# Distributed Control of Interconnected Systems in the Behavioural Framework

by

**Yitao Yan**

B.E. (Hons) in Chemical Engineering

A thesis submitted in fulfilment
of the requirements for the degree of
Doctor of Philosophy

School of Chemical Engineering
The University of New South Wales
Sydney, Australia

December, 2019

# Thesis/Dissertation Sheet

| | | |
|---|---|---|
| Surname/Family Name | : | **Yan** |
| Given Name/s | : | **Yitao** |
| Abbreviation for degree as give in the University calendar | : | **PhD** |
| Faculty | : | **Faculty of Engineering** |
| School | : | **School of Chemical Engineering** |
| Thesis Title | : | **Distributed Control of Interconnected Systems in the Behavioural Framework** |


**Abstract 350 words maximum: (PLEASE TYPE)**

The rapid development of technology has made the design, monitoring and data storage of large-scale, complex interconnected systems possible. These efficient and economical interconnected systems come with a price: the complex dynamics due to convoluted interconnections makes the effective control of such a system incredibly difficult. The behaviour of the subsystems in a network is vastly different than that when it is not, and the inherent uncertainties due to modelling errors may be amplified as a result of the strong interactions. Furthermore, the ability to collect and process large amount of data leads to the paradigm shift from model-centric description to data-centric description or hybrid model/data description of a system. These challenges necessitate the need for a unified foundation for the control of complex systems that is able to admit descriptions of systems other than the conventional differential/difference models.

Motivated by these challenges, this thesis aims to develop such a framework for the distributed control of an interconnected system using the behavioural systems theory. As a theory that focuses on analysing the dynamics of the external variables and places the trajectories admissible within the system as the central description of a dynamical system, it is perfect for the construction of a platform that unifies various classes of systems and is effective in the analysis of interconnections. The framework is eventually set up as a completely representation-free structure, allowing for free choice of representations for the systems according to the specific needs. Algorithms for several representation structures are also provided.

# Originality Statement

"I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgment is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged."

Signed ........................................................

Date ........................................................

# Copyright Statement

# Authenticity Statement

"I certify that the Library deposit digital copy is a direct equivalent of the final officially approved version of my thesis. No emendation of content has occurred and if there are any minor variations in formatting, they are the result of the conversion to digital format."

Signed ........................................................

Date ........................................................

# INCLUSION OF PUBLICATIONS STATEMENT

UNSW is supportive of candidates publishing their research results during their candidature as detailed in the UNSW Thesis Examination Procedure.

**Publications can be used in their thesis in lieu of a Chapter if:**

- The candidate contributed greater than 50% of the content in the publication and is the "primary author", ie. the candidate was responsible primarily for the planning, execution and preparation of the work for publication
- The candidate has approval to include the publication in their thesis in lieu of a Chapter from their supervisor and Postgraduate Coordinator.
- The publication is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in the thesis

Please indicate whether this thesis contains published material or not:

☐ This thesis contains no publications, either published or submitted for publication

☒ Some of the work described in this thesis has been published and it has been documented in the relevant Chapters with acknowledgement

☐ This thesis has publications (either published or submitted for publication) incorporated into it in lieu of a chapter and the details are presented below

**CANDIDATE'S DECLARATION**

I declare that:

- I have complied with the UNSW Thesis Examination Procedure
- where I have used a publication in lieu of a Chapter, the listed publication(s) below meet(s) the requirements to be included in the thesis.

| Candidate's Name | Signature | Date (dd/mm/yy) |
| --- | --- | --- |
| | | |

# Abstract

The rapid development of technology has made the design, monitoring and data storage of large-scale, complex interconnected systems possible. These efficient and economical interconnected systems come with a price: the complex dynamics due to convoluted interconnections make the effective control of such a system incredibly difficult. The behaviour of the subsystems in a network is vastly different than that when it is not, and the inherent uncertainties due to modelling errors may be amplified as a result of the strong interactions. Furthermore, the ability to collect and process large amount of data leads to the paradigm shift from model-centric description to data-centric description or hybrid model/data description of a system. These challenges necessitate the need for a unified foundation for the control of complex systems that is able to admit descriptions of systems not only limited to the conventional differential/difference models.

Motivated by these challenges, this thesis aims to develop such a framework for the distributed control of an interconnected system using the behavioural systems theory. As a theory that focuses on analysing the dynamics of the external variables and places the trajectories admissible within the system as the central role of describing a dynamical system, it is perfect for the construction of a platform that unifies various classes of systems and is effective in the analysis of interconnections. The framework is eventually set up as a completely representation-free structure, allowing for free choice of representations for the systems according to the specific needs. Algorithms for several representation structures are also provided.

For the case where the subsystems are represented as linear time-invariant differential systems while the global requirements are specified as $\mathcal{H}_\infty$ type conditions, the control design follows a two-step algorithm. Firstly, the behaviours of the subsystems, the (to-be-designed) controllers as well as the global requirements are all represented as dissipative dynamical systems with quadratic supply rates, from which the (to-be-determined) controller supply rates can be found. Secondly, parametrisations of the supply rates are carried out to search for linear time-invariant representations for the controllers. Algorithms for subsystems with various types of parametric uncertainties are given to add robustness to the controllers. The resulting framework deals with interconnections, uncertainties in the subsystems and disturbance attenuation simultaneously.

For the general framework, neither the subsystems nor the controllers have prescribed representations. The behaviours of the subsystems are denoted by their respective sets of trajectories and interconnections are interpreted entirely as variable sharing instead of signal flows. Furthermore, the network of an interconnected system is also defined as a dynamical system with its own behaviour, leading to a generic, scalable and flexible representation of the interconnected behaviour. From this structure, necessary and sufficient conditions for the existence of the controller behaviours can be given and all distributed controller behaviours can be constructed explicitly. This framework unites various representations and descriptions of the features of dynamical systems as behaviours, thereby allowing for the formation of a hybrid platform for the analysis and distributed control generically and systematically.

# Acknowledgements

This doctoral journey is undoubtedly a bumpy but fruitful one, along which I have gained not only knowledge, but also valuable mentors and friends. I'm very lucky to be surrounded by these wonderful people who have made this otherwise rather mundane, and sometimes frustrating, experience colourful and rewarding.

First and foremost, I would like to express my sincere gratitude to my supervisor Professor Jie Bao. His passion towards research is contagious, he is professional and rigorous in his research areas, and he thinks ahead of the curve. During the past few years, not only has he provided me with constructive guidances in times of need, but he has also left enough freedom for me to explore the existing theories in order to get inspirations. All new developments take time, even more so for theoretical ones, and I feel very fortunate to have a supervisor who can inspire me with new ideas and, in the mean time, has enough patience to put up with my slow progress when things didn't go quite well. Jie, I don't think the completion of this thesis would be possible had I not been guided and supported by you.

I am also grateful to have a group of great friends and colleagues as they make my routine life enjoyable. I would like to thank Yuchen for all the fun times and meaningful discussions, both work related and about all other things. I would like to thank Wangyan for the relaxing coffee times and the valuable suggestions for my research. I would also like to thank Ruigang for helping me out during my early stages, thank Chaoxu for introducing me to the realm of behavioural systems theory, thank Yifeng for showing me how to use Matlab and LaTeX properly, and thank Ryan for taking the time to proofread my thesis. Furthermore, I would like to thank Pesila, Chee Keong, Xinan, Qingyang, Jing, Parvez, Choon Jie, Lai, Guannan and Mengjie, who are or were in the process control group, for all the happy memories in the past few years. It is my honour to work in a group with members as awesome as you guys.

I don't believe I would be able to finish this thesis without the constant support from my parents, Ning Yan and Xuhong Jiang. I'm grateful to be blessed with such wise parents who always show me the way when I'm faced with obstacles and difficult life choices. Finally, I wouldn't be able to reach the end of this journey without the company of my girlfriend, Jingya Peng, who has always believed in me, encouraged me at difficult times and suffered through my night-less and weekend-less schedule during the past few years.

# List of Publications

## Journal Papers

1. **Y. Yan**, R. Wang, J. Bao and C. Zheng, "Robust Distributed Control of Plantwide Processes Based on Dissipativity," *Journal of Process Control*, vol. 77, pp. 48–60, 2019.

2. W. Li, **Y. Yan** and J. Bao, "Dissipativity-Based Distributed Fault Diagnosis for Plantwide Chemical Processes," *submitted to Journal of Process Control.*

3. W. Li, **Y. Yan**, J. Bao and B. Huang, "On Dissipativity of Data-Based Behaviors: from Finite to Infinite Horizons," *submitted to Automatica.*

4. **Y. Yan**, J. Bao and B. Huang, "Behavioral Approach to Distributed Control of Interconnected Systems," *to be submitted to Automatica.*

5. G. Xiao, **Y. Yan**, J. Bao and F. Liu, "Distributed Robust Economic MPC with Plantwide Optimization," *to be submitted to AIChE Journal.*

## Conference Papers

1. **Y. Yan**, R. Wang and J. Bao, "Robust Control Synthesis for Linear Differential Systems with Parametric Uncertainty," in *Proceedings of Australian & New Zealand Control Conference (ANZCC) 2018, Melbourne, Australia*, pp. 281–284, 2018.

2. W. Li, **Y. Yan** and J. Bao, "Vector Dissipativity-Based Distributed Fault Detection for Plantwide Chemical Processes," in *Proceedings of Asian Control Conference (ASCC) 2019, Kitakyushu, Japan*, pp. 1507–1512, 2019.

3. **Y. Yan** and J. Bao, "A Scenario Approach to Robust Distributed Control for Plantwide Process Systems," in *Proceedings of European Control Conference (ECC) 2019, Naples, Italy*, pp. 560–565, 2019.

4. **Y. Yan**, J. Bao and B. Huang, "Dissipativity Analysis for Linear Systems in the Behavioural Framework," *Australian & New Zealand Control Conference (ANZCC) 2019, Auckland, New Zealand*, pp. 152–156, 2019.

5. W. Li, **Y. Yan** and J. Bao, "Fault Detection Problems in the Behavioral Framework," *Australian & New Zealand Control Conference (ANZCC) 2019, Auckland, New Zealand*, pp. 94–98, 2019.

6. L. Wei, **Y. Yan** and J. Bao, "A Data-Driven Predictive Control Structure in the Behavioral Framework," *IFAC World Congress 2020, Berlin, Germany*, accepted.

# Publications Relevant to Thesis Scope

1. **Y. Yan**, Y. Li, M. Skyllas-Kazacos and J. Bao, "Modelling and Simulation of Thermal Behaviour of Vanadium Redox Flow Battery," *Journal of Power Sources*, vol. 322, pp. 116–128, 2016.

2. **Y. Yan**, M. Skyllas-Kazacos and J. Bao, "Effects of Battery Design, Environmental Temperature and Electrolyte Flowrate on Thermal Behaviour of a Vanadium Redox Flow Battery in Different Applications," *Journal of Energy Storage* vol. 11, pp. 104–118, 2016.

# Nomenclature

***Behavioural Systems Theory***

| | |
|---|---|
| $\mathfrak{B}$ | Behaviour |
| $\mathfrak{B}_{\|[1,T]}$ | Behaviour restricted to the interval $[1,T]$ |
| $\mathtt{L}(\mathfrak{B})$ | Lag of $\mathfrak{B}$ |
| $\mathtt{n}(\mathfrak{B})$ | State cardinality of $\mathfrak{B}$ |
| $\mathbb{R}^{\mathrm{m}\times\mathrm{n}}[\xi]$ | One-variable $\mathrm{m}\times\mathrm{n}$ polynomial matrices with real coefficients |
| $\mathbb{R}^{\bullet\times\bullet}[\xi]$ | One-variable polynomial matrices with unspecified but finite dimensions |
| $\mathfrak{L}^{\mathrm{w}}$ | The set of Linear Time-invariant Systems with dimension w |
| $\mathfrak{L}^{\bullet}$ | The set of Linear Time-invariant Systems with unspecified but finite dimension |

***Dissipativity Theory***

| | |
|---|---|
| $\mathbb{R}^{\mathrm{m}\times\mathrm{n}}[\zeta,\eta]$ | Two-variable $\mathrm{m}\times\mathrm{n}$ polynomial matrices with real coefficients |
| $\mathbb{R}^{\bullet\times\bullet}[\zeta,\eta]$ | Two-variable polynomial matrices with unspecified but finite dimensions |

| | |
|---|---|
| $\mathbb{S}^{\mathrm{m}}[\zeta,\eta]$ | Two-variable symmetric $\mathrm{m}\times\mathrm{m}$ polynomial matrices with real coefficients |
| $\mathbb{S}^{\bullet}[\zeta,\eta]$ | Two-variable symmetric polynomial matrices with uspecified but finite dimensions |
| $\Phi^{\star}$ | The dual variable, $\Phi^{\star}(\zeta,\eta)=\Phi^{T}(\eta,\zeta)$ |
| $\Phi(\zeta,\eta)\succ(\succeq)0$ | The two-variable polynomial matrix is (semi-)positive |
| $\Phi(\zeta,\eta)\overset{\mathfrak{B}}{\succ}(\overset{\mathfrak{B}}{\succeq})0$ | The two-variable polynomial matrix is (semi-)positive along $\mathfrak{B}$ |

***Set Theory***

| | |
|---|---|
| $\forall,\exists$ | For all, there exists |
| $\cap,\ \cup,\ \setminus,\ \times$ | Set intersection, union, exclusion, Cartesian product |
| $\bigtimes$ | Cartesian product of a set of sets |
| $\mathbb{W}^{\mathbb{T}}$ | The set of all functions from $\mathbb{T}$ to $\mathbb{W}$ |
| $\mathbb{Z}_N^{+}$ | The set of all positive integers no greater than $N$ |

***Other Notations***

| | |
|---|---|
| $\dagger$ | Moore-Penrose inverse |
| $\oplus$ | Minkowski sum |
| $\otimes$ | Kronecker product |
| $\perp$ | $A^{\perp}:=I-A^{\dagger}A$ |
| $\wedge$ | Interconnection of two systems with the same variables or concatenation of trajectories |

| | |
|---|---|
| $\wedge_{t_0}$ | Concatenation of two trajectories at $t = t_0$ |
| $\sqcap$ | Concatenation of two systems with different variables |
| $\prod$ | Concatenation of a set of systems with different variables |
| $\mathrm{col}(\cdot)$ | Column vector of the elements |
| $\mathrm{He}\{\cdot\}$ | $\mathrm{He}\{A\} \coloneqq A + A^\star$ |

# List of Abbreviations

| | |
|---|---|
| ARE | Algebraic Riccati Equation |
| CSTR | Continuous Stirred Tank Reactor |
| DMPC | Distributed Model Predictive Control |
| LMI | Linear Matrix Inequality |
| LPV | Linear Parameter-Varying |
| LTI | Linear Time-Invariant |
| KYP | Kalman–Yakubovich–Popov |
| (P)QDF | (Parametric) Quadratic Differential Form |
| QdF | Quadratic Difference Form |
| SDP | Semi-definite Programming |
| SOS | Sum-of-Squares |
| VRFB | Vanadium Redox Flow Battery |
| ZFC | Zermelo-Fraenkel set theory with the Axiom of Choice |

# Contents

# List of Figures

# List of Tables

# Chapter 1

# Introduction

## 1.1 Background

The advancement of technology in recent years has profound impact on the direction of control strategy and algorithm development. On the one hand, it has made possible the effective monitoring and efficient control of large-scale, complex interconnected systems. This type of system typically contains a large number of subsystems with complex, convoluted and even reconfigurable interconnections among them. For example, in the chemical industry, this type of system, more commonly known as a plantwide process [1], often consists of many process units (reactors, heat exchangers distillation columns, etc.) interconnected through material recycle (where unreacted reactants are separated from the products and reused) and energy integration loops (where excessive heat generated in process units is utilised in another process unit) [2–4]. Interconnected systems have become increasingly popular among different fields of manufacturing because they generally yield higher production rates and better energy efficiency. On the other hand, the advancements have also made rapid collection and huge storage of data of interconnected systems possible. These data sets contain rich information of the dynamics of these systems and they are the most accurate way of describing the systems dynamics. In fact, the accuracy of an empirically constructed model can at most describe the system as good as the original data set. If there exist methods to extract the dynamical features directly from the data set then they should unequivocally be implemented. As a result, a new paradigm of data-enhanced operations and data-driven control is emerging [5].

As efficient and economical as interconnected systems are, the complex dynamics caused by the incredibly convoluted interconnections among subsystems poses grand challenges. Due to the complex interconnections, the dynamics of each subsystem are vastly different from that when it is a stand-alone system. For example, material recycles and heat integration among process units can be understood as positive feedback loops within the process network, which have the potential to amplify the exogenous disturbances, leading to deteriorating effects on control performance [6, 7]. The structural evaluation and operability analysis of such a system have been carried out in [8–14]. To control such a large-scale system, centralised control, in which the entire system is treated as a single complex multivariable system, is obviously not an optimal choice because the computation becomes prohibitively complex and the lack of flexibility may even make the control problem infeasible [15]. Decentralised control [16–19], in which each subsystem is controlled by a local controller without the knowledge of the rest of the system, are much simpler to implement, but these approaches are very conservative and often lead to poor global control performance because the known interactions are treated as unknown uncertainties. To perform control design on such a system, one must look for a balance between flexibility and performance, and distributed control is the best strategy for this situation. In the past decade, distributed control strategies [20–23], which coordinate an array of controllers through a communication network to achieve the global objectives, started to attract much attention in both industry and academia. As a balance between the aforementioned approaches, distributed control structure can provide good global performance like centralised control while preserving the flexibility and fault tolerance of decentralised control.

Another issue of such a system is brought by the uncertainties in the subsystems, or robustness issue. Models are typically obtained either from physics with assumptions or empirical regressions, hence they are inherently erroneous [18, 24]. Within an interconnected system, the convoluted interconnections may well magnify such errors. Robust stability of uncertain systems has been extensively studied in literature [25–29], in which the uncertainty was described in the sense of a norm bound. However, this could lead to conservative control design due to the coarse description of model uncertainties. A less conservative uncertainty description is polytopic uncertainty, in which the "true" process model is within a convex polytopic region [30–32]. To some extent, all of the aforementioned works design controllers based on the worst case scenario, but compensating for the extreme cases can make the control design very conservative as such descriptions inevitably include regions that do not belong to the system. The reasons for this unavoidable sacrifice of control performance are (1) that convexity is imposed for easier computation and (2) that models are placed as the central role in defining a dynamical system while they are actually not. Relaxation of convexity will definitely yield much less conservative design, but more importantly, this calls for new way of thinking: a model is perhaps only to summarise some of the characteristics of a dynamical system rather than to define it. The trajectories in a data set are actually what define a dynamical system.

This thesis discusses the distributed control problems in the framework of the behavioural systems theory [33], which views a dynamical system as a set of functions mapped from a time axis to a signal space, or more commonly known as trajectories. This set, called the behaviour, is the centre of a dynamical system.

Analogous to the very nature of set theory that a set is defined by its elements, trajectories admissible through a dynamical system define the system. This means that dynamical systems in this theory are entirely representation-free. As a result, dynamical systems can choose preferred representations of their behaviours according to the perspectives needed, and systems with different representations can be united under the same framework. Furthermore, the theory excels in dealing with interconnections, in that it does not distinguish between input and output but views them as a single set of variables for the system. Interconnection of two dynamical systems is the sharing of trajectories between the two systems, hence the interconnected behaviour is simply the common trajectories in the two systems. With this rationale, control can be viewed as interconnection and controllers are essentially restricting the set of behaviour that can happen in the to-be-controlled systems. This gives a flexible and scalable representation of an interconnected system because an additional subsystem integrated is essentially an additional set of constraints on the existing behaviour [34].

## 1.2 Aim and Objectives

The aim of of this thesis is to develop a unified platform for the distributed control of interconnected systems with no prescribed representations. To achieve this ultimate goal, objectives include

1. the development of a single distributed control framework that deals with interconnection and robustness issues at the same time;

2. the development of algorithms to realise the above objective with polytopic and generic parametric uncertainty regions;

3. the formulation of a completely representation-free structure for the analysis

and distributed control of interconnected systems.

## 1.3 Thesis Structure

The rest of the chapters are organised as follows.

In Chapter 2, brief reviews on the general theories of the behavioural framework, its most well studied representations, the rationale of behavioural interconnections, notable structures of distributed control schemes as well as the theories concerning necessary aspects of robust control are provided.

Chapter 3 formulates the framework that simultaneously addresses the issues of interconnections and robustness for the case where subsystems are with polytopic uncertainties (i.e., the polytopic part of objectives 1 and 2). The framework is from the dissipativity point of view with parameter-dependant dissipativity. The control design is carried out in two steps: (1) from the dissipativity point of view, search for valid representations for the controllers that satisfies the global requirements also represented from the same point of view, and (2) obtain a linear time-invariant representation of the controllers through behaviour parametrisation.

Chapter 4 generalises the results in Chapter 3 to the case where the uncertain subsystems are with generic parametric uncertainties (i.e., the generic part of objectives 1 and 2). The two-step formulation is still adopted, but parameter-dependant dissipativity is generalised into parametric dissipativity. Due to the non-convexity of the problem, two algorithms are proposed. The first one leads to a deterministic solution but with stricter assumptions while the second one leads to a probabilistic solution with much more relaxed assumptions.

In Chapter 5, a complete set-theoretic framework for the analysis and distributed control is formulated (objective 3). The entire chapter assumes no representation of the behaviour and all results are given in terms of behavioural sets.

A unified platform is formulated for the construction of interconnected behaviour. Necessary and sufficient conditions for the existence of the controller behaviours as well as explicit constructions of the behavioural sets are provided.

Chapter 6 concludes the thesis with the summary of important results and some recommendations of the possible future work.

As a relatively new theory, the language and representations may not be user friendly. Therefore, comparisons with the conventional notations and representation are also carried out when necessary in Chapter 2. The graphical representation of the system still adopts the conventional input/output setting with signal flows. In Chapter 3 and Chapter 4, the behavioural language is used much more frequently, but comparisons to the conventional representations are still made when necessary. The interconnected system layout still adopts the input/output representation, but the notion of signal flow is abandoned. In Chapter 5, the input/output thinking is completely abandoned as well. All illustrations and graphical representations are presented in behavioural framework. The gradual shift from the conventional structure into the behavioural framework is carried out in the hope that the effectiveness and elegance of this theory can be recognised and appreciated.

# Chapter 2

# Preliminaries and Literature Review

In this chapter, a brief review of the background knowledge relative to this thesis is carried out. Section 2.1 introduces the governing theory for this thesis - the behavioural systems theory, including its key components, rationale and current developments. Section 2.2 reviews the state-of-the-art methods for distributed control of interconnected systems. Section 2.3 provides a brief review on the concepts of robust control and its key representations. Finally, the research gaps are highlighted in Section 2.4.

## 2.1 Behavioural Systems Theory

### 2.1.1 General Theory

Behavioural systems theory was proposed by Jan C. Willems in 1979 [33]. It was initially considered as another point of view of modelling a dynamical system and the relevant modelling techniques were discussed in details in [35–37]. It was then gradually developed as a theory of dynamical systems on its own [38, 39]. It views a dynamical system from set-theoretic point of view and can therefore be defined without any presumption on its structure.

**Definition 2.1** (Dynamical Systems [33])**.** A dynamical system is a triple

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}), \tag{2.1}$$

where $\mathbb{T}$ is the time axis, $\mathbb{W}$ is the signal space and $\mathfrak{B} \subset \mathbb{W}^{\mathbb{T}}$ is the behaviour.

Usually, the time axis is a subset of $\mathbb{R}$ (in the case of continuous-time systems) or $\mathbb{Z}$ (in the case of discrete-time systems). Central to this definition is the notion of behaviour, which is a subset of all mappings, or more commonly known as trajectories, from the time axis to the signal space. This definition reveals the very basic nature of a dynamical system: a dynamical system is essentially a set of trajectories and it is this set of trajectories $\mathfrak{B}$ that defines the system. This is a set-theoretic point of view and the dynamical system is defined in a completely representation-free manner. In such a dynamical system, the generic variable is commonly denoted as $w : \mathbb{T} \to \mathbb{W}$ and it is called the *manifest* variable, within which contains all variables of interest such as exogenous inputs and outputs. This naming of variable shows another feature of this framework: the idea of input/output is blurred and they are not distinguished from each other in $w$. This equal treatment of input and output variables represents the reality of a complex system more accurately because it is often impossible to tell the directions of the system flow, let alone the input or output of a particular subsystem.

While the behaviour of manifest variables are the main interest within a dynamical system, it may be insufficient to define a system with manifest variables only. To fully define a system, auxiliary variables can be introduced to aid the description. This type of variables are called *latent* variables.

**Definition 2.2** (Latent Variable Dynamical System [38])**.** A dynamical system with latent variable is a quadruple $\Sigma^{full} = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}^{full})$ where $\mathfrak{B}^{full} \subset (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$

is the *full* behaviour. The manifest behaviour of this system is

$$\mathfrak{B} = \left\{ w | \exists \ell \in \mathbb{L}^{\mathbb{T}}, (w, \ell) \in \mathfrak{B}^{full} \right\}.$$

The behavioural set defined in Definition 2.2 is a stricter version of that defined in Definition 2.1 because the latter can be viewed as a system with unspecified latent variable. One of the most well-know latent variable is the state variable. It is a special type of latent variable that has the property of state [39]. Among the elements of a manifest variable, there is a set of variables called free variables.

**Definition 2.3** (Free Variables [39])**.** For a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ with manifest variable $w = \mathrm{col}(w_1, w_2)$, $w_1$ is said to be free if for all $w_1 \in \mathbb{W}_1^{\mathbb{T}}$, there exists a $w_2 \in \mathbb{W}_2^{\mathbb{T}}$ such that $(w_1, w_2) \in \mathfrak{B}$, i.e. the set of possible trajectories for it is $\mathbb{W}_1^{\mathbb{T}}$.

Free variables include all exogenous inputs such reference and disturbance. If all variables in $w_1$ are free variables while none of the variables in $w_2$ are, then $(w_1, w_2)$ is called an input/output partition of $\mathfrak{B}$. This definition is addressing the indistinguishability among the manifest variables as input/output variables from another point of view: it is because $w_1$ is free that it is called an input variable rather than because $w_1$ being an input variable that it is free.

An important class of dynamical systems that occurs frequently in nature is time-invariant systems.

**Definition 2.4** (Time-invariance [39])**.** A dynamical system (2.1) is time-invariant if $\sigma^t \mathfrak{B} = \mathfrak{B}$ when $\mathbb{T} = \mathbb{R}$ or $\sigma \mathfrak{B} = \mathfrak{B}$ when $\mathbb{T} = \mathbb{Z}$, where $(\sigma^t f)(t_0) := f(t_0 + t)$. If $\mathbb{T} = \mathbb{R}^+$ (respectively, $\mathbb{Z}^+$), then (2.1) is time-invariant if $\sigma^t \mathfrak{B} \subset \mathfrak{B}$ (respectively, $\sigma \mathfrak{B} \subset \mathfrak{B}$).

If a system is time-invariant, then it is possible for the system to have finite memory span. That is, after a certain amount of time, the outcome is independent of the trajectory's history.

**Definition 2.5** (Memory Span [38]). A dynamical system (2.1) is said to have memory span $M$ if the concatenation of two trajectories in $\mathfrak{B}$ having a segment of identical trajectory of at least length $M$ at the point when the identical segment starts is also in $\mathfrak{B}$, i.e.,

$$\left.\begin{aligned} & w_1, w_2 \in \mathfrak{B} \\ & w_1(t) = w_2(t), \forall t \in [0, M) \end{aligned}\right\} \Rightarrow w_1 \wedge_0 w_2 \in \mathfrak{B} \tag{2.2}$$

where $\wedge_0$ means the concatenation at $t = 0$.

Controllability and observability are crucial properties of a dynamical system because they provide information on whether the system can be controlled effectively. In the behavioural framework, they are defined, respectively, as the ability to switch from one trajectory to another and the ability to deduce the trajectories of a part of its variables from the observations of that of the rest.

**Definition 2.6** (Controllability [39]). A time-invariant dynamical system with behaviour $\mathfrak{B}$ is controllable if for any $w_1, w_2 \in \mathfrak{B}$, there exists a trajectory $w \in \mathfrak{B}$ and $t' \geq 0$ such that $w(t) = w_1(t)$ when $t \leq 0$ and $w(t) = w_2(t - t')$ when $t \geq t'$.

**Definition 2.7** (Observability [38]). Given a dynamical system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$ with manifest variable $w = \mathrm{col}(w_1, w_2)$, $w_1$ is observable from $w_2$ in $\Sigma$ if $(w_1, w_2) \in \mathfrak{B}$ and $(w_1', w_2) \in \mathfrak{B}$ imply $w_1 = w_1'$ for all $w_2$.

While a behaviour is essentially a set of trajectories, it can be represented in various ways and each description reveals the insights of a dynamical system from a different perspective. In the next two sections, two most well-studied representations, namely, linear time-invariant systems and dissipative dynamical systems, will be reviewed.

### 2.1.2 Linear Time-Invariant (LTI) Systems

This class of systems is the most widely studied among literature due to its nice structure and the fact that a wide range of nonlinear systems can be locally approximated by LTI systems, thereby facilitating the development of nonlinear controller design. Before introducing the representations of LTI systems, the concept of linearity needs to be defined in the behavioural framework.

**Definition 2.8** (Linearity [38]). A dynamical system (2.1) is linear if $\mathbb{W}$ is a vector space and $\mathfrak{B}$ is a linear subspace of $\mathbb{W}^{\mathbb{T}}$.

An LTI system can therefore be defined as one whose behaviour satisfies the properties in both Definition 2.4 and Definition 2.8. The set of all LTI systems with dimension $\mathrm{w}$ is denoted by $\mathfrak{L}^{\mathrm{w}}$. If a dynamical system is an LTI system with manifest variable $w$, then it is denoted as $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathrm{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathrm{w}}$. However, since $\Sigma$ is in essence determined by $\mathfrak{B}$, LTI behaviours are also denoted as, with slight abuse of notation, $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$. A direct result from Definition 2.8 is that the trajectories in an LTI system obey superposition principle, i.e., $w_1, w_2 \in \mathfrak{B} \Rightarrow aw_1 + bw_2 \in \mathfrak{B}, \forall a, b \in \mathbb{R}$.

#### 2.1.2.1 Systems Described by Linear Differential Equations

This section introduces the description of LTI behaviour as solutions to linear differential equations. Note that while the review is carried out in continuous time, the discrete version have analogous results. The trajectories contained within an LTI system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathrm{w}}, \mathfrak{B})$ can be described by the solutions to a set of differential equations

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w := \sum_{k=0}^{L} \widetilde{R}_k \frac{\mathrm{d}^k}{\mathrm{d}t^k} w = 0, \tag{2.3}$$

with $\widetilde{R}_k \in \mathbb{R}^{\bullet\times\mathrm{w}}$. By defining the differential operator $\xi := \frac{\mathrm{d}}{\mathrm{d}t}$, the *kernel representation* of $\mathfrak{B}$ can be obtained as

$$R(\xi)w = 0, \tag{2.4}$$

where $R(\xi) = \sum_{k=0}^{L} \widetilde{R}_k \xi^k \in \mathbb{R}^{\bullet\times\mathrm{w}}[\xi]$. A behaviour described by a kernel representation (2.4) is denoted as $\mathfrak{B} = \ker(R)$. This representation is the fundamental representation of an LTI system, in that all LTI systems admit a kernel representation. Two kernel representations are said to be *equivalent* if one of them is the other one multiplied by a unimodular matrix, i.e.,

$$\ker(R) = \ker(UR) \tag{2.5}$$

if $U(\xi)$ is unimodular. It can be shown that $\mathfrak{B}$ is controllable if and only if its kernel representation satisfies $\mathrm{rank}(R(\lambda)) = \mathrm{rank}(R)$ for all $\lambda \in \mathbb{C}$. A kernel representation is minimal if and only if $R$ has full row rank.

If a system $\Sigma$ is defined with the aid of the latent variable $\ell$, then $\Sigma^{full} = (\mathbb{R}, \mathbb{R}^{\mathrm{w}}, \mathbb{R}^{\mathrm{l}}, \mathfrak{B}^{full})$ admits a *latent variable representation*

$$R(\xi)w = M(\xi)\ell, \tag{2.6}$$

where $M \in \mathbb{R}^{\bullet\times\mathrm{l}}[\xi]$ is defined in a similar way as $R(\xi)$. If the system is controllable, then it admits a special latent variable representation called *image representations*, in which $R(\xi) = I_{\mathrm{w}}$, i.e.,

$$w = M(\xi)\ell. \tag{2.7}$$

In such a case, the corresponding manifest variable is denoted as $\mathfrak{B} = \mathrm{im}(M)$. If $w$ admits an input/output partition $w = (y, u)$ and $\ell$ has the property of state, then (2.6) can be written as the well-known *state-space representation*

$$\begin{aligned} \xi x &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{2.8}$$

There are two other ways of representing an LTI behaviour, namely the input/output representation $D(\xi)y = N(\xi)u$, in which the manifest variable admits an input/output partition and the transfer function can be obtained as $G(s) = D^{-1}(s)N(s)$ should $D(\xi)$ be invertible, and the generic state-space representation $(E\xi + F)x = Gw$ [39]. The details of these two representations are omitted due to the similarity compared with the ones introduced above. A thorough discussion of the LTI systems have been carried out in [39] and a more accessible introduction can be found in [40]. The theory of LTI systems has been generalised into linear parameter-varying (LPV) systems [41], in which a space for the scheduling variable is added into the description of the system, switched linear systems [42, 43], in which a system is defined as a bank of linear behaviours with gluing conditions to ensure smooth transitions, and high-dimensional systems [44], in which $\mathbb{T} \subset \mathbb{R}^{\bullet}$ is a high-dimensional space.

Due to the similarity of the structures of continuous-time and discrete-time systems, almost all concepts introduced above can be similarly defined for the case when $\mathbb{T} \subset \mathbb{Z}$ by replacing the differential operator $\xi$ with the shift operator $\sigma$. One difference though is that the memory span is the number of steps for which the future is independent of the past, which can be defined as the order of the minimum kernel representation. This is normally called the lag of $\mathfrak{B}$ and denoted as $\mathtt{L}(\mathfrak{B})$ [45].

#### 2.1.2.2 Systems Described by Data

Since the behaviour is viewing a dynamical system from a set-theoretic point of view, it is logical to seek for a way to describe the behaviour of a system using existing data sets. However, it is also obvious that a data set can only represent the behaviour up to a finite length. It is therefore necessary to define the truncated behaviour. Since the interest is on the data set, all subsequent discussions in this particular section are in discrete time.

**Definition 2.9** ([45]). Given a behavioural set $\mathfrak{B}$, the truncated behaviour restricted to the time period $\mathbb{Z} \cap [1, T]$ is defined as

$$\mathfrak{B}_{|[1,T]} \coloneqq \left\{ w : \mathbb{Z}_T^+ \to \mathbb{R}^{\mathrm{w}} | \exists w' \in \mathfrak{B}, \forall k \in \mathbb{Z}_T^+, w_k = w_k' \right\}. \tag{2.9}$$

Although an incomplete and finite approximation, the description of $\mathfrak{B}$ using data sets become more accurate with larger amount of data. In the case of an LTI system, however, one measured trajectory can theoretically parametrise the entire behavioural space. Consider an LTI dynamical system $\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathrm{w}}, \mathfrak{B})$ where $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$ and suppose a measured trajectory $\widetilde{w}_{|[1,T]} = \mathrm{col}\,(w_1, w_2, \ldots, w_T) \in \mathfrak{B}_{|[1,T]}$ is available. Then it is possible to construct a Hankel matrix of order $L$ as

$$\mathfrak{H}_L(\widetilde{w}) = \begin{bmatrix} w_1 & w_2 & \cdots & w_{T-L+1} \\ w_2 & w_3 & \cdots & w_{T-L+2} \\ \vdots & \vdots & \ddots & \vdots \\ w_L & w_{L+1} & \cdots & w_T \end{bmatrix}. \tag{2.10}$$

If $w$ admits an input/output partition $w = (y, u)$, then the Hankel matrix (2.10) can be permuted accordingly and partitioned as $\mathrm{col}\,(\mathfrak{H}_L(\tilde{y}), \mathfrak{H}_L(\tilde{u}))$. A crucial element in the description of a behaviour using data set is the excitation of the free variables, in that no useful dynamics could be observed if the measured trajectory were at steady state. A signal is said to be persistently exciting of order $L$ if $\mathrm{rank}(\mathfrak{H}_L(\widetilde{w})) = L\mathrm{w}$ [45].

Denoting the state cardinality of a behaviour (the number of state variables in a minimal state space representation of $\mathfrak{B}$) as $\mathtt{n}(\mathfrak{B})$, the behaviour up to $L$ steps is then able to be constructed using the following theorem.

**Theorem 2.1** (Behaviour Parametrisation [45])**.** Suppose $\mathfrak{B}$ is controllable. Given a trajectory $\widetilde{w} \in \mathfrak{B}_{|[1,T]}$, if the free variable part is persistently exciting of order $L + \mathtt{n}(\mathfrak{B})$, i.e., if rank $\left(\mathfrak{H}_{L+\mathtt{n}(\mathfrak{B})}\left(\widetilde{w}_f\right)\right) = [L + \mathtt{n}(\mathfrak{B})]\mathrm{w}_\mathrm{f}$, then colspan$(\mathfrak{H}_L(\widetilde{w})) = \mathfrak{B}_{|[1,L]}$.

This theorem suggests that if the system is controllable and that the free variables are persistently exciting of order $L + \mathtt{n}(\mathfrak{B})$, then any trajectory in $\mathfrak{B}_{|[1,L]}$ can be represented as a linear combination of the columns of $\mathfrak{H}_L(\widetilde{w})$. In other words, for all $\widetilde{w}' \in \mathfrak{B}_{|[1,L]}$, there exists a $g \in \mathbb{R}^{T-L+1}$ such that $\widetilde{w}' = \mathfrak{H}_L(\widetilde{w})g$ [46]. Notice that the structure of this representation is similar to that of the image representation. Furthermore, a longer trajectory from $\mathfrak{B}$ can be formulated from two shorter trajectories provided that the final segment of one trajectory is the same as the beginning segment of the other and that the length of this segment is at least $\mathtt{L}(\mathfrak{B})$. This result is summarised by the following lemma.

**Lemma 2.2** ([47])**.** Given $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$ with lag $\mathtt{L}(\mathfrak{B})$, if $w_1 \in \mathfrak{B}_{|[1,L_1]}$, $w_2 \in \mathfrak{B}_{|[1,L_2]}$ and $\widetilde{w}_{1|[L_1-l+1,L_1]} = \widetilde{w}_{2|[1,l]}$ with $l \geq \mathtt{L}(\mathfrak{B})$, then

$$\widetilde{w}_{|[1,L_1+L_2-l]} = \widetilde{w}_{1|[1,L_1-l]} \wedge_{L_1-l+1} \widetilde{w}_{2|[1,L_2]} \in \mathfrak{B}_{|[1,L_1+L_2-l]}. \tag{2.11}$$

### 2.1.3 Dissipative Dynamical Systems

#### 2.1.3.1 Representations

Dissipative systems are those for which the change in stored energy is bounded by the amount of energy supplied by the environment. This concept was initially extended into a property of a dynamical system setting by Willems in 1972 [48].

