{23} = 0$ or $a_{34} = 0$ because, from (5.4.3), either implies the other, and both together, from (5.4.4), imply that joint 1 is locked.

Elimination of r_1 between (5.4.4) and (i) leads to

$$R_4+R_3+R_2+h_4\theta_4+h_3\theta_3+h_2\theta_2 = \frac{c\alpha_{12}}{s\alpha_{12}}(\sigma a_{34}s\theta_4+a_{23}s\theta_2). \qquad \text{(ii)}$$

Let us assume for the present that $h_3 \neq 0$. Then, multiplication of (5.4.1) by h_3 allows us to eliminate θ_3 between that equation and (ii). Consequently, we obtain

$$R_4+R_3+R_2+(h_4-h_3)\theta_4+(h_2-h_3)\theta_2+(2k + \frac{1+\sigma}{2})\pi h_3$$
$$= \frac{c\alpha_{12}}{s\alpha_{12}}(\sigma a_{34}s\theta_4+a_{23}s\theta_2).$$

Differentiation of this equation with respect to θ_2 yields

$$(h_4-h_3)\frac{d\theta_4}{d\theta_2}+(h_2-h_3) = \frac{c\alpha_{12}}{s\alpha_{12}}(\sigma a_{34}c\theta_4 \frac{d\theta_4}{d\theta_2} + a_{23}c\theta_2). \qquad \text{(iii)}$$

Now, differentiation of (5.4.3) with respect to θ_2 results in

$$\sigma a_{34}s\theta_4 \frac{d\theta_4}{d\theta_2} = a_{23}s\theta_2. \qquad \text{(iv)}$$

Elimination of $\frac{d\theta_4}{d\theta_2}$ between equations (iii) and (iv) ($a_{34}\neq 0$) yields the following result.

$$a_{23}(h_4-h_3)s\theta_2+\sigma a_{34}(h_2-h_3)s\theta_4 = \frac{c\alpha_{12}}{s\alpha_{12}}a_{23}a_{34}(s\theta_2c\theta_4+c\theta_2s\theta_4) \quad \text{(v)}$$

Rearranging terms, and squaring both sides of the rewritten equation (v), leads to

$$a_{34}{}^2(1-c^2\theta_4)(\sigma[h_2-h_3] - \frac{c\alpha_{12}}{s\alpha_{12}} a_{23}c\theta_2)^2$$
$$= (a_{23}{}^2-a_{23}{}^2c^2\theta_2)([h_4-h_3] - \frac{c\alpha_{12}}{s\alpha_{12}} a_{34}c\theta_4)^2.$$

If we now substitute for $c\theta_2$ in this equation from (5.4.3), we obtain an equation in $c\theta_4$ alone, namely

$$a_{34}^2(1-c^2\theta_4)\{(\sigma[h_2-h_3]+\frac{c\alpha_{12}}{s\alpha_{12}}[a_{12}-\sigma a_{41}])^2+(a_{34}\frac{c\alpha_{12}}{s\alpha_{12}}c\theta_4)^2$$

$$-\ 2\sigma a_{34}\ \frac{c\alpha_{12}}{s\alpha_{12}}c\theta_4(\sigma[h_2-h_3] + \frac{c\alpha_{12}}{s\alpha_{12}}[a_{12}-\sigma a_{41}])\}$$

$$= \{(a_{23}^2-[\sigma a_{41}-a_{12}]^2)-(a_{34}c\theta_4)^2-2\sigma a_{34}(\sigma a_{41}-a_{12})c\theta_4\}$$

$$\times\ ([h_4-h_3]^2 + [\frac{c\alpha_{12}}{s\alpha_{12}}a_{34}c\theta_4]^2-2a_{34}\ \frac{c\alpha_{12}}{s\alpha_{12}}[h_4-h_3]c\theta_4). \qquad \text{(vi)}$$

For joint 4 to be mobile, equation (vi) must be an identity in $c\theta_4$. Let us now equate coefficients of like powers of $c\theta_4$.

$$c^3\theta_4: \quad \sigma(\sigma[h_2-h_3] + \frac{c\alpha_{12}}{s\alpha_{12}}[a_{12}-\sigma a_{41}])=(h_4-h_3) - \sigma\frac{c\alpha_{12}}{s\alpha_{12}}(\sigma a_{41}-a_{12})$$

This result may be immediately simplified to

$$h_2 - h_3 = h_4 - h_3,$$

whence
$$h_2 = h_4. \qquad \text{(vii)}$$

We shall use the simplification of equation (vii) in the remaining equating of coefficients.

$$c^2\theta_4: \quad (a_{34}\ \frac{c\alpha_{12}}{s\alpha_{12}})^2-(\sigma[h_2-h_3] + \frac{c\alpha_{12}}{s\alpha_{12}}[a_{12}-\sigma a_{41}])^2$$

$$=-(h_2-h_3)^2+(\frac{c\alpha_{12}}{s\alpha_{12}})^2(a_{23}^2-[\sigma a_{41}-a_{12}]^2)+4\sigma\frac{c\alpha_{12}}{s\alpha_{12}}(\sigma a_{41}-a_{12})(h_2-h_3)$$

Upon expansion, this equation may be readily simplified to yield

$$\left(\frac{c\alpha_{12}}{s\alpha_{12}}\right)^2(a_{23}{}^2-a_{34}{}^2) = 2\sigma(a_{12}-\sigma a_{41})\frac{c\alpha_{12}}{s\alpha_{12}}(h_2-h_3). \qquad \text{(viii)}$$

$$c^1\theta_4: \quad a_{34}{}^2\frac{c\alpha_{12}}{s\alpha_{12}}(h_2-h_3)+\sigma a_{34}{}^2\left(\frac{c\alpha_{12}}{s\alpha_{12}}\right)^2(a_{12}-\sigma a_{41})$$
$$= a_{23}{}^2\frac{c\alpha_{12}}{s\alpha_{12}}(h_2-h_3)-\frac{c\alpha_{12}}{s\alpha_{12}}(h_2-h_3)(\sigma a_{41}-a_{12})^2+\sigma(\sigma a_{41}-a_{12})(h_2-h_3)^2 \qquad \text{(ix)}$$

$$c^0\theta_4: \quad a_{34}{}^2(h_2-h_3)^2+a_{34}{}^2\left(\frac{c\alpha_{12}}{s\alpha_{12}}\right)^2(a_{12}-\sigma a_{41})^2$$
$$+\ 2\sigma a_{34}{}^2\frac{c\alpha_{12}}{s\alpha_{12}}(h_2-h_3)(a_{12}-\sigma a_{41})=a_{23}{}^2(h_2-h_3)^2-(h_2-h_3)^2(\sigma a_{41}-a_{12})^2 \qquad \text{(x)}$$

Equations (viii)-(x) are to be satisfied simultaneously. The following consequences are apparent.

$\sigma a_{41} = a_{12}$ implies $a_{34}{}^2 = a_{23}{}^2$

$h_3 = h_2$ implies $c\alpha_{12} = 0$

$c\alpha_{12} = 0$ implies $h_3 = h_2$ or $\sigma a_{41} = a_{12}, a_{34}{}^2 = a_{23}{}^2$

Let us attempt to solve the equations subject to the restrictions

$$\sigma a_{41} \neq a_{12} \qquad h_3 \neq h_2 \qquad c\alpha_{12} \neq 0.$$

Hence, equation (viii) may be replaced by

$$\frac{c\alpha_{12}}{s\alpha_{12}}(a_{23}{}^2-a_{34}{}^2) = 2\sigma(a_{12}-\sigma a_{41})(h_2-h_3). \qquad (\alpha)$$

We may use this result in (ix) to obtain

$$2\sigma(\sigma a_{41}-a_{12})(h_2-h_3)^2+\sigma a_{34}{}^2\left(\frac{c\alpha_{12}}{s\alpha_{12}}\right)^2(a_{12}-\sigma a_{41})$$
$$+\ \frac{c\alpha_{12}}{s\alpha_{12}}(h_2-h_3)(\sigma a_{41}-a_{12})^2+\sigma(a_{12}-\sigma a_{41})(h_2-h_3)^2 = 0,$$

which may be simplified to yield

$$(h_2-h_3)^2 + \sigma\frac{c\alpha_{12}}{s\alpha_{12}}(h_2-h_3)(\sigma a_{41}-a_{12}) = a_{34}^2(\frac{c\alpha_{12}}{s\alpha_{12}})^2.$$

Using (α) in this equation yields

$$2(h_2-h_3)^2 = (\frac{c\alpha_{12}}{s\alpha_{12}})^2(a_{23}^2+a_{34}^2). \qquad (\beta)$$

We now eliminate (h_2-h_3) from (x) by means of (α) and (β):

$$a_{34}^2\{\frac{1}{2}(\frac{c\alpha_{12}}{s\alpha_{12}})^2(a_{23}^2+a_{34}^2)+(\frac{c\alpha_{12}}{s\alpha_{12}})^2(a_{12}-\sigma a_{41})^2+(\frac{c\alpha_{12}}{s\alpha_{12}})^2(a_{23}^2-a_{34}^2)\}$$

$$= \frac{1}{2}(\frac{c\alpha_{12}}{s\alpha_{12}})^2(a_{23}^2+a_{34}^2)\{a_{23}^2-(\sigma a_{41}-a_{12})^2\}$$

Simplifying and rearranging terms,

$$(\sigma a_{41}-a_{12})^2(3a_{34}^2+a_{23}^2) = (a_{34}^2-a_{23}^2)^2. \qquad (\gamma)$$

Elimination of (h_2-h_3) between (α) and (β) yields

$$(a_{34}^2-a_{23}^2)^2 = 2(\sigma a_{41}-a_{12})^2(a_{34}^2+a_{23}^2)$$

which, when substituted into (γ), leads to

$$3a_{34}^2 + a_{23}^2 = 2a_{34}^2 + 2a_{23}^2,$$

whence $$a_{34}^2 = a_{23}^2.$$

Because of equations (viii)-(x), this result implies

$$\sigma a_{41} = a_{12} \quad \text{or} \quad h_3 = h_2, \; c\alpha_{12} = 0.$$

We may therefore summarise the solutions of equations (viii)-(x) by the two possibilities,

$$h_3 = h_2 \qquad c\alpha_{12} = 0$$

and $$\sigma a_{41} = a_{12} \qquad a_{34} = a_{23}.$$

For both sets of conditions, equation (vii) also holds. We shall consider these cases separately, and then look at the case which we previously eliminated from discussion, namely that for which

$$h_3 = 0.$$

$h_2 = h_3 = h_4, c\alpha_{12} = 0$:

Under these conditions, in view of (5.4.1), equation (5.4.2) is identically satisfied, subject to the additional dimensional constraint

$$R_2 + R_3 + R_4 + (2k + \frac{1+\sigma}{2})\pi h = 0,$$

where $h_2 = h_3 = h_4 = h.$

So there are three independent closure equations, and we have isolated the Delassus linkage number d.2.

$h_2 = h_4,\ \sigma a_{41} = a_{12},\ a_{34} = a_{23}$:

In this case, equation (5.4.3) may be simplified to

$$c\theta_4 = \sigma c\theta_2,$$

whence $$\theta_4 = \tau\theta_2 + (2l + \frac{1-\sigma}{2})\pi. \quad \text{(a)}$$

$$\therefore\ s\theta_4 = \sigma\tau s\theta_2$$

Hence, equation (5.4.4) may be rewritten as

$$a_{23}(\tau+1)s\theta_2 + r_1 s\alpha_{12} = 0.$$

Now, if $\tau=-1$, this equation implies that joint 1 is locked. We conclude that $\tau=1$, and the equation may be written in the form

$$2a_{23}s\theta_2 + r_1 s\alpha_{12} = 0. \quad \text{(b)}$$

Substitution for θ_4 from (a), with $\tau=1$, into equation (5.4.1) leads to

$$\theta_3 = -2\theta_2 + (2m+\sigma)\pi. \tag{c}$$

Substitution for θ_4 from (a), with $\tau=1$, and for θ_3 from (c) into equation (i) yields

$$R_4+R_3+R_2+h_3\{-2\theta_2+(2m+\sigma)\pi\}+h_2\{2\theta_2+(2l+\tfrac{1-\sigma}{2})\pi\}+r_1 c\alpha_{12} = 0.$$

That is,

$$R_4+R_3+R_2+(2m+\sigma)\pi h_3+(2l+\tfrac{1-\sigma}{2})\pi h_2+2(h_2-h_3)\theta_2+r_1 c\alpha_{12} = 0. \tag{d}$$

Elimination of r_1 between (b) and (d) results in

$$s\alpha_{12}\{R_4+R_3+R_2+(2m+\sigma)\pi h_3+(2l+\tfrac{1-\sigma}{2})\pi h_2+2(h_2-h_3)\theta_2\}=2a_{23}c\alpha_{12}s\theta_2.$$

This equation must be an identity in θ_2 for joint 2 to be mobile. Let us expand $s\theta_2$ in powers of θ_2 and then equate coefficients of like powers.

$\theta_2{}^3$: $\quad 0 = -\frac{1}{3}a_{23}c\alpha_{12},$

whence $\quad c\alpha_{12} = 0.$

Then, clearly,

$$h_2 = h_3.$$

So we have found merely a special case of Delassus solution d.2.

$h_3 = 0$:

Equation (ii), for this case, may be replaced by

$$R_4+R_3+R_2+h_4\theta_4+h_2\theta_2 = \frac{c\alpha_{12}}{s\alpha_{12}}(\sigma a_{34}s\theta_4+a_{23}s\theta_2).$$

Then differentiation with respect to θ_2 yields

$$h_4 \frac{d\theta_4}{d\theta_2} + h_2 = \frac{c\alpha_{12}}{s\alpha_{12}}(\sigma a_{34} c\theta_4 \frac{d\theta_4}{d\theta_2} + a_{23} c\theta_2).$$

This equation is the same as (iii) above, with $(h_4 - h_3)$ and $(h_2 - h_3)$ replaced, respectively, by h_4 and h_2. The solution procedure will follow the same subsequent steps, and the only relevant linkage obtained will be the -R- derivative of d.2, the planar slider-crank chain.

We have seen that the only connectivity sum 4, mobility 1 linkage in the present sub-category is the Delassus P-H^H^H- chain.

P-H^C^H- sub-category

If we replace the cylindric joint by a slider, Θ_3 is fixed and we then have two equations, (5.4.1) and (5.4.3), in two variables, Θ_2 and Θ_4. Again, to allow mobility of joint 1, we cannot have $a_{23} = 0$ or $a_{34} = 0$. Let us rewrite equation (5.4.1) for this situation in the form

$$\Theta_2 + \Theta_4 = \gamma, \tag{i}$$

where γ is constant. Then,

$$c\Theta_2 = c\gamma c\Theta_4 + s\gamma s\Theta_4.$$

We substitute for $c\Theta_2$ into equation (5.4.3):

$$- \sigma a_{41} + c\Theta_4(a_{23}c\gamma - \sigma a_{34}) + a_{23}s\gamma s\Theta_4 + a_{12} = 0 \tag{ii}$$

Clearly, from (ii),

$$s\gamma = 0 \qquad c\gamma = \sigma \qquad a_{34} = a_{23} \qquad a_{12} = \sigma a_{41}.$$

Thus, using (i),

$$s\Theta_2 = -\ \sigma s\Theta_4.$$

Consequently, equation (5.4.4) demands that $r_1 = 0$.

Thus, there is no P-H^P^H- linkage with the given initial dimensional constraints.

Replacing the cylindric joint by a screw yields the solution we obtained in the previous sub-category.

. . . .

It might be noted that the P-H^H^C- and P-H^C^H- chains possess screw joints with completely arbitrary and independent pitches. So, the -R- derivatives are P-H^R^C-, P-R^H^C-, P-R^R^C-, P-R^C^H-, P-R^C^R-. There is, however, one restriction: if both screws are replaced by revolutes, the resulting linkage for either generating loop will be improper in the singular case for which $\alpha_{12} = \frac{\pi}{2}$ — the linkage will then be based on a planar slider-crank. In fact, if the two screw pitches are equal and $\alpha_{12} = \frac{\pi}{2}$, either chain will be improper and based on Delassus solution d.2, obtained above.

5.5 All joint axes parallel

$$\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{41} = 0$$

Equations (5.3) and (5.6)-(5.9) are identically satisfied.

Equations (5.1), (5.2), (5.4), (5.5) imply that

$$\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 = 2k\pi. \tag{5.5.1}$$

Equations (5.10), (5.11) and (5.12) become, respectively,

$$a_{41}c(\Theta_3+\Theta_4) + a_{34}c\Theta_3 + a_{23} + a_{12}c\Theta_2 = 0 \tag{5.5.2}$$

$$a_{41}s(\Theta_3+\Theta_4) + a_{34}s\Theta_3 - a_{12}s\Theta_2 = 0 \tag{5.5.3}$$

$$r_4 + r_3 + r_2 + r_1 = 0. \tag{5.5.4}$$

Since there are only four independent closure equations, any relevant linkage of mobility unity obeying the above geometrical conditions must have connectivity sum of five or less.

By transposing terms in (5.5.2) and (5.5.3), then squaring and adding, we obtain after simplification

$$a_{41}^2 + a_{34}^2 + 2a_{41}a_{34}c\Theta_4 = a_{23}^2 + a_{12}^2 + 2a_{23}a_{12}c\Theta_2. \tag{5.5.5}$$

By symmetry,

$$a_{34}^2 + a_{23}^2 + 2a_{34}a_{23}c\Theta_3 = a_{12}^2 + a_{41}^2 + 2a_{12}a_{41}c\Theta_1. \tag{5.5.6}$$

We cannot allow more than one joint to be a slider, since such a linkage would exhibit part-chain mobility. Let us suppose that joint 2, say, is prismatic. From equations (5.5.5) and (5.5.6), rotational mobility of the other three joints will require that one of the three following sets of constraints is satisfied. The possibilities are

$$a_{41} = a_{34} = 0 \qquad a_{12} = a_{23} \neq 0 \qquad (5.5.a)$$

$$a_{41} = a_{23} = 0 \qquad a_{12} = a_{34} \neq 0 \qquad (5.5.b)$$

$$a_{34} = a_{12} = 0 \qquad a_{23} = a_{41} \neq 0\,. \qquad (5.5.c)$$

If all the $a_{i\ i+1}$ were zero, there would remain only two closure equations.

The conditions (5.5.a) would eliminate closure equations (5.5.2) and (5.5.3), and so need not be further considered.
The conditions (5.5.b), from equations (5.5.2) and (5.5.3), would fix Θ_3; they are disallowed.
The set (5.5.c), from equations (5.5.1)-(5.5.3), would fix Θ_1.
Thus, there can be no eligible linkage under these constraints.

We conclude that, for a four-bar with all joint axes parallel, there is no linkage of mobility one with any joint a slider which is exempt from part-chain mobility.

. . . .

Thus, the only proper linkage of connectivity sum five, mobility unity, satisfying the above geometrical constraints is the $C\hat{}H\hat{}H\hat{}H$- chain.

It is worthy of note that this linkage has no proper $C\hat{}R\hat{}R\hat{}R$- derivative. This perhaps unexpected result is easily seen by reference to equation (5.5.4). Although the three screw pitches are nominally arbitrary and independent of each other, setting them all equal to zero will prevent translation of joint 1. We therefore have a counter-example to the hypothesis that a screw pitch may always be made zero. Here, at least one screw must have a non-zero pitch.

We may go further. Professor K. Hunt (Monash University, Victoria, Australia) has pointed out in correspondence that, when the three pitches are equal and non-zero, the C^H^H^H- linkage is still improper. Comparing equations (5.5.1) and (5.5.4) for this instance, it is seen that the cylindric joint is effectively a screw, with the same pitch as the other three joints. The linkage has degenerated to Delassus solution d.1, soon to be discussed. We reiterate in passing what was said in this connection in section 5.1. One needs to be constantly vigilant in spatial linkage analysis, because of the important consequences of geometric singularities.

. . . .

We cannot replace the cylindric joint by a slider. Any feasible linkage of connectivity sum four, then, must be of the form H^H^H^H- if it is to satisfy the geometrical restrictions for this category.

Let us now attempt to seek out the possible H-H-H-H- linkages with all joint axes parallel. Equation (5.5.4) becomes

$$(R_1+R_2+R_3+R_4) + h_1\Theta_1 + h_2\Theta_2 + h_3\Theta_3 + h_4\Theta_4 = 0. \qquad (5.5.4')$$

Clearly, one solution is given by

$$\begin{cases} h_1 = h_2 = h_3 = h_4 = h, \quad \text{say} \\ R_1 + R_2 + R_3 + R_4 + 2k\pi h = 0. \end{cases}$$

These conditions make equations (5.5.1) and (5.5.4') equivalent, and yield the Delassus solution number d.1.

. . . .

We next consider the possibility of one or more of the $a_{i\ i+1}$ being zero. Choose, for example,

$$a_{12} = 0.$$

In order that joint 4 is not locked, equation (5.5.5) implies

$$\text{either} \quad a_{34} = 0 \qquad \text{or} \qquad a_{41} = 0$$

$$a_{23} = a_{41} \qquad\qquad a_{23} = a_{34}.$$

Now the conditions

$$a_{12} = a_{41} = 0 \qquad a_{23} = a_{34} \neq 0,$$

by equation (5.5.6), will result in the locking of joint 3, and so may be excluded. We also disallow the possibility that

$$a_{12} = a_{23} = a_{34} = a_{41} = 0,$$

since it would result in a linkage of mobility 2 consisting of four coaxial screws.
On the other hand, the conditions

$$a_{12} = a_{34} = 0 \qquad a_{23} = a_{41} \neq 0$$

satisfy equation (5.5.6).

Choosing any other $a_{i\ i+1}$ zero would result in an entirely analogous situation.

We conclude that there is precisely one solution in this group, that being the Delassus linkage number d.7, expressible as H=H^H=H-. Its dimensional constraints are, for example,

$$a_{12} = a_{34} = 0 \qquad a_{23} = a_{41} \neq 0$$

$$\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{41} = 0.$$

We may replace closure equations (5.5.1)-(5.5.3) by

$$\Theta_1 + \Theta_2 = (2l+1)\pi$$

$$\Theta_3 + \Theta_4 = (2m+1)\pi.$$

Equation (5.5.4') remains unchanged. The offsets and pitches are arbitrary.

. . . .

We may henceforth exclude the possibility of a normal link-length being zero or all pitches being equal.

. . . .

Let us now consider the consequences of choosing

$$h_1 = h_3.$$

Eliminating Θ_1 and Θ_3 between (5.5.1) and (5.5.4') yields

$$(h_2-h_1)\Theta_2+(h_4-h_1)\Theta_4 + (R_1+R_2+R_3+R_4) + 2k\pi\ h_1 = 0.$$

Now, if $h_2=h_1$, in order that joint 4 is not fixed, we must have that $h_4=h_1$. The converse is also true. We should then obtain the Delassus linkage d.1.

We may therefore assume that

$$h_2 \neq h_1 \qquad h_4 \neq h_1.$$

Then we may rewrite the last equation as

$$\Theta_2 = \frac{h_4-h_1}{h_1-h_2}\ \Theta_4 + \frac{R_1+R_2+R_3+R_4+2k\pi h_1}{h_1-h_2}. \tag{i}$$

Let us put for convenience

$$\gamma = \frac{R_1+R_2+R_3+R_4+2k\pi h_1}{h_1-h_2}, \text{ a constant.}$$

By equation (i),

$$c\Theta_2 = c(\frac{h_4-h_1}{h_1-h_2}\Theta_4)\ c\gamma-s(\frac{h_4-h_1}{h_1-h_2}\Theta_4)\ s\gamma.$$

We substitute this result into (5.5.5), and expand the consequent equation in terms of powers of Θ_4:

$$a_{41}^2 + a_{34}^2 + 2a_{41}a_{34}\left\{1 - \frac{\Theta_4^2}{2!} + \frac{\Theta_4^4}{4!} - \ldots\right\}$$

$$= a_{23}^2+a_{12}^2+2a_{23}a_{12}c\gamma\left\{1 - \frac{1}{2!}\left(\frac{h_4-h_1}{h_1-h_2}\Theta_4\right)^2+\frac{1}{4!}\left(\frac{h_4-h_1}{h_1-h_2}\Theta_4\right)^4-\ldots\right\}$$

$$- 2a_{23}a_{12}s\gamma\left\{\left(\frac{h_4-h_1}{h_1-h_2}\Theta_4\right) - \frac{1}{3!}\left(\frac{h_4-h_1}{h_1-h_2}\Theta_4\right)^3 + \ldots\right\} \qquad \text{(ii)}$$

Since we have excluded the possibilities $a_{12}=0$, $a_{23}=0$, $h_4=h_1$, and since equation (ii) is to be an identity in Θ_4, by equating coefficients of odd powers of Θ_4, we find that

$$s\gamma = 0.$$

Hence, $$c\gamma = \pm 1.$$

Let us now equate coefficients of Θ_4^2 and Θ_4^4:

$$a_{41}a_{34} = a_{23}a_{12}c\gamma\left(\frac{h_4-h_1}{h_1-h_2}\right)^2$$

$$a_{41}a_{34} = a_{23}a_{12}c\gamma\left(\frac{h_4-h_1}{h_1-h_2}\right)^4$$

By division,

$$(h_4-h_1)^2 = (h_1-h_2)^2. \qquad \text{(iii)}$$

As a result,

$$c\gamma = 1 \quad \text{and} \quad a_{41}a_{34} = a_{23}a_{12}.$$

Thus,

$$\gamma = 2m\pi,$$

whence $R_1+R_2+R_3+R_4 = 2\pi\ [(m-k)h_1-mh_2]$. (iv)

Also, by equating coefficients of Θ_4^0 in (ii), we have that

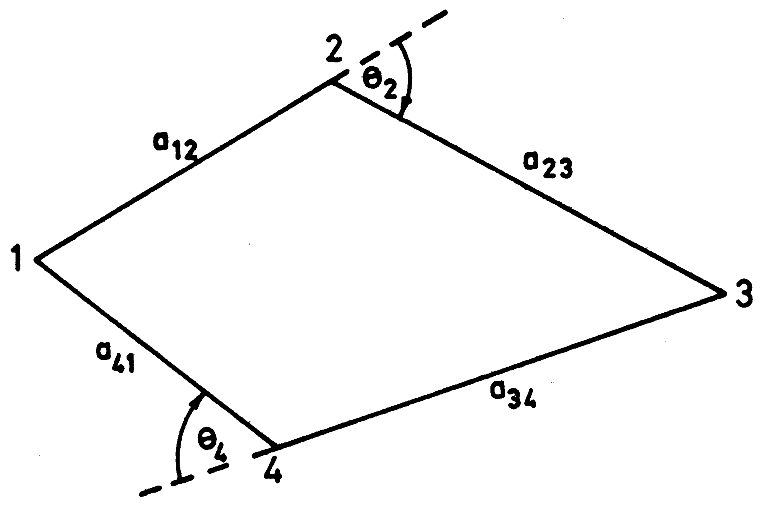


Fig. 5.5.1

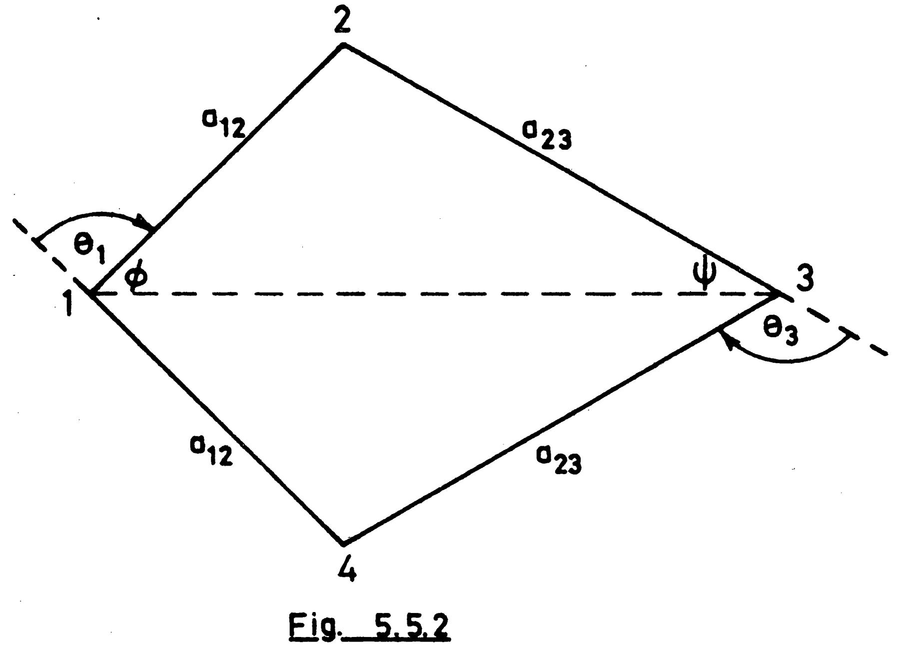


Fig. 5.5.2

$$(a_{41}+a_{34})^2 = (a_{23}+a_{12})^2, \tag{v}$$

whence $$a_{41}+a_{34} = a_{23}+a_{12}. \tag{vi}$$

The result $a_{41}a_{34}=a_{23}a_{12}$ together with (v) also implies that

$$(a_{41}-a_{34})^2 = (a_{23}-a_{12})^2,$$

whence $$a_{41}-a_{34}=a_{23}-a_{12} \quad \underline{\text{or}} \quad a_{41}-a_{34}=a_{12}-a_{23}.$$

These alternative results, with (vi), imply respectively that

$$a_{41}=a_{23},\ a_{34}=a_{12} \quad \underline{\text{or}} \quad a_{41}=a_{12},\ a_{34}=a_{23}. \tag{vii}$$

Result (iii) has two alternative consequences:

$\underline{A}$ $h_4+h_2=2h_1$. $\underline{B}$ $h_4=h_2$

We shall examine them separately.

$\underline{A}$:

Result (iv) becomes

$$R_1+R_2+R_3+R_4 = 2\pi[(m-k)\frac{h_4}{2} - (m+k)\frac{h_2}{2}].$$

Equation (i) simplifies to

$$\Theta_2 = \Theta_4 + 2m\pi.$$

In order to satisfy this last result, a 'plan view' of the linkage must exhibit an uncrossed configuration, shown in Fig. 5.5.1.

We now test the two possibilities for link-length constraints, following result (vii).

A1:

If

$$a_{41} = a_{23} \qquad a_{34} = a_{12},$$

we have, from (5.5.6), that

$$c\Theta_3 = c\Theta_1,$$

whence $$\Theta_3 = \pm\,\Theta_1 + 2n\pi.$$

(Physical requirements preclude the possibility of signing Θ_1 with ±.)

Hence, our configuration in plan is a parallelogram with the screw-pitch conditions

$$h_1 = h_3 = \frac{h_2 + h_4}{2}\ .$$

We have obtained, in fact, a special case of solution number d.10, to be discussed later in this section.

A2:

If

$$a_{41} = a_{12} \qquad a_{34} = a_{23},$$

we have, from (5.5.6), that

$$a_{23}{}^2\cos^2\frac{\Theta_3}{2} = a_{12}{}^2\cos^2\frac{\Theta_1}{2}.$$

That is, with reference to Fig. 5.5.2,

$$\left(\frac{a_{23}}{\sin\phi}\right)^2 = \left(\frac{a_{12}}{\sin\psi}\right)^2,$$

which is just the sine rule for half of the figure shown.

So we have in this case a projected configuration in the shape of a kite, Delassus's "rhomboide" and solution number d.9.

The three independent closure equations may be written in the form

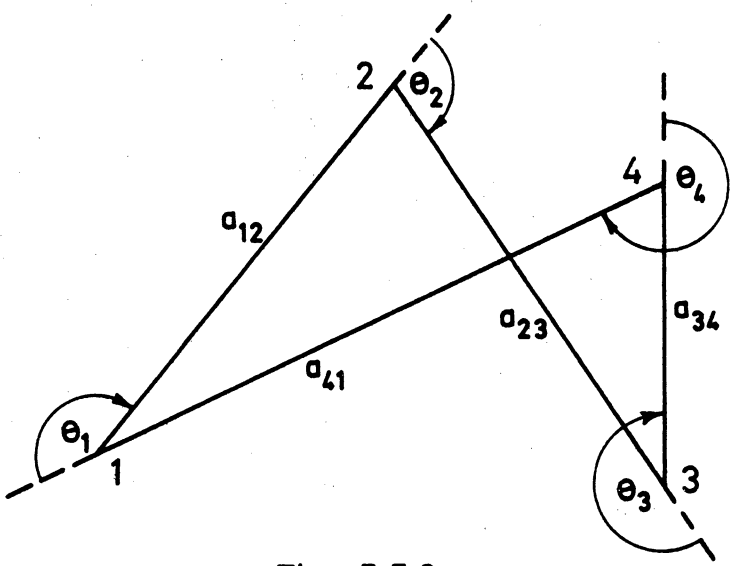


Fig. 5.5.3

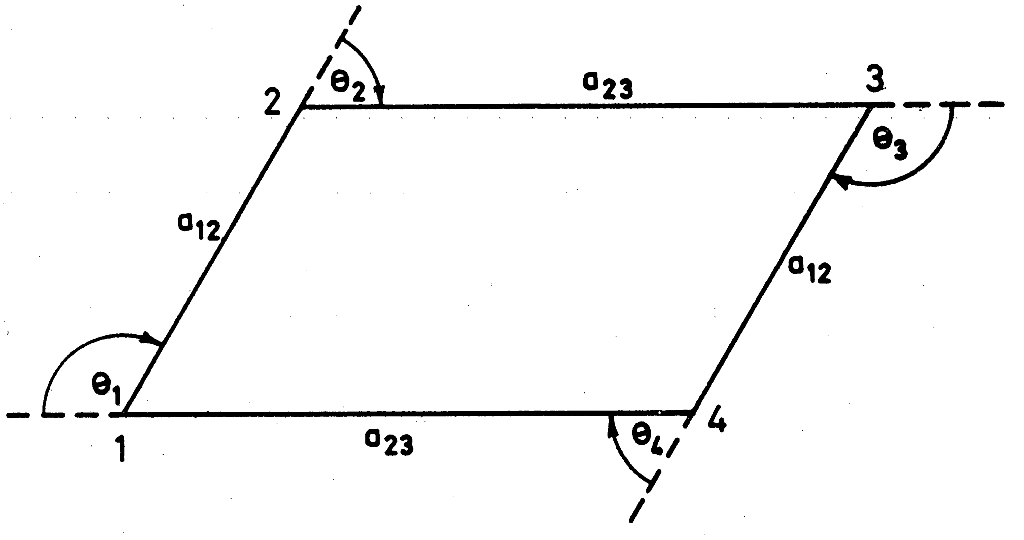


Fig. 5.5.4

$$\begin{cases} \Theta_2 = \Theta_4 + 2m\pi \\ \Theta_1 + \Theta_3 + 2\Theta_4 = 2\pi(k-m) \\ a_{23}{}^2\cos^2\dfrac{\Theta_3}{2} = a_{12}{}^2\cos^2\dfrac{\Theta_1}{2} . \end{cases}$$

The geometrical constraints are as specified above.

B:

Equation (i) here becomes

$$\Theta_2 = -\Theta_4 + 2m\pi.$$

For this last result to hold, a 'plan view' of the linkage must show a crossed configuration, as in Fig. 5.5.3.

We proceed to test the two possibilities allowed by result (vii).

B1:

If

$$a_{41} = a_{23} \qquad a_{34} = a_{12},$$

from (5.5.6) we have that

$$c\Theta_3 = c\Theta_1,$$

which is compatible with the configuration shown.

We have found Delassus's anti-parallelogram linkage, number d.11.

The independent closure equations may be given as

$$\begin{cases} \Theta_2 + \Theta_4 = 2m\pi \\ \Theta_1 + \Theta_3 = 2(k-m)\pi \\ a_{23}s(\Theta_3+\Theta_4) + a_{12}(s\Theta_3+s\Theta_4) = 0. \end{cases}$$

The dimensional conditions are as given above.

B2:

For the conditions

$$a_{41} = a_{12} \qquad a_{34} = a_{23}$$

to be satisfied, it would be necessary that a pair of alternate joints be coaxial. This is easily seen by reference to Fig. 5.5.3. Then, however, all joints would be locked.

We conclude that no solution exists with such constraints.

. . . .

Having investigated the consequences of all obvious geometric specialties for this category, we now consider the general case. It may be classified simply, in view of the foregoing, as

$$h_1 \neq h_3.$$

We can assume that $a_{i\ i+1} \neq 0$, for all i.

Eliminating Θ_1 between (5.5.1) and (5.5.4'),

$$R_1+R_2+R_3+R_4+2k\pi h_1 + (h_2-h_1)\Theta_2 + (h_3-h_1)\Theta_3 + (h_4-h_1)\Theta_4 = 0.$$

If we put

$$\gamma = \frac{R_1+R_2+R_3+R_4+2k\pi h_1}{h_1-h_3},$$

we may write the last equation as

$$\Theta_3 = \frac{h_2-h_1}{h_1-h_3}\Theta_2 + \frac{h_4-h_1}{h_1-h_3}\Theta_4 + \gamma. \qquad \text{(i)}$$

Thus,

$$\Theta_3+\Theta_4 = \frac{h_2-h_1}{h_1-h_3}\Theta_2 + \frac{h_4-h_3}{h_1-h_3}\Theta_4 + \gamma. \qquad \text{(ii)}$$

Differentiation of (5.5.2) and (5.5.3) with respect to Θ_2

yields, respectively,

$$- a_{41}s(\Theta_3+\Theta_4)\frac{d(\Theta_3+\Theta_4)}{d\Theta_2} - a_{34}s\Theta_3\frac{d\Theta_3}{d\Theta_2} - a_{12}s\Theta_2 = 0$$

$$a_{41}c(\Theta_3+\Theta_4)\frac{d(\Theta_3+\Theta_4)}{d\Theta_2} + a_{34}c\Theta_3\frac{d\Theta_3}{d\Theta_2} - a_{12}c\Theta_2 = 0.$$

By differentiating equations (i) and (ii) with respect to Θ_2, we may rewrite these last two equations as

$$-a_{41}s(\Theta_3+\Theta_4)\left\{\frac{h_2-h_1}{h_1-h_3} + \frac{h_4-h_3}{h_1-h_3}\frac{d\Theta_4}{d\Theta_2}\right\} - a_{34}s\Theta_3\left\{\frac{h_2-h_1}{h_1-h_3} + \frac{h_4-h_1}{h_1-h_3}\frac{d\Theta_4}{d\Theta_2}\right\}$$
$$- a_{12}s\Theta_2 = 0 \qquad \text{(iii)}$$

$$a_{41}c(\Theta_3+\Theta_4)\left\{\frac{h_2-h_1}{h_1-h_3} + \frac{h_4-h_3}{h_1-h_3}\frac{d\Theta_4}{d\Theta_2}\right\} + a_{34}c\Theta_3\left\{\frac{h_2-h_1}{h_1-h_3} + \frac{h_4-h_1}{h_1-h_3}\frac{d\Theta_4}{d\Theta_2}\right\}$$
$$- a_{12}c\Theta_2 = 0. \qquad \text{(iv)}$$

By adding equations (5.5.2) and (iv) and rearranging terms, we obtain

$$\frac{d\Theta_4}{d\Theta_2} = \frac{a_{23}(h_3-h_1)+(h_3-h_2)\{a_{41}c(\Theta_3+\Theta_4)+a_{34}c\Theta_3\}}{a_{41}(h_4-h_3)c(\Theta_3+\Theta_4) + a_{34}(h_4-h_1)c\Theta_3}. \qquad \text{(v)}$$

After subtracting (iii) from (5.5.3), we rearrange terms to obtain

$$\frac{d\Theta_4}{d\Theta_2} = \frac{(h_3-h_2)\{a_{41}s(\Theta_3+\Theta_4) + a_{34}s\Theta_3\}}{a_{41}(h_4-h_3)s(\Theta_3+\Theta_4) + a_{34}(h_4-h_1)s\Theta_3} \; . \qquad \text{(vi)}$$

After eliminating $\frac{d\Theta_4}{d\Theta_2}$ between equations (v) and (vi), simplification leads to

$$a_{41}a_{34}(h_3-h_2)s\Theta_4 = a_{23}\{a_{41}(h_4-h_3)s(\Theta_3+\Theta_4) + a_{34}(h_4-h_1)s\Theta_3\}. \qquad \text{(vii)}$$

Elimination of $s\Theta_3$ between (5.5.3) and (vii) results in

$$s(\Theta_3+\Theta_4) = \frac{a_{12}}{a_{41}}\frac{h_4-h_1}{h_3-h_1}s\Theta_2 + \frac{a_{34}}{a_{23}}\frac{h_2-h_3}{h_3-h_1}s\Theta_4. \qquad \text{(viii)}$$

Substitution for $s(\Theta_3+\Theta_4)$ from equation (viii) into (5.5.3) yields the result

$$s\Theta_3 = \frac{a_{12}}{a_{34}}\frac{h_3-h_4}{h_3-h_1}s\Theta_2 + \frac{a_{41}}{a_{23}}\frac{h_3-h_2}{h_3-h_1}s\Theta_4. \qquad \text{(ix)}$$

Since

$$s(\Theta_3+\Theta_4) = s\Theta_3 c\Theta_4 + c\Theta_3 s\Theta_4,$$

from equations (viii) and (ix) we find that

$$c\Theta_3 = \frac{1}{a_{23}}\frac{h_2-h_3}{h_3-h_1}\{a_{34}+a_{41}c\Theta_4\} + \frac{a_{12}}{h_3-h_1}\left\{\frac{h_4-h_1}{a_{41}} + \frac{h_4-h_3}{a_{34}}c\Theta_4\right\}\frac{s\Theta_2}{s\Theta_4}. \qquad \text{(x)}$$

We may now use equations (ix) and (x) to eliminate Θ_3 entirely from equation (5.5.2). Substituting for $s\Theta_3$ and $c\Theta_3$ in (5.5.2) yields, after simplification,

$$\frac{1}{a_{23}}\frac{h_2-h_3}{h_3-h_1}\{a_{41}^2 + a_{34}^2 + 2a_{34}a_{41}c\Theta_4\} + a_{23} + a_{12}c\Theta_2$$
$$+ \frac{s\Theta_2}{s\Theta_4}c\Theta_4 a_{12}\frac{2h_4-(h_1+h_3)}{h_3-h_1} + \frac{s\Theta_2}{s\Theta_4}\frac{a_{12}a_{34}}{a_{41}}\frac{h_4-h_1}{h_3-h_1}$$
$$+ \frac{s\Theta_2}{s\Theta_4}\frac{a_{41}a_{12}}{a_{34}}\frac{h_4-h_3}{h_3-h_1} = 0.$$

We may eliminate $c\Theta_4$ and then $s\Theta_4$ from this last equation by means of (5.5.5). Doing this and rearranging terms yields

$$\{\frac{a_{12}^2(h_2-h_3)+a_{23}^2(h_2-h_1)}{a_{23}} + a_{12}(2h_2-h_3-h_1)c\Theta_2\}^2$$

$$\times\{1 - \frac{1}{4a_{41}^2a_{34}^2}[a_{23}^2+a_{12}^2-a_{41}^2-a_{34}^2+2a_{12}a_{23}c\Theta_2]^2\}$$

$$= a_{12}^2\{1-c^2\Theta_2\}$$

$$\times\{\frac{a_{41}}{a_{34}}(h_3-h_4) + \frac{a_{34}}{a_{41}}(h_1-h_4)$$

$$+ (h_1+h_3-2h_4)\frac{1}{2a_{41}a_{34}}(a_{23}^2+a_{12}^2-a_{41}^2-a_{34}^2+2a_{12}a_{23}c\Theta_2)\}^2. \qquad \text{(xi)}$$

This last equation is in terms of $c\Theta_2$ alone and must be an identity in that variable. Equating coefficients of $c^4\Theta_2$ in (xi) leads to the requirement that

$$(2h_2-h_3-h_1)^2 = (h_1+h_3-2h_4)^2.$$

The two solutions of this result are

$$h_2 = h_4$$

$$\text{and} \qquad h_2 + h_4 = h_1 + h_3 \quad . \qquad \text{(xii)}$$

The first of these has already been treated, by analogy with our above discussion for the constraint $h_1=h_3$.
We therefore accept the second as a relevant condition and substitute it into equation (xi), simplifying it slightly:

$$\left\{\frac{a_{12}{}^2(h_2-h_3) + a_{23}{}^2(h_2-h_1)}{a_{23}} + a_{12}(h_2-h_4)c\Theta_2\right\}^2$$

$$\times \left\{1 - \frac{1}{4a_{41}{}^2a_{34}{}^2}[a_{23}{}^2+a_{12}{}^2-a_{41}{}^2-a_{34}{}^2+2a_{12}a_{23}c\Theta_2]^2\right\}$$

$$= a_{12}{}^2\{1-c^2\Theta_2\}$$

$$\times \left\{\frac{a_{41}}{a_{34}}(h_3-h_4) + \frac{a_{34}}{a_{41}}(h_1-h_4)\right.$$

$$\left. + (h_2-h_4)\frac{1}{2a_{41}a_{34}}(a_{23}{}^2+a_{12}{}^2-a_{41}{}^2-a_{34}{}^2+2a_{12}a_{23}c\Theta_2)\right\}^2 \qquad \text{(xiii)}$$

Equating coefficients of $c^3\Theta_2$ and making use of constraint (xii) yields, after simplification,

$$(a_{12}{}^2-a_{34}{}^2)(h_2-h_3) = (a_{41}{}^2-a_{23}{}^2)(h_2-h_1). \qquad \text{(a)}$$

Equating coefficients of $c^2\Theta_2$ in (xiii), using constraints (xii) and (a) and the result that $h_2 \neq h_4$, we obtain, after considerable simplification, that

$$(h_2-h_4)(a_{41}{}^2a_{34}{}^2-a_{12}{}^2a_{23}{}^2)$$
$$= (a_{23}{}^2+a_{12}{}^2-a_{41}{}^2-a_{34}{}^2)[a_{23}{}^2(h_2-h_1)+a_{12}{}^2(h_2-h_3)]. \qquad \text{(b)}$$

Using constraints (xii), (a) and (b) and the result that $h_2 \neq h_4$, equating coefficients of $c^1\Theta_2$ in (xiii) leads, after simplifying, to

$$2[a_{12}{}^2(h_2-h_3) + a_{23}{}^2(h_2-h_1)][a_{41}{}^2a_{34}{}^2-a_{12}{}^2a_{23}{}^2]$$
$$= (h_2-h_4)(a_{23}{}^2+a_{12}{}^2-a_{41}{}^2-a_{34}{}^2)(a_{12}{}^2a_{23}{}^2+a_{41}{}^2a_{34}{}^2) . \qquad \text{(c)}$$

Using constraints (xii), (a), (b) and (c) and the result that $h_2 \neq h_4$, equating coefficients of $c^0\Theta_2$ in (xiii) and simplifying yields

$$(a_{41}{}^2a_{34}{}^2-a_{12}{}^2a_{23}{}^2)^2 = a_{12}{}^2a_{23}{}^2(a_{23}{}^2+a_{12}{}^2-a_{41}{}^2-a_{34}{}^2)^2. \qquad (d)$$

If either $$a_{41}{}^2a_{34}{}^2 = a_{12}{}^2a_{23}{}^2 \qquad (e)$$

or $$a_{23}{}^2 + a_{12}{}^2 = a_{41}{}^2 + a_{34}{}^2, \qquad (f)$$

then, from (d) and the result that $a_{i\ i+1} \neq 0$ for all i, the other is also implied.

Let us assume

$$a_{41}{}^2a_{34}{}^2 \neq a_{12}{}^2a_{23}{}^2$$

$$a_{23}{}^2 + a_{12}{}^2 \neq a_{41}{}^2 + a_{34}{}^2.$$

Then, since $h_2 \neq h_4$, eliminating the h_i between results (b) and (c) yields

$$2(a_{41}{}^2a_{34}{}^2-a_{12}{}^2a_{23}{}^2)^2 = (a_{23}{}^2+a_{12}{}^2-a_{41}{}^2-a_{34}{}^2)^2 \times (a_{12}{}^2a_{23}{}^2+a_{41}{}^2a_{34}{}^2).$$

Eliminating the LHS of this last equation by use of (d), we have that

$$2a_{12}{}^2a_{23}{}^2 = a_{12}{}^2a_{23}{}^2 + a_{41}{}^2a_{34}{}^2,$$

whence $$a_{12}{}^2a_{23}{}^2 = a_{41}{}^2a_{34}{}^2.$$

This contradiction establishes the validity of results (e) and (f).

Constraints (e) and (f) will satisfy equations (b)-(d) identically.

We have to consider the satisfaction of equation (a).

From result (e),

$$a_{41}a_{34} = a_{12}a_{23}. \qquad (e')$$

Now, (e') and (f) together imply that

$$a_{41} = a_{23} \quad a_{34} = a_{12} \quad \underline{\text{or}} \quad a_{41} = a_{12} \quad a_{34} = a_{23}.$$

The first of these possibilities satisfies equation (a) without any more demands on the h_i.

The second, from (a), generally requires that $h_1=h_3$, a contradiction.

We conclude that the only remaining solution for this category is defined by the constraints

$$\begin{cases} a_{41} = a_{23} \qquad a_{34} = a_{12} \\ h_1 + h_3 = h_2 + h_4. \end{cases}$$

The linkage in 'plan view' is a parallelogram, Delassus solution number d.10. It is illustrated in Fig. 5.5.4.

