On aspects of Ramsey theory

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ON ASPECTS OF RAMSEY THEORY

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This thesis presents various types of results from Ramsey Theory, most particularly, Ramsey-type theorems concerning graphs and families of sets.

This thesis consists of 8 chapters. In Chapter 1, we give a brief historical introduction to Ramsey Theory. Then, we introduce some necessary notation and definitions that will be consistently used throughout the thesis, including some basic knowledge of Graph Theory which is particularly useful in Chapters 2 and 3.

We present Ramsey-type results about graphs in Chapters 2 and 3. In Chapter 2, we introduce the classical Ramsey's Theorem which is the Ramsey-type theorem on the edge-colouring of the complete graph. We also introduce Ramsey numbers and present some results on these, especially some upper and lower bounds. In Chapter 3, we look at Ramsey-type results for monochromatic tree graphs, cycle graphs and bipartite graphs, respectively, occurring in arbitrary edge colourings of the complete graph. Then, we present the bipartite version of Ramsey's Theorem.

Chapters 4, 5 and 6 present other famous Ramsey-type theorems, for arithmetic progressions and other, more general, structures. In Chapter 4, we introduce and prove Van der Waerden's Theorem and we also present some results on the bounds of the Van der Waerden numbers. In Chapter 5, we present Schur's Theorem and some results relating to the Schur numbers. Then, we look into some generalisations of Schur's Theorem, including Rado's Theorem and Folkman's Theorem. In Chapter 6, we prove the Hales-Jewett Theorem. We also construct a proof of Van der Waerden's Theorem by using the Hales-Jewett Theorem.

Before we end our studies, in Chapter 7, we include some application of the Ramsey Theory. We look into the application of the Ramsey Theory in various fields, including graph theory, geometry and number theory. In Chapter 8, we conclude our studies. We give some overall comment on Ramsey Theory and include some possible future work on the field.

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Abstract

This thesis presents a representative spread of results from Ramsey Theory, most particularly, Ramsey-type theorems concerning graphs, families of sets and the integers.

This thesis consists of 8 chapters. In Chapter 1, we give a brief historical introduction to Ramsey Theory. Then, we introduce some necessary notation and definitions that will be consistently used throughout the thesis, including some basic knowledge of Graph Theory which is particularly useful in Chapters 2 and 3.

We present Ramsey-type results about graphs Chapters 2 and 3. In Chapter 2, we introduce the classical Ramsey's Theorem which is the Ramsey-type theorem on the colouring of the complete graph. We also introduce Ramsey numbers and present some results on these, especially some upper and lower bounds. In Chapter 3, we look at Ramsey-type results for monochromatic tree graphs, cycle graphs and bipartite graphs, respectively, occurring in arbitrary edge colourings of the complete graph. Then, we present the bipartite version of Ramsey's Theorem.

Chapters 4, 5 and 6 present other famous Ramsey-type theorems, for arithmetic progressions and other, more general, structures. In Chapter 4, we introduce and prove Van der Waerden's Theorem and we also present some results on the bounds of the Van der Waerden numbers. In Chapter 5, we present Schur's Theorem and some results relating to the Schur numbers. Then, we look into some generalisations of Schur's Theorem, including Rado's Theorem and Folkman's Theorem. In Chapter 6, we prove the Hales-Jewett Theorem. We also construct a proof of Van der Waerden's Theorem by using the Hales-Jewett Theorem.

Before we end our studies, in Chapter 7, we include some applications of Ramsey Theory to Graph Theory, Geometry and Number Theory. In Chapter 8, we conclude our studies with overall comments on Ramsey Theory and possible future work in this field.

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List of Our Results

Listed above are the results that we contributed independently to.

The following theorems were discovered and proved independently and could not be found in previous literature:

Theorems 2.16–2.20, 3.5, 3.19–3.20, 4.14–4.16, 5.3–5.4, and 5.7.

The following theorems were discovered and proved independently but were later found in previous literature:

Theorems 3.7, 3.9, and 5.8–5.12.

The following theorems were found in the literature but we independently filled in missing details of their proofs and adapted and modified the proofs in order to provide better clarity of argument and notation:

Theorems 3.13–3.15.

Chapter 1

Introduction

1.1 Historical Background and Introduction

Ramsey Theory is a beautiful but difficult subject that, generally speaking, shows how, in certain orderly structures, patterns and order can never be completely eradicated by randomness or disarray. A typical result in Ramsey Theory states that *if some mathematical structure is cut into pieces, then at least one of the parts must have a given interesting property.* There are many interesting applications of Ramsey Theory, including the results in number theory, algebra, geometry, topology, set theory, logic and set theory [86].

Ramsey Theory is named after the British mathematician and philosopher Frank Plumpton Ramsey, who did seminal work in this area before his death at the age of 26 in 1930. However, the theory was brought to public attention by Paul Erdős, a Hungarian mathematician who made enormous contributions to the fields of combinatorics and graph theory. He contributed much to Ramsey Theory, especially on Ramsey's Theorem for complete graphs which states that *in any sufficiently large finitely coloured complete graph, one can find some large monochromatic substructure.* In the language of graph theory, Ramsey number $R(m, n)$ is the minimum number of vertices to ensure that a simple undirected graph with that number of vertices contains either a complete graph of order m or an independent set of size n. The first lower bound on Ramsey numbers were obtained by Paul Erdős using probabilistic methods [22]. Together with George Szekeres, Paul Erdős also found some upper bounds on these numbers [25].

One of the key theorems of Ramsey Theory is a result on arithmetic progressions, Van der Waerden's Theorem from 1927. This theorem is named after the Dutch mathematician Bartel Leendert Van der Waerden. Van der Waerden's Theorem states that *for every positive integer k, there exists a positive integer n such that if the set $\{1, 2, \ldots, n\}$ is partitioned into two subsets, then at least one of the subsets must contain an arithmetic progression of length k* [102]. This theorem is further proven by Ron Graham and B. L. Rothschild [44]. Terence Tao also constructed a topological proof of Van der Waerden's Theorem in 2008 [101].

Another result that is similar to Van der Waerden's Theorem is Schur's Theorem from 1916. This is a Ramsey-type result on integer solutions to equations and was proved by Issai Schur. The theorem states that *in any finite colouring of the natural numbers, there must be a pair of integers x and y, such that x, y and $x+y$ are all the same colour* [92]. This basic result was generalised by Richard Rado in 1933 to give a characterisation of the homogeneous system in which a monochromatic solution can be found in any finite colouring of the natural numbers [81]. The theorems of

Schur, Rado, Ramsey and Van der Waerden are considered to be central results of Ramsey Theory.

Another key theorem of Ramsey Theory is geometrical, namely the Hales-Jewett Theorem. It is a fundamental combinatorial result of Ramsey Theory named after Alfred W. Hales and Robert I. Jewett which states that *for $k, r \in \mathbb{N}$, if n is sufficiently large, then for any r-colouring of a cube $C_k^n = \{(x_1, \ldots, x_n) : x_i \in [0, k-1]\}$, there is a monochromatic line* [50]. Informally speaking, Hales-Jewett Theorem states that *for any positive integers n and c, there is a number H such that if the cells of an H-dimensional $n \times n \times \cdots \times n$ cube are coloured with c colours, there must be one row, column, or certain diagonal of length n, all of whose cells are the same colour.* Hales and Jewett showed that if the dimension is large enough, then one can always find an n-in-a-row tic-tac-toe that never ends in a tie [50].

1.2 Preliminaries and Definitions

In this section, we will introduce some definitions and theorems which will be frequently referred to throughout this thesis.

1.2.1 Notation

$$\begin{aligned}
\mathbb{N} &: \text{ The set of natural numbers } \{1, 2, \ldots\}. \\
[n] &: \text{ The set } \{1, 2, \ldots, n\}. \\
[m, n] &: \text{ The set } \{m, m+1, \ldots, n\}. \\
\binom{X}{k} &: \text{ The family } \{Y \subseteq X : |Y| = k\}. \\
|X| &: \text{ The cardinality of } X.
\end{aligned}$$

1.2.2 The Pigeonhole Principle

One of the basic tools used in Ramsey Theory is the *Pigeonhole Principle.* It was first formulated in 1834, by the German mathematician Peter Gustav Lejeune Dirichlet.

Theorem 1.1 (The Pigeonhole Principle)**.** [108] *If $n+1$ objects are put into n boxes, then at least one box contains two or more of the objects.*

Theorem 1.2 (Stronger Form of the Pigeonhole Principle)**.** [108]
Let $q_1, q_2, \ldots, q_n$ be positive integers. If $q_1 + q_2 + \cdots + q_n - n + 1$ objects are put into n boxes, then, for some $i \in [n]$, there are at least q_i objects in the i^{th} box.

Proof. Suppose the contrary, namely that the i^{th} box has at most $i - 1$ objects, for each $i = 1, 2, \ldots, n$. Then the total number of objects contained in the n boxes is $(q_1 - 1) + (q_2 - 1) + \cdots + (q_n - 1) = q_1 + q_2 + \cdots + q_n - n$, which is less than the number of objects allocated. This is a contradiction. □

Corollary 1.3. [108] *If $n(r-1)+1$ objects are put into n boxes, then at least one of the boxes will contains r or more objects.*

Proof. It follows from the stronger form of the Pigeonhole Principle for the special case $q_1 = q_2 = \cdots = q_n = r$. □

1.2.3 Graph Theory

In Chapters 2 and 3, we will present the graph theory results from Ramsey Theory. Here, we introduce some graph theory definitions that will be used in those chapters.

Definition 1.4 (Graph)**.** *A* graph *G is a pair of sets $(V(G), E(G))$ where $V(G)$ is a finite non-empty set of elements called* vertices *and $E(G)$ is a set of unordered pairs of elements of $V(G)$ called* edges*.*

Figure 1.1 shows a graph G with the vertex set $\{s, t, u, v, w\}$ and the edge set $\big\{\{s, t\}, \{t, u\}, \{t, w\}, \{u, v\}, \{v, w\}\big\}$.

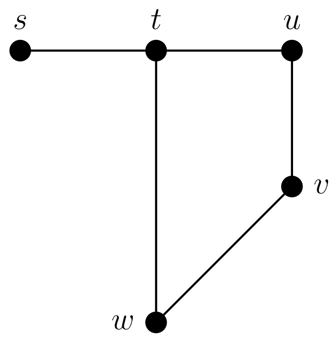
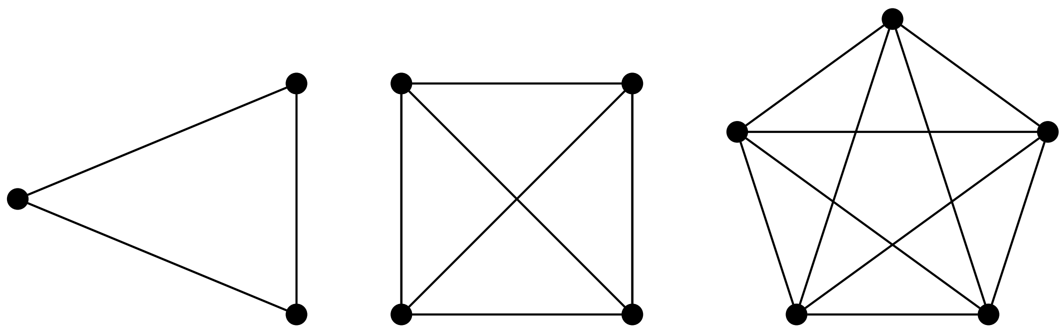


Figure 1.1: A graph G

Definition 1.5 (Complete graph)**.** *Two vertices u and v of the graph are said to be* adjacent *if they are joined by an edge e. In this case, e is* incident *to u and v. A graph in which every two vertices are adjacent to each other is called a* complete graph*. A complete graph with n vertices is denoted by K_n.*

Figure 1.2 shows some examples of complete graphs, namely K_3, K_4, and K_5.



(a) K_3 (b) K_4 (c) K_5

Figure 1.2: Complete graphs

Definition 1.6 (Degree of a vertex)**.** *The* degree *of a vertex in a graph is the number of edges incident to it. A graph where all its vertices have the same degree is known as a* regular graph*.*

Figure 1.3 shows a graph with each of the vertices labelled by their degree.

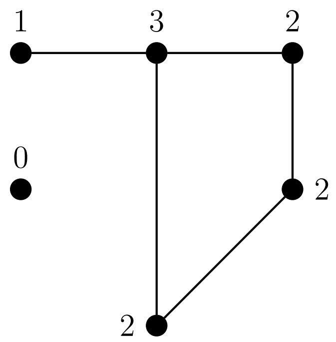
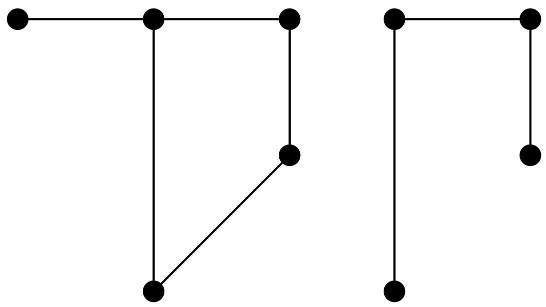


Figure 1.3: Graph G with vertices labelled by their degree

Definition 1.7 (Subgraph)**.** *A graph H is a* subgraph *of G if $V(S) \subseteq V(G)$ and $E(S) \subseteq E(G)$.*

Figure 1.4 shows an example of a subgraph H of a graph G.

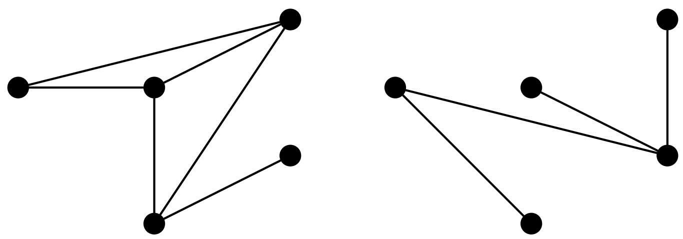


(a) G (b) H

Figure 1.4: H is a subgraph of G

Definition 1.8 (Complement of a graph)**.** *Let G be a graph with n vertices. The* complement *of G, denoted by $\overline{G}$, is the graph with vertices $V(\overline{G}) = V(G)$ and edges $E(\overline{G}) = E(K_n) - E(G)$.*

Figure 1.5 shows a graph G and its complement $\overline{G}$.