**Definition 2.10** (Dissipativity [48])**.** A continuous-time system

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{2.12}$$

is said to be dissipative if there exist a function defined on input and output, called the *supply rate* $s(y, u)$ and a positive semidefinite function defined on the state, called the *storage function* $V(x)$ such that

$$V(x(T)) - V(x(t_0)) \leq \int_{t_0}^{T} s(y(t), u(t))\, \mathrm{d}t, \quad \forall t_0 \in \mathbb{R},\ T \geq t_0. \tag{2.13}$$

This property has been used widely in the analysis and control design of a system, in that it is a generalisation and unification of many important results in control theory. To name but a few examples, Lyapunov stability criterion can be recovered by setting $s(y, u) = 0$, passivity can be recovered by setting $s(y, u) = y^T u$ and $\mathcal{L}_2$ gain condition can be recovered by choosing $s(y, u) = \gamma^2 u^T u - y^T y$. Other important results such as the positive-real lemma, bounded-real lemma and Kalman–Yakubovich–Popov (KYP) lemma are all specific results by choosing appropriate supply rates [49–52]. In all cases mentioned above, the supply rate can be written as a general form

$$s(y, u) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{2.14}$$

where $Q \in \mathbb{S}^{\mathrm{y}}$, $S \in \mathbb{R}^{\mathrm{y} \times \mathrm{u}}$ and $R \in \mathbb{S}^{\mathrm{u}}$. The matrix $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$ is called the *supply rate matrix*. The dissiaptivity property with respect to this type of supply rate is called QSR-dissipativity, or dissipativity with respect to a quadratic supply rate. QSR-dissipativity for LTI systems have been well studied in [53], and the conditions for QSR-dissipativity for nonlinear systems have been given in [54].

With the development of the behavioural systems theory, dissipative dynamical systems have been integrated into the framework, resulting in it being a representation of a distinct class of dynamical systems with its own manifest variables that depends only on time.

**Definition 2.11** (Dissipative Dynamical Systems [55])**.** A dynamical system $\Sigma = (\mathbb{R}, \mathbb{R}, \mathfrak{B})$ with manifest variable $s : \mathbb{R} \to \mathbb{R}$ representing the rate of supply absorbed by $\Sigma$ is a dissipative dynamical system if for all $s \in \mathfrak{B}$, $t_0 \in \mathbb{R}$, $T \geq t_0$, there exists $M \in \mathbb{R}$ such that

$$-\int_{t_0}^{T} s(t)\ \mathrm{d}t \leq M. \tag{2.15}$$

This suggests that in a dissipative system, the supply rate $s(t)$ is considered as the manifest variable, which is natural and reasonable considering it containing the description of the dynamical features of a system. The latent variable in this class of system is the classical storage function, and a dissipative dynamical system with latent variable can be accordingly defined.

**Definition 2.12** ([40])**.** For a dynamical system with latent variable $\Sigma^{full} = (\mathbb{R}, \mathbb{R}, \mathbb{R}, \mathfrak{B}^{full})$ whose latent variable $V : \mathbb{R} \to \mathbb{R}$ represents the supply stored in the system and satisfies

$$V(T) - V(t_0) \leq \int_{t_0}^{T} s(t)\ \mathrm{d}t \tag{2.16}$$

for all $t_0 \in \mathbb{R}$, $T \geq t_0$, its corresponding manifest system $\Sigma$ is a dissipative dynamical system if $V(t)$ is non-negative. The inequality (2.16) is call the *dissipation inequality.*

If the time axis is $\mathbb{Z}$ instead of $\mathbb{R}$, then the representations for both the manifest behaviour and that with latent variables can be constructed analogously as

$$-\sum_{k=k_0}^{K} s(k) \leq M \tag{2.17}$$

and

$$V(K+1) - V(k_0) \leq \sum_{k=k_0}^{K} s(k), \tag{2.18}$$

respectively.

#### 2.1.3.2 Quadratic Differential/Difference Forms

System analysis based on QSR-type supply rate can be conservative, as the supply rate in (2.14) can only provide a coarse description of the system's dynamic features [56]. The quadratic differential form (QDF) was introduced in [57] to represent dissipativity property of system behaviours in a more relaxed and detailed way. A QDF $Q_\Phi(w)$ takes the form

$$Q_\Phi(w) := \sum_{k=0}^{L}\sum_{l=0}^{L}\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k}w\right)^T \widetilde{\Phi}_{kl}\left(\frac{\mathrm{d}^l}{\mathrm{d}t^l}w\right), \tag{2.19}$$

where $L$ is the order of the QDF. Defining $w^T\zeta = \dot{w}^T$ and $\eta w = \dot{w}$, a QDF can also be written more compactly as

$$Q_\Phi(w) = w^T\Phi(\zeta,\eta)w \tag{2.20}$$

where

$$\Phi(\zeta,\eta) = \sum_{k=0}^{L}\sum_{l=0}^{L}\zeta^k\Phi_{kl}\eta^l \tag{2.21}$$

and is said to be *induced* by the two-variable polynomial matrix $\Phi \in \mathbb{S}^{\mathrm{w}}[\zeta,\eta]$. The corresponding coefficient matrix, denoted as $\widetilde{\Phi}$, is

$$\widetilde{\Phi} = \begin{bmatrix} \Phi_{00} & \Phi_{01} & \cdots & \Phi_{0L} \\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1L} \\ \vdots & \vdots & \ddots & \vdots \\ \Phi_{L0} & \Phi_{L1} & \cdots & \Phi_{LL} \end{bmatrix}. \tag{2.22}$$

A QDF induced by $\Phi$ is said to be positive (non-negative), denoted by $\Phi \succ 0$ ($\succeq 0$) if and only if $\widetilde{\Phi} > 0$ ($\geq 0$). The derivative of a QDF $\frac{\mathrm{d}}{\mathrm{d}t}Q_\Phi(w)$ is induced by

$$\nabla\Phi(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta). \tag{2.23}$$

The definitions for the discrete-time counterpart, quadratic difference form (QdF), is almost exactly the same [58, 59], except that the indeterminates $\zeta$ and $\eta$ represent shift operators instead of differential operators, i.e., $w_k^T\zeta = w_{k+1}^T$ and $\eta w_k = w_{k+1}$, and that instead of a derivative, a rate of change operation, $Q_\Phi(w_{k+1}) - Q_\Phi(w_k)$, is defined, and it is induced by

$$\nabla\Phi(\zeta, \eta) = (\zeta\eta - 1)\Phi(\zeta, \eta). \tag{2.24}$$

#### 2.1.3.3 QDF/QdF Dissipativity of LTI Systems

Due to the similarity between the structure of continuous-time and discrete-time systems, concepts in this sections are reviewed for continuous-time systems only. For a dynamical system (2.1) with trajectories in $\mathfrak{B}$ infinitely differentiable, dissipation inequality (2.16) can be written in a differential form and both $s(t)$ and $V(t)$ can be expressed as QDFs of $w$. As a result, the dissipativity with respect to a QDF is given in the following definition.

**Definition 2.13** (QDF dissipativity [57]). A dynamical system (2.1) with $\mathbb{T} \subset \mathbb{R}$ is said to be dissipative with respect to $Q_\Phi(w)$, or $\Phi$-dissipative, if there exists a storage function induced by $Q_\Psi(w)$ such that

$$\begin{aligned} Q_\Psi(w) &\geq 0 \\ \frac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) &\leq Q_\Phi(w) \end{aligned} \tag{2.25}$$

holds for all $w \in \mathfrak{B}$, where $Q_\Phi(w)$ is a QDF supply rate.

Since QDF dissipativity takes the extended manifest variable

$$\hat{w} \coloneqq \operatorname{col}\left(w, w^{(1)}, \ldots, w^{(L)}\right)$$

into account, it captures much more detailed features of the system dynamics compared to the QSR-type dissipativity, leading to a much less conservative dissipativity-based analysis [56]. Assuming that the manifest variable has an input-output partition $w = (y, u)$ with $u \in \mathbb{R}^{\mathrm{u}}$ and $y \in \mathbb{R}^{\mathrm{y}}$, the QDF $Q_\Phi(w)$ can be rewritten as

$$Q_\Phi(y,u) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}(\zeta,\eta) & \mathcal{S}(\zeta,\eta) \\ \mathcal{S}^\star(\zeta,\eta) & \mathcal{R}(\zeta,\eta) \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} \hat{y} \\ \hat{u} \end{bmatrix} \begin{bmatrix} \widetilde{\mathcal{Q}} & \widetilde{\mathcal{S}} \\ \widetilde{\mathcal{S}}^T & \widetilde{\mathcal{R}} \end{bmatrix} \begin{bmatrix} \hat{y} \\ \hat{u} \end{bmatrix} \tag{2.26}$$

where $\mathcal{Q} \in \mathbb{S}^{\mathrm{y}}[\zeta,\eta]$, $\mathcal{S} \in \mathbb{R}^{\mathrm{y}\times\mathrm{u}}[\zeta,\eta]$, $\mathcal{R} \in \mathbb{S}^{\mathrm{u}}[\zeta,\eta]$ and

$$\hat{y} = \mathrm{col}\left(y, y^{(1)}, \cdots, y^{(L)}\right), \hat{u} = \mathrm{col}\left(u, u^{(1)}, \cdots, u^{(L)}\right),$$

which means that $Q_\Phi(y,u)$ can be understood as a dynamic QSR supply rate. As a result, it can be used to describe frequency-weighted $\mathcal{H}_\infty$ gain bound of LTI systems.

**Proposition 2.3** ([56])**.** Assuming that an LTI system $G : u \mapsto y$ with transfer function $G(s)$ is $\Phi$-dissipative (i.e., dissipative with respect to the supply rate given in (2.26)) with $\mathcal{Q}(\zeta,\eta) \prec 0$ (i.e., $\widetilde{\mathcal{Q}} < 0$), then the system $G$ satisfies

$$\left\| \frac{N}{\alpha} G \right\|_\infty \leq \gamma \tag{2.27}$$

where, $N(j\omega) = (-\mathcal{Q})^{\frac{1}{2}}(-j\omega, j\omega)$ and a scalar $\alpha(\omega) > 0$ satisfying

$$\gamma^2\alpha^2(\omega)I \geq \mathcal{R}(-j\omega,j\omega) - \mathcal{S}^T(-j\omega,j\omega)\mathcal{Q}^{-1}(-j\omega,j\omega)\mathcal{S}(-j\omega,j\omega), \ \ \forall\omega. \tag{2.28}$$

The dissipativity analysis of a dynamical system using QDFs relies exclusively on the positivity of them along the behaviour of the system. A QDF is said to be positive (respectively, non-negative) along a behaviour $\mathfrak{B}$, denoted as $\Phi \overset{\mathfrak{B}}{\succ} 0$ (respectively, $\Phi \overset{\mathfrak{B}}{\succeq} 0$) if it is positive (respectively, non-negative) for all $w \in \mathfrak{B}$ [57]. With this definition, stability and dissipativity can be verified accordingly.

**Proposition 2.4** (Stability and dissipativity along $\mathfrak{B}$ [57])**.** For a dynamical system (2.1),

(i) $\mathfrak{B}$ is stable if there exists $\Psi \overset{\mathfrak{B}}{\succeq} 0$ such that $-\nabla\Psi \overset{\mathfrak{B}}{\succeq} 0$. The QDF induced by $\Psi \in \mathbb{S}^{\mathrm{w}}[\zeta, \eta]$ is a Lyapunov function.

(ii) $\mathfrak{B}$ is $\Phi$-dissipative if there exists $\Psi \overset{\mathfrak{B}}{\succeq} 0$ such that $\Phi - \nabla\Psi \overset{\mathfrak{B}}{\succeq} 0$. The QDFs induced by $\Phi \in \mathbb{S}^{\mathrm{w}}[\zeta, \eta]$ and $\Psi \in \mathbb{S}^{\mathrm{w}}[\zeta, \eta]$ are the supply rate and a storage function, respectively, and the pair $(\Phi, \Psi)$ is a dissipativity property of $\mathfrak{B}$.

If $\Sigma \in \mathfrak{L}^{\mathrm{w}}$ with $\mathfrak{B} = \ker(R)$, then the non-negativity of a QDF along $\mathfrak{B}$ can be validated using the following proposition.

**Proposition 2.5** (Non-negativity along a behaviour [57]). Given a QDF $\Phi \in \mathbb{S}^{\mathrm{w}}[\zeta, \eta]$ and a behaviour $\mathfrak{B} = \ker(R)$ where $R \in \mathbb{R}^{\bullet\times\mathrm{w}}[\xi]$, then $\Phi \overset{\mathfrak{B}}{\succeq} 0$ if and only if there exists $F \in \mathbb{R}^{\bullet\times\mathrm{w}}[\zeta, \eta]$ such that

$$\Phi(\zeta, \eta) + \mathrm{He}\left\{F^{\star}(\zeta, \eta)R(\eta)\right\} \succeq 0. \tag{2.29}$$

In (2.29), the two-variable polynomial matrix $F(\zeta, \eta)$ can be understood as a Lagrange multiplier for the "equality constraint" – kernel representation (2.4). With this result, the dissipativity along a behaviour $\mathfrak{B}$ can be validated using the following theorem.

**Theorem 2.6** ($\Phi$-dissipativity along $\ker(R)$ [60]). Let $\mathfrak{B}$ be given by the kernel representation in (2.4). $\mathfrak{B}$ is $\Phi$-dissipative if there exist $\Psi \in \mathbb{S}^{\mathrm{w}}[\zeta, \eta]$ with $\Psi \succeq 0$, and $F \in \mathbb{R}^{\mathrm{w}\times\bullet}[\zeta, \eta]$ such that

$$\Phi(\zeta, \eta) - \nabla\Psi(\zeta, \eta) + \mathrm{He}\left\{F^{\star}(\zeta, \eta)R(\eta)\right\} \succeq 0 \tag{2.30}$$

where $\nabla\Psi(\zeta, \eta)$ is given in (2.23) for continuous-time systems and in (2.24) for discrete-time systems.

The condition in Theorem 2.6 can be further converted into a linear matrix inequality (LMI) [61]

$$\widetilde{\Phi} - \nabla\widetilde{\Psi} + \mathrm{He}\left\{\widetilde{F}^{T}\widehat{R}\right\} \geq 0 \tag{2.31}$$

with $\widetilde{\Phi}$, $\widetilde{\Psi}$, $\widetilde{F}$ the coefficient matrices of $\Phi(\zeta,\eta)$, $\Psi(\zeta,\eta)$, $F(\zeta,\eta)$, respectively (all of which can be obtained according to (2.22)), and

$$\nabla\widetilde{\Psi} = \begin{bmatrix} 0 & \widetilde{\Psi} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \widetilde{\Psi} & 0 \end{bmatrix}, \; \widehat{R} = \begin{bmatrix} R_0 & R_1 & \cdots & R_L & 0 & \cdots & 0 \\ 0 & R_0 & R_1 & \cdots & R_L & \ddots & 0 \\ \vdots & \ddots & \cdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & R_0 & R_1 & \cdots & R_L \end{bmatrix}. \tag{2.32}$$

Conditions when $\mathfrak{B}$ is represented as a state-space model are presented in [62].

The discrete counterpart of the above concepts and operations can be formulate analogously by replacing $\nabla\Phi$ defined in (2.23) into that defined in (2.24), in which case the second condition in (2.25) as

$$Q_\Psi(w_{k+1}) - Q_\Psi(w_k) \leq Q_\Phi(w_k) \tag{2.33}$$

and $\nabla\widetilde{\Psi}$ in (2.31) should be replaced by

$$\nabla\widetilde{\Psi} = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{\Psi} \end{bmatrix} - \begin{bmatrix} \widetilde{\Psi} & 0 \\ 0 & 0. \end{bmatrix}$$

In the case when $\mathbb{T} = \mathbb{Z}_L^+$, and $\mathfrak{B}_{|[1,L]}$ is described by a data set, the operators $\zeta$ and $\eta$ can still be defined as forward-shifting operators but the "current" step is in fact the last step of the QdF. The rate of change $\nabla\Phi$ can be defined similarly as in (2.24) but the implication is the difference between the current step and the previous step (i.e., backward difference) rather than between the next step and the current step (i.e., forward difference). For the clarity of presentation, such a QdF is still denoted at $Q_\Phi(w_k)$ but the dissipation inequality (2.33) should be changed as

$$Q_\Psi(w_k) - Q_\Psi(w_{k-1}) \leq Q_\Phi(w_k). \tag{2.34}$$

Furthermore, the notion of $\Phi$-dissipativity is less useful because the time axis itself is finite. The definition of the dissipativity on a finite time axis is hence given as follows.

**Definition 2.14** ($\Phi-L$-dissipativity [63]). A dynamical system $\Sigma = \left(\mathbb{Z}_L^+, \mathbb{W}, \mathfrak{B}_{|[1,L]}\right)$ is $\Phi-L$-dissipative if

$$\sum_{k=1}^{L} Q_\Phi(w_k) \geq 0 \tag{2.35}$$

for all $w \in \mathfrak{B}_{|[1,L]}$.

$\Phi-L$-dissipativity of a system $\Sigma$ when $\mathfrak{B}_{|[1,L]}$ is represented by a the column span of a Hankel matrix are given in [63, 64].

#### 2.1.3.4 Factorisation of QDFs

An important information QDFs have is the parametrisation of behaviours and the mapping from a relatively complex behaviour to a much simpler one [65, 66]. This is done through polynomial spectral factorisation, or $J$-factorisation [67–69], in which a para-hermitian polynomial matrix (i.e., $Z^T(\xi) = Z(-\xi)$) with constant signature $(\pi_+, 0, \pi_-)$, with $\pi_+$ and $\pi_-$ being the positive and negative eigenvalues of $Z(\xi)$, is factorised as

$$Z(\xi) = K^T(-\xi) J K(\xi), \tag{2.36}$$

where $J = \begin{bmatrix} I_{\pi_+} & 0 \\ 0 & -I_{\pi_-} \end{bmatrix}$ is a signature matrix. Obviously, if $\Phi \in \mathbb{S}^\bullet[\zeta, \eta]$, then $\Phi(-\xi, \xi)$ is para-hermitian.

**Theorem 2.7** ([65]). Assuming that the polynomial matrix $\Phi \in \mathbb{S}^{\mathrm{w}}[\zeta, \eta]$ admits a $J$-factorisation (2.36), then the following statements hold:

(i) if $\mathfrak{B} = \mathrm{im}(M)$ is $\Phi$-dissipative, then $\mathfrak{B}' = \mathrm{im}(KM)$ is $J$-dissipative;

(ii) let $K^{\mathrm{A}}(\xi)$ be the adjoint matrix for $K(\xi)$, i.e., $K^{\mathrm{A}}(\xi)K(\xi) = \det(K(\xi))I$. If $\mathfrak{B} = \mathrm{im}(M)$ is $J$-dissipative, then $\mathfrak{B}' = \mathrm{im}(K^{\mathrm{A}}M)$ is $\Phi$-dissipative.

In [68], a complete algorithm to compute such a factorisation using the solution to an algebraic Riccati equation (ARE) of sufficiently high order has been formulated.

**Algorithm 2.1** ([68])**.**

(1) Find a solution $P \in \mathbb{S}^{(L+h)\mathrm{w}}$ to the ARE

$$\mathrm{He}\,\{PA(h)\}+\phi_{11}(h)-(\phi_{01}(h)+B^T(h)P)^T\phi_{00}^{-1}(\phi_{01}(h)+B^T(h)P) = 0 \quad (2.37)$$

where $h \geq 0$ is sufficiently large, $\phi_{00} = \widetilde{\Phi}_{00}$,

$$\phi_{01}(h) = \begin{bmatrix} \widetilde{\Phi}_{01} & \cdots & \widetilde{\Phi}_{0L} & 0_{\mathrm{w}\times h\mathrm{w}} \end{bmatrix},\ \phi_{11}(h) = \begin{bmatrix} \widetilde{\Phi}_{11} & \cdots & \widetilde{\Phi}_{1L} & 0_{\mathrm{w}\times h\mathrm{w}} \\ \vdots & \ddots & \vdots & \vdots \\ \widetilde{\Phi}_{1L} & \cdots & \widetilde{\Phi}_{LL} & 0_{\mathrm{w}\times h\mathrm{w}} \\ 0_{\mathrm{w}\times h\mathrm{w}} & \cdots & 0_{\mathrm{w}\times h\mathrm{w}} & 0_{\mathrm{w}\times h\mathrm{w}} \end{bmatrix},$$

$$A(h) = \begin{bmatrix} 0_{\mathrm{w}(L+h-1)\mathrm{w}} & 0_{\mathrm{w}\times\mathrm{w}} \\ I_{(L+h-1)\mathrm{w}} & 0_{(L+h-1)\mathrm{w}\times\mathrm{w}} \end{bmatrix},\ B(h) = \begin{bmatrix} I_{\mathrm{w}} \\ 0_{(L+h-1)\mathrm{w}\times\mathrm{w}} \end{bmatrix}.$$

(2) Factorise $\phi_{00} = K_0^T J K_0$ with a non-singular $K_0 \in \mathbb{S}^{\mathrm{w}}$, and $K(\xi)$ can be constructed as

$$K(\xi) = K_0 \begin{bmatrix} I_{\mathrm{w}} & \phi_{00}^{-1}(\phi_{01}(h) + B^T(h)P) \end{bmatrix} \mathcal{I}_{\mathrm{w}}^L(\xi) \quad (2.38)$$

where $\mathcal{I}_{\mathrm{w}}^L(\xi) = \mathrm{col}(I_{\mathrm{w}}, \xi I_{\mathrm{w}}, \ldots, \xi^L I_{\mathrm{w}})$.

### 2.1.4 Interconnection of Systems

One of the advantages of the behavioural approach is that the interconnection of systems can be carried out easily. Rather than viewing interconnections as signals flowing from one subsystem to another, they are viewed as two systems sharing the same variables. The rationale for the analysis of interconnections follows the roadmap of "tearing" - partitioning an interconnected system into small pieces of subsystems, "zooming" - analysing the dynamics of each piece, and "linking" - determining the sharing variables and forcing external equivalences on them [40]. For two systems

$$\Sigma^1 = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B}^1),\ \Sigma^2 = (\mathbb{T}, \mathbb{W}_2 \times \mathbb{W}_3, \mathfrak{B}^2) \tag{2.39}$$

with manifest variables $w^1 = (w_1, w_2)$ and $w^2 = (w_2, w_3)$, respectively, the interconnected system (through $w_2$) has behaviour

$$\mathfrak{B} = \left\{(w_1, w_2, w_3) | (w_1, w_2) \in \mathfrak{B}^1, (w_2, w_3) \in \mathfrak{B}^2\right\}. \tag{2.40}$$

This type of interconnection is called partial interconnection [28, 29]. If all variables are shared in the interconnection, i.e., if $\Sigma^1$ and $\Sigma^2$ have the same signal space, then the interconnection is called full interconnection [34, 70]. In this case, the interconnected behaviour is simply the common trajectories admissible to both subsystems, i.e., $\mathfrak{B} = \mathfrak{B}^1 \cap \mathfrak{B}^2$. All partial interconnections can be augmented in to full interconnections by assuming the variables belonging to the other systems as free variables. For example, the two systems in (2.39) can be augmented as

$$\begin{aligned} \Sigma^1 &= (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3, \mathfrak{B}^1 \times \mathbb{W}_3^{\mathbb{T}}), \\ \Sigma^2 &= (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2 \times \mathbb{W}_3, \mathbb{W}_1^{\mathbb{T}} \times \mathfrak{B}^2), \end{aligned} \tag{2.41}$$

after which the signal spaces become the same.

If the systems in question are LTI systems, then there exists a kernel representation for every $\mathfrak{B} \in \mathfrak{L}^{\bullet}$. This can be viewed as a set of "rules" that the trajectories in the behaviour must satisfy. If two systems were interconnected, then the admissible trajectories should be those that satisfy the "rules" of both systems. In fact, this rationale can be summarised into the lemma below.

**Lemma 2.8** (Interconnection of linear systems [28])**.** Suppose $\Sigma^1, \Sigma^2 \in \mathfrak{L}^{\mathrm{w}}$ with $\mathfrak{B}^1 = \ker(R^1)$, $\mathfrak{B}^2 = \ker(R^2)$, in which at least one of $R^1$ and $R^2$ is a polynomial matrix, then

$$\mathfrak{B}^1 \cap \mathfrak{B}^2 = \ker\left(\begin{bmatrix} R^1 \\ R^2 \end{bmatrix}\right).$$

On the other hand, if two systems are dissipative dynamical systems, then the interconnected system has a manifest variable being the linear combination of that of its subsystems [62]. For example, consider two systems ($G_1 : u_1 \mapsto y_1$ and $G_2 : u_2 \mapsto y_2$) in negative feedback configuration (as shown in Figure 2.1) with supply rate matrix $\begin{bmatrix} Q_i & S_i \\ S_i^T & R_i \end{bmatrix}$, $i = 1, 2$. The net supply rate of the closed-loop system from $\mathrm{col}(d_1, d_2)$ to $\mathrm{col}(y_1, y_2)$ can be written as

$$s(y_1, y_2, d_1, d_2) = \begin{bmatrix} y_1 \\ y_2 \\ d_1 \\ d_2 \end{bmatrix}^T \begin{bmatrix} Q_1 + R_2 & S_1 + S_2^T & S_1 & R_2 \\ S_1^T + S_2 & Q_2 + R_2 & R_1 & S_2 \\ S_1^T & R_1 & R_1 & 0 \\ R_2 & S_2^T & 0 & R_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ d_1 \\ d_2 \end{bmatrix}. \quad (2.42)$$

With the above discussions, it is obvious that since the behavioural approach is a theory focusing on the external variables with no *a-priori* partition of input and output, it is a scalable and flexible framework for analysis of large-scale networked systems [34, 55, 71].

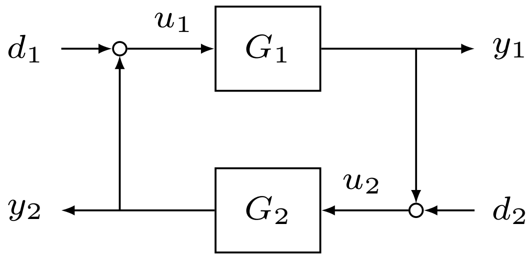



Figure 2.1: Feedback interconnection of two systems.

### 2.1.5 Control as Interconnection

One of the cases with an interconnected system is where one or more of the subsystems are controllers. This means that controllers are simply another dynamical system used to restrict the possible outcome of the system. This is another core concept in the behavioural systems theory: the controllers are not creating new trajectories nor do they shape the to-be-controlled system into a new direction. It is simply selecting trajectories that are *already* in the system that meet the design criteria. In other words, if the system originally contains no trajectory satisfying the prescribed requirements, then the effort of designing a controller to achieve the goals is futile.

Denoting the uncontrolled behaviour as $\mathfrak{B} = \{w|\exists w_c, (w, w_c) \in \mathfrak{B}^{full}\}$, the controller behaviour as $\mathfrak{B}_c$ with manifest variable $w_c$ and the controlled behaviour as $\mathfrak{B}_d = \{w|\exists w_c \in \mathfrak{B}_c, (w, w_c) \in \mathfrak{B}^{full}\}$, it is obvious that $\mathfrak{B}_d$ is a restricted behaviour from $\mathfrak{B}$ by $\mathfrak{B}_c$. Control design then becomes the search for such a controller behaviour $\mathfrak{B}_c$ that implements $\mathfrak{B}_d$. For $\mathfrak{B} \in \mathfrak{L}^{\mathrm{w}}$, an important concept is the hidden behaviour $\mathfrak{B}_h = \{w|(w, 0) \in \mathfrak{B}^{full}\}$. This set of trajectories are called "hidden" trajectories because no dynamics can be observed from $w_c$ should any of these trajectories occur. As a result, if a trajectory $w \in \mathfrak{B}$ has a corresponding trajectory $w_c$, then all trajectories in $\{w\} \oplus \mathfrak{B}_h$, where

$$A \oplus B := \{a + b|a \in A, b \in B\} \tag{2.43}$$

have the corresponding trajectory as $w_c$. Obviously, if $\mathfrak{B}_h = \{0\}$, then $w$ is observable from $w_c$ [72]. With these definitions, the implementability of a controlled behaviour can be checked using the following theorem.

**Theorem 2.9** (Controller implementability [73])**.** Given an uncontrolled behaviour $\mathfrak{B}$ and a desired controlled behaviour $\mathfrak{B}_d$, there exists a controller behaviour $\mathfrak{B}_c$ that implements $\mathfrak{B}_d$ if and only if

$$\mathfrak{B}_h \subset \mathfrak{B}_d \subset \mathfrak{B}. \tag{2.44}$$

This theorem conveys an important message: for the controlled behaviour to be implementable, the *entire* hidden behaviour must be contained in the controlled behaviour. This is because they are indistinguishable from $w_c$ point of view, and there is no way to eliminate these trajectories through control. This characterisation works only for LTI systems because trajectories in $\mathfrak{L}^{\bullet}$ follows superposition principle. In [73], necessary and sufficient conditions for the existence of an implementable LTI controlled behaviour satisfying a requirement specified using QSR-type supply rate have been given with the aid of QDF-type storage functions.

## 2.2 Distributed Control of Networked Systems

Distributed control structure is the interconnection of a network of subsystems and a network of local controllers. One of the structures is to group each subsystem with its respective controller into individual feedback loops and regroup all other dynamics as a large system. In such a way, a controlled interconnected system can be viewed as two large systems with feedback interconnection [74, 75]. This method is an intuitive way for the analysis of such a system as it has similar structure as a feedback system, but due to various assumptions on the system

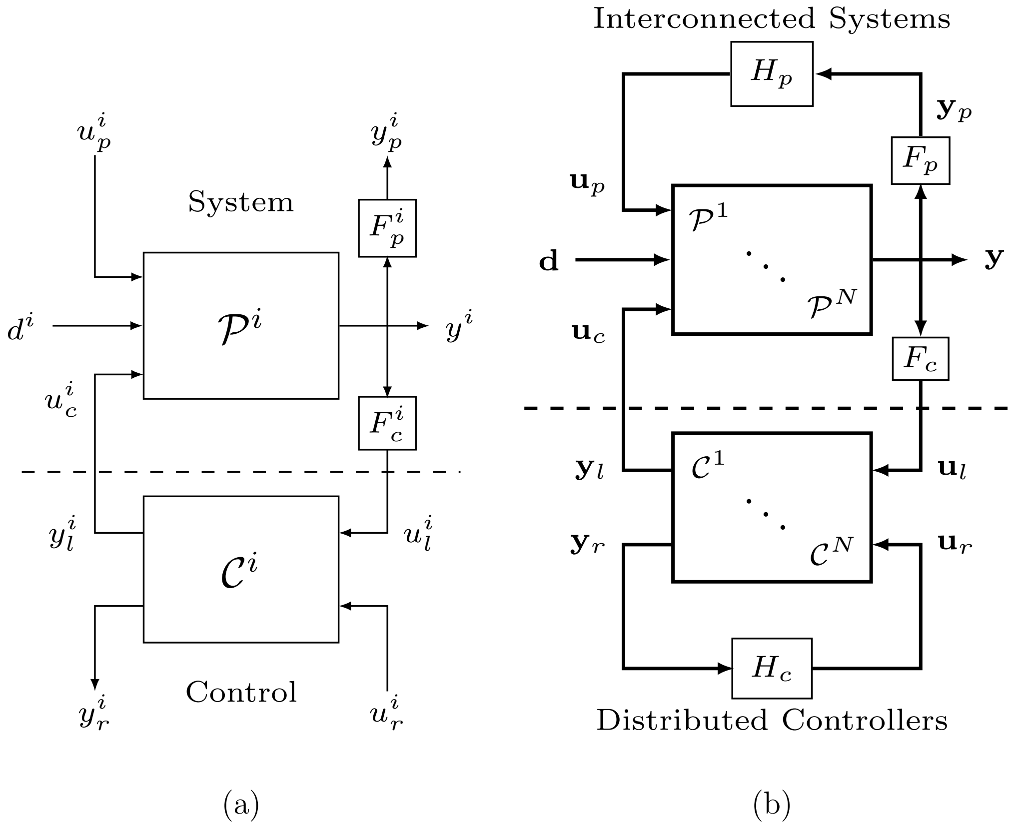



Figure 2.2: Configurations of (a) a controlled subsystem and (b) distributed control of an interconnected system

structure, flexibility becomes an issue if the interactions among subsystems and controllers become progressively complex.

An alternative structure is to represent the interconnected system as a network of subsystems and design a network of local controllers to control the interconnected system system [56, 62]. On the local level, as shown in Figure 2.2a, each subsystem $\mathcal{P}^i$, $i \in \mathbb{Z}_N^+$ is with output $y^i$, interconnecting input $u_p^i$, control input $u_c^i$, and external disturbance $d^i$. The process interconnecting output $y_p^i = F_p^i y^i$ denotes the physical flows to other subsystems. Each subsystem is equipped with a local controller $\mathcal{C}^i$, which is a dynamical system with $y_l^i$ the control action for $\mathcal{P}^i$, $u_l^i$ the process measured output and $u_r^i$, $y_r^i$ the exchanged input/output information between $\mathcal{C}^i$ and other controllers. $\mathcal{P}^i$ and $\mathcal{C}^i$ are interconnected through the relationships $u_l^i = F_c^i y^i$ and $y_l^i = u_c^i$. On the global level, as depicted in Figure 2.2b, the subsystems are arranged in two arrays. The corresponding local variables are stacked together as the corresponding variables for the entire system. Matrices $F_p = \mathrm{diag}(F_p^1, \ldots, F_p^N)$ and $F_c = \mathrm{diag}(F_c^1, \ldots, F_c^N)$ represent "selectors" for the interconnecting and measured outputs. Matrix $H_p$ with elements either 0 or 1 represents the process network topology. The interconnection relations among subsystems can be described by

$$\mathbf{u}_p = H_p \mathbf{y}_p = H_p F_p \mathbf{y} \tag{2.45}$$

where $\mathbf{y} = \mathrm{col}(y^1, y^2, \ldots, y^N)$ and $\mathbf{u}_p$ can be defined analogously. Similarly, $H_c$ defines the controller communication network, which may have any structure. A common choice is the controller network to have the same topology as the process network, $\mathbf{y}_r = \mathbf{y}_p$ and $\mathbf{u}_r = \mathbf{u}_p$. This structure also includes the fully decentralised control structure as a special case by choosing $H_c = 0$. This layout is much more flexible, in that almost no structural constraints are imposed onto the system.

Distributed controllers can mainly be synthesised in two ways. The first way is assuming controller structure and solving the undetermined coefficients [76]. Interconnection is dealt with by substituting one model into another, resulting in a multi-variable model for the entire controlled system. The to-be-designed controllers can then be found using relative LMI solving techniques. Another direction that has seen rapid developments is the distributed model predictive control (DMPC), in which an array of controllers computes the control actions for their respective subsystems in real time subject to constraints such as terminal costs, global constraints and coordination [22, 77–79].

Since dissipativity properties are easy to handle in a complex system, a method centring around dissipativity has been proposed in [56]. In this method, a dissipativity property is solved for each subsystem, and a supply rate for each controller to be satisfied is assigned as a decision variable. The controller supply rates are then solved simultaneously with that of the subsystems under the condition that the resulting supply rate for the controlled distributed system (as linear combinations of that of the subsystems and controllers) meets the design criteria, which can also be specified using dissipativity properties. In such a way, all resulting equations and inequalities are LMIs with respect to the decision variables, which are readily solvable using any convex optimisation toolboxes such as YALMIP [80]. The resulting controller supply rates can be used to deduce controller trajectories either through $J$-factorisation mentioned above or as constraints through distributed model predictive control [81–83]. It is important to know that none of the supply rates, except for the one for the entire controlled system, has any physical meaning. From the behavioural point of view, they are just another set of variables used to describe the dynamical system from a different perspective. This method inevitably introduces a certain degree of conservativeness, in that dissipativity is a sufficient condition, but by solve all supply rates and storage functions simultaneously, the level of conservativeness can be significantly reduced.

A crucial issue in the design of distributed controllers is the choice of the controller network. In many cases, there are prescribed requirements that the network design must obey. For example, in [78], it is required that all controllers must be interconnected with each other to ensure full communication. If no specific requirements are imposed, then a common choice of the controller network is to be the same as the system network [76]. It is also possible to "optimise" the network design by integrating it into the cost function [84].

## 2.3 Robust Control

Robust control has been an important and ongoing research topic for a long period of time due to its practicality in applications. It is the branch of control theory that aims to design controllers for processes with inaccurate models. This problem may arise from many causes. For example, since many models are determined empirically, inaccuracy is an inherent property of them [18, 24]. Other examples include modelling assumptions, parameter uncertainties and linearisation errors. The control design is therefore to ensure stability and a certain level of performance for the worst case scenario. Controllers designed based on this rationale are undoubtedly conservative, but the method is practical and works reasonably well if a good description of uncertainty is available.

### 2.3.1 The Norm-bound Structure

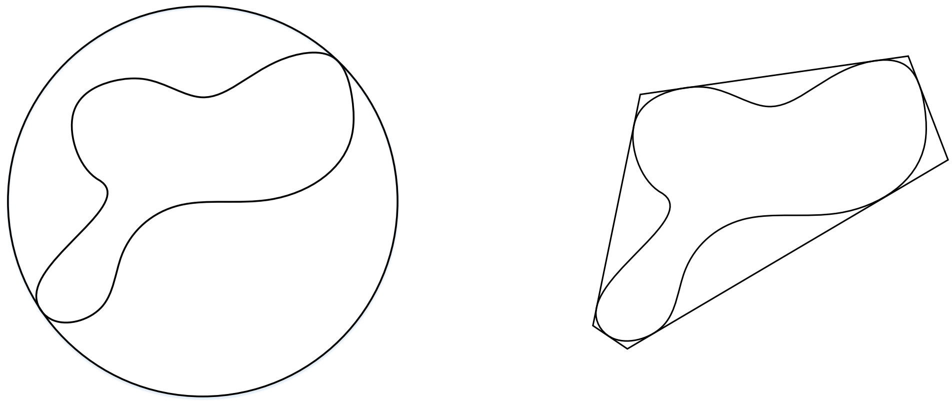
The norm-bound description of uncertainty describes the uncertain system as a theoretical model with a hyper-spherical region of uncertainty. The theoretical model is called the nominal model and it is assumed to be able to describe the system dynamics reasonably well if there were no uncertainties involved. Robust stability of this type of uncertain systems has been extensively studied in either input-output formulation or state-space representation [25–27, 85–90].

Given a nominal model with transfer function $G_0(s) = D_0^{-1}(s)N_0(s)$, depending on the structure, the uncertainty can be characterised as [91]

- additive uncertainty: $G = G_0 + w_A\Delta_A$, $\|\Delta_A\|_\infty < 1$,
- multiplicative uncertainty: $G = G_0(I+w_I\Delta_I)$, $\|\Delta_I\|_\infty < 1$ (input multiplicative uncertainty) or $G = (I + w_O\Delta_O)G_0$, $\|\Delta_D\|_\infty < 1$ (output multiplicative uncertainty), and
- co-prime factor uncertainty: $G = (D_0+\Delta_D)^{-1}(N_0+\Delta_N)$, $\|[\Delta_D \quad \Delta_N]\|_\infty < \epsilon$.

An intuitive depiction of such an uncertainty description is shown in Figure 2.3a.