We may write the independent closure equations in the form

$$\begin{cases} \Theta_3 + \Theta_4 = (2l+1)\pi \\ \Theta_1 + \Theta_2 = (2m+1)\pi \\ \Theta_4 = \Theta_2 + 2n\pi. \end{cases}$$

The fixed offsets are related by

$$R_1+R_2+R_3+R_4+ (2m+1)\pi h_1 + (2l+1)\pi h_3 + 2n\pi(h_4-h_3) = 0.$$

. . . .

We have found five solutions in this category, all of them derived from the $C^{\wedge}H^{\wedge}H^{\wedge}H$- linkage.

5.6 General C-H-C-H- derivatives (no two adjacent joint axes parallel)

The linkages in this category are isolated in references [1,4] and in section 6.2 of this work. Equations II.1 in [1] and the corresponding IIb.1 of section 6.2 provide us with relationships which may be written jointly as

$$\theta_2 = \sigma\theta_4 - (2m + \frac{1-\rho}{2})\pi. \tag{5.6.1}$$

This equation satisfies (5.9) identically, if we also make use of the corresponding relationships, II.2 in [1] and IIb.2 in section 6.2. Equations (5.1)-(5.8), only two of which are independent, become respectively, by (5.6.1),

$$\rho(c\theta_1 c\theta_4 - \sigma s\theta_1 s\theta_4 c\alpha_{12}) = c\theta_3 c\theta_4 - s\theta_3 s\theta_4 c\alpha_{34} \tag{5.6.2}$$

$$-\rho(\sigma c\theta_1 s\theta_4 c\alpha_{23} + s\theta_1 c\theta_4 c\alpha_{12} c\alpha_{23}) + s\theta_1 s\alpha_{12} s\alpha_{23} = s\theta_3 c\theta_4 + c\theta_3 s\theta_4 c\alpha_{34} \tag{5.6.3}$$

$$\rho(\sigma c\theta_1 s\theta_4 s\alpha_{23} + s\theta_1 c\theta_4 c\alpha_{12} s\alpha_{23}) + s\theta_1 s\alpha_{12} c\alpha_{23} = s\theta_4 s\alpha_{34} \tag{5.6.4}$$

$$\rho(s\theta_1 c\theta_4 + \sigma c\theta_1 s\theta_4 c\alpha_{12}) = -c\theta_3 s\theta_4 c\alpha_{41} - s\theta_3 c\theta_4 c\alpha_{34} c\alpha_{41} + s\theta_3 s\alpha_{34} s\alpha_{41} \tag{5.6.5}$$

$$\rho(-\sigma s\theta_1 s\theta_4 c\alpha_{23} + c\theta_1 c\theta_4 c\alpha_{12} c\alpha_{23}) - c\theta_1 s\alpha_{12} s\alpha_{23} = -s\theta_3 s\theta_4 c\alpha_{41} + c\theta_3 c\theta_4 c\alpha_{34} c\alpha_{41} - c\theta_3 s\alpha_{34} s\alpha_{41} \tag{5.6.6}$$

$$\rho(\sigma s\theta_1 s\theta_4 s\alpha_{23} - c\theta_1 c\theta_4 c\alpha_{12} s\alpha_{23}) - c\theta_1 s\alpha_{12} c\alpha_{23} = c\theta_4 s\alpha_{34} c\alpha_{41} + c\alpha_{34} s\alpha_{41} \tag{5.6.7}$$

$$\rho\sigma s\theta_4 s\alpha_{12} = c\theta_3 s\theta_4 s\alpha_{41} + s\theta_3 c\theta_4 c\alpha_{34} s\alpha_{41} + s\theta_3 s\alpha_{34} c\alpha_{41} \tag{5.6.8}$$

$$\rho c\theta_4 s\alpha_{12} c\alpha_{23} + c\alpha_{12} s\alpha_{23} = s\theta_3 s\theta_4 s\alpha_{41} - c\theta_3 c\theta_4 c\alpha_{34} s\alpha_{41} - c\theta_3 s\alpha_{34} c\alpha_{41}. \tag{5.6.9}$$

We use as well, for convenience, an equation obtained by cycling the indices in (5.9). This equation could be used in place of one of the two independent equations from the set above. It is

$$-c\Theta_1 s\alpha_{41} s\alpha_{12} + c\alpha_{41} c\alpha_{12} = -c\Theta_3 s\alpha_{23} s\alpha_{34} + c\alpha_{23} c\alpha_{34}. \tag{5.6.10}$$

In addition, the results II.1, II.2, II.3 from [1] and IIb.1, IIb.2, IIb.3 from section 6.2 render equations (5.10) and (5.11) equivalent.

From II.1 and IIb.1 we also deduce that

$$R_2 + h\Theta_2 = \sigma(R_4 + h\Theta_4)$$

whence, again using II.1 and IIb.1, equations (5.10) and (5.12) may be written as

$$\begin{aligned} a_{41}(c\Theta_3 c\Theta_4 - s\Theta_3 s\Theta_4 c\alpha_{34}) + (R_4 + h\Theta_4) s\Theta_3 s\alpha_{34} + a_{34} c\Theta_3 & \\ + a_{23} + \rho(a_{12} c\Theta_4 + \sigma r_1 s\Theta_4 s\alpha_{12}) = 0 & \qquad (5.6.11) \end{aligned}$$

$$\begin{aligned} a_{41} s\Theta_4 s\alpha_{34} + (R_4 + h\Theta_4) c\alpha_{34} + r_3 + \sigma(R_4 + h\Theta_4) c\alpha_{23} & \\ + \rho\sigma a_{12} s\Theta_4 s\alpha_{23} + r_1(c\alpha_{12} c\alpha_{23} - \rho c\Theta_4 s\alpha_{12} s\alpha_{23}) = 0. & \qquad (5.6.12) \end{aligned}$$

For convenience, we rewrite jointly here results II.2 and II.3 from reference [1] and IIb.2 and IIb.3 from section 6.2:

$$\left.\begin{aligned} s\alpha_{12} s\alpha_{23} &= \rho s\alpha_{34} s\alpha_{41} \\ c\alpha_{12} c\alpha_{23} &= c\alpha_{34} c\alpha_{41} \end{aligned}\right\} \qquad (5.6.\mathrm{I})$$

$$\left.\begin{aligned} a_{12}s\alpha_{12}c\alpha_{23}+a_{23}c\alpha_{12}s\alpha_{23} &= a_{34}s\alpha_{34}c\alpha_{41}+a_{41}c\alpha_{34}s\alpha_{41} \\ a_{12}c\alpha_{12}s\alpha_{23}+a_{23}s\alpha_{12}c\alpha_{23} &= \rho(a_{34}c\alpha_{34}s\alpha_{41}+a_{41}s\alpha_{34}c\alpha_{41}) \end{aligned}\right\} \quad (5.6.\mathrm{II})$$

Let us now examine derivatives with connectivity sum five. We first try to replace joint 3, say, by a slider.

From (5.6.8), if we fix Θ_3, Θ_4 will also be fixed unless

either A $\quad s\Theta_3 = 0, \quad c\Theta_3 = \sigma, \quad s\alpha_{12} = \rho s\alpha_{41}$

or B $\quad c\alpha_{34} = c\alpha_{41} = 0, \quad c\theta_3 = \rho\sigma s\alpha_{12}.$

A

By the first of constraints (5.6.I),

$$s\alpha_{23} = s\alpha_{34}.$$

$s\Theta_3 = 0$ implies that joints 2, 3, 4 are in parallel planes.

From (5.6.9), equating coefficients of powers of $c\Theta_4$,

$$c\alpha_{23} = -\sigma c\alpha_{34} \qquad c\alpha_{12} = -\sigma c\alpha_{41}.$$

Then, from (5.6.10),

$$c\Theta_1 = \rho\sigma.$$

$$\therefore \quad s\Theta_1 = 0$$

So joint 1 is also prismatic, and joints 4, 1, 2 are in parallel planes.

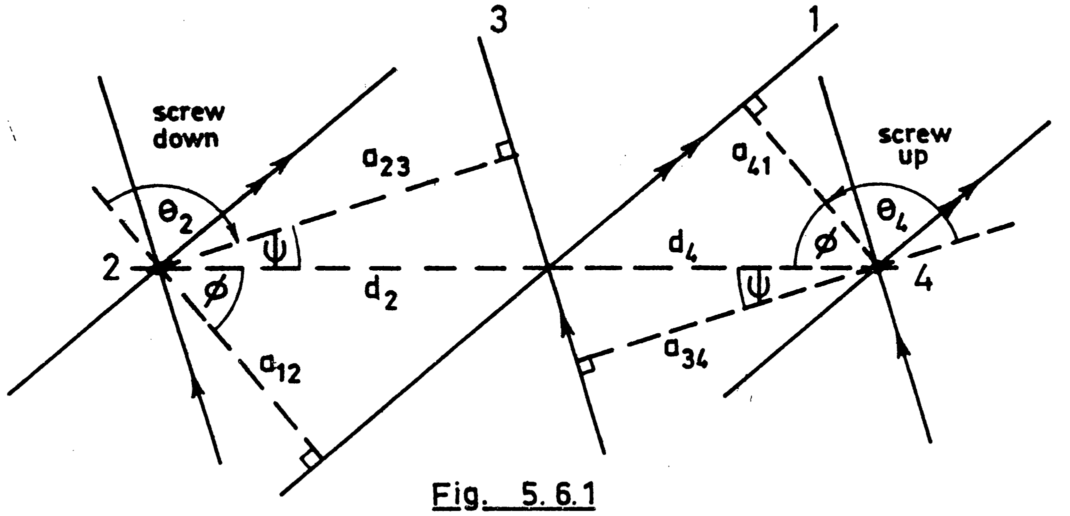


Fig. 5.6.1

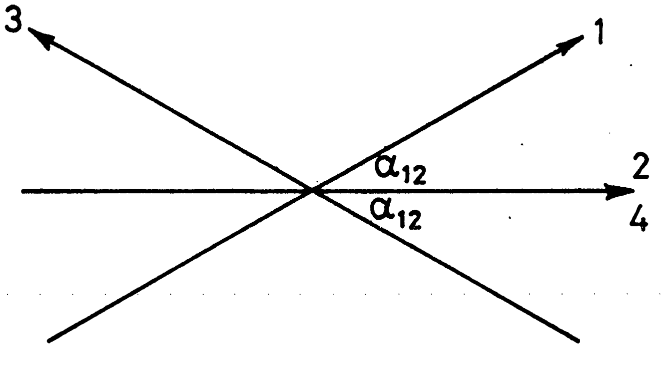


Fig. 5.6.2

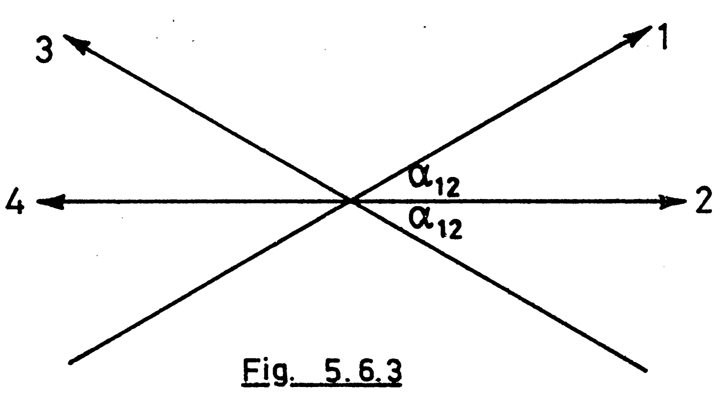


Fig. 5.6.3

Now, this set of planes cannot be always parallel to the set containing joints 2, 3, 4 because, if it were, joints 2 and 4 would be locked. Since, then, the two sets are generally non-parallel, but they both include the screw joints, these joint axes must be parallel.

We may represent an 'end view' of the intersecting planes as shown in Fig. 5.6.1. Since the line of action of a prismatic joint is arbitrary, we have here positioned the planes of action of the sliders so that they intersect on the common normal to the screw axes, between the screw axes. This has been done purely for convenience, and only the case $\rho=\sigma=1$ is illustrated.

With reference to the diagram,

$$\Theta_2 = \pi - (\phi+\psi) = \begin{cases} \Theta_4 \\ \\ 2\pi - \Theta_4 \end{cases}, \quad \text{say,}$$

depending on the relative sense of the two screw directions.

Rotational closure equations (5.6.2)-(5.6.10) are identically satisfied.

The two translational equations reduce to

$$\sigma a_{41}c\Theta_4+\sigma a_{34}+a_{23}+\rho(a_{12}c\Theta_4+\sigma r_1 s\alpha_{12}s\Theta_4) = 0 \qquad (5.6.11')$$

$$a_{41}s\alpha_{23}s\Theta_4+r_3+\rho\sigma a_{12}s\alpha_{23}s\Theta_4+r_1(c\alpha_{12}c\alpha_{23}-\rho c\Theta_4 s\alpha_{12}s\alpha_{23}) = 0. \qquad (5.6.12')$$

Let us now test each of the applicable solutions found previously for the general C-H-C-H- linkage. Since, in the present instance, joints 1 and 3 are not rotationally mobile, the relevant results from [1] are those listed on page 27 therein which do not appear in Table 1. The appropriate ones from 6.2 are those listed as [*a*]-[*i*] which do not appear in Table 6.2.1. Collectively, they are as follows.

$\rho=1$:-

A1: $a_{12} = a_{41} \quad a_{23} = a_{34} \quad \alpha_{12} = \alpha_{41} \quad \alpha_{23} = \alpha_{34} \qquad \sigma=-1$

A2: $a_{12} + a_{34} = a_{23} + a_{41} \quad \alpha_{12} = \alpha_{41} \quad \alpha_{23} = \alpha_{34} = \pi - \alpha_{12} \qquad \sigma=-1$

A3: $\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{41} = \frac{\pi}{2} \qquad \sigma=\pm1$

A4: $a_{12} + a_{23} = a_{34} + a_{41} \quad \alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{41} \qquad \sigma=-1$

A5: $a_{12} + a_{41} = a_{23} + a_{34} \quad \alpha_{12} = \alpha_{34} \quad \alpha_{23} = \alpha_{41} = \pi - \alpha_{12} \qquad \sigma=1$

A6: $a_{12} = a_{23} = a_{34} = a_{41} = 0 \quad \alpha_{12} = \pi - \alpha_{41} \quad \alpha_{23} = \pi - \alpha_{34} \qquad \sigma=1$

$\rho=-1$:-

A7: $\alpha_{12} = \frac{3\pi}{2} \quad \alpha_{23} = \alpha_{34} = \alpha_{41} = \frac{\pi}{2} \qquad \sigma=\pm1$

A8: $a_{23} = a_{34} \quad a_{12} = a_{41} = 0 \quad \alpha_{12} = 2\pi - \alpha_{41} \quad \alpha_{23} = \alpha_{34} \qquad \sigma=-1$

A9: $a_{12} = a_{41} \quad a_{23} = a_{34} = 0 \quad \alpha_{12} = \pi + \alpha_{41} \quad \alpha_{23} = \pi - \alpha_{34} \qquad \sigma=1$

A10: $a_{41} = a_{12} + a_{23} + a_{34} \quad \alpha_{23} = \alpha_{41} = \alpha_{12} - \pi = \pi - \alpha_{34} \qquad \sigma=1$

A11: $a_{23} = a_{34} + a_{41} + a_{12} \quad \alpha_{23} = \alpha_{34} = \alpha_{41} = 2\pi - \alpha_{12} \qquad \sigma=-1$

A12: $a_{34} = a_{41} + a_{12} + a_{23} \quad \alpha_{23} = \alpha_{34} = \alpha_{12} - \pi = \pi - \alpha_{41} \qquad \sigma=-1$

A13: $a_{12} = a_{23} + a_{34} + a_{41} \quad \alpha_{34} = \alpha_{41} = \pi - \alpha_{23} = \alpha_{12} - \pi \qquad \sigma=1$

A1:

Equation (5.6.11') becomes simply

$$r_1 s\Theta_4 = 0,$$

which is inadmissible.

A2:

Equation (5.6.11') reduces to

$$s\alpha_{12} r_1 s\Theta_4 = (a_{12}-a_{41})(c\Theta_4+1).$$

It is clear that

$$a_{12} \neq a_{41}.$$

Physically, $a_{12}=a_{41}$ would result in the screw joints being coaxial.

With reference to Fig. 5.6.1,

$$a_{12} - a_{41} = a_{23} - a_{34}$$

becomes

$$d_2\cos\phi - d_4\cos\phi = d_2\cos\psi - d_4\cos\psi.$$

Since $a_{12}\neq a_{41}$, $d_2\neq d_4$.

$$\therefore \qquad \cos\phi = \cos\psi$$

Now, $\phi=-\psi$ would place the sliders in planes parallel to each other. We conclude that

$$\phi = \psi.$$

Thus, since $\alpha_{23}=\pi-\alpha_{12}$ and $\sigma=-1$, the screws are directed in the same sense, and the sliders are bilaterally symmetric with respect to the plane containing the screws (See Fig. 5.6.2).

A3:

This case will be covered under sub-section B.

A4:

Equation (5.6.11') reduces to

$$s\alpha_{12}r_1 s\Theta_4 = (a_{12}-a_{41})(c\Theta_4-1).$$

As for A2, we cannot have $a_{12}=a_{41}$.
With reference to Fig. 5.6.1,

$$d_2\cos\phi - d_4\cos\phi = d_4\cos\psi - d_2\cos\psi,$$

whence $$\cos\phi = -\cos\psi.$$

The solution $\phi+\psi=\pi$ would make the slider planes parallel to each other. The solution

$$\phi = \pi + \psi$$

or $$\psi = \pi + \phi$$

yields an alternative version of the A2 result.

A5:

Equation (5.6.11') reduces to

$$s\alpha_{12}r_1 s\Theta_4 = -(a_{12}+a_{41})(c\Theta_4+1).$$

Here we must have

$$a_{12} + a_{41} \neq 0.$$

Again, joints 2 and 4 would be coaxial otherwise.
With reference to the diagram,

$$d_2\cos\phi + d_4\cos\phi = d_2\cos\psi + d_4\cos\psi.$$

Since $a_{12} + a_{41} \neq 0$, $d_2 + d_4 \neq 0$. Thus,

$$\cos\phi = \cos\psi.$$

As for A2, we disallow the solution $\phi=-\psi$.
We conclude that

$$\phi = \psi.$$

Thus, since $\sigma=1$ and $\alpha_{12}=\alpha_{34}$, the screws are directed in the opposite sense, and the slider planes are again bilaterally symmetric with respect to the plane containing the screws (See Fig. 5.6.3.).

A6:

Equation (5.6.11') becomes simply

$$r_1 s\Theta_4 = 0,$$

which is inadmissible.

A7:

This case will be covered under sub-section B.

A8:

Equation (5.6.11') again reduces to

$$r_1 s\Theta_4 = 0,$$

which yields no solution.

A9:

The same result is obtained as for A8.

A10:

Equation (5.6.11') becomes

$$r_1 s\Theta_4 s\alpha_{12} = (a_{41} - a_{12})(c\Theta_4 + 1),$$

in which $(a_{41} - a_{12})$ cannot be zero. The only solution is an alternative form of the A5 result.

A11:

Equation (5.6.11') reduces to

$$r_1 s\Theta_4 s\alpha_{12} = (a_{12} + a_{41})(c\Theta_4 - 1),$$

in which $(a_{12} + a_{41})$ cannot be zero. The solution for this case is another form of the A2 result.

A12:

Here, equation (5.6.11') becomes

$$r_1 s\Theta_4 s\alpha_{12} = (a_{12} + a_{41})(c\Theta_4 + 1).$$

$(a_{12} + a_{41})$ cannot be zero, and the solution is another form of the result for case A2.

A13:

For this case, equation (5.6.11') reduces to

$$r_1 s\theta_4 s\alpha_{12} = (a_{41}-a_{12})(c\theta_4-1).$$

Again, $(a_{41}-a_{12})$ is not zero, and the only solution is an alternative form of that found for case A5.

. . . .

So, in cases A2, A4, A5, A10-A13, we have verified the Delassus linkage number d.12.

It might be noted that the values of r_1 and r_3 are independent of the screw pitch and the screw joint offsets. This kind of independence is common to all the Delassus linkages with prismatic joints except number d.8.

In his list of four-bar linkages, Waldron [45,48] overlooked the fact that the above solution d.12 has mobile -R- derivatives free from part-chain mobility.

<u>B</u>

From equation (5.6.10),

$$c\theta_1 = \rho\sigma s\alpha_{23}.$$

So joint 1 is also a slider.
From the first of constraints (5.6.I),

$$s\alpha_{12} s\alpha_{23} = \rho,$$

whence $\alpha_{23} = \frac{\pi}{2} \qquad \alpha_{12} = \frac{\pi}{2}$ or $\frac{3\pi}{2}$.

We therefore have that

$$\alpha_{23} = \alpha_{34} = \alpha_{41} = \frac{\pi}{2} \qquad \alpha_{12} = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$c\Theta_1 = \rho\sigma \qquad c\Theta_3 = \sigma .$$

These conditions indicate precisely the two cases A3 and A7 above. By the same reasoning as in sub-section A, the screw joints are parallel to each other, but there cannot be planes containing the sliders which are always mutually parallel.

Both sliders are normal to each screw joint.

The translational equations (5.6.11) and (5.6.12) become respectively

$$\sigma a_{41}c\Theta_4+\sigma a_{34}+a_{23}+\rho a_{12}c\Theta_4+\sigma r_1 s\Theta_4 = 0 \qquad (5.6.11'')$$

$$a_{41}s\Theta_4+r_3+\rho\sigma a_{12}s\Theta_4-r_1c\Theta_4 = 0 . \qquad (5.6.12'')$$

We have thus obtained Delassus solution number d.4.

. . . .

We have also shown that there is no proper C-H-P-H- linkage with no two adjacent joint axes parallel.

. . . .

We now investigate the possible solutions which result from replacing joint 1, say, by a screw. Equations (5.6.1) to (5.6.10) remain unchanged. Equation (5.6.11) becomes

$$a_{41}(c\Theta_3 c\Theta_4 - s\Theta_3 s\Theta_4 c\alpha_{34}) + (R_4 + h\Theta_4)s\Theta_3 s\alpha_{34} + a_{34}c\Theta_3 + a_{23}$$

$$+ \rho a_{12}c\Theta_4 + \rho(R_1 + h_1\Theta_1)\sigma s\Theta_4 s\alpha_{12} = 0. \qquad (5.6.a)$$

We must establish the conditions under which this equation is obtainable from the rotational equations.

Assuming for the present that $c\alpha_{34} \neq 0$, and using equation (5.6.2), we can rewrite (5.6.a) as

$$s\Theta_4 c\alpha_{34}\{a_{41}\rho(c\Theta_1 c\Theta_4 - \sigma s\Theta_1 s\Theta_4 c\alpha_{12}) + a_{34}c\Theta_3 + a_{23} + \rho a_{12}c\Theta_4$$

$$+ \rho(R_1 + h_1\Theta_1)\sigma s\Theta_4 s\alpha_{12}\} = - s\Theta_3 s\Theta_4 c\alpha_{34}(R_4 + h\Theta_4)s\alpha_{34}.$$

Again using (5.6.2), this last equation becomes

$$s\Theta_4 c\alpha_{34}\{a_{41}\rho(c\Theta_1 c\Theta_4 - \sigma s\Theta_1 s\Theta_4 c\alpha_{12}) + a_{34}c\Theta_3 + a_{23} + \rho a_{12}c\Theta_4$$

$$+ \rho(R_1 + h_1\Theta_1)\sigma s\Theta_4 s\alpha_{12}\}$$

$$= (R_4 + h\Theta_4)s\alpha_{34}\{\rho c\Theta_1 c\Theta_4 - c\Theta_3 c\Theta_4 - \rho\sigma s\Theta_1 s\Theta_4 c\alpha_{12}\}$$

which, by use of (5.6.10), becomes

$$s\Theta_4 s\alpha_{23} s\alpha_{34} c\alpha_{34}\{a_{41}\rho(c\Theta_1 c\Theta_4 - \sigma s\Theta_1 s\Theta_4 c\alpha_{12}) + a_{23} + \rho a_{12}c\Theta_4$$

$$+ \rho(R_1 + h_1\Theta_1)\sigma s\Theta_4 s\alpha_{12}\}$$

$$+ a_{34}s\Theta_4 c\alpha_{34}\{c\Theta_1 s\alpha_{41} s\alpha_{12} + c\alpha_{23}c\alpha_{34} - c\alpha_{41}c\alpha_{12}\}$$

$$= (R_4 + h\Theta_4)\rho s\alpha_{23} s^2\alpha_{34}\{c\Theta_1 c\Theta_4 - \sigma s\Theta_1 s\Theta_4 c\alpha_{12}\}$$

$$- (R_4 + h\Theta_4)s\alpha_{34}c\Theta_4\{c\Theta_1 s\alpha_{41} s\alpha_{12} + c\alpha_{23}c\alpha_{34} - c\alpha_{41}c\alpha_{12}\}. \qquad (5.6.b)$$

We consider now the effect of increasing Θ_4 to

$$\Theta_4' = \Theta_4 + 2\pi.$$

Let the corresponding new value of Θ_1 be Θ_1'.

From equations (5.6.4) and (5.6.7), we may then write

$$\left.\begin{aligned} \rho\sigma s\Theta_4 s\alpha_{23}(c\Theta_1'-c\Theta_1)+(\rho c\Theta_4 c\alpha_{12}s\alpha_{23}+s\alpha_{12}c\alpha_{23})(s\Theta_1'-s\Theta_1) &= 0 \\ -(c\Theta_4 c\alpha_{12}s\alpha_{23}+\rho s\alpha_{12}c\alpha_{23})(c\Theta_1'-c\Theta_1)+\sigma s\Theta_4 s\alpha_{23}(s\Theta_1'-s\Theta_1) &= 0 \end{aligned}\right\}(5.6.c).$$

Let us consider the determinant of coefficients in equations (5.6.c).

$$D = \begin{vmatrix} \rho\sigma s\Theta_4 s\alpha_{23} & \rho c\Theta_4 c\alpha_{12}s\alpha_{23}+s\alpha_{12}c\alpha_{23} \\ -(c\Theta_4 c\alpha_{12}s\alpha_{23}+\rho s\alpha_{12}c\alpha_{23}) & \sigma s\Theta_4 s\alpha_{23} \end{vmatrix}$$

$$= -\rho c^2\Theta_4 s^2\alpha_{12}s^2\alpha_{23}+2c\Theta_4 c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}+\rho s^2\alpha_{12}c^2\alpha_{23}+\rho s^2\alpha_{23}\neq 0,$$

since $\qquad s\alpha_{12} \neq 0, \quad s\alpha_{23} \neq 0.$

Hence, the only solution to equations (5.6.c) is

$$c\Theta_1' = c\Theta_1 \qquad s\Theta_1' = s\Theta_1,$$

whence $\qquad \Theta_1' = \Theta_1 + 2n\pi.$

For the above changes in Θ_4 and Θ_1, we have from (5.6.b) that

$$s\Theta_4 s\alpha_{23}s\alpha_{34}c\alpha_{34}.2n\pi\ h_1\ \sigma s\Theta_4 s\alpha_{12}$$

$$= 2\pi\ h\{s\alpha_{23}s^2\alpha_{34}(c\Theta_1 c\Theta_4-\sigma s\Theta_1 s\Theta_4 c\alpha_{12})$$

$$- s\alpha_{34}c\Theta_4\rho(c\Theta_1 s\alpha_{41}s\alpha_{12}+c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})\},$$

or

$$\sigma n h_1 s\alpha_{12} s\alpha_{23} c\alpha_{34} s^2\Theta_4$$

$$= h\{c\Theta_1 c\Theta_4 (s\alpha_{23} s\alpha_{34} - \rho s\alpha_{41} s\alpha_{12})$$

$$+ c\Theta_4 \rho (c\alpha_{41} c\alpha_{12} - c\alpha_{23} c\alpha_{34}) - \sigma s\Theta_1 s\Theta_4 c\alpha_{12} s\alpha_{23} s\alpha_{34}\}. \quad (5.6.d)$$

Let us now 'solve' equations (5.6.4) and (5.6.7) to give us $c\Theta_1$ and $s\Theta_1$ in terms of $c\Theta_4$ and $s\Theta_4$.
We write as before

$$D = \begin{vmatrix} \rho\sigma s\Theta_4 s\alpha_{23} & \rho c\Theta_4 c\alpha_{12} s\alpha_{23} + s\alpha_{12} c\alpha_{23} \\ -(c\Theta_4 c\alpha_{12} s\alpha_{23} + \rho s\alpha_{12} c\alpha_{23}) & \sigma s\Theta_4 s\alpha_{23} \end{vmatrix}$$

$$= -\rho c^2\Theta_4 s^2\alpha_{12} s^2\alpha_{23} + 2c\Theta_4 c\alpha_{12} s\alpha_{12} c\alpha_{23} s\alpha_{23} + \rho s^2\alpha_{12} c^2\alpha_{23}$$

$$+ \rho s^2\alpha_{23} \neq 0,$$

as we have previously shown.

Now put

$$D_c = \begin{vmatrix} s\Theta_4 s\alpha_{34} & \rho c\Theta_4 c\alpha_{12} s\alpha_{23} + s\alpha_{12} c\alpha_{23} \\ \rho(c\Theta_4 s\alpha_{34} c\alpha_{41} + c\alpha_{34} s\alpha_{41}) & \sigma s\Theta_4 s\alpha_{23} \end{vmatrix}$$

$$= -c^2\Theta_4 (c\alpha_{12} c\alpha_{41} + \sigma) s\alpha_{23} s\alpha_{34}$$

$$- c\Theta_4 (c\alpha_{12} s\alpha_{23} c\alpha_{34} s\alpha_{41} + \rho s\alpha_{12} c\alpha_{23} s\alpha_{34} c\alpha_{41})$$

$$+ \sigma s\alpha_{23} s\alpha_{34} - \rho s\alpha_{12} c\alpha_{23} c\alpha_{34} s\alpha_{41} \quad (5.6.e)$$

and

$$D_s = \begin{vmatrix} \rho\sigma s\Theta_4 s\alpha_{23} & s\Theta_4 s\alpha_{34} \\ -(c\Theta_4 c\alpha_{12} s\alpha_{23} + \rho s\alpha_{12} c\alpha_{23}) & \rho(c\Theta_4 s\alpha_{34} c\alpha_{41} + c\alpha_{34} s\alpha_{41}) \end{vmatrix}$$

$$= s\Theta_4 c\Theta_4 (c\alpha_{12} + \sigma c\alpha_{41}) s\alpha_{23} s\alpha_{34}$$

$$+ s\Theta_4 (\rho s\alpha_{12} c\alpha_{23} s\alpha_{34} + \sigma s\alpha_{23} c\alpha_{34} s\alpha_{41}). \qquad (5.6.f)$$

Using Cramer's rule, we substitute the expressions for $c\Theta_1$ and $s\Theta_1$ obtainable from D, D_c and D_s into equation (5.6.d):

$$\sigma n h_1 s\alpha_{12} s\alpha_{23} c\alpha_{34} (1 - c^2\Theta_4)\{-c^2\Theta_4 s^2\alpha_{23} s^2\alpha_{12} + 2\rho c\Theta_4 c\alpha_{12} s\alpha_{12} c\alpha_{23} s\alpha_{23}$$

$$+ s^2\alpha_{12} c^2\alpha_{23} + s^2\alpha_{23}\}$$

$$= h[(s\alpha_{23} s\alpha_{34} - \rho s\alpha_{41} s\alpha_{12})\{-c^2\Theta_4(\rho c\alpha_{12} s\alpha_{23} c\alpha_{34} s\alpha_{41}$$

$$+ s\alpha_{12} c\alpha_{23} s\alpha_{34} c\alpha_{41})$$

$$+ c\Theta_4(\rho\sigma s\alpha_{23} s\alpha_{34} - s\alpha_{12} c\alpha_{23} c\alpha_{34} s\alpha_{41})$$

$$- c^3\Theta_4 \rho (c\alpha_{12} c\alpha_{41} + \sigma) s\alpha_{23} s\alpha_{34}\}$$

$$+ \rho(c\alpha_{41} c\alpha_{12} - c\alpha_{23} c\alpha_{34})\{-c^3\Theta_4 s^2\alpha_{12} s^2\alpha_{23}$$

$$+ 2\rho c^2\Theta_4 c\alpha_{12} s\alpha_{12} c\alpha_{23} s\alpha_{23} + c\Theta_4(s^2\alpha_{12} c^2\alpha_{23} + s^2\alpha_{23})\}$$

$$- \sigma c\alpha_{12} s\alpha_{23} s\alpha_{34}\{(1 - c^2\Theta_4) c\Theta_4 \rho (c\alpha_{12} + \sigma c\alpha_{41}) s\alpha_{23} s\alpha_{34}$$

$$+ (1 - c^2\Theta_4)(s\alpha_{12} c\alpha_{23} s\alpha_{34} + \rho\sigma s\alpha_{23} c\alpha_{34} s\alpha_{41})\}] \qquad (5.6.g)$$

Since this equation must be an identity in $c\Theta_4$, we may now equate the coefficients of powers of $c\Theta_4$.

$$
\left.
\begin{aligned}
c^4\theta_4:&\quad \sigma n h_1 s\alpha_{12}s\alpha_{23}c\alpha_{34}s^2\alpha_{23}s^2\alpha_{12} = 0\\
c^3\theta_4:&\quad -2\sigma\rho n h_1 s\alpha_{12}s\alpha_{23}c\alpha_{34}c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}\\
&\qquad = h\{-(c\alpha_{12}c\alpha_{41}+\sigma)\rho s\alpha_{23}s\alpha_{34}(s\alpha_{23}s\alpha_{34}-\rho s\alpha_{41}s\alpha_{12})\\
&\qquad\quad -\rho s^2\alpha_{12}s^2\alpha_{23}(c\alpha_{41}c\alpha_{12}-c\alpha_{23}c\alpha_{34})\\
&\qquad\quad +\rho\sigma c\alpha_{12}s^2\alpha_{23}s^2\alpha_{34}(c\alpha_{12}+\sigma c\alpha_{41})\}\\
c^2\theta_4:&\quad -\sigma n h_1 s\alpha_{12}s\alpha_{23}c\alpha_{34}(s^2\alpha_{23}s^2\alpha_{12}+s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})\\
&\qquad = h\{-(s\alpha_{23}s\alpha_{34}-\rho s\alpha_{41}s\alpha_{12})(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}\\
&\qquad\qquad\qquad + s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})\\
&\qquad\quad + 2c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}(c\alpha_{41}c\alpha_{12}-c\alpha_{23}c\alpha_{34})\\
&\qquad\quad + \sigma c\alpha_{12}s\alpha_{23}s\alpha_{34}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41})\}\\
c^1\theta_4:&\quad 2\rho\sigma n h_1 s^2\alpha_{12}s^2\alpha_{23}c\alpha_{34}c\alpha_{12}c\alpha_{23}\\
&\qquad = h\{(s\alpha_{23}s\alpha_{34}-\rho s\alpha_{41}s\alpha_{12})(\sigma\rho s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41})\\
&\qquad\quad + (c\alpha_{41}c\alpha_{12}-c\alpha_{23}c\alpha_{34})\rho(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})\\
&\qquad\quad - \rho\sigma c\alpha_{12}s^2\alpha_{23}s^2\alpha_{34}(c\alpha_{12}+\sigma c\alpha_{41})\}\\
c^0\theta_4:&\quad \sigma n h_1 s\alpha_{12}s\alpha_{23}c\alpha_{34}(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})\\
&\qquad = h\{-\sigma c\alpha_{12}s\alpha_{23}s\alpha_{34}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41})\}
\end{aligned}
\right\}
\tag{5.6.h}
$$

From the first of these equations, we see that we must have $n=0$, $h_1=0$ or $c\alpha_{34}=0$.

We first show that $n\neq 0$.

For n to be zero, as Θ_4 increases to $\Theta_4+2\pi$, it would be necessary for Θ_1 either to remain fixed or to reach an extremum and return subsequently to its original value. The first contingency is inadmissible; for the second to be the case, it would require that the differential of Θ_1 become zero at some stage. Let us differentiate equation (5.6.4) with respect to Θ_4:

$$\sigma\rho c\Theta_1 c\Theta_4 s\alpha_{23} - \rho s\Theta_1 s\Theta_4 c\alpha_{12} s\alpha_{23}$$
$$-\rho\sigma s\Theta_1 s\Theta_4 s\alpha_{23}\frac{d\Theta_1}{d\Theta_4} + \rho c\Theta_1 c\Theta_4 c\alpha_{12} s\alpha_{23}\frac{d\Theta_1}{d\Theta_4} + c\Theta_1 s\alpha_{12} c\alpha_{23}\frac{d\Theta_1}{d\Theta_4} = c\Theta_4 s\alpha_{34}$$

If $\frac{d\Theta_1}{d\Theta_4} = 0$, then

$$\sigma c\Theta_1 c\Theta_4 s\alpha_{23} - s\Theta_1 s\Theta_4 c\alpha_{12} s\alpha_{23} = \rho c\Theta_4 s\alpha_{34}. \qquad (5.6.\alpha)$$

By means of results (5.6.e) and (5.6.f), we may substitute 'solutions' for $s\Theta_1$ and $c\Theta_1$ in terms of $s\Theta_4$ and $c\Theta_4$ from equations (5.6.4) and (5.6.7) into equation (5.6.α):

$$\sigma s\alpha_{23}\{-c^3\Theta_4\rho(c\alpha_{12}c\alpha_{41}+\sigma)s\alpha_{23}s\alpha_{34}$$
$$- c^2\Theta_4(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})$$
$$+ c\Theta_4(\rho\sigma s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41})\}$$
$$-c\alpha_{12}s\alpha_{23}\{(1-c^2\Theta_4)c\Theta_4\rho(c\alpha_{12}+\sigma c\alpha_{41})s\alpha_{23}s\alpha_{34}$$
$$+ (1-c^2\Theta_4)(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41})\}$$
$$=\rho s\alpha_{34}\{-c^3\Theta_4 s^2\alpha_{12}s^2\alpha_{23}+2c^2\Theta_4\rho c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}$$
$$+ c\Theta_4(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})\}$$

This equation is to be an identity in $c\Theta_4$; we equate coefficients of powers of $c\Theta_4$.

$$\left.\begin{array}{ll}
c^3\Theta_4: & -\sigma s^2\alpha_{23}s\alpha_{34}\rho(c\alpha_{12}c\alpha_{41}+\sigma)+\rho c\alpha_{12}s^2\alpha_{23}s\alpha_{34}(c\alpha_{12}+\sigma c\alpha_{41}) \\
& \qquad = -\rho s^2\alpha_{12}s^2\alpha_{23}s\alpha_{34} \\
c^2\Theta_4: & -\sigma s\alpha_{23}(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41}) \\
& +c\alpha_{12}s\alpha_{23}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41}) \\
& \qquad =2c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}s\alpha_{34} \\
c^1\Theta_4: & \sigma s\alpha_{23}(\sigma\rho s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41}) \\
& \qquad -\rho c\alpha_{12}s^2\alpha_{23}s\alpha_{34}(c\alpha_{12}+\sigma c\alpha_{41}) \\
& \qquad =\rho s\alpha_{34}(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23}) \\
c^0\Theta_4: & -c\alpha_{12}s\alpha_{23}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41}) = 0
\end{array}\right\} \quad (5.6.\beta)$$

The second of these equations reduces to

$$-\sigma s\alpha_{23}s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41} = c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}s\alpha_{34},$$

from which we conclude that

$$\text{either} \quad c\alpha_{12} = -\sigma c\alpha_{41} \quad \text{or} \quad c\alpha_{23} = 0.$$

If $c\alpha_{12}=-\sigma c\alpha_{41}\neq 0$, from equations (5.6.I),

$$c\alpha_{23}=-\sigma c\alpha_{34} \qquad \rho s\alpha_{12}=s\alpha_{41} \qquad s\alpha_{23}=s\alpha_{34}.$$

Then equations (5.6.4) and (5.6.7) become

$$\rho c\Theta_1(\sigma s\Theta_4 s\alpha_{23})+s\Theta_1(\rho c\Theta_4 c\alpha_{12}s\alpha_{23}+s\alpha_{12}c\alpha_{23})=s\Theta_4 s\alpha_{23}$$

$$-c\Theta_1(\rho c\Theta_4 c\alpha_{12}s\alpha_{23}+s\alpha_{12}c\alpha_{23})+\rho s\Theta_1(\sigma s\Theta_4 s\alpha_{23})$$

$$=-\sigma(c\Theta_4 s\alpha_{23}c\alpha_{12}+\rho c\alpha_{23}s\alpha_{12}).$$

But simultaneous satisfaction of these equations demands the fixing of either Θ_1 or Θ_4.

If $c\alpha_{12}=c\alpha_{41}=0$, we see from equation (5.6.α) directly that Θ_1 or Θ_4 is fixed.

If $c\alpha_{23}=0$, from the last of equations (5.6.β) and the second of equations (5.6.I), we must have that

$$\text{either} \quad c\alpha_{12}=c\alpha_{41}=0 \quad \text{or} \quad c\alpha_{34}=0.$$

The former possibility has just been dealt with. For the latter to hold, from the first of (5.6.I),

$$\rho s\alpha_{12} = s\alpha_{41}.$$

Now, the third of equations (5.6.β) reduces to

$$1-c\alpha_{12}(c\alpha_{12}+\sigma c\alpha_{41}) = 1.$$

Hence, either $c\alpha_{12}=0$ or $c\alpha_{12}=-\sigma c\alpha_{41}$.

The second of these possibilities yields a sub-case of the situation we have just discussed.
The first, again by equation (5.6.α) directly, implies that Θ_1 or Θ_4 is fixed.

We conclude that $n\neq 0$.

With reference to the satisfaction of the first of equations (5.6.h), we now investigate the consequences of choosing

$$c\alpha_{34} = 0.$$

From the second of equations (5.6.I), we have

either $\quad c\alpha_{12} = 0 \quad$ or $\quad c\alpha_{23} = 0.$

From the first of equations (5.6.I),

if $c\alpha_{12} = 0$, $\qquad$ and if $c\alpha_{23} = 0$,

$s\alpha_{23} = s\alpha_{41}$ $\qquad$ $\rho s\alpha_{12} = s\alpha_{41}.$

We shall look at each of these possibilities.

$\alpha_{12} = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ $\alpha_{34} = \frac{\pi}{2}$ $s\alpha_{23} = s\alpha_{41}$:

We cannot allow $c\alpha_{23}$ or $c\alpha_{41}$ to be zero, since it would result in one of the cases referred to previously as A3 and A7.

Equation (5.6.4) becomes

$$\sigma c\Theta_1 s\Theta_4 s\alpha_{23} + s\Theta_1 c\alpha_{23} = \rho s\Theta_4,$$

whence

$$s\Theta_4 = \frac{s\Theta_1 \rho c\alpha_{23}}{1-\sigma\rho c\Theta_1 s\alpha_{23}}.$$

From equation (5.6.10),

$$c\Theta_3 = \rho c\Theta_1.$$

From equation (5.6.8),

$$\rho\sigma \frac{s\Theta_1 c\alpha_{23}}{1-\sigma\rho c\Theta_1 s\alpha_{23}} = c\Theta_1 \frac{s\Theta_1 c\alpha_{23}}{1-\sigma\rho c\Theta_1 s\alpha_{23}} s\alpha_{23} + s\Theta_3 c\alpha_{41},$$

whence

$$s\Theta_3 = \rho\frac{c\alpha_{23}}{c\alpha_{41}} \sigma s\Theta_1.$$

We now substitute for $c\Theta_3$ and $s\Theta_3$, along with the other constraints, in equation (5.6.11):

$$a_{41}\rho c\Theta_1 c\Theta_4+(R_4+h\Theta_4)\frac{c\alpha_{23}}{c\alpha_{41}}\rho\sigma s\Theta_1+a_{34}\rho c\Theta_1+a_{23}+\rho a_{12}c\Theta_4$$

$$+\ (R_1+h_1\Theta_1)\sigma s\Theta_4\ =\ 0$$

Allowing now Θ_4 to increase to $\Theta_4+2\pi$, Θ_1 to $\Theta_1+2n\pi$, we obtain from the last equation

$$2\pi h\rho\ \frac{c\alpha_{23}}{c\alpha_{41}}\ \sigma s\Theta_1+2n\pi h_1\sigma s\Theta_4\ =\ 0.$$

That is,

$$h\ \frac{c\alpha_{23}}{c\alpha_{41}}\ s\Theta_1+nh_1\ \frac{s\Theta_1 c\alpha_{23}}{1-\sigma\rho c\Theta_1 s\alpha_{23}}\ =\ 0$$

$$h(1-\sigma\rho c\Theta_1 s\alpha_{23})+nh_1 c\alpha_{41}\ =\ 0.$$

Since this equation must be an identity in $c\Theta_1$, and n and $c\alpha_{41}$ are non-zero, we deduce that

$$h\ =\ h_1\ =\ 0.$$

$\alpha_{23}=\alpha_{34}=\frac{\pi}{2}\quad \rho s\alpha_{12}=s\alpha_{41}$:

Here, we cannot allow $c\alpha_{12}$ or $c\alpha_{41}$ to be zero for the same reason given above.

Equations (5.6.4) and (5.6.7) reduce to

$$\left.\begin{aligned} &s\Theta_4(\sigma\rho c\Theta_1-1)+\rho c\Theta_4 s\Theta_1 c\alpha_{12}\ =\ 0\\ &\rho s\Theta_4\sigma s\Theta_1-c\Theta_4 c\alpha_{12}\left(\rho c\Theta_1\ +\ \frac{c\alpha_{41}}{c\alpha_{12}}\right)\ =\ 0 \end{aligned}\right\}. \qquad (5.6.\gamma)$$

In order that the joint angles remain variable, we must have that

$$\begin{vmatrix} \rho\sigma c\Theta_1 - 1 & s\Theta_1 \rho c\alpha_{12} \\ \sigma\rho s\Theta_1 & -c\alpha_{12}(\rho c\Theta_1 + \frac{c\alpha_{41}}{c\alpha_{12}}) \end{vmatrix} = 0.$$

That is,

$$\sigma - \rho c\Theta_1 + \rho\sigma c\Theta_1 \frac{c\alpha_{41}}{c\alpha_{12}} - \frac{c\alpha_{41}}{c\alpha_{12}} = 0.$$

So we must have that

$$\sigma = \frac{c\alpha_{41}}{c\alpha_{12}} .$$

Making this substitution, we find from the first of equations (5.6.γ) that

$$s\Theta_1 = \frac{s\Theta_4}{\rho c\Theta_4 c\alpha_{12}} (1 - \sigma\rho c\Theta_1)$$

which, in the second of (5.6.γ), yields

$$\sigma s^2\Theta_4 (1 - \rho\sigma c\Theta_1) - c^2\Theta_4 c^2\alpha_{12}(\rho c\Theta_1 + \sigma) = 0.$$

$$\therefore \qquad \rho c\Theta_1 = \sigma \frac{s^2\Theta_4 - c^2\Theta_4 c^2\alpha_{12}}{1 - c^2\Theta_4 s^2\alpha_{12}}$$

Now, from equation (5.6.10),

$$c\Theta_3 = \rho c\Theta_1 s^2\alpha_{12} - \sigma c^2\alpha_{12}.$$

From equation (5.6.8),

$$s\Theta_3\sigma c\alpha_{12} = \rho\sigma s\Theta_4 s\alpha_{12} - \rho s\Theta_4 s\alpha_{12} c\Theta_3 = \rho s\Theta_4 s\alpha_{12}(\sigma - \rho c\Theta_1 s^2\alpha_{12} + \sigma c^2\alpha_{12})$$

$$= \rho\sigma s\Theta_4 s\alpha_{12}(1 + c^2\alpha_{12} - s^2\alpha_{12} \frac{s^2\Theta_4 - c^2\Theta_4 c^2\alpha_{12}}{1 - c^2\Theta_4 s^2\alpha_{12}}).$$

$$\therefore s\Theta_3 = \rho\frac{s\alpha_{12}}{c\alpha_{12}}\frac{s\Theta_4}{1-c^2\Theta_4 s^2\alpha_{12}}([1+c^2\alpha_{12}][1-c^2\Theta_4 s^2\alpha_{12}]$$

$$-s^2\alpha_{12}[s^2\Theta_4 - c^2\Theta_4 c^2\alpha_{12}])$$

$$= 2\rho s\alpha_{12}c\alpha_{12}\frac{s\Theta_4}{1-c^2\Theta_4 s^2\alpha_{12}}$$

We may now write equation (5.6.11) as

$$a_{41}c\Theta_4(\rho c\Theta_1 s^2\alpha_{12}-\sigma c^2\alpha_{12})+\rho(R_4+h\Theta_4).2s\alpha_{12}c\alpha_{12}\frac{s\Theta_4}{1-c^2\Theta_4 s^2\alpha_{12}}$$

$$+ a_{34}(\rho c\Theta_1 s^2\alpha_{12}-\sigma c^2\alpha_{12})+a_{23}+\rho a_{12}c\Theta_4+\rho(R_1+h_1\Theta_1)\sigma s\Theta_4 s\alpha_{12}=0.$$

Now allowing Θ_4 to increase to $\Theta_4+2\pi$, Θ_1 to $\Theta_1+2n\pi$, we have from the last equation that

$$2\pi\rho h.2s\alpha_{12}c\alpha_{12}\frac{s\Theta_4}{1-c^2\Theta_4 s^2\alpha_{12}} + 2n\pi\rho h_1\sigma s\Theta_4 s\alpha_{12} = 0.$$

That is,

$$2hc\alpha_{12}+\sigma nh_1(1-c^2\Theta_4 s^2\alpha_{12}) = 0.$$

For this to be an identity in $c\Theta_4$, and since $c\alpha_{12}$ and n are non-zero, we must have

$$h = h_1 = 0.$$

We have thus shown the constraint $c\alpha_{34}=0$ to result in a special case of the possibility $h_1 = 0$.