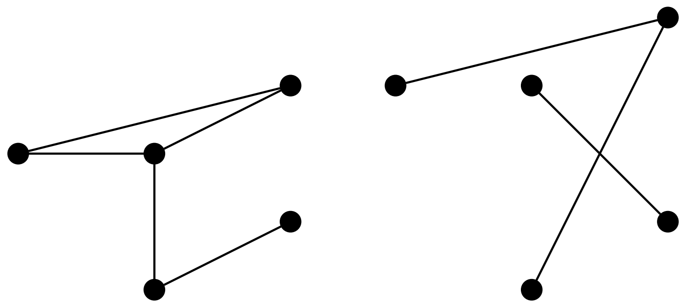
(a) G (b) $\overline{G}$

Figure 1.5: A graph G and its complement $\overline{G}$

Definition 1.9 (Walks, paths and cycles)**.** *A* walk *in a graph G is an alternating sequence of vertices and edges $v_0e_1v_1e_2v_2 \dots e_kv_k$ in which the ends of each edge e_i are v_{i-1} and v_i for $i \in [k]$. It is* closed *if $v_0 = v_k$ and is* open *otherwise. A walk in which all vertices $v_0, v_1, \dots, v_k$ are distinct is called a* path*. A closed path is called a* cycle*.*

Definition 1.10 (Connected graph)**.** *A graph G is* connected *if there exists a walk between each pair of vertices in G. If G is not connected, then it is* disconnected*.*

Figure 1.6 shows a connected graph G and a disconnected graph H.

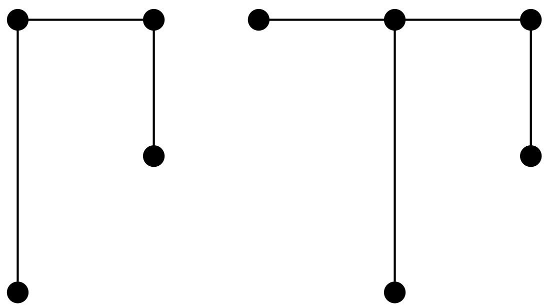


(a) G (b) H

Figure 1.6: A connected graph G and a disconnected graph H

Definition 1.11 (Trees)**.** *A* tree *is a connected graph which has no cycle subgraph.*

Figure 1.7 shows some examples of trees.

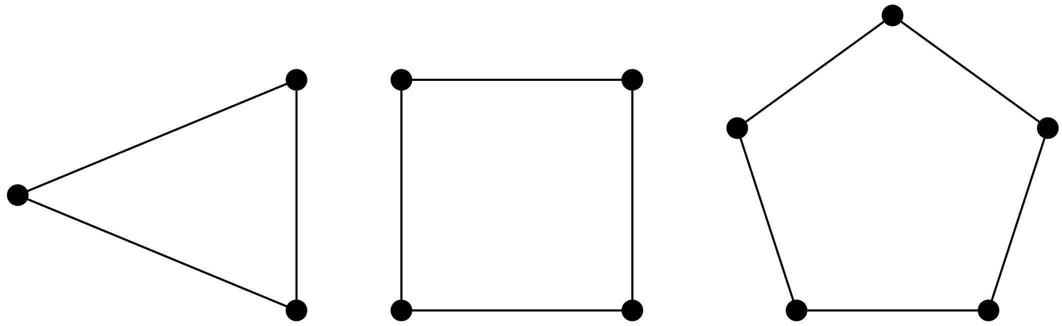


(a) T_4 (b) T_5

Figure 1.7: Trees T_4 and T_5

Definition 1.12 (Cycle graphs)**.** *A* cycle [graph] *is a graph that consists of a single cycle. A cycle with n vertices is denoted by C_n.*

Figure 1.8 shows cycle graphs C_3, C_4, and C_5.

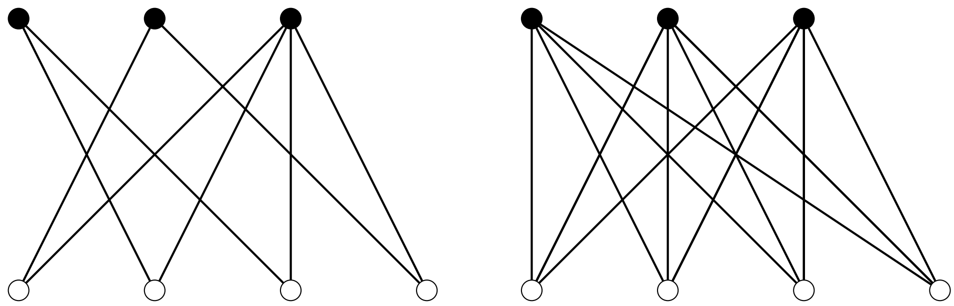


(a) C_3 (b) C_4 (c) C_5

Figure 1.8: Cycle graphs.

Definition 1.13 (Bipartite graph)**.** *A* bipartite graph *is a connected graph whose the vertex set can be partition into two disjoint subsets so that each edge joins a vertex from one subset to a vertex from the other subset. The vertices of a bipartite graph can be coloured black and white according to the subset in which they belong. A bipartite graph is* complete *if each vertex from one subset is adjacent to every vertex from another subset. A complete bipartite graph is denoted by K_{n_1,n_2} where n_1 and n_2 are the numbers of vertices in each subset, respectively.*

Figure 1.9 shows examples of bipartite graphs.



(a) A bipartite graph (b) A complete bipartite graph, $K_{3,4}$

Figure 1.9: Bipartite graphs

1.2.4 Main Theorems and Definitions

In this subsection, we list some of the main theorems and definitions that will be discussed further in this thesis.

Definition 1.14 (Colouring)**.** *A* colouring *is a type of labelling, assigning "colours" as the labels to elements of a mathematical structure such as a graph or some set under certain constraints.*

In this thesis, the mathematical structures that we colour are graphs (in Chapters 2 and 3) and various families defined via natural numbers (in Chapters 4, 5 and 6).

Example 1.15. *As mentioned in Definition 1.14, there are various type of colouring. Here, we give an example of the edge-colouring of a graph, which will be largely used in Chapters 2 and 3.*

Figure 1.10 shows an example of edge-colouring of a graph G. The edges $\{1, 5\}$, $\{1, 4\}$ and $\{2, 3\}$ are blue(———), whereas the edges $\{1, 2\}$ and $\{3, 4\}$ are red($---$). *Note: The dashed lines are used to differentiate two colours for the black and white printed version.*

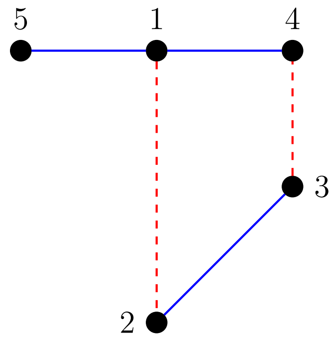


Figure 1.10: Graph G with blue-colour(———) and red-coloured($---$) edges.

Definition 1.16 (Monochromatic)**.** *A mathematical structure is* monochromatic *if all of its elements are of the same single colour.*

Theorem 1.17 (Ramsey's Theorem for 2-colouring of the Edges of the Graph)**.** [82] *Let m_1 and $m_2 \in \mathbb{N}$. There exists an integer $N \in \mathbb{N}$ such that in every edge-colouring of K_N with the colours c_1 and c_2, there is either a c_1-monochromatic K_{m_1} subgraph or a c_2-monochromatic K_{m_2} subgraph. The least such N is known as the* Ramsey number *$R(m_1, m_2)$.*

Example 1.18 (Party Problem)**.** One of the typical results in Ramsey's Theorem is the Party Problem. In the Party Problem, we are asked to find the minimum number of guests to be invited to ensure that at least m of them know each other or at least n of them do not know each other. This problem is equivalent to Ramsey's Theorem for two colours.

Theorem 1.19 (Ramsey's Theorem for r-colouring of the Edges of the Graph)**.** [82] *If $r, m \in \mathbb{N}$ and n is sufficiently large, then each r-colouring of the edges of K_n gives a complete subgraph K_m with monochromatic edges.*

Theorem 1.20 (Ramsey's Theorem)**.** [82] *If $m_1, m_2, \ldots, m_r, k \in \mathbb{N}$ and n is sufficiently large, then for each colouring of $\binom{[n]}{k}$ with colours $c_1, c_2, \ldots, c_r$, there is an m_i-subset $S \subseteq [n]$ such that the subfamily $\binom{S}{k}$ is coloured c_i for some $i \in [r]$. The least such n is denoted by $R_k(m_1, m_2, \ldots, m_r)$.*

In Chapter 4, we will present *Van der Waerden's Theorem*. Here, we introduce some definitions that will be used.

Definition 1.21 (Arithmetic Progression)**.** *An* arithmetic progression *is a sequence of numbers such that the differences between consecutive terms is constant.*

Theorem 1.22 (Van der Waerden's Theorem)**.** [102] *If $k, r \in (N)$, and N is sufficiently large, then each r-colouring of $[N]$ gives a monochromatic arithmetic progression of length k. The least such N is known as the* Van der Waerden number *$W(k, r)$.*

In Chapter 5, we will introduce *Schur's Theorem*. Here, we give some main theorems related to it that will be used in that chapter.

Theorem 1.23 (Schur's Theorem)**.** [92] *Let $r \in \mathbb{N}$. If $\mathbb{N}$ is r-coloured, then there are some same-coloured $a, b, c \in \mathbb{N}$ such that $a + b = c$.*

Theorem 1.24 (Schur's Theorem (finite))**.** [92] *Let $r \in \mathbb{N}$ and N is sufficiently large, then for any r-colouring of $[N]$, there are some same-coloured $a, b, c \in [N]$ such that $a + b = c$. The least such N is known as the* Schur number *$S(r)$.*

In Chapter 6, we will discuss the Hales-Jewett Theorem. Now, we introduce the main definition and theorem that will be used in the chapter.

Definition 1.25 (n-cube over t elements)**.** *We define the n-cube over t elements by*

$$C_k^n = \{(x_1, \ldots, x_n) : x_i \in [0, t-1]\} .$$

Definition 1.26 (Line)**.** *A line in C_k^n is a set of points $x_0, \ldots, x_{k-1}$, where $x_i = (x_{i1}, \ldots, x_{in})$ so that in each coordinate $j \in [n]$, either*

$$x_{0j} = \cdots = x_{k-1,j}$$

or

$$x_{sj} = s\,, \text{ where } s \in [0, k-1]\,, \text{ for some } j\,.$$

Theorem 1.27 (Hales-Jewett Theorem)**.** [50] *For all $r, t \in \mathbb{N}$ and N is sufficiently large, if the vertices of C_t^N are r-coloured, then there exists a monochromatic line.*

Chapter 2
Ramsey's Theorem

In this chapter, we present the main theorem in Ramsey Theory, which is Ramsey's Theorem, first proved by Flank Plumpton Ramsey in 1930 [82]. In Section 2.1, we will first present and prove a special case of it, namely that for the edge colouring of complete graphs. In doing so, we introduce the Ramsey number terminology. In Section 2.2, we then introduce and prove Ramsey's Theorem in full. In Section 2.3, we will also present some results and theorems on Ramsey numbers, including some known Ramsey numbers and bounds on them in general.

2.1 Ramsey's Theorem for Edge-Colouring a Graph

In this section, we state Ramsey's Theorem on the edge-colouring of the complete graph and we construct a proof of the theorem by induction. We also introduce the Ramsey Number terminology which is particularly useful in the studies of Ramsey Theory.

Theorem 2.1 (Ramsey's Theorem for 2-colouring the Complete Graph)**.** [82]
Let m_1 and $m_2 \in \mathbb{N}$. There exists an integer $N \in \mathbb{N}$ such that in every edge-colouring of K_N with the colours c_1 and c_2, there is either a c_1-monochromatic K_{m_1} subgraph or a c_2-monochromatic K_{m_2} subgraph.
The least such N is known as the Ramsey number $R(m_1, m_2)$.

To prove Theorem 2.1, we first prove some auxiliary lemmas.

Lemma 2.2. [82]
(1) $R(m, 1) = 1 = R(1, m)$
(2) $R(m, 2) = m = R(2, m)$

Proof.
(1) The graph K_1 which only has a single vertex is trivially monochromatic.
(2) Suppose we colour all of the edges of K_m with the colours c_1 and c_2. Then either there is a c_1-coloured K_2 (just a single edge) or else all the edges are c_2-coloured, forming K_m, or vice versa. □

Note that Theorem 2.1 is proven if we can show that the Ramsey Number $R(m_1, m_2)$ exists for all $m_1, m_2 \in \mathbb{N}$. Such a result has been proven by P. Erdős and G. Szekeres in 1935, as follows.

Lemma 2.3. [25] *For all $m_1, m_2 \geq 2$, $R(m_1, m_2) \leq R(m_1 - 1, m_2) + R(m_1, m_2 - 1)$.*

Proof. Let v be any vertex of $K_{R(m_1-1,m_2)+R(m_1,m_2-1)}$. Partition the remaining $R(m_1 - 1, m_2) + R(m_1, m_2 - 1) - 1$ vertices into two sets M_1 and M_2, in such the way that for every vertex w, w is in M_1 if the edge $\{v, w\}$ is coloured with c_1 and M_2

otherwise. Note that either $|M_1| \geq R(m_1-1, m_2)$ or $|M_2| \geq R(m_1, m_2-1)$ because otherwise $|M_1|+|M_2| \leq R(m_1-1, m_2)-1+R(m_1, m_2-1)-1$ which is impossible.