In the behavioural framework, robustness has also been discussed with uncertainties described in this form, except that instead of the specification represented



(a) (b)

Figure 2.3: Description of uncertainties by (a) norm bound (b) a convex polytope

as transfer functions, it is represented as kernel representations or image representations. Suppose the uncertainty-free behaviour can be represented using a nominal kernel representation $\mathfrak{B}_0 = \ker(R_0)$, then the uncertain region can be represented as a ball centring at $\mathfrak{B}_0$ with radius $\epsilon$ [28], i.e.,

$$\|R - R_\Delta\|_\infty \leq \epsilon. \tag{2.46}$$

A storage function (represented by a QDF) of the nominal plant can also be used to describe the smallest upper bound on the radii of the neighbourhoods [28]. If, on the other hand, $\mathfrak{B}$ is representation-free, then uncertainty can be described using the concept of distance between behaviours [29, 92].

### 2.3.2 Polytopic Description of Uncertainty

Polytopic uncertainty describes the uncertain system in an entirely different way. Rather than assuming a nominal model and then formulating the uncertain region around it, this description of uncertainty assumes that all models in the uncertain region are equally likely to happen, and therefore it is un-necessary to assume any particular model to be more important than others. Therefore, the original uncertain region can be bounded by a convex polytope [30, 31], as shown in the depiction in Figure 2.3b. The advantage of this uncertainty description compared with the one in the previous section is obvious: the uncertain region can be much smaller and therefore controllers synthesised according to this description is much less conservative than that according to a norm-bound description. This is especially true if the chosen nominal model is at a skewed point in the uncertain region because the radius of the uncertain region is the distance between the nominal model and the model furthest from it. This description is particularly useful in the case where uncertain behaviour is caused by parameter uncertainty. An uncertain system LTI system $\Sigma_\theta = (\mathbb{R}, \mathbb{R}^{\mathrm{w}}, \mathfrak{B}_\theta)$ with generic parametric uncertainty can be represented as a parametric kernel representation

$$R(\theta, \xi)w := \sum_{k=0}^{L} R_k(\theta)\xi^k w = 0 \tag{2.47}$$

in the case of polytopic uncertainty, the uncertain representation can be written as

$$R(\theta, \xi)w = \sum_{j=1}^{M} \theta_j R_j(\xi)w = 0, \quad \theta \in \Theta := \left\{ \theta \in [0,1]^M \mid \sum_{j=1}^{M} \theta_j = 1 \right\}. \tag{2.48}$$

In such a case, the uncertain behaviour is denoted as $\mathfrak{B}_\theta = \ker(R_\theta)$ and the behaviour on each vertex is denoted as $\mathfrak{B}_j = \ker(R_j)$. If $\mathfrak{B}_\theta$ is the manifest behaviour of a state space representation, then $\mathfrak{B}_\theta^{full}$ admits generic and polytopic coefficient matrices $A(\theta)$, $B(\theta)$, $C(\theta)$ and $D(\theta)$ analogous to (2.47) and (2.48), respectively [93, 94].

For control design, since the system has parametric uncertainty, it is reasonable to associate it with a parametric Lyapunov function, after which parametric LMIs can be obtained. In particular, if $\mathfrak{B}_\theta = \ker(R(\theta))$ with $\mathfrak{B}_j = \ker(R_j)$, then the stability of *any* behaviour within the polytope can be guaranteed by guaranteeing that of the vertices.

**Theorem 2.10** (Stability along $\ker(R(\theta))$ [32, 61])**.** Assuming $\mathfrak{B}_\theta$ can be represented as (2.48), then all behaviours within the region are stable if there exists an array of QDFs $\Psi_1, \Psi_2, \ldots, \Psi_M \in \mathbb{S}^{\mathrm{w}}[\zeta, \eta]$ such that $\Psi_j \overset{\mathfrak{B}_j}{\succeq} 0$ and $-\nabla\Psi_j \overset{\mathfrak{B}_j}{\succeq} 0$ for all $j \in \mathbb{Z}_M^+$.

## 2.4 Summary and Research Gaps

In this chapter, brief reviews on the behavioural systems theory, the distributed control structures and design methods and the main types of uncertainty descriptions in robust control have been carried out. It can be seen that by placing behaviour as the centre of a dynamical system, analysis and control of systems becomes much more flexible because more than one representation can be assigned to each system and each representation provides different insights of the dynamical features of a system. As a relatively new framework, many aspects of control in the behavioural framework remain largely underdeveloped, or even undeveloped. Furthermore, despite the popularity of distributed control and robust control, there are still topics that fail to be covered in the literature. Specifically, there are several gaps awaiting to be filled:

1. To the best of the author's knowledge, little has been reported in the literature on robust distributed control for interconnected systems, and even less so within the behavioural framework. This is a crucial aspect in distributed control because the interactions between process units may exacerbate the effects of uncertainties, leading to poor performance or even instability. There are a number of robust decentralised control approaches for interconnected systems including multi-unit decentralised control (with block diagonal controllers) [16, 18, 95–98]. In these approaches, both interactions between process units and model-plant mismatch are modelled as uncertainties and dealt with using a robust control framework. Obviously this is not the best way to control an interconnected system because interactions among sub-

systems limits the possible outcomes of each subsystem significantly, and controllers design by ignoring this information is undoubtedly conservative. Some results have been reported for the distributed robust control of multi-agent systems [99–101], but the setup is different to the scope of this thesis, in that multi-agent systems contain identical subsystems while this thesis considers the interconnection of subsystems different from each other. In Chapters 3 and 4, this issue will be discussed in details.

2. Robust control design based on generic description of uncertainty in the behavioural framework remains undeveloped. In fact, the generic description of uncertainty is less preferable among all fields of control because the problem is not convex. Nevertheless, it is still the most accurate way in describing the actual behaviour of the system with parametric uncertainties, hence relative control design techniques and algorithms should be developed. In Chapter 4, two algorithms will be developed to address this issue. The first algorithm targets at relatively small-scaled systems with reasonably well-behaved generic uncertainty and the second algorithm targets at a much more relaxed description of uncertainty.

3. Despite being a set-theoretic centric theory, a complete set-theoretic framework for the analysis and distributed control of an interconnected system using the behavioural language is still undeveloped. In existing literature, all behaviours have some prescribed representations such as linear differential systems [28, 34–39, 45, 46, 53, 56, 57, 60, 70, 72, 102], dissipative dynamical systems [48, 55] or other representations [41–43, 103]. Even the ones assuming no representations of behaviours still impose linearity [29, 73]. In Chapter 5, a framework for the analysis and distributed control of interconnected system will be formulated in a purely set-theoretic point of view.

While the discussions in all of the rest of the chapters will use the behavioural language, it is still assumed in Chapters 3 and 4 that the systems in consideration are LTI system and admit input/output partitions. Chapter 5, on the other hand, will demonstrate the essence of the behavioural systems theory: trajectories as the foundation of a system, complete blurry of input and output, and interconnection as restriction on the set of possible trajectories.

# Chapter 3

# Distributed Robust Control with Polytopic Uncertainty

In this chapter, a novel robust distributed control approach based on dissipativity analysis is proposed. In this approach, the global robust stability, robust performance and distributed control (i.e., the model uncertainties, the interaction effects as well as the external disturbances) are achieved using a *single* framework: they are all viewed from a dissipativity perspective. This approach is made possible by the use of supply rates and storage functions in QDFs [57], which capture more detailed features of process dynamics compared to conventional QSR-dissipativity [56]. The interconnected system and the distributed control system are represented as two interacting networks. The subsystems are with parametric uncertainties and described by uncertain polytopic kernel representation (2.48). Parameter-dependent QDF dissipativity, which is also a convex combination of QDFs associated with vertex behaviours, is proposed to describe the dynamic features of uncertain subsystems. By associating each controller with a QDF dissipativity property, the controlled interconnected behaviour can be constructed according to the method of interconnecting dissipative dynamical systems. Sta-

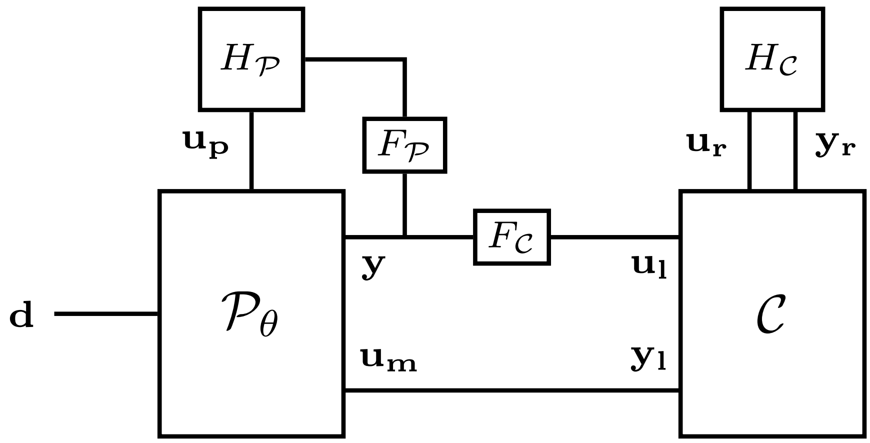



Figure 3.1: An interconnected Uncertain System

bility and performance conditions of the controlled behaviour are also represented using a global dissipativity condition, which in turn are translated into the dissipativity conditions that individual controllers must satisfy. The controller behaviour can then be parametrised through $J$-factorisation according to Theorem 2.7.

*The key results in this chapter have been published in:*

- **Y. Yan**, R. Wang, J. Bao and C. Zheng, "Robust Distributed Control of Plantwide Processes Based on Dissipativity," *Journal of Process Control*, vol. 77, pp. 48–60, 2019.

## 3.1 Controlled System Layout

Since this thesis is built on behavioural systems theory, the description of the configuration of the interconnected system with distributed control system is given from the context of the behavioural systems theory. A depiction of the overall control system is in Figure 3.1. The uncertain system $\mathcal{P}_\theta$ consists of $N$ diagonally stacked subsystems, i.e., $\mathcal{P}_\theta = \left\{\mathcal{P}_\theta^1, \mathcal{P}_\theta^2, \ldots, \mathcal{P}_\theta^N\right\}$ where $\mathcal{P}_\theta^i = (\mathbb{R}, \mathbb{R}^{\mathrm{w}^i}, \mathfrak{B}_\theta^i)$. In this chapter, each subsystem is assumed to have polytopic uncertainty described by

(2.48), i.e.,

$$R^i_{\theta^i}(\xi)w^i = \sum_{j=1}^{M^i} \theta^i_j R^i_j(\xi)w^i = 0, \quad \theta^i \in \Theta^i = \left\{ \theta^i \in [0,1]^{M^i} \mid \sum_{j=1}^{M^i} \theta^i_j = 1 \right\}, \tag{3.1}$$

where $w^i = \mathrm{col}\left(y^i, u^i_p, u^i_m, d^i\right)$, whose four partitions denote, respectively, the $i$th system output, interconnection input, manipulated input and disturbance. As such, each vertex behaviour can be denoted as $\mathfrak{B}^i_\theta = \ker(R^i_\theta)$. The controller network is set up in a similar way: it consists of $N$ diagonally stacked distributed controllers so that $\mathcal{C} = \left\{\mathcal{C}^1, \mathcal{C}^2, \cdots, \mathcal{C}^N\right\}$, $\mathcal{C}^i = (\mathbb{R}, \mathbb{R}^{c^i}, \mathfrak{B}^i_c)$, in which $\mathfrak{B}^i_c$ are to be determined, and $c^i = \mathrm{col}\,(y^i_l, y^i_r, u^i_l, u^i_r)$ in which the four partitions of $c^i$ denote, respectively, the $i$th local output, remote output, local input and remote input. Similar to the construction in Figure 2.2, the manifest variables in the system on the global level are column vectors of their respective local counterparts in all subsystems, i.e., $\mathbf{w} = \mathrm{col}(w^1, w^2, \ldots, w^N)$, $\mathbf{c} = \mathrm{col}(c^1, c^2, \ldots, c^N)$ and similarly for other variables.

The network for the subsystems, the network for the controllers and the interconnection between the system and controller networks are constructed in similar ways as (2.45), with system topology $H_{\mathcal{P}}$, controller topology $H_{\mathcal{C}}$, interconnection variable selection matrix $F_{\mathcal{P}}$ and control variable selection matrix $F_{\mathcal{C}}$. This structure permits a wide range of configurations, whereby the controller topology matrix $H_{\mathcal{C}}$ can be designed according to the requirements. If the criterion were to achieve the best performance possible, then $H_{\mathcal{C}}$ should be chosen such that the controller topology is the same as that of the plant. For other requirements, control design can be carried out repeatedly with different choices of $H_{\mathcal{C}}$ until a satisfactory result is obtained. In this chapter, the interconnections are represented as

$$\begin{bmatrix} H_{\mathcal{P}}F_{\mathcal{P}} & -I \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u_p} \end{bmatrix} = 0, \quad \begin{bmatrix} H_{\mathcal{C}} & -I \end{bmatrix} \begin{bmatrix} \mathbf{y_r} \\ \mathbf{u_r} \end{bmatrix} = 0, \quad \begin{bmatrix} F_{\mathcal{C}} & 0 & 0 & -I \\ 0 & I & -I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{u_m} \\ \mathbf{y_l} \\ \mathbf{u_l} \end{bmatrix} = 0$$

Although this description is with similar rationale to that depicted in Figure 2.2, the viewpoint is rather different - while it is assumed that *a priori* input/output partition of the system is available, they are treated entirely equally. Furthermore, these representations are structurally identical to kernel representations. In fact, they can be thought of as kernel representations with constant coefficients. This perspective is a crucial aid for the development and understanding of the much more abstract layout that will be presented in Chapter 5. The problem to be addressed can be summarised as follows.

**Problem 3.1.** As shown in Figure 3.1, given $N$ uncertain subsystems with topology given by $H_{\mathcal{P}}$, in which the $i$th subsystem $\mathcal{P}^i_{\theta^i}$ has behaviour described by $\mathfrak{B}^i_{\theta^i} = \ker\left(R^i_{\theta^i}\right)$, where $R^i_{\theta^i}(\xi)$ is given in (3.1), design $N$ distributed controllers $\mathcal{C}^i$ with controller network topology $H_{\mathcal{C}}$ such that the controlled interconnected system is robustly stable and that the frequency-weighted $\mathcal{H}_\infty$ norm from disturbances $\mathbf{d}$ to outputs $\mathbf{y}$ satisfies

$$\|\mathbf{W}\mathbf{T_{yd}}\|_\infty \leq \gamma \tag{3.2}$$

for all $\theta^i \in \Theta^i$, $i \in \mathbb{Z}^+_N$, where $\mathbf{W}(s)$ is a weighting function, $\mathbf{T_{yd}}(s)$ is the sensitivity function from disturbance to output, and $\gamma$ is the desired gain bound.

If $\mathbf{W}(s) = \mathbf{N}(s)/\mathbf{p}(s)$ where $\mathbf{W} \in \mathbb{R}^{\mathbf{y}\times\mathbf{d}}[s]$ and $\mathbf{p}(s) \in \mathbb{R}[s]$, then (3.2) can be

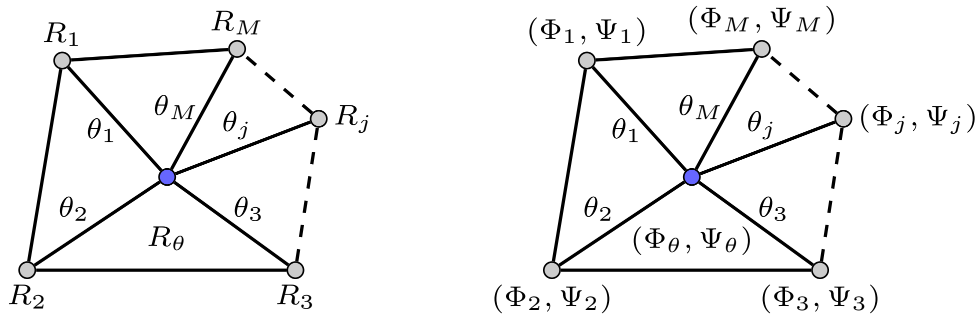



Figure 3.2: Corresponding dissipativity of a behaviour with polytopic uncertainty

easily translated into a desired supply rate induced by the QDF

$$\Phi_{desired}(\zeta,\eta) = \begin{bmatrix} -\mathbf{N}^T(\zeta)\mathbf{N}(\eta) & 0 \\ 0 & \gamma^2 \mathbf{p}(\zeta)\mathbf{p}(\eta) \end{bmatrix}. \tag{3.3}$$

The goal is then to design an array of controllers to render the controlled interconnected system dissipative with respect to the supply rate induced by (3.3) for all $\theta^i \in \Theta^i$, $i \in \mathbb{Z}_N^+$.

## 3.2 Control via Parameter-dependant QDFs

### 3.2.1 Parameter-dependant QDF Dissipativity

For a system $\Sigma_\theta = (\mathbb{R}, \mathbb{R}^{\mathrm{w}}, \mathfrak{B}_\theta)$ where $\mathfrak{B}_\theta \in \mathfrak{L}^{\mathrm{w}}$ is described by (2.48), a QDF-dissipativity property $(\Phi_j, \Psi_j)$ can be associated with each vertex behaviour $\mathfrak{B}_j$. In such a way, an uncertain region for the dissipativity property can be constructed accordingly, as shown in Figure 3.2. Similar to the construction of $\mathfrak{B}_\theta$ from $\mathfrak{B}_j$, a parameter-dependant QDF-dissipativity property $(\Phi_\theta, \Psi_\theta)$ can also be constructed from $(\Phi_j, \Psi_j)$ as

$$\Phi_\theta(\zeta,\eta) = \sum_{j=1}^{M} \theta_j \Phi_j(\zeta,\eta),\ \ \Psi_\theta(\zeta,\eta) = \sum_{j=1}^{M} \theta_j \Psi_j(\zeta,\eta),\ \ \theta \in \Theta. \tag{3.4}$$

An immediate question following this construction is that whether $(\Phi_\theta, \Psi_\theta)$ is a valid dissipativity property for $\mathfrak{B}_\theta$. The answer is affirmative, provided that $\mathfrak{B}_j$ are described by kernel representations and that there is a common "Lagrange multiplier" (similar to the polynomial matrix $F(\zeta, \eta)$ in (2.30)) among all vertex behaviours.

**Proposition 3.1.** Given an uncertain LTI system $\Sigma_\theta = (\mathbb{R}, \mathbb{R}^{\mathrm{w}}, \mathfrak{B}_\theta)$ in which $\mathfrak{B}_\theta$ is given by (2.48) and $\theta \in \Theta$, if there exist $\Phi_j, \Psi_j \in \mathbb{S}^{\mathrm{w}}[\zeta, \eta]$ for each vertex behaviour $\mathfrak{B}_j$ and a two variable polynomial matrix $F \in \mathbb{R}^{\mathrm{w}\times\bullet}[\zeta, \eta]$ such that

$$\Psi_j \succeq 0, \tag{3.5a}$$

$$\Phi_j(\zeta, \eta) - \nabla\Psi_j(\zeta, \eta) + \mathrm{He}\left\{F^\star(\zeta, \eta)R_j(\eta)\right\} \succeq 0, \tag{3.5b}$$

for all $\forall j \in \mathbb{Z}_M^+$, then $(\Phi_\theta, \Psi_\theta)$ defined in (3.4) is a dissipativity property for the uncertain behaviour $\mathfrak{B}_\theta$, i.e., $\mathfrak{B}_\theta$ is $\Phi_\theta$-dissipative with $\Psi_\theta$ as a storage function.

*Proof.* Since every $\theta_j$ is a constant, it is easy to see that $\theta_j \nabla\Psi_j = \nabla(\theta_j\Psi_j)$ and that $\mathrm{He}\left\{\theta_j F^\star(\zeta, \eta)R_j(\eta)\right\} = \mathrm{He}\left\{F^\star(\zeta, \eta)\theta_j R_j(\eta)\right\}$. Since $\theta_j \geq 0$ for all $j$, multiplying each vertex dissipativity property in (3.5) by their respective weightings and adding them up lead to the condition

$$\Psi_\theta \succeq 0, \tag{3.6a}$$

$$\Phi_\theta(\zeta, \eta) - \nabla\Psi_\theta(\zeta, \eta) + \mathrm{He}\left\{F^\star(\zeta, \eta)R_\theta(\eta)\right\} \succeq 0. \tag{3.6b}$$

Dissipativity follows readily from Proposition 2.4(ii), Proposition 2.5 and Theorem 2.6. This completes the proof. □

Although the value of $\theta$ is unknown, it does not impede the construction of the parameter-dependant dissipativity condition because none of the conditions use any information of $\theta$. Furthermore, this argument is only valid for kernel representations due to the fact that the non-negativity of a QDF along a behaviour

$\mathfrak{B} = \ker(R)$ is a linear inequality in $R$. Similar to (2.31), dissipativity properties for the vertices can be found by solving the following LMIs

$$\widetilde{\Psi}_j \geq 0, \tag{3.7a}$$

$$\widetilde{\Phi}_j - \nabla\widetilde{\Psi}_j + \mathrm{He}\left\{\widetilde{F}^T \widehat{R}_j\right\} \geq 0, \tag{3.7b}$$

for all $j \in \mathbb{Z}_M^+$ simultaneously.

### 3.2.2 Global Dissipativity Synthesis

The central idea for the analysis of the interconnected system is to search for parametric dissipativity properties of individual subsystems and controllers such that the global stability and performance condition is satisfied. In such a way, the description of the behaviour of each subsystem change from LTI systems to dissipative dynamical systems. As illustrated in Section 2.1.4, interconnecting dissipative dynamical systems is much less demanding computationally than interconnecting differential systems, and such reduction in computational burden is crucial in the efficient search for controllers. The distributed controllers are then synthesised individually based on their respective supply rates through $J$-factorisation (see Algorithm 2.1).

For each subsystem, a parameter-dependant supply rate $Q_{\Phi^i_{\theta^i}}(w^i) = (w^i)^T\, \Phi^i_{\theta^i} w^i$ can be associated with it. A $\theta$-free supply rate $Q_{\Phi^i_c}(c^i) = (c^i)^T\, \Phi^i_c c^i$ can also be assigned to each corresponding controller because the controllers have no information of $\theta$. Denoting the collective manifest variables, input/output partition of the variables and the input/output partition of the overall controlled system after interconnection as

$$\mathbf{w}_{\mathcal{P}} = \mathrm{col}(\mathbf{w}, \mathbf{c}),\ \ \mathbf{w}'_{\mathcal{P}} = \mathrm{col}(\mathbf{y}, \mathbf{y_l}, \mathbf{y_r}, \mathbf{u_p}, \mathbf{d}, \mathbf{u_m}, \mathbf{u_l}, \mathbf{u_r}),\ \ \mathbf{w}_{\mathcal{PC}} = \mathrm{col}(\mathbf{y}, \mathbf{y_l}, \mathbf{y_r}, \mathbf{d}), \tag{3.8}$$

then

$$\mathbf{w}_{\mathcal{P}} = P_{\pi}\mathbf{w}'_{\mathcal{P}} = P_{\pi}H_{\pi}\mathbf{w}_{\mathcal{PC}}, \tag{3.9}$$

where $P_{\pi}$ is a permutation matrix with exactly one entry of 1 in each row and each column and 0's elsewhere, and $H_{\pi}$ is defined as

$$H_{\pi} = \begin{bmatrix} I & 0 & 0 & F_{\mathcal{P}}^T H_{\mathcal{P}}^T & 0 & 0 & F_{\mathcal{C}}^T & 0 \\ 0 & I & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 & H_{\mathcal{C}}^T \end{bmatrix}^T . \tag{3.10}$$

Assuming that the $i$-th subsystem $\mathcal{P}_{\theta}^i$ and the corresponding controller $\mathcal{C}^i$ are $\Phi_{\theta}^i$- and $\Phi_c^i$-dissipative, respectively, then the net supply rate of the controlled interconnected system can be written as

$$\begin{aligned} Q_{\Phi_{\mathcal{PC}}}(\mathbf{w}_{\mathcal{PC}}) &= \sum_{i=1}^{N} Q_{\Phi_{\theta}^i}(w^i) + Q_{\Phi_c^i}(c^i) \\ &= \mathbf{w}_{\mathcal{P}}^T \underbrace{\mathrm{diag}(\Phi_{\theta}^1, \ldots, \Phi_{\theta}^N, \Phi_c^1, \ldots, \Phi_c^N)}_{\Phi_{\mathcal{P}}} \mathbf{w}_{\mathcal{P}} \\ &= \mathbf{w}_{\mathcal{PC}}^T \underbrace{H_{\pi}^T P_{\pi}^T \Phi_{\mathcal{P}} P_{\pi} H_{\pi}}_{\Phi_{\mathcal{PC}}} \mathbf{w}_{\mathcal{PC}} \\ &= \begin{bmatrix} \mathbf{y} \\ \mathbf{y_l} \\ \mathbf{y_r} \\ \mathbf{d} \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}_{\mathbf{yy}} & \mathcal{Q}_{\mathbf{yl}} & \mathcal{Q}_{\mathbf{yr}} & \mathcal{S}_{\mathbf{yd}} \\ \star & \mathcal{Q}_{\mathbf{ll}} & \mathcal{Q}_{\mathbf{lr}} & \mathcal{S}_{\mathbf{ld}} \\ \star & \star & \mathcal{Q}_{\mathbf{rr}} & \mathcal{S}_{\mathbf{rd}} \\ \star & \star & \star & \mathcal{R}_{\mathbf{dd}} \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{y_l} \\ \mathbf{y_r} \\ \mathbf{d} \end{bmatrix}, \end{aligned} \tag{3.11}$$

where $\star$ denotes elements inferred by symmetry. Since the resulting supply rate is a description of the dissipativity of the controlled interconnected system, it can be used to impose conditions on the controller supply rates that lead to the desired behaviour.

**Theorem 3.2.** Let $\mathcal{P}_\theta^i$ and $\mathcal{C}^i$ be $\Phi_\theta^j$- and $\Phi_c^j$-dissipative, respectively. If the net supply rate in (3.11) satisfies

$$\begin{bmatrix} \mathcal{Q}_{\mathbf{yl}}(\zeta,\eta) & \mathcal{Q}_{\mathbf{yr}}(\zeta,\eta) \end{bmatrix} = 0, \tag{3.12a}$$

$$\begin{bmatrix} \mathcal{S}^\star_{\mathbf{yd}}(\zeta,\eta) & \mathcal{S}^\star_{\mathbf{ld}}(\zeta,\eta) & \mathcal{S}^\star_{\mathbf{rd}}(\zeta,\eta) \end{bmatrix} = 0, \tag{3.12b}$$

$$\begin{bmatrix} \mathcal{Q}_{\mathbf{ll}}(\zeta,\eta) & \mathcal{Q}_{\mathbf{lr}}(\zeta,\eta) \\ \star & \mathcal{Q}_{\mathbf{rr}}(\zeta,\eta) \end{bmatrix} \preceq 0, \tag{3.12c}$$

$$\mathcal{Q}_{\mathbf{yy}}(\zeta,\eta) + \mathbf{N}^T(\zeta)\mathbf{N}(\eta) \preceq 0, \tag{3.12d}$$

$$\mathcal{R}_{\mathbf{dd}}(\zeta,\eta) - \gamma^2\mathbf{D}^T(\zeta)\mathbf{D}(\eta) \preceq 0, \tag{3.12e}$$

with $\mathbf{D}$ Hurwitz, then the controlled interconnected system is robustly stable and satisfies the frequency-weighted $\mathcal{H}_\infty$ norm condition (3.2) with weighting function $\mathbf{W}(s) = \mathbf{D}^{-1}(s)\mathbf{N}(s)$.

*Proof.* Condition in (3.12) implies that

$$\begin{aligned} &Q_{\Phi_{\mathcal{PC}}}(\mathbf{w}_{\mathcal{PC}}) \\ &\leq \begin{bmatrix} \mathbf{y} \\ \mathbf{d} \end{bmatrix}^T \begin{bmatrix} -\mathbf{N}^T(\zeta)\mathbf{N}(\eta) & 0 \\ 0 & \gamma^2\mathbf{D}^T(\zeta)\mathbf{D}(\eta) \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{d} \end{bmatrix} + \begin{bmatrix} \mathbf{y_l} \\ \mathbf{y_r} \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}_{\mathbf{ll}}(\zeta,\eta) & \mathcal{Q}_{\mathbf{lr}}(\zeta,\eta) \\ \star & \mathcal{Q}_{\mathbf{rr}}(\zeta,\eta) \end{bmatrix} \begin{bmatrix} \mathbf{y_l} \\ \mathbf{y_r} \end{bmatrix} \\ &\leq \begin{bmatrix} \mathbf{y} \\ \mathbf{d} \end{bmatrix}^T \begin{bmatrix} -\mathbf{N}^T(\zeta)\mathbf{N}(\eta) & 0 \\ 0 & \gamma^2\mathbf{D}^T(\zeta)\mathbf{D}(\eta) \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{d} \end{bmatrix}. \end{aligned} \tag{3.13}$$

Since the controlled interconnected system is dissipative with respect to $Q_{\Phi_{\mathcal{PC}}}(\mathbf{w}_{\mathcal{PC}})$, there exists a storage function $Q_{\Psi_{\mathcal{PC}}}(\mathbf{w}_{\mathcal{PC}})$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{\Psi_{\mathcal{PC}}}(\mathbf{w}_{\mathcal{PC}}) \leq Q_{\Phi_{\mathcal{PC}}}(\mathbf{w}_{\mathcal{PC}}).$$

Thus, the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_{\Psi_{\mathcal{PC}}}(\mathbf{w}_{\mathcal{PC}}) \leq \begin{bmatrix} \mathbf{y} \\ \mathbf{d} \end{bmatrix}^T \begin{bmatrix} -\mathbf{N}^T(\zeta)\mathbf{N}(\eta) & 0 \\ 0 & \gamma^2 \mathbf{D}^T(\zeta)\mathbf{D}(\eta) \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \mathbf{d} \end{bmatrix}$$

is also true, meaning that the controlled interconnected system is also dissipative with respect to the supply rate induced by diag $\left(-\mathbf{N}^T(\zeta)\mathbf{N}(\eta),\ \gamma^2\mathbf{D}^T(\zeta)\mathbf{D}(\eta)\right)$, which, according to the definition of dissipativity [48], means that

$$\int_{-\infty}^{\infty} \hat{\mathbf{y}}^T \widetilde{\mathbf{N}}^T \widetilde{\mathbf{N}} \hat{\mathbf{y}} \ \mathrm{d}t \leq \int_{-\infty}^{\infty} \gamma^2 \hat{\mathbf{d}}^T \widetilde{\mathbf{D}}^T \widetilde{\mathbf{D}} \hat{\mathbf{d}} \ \mathrm{d}t, \tag{3.14}$$

or equivalently

$$\left\| \widetilde{\mathbf{N}} \hat{\mathbf{y}} \right\|_2^2 \leq \gamma^2 \left\| \widetilde{\mathbf{D}} \hat{\mathbf{d}} \right\|_2^2. \tag{3.15}$$

Since $\widetilde{\mathbf{N}}\hat{\mathbf{y}} = \mathbf{N}(s)\mathbf{y}$, $\widetilde{\mathbf{D}}\hat{\mathbf{d}} = \mathbf{D}(s)\mathbf{d}$, it then follows that

$$\frac{\|\mathbf{N}(s)\mathbf{y}\|_2^2}{\|\mathbf{D}(s)\mathbf{d}\|_2^2} = \frac{\|\mathbf{D}^{-1}(s)\mathbf{N}(s)\mathbf{y}\|_2^2}{\|\mathbf{d}\|_2^2} = \frac{\|\mathbf{W}\mathbf{y}\|_2^2}{\|\mathbf{d}\|_2^2} \leq \gamma^2. \tag{3.16}$$

Since $\mathbf{W}\mathbf{y} = \mathbf{W}\mathbf{T}_{\mathbf{yd}}\mathbf{d}$ and the above inequality holds true for any $\mathbf{d}$, it then follows that

$$\max_{\mathbf{d}\neq 0} \frac{\|\mathbf{W}\mathbf{y}\|_2^2}{\|\mathbf{d}\|_2^2} = \|\mathbf{W}\mathbf{T}_{\mathbf{yd}}\|_\infty^2 \leq \gamma^2. \tag{3.17}$$

Taking the square roots of both sides gives the result. Note that this is an extension to Proposition 2.3. □

In the proposed approach, the parameter-dependant dissipativity of all subsystems $(\Phi_j^i, \Psi_j^i)$, $\forall j \in \mathbb{Z}_{M^i}^+,\ i \in \mathbb{Z}_N^+$, and supply rate matrices for all controllers

$\Phi_c^i$, $\forall i \in \mathbb{Z}_N^+$ need to be solved simultaneously to satisfy the *global robust performance* condition (3.2). The above dissipativity synthesis takes into account all interactions among process units/controllers to achieve a high level of global robust performance and is carried out *offline*. Real-time control is implemented in a fully distributed manner.

As the weighting parameter $\theta^i$ is unknown, all possible combinations of subsystem supply rates on the vertices need to be checked, in which case the number of LMIs increases rapidly with the increase in the numbers of subsystems and polytopic vertices. One way to resolve this issue is to find *a common supply rate* $Q_{\Phi^i}(w^i)$ shared by all vertex behaviours, i.e., a single supply rate for the uncertain behaviour $\mathfrak{B}_\theta$. This only requires the solution to *one* set of LMIs for (3.12). On the other hand, *different* storage functions are still determined for each vertex behaviour $\mathfrak{B}_j$. This significantly reduces the conservativeness caused by the common supply rate, thereby preserving the flexibility of the polytopic conditions whilst keeping the computational complexity within a manageable level.

**Remark 3.1.** Equations in (3.12) represent the conditions that ensure global robust stability and robust performance requirements. Often the robust performance conditions competes with that of robust stability. If a robust performance requirement were not achievable, then the LMIs in (3.12) would not be feasible. In this case, the weighting function $\mathbf{W}(s)$ could be adjusted to reduce the performance requirement.

**Remark 3.2.** Generally speaking, all robust control designs bear a certain level of conservatism. The proposed approach reduces the conservatism by: (a) describing the uncertain region using a convex hull to provide a tighter bound; (b) adopting QDF type dissipativity to capture detailed dynamic features of the processes and (c) allowing the use of different storage functions for each vertex of the polytope as well as using the multiplier $F^i(\zeta, \eta)$, $\forall i \in \mathbb{Z}_N^+$.

### 3.2.3 Distributed Robust Controller Synthesis

Once the controller supply rates for the controllers are obtained, behaviour parametrisation can be carried out using $J$-factorisation according to Theorem 2.7 and Algorithm 2.1. However, a $J$-factorisation for a polynomial $\Phi(-\xi,\xi)$ exists if and only if it has constant signature and no zeros on the imaginary axis, i.e., the signature of $\Phi(-j\omega, j\omega)$ is constant for all $\omega \in \mathbb{R}$ [68]. The following proposition gives a sufficient condition for the existence of $J$-factorisation.

**Proposition 3.3.** If the supply rate $Q_\Phi(w) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} \mathcal{Q}(\zeta,\eta) & \mathcal{S}(\zeta,\eta) \\ \star & \mathcal{R}(\zeta,\eta) \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$ with $y \in \mathbb{R}^{\mathrm{y}}$, $u \in \mathbb{R}^{\mathrm{u}}$ satisfies

$$\mathcal{Q}(\zeta,\eta) \prec 0, \quad \mathcal{R}(\zeta,\eta) \succ 0, \tag{3.18}$$

then $\Phi$ admits a $J$-factorisation

$$\Phi(-\xi,\xi) = K^T(-\xi) J K(\xi), \quad K \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]. \tag{3.19}$$

*Proof.* Firstly, observe that

$$\Phi(\zeta,\eta) = \begin{bmatrix} \mathcal{Q}(\zeta,\eta) & \mathcal{S}(\zeta,\eta) \\ \star & \mathcal{R}(\zeta,\eta) \end{bmatrix} = \Gamma^\star(\zeta,\eta)\Omega(\zeta,\eta)\Gamma(\zeta,\eta), \tag{3.20}$$

where

$$\Gamma(\zeta,\eta) = \begin{bmatrix} I & \mathcal{Q}^{-1}(\zeta,\eta)\mathcal{S}(\zeta,\eta) \\ 0 & I \end{bmatrix},$$

$$\Omega(\zeta,\eta) = \begin{bmatrix} \mathcal{Q}(\zeta,\eta) & 0 \\ 0 & \mathcal{R}(\zeta,\eta) - \mathcal{S}^\star(\zeta,\eta)\mathcal{Q}^{-1}(\zeta,\eta)\mathcal{S}(\zeta,\eta) \end{bmatrix}.$$

Setting $\zeta = -j\omega$ and $\eta = j\omega$, it can be seen that $\Phi(-j\omega, j\omega)$ is congruent to $\Omega(-j\omega, j\omega)$ because $\Gamma(-j\omega, j\omega)$ is a congruent transformation. Since conditions in (3.18) imply

$$\mathcal{Q}(\zeta,\eta) \prec 0 \Leftrightarrow \widetilde{\mathcal{Q}} < 0 \Leftrightarrow \mathcal{Q}(-j\omega, j\omega) < 0,$$
$$\mathcal{R}(\zeta,\eta) \succ 0 \Leftrightarrow \widetilde{\mathcal{R}} > 0 \Leftrightarrow \mathcal{R}(-j\omega, j\omega) > 0,$$

for all $\omega$ such that $\det(\mathcal{Q}(-j\omega, j\omega)) \neq 0$ and $\det(\mathcal{R}(-j\omega, j\omega)) \neq 0$, it follows that the signature of $\Omega(-\xi,\xi)$ on the imaginary axis, hence that of $\Phi(-\xi,\xi)$, is constant at $(\mathrm{y}, 0, \mathrm{u})$. According to Proposition 3.1 in [68], $\Phi(-\xi,\xi)$ admits factorisation (3.19) where $J = \begin{bmatrix} I_{\mathrm{u}} & 0 \\ 0 & -I_{\mathrm{y}} \end{bmatrix}$. This completes the proof. □

In summary, the proposed dissipativity-based distributed robust control approach can be carried out using the following algorithm.

**Algorithm 3.1.**

(1) Dissipativity synthesis: search for the parametric dissipativity $(\Psi^i_j, \Phi^i)$ with $i \in \mathbb{Z}^+_N$, $j \in \mathbb{Z}^+_{M^i}$ and supply rate matrix $\Phi^i_c$ of $i$-th controller by solving the following LMIs *simultaneously*:

- the dissipativity condition in (3.7) (for an individual process unit) for all $N$ process units;
- plantwide stability and performance condition in (3.12);
- controller feasibility condition in (3.18).

(2) Distributed robust control synthesis for each controller $\mathcal{C}^i$,

- compute the $J$-factorisation of $\Phi^i_c$ as $\Phi^i_c(-\xi,\xi) = {K^i}^T(-\xi)J^iK^i(\xi)$;
- construct a $J^i$-dissipative system with image representation $z^i = M^i(\xi)\ell^i$;
- obtain the image representation of the $i$-th controller as $v^i = L^i(\xi)M^i(\xi)\ell^i$.