We may now assume that $c\alpha_{34} \neq 0$ and satisfy the first of equations (5.6.h) by the constraint

$$h_1 = 0.$$

Let us suppose that $h \neq 0$.

Then, from the last of equations (5.6.h), we have

$$\text{either} \quad c\alpha_{12} = 0 \quad \text{or} \quad c\alpha_{34} = -\sigma c\alpha_{23} \frac{\rho s\alpha_{12} s\alpha_{34}}{s\alpha_{23} s\alpha_{41}}.$$

We proceed to examine both of these possibilities.

$c\alpha_{12} = 0$:

Since $c\alpha_{34} \neq 0$, from equations (5.6.I), we have that

$$c\alpha_{41} = 0 \qquad s\alpha_{23} = s\alpha_{34}.$$

Then, from the second of equations (5.6.h),

$$0 = -\sigma\rho s^2\alpha_{23}(s^2\alpha_{23} - 1) + \rho s^2\alpha_{23} c\alpha_{23} c\alpha_{34}.$$

Now, we cannot allow $c\alpha_{23} = 0$, since it would result in $c\alpha_{34} = 0$. Therefore,

$$c\alpha_{34} = -\sigma c\alpha_{23}.$$

Equations (5.6.4) and (5.6.7) become

$$\sigma\rho c\Theta_1 s\Theta_4 s\alpha_{23} + \rho s\Theta_1 c\alpha_{23} = s\Theta_4 s\alpha_{23}$$

$$-\rho c\Theta_1 c\alpha_{23} + \rho\sigma s\Theta_1 s\Theta_4 s\alpha_{23} = -\sigma c\alpha_{23},$$

from which we deduce that either joint 1 or joint 4 is locked.

$$c\alpha_{34} = -\sigma c\alpha_{23}\frac{\rho s\alpha_{12}s\alpha_{34}}{s\alpha_{23}s\alpha_{41}}:$$

From the second of equations (5.6.I), we have

$$c\alpha_{41} = -\sigma c\alpha_{12}\frac{s\alpha_{23}s\alpha_{41}}{\rho s\alpha_{12}s\alpha_{34}}.$$

From the second of equations (5.6.h),

$$0 = -\left(-\sigma c^2\alpha_{12}\frac{s\alpha_{23}s\alpha_{41}}{\rho s\alpha_{12}s\alpha_{34}}+\sigma\right)\rho s\alpha_{23}s\alpha_{34}(s\alpha_{23}s\alpha_{34}-\rho s\alpha_{41}s\alpha_{12})$$

$$-s^2\alpha_{12}s^2\alpha_{23}\rho\left(-\sigma c^2\alpha_{12}\frac{s\alpha_{23}s\alpha_{41}}{\rho s\alpha_{12}s\alpha_{34}}+\sigma c^2\alpha_{23}\frac{\rho s\alpha_{12}s\alpha_{34}}{s\alpha_{23}s\alpha_{41}}\right)$$

$$+\rho\sigma c\alpha_{12}s^2\alpha_{23}s^2\alpha_{34}\left(c\alpha_{12}-c\alpha_{12}\frac{s\alpha_{23}s\alpha_{41}}{\rho s\alpha_{12}s\alpha_{34}}\right).$$

Therefore,

$$c^2\alpha_{12}\left\{\frac{s\alpha_{23}s\alpha_{41}}{\rho s\alpha_{12}s\alpha_{34}}(s^2\alpha_{23}s^2\alpha_{34}-\rho s\alpha_{12}s\alpha_{23}s\alpha_{34}s_{41}\right.$$

$$\left.+s^2\alpha_{12}s^2\alpha_{23}-s^2\alpha_{23}s^2\alpha_{34})+s^2\alpha_{23}s^2\alpha_{34}\right\}$$

$$-(s^2\alpha_{23}s^2\alpha_{34}-\rho s\alpha_{12}s\alpha_{23}s\alpha_{34}s\alpha_{41})-c^2\alpha_{23}s^2\alpha_{12}s^2\alpha_{23}\frac{\rho s\alpha_{12}s\alpha_{34}}{s\alpha_{23}s\alpha_{41}} = 0.$$

Hence, using the first of equations (5.6.I),

$$s^2\alpha_{23}s^2\alpha_{34}(c^2\alpha_{12}-1)+s^2\alpha_{12}s^2\alpha_{23}-c^2\alpha_{23}s^2\alpha_{12}s^2\alpha_{23}\frac{\rho s\alpha_{12}s\alpha_{34}}{s\alpha_{23}s\alpha_{41}} = 0.$$

Therefore,

$$c^2\alpha_{34}-c^2\alpha_{23}\frac{\rho s\alpha_{12}s\alpha_{34}}{s\alpha_{23}s\alpha_{41}} = 0$$

whence $\quad c^2\alpha_{23}\left(\frac{s\alpha_{12}s\alpha_{34}}{s\alpha_{23}s\alpha_{41}}\right)^2 - c^2\alpha_{23}\frac{\rho s\alpha_{12}s\alpha_{34}}{s\alpha_{23}s\alpha_{41}} = 0.$

We cannot have $c\alpha_{23}=0$, as it would imply that $c\alpha_{34} = 0$.

Hence,

$$\rho s\alpha_{12}s\alpha_{34} = s\alpha_{23}s\alpha_{41}.$$

Thus,

$$c\alpha_{34} = -\sigma c\alpha_{23} \quad \text{and} \quad c\alpha_{41} = -\sigma c\alpha_{12}.$$

But we have already shown that such a set of conditions will lock either joint 1 or joint 4.

We conclude that

$$h = 0.$$

. . . .

We have thus shown that the H-H-C-H- loop with no two adjacent joint axes parallel degenerates immediately to a R-R-C-R- chain. We have yet to examine the cylindric joint for mobility and to determine any other consequent dimensional conditions.

We shall first investigate the geometrical constraints concomitant with the result above that h_1=h=0. To do this, we again look at equation (5.6.11), under now five separate headings:

$$\alpha_{12}=\alpha_{34}=\frac{\pi}{2},\ s\alpha_{23}=s\alpha_{41};\quad \alpha_{23}=\alpha_{34}=\frac{\pi}{2},\ s\alpha_{12}=s\alpha_{41};$$

$$\alpha_{12}=\frac{3\pi}{2},\ \alpha_{34}=\frac{\pi}{2},\ s\alpha_{23}=s\alpha_{41};\quad \alpha_{23}=\alpha_{34}=\frac{\pi}{2},\ -s\alpha_{12}=s\alpha_{41};\quad c\alpha_{34}\neq 0$$

$\alpha_{12}=\alpha_{34}=\frac{\pi}{2}$, $\quad s\alpha_{23}=s\alpha_{41}$:

From Table 1 of reference [1] we see that there are only two possibilities, remembering that $\alpha_{12}=\alpha_{23}=\alpha_{34}=\alpha_{41}=\frac{\pi}{2}$ is not admissible.

They are

$$\text{(a)}: \quad a_{12}=a_{34} \quad a_{23}=a_{41} \quad \alpha_{23}=\alpha_{41} \qquad \sigma=\pm 1$$

$$\text{(b)}: \quad a_{12} = a_{23} = a_{34} = a_{41} = 0 \quad \alpha_{23}=\pi-\alpha_{41} \quad \sigma=\pm 1.$$

(a):

For this case, equation (5.6.11) becomes

$$a_{23}c\Theta_3 c\Theta_4+R_4 s\Theta_3+a_{12}c\Theta_3+a_{23}+a_{12}c\Theta_4+R_1\sigma s\Theta_4 = 0.$$

From previous results, we have that

$$s\Theta_3 = \sigma s\Theta_1 \qquad s\Theta_4 = \frac{s\Theta_1 c\alpha_{23}}{1-\sigma c\Theta_1 s\alpha_{23}}$$

$$c\Theta_3 = c\Theta_1 \qquad c\Theta_4 = \frac{1}{c\alpha_{23}}\left\{\sigma s\alpha_{23}\frac{s^2\Theta_1 c\alpha_{23}}{1-\sigma c\Theta_1 s\alpha_{23}} - c\Theta_1 c\alpha_{23}\right\}$$

$$= \frac{\sigma s\alpha_{23}-c\Theta_1}{1-\sigma c\Theta_1 s\alpha_{23}} .$$

So equation (5.6.11) may be further rewritten as

$$a_{23}(\sigma s\alpha_{23}c\Theta_1-c^2\Theta_1)+\sigma R_4 s\Theta_1(1-\sigma c\Theta_1 s\alpha_{23})+a_{12}(c\Theta_1-\sigma c^2\Theta_1 s\alpha_{23})$$

$$+ a_{23}(1-\sigma c\Theta_1 s\alpha_{23})+a_{12}(\sigma s\alpha_{23}-c\Theta_1)+R_1\sigma s\Theta_1 c\alpha_{23} = 0.$$

That is,

$$a_{12}\sigma s\alpha_{23}(1-c^2\Theta_1)+a_{23}(1-c^2\Theta_1)+\sigma s\Theta_1(R_1 c\alpha_{23}+R_4[1-\sigma c\Theta_1 s\alpha_{23}]) = 0.$$

By equating coefficients of like terms in $s\Theta_1$ and $c\Theta_1$, since $c\alpha_{23}\neq 0$,

$$R_1 = R_4 = 0 \qquad a_{12} = -\frac{a_{23}}{\sigma s\alpha_{23}} \ .$$

So, in this case, it is clear that

$$\sigma = -1 \quad \text{and} \quad a_{12} = \frac{a_{23}}{s\alpha_{23}} \ .$$

We thus have a sub-case of what will be later referred to as solution D.

(b):

Here, equation (5.6.11) reduces to

$$R_4 s\Theta_3 + R_1 \sigma s\Theta_4 = 0$$

which, from previous results, may be written as

$$-R_4 \sigma s\Theta_1 + R_1 \sigma \frac{s\Theta_1 c\alpha_{23}}{1-\sigma c\Theta_1 s\alpha_{23}} = 0.$$

Hence,

$$R_1 c\alpha_{23} + R_4(\sigma c\Theta_1 s\alpha_{23} - 1) = 0.$$

For this equation to be an identity in $c\Theta_1$, since we cannot have $c\alpha_{23}=0$, we deduce that

$$R_4 = R_1 = 0.$$

Thus is yielded a special case of the solution later called C.

$\alpha_{23} = \alpha_{34} = \frac{\pi}{2}$, $s\alpha_{12} = s\alpha_{41}$:

From reference [1], Table 1, there are again only two possibilities, which are

(a): $a_{12}=a_{41}$ $\quad a_{23}=a_{34}$ $\quad \alpha_{12}=\alpha_{41}$ $\quad \sigma=1$

(b): $a_{12} = a_{23} = a_{34} = a_{41} = 0$ $\quad \alpha_{12} = \pi - \alpha_{41}$ $\quad \sigma=-1.$

(a):

Using prior results, equation (5.6.11) becomes

$$a_{12}c\Theta_4(c\Theta_1 s^2\alpha_{12}-c^2\alpha_{12})+R_4.2s\alpha_{12}c\alpha_{12}\frac{s\Theta_4}{1-c^2\Theta_4 s^2\alpha_{12}}$$

$$+\ a_{23}(c\Theta_1 s^2\alpha_{12}-c^2\alpha_{12})\ +\ a_{23}+a_{12}c\Theta_4+R_1 s\Theta_4 s\alpha_{12}\ =\ 0.$$

Hence,

$$a_{12}c\Theta_4 s^2\alpha_{12}(c\Theta_1+1)+a_{23}s^2\alpha_{12}(c\Theta_1+1)+s\alpha_{12}s\Theta_4(R_1+\frac{2R_4c\alpha_{12}}{1-c^2\Theta_4 s^2\alpha_{12}})\ =\ 0.$$

By the same prior results, this equation becomes

$$s\alpha_{12}(s^2\Theta_4-c^2\Theta_4c^2\alpha_{12}+1-c^2\Theta_4 s^2\alpha_{12})(a_{12}c\Theta_4+a_{23})$$

$$+\ s\Theta_4(R_1[1-c^2\Theta_4 s^2\alpha_{12}]+2R_4c\alpha_{12})\ =\ 0,$$

or

$$2s\alpha_{12}s^2\Theta_4(a_{12}c\Theta_4+a_{23})+s\Theta_4(R_1[1-c^2\Theta_4 s^2\alpha_{12}]+2R_4c\alpha_{12})\ =\ 0.$$

Since $c\alpha_{12}\neq 0$, by equating coefficients of like terms in $s\Theta_4$ and $c\Theta_4$, we find that

$$R_1\ =\ R_4\ =\ a_{12}\ =\ a_{23}\ =\ 0.$$

We then have a special case of the chain treated later under the heading C.

(b):

Equation (5.6.11) reduces to

$$R_4 s\Theta_3\ -\ R_1 s\Theta_4 s\alpha_{12}\ =\ 0.$$

From previous results, we may rewrite this equation as

$$2R_4 s\alpha_{12} c\alpha_{12} \frac{s\Theta_4}{1-c^2\Theta_4 s^2\alpha_{12}} - R_1 s\alpha_{12} s\Theta_4 = 0,$$

or

$$2R_4 c\alpha_{12} + R_1 (c^2\Theta_4 s^2\alpha_{12} - 1) = 0.$$

Thus, since $c\alpha_{12} \neq 0$,

$$R_4 = R_1 = 0.$$

Again we have a special case of the solution later referred to as C.

$\alpha_{12} = \frac{3\pi}{2}, \quad \alpha_{34} = \frac{\pi}{2}, \; s\alpha_{23} = s\alpha_{41}$:

From Table 6.2.1 of section 6.2, we see that there are two cases to consider, namely

(a): $a_{23} = a_{41} \quad a_{12} = a_{34} = 0 \quad \alpha_{23} = \alpha_{41} \quad \sigma = \pm 1$

(b): $a_{12} = a_{34} \quad a_{23} = a_{41} = 0 \quad \alpha_{23} = \pi - \alpha_{41} \; \sigma = \pm 1$.

(a):

Equation (5.6.11) becomes

$$a_{23} c\Theta_3 c\Theta_4 + R_4 s\Theta_3 + a_{23} + \sigma R_1 s\Theta_4 = 0.$$

But, from previous results,

$$s\Theta_3 = -\sigma s\Theta_1 \qquad s\Theta_4 = \frac{-s\Theta_1 c\alpha_{23}}{1+\sigma c\Theta_1 s\alpha_{23}}$$

$$c\Theta_3 = -c\Theta_1 \qquad c\Theta_4 = \frac{\sigma s^2\Theta_1 s\alpha_{23}}{1+\sigma c\Theta_1 s\alpha_{23}} + c\Theta_1$$

$$= \frac{\sigma s\alpha_{23} + c\Theta_1}{1+\sigma c\Theta_1 s\alpha_{23}} .$$

Substituting into the above equation, we obtain

$$a_{23}s^2\Theta_1-\sigma s\Theta_1(R_1c\alpha_{23}+R_4+\sigma R_4c\Theta_1s\alpha_{23}) = 0.$$

By equating coefficients of like powers of Θ_1, since $c\alpha_{23}$ cannot be zero, we deduce that

$$R_4 = R_1 = a_{23} = 0.$$

This will be seen as a special case of solution C.

(b):

Equation (5.6.11) may be written as

$$R_4s\Theta_3+a_{12}c\Theta_3-a_{12}c\Theta_4+\sigma R_1s\Theta_4 = 0.$$

From prior results,

$$s\Theta_3 = \sigma s\Theta_1 \qquad s\Theta_4 = \frac{-s\Theta_1c\alpha_{23}}{1+\sigma c\Theta_1s\alpha_{23}}$$

$$c\Theta_3 = -c\Theta_1 \qquad c\Theta_4 = -\frac{\sigma s^2\Theta_1s\alpha_{23}}{1+\sigma c\Theta_1s\alpha_{23}} - c\Theta_1$$

$$= -\frac{\sigma s\alpha_{23}+c\Theta_1}{1+\sigma c\Theta_1s\alpha_{23}}.$$

Substituting into the above equation leads to

$$a_{12}s\alpha_{23}s^2\Theta_1+s\Theta_1(\sigma R_4c\Theta_1s\alpha_{23}+R_4-R_1c\alpha_{23}) = 0.$$

Again, since $c\alpha_{23}\neq 0$, equating coefficients of like powers of Θ_1 allows us to deduce that

$$R_4 = R_1 = a_{12} = 0.$$

We have a special case of solution C.

$\alpha_{23} = \alpha_{34} = \frac{\pi}{2}$, $-s\alpha_{12} = s\alpha_{41}$:

From Table 6.2.1, we see that there are two possibilities, namely

(a): $a_{23} = a_{34}$ $a_{12} = a_{41} = 0$ $\alpha_{12} = 2\pi - \alpha_{41}$ $\sigma = 1$

(b): $a_{12} = a_{41}$ $a_{23} = a_{34} = 0$ $\alpha_{12} = \pi + \alpha_{41}$ $\sigma = -1$.

(a):

Using previous results, equation (5.6.11) may be given as

$$-2R_4 s\alpha_{12} c\alpha_{12} \frac{s\Theta_4}{1 - c^2\Theta_4 s^2\alpha_{12}} - a_{23}(c\Theta_1 s^2\alpha_{12} + c^2\alpha_{12}) + a_{23} - R_1 s\Theta_4 s\alpha_{12} = 0,$$

where $$c\Theta_1 = \frac{c^2\Theta_4 c^2\alpha_{12} - s^2\Theta_4}{1 - c^2\Theta_4 s^2\alpha_{12}} .$$

Substituting, and simplifying,

$$2a_{23} s^2\alpha_{12} s^2\Theta_4 - s\alpha_{12} s\Theta_4 (2R_4 c\alpha_{12} + R_1 - R_1 c^2\Theta_4 s^2\alpha_{12}) = 0.$$

Therefore, since $c\alpha_{12}$ cannot be zero,

$$R_1 = R_4 = a_{23} = 0,$$

thus yielding a sub-case of solution C.

(b):

Here, from prior results, equation (5.6.11) may be expressed as

$$-a_{12} c\Theta_4 (c\Theta_1 s^2\alpha_{12} - c^2\alpha_{12}) - 2R_4 s\alpha_{12} c\alpha_{12} \frac{s\Theta_4}{1 - c^2\Theta_4 s^2\alpha_{12}} - a_{12} c\Theta_4$$
$$+ R_1 s\Theta_4 s\alpha_{12} = 0,$$

where $$c\Theta_1 = \frac{s^2\Theta_4 - c^2\Theta_4 c^2\alpha_{12}}{1 - c^2\Theta_4 s^2\alpha_{12}} .$$

After substituting for $c\Theta_1$, and simplifying, we find that

$$2a_{12}c\Theta_4 s^2\Theta_4 s^2\alpha_{12}+s\alpha_{12}s\Theta_4\{R_1c^2\Theta_4 s^2\alpha_{12}-R_1+2R_4c\alpha_{12}\} = 0.$$

Thence, since $c\alpha_{12}\neq 0$, equating of coefficients of like powers of Θ_4 leads to

$$R_1 = R_4 = a_{12} = 0,$$

which imply a special case of solution C.

$c\alpha_{34} \neq 0$:

We substitute h_1=h=0 into equation (5.6.b), along with the results for $c\Theta_1$ and $s\Theta_1$ from equations (5.6.e) and (5.6.f):

$$\begin{aligned}
&s\Theta_4 s\alpha_{23}s\alpha_{34}c\alpha_{34}\{a_{41}\rho[-c^3\Theta_4\rho(c\alpha_{12}c\alpha_{41}+\sigma)s\alpha_{23}s\alpha_{34}\\
&\qquad - c^2\Theta_4(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})\\
&\qquad + c\Theta_4(\sigma\rho s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41})\\
&\qquad - \sigma c\alpha_{12}s\alpha_{23}s\alpha_{34}s^2\Theta_4c\Theta_4\rho(c\alpha_{12}+\sigma c\alpha_{41})\\
&\qquad - \sigma c\alpha_{12}s^2\Theta_4(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\sigma\rho s\alpha_{23}c\alpha_{34}s\alpha_{41})]\\
&\ + [a_{23}+\rho a_{12}c\Theta_4+\rho R_1\sigma s\Theta_4 s\alpha_{12}]\\
&\ \times [-c^2\Theta_4 s^2\alpha_{12}s^2\alpha_{23}+2\rho c\Theta_4c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}+s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23}]\}\\
&+ a_{34}s\Theta_4c\alpha_{34}\{s\alpha_{41}s\alpha_{12}[-c^2\Theta_4\rho(c\alpha_{12}c\alpha_{41}+\sigma)s\alpha_{23}s\alpha_{34}\\
&\qquad - c\Theta_4(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})\\
&\qquad + \rho\sigma s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41}] + [c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12}]\\
&\ \times[-c^2\Theta_4 s^2\alpha_{12}s^2\alpha_{23}+2\rho c\Theta_4c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}+s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23}]\}
\end{aligned}$$

$$= R_4 \rho s\alpha_{23} s^2\alpha_{34} \{-c^3\Theta_4 \rho(c\alpha_{12}c\alpha_{41}+\sigma)s\alpha_{23}s\alpha_{34}$$

$$-c^2\Theta_4(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})$$

$$+ c\Theta_4(\sigma\rho s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41})$$

$$-\sigma c\alpha_{12}s^2\Theta_4 c\Theta_4 \rho(c\alpha_{12}+\sigma c\alpha_{41})s\alpha_{23}s\alpha_{34}$$

$$-\sigma c\alpha_{12}s^2\Theta_4(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41})\}$$

$$-R_4 s\alpha_{34}c\Theta_4\{s\alpha_{41}s\alpha_{12}[-c^2\Theta_4\rho(c\alpha_{12}c\alpha_{41}+\sigma)s\alpha_{23}s\alpha_{34}$$

$$- c\Theta_4(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})$$

$$+ \rho\sigma s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41}]$$

$$+ [c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12}]$$

$$\times [-c^2\Theta_4 s^2\alpha_{12}s^2\alpha_{23}+2\rho c\Theta_4 c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}$$

$$+s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23}]\} \qquad (5.6.\text{i})$$

We may separate equation (5.6.i) into two parts by thinking of the terms involving powers of $c\Theta_4$ and $s\Theta_4$ as expanded in terms of powers of Θ_4. Taking first those terms which give rise to only even powers of Θ_4, we have

$$\rho\sigma R_1 s\alpha_{12}s\alpha_{23}s\alpha_{34}c\alpha_{34}(1-c^2\Theta_4)$$

$$\times \{-c^2\Theta_4 s^2\alpha_{12}s^2\alpha_{23}+2\rho c\Theta_4 c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}+s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23}\}$$

$$= R_4 s\alpha_{34}\{c^3\Theta_4[-s^2\alpha_{23}s^2\alpha_{34}(c\alpha_{12}c\alpha_{41}+\sigma)+\sigma c\alpha_{12}s^2\alpha_{23}s^2\alpha_{34}(c\alpha_{12}+\sigma c\alpha_{41})$$

$$+ \rho s\alpha_{12}s\alpha_{23}s\alpha_{34}s\alpha_{41}(c\alpha_{12}c\alpha_{41}+\sigma)$$

$$+s^2\alpha_{12}s^2\alpha_{23}(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})]$$

$$+c^2\Theta_4[-\rho s\alpha_{23}s\alpha_{34}(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})$$

$$+\rho\sigma c\alpha_{12}s\alpha_{23}s\alpha_{34}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\sigma\rho s\alpha_{23}c\alpha_{34}s\alpha_{41})$$

$$+(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})s\alpha_{41}s\alpha_{12}$$

$$-2\rho c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})]$$

$$+c\theta_4[\rho s\alpha_{23}s\alpha_{34}(\sigma\rho s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41})$$

$$-\sigma c\alpha_{12}s^2\alpha_{23}s^2\alpha_{34}(c\alpha_{12}+\sigma c\alpha_{41})$$

$$-s\alpha_{41}s\alpha_{12}(\rho\sigma s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41})$$

$$-(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})]$$

$$-\sigma\rho c\alpha_{12}s\alpha_{23}s\alpha_{34}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41})\}.$$

By inspection of the coefficient of $c^4\theta_4$ in this equation, since $c\alpha_{34}\neq 0$, we deduce that

$$R_1 = 0.$$

Then, using the first of equations (5.6.I), the last equation becomes

$$0 = R_4 s\alpha_{34}\{c^3\theta_4[-\sigma s^2\alpha_{23}s^2\alpha_{34}+\sigma c^2\alpha_{12}s^2\alpha_{23}s^2\alpha_{34}+\sigma s^2\alpha_{12}s^2\alpha_{23}$$
$$+s^2\alpha_{12}s^2\alpha_{23}c\alpha_{23}c\alpha_{34}]$$

$$+c^2\theta_4[\rho s\alpha_{12}s\alpha_{23}c\alpha_{23}s^2\alpha_{34}(\sigma c\alpha_{12}-c\alpha_{41})$$

$$+s\alpha_{12}s\alpha_{41}(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})$$

$$-2\rho c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})]$$

$$+c\theta_4[\sigma s^2\alpha_{23}s^2\alpha_{34}-\sigma s^2\alpha_{23}s^2\alpha_{34}c^2\alpha_{12}-\sigma s^2\alpha_{12}s^2\alpha_{23}$$

$$-s^2\alpha_{12}s^2\alpha_{23}c\alpha_{23}c\alpha_{34}-s^2\alpha_{12}c^2\alpha_{23}c\alpha_{23}c\alpha_{34}$$

$$+s^2\alpha_{12}s^2\alpha_{41}c\alpha_{23}c\alpha_{34}-s^2\alpha_{23}c\alpha_{23}c\alpha_{34}+s^2\alpha_{23}c\alpha_{41}c\alpha_{12}$$

$$-s^2\alpha_{23}s^2\alpha_{34}c\alpha_{12}c\alpha_{41}+s^2\alpha_{12}c^2\alpha_{23}c\alpha_{41}c\alpha_{12}]$$

$$-\rho\sigma c\alpha_{12}s\alpha_{23}s\alpha_{34}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41})\}$$

$$=R_4 s\alpha_{34}\{c^3\Theta_4[s^2\alpha_{12}s^2\alpha_{23}c\alpha_{34}(c\alpha_{23}+\sigma c\alpha_{34})]$$

$$+c^2\Theta_4[s\alpha_{12}s\alpha_{23}c\alpha_{23}s^2\alpha_{34}\rho(\sigma c\alpha_{12}-c\alpha_{41})$$

$$+s\alpha_{12}s\alpha_{41}(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})$$

$$-2\rho c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})]$$

$$+c\Theta_4[-s^2\alpha_{12}s^2\alpha_{23}c\alpha_{34}(c\alpha_{23}+\sigma c\alpha_{34})+s^2\alpha_{12}s^2\alpha_{23}c\alpha_{23}c\alpha_{34}$$

$$-c\alpha_{23}c\alpha_{34}(s^2\alpha_{12}+s^2\alpha_{23}-s^2\alpha_{12}s^2\alpha_{41})$$

$$+c\alpha_{12}c\alpha_{41}(s^2\alpha_{23}c^2\alpha_{34}+s^2\alpha_{12}c^2\alpha_{23})]$$

$$-\rho\sigma c\alpha_{12}s\alpha_{23}s\alpha_{34}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41})\}.$$

If we assume that $R_4 \neq 0$, by inspection of the coefficient of $c^3\Theta_4$ in the last equation, since $c\alpha_{34} \neq 0$, we must have that $c\alpha_{23}=-\sigma c\alpha_{34}$.

But we have already shown that this constraint would lead to the locking of joint 1 or joint 4.

We conclude that

$$R_4 = 0.$$

We now examine that part of equation (5.6.i) consisting only of terms which give rise to odd powers of Θ_4. We have, since $c\alpha_{34} \neq 0$,

$$\rho a_{41}s\alpha_{23}s\alpha_{34}\{c^3\Theta_4\rho[-c\alpha_{12}c\alpha_{41}-\sigma+\sigma c^2\alpha_{12}+c\alpha_{12}c\alpha_{41}]s\alpha_{23}s\alpha_{34}$$

$$+c^2\Theta_4[-\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}-s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41}$$

$$+\sigma c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}]$$

$$+c\Theta_4[\sigma\rho s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41}-\rho\sigma c^2\alpha_{12}s\alpha_{23}s\alpha_{34}$$

$$-\rho c\alpha_{12}s\alpha_{23}s\alpha_{34}c\alpha_{41}]$$

$$-\sigma c\alpha_{12}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41})\}$$

$$+a_{23}s\alpha_{23}s\alpha_{34}\{-c^2\Theta_4s^2\alpha_{12}s^2\alpha_{23}+2\rho c\Theta_4c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}$$

$$+s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23}\}$$

$$+\rho a_{12}s\alpha_{23}s\alpha_{34}\{-c^3\Theta_4s^2\alpha_{12}s^2\alpha_{23}+2c^2\Theta_4\rho c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}$$

$$+c\Theta_4(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})\}$$

$$+a_{34}\{c^2\Theta_4[-\rho s\alpha_{12}s\alpha_{23}s\alpha_{34}s\alpha_{41}(c\alpha_{12}c\alpha_{41}+\sigma)$$

$$-s^2\alpha_{12}s^2\alpha_{23}(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})]$$

$$+c\Theta_4[-s\alpha_{12}s\alpha_{41}(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})$$

$$+2\rho c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})]$$

$$+s\alpha_{41}s\alpha_{12}(\rho\sigma s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41})$$

$$+(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})\}$$

$$= 0.$$

Then, using the first of equations (5.6.I),

$$a_{41}s\alpha_{23}s\alpha_{34}\{-c^3\Theta_4.\sigma s^2\alpha_{12}s\alpha_{23}s\alpha_{34}+c^2\Theta_4.\rho s\alpha_{12}s\alpha_{34}c\alpha_{23}(\sigma c\alpha_{12}-c\alpha_{41})$$

$$+c\Theta_4\rho(\sigma\rho s^2\alpha_{12}s\alpha_{23}s\alpha_{34}-s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41}-\rho c\alpha_{12}s\alpha_{23}s\alpha_{34}c\alpha_{41})$$

$$-\sigma c\alpha_{12}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41})\}$$

$$+a_{34}\{-c^2\Theta_4.s^2\alpha_{12}s^2\alpha_{23}(\sigma+c\alpha_{23}c\alpha_{34})$$

$$+c\Theta_4[-s\alpha_{12}s\alpha_{41}(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})$$

$$+2\rho c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})]$$

$$+\sigma s^2\alpha_{12}s^2\alpha_{23}-s^2\alpha_{12}s^2\alpha_{41}c\alpha_{23}c\alpha_{34}$$

$$+(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})\}$$

$$+a_{23}s\alpha_{23}s\alpha_{34}\{-c^2\Theta_4s^2\alpha_{12}s^2\alpha_{23}+2\rho c\Theta_4c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}$$

$$+s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23}\}$$

$$+\rho a_{12}s\alpha_{23}s\alpha_{34}\{-c^3\Theta_4s^2\alpha_{12}s^2\alpha_{23}+2\rho c^2\Theta_4c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}$$

$$+c\Theta_4(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})\}$$

$$= 0. \qquad (5.6.j)$$

This equation is to be an identity in $c\Theta_4$. Clearly, one solution is

$$a_{12} = a_{23} = a_{34} = a_{41} = 0.$$

We shall denote it by C̲.

This solution demands no constraints on the twist angles above those already established by equations (5.6.I). Equations (5.6.II) are identically satisfied. Hence, any of the C-H-C-H-solutions listed in Table 1 of reference [1] or Table 6.2.1 of section 6.2 is applicable here, after appropriate simplifications.

To seek out other solutions, we equate coefficients of powers of $c\theta_4$ in equation (5.6.j).

$$\left.\begin{array}{ll}
c^3\theta_4: & a_{41}s\alpha_{34} = -\sigma\rho a_{12}s\alpha_{23} \\
c^2\theta_4: & \rho a_{41}s^2\alpha_{34}c\alpha_{23}(\sigma c\alpha_{12}-c\alpha_{41})-a_{34}s\alpha_{12}s\alpha_{23}(\sigma+c\alpha_{23}c\alpha_{34}) \\
 & \quad -a_{23}s\alpha_{12}s^2\alpha_{23}s\alpha_{34}+2a_{12}c\alpha_{12}c\alpha_{23}s\alpha_{23}s\alpha_{34} = 0 \\
c^1\theta_4: & a_{41}s\alpha_{23}s\alpha_{34}(\sigma s^2\alpha_{12}s\alpha_{23}s\alpha_{34}-\rho s\alpha_{12}c\alpha_{23}c\alpha_{34}s\alpha_{41} \\
 & \qquad -c\alpha_{12}s\alpha_{23}s\alpha_{34}c\alpha_{41}) \\
 & +a_{34}[2\rho c\alpha_{12}s\alpha_{12}c\alpha_{23}s\alpha_{23}(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12}) \\
 & \quad -s\alpha_{12}s\alpha_{41}(\rho c\alpha_{12}s\alpha_{23}c\alpha_{34}s\alpha_{41}+s\alpha_{12}c\alpha_{23}s\alpha_{34}c\alpha_{41})] \\
 & +2a_{23}\rho c\alpha_{12}s\alpha_{12}c\alpha_{23}s^2\alpha_{23}s\alpha_{34} \\
 & \qquad +a_{12}\rho s\alpha_{23}s\alpha_{34}(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23}) \\
 & = 0 \\
c^0\theta_4: & -\sigma a_{41}c\alpha_{12}s\alpha_{23}s\alpha_{34}(s\alpha_{12}c\alpha_{23}s\alpha_{34}+\rho\sigma s\alpha_{23}c\alpha_{34}s\alpha_{41}) \\
 & +a_{34}[\sigma s^2\alpha_{12}s^2\alpha_{23}-s^2\alpha_{12}s^2\alpha_{41}c\alpha_{23}c\alpha_{34} \\
 & \quad +(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23})(c\alpha_{23}c\alpha_{34}-c\alpha_{41}c\alpha_{12})] \\
 & +a_{23}s\alpha_{23}s\alpha_{34}(s^2\alpha_{12}c^2\alpha_{23}+s^2\alpha_{23}) = 0
\end{array}\right\} \quad (5.6.k)$$

Let us first assume that $\rho\sigma=1$. Then, from the first of equations (5.6.k), it follows that $a_{41}=a_{12}=0$.

Then, by inspection of Table 1 in [1] and Table 6.2.1 in section 6.2, and excepting those cases covered by solution C, we must have that $a_{23}=a_{34}\neq 0$.

Substituting these results into the second of equations (5.6.k), we find that

$$(\sigma + c\alpha_{23}c\alpha_{34}) + s\alpha_{23}s\alpha_{34} = 0.$$

That is,

$$c\alpha_{23}c\alpha_{34} = -(\sigma + s\alpha_{23}s\alpha_{34}).$$

Checking the relevant entries in Table 1 of [1] and Table 6.2.1 of 6.2 yields the information that, in all cases, $\sigma=1$ (There are, therefore, actually no eligible cases in Table 6.2.1.). Hence, this last requirement implies that

$$c\alpha_{23}c\alpha_{34} < -1,$$

which is impossible.

We therefore conclude that

$$\rho\sigma = -1.$$

Let us first examine the relevant entries in Table 6.2.1 of section 6.2. Since $\rho=-1$ for this table, we need only look at those cases for which $\sigma=1$ not already covered by C. We shall investigate each relevant case in turn.

$a_{23}=a_{41}\neq 0$, $a_{12}=a_{34}=0$, $\alpha_{12}=2\pi-\alpha_{34}$, $\alpha_{23}=\alpha_{41}$:

From the first of equations (5.6.k), $a_{41}=0$. So this case will be included under C.

$a_{12}=a_{34}\neq 0$, $a_{23}=a_{41}=0$, $\alpha_{12}=\pi+\alpha_{34}$, $\alpha_{23}=\pi-\alpha_{41}$:

From the first of equations (5.6.k), $a_{12}=0$. This case will also be included under C.

$a_{23}=a_{34}\neq 0$, $a_{12}=a_{41}=0$, $\alpha_{12}=2\pi-\alpha_{41}$, $\alpha_{23}=\alpha_{34}$:

The second of equations (5.6.k) implies that

$$1 + c^2\alpha_{23} + s^2\alpha_{23} = 0.$$

So there is no solution for this case.

$a_{23}=a_{34}+a_{41}+a_{12}$, $\alpha_{23}=\alpha_{34}=\alpha_{41}=2\pi-\alpha_{12}$:

The first of equations (5.6.k) implies that

$$a_{41} = a_{12},$$

whence $$a_{23} = a_{34} + 2a_{12}.$$

Then, in the second of equations (5.6.k),

$$a_{34}s^2\alpha_{23}(1+c^2\alpha_{23})+(a_{34}+2a_{12})s^4\alpha_{23}+2a_{12}c^2\alpha_{23}s^2\alpha_{23} = 0.$$

$$\therefore \quad 2a_{34} + 2a_{12} = 0,$$

whence $$a_{34} = a_{12} = 0.$$

Again, a special case of C is indicated.

$a_{34}=a_{41}+a_{12}+a_{23}$, $\alpha_{12}-\pi=\alpha_{23}=\alpha_{34}=\pi-\alpha_{41}$:

From the first of equations (5.6.k), we see that

$$a_{41} = a_{12},$$

whence $$a_{34} = 2a_{12} + a_{23}.$$

In the second of equations (5.6.k), then,

$$(2a_{12}+a_{23})s^2\alpha_{23}(1+c^2\alpha_{23})+a_{23}s^4\alpha_{23}-2a_{12}c^2\alpha_{23}s^2\alpha_{23} = 0.$$

$$\therefore \quad 2a_{12} + 2a_{23} = 0,$$

whence $$a_{12} = a_{23} = 0.$$

A special case of C is once again indicated.

It only remains to test the entries in Table 1 of reference [1] not already covered by C for which $\sigma=-1$. We find that there are only two possibilities:

(a) $a_{12}=a_{34}$ $a_{23}=a_{41}$ $\alpha_{12}=\alpha_{34}$ $\alpha_{23}=\alpha_{41}$

(b) $a_{12}+a_{41} = a_{23}+a_{34}$ $\alpha_{12}=\alpha_{34}$ $\alpha_{23}=\alpha_{41}=\pi-\alpha_{12}$

(a)

The first of equations (5.6.k) may be rewritten as

$$a_{23}s\alpha_{12} = a_{12}s\alpha_{23}.$$

This result will be an extra constraint on the linkage.

The other three of equations (5.6.k), by substitution of the various constraint relations, are all identically satisfied.

We call this solution <u>D</u>.

(b)

The first of equations (5.6.k) yields

$$a_{41} = a_{12},$$

whence $$a_{23} + a_{34} = 2a_{12}.$$

The second of equations (5.6.k) becomes

$$-a_{12}s^2\alpha_{12}c\alpha_{12}(-c\alpha_{12}+c\alpha_{12})-a_{34}s^2\alpha_{12}(-1-c^2\alpha_{12})$$

$$-a_{23}s^4\alpha_{12}-2a_{12}c^2\alpha_{12}s^2\alpha_{12} = 0,$$

whence

$$-a_{34}(1+c^2\alpha_{12})+a_{23}s^2\alpha_{12}+2a_{12}c^2\alpha_{12} = 0.$$

Thus,

$$a_{23}-a_{34}-(a_{23}+a_{34})c^2\alpha_{12}+2a_{12}c^2\alpha_{12} = 0,$$

so that

$$a_{23} = a_{34}.$$

Hence,

$$a_{12} = a_{23} = a_{34} = a_{41}.$$

We therefore obtain a special case of solution D.

The final step is to determine what further conditions, if any, are necessary for mobility of joint 3, by testing solutions C and D in equation (5.6.12).

For both solutions, since R_4=h=0, R_2 is also zero.

<u>C</u>

Equation (5.6.12) reduces to

$$r_3 = 0.$$

That is, we have a spherical four-bar, Delassus solution number d.5.

<u>D</u>

Equation (5.6.12) becomes

$$a_{41}s\Theta_4 s\alpha_{34} + r_3 + a_{12}\sigma s\Theta_4 s\alpha_{23} = 0$$

which, by the first of equations (5.6.k), further simplifies to

$$r_3 = 0.$$

Hence, we have the Bennett linkage, Delassus solution number d.13.

. . . .

We have demonstrated that the general C-H-C-H- linkage has no proper derivatives of connectivity sum five and mobility one. Further, the only derivatives of connectivity sum four and mobility one are the relevant Delassus solutions, here given as A, B, C and D.

Solution C, it might be noted, does not include all spherical four-bars, but is limited to those governed by equations (5.6.I). The generalised d.5 linkage is a degeneracy of the H-C-C-H- loop, as already pointed out in section 5.1.

The preceding analysis gives some indication of the very special nature of the Bennett linkage, solution D. Of all the C-H-C-H- solutions, only two could be identified with d.13. Again, the Bennett linkage is obtainable as a degeneracy of H-C-C-H-.

. . . .

We have also, in passing, taken the final step in establishing the standing of the general C-H-C-H- linkage, for the following reason. In references [1] and [4] and section 6.2, rotational mobility of all four joints of the C-H-C-H- linkage is confirmed, but it would be necessary, strictly, to demonstrate that the cylindric joints were translationally mobile. In other words, we should need to show that neither cylindric joint could be replaced by a screw or revolute without introducing further dimensional constraints into the linkage, or locking joints.

In the foregoing analysis, as preparation for solutions C and D, we tried to replace a cylindric joint by a screw. We found that the C-H-C-H- linkage immediately degenerated to either a spherical or Bennett chain. It is clear that the question of revolute replacement of a cylindric joint was included in the above process. Thus, the checking is complete and the C-H-C-H- linkage solutions are vindicated as proper, mobility one.

5.7 The governing equations

Detailed below, for each Delassus linkage and each parent linkage encountered in the preceding analysis, is a set of corresponding independent closure equations and dimensional constraint relations. All of the equations given are readily derivable from information contained in the relevant sections above, possibly augmented by closure equations (5.1)-(5.9).

Whether or not a particular linkage has -R- derivatives free from part-chain mobility can be checked by setting any desired screw pitch equal to zero in the dimensional conditions and, less often, in the closure equations.