If $|M_1| \geq R(m_1-1, m_2)$, then we either have a c_1-coloured subgraph K_{m_1-1} or a c_2-coloured K_{m_2}. For the latter case, we are done. Suppose we have a c_1-coloured subgraph K_{m_1-1}. Then take the subgraph with the vertex v and all the c_1-coloured edges between them, we will get a c_1-coloured subgraph K_{m_1}.

Similarly, if $|M_2| \geq R(m_1, m_2-1)$, then we either have a c_1-coloured subgraph K_{m_1}, in which a case the theorem is proven, or else we have a c_2-coloured subgraph K_{m_2-1}. This subgraph together with the vertex v and all the c_2-coloured edges between them will form a c_2-coloured subgraph K_{m_2}. In all cases, we either have a c_1-coloured subgraph K_{m_1} or a c_2-coloured subgraph K_{m_2}. □

Example 2.4. Any 2-colouring of the complete graph K_6 will give us a monochromatic K_3 subgraph. Furthermore, if we colour the complete graph K_5 as in Figure 2.2 (Section 2.3), then no monochromatic K_3 subgraph can be found. Hence, we can conclude that $R(3,3)=6$. A detailed proof of $R(3,3)=6$ will be given in Section 2.3. Figure 2.1 shows an example of monochromatic K_3 subgraph in a c_1(———) and c_2(− − −) colouring of K_6.

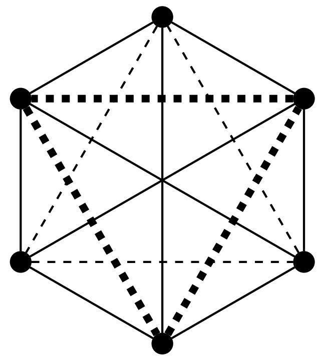


Figure 2.1: Monochromatic c_2-coloured K_3 in 2-colouring of K_6

Theorem 2.5 (Ramsey's Theorem for r-colouring of the Complete Graph). [82] *If $r, m \in \mathbb{N}$, and n is sufficiently large, then each r-colouring of the edges of K_n gives a complete subgraph K_m with monochromatic edges.*

Proof. We prove by induction on r.

For $r=1$, it is clear that we can always take any $n \geq m$ and we can find a complete subgraph K_m with monochromatic edges. Suppose that the theorem is valid for $r-1$ colours. Now, we consider the r-colouring case. Colour each edge of K_n in colours $c_1, c_2, \ldots, c_r$. Recolour each c_{r-1}-coloured and c_r-coloured edges with a new colour $c_{r-1'}$. From the induction hypothesis, for a big enough n, we can get a subgraph $K_{R(m,m)}$ with monochromatic edges. If the edges of this subgraph $K_{R(m,m)}$ is coloured with c_i for some $i \in [r-2]$, then we can always get a c_i-coloured K_m subgraph, and we are done. Suppose that $K_{R(m,m)}$ is coloured with $c_{r-1'}$. Since the edges of this subgraph is originally coloured with colour c_{r-1} and c_r, thus, by the definition of $R(m,m)$, we can always get a c_{r-1}-coloured or c_r-coloured K_m, in which we are done in either case.

Hence by induction, the theorem is proven. □

Definition 2.6. *The* Ramsey Number $R(m_1, m_2, \ldots, m_r)$ *is the least integer* N *such that for all* $n \geq N$*, if all the edges of* K_n *are* r*-coloured, then there is always a monochromatic* K_{m_i}*, for some* $i \in [r]$*.*

2.2 Ramsey's Theorem

In this section, we give the full version of Ramsey's Theorem and construct a proof for it.

Theorem 2.7 (Ramsey's Theorem)**.** [82]
If $m_1, m_2, \ldots, m_r, k \in \mathbb{N}$ *and* n *is sufficiently large, then for each colouring of* $\binom{[n]}{k}$ *with colours* $c_1, c_2, \ldots, c_r$*, there is an* m_i*-subset* $S \subseteq [n]$ *such that the subfamily* $\binom{S}{k}$ *is coloured* c_i *for some* $i \in [r]$*. The least such* n *is denoted by* $R_k(m_1, m_2, \ldots, m_r)$*.*

Proof. To prove the theorem, we first show that the theorem holds for 2 colours. Then, by induction on r, we prove that the theorem is valid for any r-colouring. For 2-colouring, we need to show the existence of $R_k(m_1, m_2)$, for all $k, m_1, m_2 \in \mathbb{N}$. Note that the case $k = 2$ is none other than Theorem 2.1. Further notice that, if $m_i = k$, for some $i = 1, 2$, $R_k(m_1, m_2) = R_k(m_2, m_1) = k$ is trivial because in any 2-colouring of $\binom{k}{k}$, we will have a k-subset S such that the subfamily $\binom{S}{k}$ is monochromatic. Now, suppose that $R_{k-1}(m_1, m_2)$,$R_k(m_1 - 1, m_2)$ and $R_k(m_1, m_2 - 1)$ exist. We want to show the existence of $R_k(m_1, m_2)$.
Take $N = R_{k-1}(R_k(m_1 - 1, m_2), R_k(m_1, m_2 - 1)) + 1$ and consider the set $\binom{[N]}{k}$. Colour all the k-subsets with colours c_1 and c_2 and we denote this colouring as χ.

Now choose an element x and consider all $(k-1)$-subsets not containing the element x. We call this family S. Note that S is equivalent to the family $\binom{[N-1]}{k-1}$. Let S be 2-coloured by a c_1 and c_2, by a colouring χ^* induced in such a way that $\chi^*(T) = \chi(T \cup x)$, for all $T \in S$. By the induction hypothesis, we are guaranteed one of the following cases:

(1) S has a subset M, where $|M| = R_k(m_1 - 1, m_2)$ and all the $(k-1)$-subset of M is c_1-coloured.
(2) S has a subset N, where $|N| = R_k(m_1, m_2 - 1)$ and all the $(k-1)$-subset of N is c_2-coloured.

Suppose that Case 1 holds. Then by induction hypothesis, we assume that $R_k(m_1 - 1, m_2)$ exists. Therefore, M has either a subset M_1 with $m_1 - 1$ elements where all k-subset of M_1 are c_1-coloured or a subset M_2 with m_2 elements where all k-subset of M_2 are c_2-coloured. For the latter, we are done. Suppose that there is such a subset M_1, and consider $M^* = M_1 \cup x$, $|M^*| = m_1$. Take k-subset of M^* if the k-subset contains element x. Then it is c_1-coloured by the induced colour of $(k-1)$-subset of M. On the other hand, if the k-subset does not contain the element x, then it is actually a k-subset of M which is then c_1-coloured. Either way, we are done.

On the other hand, suppose that Case 2 holds. From the induction hypothesis, we assume that $R_k(m_1, m_2 - 1)$ exists. Thus N has either subset N_1 with m_1 elements where all k-subset of N_1 are c_1-coloured, where we are done; or else, subset N_2 with $m_2 - 1$ elements which all k-subset of N_2 are c_2-coloured. Suppose the latter, and consider $N^* = N_1 \cup x$, $|N^*| = m_2$. Take k-subset of M^*. If the k-subset contains the element x, then it is c_2-coloured by the induced colouring of $(k-1)$-subset of N. If the k-subset does not contain the element x, then it is actually the c_2-coloured k-subset of N. Then, we are done.

Now, we have shown $R_k(m_1, m_2)$ exists for all $m_1, m_2 \in \mathbb{N}$. By induction on r, we want to show the theorem holds for any r-colouring. Assume that $R_k(m_1, m_2, \ldots, m_{r-1})$ exists. Since the theorem holds for 2 colours, $R_k(m_{r-1}, m_r)$ exists. We take $N = R_k(m_1, m_2, \ldots, m_{r-2}, R_k(m_{r-1}, m_r))$. By the induction hypothesis, we either have a m_i-subset S of $\binom{[N]}{k}$ in which all k-subsets of S are c_i-coloured, for some $i \in [r-2]$, in which we are done. Otherwise, we have a $(R_k(m_{r-1}, m_r))$-subset S of $\binom{[N]}{k}$ in which all k-subset of S are c_{r-1}-coloured or c_r-coloured. By the definition of $R_k(m_{r-1}, m_r)$, we have a set $S_1 \subseteq S$ in which all the k-subset of S_1 are c_{r-1}-coloured or a set $S_2 \subseteq S$ where all the k-subset of S_2 are c_r-coloured. In either case, we are done.

Hence by induction, Ramsey's Theorem holds for all $m_1, m_2, \ldots, m_r, k, r \in \mathbb{N}$. $\square$

2.3 Ramsey Numbers

In this section, we present some known results, from old to recent, of the Ramsey numbers.

Example 2.8. In any group of 6 people, there are either 3 mutual friends or 3 mutual strangers.

This example is equivalent to the following statements:

(1) In any 2-colouring of K_6, there is a monochromatic K_3 subgraph.
(2) $R(3,3) \leq 6$.

Proof. Let A be one of the groups of six. The remaining 5 people fall into one of the two classes: F, a set of friends of A and S, a set of strangers to A. Now by the Pigeonhole Principle, one of the classes must have at least 3 people.

Case (i) : $|F| \geq 3$.
If F has 3 mutual strangers, then we are done. Otherwise, F has a pair of friends. This pair of friends together with A will form a group of 3 mutual friends.

Case (ii): $|S| \geq 3$.
If S has 3 mutual friends, then we are done. Otherwise, S has a pair of strangers. This pair of strangers together with A will form a group of 3 mutual strangers.

In all cases, we either have 3 mutual friends or else 3 mutual strangers. $\square$

From Example 2.8, we have shown that $R(3,3) \leq 6$. We need to show that $R(3,3) \geq 6$ to prove $R(3,3) = 6$. In doing so, we give a counterexample. Suppose it is possible to colour the edges of the graph K_n with the colour c_1 and c_2 so that K_n contains neither c_1-coloured K_{m_1} nor c_2-coloured K_{m_2}. Then we can conclude that $R(m_1, m_2) > n$.

Example 2.9. The figure below shows an example of K_5 coloured with colours c_1(———) and c_2(− − −) in such a way that there is no monochromatic K_3 subgraph.

With the construction of K_5 as shown in Figure 2.2, we have shown that $R(3,3) > 5$. Since $R(3,3)$ must be an integer, we can deduce that $R(3,3) \geq 6$.

Thus, we have shown that $R(3,3) = 6$.

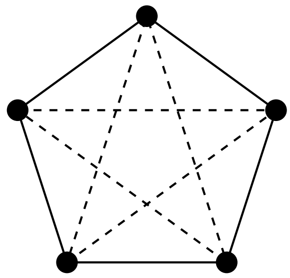


Figure 2.2: $R(3,3) > 5$

Theorem 2.10. [25] $R(m_1, m_2) \leq \binom{m_1+m_2-2}{m_1-1}$.

Proof. We prove the theorem by induction. Note that $R(1,1) = 1 \leq \binom{1+1-2}{1-1}$. For the induction hypothesis, assume that the theorem holds for $R(m_1 - 1, m_2)$ and $R(m_1, m_2 - 1)$. Now, we have $R(m_1 - 1, m_2) = \binom{m_1+m_2-3}{m_1-2}$ and $R(m_1, m_2 - 1) = \binom{m_1+m_2-3}{m_1-1}$. By Lemma 2.3, we have $R(m_1, m_2) \leq R(m_1 - 1, m_2) + R(m_1, m_2 - 1) \leq \binom{m_1+m_2-3}{m_1-2} + \binom{m_1+m_2-3}{m_1-1}$. By the recursive formula of the binomial coefficient, we get $R(m_1, m_2) \leq \binom{m_1+m_2-2}{m_1-1}$. Then, by induction, we are done. □

Under certain circumstances, Lemma 2.3 has been improved by R.E. Greenwood and A.M. Gleason in 1955, as follows.

Theorem 2.11. [48] *If both $R(m_1 - 1, m_2)$ and $R(m_1, m_2 - 1)$ are even, then $R(m_1, m_2) \leq R(m_1 - 1, m_2) + R(m_1, m_2 - 1) - 1$.*

Proof. Set $N := R(m_1 - 1, m_2) + R(m_1, m_2 - 1) - 1$ and colour the edges of K_N with colours c_1 and c_2. Select a vertex v and partition the remaining $N - 1$ vertices into two sets M_1 and M_2 in such the way that, for every vertex w, w is in M_1 if $\{v, w\}$ is coloured with c_1 and w is in M_2 otherwise. Then, one of the following cases will hold:

(1) $|M_1| = R(m_1 - 1, m_2) - 1$ and $|M_2| = R(m_1, m_2 - 1) - 1$
(2) $|M_1| \geq R(m_1 - 1, m_2)$
(3) $|M_2| \geq R(m_1, m_2 - 1)$

Assume that (1) is true for all vertices in K_N. Then, the c_1-coloured subgraph will contain $c := \frac{1}{2}N(R(m_1 - 1, m_2) - 1)$ c_1-coloured edges, a contradiction since since c is not an integer. Thus, (1) is not always true and we can therefore always choose a vertex v, so that either (2) or (3) holds.

Now, suppose that (2) holds. We have either a c_1-coloured subgraph K_{m_1-1} or a c_2-coloured K_{m_2}. For the latter, we are done. Suppose there is a c_1-coloured subgraph K_{m_1-1}. Then the subgraph K_{m_1-1} together with the vertex v, and all the c_1-coloured edges incident to them will form a c_1-coloured K_{m_1}.

Suppose that (3) holds. We have either a c_1-coloured subgraph K_{m_1} or a c_2-coloured K_{m_2-1}. If there is c_1-coloured subgraph K_{m_1}, we are done. Suppose the latter. Then the subgraph K_{m_2-1} together with the vertex v and all the c_2-coloured edges incident to them will form a c_2-coloured K_{m_2}. □

Example 2.12. Note that $R(2,4) = 4$ and $R(3,3) = 6$ are both even; therefore, Theorem 2.11 implies that $R(3,4) \leq R(2,4) + R(3,3) = 9$. Figure 2.3 shows a (c_1, c_2)-colouring of K_8 without a c_1-coloured (——) K_3 or a c_2-coloured $(- - -)$ K_4 as subgraph. We thus have $R(3,4) \geq 9$. Hence, $R(3,4) = 9$.