## 3.3 Case Study

A case study of an interconnected system consisting of two chemical reactors and one separator, with configuration depicted in Figure 3.3, is carried out to illustrate the proposed controller design procedure. The chemical reaction involved is A $\rightarrow$ B $\rightarrow$ C where A is the reactant, B is the product and $C$ is the undesirable by-product [22]. The aim is to control the temperature in the two reactors and the separator under the disturbances from the temperature of the feed streams 1 and 2 by manipulating the heat supplies $Q_1$, $Q_2$ and $Q_3$. The subsystems can be described by

$$\mathcal{P}^1 : \begin{cases} \frac{\mathrm{d}x_{A_1}}{\mathrm{d}t} = \frac{F_{1f}}{V_1}(x_{A_{1f}} - x_{A_1}) + \frac{F_r}{V_1}(x_{A_r} - x_{A_1}) - k_1 e^{\frac{-E_1}{RT_1}} x_{A_1} \\ \frac{\mathrm{d}x_{B_1}}{\mathrm{d}t} = \frac{F_{1f}}{V_1}(x_{B_{1f}} - x_{B_1}) + \frac{F_r}{V_1}(x_{B_r} - x_{B_1}) + k_1 e^{\frac{-E_1}{RT_1}} x_{A_1} - k_2 e^{\frac{-E_2}{RT_1}} x_{B_1} \\ \frac{\mathrm{d}T_1}{\mathrm{d}t} = \frac{F_{1f}}{V_1}(T_{1f} - T_1) + \frac{F_r}{V_1}(T_3 - T_1) + \frac{-\Delta H_1}{C_p} k_1 e^{\frac{-E_1}{RT_1}} x_{A_1} + \frac{-\Delta H_2}{C_p} k_2 e^{\frac{-E_2}{RT_1}} x_{B_1} + \frac{Q_1}{\rho C_p V_1} \end{cases}$$

$$\mathcal{P}^2 : \begin{cases} \frac{\mathrm{d}x_{A_2}}{\mathrm{d}t} = \frac{F_1}{V_2}(x_{A_1} - x_{A_2}) + \frac{F_{2f}}{V_2}(x_{A_{2f}} - x_{A_2}) - k_1 e^{\frac{-E_1}{RT_2}} x_{A_2} \\ \frac{\mathrm{d}x_{B_2}}{\mathrm{d}t} = \frac{F_1}{V_2}(x_{B_1} - x_{B_2}) + \frac{F_{2f}}{V_2}(x_{B_{2f}} - x_{B_2}) + k_1 e^{\frac{-E_1}{RT_2}} x_{A_2} - k_2 e^{\frac{-E_2}{RT_2}} x_{B_2} \\ \frac{\mathrm{d}T_2}{\mathrm{d}t} = \frac{F_1}{V_2}(T_1 - T_2) + \frac{F_{2f}}{V_2}(T_{2f} - T_2) + \frac{-\Delta H_1}{C_p} k_1 e^{\frac{-E_1}{RT_2}} x_{A_2} + \frac{-\Delta H_2}{C_p} k_2 e^{\frac{-E_2}{RT_2}} x_{B_2} + \frac{Q_2}{\rho C_p V_2} \end{cases}$$

$$\mathcal{P}^3 : \begin{cases} \frac{\mathrm{d}x_{A_3}}{\mathrm{d}t} = \frac{F_2}{V_3}(x_{A_2} - x_{A_3}) - \frac{F_r}{V_3}(x_{A_r} - x_{A_3}) \\ \frac{\mathrm{d}x_{B_3}}{\mathrm{d}t} = \frac{F_2}{V_3}(x_{B_2} - x_{B_3}) - \frac{F_r}{V_3}(x_{B_r} - x_{B_3}) \\ \frac{\mathrm{d}T_3}{\mathrm{d}t} = \frac{F_2}{V_3}(T_2 - T_3) + \frac{Q_3}{\rho C_p V_3} \end{cases}$$

In the models, $x_{A_1}$, $x_{A_2}$, $x_{A_3}$, $x_{A_{1f}}$ and $x_{A_{2f}}$ denote, respectively, mass fraction of $A$ in the first reactor, second reactor, separator, the first feed stream and the second feed stream. Similar counterparts can be defined for $x_B$ and $T$, respectively, for the mass fraction of $B$ and temperature. $x_{A_r}$ and $x_{B_r}$ denote, respectively, the mass fraction of $A$ and $B$ in the recycle stream. $F_{1f}$ and $F_{2f}$ denote, respectively,

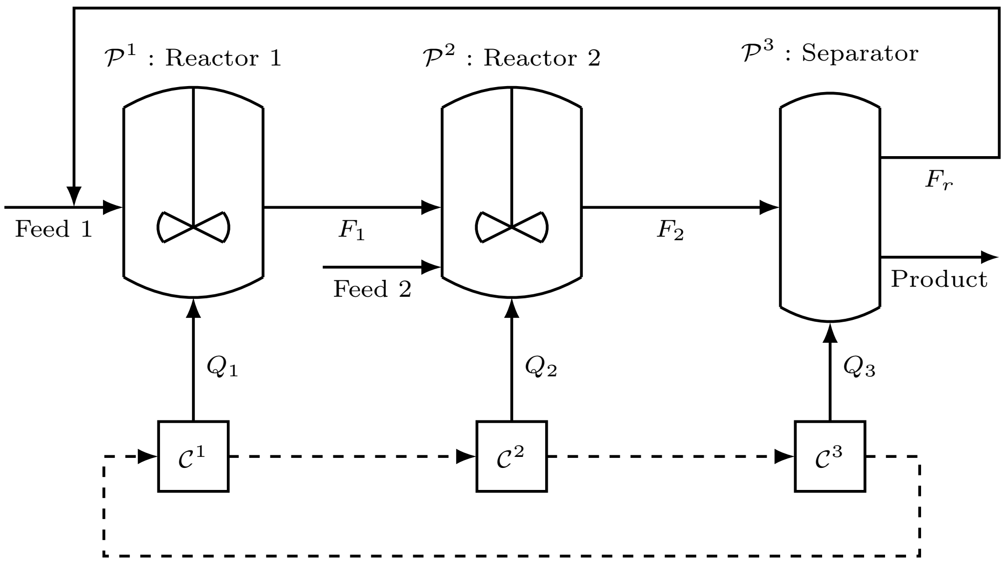



Figure 3.3: Configuration of the reactor-separator process

the volumetric flowrates in Feed 1 and Feed 2. Since the mass fraction of C, $x_C$, is not independent to $x_A$ and $x_B$ ($x_A + x_B + x_C = 1$ always holds for all streams), a model for it is not included. For this process, the manifest and interconnection variables can be defined as

$$y^i = \mathrm{col}(x_{A_i}, x_{B_i}, T_i),\ u_c^i = y_l^i = Q_i,\ u_l^i = T_i\ , d^i = T_{jf},\ i = 1, 2, 3,\ j = 1, 2,$$
$$u_p^1 = y^3,\ u_p^2 = y^1,\ u_p^3 = y^2,\ u_r^1 = y_r^3,\ u_r^2 = y_r^1,\ u_r^3 = y_r^2.$$

The operating point and parameters in the process are given in Table 3.1. For illustration purposes, it is assumed that there are variations on $k_1$ and $k_2$ up to 10% and that $\alpha_C$ varies between 0.5 and 0.7. The models are linearised around the operating point and transformed into kernel representation. Polytopic regions can capture a variety of uncertainties. In this case, the polytopic regions are used to capture the effects of uncertain parameters in the nonlinear process models and linearisation errors. The manifest variables considered for each unit are

$$w^1 = \mathrm{col}\left(x_{A_1}, x_{B_1}, T_1, x_{A_3}, x_{B_3}, T_3, Q_1, T_{1f}\right),$$
$$w^2 = \mathrm{col}\left(x_{A_2}, x_{B_2}, T_2, x_{A_1}, x_{B_1}, T_1, Q_2, T_{2f}\right),$$
$$w^3 = \mathrm{col}\left(x_{A_3}, x_{B_3}, T_3, x_{A_2}, x_{B_2}, T_2, Q_3\right).$$

Table 3.1: Operating point and parameters

| Operating Point | | Parameters | | | |
|---|---|---|---|---|---|
| $x_{A_1}$ | 0.292 | $x_{A_{1f}}$ | 0.846 | $\rho$ | $1000\ kg\ m^{-3}$ |
| $x_{B_1}$ | 0.666 | $x_{B_{1f}}$ | 0.154 | $C_p$ | $4.2\ kJ\ kg^{-1}\ K^{-1}$ |
| $T_1$ | $447.6\ K$ | $x_{A_{2f}}$ | 0.610 | $R$ | $8.314\ kJ\ kmol^{-1}\ K^{-1}$ |
| $x_{A_2}$ | 0.287 | $x_{B_{2f}}$ | 0.390 | $k_1$ | $2.77 \times 10^3\ s^{-1}$ |
| $x_{B_2}$ | 0.670 | $V_1$ | $1\ m^3$ | $k_2$ | $2.5 \times 10^3\ s^{-1}$ |
| $T_2$ | $444.3\ K$ | $V_2$ | $0.5\ m^3$ | $E_1$ | $5 \times 10^4\ kJ\ kmol^{-1}$ |
| $x_{A_3}$ | 0.114 | $V_3$ | $1\ m^3$ | $E_2$ | $6 \times 10^4\ kJ\ kmol^{-1}$ |
| $x_{B_3}$ | 0.800 | $F_{1f}$ | $5\ m^3\ h^{-1}$ | $\Delta H_1$ | $-6 \times 10^4\ kJ\ kmol^{-1}$ |
| $T_3$ | $449.3\ K$ | $F_{2f}$ | $5\ m^3\ h^{-1}$ | $\Delta H_2$ | $7 \times 10^4\ kJ\ kmol^{-1}$ |
| $T_{1f}$ | $300\ K$ | $F_r$ | $50\ m^3\ h^{-1}$ | | |
| $T_{2f}$ | $300\ K$ | $\alpha_A$ | 3.5 | | |
| $Q_1$ | $12.6 \times 10^5\ kJ\ h^{-1}$ | $\alpha_B$ | 1 | | |
| $Q_2$ | $16.2 \times 10^5\ kJ\ h^{-1}$ | $\alpha_C$ | 0.5 | | |
| $Q_3$ | $12.6 \times 10^5\ kJ\ h^{-1}$ | | | | |

After linearisation and setting up the polytopic region, the first process is within a region of 8 vertices, whose kernel representations are

$$R_1^1(\xi) = \begin{bmatrix} \xi + 65.97 & 0 & 0.128 & -102.2 & 6.46 & 0 & 0 & 0 \\ -14.57 & \xi + 55.90 & -0.106 & 77.83 & -27.3 & 0 & 0 & 0 \\ -2082 & 383.6 & \xi + 45.97 & 0 & 0 & -50 & -0.0002 & -5 \end{bmatrix},$$

$$R_2^1(\xi) = \begin{bmatrix} \xi + 65.97 & 0 & 0.128 & -102.2 & 6.46 & 0 & 0 & 0 \\ -16.03 & \xi + 55.98 & -0.104 & 77.83 & -27.3 & 0 & 0 & 0 \\ -2290 & 422.0 & \xi + 46.90 & 0 & 0 & -50 & -0.0002 & -5 \end{bmatrix},$$

$$R_3^1(\xi) = \begin{bmatrix} \xi + 71.03 & 0 & 0.140 & -102.2 & 6.46 & 0 & 0 & 0 \\ -16.03 & \xi + 55.90 & -0.119 & 77.83 & -27.3 & 0 & 0 & 0 \\ -2390 & 383.6 & \xi + 44.15 & 0 & 0 & -50 & -0.0002 & -5 \end{bmatrix},$$

$$R_4^1(\xi) = \begin{bmatrix} \xi + 71.03 & 0 & 0.140 & -102.2 & 6.46 & 0 & 0 & 0 \\ -14.57 & \xi + 55.98 & -0.117 & 77.83 & -27.3 & 0 & 0 & 0 \\ -2082 & 422.0 & \xi + 45.07 & 0 & 0 & -50 & -0.0002 & -5 \end{bmatrix},$$

$$R_5^1(\xi) = \begin{bmatrix} \xi + 65.97 & 0 & 0.128 & -103.8 & 3.77 & 0 & 0 & 0 \\ -14.57 & \xi + 55.90 & -0.106 & 70.66 & -32.2 & 0 & 0 & 0 \\ -2082 & 383.6 & \xi + 45.97 & 0 & 0 & -50 & -0.0002 & -5 \end{bmatrix},$$

$$R_6^1(\xi) = \begin{bmatrix} \xi + 71.03 & 0 & 0.128 & -103.8 & 3.77 & 0 & 0 & 0 \\ -16.03 & \xi + 55.98 & -0.104 & 70.66 & -32.2 & 0 & 0 & 0 \\ -2290 & 422.0 & \xi + 46.90 & 0 & 0 & -50 & -0.0002 & -5 \end{bmatrix},$$

$$R_7^1(\xi) = \begin{bmatrix} \xi + 71.03 & 0 & 0.140 & -103.8 & 3.77 & 0 & 0 & 0 \\ -16.03 & \xi + 55.90 & -0.119 & 70.66 & -32.2 & 0 & 0 & 0 \\ -2290 & 383.6 & \xi + 44.15 & 0 & 0 & -50 & -0.0002 & -5 \end{bmatrix},$$

$$R_8^1(\xi) = \begin{bmatrix} \xi + 65.97 & 0 & 0.140 & -103.8 & 3.77 & 0 & 0 & 0 \\ -14.57 & \xi + 55.98 & -0.117 & 70.66 & -32.2 & 0 & 0 & 0 \\ -2082 & 422.0 & \xi + 45.07 & 0 & 0 & -50 & -0.0002 & -5 \end{bmatrix}.$$

The second process is in a polytopic region with 4 vertices, namely

$$R_1^2(\xi) = \begin{bmatrix} \xi + 133.2 & 0 & 0.115 & -110 & 0 & 0 & 0 & 0 \\ -13.20 & \xi + 120.8 & -0.096 & 0 & -110 & 0 & 0 & 0 \\ -1886 & 340.8 & \xi + 111.9 & 0 & 0 & -110 & -0.0005 & -10 \end{bmatrix},$$

$$R_2^2(\xi) = \begin{bmatrix} \xi + 133.2 & 0 & 0.115 & -110 & 0 & 0 & 0 & 0 \\ -13.20 & \xi + 120.9 & -0.094 & 0 & -110 & 0 & 0 & 0 \\ -1886 & 374.8 & \xi + 112.7 & 0 & 0 & -110 & -0.0005 & -10 \end{bmatrix},$$

$$R_3^2(\xi) = \begin{bmatrix} \xi + 134.5 & 0 & 0.127 & -110 & 0 & 0 & 0 & 0 \\ -14.52 & \xi + 120.8 & -0.107 & 0 & -110 & 0 & 0 & 0 \\ -2074 & 340.8 & \xi + 110.2 & 0 & 0 & -110 & -0.0005 & -10 \end{bmatrix},$$

$$R_4^2(\xi) = \begin{bmatrix} \xi + 134.5 & 0 & 0.127 & -110 & 0 & 0 & 0 & 0 \\ -14.52 & \xi + 120.9 & -0.105 & 0 & -110 & 0 & 0 & 0 \\ -2074 & 374.8 & \xi + 111.1 & 0 & 0 & -110 & -0.0005 & -10 \end{bmatrix}.$$

The last process is within a 2-vertex region whose vertices are

$$R_1^3(\xi) = \begin{bmatrix} \xi + 112.2 & -6.46 & 0 & -60 & 0 & 0 & 0 \\ -77.83 & \xi + 37.30 & 0 & 0 & -60 & 0 & 0 \\ 0 & 0 & \xi + 60 & 0 & 0 & -60 & -0.0002 \end{bmatrix},$$

$$R_2^3(\xi) = \begin{bmatrix} \xi + 113.8 & -3.77 & 0 & -60 & 0 & 0 & 0 \\ -70.66 & \xi + 42.15 & 0 & 0 & -60 & 0 & 0 \\ 0 & 0 & \xi + 60 & 0 & 0 & -60 & -0.0002 \end{bmatrix}.$$

The objective of the distributed control system is to control the purity of product B in all units. As $T_3$ is the control error of the temperature of the final product, higher level of disturbance attenuation should be achieved compared with $T_1$ and $T_2$. The weighting functions for plantwide performance (as in (3.2)) is therefore chosen as $\mathbf{W}(s) = \mathrm{diag}\,(W_1(s), W_2(s), W_3(s))$, where

$$\begin{aligned} W_1(s) = W_2(s) &= \frac{1}{0.2s+1}\,\mathrm{diag}\,(0,0,20)\,, \\ W_3(s) &= \frac{1}{0.2s+1}\,\mathrm{diag}\,(0,0,50)\,, \end{aligned} \tag{3.21}$$

are the weighting functions for process units 1 to 3 respectively. This specification requires a new steady state to be reached within approximately 1 hour after the introduction of the disturbance and the disturbance effect to be attenuated by 20 times in the reactors and 50 times in the separator. To show how to transform the weighting function into a supply rate, $W_1(s)$ is used as an example to illustrate the procedure and the other ones follow analogously. $W_1(s)$ can be written as $W_1(s) = N_1(s)/p_1(s)$ where $N_1(s) = \text{diag}(0, 0, 20)$ and $p_1(s) = 0.2s + 1$. Since in linearized systems all variables are deviation variables, it then follows that $s = \xi$. Therefore, $N(\xi) = \text{diag}(0, 0, 20)$ and $p(\xi) = 0.2\xi + 1$. It then follows that

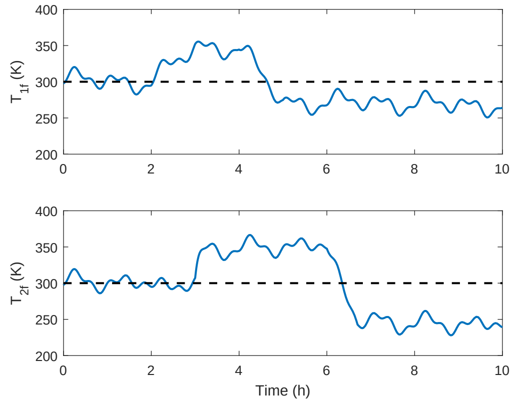
$$N^T(\zeta)N(\eta) = \begin{bmatrix} 0 & 0 & 0 \\ * & 0 & 0 \\ * & * & 400 \end{bmatrix}, \tag{3.22}$$

$$p(\zeta)p(\eta)I_3 = [1 + 0.2(\zeta + \eta) + 0.04\zeta\eta]\, I_3 = \begin{bmatrix} I_3 \\ \zeta I_3 \end{bmatrix}^T \begin{bmatrix} I_3 & 0.2I_3 \\ * & 0.04I_3 \end{bmatrix} \begin{bmatrix} I_3 \\ \eta I_3 \end{bmatrix}. \tag{3.23}$$

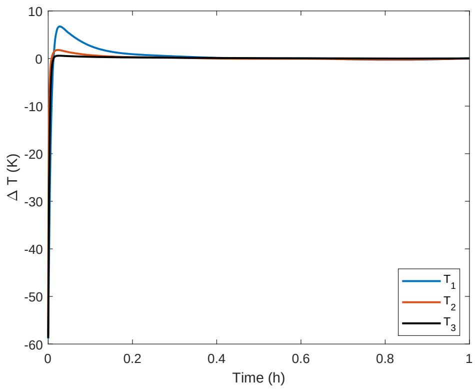
In most cases, the order of the QDFs in an inequality are not necessarily of the same order. To match the dimensions, these QDFs can be augmented. For example, to represent (3.23) with second order, one can write it as

$$p(\zeta)p(\eta)I_3 = \begin{bmatrix} I_3 \\ \zeta I_3 \\ \zeta^2 I_3 \end{bmatrix}^T \begin{bmatrix} I_3 & 0.2I_3 & 0_{3\times3} \\ * & 0.04I_3 & 0_{3\times3} \\ * & * & 0_{3\times3} \end{bmatrix} \begin{bmatrix} I_3 \\ \eta I_3 \\ \eta^2 I_3 \end{bmatrix}. \tag{3.24}$$

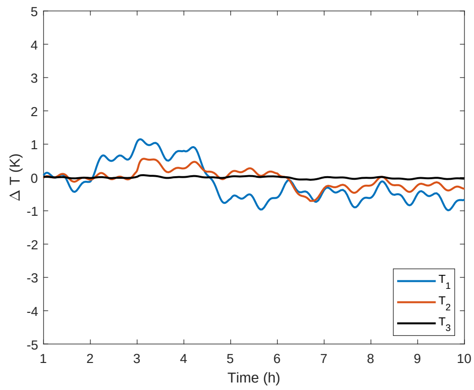
Simulation studies were carried out for a “true” process behaviour randomly chosen within the polytopic regions (i.e., with different $\theta^i$, $i \in \mathbb{Z}_3^+$). Following the two-step design, inequalities (3.7), (3.12) and (3.18) were solved simultaneously, resulting in a total of 39 LMIs. The $J$-factorisation process (2.37) – (2.38) is then




(a) Disturbance Profile




(b) Transition




(c) Steady State

Figure 3.4: Simulation result

used to determine the final controllers that attain the image representation

$$\begin{bmatrix} Q_i \\ y_r^i \\ \hline T_i \\ u_r^i \end{bmatrix} = \begin{bmatrix} M_y^i(\xi) \\ \hline M_u^i(\xi) \end{bmatrix} \ell^i \tag{3.25}$$

for some polynomial matrix $M^i(\xi)$ and latent variable $\ell^i$.

The simulation result is depicted in Figure 3.4. As shown, the performance specified by the weighting matrices (3.21) has been achieved. In fact, the actual performance is much better than specified. Most notably, temperature in the separator almost fully rejects the disturbance effect.

## 3.4 Summary

In this chapter, a distributed robust control approach has been proposed for interconnected systems with uncertain subsystems from the dissipativity perspective. Uncertainties in the subsystems are described using polytopic uncertainty. The global robust stability and performance conditions have been developed based on global dissipativity conditions, which in turn have been converted into supply rates that the distributed controllers must satisfy. The controllers can be synthesised individually in parallel by computing $J$-factorisation of their corresponding supply rates.

While this approach provides a much less conservative design method than the norm-bound design and the computational burden is quite manageable for a distributed uncertain system, it is still based on describing the uncertain region using a convex region. In this case, extra degrees of conservativeness would be inevitable if the original region is non-convex by nature. In the next chapter, the non-convex region will be handled directly. All representations and properties in this chapter will be generalised accordingly.

# Chapter 4

# Distributed Robust Control with Generic Uncertainties

In this chapter, a more generic control design approach is formulated for interconnected systems. Unlike in Chapter 3 in which convex polytopes are constructed for the subsystems, in this chapter all subsystems are described by a kernel representation with *generic* parameter uncertainty similar to (2.47), i.e.,

$$R^i(\theta, \xi)w^i := \sum_{k=0}^{L^i} R^i_k(\theta)\xi^k w^i = 0,\ i \in \mathbb{Z}^+_N. \tag{4.1}$$

The uncertain behaviour in each subsystem is then denoted as $\mathfrak{B}^i_\theta = \ker\left(R^i(\theta)\right)$ and Problem 3.1 can also be modified accordingly - to achieve (3.2) for subsystems with generic uncertainties.

Mathematically, the main focus in Chapter 3, i.e., interconnected systems whose subsystems are represented by (3.1), can be viewed as a special case of (4.1) by setting $\theta = \mathrm{col}(\theta^1, \theta^2, \ldots, \theta^N)$ and $R^i(\theta, \xi) = \sum_{j=1}^{M^i} \theta^i_j R^i(\xi)$. The physical implication for such generalisation is, on the other hand, more profound. In practice, many parameters in the models are with inexact values but rather well understood ranges. These parameters rarely occur naturally linear in the models.

By solving the uncertain problem directly, the physical meanings of the results become much clearer and the design is much less conservative. However, as the problem becomes non-convex and more generic, new tools and algorithms are needed.

*The key results in this chapter have been published in:*

- **Y. Yan**, R. Wang and J. Bao, "Robust Control Synthesis for Linear Differential Systems with Parametric Uncertainty," in *Proceedings of Australian & New Zealand Control Conference (ANZCC) 2018, Melbourne, Australia*, pp. 281–284, 2018.

- **Y. Yan** and J. Bao, "A Scenario Approach to Robust Distributed Control for Plantwide Process Systems," in *Proceedings of European Control Conference (ECC) 2019, Naples, Italy*, pp. 560–565, 2019.

*The model in Section 4.1 is a simplified version of that published in:*

- **Y. Yan**, Y. Li, M. Skyllas-Kazacos and J. Bao, "Modelling and simulation of thermal behaviour of vanadium redox flow battery," *Journal of Power Sources*, vol. 322, pp. 116–128, 2016.

- **Y. Yan**, M. Skyllas-Kazacos and J. Bao, "Effects of battery design, environmental temperature and electrolyte flowrate on thermal behaviour of a vanadium redox flow battery in different applications," *Journal of Energy Storage* vol. 11, pp. 104–118, 2016.

## 4.1 A Motivating Example

To demonstrate the necessity of the considered problem, a motivating example of a multi-cell vanadium redox flow battery (VRFB) stack is given.

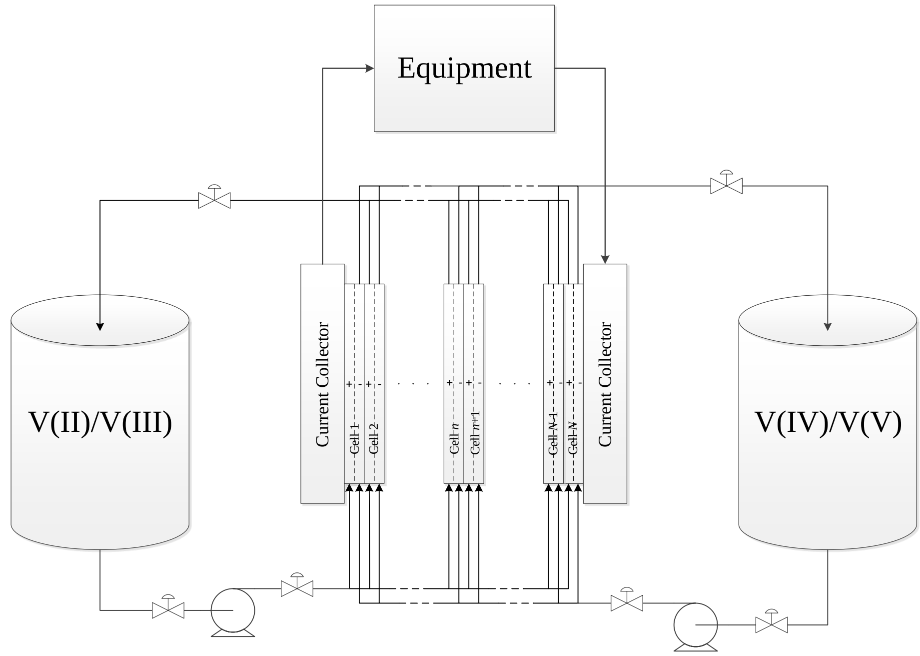



Figure 4.1: A VRFB Stack

**Example 4.1.** A VRFB is a type of redox flow battery whose electrolyte contains vanadium ions only. The reactions taking place within a cell are

$$\mathrm{V^{3+} + e^- \rightleftharpoons V^{2+}},$$

$$\mathrm{VO^{2+} + H_2O \rightleftharpoons VO_2^+ + 2H^+ + e^-},$$

where the first reaction takes place in the negative half-cell and the second reaction happens in the positive half-cell. The ions $V^{2+}$, $V^{3+}$, $VO^{2+}$ and $VO_2^+$ are normally denoted as V(II), V(III), V(IV) and V(VI), respectively, to indicate the type of vanadium ions. The two half-cells are separated by a membrane to allow electrons to flow between half-cells, but small amounts of vanadium will pass through the membrane and react directly with the ions on the other side, causing capacity loss (self-discharge). More details regarding the setup and properties of the battery can be found in [104].

Consider the VRFB system depicted in Figure 4.1, in which the stack consists of $N$ cells. For simplicity, only material dynamics are discussed and all physical properties are assumed to be temperature independent. The mass balance models for each cell can be constructed from first principle as

$$\begin{aligned}
\frac{V_c}{2}\frac{\mathrm{d}c_2^n}{\mathrm{d}t} &= Q_c\left(c_2^t - c_2^n\right) + \frac{i}{F} - k_2\frac{c_2^n}{d}S - k_4\frac{c_4^n}{d}S - 2k_5\frac{c_5^n}{d}S,\\
\frac{V_c}{2}\frac{\mathrm{d}c_3^n}{\mathrm{d}t} &= Q_c\left(c_3^t - c_3^n\right) - \frac{i}{F} - k_3\frac{c_3^n}{d}S + 2k_4\frac{c_4^n}{d}S + 3k_5\frac{c_5^n}{d}S,\\
\frac{V_c}{2}\frac{\mathrm{d}c_4^n}{\mathrm{d}t} &= Q_c\left(c_4^t - c_4^n\right) - \frac{i}{F} + 3k_2\frac{c_2^n}{d}S + 2k_3\frac{c_3^n}{d}S - k_4\frac{c_4^n}{d}S,\\
\frac{V_c}{2}\frac{\mathrm{d}c_5^n}{\mathrm{d}t} &= Q_c\left(c_5^t - c_5^n\right) + \frac{i}{F} - 2k_2\frac{c_2^n}{d}S - k_3\frac{c_3^n}{d}S - k_5\frac{c_5^n}{d}S,
\end{aligned}$$

where $c_2$, $c_3$, $c_4$ and $c_5$ are the concentrations of $V$(II), $V$(III), $V$(IV) and $V$(VI), respectively. The superscripts $n$ and $t$ denote, respectively the $n$-th cell and the storage tank. The current applied to the stack is denoted as $i$, and $k_2$, $k_3$, $k_4$ and $k_5$ are the diffusivity of each vanadium ion. $V_c$ is the volume of each cell, $Q_c$ is the volumetric flowrate of the electrolyte in each cell, $S$ is the surface area of the membrane, $d$ is the thickness of the membrane and $F$ is the Faraday's constant. The resulting manifest variables in each cell is then $w^n = \mathrm{col}\left(c_2^n, c_3^n, c_4^n, c_5^n, c_2^t, c_3^t, c_4^t, c_5^t, i\right)$. By appropriate input/output partition, the first principle models can be rearranged as

$$\begin{bmatrix} R_y^n(\theta,\xi) & R_u^n(\theta,\xi) \end{bmatrix} \begin{bmatrix} y^n \\ u^n \end{bmatrix} = 0, \tag{4.2}$$

where

$$y^n = \begin{bmatrix} c_2^n & c_3^n & c_4^n & c_5^n \end{bmatrix}^T, \; u^n = \begin{bmatrix} c_2^t & c_3^t & c_4^t & c_5^t & i \end{bmatrix}^T,$$

$$R_y^n(\theta,\xi) = \begin{bmatrix} \frac{V_c}{2}\xi + Q_c + k_2\frac{S}{d} & 0 & k_4\frac{S}{d} & 2k_5\frac{S}{d} \\ 0 & \frac{V_c}{2}\xi + Q_c + k_3\frac{S}{d} & -2k_4\frac{S}{d} & -3k_5\frac{S}{d} \\ -3k_2\frac{S}{d} & -2k_3\frac{S}{d} & \frac{V_c}{2}\xi + Q_c + k_4\frac{S}{d} & 0 \\ 2k_2\frac{S}{d} & k_3\frac{S}{d} & 0 & \frac{V_c}{2}\xi + Q_c + k_5\frac{S}{d} \end{bmatrix},$$

$$R_u^n(\theta,\xi) = \begin{bmatrix} -Q_c & 0 & 0 & 0 & -\frac{1}{F} \\ 0 & -Q_c & 0 & 0 & \frac{1}{F} \\ 0 & 0 & -Q_c & 0 & \frac{1}{F} \\ 0 & 0 & 0 & -Q_c & -\frac{1}{F} \end{bmatrix}.$$

From the above representation, it is clear to see that the dynamics are linear. However, in practice all physical properties of the system (i.e., $k_2$, $k_3$, $k_4$, $k_5$, $V_c$, $Q_c$, $S$ and $d$) are uncertain. It is possible to obtain a linear uncertain model by lumping certain variables into a new uncertain variable. For example, by choosing

$$\theta = \mathrm{col}\left(V_c, Q_c, k_2\frac{S}{d}, k_3\frac{S}{d}, k_4\frac{S}{d}, k_5\frac{S}{d}\right),$$

the original representation immediately becomes linear with respect to $\theta$ and a polytopic region can be easily formulated. However, such choice of the uncertain parameter leads to conservative design, as it is neglecting the inherent dependencies among the parameters. For example, it is obvious that $k_2\frac{S}{d}$ and $k_3\frac{S}{d}$ are not independent, but by naming them as $\theta_3$ and $\theta_4$, they are treated as two completely independent variables. Therefore, by making the region convex, conservatism is introduced. Such conservatism may also accumulate due to the scale of the problem and even be amplified due to interactions. It is therefore more appropriate to deal with the uncertainties of the parameters directly instead of forcing linearity and convexity by lumping variables together. ■

While the uncertain region is not assumed to be convex, some mild assumptions are still needed.

**Assumption 4.1.** All uncertain parameters in all subsystems are uniformly distributed with their respective bounds, i.e.,

$$\theta \in \Theta = \left\{\theta \in \mathbb{R}^M \mid \theta_j \sim \mathcal{U}\left(\underline{\theta}_j, \overline{\theta}_j\right),\ \forall j \in \mathbb{Z}_M^+\right\}. \tag{4.3}$$

**Assumption 4.2.** All entries in the coefficient matrix of the kernel representations of the subsystems, $R^i(\theta^i, \xi)$, are finite for all $\theta^i \in \Theta^i$.

Assumption 4.1 states that the uncertain parameters are within certain ranges, and all values in the ranges are equally likely to happen. This is reasonable assumption since the region of the uncertain parameters are mostly determined empirically, in which case the range should not have particular distributions. Assumption 4.2 states that there will not be a case where any coefficient will be infinity for any parameter within their regions. This is also a natural assumption because it physically cannot happen.

As the complexity of the problem increases, it is obvious that new tools and concepts are required to accommodate it. In the next section, a relaxed version of QDFs will be defined so that dynamic features of systems with non-convex uncertainty regions can be described in more details.

## 4.2 Dissipativity Analysis via Parametric QDFs

Much like the coefficient matrix in a kernel representation, QDFs themselves can also have uncertainties in their coefficient matrices, making them more flexible in dealing with uncertainties in the subsystems. A parametric QDF (PQDF) can be defined as

$$\Phi(\theta, \zeta, \eta) := \sum_{k=0}^{L} \sum_{l=0}^{L} \zeta^k \Phi_{kl}(\theta) \eta^l, \tag{4.4}$$

where the coefficient matrix depends on the uncertain parameter $\theta$. Analogous to (2.48) being the special case of (2.47), parameter-dependant QDF defined in (3.4) is a special case of the PQDF defined in (4.4). Although $\theta$ is unknown, it is still constant, and therefore the operator $\nabla$ can be defined in a similar way, i.e., $\nabla\Phi(\theta, \zeta, \eta) = (\zeta + \eta)\Phi(\theta, \zeta, \eta)$. However, since the uncertain parameters are not assumed to be of any special form, PQDF-dissipativity conditions are also generic. The following theorem gives the conditions of PQDF-dissipativity along an uncertain behaviour.

**Theorem 4.1** (PQDF-Dissipativity along $\ker(R(\theta))$)**.** A dynamical system $\Sigma_\theta = (\mathbb{R}, \mathbb{R}^{\mathrm{w}}, \mathfrak{B}_\theta)$ with behaviour $\mathfrak{B}_\theta$ represented by (2.47) is dissipative with respect to a PQDF induced by $\Phi(\theta, \zeta, \eta)$ if there exist PQDFs $\Psi \in \mathbb{S}^{\mathrm{w}}[\zeta, \eta]$ and $F \in \mathbb{R}^{\bullet \times \mathrm{w}}[\zeta, \eta]$ such that

$$\Psi(\theta, \zeta, \eta) \succeq 0, \tag{4.5a}$$

$$\Phi(\theta, \zeta, \eta) - \nabla\Psi(\theta, \zeta, \eta) + \mathrm{He}\left\{F^\star(\theta, \zeta, \eta) R(\theta, \eta)\right\} \succeq 0, \tag{4.5b}$$

hold for all $\theta \in \Theta$.

*Proof.* This proof is analogous to the proof of Proposition 3.1 and is therefore omitted. □

Similar to (3.5), inequalities in (4.5) can be transformed into ($\theta$-dependant) LMIs

$$\widetilde{\Psi}(\theta) \geq 0, \tag{4.6a}$$

$$\widetilde{\Phi}(\theta) - \nabla\widetilde{\Psi}(\theta) + \mathrm{He}\left\{\widetilde{F}^T(\theta)\widehat{R}(\theta)\right\} \geq 0, \tag{4.6b}$$

where

$$\nabla\widetilde{\Psi}(\theta) = \begin{bmatrix} 0 & 0 \\ \widetilde{\Psi}(\theta) & 0 \end{bmatrix} + \begin{bmatrix} 0 & \widetilde{\Psi}(\theta) \\ 0 & 0 \end{bmatrix},$$

$$\widehat{R}(\theta) = \begin{bmatrix} R_0(\theta) & \cdots & R_N(\theta) & 0 & \cdots & 0 \\ 0 & R_0(\theta) & \cdots & R_N(\theta) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & R_0(\theta) & \cdots & R_N(\theta) \end{bmatrix}.$$

The supply rate for the controlled interconnected system can be constructed in similar fashion to (3.11). The requirements to ensure the frequency-weighted condition (3.2) is similar to that in Theorem 3.2, but they are no longer linear with respect to the uncertain parameter $\theta$. To highlight this difference, the following theorem gives the requirements in terms of parametric LMIs.

**Theorem 4.2.** Given the controlled system depicted in Figure 3.1, whose subsystems $\mathcal{P}^i_{\theta^i}$ and corresponding controllers $\mathcal{C}^i$ are dissipative with respect to PQDFs induced by $\Phi^i(\theta^i, \zeta, \eta)$ and QDFs induced by $\Phi^i_c(\zeta, \eta)$, if

$$\begin{bmatrix} \mathcal{Q}_{\mathbf{yl}}(\theta,\zeta,\eta) & \mathcal{Q}_{\mathbf{yr}}(\theta,\zeta,\eta) \end{bmatrix} = 0, \tag{4.7a}$$

$$\begin{bmatrix} \mathcal{S}^\star_{\mathbf{yd}}(\theta,\zeta,\eta) & \mathcal{S}^\star_{\mathbf{ld}}(\theta,\zeta,\eta) & \mathcal{S}^\star_{\mathbf{rd}}(\theta,\zeta,\eta) \end{bmatrix} = 0, \tag{4.7b}$$

$$\begin{bmatrix} \mathcal{Q}_{\mathbf{ll}}(\theta,\zeta,\eta) & \mathcal{Q}_{\mathbf{lr}}(\theta,\zeta,\eta) \\ \star & \mathcal{Q}_{\mathbf{rr}}(\theta,\zeta,\eta) \end{bmatrix} \preceq 0, \tag{4.7c}$$

$$\mathcal{Q}_{\mathbf{yy}}(\theta,\zeta,\eta) + \mathbf{N}^T(\zeta)\mathbf{N}(\eta) \preceq 0, \tag{4.7d}$$

$$\mathcal{R}_{\mathbf{dd}}(\theta, \zeta, \eta) - \gamma^2 \mathbf{D}^T(\zeta)\mathbf{D}(\eta) \preceq 0, \tag{4.7e}$$

are satisfied for all $\theta \in \Theta$, then the controlled behaviour is robustly stable and satisfies the performance condition (3.2) with weighting function $\mathbf{W}(s) = \mathbf{D}^{-1}(s)\mathbf{N}(s)$.