Delassus linkages

d.1

H-H-H-H-: screws parallel, pitches equal

$$\Theta_1+\Theta_2+\Theta_3+\Theta_4 = 2k\pi \qquad \text{(i)}$$

$$a_{41}c(\Theta_3+\Theta_4)+a_{34}c\Theta_3+a_{23}+a_{12}c\Theta_2 = 0 \qquad \text{(ii)}$$

$$a_{41}s(\Theta_3+\Theta_4)+a_{34}s\Theta_3-a_{12}s\Theta_2 = 0 \qquad \text{(iii)}$$

where

$$R_1+R_2+R_3+R_4+2k\pi h = 0$$

d.2

P-H-H-H-: screws parallel, pitches equal, slider normal to screws

$$\Theta_2+\Theta_3+\Theta_4 = (2k+\frac{1+\sigma}{2})\pi \qquad \text{(i)}$$

$$-\sigma a_{41}-\sigma a_{34}c\Theta_4+a_{23}c\Theta_2+a_{12} = 0 \qquad \text{(ii)}$$

$$\sigma a_{34}s\Theta_4+a_{23}s\Theta_2+r_1 s\alpha_{12} = 0 \qquad \text{(iii)}$$

where

$$\Theta_1 = \frac{1+\sigma}{2}\pi \quad \text{and} \quad R_2 + R_3 + R_4 + (2k+\frac{1+\sigma}{2})\pi h = 0$$

d.3

P-P-H-H-: screws parallel, pitches equal, sliders normal to screws

$$\Theta_3 + \Theta_4 = (2k+1)\pi - \sigma\alpha_{12} \qquad \text{(i)}$$

$$-a_{41}c\alpha_{12}+a_{34}c\Theta_3+a_{23}+\sigma r_1 s\alpha_{12} = 0 \qquad \text{(ii)}$$

$$\sigma a_{41}s\alpha_{12}+a_{34}s\Theta_3+r_2+r_1c\alpha_{12} = 0 \qquad \text{(iii)}$$

where

$$s\Theta_1 = s\Theta_2 = \sigma$$

and $R_3 + R_4 + \sigma a_{12} + [(2k+1)\pi-\sigma\alpha_{12}]h = 0$

d.4

P-H-P-H-: screws parallel, pitches equal, sliders normal to screws

$$\Theta_2 = \sigma\Theta_4 - (2m+\frac{1-\rho}{2})\pi \qquad \text{(i)}$$

$$\sigma a_{41}c\Theta_4+\sigma a_{34}+a_{23}+\rho a_{12}c\Theta_4+\sigma r_1 s\Theta_4 = 0 \qquad \text{(ii)}$$

$$a_{41}s\Theta_4+r_3+\rho\sigma a_{12}s\Theta_4-r_1c\Theta_4 = 0 \qquad \text{(iii)}$$

where

$$c\Theta_1 = \rho\sigma \qquad c\Theta_3 = \sigma$$

and $R_2 = \sigma R_4 + (2m+\frac{1-\rho}{2})\pi h$

d.5

R-R-R-R-: spherical linkage

Here, since

$$R_1 = R_2 = R_3 = R_4 = 0$$

$$a_{12} = a_{23} = a_{34} = a_{41} = 0,$$

the translational closure equations (5.10)-(5.12) are identically satisfied. Three independent equations may be chosen from (5.1)-(5.9) to form the governing set.

d.6

P-P-P-P-: spatial four-slider

The three translational closure equations (5.10)-(5.12) are here the three independent equations. The Θ_i are fixed; one of them may be chosen freely, and the other three are determined in accordance with equations (5.1)-(5.9), assuming given angles of twist.

d.7

H-H-H-H-: two parallel pairs of coaxial screws

$$\Theta_1 + \Theta_2 = (2l+1)\pi \tag{i}$$

$$\Theta_3 + \Theta_4 = (2m+1)\pi \tag{ii}$$

$$R_1+R_2+R_3+R_4+h_1\Theta_1+h_2\Theta_2+h_3\Theta_3+h_4\Theta_4 = 0 \tag{iii}$$

where

$$a_{12} = a_{34} = 0 \qquad a_{23} = a_{41} \neq 0$$

d.8

P-P-H-H-: screws coaxial, sliders lying in planes parallel to each other and to screws

$$\Theta_3 + \Theta_4 = 2k\pi + \frac{1-\sigma\tau}{2}\pi \qquad \text{(i)}$$

$$r_2 s\alpha_{23} = \sigma\tau r_1 s\alpha_{41} \qquad \text{(ii)}$$

$$R_3+R_4+h_3\Theta_3+h_4\Theta_4+r_2 c\alpha_{23}+r_1 c\alpha_{41} = 0 \qquad \text{(iii)}$$

where

$$c\Theta_1 = \sigma \qquad c\Theta_2 = \tau$$

$$\alpha_{12} + \tau\alpha_{23} + \sigma\alpha_{41} = 0$$

$$a_{12} + \tau a_{23} + \sigma a_{41} = 0$$

d.9

H-H-H-H-: screws parallel, kite shape in projection, pitches of screws in plane of symmetry equal to each other and to arithmetic mean of remaining two

$$\Theta_2 = \Theta_4 + 2m\pi \qquad \text{(i)}$$

$$\Theta_1 + \Theta_3 + 2\Theta_4 = 2(k-m)\pi \qquad \text{(ii)}$$

$$a_{23}{}^2\cos^2\frac{\Theta_3}{2} = a_{12}{}^2\cos^2\frac{\Theta_1}{2} \qquad \text{(iii)}$$

where

$$a_{41} = a_{12} \qquad a_{34} = a_{23}$$

$$h_1 = h_3 = \frac{h_2+h_4}{2}$$

$$R_1+R_2+R_3+R_4=2\pi[(m-k)\frac{h_4}{2} - (m+k)\frac{h_2}{2}]$$

d.10

H-H-H-H-: screws parallel, parallelogram in projection, sums of pitches of diagonally opposite screws equal

$$\Theta_3 + \Theta_4 = (2l+1)\pi \qquad \text{(i)}$$

$$\Theta_1 + \Theta_2 = (2m+1)\pi \qquad \text{(ii)}$$

$$\Theta_4 = \Theta_2 + 2n\pi \qquad \text{(iii)}$$

where

$$a_{41} = a_{23} \qquad a_{34} = a_{12}$$

$$h_1 + h_3 = h_2 + h_4$$

$$R_1+R_2+R_3+R_4+(2m+1)\pi h_1+(2l+1)\pi h_3$$
$$+ 2n\pi(h_4-h_3) = 0$$

d.11

H-H-H-H-: screws parallel, anti-parallelogram in projection, pitches of alternate screws equal

$$\Theta_2 + \Theta_4 = 2m\pi \qquad \text{(i)}$$

$$\Theta_1 + \Theta_3 = 2(k-m)\pi \qquad \text{(ii)}$$

$$a_{23}s(\Theta_3+\Theta_4)+a_{12}(s\Theta_3+s\Theta_4) = 0 \qquad \text{(iii)}$$

where

$$a_{41} = a_{23} \qquad a_{34} = a_{12}$$

$$h_4 = h_2 \qquad h_3 = h_1$$

$$R_1+R_2+R_3+R_4 = 2\pi[(m-k)h_1-mh_2]$$

d.12

P-H-P-H-: screws parallel, pitches equal, sliders bilaterally symmetric with respect to the plane containing the screws

$$\Theta_2 = \sigma\Theta_4 - (2m + \frac{1-\rho}{2})\pi \qquad \text{(i)}$$

$$\sigma a_{41}c\Theta_4 + \sigma a_{34} + a_{23} + \rho(a_{12}c\Theta_4 + \sigma r_1 s\alpha_{12}s\Theta_4) = 0 \qquad \text{(ii)}$$

$$a_{41}s\alpha_{23}s\Theta_4 + r_3 + \rho\sigma a_{12}s\alpha_{23}s\Theta_4$$
$$+ r_1(c\alpha_{12}c\alpha_{23} - \rho c\Theta_4 s\alpha_{12}s\alpha_{23}) = 0 \qquad \text{(iii)}$$

where

$$c\Theta_1 = \rho\sigma \qquad c\Theta_3 = \sigma$$

$$c\alpha_{12} = -\sigma c\alpha_{41} \qquad c\alpha_{23} = -\sigma c\alpha_{34}$$

$$R_2 = \sigma R_4 + (2m + \frac{1-\rho}{2})\pi h$$

further constraints demanded by each of the particular solutions - see 5.6A for details

d.13

R-R-R-R-: Bennett linkage

$$\Theta_2 + \Theta_4 = 2k\pi \qquad \text{(i)}$$

$$\Theta_1 + \Theta_3 = 2l\pi \qquad \text{(ii)}$$

AND

one equation from the following set, for example

$$c\Theta_1 s\Theta_2 s\alpha_{23} + s\Theta_1 c\Theta_2 c\alpha_{12} s\alpha_{23} + s\Theta_1 s\alpha_{12} c\alpha_{23} = - s\Theta_2 s\alpha_{12}$$

$$s\Theta_1 s\Theta_2 s\alpha_{23} - c\Theta_1 c\Theta_2 c\alpha_{12} s\alpha_{23} - c\Theta_1 s\alpha_{12} c\alpha_{23} = c\Theta_2 s\alpha_{12} c\alpha_{23} + c\alpha_{12} s\alpha_{23}$$

$$c\Theta_1 c\Theta_2 s\alpha_{23} - s\Theta_1 s\Theta_2 c\alpha_{12} s\alpha_{23} + c\Theta_1 s\alpha_{12} + s\alpha_{23} + c\Theta_2 s\alpha_{12} = 0$$

$$-s\Theta_1 c\Theta_2 s\alpha_{23} - c\Theta_1 s\Theta_2 c\alpha_{12} s\alpha_{23} - s\Theta_1 s\alpha_{12} - s\Theta_2 c\alpha_{23} s\alpha_{12} = 0$$

$$s\Theta_1 s\Theta_2 s\alpha_{12} s\alpha_{23} = (c\Theta_1 + c\Theta_2)(c\alpha_{23} - c\alpha_{12})$$

where

$$R_1 = R_2 = R_3 = R_4 = 0$$

$$a_{12} = a_{34} \qquad a_{23} = a_{41}$$

$$\alpha_{12} = \alpha_{34} \qquad \alpha_{23} = \alpha_{41} \neq \alpha_{12}$$

$$a_{23} s\alpha_{12} = a_{12} s\alpha_{23}$$

Parent linkages

P-P-H-C-: (d.3, d.8)

screw and cylindric joints parallel

$$c(\Theta_3 + \Theta_4) = \frac{c\alpha_{23} c\alpha_{41} - c\alpha_{12}}{s\alpha_{23} s\alpha_{41}} \tag{i}$$

$$a_{41} s\alpha_{12} (c\alpha_{23} c\alpha_{41} - c\alpha_{12}) + a_{12} s\alpha_{41} (c\alpha_{12} c\alpha_{23} - c\alpha_{41})$$
$$+ a_{23} s\alpha_{41} s\alpha_{12} s\alpha_{23}$$
$$+ a_{34} s\alpha_{41} s\alpha_{12} s\alpha_{23} c\Theta_3 + r_1 s\alpha_{41} s^2\alpha_{12} s\alpha_{23} s\Theta_2 = 0 \tag{ii}$$

$$a_{41} s\alpha_{12} s\alpha_{23} s\Theta_2 - a_{12} s\alpha_{23} s\alpha_{41} c\alpha_{23} s\Theta_2 + a_{34} s\alpha_{23} s\alpha_{41} s\Theta_3$$
$$+ r_2 s^2\alpha_{23} s\alpha_{41} + r_1 s\alpha_{41} (c\alpha_{12} - c\alpha_{23} c\alpha_{41}) = 0 \tag{iii}$$

$$a_{12} s\alpha_{23} s\Theta_2 + r_4 + (R_3 + h_3 \Theta_3) + r_2 c\alpha_{23} + r_1 c\alpha_{41} = 0 \tag{iv}$$

where

$$c\Theta_1 = \frac{c\alpha_{41}c\alpha_{12}-c\alpha_{23}}{s\alpha_{41}s\alpha_{12}} \qquad c\Theta_2 = \frac{c\alpha_{12}c\alpha_{23}-c\alpha_{41}}{s\alpha_{12}s\alpha_{23}}$$

C-H-C-H-: two parallel C-H pairs

$$\Theta_1 + \Theta_2 = (2k + \frac{1+\sigma}{2})\pi \qquad \text{(i)}$$

$$\Theta_3 + \Theta_4 = (2l + \frac{1+\sigma}{2})\pi \qquad \text{(ii)}$$

$$-\sigma a_{41}+a_{34}c\Theta_3+a_{23}+a_{12}c\Theta_2 = 0 \qquad \text{(iii)}$$

$$a_{34}s\Theta_3+(R_2+h_2\Theta_2)s\alpha_{23}-a_{12}s\Theta_2c\alpha_{23}+r_1s\alpha_{23} = 0 \qquad \text{(iv)}$$

$$(R_4+h_4\Theta_4)+r_3+(R_2+h_2\Theta_2)c\alpha_{23}+a_{12}s\Theta_2s\alpha_{23}+r_1c\alpha_{23}=0 \qquad \text{(v)}$$

H-C-C-H-: two parallel C-H pairs

$$\Theta_1 + \Theta_2 = (2k + \frac{1+\sigma}{2})\pi \qquad \text{(i)}$$

$$\Theta_3 + \Theta_4 = (2l + \frac{1+\sigma}{2})\pi \qquad \text{(ii)}$$

$$-\sigma a_{41}+a_{34}c\Theta_3+a_{23}+a_{12}c\Theta_2 = 0 \qquad \text{(iii)}$$

$$a_{34}s\Theta_3+r_2s\alpha_{23}-a_{12}s\Theta_2c\alpha_{23}+(R_1+h_1\Theta_1)s\alpha_{23}=0 \qquad \text{(iv)}$$

$$(R_4+h_4\Theta_4)+r_3+r_2c\alpha_{23}+a_{12}s\Theta_2s\alpha_{23}$$

$$+ (R_1+h_1\Theta_1)c\alpha_{23} = 0 \qquad \text{(v)}$$

P-H-H-C-: (d.2)

screw and cylindric joints parallel

$$\Theta_2 + \Theta_3 + \Theta_4 = (2k + \frac{1+\sigma}{2})\pi \qquad \text{(i)}$$

$$-\sigma a_{41}c\Theta_2+a_{34}c\Theta_3+a_{23}+a_{12}c\Theta_2+r_1s\Theta_2s\alpha_{12} = 0 \qquad \text{(ii)}$$

$$\sigma a_{41}s\Theta_2+a_{34}s\Theta_3-a_{12}s\Theta_2+r_1c\Theta_2s\alpha_{12} = 0 \qquad \text{(iii)}$$

$$r_4+(R_3+h_3\Theta_3)+(R_2+h_2\Theta_2)+r_1c\alpha_{12} = 0 \qquad \text{(iv)}$$

where

$$\Theta_1 = \frac{1+\sigma}{2}\pi$$

P-H-C-H-: (d.2)

screw and cylindric joints parallel

$$\Theta_2 + \Theta_3 + \Theta_4 = (2k + \frac{1+\sigma}{2})\pi \qquad \text{(i)}$$

$$-\sigma a_{41}c\Theta_2+a_{34}c\Theta_3+a_{23}+a_{12}c\Theta_2+r_1s\Theta_2s\alpha_{12} = 0 \qquad \text{(ii)}$$

$$\sigma a_{41}s\Theta_2+a_{34}s\Theta_3-a_{12}s\Theta_2+r_1c\Theta_2s\alpha_{12} = 0 \qquad \text{(iii)}$$

$$(R_4+h_4\Theta_4)+r_3+(R_2+h_2\Theta_2)+r_1c\alpha_{12} = 0 \qquad \text{(iv)}$$

where

$$\Theta_1 = \frac{1+\sigma}{2}\pi$$

C-H-H-H-: (d.1, d.7, d.9-d.11)

all joint axes parallel

$$\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 = 2k\pi \qquad \text{(i)}$$

$$a_{41}c(\Theta_3+\Theta_4)+a_{34}c\Theta_3+a_{23}+a_{12}c\Theta_2 = 0 \qquad \text{(ii)}$$

$$a_{41}s(\Theta_3+\Theta_4)+a_{34}s\Theta_3-a_{12}s\Theta_2 = 0 \qquad \text{(iii)}$$

$$(R_4+h_4\Theta_4)+(R_3+h_3\Theta_3)+(R_2+h_2\Theta_2)+r_1 = 0 \qquad \text{(iv)}$$

C-H-C-H-: (d.4, d.5, d.12, d.13)

no two adjacent joint axes parallel

governed by equations (5.6.1)-(5.6.12) and constraints (5.6.I) and (5.6.II)

6

FOUR-BAR LINKAGE ANALYSIS

6.1 The task

Apart from some relatively minor errors (most of which are mentioned in this section and 6.2 below), Waldron [45, 47,48] has listed the existence criteria for apparently all proper, single closed loop four-bars of mobility one, except where screw pairs are present. To complete the analysis of overconstrained four-bars, we shall consider in this chapter those remaining loops which contain screw joints. Elsewhere in reference [45], and in [42,44], Waldron has actually treated certain classes of four-bars containing screw pairs. Other relevant references for two of these classes are Voinea and Atanasiu [40] and Hunt [27].

In chapter 5 of this work, following Delassus [11-13], we were able to isolate all four-bar linkages with mobility one and connectivity sum four, along with some other loops of higher connectivity sum. It remains, therefore, to consider systematically only those linkages with screw joints of connectivity sum five, six and seven. In doing so, we shall make use of some of the references given above.

Connectivity sum five loops

The relevant four-bars in this category are the C-H-H-H-, C-H-H-P-, C-H-P-H-, C-H-P-P-, C-P-H-P- chains. Using the spherical indicatrix for each of the last four loops, we see that the cylindric and screw joints must be parallel, for mobility, in every case. Each of the four loops, then, will be a special case of the parallel-screw linkages [27,40,42,45] detailed in the Introduction to chapter 7. In addition, the C-H-H-P-, C-H-P-H-, C-H-P-P- chains are 'parent' linkages for some of the Delassus solutions, and are treated at some length in chapter 5, where their independent closure equations are given.

We now consider the C-H-H-H- chain. It is convenient to investigate all possibilities by checking through sections 5.2-5.6. Any mobile C-H-H-H- loop must appear among the sub-categories or derivatives developed in those sections. In so doing, we find that the only proper solution is the parent linkage isolated in section 5.5. This linkage has all joint axes parallel, and is therefore a special parallel-screw linkage. For this loop to be proper, the three screw pitches cannot be equal.

It is interesting to note that all the loops in this category have solutions and that those solutions are all parallel-screw linkages. Their -R- derivatives (except for the C-H-H-H- case), which were isolated by Waldron [45,47,48], are also proper parallel-screw linkages. Even the improper C-P-P-P-, which we did not have to consider here, can be regarded as a special parallel-screw linkage.

Connectivity sum six loops

The linkages to be examined here are the C-C-H-P-, C-H-C-P-, C-H-C-H-, C-C-H-H-, S-H-P-P-, S-P-H-P-, S-H-P-H-, S-P-H-H-, S-H-H-H-, F-H-P-P-, F-P-H-P-, F-H-P-H-, F-P-H-H-, F-H-H-H- chains. For the first two loops, the spherical indicatrix allows us to conclude that, for rotational mobility, the screw and cylindric joints are parallel. Then, each of these linkages will have part-chain mobility, based on the motion of a parallel-screw linkage, and so may be excluded. (In fact, because the cylindric joints are parallel, there will be part-chain mobility based on the P$\hat{-}$P- linkage.) We may replace the spherical joint in each of the S-H-P-P- and S-P-H-P- loops by three concurrent revolutes and set one of them parallel to the screw joint axis. The spherical indicatrix then shows that all three revolutes are parallel or that one or more are locked with the remainder parallel to the screw. In any case, we see that the linkage must be improper, the spherical joint being replaceable by a single revolute. The only solutions will be based on Delassus four-bars. Thus, these two chains may also be eliminated. If we replace the planar joint in each of the F-H-P-P- and F-P-H-P- loops by three distinct revolutes all normal to the plane, the spherical indicatrix requires, for mobility, that the screw joint must also be normal to the plane. Each loop will then possess part-chain mobility based on the motion of a parallel-screw chain. We may therefore exclude these two linkages.

We have eliminated six of the fourteen potential linkages in this category. We shall now consider the two loops F-H-P-H- and F-P-H-H-. In each case, we may again replace the -F- joint by -R-R-R- normal to the plane. For a proper, mobility one linkage with no locked joints to result, the spherical indicatrix requires that the two screw pairs be parallel to each other, but not to the revolutes. We then have a linkage which is a special parallel-screw chain.

The S-H-P-H- and S-P-H-H- loops require extended analyses and are fully treated below, in sections 6.3 and 6.4 respectively. The groundwork for the C-H-C-H- linkage and about half of its many solutions were due largely to Baker and Waldron [1,4]. The remainder of the analysis is given in the next section. To analyse the C-C-H-H- linkage, it is convenient to make use of a C-C-R-R- analysis and the theorem of section 4.3. Unfortunately, the known results for the C-C-R-R- loop are doubtful. For this reason, a fresh treatment is presented in section 6.5.

The remaining three chains, C-C-H-H-, S-H-H-H- and F-H-H-H-, have, at this time, defied complete analysis. Because of the screw joints and the absence of simplifying geometrical properties, the general forms of these linkages give rise to relatively intractable algebraic equations. The extent to which the loops have been investigated is presented in section 6.6.

Connectivity sum seven loops

General connectivity sum seven loops, we know, have mobility of unity. But not all linkages with connectivity sum seven can be regarded as general. The presence of joints with connectivity greater than one makes such linkages special, and often results in part-chain mobility. The eleven linkages to be considered in this category are the S-C-H-H-, S-H-C-H-, S-H-C-P-, S-C-P-H-, S-C-H-P-, F-C-H-H-, F-H-C-H-, F-H-C-P-, F-C-P-H-, F-C-H-P-, C-C-C-H- chains.

Using the spherical indicatrix, we see that the F-H-C-P-, F-C-P-H- and F-C-H-P- chains have part-chain mobility, being based on special parallel-screw linkages of the connectivity sum six type, since the screw and cylindric joints must be parallel in each loop. It is noted that Waldron [45,48], who defines "proper" differently, has listed the -R- derivatives of these loops as proper solutions. The S-H-C-P-, S-C-P-H- and S-C-H-P- chains are seen, through the spherical indicatrix, to have mobility one. They are based on special parallel-screw linkages of the connectivity sum six variety, the spherical joint being equivalent in each case to only two revolutes. The remaining five linkages, namely S-C-H-H-, S-H-C-H-, F-C-H-H-, F-H-C-H- and C-C-C-H-, are all of mobility one and true connectivity sum seven, generally. Certain dimensional conditions could, of course, produce greater mobility, locked joints or smaller effective connectivity sum.

6.2 On the C-H-C-H- linkage

Recently, Baker and Waldron [1,4] claimed to have completely solved the overconstrained C-H-C-H- linkage existence problem, and to have listed all the individual cases, in terms of the differing constraints on linkage parameters.* Unfortunately, as pointed out in an errata sheet for [2], all solutions had not been isolated. Inequalities (7) in [1,4] were over-restrictive. They may, however, be easily replaced by (1.1) of this work together with the constraints

$$s\alpha_{ii+1} \neq 0, \quad i = 1,\ldots,4,$$

since the case of parallel adjacent axes has been dealt with. If we now select I=1 in (1.1), and take into account the solutions found in [1,4], any remaining cases will be governed by the restrictions (6.2.1).

$$\left.\begin{array}{ll} 0 \leqslant a_{ii+1}, & i=1,\ldots,4 \\ 0 < \alpha_{ii+1} < \pi, & i=2,3,4 \\ \pi < \alpha_{12} < 2\pi & \end{array}\right\} \qquad (6.2.1)$$

On reworking the analysis presented in [1,4] which leads to the preliminary solutions, we find that the wider set of constraints (1.1) requires us to examine two additional possibilities which are subject to the respective conditions given below.

*The $\mathcal{R}$- derivatives of the C-H-C-H- solutions are also listed in [1]. This list should be consulted in preference to a corresponding, but incomplete, set in [45].

IIb: $$h_4 = \rho N h_2, \quad N>0$$

$$c\alpha_{12}c\alpha_{23} = c\alpha_{34}c\alpha_{41} \qquad -s\alpha_{12}s\alpha_{23} = s\alpha_{34}s\alpha_{41}$$

$$s\Theta_4 = \pm\, s\Theta_2 \qquad c\Theta_4 = -\, c\Theta_2$$

IVb: $$h_4 = \rho(\tau N h_2 + \frac{R_2}{\pi}), \quad N>0$$

$$c\alpha_{12}c\alpha_{23} = c\alpha_{34}c\alpha_{41} \qquad -s\alpha_{12}s\alpha_{23} = s\alpha_{34}s\alpha_{41}$$

$$s\Theta_4 = \pm\, s\Theta_2 \qquad c\Theta_4 = -\, c\Theta_2 \;.$$

Each of the two types is also governed directly by (6.2.1).

As in [1,4], we may test IIb and IVb simultaneously by considering only the constraints common to them, namely

$$\left.\begin{aligned} c\alpha_{12}c\alpha_{23} &= c\alpha_{34}c\alpha_{41} \\ -s\alpha_{12}s\alpha_{23} &= s\alpha_{34}s\alpha_{41} \\ \Theta_4 &= \sigma\Theta_2 + (2k+1)\pi \end{aligned}\right\}, \qquad \text{(i)}$$

together with (6.2.1).

Recalling closure equation (5.9), by taking the secondary part of its dual, we obtain

$$a_{12}(c\Theta_2 c\alpha_{12}s\alpha_{23}+s\alpha_{12}c\alpha_{23})+a_{23}(c\Theta_2 s\alpha_{12}c\alpha_{23}+c\alpha_{12}s\alpha_{23})$$
$$-a_{34}(c\Theta_4 c\alpha_{34}s\alpha_{41}+s\alpha_{34}c\alpha_{41})-a_{41}(c\Theta_4 s\alpha_{34}c\alpha_{41}+c\alpha_{34}s\alpha_{41})$$
$$=(R_2+h_2\Theta_2)s\Theta_2 s\alpha_{12}s\alpha_{23}-(R_4+h_4\Theta_4)s\Theta_4 s\alpha_{34}s\alpha_{41}, \qquad (6.2.2)$$

which is given as equation (8) in [1,4]. Using the third of equations (i), equation (6.2.2) may be replaced by

$$a_{12}(c\Theta_2 c\alpha_{12} s\alpha_{23}+s\alpha_{12} c\alpha_{23})+a_{23}(c\Theta_2 s\alpha_{12} c\alpha_{23}+c\alpha_{12} s\alpha_{23})$$

$$+a_{34}(c\Theta_2 c\alpha_{34} s\alpha_{41}-s\alpha_{34} c\alpha_{41})+a_{41}(c\Theta_2 s\alpha_{34} c\alpha_{41}- c\alpha_{34} s\alpha_{41})$$

$$=(R_2+h_2\Theta_2) s\Theta_2 s\alpha_{12} s\alpha_{23}-(R_4+h_4\Theta_4) s\Theta_4 s\alpha_{34} s\alpha_{41}. \qquad \text{(ii)}$$

Differentiation of (ii) with respect to Θ_2 leads to

$$-s\Theta_2(a_{12} c\alpha_{12} s\alpha_{23}+a_{23} s\alpha_{12} c\alpha_{23}+a_{34} c\alpha_{34} s\alpha_{41}+a_{41} s\alpha_{34} c\alpha_{41})$$

$$=h_2 s\Theta_2 s\alpha_{12} s\alpha_{23}-\sigma h_4 s\Theta_4 s\alpha_{34} s\alpha_{41}$$

$$+(R_2+h_2\Theta_2) c\Theta_2 s\alpha_{12} s\alpha_{23}-\sigma(R_4+h_4\Theta_4) c\Theta_4 s\alpha_{34} s\alpha_{41}.$$

Rearranging terms, multiplying throughout by $s\Theta_2$ and applying the second and third of equations (i) then yields

$$-s^2\Theta_2\{a_{12} c\alpha_{12} s\alpha_{23}+a_{23} s\alpha_{12} c\alpha_{23}+a_{34} c\alpha_{34} s\alpha_{41}+a_{41} s\alpha_{34} c\alpha_{41}$$

$$+[h_2-h_4] s\alpha_{12} s\alpha_{23}\}$$

$$=(R_2+h_2\Theta_2) s\Theta_2 c\Theta_2 s\alpha_{12} s\alpha_{23}-(R_4+h_4\Theta_4) s\Theta_4 c\Theta_2 s\alpha_{34} s\alpha_{41}.$$

Substitution of (ii) then results in

$$(c^2\Theta_2-1)\{a_{12} c\alpha_{12} s\alpha_{23}+a_{23} s\alpha_{12} c\alpha_{23}+a_{34} c\alpha_{34} s\alpha_{41}+a_{41} s\alpha_{34} c\alpha_{41}$$

$$+[h_2-h_4] s\alpha_{12} s\alpha_{23}\}$$

$$=c\Theta_2\{a_{12}(c\Theta_2 c\alpha_{12} s\alpha_{23}+s\alpha_{12} c\alpha_{23})+a_{23}(c\Theta_2 s\alpha_{12} c\alpha_{23}+c\alpha_{12} s\alpha_{23})$$

$$+a_{34}(c\Theta_2 c\alpha_{34} s\alpha_{41}-s\alpha_{34} c\alpha_{41})+a_{41}(c\Theta_2 s\alpha_{34} c\alpha_{41}-c\alpha_{34} s\alpha_{41})\}.$$

This equation may be simplified to

$$[h_2-h_4]s\alpha_{12}s\alpha_{23}c^2\Theta_2$$
$$-(a_{12}s\alpha_{12}c\alpha_{23}+a_{23}c\alpha_{12}s\alpha_{23}-a_{34}s\alpha_{34}c\alpha_{41}-a_{41}c\alpha_{34}s\alpha_{41})c\Theta_2$$
$$-(a_{12}c\alpha_{12}s\alpha_{23}+a_{23}s\alpha_{12}c\alpha_{23}+a_{34}c\alpha_{34}s\alpha_{41}+a_{41}s\alpha_{34}c\alpha_{41})$$
$$-[h_2-h_4]s\alpha_{12}s\alpha_{23}$$
$$= 0. \qquad \text{(iii)}$$

For (iii) to be identically satisfied, we require, in view of (6.2.1), that

$$\left.\begin{aligned} h_4 = h_2 = h, \quad \text{say} & \\ a_{12}s\alpha_{12}c\alpha_{23}+a_{23}c\alpha_{12}s\alpha_{23}-a_{34}s\alpha_{34}c\alpha_{41}-a_{41}c\alpha_{34}s\alpha_{41} &= 0 \\ a_{12}c\alpha_{12}s\alpha_{23}+a_{23}s\alpha_{12}c\alpha_{23}+a_{34}c\alpha_{34}s\alpha_{41}+a_{41}s\alpha_{34}c\alpha_{41} &= 0 \end{aligned}\right\} . \qquad \text{(iv)}$$

For type IIb, the first of conditions (iv) requires only that $\rho N=1$. For type IVb, it is necessary that

$$\rho h = \tau Nh + \frac{R_2}{\pi},$$

or $$R_2 = \pi nh, \quad n=0, \pm1, \pm2, \ldots.$$

Thus, IVb is a sub-type of IIb.

Substitution of (i) and (iv) into equation (6.2.2) yields

$$\{R_2+h\Theta_2\}s\Theta_2 s\alpha_{12}s\alpha_{23}=\{R_4+h[\sigma\Theta_2+(2k+1)\pi]\}\sigma s\Theta_2 s\alpha_{12}s\alpha_{23}.$$

Because of (6.2.1), this result is easily reduced to

$$R_2 = \sigma\{R_4+(2k+1)\pi h\},$$

or $$R_2 - \sigma R_4 = (2m+1)\pi h, \quad m=0,\pm1,\pm2,\ldots.$$

This relation together with equations (i) and (iv) provides us with the complete set of existence criteria for solution type IIb. Rearranging them into the form used in [1,4], we have the following groups of constraints.

$$\left.\begin{array}{lr} & R_2 - \sigma R_4 = (2m+1)\pi h \\ \text{where} & h_2 = h_4 = h \\ \text{and} & \Theta_4 = \sigma\Theta_2 + (2k+1)\pi \end{array}\right\} \quad \text{IIb.1}$$

$$\left.\begin{array}{r} -s\alpha_{12}s\alpha_{23} = s\alpha_{34}s\alpha_{41} \\ c\alpha_{12}c\alpha_{23} = c\alpha_{34}c\alpha_{41} \end{array}\right\} \quad \text{IIb.2}$$

$$\left.\begin{array}{l} a_{12}s\alpha_{12}c\alpha_{23}+a_{23}c\alpha_{12}s\alpha_{23}=a_{34}s\alpha_{34}c\alpha_{41}+a_{41}c\alpha_{34}s\alpha_{41} \\ a_{12}c\alpha_{12}s\alpha_{23}+a_{23}s\alpha_{12}c\alpha_{23}=-a_{34}c\alpha_{34}s\alpha_{41}-a_{41}s\alpha_{34}c\alpha_{41} \end{array}\right\} \quad \text{IIb.3}$$

We shall also find it convenient to use the following result as an alternative to the first or third of IIb.1.

$$R_4 + h_4\Theta_4 = \sigma(R_2+h_2\Theta_2) \qquad \text{(v)}$$

Determination of individual possibilities

Having isolated the one extra preliminary solution IIb, we may now proceed to enumerate the separate potential solutions contained in it. It will be first necessary to determine the possible linkage constructions which satisfy both IIb.2 and IIb.3. We shall begin by solving equations IIb.2, and then consider what further constraints are imposed in order that IIb.3 be satisfied.

By analogy with the result in [1], we may deduce that the two means of satisfying IIb.2 are given by the following relations.

$$-s\alpha_{12} = s\alpha_{34} \qquad s\alpha_{23} = s\alpha_{41} \qquad c\alpha_{12}c\alpha_{23} = c\alpha_{34}c\alpha_{41}$$

$$-s\alpha_{12} = s\alpha_{41} \qquad s\alpha_{23} = s\alpha_{34} \qquad c\alpha_{12}c\alpha_{23} = c\alpha_{34}c\alpha_{41}$$

Now each of these possibilities possesses two solutions, giving us in all the following four.

$$\left.\begin{array}{ll} \alpha_{12} = 2\pi - \alpha_{34} & \alpha_{23} = \alpha_{41} \\ \alpha_{12} = \pi + \alpha_{34} & \alpha_{23} = \pi - \alpha_{41} \\ \alpha_{12} = 2\pi - \alpha_{41} & \alpha_{23} = \alpha_{34} \\ \alpha_{12} = \pi + \alpha_{41} & \alpha_{23} = \pi - \alpha_{34} \end{array}\right\} \qquad \text{IIb.2.(i)-(iv)}$$

We proceed to substitute in turn each of the solutions IIb.2.(i)-(iv) into equations IIb.3, to determine the corresponding sets of constraints for the a_i.

IIb.2.(i) with IIb.3:

$$\left.\begin{array}{l} a_{12}s\alpha_{12}c\alpha_{23}+a_{23}c\alpha_{12}s\alpha_{23}=-a_{34}s\alpha_{12}c\alpha_{23}+a_{41}c\alpha_{12}s\alpha_{23} \\ a_{12}c\alpha_{12}s\alpha_{23}+a_{23}s\alpha_{12}c\alpha_{23}=-a_{34}c\alpha_{12}s\alpha_{23}+a_{41}s\alpha_{12}c\alpha_{23} \end{array}\right\}$$

imply

$$\left.\begin{array}{l} (a_{12}+a_{34})s\alpha_{12}c\alpha_{23} = (a_{41}-a_{23})c\alpha_{12}s\alpha_{23} \\ (a_{12}+a_{34})c\alpha_{12}s\alpha_{23} = (a_{41}-a_{23})s\alpha_{12}c\alpha_{23} \end{array}\right\} .$$

This pair of equations yields the four solutions given below.

$$a_{23} = a_{41} \quad a_{12} = a_{34} = 0 \quad \alpha_{12} = 2\pi - \alpha_{34} \quad \alpha_{23} = \alpha_{41}$$

$$\alpha_{12} = \frac{3\pi}{2} \qquad \alpha_{23} = \alpha_{34} = \alpha_{41} = \frac{\pi}{2}$$

$$a_{41} - a_{23} = a_{12} + a_{34} \quad \alpha_{12} = 2\pi - \alpha_{34} \quad \alpha_{23} = \alpha_{41} = \alpha_{12} - \pi$$

$$a_{23} - a_{41} = a_{12} + a_{34} \quad \alpha_{12} = 2\pi - \alpha_{34} \quad \alpha_{23} = \alpha_{41} = \alpha_{34}$$

IIb.2.(ii) with IIb.3:

$$\left.\begin{aligned} a_{12}s\alpha_{12}c\alpha_{23}+a_{23}c\alpha_{12}s\alpha_{23}&=a_{34}s\alpha_{12}c\alpha_{23}-a_{41}c\alpha_{12}s\alpha_{23} \\ a_{12}c\alpha_{12}s\alpha_{23}+a_{23}s\alpha_{12}c\alpha_{23}&=a_{34}c\alpha_{12}s\alpha_{23}-a_{41}s\alpha_{12}c\alpha_{23} \end{aligned}\right\}$$

imply

$$\left.\begin{aligned} (a_{23}+a_{41})c\alpha_{12}s\alpha_{23} &= (a_{34}-a_{12})s\alpha_{12}c\alpha_{23} \\ (a_{23}+a_{41})s\alpha_{12}c\alpha_{23} &= (a_{34}-a_{12})c\alpha_{12}s\alpha_{23} \end{aligned}\right\}.$$

Here, the four solutions are as follows.

$$a_{34} = a_{12} \quad a_{23} = a_{41} = 0 \quad \alpha_{12} = \pi + \alpha_{34} \quad \alpha_{23} = \pi - \alpha_{41}$$

$$\alpha_{12} = \frac{3\pi}{2} \qquad \alpha_{23} = \alpha_{34} = \alpha_{41} = \frac{\pi}{2}$$

$$a_{34} - a_{12} = a_{23} + a_{41} \quad \alpha_{12} = 2\pi - \alpha_{41} \quad \alpha_{23} = \alpha_{34} = \alpha_{12} - \pi$$

$$a_{12} - a_{34} = a_{23} + a_{41} \quad \alpha_{12} = 2\pi - \alpha_{23} \quad \alpha_{34} = \alpha_{41} = \alpha_{12} - \pi$$

IIb.2.(iii) with IIb.3:

$$\left.\begin{aligned} a_{12}s\alpha_{12}c\alpha_{23}+a_{23}c\alpha_{12}s\alpha_{23}&=a_{34}c\alpha_{12}s\alpha_{23}-a_{41}s\alpha_{12}c\alpha_{23} \\ a_{12}c\alpha_{12}s\alpha_{23}+a_{23}s\alpha_{12}c\alpha_{23}&=a_{34}s\alpha_{12}c\alpha_{23}-a_{41}c\alpha_{12}s\alpha_{23} \end{aligned}\right\}$$

imply

$$\left.\begin{aligned}(a_{12}+a_{41})s\alpha_{12}c\alpha_{23}&=(a_{34}-a_{23})c\alpha_{12}s\alpha_{23}\\(a_{12}+a_{41})c\alpha_{12}s\alpha_{23}&=(a_{34}-a_{23})s\alpha_{12}c\alpha_{23}\end{aligned}\right\} .$$

The four solutions here are listed below.

$$a_{23} = a_{34} \quad a_{12} = a_{41} = 0 \quad \alpha_{12} = 2\pi - \alpha_{41} \quad \alpha_{23} = \alpha_{34}$$

$$\alpha_{12} = \frac{3\pi}{2} \quad \alpha_{23} = \alpha_{34} = \alpha_{41} = \frac{\pi}{2}$$

$$a_{34} - a_{23} = a_{12} + a_{41} \quad \alpha_{12} = 2\pi - \alpha_{41} \quad \alpha_{23} = \alpha_{34} = \alpha_{12} - \pi$$

$$a_{23} - a_{34} = a_{12} + a_{41} \quad \alpha_{12} = 2\pi - \alpha_{41} \quad \alpha_{23} = \alpha_{34} = \alpha_{41}$$

IIb.2.(iv) with IIb.3:

$$\left.\begin{aligned}a_{12}s\alpha_{12}c\alpha_{23}+a_{23}c\alpha_{12}s\alpha_{23}&=-a_{34}c\alpha_{12}s\alpha_{23}+a_{41}s\alpha_{12}c\alpha_{23}\\a_{12}c\alpha_{12}s\alpha_{23}+a_{23}s\alpha_{12}c\alpha_{23}&=-a_{34}s\alpha_{12}c\alpha_{23}+a_{41}c\alpha_{12}s\alpha_{23}\end{aligned}\right\}$$

imply

$$\left.\begin{aligned}(a_{23}+a_{34})c\alpha_{12}s\alpha_{23} &= (a_{41}-a_{12})s\alpha_{12}c\alpha_{23}\\(a_{23}+a_{34})s\alpha_{12}c\alpha_{23} &= (a_{41}-a_{12})c\alpha_{12}s\alpha_{23}\end{aligned}\right\} .$$

In this case, the four solutions are as follows.

$$a_{41} = a_{12} \quad a_{23} = a_{34} = 0 \quad \alpha_{12} = \pi + \alpha_{41} \quad \alpha_{23} = \pi - \alpha_{34}$$

$$\alpha_{12} = \frac{3\pi}{2} \quad \alpha_{23} = \alpha_{34} = \alpha_{41} = \frac{\pi}{2}$$

$$a_{41} - a_{12} = a_{23} + a_{34} \quad \alpha_{12} = 2\pi - \alpha_{34} \quad \alpha_{23} = \alpha_{41} = \alpha_{12} - \pi$$

$$a_{12} - a_{41} = a_{23} + a_{34} \quad \alpha_{12} = 2\pi - \alpha_{23} \quad \alpha_{34} = \alpha_{41} = \alpha_{12} - \pi$$

Only nine of the above sixteen potential solutions are distinct. They are listed below in a more systematic manner.

[*a*] $\alpha_{12} = \frac{3\pi}{2} \quad \alpha_{23} = \alpha_{34} = \alpha_{41} = \frac{\pi}{2}$

[*b*] $a_{23} = a_{41} \quad a_{12} = a_{34} = 0 \quad \alpha_{12} = 2\pi - \alpha_{34} \quad \alpha_{23} = \alpha_{41}$

[*c*] $a_{34} = a_{12} \quad a_{23} = a_{41} = 0 \quad \alpha_{12} = \pi + \alpha_{34} \quad \alpha_{23} = \pi - \alpha_{41}$

[*d*] $a_{23} = a_{34} \quad a_{12} = a_{41} = 0 \quad \alpha_{12} = 2\pi - \alpha_{41} \quad \alpha_{23} = \alpha_{34}$

[*e*] $a_{12} = a_{41} \quad a_{23} = a_{34} = 0 \quad \alpha_{12} = \pi + \alpha_{41} \quad \alpha_{23} = \pi - \alpha_{34}$

[*f*] $a_{41} = a_{12} + a_{23} + a_{34} \quad \alpha_{23} = \alpha_{41} = \alpha_{12} - \pi = \pi - \alpha_{34}$

[*g*] $a_{23} = a_{34} + a_{41} + a_{12} \quad \alpha_{23} = \alpha_{34} = \alpha_{41} = 2\pi - \alpha_{12}$

[*h*] $a_{34} = a_{41} + a_{12} + a_{23} \quad \alpha_{23} = \alpha_{34} = \alpha_{12} - \pi = \pi - \alpha_{41}$

[*i*] $a_{12} = a_{23} + a_{34} + a_{41} \quad \alpha_{34} = \alpha_{41} = \alpha_{12} - \pi = \pi - \alpha_{23}$

These are the solutions of equations IIb.2 and IIb.3. Equations IIb.1 also apply to each of them.

Testing of potential solutions

The nine different linkage arrangements isolated above, together with equations IIb.1, represent possible C-H-C-H-solutions. They all permit mobility of joints 2 and 4. We shall now test them further by checking for rotational mobility of joint 3 (and, by symmetry, joint 1). This is achieved by substituting each of them in turn into closure equations (5.7) and (5.8). These equations may be simplified to the following two, respectively, by using the last of IIb.1.

$$s\theta_2 s\alpha_{12} = -\sigma s\theta_2 c\theta_3 s\alpha_{41} - c\theta_2 s\theta_3 c\alpha_{34} s\alpha_{41} + s\theta_3 s\alpha_{34} c\alpha_{41} \tag{6.2.3}$$

$$c\theta_2 s\alpha_{12} c\alpha_{23} + c\alpha_{12} s\alpha_{23} = -\sigma s\theta_2 s\theta_3 s\alpha_{41} + c\theta_2 c\theta_3 c\alpha_{34} s\alpha_{41} - c\theta_3 s\alpha_{34} c\alpha_{41} \tag{6.2.4}$$

For each potential solution, we shall eliminate θ_2 between (6.2.3) and (6.2.4), and require that the resulting equation is an identity in Θ_3. We shall thereby determine for which values, if any, of σ rotational mobility of joint 3 is possible. Since equations (6.2.3) and (6.2.4) are rotational closure equations, the constraints on the link lengths in our potential solutions are here irrelevant.

Potential solution [a]:

Equations (6.2.3) and (6.2.4) are considerably simplified to the following two.

$$\left.\begin{aligned} \sigma &= c\Theta_3 \\ 0 &= s\Theta_3 \end{aligned}\right\}$$

Clearly, no solution is possible.

Potential solution [b]:

Equations (6.2.3) and (6.2.4) may be here written as follows.

$$\left.\begin{aligned} &s\Theta_2(s\alpha_{12}+\sigma c\Theta_3 s\alpha_{23})+c\Theta_2(s\Theta_3 c\alpha_{12}s\alpha_{23})=-s\Theta_3 s\alpha_{12}c\alpha_{23} \\ &s\Theta_2(\sigma s\Theta_3 s\alpha_{23})+c\Theta_2(s\alpha_{12}c\alpha_{23}-c\Theta_3 c\alpha_{12}s\alpha_{23})=-c\alpha_{12}s\alpha_{23}+c\Theta_3 s\alpha_{12}c\alpha_{23} \end{aligned}\right\}$$

We proceed to eliminate Θ_2 between the equations by means of Cramer's rule. In this context, the "determinant of coefficients" is

$$D = (s\alpha_{12}+\sigma c\Theta_3 s\alpha_{23})(s\alpha_{12}c\alpha_{23}-c\Theta_3 c\alpha_{12}s\alpha_{23})-\sigma s^2\Theta_3 c\alpha_{12}s^2\alpha_{23}$$

$$= c\Theta_3 s\alpha_{12}s\alpha_{23}(\sigma c\alpha_{23}-c\alpha_{12})+s^2\alpha_{12}c\alpha_{23}-\sigma c\alpha_{12}s^2\alpha_{23}.$$

We shall assume that $D\neq 0$, since this contingency will be covered under later headings, namely [*f*] and [*g*].

With a view to 'solving' the equations for $s\Theta_2$ and $c\Theta_2$, we now define D_s and D_c as follows.

$$D_s = -s\Theta_3 s\alpha_{12} c\alpha_{23}(s\alpha_{12}c\alpha_{23} - c\Theta_3 c\alpha_{12} s\alpha_{23})$$

$$-s\Theta_3 c\alpha_{12} s\alpha_{23}(-c\alpha_{12} s\alpha_{23} + c\Theta_3 s\alpha_{12} c\alpha_{23})$$

$$= s\Theta_3(c^2\alpha_{12}s^2\alpha_{23} - s^2\alpha_{12}c^2\alpha_{23})$$

$$D_c = (s\alpha_{12} + \sigma c\Theta_3 s\alpha_{23})(-c\alpha_{12}s\alpha_{23} + c\Theta_3 s\alpha_{12}c\alpha_{23}) + \sigma s^2\Theta_3 s\alpha_{12}s\alpha_{23}c\alpha_{23}$$

$$= c\Theta_3(s^2\alpha_{12}c\alpha_{23} - \sigma c\alpha_{12}s^2\alpha_{23}) + s\alpha_{12}s\alpha_{23}(\sigma c\alpha_{23} - c\alpha_{12})$$

Since $s\Theta_2 = \frac{D_s}{D}$ and $c\Theta_2 = \frac{D_c}{D}$, and $s^2\Theta_2 + c^2\Theta_2 = 1$, we have that

$$D_s{}^2 = D^2 - D_c{}^2.$$

That is, we require

$$s^2\Theta_3(c^2\alpha_{12}s^2\alpha_{23} - s^2\alpha_{12}c^2\alpha_{23})^2$$

$$= \{c\Theta_3 s\alpha_{12}s\alpha_{23}(\sigma c\alpha_{23} - c\alpha_{12}) + (s^2\alpha_{12}c\alpha_{23} - \sigma c\alpha_{12}s^2\alpha_{23})\}^2$$

$$-\{c\Theta_3(s^2\alpha_{12}c\alpha_{23} - \sigma c\alpha_{12}s^2\alpha_{23}) + s\alpha_{12}s\alpha_{23}(\sigma c\alpha_{23} - c\alpha_{12})\}^2$$

$$= (1 - c^2\Theta_3)\{(s^2\alpha_{12}c\alpha_{23} - \sigma c\alpha_{12}s^2\alpha_{23})^2 - s^2\alpha_{12}s^2\alpha_{23}(\sigma c\alpha_{23} - c\alpha_{12})^2\}$$

$$= s^2\Theta_3\{s^2\alpha_{12}c^2\alpha_{23}(s^2\alpha_{12} - s^2\alpha_{23}) + c^2\alpha_{12}s^2\alpha_{23}(s^2\alpha_{23} - s^2\alpha_{12})\}$$

$$= s^2\Theta_3\ (s^2\alpha_{23} - s^2\alpha_{12})(c^2\alpha_{12}s^2\alpha_{23} - s^2\alpha_{12}c^2\alpha_{23}).$$

Now,

$$c^2\alpha_{12}s^2\alpha_{23} - s^2\alpha_{12}c^2\alpha_{23} = (1 - s^2\alpha_{12})s^2\alpha_{23} - s^2\alpha_{12}(1 - s^2\alpha_{23})$$

$$= s^2\alpha_{23} - s^2\alpha_{12}.$$

We may conclude that [*b*] permits full rotational mobility of the linkage.

Potential solution [*c*]:

We may write equations (6.2.3) and (6.2.4) as follows, for this case.

$$\left.\begin{aligned} s\Theta_2(s\alpha_{12}+\sigma c\Theta_3 s\alpha_{23})-c\Theta_2(s\Theta_3 c\alpha_{12}s\alpha_{23}) &= s\Theta_3 s\alpha_{12}c\alpha_{23} \\ s\Theta_2(\sigma s\Theta_3 s\alpha_{23})+c\Theta_2(s\alpha_{12}c\alpha_{23}+c\Theta_3 c\alpha_{12}s\alpha_{23}) &= -c\alpha_{12}s\alpha_{23}-c\Theta_3 s\alpha_{12}c\alpha_{23} \end{aligned}\right\}$$

As for [*b*], we eliminate Θ_2 between the two equations. To this end, we define the following determinants.

$$D = (s\alpha_{12}+\sigma c\Theta_3 s\alpha_{23})(s\alpha_{12}c\alpha_{23}+c\Theta_3 c\alpha_{12}s\alpha_{23})+\sigma s^2\Theta_3 c\alpha_{12}s^2\alpha_{23}$$

$$= c\Theta_3 s\alpha_{12}s\alpha_{23}(c\alpha_{12}+\sigma c\alpha_{23})+s^2\alpha_{12}c\alpha_{23}+\sigma c\alpha_{12}s^2\alpha_{23}$$

$$D_S = s\Theta_3 s\alpha_{12}c\alpha_{23}(s\alpha_{12}c\alpha_{23}+c\Theta_3 c\alpha_{12}s\alpha_{23})$$

$$-s\Theta_3 c\alpha_{12}s\alpha_{23}(c\alpha_{12}s\alpha_{23}+c\Theta_3 s\alpha_{12}c\alpha_{23})$$

$$= s\Theta_3(s^2\alpha_{12}c^2\alpha_{23}-c^2\alpha_{12}s^2\alpha_{23})$$

$$D_C = -(s\alpha_{12}+\sigma c\Theta_3 s\alpha_{23})(c\alpha_{12}s\alpha_{23}+c\Theta_3 s\alpha_{12}c\alpha_{23})-\sigma s^2\Theta_3 s\alpha_{12}s\alpha_{23}c\alpha_{23}$$

$$= -c\Theta_3(s^2\alpha_{12}c\alpha_{23}+\sigma c\alpha_{12}s^2\alpha_{23})-s\alpha_{12}s\alpha_{23}(c\alpha_{12}+\sigma c\alpha_{23})$$

We may assume $D\neq 0$, because such an eventuality is covered under [*h*] and [*i*].

As for [*b*], since $s\Theta_2 = \frac{D_S}{D}$ and $c\Theta_2 = \frac{D_C}{D}$, we must have

$$D_S{}^2 = D^2 - D_C{}^2.$$

Substituting the respective expressions above, we require that

$$s^2\Theta_3(s^2\alpha_{12}c^2\alpha_{23}-c^2\alpha_{12}s^2\alpha_{23})^2$$

$$=\{c\Theta_3 s\alpha_{12}s\alpha_{23}(c\alpha_{12}+\sigma c\alpha_{23})+(s^2\alpha_{12}c\alpha_{23}+\sigma c\alpha_{12}s^2\alpha_{23})\}^2$$

$$-\{c\Theta_3(s^2\alpha_{12}c\alpha_{23}+\sigma c\alpha_{12}s^2\alpha_{23})+s\alpha_{12}s\alpha_{23}(c\alpha_{12}+\sigma c\alpha_{23})\}^2$$

$$=(1-c^2\Theta_3)\{(s^2\alpha_{12}c\alpha_{23}+\sigma c\alpha_{12}s^2\alpha_{23})^2-s^2\alpha_{12}s^2\alpha_{23}(c\alpha_{12}+\sigma c\alpha_{23})^2\}$$

$$=s^2\Theta_3\{s^2\alpha_{12}c^2\alpha_{23}(s^2\alpha_{12}-s^2\alpha_{23})+c^2\alpha_{12}s^2\alpha_{23}(s^2\alpha_{23}-s^2\alpha_{12})\}.$$

By comparison with the relevant result for [*b*], we may conclude that [*c*] permits rotational motion about joint 3.

Potential solution [*d*]:

Equations (6.2.3) and (6.2.4) are reduced in this case to the following two.

$$\left.\begin{aligned} &s\Theta_2(1-\sigma c\Theta_3)s\alpha_{12}-c\Theta_2(s\Theta_3 c\alpha_{23})s\alpha_{12}=(s\Theta_3 c\alpha_{12})s\alpha_{23} \\ &s\Theta_2(\sigma s\Theta_3)s\alpha_{12}-c\Theta_2(c\alpha_{23}+c\Theta_3 c\alpha_{23})s\alpha_{12}=(c\alpha_{12}+c\Theta_3 c\alpha_{12})s\alpha_{23} \end{aligned}\right\}$$

If $\sigma=1$, the equations are equivalent; Θ_3 is determined in terms of Θ_2, which fact is compatible with linkage mobility 1. If $\sigma=-1$, multiplication of the first equation by $(1+c\Theta_3)$, the second by $s\Theta_3$, and subtraction yield the result

$$1 + c\Theta_3 = 0,$$

which would indicate locking of joint 3 in rotation. We may conclude that [*d*] is a solution only for $\sigma=1$.

Potential solution [e]:

Here, equations (6.2.3) and (6.2.4) are reduced as follows.

$$\left.\begin{aligned} &s\Theta_2(1-\sigma c\Theta_3)s\alpha_{12}+c\Theta_2(s\Theta_3 c\alpha_{23})s\alpha_{12}=-(s\Theta_3 c\alpha_{12})s\alpha_{23} \\ &s\Theta_2(\sigma s\Theta_3)s\alpha_{12}-c\Theta_2(c\alpha_{23}-c\Theta_3 c\alpha_{23})s\alpha_{12}=(c\alpha_{12}-c\Theta_3 c\alpha_{12})s\alpha_{23} \end{aligned}\right\}$$

If $\sigma=1$, multiplication of the first equation by $(1-c\Theta_3)$, the second by $s\Theta_3$, and addition yield the result

$$1 - c\Theta_3 = 0,$$

which would imply that joint 3 were locked in rotation.
If $\sigma=-1$, the equations are equivalent; Θ_3 is expressible in terms of Θ_2 in the normal way for a mobility one linkage.
For [*e*] to be a solution, then, we require that $\sigma=-1$.