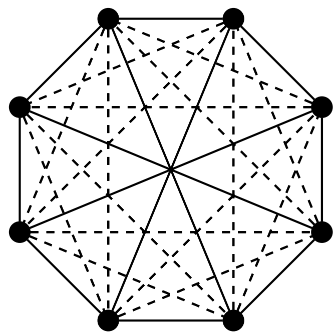


Figure 2.3: $R(3,4) > 9$.

Theorem 2.13. $R(m_1, m_2) = R(m_2, m_1)$.

Proof. By the definition of $R(m_2, m_1)$, $R(m_2, m_1)$ is the minimum number of vertices in the complete graph, such that in any edge-colouring of the complete graph $K_{R(m_2,m_1)}$ with colours c_1' and c_2', there is either a c_1'-coloured K_{m_2} or c_2'-coloured K_{m_1}. Now, consider that we recoloured every edges of the graph in such a way that colour c_1' will be replaced by colour c_2 and the colour c_2' will be replaced by colour c_1. Then, $R(m_2, m_1)$ will be indeed the minimum number of vertices in the complete graph, such that in any edge-colouring of the complete graph $K_{R(m_2,m_1)}$ with colours c_1 and c_2, there is either a c_1-coloured K_{m_1} or c_2-coloured K_{m_2}. Hence, $R(m_1, m_2) = R(m_2, m_1)$. □

Example 2.14. In Example 2.12, we have shown $R(3,4) = 9$. By Theorem 2.13, we get that $R(4,3) = 9$.

Theorem 2.15. [48]

(1) $R(3,5) = 14$.
(2) $R(4,4) = 18$.

Proof.

(1) We have $R(3,4) = 9$ by Example 2.12, and $R(2,5) = 5$ by Lemma 2.2. Hence by Lemma 2.3, we have $R(3,5) \leq 14$. Now, by the colouring of the complete graph K_{13} as shown in Figure 2.4, we get $R(3,5) > 13$, and hence, we can conclude that $R(3,5) = 14$.
(2) We have $R(3,4) = 9$ by Example 2.12. By Lemma 2.3, we have $R(4,4) \leq 18$. Now, by the colouring of the complete graph K_{17} shown in Figure 2.5, we get $R(4,4) > 17$, and hence, we can conclude that $R(4,4) = 18$.

□

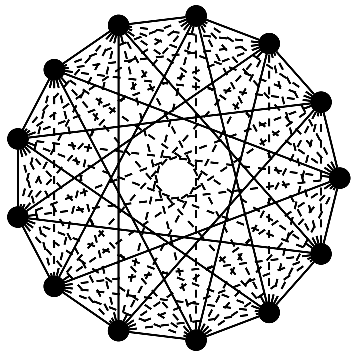


Figure 2.4: $R(3,5) > 13$.

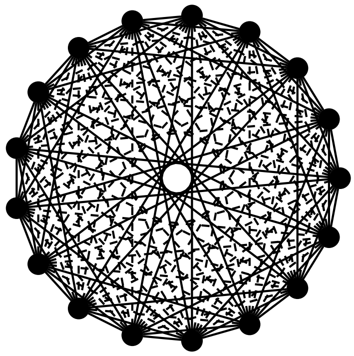


Figure 2.5: $R(4,4) > 18$.

Table 2.1 shows some of the known Ramsey numbers, with the references as cited in the table and previous discussion.

(m_1, m_2)	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9
3	1	3	6	9	14	18 [47]	23 [59]	28 [73]	36 [49]
4	1	4	9	18	25 [74]				
5	1	5	14	25					
6	1	6	18						
7	1	7	23						
8	1	8	28						
9	1	9	36						

Table 2.1: Known Ramsey numbers

These are some of the presently known Ramsey numbers. Finding bounds on Ramsey-type numbers is a major area of research in Ramsey Theory. We now prove some bounds on Ramsey numbers that we were able to find independently, and could not be found in previous literature, namely the bounds given in Theorems 2.16–2.20 below.

Theorem 2.16. $R(3,m) \leq \frac{(m)(m+1)}{2}$.

Proof. By Lemma 2.3, we have

$$\begin{aligned}
R(3,m) &\leq R(2,m) + R(3,m-1) \\
&\leq R(2,m) + R(2,m-1) + R(3,m-2) \\
&\quad\vdots \\
&\leq R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,2) + R(3,1) \\
&= m + (m-1) + (m-2) + \cdots + 4 + 3 + 2 + 1 \\
&= \frac{m(m+1)}{2}. \qquad \square
\end{aligned}$$

In fact, since $\frac{m(m+1)}{2} = \frac{m(m+1)(m-1)!}{2(m-1)!} = \frac{(m+1)!}{2![(m+1)-2]!} = \binom{m+1}{2}$, Theorem 2.16 coincides with Theorem 2.10. Further notice that,

$$\begin{aligned}
\frac{m(m+1)}{2} &= \frac{m^2+m}{2} \\
&= \frac{2m+m^2-m}{2} \\
&= m + \frac{m^2-m}{2} \\
&= m + \frac{m(m-1)}{2} \\
&= |V(K_m)| + |E(K_m)|
\end{aligned}$$

where $|V(K_m)|$ and $|E(K_m)|$ are the number of vertices and edges in K_m, respectively. This makes Theorem 2.16 a special case of a conjecture by Sidorenko [93]. Note also that $R(3,3), R(2,4), R(3,5), R(2,6), R(3,9)$ and $R(2,10)$ are all even, so by Theorem 2.11, we can further improve Theorem 2.16.

Theorem 2.17.

(1) *For* $m \geq 4, R(3,m) \leq \frac{(m)(m+1)}{2} - 1$.
(2) *For* $m \geq 6, R(3,m) \leq \frac{(m)(m+1)}{2} - 2$.
(3) *For* $m \geq 10, R(3,m) \leq \frac{(m)(m+1)}{2} - 3$.

Proof.

(1)
$$\begin{aligned}
R(3,m) &\leq R(2,m) + R(3,m-1) \\
&\leq R(2,m) + R(2,m-1) + R(3,m-2) \\
&\quad\vdots \\
&\leq R(2,m) + R(2,m-1) + R(2,m-2) + \cdots + R(2,4) \\
&\quad + R(3,3) - 1 \\
&\leq m + (m-1) + (m-2) + \cdots + R(2,4) + R(2,3) + R(2,3) - 1 \\
&= m + (m-1) + (m-2) + \cdots + 4 + 3 + 2 + 1 - 1 \\
&= \frac{m(m+1)}{2} - 1.
\end{aligned}$$

(2)
$$\begin{aligned}
R(3,m) &\le R(2,m)+R(3,m-1)\\
&\le R(2,m)+R(2,m-1)+R(3,m-2)\\
&\qquad\vdots\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,7)+R(3,6)\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,7)+R(2,6)\\
&\quad +R(3,5)-1\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,5)+R(3,4)-1\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,5)+R(2,4)\\
&\quad +R(3,3)-1-1\\
&\le m+(m-1)+(m-2)+\cdots+R(2,4)+R(2,3)+R(2,3)-2\\
&= m+(m-1)+(m-2)+\cdots+4+3+3-2\\
&= m+(m-1)+(m-2)+\cdots+4+3+2+1-2\\
&= \frac{m(m+1)}{2}-2\,.
\end{aligned}$$

(3)

$$\begin{aligned}
R(3,m) &\le R(2,m)+R(3,m-1)\\
&\le R(2,m)+R(2,m-1)+R(3,m-2)\\
&\qquad\vdots\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,11)+R(3,10)\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,11)+R(2,10)\\
&\quad +R(3,9)-1\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,7)+R(3,6)-1\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,7)+R(2,6)\\
&\quad +R(3,5)-1-1\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,5)+R(3,4)-2\\
&\le R(2,m)+R(2,m-1)+R(2,m-2)+\cdots+R(2,5)+R(2,4)\\
&\quad +R(3,3)-1-2\\
&\le m+(m-1)+(m-2)+\cdots+R(2,4)+R(2,3)+R(2,3)-3\\
&= m+(m-1)+(m-2)+\cdots+4+3+3-3\\
&= m+(m-1)+(m-2)+\cdots+4+3+2+1-3\\
&= \frac{m(m+1)}{2}-3\,.
\end{aligned}$$

□

Theorem 2.18. $R(4,m) \leq \frac{m(m+1)(m+2)}{6}$.
Furthermore, for $m \geq 5$, $R(4,m) \leq \frac{m(m+1)(m+2)}{6} - 1$.

Proof. By Lemma 2.3, we have

$$\begin{aligned}
R(4,m) &\leq R(3,m) + R(4,m-1) \\
&\leq R(3,m) + R(3,m-1) + R(4,m-2) \\
&\quad\vdots \\
&\leq R(3,m) + R(3,m-1) + R(3,m-2) + \cdots + R(3,2) + R(4,1) \\
&\leq \frac{(m)(m+1)}{2} + \frac{(m-1)(m)}{2} + \cdots + \frac{(4)(5)}{2} + \frac{(3)(4)}{2} + \frac{(2)(3)}{2} + 1 \\
&\leq \frac{(m)(m+1)}{2} + \frac{(m-1)(m)}{2} + \cdots + \frac{(4)(5)}{2} + \frac{(3)(4)}{2} + \frac{(2)(3)}{2} + \frac{(1)(2)}{2} \\
&= \frac{m(m+1)(m+2)}{6}\,.
\end{aligned}$$

Now, note that $R(3,5)$ and $R(4,4)$ are both even. Then, for $m \geq 5$, by Theorem 2.11, we have

$$\begin{aligned}
R(4,m) &\leq R(3,m) + R(4,m-1) \\
&\leq R(3,m) + R(3,m-1) + R(4,m-2) \\
&\quad\vdots \\
&\leq R(3,m) + R(3,m-1) + R(3,m-2) + \cdots + R(4,5) \\
&\leq R(3,m) + R(3,m-1) + \cdots + R(3,5) + R(4,4) - 1 \\
&\leq R(3,m) + R(3,m-1) + \cdots + R(3,5) + R(3,4) + R(4,3) - 1 \\
&\leq \frac{(m)(m+1)}{2} + \frac{(m-1)(m)}{2} + \cdots + \frac{(4)(5)}{2} + \frac{(4)(5)}{2} - 1 \\
&= \frac{(m)(m+1)}{2} + \frac{(m-1)(m)}{2} + \cdots + \frac{(4)(5)}{2} \\
&\quad + \frac{(3)(4)}{2} + \frac{(2)(3)}{2} + \frac{(1)(2)}{2} - 1 \\
&= \frac{m(m+1)(m+2)}{6} - 1\,.
\end{aligned}$$

□

Theorem 2.19. *For $m \geq 3$,*

$$\begin{aligned}
R(4,m) \leq {} & \frac{m(m+1)(m+2)}{6} - 3\max\{0, m-9\} \\
& - 2\max\{0, \min\{m,9\} - 5\} - \max\{0, \min\{m,5\} - 3\} - 1\,.
\end{aligned}$$

Proof. We use Theorem 2.17. First, divide the value of m into 3 cases.
Case (i) : $3 \leq m \leq 5$.

Note that both $\max\{0, m-9\}$ and $\max\{0, \min\{m, 9\}-5\}$ are equal to 0 and $\max\{0, \min\{m, 5\}-3\} = m-3$. Hence, Theorem 2.19 will become

$$R(4, m) \leq \frac{m(m+1)(m+2)}{6} - (m-3) - 1\,.$$

$$\begin{aligned}
m = 3: \quad R(4,3) &\leq R(3,3) + R(4,2) - 1 \\
&\leq R(3,3) + R(3,2) + R(4,1) - 1 \\
&\leq \frac{3(4)}{2} + \frac{2(3)}{2} + \frac{1(2)}{2} - 1 \\
&= \frac{3(4)(5)}{6} - 1. \\
m = 4: \quad R(4,4) &\leq R(3,4) + R(4,3) \\
&= 2R(4,3) \\
&\leq 2[\frac{3(4)(5)}{6} - 1] \\
&= \frac{2(3)(4)(5)}{6} - 2 \\
&= \frac{4(5)(6)}{6} - 1 - 1. \\
m = 5: \quad R(4,5) &\leq R(3,5) + R(4,4) \\
&\leq \frac{(5)(6)}{2} - 1 + \frac{(4)(5)(6)}{6} - 1 - 1 \\
&= \frac{(3)(5)(6)}{6} - 1 + \frac{(4)(5)(6)}{6} - 1 - 1 \\
&= \frac{5(6)(7)}{6} - 2 - 1
\end{aligned}$$

Case (ii) : $6 \leq m \leq 9$.
Note that $\max\{0, m-9\} = 0$, that $\max\{0, \min\{m, 9\}-5\} = m-5$ and that $\max\{0, \min\{m, 5\}-3\} = 2$. Hence, Theorem 2.19 will reduce to

$$R(4, m) \leq \frac{m(m+1)(m+2)}{6} - 2(m-5) - 3\,.$$

Hence,

$$\begin{aligned}
m=6: \quad R(4,6) &\le R(3,6)+R(4,5)\\
&\le \frac{6(7)}{2}-2+\frac{5(6)(7)}{6}-2-1\\
&= \frac{3(6)(7)}{6}-2+\frac{5(6)(7)}{6}-2-1\\
&= \frac{6(7)(8)}{6}-2-3\,.\\
m=7: \quad R(4,7) &\le R(3,7)+R(4,6)\\
&\le \frac{7(8)}{2}-2+\frac{6(7)(8)}{6}-2-3\\
&= \frac{3(7)(8)}{6}-2+\frac{6(7)(8)}{6}-2-3\\
&= \frac{7(8)(9)}{6}-2(2)-3\,.\\
m=8: \quad R(4,8) &\le R(3,8)+R(4,7)\\
&\le \frac{8(9)}{2}-2+\frac{7(8)(9)}{6}-2(2)-3\\
&= \frac{3(8)(9)}{6}-2+\frac{7(8)(9)}{6}-2(2)-1\\
&= \frac{8(9)(10)}{6}-2(3)-3.\\
m=9: \quad R(4,9) &\le R(3,9)+R(4,8)\\
&\le \frac{9(10)}{2}-2+\frac{8(9)(10)}{6}-2(3)-3\\
&= \frac{3(9)(10)}{6}-2+\frac{8(9)(10)}{6}-2(3)-3\\
&= \frac{9(10)(11)}{6}-2(4)-3\,.
\end{aligned}$$