*Proof.* The proof is analogous to the proof of Theorem 3.2 and is omitted. □

Since the controller supply rates are QDFs, they can still be used to parametrise the controller behaviours $\mathfrak{B}_c^i$ through $J$-factorisation. For each subsystem, in order for the controller supply rate matrices $\Phi_c^i(\zeta, \eta)$ to be factorisable, they must have constant signatures. Defining

$$y_c^i := \operatorname{col}\left(y_l^i, y_r^i\right), \; u_c^i := \operatorname{col}\left(u_l^i, u_r^i\right), \; \mathcal{Q}_c^i := \begin{bmatrix} \mathcal{Q}_l^i & \mathcal{Q}_{lr}^i \\ \star & \mathcal{Q}_r^i \end{bmatrix}, \; \mathcal{R}_c^i := \begin{bmatrix} \mathcal{R}_l^i & \mathcal{R}_{lr}^i \\ \star & \mathcal{R}_r^i \end{bmatrix},$$

then either $\pi_+^i = \dim(u_c^i)$, $\pi_-^i = \dim(y_c^i)$ or $\pi_+^i = \dim(y_c^i)$, $\pi_-^i = \dim(u_c^i)$ need to be satisfied to ensure factorisability. Sufficient conditions for these two cases are, respectively,

$$\mathcal{Q}_c^i \prec 0, \quad \mathcal{R}_c^i \succ 0 \Rightarrow \pi^i = \left(\dim(y_c^i), 0, \dim(u_c^i)\right), \text{ or} \tag{4.8a}$$

$$\mathcal{Q}_c^i \succ 0, \quad \mathcal{R}_c^i \prec 0 \Rightarrow \pi^i = \left(\dim(u_c^i), 0, \dim(y_c^i)\right). \tag{4.8b}$$

Whilst the former is preferred over the latter because it guarantees stable controllers, it is possible that there is no solution if the processes are unstable. In this case, the latter condition can be used.

To sum up, conditions in (4.5), (4.7) and one of (4.8) should be solved simultaneously to determine $\Phi_c^i$. If a solution exists, behaviours $\mathfrak{B}_c^i$ can be constructed in image representation through $J$-factorisation. It is easy to recognise that the parametric LMIs can be non-convex, hence a solution is much harder to find, and in many cases a deterministic solution may not exist at all. In the next section, two possible methods are proposed to solve the problem. The first approach results in deterministic solutions via sum-of-square (SOS) programming [105–107] and robust optimisation [108]. To this effect, additional assumptions are required and the scale of the problem cannot be too large. The second approach results in a probabilistic solution via the scenario approach [109–111], which can handle larger systems with no further assumptions.

## 4.3 The Search for Controller Supply Rates

### 4.3.1 Deterministic Solution via SOS

SOS programming is a computationally tractable algorithm for the validation of the positivity of a polynomial. A polynomial $p \in \mathbb{R}[x]$ with real coefficients and indeterminate vector $x$ is called an SOS polynomial if it can be written as

$$p(x) = m^T(x)Qm(x) \tag{4.9}$$

where $Q \in \mathbb{R}^{\bullet \times \bullet}$ is a positive semidefinite matrix and $m(x)$ is a vector of monomials [105–107, 112]. The set of all SOS polynomials are denoted as

$$\mathbb{P}[x] = \left\{ p \in \mathbb{R}[x] \;\middle|\; p = m^T Q m,\ m \in \mathbb{R}^{\bullet}[x],\ Q \in \mathbb{R}^{\bullet \times \bullet},\ Q \geq 0 \right\}. \tag{4.10}$$

While $\Sigma[x]$ is a more popular notation for the set of SOS polynomials, $\mathbb{P}[x]$ is used here since $\Sigma$ is exclusively reserved to represent a dynamical system throughout this thesis.

To use this algorithm, obviously all uncertain matrices in question should be polynomials of the indeterminates. While it is not always necessarily the case, this algorithm can still be used to achieve rather satisfactory results if additional regularity assumptions can be made.

**Assumption 4.3.** The coefficient matrices $R^i(\theta^i, \xi)$ are smooth matrix functions on $\Theta^i$ for all $i \in \mathbb{Z}_N^+$.

From Assumption 4.3, all coefficient matrices $R^i(\theta^i, \xi)$, $i \in \mathbb{Z}_N^+$ can be represented or approximated by polynomials of $\theta$. Then SOS programming can be implemented to solve the parametric LMIs in (4.5) and (4.7). Assuming (4.8b) is used to ensure feasibility, the LMI and SOS conditions to be satisfied can then be summarised as

$$\left(\widehat{w}^i\right)^T \widetilde{\Psi}^i(\theta^i)\widehat{w}^i \in \mathbb{P}\left[\widehat{w}^i, \theta^i\right], \tag{4.11a}$$

$$\left(\widehat{w}^i\right)^T \left(\widetilde{\Phi}^i(\theta^i) - \nabla\widetilde{\Psi}^i(\theta^i) + \mathrm{He}\left\{\left(\widetilde{F}^i\right)^T(\theta^i)\widehat{R}^i(\theta^i)\right\}\right)\widehat{w}^i \in \mathbb{P}\left[\widehat{w}^i, \theta^i\right], \tag{4.11b}$$

$$\left(\hat{c}^i\right)^T \widetilde{\mathcal{Q}}_c^i \hat{c}^i \in \mathbb{P}\left[\hat{c}^i\right], \tag{4.11c}$$

$$-\left(\hat{c}^i\right)^T \widetilde{\mathcal{R}}_c^i \hat{c}^i \in \mathbb{P}\left[\hat{c}^i\right], \tag{4.11d}$$

$$\begin{bmatrix} \widetilde{\mathcal{Q}}_{\mathbf{yl}}(\theta) & \widetilde{\mathcal{Q}}_{\mathbf{yr}}(\theta) \end{bmatrix} = 0, \tag{4.11e}$$

$$\begin{bmatrix} \widetilde{\mathcal{S}}_{\mathbf{yd}}^T(\theta) & \widetilde{\mathcal{S}}_{\mathbf{ld}}^T(\theta) & \widetilde{\mathcal{S}}_{\mathbf{rd}}^T(\theta) \end{bmatrix} = 0, \tag{4.11f}$$

$$-\begin{bmatrix} \hat{\mathbf{y}}_{\mathbf{l}} \\ \hat{\mathbf{y}}_{\mathbf{r}} \end{bmatrix}^T \begin{bmatrix} \widetilde{\mathcal{Q}}_{\mathbf{ll}}(\theta) & \widetilde{\mathcal{Q}}_{\mathbf{lr}}(\theta), \\ * & \widetilde{\mathcal{Q}}_{\mathbf{rr}}(\theta) \end{bmatrix} \begin{bmatrix} \hat{\mathbf{y}}_{\mathbf{l}} \\ \hat{\mathbf{y}}_{\mathbf{r}} \end{bmatrix} \in \mathbb{P}\left[\hat{\mathbf{y}}_{\mathbf{l}}, \hat{\mathbf{y}}_{\mathbf{r}}, \theta\right], \tag{4.11g}$$

$$-\hat{\mathbf{y}}^T \left(\widetilde{\mathcal{Q}}_{\mathbf{yy}}(\theta) + \widetilde{\mathbf{N}}^T\widetilde{\mathbf{N}}\right)\hat{\mathbf{y}} \in \mathbb{P}\left[\hat{\mathbf{y}}, \theta\right], \tag{4.11h}$$

$$\hat{\mathbf{d}}^T \left(\widetilde{\mathbf{D}}^T\widetilde{\mathbf{D}} - \widetilde{\mathcal{R}}_{\mathbf{dd}}(\theta)\right)\hat{\mathbf{d}} \in \mathbb{P}\left[\hat{\mathbf{d}}, \theta\right], \tag{4.11i}$$

for all $\theta \in \Theta$ and $i \in \mathbb{Z}_N^+$. The SOS decomposition (4.9) for the conditions in (4.11) can be obtained through semidefinite programming (SDP) [112]. In this case, open source Matlab toolbox – YALMIP [80, 113] and SDP solver – SeDuMi [114] are used to find the $Q$ matrix. The resulting controller supply rates can then parametrise the controller behaviour through $J$-factorisation. Observe that

(4.11) solves for feasible supply rates for any $\theta$, which is much more than required and can easily lead to infeasibility. The uncertain parameters can be specified as uncertain parameters within their respective bounds and the supply rates can be determined via robust optimisation [108].

**Example 4.2.** Consider a system with transfer function

$$y = \frac{1}{s+\theta}u + \frac{2}{2s+1}d, \tag{4.12}$$

where $\theta \in [-1, 1]$. Note that this process can potentially be unstable. The transfer function can be transferred into a $\theta$-dependant kernel representation

$$\begin{bmatrix} 2\xi^2 + (2\theta+1)\xi + \theta & -2\xi - 1 & -2\xi - 2\theta \end{bmatrix} w = 0, \tag{4.13}$$

where $w = \mathrm{col}(y, u, d)$. The weighting function chosen is

$$W(s) = \frac{5}{s+1}, \tag{4.14}$$

which requires the disturbance to be attenuated by 5 times in approximately 5 seconds.

A step change disturbance with a magnitude of 1 is introduced into the process and the output for $\theta = 0,\ \pm 0.2,\ \pm 0.5$ and $\pm 1$ are shown in Figure 4.2. It clearly shows that, for all of the tested $\theta$ values, the closed-loop system is stable and the performance requirement is achieved.

### 4.3.2 Probabilistic Solution via the Scenario Approach

Since the problem concerned in this chapter is not necessarily convex and the scale of the problem can be rather large, it is difficult, sometimes impossible, to find distributed controllers to solve the problem definitely. The basic idea introduced in this section is to use the scenario approach, a randomised algorithm, to search for QDF supply rates for the controllers $\mathcal{C}^i$ and PQDF dissipativity conditions

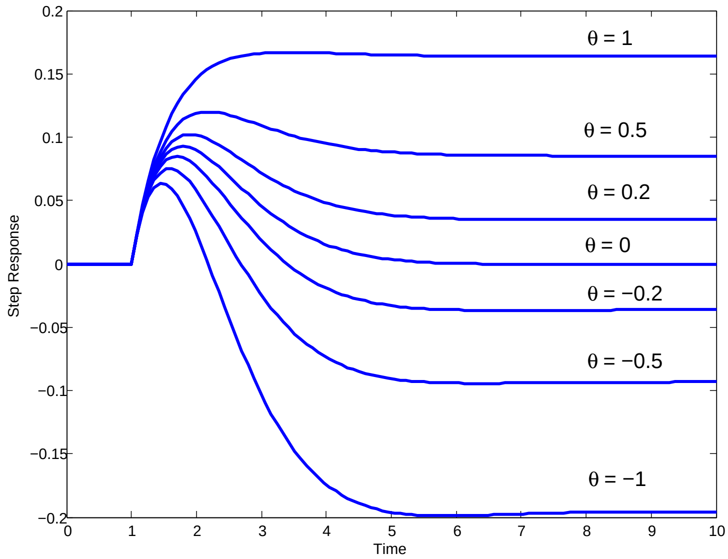



Figure 4.2: Step Change Response with Various $\theta$ Values

for the uncertain subsystem $\Sigma_\theta^i$ such that the supply rate of the controlled system satisfies the $\mathcal{H}_\infty$ performance condition (3.2) for *almost* all of the uncertain behaviours [109, 110]. The distributed controllers can then be synthesised explicitly from the supply rates through $J$-factorisation. Note that while another randomised algorithm, namely the sequential method [115], is effective as well, it is not applicable to the problem in this section because it is impossible to compute stochastic gradients without *a priori* knowledge of the forms of the matrices. For the clarity of presentation, dependencies on $\zeta$ and $\eta$ for polynomial matrices are omitted.

The scenario approach involves finding an optimal solution to the problem

$$\begin{aligned} &\min_q \; C^T q \\ &\text{s.t. } \mathcal{J}(X^\alpha, q) \le \beta, \qquad \alpha \in \mathbb{Z}_K^+, \end{aligned} \tag{4.15}$$

where $q$ is a vector of design parameters, $\mathcal{J}(\cdot)$ is the performance function, $\alpha$ is the index from 1 to $K$ and $X^\alpha$ are the scenarios. Under this setting, the original uncertain problem that is NP-hard is converted to a finite, potentially solvable problem. The requirement for this approach to be applicable is that $\mathcal{J}(X,q)$ is convex in $q$ for any fixed value of $X$ [111]. Denoting the optimal solution for a set of scenarios $X$ as $q_X^s$, the probability of violation is then defined as

$$\mathcal{Y}(q_X^s) = \Pr\left\{\mathcal{J}(X, q_X^s) > \beta\right\}. \tag{4.16}$$

The following lemma provides a method to choose a lower bound on the required number of samples.

**Lemma 4.3** ([109])**.** Let $\epsilon, \rho \in (0,1)$. Assuming a unique solution exists and $K \geq q+1$, if $K$ satisfies

$$K \geq \frac{2}{\epsilon}(q - \ln \rho), \tag{4.17}$$

then $\Pr\left\{\mathcal{Y}(q_X^s) > \epsilon\right\} \leq \rho$. In other words, with high probability $1-\rho$, the probability of violation is no greater than $\epsilon$.

This lemma suggests that the solution obtained from the scenario approach is a nested probabilistic solution: the probability that the solution works more than $1-\epsilon$ of the time is $1-\rho$. However, since $K$ is related with $\rho$ in a logarithmic way, $\rho$ can be set to an extremely small number without increasing the complexity significantly. As a result, the upper bound on the probability of violation $\epsilon$ can be almost definitely guaranteed. In the case where the scale of the problem is larger, the coefficient 2 in (4.17) can be replaced by $\frac{e}{e-1}$ [111], which reduces the number of samples by approximately 21%.

Suppose $K_c$ samples have been generated, then for each scenario $\alpha$, the constraints to be satisfied are

$$\Psi_\alpha^i \succeq 0, \tag{4.18a}$$

$$\Phi_\alpha^i - \nabla\Psi_\alpha^i + \mathrm{He}\left\{(F_\alpha^i)^\star R^i(\hat{\theta}^\alpha)\right\} \succeq 0, \tag{4.18b}$$

$$\begin{bmatrix} \mathcal{Q}_{\mathbf{yl}_\alpha} & \mathcal{Q}_{\mathbf{yr}_\alpha} \end{bmatrix} = 0, \tag{4.18c}$$

$$\begin{bmatrix} \mathcal{S}^{\star}_{\mathbf{yd}_\alpha} & \mathcal{S}^{\star}_{\mathbf{ld}_\alpha} & \mathcal{S}^{\star}_{\mathbf{rd}_\alpha} \end{bmatrix} = 0, \tag{4.18d}$$

$$\begin{bmatrix} \mathcal{Q}_{\mathbf{ll}_\alpha} & \mathcal{Q}_{\mathbf{lr}_\alpha} \\ \star & \mathcal{Q}_{\mathbf{rr}_\alpha} \end{bmatrix} \preceq 0, \tag{4.18e}$$

$$\mathcal{Q}_{\mathbf{yy}_\alpha} + \mathbf{N}^{\star}\mathbf{N} \preceq 0, \tag{4.18f}$$

$$\mathcal{R}_{\mathbf{dd}_\alpha} - \gamma^2 \mathbf{D}^{\star}\mathbf{D} \preceq 0, \tag{4.18g}$$

and one of

$$\mathcal{Q}^i_c \prec 0, \quad \mathcal{R}^i_c \succ 0, \text{ or} \tag{4.18h}$$

$$\mathcal{Q}^i_c \succ 0, \quad \mathcal{R}^i_c \prec 0. \tag{4.18i}$$

for all $i \in \mathbb{Z}^+_N$ and $\alpha \in \mathbb{Z}^+_{K_c}$. In (4.18), all polynomial matrices with a subscript or superscript $\alpha$ attain different values for each scenario, all of those without stay the same for all scenarios. For each scenario, a different kernel representation for each subsystem is generated as $R^i(\hat{\theta}^\alpha, \xi)w^i = 0$, making the coefficient matrices $\theta$-free. This means that all LMIs in (4.18) are deterministic LMIs. All of the LMIs in the $K_c$ scenarios are then solved simultaneously. A probabilistic solution to the problem can be obtained using the following proposition.

**Proposition 4.4.** For the interconnected system depicted in Figure 3.1, assume the number of design variables in the controller supply rate $Q_{\Phi_c}(\mathbf{c})$ is $q_c$. If there exists a solution to the simultaneous LMIs (4.18) for $K_c$ randomly generated samples, where

$$K_c \geq \frac{2}{\epsilon}(q_c - \ln \rho), \tag{4.19}$$

then with high probability $1 - \rho$, there exist controller supply rates (and hence an array of distributed controllers) such that the closed-loop system is robustly stable and satisfies the performance condition (3.2) with the probability of violation no greater than $\epsilon$.

*Proof.* Following Lemma 4.3, if a solution $q^s_{cX}$ exists with $K_c$ scenarios, then $\Pr\{\mathcal{Y}(q^s_{cX}) > \epsilon\} \le \rho$. This means that with high probability $1-\rho$, there exist $\theta$-dependant process supply rates $\Phi^i(\theta)$, storage functions $\Psi^i(\theta)$ and multipliers $F^i(\theta)$ such that $\Phi^i(\hat{\theta}^\alpha) = \Phi^i_\alpha$, $\Psi^i(\hat{\theta}^\alpha) = \Psi^i_\alpha$ and $F^i(\hat{\theta}^\alpha) = F^i_\alpha$ for $\alpha = 1, \dots, N$, as well as controller supply rates $\Phi^i_c$ such that inequalities (4.5), (4.7) and one of (4.8) hold with a probability of violation no greater than $\epsilon$. The satisfaction of robust stability and performance conditions follow from Lemma 4.1 and Theorem 4.2. The controllers can be subsequently synthesised using $J$-factorisation because the supply rates are $\theta$-free and have constant signatures. This completes the proof. □

**Remark 4.1.** Notice that closed forms of $\Phi^i(\theta)$, $\Psi^i(\theta)$ or $F^i(\theta)$ as matrix functions of $\theta$ are *not* obtained eventually. What can be concluded is that with a high probability, there exist these polynomial matrices that take the values of the $\alpha$th solution when substituting $\theta$ with $\hat{\theta}^\alpha$. Since they are auxiliary variables to aid the search for the controller supply rates and are not used in control design, obtaining closed forms for them is not necessary.

**Example 4.3.** Consider a plantwide system with 2 subsystems whose behaviours are described by

$$\mathcal{P}^1_\theta: \qquad \frac{dy^1}{dt} + \theta_1\theta_2 y^1 = \sin(\theta_1)u^1_p + u^1_m + d, \tag{4.20a}$$

$$\mathcal{P}^2_\theta: \qquad \theta_3\frac{dy^2}{dt} + y^2 = e^{-\theta_4}u^2_p + (\theta_1 - \theta_4)u^2_m, \tag{4.20b}$$

where $\theta_m \in [0.8, 1.2]$, $m = 1, \dots, 4$. The time base is seconds. Converting differentials to $\xi$ leads to the following kernel representations

$$R^1(\theta,\xi) = \begin{bmatrix} \xi + \theta_1\theta_2 & -\sin(\theta_1) & -1 & -1 \end{bmatrix}, \tag{4.21a}$$

$$R^2(\theta,\xi) = \begin{bmatrix} \theta_3\xi + 1 & -e^{-\theta_4} & \theta_4 - \theta_1 \end{bmatrix}. \tag{4.21b}$$

In this example, $\mathcal{P}_\theta^1$ and $\mathcal{P}_\theta^2$ are connected in series and the output of $\mathcal{P}_\theta^2$ is fed into $\mathcal{P}_\theta^1$. Hence, $F_\mathcal{P} = I_2$ and $H_\mathcal{P} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since all outputs are sent to the controllers, $F_\mathcal{C} = I_2$. The controller topology in this example is assumed to be the same as that of the interconnected system. Therefore, $H_\mathcal{P} = H_\mathcal{C}$. In this case $\mathbf{d} = d$ and $\mathbf{y} = \mathrm{col}(y^1, y^2)$. The goal is to attenuate the effect of disturbance $d$ on $y^1$ by 5 times and $y^2$ by 10 times at the steady state with the attenuation achieved in approximately 5 seconds. The weighting function in (3.2) can then be chosen as

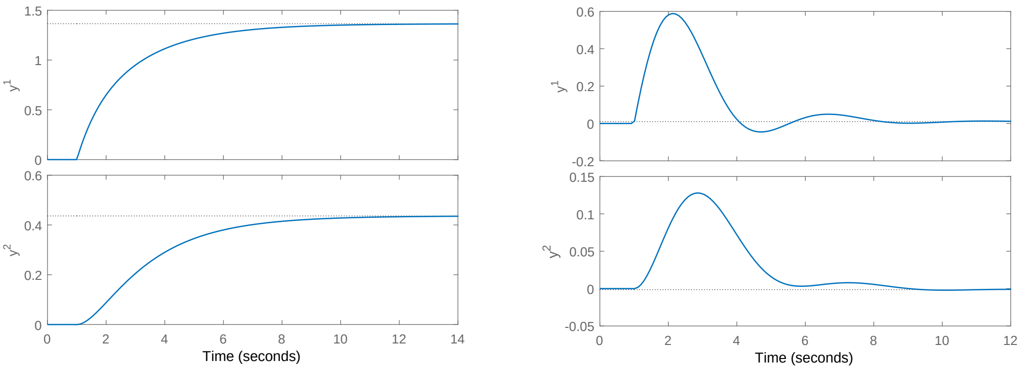
$$\mathbf{W}(s) = \frac{1}{s+1}\begin{bmatrix} 5 & 10 \end{bmatrix}. \tag{4.22}$$

Setting $\epsilon = 0.1$, $\rho = 10^{-6}$ and the QDF order $L = 2$, $K_c$ can be calculated from (4.17) to be at least 2677.

To test the probability of feasibility, *a posteriori* tests using $25,000$ samples with a step change disturbance of magnitude 1 introduced at $t = 1$ have been carried out. The test shows the probability of feasibility of 99.2%, which is much higher than specified (90%). One of the test samples is shown in Figure 4.3 with $\theta = [1.15,\ 0.89,\ 0.96,\ 1.14]$, including the process behaviours with and without the controllers, from which it can be seen that the disturbance is effectively rejected.

## 4.4 Summary

In this chapter, a distributed robust control design approach for interconnected systems whose subsystems have generic parametric uncertainties based on PQDF have been proposed. The resulting LMI conditions to be satisfied can be thought of as the parametric versions of those formulated in Chapter 3. For smaller-scaled systems whose subsystems have coefficient matrices that are smooth with




(a) Uncontrolled behaviour (b) Controlled behaviour

Figure 4.3: Step Change Response with $d = 1$

respect to the uncertain parameters, SOS programming combined with robust optimisation can be carried out to obtain a deterministic solution; for larger-scaled systems whose subsystems have coefficient matrices that are less well-behaved with respect to the uncertain parameters, the scenario approach can be used to obtain a probabilistic solution.

While the uncertain description in this chapter is much less conservative than that in Chapter 3, it is still only an inaccurate representation of the actual behaviour of the systems. The main idea behind robust control is also to control the system subject to an inaccurate description. The most accurate way to represent a system is through its behaviour as a collection of trajectories, in which case the idea of robustness is non-existent. In the next chapter, analysis and control design will be carried out from a set-theoretic point of view, leading to a representation-free framework.

# Chapter 5

# A Set-theoretic Approach to Distributed Control

As pointed out in Chapter 4, analysis of a dynamical system with prescribed representation is only applicable to *that particular class* of system. This chapter formulates a framework for the analysis and distributed control of interconnected systems from the perspective of the essence of the behavioural systems theory - the set-theoretic perspective. The core of a dynamical system is the set of all trajectories admissible through the system and interconnections are interpreted as constraints on the choice of trajectories. Under this setting, the discussion of system dynamics and control can be carried out entirely representation-free. From this view, it will be shown that the interconnected behaviour can be directly built from the behaviours of the subsystems in an explicit way. Furthermore, necessary and sufficient conditions for the existence of distributed controller behaviours as well as the explicit construction of the distributed controller behaviours will be presented. For each set-theoretic development, the realisation in the context of LTI systems described by image of Hankel matrices will be performed. This framework unites various representations and descriptions of features of dynamical systems (e.g. models, dissipativity, data, etc.) as behaviours, thereby allowing the formation of a hybrid platform for the analysis and distributed control generically and systematically.

*Some results in this chapter have been/will be published in:*

- **Y. Yan**, J. Bao and B. Huang, "Dissipativity Analysis for Linear Systems in the Behavioural Framework," in *Proceedings of Australian & New Zealand Control Conference (ANZCC) 2019, Auckland, New Zealand*, pp. 152–156, 2019.

- **Y. Yan**, J. Bao and B. Huang, "Behavioral Approach to Distributed Control of Interconnected Systems," *to be submitted to Automatica.*

## 5.1 Basic Set Theory and Algebra of Sets

The discussions carried out in this chapter are from the set-theoretic point of view under the Zermelo-Fraenkel axiomatic set theory with the Axiom of Choice (ZFC). The conventional operators $\cap$, $\cup$ and $\setminus$ are used to denote set intersection, union and difference, respectively. The algebra of sets can be abstracted into the Boolean algebra [116], which is a set $S$ that has at least elements 0 and 1 with binary operations $+$ and $\cdot$ and a unary operation $-$ that satisfy

| | | |
|---|---|---|
| Commutativity | $a+b=b+a$, | $a\cdot b=b\cdot a$, |
| Associativity | $(a+b)+c=a+(b+c)$, | $(a\cdot b)\cdot c=a\cdot(b\cdot c)$, |
| Distributivity | $a+(b\cdot c)=(a+b)\cdot(a+c)$, | $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$, |
| Absorption | $a\cdot(a+b)=a$, | $a+(a\cdot b)=a$, |
| Complementation | $a+(-a)=1$, | $a\cdot(-a)=0$. |

Note that the 0 and 1 are, respectively, the additive identity ($a + 0 = a$) and multiplicative identity ($a \cdot 1 = a$) rather than numbers. The difference of two elements in $S$ can be defined as

$$a - b = a \cdot (-b).$$

From these basic operations, some important rules can be derived, such as

$$a + a = a \cdot a = a,\ a \cdot 0 = 0,\ a + 1 = 1$$

and the well-known De Morgan's Laws

$$-(a + b) = (-a) \cdot (-b),\ -(a \cdot b) = (-a) + (-b).$$

It can be shown that every Boolean algebra is isomorphic to an algebra of sets. In this particular case, the elements 0 and 1 are, respectively $\varnothing$ and $\mathbb{W}^{\mathbb{T}}$, and the operations $+$, $\cdot$ and $-$ are, respectively, $\cap$, $\cup$ and $\setminus$.

The Cartesian product $\times$ will also be used extensively in this chapter. The formal definition of the Cartesian product of two sets $A$ and $B$, with their elements denoted by $a$ and $b$, respectively, is

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

It is important to make clear distinction between $\cap$ and $\times$. Specifically, $\cap$ operates on sets with the same elements whereas $\times$ operates on those with different ones. Furthermore, $\cap$ is commutative and associative while $\times$ is neither. Other than the basic operations listed above, the rest of the operations as well as their relations with $\times$ used in this paper are summarised in the following lemma.

**Lemma 5.1** (Set Operations [117–119])**.**

(i) Let $A$ be a set and let $A_1, A_2 \subset A$, then

(a) $A_1 \subset A_2$ if and only if $A_1 \cap A_2 = A_1$ if and only if $A_1 \setminus A_2 = \varnothing$;

(b) $A_1 \cap A_2 = \varnothing$ if and only if $A_1 \setminus A_2 = A_1$;

(c) $A_1 = A \setminus A_2$ if and only if $A_1 \cap A_2 = \varnothing$ and $A_1 \cup A_2 = A$.

(ii) Let $A^1, A^2, \ldots, A^N$ be $N$ sets with different elements. Let $A_1^i, A_2^i \subset A^i, \forall i \in \mathbb{Z}_N^+$ and define

$$\bigtimes_{i=1}^{N} A^i = A^1 \times A^2 \times \ldots \times A^N,$$

then

$$\bigtimes_{i=1}^{N} \left(A_1^i \cap A_2^i\right) = \left(\bigtimes_{i=1}^{N} A_1^i\right) \cap \left(\bigtimes_{i=1}^{N} A_2^i\right). \tag{5.1}$$

Notice that (5.1) illustrates the distributivity of $\times$ over $\cap$. Furthermore, setting $N = 2$ in (5.1) yields the important identity

$$\left(A_1^1 \cap A_2^1\right) \times \left(A_1^2 \cap A_2^2\right) = \left(A_1^1 \times A_1^2\right) \cap \left(A_2^1 \times A_2^2\right). \tag{5.2}$$

## 5.2 Representation of Interconnected Systems

This section proposes a new structure to obtain interconnected behaviour directly from the behaviours of the subsystems. By introducing the concept of abstracting the network of an interconnected system as a dynamical system, the interconnection network becomes a distinct object with its own trajectories instead of being a feature of the interconnected system. This viewpoint leads to the representation of the interconnected behaviour in a general form built up from its components explicitly.

### 5.2.1 Network as a Dynamical System

As proposed by Willems [38], interconnection of dynamical systems can be thought of as variable sharing, and henceforth one of the two variables interconnected together can be eliminated. This procedure yields a compact representation of an

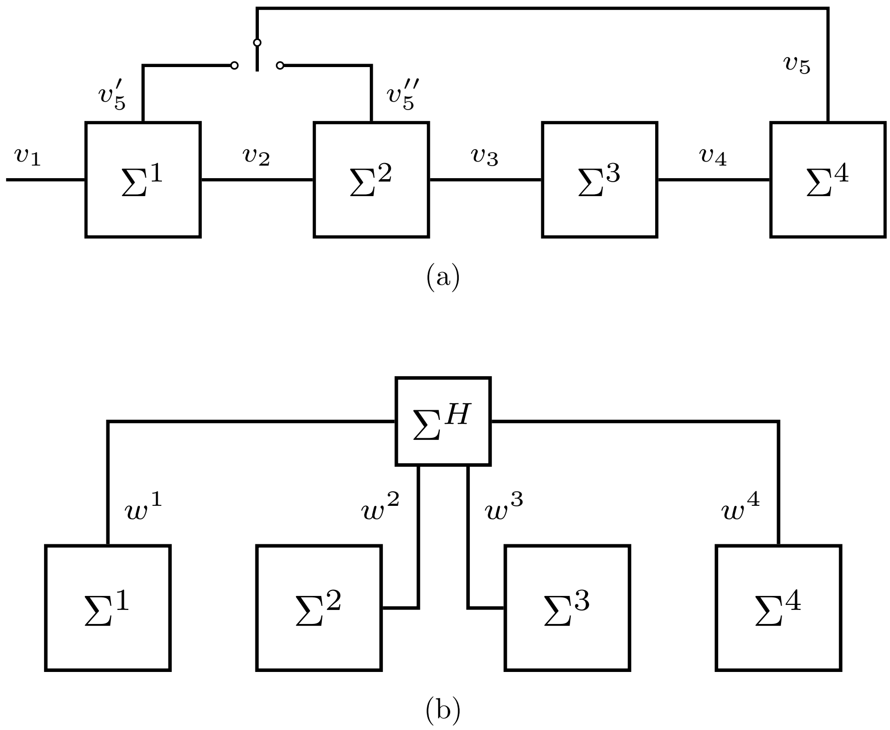



Figure 5.1: Four Systems with Switched Network

interconnected system and shows clearly what variables are left unconnected (i.e., the free variables). However, after this process, the membership of the interconnected variables to the subsystems becomes ambiguous: one variable is shared between two subsystems while it is in fact two distinct variables that happen to share the same value. It is only when *all* variables of the interconnected subsystems are shared variables that the ambiguity vanishes. It is therefore natural to propose a structure in which all subsystems are isolated but sharing all of their variables to a central system which can be seen as a generalised "topology". To explain the rationale, an example is given to illustrate the concept.

**Example 5.1.** Consider a network depicted in Figure 5.1a, in which four dynamical systems are interconnected in a network with a switch. Based on the outcome of the rest of the plant, $v_5$ can connect with either $v_5'$ or $v_5''$. Assuming $v_5$, $v_5'$ and $v_5''$ are of the same dimension, the interconnected system can be represented as

$\Sigma = (\mathbb{T}, \mathbb{V}_1 \times \mathbb{V}_2 \times \mathbb{V}_3 \times \mathbb{V}_4 \times \mathbb{V}_5 \times \mathbb{V}_5, \mathfrak{B})$ where

$$\begin{aligned}\mathfrak{B} = \big\{&(v_1, v_2, v_3, v_4, v_5, v_5'') \mid (v_1, v_2, v_5) \in \mathfrak{B}^1,\\ &(v_2, v_3, v_5'') \in \mathfrak{B}^2, (v_3, v_4) \in \mathfrak{B}^3, (v_4, v_5) \in \mathfrak{B}^4\big\}\end{aligned} \tag{5.3}$$

if $v_5$ is interconnected with $v_5'$ and

$$\begin{aligned}\mathfrak{B} = \big\{&(v_1, v_2, v_3, v_4, v_5, v_5') \mid (v_1, v_2, v_5') \in \mathfrak{B}^1,\\ &(v_2, v_3, v_5) \in \mathfrak{B}^2, (v_3, v_4) \in \mathfrak{B}^3, (v_4, v_5) \in \mathfrak{B}^4\big\}\end{aligned} \tag{5.4}$$

if $v_5$ is interconnected with $v_5''$.

While an elegant and insightful description of the interconnected behaviour, representing the behaviour in this slightly over-compacted fashion creates two obstacles towards the analysis of the system. Firstly, due to variable elimination, only one variable is used to represent variables that originally come from several different systems, making it difficult to construct the interconnected behaviour from that of its subsystems directly and explicitly. A more natural representation is that each subsystem has its own manifest variables and several of them 'happen to' coincide during interconnection. Secondly, for different networks, variables shared among subsystems are different. If the above method of representation were adopted, a new representation is needed every time the interconnection changes. What is actually changing is the network itself, and pushing the 'dynamics' of the network into the subsystems makes the analysis of the interconnected behaviour much more complicated. It is therefore reasonable to treat the network as a dynamical system itself with its own behaviour. With this thinking, the interconnected system in Figure 5.1a can be equivalently depicted as in Figure 5.1b, in which four (isolated) dynamical systems $\Sigma^i = (\mathbb{T}, \mathbb{W}^i, \mathfrak{B}^i)$ are 'plugged' into a dynamical system $\Sigma^H = (\mathbb{T}, \mathbb{W}^1 \times \mathbb{W}^2 \times \mathbb{W}^3 \times \mathbb{W}^4, \mathfrak{B}^H)$ that is the network. By setting $w = \mathrm{col}(w^1, w^2, w^3, w^4)$, the interconnected system can be described as $\Sigma = (\Sigma^1 \sqcap \Sigma^2 \sqcap \Sigma^3 \sqcap \Sigma^4) \wedge \Sigma^H = (\mathbb{T}, \mathbb{W}^1 \times \mathbb{W}^2 \times \mathbb{W}^3 \times \mathbb{W}^4, \mathfrak{B}^{int})$ where

$$\mathfrak{B} = \left\{ w \mid w^i \in \mathfrak{B}^i, i \in \mathbb{Z}_4^+ \text{ and } w \in \mathfrak{B}^H \right\}. \tag{5.5}$$

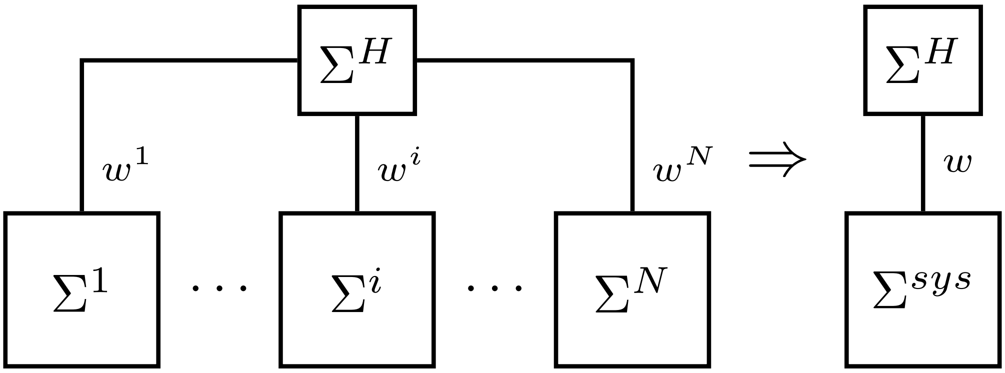



Figure 5.2: Network Viewed as a Dynamical System

In this way, the interconnected system can be constructed directly from its components. ■

Note that this description of an interconnected behaviour can be generalised into arbitrary numbers of subsystems. This leads to the definition of a network as a dynamical system.

**Definition 5.1.** The *network* of an interconnected system consisting of $N$ subsystems with the $i$th subsystem denoted as $\Sigma^i = (\mathbb{T}, \mathbb{W}^i, \mathfrak{B}^i)$ is itself a dynamical system

$$\Sigma^H = \left( \mathbb{T}, \bigtimes_{i=1}^{N} \mathbb{W}^i, \mathfrak{B}^H \right) \tag{5.6}$$

where $\mathfrak{B}^H$ is the *network behaviour.*

The hint of representing the network as a dynamical system has already appeared in literature. In [48], the network was represented by a static interconnection function and in [14, 81], it was represented by a static LTI system using input/output representation and was translated into a static kernel representation in Chapter 3. The above definition encompasses these cases and generalises the network to be a dynamical system on its own, which allows much more systematic description of an interconnected system with an arbitrary and probably time varying topology. It is, however, important to realise the difference between the network behaviour defined in Definition 5.1 and the topology matrix/interconnecting function in the literature: the network behaviour has it *own* set of behaviour whereas those in the literature are entirely defined by the interconnection inputs and outputs. To name but one distinct difference, if all subsystems were isolated, the topology matrix $H_{\mathcal{P}}$ in Figure 3.1, for example, would be 0, whereas the behaviour defined in Definition 5.1 would be $\mathfrak{B}^H = \mathbb{W}^{\mathbb{T}}$ where $\mathbb{W} = \bigtimes_{i=1}^{N} \mathbb{W}^i$, which means that all mappings from the time axis to the signal space are admissible in $\Sigma^H$. A closer observation reveals that in this particular case, the former is in fact a representation of $\mathfrak{B}^H$. In this way, variables with no physical interconnection can also be interpreted as "interconnected" with the network. In the next section, it will be shown that the proposed structure provides a clean and flexible representation of the interconnected system which can be constructed from its subsystems explicitly.