Potential solution [f]:

For our present purposes, this case may be regarded as a special form of either [*b*] or [*e*]. Using the result for [*e*], the equation relating Θ_2 and Θ_3 reduces to

$$s\Theta_2(1+c\Theta_3) = s\Theta_3(1+c\Theta_2)c\alpha_{12},$$

but the main consequence for [*e*] is here unaltered.
A solution is possible only for $\sigma=-1$.

Potential solution [g]:

We may treat this case as a special form of either [*b*] or [*d*]. Using the result for the latter, the relationship between Θ_2 and Θ_3 simplifies to

$$s\Theta_2(1-c\Theta_3) = - s\Theta_3(1-c\Theta_2)c\alpha_{12}.$$

We may still conclude that a solution is possible only for $\sigma=1$.

Potential solution [h]:

This case may be treated as a special form of either [*c*] or [*d*]. Again, we may conclude that a solution here is possible only for $\sigma=1$. The simplified relationship between Θ_2 and Θ_3 is

$$s\Theta_2(1-c\Theta_3) = -\ s\Theta_3(1+c\Theta_2)c\alpha_{12}.$$

Potential solution [i]:

We may here regard this potential solution as a special case of either [*c*] or [*e*]. Again, we may conclude that a solution is possible only for $\sigma=-1$. The equation relating Θ_2 and Θ_3 may be reduced to

$$s\Theta_2(1+c\Theta_3) = s\Theta_3(1-c\Theta_2)c\alpha_{12}.$$

The results of the above tests for potential solutions [*a*]-[*i*] are summarised in Table 6.2.1 below. The solutions given there all permit rotation of the four linkage joints. We shall not demonstrate here that the solutions are proper, because this fact is established in section 5.6. That is, we may assume a true and active connectivity of two in each of the cylindric joints.

Table 6.2.1

$a_{23} = a_{41}$	$a_{12} = a_{34} = 0$	$\alpha_{12} = 2\pi - \alpha_{34}$	$\alpha_{23} = \alpha_{41}$	$\sigma=\pm1$
$a_{12} = a_{34}$	$a_{23} = a_{41} = 0$	$\alpha_{12} = \pi + \alpha_{34}$	$\alpha_{23} = \pi - \alpha_{41}$	$\sigma=\pm1$
$a_{23} = a_{34}$	$a_{41} = a_{12} = 0$	$\alpha_{12} = 2\pi - \alpha_{41}$	$\alpha_{23} = \alpha_{34}$	$\sigma=+1$
$a_{41} = a_{12}$	$a_{23} = a_{34} = 0$	$\alpha_{12} = \pi + \alpha_{41}$	$\alpha_{23} = \pi - \alpha_{34}$	$\sigma=-1$
$a_{41} = a_{12} + a_{23} + a_{34}$		$\alpha_{12} - \pi = \alpha_{23} =$	$\alpha_{41} = \pi - \alpha_{34}$	$\sigma=-1$
$a_{34} = a_{41} + a_{12} + a_{23}$		$\alpha_{12} - \pi = \alpha_{23} =$	$\alpha_{34} = \pi - \alpha_{41}$	$\sigma=+1$
$a_{23} = a_{34} + a_{41} + a_{12}$		$2\pi - \alpha_{12} = \alpha_{23} =$	$\alpha_{34} = \alpha_{41}$	$\sigma=+1$
$a_{12} = a_{23} + a_{34} + a_{41}$		$\alpha_{12} - \pi = \alpha_{34} =$	$\alpha_{41}=\pi - \alpha_{23}$	$\sigma=-1$

For each of the ten solutions in the table, equations IIb.1 hold, with the relevant value(s) of σ. The last of equations IIb.1 is the closure equation relating Θ_2 and Θ_4. A suitable closure equation relating Θ_2 and Θ_3 may be gleaned from the appropriate place among the tests above. A convenient closure equation relating Θ_1 and Θ_3 is that obtained by cycling the indices in (5.9), namely

$$-c\Theta_1 s\alpha_{41} s\alpha_{12} + c\alpha_{41} c\alpha_{12} = -c\Theta_3 s\alpha_{23} s\alpha_{34} + c\alpha_{23} c\alpha_{34}. \qquad (6.2.5)$$

Two independent translational closure equations may be obtained from equation (5.10), one directly, the other by advancing the subscripts by 2. They are

$$a_{41}(-c\Theta_2 c\Theta_3 + \sigma s\Theta_2 s\Theta_3 c\alpha_{34}) + \sigma(R_2 + h\Theta_2) s\Theta_3 s\alpha_{34} + a_{34} c\Theta_3 + a_{23}$$
$$+ a_{12} c\Theta_2 + r_1 s\Theta_2 s\alpha_{12} = 0 \qquad (6.2.6)$$

and

$$a_{23}(c\Theta_1 c\Theta_2 - s\Theta_1 s\Theta_2 c\alpha_{12}) + (R_2 + h\Theta_2) s\Theta_1 s\alpha_{12} + a_{12} c\Theta_1 + a_{41}$$

$$-a_{34} c\Theta_2 - \sigma r_3 s\Theta_2 s\alpha_{34} = 0. \qquad (6.2.7)$$

Clearly, alternative equations may be used. One such alternative is that equation obtained by taking the secondary part of the dual of (6.2.5) to yield a relationship between r_1 and r_3.

Just as in [1], we may deduce the -R- derivatives of the solutions found above. Table 6.2.1 will remain unchanged, but equations (6.2.6), (6.2.7) and the first two of IIb.1 will be simplified by putting h=0. These C-R-C-R- linkages, of course, do not appear in Waldron's [45] list.

A comparative note

In the Introduction to chapter 7, we shall remark at length upon the intrinsic value of algebraic techniques in linkage analysis, whilst showing that the earlier five-bar chains isolated were due to basically geometrical work. It is in order here, perhaps, to draw attention more briefly to a similar situation. The above results, coupled with those of [1], stand as a demonstration of the thoroughness of the algebraic approach. It would have to be conceded that, in comparison with a geometrical development, the algebraic analysis does not highlight the kinematic essence of the C-H-C-H-linkage, the spatial relationships among axes and links.

Waldron, in [44], was not seeking specifically C-H-C-H-linkages. He was considering the conditions for mobility of six-bar line-symmetric linkages which, in their most general form, must contain three symmetric pairs of screw joints. By allowing the screw axes of one pair to be coaxial with the screw axes of an adjacent pair, he produced a particular case of the C-H-C-H- solution given in [1] by the constraints

$$a_{12} = a_{34} \quad a_{23} = a_{41} \quad \alpha_{12} = \alpha_{34} \quad \alpha_{23} = \alpha_{41} \quad R_2 - R_4 = 2m\pi h.$$

Waldron's approach was through algebraic screw system theory, which allowed him to study definitively symmetric linkages of the generalised types which he sought to investigate, and which gave rise to by-products such as the C-H-C-H- chain mentioned above. The method certainly provided insights, and described some linkage motions elegantly, but it is incapable of unearthing the detailed, exhaustive results obtained in this section.

In the discussion to [44], Hunt also obtained the particular C-H-C-H- chain given above, but by pure geometrical means, and independently of the kind of general line-symmetry propounded by Waldron. Indeed, Hunt, by screw geometry alone, was able to offer a second C-H-C-H- solution, a special case of the solution listed in [1] with the constraints

$$a_{12} = a_{41} \quad a_{23} = a_{34} \quad \alpha_{12} = \alpha_{41} \quad \alpha_{23} = \alpha_{34} \quad R_2 - R_4 = 2m\pi h.$$

This linkage, whilst possessing a line of quasi-symmetry, could not be included in the line-symmetric class of linkages examined by Waldron. Hunt's geometrical treatment of mobile linkages is also of great value in exposing spatial relationships for certain classes of linkages, but is, again, completely unsuitable for comprehensive analysis. One feels inclined, nevertheless, to liken the geometrician and algebraicist, respectively, to the artist and artisan; one hopes that their works are always complementary.

6.3 The S-H-P-H- linkage

The spherical joint in the S-H-P-H- linkage may be replaced by three concurrent revolutes. If two of the revolutes are arranged to be parallel to the two screws, the spherical indicatrix of the linkage shows that the third revolute must be either locked or parallel to one of the others. In either case, we conclude that the -S- joint is replaceable by only two revolutes, making the linkage kinematically equivalent to a R^H-P-H^R- five-bar chain. By direct reference to section 7.5, then, we may isolate any solutions for the present linkage. We need to impose the additional constraints,

$$h_5 = R_5 = a_{51} = R_1 = h_1 = 0. \tag{6.3.1}$$

There are three cases to consider.

A

We may apply directly the results of part A of section 7.5 here, with the added simplifications due to (6.3.1). We find that equations (7.5.4) and (7.5.5) are further reduced to the following two, respectively.

$$s\alpha_{23}(R_2+h_2\Theta_2) = \sigma s\alpha_{34}(R_4+h_4\Theta_4)$$

$$c\alpha_{23}(R_2+h_2\Theta_2) + c\alpha_{34}(R_4+h_4\Theta_4) + r_3 = 0$$

These, together with (7.5.1') and (7.5.2'), are the four independent closure equations of the linkage, which therefore has mobility one. The screws are concurrent at the centre of the spherical joint, and the plane containing them is parallel to a plane containing the slider.

It is clear from the geometry of the linkage that $\sigma=-1$. If the slider is in the plane of the screws, its line of action must also pass through the centre of the spherical joint, and the linkage will be locked. In any case, it is evident that the present linkage is equivalent to a special case of the mobile P-P-P- three-bar.

It is important to note that this solution does not possess R- derivatives, for such linkages would exhibit part-chain mobility.

B

Using the results of part B of section 7.5, we find a second solution. It is a generalisation of the only S-R-P-R- linkage, which was isolated by Waldron [45,48]. The first three closure equations remain unchanged, but the fourth, using (6.3.1), is reduced to

$$r_3+2a_{12}s\alpha_{23}s\Theta_2 + 2(R_2+h_2\Theta_2)c\alpha_{23} = 0.$$

The linkage is a special case of Waldron's plane-symmetric five-bar, and we have

$$\sigma = -1 \quad a_{34} = a_{23} \quad h_4 = -h_2.$$

The two screw-revolute pairs of axes are symmetrically disposed with respect to the plane of symmetry, and the slider is normal to it. If $a_{12}=0$, the linkage degenerates to a special case of solution A. If the screws are replaced by revolutes, we obtain the S-R-P-R- linkage.

C

A third solution is obtained by considering the results of part C of section 7.5. The first two closure equations do not change, but (6.3.1) allows us to simplify the other two. They become

$$a_{23}+a_{12}c\Theta_2+\sigma s\alpha_{34}(R_4+h_4\Theta_4) = 0$$

and $$a_{12}s\Theta_2+c\alpha_{34}(R_4+h_4\Theta_4) + r_3 = 0.$$

For this solution, joint 2 is a revolute and the slider is perpendicular to axes 1 and 2. Axes 4 and 5 are coaxial and also perpendicular to joints 1 and 2. The linkage is kinematically equivalent to a special case of the planar double-slider with adjacent revolutes.

Replacement of joint 4 by a revolute would result in part-chain mobility.

. . . .

It might have been noted that, in the foregoing, we did not consider at all the possibility of any two of joints 2, 3 and 4 being parallel. Let us now determine the consequences of such an eventuality. If joints 2 and 3 or joints 3 and 4 were parallel, the linkage would be equivalent to a R-C-H-R- loop, which is a special case of a C-H-H-H- linkage. But we found in section 6.1 that the only proper C-H-H-H- loop, mobility unity, is that with all joint axes parallel. That case cannot be applied here, because making the two revolutes parallel would reduce the spherical joint to a single revolute.

We can see from the spherical indicatrix that parallelism of the screw joints 2 and 4 would demand parallelism of the three spherical joint axes (or locking of one or more, with parallelism of the others). This fact would imply that the linkage was improper, the spherical joint being replaceable by a revolute, the linkage thereby degenerating to a four-bar chain with connectivity sum four. Such a linkage has, of course, been thoroughly investigated in chapter 5.

In this context of improperness, it should perhaps be pointed out that replacement of a spherical joint by two concurrent revolutes, as we have done above, does not imply that the linkage is improper. As defined in chapter 1, a linkage is improper if a joint may be replaced by one other of lower connectivity without affecting the mobility of the chain. In our case, we replaced the -S- joint by a -R-R- combination of two other joints. Thus, the spherical joint of the S-H-P-H- linkage has true connectivity two.

6.4 The S-P-H-H- linkage

As in section 6.3, we may replace the spherical joint in the present linkage by three concurrent revolutes. We again arrange two of the revolutes to be parallel to the screws. From the spherical indicatrix of the resulting linkage, we see that the remaining revolute must be either locked or parallel to one of the others. The spherical joint is therefore replaceable by only two turning pairs, and the linkage equivalent to a R-P-H-H^R- five-bar. The following dimensional constraints may be applied.

$$\left.\begin{array}{ll} R_1 = \alpha_{45} = R_5 = a_{51} = 0 & s\alpha_{23} = s\alpha_{12} \\ & s\Theta_2 = 0 \end{array}\right\} \qquad (6.4.1)$$

Let us first consider the consequences of other joint axis parallelisms. If the two screws were parallel, parallelism of joints 1 and 5 would be implied; the spherical joint would then degenerate to a single revolute. If joints 2 and 3 were parallel, the linkage would be equivalent to a R-C-H^R- chain, a special case of a C-H-H-H- four-bar; as shown in the previous section, such a possibility would again result in the -S- joint degenerating into a revolute. We may henceforth assume that

$$s\alpha_{23} \neq 0 \quad , \quad s\alpha_{34} \neq 0 .$$

Now, Waldron [45,48] has solved the S-P-R-R- over-constrained linkage problem. We may take advantage of this fact by employing the theorem of section 4.3. Apart from the singular case of coaxial screws, the theorem allows us to conclude that S-P-H-H- solutions are possible only for sets of constraints which allow mobile S-P-R-R- loops. Waldron found that there are no proper S-P-R-R- chains with mobility one. He did, however, eliminate the possibility that $a_{45}=0$, since it led to a linkage with part-chain mobility. We must investigate this case.

For the present linkage with $a_{45}=0$, the chain becomes equivalent to a R-$\widehat{\mathrm{P}}$-H-C- four-bar. From chapter 5, we see that there are no proper, mobility one chains of this form. There are, however, two degeneracies of the form R-$\widehat{\mathrm{P}}$-R-P-, special cases of Delassus solutions d.4 and d.12. Both of these cases are applicable here, since our five-bar chain can assume the form R-$\widehat{\mathrm{P}}$-R-H=R-, where the H=R combination acts as a slider. We therefore have two S-P-H-H- solutions, in both of which the screw adjacent to the slider has zero pitch, and the other screw axis passes through the centre of the spherical joint. For the d.4 case, joints 2 and 4 are perpendicular to joint axis 3, and the linkage is equivalent to a special planar double-slider.

6.5 R-R-C-C- linkages?

During 1969, two researchers independently completed their doctoral work, which included attempting a search for four-bar, single closed loop linkages of mobility one. Both searches were later published. Waldron's [45,47,48] attempt was the more ambitious, seeking all such linkages which contained R,P,C,S and F joints, whilst Savage [35,36] confined himself to chains containing R,P and C joints. The two approaches differed from each other and, on comparison, it seems that, within his chosen area, Savage has overlooked some solutions. Waldron's results include all of Savage's with two apparent exceptions. They are particular R-R-R-P- and R-R-C-C- chains, the existence of which Savage has claimed to show. The first of them, having non-parallel adjacent joint axes, clearly cannot exist as a proper, mobility one linkage; reference to the spherical indicatrix or to Delassus's results (chapter 5) settles the matter. Savage evidently included the chain as one having a passive degree of freedom for most of its motion. This fact explains the first discrepancy.

The second discrepancy is more serious, and both authors referred to it. Waldron found that, apart from the case of two $C\hat{}R$ groups, there were no proper solutions. He did point out, however, the existence of two improper solutions in which both cylindric joints had passive degrees of freedom; one solution was based on a Bennett linkage and the other on a four-bar spherical chain. Savage, on the other hand, listed several equations which he claimed represented the existence criteria for additional R-R-C-C- solutions. Neither author tested the authenticity of the other's work. In the interest of making available an authoritative list of four-bar solutions, a third, independent, analysis is here presented for the R-R-C-C-chain. The approach is again algebraic and, although similar in some respects to the previous two, it makes special use of symmetry and the notion of the dual.

Let us number the revolutes 1 and 2 and the cylindric joints 3 and 4. If joints 3 and 4 were parallel, the linkage would possess part-chain mobility, and so may be disallowed. Parallelism of joints 1 and 2 implies, using the spherical indicatrix, parallelism of joints 3 and 4. Again using the spherical indicatrix, for joint axes 2 and 3 to be parallel, joints 4 and 1 must also be parallel, and vice-versa. This yields the only solution with parallel adjacent joint axes, and is a special case of Waldron's [42,45] and Hunt's [27,30] parallel screw linkages. Both Waldron and Savage acknowledged this solution. From this point, however, their findings diverged.

Having dealt with the only parallel adjacent axes case, we may henceforth make the following assumption.

$$s\alpha_{ii+1} \neq 0, \qquad i = 1,\ldots,4 \tag{i}$$

Let us also choose, in accordance with (1.1),

$$\left.\begin{aligned} &0 < \alpha_{ii+1} < \pi \text{ , } i = 4,1,2 \\ &0 < \alpha_{34} < 2\pi. \end{aligned}\right\} \tag{ii}$$

Since each of joints 1 and 2 has connectivity one, we select from among all possible closure equations two nominally independent ones, both of which should contain precisely the two variables Θ_1 and Θ_2. We then attempt to find the conditions under which the two equations will be equivalent. These constraints can be interpreted as necessary existence criteria for R-R-C-C- chains.

Elimination of $c\Theta_4$ between equations (5.6) and (5.9) results in

$$s\Theta_1 s\Theta_2 s\alpha_{23} s\alpha_{41} - c\Theta_1 c\Theta_2 c\alpha_{12} s\alpha_{23} s\alpha_{41} - c\Theta_1 s\alpha_{12} c\alpha_{23} s\alpha_{41} - c\Theta_2 s\alpha_{12} s\alpha_{23} c\alpha_{41}$$
$$= c\alpha_{34} - c\alpha_{12} c\alpha_{23} c\alpha_{41}.$$

It is very important, in view of a later stage in the analysis, to observe the symmetrical roles played by joints 1 and 2 in this equation. We now rewrite the equation in the form

$$s\Theta_1 (s\Theta_2 s\alpha_{23}) s\alpha_{41} - c\Theta_1 (c\Theta_2 c\alpha_{12} s\alpha_{23} + s\alpha_{12} c\alpha_{23}) s\alpha_{41}$$
$$= c\Theta_2 s\alpha_{12} s\alpha_{23} c\alpha_{41} - c\alpha_{12} c\alpha_{23} c\alpha_{41} + c\alpha_{34} . \qquad (6.5.1)$$

If we now write the dual of (6.5.1), take the secondary part, and rearrange terms, we can obtain the following equation.

$$s\Theta_1 (R_2 c\Theta_2 s\alpha_{23} s\alpha_{41} + a_{23} s\Theta_2 c\alpha_{23} s\alpha_{41} + a_{41} s\Theta_2 s\alpha_{23} c\alpha_{41}$$
$$+ R_1 c\Theta_2 c\alpha_{12} s\alpha_{23} s\alpha_{41} + R_1 s\alpha_{12} c\alpha_{23} s\alpha_{41})$$
$$+ c\Theta_1 (R_1 s\Theta_2 s\alpha_{23} s\alpha_{41} + R_2 s\Theta_2 c\alpha_{12} s\alpha_{23} s\alpha_{41} + a_{12} c\Theta_2 s\alpha_{12} s\alpha_{23} s\alpha_{41}$$
$$- a_{23} c\Theta_2 c\alpha_{12} c\alpha_{23} s\alpha_{41} - a_{41} c\Theta_2 c\alpha_{12} s\alpha_{23} c\alpha_{41} - a_{12} c\alpha_{12} c\alpha_{23} s\alpha_{41}$$
$$+ a_{23} s\alpha_{12} s\alpha_{23} s\alpha_{41} - a_{41} s\alpha_{12} c\alpha_{23} c\alpha_{41})$$
$$= -R_2 s\Theta_2 s\alpha_{12} s\alpha_{23} c\alpha_{41} + a_{12} c\Theta_2 c\alpha_{12} s\alpha_{23} c\alpha_{41} + a_{23} c\Theta_2 s\alpha_{12} c\alpha_{23} c\alpha_{41}$$
$$- a_{41} c\Theta_2 s\alpha_{12} s\alpha_{23} s\alpha_{41} + a_{12} s\alpha_{12} c\alpha_{23} c\alpha_{41} + a_{23} c\alpha_{12} s\alpha_{23} c\alpha_{41}$$
$$+ a_{41} c\alpha_{12} c\alpha_{23} s\alpha_{41} - a_{34} s\alpha_{34} \qquad (6.5.2)$$

Equations (6.5.1) and (6.5.2) contain only the two joint variables Θ_1 and Θ_2. We shall eliminate Θ_1 between them by Cramer's rule, and proceed to find the conditions under which the resulting equation, in Θ_2 alone, is an identity in that variable.

The 'determinant of coefficients' is given by D in

$$\frac{1}{s^2\alpha_{41}} D = R_1(-c^2\Theta_2 s^2\alpha_{12}s^2\alpha_{23}+1-c^2\alpha_{12}c^2\alpha_{23})+a_{12}s\Theta_2 c\Theta_2 s\alpha_{12}s^2\alpha_{23}$$
$$+ 2R_1 c\Theta_2 s\alpha_{12}c\alpha_{12}s\alpha_{23}c\alpha_{23}+R_2 c\Theta_2 s\alpha_{12}s\alpha_{23}c\alpha_{23}$$
$$- a_{12}s\Theta_2 c\alpha_{12}s\alpha_{23}c\alpha_{23}+a_{23}s\Theta_2 s\alpha_{12}+R_2 c\alpha_{12}s^2\alpha_{23},$$

which is obtained after some simplification. Before continuing, it is first necessary to consider the possibility that D is identically zero. For such to be the case, we should require that the coefficients of $c^2\Theta_2$, $s\Theta_2 c\Theta_2$, $c\Theta_2$ and $s\Theta_2$ and the constant term are all zero. We are therefore led to the results, in view of (i),

$$a_{12} = a_{23} = R_1 = 0$$

and either $R_2 = 0$ or $c\alpha_{12} = c\alpha_{23} = 0$.

If $R_2=0$, equation (6.5.2) reduces to

$$s\Theta_1(a_{41}s\Theta_2 s\alpha_{23}c\alpha_{41})+c\Theta_1(-a_{41}c\Theta_2 c\alpha_{12}s\alpha_{23}c\alpha_{41}-a_{41}s\alpha_{12}c\alpha_{23}c\alpha_{41})$$
$$=-a_{41}c\Theta_2 s\alpha_{12}s\alpha_{23}s\alpha_{41}+a_{41}c\alpha_{12}c\alpha_{23}s\alpha_{41}-a_{34}s\alpha_{34}.$$

Eliminating the Θ_1 terms between this equation and (6.5.1) results in

$$a_{41}c\alpha_{41}(c\Theta_2 s\alpha_{12}s\alpha_{23}c\alpha_{41}-c\alpha_{12}c\alpha_{23}c\alpha_{41}+c\alpha_{34})$$
$$=s\alpha_{41}(-a_{41}c\Theta_2 s\alpha_{12}s\alpha_{23}s\alpha_{41}+a_{41}c\alpha_{12}c\alpha_{23}s\alpha_{41}-a_{34}s\alpha_{34}),$$

whence

$$a_{41}c\Theta_2 s\alpha_{12}s\alpha_{23}-a_{41}c\alpha_{12}c\alpha_{23}+a_{41}c\alpha_{34}c\alpha_{41}+a_{34}s\alpha_{34}s\alpha_{41} = 0.$$

Considering the coefficient of $c\Theta_2$ and the constant term in this equation leads us to the conclusion that

$$a_{41} = a_{34} = 0.$$

Substitution of the complete set of constraints

$$a_{12} = a_{23} = a_{34} = a_{41} = R_1 = R_2 = 0$$

into equation (5.10) requires that either joint 4 is locked in translation or joint 3 is locked in rotation. Neither contingency is acceptable. In fact, under the constraints listed above, the linkage is mobile but improper, and is based on a spherical four-bar.

If, on the other hand, $c\alpha_{12}=c\alpha_{23}=0$, equations (6.5.1) and (6.5.2) reduce respectively to the two following.

$$s\Theta_1 s\Theta_2 s\alpha_{41} = c\Theta_2 c\alpha_{41} + c\alpha_{34}$$

$$s\Theta_1 (R_2 c\Theta_2 s\alpha_{41} + a_{41} s\Theta_2 c\alpha_{41}) = -R_2 s\Theta_2 c\alpha_{41} - a_{41} c\Theta_2 s\alpha_{41} - a_{34} s\alpha_{34}$$

Eliminating $s\Theta_1$ between these two equations leads to, after some simplification,

$$a_{41} s\Theta_2 c\Theta_2 + R_2 c\Theta_2 c\alpha_{34} s\alpha_{41} + a_{34} s\Theta_2 s\alpha_{34} s\alpha_{41} + a_{41} s\Theta_2 c\alpha_{34} c\alpha_{41}$$
$$+ R_2 s\alpha_{41} c\alpha_{41} = 0.$$

From inspection of the coefficients of $s\Theta_2 c\Theta_2$, $c\Theta_2$ and $s\Theta_2$ and the constant term, we conclude that

$$a_{34} = a_{41} = 0$$

and either $R_2 = 0$ or $c\alpha_{34} = c\alpha_{41} = 0.$

In either case, substitution of the set

$$a_{12} = a_{23} = a_{34} = a_{41} = R_1 = 0$$

into equation (5.10) leads to the same result as before, namely an improper linkage based on a spherical four-bar, or one with otherwise locked joints.

We may therefore conclude that $D \neq 0$.

With a view to 'solving' equations (6.5.1) and (6.5.2) for $s\Theta_1$ and $c\Theta_1$, we now write down the appropriate determinants. Let us put

$$\begin{aligned}
D_s = {} & (c\Theta_2 s\alpha_{12} s\alpha_{23} c\alpha_{41} - c\alpha_{12} c\alpha_{23} c\alpha_{41} + c\alpha_{34}) \\
& \times(R_1 s\Theta_2 s\alpha_{23} s\alpha_{41} + R_2 s\Theta_2 c\alpha_{12} s\alpha_{23} s\alpha_{41} + a_{12} c\Theta_2 s\alpha_{12} s\alpha_{23} s\alpha_{41} \\
& \quad - a_{23} c\Theta_2 c\alpha_{12} c\alpha_{23} s\alpha_{41} - a_{41} c\Theta_2 c\alpha_{12} s\alpha_{23} c\alpha_{41} - a_{12} c\alpha_{12} c\alpha_{23} s\alpha_{41} \\
& \qquad + a_{23} s\alpha_{12} s\alpha_{23} s\alpha_{41} - a_{41} s\alpha_{12} c\alpha_{23} c\alpha_{41}) \\
& + s\alpha_{41}(c\Theta_2 c\alpha_{12} s\alpha_{23} + s\alpha_{12} c\alpha_{23}) \\
& \times(-R_2 s\Theta_2 s\alpha_{12} s\alpha_{23} c\alpha_{41} + a_{12} c\Theta_2 c\alpha_{12} s\alpha_{23} c\alpha_{41} + a_{23} c\Theta_2 s\alpha_{12} c\alpha_{23} c\alpha_{41} \\
& \quad - a_{41} c\Theta_2 s\alpha_{12} s\alpha_{23} s\alpha_{41} + a_{12} s\alpha_{12} c\alpha_{23} c\alpha_{41} + a_{23} c\alpha_{12} s\alpha_{23} c\alpha_{41} \\
& \qquad + a_{41} c\alpha_{12} c\alpha_{23} s\alpha_{41} - a_{34} s\alpha_{34})
\end{aligned}$$

$$=R_1 s\Theta_2 c\Theta_2 s\alpha_{12} s^2\alpha_{23} s\alpha_{41} c\alpha_{41} + a_{12} c^2\Theta_2 s^2\alpha_{23} s\alpha_{41} c\alpha_{41}$$

$$-a_{41} c^2\Theta_2 s\alpha_{12} c\alpha_{12} s^2\alpha_{23} + a_{23} c\Theta_2 s\alpha_{41} c\alpha_{41} - a_{41} c\Theta_2 s^2\alpha_{12} s\alpha_{23} c\alpha_{23}$$

$$-R_1 s\Theta_2 c\alpha_{12} s\alpha_{23} c\alpha_{23} s\alpha_{41} c\alpha_{41} - R_2 s\Theta_2 s\alpha_{23} c\alpha_{23} s\alpha_{41} c\alpha_{41}$$

$$+a_{41} c\Theta_2 c^2\alpha_{12} s\alpha_{23} c\alpha_{23} + a_{12} c^2\alpha_{23} s\alpha_{41} c\alpha_{41} + a_{41} s\alpha_{12} c\alpha_{12} c^2\alpha_{23}$$

$$-a_{34} c\Theta_2 c\alpha_{12} s\alpha_{23} s\alpha_{34} s\alpha_{41} - a_{34} s\alpha_{12} c\alpha_{23} s\alpha_{34} s\alpha_{41}$$

$$+c\alpha_{34}(R_1 s\Theta_2 s\alpha_{23} s\alpha_{41} + R_2 s\Theta_2 c\alpha_{12} s\alpha_{23} s\alpha_{41} + a_{12} c\Theta_2 s\alpha_{12} s\alpha_{23} s\alpha_{41}$$

$$-a_{23} c\Theta_2 c\alpha_{12} c\alpha_{23} s\alpha_{41} - a_{41} c\Theta_2 c\alpha_{12} s\alpha_{23} c\alpha_{41} - a_{12} c\alpha_{12} c\alpha_{23} s\alpha_{41}$$

$$+a_{23} s\alpha_{12} s\alpha_{23} s\alpha_{41} - a_{41} s\alpha_{12} c\alpha_{23} c\alpha_{41}),$$

and

$$D_C = s\Theta_2 s\alpha_{23} s\alpha_{41}(-R_2 s\Theta_2 s\alpha_{12} s\alpha_{23} c\alpha_{41} + a_{12} c\Theta_2 c\alpha_{12} s\alpha_{23} c\alpha_{41}$$

$$+a_{23} c\Theta_2 s\alpha_{12} c\alpha_{23} c\alpha_{41} - a_{41} c\Theta_2 s\alpha_{12} s\alpha_{23} s\alpha_{41}$$

$$+a_{12} s\alpha_{12} c\alpha_{23} c\alpha_{41} + a_{23} c\alpha_{12} s\alpha_{23} c\alpha_{41}$$

$$+a_{41} c\alpha_{12} c\alpha_{23} s\alpha_{41} - a_{34} s\alpha_{34})$$

$$-(c\Theta_2 s\alpha_{12} s\alpha_{23} c\alpha_{41} - c\alpha_{12} c\alpha_{23} c\alpha_{41} + c\alpha_{34})$$

$$\times(R_2 c\Theta_2 s\alpha_{23} s\alpha_{41} + a_{23} s\Theta_2 c\alpha_{23} s\alpha_{41} + a_{41} s\Theta_2 s\alpha_{23} c\alpha_{41}$$

$$+R_1 c\Theta_2 c\alpha_{12} s\alpha_{23} s\alpha_{41} + R_1 s\alpha_{12} c\alpha_{23} s\alpha_{41})$$

$$=-R_2 s\alpha_{12} s^2\alpha_{23} s\alpha_{41} c\alpha_{41} + a_{12} s\Theta_2 c\Theta_2 c\alpha_{12} s^2\alpha_{23} s\alpha_{41} c\alpha_{41}$$

$$-a_{41} s\Theta_2 c\Theta_2 s\alpha_{12} s^2\alpha_{23} + a_{12} s\Theta_2 s\alpha_{12} s\alpha_{23} c\alpha_{23} s\alpha_{41} c\alpha_{41}$$

$$+a_{23} s\Theta_2 c\alpha_{12} s\alpha_{41} c\alpha_{41} - a_{34} s\Theta_2 s\alpha_{23} s\alpha_{34} s\alpha_{41} + a_{41} s\Theta_2 c\alpha_{12} s\alpha_{23} c\alpha_{23}$$

$$-R_1 c^2\Theta_2 s\alpha_{12} c\alpha_{12} s^2\alpha_{23} s\alpha_{41} c\alpha_{41} + R_1 c\Theta_2 s\alpha_{23} c\alpha_{23} s\alpha_{41} c\alpha_{41} (c^2\alpha_{12} - s^2\alpha_{12})$$

$$+R_2 c\Theta_2 c\alpha_{12} s\alpha_{23} c\alpha_{23} s\alpha_{41} c\alpha_{41} + R_1 s\alpha_{12} c\alpha_{12} c^2\alpha_{23} s\alpha_{41} c\alpha_{41}$$

$$-c\alpha_{34} (R_2 c\Theta_2 s\alpha_{23} s\alpha_{41} + a_{23} s\Theta_2 c\alpha_{23} s\alpha_{41} + a_{41} s\Theta_2 s\alpha_{23} c\alpha_{41}$$

$$+R_1 c\Theta_2 c\alpha_{12} s\alpha_{23} s\alpha_{41} + R_1 s\alpha_{12} c\alpha_{23} s\alpha_{41}).$$

Since $c^2\Theta_1 + s^2\Theta_1 = 1$, we must have $D^2 = D_S{}^2 + D_C{}^2$.
That is,

$$s^4\alpha_{41} [-c^2\Theta_2 . R_1 s^2\alpha_{12} s^2\alpha_{23} + s\Theta_2 c\Theta_2 . a_{12} s\alpha_{12} s^2\alpha_{23}$$

$$+c\Theta_2 \{2R_1 c\alpha_{12} + R_2\} s\alpha_{12} s\alpha_{23} c\alpha_{23}$$

$$+s\Theta_2 \{a_{23} s\alpha_{12} - a_{12} c\alpha_{12} s\alpha_{23} c\alpha_{23}\}$$

$$+\{R_2 c\alpha_{12} s^2\alpha_{23} + R_1 (1 - c^2\alpha_{12} c^2\alpha_{23})\}]^2$$

$$=[c^2\Theta_2 \{a_{12} s\alpha_{41} c\alpha_{41} - a_{41} s\alpha_{12} c\alpha_{12}\} s^2\alpha_{23} + s\Theta_2 c\Theta_2 . R_1 s\alpha_{12} s^2\alpha_{23} s\alpha_{41} c\alpha_{41}$$

$$+c\Theta_2 \{a_{23} s\alpha_{41} c\alpha_{41} + a_{41} (c^2\alpha_{12} - s^2\alpha_{12}) s\alpha_{23} c\alpha_{23} - a_{34} c\alpha_{12} s\alpha_{23} s\alpha_{34} s\alpha_{41}$$

$$+a_{12} s\alpha_{12} s\alpha_{23} c\alpha_{34} s\alpha_{41} - a_{23} c\alpha_{12} c\alpha_{23} c\alpha_{34} s\alpha_{41} - a_{41} c\alpha_{12} s\alpha_{23} c\alpha_{34} c\alpha_{41}\}$$

$$+s\Theta_2 \{-R_1 c\alpha_{12} c\alpha_{23} c\alpha_{41} - R_2 c\alpha_{23} c\alpha_{41} + R_1 c\alpha_{34} + R_2 c\alpha_{12} c\alpha_{34}\} s\alpha_{23} s\alpha_{41}$$

$$+\{(a_{12} s\alpha_{41} c\alpha_{41} + a_{41} s\alpha_{12} c\alpha_{12}) c^2\alpha_{23} - a_{12} c\alpha_{12} c\alpha_{23} c\alpha_{34} s\alpha_{41}$$

$$+a_{23} s\alpha_{12} s\alpha_{23} c\alpha_{34} s\alpha_{41} - a_{41} s\alpha_{12} c\alpha_{23} c\alpha_{34} c\alpha_{41} - a_{34} s\alpha_{12} c\alpha_{23} s\alpha_{34} s\alpha_{41}\}]^2$$

$$+[-c^2\Theta_2.R_1 s\alpha_{12}c\alpha_{12}s^2\alpha_{23}s\alpha_{41}c\alpha_{41}$$

$$+s\Theta_2 c\Theta_2\{a_{12}c\alpha_{12}s\alpha_{41}c\alpha_{41}-a_{41}s\alpha_{12}\}s^2\alpha_{23}$$

$$+c\Theta_2\{R_1(c^2\alpha_{12}-s^2\alpha_{12})c\alpha_{23}c\alpha_{41}$$

$$+R_2 c\alpha_{12}c\alpha_{23}c\alpha_{41}-R_1 c\alpha_{12}c\alpha_{34}-R_2 c\alpha_{34}\}s\alpha_{23}s\alpha_{41}$$

$$+s\Theta_2\{a_{12}s\alpha_{12}s\alpha_{23}c\alpha_{23}s\alpha_{41}c\alpha_{41}+a_{23}c\alpha_{12}s\alpha_{41}c\alpha_{41}-a_{34}s\alpha_{23}s\alpha_{34}s\alpha_{41}$$

$$+a_{41}c\alpha_{12}s\alpha_{23}c\alpha_{23}-a_{23}c\alpha_{23}c\alpha_{34}s\alpha_{41}-a_{41}s\alpha_{23}c\alpha_{34}c\alpha_{41}\}$$

$$+\{-R_2 s\alpha_{12}s^2\alpha_{23}c\alpha_{41}+R_1 s\alpha_{12}c\alpha_{12}c^2\alpha_{23}c\alpha_{41}$$

$$-R_1 s\alpha_{12}c\alpha_{23}c\alpha_{34}\}s\alpha_{41}]^2. \qquad \text{(iii)}$$

Equating coefficients of the terms in $s\Theta_2 c^3\Theta_2$, we have

$$-R_1 a_{12}s^3\alpha_{12}s^4\alpha_{23}s^4\alpha_{41}$$

$$=R_1 s\alpha_{12}s^2\alpha_{23}s\alpha_{41}c\alpha_{41}\{a_{12}s\alpha_{41}c\alpha_{41}-a_{41}s\alpha_{12}c\alpha_{12}\}s^2\alpha_{23}$$

$$-R_1 s\alpha_{12}c\alpha_{12}s^2\alpha_{23}s\alpha_{41}c\alpha_{41}\{a_{12}c\alpha_{12}s\alpha_{41}c\alpha_{41}-a_{41}s\alpha_{12}\}s^2\alpha_{23}$$

from which we conclude, after simplification and using (i), that either $R_1=0$ or $a_{12}=0$.

Now equating coefficients of terms in $c^4\Theta_2$ leads to

$$R_1{}^2 s^4\alpha_{12}s^4\alpha_{41}-a_{12}{}^2 s^2\alpha_{12}s^4\alpha_{41}$$

$$=\{a_{12}s\alpha_{41}c\alpha_{41}-a_{41}s\alpha_{12}c\alpha_{12}\}^2-R_1{}^2 s^2\alpha_{12}s^2\alpha_{41}c^2\alpha_{41}$$

$$+R_1{}^2 s^2\alpha_{12}c^2\alpha_{12}s^2\alpha_{41}c^2\alpha_{41}-\{a_{12}c\alpha_{12}s\alpha_{41}c\alpha_{41}-a_{41}s\alpha_{12}\}^2.$$

Putting $a_{12}=0$ in this equation results in

$$R_1{}^2 s^2\alpha_{41} = -\,a_{41}{}^2,$$

which can only hold if $R_1 = a_{41} = 0$. We may therefore conclude that

$$R_1 = 0. \tag{iv}$$

Putting this result into the above equation yields

$$a_{41}^2 s^2\alpha_{12} = a_{12}^2 s^2\alpha_{41}$$

which, in view of equations (1.1) and (ii), implies

$$a_{41}s\alpha_{12} = a_{12}s\alpha_{41}. \tag{v}$$

Equating coefficients of terms in $s\Theta_2 c^2\Theta_2$ and substituting results (iv) and (v) leads to

$$R_2a_{12}s^2\alpha_{12}c\alpha_{23}s^2\alpha_{41}=R_2a_{12}\{c\alpha_{41}-c\alpha_{12}\}\{c\alpha_{12}c\alpha_{34}-c\alpha_{23}c\alpha_{41}\}$$
$$+R_2a_{12}\{c\alpha_{12}c\alpha_{41}-1\}\{c\alpha_{12}c\alpha_{23}c\alpha_{41}-c\alpha_{34}\},$$

which implies $R_2=0$, $a_{12}=0$ or $c\alpha_{23}=c\alpha_{34}$.

By (v), if $a_{12} = 0$, then $a_{41}=0$. Substituting $R_1=a_{12}=a_{41}=0$ into equations (5.10) and (5.11) results in

$$r_4s\Theta_3s\alpha_{34}+a_{34}c\Theta_3+a_{23} = 0$$

and $$-r_4c\Theta_3s\alpha_{34}+a_{34}s\Theta_3+R_2s\alpha_{23} = 0.$$

Eliminating r_4 between these equations yields

$$a_{34}+a_{23}c\Theta_3+R_2s\alpha_{23}s\Theta_3 = 0.$$

For mobility of joint 3 in rotation, we then require that $R_2=a_{23}=a_{34}=0$. As previously discussed, these six constraints result in a linkage with locked joints. We may therefore assume that $a_{12}\neq 0$.

Now, the above results (iv) and (v) and the two remaining possibilities $R_2=0$ or $c\alpha_{23}=c\alpha_{34}$ were obtained from equation (iii), by equating coefficients of certain powers of Θ_2. Equation (iii) itself was produced by eliminating Θ_1 between (6.5.1) and (6.5.2). It was previously remarked that the variables Θ_1 and Θ_2 played symmetrical roles in the first of these equations; the second is derived from the dual of the first, and exhibits the same symmetry. If, then, we had chosen to eliminate Θ_2 between (6.5.1) and (6.5.2), we should have obtained constraints analogous with those just cited. In particular, by analogy with the set

$$R_1 = 0 \quad a_{41}s\alpha_{12} = a_{12}s\alpha_{41} \quad c\alpha_{23} = c\alpha_{34},$$

we may infer that

$$R_2 = 0 \quad a_{23}s\alpha_{12} = a_{12}s\alpha_{23} \quad c\alpha_{41} = c\alpha_{34}.$$

This potential solution is therefore a special case of the other, for which we have already established $R_2=0$. We need consider it no further. Regarding now the alternative set

$$R_1 = R_2 = 0 \quad a_{41}s\alpha_{12} = a_{12}s\alpha_{41},$$

by analogy we have that

$$R_2 = R_1 = 0 \quad a_{23}s\alpha_{12} = a_{12}s\alpha_{23}. \qquad \text{(vi)}$$

We now have only one potential solution to consider, that summarised by equations (v) and (vi). Substitution of these results into (iii) reduces it to the following equation, after some simplification.

$$(1-c^2\theta_2)a_{12}{}^2 s^2\alpha_{23}s^2\alpha_{41}[c\theta_2 s\alpha_{12}s\alpha_{23}+\{1-c\alpha_{12}c\alpha_{23}\}]^2$$

$$=[c^2\theta_2\{c\alpha_{41}-c\alpha_{12}\}a_{12}s^2\alpha_{23}$$

$$+c\theta_2\{a_{12}\frac{c\alpha_{41}}{s\alpha_{12}}+a_{12}\frac{c\alpha_{23}}{s\alpha_{12}}(c^2\alpha_{12}-s^2\alpha_{12})-a_{34}c\alpha_{12}s\alpha_{34}$$

$$+a_{12}s\alpha_{12}c\alpha_{34}-a_{12}\frac{c\alpha_{12}}{s\alpha_{12}}c\alpha_{23}c\alpha_{34}-a_{12}\frac{c\alpha_{12}}{s\alpha_{12}}c\alpha_{34}c\alpha_{41}\}s\alpha_{23}$$

$$+\{(c\alpha_{41}+c\alpha_{12})a_{12}c^2\alpha_{23}-a_{12}c\alpha_{12}c\alpha_{23}c\alpha_{34}$$

$$+a_{12}s^2\alpha_{23}c\alpha_{34}-a_{12}c\alpha_{23}c\alpha_{34}c\alpha_{41}-a_{34}s\alpha_{12}c\alpha_{23}s\alpha_{34}\}]^2$$

$$+(1-c^2\theta_2)s^2\alpha_{23}[c\theta_2\{c\alpha_{12}c\alpha_{41}-1\}a_{12}s\alpha_{23}$$

$$+\ \{a_{12}s\alpha_{12}c\alpha_{23}c\alpha_{41}+a_{12}\frac{c\alpha_{12}}{s\alpha_{12}}c\alpha_{41}-a_{34}s\alpha_{34}$$

$$+\ a_{12}\frac{c\alpha_{12}}{s\alpha_{12}}c\alpha_{23}-a_{12}\frac{c\alpha_{23}}{s\alpha_{12}}c\alpha_{34}-a_{12}\frac{c\alpha_{34}}{s\alpha_{12}}c\alpha_{41}\}]^2 \qquad \text{(vii)}$$

Equating coefficients of $c^3\theta_2$ in (vii) and simplifying considerably lead to the following result, since we have shown that $a_{12}\neq 0$.

$$a_{34}s\alpha_{12}s\alpha_{34}=a_{12}\{1-c\alpha_{12}c\alpha_{34}+c\alpha_{12}c\alpha_{23}-c\alpha_{23}c\alpha_{34}\} \qquad \text{(viii)}$$

By symmetry, therefore, we also have that

$$a_{34}s\alpha_{12}s\alpha_{34}=a_{12}\{1-c\alpha_{12}c\alpha_{34}+c\alpha_{12}c\alpha_{41}-c\alpha_{41}c\alpha_{34}\}.$$

From the last two equations it is clear that

$$c\alpha_{12}c\alpha_{23}-c\alpha_{23}c\alpha_{34}=c\alpha_{12}c\alpha_{41}-c\alpha_{41}c\alpha_{34},$$

whence

$$(c\alpha_{23}-c\alpha_{41})(c\alpha_{12}-c\alpha_{34}) = 0.$$

Therefore, either $c\alpha_{41}=c\alpha_{23}$ or $c\alpha_{34}=c\alpha_{12}$.

Let us first consider the former possibility. Because of (ii) we may conclude that $\alpha_{41} = \alpha_{23}$. Substitution in (v) and subsequent division by the relevant equation of (vi), in view of (i) and the fact that $a_{12} \neq 0$, yield the result that $a_{41} = a_{23}$. If we now substitute $\alpha_{41} = \alpha_{23}$ and equation (viii) into (vii) we obtain, after simplification,

$$(1-c^2\theta_2)s^2\alpha_{23}[c\theta_2 s\alpha_{12}s\alpha_{23}+\{1-c\alpha_{12}c\alpha_{23}\}]^2$$

$$= [c^2\theta_2\{c\alpha_{23}-c\alpha_{12}\}s\alpha_{23}$$

$$+ c\theta_2\{\frac{c\alpha_{23}}{s\alpha_{12}}c^2\alpha_{12}-\frac{c\alpha_{12}}{s\alpha_{12}}+\frac{c\alpha_{34}}{s\alpha_{12}}-\frac{c\alpha_{12}}{s\alpha_{12}}c\alpha_{34}c\alpha_{23}\}$$

$$+\{c\alpha_{34}-c\alpha_{23}\}s\alpha_{23}]^2$$

$$+(1-c^2\theta_2)[c\theta_2\{c\alpha_{12}c\alpha_{23}-1\}s\alpha_{23}$$

$$+\{s\alpha_{12}c^2\alpha_{23}+\frac{c\alpha_{12}}{s\alpha_{12}}c\alpha_{23}-\frac{1}{s\alpha_{12}}+\frac{c\alpha_{12}}{s\alpha_{12}}c\alpha_{34}-\frac{c\alpha_{34}}{s\alpha_{12}}c\alpha_{23}\}]^2.$$

Equating coefficients of $c\theta_2$ in this equation leads eventually to the result

$$(c\alpha_{12}-c\alpha_{34})^2(1-c\alpha_{12}c\alpha_{23}) = 0,$$

from which we must have either $c\alpha_{34}=c\alpha_{12}$ or $c\alpha_{12}c\alpha_{23}=1$. Now, the second possibility can hold only if $c\alpha_{12}=c\alpha_{23}=\pm 1$. But such a result is disallowed by (ii). We therefore conclude that $c\alpha_{34}=c\alpha_{12}$, which consequence is precisely the second case from above which we need to consider.