Case (iii) : $m \ge 10$.
Note that $\max\{0, m-9\} = m-9$, that $\max\{0, \min\{m,9\}-5\} = 4$ and that $\max\{0, \min\{m,5\}-3\} = 2$. Hence, Theorem 2.19 reduces to

$$R(4,m) \le \frac{m(m+1)(m+2)}{6} - 3(m-9) - 11\,.$$

We prove this case by induction.

$$
\begin{aligned}
m = 10: \quad R(4,10) &\leq R(3,10) + R(4,9) \\
&\leq \frac{10(11)}{2} - 3 + \frac{9(10)(11)}{6} - 2(4) - 3 \\
&= \frac{3(10)(11)}{6} - 3 + \frac{9(10)(11)}{6} - 11 \\
&= \frac{10(11)(12)}{6} - 3 - 11\,.
\end{aligned}
$$

Now, assume that the theorem valid for $R(4, m_1 - 1)$ for some $m_1 - 1 \geq 10$. We want to show that the theorem also valid for $R(4, m_1)$. Notice that, by the assumption, we have $R(4, m_1 - 1) \leq \frac{m_1(m_1-1)(m_1+1)}{6} - 3[(m_1 - 1) - 9] - 11$.

$$
\begin{aligned}
R(4, m_1) &\leq R(3, m_1) + R(4, m_1 - 1) \\
&\leq \frac{m_1(m_1+1)}{2} - 3 + \frac{m_1(m_1-1)(m_1+1)}{6} - 3(m_1 - 10) - 11 \\
&= \frac{3(m_1)(m_1+1)}{6} - 3 + \frac{m_1(m_1-1)(m_1+1)}{6} - 3(m_1 - 10) - 11 \\
&= \frac{m_1(m_1+1)(m_1+2)}{6} - 3(m_1 - 9) - 11\,.
\end{aligned}
$$

Hence by induction, the theorem is valid for any $m \geq 10$. □

Theorem 2.20. *For $m \geq 4$, $R(5,m) \leq \frac{m(m+1)(m+2)(m+3)}{24} - 1$.*

Proof. By Lemma 2.3, we have

$$
\begin{aligned}
R(5,m) &\leq R(4,m) + R(5,m-1) \\
&\leq R(4,m) + R(4,m-1) + R(5,m-2) \\
&\ \ \vdots \\
&\leq R(4,m) + R(4,m-1) + \cdots + R(4,5) + R(5,4) \\
&\leq R(4,m) + R(4,m-1) + \cdots + R(4,5) + R(4,4) + R(5,3) - 1 \\
&\quad \text{since both } R(4,4) \text{ and } R(5,3) \text{ are even} \\
&\leq R(4,m) + R(4,m-1) + \cdots + R(4,3) + R(4,2) + R(5,1) - 1 \\
&\leq \frac{(m)(m+1)(m+2)}{6} + \frac{(m-1)(m)(m+1)}{6} + \cdots + \frac{(3)(4)(5)}{6} \\
&\quad + \frac{(2)(3)(4)}{6} + 1 - 1 \\
&\leq \frac{(m)(m+1)(m+2)}{6} + \frac{(m-1)(m)(m+1)}{6} + \cdots + \frac{(3)(4)(5)}{6} \\
&\quad + \frac{(2)(3)(4)}{6} + \frac{(1)(2)(3)}{6} - 1 \\
&= \frac{m(m+1)(m+2)(m+3)}{24} - 1\,. \qquad \square
\end{aligned}
$$

A more general result on the upper bound of the Ramsey numbers was established by Ajtai, Komlós and Szemerédi [2], who showed that

$$R(m_1, m_2) \leq \frac{c_{m_1} m_2^{m_1-1}}{(\ln m_2)^(m_1-2)}, \text{ for some constant } c_k > 0.$$

Interested reader is referred to [2] for more details and proofs.

Next, we will discuss on the lower bound of the Ramsey numbers.

Theorem 2.21. [10] $R(m_1, m_2) \geq R(m_1, m_2 - 1) + 2m_1 - 3$.

Proof. Consider the graph $G = K_{R(m_1,m_2-1)-1}$. By the definition of $R(m_1, m_2 - 1)$, there is a 2-colouring of the edges of the graph G which neither has a c_1-coloured K_{m_1} subgraph nor a c_2-coloured K_{m_2-1} subgraph. Consider this colouring. Note that G must contain a c_1-coloured K_{m_1-1} subgraph, for otherwise, if we add a new vertex and join it to all the edges in G with c_1-coloured edges, we get a colouring of $K_{R(m_1,m_2-1)}$ without any monochromatic K_{m_1} or K_{m_2-1} subgraphs, violating the definition of $R(m_1, m_2 - 1)$. In fact, we only need to consider some c_1-coloured K_{m_1-2} subgraph of G. Denote the vertices in this K_{m_1-2} by $u_1, u_2, \ldots, u_{m_1-2}$.

Now, add $m_1 - 2$ more vertices to G and denote them by $v_1, v_2, \ldots, v_{m_1-2}$. For each $i \in [m_1 - 2]$, join the vertices u_i and v_i with a c_2-coloured edge. Join the vertex v_i to each of the other vertices x in G with the edges in the same colour as the edges joining u_i to x. Let $H = K_{R(m_1,m_2-1)+m_1-3}$ be the resulting graph.

Note that in graph H, there is no c_1-coloured K_{m_1}. For suppose that there is one; then u_i and v_i cannot both be in that K_{m_1}. Since v_i's were added to the graph by duplicating the u_i's, any v_i involved in the K_{m_1} is isomorphic to the K_{m_1} obtained by replacing v_i with u_i. However, this contradicts with the original colouring of graph G. On the other hand, in the initial colouring of graph G, there is no c_2-coloured K_{m_2-1}. For graph H, the biggest degree of c_2-coloured complete graph is $m_2 - 1$ but any c_2-coloured K_{m_2-1} must involve a pair of u_i and v_i and no other u and v.

We then adjoin m_1 more vertices, denoting them by $w_1, w_2, \ldots, w_{m_1}$. Colour the edges $\{w_i, w_j\}$ with colour c_1 for all $i \neq j$ and the edges $\{w_i, y\}$ with colour c_2 for all $y \in \{u_1, u_2, \ldots, u_{m_1-2}, v_1, v_2, \ldots, v_{m_1-2}, w_1, w_2, \ldots, w_{m_1}\}$. For the edges $\{u_i, w_j\}$, colour with c_1 if $i \geq j$ and c_2 otherwise. For the edges $\{v_i, w_j\}$, we colour the other way round: colour c_2 if $i \geq j$ and c_1 otherwise.

We first prove that $K_{R(m_1,m_2-1)+2m_1-4}$ contains no c_1-coloured K_{m_1}. Assume the contrary. Since there is no such K_{m_1} in H, the subgraph K_{m_1} must involve some w_i. Hence, the subgraph K_{m_1} must contain only vertices from the set $\{u_1, \ldots, u_{m_1-2}, v_1, \ldots, v_{m_1-2}, w_1, \ldots, w_{m_1}\}$. There are two cases to consider.
Case 1: Only one of the w_i's is involved, say w_a. The vertices connected to w_a with c_1-coloured edges are $\{v_1, \ldots, v_{a-1}, u_a, \ldots, u_{m_1-2}\}$. Thus, the maximum degree of the monochromatic c_1-coloured complete subgraph is m_1-1, which is a contradiction.
Case 2: There are two or more of the w_i's involved. Without loss of generality, we may assume there are k of them, $\{w_{a_1}, \ldots, w_{a_k}\}$. Note that $k \leq a_k - a_1$. Further note that the only vertices that are connected to all of them are $\{u_{a_k}, \ldots, u_{m_1-2}, v_1, v_{a_1-1}\}$. Thus, the maximum degree of the monochromatic c_1-coloured complete subgraph is $k + (m_1 - 2 - a_k + 1) + (a_1 - 1 + 1) \leq m_1 - 1$, which then is a contradiction.

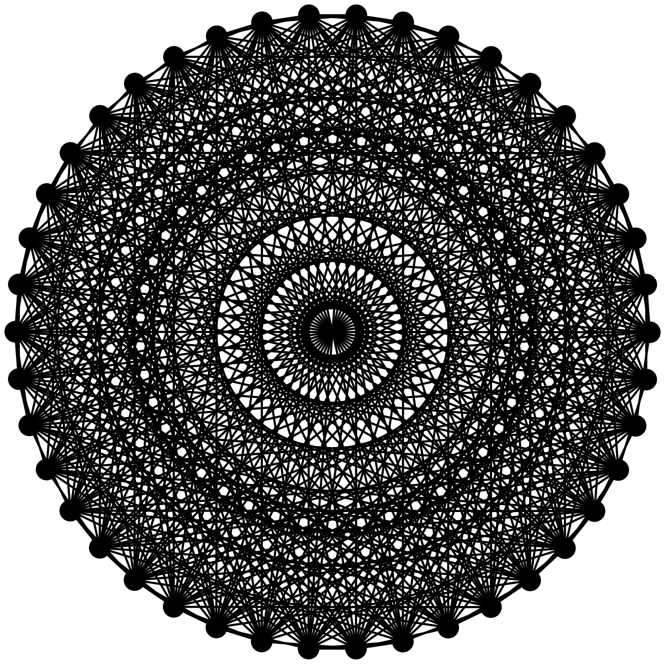
Now, we want to prove that the graph $K_{R(m_1,m_2-1)+2m_1-4}$ does not contain a c_2-coloured K_{m_2} subgraph. Assume the contrary. Since there is no such K_{m_2} in graph H, again, the subgraph K_{m_2} must involve some w_i. Because the edges $\{w_i, w_j\}$ are coloured with c_1 for all $i \neq j$, there is exactly one of the w_i's involved, say w_b. Hence, the K_{m_2} must include K_{m_2-1} in H, which must use exactly one pair u_i, v_i. However, one of the edges $\{u_i, w_b\}$ and $\{v_i, w_b\}$ must be c_1-coloured, depending on the value of i and b, which is then a contradiction.

This colouring of $K_{R(m_1,m_2-1)+2m_1-4}$ does not contain a c_1-coloured K_{m_1} subgraph or a c_2-coloured K_{m_2} subgraph, so the proof is complete. □

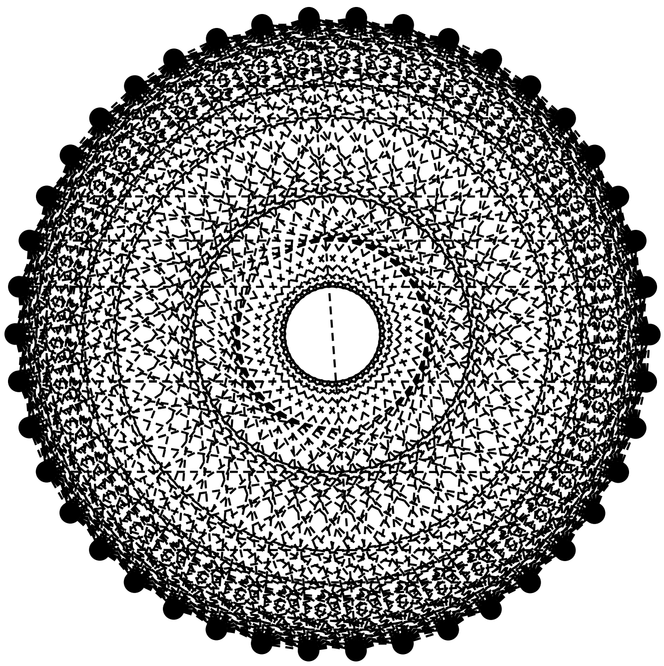
Theorem 2.22.
(1) $R(5,5) \leq 50$.
(2) $R(5,5) \geq 43$. [27]

Proof.
(1) In [74], it is proven that $R(4,5) = 25$. By Theorem 2.11, we have $R(5,5) \leq R(4,5) + R(5,4) = 25 + 25 = 50$.
(2) Figure 2.6 shows K_{42} with the edge-colouring with 2 colours c_1(———) and c_2(− − −) as demonstrated by Exoo in [27] which contains no monochromatic copies of K_5 in the colouring. The interested reader is referred to [27] for more details. □

Hence, we have now proven that $43 \leq R(5,5) \leq 50$. However, in [3], by using a technique of gluing induced subgraphs and verification by checking approximately two trillion separate cases by computer, Angeltveit and McKay proved that $R(5,5) \leq 48$. In fact, this is the best upper bound known for $R(5,5)$ today.



(a) No c_1-coloured K_5



(b) No c_2-coloured K_5

Figure 2.6: $R(5,5) > 42$

Table 2.2 shows some known bounds of the Ramsey numbers, after compiling the known results in the bounds of Ramsey numbers, with the references as cited, and referring to the table compiled by Radziszowski in [83].