### 5.2.2 The Interconnected Behaviour

As illustrated above, the proposed structure allows for the representation of all interconnections as full interconnections. A result of this is that the behaviour of a dynamical system formed by the interconnection of two subsystems, denoted by $\Sigma = \Sigma^1 \wedge \Sigma^2$, can be simply constructed as $\mathfrak{B} = \mathfrak{B}^1 \cap \mathfrak{B}^2$. One special case associated with this is when $\Sigma^1$ and $\Sigma^2$ have two distinct signal spaces $\mathbb{W}^1$ and $\mathbb{W}^2$. In this case, the behaviour of $\Sigma$ can be constructed as

$$\mathfrak{B} = \left[\mathfrak{B}^1 \times \left(\mathbb{W}^2\right)^{\mathbb{T}}\right] \cap \left[\left(\mathbb{W}^1\right)^{\mathbb{T}} \times \mathfrak{B}^2\right] = \mathfrak{B}^1 \times \mathfrak{B}^2. \tag{5.7}$$

Since in this case Cartesian product represents the interconnected behaviour much clearer than augmentation, it will be used for this particular case, and, with a little abuse of notation, the interconnected system constructed through Cartesian product is expressed as $\Sigma = \Sigma^1 \sqcap \Sigma^2$. Note that the relationship between $\sqcap$ and

$\wedge$ is similar to that between $\times$ and $\cap$. It is also straightforward to construct the interconnected behaviour from the relationship among different components if all behaviours were fully known by replacing $\Sigma$ with $\mathfrak{B}$, $\wedge$ with $\cap$ and $\sqcap$ with $\times$. For example, a dynamical system $\Sigma = (\Sigma^1 \sqcap \Sigma^2) \wedge \Sigma^3$ has behaviour $\mathfrak{B} = (\mathfrak{B}^1 \times \mathfrak{B}^2) \cap \mathfrak{B}^3$.

As shown in Figure 5.2, assuming that the plant consists of $N$ subsystems with the $i$th subsystem denoted as $\Sigma^i = (\mathbb{T}, \mathbb{W}^i, \mathfrak{B}^i)$, then all subsystems can be written together compactly as a large system $\Sigma^{sys}$ with an array of isolated subsystems, i.e.,

$$\Sigma^{sys} = \prod_{i=1}^{N} \Sigma^i = \left( \mathbb{T}, \bigtimes_{i=1}^{N} \mathbb{W}^i, \mathfrak{B}^{sys} \right). \tag{5.8}$$

Since all subsystems are isolated at this point, the behaviour $\mathfrak{B}^{sys}$ is simply the Cartesian product of that of all of its subsystems, i.e.,

$$\mathfrak{B}^{sys} = \bigtimes_{i=1}^{N} \mathfrak{B}^i. \tag{5.9}$$

The final interconnection is the interconnection of $\Sigma^{sys}$ with $\Sigma^H$. Since they have exactly the same signal space and they have full interconnection, the final interconnected system can be obtained as $\Sigma = \Sigma^{sys} \wedge \Sigma^H = \left( \mathbb{T}, \bigtimes_{i=1}^{N} \mathbb{W}^i, \mathfrak{B} \right)$ where

$$\mathfrak{B} = \left( \bigtimes_{i=1}^{N} \mathfrak{B}^i \right) \cap \mathfrak{B}^H. \tag{5.10}$$

Note that in the last part the variable elimination procedure is still adopted by naming the manifest variable of the network $\Sigma^H$ as $w$, the collection of all manifest variables in the subsystems, because $\Sigma^{sys}$ and $\Sigma^H$ are indeed sharing all variables. In this way, the interconnected behaviour is constructed from that of its subsystems and network directly and explicitly in a very simple fashion, which allows the manipulation of the behaviours to be carried out easily. Furthermore, the change of network behaviour does not affect that of the subsystems at all, making this representation a very general one. Lastly, and perhaps more importantly, the manifest variables of each subsystem is defined unambiguously - there is no overlapping of variables due to variable elimination, the "sharing of variables" happen inside the network $\Sigma^H$. As a result, observation of the trajectories of specific variables within the interconnected system can be easily obtained by using the projection operation. In the next section, a detailed treatment of the projection operation will be carried out. It will be shown that all trajectories that are admissible within an interconnected system can be obtained from the projections, hence it is not necessary to know the complete behavioural set of all subsystems in order to obtain the complete interconnected behaviour.

## 5.3 The Projection Operation

### 5.3.1 Projection onto Subspaces

Given a dynamical system (2.1), the projection of the behaviour $\mathfrak{B}$ onto the space $\mathbb{W}_i$ is a map $\pi_{w_i} : \mathbb{W}^{\mathbb{T}} \to \mathbb{W}_i^{\mathbb{T}}$ defined by

$$\pi_{w_i}(\mathfrak{B}) = \left\{ w_i \mid \exists \ell_j, j \in \mathbb{Z}_{\mathrm{w}}^+ \setminus \{i\}, (\ell_1, \ldots, \ell_{i-1}, w_i, \ell_{i+1}, \ldots, \ell_{\mathrm{w}}) \in \mathfrak{B} \right\}. \tag{5.11}$$

This map allows for the extraction of the set of trajectories of any specific manifest variable from $\mathfrak{B}$. Obviously, if the dynamical system is already one with latent variables $\Sigma^{full} = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}^{full})$, then the projection $\pi_w\left(\mathfrak{B}^{full}\right)$ is the manifest behaviour of $\Sigma^{full}$. The following lemma provides the relationship when distributing the projection operation over intersection, union and difference.

**Lemma 5.2.** Given $\mathfrak{B}^1, \mathfrak{B}^2 \in (\mathbb{W}_1 \times \mathbb{W}_2)^{\mathbb{T}}$, we have

(i) $\pi_{w_1}(\mathfrak{B}^1 \cap \mathfrak{B}^2) \subset \pi_{w_1}(\mathfrak{B}^1) \cap \pi_{w_1}(\mathfrak{B}^2)$;

(ii) $\pi_{w_1}(\mathfrak{B}^1 \cup \mathfrak{B}^2) = \pi_{w_1}(\mathfrak{B}^1) \cup \pi_{w_1}(\mathfrak{B}^2)$;

(iii) $\pi_{w_1}(\mathfrak{B}^1 \setminus \mathfrak{B}^2) \supset \pi_{w_1}(\mathfrak{B}^1) \setminus \pi_{w_1}(\mathfrak{B}^2)$.

*Proof.* (i) Notice that $\mathfrak{B}^1 \cap \mathfrak{B}^2 \subset \mathfrak{B}^1$, hence $\pi_{w_1}(\mathfrak{B}^1 \cap \mathfrak{B}^2) \subset \pi_{w_1}(\mathfrak{B}^1)$. Similar argument can be made such that $\pi_{w_1}(\mathfrak{B}^1 \cap \mathfrak{B}^2) \subset \pi_{w_1}(\mathfrak{B}^2)$. The two relationships give the result in (i). On the other hand, it is easy to find counterexamples to show that the two sets are not equal. For example, if $\pi_{w_2}(\mathfrak{B}^1 \cap \mathfrak{B}^2) = \varnothing$ but $\pi_{w_1}(\mathfrak{B}^1) \cap \pi_{w_1}(\mathfrak{B}^2)$ (the same trajectory of $w_1$ corresponding to different trajectories in of $w_2$ in different behaviours) is not, then obviously $\pi_{w_1}(\mathfrak{B}^1 \cap \mathfrak{B}^2) = \varnothing \neq \pi_{w_1}(\mathfrak{B}^1) \cap \pi_{w_1}(\mathfrak{B}^2)$.

(ii) This is a standard result (see [120], for example).

(iii) The set on the left hand side is

$$\pi_{w_1}\left(\mathfrak{B}^1 \setminus \mathfrak{B}^2\right) = \left\{w^1 | \exists \ell_2, (w_1, \ell_2) \in \mathfrak{B}^1, (w_1, \ell_2) \notin \mathfrak{B}^2\right\}. \tag{5.12}$$

while the one on the right hand side is

$$\pi_{w^1}\left(\mathfrak{B}^1\right) \setminus \pi_{w_1}\left(\mathfrak{B}^2\right) = \left\{w_1 | \exists \ell_{21} \forall \ell_{22}, (w_1, \ell_{21}) \in \mathfrak{B}^1, (w_1, \ell_{22}) \notin \mathfrak{B}^2\right\} \tag{5.13}$$

Obviously the latter includes the former with $\ell_{21} = \ell_{22}$. □

The projected behavioural set can be understood in two ways: from the point of view of the dynamical system $\Sigma$, $\pi_{w_i}(\mathfrak{B})$ can be interpreted as the observation of all possible trajectories of $w_i$; from the point of view of the manifest variable $w_i$ itself, $\pi_{w_i}(\mathfrak{B})$ can also be interpreted as a virtual "dynamical system" with manifest variable $w_i$ having full interconnection with another virtual "dynamical system" with $w_i$ as manifest variable and all other manifest variables in $\Sigma$ as "latent variables". Note that the definition of $\pi_{w_i}(\mathfrak{B})$ in (5.11) deliberately uses $\ell_j$ instead of $w_j$ to emphasise that the choices of $\ell_j$ may not be unique and that $w_j$ may be only one of the choices. If, however, the choices of $w_j$ are actually unique, then the system is said to be observable, as is defined in Definition 2.7.

From the projection point of view, if $w_1$ is observable from $w_2$ in $\Sigma$, then for a given trajectory of $w_2 \in \pi_{w_2}(\mathfrak{B})$, there exists *only one* trajectory of $w_1 \in \pi_{w_1}(\mathfrak{B})$ such that $(w_1, w_2)$ is an admissible trajectory in $\mathfrak{B}$. In other words, knowing the complete information of $w_2$ is equivalent to knowing the complete information of $w_1$. It is, however, important to see that $w_1$ being observable from $w_2$ does not necessarily mean that each trajectory of $w_2$ corresponds to a distinct trajectory of $w_1$. It is perfectly possible for different trajectories of $w_2$ to have the same corresponding trajectory of $w_1$.

The projection operation leads to a crucial result on how to construct the projected behaviour from an interconnected system and what observability implies in an interconnected system. It shows explicitly how to reconstruct an interconnected system from its projections. Note that in this proposition, and in many subsequent results, the interconnected systems are only given in terms of their subsystems $\Sigma^i$ and not in terms of their behaviours $\mathfrak{B}^i$ because although the interconnected systems are physically configured in the way described by $\Sigma^i$, the interconnected behaviour may be obtained through other methods.

**Proposition 5.3.** Given a dynamical system

$$\Sigma = \left[\left(\Sigma^1 \wedge \Sigma^2\right) \sqcap \Sigma^3\right] \wedge \Sigma^4 = (\mathbb{T}, \mathbb{W}^1 \times \mathbb{W}^2, \mathfrak{B}), \tag{5.14}$$

then

(i) $\pi_{w^1}(\mathfrak{B}) = \mathfrak{B}^1 \cap \pi_{w^1}\left((\mathfrak{B}^2 \times \mathfrak{B}^3) \cap \mathfrak{B}^4\right) = \mathfrak{B}^1 \cap \mathfrak{B}^2 \cap \pi_{w^1}\left(\left[(\mathbb{W}^1)^{\mathbb{T}} \times \mathfrak{B}^3\right] \cap \mathfrak{B}^4\right)$;

(ii) $\mathfrak{B} = \left[\mathfrak{B}^1 \times \pi_{w^2}(\mathfrak{B})\right] \cap \mathfrak{B}^4 = \left[(\mathbb{W}^1)^{\mathbb{T}} \times \pi_{w^2}(\mathfrak{B})\right] \cap \mathfrak{B}^4$ if and only if $w^1$ is observable from $w^2$ in $\Sigma$.

*Proof.* (i) This is obvious because the projection of the overall system onto certain spaces is the same as the intersection of the behaviour(s) containing the variables and the rest of the system with the said variables regarded as "manifest" variables.

(ii) For this part, only the proof of the equivalence of the observability of the system and $\mathfrak{B} = (\mathfrak{B}^1 \times \pi_{w^2}(\mathfrak{B})) \cap \mathfrak{B}^4$ is given. The other equivalence follows analogously. The definitions of the two behaviours are, respectively,

$$\begin{aligned} &\left[\left(\mathfrak{B}^1 \cap \mathfrak{B}^2\right) \times \mathfrak{B}^3\right] \cap \mathfrak{B}^4 \\ &= \left\{(w^1, w^2) \mid w^1 \in \mathfrak{B}^1, w^1 \in \mathfrak{B}^2, w^2 \in \mathfrak{B}^3, (w^1, w^2) \in \mathfrak{B}^4\right\}, \end{aligned} \tag{5.15a}$$

$$\begin{aligned} &\left(\mathfrak{B}^1 \times \pi_{w^2}(\mathfrak{B})\right) \cap \mathfrak{B}^4 \\ &= \left\{(w^1, w^2) \mid \exists \ell, \ell \in \mathfrak{B}^2, (\ell, w^2) \in \mathfrak{B}^4, w^1 \in \mathfrak{B}^1, w^2 \in \mathfrak{B}^3, (w^1, w^2) \in \mathfrak{B}^4\right\}. \end{aligned} \tag{5.15b}$$

The two sets are equal if and only if $\ell = w^1$, and this should hold true for all $w^1$, which is true if and only if $w^1$ is observable from $w^2$. □

The first statement is a mathematical representation of the projection operation illustrated above. Other than this, it also provides a very useful operation: it is possible to move one or more sets of behaviours into or out of the projection operator if the set of behaviour to be moved has the same signal space as that the interconnected system is projected onto. The second statement is a powerful result due to the definition of observability. It states that if $w^1$ is observable from $w^2$, then the projection of $w^2$, together with the network, determines the entire behaviour completely. As will be shown in the next section, this result is one of the foundations of the construction of the desired controlled behaviour.

### 5.3.2 Construction of the Interconnected Behaviour

One of the physical interpretations of a trajectory belonging to the projected behaviour is a measured trajectory. The projected behaviour is therefore the set of all measured trajectories. The question that naturally arises is that whether this behaviour is enough to construct the complete interconnected behaviour. The following theorem claims that the behaviour of an interconnected system can be reconstructed from the projections of the behaviour onto the signal space of each subsystem combined with the network behaviour.

**Theorem 5.4.** Given an interconnected system

$$\Sigma = \left(\prod_{i=1}^{N} \Sigma^i\right) \wedge \Sigma^H = \left(\mathbb{T}, \bigtimes_{i=1}^{N} \mathbb{W}^i, \mathfrak{B}\right), \tag{5.16}$$

then,

(i) assuming that behaviours $\mathfrak{B}^i$ are not known but the projections on their manifest variables are known, the interconnected behaviour can be fully obtained as

$$\mathfrak{B} = \left(\bigtimes_{i=1}^{N} \pi_{w^i}(\mathfrak{B})\right) \cap \mathfrak{B}^H; \tag{5.17}$$

(ii) assuming, without loss of generality, that the first $n$ behaviours are fully known while the rest only have information of the projections, the interconnected behaviour can be fully obtained as

$$\mathfrak{B} = \left[\left(\bigtimes_{i=1}^{n} \mathfrak{B}^i\right) \times \left(\bigtimes_{i=n+1}^{N} \pi_{w^i}(\mathfrak{B})\right)\right] \cap \mathfrak{B}^H. \tag{5.18}$$

*Proof.* To prove this theorem, an auxiliary result is needed, which is stated in the following lemma.

**Lemma 5.5.** Suppose an interconnected behaviour is of the form (5.10), then

(i) $\pi_{w^i}(\mathfrak{B}) \subset \mathfrak{B}^i$, $\forall i \in \mathbb{Z}_N^+$;

(ii) $\mathfrak{B} \subset \bigtimes_{i=1}^{N} \pi_{w^i}(\mathfrak{B})$.

*Proof.* (i) This is a direct generalisation from Proposition 5.3(i).

(ii) Using the definitions of the two sets:

$$\mathfrak{B} = \left\{w \mid \forall i, w^i \in \mathfrak{B}^i, w \in \mathfrak{B}^H\right\} \tag{5.19}$$

and

$$\bigtimes_{i=1}^{N} \pi_{w^i}(\mathfrak{B}) = \big\{ w \mid \forall i \exists \ell_j^i \forall j \neq i, w^i \in \mathfrak{B}^i, \ell_j^i \in \mathfrak{B}^j, \\ (\ell_1^i, \ldots, \ell_{i-1}^i, w^i, \ell_{i+1}^i, \ldots, \ell_N^i) \in \mathfrak{B}^H \big\}, \tag{5.20}$$

it is obvious that the former is the latter with extra condition $\ell_j^i = w^j, \forall i, j$. Note that the two behaviours are the same when $\mathfrak{B}^H = \mathbb{W}^{\mathbb{T}}$ (i.e., $\pi_{w^i}(\mathfrak{B}) = \mathfrak{B}^i$, $\forall i$). This happens when all subsystems are isolated. □

We are now in the position to prove the theorem.

(i) From Proposition 5.3(i),

$$\begin{aligned} &\pi_{w^i}(\mathfrak{B}) \\ =&\mathfrak{B}^i \cap \pi_{w^i}\left(\left[\left(\bigtimes_{j=1}^{i-1} \mathfrak{B}^j\right) \times \left(\mathbb{W}^i\right)^{\mathbb{T}} \times \left(\bigtimes_{j=i+1}^{N} \mathfrak{B}^j\right)\right] \cap \mathfrak{B}^H\right) \\ :=&\mathfrak{B}^i \cap \mathfrak{B}_\pi^i \end{aligned} \tag{5.21}$$

It follows from Lemma 5.1(iii) that

$$\begin{aligned} &\left(\bigtimes_{i=1}^{N} \pi_{w^i}(\mathfrak{B})\right) \cap \mathfrak{B}^H \\ =&\left(\bigtimes_{i=1}^{N} \mathfrak{B}^i \cap \mathfrak{B}_\pi^i\right) \cap \mathfrak{B}^H \\ =&\left(\bigtimes_{i=1}^{N} \mathfrak{B}^i\right) \cap \left(\bigtimes_{i=1}^{N} \mathfrak{B}_\pi^i\right) \cap \mathfrak{B}^H \\ =&\mathfrak{B} \cap \left(\bigtimes_{i=1}^{N} \mathfrak{B}_\pi^i\right). \end{aligned} \tag{5.22}$$

According to Lemma 5.1(i), it suffices to prove that $\mathfrak{B} \subset \bigtimes_{i=1}^{N} \mathfrak{B}_\pi^i$. Obviously, for all $i$,

$$\begin{aligned}
\mathfrak{B}^i_\pi =&\pi_{w^i}\left(\left[\left(\bigtimes_{j=1}^{i-1}\mathfrak{B}^j\right)\times\left(\mathbb{W}^i\right)^{\mathbb{T}}\times\left(\bigtimes_{j=i+1}^{N}\mathfrak{B}^j\right)\right]\cap\mathfrak{B}^H\right)\\
\supset&\pi_{w^i}\left(\left[\left(\bigtimes_{j=1}^{i-1}\mathfrak{B}^j\right)\times\mathfrak{B}^i\times\left(\bigtimes_{j=i+1}^{N}\mathfrak{B}^j\right)\right]\cap\mathfrak{B}^H\right)\\
=&\pi_{w^i}\left(\left(\bigtimes_{i=1}^{N}\mathfrak{B}^i\right)\cap\mathfrak{B}^H\right)\\
=&\pi_{w^i}\left(\mathfrak{B}\right).
\end{aligned} \tag{5.23}$$

It then follows that

$$\bigtimes_{i=1}^{N}\mathfrak{B}^i_\pi\supset\bigtimes_{i=1}^{N}\pi_{w^i}\left(\mathfrak{B}\right)\supset\mathfrak{B}. \tag{5.24}$$

To prove (ii), note that $\pi_{w^i}\left(\mathfrak{B}\right)\subset\mathfrak{B}^i$, $\forall i$. Combining with the result in (i), it then follows that

$$\begin{aligned}
\mathfrak{B}=&\left(\bigtimes_{i=1}^{N}\pi_{w^i}\left(\mathfrak{B}\right)\right)\cap\mathfrak{B}^H\\
\subset&\left[\left(\bigtimes_{i=1}^{n}\mathfrak{B}^i\right)\times\left(\bigtimes_{i=n+1}^{N}\pi_{w^i}\left(\mathfrak{B}\right)\right)\right]\cap\mathfrak{B}^H\\
\subset&\left(\bigtimes_{i=1}^{N}\mathfrak{B}^i\right)\cap\mathfrak{B}^H\\
=&\,\mathfrak{B}.
\end{aligned} \tag{5.25}$$

This completes the proof of Theorem 5.4. □

The first claim of Theorem 5.4 is that all of the projections of the interconnected behaviour, together with the network behaviour, determine the interconnected behaviour completely. This provides an insight to an interconnected system: each subsystem within an interconnected system contains a set of trajectories that can *never* happen. Therefore, it is in fact not necessary to know the complete information of each subsystem, knowing the behaviour of each subsystem *as an integrated part of the interconnected system* is enough. The most interesting part of this statement is that the network behaviour is still needed to reconstruct the interconnected behaviour even though the projections already contain the network information. This can be understood from the property of the projection operation: by projecting $\mathfrak{B}$ onto $\mathbb{W}^i$, the manifest variables of all other subsystems are essentially set to latent variables with respect to $w^i$. As such, there may be trajectories that are not admissible through the interconnected system but indistinguishable from $w^i$ point of view. The network behaviour precisely eliminates this problem because any trajectory that is admissible in the interconnected system must be admissible through the network behaviour. In most cases, $\pi_{w^i}(\mathfrak{B})$ can be obtained to a high level of completeness with a large data bank of measured trajectories and $\mathfrak{B}^H$ is essentially known completely, which gives a possibility for data-driven control design under this framework.

The second claim of Theorem 5.4, on the other hand, is a much more powerful result. It shows that if an interconnected system contains several subsystems with fully known behaviour (e.g., behaviour described by models), using these fully known behaviours, the observations of the interconnected system from the manifest variables of the subsystems with unknown behaviours and the network behaviour also recovers the complete interconnected behaviour. This means that the proposed construction allows for a unified platform for a hybrid interconnected system.

If the number of subsystem is 2, then a special, simplified result can be obtained, which is given in the corollary below.

**Corollary 5.6.** Given a system $\Sigma = (\Sigma^1 \sqcap \Sigma^2) \wedge \Sigma^H = (\mathbb{T}, \mathbb{W}^1 \times \mathbb{W}^2, \mathfrak{B})$, then

$$\mathfrak{B} = \left[\pi_{w^1}(\mathfrak{B}) \times \mathfrak{B}^2\right] \cap \mathfrak{B}^H \tag{5.26a}$$

$$= \left[\mathfrak{B}^1 \times \pi_{w^2}(\mathfrak{B})\right] \cap \mathfrak{B}^H \tag{5.26b}$$

$$= \left[\pi_{w^1}(\mathfrak{B}) \times \pi_{w^2}(\mathfrak{B})\right] \cap \mathfrak{B}^H . \tag{5.26c}$$

*Proof.* This is a direct result from Theorem 5.4 by setting $N = 2$ and $n = 1$. □

This result is crucial in the synthesis of controlled behaviour as the interconnected system and the distributed controllers can be seen as two large subsystems.

### 5.3.3 The Linear Time-Invariant Case

This section presents the realisation of the behaviour constructed in the previous section with the case where each subsystem is represented by the image of Hankel matrices. Consider an interconnected system consisting of $N$ controllable LTI subsystems. Assume that the network is described by kernel representation with a constant coefficient matrix $Hw = 0$, or $\mathfrak{B}^H = \ker(H)$. Note that if $w$ admits an input/output partition, then $\ker(H)$ is the same as the topology matrix described in [56]. Suppose the measured trajectories of the subsystems are available. It is then possible to construct Hankel matrices $\mathfrak{H}_L\left(\tilde{v}^i\right)$ where $\tilde{v}^i$ is a measured trajectory for the $i$th subsystem (see Eq. (2.10)) and it is possible to express the behaviour of the rest of the subsystems from time steps 1 to $L$ using the column span of the Hankel matrices. In this case, the order of excitation for the free variables should be at least $L + \sum_{i=1}^{N} \mathtt{n}\left(\mathfrak{B}^i\right)$ because

$$\mathtt{n}(\mathfrak{B}) = \mathtt{n}\left(\mathfrak{B}^H\right) + \sum_{i=1}^{N} \mathtt{n}\left(\mathfrak{B}^i\right) = \sum_{i=1}^{N} \mathtt{n}\left(\mathfrak{B}^i\right). \tag{5.27}$$

For a Hankel matrix $\mathfrak{H}_L(\widetilde{w})$, its sub-matrix with depth $l$ starting from the $k$th block row is denoted as $\mathfrak{H}_{l,k}(\widetilde{w})$. Assuming sufficient excitation, then according to Theorem 2.1,

$$\operatorname{colspan}\left(\mathfrak{H}_L\left(\tilde{v}^i\right)\right) = \pi_{w^i}\left(\mathfrak{B}\right)_{|[1,L]}. \tag{5.28}$$

According to Theorem 5.4(i) and following Lemma 2.8, a trajectory $w \in \mathfrak{B}_{|[1,L]}$ can be generated by

$$\begin{bmatrix} I_{L\mathrm{w}} \\ \\ I_L \otimes H \end{bmatrix} \widetilde{w}_{|[1,L]} = \begin{bmatrix} \mathfrak{H}_L(\tilde{v}) \\ \\ 0_{\bullet\times\mathrm{g}} \end{bmatrix} g, \tag{5.29}$$

where $g$ is an arbitrary vector, $\otimes$ is the Kronecker product, and

$$\mathfrak{H}_L(\tilde{v}) = \operatorname{col}\left(\mathfrak{H}_{1,1}\left(\tilde{v}\right), \mathfrak{H}_{1,2}\left(\tilde{v}\right), \ldots, \mathfrak{H}_{1,L}\left(\tilde{v}\right)\right), \tag{5.30a}$$

$$\mathfrak{H}_{1,k}(\tilde{v}) = \operatorname{col}\left(\mathfrak{H}_{1,k}\left(\tilde{v}^1\right), \mathfrak{H}_{1,k}\left(\tilde{v}^2\right), \ldots, \mathfrak{H}_{1,k}\left(\tilde{v}^N\right)\right), \tag{5.30b}$$

$$w = \operatorname{col}\left(w^1, w^2, \ldots, w^N\right). \tag{5.30c}$$

Defining $A^\perp \coloneqq I - A^\dagger A$ where $\dagger$ denotes the Moore-Penrose inverse and $H^{\otimes_L} \coloneqq I_L \otimes H$ where $\otimes$ is the Kronecker product, some simple manipulations of (5.29) will lead to the representation

$$\widetilde{w}_{|[1,L]} = \mathfrak{H}_L(\tilde{v}) \left[H^{\otimes_L} \mathfrak{H}_L(\tilde{v})\right]^\perp z, \tag{5.31}$$

where $z \in \mathbb{R}^{\mathrm{g}}$ is an arbitrary vector.

Through this construction, the complete behaviour of the interconnected system from local measurements can be generated. On the other hand, in order to obtain the behaviour up to step $L$, *all* free variables must be persistently exciting of order the *sum* of the state cardinalities of all subsystems.

## 5.4 Distributed Control Design

This section presents the procedure of obtaining the behavioural sets for distributed controllers that, when integrated into the system, yield the desired behaviour for the variables of interest. The control structure is firstly set up using the behaviours of the subsystems and problem to be solved can be formulated. It is then shown that under the behavioural framework, the controller behaviours can be obtained in a very intuitive way.

### 5.4.1 Control Structure and Problem Formulation

Consider an interconnected system $\Sigma_p = (\mathbb{T}, \mathbb{W}_p, \mathfrak{B}_p)$ consisting of $N$ subsystems, where the $i$th subsystem is denoted as $\Sigma_p^i = \left(\mathbb{T}, \mathbb{W}_p^i, \mathfrak{B}_p^i\right)$. The subsystems are interconnected with a network $\Sigma_p^H = \left(\mathbb{T}, \mathbb{W}_p, \mathfrak{B}_p^H\right)$. As a result, the interconnected, uncontrolled behaviour can be constructed according to (5.10) as

$$\mathfrak{B}_p = \left(\bigtimes_{i=1}^{N} \mathfrak{B}_p^i\right) \cap \mathfrak{B}_p^H. \tag{5.32}$$

Suppose that an array of $N_c$ controllers $\Sigma_c^j = (\mathbb{T}, \mathbb{W}_c^j, \mathfrak{B}_c^j)$ are employed to control $\Sigma_p$ and the controllers have their own network $\Sigma_c^H = \left(\mathbb{T}, \mathbb{W}_c, \mathfrak{B}_c^H\right)$. Then the interconnected controllers can be represented as an interconnected system $\Sigma_c = (\mathbb{T}, \mathbb{W}_c, \mathfrak{B}_c)$ where

$$\mathfrak{B}_c = \left(\bigtimes_{j=1}^{N_c} \mathfrak{B}_c^j\right) \cap \mathfrak{B}_c^H. \tag{5.33}$$

Note that the number of controllers is not necessarily the same as that of the subsystems, nor is it assumed that the manipulated variables have any relationship with the system variables at this stage.

When interconnecting the system with the distributed controllers, another network is needed. This network is defined as $\Sigma_{pc}^H = \left(\mathbb{T}, \mathbb{W}_p \times \mathbb{W}_c, \mathfrak{B}_{pc}^H\right)$. With these building blocks, a controlled system can be constructed, which is depicted in Figure 5.3. As is depicted, the controlled system can be viewed as the interconnection of two interconnected systems, which defines a latent variable dynamical system $\Sigma_{pc}^{full} = (\Sigma_p \sqcap \Sigma_c) \wedge \Sigma_{pc}^H = \left(\mathbb{T}, \mathbb{W}_p, \mathbb{W}_c, \mathfrak{B}_{pc}^{full}\right)$ with full behaviour

$$\mathfrak{B}_{pc}^{full} = (\mathfrak{B}_p \times \mathfrak{B}_c) \cap \mathfrak{B}_{pc}^H = \left[\left(\bigtimes_{i=1}^{N} \mathfrak{B}_p^i\right) \times \left(\bigtimes_{j=1}^{N_c} \mathfrak{B}_c^j\right)\right] \cap \left(\mathfrak{B}_p^H \times \mathfrak{B}_c^H\right) \cap \mathfrak{B}_{pc}^H. \tag{5.34}$$

The controlled system can then be expressed as the triple $\Sigma_{pc} = (\mathbb{T}, \mathbb{W}_p, \mathfrak{B}_{pc})$ where

$$\mathfrak{B}_{pc} = \pi_{w_p}\left(\mathfrak{B}_{pc}^{full}\right). \tag{5.35}$$

Note that by defining

$$\mathfrak{B}'_p := \left(\mathfrak{B}_p \times \mathfrak{B}_c^H\right) \cap \mathfrak{B}_{pc}^H, \tag{5.36}$$

it follows that

$$\mathfrak{B}_{pc}^{full} = \mathfrak{B}'_p \cap \left[\mathbb{W}_p^T \times \left(\bigtimes_{j=1}^{N_c} \mathfrak{B}_c^j\right)\right], \tag{5.37}$$

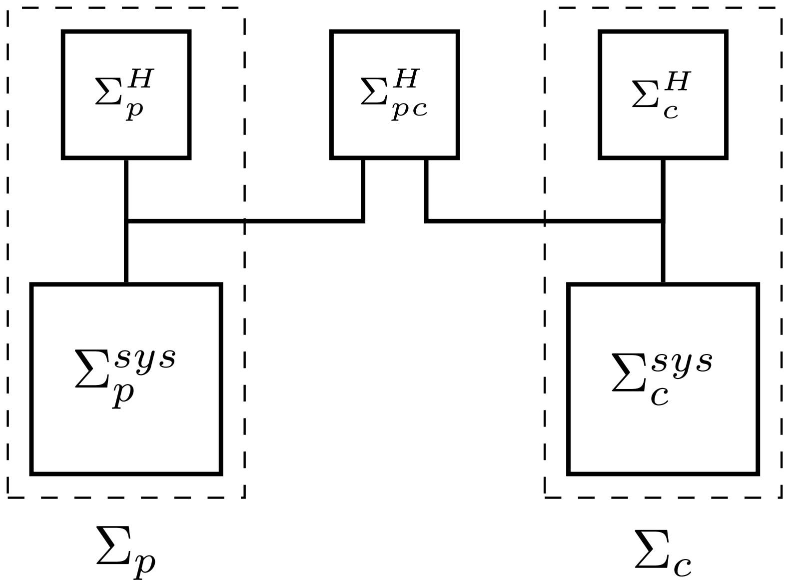



Figure 5.3: The Interconnected System Layout

which shows that the distributed control design is equivalent to decentralised control design to an augmented system with controller network and system-controller network integrated. By treating the networks as dynamical systems, they have their own behavioural sets and can thus be treated and rearranged like physical subsystems.

The objective of control is to select the set of behaviour from the uncontrolled system such that all selected trajectories meet certain specifications. These specifications can be formulated as a behavioural set $\mathfrak{B}_{ps}$ imposed on all manifest variables $w_p$. As a result, the objective for the control design is to implement $\mathfrak{B}_p \cap \mathfrak{B}_{ps}$. On the other hand, the controllers themselves may have restrictions and objectives such as control saturation and minimum gain requirement, which can also be formulated as a set of behaviour $\mathfrak{B}_{cr}$ on the control variables $w_c$. Although for illustration purpose it is assumed that the system and specifications share the same signal space, it is easy to formulate such a $\mathfrak{B}_{ps}$ even if the requirements are specified otherwise. Suppose the desired requirements are described by a set $\mathfrak{B}_s \subset \mathbb{W}_s^{\mathbb{T}}$ with manifest variable $w_s$, then for $\mathfrak{B}_s$ to be able to restrict

$\mathfrak{B}_p$, there must exist a network behaviour $\mathfrak{B}_{ps}^H \subset (\mathbb{W}_p \times \mathbb{W}_s)^{\mathbb{T}}$ such that for all $w_p \in \mathfrak{B}_p$, there exists $w_s \in \mathfrak{B}_s$ such that $(w_p, w_s) \in \mathfrak{B}_{ps}^H$. In this case, the set describing the requirements can be constructed as

$$\mathfrak{B}_{ps} = \pi_{w_p}\left(\left(\mathbb{W}_p^T \times \mathfrak{B}_s\right) \cap \mathfrak{B}_{ps}^H\right). \tag{5.38}$$

Similarly, if the restrictions are imposed on the variable $w_r \in \mathfrak{B}_r \subset \mathbb{W}_r^{\mathbb{T}}$, then there must exsit a network behaviour $\mathfrak{B}_{cr}$ linking $w_c$ to $w_r$. In such a case, $\mathfrak{B}_{cr}$ can be constructed as

$$\mathfrak{B}_{cr} = \pi_{w_c}\left(\left(\mathbb{W}_c^T \times \mathfrak{B}_r\right) \cap \mathfrak{B}_{cr}^H\right). \tag{5.39}$$

For the clarity of presentation, the notations $\mathfrak{B}_{ps}$ and $\mathfrak{B}_{cr}$ will be used and it is assumed that $\mathfrak{B}_{ps}$ and $\mathfrak{B}_{cr}$ always have the same signal spaces as $\mathfrak{B}_p$ and $\mathfrak{B}_c^H$, respectively.

With these components, the problem to be solve in this paper can be formulated as follows:

**Problem 5.1.** Given an interconnected dynamical system $\Sigma_p = (\mathbb{T}, \mathbb{W}_p, \mathfrak{B}_p)$ with $N$ subsystems $\Sigma_p^i = \left(\mathbb{T}, \mathbb{W}_p^i, \mathfrak{B}_p^i\right)$ and a network $\Sigma_p^H = \left(\mathbb{T}, \mathbb{W}_p, \mathfrak{B}_p^H\right)$, as well as the control objectives described by $\mathfrak{B}_{ps}$, design, if possible, a distributed control system $\Sigma_c = (\mathbb{T}, \mathbb{W}_c, \mathfrak{B}_c)$ with $N_c$ controllers $\Sigma_c^j = (\mathbb{T}, \mathbb{W}_c^j, \mathfrak{B}_c^j)$ and a network $\Sigma_c^H = \left(\mathbb{T}, \mathbb{W}_c, \mathfrak{B}_c^H\right)$ and restrictions $\mathfrak{B}_{cr}$ such that after the interconnection of $\Sigma_p$ and $\Sigma_c$ with network $\Sigma_{pc}^H = \left(\mathbb{T}, \mathbb{W}_p \times \mathbb{W}_c, \mathfrak{B}_{pc}^H\right)$, the controlled system $\Sigma_{pc} = (\mathbb{T}, \mathbb{W}_p \times \mathbb{W}_c, \mathfrak{B}_{pc})$ shown in Figure 5.3 satisfies

$$\mathfrak{B}_{pc} = \mathfrak{B}_p \cap \mathfrak{B}_{ps} \tag{5.40a}$$

$$\pi_{w_f}\left(\mathfrak{B}_{pc}\right) = \mathbb{W}_f^{\mathbb{T}}, \tag{5.40b}$$

where $\mathfrak{B}_{pc}$ is given in (5.35) and $w_f$ denotes the chosen free variables after interconnection.

The meaning of (5.40a) is straightforward: the control design should result in the manifest controlled behaviour to be the set of common trajectories from $\mathfrak{B}_p$ and $\mathfrak{B}_{ps}$. In other words, the controlled behaviour should be the subset of the uncontrolled system behaviour whose trajectories satisfy the requirements. The implication of (5.40b) is that after the integration of the controller network, $w_f$ (which normally contains exogenous inputs such as disturbance and reference signals) should still be able to chose whatever trajectories it prefers. This objective was specified in [73] as a cardinality condition for LTI systems in relation to the signature of the desired QSR supply rate and was termed "liveness". Here it is specified more generically so that the behaviours are representation-free.

### 5.4.2 Controller Behaviour Synthesis

This section gives the main result of this chapter: the construction of behaviours of the distributed controllers. Necessary and sufficient conditions for the existence of the controllers that achieve the objectives in (5.40) will be provided and it will be shown that the controller behavioural sets can be constructed in a very simple way.