Now, $c\alpha_{34}=c\alpha_{12}$ implies that $s\alpha_{34}=\pm s\alpha_{12}$. Substituting for $c\alpha_{34}$ and $s\alpha_{34}$ in (viii) yields $a_{34}=\pm a_{12}$. Then, in view of (1.1) and the fact that $a_{12}\neq 0$, we may conclude that $a_{34}=a_{12}$; subsequently, $\alpha_{34}=\alpha_{12}$. Substitution of these results into (vii) and simplification lead to the equation

$$(1-c^2\theta_2)s^2\alpha_{23}s^2\alpha_{41}[c\theta_2 s\alpha_{12}s\alpha_{23}+\{1-c\alpha_{12}c\alpha_{23}\}]^2$$

$$=[c^2\theta_2\{c\alpha_{41}-c\alpha_{12}\}s^2\alpha_{23}$$

$$+c\theta_2\{c\alpha_{41}-c\alpha_{23}\}s\alpha_{12}s\alpha_{23}$$

$$+\{c\alpha_{41}c^2\alpha_{23}+c\alpha_{12}-c\alpha_{23}-c\alpha_{23}c\alpha_{12}c\alpha_{41}\}]^2$$

$$+(1-c^2\theta_2)s^2\alpha_{23}[c\theta_2\{c\alpha_{12}c\alpha_{41}-1\}s\alpha_{23}+\{c\alpha_{23}c\alpha_{41}-1\}s\alpha_{12}]^2.$$

Equating coefficients of $c\theta_2$ and considerable simplification result in the equation

$$(c\alpha_{41}-c\alpha_{23})^2(1-c\alpha_{23}c\alpha_{12}) = 0.$$

As before, we can show that $c\alpha_{23}c\alpha_{12}\neq 1$. Therefore $c\alpha_{41}=c\alpha_{23}$, and we return to the first case considered.

Collecting the consequences of examining both possibilities, we have the following results.

$$a_{34} = a_{12} \qquad a_{41} = a_{23}$$

$$\alpha_{34} = \alpha_{12} \qquad \alpha_{41} = \alpha_{23}$$

These relations together with (vi) are precisely the Bennett linkage constraints. What we have demonstrated in this section, then, is that the only two forms of mobile R-R-C-C-linkage, apart from the parallel adjacent axes cases, are both improper. One of them is based on the four-bar spherical linkage and the other on the Bennett linkage. Waldron's findings are thereby vindicated.

6.6 The remaining connectivity sum six loops

It was indicated in section 6.1 that we are unable at this stage to complete the analysis of three of the four-bars with connectivity sum six. In this section, we shall set down the work which has been done and explain what further considerations must be given.

The H-H-C-C- linkage

If the screw joints of this linkage are parallel, the spherical indicatrix tells us that the cylinders must also be parallel to each other. But, if the cylindric joints are parallel, the linkage will have part-chain mobility, based on the motion of the P^P- two-bar. If one of the screws is parallel to its adjacent cylindric joint, we can see from the spherical indicatrix that, for rotational mobility, the other two joints must be parallel. Hence, there is one solution with parallel adjacent joint axes, given by H^C-C^H-. This loop is a special parallel-screw linkage, references for which are given in the Introduction to chapter 7.

Let us now consider the general case, in which no two adjacent joint axes are parallel. Recalling the theorem of section 4.3, we see that rotational joint mobility of the present linkage is possible only under the necessary dimensional conditions required for similar mobility of the same linkage with tied screw pitches. In particular, we think of the loop in which both screw pitches are set to zero, namely the R-R-C-C- linkage. Now, in the last section, we showed that the only two mobile R-R-C-C- chains were improper, and based on either the Bennett or spherical four-bar. Hence, we conclude that any H-H-C-C- solution must be bound by at least one of the following two sets of dimensional constraints.

A: $\quad a_{34} = a_{12} \quad a_{41} = a_{23} \quad \alpha_{34} = \alpha_{12} \quad \alpha_{41} = \alpha_{23}$

$$a_{12}s\alpha_{23} = a_{23}s\alpha_{12}$$

B: $$a_{12} = a_{23} = a_{34} = a_{41} = 0$$

Since the linkage has connectivity sum six, a gross mobility one solution will have joint ISAs which belong to a fifth order screw system. Hence, the mobile linkage will have a unique reciprocal screw in all of its configurations, with the possible exception of a few discrete positions. Let us attempt to isolate this reciprocal screw by the technique used in section 2.3.

If we position the origin on joint 4 in the usual way, the linkage velocities will be given as follows.

$$\underset{\sim}{\omega}_4 = \omega_4\underset{\sim}{k} \qquad \underset{\sim}{\mu}_4 = \underset{\sim}{0}$$

$$\underset{\sim}{\omega}_{4'} = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_{4'} = \mu_{4'}\underset{\sim}{k}$$

$$\underset{\sim}{\omega}_3 = \omega_3(s\alpha_{34}\underset{\sim}{j}+c\alpha_{34}\underset{\sim}{k}) \qquad \underset{\sim}{\mu}_3 = a_{34}\omega_3(c\alpha_{34}\underset{\sim}{j}-s\alpha_{34}\underset{\sim}{k})$$

$$\underset{\sim}{\omega}_{3'} = \underset{\sim}{0} \qquad \underset{\sim}{\mu}_{3'} = \mu_{3'}(s\alpha_{34}\underset{\sim}{j}+c\alpha_{34}\underset{\sim}{k})$$

$$\underset{\sim}{\omega}_1 = \omega_1(s\Theta_4 s\alpha_{41}\underset{\sim}{i}-c\Theta_4 s\alpha_{41}\underset{\sim}{j}+c\alpha_{41}\underset{\sim}{k})$$

$$\underset{\sim}{\mu}_1 = \omega_1([a_{41}c\alpha_{41}s\Theta_4+r_4 s\alpha_{41}c\Theta_4]\underset{\sim}{i}$$
$$-[a_{41}c\alpha_{41}c\Theta_4-r_4 s\alpha_{41}s\Theta_4]\underset{\sim}{j}-a_{41}s\alpha_{41}\underset{\sim}{k})+h_1\underset{\sim}{\omega}_1$$

$$\underset{\sim}{\omega}_2 = \omega_2(s\Theta_3 s\alpha_{23}\underset{\sim}{i}+[c\Theta_3 s\alpha_{23}c\alpha_{34}+c\alpha_{23}s\alpha_{34}]\underset{\sim}{j}$$
$$+[-c\Theta_3 s\alpha_{23}s\alpha_{34}+c\alpha_{23}c\alpha_{34}]\underset{\sim}{k})$$

$$\underset{\sim}{\mu}_2 = \omega_2([r_3 s\alpha_{23} c\Theta_3 + a_{23} c\alpha_{23} s\Theta_3]\underset{\sim}{i}$$

$$-[a_{23} s\alpha_{23} s\alpha_{34} + r_3 s\alpha_{23} c\alpha_{34} s\Theta_3 + a_{34} s\alpha_{23} s\alpha_{34} c\Theta_3$$

$$-a_{23} c\alpha_{23} c\alpha_{34} c\Theta_3 - a_{34} c\alpha_{23} c\alpha_{34}]\underset{\sim}{j}$$

$$+[-a_{23} s\alpha_{23} c\alpha_{34} + r_3 s\alpha_{23} s\alpha_{34} s\Theta_3 - a_{34} s\alpha_{23} c\alpha_{34} c\Theta_3$$

$$-a_{23} c\alpha_{23} s\alpha_{34} c\Theta_3 - a_{34} c\alpha_{23} s\alpha_{34}]\underset{\sim}{k}) + h_2 \underset{\sim}{\omega}_2$$

Suppose the abovementioned reciprocal screw is expressed as

$$(\underset{\sim}{\Omega}, \underset{\sim}{M}) \equiv (A, B, C, D, E, F).$$

Using equation (1.3) on each joint ISA in turn, we obtain the following results.

$\$_4$: $F = 0$

$\$_{4'}$: $C = 0$

$\$_3$: $Ba_{34} c\alpha_{34} + Es\alpha_{34} = 0$

$\$_{3'}$: $B = 0$

$\therefore \quad E = 0$

$\$_1$: $A(a_{41} c\alpha_{41} s\Theta_4 + r_4 s\alpha_{41} c\Theta_4 + h_1 s\Theta_4 s\alpha_{41}) + Ds\Theta_4 s\alpha_{41} = 0$

$\$_2$: $A(a_{23} c\alpha_{23} s\Theta_3 + r_3 s\alpha_{23} c\Theta_3 + h_2 s\Theta_3 s\alpha_{23}) + Ds\Theta_3 s\alpha_{23} = 0$

It is now clear that the ISA of the reciprocal screw has components

$$\hat{\underset{\sim}{\omega}} = (1, 0, 0) \qquad \hat{\underset{\sim}{\mu}} = (\tfrac{D}{A}, 0, 0).$$

By means of equations (1.2), we see that, for this ISA,

$$\underset{\sim}{\rho}_n = \underset{\sim}{0} \qquad \text{and} \qquad h = \frac{D}{A}\,.$$

So the screw lies along the common normal between the cylindric pairs.

We may easily write down the condition for a non-trivial solution for A and D from the above two equations containing them. It is

$$s\Theta_4 s\alpha_{41}(a_{23}s\Theta_3 c\alpha_{23}+r_3 c\Theta_3 s\alpha_{23}+h_2 s\Theta_3 s\alpha_{23})$$

$$=s\Theta_3 s\alpha_{23}(a_{41}s\Theta_4 c\alpha_{41}+r_4 c\Theta_4 s\alpha_{41}+h_1 s\Theta_4 s\alpha_{41}). \qquad \text{(i)}$$

We have already shown that a H-H-C-C- solution of the required type must satisfy at least one of the sets of dimensional constraints given above as A and B. Let us determine the simplifications produced in (i) by applying each of the two sets of constraints. We find immediately that, for each of A and B, equation (i) is reduced to

$$s\Theta_4(r_3 c\Theta_3+h_2 s\Theta_3) = s\Theta_3(r_4 c\Theta_4+h_1 s\Theta_4). \qquad \text{(ii)}$$

For a satisfactory solution to exist, this equation must be compatible with the closure equations of the linkage.

We again recall the theorem of section 4.3 and consider the possibility of a solution for the case

$$h_1 = h_2 = h.$$

If a solution exists for this case, any requisite dimensional constraints (in addition to those given under A and B) must also apply to the more general linkage. With the screw pitches so tied, equation (ii) becomes, simply,

$$r_3 c\Theta_3 s\Theta_4 = r_4 s\Theta_3 c\Theta_4. \qquad \text{(iii)}$$

Let us consider, separately for the quasi-Bennett and quasi-spherical linkage conditions, the implications of equation (iii).

A:

Advancing the indices in closure equation (5.10) by 3 and using the quasi-Bennett constraints lead to the following equation.

$$a_{12}(c\Theta_2 c\Theta_3 - s\Theta_2 s\Theta_3 c\alpha_{23} + \frac{s\alpha_{23}}{s\alpha_{12}}c\Theta_2 + 1 + \frac{s\alpha_{23}}{s\alpha_{12}}c\Theta_1)$$

$$+r_3 s\Theta_2 s\alpha_{23} + r_4 s\Theta_1 s\alpha_{23} = 0$$

The simplified (5.9), with indices advanced by 1, allows us to rewrite this result as

$$a_{12}(c\Theta_2 c\Theta_3 s\alpha_{12} - s\Theta_2 s\Theta_3 s\alpha_{12} c\alpha_{23} + c\Theta_2 s\alpha_{23} + s\alpha_{12} + c\Theta_3 s\alpha_{23})$$

$$+s\alpha_{12} s\alpha_{23}(r_3 s\Theta_2 + r_4 s\Theta_1) = 0.$$

Now the first expression enclosed by parentheses is zero. This result may be seen by, for example, advancing by 1 the indices of the third unnumbered closure equation given for the Bennett linkage (d.13) in section 5.7. Hence, the equation is reduced to

$$r_3 s\Theta_2 + r_4 s\Theta_1 = 0.$$

We may substitute in this equation for $s\Theta_2$ from (5.7). We may also substitute for $s\Theta_1$ by using a relationship analogous to (5.7), obtained by proceeding around the linkage in the opposite direction; to retain our sign convention in this relationship, the only change necessary is to give the opposite sign to each $s\Theta_j$. Making the substitutions, we obtain the following equation.

$$r_3(c\theta_3 s\theta_4 s\alpha_{23}+s\theta_3 c\theta_4 c\alpha_{12} s\alpha_{23}+s\theta_3 s\alpha_{12} c\alpha_{23})$$

$$= r_4(c\theta_4 s\theta_3 s\alpha_{23}+s\theta_4 c\theta_3 c\alpha_{12} s\alpha_{23}+s\theta_4 s\alpha_{12} c\alpha_{23})$$

We have used the quasi-Bennett constraints to write the equation in this form. If we now eliminate r_3 and r_4 between this equation and (iii), we find the following relationship.

$$s\theta_3 c\theta_4(c\theta_3 s\theta_4 s\alpha_{23}+s\theta_3 c\theta_4 c\alpha_{12} s\alpha_{23}+s\theta_3 s\alpha_{12} c\alpha_{23})$$

$$= c\theta_3 s\theta_4(s\theta_3 c\theta_4 s\alpha_{23}+c\theta_3 s\theta_4 c\alpha_{12} s\alpha_{23}+s\theta_4 s\alpha_{12} c\alpha_{23})$$

Now, we may substitute for each of the two expressions in parentheses by advancing by 2 respectively the first and fourth unnumbered closure equations given for the Bennett linkage in section 5.7. On doing so, we find that

$$s\theta_3 c\theta_4(-s\theta_4 s\alpha_{12}) = c\theta_3 s\theta_4(-s\theta_3 s\alpha_{12}),$$

whence $\qquad c\theta_4 = c\theta_3, \quad \text{and} \quad s\theta_4 = \sigma s\theta_3.$

The existence or non-existence of a solution depends on whether or not this result is compatible with the established closure equations. If we substitute the relationship into the last unnumbered closure equation given for d.13, indices advanced by 2, we obtain

$$\sigma s^2\theta_3 s\alpha_{12} s\alpha_{23} = 2c\theta_3(c\alpha_{23}-c\alpha_{12}),$$

whence $\quad \sigma c^2\theta_3 s\alpha_{12} s\alpha_{23}+2c\theta_3(c\alpha_{23}-c\alpha_{12})-\sigma s\alpha_{12} s\alpha_{23} = 0.$

But, by inspecting the coefficients of powers of $c\theta_3$ in this equation, we can only conclude that θ_3 is fixed, so that mobility of the linkage is precluded.

We have therefore shown that there is no proper, mobility 1 H-H-C-C- solution in this category with both screw pitches equal. By the provisions contained in the theorem of section 4.3, then, there is no relevant H-H-C-C- solution under the quasi-Bennett dimensional constraints.

B:

Advancing the indices in closure equation (5.10) by 3 and using the quasi-spherical constraints leads immediately to

$$r_3 s\Theta_2 s\alpha_{23} + r_4 s\Theta_1 s\alpha_{41} = 0.$$

As in A, we may substitute for $s\Theta_2$ and $s\Theta_1$ by means of equations based on (5.7), and then eliminate r_3 and r_4 by using (iii). Hence, we obtain

$$s\Theta_3 c\Theta_4 s\alpha_{23}(c\Theta_3 s\Theta_4 s\alpha_{41}+s\Theta_3 c\Theta_4 c\alpha_{34} s\alpha_{41}+s\Theta_3 s\alpha_{34} c\alpha_{41})$$

$$= c\Theta_3 s\Theta_4 s\alpha_{41}(s\Theta_3 c\Theta_4 s\alpha_{23}+c\Theta_3 s\Theta_4 s\alpha_{23} c\alpha_{34}+s\Theta_4 c\alpha_{23} s\alpha_{34}),$$

which may be simplified to

$$s^2\Theta_3 s\alpha_{23}(c\alpha_{34} s\alpha_{41}+c\Theta_4 s\alpha_{34} c\alpha_{41})$$

$$=s^2\Theta_4 s\alpha_{41}(s\alpha_{23} c\alpha_{34}+c\Theta_3 c\alpha_{23} s\alpha_{34}). \qquad (a)$$

We can obtain an independent equation linking Θ_3 and Θ_4 by advancing by 2 the indices in a closure equation produced in section 6.5, namely (6.5.1). Thus, we find

$$s\Theta_3 s\Theta_4 s\alpha_{23} s\alpha_{41} - c\Theta_3 c\Theta_4 s\alpha_{23} c\alpha_{34} s\alpha_{41} - c\Theta_3 s\alpha_{23} s\alpha_{34} c\alpha_{41} - c\Theta_4 c\alpha_{23} s\alpha_{34} s\alpha_{41}$$

$$= c\alpha_{12} - c\alpha_{23} c\alpha_{34} c\alpha_{41}. \qquad (b)$$

Existence or non-existence of a satisfactory solution in this category depends on whether or not equations (a) and (b) are compatible. We have not been able, at this time, to reach a firm conclusion on the matter.

We have shown that, if there is a proper, mobility one H-H-C-C- linkage, it must be governed by the quasi-spherical constraints. Further, if equations (a) and (b) of B are incompatible, there will be no solution. If a solution with $h_1=h_2$ is found, using (a) and (b), further testing will be required to determine whether or not there is a solution for which the screw pitches are not equal.

The S-H-H-H- linkage

As we did in other places, we may replace the spherical joint by three concurrent revolutes and set two of them parallel to the adjacent screw joint axes. We thus obtain a R-R^H-H-H^R- six-bar loop with the following dimensional constraints.

$$\alpha_{56} = R_6 = a_{61} = R_1 = a_{12} = R_2 = \alpha_{23} = 0$$

We shall consider the linkage under three separate headings.

Three screw joint axes parallel:

We see immediately from the spherical indicatrix that, since parallelism of any two of the revolutes is not feasible, joint 1 and one of joints 2 and 6 are locked. The spherical joint therefore acts as a single revolute, and any mobile loop will be improper, based on a Delassus four-bar.

Two adjacent screw joint axes parallel:

Suppose, for example, that

$$\alpha_{34} = 0.$$

The spherical indicatrix shows that joint 1 is locked. The linkage degenerates to a five-bar and may be placed in the category of section 7.4. The results of that section indicate that the only possible solution is given as $R^{\frown}R^{\frown}R$-H=R-. That is, joints 3 and 4 take zero pitch and

$$\alpha_{45} = \frac{\pi}{2}.$$

The axis of screw joint 5 passes through the centre of the spherical joint and is perpendicular to the axes of joints 3 and 4. The solution is kinematically equivalent to a planar slider-crank chain.

No two adjacent screw joint axes parallel:

The linkage will be governed by the six-bar closure equations. For mobility, it is necessary that the equations be interdependent. Let us consider the translational equations which, under the present constraints, may be written as follows.

$$-(R_5+h_5\Theta_5)s\Theta_4 s\alpha_{45}=a_{23}c\Theta_3+a_{34}+a_{45}c\Theta_4$$
$$+a_{56}(c\Theta_4 c\Theta_5-s\Theta_4 s\Theta_5 c\alpha_{45}) \qquad \text{(i)}$$

$$-(R_3+h_3\Theta_3)s\Theta_4 s\alpha_{34}=a_{23}(c\Theta_3 c\Theta_4-s\Theta_3 s\Theta_4 c\alpha_{34})$$
$$+a_{34}c\Theta_4+a_{45}+a_{56}c\Theta_5 \qquad \text{(ii)}$$

$$(R_4+h_4\Theta_4)s\Theta_4 s\alpha_{34}s\alpha_{45}=a_{23}(c\Theta_3 c\Theta_4 c\alpha_{34}s\alpha_{45}-s\Theta_3 s\Theta_4 s\alpha_{45}+c\Theta_3 s\alpha_{34}c\alpha_{45})$$
$$+a_{34}(c\Theta_4 c\alpha_{34}s\alpha_{45}+s\alpha_{34}c\alpha_{45})$$
$$+a_{45}(c\Theta_4 s\alpha_{34}c\alpha_{45}+c\alpha_{34}s\alpha_{45})$$
$$+a_{56}(c\Theta_4 c\Theta_5 s\alpha_{34}c\alpha_{45}-s\Theta_4 s\Theta_5 s\alpha_{34}$$
$$+c\Theta_5 c\alpha_{34}s\alpha_{45}) \qquad \text{(iii)}$$

Now, there are solutions for which

$$a_{23} = a_{56} = 0.$$

We may deduce the result by applying the constraints to equation (iii), which then contains Θ_4 as its only variable. Let us equate coefficients of powers of Θ_4 in the simplified equation, after expanding $s\Theta_4$ and $c\Theta_4$ appropriately.

$$\Theta_4{}^1: \qquad R_4 = 0$$

$$\Theta_4{}^0: \quad a_{34}(c\alpha_{34}s\alpha_{45}+s\alpha_{34}c\alpha_{45})+a_{45}(s\alpha_{34}c\alpha_{45}+c\alpha_{34}s\alpha_{45}) = 0 \qquad \text{(a)}$$

$$\Theta_4{}^2: \quad h_4 s\alpha_{34}s\alpha_{45}=-\frac{1}{2!}(a_{34}c\alpha_{34}s\alpha_{45}+a_{45}s\alpha_{34}c\alpha_{45}) \qquad \text{(b)}$$

$$\Theta_4{}^4: \quad -\frac{1}{3!}h_4 s\alpha_{34}s\alpha_{45}=\frac{1}{4!}(a_{34}c\alpha_{34}s\alpha_{45}+a_{45}s\alpha_{34}c\alpha_{45}) \qquad \text{(c)}$$

From equations (b) and (c), it is clear that

$$h_4 = 0$$

and $\quad a_{34}c\alpha_{34}s\alpha_{45} + a_{45}s\alpha_{34}c\alpha_{45} = 0.$ (d)

Hence, from (a),

$$a_{34}s\alpha_{34}c\alpha_{45} + a_{45}c\alpha_{34}s\alpha_{45} = 0. \qquad \text{(e)}$$

Equations (d) and (e) have three solutions, namely

$$a_{34} = a_{45} = 0,$$

$$\alpha_{34} = \alpha_{45} = \frac{\pi}{2}$$

and $$a_{34} = a_{45} \qquad \alpha_{45} = \pi - \alpha_{34}.$$

(We are free to choose both α_{34} and α_{45} less than π.) In view of the other constraints applying, the first of the three solutions will yield only a linkage with part-chain mobility, in which the axis of joint 4 passes through the centre of the spherical joint.

The other two solutions do yield proper, mobility one linkages, however. In each of them, joint 4 has degenerated into a revolute with zero offset, and the axes of joints 3 and 5 both pass through the centre of the spherical joint. Each of the linkages may be represented by R-R^H-R-H^R-, so that the loop is kinematically equivalent to a certain R-C-R-C- chain, since the two joint axes in each R^H combination are coaxial. The first of them may only be obtained as a special case of the solutions governed by equations II.b in reference [1], or given as linkage no. 4a.15 in [45] or no. 16 in [48]. The second may be obtained as a special case of any one of several of the solutions listed in Table 2 of reference [1], or of the -R- derivatives of the solutions listed in Table 6.2.1 of this work.

We have found above three distinct S-H-H-H- linkages of the required type; the analysis stands there at the time of writing this thesis. The next stage is to consider the implications of an S-R-R-R- analysis, by invoking the theorem of section 4.3, just as we proceeded from R-R-C-C- to H-H-C-C-above. The S-R-R-R- analysis has been carried out by Waldron [45,48]. That work will be a suitable starting-point, after all the solutions with part-chain mobility have been gathered.

The F-H-H-H- linkage

We here replace the planar joint by three revolutes normal to the plane. The consequent six-bar chain R^R-H-H-H-R^ may be assumed to have the following dimensional constraints.

$$a_{56} = R_6 = \alpha_{61} = R_1 = \alpha_{12} = R_2 = a_{23} = 0$$

Let us consider the loop under four different headings.

Three screw joint axes parallel:

In this case, the linkage is proper with mobility 1, being a special parallel-screw linkage [27,42,45] and a generalisation of the Sarrut linkage.

Two adjacent screw joint axes parallel:

Suppose, for example, joints 3 and 4 have parallel axes. By means of the spherical indicatrix, we see that joint 5 must be either parallel to 3 and 4 or parallel to the revolutes. The former possibility has just been dealt with. For the latter, we again obtain a proper, mobility 1 loop, a special parallel-screw linkage [42,45].

Screw joints parallel to revolutes:

If one of joints 3 and 5 is parallel to the revolutes, the spherical indicatrix requires, for mobility, that the two remaining screws be parallel to each other. We then obtain the linkage just dealt with.

If both joints 3 and 5 are parallel to the revolutes, joint 4 will be locked and the linkage will have part-chain mobility, based on the motion of a special parallel-screw linkage [27,40,42,45].

The general case:

The linkage will be governed by the six-bar closure equations which, for mobility, must be interdependent. Under the present constraints, and after some algebraic manipulation, the translational closure equations may be written as follows.

$$
\begin{aligned}
-(R_5+h_5\theta_5)s\theta_4 s\alpha_{45}=a_{12}(c\theta_2 c\theta_3-s\theta_2 s\theta_3 c\alpha_{23})+a_{34}+a_{45}c\theta_4 \\
+a_{61}(c\theta_4 c\theta_5 c\theta_6-s\theta_4 s\theta_5 c\theta_6 c\alpha_{45} \\
-c\theta_4 s\theta_5 s\theta_6 c\alpha_{56}-s\theta_4 c\theta_5 s\theta_6 c\alpha_{45}c\alpha_{56} \\
+s\theta_4 s\theta_6 s\alpha_{45}s\alpha_{56}) \qquad \text{(i)}
\end{aligned}
$$

$$
\begin{aligned}
-(R_3+h_3\theta_3)s\theta_4 s\alpha_{34}=a_{12}(c\theta_2 c\theta_3 c\theta_4-s\theta_2 s\theta_3 c\theta_4 c\alpha_{23} \\
-c\theta_2 s\theta_3 s\theta_4 c\alpha_{34}-s\theta_2 c\theta_3 s\theta_4 c\alpha_{23}c\alpha_{34} \\
+s\theta_2 s\theta_4 s\alpha_{23}s\alpha_{34}) \\
+a_{34}c\theta_4+a_{45}+a_{61}(c\theta_5 c\theta_6-s\theta_5 s\theta_6 c\alpha_{56}) \qquad \text{(ii)}
\end{aligned}
$$

$$
\begin{aligned}
(R_4+h_4\theta_4)s\theta_4 s\alpha_{34}s\alpha_{45}=a_{12}(c\theta_2 c\theta_3 c\theta_4 c\alpha_{34}s\alpha_{45}-s\theta_2 c\theta_3 s\theta_4 c\alpha_{23}s\alpha_{45} \\
-s\theta_2 s\theta_3 c\theta_4 c\alpha_{23}c\alpha_{34}s\alpha_{45}-c\theta_2 s\theta_3 s\theta_4 s\alpha_{45} \\
+c\theta_2 c\theta_3 s\alpha_{34}c\alpha_{45}-s\theta_2 s\theta_3 c\alpha_{23}s\alpha_{34}c\alpha_{45}) \\
+a_{34}(c\theta_4 c\alpha_{34}s\alpha_{45}+s\alpha_{34}c\alpha_{45}) \\
+a_{45}(c\theta_4 s\alpha_{34}c\alpha_{45}+c\alpha_{34}s\alpha_{45})
\end{aligned}
$$

$$+a_{61}(c\Theta_4 c\Theta_5 c\Theta_6 s\alpha_{34} c\alpha_{45} - s\Theta_4 c\Theta_5 s\Theta_6 s\alpha_{34} c\alpha_{56}$$

$$-c\Theta_4 s\Theta_5 s\Theta_6 s\alpha_{34} c\alpha_{45} c\alpha_{56} - s\Theta_4 s\Theta_5 c\Theta_6 s\alpha_{34}$$

$$+c\Theta_5 c\Theta_6 c\alpha_{34} s\alpha_{45} - s\Theta_5 s\Theta_6 c\alpha_{34} s\alpha_{45} c\alpha_{56}) \qquad \text{(iii)}$$

Two more equations worth recording because of their simplicity are obtained from the primary and secondary parts of the dual of a rotational closure equation. After applying the dimensional conditions, they may be written as follows.

$$-c\Theta_3 s\alpha_{23} s\alpha_{34} + c\alpha_{23} c\alpha_{34} = -c\Theta_5 s\alpha_{45} s\alpha_{56} + c\alpha_{45} c\alpha_{56} \qquad \text{(iv)}$$

$$(R_3 + h_3\Theta_3) s\Theta_3 s\alpha_{23} s\alpha_{34} - a_{34}(c\Theta_3 s\alpha_{23} c\alpha_{34} + c\alpha_{23} s\alpha_{34})$$

$$+a_{12}(s\Theta_2 s\Theta_3 s\alpha_{34} - c\Theta_2 c\Theta_3 c\alpha_{23} s\alpha_{34} - c\Theta_2 s\alpha_{23} c\alpha_{34})$$

$$=(R_5 + h_5\Theta_5) s\Theta_5 s\alpha_{45} s\alpha_{56} - a_{45}(c\Theta_5 c\alpha_{45} s\alpha_{56} + s\alpha_{45} c\alpha_{56})$$

$$+a_{61}(s\Theta_5 s\Theta_6 s\alpha_{45} - c\Theta_5 c\Theta_6 s\alpha_{45} c\alpha_{56} - c\Theta_6 c\alpha_{45} s\alpha_{56}) \qquad \text{(v)}$$

At this time, we have not been able to proceed satisfactorily beyond expressing the appropriate equations in an operative form.

We have isolated two proper, mobility one solutions. Probably, the best next step is to consider the implications of Waldron's [45,48] F-R-R-R- analysis. In the same way as suggested for the S-H-H-H- linkage, we should make use of the theorem of section 4.3 to proceed to a final analysis of the F-H-H-H- loop.

7

FIVE-BAR LINKAGE ANALYSIS

Introduction

We have reached the stage, in the science of determining existence criteria for overconstrained linkages, of being able to consider five-bar loops more generally. Waldron [45,47,48] and Savage [35,36] have, between them, published substantially exhaustive lists of existence criteria for all four-bar linkages not containing screw joints. The two works are largely in agreement about their common material; the only notable exception is dealt with in section 6.5 of this thesis. Almost all of the four-bars with screw joints have also been fully analysed by Delassus [11-13], Waldron [42,45], Hunt [27] and Baker [1,2,4,5], and in chapters 5 and 6 of this work. The only four-bars awaiting a complete analysis are specified in sections 6.1 and 6.6. At least one of their particular cases can be regarded as a special form of a five-bar chain. In any case, with only three four-bars yet unsolved, it is reasonable to begin an analysis of the linkages of one order higher.

Rather few five-bars, by comparison, have so far been isolated. Symmetry plays a significant role in linkage mobility analysis and, against four-bars and six-bars, linkages with five joints are wanting in obvious geometrical niceties. The known five-bar loops are listed below.

1. *parallel-screw linkages*

(Refer to Voinea and Atanasiu [40], Waldron [42,45] and Hunt [27].)

(a) *derived from five-bar loops*

The basic linkage of this type is made up precisely of five parallel screw joints of arbitrary pitch. Up to three of the joints may be replaced by prismatic pairs, arbitrarily oriented. As well as the several general forms of mobility one, many special forms are possible, along with cases of greater mobility, depending on what additional geometrical constraints apply.

(b) *derived from six-bar loops*

Here, the fundamental linkage consists of two groups of parallel screw joints, there being at least two screws in each group. Screw pitches are arbitrary, and up to two of the screws may be replaced by arbitrarily oriented sliders. Again, there are several general forms of mobility one, as well as special forms and cases with greater mobility. The significance of this type for five-bar loops is that certain combinations of two joints may be replaced by single joints. Thus, a pair of coaxial screws of differing pitch is kinematically equivalent to a cylindric joint, and the six-bar loop becomes a five-bar of connectivity sum six.

2. *plane-symmetric linkages*

(Refer to Waldron [44,45].)

If we exclude reference to the form which is equivalent to a special parallel-screw linkage and to the Myard (for which, see 4.) plane-symmetric five-bar, this type of chain has the following properties. The linkage has a plane of symmetry, and consists of two pairs of finite pitch screw joints and a slider. The two screws in each pair are symmetrically disposed with respect to the plane of symmetry, and the sum of their pitches is zero. The two screws on either side of the plane of symmetry are parallel to each other. The prismatic joint is normal to the plane of symmetry. The common normals between consecutive joints are also in symmetrically disposed pairs, and the members of a pair have the same projection in the plane of symmetry.

3. *Delassus hybrid five-bars*

These chains, obtained by combining Delassus three- and four-bar solutions, are detailed in section 4.1 above.

4. *Goldberg five-bar*

(Refer to Goldberg [25] and Myard [32].)

This linkage is produced by combining a pair of Bennett chains in such a way that a link common to each may be removed and a pair of adjacent links may be rigidly attached to each other. The resultant loop will then have a pair of corresponding links with equal length and twist angle; two other adjacent links will have a combined length and angle of twist equal to those of the last link

(which is that one derived from two links of the original chains).
The Myard plane-symmetric five-bar is the special case of the Goldberg five-bar for which the two Bennett chains are symmetrically disposed before combining them. They must therefore be mirror images of each other, the mirror being coincident with the plane of symmetry of the resultant linkage.

5. *coaxial screw substitutes*

In section 4.2, it was shown that, under certain conditions, a slider or a revolute in a mobile linkage could be replaced by a H=H or H=P combination, without altering the kinematics of the loop as a whole. Thus, eligible four-bar linkages may be converted into five-bars by such a replacement.

The five types of linkage summarised above, whether singular cases or groups, have something in common. They were all discovered by researchers following a very special line of enquiry in each case. They were not isolated because they were five-bars as such, or because of some particular properties exhibited by five-bars. It was just that five happened to be the appropriate number of links for mobility one for the kinds of motion characteristics being investigated. That is, the workers concerned did not begin with five links and look at the potentialities for mobility; they proceeded in the opposite direction. Whilst types 1. to 5. above must figure largely in a direct treatment of five-bar linkages, a progressive algebraic analysis is required in order to check out all possibilities.

The other advantage of an algebraic approach is that it yields the independent closure equations of the linkage under consideration. In the past, workers have not concerned themselves with setting out the final governing equations of mobile linkages. Such a step, however, will be of increasing importance in the establishment of input-output relationships, limit positions and even mere computer-produced pathplots. Geometry provides us with valuable insights, but 'number production' depends solely on algebraic results.

In chapter 5, we were able to isolate many mobile four-bar chains by examining groupings of linkages with parallel adjacent joint axes. There is no reason why we cannot adopt the same general procedure for five-bars, without specifically looking for mobility one loops with joints of connectivity one. Even if not as fruitful, in terms of the number of linkages uncovered, as for the four-bars, it is a suitable starting-point in a systematic treatment of five-link chains.

Within the following sections, then, we shall largely follow the same approach used in chapter 5, although confining ourselves to just those cases where there is parallelism between adjacent joint axes. An overview of what remains to be attempted will be given in the Conclusion to this chapter. It might be noted here that, for linkages with five members, any proper, mobile, overconstrained loop can have connectivity sum of only five or six.

For a chain containing five members, closure equations (iv) and (v) from the first section of chapter 1 become

$$\underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{\underline{V}}_{23}\underline{\underline{U}}_3\underline{\underline{V}}_{34}\underline{\underline{U}}_4\underline{\underline{V}}_{45}\underline{\underline{U}}_5\underline{\underline{V}}_{51} = \underline{\underline{I}}$$

and

$$\underline{\underline{U}}_1\underline{S}_1+\underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{S}_2+\underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{\underline{V}}_{23}\underline{\underline{U}}_3\underline{S}_3+\underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{\underline{V}}_{23}\underline{\underline{U}}_3\underline{\underline{V}}_{34}\underline{\underline{U}}_4\underline{S}_4$$

$$+\underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{\underline{V}}_{23}\underline{\underline{U}}_3\underline{\underline{V}}_{34}\underline{\underline{U}}_4\underline{\underline{V}}_{45}\underline{\underline{U}}_5\underline{S}_5 = \underline{0} .$$

As in chapter 5, we follow Waldron's [45,47] procedure and rewrite these equations as

$$\underline{\underline{U}}_1\underline{\underline{V}}_{12}\underline{\underline{U}}_2\underline{\underline{V}}_{23}\underline{\underline{U}}_3 = \underline{\underline{V}}_{51}{}^T\underline{\underline{U}}_5{}^T\underline{\underline{V}}_{45}{}^T\underline{\underline{U}}_4{}^T\underline{\underline{V}}_{34}{}^T$$

and

$$\underline{\underline{V}}_{34}\underline{\underline{U}}_4\underline{\underline{V}}_{45}\underline{\underline{U}}_5\underline{S}_5+\underline{\underline{V}}_{34}\underline{\underline{U}}_4\underline{S}_4+\underline{S}_3+\underline{\underline{U}}_3{}^T\underline{\underline{V}}_{23}{}^T\underline{S}_2+\underline{\underline{U}}_3{}^T\underline{\underline{V}}_{23}{}^T\underline{\underline{U}}_2{}^T\underline{\underline{V}}_{12}{}^T\underline{S}_1 = \underline{0}.$$

We may now expand the two matrix equations to yield the equivalent twelve scalar closure equations, as follows.

$$c\theta_1c\theta_2c\theta_3-s\theta_1s\theta_2c\theta_3c\alpha_{12}-c\theta_1s\theta_2s\theta_3c\alpha_{23}-s\theta_1c\theta_2s\theta_3c\alpha_{12}c\alpha_{23}$$
$$+s\theta_1s\theta_3s\alpha_{12}s\alpha_{23}=c\theta_4c\theta_5-s\theta_4s\theta_5c\alpha_{45} \tag{7.1}$$

$$-c\theta_1c\theta_2s\theta_3+s\theta_1s\theta_2s\theta_3c\alpha_{12}-c\theta_1s\theta_2c\theta_3c\alpha_{23}-s\theta_1c\theta_2c\theta_3c\alpha_{12}c\alpha_{23}$$
$$+s\theta_1c\theta_3s\alpha_{12}s\alpha_{23}=s\theta_4c\theta_5c\alpha_{34}+c\theta_4s\theta_5c\alpha_{34}c\alpha_{45}-s\theta_5s\alpha_{34}s\alpha_{45} \tag{7.2}$$

$$c\theta_1s\theta_2s\alpha_{23}+s\theta_1c\theta_2c\alpha_{12}s\alpha_{23}+s\theta_1s\alpha_{12}c\alpha_{23}$$
$$=s\theta_4c\theta_5s\alpha_{34}+c\theta_4s\theta_5s\alpha_{34}c\alpha_{45}+s\theta_5c\alpha_{34}s\alpha_{45} \tag{7.3}$$

$$s\theta_1c\theta_2c\theta_3+c\theta_1s\theta_2c\theta_3c\alpha_{12}-s\theta_1s\theta_2s\theta_3c\alpha_{23}+c\theta_1c\theta_2s\theta_3c\alpha_{12}c\alpha_{23}$$
$$-c\theta_1s\theta_3s\alpha_{12}s\alpha_{23}=-c\theta_4s\theta_5c\alpha_{51}-s\theta_4c\theta_5c\alpha_{45}c\alpha_{51}$$
$$+s\theta_4s\alpha_{45}s\alpha_{51} \tag{7.4}$$

$$-s\theta_1 c\theta_2 s\theta_3 - c\theta_1 s\theta_2 s\theta_3 c\alpha_{12} - s\theta_1 s\theta_2 c\theta_3 c\alpha_{23} + c\theta_1 c\theta_2 c\theta_3 c\alpha_{12} c\alpha_{23}$$
$$- c\theta_1 c\theta_3 s\alpha_{12} s\alpha_{23}$$
$$= -s\theta_4 s\theta_5 c\alpha_{34} c\alpha_{51} + c\theta_4 c\theta_5 c\alpha_{34} c\alpha_{45} c\alpha_{51} - c\theta_4 c\alpha_{34} s\alpha_{45} s\alpha_{51}$$
$$- c\theta_5 s\alpha_{34} s\alpha_{45} c\alpha_{51} - s\alpha_{34} c\alpha_{45} s\alpha_{51} \quad (7.5)$$

$$s\theta_1 s\theta_2 s\alpha_{23} - c\theta_1 c\theta_2 c\alpha_{12} s\alpha_{23} - c\theta_1 s\alpha_{12} c\alpha_{23}$$
$$= -s\theta_4 s\theta_5 s\alpha_{34} c\alpha_{51} + c\theta_4 c\theta_5 s\alpha_{34} c\alpha_{45} c\alpha_{51} - c\theta_4 s\alpha_{34} s\alpha_{45} s\alpha_{51}$$
$$+ c\theta_5 c\alpha_{34} s\alpha_{45} c\alpha_{51} + c\alpha_{34} c\alpha_{45} s\alpha_{51} \quad (7.6)$$

$$s\theta_2 c\theta_3 s\alpha_{12} + c\theta_2 s\theta_3 s\alpha_{12} c\alpha_{23} + s\theta_3 c\alpha_{12} s\alpha_{23}$$
$$= c\theta_4 s\theta_5 s\alpha_{51} + s\theta_4 c\theta_5 c\alpha_{45} s\alpha_{51} + s\theta_4 s\alpha_{45} c\alpha_{51} \quad (7.7)$$

$$-s\theta_2 s\theta_3 s\alpha_{12} + c\theta_2 c\theta_3 s\alpha_{12} c\alpha_{23} + c\theta_3 c\alpha_{12} s\alpha_{23}$$
$$= s\theta_4 s\theta_5 c\alpha_{34} s\alpha_{51} - c\theta_4 c\theta_5 c\alpha_{34} c\alpha_{45} s\alpha_{51} - c\theta_4 c\alpha_{34} s\alpha_{45} c\alpha_{51}$$
$$+ c\theta_5 s\alpha_{34} s\alpha_{45} s\alpha_{51} - s\alpha_{34} c\alpha_{45} c\alpha_{51} \quad (7.8)$$

$$-c\theta_2 s\alpha_{12} s\alpha_{23} + c\alpha_{12} c\alpha_{23}$$
$$= s\theta_4 s\theta_5 s\alpha_{34} s\alpha_{51} - c\theta_4 c\theta_5 s\alpha_{34} c\alpha_{45} s\alpha_{51} - c\theta_4 s\alpha_{34} s\alpha_{45} c\alpha_{51}$$
$$- c\theta_5 c\alpha_{34} s\alpha_{45} s\alpha_{51} + c\alpha_{34} c\alpha_{45} c\alpha_{51} \quad (7.9)$$

$$a_{51}(c\theta_4 c\theta_5 - s\theta_4 s\theta_5 c\alpha_{45}) + r_5 s\theta_4 s\alpha_{45} + a_{45} c\theta_4 + a_{34} + a_{23} c\theta_3$$
$$+ r_2 s\theta_3 s\alpha_{23} + a_{12}(c\theta_2 c\theta_3 - s\theta_2 s\theta_3 c\alpha_{23})$$
$$+ r_1 (s\theta_2 c\theta_3 s\alpha_{12} + c\theta_2 s\theta_3 s\alpha_{12} c\alpha_{23} + s\theta_3 c\alpha_{12} s\alpha_{23}) = 0 \quad (7.10)$$

$$a_{51}(s\Theta_4c\Theta_5c\alpha_{34}+c\Theta_4s\Theta_5c\alpha_{34}c\alpha_{45}-s\Theta_5s\alpha_{34}s\alpha_{45})$$

$$-r_5(c\Theta_4c\alpha_{34}s\alpha_{45}+s\alpha_{34}c\alpha_{45})+a_{45}s\Theta_4c\alpha_{34}-r_4s\alpha_{34}-a_{23}s\Theta_3$$

$$+r_2c\Theta_3s\alpha_{23}-a_{12}(c\Theta_2s\Theta_3+s\Theta_2c\Theta_3c\alpha_{23})$$

$$+r_1(-s\Theta_2s\Theta_3s\alpha_{12}+c\Theta_2c\Theta_3s\alpha_{12}c\alpha_{23}+c\Theta_3c\alpha_{12}s\alpha_{23}) = 0 \qquad (7.11)$$

$$a_{51}(s\Theta_4c\Theta_5s\alpha_{34}+c\Theta_4s\Theta_5s\alpha_{34}c\alpha_{45}+s\Theta_5c\alpha_{34}s\alpha_{45})$$

$$+r_5(c\alpha_{34}c\alpha_{45}-c\Theta_4s\alpha_{34}s\alpha_{45})+a_{45}s\Theta_4s\alpha_{34}+r_4c\alpha_{34}+r_3+r_2c\alpha_{23}$$

$$+a_{12}s\Theta_2s\alpha_{23}+r_1(c\alpha_{12}c\alpha_{23}-c\Theta_2s\alpha_{12}s\alpha_{23}) = 0 \qquad (7.12)$$

Again as in chapter 5, we may obtain alternative forms of the above twelve equations by cyclic advancing of the subscripts.

Armed with the governing equations, we now proceed to isolate the proper, mobility one five-bar solutions with the various combinations of parallel adjacent joint axes. We achieve complete success in all cases except that for which there is only one pair of adjacent joints parallel (section 7.6).

7.1 All joint axes parallel

$$\alpha_{12} = \alpha_{23} = \alpha_{34} = \alpha_{45} = \alpha_{51} = 0$$

Closure equations (7.3) and (7.6)-(7.9) are identically satisfied. Equations (7.1) and (7.5) yield

$$c(\Theta_1+\Theta_2+\Theta_3) = c(\Theta_4+\Theta_5),$$

and equations (7.2) and (7.4) yield

$$-s(\Theta_1+\Theta_2+\Theta_3) = s(\Theta_4+\Theta_5).$$

These results together imply that

$$\Theta_1 + \Theta_2 + \Theta_3 + \Theta_4 + \Theta_5 = 2k\pi. \qquad (7.1.1)$$

Equations (7.10)-(7.12) become, respectively,

$$a_{51}c(\Theta_4+\Theta_5)+a_{45}c\Theta_4+a_{34}+a_{23}c\Theta_3+a_{12}c(\Theta_2+\Theta_3) = 0 \qquad (7.1.2)$$

$$a_{51}s(\Theta_4+\Theta_5)+a_{45}s\Theta_4-a_{23}s\Theta_3-a_{12}s(\Theta_2+\Theta_3) = 0 \qquad (7.1.3)$$

$$r_5 + r_4 + r_3 + r_2 + r_1 = 0. \qquad (7.1.4)$$

Since there are only four independent closure equations, all joints must have connectivity one if the linkage is to possess mobility unity. Only one of the parallel joints can be prismatic without making the linkage improper. Hence, the only possible linkages are H^H^H^H^H- and H^H^H^H^P-. The first of these is a general parallel-screw linkage, whilst the other is a special case of a parallel-screw type.

7.2 Four joint axes parallel

We may suppose that

$$\alpha_{34} = \alpha_{45} = \alpha_{51} = 0,$$

and $\alpha_{12}=\alpha_{23}$ or $2\pi-\alpha_{23}$, where $0<\alpha_{12}<2\pi$, $0<\alpha_{23}<\pi$.

We put $$s\alpha_{12} = \sigma s\alpha_{23}.$$

Equation (7.9) yields the result that

$$c\Theta_2 = -\sigma.$$

That is, joint 2 is prismatic, and $s\Theta_2=0$.

Equations (7.3), (7.6)-(7.8) are identically satisfied. Equations (7.1) and (7.5) yield

$$-\sigma c(\Theta_1+\Theta_3) = c(\Theta_4+\Theta_5),$$

whilst equations (7.2) and (7.4) yield

$$\sigma s(\Theta_1+\Theta_3) = s(\Theta_4+\Theta_5).$$

Together, these results imply that

$$\Theta_3+\Theta_4+\Theta_5+\Theta_1 = 2k\pi + \frac{1+\sigma}{2}\pi . \qquad (7.2.1)$$

Equations (7.10)-(7.12), respectively, reduce to

$$a_{51}c(\Theta_4+\Theta_5)+a_{45}c\Theta_4+a_{34}+a_{23}c\Theta_3+r_2s\Theta_3s\alpha_{23}-\sigma a_{12}c\Theta_3 = 0 \qquad (7.2.2)$$

$$a_{51}s(\Theta_4+\Theta_5)+a_{45}s\Theta_4-a_{23}s\Theta_3+r_2c\Theta_3s\alpha_{23}+\sigma a_{12}s\Theta_3 = 0 \qquad (7.2.3)$$

$$r_5 + r_4 + r_3 + r_2c\alpha_{23} + r_1 = 0. \qquad (7.2.4)$$

Again, there are four independent equations. The only possible proper linkages with mobility one here are $P\text{-}H\hat{-}H\hat{-}H\hat{-}H\text{-}$, $P\text{-}P\hat{-}H\hat{-}H\hat{-}H\text{-}$ and $P\text{-}H\hat{-}P\hat{-}H\hat{-}H\text{-}$. The first of them is a general parallel-screw linkage, and the other two are special cases.

7.3 Three adjacent axes only parallel

We choose

$$\alpha_{23} = \alpha_{34} = 0.$$

Equation (7.9) reduces to

$$c\theta_5 = \frac{c\alpha_{45}c\alpha_{51}-c\alpha_{12}}{s\alpha_{45}s\alpha_{51}},$$

establishing joint 5 as a slider.

Decreasing the indices in equation (7.9) by 1 results in an equation which reduces to

$$c\theta_1 = \frac{c\alpha_{51}c\alpha_{12}-c\alpha_{45}}{s\alpha_{51}s\alpha_{12}}.$$

As expected from symmetry, joint 1 is then also a slider.