$m_1 \backslash m_2$	3	4	5	6
4	9	18	25	[36, 41][31, 75]
5	14	25	[43, 48][27, 3]	[58, 87][29, 58]
6	18	[36,41]	[58,87]	[102, 165][60, 71]
7	23	[49, 61][28, 71]	[80, 143][11, 95]	[115, 298][34, 71]
8	28	[59, 84][34, 71]	[101, 216][52, 95]	[134, 495][34, 71]
9	36	[73, 115][83, 71]	[133, 316][68, 71]	[183, 780][68, 71]
10	[40, 42][32, 39]	[92, 149][52, 71]	[149, 442][34, 71]	[204, 1171][68, 71]
11	[47, 50][30, 39]	[102, 191][34, 95]	[183, 633][68, 56]	[256, 1804][68, 56]
12	[53, 59][64, 70]	[128, 238][97, 95]	[203, 848][68, 56]	[294, 2566][68, 56]
13	[60, 68][64, 39]	[138, 291][34, 95]	[233, 1138][68, 56]	[347, 3703][68, 56]
14	[67, 77][64, 39]	[147, 349][34, 95]	[267, 1461][68, 56]	[326, 5033][Thm 2.21,[56]]
15	[74, 87][64, 39]	[155, 417][34, 95]	[269, 1878][34, 56]	[401, 6911][83, 56]

Table 2.2: Bounds for Ramsey number $R(m_1, m_2)$ for $m_1 \leq 6$ and $m_2 \leq 15$

Let us introduce one very interesting special type of Ramsey numbers, known as diagonal Ramsey number, $R(m, m)$, or also denoted by $R(m)$. Despite much research on these numbers, $R(3) = 6$ and $R(4) = 18$ are the only known exact diagonal Ramsey numbers. The first upper bound on the $R(m)$ is a consequence of the Erdős and Szekeres proof of Ramsey theorem in 1935 [25]:

$$R(m) \leq \binom{2m-2}{m-1} \leq 4^m.$$

In 1968, Walker had established a recurrence result on $R(m, m)$ [104], proved that

$$R(m, m) \leq 4R(m, m-2) + 2.$$

The best until now upper bound was provided by Conlon in 2009 [17]:

$$R(m+1) \leq \binom{2m}{m} m^{-C\frac{\log m}{\log \log m}} \text{ , for some constant C .}$$

On the other hand, the first lower bound of the diagonal Ramsey numbers was provided by Erdős in 1947 [22], who gave a simple probabilistic proof of $R(m) \geq cm2^{\frac{m}{2}}$, for some constant c. In 1975, Spencer [94] improved the result to

$$R(m) \geq m2^{\frac{m}{2}}[\frac{\sqrt{2}}{e} + o(1)].$$

Before we end this chapter, we look at some results on the colouring of graph edges with more than 2 colours.

Theorem 2.23. $R(3,3,3) = 17$

Proof. We first prove that $R(3,3,3) \leq 17$. Let A be any vertex in K_{17}. Then there are 16 edges incident with it. By the Pigeonhole Principle, at least 6 of them must be of the same colour, say c_1. Among these 6 vertices, if there are two vertices whose edge connecting them is c_1-coloured, then we have a monochromatically c_1-coloured K_3. Otherwise, all the edges in this K_6 are coloured with the other 2 colours, c_2 and c_3. By Example 2.8, $R(3,3) = 6$. Hence, we will get a monochromatic K_3. Therefore, $R(3,3,3) \leq 17$.

Now, we need to prove that $R(3,3,3) \geq 17 > 16$. To do so, we need to construct a 3-colouring of edges of K_{16} which does not contain monochromatic K_3. We first label the vertices in K_{16} as $u_1, u_2, \ldots, u_{16}$. Define the following adjacency matrix:

$$M = \{a_{ij}\} = \begin{cases} 0 & , \{u_i, u_j\} \notin E(K_{16}); \\ 1 & , \{u_i, u_j\} \text{ is } c_1\text{-coloured}; \\ 2 & , \{u_i, u_j\} \text{ is } c_2\text{-coloured}; \\ 3 & , \{u_i, u_j\} \text{ is } c_3\text{-coloured}. \end{cases}$$

Now, consider a colouring of K_{16} defined by the following adjacency matrix [99].

$$M = \begin{bmatrix}
0 & 3 & 3 & 1 & 2 & 2 & 2 & 3 & 1 & 1 & 2 & 3 & 1 & 2 & 1 & 3 \\
3 & 0 & 1 & 3 & 2 & 2 & 3 & 2 & 1 & 1 & 3 & 2 & 2 & 1 & 3 & 1 \\
3 & 1 & 0 & 3 & 2 & 3 & 2 & 2 & 2 & 3 & 1 & 1 & 1 & 3 & 1 & 2 \\
1 & 3 & 3 & 0 & 3 & 2 & 2 & 2 & 3 & 2 & 1 & 1 & 3 & 1 & 2 & 1 \\
2 & 2 & 2 & 3 & 0 & 3 & 3 & 1 & 1 & 2 & 1 & 3 & 1 & 1 & 2 & 3 \\
2 & 2 & 3 & 2 & 3 & 0 & 1 & 3 & 2 & 1 & 3 & 1 & 1 & 1 & 3 & 2 \\
2 & 3 & 2 & 2 & 3 & 1 & 0 & 3 & 1 & 3 & 1 & 2 & 2 & 3 & 1 & 1 \\
3 & 2 & 2 & 2 & 1 & 3 & 3 & 0 & 3 & 1 & 2 & 1 & 3 & 2 & 1 & 1 \\
1 & 1 & 2 & 3 & 1 & 2 & 1 & 3 & 0 & 3 & 3 & 1 & 2 & 2 & 2 & 3 \\
1 & 1 & 3 & 2 & 2 & 1 & 3 & 1 & 3 & 0 & 1 & 3 & 2 & 2 & 3 & 2 \\
2 & 3 & 1 & 1 & 1 & 3 & 1 & 2 & 3 & 1 & 0 & 3 & 2 & 3 & 2 & 2 \\
3 & 2 & 1 & 1 & 3 & 1 & 2 & 1 & 1 & 3 & 3 & 0 & 3 & 2 & 2 & 2 \\
1 & 2 & 1 & 3 & 1 & 1 & 2 & 3 & 2 & 2 & 2 & 3 & 0 & 3 & 3 & 1 \\
2 & 1 & 3 & 1 & 1 & 1 & 3 & 2 & 2 & 2 & 3 & 2 & 3 & 0 & 1 & 3 \\
1 & 3 & 1 & 2 & 2 & 3 & 1 & 1 & 2 & 3 & 2 & 2 & 3 & 1 & 0 & 3 \\
3 & 1 & 2 & 1 & 3 & 2 & 1 & 1 & 3 & 2 & 2 & 2 & 1 & 3 & 3 & 0
\end{bmatrix}$$

There is no monochromatic K_3 in this colouring of K_{16}. Thus, $R(3,3,3) > 16 \geq 17$. Therefore, we have $R(3,3,3) = 17$. $\square$

Let $R(k;r) = R(k_1, k_2, \ldots, k_r)$, where $k_1 = k_2 = \cdots = k_r = k$. There is a result regarding $R(3;r)$ which has been proven by Wan in 1997 [106]. Here we state the theorem; the interested reader is referred to [106] for the detailed proof.

Theorem 2.24. [106] *For* $r \geq 4$, $R(3;r) \leq r!(\frac{e-e^{-1}+3}{2}) + 1 \approx 2.68r! + 1$.

Suppose that $G_1, G_2, \ldots, G_r$ are graphs. We can further extend the concept of Ramsey numbers $R(m_1, m_2, \ldots, m_r)$ by defining $R(G_1, G_2, \ldots, G_r)$ to be the least n such that if the edges of a complete graph K_n is r-coloured, then there is always a monochromatic subgraph G_i, for some $i \in [r]$. In next chapter, we will present some Ramsey results on certain types of graphs.

Chapter 3
Ramsey-type Theorems for Graphs

In this chapter, we will introduce some graph analogues of Ramsey's Theorem. In Section 3.1, we will present some Ramsey-type results for general graphs. In Section 3.2, we will present some Ramsey-type results for tree graphs. In Section 3.3, we will introduce some Ramsey-type results for cycle graphs. In Section 3.4, we will look into Ramsey's Theorem for bipartite graphs. In Section 3.5, we will present more on the result of bipartite Ramsey Number.

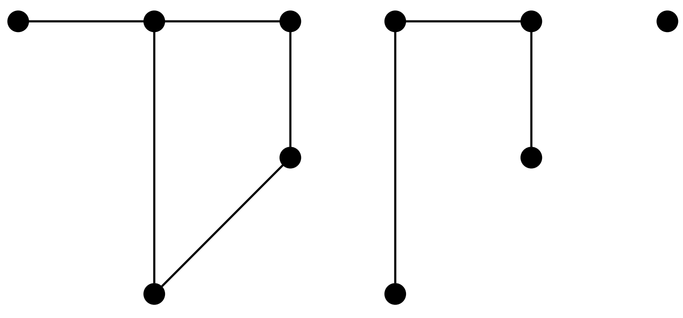
3.1 Ramsey-type Results for General Graphs

Before presenting Ramsey-type results for more specific types of graphs in the later subsections, we present some results that are valid for graphs in general. We first introduce some definitions that will be useful in our discussion.

Definition 3.1 (Chromatic index)**.** *The* chromatic index *is the smallest number of colours needed to colour the edges of a graph in such a way that no two edges incident to the same vertex share the same colour. The chromatic index of the graph G is denoted by $\chi(G)$.*

Definition 3.2 (Connected component)**.** *A* connected component *of a graph is a subgraph in which any two vertices of the subgraph are connected to each other. In our study, $c(G)$ denotes the largest size of a connected component of the graph G.*

Example 3.3. Let G and H be graphs shown below.



(a) G (b) H

Figure 3.1: Graphs G and H.

Then $\chi(G) = 3, \chi(H) = 2$ and $c(G) = 5, c(H) = 4$.

Theorem 3.4. [16] *Let G and H be any graphs.* $R(G, H) \geq (c(G)-1)(\chi(H)-1)+1$.

Proof. Consider the complete graph $K_{(c(G)-1)(\chi(H)-1)}$. Note that we can find $\chi(H)-1$ disjointed copies of $K_{c(G)-1}$ as subgraphs of $K_{(c(G)-1)(\chi(H)-1)}$. Colour the edges of these subgraphs with colour c_1 and all the rest of the edges with colour c_2. In this way, we have no c_1-coloured G as the largest size of the c_1-coloured component is $c(G)-1$. On the other hand, the c_2-coloured subgraph has the chromatic index of $\chi(H)-1$ and hence it is impossible to have a c_2-coloured H. Hence, $R(G,H) \geq (c(G)-1)(\chi(H)-1)+1$. □

We discovered and proved the following improvement of Theorem 3.4.

Theorem 3.5. *Let $k \geq 2$ and $R(G;k)$ be the least n such that k-colouring of the complete graph K_n will give us a monochromatic subgraph G.*
Then $R(G;k) \geq (\chi(G)-1)(R(G;k-1)-1)+1$.

Proof. Consider the complete graph $K_{(\chi(G)-1)(R(G;k-1)-1)}$. Note that we can find $(\chi(G)-1)$ disjointed copies of $K_{R(G;k-1)-1}$ as subgraphs of $K_{(\chi(G)-1)(R(G;k-1)-1)}$. Colour the edges of these subgraphs with colour c_i for $1 \leq i \leq k-1$. By the definition of $R(G;k-1)$, there is a colouring of these edges such that there is no monochromatic G. On the other hand, we colour all the remaining edges with colour c_k. Since the c_k-coloured subgraph has the chromatic index of $\chi(G)-1$ and hence it is impossible to have a c_k-coloured G. Therefore, we have $R(G;k) \geq (\chi(G)-1)(R(G;k-1)-1)+1$. □

3.2 Ramsey-type Results for Trees

In this section, we will present Ramsey-type results on tree graphs.

Theorem 3.6. [15] $R(T_m, K_n) = (m-1)(n-1)+1$.

Proof. Note that $c(T_m) = m$ and $\chi(K_n) = n-1$. By Theorem 3.4, we have $R(T_m, K_n) \geq (c(T_m)-1)(\chi(K_n)-1)+1 = (m-1)(n-1)+1$. We now wish to prove that $R(T_m, K_n) \leq (m-1)(n-1)+1$. Note that $R(T_1, K_1) = R(K_1, K_1) = R(1,1) = 1$ by Lemma 2.2. Assume that $R(T'_m, K'_n) \leq (m'-1)(n'-1)+1$ for all values of m' and n' such that $m'+n' < m+n$. Consider any colouring of $K_{(m-1)(n-1)+1}$ with colour c_1 and c_2. By the induction assumption, we have $R(T_{m-1}, K_n) \leq (m-2)(n-1)+1 < (m-1)(n-1)+1$. Hence, in any colouring of $K_{(m-1)(n-1)+1}$, we have either c_1-coloured T_{m-1} or c_2-coloured K_n. In latter case, we are done. Suppose that we have a c_1-coloured subgraph T_{m-1} Remove this c_1-coloured T_{m-1} and all the edges incident to it from the graph. Then, we will get a complete graph $K_{(m-1)(n-2)+1}$. By the induction assumption, we have $R(T_m, K_{n-1}) \leq (m-1)(n-2)+1$. Therefore, in this resulting $K_{(m-1)(n-2)+1}$, either we have some c_1-coloured T_m, in which case we are done by adding back the c_1-coloured T_{m-1} and the other removed edges, or we have a c_2-coloured subgraph K_{n-1}. Suppose the latter case and add back T_{m-1} to $K_{(m-1)(n-2)+1}$. Let u be any end vertex of T_{m-1}. Consider all the edges joining u to $K_{(m-1)(n-2)+1}$. If one of the edges is c_1-coloured, then a c_1-coloured T_m is formed and we are done. Otherwise, all the edges will be c_2-coloured including those joining vertex u to the c_2-coloured K_{n-1} in $K_{(m-1)(n-2)+1}$, which will give us a c_2-coloured K_n. Hence, in either way, we can find either a c_1-coloured T_m subgraph or a c_2-coloured K_n subgraph in the colouring of $K_{(m-1)(n-1)+1}$. Thus, $R(T_m, K_n) \leq (m-1)(n-1)+1$. Therefore, we have $R(T_m, K_n) = (m-1)(n-1)+1$. □

Theorem 3.7. *Let T_m be a tree graph with m vertices. Then $m \leq R(T_3, T_m) \leq m+1$.*

Proof. First, note that $\chi(T_3) = 2$ and $c(T_m) = m$. Hence by Theorem 3.4, $R(T_3, T_m) = R(T_m, T_3) \geq (m-1)(2-1)+1 = m$. Now, consider any 2-colouring of K_{m+1} and assume that there is no c_1-coloured T_3 in the colouring. Note that there is a T_m as a subgraph in K_{m+1}. If all edges in this subgraph T_m are c_2-coloured, then we are done. Suppose that is not the case. Then there is at least an edge, say $\{u, v\}$, is c_1-coloured. Now, let w be the vertex that not included in the T_m and let $a_1, \ldots, a_i$ and $b_1, \ldots, b_j$ be the vertices adjacent to u and v respectively. Note that the vertices u and v must be adjacent to all these vertices with c_2-coloured edges, or else we will have a c_1-coloured T_3, which is a contradiction. Consider the edges connecting the vertex w and all the vertices $a_1, \ldots, a_i$. If all are c_2-coloured, then replace the vertex u with the vertex w. If any of the edges is c_1-coloured, then all the edges connecting the vertex w and the vertices $b_1, \ldots, b_j$ must be c_2-coloured, so replace the vertex v with the vertex w. Now, we will get a new T_m. If all the edges in this T_m are c_2-coloured, then we are done, or else, repeat the process and we can get one eventually. Hence, we have $R(T_3, T_m) \leq m+1$. Thus, we have $m \leq R(T_3, T_m) \leq m+1$. □

After finding and proving Theorem 3.7, we discovered that Chartrand, Gould and Polimeni [12] had proved a more complete result on $R(T_3, T_m)$, below. Interested readers are referred to [12] for a proof.