Before stating the main result, an intuitive explanation is provided to aid the understanding of the rationale of the control design. As stated in Problem 5.1, the given components are the subsystems $\Sigma_p^i$, the system network $\Sigma_p^H$, the controller network $\Sigma_c^H$ and the system-controller network $\Sigma_{pc}^H$. The specifications on the system and the restriction on the controllers can be constructed as two virtual "systems" $\Sigma_{ps} = (\mathbb{T}, \mathbb{W}_p, \mathfrak{B}_{ps})$ and $\Sigma_{cr} = (\mathbb{T}, \mathbb{W}_c, \mathfrak{B}_{cr})$, respectively. By doing so, the two virtual systems can be integrated into the given components, resulting in a desired objective dynamical system as shown in Figure 5.4. Obviously, the full behaviour of this system, denoted by $\mathfrak{B}_d$, is

$$\mathfrak{B}_d := \left[(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \times \left(\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H. \tag{5.41}$$

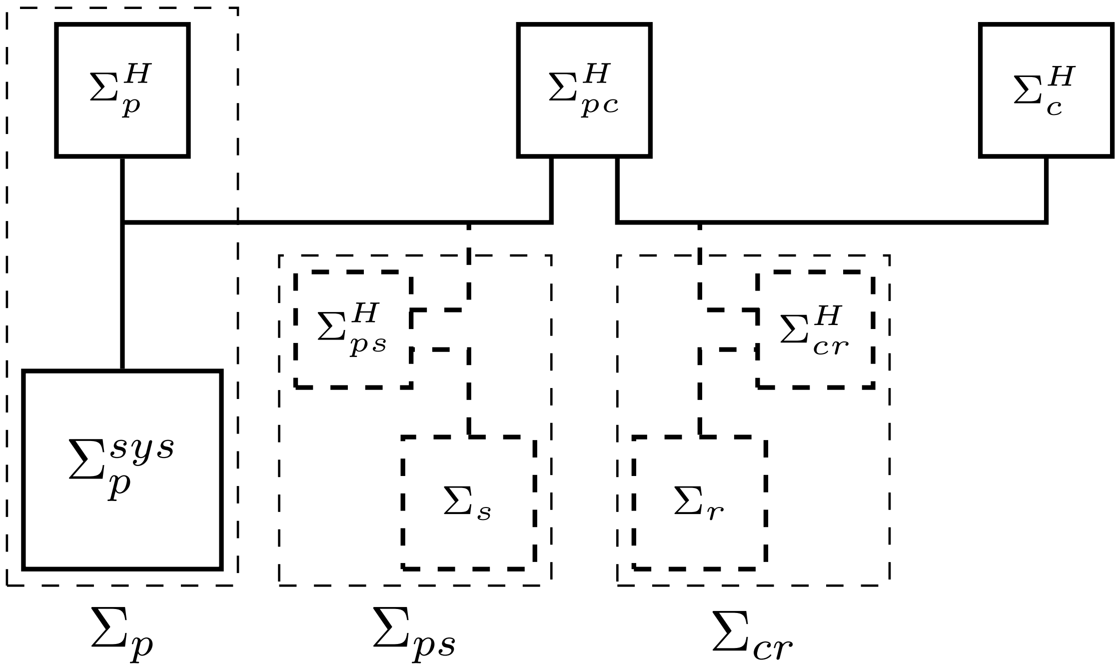



Figure 5.4: The Desired System Behaviour

The projection of the behaviour of this dynamical system on the control variables $w_c$ gives the largest possible set of behaviour for the controllers. As shown in (5.37), since the controllers can be viewed as decentralised, the projection on the control variables of each controller $w_c^j$ gives the behavioural set for the corresponding controller $\Sigma_c^j$. Therefore, if the controllers with the aforementioned behavioural sets are integrated into the system depicted in Figure 5.3, the resulting behaviour should, in some sense, resemble that shown in Figure 5.4.

The above illustration is now formulated rigorously with the following theorem. It shows that under certain conditions, the control design can indeed be carried out using this philosophy, but the resulting controller behaviours require much more careful descriptions.

**Theorem 5.7.** For an interconnected system $\Sigma_p = (\mathbb{T}, \mathbb{W}_p, \mathfrak{B}_p)$ whose stucture is of the form (5.16), a desirable behavioural set $\mathfrak{B}_{ps}$, a controller network $\Sigma_c^H = (\mathbb{T}, \mathbb{W}_c, \mathfrak{B}_c^H)$, a set of controller constraints $\mathfrak{B}_{cr}$ and a system-controller network

$\Sigma_{pc}^H = (\mathbb{T}, \mathbb{W}_p \times \mathbb{W}_c, \mathfrak{B}_{pc}^H)$, there exists a set of controllers $\Sigma_c^j = (\mathbb{T}, \mathbb{W}_c^j, \mathfrak{B}_c^j)$ such that $\mathfrak{B}_c \subset \mathfrak{B}_c^H \cap \mathfrak{B}_{cr}$, where $\mathfrak{B}_c$ is given by (5.33), that achieves (5.40) if and only if

$$\mathfrak{B}_p \cap \mathfrak{B}_{ps} \subset \pi_{w_p}\left(\left[\mathfrak{B}_p \times \left(\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H\right) \tag{5.42a}$$

$$\pi_{w_p}(\mathfrak{B}_{ex}) \subset \pi_{w_p}\left(\left[\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})\right] \cap \mathfrak{B}_{pc}^H\right) \tag{5.42b}$$

$$\pi_{w_f}(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}) \subset \pi_{w_f}(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \tag{5.42c}$$

$$\pi_{w_f}(\mathfrak{B}_p) = \mathbb{W}_f^{\mathbb{T}} \tag{5.42d}$$

where

$$\begin{aligned}
\mathfrak{B}_{in} &= \left[\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps} \setminus \pi_{w_p}(\mathfrak{B}_{ex})\right) \times \left(\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H \\
\mathfrak{B}_{ex} &= \left[(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \times \pi_{w_c}(\mathfrak{B}_{out})\right] \cap \mathfrak{B}_{pc}^H \\
\mathfrak{B}_{out} &= \left[(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}) \times \left(\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H.
\end{aligned}$$

In such a case, all controller trajectories that implement $\mathfrak{B}_p \cap \mathfrak{B}_{ps}$ are given by

$$\mathfrak{B}_c^j = \pi_{w_c^j}(\mathfrak{B}_{in}) \tag{5.43a}$$

$$= \pi_{w_c^j}\left(\pi_{w_c}(\mathfrak{B}_d) \setminus \pi_{w_c}(\mathfrak{B}_{ex})\right) \tag{5.43b}$$

$$= \pi_{w_c^j}\left(\pi_{w_c}(\mathfrak{B}_d) \setminus \pi_{w_c}(\mathfrak{B}_{out})\right) \tag{5.43c}$$

where $\mathfrak{B}_d$ is given in (5.41).

*Proof.* (*only if*): Suppose (5.40) is achieved, we then have

$$\mathfrak{B}_p \cap \mathfrak{B}_{ps} = \pi_{w_p}\left((\mathfrak{B}_p \times \mathfrak{B}_c) \cap \mathfrak{B}_{pc}^H\right).$$

Then (5.42a) is obvious because $\mathfrak{B}_c \subset \mathfrak{B}_c^H \cap \mathfrak{B}_{cr}$. (5.42c) is trivially true because $\pi_{w_f}(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) = \mathbb{W}_f^{\mathbb{T}}$. Also, according to Lemma 5.2(i), $\pi_{w_f}(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \subset \pi_{w_f}(\mathfrak{B}_p) \cap \pi_{w_f}(\mathfrak{B}_{ps})$, which means $\pi_{w_f}(\mathfrak{B}_p) \cap \pi_{w_f}(\mathfrak{B}_{ps}) = \mathbb{W}_f^{\mathbb{T}}$. Since $\mathbb{W}_f^{\mathbb{T}}$ is the largest possible set, it follows that (5.42d) is true. To prove (5.42b), we begin by explaining the construction of the various sets in the theorem. For a given trajectory of $w_c$, there may be a set of trajectories of $w_p$ such that $(w_p, w_c) \in \mathfrak{B}_{pc}^{full}$. We call these trajectories the *multiplicities* of a trajectory $w_p$ and we denote the set of all these trajectories as

$$\mathfrak{B}_p^m = \left\{ w_p \in \mathfrak{B}_p | \exists w_c \in \mathfrak{B}_c^H \cap \mathfrak{B}_{cr}, (w_p, w_c) \in \mathfrak{B}_{pc}^{full} \right\}.$$

In the situation where $\mathfrak{B}_p^m \cap \mathfrak{B}_p \setminus \mathfrak{B}_{ps} \neq \varnothing$, in other words, where some trajectories of $w_p \in \mathfrak{B}_p \cap \mathfrak{B}_{ps}$ has multiplicities outside of the desired behavioural set, these trajectories need to be excluded from control design because all multiplicities are exactly the same from $w_c$ point of view. To do so, note that these trajectories can be obtained in two ways: either as trajectories within the desired set with multiplicities outside, or as trajectories outside with multiplicities within the desired set. While the former is hard to construct, the latter can be constructed in the following way:

1. Find all trajectories of $w_c$ projected from integrating a dynamical system containing all trajectories of $w_p$ that belongs to $\mathfrak{B}_p$ but not belong to $\mathfrak{B}_{ps}$ into the system. Note that the resulting behaviour for $w_c$ is precisely $\pi_{w_c}(\mathfrak{B}_{out})$;

2. All excluded trajectories of $w_p$ can be found by projecting all of $w_c$ found from the previous step to $w_p$ through the system and intersecting with $\mathfrak{B}_p \cap \mathfrak{B}_{ps}$. This gives the set $\pi_{w_p}(\mathfrak{B}_{ex})$.

The largest possible set of the controller behaviour $\mathfrak{B}_c$ is hence the subset of $\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}$ containing all trajectories of $w_c$ projected from implementing $\mathfrak{B}_p \cap \mathfrak{B}_{ps}$ excluding $\pi_{w_p}(\mathfrak{B}_{ex})$ into the network, which is precisely $\pi_{w_c}(\mathfrak{B}_{in})$. Therefore, if there exists $\mathfrak{B}_c$ such that (5.40) holds, we must have $\mathfrak{B}_c \subset \pi_{w_c}(\mathfrak{B}_{in})$. As a result,

$$\begin{aligned}
\pi_{w_p}(\mathfrak{B}_{ex}) \subset \mathfrak{B}_p \cap \mathfrak{B}_{ps} \\
&= \pi_{w_c}\left((\mathfrak{B}_p \times \mathfrak{B}_c) \cap \mathfrak{B}_{pc}^H\right) \\
&\subset \pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right).
\end{aligned}$$

This completes the *only if* part of the proof.

(*if*): Suppose that conditions in (5.42) hold. According to Lemma 5.1(i), (5.42a) is true if and only if

$$\begin{aligned}&\mathfrak{B}_p \cap \mathfrak{B}_{ps} \cap \pi_{w_p}\left(\left[\mathfrak{B}_p \times \left(\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H\right)\\ =&\pi_{w_p}\left(\left[(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \times \left(\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H\right)\\ =&\mathfrak{B}_p \cap \mathfrak{B}_{ps}.\end{aligned}$$

This precisely means that

$$\pi_{w_p}(\mathfrak{B}_d) = \mathfrak{B}_p \cap \mathfrak{B}_{ps}$$

where $\mathfrak{B}_d$ is defined in (5.41). The goal then is to construct $\mathfrak{B}_c$ such that

$$\mathfrak{B}_{pc} = \pi_{w_p}(\mathfrak{B}_d).$$

We now show that if (5.42b) holds, we have

$$\pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right) = \pi_{w_p}(\mathfrak{B}_d). \tag{5.44}$$

In other words, $\pi_{w_p}(\mathfrak{B}_d)$ can be achieved by choosing

$$\mathfrak{B}_c = \pi_{w_c}(\mathfrak{B}_{in}). \tag{5.45}$$

Firstly, notice that

$$\begin{aligned}&[(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}) \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\\ =&\left[(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}) \times \left(\pi_{w_c}(\mathfrak{B}_{in}) \cap \mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H\\ =&\mathfrak{B}_{out} \cap [(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}) \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\\ =&[(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}) \times (\pi_{w_c}(\mathfrak{B}_{in}) \cap \pi_{w_c}(\mathfrak{B}_{out}))] \cap \mathfrak{B}_{pc}^H.\end{aligned}$$

We now show that

$$\pi_{w_c}(\mathfrak{B}_{in}) \cap \pi_{w_c}(\mathfrak{B}_{out}) = \varnothing. \tag{5.46}$$

Notice that $\pi_{w_c}(\mathfrak{B}_{out}) \subset \mathfrak{B}_c^H \cap \mathfrak{B}_{cr}$, then according to Proposition 5.3(i) and Theorem 5.4(ii), we have

$$\begin{aligned}
&\pi_{w_c}(\mathfrak{B}_{in}) \cap \pi_{w_c}(\mathfrak{B}_{out}) \\
=&\pi_{w_c}\left(\left[\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps} \setminus \pi_{w_p}(\mathfrak{B}_{ex})\right) \times \pi_{w_c}(\mathfrak{B}_{out})\right] \cap \mathfrak{B}_{pc}^H\right) \\
=&\pi_{w_c}(\mathfrak{B}_{ex} \setminus \mathfrak{B}_{ex}) = \varnothing.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right) \\
=&\pi_{w_p}\left([((\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \cup (\mathfrak{B}_p \setminus \mathfrak{B}_{ps})) \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right) \\
=&\pi_{w_p}\left([(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right) \\
\subset&\mathfrak{B}_p \cap \mathfrak{B}_{ps} = \pi_{w_p}(\mathfrak{B}_d).
\end{aligned} \tag{5.47}$$

For the reverse inclusion, we first show that

$$\pi_{w_c}(\mathfrak{B}_{in}) = \pi_{w_c}(\mathfrak{B}_d) \setminus \pi_{w_c}(\mathfrak{B}_{ex}), \tag{5.48}$$

which, according to Lemma 5.1(i), is equivalent to proving that

$$\pi_{w_c}(\mathfrak{B}_{in}) \cap \pi_{w_c}(\mathfrak{B}_{ex}) = \varnothing, \tag{5.49a}$$

$$\pi_{w_c}(\mathfrak{B}_{in}) \cup \pi_{w_c}(\mathfrak{B}_{ex}) = \pi_{w_c}(\mathfrak{B}_d). \tag{5.49b}$$

Since $\pi_{w_c}(\mathfrak{B}_{ex}) \subset \pi_{w_c}(\mathfrak{B}_{out})$, (5.49a) can be directly deduced from (5.46). We prove (5.49b) using contradiction. To begin with, it is easy to see that

$$\pi_{w_c}(\mathfrak{B}_{in}) \cup \pi_{w_c}(\mathfrak{B}_{ex}) \subset \pi_{w_c}(\mathfrak{B}_d) \cup \pi_{w_c}(\mathfrak{B}_d) = \pi_{w_c}(\mathfrak{B}_d).$$

Suppose that there exists a subset of $\pi_{w_c}(\mathfrak{B}_d)$, call it the residual set $\pi_{w_c}(\mathfrak{B}_{res})$, such that

$$\pi_{w_c}(\mathfrak{B}_{res}) = \pi_{w_c}(\mathfrak{B}_d) \setminus \pi_{w_c}(\mathfrak{B}_{in}) \setminus \pi_{w_c}(\mathfrak{B}_{ex})$$

and suppose that $\pi_{w_c}(\mathfrak{B}_{res}) \neq \varnothing$. Since $\pi_{w_p}(\mathfrak{B}_{ex}) \subset \pi_{w_p}(\mathfrak{B}_d) = \mathfrak{B}_p \cap \mathfrak{B}_{ps}$, it follows that

$$\begin{aligned}
&\pi_{w_p}(\mathfrak{B}_{in}) \cup \pi_{w_p}(\mathfrak{B}_{ex}) \\
=&\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps} \setminus \pi_{w_p}(\mathfrak{B}_{ex})\right) \cup \pi_{w_p}(\mathfrak{B}_{ex}) \\
=&\mathfrak{B}_p \cap \mathfrak{B}_{ps} = \pi_{w_p}(\mathfrak{B}_d).
\end{aligned}$$

Furthermore, $\pi_{w_p}(\mathfrak{B}_{in}) \cap \pi_{w_p}(\mathfrak{B}_{ex}) = \varnothing$. Therefore, for all $w_c \in \pi_{w_c}(\mathfrak{B}_d)$ there must exist at least one $w_p$ that belongs to either $\pi_{w_p}(\mathfrak{B}_{in})$ or $\pi_{w_p}(\mathfrak{B}_{ex})$ such that $(w_p, w_c) \in \mathfrak{B}_{pc}^H$. We first show that for $w_c \in \pi_{w_c}(\mathfrak{B}_{res})$, the corresponding trajectories of $w_p$ cannot come from $\pi_{w_p}(\mathfrak{B}_{in})$, i.e.,

$$\left[\pi_{w_p}(\mathfrak{B}_{in}) \times \pi_{w_c}(\mathfrak{B}_{res})\right] \cap \mathfrak{B}_{pc}^H = \varnothing.$$

Notice that

$$\left[\pi_{w_p}(\mathfrak{B}_{in}) \times \pi_{w_c}(\mathfrak{B}_{ex})\right] \cap \mathfrak{B}_{pc} = \mathfrak{B}_{ex} \setminus \mathfrak{B}_{ex} = \varnothing.$$

Also, we have

$$\mathfrak{B}_d \setminus \mathfrak{B}_{in} \subset \mathfrak{B}_d = \left[\pi_{w_p}(\mathfrak{B}_d) \times \pi_{w_c}(\mathfrak{B}_d)\right] \cap \mathfrak{B}_{pc}^H,$$

which means that

$$\begin{aligned}
&\mathfrak{B}_d \setminus \mathfrak{B}_{in} \\
=&\mathfrak{B}_d \setminus \mathfrak{B}_{in} \cap \left[\pi_{w_p}(\mathfrak{B}_d) \times \pi_{w_c}(\mathfrak{B}_d)\right] \cap \mathfrak{B}_{pc}^H \\
=&\left[\pi_{w_p}(\mathfrak{B}_{ex}) \times \left(\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H \cap \left[\pi_{w_p}(\mathfrak{B}_d) \times \pi_{w_c}(\mathfrak{B}_d)\right] \cap \mathfrak{B}_{pc}^H \\
=&\left[\pi_{w_p}(\mathfrak{B}_{ex}) \times \pi_{w_c}(\mathfrak{B}_d)\right] \cap \mathfrak{B}_{pc}^H.
\end{aligned}$$

As a result,

$$\begin{aligned}
&\left[\pi_{w_p}(\mathfrak{B}_{in}) \times \pi_{w_c}(\mathfrak{B}_{res})\right] \cap \mathfrak{B}_{pc}^H \\
\subset&\left[\pi_{w_p}(\mathfrak{B}_{in}) \times \left(\pi_{w_c}(\mathfrak{B}_d) \setminus \pi_{w_c}(\mathfrak{B}_{in})\right)\right] \cap \mathfrak{B}_{pc}^H \\
=&\left(\left[\pi_{w_p}(\mathfrak{B}_{in}) \times \pi_{w_c}(\mathfrak{B}_d)\right] \cap \mathfrak{B}_{pc}^H\right) \setminus \mathfrak{B}_{in} \\
=&\left(\left[\left(\pi_{w_p}(\mathfrak{B}_d) \setminus \pi_{w_p}(\mathfrak{B}_{ex})\right) \times \pi_{w_c}(\mathfrak{B}_d)\right] \cap \mathfrak{B}_{pc}^H\right) \setminus \mathfrak{B}_{in} \\
=&\mathfrak{B}_d \setminus \mathfrak{B}_{in} \setminus \left(\left[\pi_{w_p}(\mathfrak{B}_{ex}) \times \pi_{w_c}(\mathfrak{B}_d)\right] \cap \mathfrak{B}_{pc}^H\right) = \varnothing.
\end{aligned}$$

Therefore, all corresponding trajectories of $w_p \in \pi_{w_p}(\mathfrak{B}_d)$ for $w_c \in \pi_{w_c}(\mathfrak{B}_{res})$ must belong to $\pi_{w_p}(\mathfrak{B}_{ex})$. On the other hand, all trajectories in $\pi_{w_p}(\mathfrak{B}_{ex})$ has multiplicities in $\mathfrak{B}_p \setminus \mathfrak{B}_{ps}$, it follows that

$$\pi_{w_c}(\mathfrak{B}_{res}) \subset \pi_{w_c}\left(\left[(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}) \times \left(\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H\right) = \pi_{w_c}(\mathfrak{B}_{out}),$$

but $\pi_{w_c}(\mathfrak{B}_{res}) \subset \pi_{w_c}(\mathfrak{B}_d)$ by definition. We therefore have

$$\begin{aligned}\pi_{w_c}(\mathfrak{B}_{res}) &\subset \pi_{w_c}(\mathfrak{B}_d) \cap \pi_{w_c}(\mathfrak{B}_{out})\\ &= \pi_{w_c}\left([(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \times \pi_{w_c}(\mathfrak{B}_{out})] \cap \mathfrak{B}_{pc}^H\right)\\ &= \pi_{w_c}(\mathfrak{B}_{ex}).\end{aligned}$$

But $\pi_{w_c}(\mathfrak{B}_{res}) \not\subset \pi_{w_c}(\mathfrak{B}_{ex})$ by construction, which leads to a contradiction. This means that $\pi_{w_c}(\mathfrak{B}_{res}) = \varnothing$. Therefore, (5.49b), hence (5.48), is satisfied.

Now, from (5.47), (5.48) and according to Proposition 5.3(i), Lemma 5.2(iii), we have

$$\begin{aligned}&\pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right)\\ =&\pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right) \cap \mathfrak{B}_p \cap \mathfrak{B}_{ps}\\ =&\pi_{w_p}\left([(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right)\\ =&\pi_{w_p}\left([(\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \times (\pi_{w_c}(\mathfrak{B}_d) \setminus \pi_{w_c}(\mathfrak{B}_{ex}))] \cap \mathfrak{B}_{pc}^H\right)\\ =&\pi_{w_p}(\mathfrak{B}_d \setminus \mathfrak{B}_{ex})\\ \supset&\pi_{w_p}(\mathfrak{B}_d) \setminus \pi_{w_p}(\mathfrak{B}_{ex}).\end{aligned}$$

Since (5.42b) holds, we then have

$$\pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right) \supset \pi_{w_p}(\mathfrak{B}_d) \setminus \pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right)$$

This is true if and only if

$$\pi_{w_p}(\mathfrak{B}_d) \setminus \pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right) = \varnothing,$$

which, according to Lemma 5.1(i), is true if and only if

$$\pi_{w_p}(\mathfrak{B}_d) \subset \pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right). \tag{5.50}$$

The combination of (5.47) and (5.50) gives (5.44). This completes the proof of achieving (5.40a).

For achieving (5.40b), if (5.42c) holds, then according to Lemma 5.2(ii)

$$\begin{aligned}\pi_{w_f}\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps}\right) &= \pi_{w_f}\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps}\right) \cup \pi_{w_f}\left(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}\right) \\ &= \pi_{w_f}\left(\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps}\right) \cup \left(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}\right)\right) \\ &= \pi_{w_f}\left(\mathfrak{B}_p\right)\end{aligned}$$

Since $\pi_{w_f}\left(\mathfrak{B}_p\right) = \mathbb{W}_f^{\mathbb{T}}$ due to (5.42d), it follows that

$$\pi_{w_f}\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps}\right) = \mathbb{W}_f^{\mathbb{T}}. \tag{5.51}$$

Since other conditions guarantee (5.40a), (5.51) implies (5.40b).

Finally, we show that choosing individual controllers as (5.43) will result in (5.45). Since $\mathfrak{B}_c$ is of the form (5.10) and $\pi_{w_c}\left(\mathfrak{B}_{in}\right) \subset \mathfrak{B}_c^H$, we have

$$\mathfrak{B}_c = \left(\bigtimes_{j=1}^{N_c} \mathfrak{B}_c^j\right) \cap \mathfrak{B}_c^H = \left(\bigtimes_{j=1}^{N_c} \pi_{w_c^j}\left(\mathfrak{B}_c\right)\right) \cap \mathfrak{B}_c^H.$$

By choosing (5.43), we can construct $\mathfrak{B}_c$ according to Theorem 5.4(i) as

$$\mathfrak{B}_c = \left(\bigtimes_{j=1}^{N_c} \pi_{w_c^j}\left(\mathfrak{B}_{in}\right)\right) \cap \mathfrak{B}_c^H = \left(\bigtimes_{j=1}^{N_c} \pi_{w_c^j}\left(\pi_{w_c}\left(\mathfrak{B}_{in}\right)\right)\right) \cap \mathfrak{B}_c^H = \pi_{w_c}\left(\mathfrak{B}_{in}\right).$$

Furthermore,

$$\begin{aligned}&\pi_{w_c}\left(\mathfrak{B}_d\right) \setminus \pi_{w_c}\left(\mathfrak{B}_{ex}\right) \\ =&\pi_{w_c}\left(\mathfrak{B}_d\right) \setminus \pi_{w_c}\left(\left[\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps}\right) \times \pi_{w_c}\left(\mathfrak{B}_{out}\right)\right] \cap \mathfrak{B}_{pc}^H\right) \\ =&\pi_{w_c}\left(\mathfrak{B}_d\right) \setminus \left[\pi_{w_c}\left(\left[\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps}\right) \times \mathbb{W}_c^{\mathbb{T}}\right] \cap \mathfrak{B}_{pc}^H\right) \cap \pi_{w_c}\left(\mathfrak{B}_{out}\right)\right] \\ =&\left(\pi_{w_c}\left(\mathfrak{B}_d\right) \setminus \left[\pi_{w_c}\left(\left[\left(\mathfrak{B}_p \cap \mathfrak{B}_{ps}\right) \times \mathbb{W}_c^{\mathbb{T}}\right] \cap \mathfrak{B}_{pc}^H\right)\right]\right) \cup \left[\pi_{w_c}\left(\mathfrak{B}_d\right) \setminus \pi_{w_c}\left(\mathfrak{B}_{out}\right)\right] \\ =&\pi_{w_c}\left(\mathfrak{B}_d\right) \setminus \pi_{w_c}\left(\mathfrak{B}_{out}\right).\end{aligned} \tag{5.52}$$

Equations (5.48) and (5.52) give the equivalence relationships in (5.43).

Notice that (5.43) is the smallest controller behavioural set to achieve (5.45). $\mathfrak{B}_c^j$ may contain other trajectories but they must not be admissible through $\mathfrak{B}_c^H$.

This completes the proof of Theorem 5.7. □

The meanings of (5.42a) and (5.42d) are easy to understand. (5.42a) claims that the desired behaviour should be admissible through the uncontrolled behaviour and (5.42d) states that the free variables must already be free in the interconnected system. The rationales of conditions (5.42b) and (5.42c) are the same so they are explained together. Although the process of design is essentially elimination of a subset of trajectories from the original set from the point of view of the design variables, there will be corresponding trajectories of the other variables eliminated as well during the process. On the other hand, it is desirable to have the behaviours of a set of variables unchanged during the process. In order to achieve this, trajectories that are eliminated for this set of variables must have other corresponding trajectories within the desired behaviours of the design variables so that these trajectories can still be implemented. For (5.42b), the eliminated trajectories of $w_p$ from $\mathfrak{B}_p \cap \mathfrak{B}_{ps}$ must have corresponding trajectories of $w_c$ that will be used for implementation; for (5.42c), the eliminated trajectories of $w_f$ due to the restriction of $\mathfrak{B}_p$ by $\mathfrak{B}_{ps}$ should also be able to be implemented somehow by the trajectories of other variables within the set $\mathfrak{B}_p \cap \mathfrak{B}_{ps}$.

The most notable feature of Theorem 5.7 is that it carries out control design in a completely balanced way: instead of assuming dominance of the to-be-controlled variable over the manipulated variables, they are simply two sets of variables whose trajectories need to be admissible through the system. The controllers are simply restricting the trajectories of the system to a subset that is also a subset of the behaviour describing the desired requirements rather than inverting the system dynamics in any way. Another interesting observation is that the final set of controller trajectories is only a subset of the desired controlled behaviour

$\pi_{w_c}(\mathfrak{B}_d)$. This is because all trajectories in $\pi_{w_c}(\mathfrak{B}_d)$ are coming form the desired controlled behaviour, trajectories in $\mathfrak{B}_p$ that are admissible with those in $\pi_{w_c}(\mathfrak{B}_d)$ may not come from $\pi_{w_p}(\mathfrak{B}_d)$. To make sure that these trajectories are not going to happen, the size of the possible choice of $w_c$ must be reduced to the point that all corresponding trajectories of $w_p$ cover exactly the whole desired behavioural set. However, under certain regularity assumptions, the controller behaviour is in fact the exact behaviour projected from the desired controlled behaviour.

**Corollary 5.8.** For the setup in Theorem 5.7, if, additionally, $w_p$ is observable from $w_c$ and/or $w_c$ is observable from $w_p$, then the controller behaviours can be obtained as

$$\mathfrak{B}_c^j = \pi_{w_c^j}(\mathfrak{B}_d). \tag{5.53}$$

*Proof.* We prove this corollary by showing that $\pi_{w_p}(\mathfrak{B}_{ex}) = \varnothing$ for both cases, thereby deducing that $\mathfrak{B}_d = \mathfrak{B}_{in}$.

*Case I: $w_p$ observable from $w_c$*

According to Proposition 5.3(ii),

$$\begin{aligned}\pi_{w_p}(\mathfrak{B}_{ex}) &= (\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \cap \pi_{w_p}\left(\left[\mathbb{W}_p^{\mathbb{T}} \times \pi_{w_c}(\mathfrak{B}_{out})\right] \cap \mathfrak{B}_{pc}^H\right) \\ &= \mathfrak{B}_p \cap \mathfrak{B}_{ps} \cap \pi_{w_p}(\mathfrak{B}_{out}).\end{aligned}$$

Now, notice that

$$\pi_{w_p}(\mathfrak{B}_{out}) = \pi_{w_p}\left(\left[(\mathfrak{B}_p \setminus \mathfrak{B}_{ps}) \times \left(\mathfrak{B}_c^H \cap \mathfrak{B}_{cr}\right)\right] \cap \mathfrak{B}_{pc}^H\right) \subset \mathfrak{B}_p \setminus \mathfrak{B}_{ps}$$

it follows that

$$\pi_{w_p}(\mathfrak{B}_{ex}) \subset (\mathfrak{B}_p \cap \mathfrak{B}_{ps}) \cap (\mathfrak{B}_p \setminus \mathfrak{B}_{ps}) = \varnothing. \tag{5.54}$$

Note that for this case, (5.42b) becomes redundant.

*Case II: $w_c$ observable from $w_p$*

To begin with, (5.42b) implies that

$$\pi_{w_p}(\mathfrak{B}_{ex}) = \pi_{w_p}(\mathfrak{B}_{ex}) \cap \pi_{w_p}\left([\mathfrak{B}_p \times \pi_{w_c}(\mathfrak{B}_{in})] \cap \mathfrak{B}_{pc}^H\right)$$

$$= \pi_{w_p}\left(\left[\pi_{w_p}\left(\mathfrak{B}_{ex}\right) \times \pi_{w_c}\left(\mathfrak{B}_{in}\right)\right] \cap \mathfrak{B}_{pc}^{H}\right).$$

Then, according to Proposition 5.3(ii) and Corollary 5.6, we have

$$\begin{aligned}\pi_{w_p}\left(\mathfrak{B}_{ex}\right) =& \pi_{w_p}\left(\left[\pi_{w_p}\left(\mathfrak{B}_{ex}\right) \times \mathbb{W}_c^{\mathbb{T}}\right] \cap \left[\mathbb{W}_p^{\mathbb{T}} \times \pi_{w_c}\left(\mathfrak{B}_{in}\right)\right] \cap \mathfrak{B}_{pc}^{H}\right)\\ =& \pi_{w_p}\left(\mathfrak{B}_{ex} \cap \left[\mathbb{W}_p^{\mathbb{T}} \times \pi_{w_c}\left(\mathfrak{B}_{in}\right)\right]\right)\\ =& \pi_{w_p}\left(\pi_{w_p}\left(\mathfrak{B}_{ex}\right) \times \left[\pi_{w_c}\left(\mathfrak{B}_{in}\right) \cap \pi_{w_c}\left(\mathfrak{B}_{ex}\right)\right]\right).\end{aligned}$$

The emptiness of the last line above follows from (5.49a). □

This corollary suggests that with the regularity assumptions, the controller behaviour can be found easily using the rationale illustrated before the introduction of Theorem 5.7: construct the desired controlled behaviour and see how $w_c$ behaves. The trajectories obtained are the ones giving the desired behaviour for $w_p$. While the case where $w_c$ is observable from $w_p$ is rather rare, the case where $w_p$ is observable from $w_c$ appears much more frequently. For example, $w_c$ is generally observable from $w_p$ in decentralised control. Therefore, if one were to search for the controller trajectories, one should begin by checking if either special case in Corollary 5.8 applies, in which case fewer conditions or relaxed conditions were needed to be verified. The general conditions in Theorem 5.7 should be used if neither of the special cases would apply.

### 5.4.3 The Linear Time-Invariant Case

The discussion in this section is a continuation of that in Section 5.3.3. Suppose that $\mathfrak{B}_p^H = \ker(H_p)$, then following Section 5.3.3, $\mathfrak{B}_{p|[1,L]}$ can represented by

$$\widetilde{w}_{p|[1,L]} = \mathfrak{H}_L\left(\tilde{v}_p\right)\left[H_p^{\otimes L}\mathfrak{H}_L\left(\tilde{v}_p\right)\right]^{\perp} z_p := \mathfrak{M}_p z_p. \tag{5.55}$$

Suppose now that a network of distributed controllers is to be designed to render the controlled system dissipative with respect to a supply rate $Q_\Phi(w_p)$. The dissipativity can be verified according to the following proposition. Note that a similar formulation has been given in the context of stand-alone systems with forward shifts in [64].

**Proposition 5.9.** Given $\mathfrak{B}_{p|[1,L]}$ represented by (5.55) and a QdF induced by $\Phi \in \mathbb{S}^{\mathtt{w}}[\zeta,\eta]$ with order $K$ with $K \leq \mathtt{L}(\mathfrak{B})$, then $\mathfrak{B}_{p|[K+1,L]}$ is $\Phi$-dissipative if

$$\mathfrak{M}_p^T \Pi^T \widetilde{\Phi}^{\otimes_{L-K}} \Pi \mathfrak{M}_p \geq 0, \tag{5.56}$$

where

$$\Pi \widetilde{w}_{p|[1,L]} = \begin{bmatrix} \widetilde{w}_{p|[1,K+1]} \\ \vdots \\ \widetilde{w}_{p|[L-K,L]} \end{bmatrix}.$$

*Proof.* The proof is relatively straightforward by seeing that (5.56) is a compact representation of (2.35) with $w_p$ represented by (5.55) for $L-K$ steps and therefore omitted. Note that the dissipativity of the first $K$ steps cannot be verified because there is no previous trajectory for them. □

Similar to the representation of $\mathfrak{B}_p^H$, it is assumed in this section that $\mathfrak{B}_{pc}^H = \ker(H_{pc})$ and $\mathfrak{B}_c^H = \ker(H_c)$. For now, no further requirements for the controllers are imposed, i.e., $\mathfrak{B}_{cr} = \mathbb{W}_c^{\mathbb{T}}$. The effect of $\mathfrak{B}_{cr}$ will be discussed by the end of this section. In general, $w_p$ and $w_c$ are not observable from each other since it is normally not possible to deduce the trajectories *within* the network from partial information of the external variables. Therefore, conditions in (5.42) need to be verified.

In this setting, (5.42a) means that all trajectories from $\mathfrak{B}_p$ that is dissipative with respect to the supply rate induced by $\Phi(\zeta,\eta)$ should be admissible to the system-controller network and the corresponding trajectories of the control variables should be admissible to the controller network. The representation of $\mathfrak{B}_{pc}^H$

can be written as

$$\begin{bmatrix} H_{pcp}^{\otimes L} & -H_{pcc}^{\otimes L} \end{bmatrix} \begin{bmatrix} \widetilde{w}_{p|[1,L]} \\ \\ \widetilde{w}_{c|[1,L]} \end{bmatrix} = 0 \tag{5.57}$$

Combining with the kernel representation of $\mathfrak{B}_c^H$, i.e.,

$$H_c^{\otimes L} \widetilde{w}_{c|[1,L]} = 0, \tag{5.58}$$

the behaviour of $\pi_{w_p}\left(\left(\mathfrak{B}_p \times \mathfrak{B}_c^H\right) \cap \mathfrak{B}_{pc}^H\right)$ can be constructed as the image of $\mathfrak{M}_p$ with $z_p$ satisfying

$$H_{pcp}^{\otimes L} \mathfrak{M}_p z_p = \left(H_{pcp} H_c^{\perp}\right)^{\otimes L} z_{c1}. \tag{5.59}$$

Assuming, for illustration purpose, that $H_{pcp}^{\otimes L} \mathfrak{M}_p$ is right invertible, the representation for $z_p$ is then

$$z_p = \begin{bmatrix} \left(H_{pcp}^{\otimes L} \mathfrak{M}_p\right)^{\dagger} \left(H_{pcp} H_c^{\perp}\right)^{\otimes L} & \left(H_{pcp}^{\otimes L} \mathfrak{M}_p\right)^{\perp} \end{bmatrix} \begin{bmatrix} z_{c1} \\ \\ z_{c2} \end{bmatrix} := M_z z_c \tag{5.60}$$

where $z_c$ and $z_c'$ are arbitrary and $M_z \in \mathbb{R}^{\mathrm{z_p} \times (\mathrm{z_p} + L \mathrm{w_c})}$. The condition is then that (5.60) holds for all $z_p$ such that

$$z_p^T \mathfrak{M}_p^T \Pi^T \widetilde{\Phi}^{\otimes_{L-K}} \Pi \mathfrak{M}_p z_p \geq 0. \tag{5.61}$$

This is equivalent to stating that (5.61) does not hold for all $z_p$ such that (5.60) does not hold. It is a well-known fact that the orthogonal complement of (5.60) is given by

$$M_z^T z_p^{\perp} = 0. \tag{5.62}$$

According to the well-known Finsler Lemma (see [121], for example), the negation of (5.61) for all $z_p^{\perp}$ satisfying (5.62) is equivalent to the existence of a matrix $F \in \mathbb{R}^{(\mathrm{z_p} + L \mathrm{w_c}) \times \mathrm{z_p}}$ such that

$$-\mathfrak{M}_p^T \Pi^T \widetilde{\Phi}^{\otimes_{L-K}} \Pi \mathfrak{M}_p + \mathrm{He}\left\{F^T M_z^T\right\} > 0. \tag{5.63}$$

As $\Sigma_p^H$, $\Sigma_{pc}^H$ and all subsystems in $\Sigma_p^{sys}$ are LTI systems, (5.42b) can be formulated more uniformly in terms of the hidden behaviour. Since the resulting system by interconnecting $\Sigma_p^H$, $\Sigma_{pc}^H$ and $\Sigma_p^{sys}$ is still linear, the requirement (5.42b) becomes

$$\pi_{w_p}(\mathfrak{B}_{ex}) \subset \pi_{w_p}(\mathfrak{B}_{in}) \oplus \mathfrak{B}_{hc} \tag{5.64}$$

where $\mathfrak{B}_{hc}$ is the behaviour "hidden" from $w_c$. Since $\pi_{w_p}(\mathfrak{B}_{in})$ and $\pi_{w_p}(\mathfrak{B}_{in}) \oplus \mathfrak{B}_{hc}$ are indistinguishable from $w_c$ point of view, it follows that the distributed controllers exist if and only if $\mathfrak{B}_{hc} \subset \pi_{w_p}(\mathfrak{B}_d) = \mathfrak{B}_p \cap \mathfrak{B}_{ps}$ (as guaranteed by (5.42a)). Since $\mathfrak{B}_{hc}$ is already a subset of $\mathfrak{B}_p$, this condition reduces to $\mathfrak{B}_{hc} \subset \mathfrak{B}_{ps}$. This result is the generalised version of that in (2.44).