Equations (7.1) and (7.2) become respectively

$$c\theta_1 c(\theta_2+\theta_3)-c\alpha_{12}s\theta_1 s(\theta_2+\theta_3)=c\theta_4 c\theta_5-c\alpha_{45}s\theta_4 s\theta_5 \qquad \text{(i)}$$

and

$$-c\theta_1 s(\theta_2+\theta_3)-c\alpha_{12}s\theta_1 c(\theta_2+\theta_3)=s\theta_4 c\theta_5+c\alpha_{45}c\theta_4 s\theta_5.$$

Squaring and adding these two equations results in an identity, after substituting for $c\theta_1$ and $c\theta_5$ from above. That is, the equations are dependent.

Equation (7.3) reduces to

$$s\theta_1 s\alpha_{12} = s\theta_5 s\alpha_{45},$$

which is compatible with the above results for $c\theta_1$ and $c\theta_5$.

Equations (7.4) and (7.5), respectively, reduce to

$$s\Theta_1 c(\Theta_2+\Theta_3)+c\alpha_{12}c\Theta_1 s(\Theta_2+\Theta_3)=c\alpha_{51}\{-c\Theta_4 s\Theta_5-c\alpha_{45}s\Theta_4 c\Theta_5\}$$
$$+s\alpha_{45}s\alpha_{51}s\Theta_4 \qquad \text{(ii)}$$

and

$$-s\Theta_1 s(\Theta_2+\Theta_3)+c\alpha_{12}c\Theta_1 c(\Theta_2+\Theta_3)=c\alpha_{51}\{-s\Theta_4 s\Theta_5+c\alpha_{45}c\Theta_4 c\Theta_5\}$$
$$-s\alpha_{45}s\alpha_{51}c\Theta_4.$$

Squaring and adding these equations, and substituting for $c\Theta_1$ and $c\Theta_5$, results, after some manipulation, in an identity. The equations are therefore dependent.

Equation (7.6) is identically satisfied by substitution for $c\Theta_1$ and $c\Theta_5$.

Equations (7.7) and (7.8) become, respectively,

$$s\alpha_{12}s(\Theta_2+\Theta_3)=s\alpha_{51}(c\Theta_4 s\Theta_5+c\alpha_{45}s\Theta_4 c\Theta_5)+s\alpha_{45}c\alpha_{51}s\Theta_4 \qquad \text{(iii)}$$

and

$$s\alpha_{12}c(\Theta_2+\Theta_3)=s\alpha_{51}(s\Theta_4 s\Theta_5-c\alpha_{45}c\Theta_4 c\Theta_5)-s\alpha_{45}c\alpha_{51}c\Theta_4. \qquad \text{(iv)}$$

Again, squaring and adding results in an identity, showing that equations (iii) and (iv) are dependent.

If we now substitute for $s(\Theta_2+\Theta_3)$ and $c(\Theta_2+\Theta_3)$ from (iii) and (iv) into equation (i) we find, after manipulation, that an identity results.
An identity is also the consequence of a similar substitution from (iii) and (iv) into equation (ii).

We therefore conclude that equations (7.1)-(7.8) reduce to only one independent closure equation. The most convenient form of this equation is obtained by subtracting (iv) × $c\theta_4$ from (iii) × $s\theta_4$ to produce

$$-s\alpha_{12}c(\theta_2+\theta_3+\theta_4) = c\alpha_{45}s\alpha_{51}c\theta_5+s\alpha_{45}c\alpha_{51}$$

$$= \frac{c\alpha_{45}}{s\alpha_{45}}(c\alpha_{45}c\alpha_{51}-c\alpha_{12})+s\alpha_{45}c\alpha_{51},$$

whence

$$c(\theta_2+\theta_3+\theta_4) = \frac{c\alpha_{45}c\alpha_{12}-c\alpha_{51}}{s\alpha_{45}s\alpha_{12}} . \qquad (7.3.1)$$

If, instead, we eliminate $c\theta_5$ from equations (iii) and (iv), we are led to the useful subsidiary relationship

$$\frac{s(\theta_2+\theta_3+\theta_4)}{s\alpha_{51}} = \frac{s\theta_5}{s\alpha_{12}} = \frac{s\theta_1}{s\alpha_{45}}.$$

Translational closure equations (7.10) and (7.11) become, respectively,

$$a_{51}(c\theta_4c\theta_5-s\theta_4s\theta_5c\alpha_{45})+r_5s\theta_4s\alpha_{45}+a_{45}c\theta_4+a_{34}+a_{23}c\theta_3$$

$$+a_{12}c(\theta_2+\theta_3)+r_1s\alpha_{12}s(\theta_2+\theta_3) = 0$$

and

$$a_{51}(s\theta_4c\theta_5+c\theta_4s\theta_5c\alpha_{45})-r_5c\theta_4s\alpha_{45}+a_{45}s\theta_4-a_{23}s\theta_3$$

$$-a_{12}s(\theta_2+\theta_3)+r_1s\alpha_{12}c(\theta_2+\theta_3) = 0 .$$

Elimination of $s\theta_5$ and $c\theta_5$ between these two equations results in the more convenient alternative forms

$$a_{51}c\theta_5+a_{45}+a_{34}c\theta_4+a_{23}c(\theta_3+\theta_4)+a_{12}c(\theta_2+\theta_3+\theta_4)$$

$$+r_1s\alpha_{12}s(\theta_2+\theta_3+\theta_4) = 0 \qquad (7.3.2)$$

and

$$a_{51}s\Theta_5 c\alpha_{45} - r_5 s\alpha_{45} - a_{34}s\Theta_4 - a_{23}s(\Theta_3+\Theta_4) - a_{12}s(\Theta_2+\Theta_3+\Theta_4)$$

$$+r_1 s\alpha_{12}c(\Theta_2+\Theta_3+\Theta_4) = 0. \qquad (7.3.3)$$

Equation (7.12) reduces to

$$a_{51}s\alpha_{45}s\Theta_5 + r_5 c\alpha_{45} + r_4 + r_3 + r_2 + r_1 c\alpha_{12} = 0. \qquad (7.3.4)$$

Because there remain only four independent closure equations, any linkage of mobility one must have connectivity sum five. If more than one of joints 2, 3, 4 are sliders, we have part-chain mobility and, by equation (7.3.1), all joints are sliders. Any one of 2, 3, 4 can be a slider to yield a special case of a parallel-screw linkage of form P-P$\hat{}$H$\hat{}$H-P- or P-H$\hat{}$P$\hat{}$H-P-. The only other possibility is a general parallel-screw linkage of the form P-H$\hat{}$H$\hat{}$H-P-.

7.4 One group of three and one group of two adjacent axes parallel

We may choose

$$\alpha_{12} = \alpha_{34} = \alpha_{45} = 0,$$

and $\alpha_{23}=\alpha_{51}$ or $2\pi-\alpha_{51}$ where $0<\alpha_{23}<2\pi$, $0<\alpha_{51}<\pi$.

Let us put

$$s\alpha_{23} = \sigma s\alpha_{51}.$$

Equation (7.9) is identically satisfied.

Equations (7.3) and (7.6), respectively, reduce to

$$s(\Theta_1+\Theta_2) = 0$$

and
$$c(\Theta_1+\Theta_2) = -\frac{s\alpha_{51}}{s\alpha_{23}} = -\sigma.$$

Together, these equations imply the result

$$\Theta_1 + \Theta_2 = 2k\pi + \frac{1+\sigma}{2}\pi . \tag{7.4.1}$$

Then, equations (7.1), (7.5) and (7.8) each reduce finally to

$$c\Theta_3 = -\sigma c(\Theta_4+\Theta_5).$$

Similarly, each of equations (7.2), (7.4) and (7.7) reduces ultimately to

$$s\Theta_3 = \sigma s(\Theta_4+\Theta_5).$$

These two results may be summarised as

$$\Theta_3 + \Theta_4 + \Theta_5 = 2l\pi + \frac{1+\sigma}{2}\pi . \tag{7.4.2}$$

By equation (7.4.1), neither of joints 1 and 2 may be a slider without the other also being prismatic, and so making the linkage improper through part-chain mobility.
By equation (7.4.2), precisely one of joints 3, 4 and 5 may be a slider.

The first two translational closure equations reduce to

$$a_{51}c(\Theta_4+\Theta_5)+a_{45}c\Theta_4+a_{34}+a_{23}c\Theta_3+r_2s\Theta_3s\alpha_{23}$$
$$+a_{12}(c\Theta_2c\Theta_3-s\Theta_2s\Theta_3c\alpha_{23})+r_1s\Theta_3s\alpha_{23} = 0$$

$$a_{51}s(\Theta_4+\Theta_5)+a_{45}s\Theta_4-a_{23}s\Theta_3+r_2c\Theta_3s\alpha_{23}$$
$$-a_{12}(c\Theta_2s\Theta_3+s\Theta_2c\Theta_3c\alpha_{23})+r_1c\Theta_3s\alpha_{23} = 0.$$

By eliminating either r_1 and r_2 or a_{23} and a_{51} between them, these results may be alternatively expressed as

$$a_{23}-\sigma a_{51}+a_{45}c(\Theta_3+\Theta_4)+a_{34}c\Theta_3+a_{12}c\Theta_2 = 0 \qquad (7.4.3)$$

$$r_1s\alpha_{23}+r_2s\alpha_{23}+a_{45}s(\Theta_3+\Theta_4)+a_{34}s\Theta_3-a_{12}s\Theta_2c\alpha_{23} = 0. \qquad (7.4.4)$$

Equation (7.12) simplifies to

$$r_1c\alpha_{23}+r_2c\alpha_{23}+r_3+r_4+r_5+a_{12}s\Theta_2s\alpha_{23} = 0. \qquad (7.4.5)$$

Since there are five independent closure equations, a linkage of mobility one governed by them must have joint connectivity sum six. The six eligible chains are C^H-H^H^H-, C^H-H^H^P-, C^H-H^P^H-, C^H-P^H^H-, H^H-H^H^C-, H^H-H^C^H-. All of these linkages have been mentioned by Hunt [27] and Waldron [42,45].

To complete this category, we must now investigate the possibility of mobile chains with connectivity sum five. There are three sub-categories, listed below as A, B, C. We may cover all contingencies by looking at, in turn, the cases where joint 3 is prismatic, joint 4 is prismatic and all five joints are screws. In the first two cases, the other four joints must be screws.

A

Let us assume that Θ_3 is fixed and consider the possibility of mobile $H\hat{}H\text{-}P\hat{}H\hat{}H\text{-}$ linkages. Elimination of Θ_4 from equations (7.10) and (7.11) yields, using (7.4.1) and (7.4.2),

$$a_{45}{}^2=a_{51}{}^2+a_{34}{}^2+a_{23}{}^2+\{R_2+R_1+(h_2-h_1)\Theta_2+h_1\pi(2k+\tfrac{1+\sigma}{2})\}^2s^2\alpha_{23}$$

$$+a_{12}{}^2\{c^2\Theta_2+s^2\Theta_2c^2\alpha_{23}\}-2\sigma a_{34}a_{51}c\Theta_3+2a_{23}a_{34}c\Theta_3$$

$$+2a_{34}\{R_2+R_1+(h_2-h_1)\Theta_2+h_1\pi(2k+\tfrac{1+\sigma}{2})\}s\Theta_3s\alpha_{23}$$

$$+2a_{12}a_{34}(c\Theta_2c\Theta_3-s\Theta_2s\Theta_3c\alpha_{23})-2\sigma a_{23}a_{51}-2\sigma a_{12}a_{51}c\Theta_2$$

$$+2a_{12}a_{23}c\Theta_2-2a_{12}\{R_2+R_1+(h_2-h_1)\Theta_2+h_1\pi(2k+\tfrac{1+\sigma}{2})\}s\Theta_2s\alpha_{23}c\alpha_{23}.$$

Taking only that portion of the equation which contains even powers of Θ_2, we have

$$a_{45}{}^2=a_{51}{}^2+a_{34}{}^2+a_{23}{}^2+\{R_2+R_1+h_1\pi(2k+\tfrac{1+\sigma}{2})\}^2s^2\alpha_{23}$$

$$+(h_2-h_1)^2\Theta_2{}^2s^2\alpha_{23}+a_{12}{}^2(1-s^2\Theta_2s^2\alpha_{23})-2\sigma a_{34}a_{51}c\Theta_3$$

$$+2a_{23}a_{34}c\Theta_3+2a_{34}\{R_2+R_1+h_1\pi(2k+\tfrac{1+\sigma}{2})\}s\Theta_3s\alpha_{23}$$

$$+2a_{12}a_{34}c\Theta_2c\Theta_3-2\sigma a_{23}a_{51}-2\sigma a_{12}a_{51}c\Theta_2+2a_{12}a_{23}c\Theta_2$$

$$-2a_{12}(h_2-h_1)\Theta_2s\Theta_2s\alpha_{23}c\alpha_{23}.$$

Now, equating the coefficients of some even powers of Θ_2 yields the following results.

$$\Theta_2^{\,2}:\quad 0 = (h_2-h_1)^2 s^2\alpha_{23} - a_{12}^{\,2} s^2\alpha_{23} - \frac{1}{2}\{2a_{12}a_{34}c\Theta_3 - 2\sigma a_{12}a_{51} + 2a_{12}a_{23}\}$$

$$-2a_{12}(h_2-h_1)s\alpha_{23}c\alpha_{23} \qquad \text{(i)}$$

$$\Theta_2^{\,4}:\quad 0 = \frac{a_{12}^{\,2}}{3}s^2\alpha_{23} + \frac{1}{4!}\{2a_{12}a_{34}c\Theta_3 - 2\sigma a_{12}a_{51} + 2a_{12}a_{23}\}$$

$$+2\frac{a_{12}}{3!}(h_2-h_1)s\alpha_{23}c\alpha_{23} \qquad \text{(ii)}$$

$$\Theta_2^{\,6}:\quad 0 = -\frac{2a_{12}^{\,2}}{45}s^2\alpha_{23} - \frac{1}{6!}\{2a_{12}a_{34}c\Theta_3 - 2\sigma a_{12}a_{51} + 2a_{12}a_{23}\}$$

$$-2\frac{a_{12}}{5!}(h_2-h_1)s\alpha_{23}c\alpha_{23} \qquad \text{(iii)}$$

$$\Theta_2^{\,8}:\quad 0 = \frac{a_{12}^{\,2}}{315}s^2\alpha_{23} + \frac{1}{8!}\{2a_{12}a_{34}c\Theta_3 - 2\sigma a_{12}a_{51} + 2a_{12}a_{23}\}$$

$$+2\frac{a_{12}}{7!}(h_2-h_1)s\alpha_{23}c\alpha_{23} \qquad \text{(iv)}$$

Elimination of the terms in braces between equations (ii) and (iii) leads to

$$6a_{12}^{\,2}s^2\alpha_{23} + a_{12}(h_2-h_1)s\alpha_{23}c\alpha_{23} = 0.$$

From (i), if $a_{12}=0$, $h_2=h_1$; then joints 1 and 2 would be coaxial screws of equal pitch, the linkage consequently possessing part-chain mobility. Hence, $a_{12}\neq 0$ and

$$6a_{12}s\alpha_{23} + (h_2-h_1)c\alpha_{23} = 0. \qquad \text{(v)}$$

Clearly, $h_2\neq h_1$ and $c\alpha_{23}\neq 0$, for otherwise we should have that $a_{12}=0$.

Elimination of the term in braces between (i) and (ii) results in

$(h_2-h_1)^2s^2\alpha_{23}+3a_{12}{}^2s^2\alpha_{23}=-2a_{12}(h_2-h_1)s\alpha_{23}c\alpha_{23}$.

By means of (v), this equation reduces to

$$t^2\alpha_{23} = \frac{1}{4},$$

whence $\qquad c^2\alpha_{23} = \frac{4}{5}$ and $s^2\alpha_{23} = \frac{1}{5}$.

Therefore, in equation (i), using (v),

$$\{2a_{12}a_{34}c\Theta_3-2\sigma a_{12}a_{51}+2a_{12}a_{23}\}=2(h_2-h_1)^2s^2\alpha_{23}-2a_{12}{}^2s^2\alpha_{23}$$
$$-4a_{12}(h_2-h_1)s\alpha_{23}c\alpha_{23}$$
$$= \frac{72}{5}a_{12}{}^2t^2\alpha_{23}-\frac{2}{5}a_{12}{}^2+24a_{12}{}^2s^2\alpha_{23}$$
$$= 8a_{12}^2.$$

Substituting this result into equation (iv), and again using (v), yields

$$0 = \frac{a_{12}^2}{1575} + \frac{a_{12}^2}{7!} - \frac{12a_{12}^2}{7!}s^2\alpha_{23}$$
$$= \frac{a_{12}^2}{5\times 7!}(16+5-12),$$

a contradiction.

So there is no linkage of the type H^H-P^H^H- with mobility one. Neither will there be any with a revolute in place of any screw, since such a solution would have appeared in the foregoing under the dimensional condition $h_i=0$, some i.

<u>B</u>

We next assume that Θ_4 is fixed, and look for mobile chains of the form H^H-H^P^H-. Differentiating equations (7.4.3) and (7.4.4) with respect to Θ_2, we obtain

$$\{a_{45}s(\Theta_3+\Theta_4)+a_{34}s\Theta_3\}\frac{d\Theta_3}{d\Theta_2}+a_{12}s\Theta_2 = 0$$

and

$$\{a_{45}c(\Theta_3+\Theta_4)+a_{34}c\Theta_3\}\frac{d\Theta_3}{d\Theta_2}+(h_2-h_1)s\alpha_{23}-a_{12}c\Theta_2c\alpha_{23} = 0.$$

Now, if $\{a_{45}s(\Theta_3+\Theta_4)+a_{34}s\Theta_3\} = 0$, from the first of these equations, $a_{12}=0$. Then, from (7.4.4), considering the coefficient of Θ_2, $(h_2-h_1)=0$. We should therefore have part-chain mobility.

Also, if $\{a_{45}c(\Theta_3+\Theta_4)+a_{34}c\Theta_3\}=0$, from the second of the equations, $(h_2-h_1)=0$ and, from equation (7.4.3), $a_{12}=0$. Part-chain mobility is again implied.

We may hence conclude that the coefficients of $\frac{d\Theta_3}{d\Theta_2}$ in the equations are not zero. Eliminating the derivative between the equations,

$$a_{12}s\Theta_2\{a_{45}c(\Theta_3+\Theta_4)+a_{34}c\Theta_3\}$$

$$=\{(h_2-h_1)s\alpha_{23}-a_{12}c\Theta_2c\alpha_{23}\}\{a_{45}s(\Theta_3+\Theta_4)+a_{34}s\Theta_3\}.$$

Using now equations (7.4.3) and (7.4.4),

$$a_{12}s\Theta_2\{\sigma a_{51}-a_{23}-a_{12}c\Theta_2\}$$

$$=\{(h_2-h_1)s\alpha_{23}-a_{12}c\Theta_2c\alpha_{23}\}\{a_{12}s\Theta_2c\alpha_{23}-s\alpha_{23}[R_2+R_1+(h_2-h_1)\Theta_2 +h_1\pi(2k+\frac{1+\sigma}{2})]\}.$$

We shall now equate coefficients of some odd powers of Θ_2.

$$\Theta_2{}^1:\quad a_{12}(\sigma a_{51}-a_{23})=a_{12}{}^2s^2\alpha_{23}+2a_{12}(h_2-h_1)s\alpha_{23}c\alpha_{23}-(h_2-h_1)^2s^2\alpha_{23} \quad \text{(i)}$$

$$\Theta_2{}^3:\quad -\frac{a_{12}}{3!}(\sigma a_{51}-a_{23})=-\frac{2}{3}a_{12}{}^2s^2\alpha_{23}-\frac{a_{12}}{6}(h_2-h_1)s\alpha_{23}c\alpha_{23} -\frac{a_{12}}{2}(h_2-h_1)s\alpha_{23}c\alpha_{23} \quad \text{(ii)}$$

$$\Theta_2{}^5:\quad \frac{a_{12}}{5!}(\sigma a_{51}-a_{23})=\frac{16}{5!}a_{12}{}^2s^2\alpha_{23}+\frac{a_{12}}{5!}(h_2-h_1)s\alpha_{23}c\alpha_{23}$$

$$+\frac{a_{12}}{4!}(h_2-h_1)s\alpha_{23}c\alpha_{23} \qquad \text{(iii)}$$

$$\Theta_2{}^7:\quad -\frac{a_{12}}{7!}(\sigma a_{51}-a_{23})=-\frac{64}{7!}a_{12}{}^2s^2\alpha_{23}-\frac{a_{12}}{7!}(h_2-h_1)s\alpha_{23}c\alpha_{23}$$

$$-\frac{a_{12}}{6!}(h_2-h_1)s\alpha_{23}c\alpha_{23} \qquad \text{(iv)}$$

From equation (i), if $a_{12}=0$, $h_2=h_1$. Since this would indicate part-chain mobility, we conclude that $a_{12}\neq 0$.
Eliminating the LHS of equations (ii) and (iii) between them,

$$6a_{12}{}^2s^2\alpha_{23}+a_{12}s\alpha_{23}c\alpha_{23}(h_2-h_1) = 0.$$

Since $a_{12}\neq 0$,

$$6a_{12}s\alpha_{23}+c\alpha_{23}(h_2-h_1) = 0. \qquad \text{(v)}$$

We conclude that $c\alpha_{23}\neq 0$ and $h_2\neq h_1$ since, otherwise, a_{12} would be zero.
Eliminating the LHS of equations (i) and (ii) between them,

$$3a_{12}{}^2s^2\alpha_{23}+(h_2-h_1)^2s^2\alpha_{23}=-2a_{12}s\alpha_{23}c\alpha_{23}(h_2-h_1).$$

Using equation (v), this reduces to

$$t^2\alpha_{23} = \frac{1}{4},$$

whence $$c^2\alpha_{23} = \frac{4}{5} \quad \text{and} \quad s^2\alpha_{23} = \frac{1}{5},$$

as in part A. Hence, in (i), using (v),

$$a_{12}(\sigma a_{51}-a_{23})=\frac{a_{12}{}^2}{5}-2a_{12}s\alpha_{23}\times 6a_{12}s\alpha_{23}-\frac{1}{5}\times 36\ \frac{a_{12}{}^2}{4}$$

$$= -\ 4a_{12}{}^2.$$

In (iv) then, again using (v),

$$0=-\frac{4}{7!}a_{12}^{\ 2}-\frac{64}{5\times 7!}a_{12}^{\ 2}+\frac{a_{12}}{7!}s\alpha_{23}\times 6a_{12}s\alpha_{23}+\frac{a_{12}}{6!}s\alpha_{23}\times 6a_{12}s\alpha_{23}$$

$$= -\frac{36}{5\times 7!}a_{12}^{\ 2},$$

a contradiction.

Hence, there is no linkage with mobility unity of the type H^H-H^P^H-. For the same reason as given in part A, neither are there any solutions with revolutes in place of screws.

C

We consider finally possible mobile linkages of the form H^H-H^H^H-. We may eliminate Θ_1 and Θ_5 among the closure equations (7.4.1)-(7.4.5) to rewrite the last three in the form below.

$$a_{23}-\sigma a_{51}+a_{45}c(\Theta_3+\Theta_4)+a_{34}c\Theta_3+a_{12}c\Theta_2 = 0 \tag{i}$$

$$a_{34}s\Theta_3+a_{45}s(\Theta_3+\Theta_4)+s\alpha_{23}\{R_1+R_2+h_1\pi(2k+\frac{1+\sigma}{2})+(h_2-h_1)\Theta_2\}$$
$$-a_{12}s\Theta_2c\alpha_{23} = 0 \tag{ii}$$

$$R_3+R_4+R_5+h_5\pi(2l+\frac{1+\sigma}{2})+(h_3-h_4)\Theta_3+(h_4-h_5)(\Theta_3+\Theta_4)$$
$$+c\alpha_{23}\{R_1+R_2+h_1\pi(2k+\frac{1+\sigma}{2})+(h_2-h_1)\Theta_2\}+a_{12}s\Theta_2s\alpha_{23} = 0 \tag{iii}$$

Now there is a solution for which joints 1 and 2 are coaxial and the pitches of the other three screws are equal, given by H=H-H^H^H-. This linkage is kinematically equivalent to Delassus four-bar d.2. In terms of the above closure equations, (iii) is reduced to a mere relationship among linkage constants by putting

$$h_3 = h_4 = h_5 \qquad c\alpha_{23} = a_{12} = 0.$$

We may, and must, exclude this solution from the following discussion.

In accordance with the theorem of section 4.3, let us suppose temporarily that $h_2=h_1$ and $h_3=h_4=h_5$. Equation (iii) becomes

$$R_3+R_4+R_5+h_5\pi(2l+\frac{1+\sigma}{2})+c\alpha_{23}\{R_1+R_2+h_1\pi(2k+\frac{1+\sigma}{2})\}$$
$$+a_{12}s\Theta_2 s\alpha_{23} = 0.$$

By considering the coefficient of $s\Theta_2$, it is clear that $a_{12}=0$. Under the present conditions relating the screw pitches, this would imply part-chain mobility, and hence no solution.

We may now revert to the general case of unrelated screw pitches. In so doing, to ensure mobility, we must retain the dimensional constraint just determined. That is,

$$a_{12} = 0.$$

But then joints 1 and 2 will be kinematically equivalent to a cylindric pair and the chain may be regarded as a C-H$\hat{}$H$\hat{}$H- four-bar. From the results of chapter 5, however, we know that there is no such linkage with mobility unity.

Hence, apart from the H=H-H$\hat{}$H$\hat{}$H- linkage, there is no solution of the form H$\hat{}$H-H$\hat{}$H$\hat{}$H-, and consequently no other five-bar of the form J$\hat{}$J-J$\hat{}$J$\hat{}$J- with connectivity sum five and mobility one.

7.5 Two pairs of adjacent axes parallel

We take

$$\alpha_{12} = \alpha_{45} = 0.$$

Equations (7.6), (7.8) and (7.9) may be reduced, respectively, to the following three results.

$$c(\theta_1+\theta_2) = \frac{c\alpha_{51}c\alpha_{23}-c\alpha_{34}}{s\alpha_{51}s\alpha_{23}} \tag{7.5.1}$$

$$c\theta_3 = \frac{c\alpha_{23}c\alpha_{34}-c\alpha_{51}}{s\alpha_{23}s\alpha_{34}} \tag{7.5.a}$$

$$c(\theta_4+\theta_5) = \frac{c\alpha_{34}c\alpha_{51}-c\alpha_{23}}{s\alpha_{34}s\alpha_{51}} \tag{7.5.2}$$

Hence, joint 3 is prismatic, and we may think of relations (7.5.1) and (7.5.2) as implying that

$$\theta_1 + \theta_2 = \text{constant}$$

and

$$\theta_4 + \theta_5 = \text{constant}.$$

Closure equations (7.3) and (7.7) lead to the subsidiary result that

$$\frac{s(\theta_1+\theta_2)}{s\alpha_{34}} = \frac{s\theta_3}{s\alpha_{51}} = \frac{s(\theta_4+\theta_5)}{s\alpha_{23}} . \tag{7.5.b}$$

The remaining rotational closure equations, (7.1), (7.2), (7.4) and (7.5), are satisfied by the above results. The translational closure equations (7.10)-(7.12) reduce, respectively, to the following three equations.

$$a_{51}c(\Theta_4+\Theta_5)+a_{45}c\Theta_4+a_{34}+a_{23}c\Theta_3+a_{12}(c\Theta_2c\Theta_3-s\Theta_2s\Theta_3c\alpha_{23})$$

$$+r_2s\Theta_3s\alpha_{23}+r_1s\Theta_3s\alpha_{23} = 0 \qquad (7.5.3)$$

$$a_{51}s(\Theta_4+\Theta_5)c\alpha_{34}+a_{45}s\Theta_4c\alpha_{34}-a_{23}s\Theta_3-a_{12}(c\Theta_2s\Theta_3+s\Theta_2c\Theta_3c\alpha_{23})$$

$$-r_5s\alpha_{34}-r_4s\alpha_{34}+r_2c\Theta_3s\alpha_{23}+r_1c\Theta_3s\alpha_{23} = 0$$

$$(7.5.4)$$

$$a_{51}s(\Theta_4+\Theta_5)s\alpha_{34}+a_{45}s\Theta_4s\alpha_{34}+a_{12}s\Theta_2s\alpha_{23}$$

$$+r_5c\alpha_{34}+r_4c\alpha_{34}+r_3+r_2c\alpha_{23}+r_1c\alpha_{23} = 0 \qquad (7.5.5)$$

Since there are at most five independent closure equations, a chain in this category with mobility one must have a connectivity sum of six or five. Now, none of joints 4, 5, 1 and 2 can be prismatic for, by equation (7.5.1) or (7.5.2), the joint parallel to it would also be locked in rotation. This would result in part-chain mobility.

The only solutions of connectivity sum six are C^H-P-H^H- and H^C-P-H^H-. Both of these linkages are derivatives of the six-bar parallel-screw linkages discovered by Waldron [42,45] and Hunt [27].

To find solutions of connectivity sum five, we need examine only those chains in which all joints, except 3, are screws. We must determine the conditions under which equations (7.5.1)-(7.5.5) may be reduced to four independent equations. We note that (7.5.5) is the only equation which contains r_3; it would be sensible, then, to concentrate on the other four equations. We shall first consider, in a direct way, the possibilities when $s\Theta_3=0$. We shall then investigate the other cases by means of the method of tying of screw pitches.

$\underline{s\Theta_3 = 0}$

From (7.5.b), we have

$$s(\Theta_1+\Theta_2) = s(\Theta_4+\Theta_5) = 0.$$

If we put $$c\Theta_3 = \sigma$$

and $$c(\Theta_1+\Theta_2) = \tau,$$

then $$c(\Theta_4+\Theta_5) = \sigma\tau.$$

Hence, we may write

$$\Theta_3 = \frac{1-\sigma}{2}\pi \qquad (7.5.a')$$

$$\Theta_1 + \Theta_2 = \frac{1-\tau}{2}\pi + 2m\pi \qquad (7.5.1')$$

$$\Theta_4 + \Theta_5 = \frac{1-\sigma\tau}{2}\pi + 2n\pi \qquad (7.5.2')$$

We also have, for example, from (7.5) and (7.6),

$$\left.\begin{aligned} c\alpha_{23} &= c\alpha_{34}c\alpha_{51}-\sigma\tau s\alpha_{34}s\alpha_{51} \\ -\tau s\alpha_{23} &= c\alpha_{34}s\alpha_{51}+\sigma\tau s\alpha_{34}c\alpha_{51} \end{aligned}\right\}, \qquad (7.5.i')$$

whence we conclude that

$$\alpha_{23} +\sigma\alpha_{34}+\tau\alpha_{51} = 2M\pi, \quad M=-1,\ldots,2. \qquad (7.5.c')$$

Equations (7.5.3) and (7.5.4) simplify to

$$\sigma\tau a_{51}+a_{45}c\Theta_4+a_{34}+\sigma a_{23}+\sigma a_{12}c\Theta_2 = 0 \qquad (7.5.3')$$

$$a_{45}s\Theta_4c\alpha_{34}-\sigma a_{12}s\Theta_2c\alpha_{23}-(R_5+h_5\Theta_5)s\alpha_{34}-(R_4+h_4\Theta_4)s\alpha_{34}$$

$$+\sigma(R_2+h_2\Theta_2)s\alpha_{23}+\sigma(R_1+h_1\Theta_1)s\alpha_{23} = 0. \qquad (7.5.4')$$

From (7.5.3'), if either a_{12} or a_{45} is zero, then so is the other, in order that locking of joint 4 or joint 2 is not implied. We shall investigate this case and the alternative separately, in parts A and B below.

A

We assume that

$$a_{12} = a_{45} = 0,$$

which results in the number of independent closure equations being reduced to four, along with the dimensional constraint

$$a_{23} + \sigma a_{34} + \tau a_{51} = 0.$$

The linkage indicated has mobility one and consists of two pairs of coaxial screws and a slider, all in parallel planes. It may be produced as a hybrid from Delassus's four-bar d.8 and three-bar H=H-P-, as shown in section 4.1; it may alternatively be seen (section 4.2) as the result of a -P- replacement in d.8.

Equations (7.5.4) and (7.5.5) simplify to

$$\sigma s\alpha_{23}(R_1+h_1\Theta_1+R_2+h_2\Theta_2)=s\alpha_{34}(R_4+h_4\Theta_4+R_5+h_5\Theta_5)$$

$$c\alpha_{23}(R_1+h_1\Theta_1+R_2+h_2\Theta_2)+c\alpha_{34}(R_4+h_4\Theta_4+R_5+h_5\Theta_5)+r_3 = 0.$$

From these last two closure equations, if

$$\alpha_{23} + \sigma\alpha_{34} = 2N\pi + \frac{1+\rho}{2}\pi \quad , \quad N=-1,0 \text{ or } 1,$$

joint 3 becomes locked and the linkage degenerates to Delassus's four-bar d.7. But then, of course, $s\alpha_{51}=0$, so that the chain does not belong in the category currently being investigated.

B

$$a_{12} \neq 0 \qquad a_{45} \neq 0$$

Using (7.5.1') and (7.5.2') to eliminate Θ_1 and Θ_5 from (7.5.4') results in

$$\sigma s\alpha_{23}(R_2+R_1+h_1[2m+\frac{1-\tau}{2}]\pi+[h_2-h_1]\Theta_2)-\sigma a_{12}c\alpha_{23}s\Theta_2$$

$$=s\alpha_{34}(R_4+R_5+h_5[2n+\frac{1-\sigma\tau}{2}]\pi+[h_4-h_5]\Theta_4)-a_{45}c\alpha_{34}s\Theta_4. \qquad \text{(i)}$$

Differentiating w.r.t. Θ_2,

$$\sigma s\alpha_{23}[h_2-h_1]-\sigma a_{12}c\alpha_{23}c\Theta_2=\frac{d\Theta_4}{d\Theta_2}(s\alpha_{34}[h_4-h_5]-a_{45}c\alpha_{34}c\Theta_4). \qquad \text{(ii)}$$

But differentiation of (7.5.3') yields, since $a_{45} \neq 0$,

$$\frac{d\Theta_4}{d\Theta_2} = -\frac{\sigma a_{12}}{a_{45}}\frac{s\Theta_2}{s\Theta_4}.$$

Substitution of this result into (ii) and subsequent squaring leads to

$$a_{45}{}^2 s^2\Theta_4(s\alpha_{23}[h_2-h_1]-a_{12}c\alpha_{23}c\Theta_2)^2$$

$$=a_{12}{}^2 s^2\Theta_2(s\alpha_{34}[h_4-h_5]-a_{45}c\alpha_{34}c\Theta_4)^2. \qquad \text{(iii)}$$

But equation (7.5.3') may be rewritten as

$$-c\Theta_4 = \frac{1}{a_{45}}(\sigma a_{12}c\Theta_2+a_{34}+\sigma a_{23}+\sigma\tau a_{51}).$$

Using this result in (iii), we have

$$\{a_{45}{}^2-[a_{23}+\sigma a_{34}+\tau a_{51}]^2-a_{12}{}^2c^2\Theta_2-2a_{12}[a_{23}+\sigma a_{34}+\tau a_{51}]c\Theta_2\}$$

$$\times\{s^2\alpha_{23}[h_2-h_1]^2+a_{12}{}^2c^2\alpha_{23}c^2\Theta_2-2a_{12}s\alpha_{23}c\alpha_{23}[h_2-h_1]c\Theta_2\}$$

$$=a_{12}{}^2(1-c^2\Theta_2)$$

$$\times\{(s\alpha_{34}[h_4-h_5]+\sigma c\alpha_{34}[a_{23}+\sigma a_{34}+\tau a_{51}])^2+a_{12}{}^2c^2\alpha_{34}c^2\Theta_2$$

$$+2\sigma a_{12}c\alpha_{34}(s\alpha_{34}[h_4-h_5]+\sigma c\alpha_{34}[a_{23}+\sigma a_{34}+\tau a_{51}])c\Theta_2\}. \qquad \text{(iv)}$$

Since this is an equation in powers of $c\Theta_2$ alone, we may equate coefficients to zero.

$c^4\Theta_2$: $$-a_{12}{}^4c^2\alpha_{23} = -\,a_{12}{}^4c^2\alpha_{34}$$

Then, since $a_{12}\neq 0$,

$$c^2\alpha_{23} = c^2\alpha_{34}.$$

$c^3\Theta_2$: $$2a_{12}{}^3s\alpha_{23}c\alpha_{23}[h_2-h_1]-2a_{12}{}^3c^2\alpha_{23}[a_{23}+\sigma a_{34}+\tau a_{51}]$$

$$=-2\sigma a_{12}{}^3c\alpha_{34}(s\alpha_{34}[h_4-h_5]+\sigma c\alpha_{34}[a_{23}+\sigma a_{34}+\tau a_{51}])$$

$$\therefore \quad s\alpha_{23}c\alpha_{23}[h_2-h_1]=\sigma s\alpha_{34}c\alpha_{34}[h_5-h_4]$$

Now, since $$c^2\alpha_{23} = c^2\alpha_{34},$$

then $$s^2\alpha_{23} = s^2\alpha_{34}.$$

But, by allowing $0<\alpha_{51}<2\pi$, we may constrain α_{23} and α_{34} to be both less than π. Hence,

$$s\alpha_{23} = s\alpha_{34}. \qquad \text{(v)}$$

$$\therefore \quad c\alpha_{23}[h_2-h_1] = \sigma c\alpha_{34}[h_5-h_4] \qquad \text{(vi)}$$

$$c^2\Theta_2:\ -a_{12}{}^2s^2\alpha_{23}[h_2-h_1]^2+a_{12}{}^2c^2\alpha_{23}(a_{45}{}^2-[a_{23}+\sigma a_{34}+\tau a_{51}]^2)$$

$$+4a_{12}{}^2s\alpha_{23}c\alpha_{23}[h_2-h_1][a_{23}+\sigma a_{34}+\tau a_{51}]$$

$$=-a_{12}{}^2(s\alpha_{34}[h_4-h_5]+\sigma c\alpha_{34}[a_{23}+\sigma a_{34}+\tau a_{51}])^2+a_{12}{}^4c^2\alpha_{34}$$

$$\therefore\ -s^2\alpha_{23}[h_2-h_1]^2+a_{45}{}^2c^2\alpha_{23}$$

$$+4s\alpha_{23}c\alpha_{23}[h_2-h_1][a_{23}+\sigma a_{34}+\tau a_{51}]$$

$$=a_{12}{}^2c^2\alpha_{34}-s^2\alpha_{34}[h_4-h_5]^2-2\sigma s\alpha_{34}c\alpha_{34}[h_4-h_5][a_{23}+\sigma a_{34}+\tau a_{51}]$$

Using results (v) and (vi), this equation reduces to

$$a_{45}{}^2c^2\alpha_{23}-s^2\alpha_{23}[h_2-h_1]^2+2s\alpha_{23}c\alpha_{23}[h_2-h_1][a_{23}+\sigma a_{34}+\tau a_{51}]$$

$$=a_{12}{}^2c^2\alpha_{34}-s^2\alpha_{34}[h_4-h_5]^2. \qquad \text{(vii)}$$

Now, from (vi),

$$c^2\alpha_{23}[h_2-h_1]^2=c^2\alpha_{34}[h_5-h_4]^2.$$

We cannot have $c^2\alpha_{23}=c^2\alpha_{34}=0$ since, by (7.5.c'), we should then have that $s\alpha_{51}=0$. Such a result is disallowed in the present category. Hence,

$$[h_2-h_1]^2 = [h_5-h_4]^2. \qquad \text{(viii)}$$

Therefore, in (vii),

$$a_{45}{}^2c^2\alpha_{23}+2s\alpha_{23}c\alpha_{23}[h_2-h_1][a_{23}+\sigma a_{34}+\tau a_{51}] = a_{12}{}^2c^2\alpha_{34}. \qquad \text{(ix)}$$

$$c^1\Theta_2:\ -2a_{12}s^2\alpha_{23}[h_2-h_1][a_{23}+\sigma a_{34}+\tau a_{51}]$$

$$-2a_{12}s\alpha_{23}c\alpha_{23}[h_2-h_1](a_{45}{}^2-[a_{23}+\sigma a_{34}+\tau a_{51}]^2)$$

$$= 2\sigma a_{12}{}^3c\alpha_{34}(s\alpha_{34}[h_4-h_5]+\sigma c\alpha_{34}[a_{23}+\sigma a_{34}+\tau a_{51}])$$

Using (ix), (v) and (vi), this result may be re-expressed as

$$[a_{23}+\sigma a_{34}+\tau a_{51}]^2 s\alpha_{23}c\alpha_{23}[h_2-h_1]$$

$$=[a_{23}+\sigma a_{34}+\tau a_{51}](a_{12}{}^2c^2\alpha_{23}-s^2\alpha_{23}[h_2-h_1]^2). \qquad \text{(x)}$$

$c^0\theta_2$: $s^2\alpha_{23}[h_2-h_1]^2(a_{45}{}^2-[a_{23}+\sigma a_{34}+\tau a_{51}]^2)$

$$=a_{12}{}^2(s^2\alpha_{23}[h_4-h_5]^2+2\sigma s\alpha_{34}c\alpha_{34}[h_4-h_5][a_{23}+\sigma a_{34}+\tau a_{51}]$$
$$+c^2\alpha_{34}[a_{23}+\sigma a_{34}+\tau a_{51}]^2)$$

Using (ix), (v), (vi) and (viii), this equation may be simplified to

$$[a_{23}+\sigma a_{34}+\tau a_{51}]^2(a_{12}{}^2c^2\alpha_{23}+s^2\alpha_{23}[h_2-h_1]^2)$$

$$=[a_{23}+\sigma a_{34}+\tau a_{51}](2\frac{s^3\alpha_{23}}{c\alpha_{23}}[h_1-h_2]^2-2a_{12}{}^2s\alpha_{23}c\alpha_{23})[h_1-h_2]. \qquad \text{(xi)}$$

From (x) or (xi), if $h_1=h_2$, $[a_{23}+\sigma a_{34}+\tau a_{51}]=0$. Let us assume that $[a_{23}+\sigma a_{34}+\tau a_{51}]\neq 0$ (and therefore that $h_1\neq h_2$). Then, from (x) and (xi) respectively,

$$a_{23}+\sigma a_{34}+\tau a_{51} = \frac{a_{12}{}^2c^2\alpha_{23}-[h_2-h_1]^2s^2\alpha_{23}}{[h_2-h_1]s\alpha_{23}c\alpha_{23}}$$

$$a_{23}+\sigma a_{34}+\tau a_{51} = \frac{2([h_1-h_2]^2\,\dfrac{s^3\alpha_{23}}{c\alpha_{23}}-a_{12}{}^2s\alpha_{23}c\alpha_{23})[h_1-h_2]}{a_{12}{}^2c^2\alpha_{23}+[h_2-h_1]^2s^2\alpha_{23}}.$$

Hence, by equating the two results,

$$2[h_2-h_1]^2s^2\alpha_{23}(a_{12}{}^2c^2\alpha_{23}-[h_2-h_1]^2s^2\alpha_{23})=a_{12}{}^4c^4\alpha_{23}-[h_2-h_1]^4s^4\alpha_{23}$$

whence, after transposing terms,

$$(a_{12}{}^2c^2\alpha_{23}-[h_2-h_1]^2s^2\alpha_{23})^2 = 0.$$

$$\therefore \quad a_{12}{}^2c^2\alpha_{23} = [h_2-h_1]^2s^2\alpha_{23}$$

Then, in (x) say,

$$[a_{23}+\sigma a_{34}+\tau a_{51}]^2 s\alpha_{23} c\alpha_{23}[h_2-h_1] = 0,$$

a contradiction. Therefore, we conclude that

$$a_{23} + \sigma a_{34} + \tau a_{51} = 0. \qquad \text{(xii)}$$

Hence, from result (ix),

$$a_{12} = a_{45}. \qquad \text{(xiii)}$$

Now reconsider equations (7.5.i'), using (v). From the first,

$$s\alpha_{23} = \frac{c\alpha_{34}c\alpha_{51}-c\alpha_{23}}{\sigma\tau s\alpha_{51}}, \quad \text{since} \quad s\alpha_{51} \neq 0.$$

In the second, then,

$$\frac{c\alpha_{23}-c\alpha_{34}c\alpha_{51}}{\sigma s\alpha_{51}} = c\alpha_{34}s\alpha_{51} + c\alpha_{51}\left(\frac{c\alpha_{34}c\alpha_{51}-c\alpha_{23}}{s\alpha_{51}}\right).$$

$$\therefore \quad \sigma(c\alpha_{23}-c\alpha_{34}c\alpha_{51})=c\alpha_{34}s^2\alpha_{51}+c\alpha_{34}c^2\alpha_{51}-c\alpha_{23}c\alpha_{51}$$

Transposing terms,

$$c\alpha_{51}(c\alpha_{23}-\sigma c\alpha_{34}) = -\sigma(c\alpha_{23}-\sigma c\alpha_{34}).$$

If $c\alpha_{23}\neq\sigma c\alpha_{34}$, then $c\alpha_{51}=-\sigma$. But then $s\alpha_{51}=0$, which is disallowed. Therefore,

$$c\alpha_{23} = \sigma c\alpha_{34}. \qquad \text{(xiv)}$$

Hence, from result (vi),

$$h_1 - h_2 = -(h_5-h_4). \qquad \text{(xv)}$$

We may now check on the sufficiency of the above results, (v) and (xii)-(xv), by substituting them into equations (7.5.3')

and (i), which we require to be equivalent.
Equation (7.5.3') reduces to

$$c\Theta_4 + \sigma c\Theta_2 = 0,$$

whence
$$\Theta_4 = \rho\Theta_2 + 2k\pi + \frac{1+\sigma}{2}\pi,$$

and (i) consequently becomes

$$\sigma s\alpha_{23}(R_2+R_1+h_1[2m+\frac{1-\tau}{2}]\pi+[h_2-h_1]\Theta_2)-\sigma a_{12}c\alpha_{23}s\Theta_2$$

$$=s\alpha_{23}(R_4+R_5+h_5[2n+\frac{1-\sigma\tau}{2}]\pi+[h_1-h_2][\rho\Theta_2+2k\pi+\frac{1+\sigma}{2}\pi])$$

$$-a_{12}\sigma c\alpha_{23}\rho s\Theta_2(-\sigma).$$

That is,

$$s\alpha_{23}(R_4+R_5-\sigma[R_1+R_2]+h_5[2n+\frac{1-\sigma\tau}{2}]\pi-\sigma h_1[2m+\frac{1-\tau}{2}]\pi$$

$$+[h_1-h_2][2k+\frac{1+\sigma}{2}]\pi)+s\alpha_{23}[h_1-h_2]\Theta_2(\rho+\sigma)$$

$$+c\alpha_{23}a_{12}s\Theta_2(\rho+\sigma) = 0.$$

For this equation to be an identity in Θ_2, we must have

$$\rho = -\sigma.$$

Hence, one of the independent closure equations will be

$$\Theta_4 + \sigma\Theta_2 = 2k\pi + \frac{1+\sigma}{2}\pi, \qquad \text{(xvi)}$$

along with the dimensional condition

$$R_4+R_5+h_5[2n+\frac{1-\sigma\tau}{2}]\pi-\sigma(R_2+R_1+h_1[2m+\frac{1-\tau}{2}]\pi)$$

$$= [h_2-h_1][2k + \frac{1+\sigma}{2}]\pi. \qquad \text{(xvii)}$$

Let us now reconsider closure equation (7.5.5) which, using results (7.5.2'), (v), (xiii), (xiv) and (xvi), may be written as

$$a_{12}s\alpha_{23}s\Theta_2+a_{12}s\alpha_{23}s\Theta_2+(R_5+h_5\Theta_5)\sigma c\alpha_{23}+(R_4+h_4\Theta_4)\sigma c\alpha_{23}$$

$$+r_3+(R_2+h_2\Theta_2)c\alpha_{23}+(R_1+h_1\Theta_1)c\alpha_{23} = 0.$$

Using (7.5.1') and (7.5.2') to eliminate Θ_5 and Θ_1, the last equation becomes

$$r_3+2a_{12}s\alpha_{23}s\Theta_2+c\alpha_{23}(R_2+R_1+h_1[2m+\tfrac{1-\tau}{2}]\pi+[h_2-h_1]\Theta_2)$$

$$+\sigma c\alpha_{23}(R_5+R_4+h_5[2n+\tfrac{1-\sigma\tau}{2}]\pi+[h_4-h_5]\Theta_4) = 0.$$

We again use result (xvi) with (xv) to convert this last equation to the form

$$r_3+2a_{12}s\alpha_{23}s\Theta_2+c\alpha_{23}(R_2+R_1+h_1[2m+\tfrac{1-\tau}{2}]\pi)+c\alpha_{23}[h_2-h_1]\Theta_2$$

$$+\sigma c\alpha_{23}(R_5+R_4+h_5[2n+\tfrac{1-\sigma\tau}{2}]\pi)+\sigma c\alpha_{23}[h_1-h_2]([2k+\tfrac{1+\sigma}{2}]\pi-\sigma\Theta_2)$$

$$= 0.$$

Using now condition (xvii), the last equation reduces to

$$r_3+2a_{12}s\alpha_{23}s\Theta_2+2[h_2-h_1]c\alpha_{23}\Theta_2+2(R_2+R_1+h_1[2m+\tfrac{1-\tau}{2}]\pi)c\alpha_{23}$$

$$= 0, \qquad \text{(xviii)}$$

which is independent of σ. The final set of four independent closure equations is then (7.5.1'), (7.5.2'), (xvi) and (xviii). If we summarise results (v) and (xiv) by the equation

$$\alpha_{34} = \sigma\alpha_{23} + \frac{1-\sigma}{2}\pi, \qquad \text{(xix)}$$

then (7.5.c') simplifies to

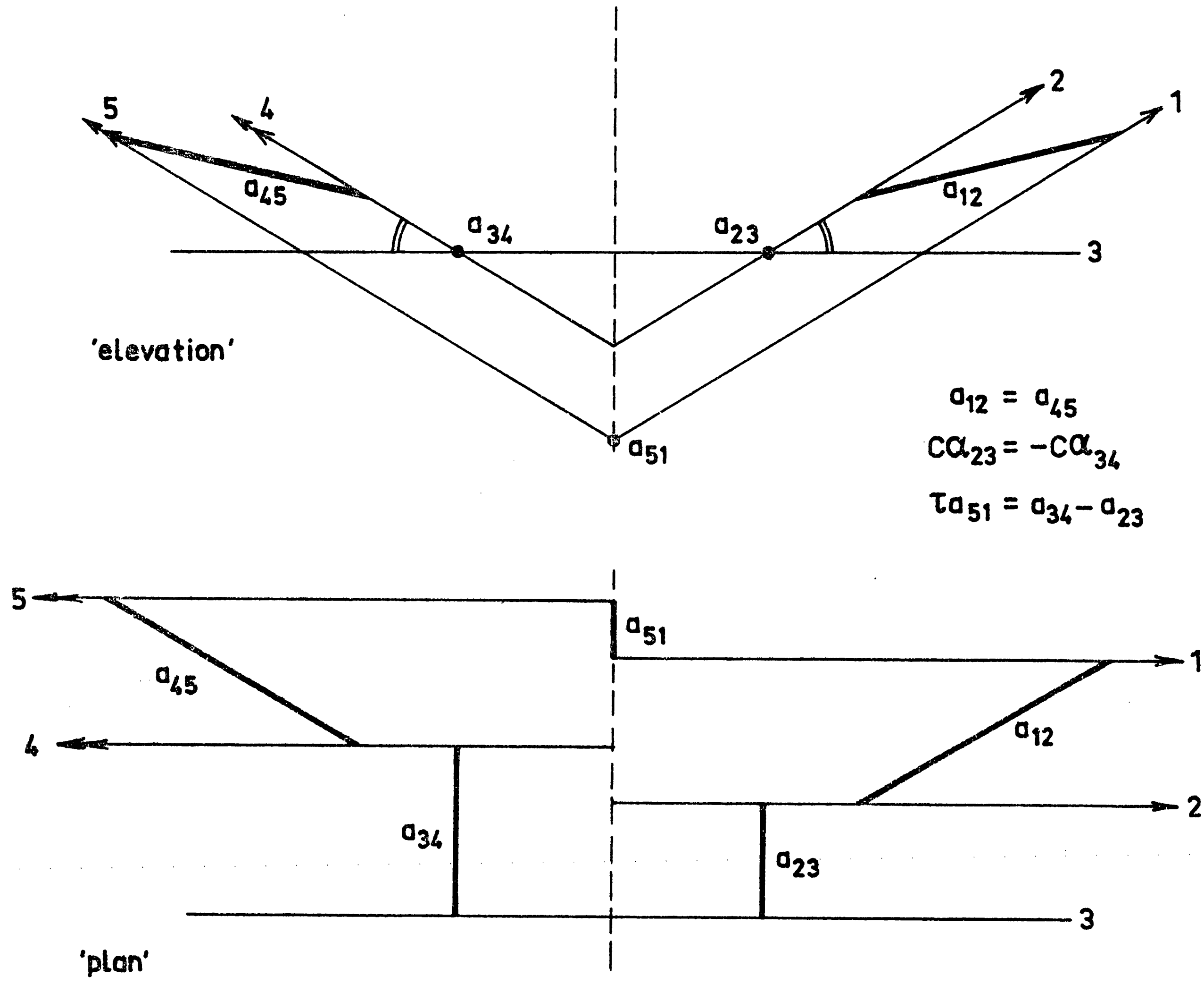


Fig. 7.5.1

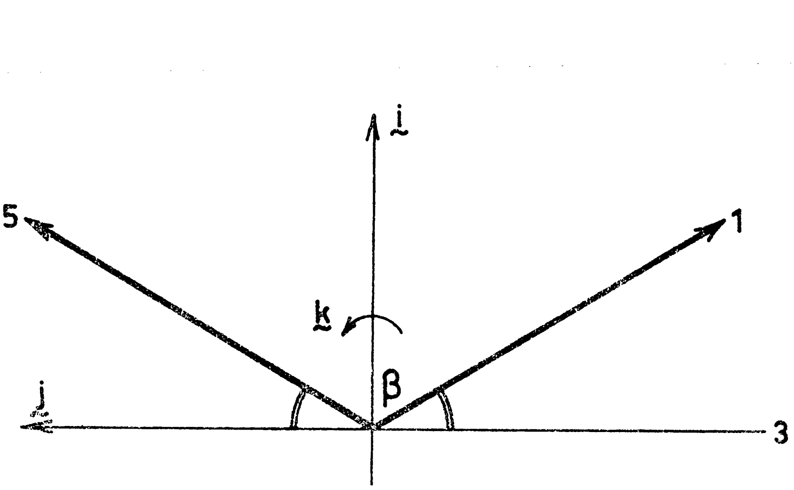


Fig. 7.5.3

$$2\alpha_{23} + \tau\alpha_{51} = (2M + \frac{1-\sigma}{2})\pi. \qquad \text{(xx)}$$

The dimensional constraints for this case are now given by equations (7.5.a'), (xii), (xiii), (xv), (xvii), (xix) and (xx). For convenience, we list them together below.

$$\theta_3 = \frac{1-\sigma}{2}\pi$$

$$a_{23} + \sigma a_{34} \pm \tau a_{51} = 0 \qquad a_{12} = a_{45}$$

$$h_1 - h_2 = -(h_5 - h_4)$$

$$R_4 + R_5 - \sigma(R_1 + R_2)$$

$$= -h_5[2n + \frac{1-\sigma\tau}{2}]\pi + \sigma h_1[2m + \frac{1-\tau}{2}]\pi + [h_2 - h_1][2k + \frac{1+\sigma}{2}]\pi$$

$$2(\sigma\alpha_{34} + \frac{1-\sigma}{2}\pi) = 2\alpha_{23} = -\tau\alpha_{51} + (2M + \frac{1-\sigma}{2})\pi$$

We note here, in particular, the mutual independence of the various dimensional conditions; for example, prescribing stronger constraints on the normal link-lengths within (xii) makes no further demands on the screw pitches under (xv). We also see from result (xii) that, if $\sigma=+1$, we must have $\tau=-1$, in order that a_{51} be positive.