Theorem 3.8. [12] *If T_m is any tree of order $m \geq 3$, then*

$$R(T_3, T_m) = \begin{cases} m+1, & \text{if } T_m \text{ is a complete bipartite graph } K_{1,m-1} \text{ and } m \text{ is even.} \\ m, & \text{otherwise.} \end{cases}$$

3.3 Ramsey-type Results for Cycles

In this section, we will present some Ramsey-type results for cycle graphs. We first derive Ramsey number for small cycles.

Theorem 3.9. *Let C_3 and C_4 be the cycle graph with 3 and 4 vertices, respectively. Then $R(C_3, C_3) = R(C_4, C_4) = 6$.*

Proof. Note that C_3 is isomorphic to K_3, so $R(C_3, C_3) = R(K_3, K_3) = R(3, 3) = 6$. Now, we need to prove that $R(C_4, C_4) = 6$. We first prove that $R(C_4, C_4) \leq 6$. Suppose the contrary, that there is no monochromatic C_4 in any 2-colouring of the edges of the complete graph K_6. As we have previously shown, there exists a monochromatic K_3 in any colouring of K_6. Without loss of generality, let the edges of the subgraph K_3 be c_1-coloured and denote the vertices of K_3 by u_1, u_2 and u_3. Let the remaining vertices be denoted by v_1, v_2 and v_3, respectively. For each v_i, there is at most one c_1-coloured edge connecting v_i to the c_1-coloured K_3 or else some c_1-coloured C_4 is formed. This means, for each v_i, that there are at least 2 c_2-coloured edges to the subgraph K_3.

Now suppose that one of the vertices v_i, say v_1, has no c_1-coloured edge to the subgraph K_3. Then, there can be at most one c_2-coloured edge from each v_2 and

v_3 to the c_1-coloured K_3, or else a c_2-coloured C_4 will be formed, a contradiction. Therefore, there is exactly one c_1-coloured edge from each v_i to the K_3.

Now, suppose there is more than one c_1-coloured edges from one of the u_i's, say u_1, to the some v_j. Without loss of generality, we let the vertex u_1 be joined to v_1 and v_2 by c_1-coloured edges. Then, both v_1 and v_2 are adjacent to u_2 and u_3 by c_2-coloured edges, and this gives us a c_2-coloured C_4, a contradiction. Hence, each v_i is adjacent to a different vertex of the c_1-coloured K_3 by a c_1-coloured edge. Without loss of generality, let the vertex u_i be joined to v_i by a c_1-coloured edge. If any of the edges v_iv_j is c_1-coloured, then we will have a c_1-coloured C_4. If all the edges v_iv_j are c_2-coloured, then we will get a c_2-coloured C_4. Either way, it will lead us to a contradiction. Therefore, we have $R(C_4, C_4) \leq 6$.

Now by Figure 3.2, $R(C_4, C_4) > 5$. Thus, we have $R(C_4, C_4) = 6$. □

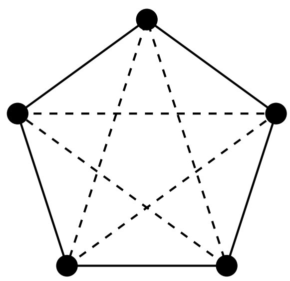


Figure 3.2: $R(C_4, C_4) > 5$.

We have looked at some Ramsey numbers on small cycles. Now, we present some Ramsey-type results on cycles more generally. In order to do prove those results, we will need the following useful proposition.

Proposition 3.10. [23] *Suppose that G is a graph with n vertices and at least $\frac{1}{2}[(c-1)(n-1)+1]$ edges. Then G contains a cycle of length at least c.*

Theorem 3.11. [8] *Let $m \geq 5$ be an odd integer. Then $R(C_m, C_m) = 2m-1$.*

Proof. Note that $\chi(C_m) = 3$ for odd m and $c(C_m) = m$. By Theorem 3.4, $R(C_m, C_m) \geq (\chi(C_m)-1)(c(C_m)-1)+1 = 2m-1$. Now, we need to prove that $R(C_m, C_m) \leq 2m-1$. Note that K_{2m-1} has $\binom{2m-1}{2}$ edges. By the Pigeonhole Principle, at least $\frac{1}{2}\binom{2m-1}{2} = \frac{1}{4}(2m-1)(2m-2) > \frac{1}{2}[(m-1)((2m-1)-1)+1]$ of the edges are of the same colour in any 2-colouring of the edges of K_{2m-1}. By Proposition 3.10, there is a monochromatic cycle of the length at least m. Now, if we can show that the existence of a monochromatic C_k in the colouring will imply the existence of a monochromatic C_{k-1}, then the theorem is proven.

Let $(v_0, \ldots, v_{k-1}, v_0)$ be a monochromatic C_k in the edge colouring of K_{2m-1}. Without loss of generality, let C_k be c_1-coloured. Suppose the contrary, that there is no monochromatic C_{k-1}. Now, consider the indices modulo k. Note that the edges of C_k, $\{v_i, v_{i+1}\}, 0 \leq i \leq k-1$ are c_1-coloured. Since there is no c_1-coloured $C_{k-1} = (v_0, \ldots, v_i, v_{i+2}, \ldots, v_k, v_0)$, we have the edges $\{v_i, v_{i+2}\}, 0 \leq i \leq k-1$ must be c_2-coloured. Since there is no c_2-coloured $C_{k-1} = (v_i, v_{i+4}, v_{i+6}, \ldots, v_{i-2}, v_i)$, we have the edges $\{v_i, v_{i+4}\}, 0 \leq i \leq k-1$ must be c_1-coloured. Now, since there is no c_1-coloured $C_{k-1} = (v_i, v_{i+3}, v_{i+4}, \ldots, v_{i-2}, v_{i+2}, v_{i+1}, v_i)$, we have the edges $\{v_i, v_{i+3}\}, 0 \leq i \leq k-1$ must be c_2-coloured. But this will give us a c_2-coloured $C_{k-1} = (v_1, v_3, \ldots, v_{k-6}, v_{k-3}, v_{k-5}, \ldots, v_2, v_{k-1}, v_{k-4}, v_{k-2}, v_1)$ if k is odd or c_2-coloured C_{k-1}, $(v_1, v_3, \ldots, v_{k-5}, v_{k-2}, v_{k-4}, \ldots, v_2, v_k, v_{k-3}, v_{k-1}, v_1)$ if k is even. In either case, we are done.

Hence by induction, we have a monochromatic cycle C_m of length m in any edge-colouring of K_{2m-1} with 2 colours. Thus, $R(C_m, C_m) \leq 2m - 1$.

Therefore, we have $R(C_m, C_m) = 2m - 1$. $\square$

Theorem 3.12. [85] *Let $m \geq 6$ be an even integer. Then $R(C_m, C_m) = \frac{3m}{2} - 1$.*

Proof. Consider a c_1-coloured complete graph K_{m-1}. Note that the complete graph does not contain C_m. Now, consider a c_2-coloured complete graph $K_{\frac{m}{2}-1}$. Join both complete graphs with c_2-coloured edges and form the complete graph $K_{\frac{3m}{2}-2}$. Clearly, there is no monochromatic C_m. Thus, we have $R(C_m, C_m) \geq \frac{3m}{2} - 1 > \frac{3m}{2} - 2$.

Now, we wish to show that $R(C_m, C_m) \leq \frac{3m}{2} - 1$. Let D be the largest monochromatic cycle with s vertices in the 2-colouring of $K_{\frac{3m}{2}-1}$. Let G be c_1-coloured subgraph of $K_{\frac{3m}{2}-1}$ and $\overline{G}$ be c_2-coloured subgraph of $K_{\frac{3m}{2}-1}$. Without loss of generality, we let D be the subgraph of G.

If $s < m$, by Proposition 3.10, then the number of edges of G, $|E(G)|$ is less than $\frac{1}{2}((m-1)(\frac{3m}{2} - 1 - 1) + 1)$. Then, we have the number of edges in $\overline{G}$,

$$\begin{aligned}
|E(\overline{G})| &= |E(K_{\frac{3m}{2}-1}| - |E(G)| \\
&> \frac{(\frac{3m}{2}-1)(\frac{3m}{2}-2)}{2} - \frac{1}{2}\Big((m-1)\Big(\frac{3m}{2} - 1 - 1\Big) + 1\Big) \\
&= \frac{6m^2 - 7m - 1}{2} \\
&> \frac{\frac{3m^2}{2} - \frac{7m}{2} + 3}{2} \\
&= \frac{1}{2}\Big((m-1)\Big(\frac{3m}{2} - 1 - 1\Big) + 1\Big).
\end{aligned}$$

By Proposition 3.10, $\overline{G}$ contains a cycle of length at least $m > s$, which is then a contradiction since s is the largest size of monochromatic cycle in the colouring. Therefore, we have $s \geq m$.

Now, if $s = m$, then we are done. If $s > m$, then by the similar method of construction in Theorem 3.11, it can be shown that either G or $\overline{G}$ will contains C_m as the subgraph and we are done. Hence, we have $R(C_m, C_m) \leq \frac{3m}{2} - 1$. Therefore, $R(C_m, C_m) = \frac{3m}{2} - 1$, for $m \geq 6$ is an even integer. $\square$

Theorem 3.13. [13] *Let $m \geq 3$,*

$$R(C_3, C_m) = \begin{cases} 6 & , \textit{if } m = 3; \\ 2m - 1 & , \textit{otherwise.} \end{cases}$$

Proof. The outline of this proof is found in [13]; we have filled in the explicit details. From Theorem 3.9, it is known that $R(C_3, C_3) = 6$. Now for $m \geq 4$, we first need to prove that $R(C_3, C_m) \leq 2m - 1$. We use induction on m.

Let $m = 4$. We have to show that $R(C_3, C_4) \leq 2(4) - 1 = 7$. Suppose to the contrary that there is neither c_1-coloured C_3 nor c_2-coloured C_4 in any 2-colouring of K_7. Note that $R(C_3, C_3) = 6 < 7$; hence, there is a monochromatic C_3, say (u_1, u_2, u_3, u_1) in the colouring of K_7. Since there is no c_1-coloured C_3, the monochromatic C_3 must be c_2-coloured. Let the remaining vertices in K_7 be v_1, v_2, v_3 and v_4. Consider the set of vertices $\{u_2, u_3, v_1, v_2, v_3, v_4\}$. Note that these 6 vertices form K_6 as a subgraph of K_7. Hence, there is a monochromatic C_3 in the subgraph K_6 and the C_3 must be c_2-coloured. If both u_2 and u_3 are vertices in the monochromatic C_3, say (u_2, u_3, v_i, u_2) for some $1 \leq i \leq 4$, then the vertices $\{u_2, u_1, u_3, v_i\}$ will form a c_2-coloured C_4, hence a contradiction. Now, suppose that both u_2 and u_3 are not vertices in the monochromatic C_3. Without the loss of generality, assume that the c_2-coloured C_3 is (v_1, v_2, v_3, v_1). Consider the vertex v_4. Note that if there are two c_2-coloured edges connecting v_4 to the vertices $\{u_1, u_2, u_3\}$, then there will be c_2-coloured C_4, a contradiction. Hence, at least two of the vertices $\{u_1, u_2, u_3\}$, say u_1 and u_2, are connected to v_4 with c_1-coloured edges. Similarly, there at least two of the vertices $\{v_1, v_2, v_3\}$, say v_1 and v_2, are connected to v_4 with c_1-coloured edges. Now, look at the edges $\{u_1, v_1\}$ and $\{u_2, v_2\}$. If either of the edges is c_1-coloured, then we will have a c_1-coloured C_3. If both of the edges are c_2-coloured, then we will have a c_2-coloured C_4. On the other hand, suppose that one of the vertices u_2 and u_3, say u_2, is a vertex in the monochromatic C_3. Without loss of generality, let the c_2-coloured C_3 be (u_2, v_1, v_2, u_2). Note that the edges $\{u_1, v_1\}$, $\{u_1, v_2\}$, $\{u_3, v_1\}$ and $\{u_3, v_2\}$ must be c_1-coloured, or else we will have a c_2-coloured C_4. Now, consider another subgraph K_6, with the vertices $\{u_1, u_3, v_1, v_2, v_3, v_4\}$. Note that there is a monochromatic C_3 in this subgraph and must be c_2-coloured since there is no c_1-coloured C_3. Further note that only one of the vertices u_1, u_3, v_1 and v_2 will be involved in the c_2-coloured C_3, or else we will have a c_2-coloured C_4. Without loss of generality, assume that the c_2-coloured C_3 is (u_1, v_3, v_4, u_1). Consider these edges, $\{u_3, v_3\}$ and $\{v_2, v_3\}$. If any of these edges is c_2-coloured, then we will have a c_2-coloured C_4. If both of the edges are c_1-coloured, then we will have a c_1-coloured C_3. Either way, it will lead us to a contradiction. Hence, $R(C_3, C_4) \leq 7$.