From (5.57), it is obvious that the hidden behaviour of $\mathfrak{B}_{pc}^H$ is $H_{pcp}w_p = 0$ and that the behaviour "hidden" from $w_c$ can be constructed similarly as that in (5.29), with $H$ replaced by $\operatorname{col}(H_p, H_{pcp})$. Leading to the representation of the behaviour as

$$\widetilde{w}_{p|[1,L]} = \mathfrak{H}_L(\tilde{v}_p)\left(\begin{bmatrix} H_p \\ \\ H_{pcp}\end{bmatrix}^{\otimes_L} \mathfrak{H}_L(\tilde{v}_p)\right)^{\perp} z_h := \mathfrak{M}_h z_h, \tag{5.65}$$

where $z_h \in \mathbb{R}^{\mathrm{z_p}}$ is an arbitrary vector. Although not all trajectories in $\mathfrak{B}_p$ are supposed to be dissipative with respective to $\Phi$, the hidden behaviour described in (5.65) should. According to Proposition 5.9, the realisation of (5.42b) is hence

$$\mathfrak{M}_h^T \Pi^T \widetilde{\Phi}^{\otimes_{L-K}} \Pi \mathfrak{M}_h \geq 0, \tag{5.66}$$

Conditions (5.42c) and (5.42d) are related with the internal dynamics of the interconnected system $\Sigma_p$. In fact, every dynamical system can be viewed as a miniature interconnected system. As an example, consider an LTI system described by a latent variable representation $R(\sigma)w = M(\sigma)\ell$. According to [39], there always exist unimodular matrices $U(\sigma)$ and $V(\sigma)$ such that

$$\begin{aligned} U(\sigma)R(\sigma) &= \begin{bmatrix} R_1(\sigma) \\ R_2(\sigma) \end{bmatrix}, \; U(\sigma)M(\sigma) = \begin{bmatrix} 0 \\ M_2(\sigma) \end{bmatrix}, \\ V(\sigma)R(\sigma) &= \begin{bmatrix} 0 \\ R_4(\sigma) \end{bmatrix}, \; V(\sigma)M(\sigma) = \begin{bmatrix} M_3(\sigma) \\ M_4(\sigma) \end{bmatrix}. \end{aligned} \tag{5.67}$$

Now, by defining $\Sigma_w = (\mathbb{T}, \mathbb{W}, \mathfrak{B}_w)$, $\Sigma_\ell = (\mathbb{T}, \mathbb{L}, \mathfrak{B}_\ell)$ and $\Sigma^H = (\mathbb{T}, \mathbb{W} \times \mathbb{L}, \mathfrak{B}^H)$, it follows that $(\Sigma_w \sqcap \Sigma_\ell) \wedge \Sigma^H = \Sigma$. Although impossible to get $\mathfrak{B}_w$ or $\mathfrak{B}_\ell$, it is true that $\pi_w(\mathfrak{B}) = \ker(R_1)$, $\pi_\ell(\mathfrak{B}) = \ker(M_3)$ and $\mathfrak{B}^H = \ker\left(\begin{bmatrix} R_2 & -M_2 \\ R_4 & -M_4 \end{bmatrix}\right)$. According to Theorem 5.4, $(\mathfrak{B}_w \times \mathfrak{B}_\ell) \cap \mathfrak{B}^H = \mathfrak{B}$, and it is not hard to verify that this is the case by constructing $\mathfrak{B}$ according to Lemma 2.8.

Similar to the realisation of (5.42b), (5.42c) can be formulated as a condition on $\mathfrak{B}_{hd}$, the behaviour with manifest variable $w_f$ that is "hidden" from $w_d$, i.e., all trajectories of $w_f$ such that $(w_f, 0) \in \mathfrak{B}_p$. Following the logic of realising (5.42b) discussed above, the condition for (5.42c) is then $\mathfrak{B}_{hd} \subset \pi_{w_f}(\mathfrak{B}_p \cap \mathfrak{B}_{ps})$. By partitioning $w_p$ as $w_p = (w_d, w_f)$ where $w_f$ contains all of the components that should remain free in the interconnected system and permuting $\mathfrak{M}_p$ accordingly to $\mathrm{col}(\mathfrak{M}_d, \mathfrak{M}_f)$, (5.55) can be represented equivalently as,

$$\widetilde{w}_{d|[1,L]} = \mathfrak{M}_d z_p, \tag{5.68a}$$

$$\widetilde{w}_{f|[1,L]} = \mathfrak{M}_f z_p. \tag{5.68b}$$

The trajectories of $w_f$ that corresponds to the trivial trajectory in $w_d$ can be obtained by all choices of $z_p$ such that $\widetilde{w}_{d|[1,L]} = 0$, or

$$\widetilde{w}_{f|[1,L]} = \mathfrak{M}_f \mathfrak{M}_d^\perp z_{hd}. \tag{5.69}$$

Following Proposition 5.9, (5.42c) in this context is then

$$\begin{bmatrix} 0 \\ \mathfrak{M}_f \mathfrak{M}_d^{\perp} \end{bmatrix}^T \Pi^T \widetilde{\Phi}^{\otimes_{L-K}} \Pi \begin{bmatrix} 0 \\ \mathfrak{M}_f \mathfrak{M}_d^{\perp} \end{bmatrix} \geq 0. \tag{5.70}$$

Lastly, (5.42d) can be formulated as

$$\operatorname{rank}(\mathfrak{M}_f) = L\mathrm{w}_\mathrm{f} \tag{5.71}$$

because in this way there always exists a $z_p$, hence a trajectory $\widetilde{w}_{d|[1,L]}$ for any given trajectory $\widetilde{w}_{f|[1,L]} \in \mathbb{W}_f^{\mathbb{Z}_L^+}$.

If (5.63), (5.66), (5.70) and (5.71) are satisfied, then the controller behaviours can be constructed based on one of the ways in (5.43). In this case, (5.43c) is used because $\mathfrak{B}_d$ and $\mathfrak{B}_{out}$ share the component $\left(\mathfrak{B}_p \times \mathfrak{B}_c^H\right) \cap \mathfrak{B}_{pc}^H$. In this case, they are the set of trajectories $(w_p, w_c)$ satisfying (5.55), (5.57) and (5.58). The $w_c$ component of the combination of (5.55) and (5.57) can be obtained with some linear algebra as

$$\widetilde{w}_{c|[1,L]} = \begin{bmatrix} \left(H_{pcc}^{\dagger} H_{pcp}\right)^{\otimes_L} \mathfrak{M}_p & \left(H_{pcc}^{\perp}\right)^{\otimes_L} \end{bmatrix} \begin{bmatrix} z_p \\ z_c' \end{bmatrix}. \tag{5.72}$$

Substituting into (5.58) gives

$$\begin{bmatrix} z_p \\ z_c' \end{bmatrix} = \begin{bmatrix} \left(H_c H_{pcc}^{\dagger} H_{pcp}\right)^{\otimes_L} \mathfrak{M}_p & \left(H_c H_{pcc}^{\perp}\right)^{\otimes_L} \end{bmatrix}^{\perp} z_{pc} := \begin{bmatrix} M_p \\ M_c \end{bmatrix} z_{pc}. \tag{5.73}$$

The set of trajectories for the control variable $w_c$ are generated by the choices of $z_{pc}$ such that the corresponding choice of $w_p$, *in addition to all of its multiplicities*, are within $\mathfrak{B}_{ps}$. For a given $z_{pc}^*$, all choices of $w_p^*$ that leads to this particular $w_c^*$ can be parametrised as

$$w_{p|[1,L]}^* = \mathfrak{M}_p M_p z_{pc}^* + \mathfrak{M}_h z_h, \tag{5.74}$$

where the latter part is the hidden behaviour defined in (5.65). The valid choices of $z_{pc}$ are then those such that

$$\left(\mathfrak{M}_p M_p z_{pc} + \mathfrak{M}_h z_h\right)^T \Pi^T \widetilde{\Phi}^{\otimes_{L-K}} \Pi \left(\mathfrak{M}_p M_p z_{pc} + \mathfrak{M}_h z_h\right) \geq 0 \qquad (5.75)$$

for all $z_h \in \mathbb{R}^{\mathtt{z}_\mathtt{p}}$. The trajectories for $w_c$ can then be generated as

$$\widetilde{w}_{c|[1,L]} = \left[\left(H_{pcc}^{\dagger} H_{pcp}\right)^{\otimes_L} \mathfrak{M}_p M_p + \left(H_{pcc}^{\perp}\right)^{\otimes_L} M_c\right] z_{pc}, \qquad (5.76)$$

and the controller trajectories for each distributed controller $\widetilde{w}_{c|[1,L]}^{j}$ are simply the corresponding elements because, with the network behaviour embedded already, the controllers are essentially "decentralised" at this point.

At this point, there is no further restriction on the controller behaviour, hence $z_{pc}$ can be chosen arbitrarily. However, further conditions can be posed onto the controller behaviour by adding restriction through $\mathfrak{B}_{cr}$. For example, the representation of $\mathfrak{B}_{cr}$ can be a min function with some argument (e.g. a positive semi-definite QdF with respective to $w_c$), in which case $z_{pc}$ should be chosen such that the argument is minimised.

## 5.5 Summary

In this chapter, a framework for the analysis and distributed control of interconnected systems from set-theoretic point of view using behavioural systems theory has been proposed. The network of an interconnected system have been viewed as a dynamical system with its own internal dynamics, which enables the representation of interconnected behaviour to be constructed explicitly from its components, regardless of their respective representations. Furthermore, it has been shown that the interconnected behaviour can be completely constructed using the projections of the behaviours of the subsystems from the interconnected system and the behaviour of the network. It has also been shown that the same effect can be achieved with any numbers of complete behavioural sets for some subsystems, the projections of the others and the behaviour of the network, allowing for a hybrid platform for model-based/data-driven interconnected systems. Necessary and sufficient conditions for the existence of distributed controllers have been provided and controller behaviours have been constructed explicitly. It is believed that this is a more natural view of a dynamical system and is a promising direction for the development of data-driven and hybrid control methods.

# Chapter 6

# Conclusions and Recommendations

## 6.1 Summary and Conclusions

The rapid development of technology has promoted new paradigms in the design and monitoring of systems. The scale of the systems has become larger, the interconnections among subsystems has becomes increasingly complex and the main focus on the analysis of a systems has shifted from model-based analysis to hybrid model/data structure or even purely data driven. While these new paradigms lead to much higher production rate and more efficient energy usage, the complex dynamical features due to the strong interactions among subsystems make the control of such a system a challenging task. This thesis amounts to provide a unified platform for the design of distributed controllers under the framework of the behavioural systems theory. The set-theoretic view of dynamical systems, the free choice of representations of subsystems and the effectiveness in representing interconnections make this framework ideal in the analysis and control of large-scale interconnected system with components represented in a variety of ways.

In Chapter 3, the framework of distributed robust control of an interconnected system whose subsystems are LTI but have uncertain regions described by convex polytopes has been formulated. By representing the behaviours of the vertices of the uncertain regions as kernel representations, the uncertain behaviours can also be represented by a kernel representation whose coefficient matrices are represented by the linear combinations of the vertices of their respective uncertain polytopes. Parameter-dependant QDF dissipativity properties of the uncertain systems can then be found by finding that of the vertices and combining them using the same uncertain weightings. In such a way, the uncertain LTI systems are viewed from the perspective of dissipative dynamical systems, which makes the representation of the interconnected system much easier to construct. The controllers can also be initially described as dissipative dynamical systems with to-be-determined manifest variables. As such, the controlled interconnected behaviour can be represented as a dissipative dynamical system whose supply rate is the linear combination of that of the subsystems and the controllers. Restrictions can then be posed on the supply rate of the controlled interconnected behaviour as a supply rate as well, from which the controller supply rates can be found. The controllers can then be described by LTI differential systems through $J$-factorisation of the controller supply rates and parametrisation of the behaviour.

In Chapter 4, the philosophy described above has been extended into interconnected systems whose subsystems have parametric uncertainties. The resulting uncertain region may no longer be convex and polytopic uncertainty illustrated above can be viewed as a special case of this. The notion of PQDF has been proposed accordingly as a generalisation of the parameter-dependant QDF. As such, the entire framework can be generalised from polytopic description to generic description of uncertainties in the subsystems. However, due to the non-convexity of the description, convex optimisation tools can no longer be directly applied to search for the solutions and other methods needs to be proposed. Two methods have been proposed in Chapter 4. If the scale of the problem is relatively small and that the uncertain systems have relatively smooth representations in terms of the uncertain parameters, then SOS programming can be used by approximating the uncertain coefficient matrices using polynomials of the uncertain parameters and assuming PQDFs as polynomials of the uncertain parameters as well. If the uncertain coefficient matrices are naturally polynomial with respect to the uncertain parameters, then this method yields a deterministic solution. On the other hand, if the scale of the problem is large and the uncertain coefficients are not smooth enough, then a probabilistic solution can be obtained through the scenario approach. This approach yields a solution that is highly possible to achieve the control goals almost all the time.

In Chapter 5, the distributed control problem have been discussed from a purely set-theoretic point of view. This is a further, and a rather big, leap of generalisation of the issues discussed in the previous two chapters. In fact, this framework is a generalisation and unification of a wide range of control problems, in that the discussions are not restricted to any type of systems but systems in general. Dynamical systems are viewed purely as a set of trajectories with no prescribed representations and interconnections are viewed as the restriction on the possible admissible trajectories by posing additional constraints on the system. Control is hence not so much a trajectory making process as a trajectory selection process. By abstracting the network of an interconnected system as a dynamical system, the interconnected behaviour can be constructed explicitly from the behaviour of the subsystems and the network. It has also been shown, through the notion of projection, that the projected behaviour of each subsystem, or any combinations of the projected and complete behaviours of the subsystems, together with the behaviour of the network, can also recover the complete interconnected behaviour.

The control specifications and controller restrictions can also be abstracted as dynamical systems, in which case the controlled interconnected system can be viewed as a dynamical system whose trajectories are the common ones in the interconnected system, the distributed controller, the system describing the control specification and the system describing controller restriction. With this setting, necessary and sufficient conditions for the existence of controller behaviour as well as the explicit construction of the controller behavioural sets have been given.

## 6.2 Main Contributions

The main contribution of this thesis is to the distributed control of interconnected system in the behavioural systematic settings. Specifically, the following contributions have been made:

1. A *single* control framework that deals with *both* interconnections among subsystems and uncertainties within the subsystems has been formulated;

2. Algorithms for the design of distributed controllers with subsystems having polytopic uncertainties, smooth generic uncertainties and non-smooth generic uncertainties have been proposed;

3. A completely representation-free framework for the analysis and distributed control of interconnected systems has been developed;

4. Necessary and sufficient conditions for the existence of controller behaviours have been given;

5. The controller behavioural sets that achieve desired control goals have been constructed explicitly.

## 6.3 Future Work

As a new framework, it opens up numerous questions and there is considerable scope for further developments. In this section, a few of the immediate future directions are listed.

### 6.3.1 Data-driven/Hybrid Control Algorithms

The primary characteristics of a data-driven system is that its behaviour is described entirely based on data sets, and data-driven control stems to control such a system without explicitly identifying the model. This philosophy fits into the behavioural framework perfectly, as the framework sees behaviour as a set of trajectories. For LTI systems, the behaviour restricted to a finite length can be well-represented by the column span of the Hankel matrix generated from one of its measured trajectories (see Theorem 2.1) and some preliminary studies on describing such a system from dissipativity point of view have already been carried out [64]. However, for a system with no prescribed structure, there is generally no uniform way, or even a regressive way, to describe its behaviour. Two immediate directions that deserve further research can be summarised as follows.

1. Theorem 5.4 has already shown the possibility of constructing the interconnected behaviour from the measurements of the subsystems and Theorem 5.7 has provided set-theoretic guidelines for the validation of the existence of the controllers as well as the controller behaviours. However, efficient algorithm to actually implement the guidelines and controllers are yet to be formulated.

2. Since data sets are collections of finite numbers of finite trajectories of a behaviour, it can only describe a finite section of the behaviour up to a certain degree of accuracy. Therefore, it is reasonable to assume that the

solution to such a problem is going to be a probabilistic one, and that online recursive training should be carried out when possible.

### 6.3.2 Partial Measurements

Throughout this thesis, it has been assumed, when necessary, that the required measurements are always available. This is not necessarily the case in reality. In such a case, the required information may need to be estimated from local partial measurements and information sent from neighbour subsystems. Observability (or detectability) in this case becomes even more important than it already is. Observability problem can be formulated in two ways: either the required variables are observable from their own measurements, in which case local observability can be achieved, or all required variables are observable from all of the measurements through the network, in which case global observability can still be guaranteed even though local observability cannot be achieved [122]. Due to the duality of estimation and control [123], observer trajectories may still be able to be obtained from implementing (possibly modified version of) the procedure in Theorem 5.7, but much more work is needed for the case where only global observability can be guaranteed than the case where local observability can be guaranteed. The discussion of the internal dynamics of a system in Section 5.4.3 may be of great importance.

### 6.3.3 Flexible Manufacturing

Due to the current trend in designing multifunctional plants and factories, the concept of flexible manufacturing becomes increasingly popular. However, the dynamics of such a system becomes complex as it is hard to change the representation of the interconnected behaviour with the reconfiguration of the interconnection. The key issue here is that the dynamics of the network is pushed into that of the subsystems. By abstracting the network of an interconnected system into a dynamical system itself according to Definition 5.1, the interconnected behaviour can be constructed much more easily. One of the precautions though is that since the network is with its own dynamics, its own memory span must be considered should the control is carried out in a data-driven context.

# Bibliography

[1] W. L. Luyben, B. D. Tyréus, and M. L. Luyben, *Plantwide process control*. McGraw-Hill, 1999.

[2] M. Baldea and P. Daoutidis, "A general analysis and control framework for process systems with inventory recycling," *International Journal of Robust and Nonlinear Control*, vol. 24, no. 17, pp. 2852–2866, 2014.

[3] A. A. Alonso and B. E. Ydstie, "Process systems, passivity and the second law of thermodynamics," *Computers & Chemical Engineering*, vol. 20, pp. S1119–S1124, 1996.

[4] L. T. Antelo, I. Otero-Muras, J. R. Banga, and A. A. Alonso, "A systematic approach to plant-wide control based on thermodynamics," *Computers & Chemical Engineering*, vol. 31, no. 5-6, pp. 677–691, 2007.

[5] G. Stanley, "Big data approximating control (BDAC) - a new model-free estimation and control paradigm based on pattern matching and approximation," *Journal of Process Control*, vol. 67, pp. 141–159, 2018.

[6] A. Kumar and P. Daoutidis, "Nonlinear dynamics and control of process systems with recycle," *Journal of Process Control*, vol. 12, pp. 475 – 484, 2002.

[7] W. L. Luyben, "Snowball effects in reactor/separator processes with recycle," *Industrial & Engineering Chemistry Research*, vol. 33, no. 2, pp. 299–305, 1994.

[8] M. L. Luyben, B. D. Tyreus, and W. L. Luyben, "Plantwide control design procedure," *AIChE Journal*, vol. 43, no. 12, pp. 3161–3174, 1997.

[9] C. K. Yi and W. L. Luyben, "Evaluation of plant-wide control structures by steady-state disturbance sensitivity analysis," *Industrial & Engineering Chemistry Research*, vol. 34, no. 7, pp. 2393–2405, 1995.

[10] S. Subramanian and C. Georgakis, "Methodology for the steady-state operability analysis of plantwide systems," *Industrial & Engineering Chemistry Research*, vol. 44, no. 20, pp. 7770–7786, 2005.

[11] C. Georgakis, "On the use of extensive variables in process dynamics and control," *Chemical Engineering Science*, vol. 41, no. 6, pp. 1471–1484, 1986.

[12] P. R. Lyman and C. Georgakis, "Plant-wide control of the Tennessee Eastman problem," *Computers & Chemical Engineering*, vol. 19, no. 3, pp. 321–331, 1995.

[13] O. J. Rojas, J. Bao, and P. L. Lee, "On dissipativity, passivity and dynamic operability of nonlinear processes," *Journal of Process Control*, vol. 18, no. 5, pp. 515–526, 2008.

[14] O. J. Rojas, R. Setiawan, J. Bao, and P. L. Lee, "Dynamic operability analysis of nonlinear process networks based on dissipativity," *AIChE Journal*, vol. 55, no. 4, pp. 963–982, 2009.

[15] S. Skogestad, "Control structure design for complete chemical plants," *Computers & Chemical Engineering*, vol. 28, no. 1-2, pp. 219–234, 2004.

[16] P. Grosdidier and M. Morari, "Interaction measures for systems under decentralized control," *Automatica*, vol. 22, no. 3, pp. 309–319, 1986.

[17] A. Swarnakar, H. Marquez, and T. Chen, "Multi-loop control synthesis for unstable systems and its application: An approach based on $\mu$ interaction measure," *International Journal of Robust and Nonlinear Control*, vol. 19, no. 15, pp. 1721–1744, 2009.

[18] M. Morari and E. Zafiriou, *Robust Process Control*. Prentice-Hall, Englewood Cliffs, NJ, 1989.

[19] Y. Samyudia, P. L. Lee, I. T. Cameron, and M. Green, "A new approach to decentralised control design," *Chemical Engineering Science*, vol. 50, no. 11, pp. 1695–1706, 1995.

[20] J. B. Rawlings and B. T. Stewart, "Coordinating multiple optimization-based controllers: New opportunities and challenges," *Journal of Process Control*, vol. 18, no. 9, pp. 839–845, 2008.

[21] R. Scattolini, "Architectures for distributed and hierarchical model predictive control–a review," *Journal of Process Control*, vol. 19, no. 5, pp. 723–731, 2009.

[22] P. D. Christofides, J. Liu, and D. Muñoz de la Peña, *Networked and distributed predictive control: Methods and nonlinear process network applications*. Springer, 2011.

[23] P. D. Christofides, R. Scattolini, D. Muñoz de la Peña, and J. Liu, "Distributed model predictive control: A tutorial review and future research directions," *Computers & Chemical Engineering*, vol. 51, pp. 21–41, 2013.

[24] G. E. Box, "Robustness in the strategy of scientific model building," in *Robustness in Statistics*. Elsevier, 1979, pp. 201–236.

[25] D. C. McFarlane and K. Glover, *Robust controller design using normalized coprime factor plant descriptions*. Springer, 1990.

[26] K. Zhou, J. C. Doyle, and K. Glover, *Robust and Optimal Control*. Prentice-Hall, New Jersey, 1996.

[27] I. Petersen, V. A. Ugrinovskii, and A. V. Savkin, *Robust Control Design Using $\mathcal{H}_\infty$ Methods*. Springer Science & Business Media, 2012.

[28] H. L. Trentelman, S. Fiaz, and K. Takaba, "Optimal robust stabilization and dissipativity synthesis by behavioral interconnection," *SIAM Journal on Control and Optimization*, vol. 49, no. 1, pp. 288–314, 2011.

[29] S. V. Gottimukkala and H. L. Trentelman, "Rational representations and robust stabilization in the behavioral framework," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 17, pp. 3252–3269, 2015.

[30] T. A. Badgwell, "Robust model predictive control of stable linear systems," *International Journal of Control*, vol. 68, no. 4, pp. 797–818, 1997.

[31] M. V. Kothare, V. Balakrishnan, and M. Morari, "Robust constrained model predictive control using linear matrix inequalities," *Automatica*, vol. 32, no. 10, pp. 1361–1379, 1996.

[32] K. Takaba, "LMI conditions for robust stability against parametric uncertainty: A behavioral approach," in *Proceedings of the IEEE International Symposium on Computer-Aided Control System Design*. IEEE, 2010, pp. 755–760.

[33] J. C. Willems, “System theoretic models for the analysis of physical systems,” *Ricerche di Automatica*, vol. 10, no. 2, pp. 71–106, 1979.

[34] ——, “On interconnections, control, and feedback,” *IEEE Transactions on Automatic Control*, vol. 42, no. 3, pp. 326–339, 1997.

[35] ——, “From time series to linear system - Part I. Finite dimensional linear time invariant systems,” *Automatica*, vol. 22, no. 5, pp. 561–580, 1986.

[36] ——, “From time series to linear system - Part II. Exact modelling,” *Automatica*, vol. 22, no. 6, pp. 675–694, 1986.

[37] ——, “From time series to linear system - Part III. Approximate modelling,” *Automatica*, vol. 23, no. 1, pp. 87–115, 1987.

[38] ——, “Paradigms and puzzles in the theory of dynamical systems,” *IEEE Transactions on Automatic Control*, vol. 36, no. 3, pp. 259–294, 1991.

[39] J. W. Polderman and J. C. Willems, *Introduction to mathematical systems theory: A behavioral approach.* Springer, 1998.

[40] J. C. Willems, “The behavioral approach to open and interconnected systems,” *IEEE Control Systems Magazine*, vol. 27, no. 6, pp. 46–99, 2007.

[41] R. Tóth, J. C. Willems, P. S. Heuberger, and P. M. Van den Hof, “The behavioral approach to linear parameter-varying systems,” *IEEE Transactions on Automatic Control*, vol. 56, no. 11, pp. 2499–2514, 2011.

[42] J. C. Mayo-Maldonado, P. Rapisarda, and P. Rocha, “Stability of switched linear differential systems,” *IEEE Transactions on Automatic Control*, vol. 59, no. 8, pp. 2038–2051, 2014.

[43] J. C. Mayo-Maldonado and P. Rapisarda, "Dissipative switched linear differential systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 12, pp. 3813–3825, 2016.

[44] J. C. Willems, "Hidden variables in dissipative systems," in *Proceedings of the 43rd IEEE Conference on Decision and Control*, vol. 1. IEEE, 2004, pp. 358–363.

[45] J. C. Willems, P. Rapisarda, I. Markovsky, and B. L. De Moor, "A note on persistency of excitation," *Systems & Control Letters*, vol. 54, no. 4, pp. 325–329, 2005.

[46] I. Markovsky and P. Rapisarda, "Data-driven simulation and control," *International Journal of Control*, vol. 81, no. 12, pp. 1946–1959, 2008.

[47] I. Markovsky, J. C. Willems, P. Rapisarda, and B. L. De Moor, "Algorithms for deterministic balanced subspace identification," *Automatica*, vol. 41, no. 5, pp. 755–766, 2005.

[48] J. C. Willems, "Dissipative dynamical systems part I: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 321–351, 1972.

[49] R. E. Kalman, "Lyapunov functions for the problem of lur'e in automatic control," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 49, no. 2, p. 201, 1963.

[50] V. A. Yakubovich, "The solution of some matrix inequalities encountered in automatic control theory," in *Doklady Akademii Nauk*, vol. 143, no. 6. Russian Academy of Sciences, 1962, pp. 1304–1307.

[51] V. Popov, "On absolute stability of non-linear automatic control systems," *Automatika i Telemekhanika*, vol. 22, no. 8, pp. 961–979, 1961.

[52] C. Scherer and S. Weiland, "Linear matrix inequalities in control," *Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands*, vol. 3, no. 2, 2000.

[53] J. C. Willems, "Dissipative dynamical systems part II: Linear systems with quadratic supply rates," *Archive for Rational Mechanics and Analysis*, vol. 45, no. 5, pp. 352–393, 1972.

[54] D. Hill and P. Moylan, "Stability results for nonlinear feedback systems," *Automatica*, vol. 13, pp. 377–382, 1977.

[55] J. C. Willems and K. Takaba, "Dissipativity and stability of interconnections," *International Journal of Robust and Nonlinear Control*, vol. 17, no. 5-6, pp. 563–586, 2007.

[56] M. J. Tippett and J. Bao, "Control of plant-wide systems using dynamic supply rates," *Automatica*, vol. 50, no. 1, pp. 44–52, 2014.

[57] J. C. Willems and H. L. Trentelman, "On quadratic differential forms," *SIAM Journal on Control and Optimization*, vol. 36, no. 5, pp. 1703–1749, 1998.

[58] O. Kaneko and T. Fujii, "Discrete-time average positivity and spectral factorization in a behavioral framework," *Systems & Control Letters*, vol. 39, no. 1, pp. 31–44, 2000.

[59] C. Kojima and K. Takaba, "A generalized lyapunov stability theorem for discrete-time systems based on quadratic difference forms," *Transactions of*

*the Society of Instrument and Control Engineers*, vol. 42, no. 5, pp. 493–502, 2006.

[60] M. N. Belur and H. L. Trentelman, "Algorithmic issues in the synthesis of dissipative systems," *Mathematical and Computer Modelling of Dynamical Systems*, vol. 8, no. 4, pp. 407–428, 2002.

[61] K. Takaba, "Robust stability analysis via quadratic differential forms," *IFAC Proceedings Volumes*, vol. 40, no. 20, pp. 308–313, 2007.

[62] M. J. Tippett and J. Bao, "Dissipativity based distributed control synthesis," *Journal of Process Control*, vol. 23, no. 5, pp. 755–766, 2013.

[63] T. Maupong, J. C. Mayo-Maldonado, and P. Rapisarda, "On lyapunov functions and data-driven dissipativity," *IFAC-PapersOnLine*, vol. 50, no. 1, pp. 7783–7788, 2017.

[64] A. Romer, J. Berberich, J. Köhler, and F. Allgöwer, "One-shot verification of dissipativity properties from input–output data," *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 709–714, 2019.

[65] I. Pendharkar and H. K. Pillai, "A parametrization for dissipative behaviors," *Systems & Control Letters*, vol. 51, no. 2, pp. 123–132, 2004.

[66] ——, "A parametrisation for dissipative behaviours - the matrix case," *International Journal of Control*, vol. 82, no. 6, pp. 1006–1017, 2009.

[67] H. Kwakernaak and M. Sebek, "Polynomial J-spectral factorization," *IEEE Transactions on Automatic Control*, vol. 39, no. 2, pp. 315–328, 1994.

[68] H. L. Trentelman and P. Rapisarda, "New algorithms for polynomial J-spectral factorization," *Mathematics of Control, Signals and Systems*, vol. 12, no. 1, pp. 24–61, 1999.

[69] D. J. Clements, "A state-space approach to indefinite spectral factorization," *SIAM Journal on Matrix Analysis and Applications*, vol. 21, no. 3, pp. 743–767, 2000.

[70] H. L. Trentelman and J. C. Willems, "$\mathcal{H}_\infty$ control in a behavioral context: the full information case," *IEEE Transactions on Automatic Control*, vol. 44, no. 3, pp. 521–536, 1999.

[71] P. Moylan and D. Hill, "Stability criteria for large-scale systems," *IEEE Transactions on Automatic Control*, vol. 23, no. 2, pp. 143–149, 1978.

[72] H. L. Trentelman and J. C. Willems, "Synthesis of dissipative systems using quadratic differential forms: Part II," *IEEE Transactions on Automatic Control*, vol. 47, no. 1, pp. 70–86, 2002.

[73] J. C. Willems and H. L. Trentelman, "Synthesis of dissipative systems using quadratic differential forms: Part I," *IEEE Transactions on Automatic Control*, vol. 47, no. 1, pp. 53–69, 2002.

[74] S. Xu and J. Bao, "Distributed control of plantwide chemical processes," *Journal of Process Control*, vol. 19, no. 10, pp. 1671–1687, 2009.

[75] ——, "Control of chemical processes via output feedback controller networks," *Industrial & Engineering Chemistry Research*, vol. 49, no. 16, pp. 7421–7445, 2010.

[76] R. D'Andrea and G. E. Dullerud, "Distributed control design for spatially interconnected systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 9, pp. 1478–1495, 2003.

[77] B. T. Stewart, A. N. Venkat, J. B. Rawlings, S. J. Wright, and G. Pannoc-

chia, "Cooperative distributed model predictive control," *Systems & Control Letters*, vol. 59, no. 8, pp. 460–469, 2010.

[78] B. T. Stewart, S. J. Wright, and J. B. Rawlings, "Cooperative distributed model predictive control for nonlinear systems," *Journal of Process Control*, vol. 21, no. 5, pp. 698–704, 2011.

[79] M. A. Müller, M. Reble, and F. Allgöwer, "Cooperative control of dynamically decoupled systems via distributed model predictive control," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 12, pp. 1376–1397, 2012.

[80] J. Löfberg, "YALMIP: A toolbox for modeling and optimization in MATLAB," in *Proceedings of the CACSD Conference*, vol. 3. IEEE, 2004.

[81] M. J. Tippett and J. Bao, "Distributed model predictive control based on dissipativity," *AIChE Journal*, vol. 59, no. 3, pp. 787–804, 2013.

[82] C. Zheng, M. J. Tippett, J. Bao, and J. Liu, "Multirate dissipativity-based distributed mpc," in *Proceedings of the Australian Control Conference*. IEEE, 2013, pp. 325–330.

[83] ——, "Dissipativity-based distributed model predictive control with low rate communication," *AIChE Journal*, vol. 61, no. 10, pp. 3288–3303, 2015.

[84] C. Langbort and V. Gupta, "Minimal interconnection topology in distributed control design," *SIAM Journal on Control and Optimization*, vol. 48, no. 1, pp. 397–413, 2009.

[85] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 301–320, 1981.

[86] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems," *IEEE Transactions on Automatic Control*, vol. 34, no. 8, pp. 831–847, 1989.

[87] B. A. Francis, "A course in $\mathcal{H}_\infty$ control theory," 1987.

[88] I. R. Petersen and R. Tempo, "Robust control of uncertain systems: Classical results and recent developments," *Automatica*, vol. 50, no. 5, pp. 1315–1335, 2014.

[89] M. Green and D. J. N. Limebeer, *Linear Robust Control*. Prentice Hall, New Jersey, 1995.

[90] I. R. Petersen and C. V. Hollot, "A Riccati equation approach to the stabilization of uncertain linear systems," *Automatica*, vol. 22, no. 4, pp. 397–411, 1986.

[91] S. Skogestad and I. Postlethwaite, *Multivariable feedback control: analysis and design*. Wiley New York, 2007, vol. 2.

[92] H. L. Trentelman and S. V. Gottimukkala, "Distance between behaviors and rational representations," *SIAM Journal on Control and Optimization*, vol. 51, no. 6, pp. 4211–4241, 2013.

[93] Y. Ebihara, D. Peaucelle, D. Arzelier, and T. Hagiwara, "Robust $\mathcal{H}_2$ performance analysis of uncertain LTI systems via polynomially parameter-dependent lyapunov functions," *IFAC Proceedings Volumes*, vol. 39, no. 9, pp. 435–440, 2006.

[94] R. C. L. F. Oliveira, M. C. de Oliveira, and P. L. D. Peres, "LMI relaxations for robust $\mathcal{H}_2$ performance analysis of polytopic linear systems," in *Proceed-*

*ings of the 45th IEEE Conference on Decision and Control.* IEEE, 2006, pp. 2907–2912.

[95] J. Bao, P. L. Lee, F. Wang, W. Zhou, and Y. Samyudia, "A new approach to decentralised process control using passivity and sector stability conditions," *Chemical Engineering Communications*, vol. 182, pp. 213–237, 2000.

[96] Y. Samyudia and K. Kadiman, "Control design for recycled, multi unit processes," *Journal of Process Control*, vol. 13, no. 4, pp. 311–324, 2003.

[97] S. Skogestad and M. Morari, "Robust performance of decentralized control systems by independent designs," *Automatica*, vol. 25, no. 1, pp. 119–125, 1989.

[98] D. Peaucelle, D. Arzelier, O. Bachelier, and J. Bernussou, "A new robust $\mathcal{D}$-stability condition for real convex polytopic uncertainty," *Systems & Control Letters*, vol. 40, no. 1, pp. 21–30, 2000.

[99] Y. Wan and J. Cao, "Distributed robust stabilization of linear multi-agent systems with intermittent control," *Journal of the Franklin Institute*, vol. 352, no. 10, pp. 4515–4527, 2015.

[100] J. Wang, Z. Duan, G. Wen, and G. Chen, "Distributed robust control of uncertain linear multi-agent systems," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 13, pp. 2162–2179, 2015.

[101] Z. Li, Z. Duan, L. Xie, and X. Liu, "Distributed robust control of linear multi-agent systems with parameter uncertainties," *International Journal of Control*, vol. 85, no. 8, pp. 1039–1050, 2012.

[102] T. Maupong and P. Rapisarda, "Data-driven control: A behavioral approach," *Systems & Control Letters*, vol. 101, pp. 37–43, 2017.

[103] R. Tóth, C. Lyzell, M. Enqvist, P. S. Heuberger, and P. M. Van den Hof, "Order and structural dependence selection of LPV-ARX models using a nonnegative garrote approach," in *Proceedings of the 48th IEEE Conference on Decision and Control.* IEEE, 2009, pp. 7406–7411.

[104] A. Tang, J. Bao, and M. Skyllas-Kazacos, "Thermal modelling of battery configuration and self-discharge reactions in vanadium redox flow battery," *Journal of Power Sources*, vol. 216, pp. 489–501, 2012.

[105] C. Ebenbauer and F. Allgöwer, "Analysis and design of polynomial control systems using dissipation inequalities and sum of squares," *Computers & Chemical Engineering*, vol. 30, no. 10-12, pp. 1590–1602, 2006.

[106] A. Packard, U. Topcu, P. Seiler, and G. Balas, "Help on SOS," *IEEE Control Systems Magazine*, vol. 30, no. 4, pp. 18–23, 2010.

[107] A. Papachristodoulou, J. Anderson, G. Valmorbida, S. Prajna, P. Seiler, and P. Parrilo, "SOSTOOLS version 3.00 sum of squares optimization toolbox for MATLAB," *arXiv preprint arXiv:1310.4716*, 2013.

[108] J. Löfberg, "Automatic robust convex programming," *Optimization Methods and Software*, vol. 27, no. 1, pp. 115–129, 2012.

[109] G. C. Calafiore, "Random convex programs," *SIAM Journal on Optimization*, vol. 20, no. 6, pp. 3427–3464, 2010.

[110] G. C. Calafiore and M. C. Campi, "The scenario approach to robust control design," *IEEE Transactions on Automatic Control*, vol. 51, no. 5, pp. 742–753, 2006.

[111] R. Tempo, G. Calafiore, and F. Dabbene, *Randomized algorithms for anal-*

*ysis and control of uncertain systems: with applications.* Springer Science & Business Media, 2012.

[112] P. A. Parrilo, "Structured semidefinite programs and semialgebraic geometry methods in robustness and optimization," Ph.D. dissertation, California Institute of Technology, Pasadena, CA, 2000.

[113] J. Löfberg, "Pre- and post-processing sum-of-squares programs in practice," *IEEE Transactions on Automatic Control*, vol. 54, no. 5, pp. 1007–1011, 2009.

[114] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones," *Optimization Methods and Software*, vol. 11, no. 1-4, pp. 625–653, 1999.

[115] Y. Fujisaki, F. Dabbene, and R. Tempo, "Probabilistic design of lpv control systems," *Automatica*, vol. 39, no. 8, pp. 1323–1337, 2003.

[116] G. Boole, *An investigation of the laws of thought: on which are founded the mathematical theories of logic and probabilities.* Dover Publications, 1854.

[117] A. Clark, *Elements of abstract algebra.* Courier Corporation, 1984.

[118] T. Jech, *Set theory.* Springer Science & Business Media, 2013.

[119] T. S. Blyth, *Set theory and abstract algebra.* Longman Inc., New York, 1975.

[120] T. Imieliński and W. Lipski Jr, "The relational model of data and cylindric algebras," *Journal of Computer and System Sciences*, vol. 28, no. 1, pp. 80–102, 1984.

[121] R. E. Skelton, T. Iwasaki, and D. E. Grigoriadis, *A unified algebraic approach to control design.* CRC Press, 1997.

[122] W. Li, G. Wei, D. W. C. Ho, and D. Ding, "A weightedly uniform detectability for sensor networks," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 29, no. 11, pp. 5790–5796, 2018.

[123] R. E. Kalman, "A new approach to linear filtering and prediction problems," *ASME Journal of Basic Engineering*, vol. 82, no. 1, pp. 35–45, 1960.