Recalling the results

$$s(\theta_1+\theta_2) = s\theta_3 = s(\theta_4+\theta_5) = 0,$$

it is clear that all five joint axes lie in parallel planes. The two varieties of the solution are depicted, with the accompanying relationships, in Figs. 7.5.1 and 7.5.2.
The pairs of parallel screws are symmetrically disposed, in the lateral sense, from a plane perpendicular to the slider. It is now evident that Waldron's plane-symmetric five-bar

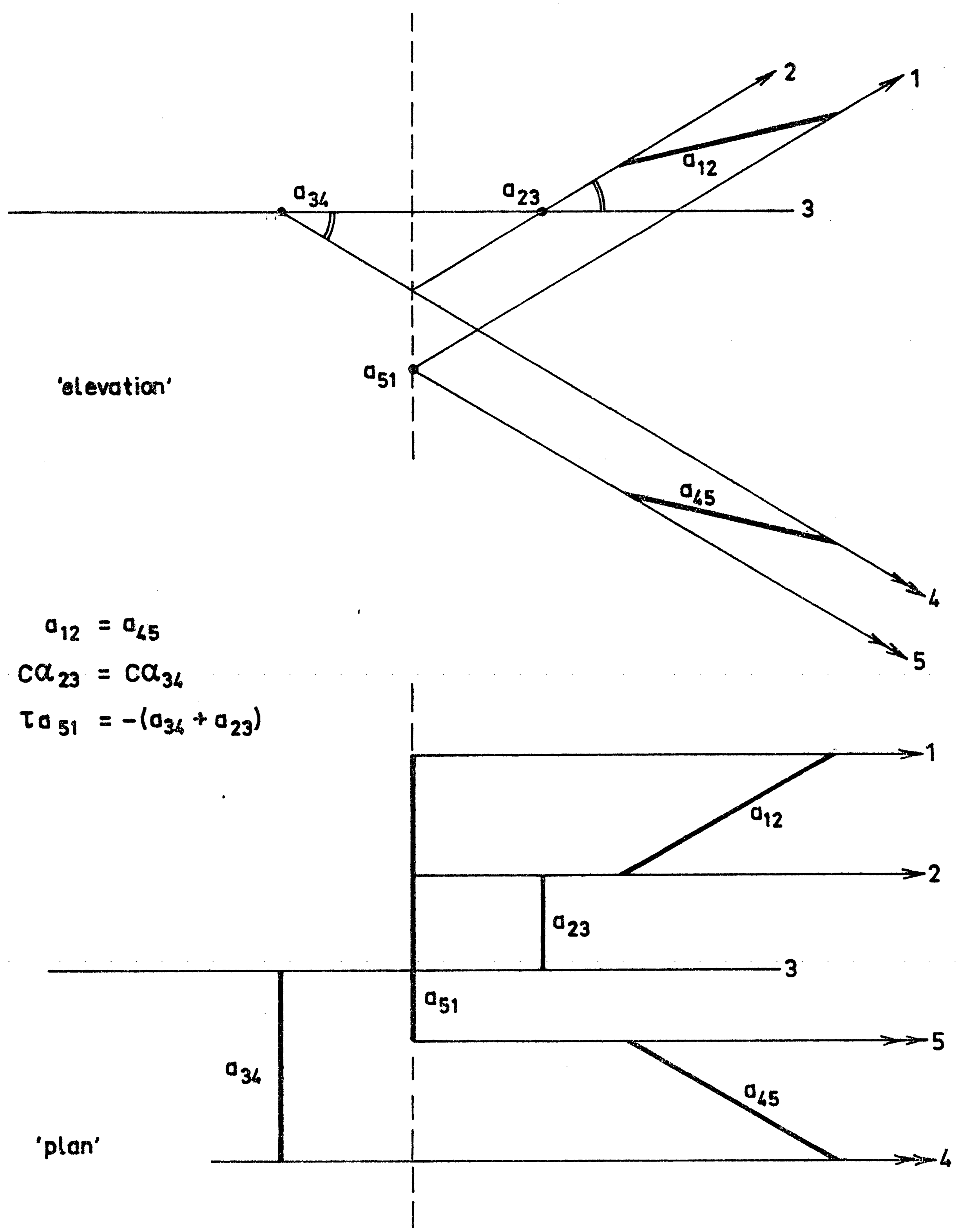

'elevation'
a_{34}
a_{23}
a_{12}
a_{51}
a_{45}
1
2
3
4
5
$a_{12} = a_{45}$
$C\alpha_{23} = C\alpha_{34}$
$\tau a_{51} = -(a_{34} + a_{23})$
'plan'

Fig. 7.5.2

(Refer to Introduction.) is a very special case of the present solution. The added constraints are

$$\sigma = -1 \qquad a_{51} = 0 \qquad a_{34} = a_{23},$$

and
$$h_5 = -h_1 \qquad h_4 = -h_2.$$

As already mentioned above, these two sets of restrictions may be applied independently of each other.

In view of Waldron's [44,45] screw-system analysis of his plane-symmetric five-bar, it seems to be worthwhile to pause here and present a similar analysis for the more general linkage. Referring to Fig. 7.5.3, we shall locate our origin of co-ordinates at the point of intersection of screw axis 1 with the hypothetical plane of quasi-symmetry. Then the ISA vector components for screw axis 1 may be given by (Refer to chapter 1.)

$$\underset{\sim}{\hat{\omega}}_1 = c\beta \underset{\sim}{i} - s\beta \underset{\sim}{j} \qquad \underset{\sim}{\hat{\mu}}_1 = h_1(c\beta \underset{\sim}{i} - s\beta \underset{\sim}{j}).$$

Also, since

$$\underset{\sim}{\rho}_5 = -\tau a_{51} \underset{\sim}{k},$$

we have that

$$\underset{\sim}{\hat{\omega}}_5 = c\beta \underset{\sim}{i} + s\beta \underset{\sim}{j} \qquad \underset{\sim}{\hat{\mu}}_5 = h_5(c\beta \underset{\sim}{i} + s\beta \underset{\sim}{j}) + \tau a_{51}(s\beta \underset{\sim}{i} - c\beta \underset{\sim}{j}).$$

Now, for equal and opposite relative angular velocities about these two joint axes, their resultant screw is obtained by simply adding the corresponding ISA components. Doing this, and normalising, the resultant screw is given by

$$\hat{\underset{\sim}{\omega}}_{15} = \underset{\sim}{i} \quad \hat{\underset{\sim}{\mu}}_{15} = \frac{h_5+h_1}{2}\underset{\sim}{i} + \frac{h_5-h_1}{2}t\beta\underset{\sim}{j} + \frac{\tau a_{51}}{2}t\beta\underset{\sim}{i} - \frac{\tau a_{51}}{2}\underset{\sim}{j}.$$

$$\therefore h_{15} = \frac{h_5+h_1}{2} + \frac{\tau a_{51}}{2}t\beta \qquad \underset{\sim}{\rho}_{15} = \left(\frac{h_5-h_1}{2}t\beta - \frac{\tau a_{51}}{2}\right)\underset{\sim}{k}$$

In the same way, for screws 2 and 4, we should obtain the following results.

$$\hat{\underset{\sim}{\omega}}_{24} = \underset{\sim}{i} \quad \hat{\underset{\sim}{\mu}}_{24} = \frac{h_4+h_2}{2}\underset{\sim}{i} + \frac{h_4-h_2}{2}t\beta\underset{\sim}{j} + \frac{\tau a_{51}}{2}t\beta\underset{\sim}{i} - \frac{\tau a_{51}}{2}\underset{\sim}{j}$$

$$h_{24} = \frac{h_4+h_2}{2} + \frac{\tau a_{51}}{2}t\beta \qquad \underset{\sim}{\rho}_{24} = \left(\frac{h_4-h_2}{2}t\beta - \frac{\tau a_{51}}{2}\right)\underset{\sim}{k}$$

Now, from equation (xv),

$$h_5 + h_1 = h_4 + h_2,$$

whence $\qquad h_{15} = h_{24}.$

So the two resultant screws are parallel and coplanar (in the plane of quasi-symmetry), and have the same pitch. The "simplified screw system" [44,45] of the linkage therefore consists of all screws of pitch h_{15} parallel to $\underset{\sim}{i}$ and in the plane of quasi-symmetry, together with all infinite pitch screws perpendicular to the plane. It is a screw system of order two. We may demonstrate these results easily as follows.

Consider the screw system defined by the two screws

$$\hat{\underset{\sim}{\omega}}_1 = \underset{\sim}{i} \quad \hat{\underset{\sim}{\mu}}_1 = h\underset{\sim}{i} \quad \text{and} \quad \hat{\underset{\sim}{\omega}}_2 = \underset{\sim}{0} \quad \hat{\underset{\sim}{\mu}}_2 = \underset{\sim}{j}.$$

A general screw of the system may be described by the motor

$$\underset{\sim}{S} = (\omega_1\underset{\sim}{i}, h\omega_1\underset{\sim}{i}+\mu_2\underset{\sim}{j}).$$

For $\omega_1 \neq 0$,

$$\underset{\sim}{\$} = (\underset{\sim}{i}, h\underset{\sim}{i} + \frac{\mu_2}{\omega_1}\underset{\sim}{j}),$$

whence pitch=h and $\underset{\sim}{\rho} = \frac{\mu_2}{\omega_1}\underset{\sim}{k}$.

For $\omega_1 = 0$,

$$\underset{\sim}{\$} = (\underset{\sim}{0}, \underset{\sim}{j}).$$

Hence, all screws of the system are screws of finite pitch h parallel to and coplanar with $(\hat{\underset{\sim}{\omega}}_1, \hat{\underset{\sim}{\mu}}_1)$, or infinite pitch screws parallel to $(\underset{\sim}{0}, \hat{\underset{\sim}{\mu}}_2)$. It is also clear that all screws of either type will be contained in the system. The particular screw system applying here has been categorised by Hunt [30] as the "second special form" of the two-system. As Waldron has pointed out, the results of this latter screw system approach alone to a mobility study of the linkage would provide a necessary, but not sufficient, basis for inferring a mobility of one.

$\underline{s\Theta_3 \neq 0}$

We find it convenient here to reframe equations (7.5.1) and (7.5.2) in the forms

$$\Theta_1 + \Theta_2 = K_{12} \qquad (7.5.1'')$$

$$\Theta_4 + \Theta_5 = K_{45}, \qquad (7.5.2'')$$

where K_{12} and K_{45} are constants. Using these equations, we may re-express equations (7.5.3) and (7.5.4) as, respectively,

$$a_{51}cK_{45} + a_{34} + a_{23}c\Theta_3 + a_{45}c\Theta_4 + a_{12}(c\Theta_2 c\Theta_3 - s\Theta_2 s\Theta_3 c\alpha_{23})$$
$$+ s\Theta_3 s\alpha_{23}(R_2 + R_1 + [h_2 - h_1]\Theta_2 + h_1 K_{12}) = 0, \qquad (7.5.ii'')$$

$a_{51}sK_{45}c\alpha_{34}-a_{23}s\Theta_3+a_{45}s\Theta_4c\alpha_{34}-a_{12}(c\Theta_2s\Theta_3+s\Theta_2c\Theta_3c\alpha_{23})$

$-s\alpha_{34}(R_4+R_5+[h_4-h_5]\Theta_4+h_5K_{45})$

$$+c\Theta_3s\alpha_{23}(R_2+R_1+[h_2-h_1]\Theta_2+h_1K_{12}) = 0. \qquad (7.5.iii'')$$

Let us first consider the possibility that $a_{12}=a_{45}=0$. From (7.5.ii"), since Θ_3 is constant, we may conclude that $h_2=h_1$. Then, from (7.5.iii"), $h_4=h_5$. The resulting improper linkage has part-chain mobility. It consists of two mobile two-bars rigidly connected by a locked slider.

We now look at the implications of only one of a_{12} and a_{45} being zero. The two alternatives are entirely analogous with each other. We shall subsequently consider the general situation.

C

We choose

$$a_{45} = 0.$$

Since $a_{12}\neq 0$ and Θ_3 is constant, we conclude from (7.5.ii") that

$$c\Theta_3 = c\alpha_{23} = 0 \qquad h_2 = h_1 = h, \text{ say}.$$

Let us put

$$s\Theta_3 = \sigma,$$

and we are free to choose $\alpha_{23} = \frac{\pi}{2}$. From (7.5.a),

$$c\alpha_{51} = 0,$$

and we are free to choose $\alpha_{51} = \frac{\pi}{2}$ by allowing $0<\alpha_{34}<2\pi$. Then, from (7.5.1), (7.5.2) and (7.5.b), we are led to the closure equations

$$\Theta_1 + \Theta_2 + \sigma\alpha_{34} = (2m+1)\pi$$

and
$$\Theta_4 + \Theta_5 = 2n\pi+\sigma\frac{\pi}{2}.$$

We also have the result

$$\Theta_3 = \sigma\frac{\pi}{2}.$$

Equation (7.5.ii") reduces to the dimensional condition

$$\sigma a_{34}+R_2+R_1+h[(2m+1)\pi-\sigma\alpha_{34}] = 0.$$

Equation (7.5.iii") simplifies to

$$a_{51}c\alpha_{34}-a_{23}-a_{12}c\Theta_2-\sigma s\alpha_{34}(R_4+R_5+h_4\Theta_4+h_5\Theta_5) = 0.$$

The last closure equation, obtained from (7.5.5), is

$$\sigma a_{51}s\alpha_{34}+a_{12}s\Theta_2+c\alpha_{34}(R_4+R_5+h_4\Theta_4+h_5\Theta_5) + r_3 = 0.$$

The mobile linkage here represented has screw joints 1 and 2 parallel with equal pitch, the slider perpendicular to them, and screws 4 and 5 coaxial and also perpendicular to screws 1 and 2. This chain, $H\frown H$-P-H=H-, may be obtained from the Delassus linkage d.3 by means of one -P- replacement (Refer to section 4.2.).

D

We now attempt the most general case, having dealt with the more obvious potential singularities. To proceed, we shall use the approach of tying of screw pitches. We suppose temporarily that $h_2=h_1$ and $h_4=h_5$. Equations (7.5.ii") and (7.5.iii") will then reduce to, respectively, the two following equations.

$$-a_{45}c\Theta_4=a_{51}cK_{45}+a_{34}+a_{23}c\Theta_3+a_{12}(c\Theta_2c\Theta_3-s\Theta_2s\Theta_3c\alpha_{23})$$
$$+s\Theta_3s\alpha_{23}(R_2+R_1+h_1K_{12}) \tag{i}$$

$$-a_{45}s\Theta_4c\alpha_{34}=a_{51}sK_{45}c\alpha_{34}-a_{23}s\Theta_3-a_{12}(c\Theta_2s\Theta_3+s\Theta_2c\Theta_3c\alpha_{23})$$
$$-s\alpha_{34}(R_4+R_5+h_5K_{45})+c\Theta_3s\alpha_{23}(R_2+R_1+h_1K_{12}) \tag{ii}$$

We now differentiate equations (i) and (ii) with respect to Θ_2 to obtain the following results.

$$a_{45}s\Theta_4\,\frac{d\Theta_4}{d\Theta_2} = -\,a_{12}(s\Theta_2c\Theta_3+c\Theta_2s\Theta_3c\alpha_{23}) \tag{iii}$$

$$a_{45}c\Theta_4c\alpha_{34}\,\frac{d\Theta_4}{d\Theta_2} = -\,a_{12}(s\Theta_2s\Theta_3-c\Theta_2c\Theta_3c\alpha_{23}) \tag{iv}$$

We have disallowed $a_{45}=0$ and $a_{12}=0$. From (ii), if $c\alpha_{34}=0$, for the equation to be an identity in Θ_2, we must have $s\Theta_3=c\alpha_{23}=0$. This possibility does not apply here. We may therefore conclude that $c\alpha_{34}\neq 0$. Hence, we are able to eliminate $\frac{d\Theta_4}{d\Theta_2}$ between (iii) and (iv) to obtain

$$(s\Theta_2c\Theta_3+c\Theta_2s\Theta_3c\alpha_{23})c\Theta_4c\alpha_{34}=(s\Theta_2s\Theta_3-c\Theta_2c\Theta_3c\alpha_{23})s\Theta_4.$$

Squaring this equation and substituting for $c\Theta_4$ from (i) results in

$$(s\Theta_2c\Theta_3+c\Theta_2s\Theta_3c\alpha_{23})^2c^2\alpha_{34}[C+a_{12}(c\Theta_2c\Theta_3-s\Theta_2s\Theta_3c\alpha_{23})]^2$$
$$=(s\Theta_2s\Theta_3-c\Theta_2c\Theta_3c\alpha_{23})^2\{a_{45}{}^2-[C+a_{12}(c\Theta_2c\Theta_3-s\Theta_2s\Theta_3c\alpha_{23})]^2\},$$

where $\quad C = a_{51}cK_{45}+a_{34}+a_{23}c\Theta_3+s\Theta_3s\alpha_{23}(R_2+R_1+h_1K_{12}).$ (v)

That is,

$$a_{45}^2(s^2\Theta_2 s^2\Theta_3+c^2\Theta_2 c^2\Theta_3 c^2\alpha_{23}-2s\Theta_2 s\Theta_3 c\Theta_2 c\Theta_3 c\alpha_{23})$$

$$=[s^2\Theta_2 s^2\Theta_3+c^2\Theta_2 c^2\Theta_3 c^2\alpha_{23}-2s\Theta_2 s\Theta_3 c\Theta_2 c\Theta_3 c\alpha_{23}$$

$$+c^2\alpha_{34}(s^2\Theta_2 c^2\Theta_3+c^2\Theta_2 s^2\Theta_3 c^2\alpha_{23}+2s\Theta_2 s\Theta_3 c\Theta_2 c\Theta_3 c\alpha_{23})]\times$$

$$[C^2+a_{12}^2(c^2\Theta_2 c^2\Theta_3+s^2\Theta_2 s^2\Theta_3 c^2\alpha_{23}-2s\Theta_2 c\Theta_2 s\Theta_3 c\Theta_3 c\alpha_{23})$$

$$+2a_{12}C(c\Theta_2 c\Theta_3-s\Theta_2 s\Theta_3 c\alpha_{23})]. \qquad \text{(vi)}$$

Equating only even powers of Θ_2 in equation (vi) leads to

$$a_{45}^2(s^2\Theta_2 s^2\Theta_3+c^2\Theta_2 c^2\Theta_3 c^2\alpha_{23})$$

$$=[s^2\Theta_2 s^2\Theta_3+c^2\Theta_2 c^2\Theta_3 c^2\alpha_{23}+c^2\alpha_{34}(s^2\Theta_2 c^2\Theta_3+c^2\Theta_2 s^2\Theta_3 c^2\alpha_{23})]\times$$

$$[C^2+a_{12}^2(c^2\Theta_2 c^2\Theta_3+s^2\Theta_2 s^2\Theta_3 c^2\alpha_{23})+2a_{12}Cc\Theta_2 c\Theta_3]$$

$$-4[s\Theta_2 s\Theta_3 c\Theta_2 c\Theta_3 c\alpha_{23} c^2\alpha_{34}-s\Theta_2 s\Theta_3 c\Theta_2 c\Theta_3 c\alpha_{23}]\times$$

$$[a_{12}^2 s\Theta_2 c\Theta_2 s\Theta_3 c\Theta_3 c\alpha_{23}+a_{12}Cs\Theta_2 s\Theta_3 c\alpha_{23}]. \qquad \text{, (vii)}$$

Equating only odd powers of Θ_2 in equation (vi) results in

$$2a_{45}^2 s\Theta_3 c\Theta_2 c\Theta_3 c\alpha_{23}$$

$$=[s^2\Theta_2 s^2\Theta_3+c^2\Theta_2 c^2\Theta_3 c^2\alpha_{23}+c^2\alpha_{34}(s^2\Theta_2 c^2\Theta_3+c^2\Theta_2 s^2\Theta_3 c^2\alpha_{23})]\times$$

$$[2a_{12}^2 c\Theta_2 s\Theta_3 c\Theta_3 c\alpha_{23}+2a_{12}Cs\Theta_3 c\alpha_{23}]$$

$$+[2s\Theta_3 c\Theta_2 c\Theta_3 c\alpha_{23}-2s\Theta_3 c\Theta_2 c\Theta_3 c\alpha_{23} c^2\alpha_{34}]\times$$

$$[C^2+a_{12}^2(c^2\Theta_2 c^2\Theta_3+s^2\Theta_2 s^2\Theta_3 c^2\alpha_{23})+2a_{12}Cc\Theta_2 c\Theta_3]. \qquad \text{(viii)}$$

In equation (vii), equating the coefficients of $c^4\Theta_2$ leads, after some algebraic manipulation, to

$$c^2\Theta_3 s^2\Theta_3(1+c^2\alpha_{23})^2-(c^2\Theta_3-s^2\Theta_3)^2 c^2\alpha_{23}$$

$$=\{4c^2\Theta_3 s^2\Theta_3 c^2\alpha_{23}-(c^2\Theta_3-s^2\Theta_3 c^2\alpha_{23})^2\}c^2\alpha_{34}. \qquad \text{(ix)}$$

In equation (viii), equating coefficients of $c^3\Theta_2$ will lead to the result

$$c\Theta_3 c\alpha_{23}(2c^2\Theta_3 s^2\alpha_{34}+2c^2\Theta_3 c^2\alpha_{23}s^2\alpha_{34}-2c^2\alpha_{23}s^2\alpha_{34}-s^2\alpha_{23})=0. \quad \text{(x)}$$

From the last equation, we must have $c\Theta_3=0$, $c\alpha_{23}=0$ or the expression in parentheses zero. In equation (ix), if $c\Theta_3=0$, since we do not allow $s\alpha_{23}=s\alpha_{34}=0$, we must have that $c\alpha_{23}=0$. Conversely, if $c\alpha_{23}=0$ in equation (ix), because we disallow $s\Theta_3=0$, we conclude that $c\Theta_3=0$.
If the expression in parentheses in equation (x) is zero, we have that

$$c^2\Theta_3 = \frac{s^2\alpha_{23}+2c^2\alpha_{23}s^2\alpha_{34}}{2s^2\alpha_{34}(1+c^2\alpha_{23})}.$$

Substitution of this result into (ix) yields, after considerable algebraic manipulation,

$$(s^4\alpha_{23}+4s^2\alpha_{34}c^2\alpha_{23})(4c^2\alpha_{23}c^2\alpha_{34}-[1+c^2\alpha_{23}]^2) = 0.$$

Since the first parenthesised expression cannot be zero, we conclude that

$$4c^2\alpha_{23}c^2\alpha_{34} = [1+c^2\alpha_{23}]^2.$$

It is easily seen that the only solution of this equation is

$$c^2\alpha_{23} = c^2\alpha_{34} = 1.$$

This result would imply that $s\alpha_{23}=s\alpha_{34}=0$, and so is unacceptable.

We therefore find that

$$c\Theta_3 = c\alpha_{23} = 0. \quad \text{(xi)}$$

Equation (viii) is then identically satisfied, and equation (vii) requires only that

$$a_{45}{}^2 = C^2.$$

Attempting to substitute result (xi) into equation (i), since $a_{45} \neq 0$, would require joint 4 to be locked. Hence there is no solution under the present circumstances of tied screw pitches. We may therefore conclude that, in the general case, neither is there any solution.

The only connectivity sum five linkages in this category with mobility one, then, are those found in parts A, B and C.

7.6 One pair of adjacent axes parallel

We have been unable, at this time, to complete the analysis for this category. We shall set out below the present extent of the analysis, and indicate what remains to be done.

We begin by putting

$$\alpha_{51} = 0.$$

We can consider the problem under three sub-categories, given below as A, B and C.

A

Equation (7.9) is simplified to

$$-c\theta_2 s\alpha_{12} s\alpha_{23} + c\alpha_{12} c\alpha_{23} = -c\theta_4 s\alpha_{34} s\alpha_{45} + c\alpha_{34} c\alpha_{45}. \qquad (7.6.1)$$

It is clear that, if either of joints 2 and 4 is locked in rotation, so is the other.

We assume in this sub-category that both of joints 2 and 4 are sliders. We therefore look for mobile linkages of the form J-P-J-P-J-.

All possibilities may be checked out as follows.

(i) If joints 1 and 5 are both screws, and rotation is possible about joint 3, we see from the spherical indicatrix that joints 1, 3, 5 are all parallel. We then have the solution H-P-H-P-H-, a general parallel-screw linkage. If rotation is not possible about joint 3, we have the solution H-P-P-P-H-, also a general parallel-screw linkage. In this latter case, if the screws are coaxial, the linkage can be expressed by P-P-P-H=H-, and is kinematically equivalent to the Delassus four-bar d.6.

(ii) If one of joints 1 and 5 is a screw and the other is cylindric, in view of (i), a linkage with part-chain mobility will result.

(iii) If both of joints 1 and 5 are sliders, or if one is prismatic and the other cylindric, the linkage will have part-chain mobility, based on the Delassus four-slider loop.

(iv) If one of joints 1 and 5 is helical and the other prismatic, the spherical indicatrix implies that, if rotation is possible about joint 3, then that axis is parallel to the screw's axis. We then have the solution, say, H-P-H-P-P-, a special parallel-screw chain. If rotation is not possible about joint 3, a linkage with part-chain mobility, based on d.6, will result.

B

We may now assume that neither of joints 2 and 4 is a slider. Let us consider here the possible solutions for which joint 3 is prismatic. We therefore seek mobile linkages of the form J-J-P-J-J-.

If both of joints 1 and 5 are prismatic, the linkage will have part-chain mobility, based on the P-P- chain.

If one of joints 1 and 5 is prismatic, we see, from the spherical indicatrix, that the three non-prismatic joints must be parallel. But then four adjacent axes are parallel and any solutions will have already been isolated in section 7.2.

We conclude that the only slider present is joint 3. Further, by the spherical indicatrix, joints 2 and 4 must be parallel.

We see immediately that there are two solutions with connectivity sum six, namely $\text{H-}\widehat{\text{H-P-C}}\text{-H-}$ and $\text{H-}\widehat{\text{H-P-H}}\text{-C-}$. Both of them are special parallel-screw loops.

It only remains to seek out solutions, in this sub-category, with connectivity sum five. Any such linkage must have the form $\text{H-}\widehat{\text{H-P-H}}\text{-H-}$. We thus have the additional constraint that

$$s\Theta_3 = 0 \qquad c\Theta_3 = \sigma.$$

Advancing the indices in equation (7.9) by 4 and substituting the dimensional conditions results in

$$\begin{aligned} c\alpha_{12} = & -\sigma c\Theta_4 s\alpha_{23} c\alpha_{34} s\alpha_{45} - \sigma s\alpha_{23} s\alpha_{34} c\alpha_{45} \\ & - c\Theta_4 c\alpha_{23} s\alpha_{34} s\alpha_{45} + c\alpha_{23} c\alpha_{34} c\alpha_{45}. \end{aligned} \tag{i}$$

We may assume that

$$0 < \alpha_{23}, \ \alpha_{34}, \ \alpha_{45} < \pi.$$

Then, from (i), in order that Θ_4 be not fixed, we conclude that

$$\text{either} \quad \alpha_{23} = \alpha_{34} = \frac{\pi}{2} \qquad \text{or} \qquad \alpha_{34} = \frac{1+\sigma}{2}\pi - \sigma\alpha_{23} \neq \frac{\pi}{2}$$

where, in both cases,

$$c\alpha_{12} = -\sigma c\alpha_{45}.$$

Advancing the indices in equation (7.9) by 1 and substituting the conditions on α_{51} and Θ_3 leads to

$$\begin{aligned} & -\sigma s\alpha_{23} s\alpha_{34} + c\alpha_{23} c\alpha_{34} \\ & \quad = s\Theta_5 s\Theta_1 s\alpha_{45} s\alpha_{12} - c\Theta_5 c\Theta_1 s\alpha_{45} s\alpha_{12} + c\alpha_{45} c\alpha_{12}. \end{aligned}$$

Putting $s\alpha_{12} = \rho s\alpha_{45}$,

this equation, in view of the additional constraints just established, can be written as

$$\rho s^2\alpha_{45}c(\Theta_5+\Theta_1)+\sigma c^2\alpha_{45} = \sigma.$$

$$\therefore \quad c(\Theta_5+\Theta_1) = \sigma\rho \tag{ii}$$

Equation (7.6.1) is, because of the same constraints, simplified to

$$c\Theta_4 = \rho c\Theta_2, \tag{iii}$$

whilst equation (7.7) becomes

$$\rho s\Theta_4 = \sigma s\Theta_2. \tag{iv}$$

Results (ii), (iii) and (iv) may be summarised by

$$\left.\begin{aligned} \Theta_5 &= 2k\pi + \frac{1-\sigma\rho}{2}\pi - \Theta_1 \\ \Theta_4 &= 2l\pi + \frac{1-\rho}{2}\pi + \sigma\Theta_2 \end{aligned}\right\} . \tag{v}$$

Equations (7.1)-(7.6), (7.8) are identically satisfied by relations (v), which are therefore the only two independent rotational closure equations.

Equation (7.12) is the only translational closure equation which contains the variable r_3. Equations (7.10) and (7.11), respectively, are reducible to the following two.

$$\sigma a_{51}(c\Theta_1 c\Theta_2 - s\Theta_1 s\Theta_2 c\alpha_{12}) + \sigma s\Theta_2 s\alpha_{12}(R_5+h_5\Theta_5+R_1+h_1\Theta_1)$$

$$+\rho a_{45}c\Theta_2 + a_{34} + \sigma a_{23} + \sigma a_{12}c\Theta_2 = 0 \tag{vi}$$

$$-\sigma a_{51}(c\Theta_1 s\Theta_2 c\alpha_{23}+s\Theta_1 c\Theta_2 c\alpha_{12} c\alpha_{23}-s\Theta_1 s\alpha_{12} s\alpha_{23})-\rho a_{45} s\Theta_2 c\alpha_{23}$$

$$-\sigma a_{12} s\Theta_2 c\alpha_{23}+(\sigma[R_2+h_2\Theta_2]-[R_4+h_4\Theta_4])s\alpha_{23}$$

$$+\sigma(R_5+h_5\Theta_5+R_1+h_1\Theta_1)(c\Theta_2 s\alpha_{12} c\alpha_{23}+c\alpha_{12} s\alpha_{23}) = 0 \qquad \text{(vii)}$$

We need to find under what conditions equations (v)-(vii) are reducible to three independent equations.

At this time, we have been unable to complete the analysis for this sub-category. It might be noted that there are certainly two solutions of equations (v)-(vii), both of which take the form $H\widehat{-P-}H-H=H-$. One of them is kinematically equivalent to the Delassus loop d.4, so that $\alpha_{23}=\alpha_{34}=\alpha_{45}=\frac{\pi}{2}$; the other functions as d.12, in which the sliders are not necessarily perpendicular to the screws. These solutions are the only two which contain a H=H group.

C

The last sub-category which needs to be considered is that for which none of joints 2, 3 and 4 is prismatic.

Again, we have not been able to conclude the analysis at this point. It is evident that several forms of parallel-screw linkage (connectivity sum six) will be solutions here, but the algebra required to isolate any other mobile loops will be quite difficult.

Conclusion

We have attempted, in sections 7.1-7.6, to begin a systematic search for all mobile over-constrained five-bar linkages free from part-chain mobility. We have succeeded in isolating all solutions for five-bars with parallel adjacent joint axes, except for the case where only one pair of adjacent axes is parallel. Even there, we were able to find several of the solutions (and quite possibly all of them).

Most of the solutions of the five-bars investigated were general or special parallel-screw linkages, connectivity sums 5 and 6. Other solutions isolated were kinematically equivalent to Delassus four-bars d.2, d.3, d.4, d.6, d.8, d.12. It is also clear that linkages equivalent to d.5 and d.13 would eventually be found in the analysis of part C, section 7.6. The only really new linkage which emerged was, perhaps surprisingly, a considerable generalisation of Waldron's plane-symmetric five-bar (solution B of section 7.5).

Once having completed the analysis for loops containing parallel adjacent joint axes, there is remarkably little involved in investigating the remaining five-bars. We may proceed as follows.

Chains with 4 or 5 sliders will have part-chain mobility, and may therefore be eliminated.
Linkages with 3 sliders must, by the spherical indicatrix, have the other two joints parallel. Hence, to preclude part-chain mobility, both joints must be screws, and the linkage will be a parallel-screw loop.

Linkages containing 2 prismatic pairs must, by the spherical indicatrix, have the other three joint axes parallel. Such a chain will have been already covered, since at least two of the parallel axes must be adjacent. Again, solutions will be parallel-screw linkages.

Hence, the only linkages to be examined are those for which no two adjacent joints are parallel and which contain a maximum of one prismatic pair. Specifically, we should be concerned with C-H-H-H-H-, P-C-H-H-H-, P-H-C-H-H- and their -H- (including -R-) and relevant -P- derivatives.

REFERENCES

1. BAKER, J.E. *The C-H-C-H- Linkage* M. Eng. Sc. Thesis, School of Mechanical and Industrial Engineering, The University of New South Wales, Jan. 1973

2. BAKER, J.E. *The Delassus Linkages* Research Report No. 1973/AM/1, School of Mechanical and Industrial Engineering, The University of New South Wales, Jun. 1973

3. BAKER, J.E. and WALDRON, K.J. *Limit Positions of Spatial Linkages via Screw System Theory* A.S.M.E. Paper No. 74-DET-107, Oct. 1974

4. BAKER, J.E. and WALDRON, K.J. *The C-H-C-H- Linkage* Mechanism and Machine Theory, 9 (1974), 3, 285-297

5. BAKER, J.E. *The Delassus Linkages* Proc. Fourth World Congress on the Theory of Machines and Mechanisms, Newcastle upon Tyne, England, Sep. 8-13, 1975, Paper no. 9, 45-49

6. BALL, R.S. *A Treatise on the Theory of Screws* C.U.P., Cambridge, 1900

7. BENNETT, G.T. *A New Mechanism* Engineering, 76 (1903), 777-778

8. BENNETT, G.T. *Deformable Octahedra* Proc. Lond. Math. Soc., 2s, 10 (1911), 1, 309-343

9. BENNETT, G.T. *The Skew Isogram Mechanism* Proc. Lond. Math. Soc., 2s, 13 (1914), 151-173

10. BRICARD, R. *Leçons de Cinématique* Vol.2, Gauthier-Villars, Paris, 1927

11. DELASSUS, E. *Sur les Systèmes Articulés Gauches, Première Partie* Paris Ecole Normale Superieure, Ann. Sci., 3s, XVII (Oct. 1900), 445-499

12. DELASSUS, E. *Sur les Systèmes Articulés Gauches, Deuxième Partie* Paris Ecole Normale Superieure, Ann. Sci., 3s, XIX (Mar. 1902), 119-152

13. DELASSUS, E. *Les Chaînes Articulés Fermées et Déformables à Quatre Membres* Bull. Sci. Math., 2s, 46 (1922), 1, 283-304

14. DIMENTBERG, F.M. *Determination of the Motions of Spatial Mechanisms* (In Russian) IZD-VO Akad. Nauk., U.S.S.R. 1950

15. DIMENTBERG, F.M. and YOSLOVICH, I.V. *A Spatial Four-Link Mechanism having Two Prismatic Pairs* (Trans. C.W. McLarnan) J. Mechanisms, 1 (1966), 291-300

16. DUFFY, J. and GILMARTIN, M.J. *Limit Positions of Four-Link Spatial Mechanisms - 1. Mechanisms Having Revolute and Cylindric Pairs* J. Mechanisms 4 (1969), 261-272

17. DUFFY, J. and HABIB-OLAHI, H.Y. *A Displacement Analysis of Spatial Five-Link 3R-2C Mechanisms - 1. On the Closures of the RCRCR Mechanism* J. Mechanisms, 6 (1971), 289-301

18. DUFFY, J. and HABIB-OLAHI, H.Y. *A Displacement Analysis of Spatial Five-Link 3R-2C Mechanisms - Part 2: Analysis of the RRCRC Mechanism* J. Mechanisms, 6 (1971), 463-473

19. DUFFY, J. and HABIB-OLAHI, H.Y. *A Displacement Analysis of Spatial Five-Link 3R-2C Mechanisms. Part 3. Analysis of the RCRRC Mechanism* Mechanism and Machine Theory, 7 (1972), 71-84

20. DUFFY, J. and ROONEY, J. *A Displacement Analysis of Spatial Six-Link 4R-P-C Mechanisms. Part 1: Analysis of RCRPRR Mechanism* A.S.M.E. Paper No. 72-Mech-85

21. DUFFY, J. and ROONEY, J. *A Displacement Analysis of Spatial Six-Link 4R-P-C Mechanisms. Part 2: Derivation of Input-Output Displacement Equation for RCRRPR Mechanism* A.S.M.E. Paper No. 72-Mech-86

22. DUFFY, J. and ROONEY, J. *A Displacement Analysis of Spatial Six-Link 4R-P-C Mechanisms. Part 3: Derivation of Input-Output Displacement Equation for RRRPCR Mechanism* A.S.M.E. Paper No. 72-Mech-87

23. EVERETT, J.D. *On a New Method in Statics and Kinematics* Mess. Math., 45 (1875), 36-57

24. GILMARTIN, M.J. and DUFFY, J. *Limit Positions of Four-Link Spatial Mechanisms - 2. Mechanisms Having Revolute, Cylindric and Prismatic Pairs* J. Mechanisms, 4 (1969), 273-281

25. GOLDBERG, M. *New Five-Bar and Six-Bar Linkages in Three Dimensions* Trans. A.S.M.E., 65 (1943), 649-661

26. GUPTA, V.K. and RADCLIFFE, C.W. *Mobility Analysis of Plane and Spatial Mechanisms* J. Eng. for Industry, Feb. 1971, (A.S.M.E. Trans., B, 93, 1) 125-130

27. HUNT, K.H. *Prismatic Pairs in Spatial Linkages* J. Mechanisms, 2 (1967), 213-230

28. HUNT, K.H. *Screw Axes and Mobility in Spatial Mechanisms via the Linear Complex* J. Mechanisms, 2 (1967), 307-327

29. HUNT, K.H. *Note on Complexes and Mobility* J. Mechanisms, 3 (1968), 3, 199-202

30. HUNT, K.H. *Screw Systems in Spatial Kinematics* Report No. MMERS 3, Department of Mechanical Engineering, Monash University, Jun. 1970

31. JENKINS, E.M., CROSSLEY, F.R.E. and HUNT, K.H. *Gross Motion Attributes of Certain Spatial Mechanisms* J. Eng. for Industry, Feb. 1969, 83-90

32. MYARD, F.E. *Contribution à la Géométrie des Systèmes Articulés* Bull. de la Soc. Math. de France, 59 (1931), 183-210

33. PAMIDI, P.R., SONI, A.H. and DUKKIPATI, R.V. *Existence Criteria of an Over-constrained R-R-R-P-R Five-Link Spatial Mechanism* Proc. Third World Congress for the Theory of Machines and Mechanisms, Kupari, Yugoslavia, Sep. 13-20, 1971, Vol.D, Paper D-13, 189-198

34. PAMIDI, P.R., SONI, A.H. and DUKKIPATI, R.V. *Necessary and Sufficient Existence Criteria of Overconstrained Five-Link Spatial Mechanisms with Helical, Cylinder, Revolute and Prism Pairs* A.S.M.E. Paper No. 72-Mech-47

35. SAVAGE, M. Doctoral Dissertation, Purdue University, Jun. 1969

36. SAVAGE, M. *Four-Link Mechanisms with Cylindric, Revolute and Prismatic Pairs* Mechanism and Machine Theory, 7 (1972), 191-210

37. SKREINER, M. *Methods to Identify the Mobility Regions of a Spatial Four-Link Mechanism* J. Mechanisms, 2 (1967), 415-427

38. SONI, A.H. *Existence Criteria of an Overconstrained R-P-R-C-R- Five-Link Spatial Mechanism* Proc. Third World Congress for the Theory of Machines and Mechanisms, Kupari, Yugoslavia, Sep. 13-20, 1971, Vol.C, Paper C-14, 179-188

39. SONI, A.H. and HARRISBERGER, L. *Existence Criteria of Mechanisms* A.S.M.E. Paper No. 68-MECH-33

40. VOINEA, R.P. et ATANASIU, M.C. *Théorie Géométrique des Vis et Quelques Applications à la Théorie des Mécanismes* Revue Roumaine des sci. tech.: serie de mec. appl., 7 (1962), 4, 845-860

41. WALDRON, K.J. *The Constraint Analysis of Mechanisms* J. Mechanisms, 1 (1966), 101-114

42. WALDRON, K.J. *A Family of Overconstrained Linkages* J. Mechanisms, 2 (1967), 201-211

43. WALDRON, K.J. *Hybrid Overconstrained Linkages* J. Mechanisms, 3 (1968), 73-78

44. WALDRON, K.J. *Symmetric Overconstrained Linkages* J. Eng. for Industry, Feb. 1969, (A.S.M.E. Trans., B, 91, 1) 158-164

45. WALDRON, K.J. *The Mobility of Linkages* Doctoral Dissertation, Stanford University, Jul. 1969

46. WALDRON, K.J. *A Method of Studying Joint Geometry* Mechanism and Machine Theory, 7 (1972), 347-353

47. WALDRON, K.J. *A Study of Overconstrained Linkage Geometry by Solution of Closure Equations - Part 1. Method of Study* Mechanism and Machine Theory, 8 (1973), 95-104

48. WALDRON, K.J. *A Study of Overconstrained Linkage Geometry by Solution of Closure Equations - Part 2. Four-bar Linkages with Lower Pair Joints other than Screw Joints* Mechanism and Machine Theory, 8 (1973), 233-247

49. WALDRON, K.J. *The Use of Motors in Spatial Kinematics* IFToMM International Symposium on Linkages and Computer Design Methods, Bucharest, Romania, Jun. 7-13, 1973, Vol. B, Paper B-41, 535-545

Closing Remarks

Linkage analysis remains, to some extent, an art in search of general scientific techniques. It appears evident with each new linkage isolated, or each set of limit positions found, or whatever, that some general principle is being slowly revealed. An estimate of 'the general principle' sometimes allows us to proceed to another linkage or class of linkages, but . . . no farther.

Whilst modern techniques are powerful by comparison of some sort, their very number is an indication of our essential lack of perception of the pervading principles of linkage motion: we have as yet no universal laws to apply. I indicated in the Introduction to chapter 2 that there are several approaches available for solving limit position problems. Other workers have commented on the number of ways of analysing the closure of a linkage. The one clear emergent fact is the precedence one method has over the others for a particular linkage, or linkage class. I believe, however, that the technique introduced and used in chapter 2 is the nearest we have come to a universal, deep-rooted analytical tool. One feels that geometry and screw system theory are of the essence in linkage analysis, and that raw algebra, whilst admittedly powerful, thorough and often short-cutting, is secondary.

It is conceivable that there will be no 'universal method', that our analyses must necessarily be piecemeal and opportunist. It is my hope, however, that dedicated linkage kinematicians will be spurred on to vindicate their belief in the existence of an underlying, albeit well-shrouded, body of truth to which all linkages are subservient.