Now, assume that $R(C_3, C_m) \leq 2m - 1$. We wish to show that $R(C_3, C_{m+1}) \leq 2(m + 1) - 1 = 2m + 1$. Consider any 2-colouring of K_{2m+1} with colour c_1 and c_2. Let G be a c_1-coloured subgraph of K_{2m+1} and $\overline{G}$ be a c_2-coloured subgraph of K_{2m+1}. Suppose there is no C_3 in the subgraph G. We need to show there is a C_{m+1} in the subgraph $\overline{G}$. Since $R(C_3, C_m) \leq 2m - 1$, therefore there must be a C_m, say $(u_1, \ldots, u_m, u_1)$, in the subgraph $\overline{G}$. Denote the remaining vertices of K_{2m+1} by $v_1, v_2, \ldots, v_m, v_{m+1}$. Note that if any of the v_i's is connected to two consecutive vertices in C_m in $\overline{G}$, then we will have a C_{m+1} in $\overline{G}$. Suppose there is no such v_i. There are then two cases to be considered.

First, assume that there exist two alternate vertices of C_m, say u_j and u_{j+2}, which are respectively adjacent in $\overline{G}$ to two distinct v_i. Without loss of generality, let u_j be adjacent to v_1 in $\overline{G}$ and u_{j+2} be adjacent to v_2 in $\overline{G}$. If v_1 is adjacent to v_2 in $\overline{G}$, then $(u_1, \ldots, u_j, v_1, v_2, u_{j+2}, \ldots, u_m, u_1)$ will form a C_{m+1} in $\overline{G}$. If v_1 and v_2 are adjacent in G, then consider the edges $\{v_1, u_{j+1}\}$ and $\{v_2, u_{j+1}\}$. If either of these two edges lies in $\overline{G}$, then we will have C_{m+1} in $\overline{G}$. Otherwise, we will have a C_3 in G, which is a contradiction.

On the other hand, suppose that there are no two alternate vertices of C_m adjacent in $\overline{G}$ to distinct v_i. Before we proceed to the case, note that since there is no C_3 in G, if there is a vertex in the C_m that is not adjacent to any v_i in $\overline{G}$ and hence adjacent to every single v_i's in G, then every v_i's must connect with each other in $\overline{G}$. However, this will form a C_{m+1} (in fact a K_{m+1}) in $\overline{G}$. Hence, every vertices in C_m must be adjacent to some v_i in $\overline{G}$. Without loss of the generality, suppose that u_j is adjacent to v_1 in $\overline{G}$. Then, u_{j+2} must also adjacent to v_1 in $\overline{G}$ as we have assumed that there are no two alternate vertices of C_m adjacent in $\overline{G}$ to distinct v_i. Since there is no v_i connected to two consecutive vertices in C_m in $\overline{G}$, v_1 must be adjacent to u_{j-1} and u_{j+1} in G. This will then force u_{j-1} and u_{j+1} to be adjacent to each other in $\overline{G}$. Then, $(u_1, \ldots, u_{j-1}, u_{j+1}, u_j, v_1, u_{j+2}, u_{j+3}, \ldots, u_m, u_1)$ will form a C_{m+1} in $\overline{G}$. In either case, we will have a C_{m+1} in $\overline{G}$. Thus, $R(C_3, C_{m+1}) \leq 2m+1 = 2(m+1)-1$. By induction, we have $R(C_3, C_m) \leq 2m-1$ for $m \geq 4$.

Now, we need to prove that $R(C_3, C_m) \geq 2m-1$ for $m \geq 4$. Note that there is a complete bipartite graph $K_{m-1,m-1}$ as a subgraph of K_{2m-2}. Colour the edges of this subgraph in colour c_1 and the remaining edges in colour c_2. In this way, we will get a c_1-coloured $K_{m-1,m-1}$ and c_2-coloured $K_{m-1} \cup K_{m-1}$. Hence, there is no c_1-coloured C_3 and c_2-coloured C_m in this colouring. Thus, we have $R(C_3, C_m) \geq 2m-1 > 2m-2$ for $m \geq 4$.

Therefore, $R(C_3, C_m) = 2m-1$ for $m \geq 4$. Thus, the theorem is valid. $\square$

Theorem 3.14. [13] *Let $m \geq 4$,*

$$R(C_4, C_m) = \begin{cases} 6, & \text{if } m = 4\,; \\ 7, & \text{if } m = 5\,; \\ m+1, & \text{otherwise}\,. \end{cases}$$

Proof. The outline of this proof is found in [13]; we have filled in the explicit details.

By Theorem 3.9, it is known that $R(C_4, C_4) = 6$.

For $m = 5$, consider any 2-colouring of K_7. Let s be the largest size of the monochromatic cycle in the colouring. Let G be a c_1-coloured subgraph of K_7 and $\overline{G}$ be a c_2-coloured subgraph of K_7. Suppose that $s \leq 3$. Then the largest cycle size in both G and $\overline{G}$ is at most 3. By Proposition 3.10, the number of edges of G and $\overline{G}$, $|E(G)|$ and $|E(\overline{G})|$, are both less than $\frac{1}{2}((3-1)(7-1)+1) = 6.5 < 7$. Therefore, $21 = |E(K_7|) = |E(G)| + |E(\overline{G})| < 14$, a contradiction. Hence, $s \geq 4$. If $4 \leq s \leq 5$, then we are done. If $s \geq 6$, then by the similar method of construction to that in Theorem 3.11, it can be shown that either G or $\overline{G}$ will contains C_5 as the subgraph and we are done since $R(C_4, C_5) = R(C_5, C_4)$. Hence, we have $R(C_4, C_5) \leq 7$. Furthermore, Figure 3.3 shows that $R(C_4, C_5) \geq 7 > 6$. Hence, we have $R(C_4, C_5) = 7$.

For $m \geq 6$, note that there is a complete bipartite graph $K_{1,m-1}$ as a subgraph of K_m. Colour the edges of this subgraph in colour c_1 and the remaining edges in colour c_2. In this way, we will get a c_1-coloured $K_{1,m-1}$ and c_2-coloured $K_1 \cup K_{m-1}$. Hence, there is no c_1-coloured C_4 and c_2-coloured C_m in this colouring. Thus for $m \geq 6$, we have $R(C_4, C_m) > m$ and so $R(C_4, C_m) \geq m+1$.

Now, we want to show that $R(C_4, C_m) \leq m+1$ for $m \geq 6$. We will proceed with induction on m. Let $m = 6$, consider any 2-colouring of K_7. Since $R(C_4, C_5) = 7$, we will have either a c_1-coloured C_4 or a c_2-coloured C_5. If there is a c_1-coloured C_4, then we are done. Suppose that is not the case. Then we have a c_2-coloured C_5; denote its vertices by u_1, u_2, u_3, u_4 and u_5, respectively. Let the remaining vertices be v_1 and v_2. If there are any consecutive vertices in C_5 that are connected to a same vertex, v_1 or v_2, by c_2-coloured edges, then we are done. Suppose there is no such vertex in C_5. Then each of the v_1 and v_2 must be adjacent to at least three of the vertices in C_5 via c_1-coloured edges. By the Pigeonhole Principle, one of the vertices in C_5, say u_1, must be adjacent to both v_1 and v_2 via c_1-coloured edges. Note that if any other vertex of C_5 join to both v_1 and v_2 by c_1-coloured edges, then we will have a c_1-coloured C_4. Assume that no such vertex exists. Every other vertex in C_5 must be adjacent to at least one of the v_1 and v_2 via a c_2-coloured edge. Without loss of generality, suppose that u_2 is adjacent to v_1 via a c_2-coloured edge. Then, u_3 must be adjacent to v_1 via a c_1-coloured edge and must be adjacent to v_2 via a c_2-coloured edge. Then, u_2 and u_4 must be both adjacent to v_2 via c_1-coloured edges. This will force u_4 to be adjacent to v_1 via a c_2-coloured edge. Then, again, u_5 must be connected to v_1 by a c_1-coloured edge and connected to v_2 by a c_2-coloured edge. Then, the edge $\{v_1, v_2\}$ must be c_1-coloured, or else we will have a c_2-coloured C_6. Now, look at the edge $\{u_2, u_5\}$. If the edge $\{u_2, u_5\}$ is c_1-coloured, then $(u_2, u_5, v_1, v_2, u_2)$ will form a c_1-coloured C_4. If the edge $\{u_2, u_5\}$ is c_2-coloured, then $(u_2, u_5, v_2, u_3, u_4, v_1, u_2)$ will form a c_2-coloured C_6. In either case, we can conclude that $R(C_4, C_6) \leq 7$.

Now, assume that $R(C_4, C_m) \leq m+1$ for some $m \geq 6$. We need to show that $R(C_4, C_{m+1}) \leq (m+1)+1 = m+2$. Consider any 2-colouring of K_{m+2}. Let H be a c_1-coloured subgraph of K_{m+1} and $\overline{H}$ be a c_2-coloured subgraph of K_{m+2}. Suppose there is no C_4 in H. Since $R(C_4, C_m) \leq m+1 < m+2$, there must be a C_m in $\overline{H}$. Label the vertices in C_m by $u_1, \ldots, u_m$. Let v_1 and v_2 be the two vertices that are not in C_m. Note that if any of v_1 and v_2 is joined to two consecutive vertices of C_m in $\overline{H}$, then we will have a C_m in $\overline{H}$ and we are done. Suppose there is no such vertex.

Then each of v_1 and v_2 are adjacent in H to at least $\frac{m}{2}$ vertices of C_m. If v_1 and v_2 are mutually adjacent in H to two or more vertices in C_m, then H will contain C_4. Hence, there are only two cases to be considered. First, v_1 and v_2 are mutually adjacent to no vertex of C_m in H. Without loss of generality, assume v_1 is adjacent to u_1 in H. Then v_2 must be adjacent to u_1 in $\overline{H}$, and hence adjacent to u_2 and u_m in H. In this case, v_1 must be adjacent to u_2 in $\overline{H}$ and u_3 in H. Continuing this process of reasoning, we notice that v_1 will be adjacent to u_i, where i is odd, in H. If m is odd, then both v_1 and v_2 are adjacent to u_m in H, a contradiction. Hence, this case can only occur when m is even, where v_1 is adjacent to u_i in H for odd i and v_2 is adjacent to u_i in H for even i. Now, suppose m is even. Consider these edges $\{u_2, u_m\}$ and $\{u_2, u_4\}$. If both edges are in H, then $(u_2, u_4, v_1, u_m, u_2)$ will form a C_4 in H. Therefore, at least one of these two edges must be in $\overline{H}$. Then, we will have a C_m in $\overline{H}$. For example, let the edges $\{u_2, u_4\}$ in $\overline{H}$. Then, $(u_2, u_4, u_3, v_1, u_5, \ldots, u_m, u_1, u_2)$ forms a C_{m+1}. On the other hand, v_1 and v_2 are mutually adjacent to one vertex of C_m, say u_1, in H. Then u_2 must be adjacent to one of the vertices v_1 and v_2 in $\overline{H}$, without loss of generality, say v_1. Then v_1 must be adjacent to u_3 in H and v_2 must be adjacent to u_2 in H and u_3 in $\overline{H}$. Continuing in this way, we see that v_1 must be adjacent to u_i for odd i in H and v_2 must be adjacent to u_i for even i and u_1 in H. If u_m is odd, then $(u_m, v_2, u_3, u_2, v_1, u_4, \ldots, u_m)$ will form a C_{m+1} in $\overline{H}$. If u_m is even, then consider the edges $\{u_1, u_{m-1}\}$ and $\{u_1, u_3\}$. Note that one of these two edges must exist in $\overline{H}$, or else $(u_1, u_3, v_1, u_{m-1}, u_1)$ will form a C_4 in H. Then, we will have a C_m in $\overline{H}$. For example, if the edge $\{u_1, u_{m-1}\}$ lies in $\overline{H}$, then $(v_1, u_m, u_{m-1}, u_1, u_2, \ldots, u_{m-2}, v_1)$ will form a C_{m+1} in $\overline{H}$. In either way, we will have a C_m in $\overline{H}$. Hence, we have $R(C_4, C_{m+1}) \leq m + 2 = (m + 1) + 1$. By induction, we have $R(C_4, C_m) \leq m + 1$ for $m \geq 6$.

Therefore, the theorem is valid. □

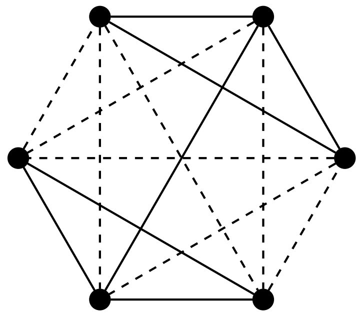


Figure 3.3: $R(C_4, C_5) > 6$.

Theorem 3.15. [13] *Let* $m \geq 5, R(C_5, C_m) = 2m - 1$.

Proof. The outline of this proof is found in [13]; we have filled in the details.

First, we will show by induction on m that $R(C_5, C_m) \leq 2m - 1$ for $m \geq 5$. For $m = 5$, Theorem 3.11 implies that $R(C_5, C_5) \leq 2(5) + 1 = 10$. Now, assume that $R(C_m, C_m) \leq 2m - 1$. We wish to show that $R(C_5, C_{m+1}) \leq 2(m + 1) - 1 = 2m + 1$. Consider any colouring of K_{2m+1}. Let L be the c_1-coloured subgraph of K_{2m+1} and $\overline{L}$ be the c_2-coloured subgraph of K_{2m+1}. Suppose there is no C_5 in L. Since $R(5, m) \leq 2m - 1 < 2m + 1$, there must be a C_m in $\overline{L}$. Denote the vertices in C_m by $u_1, \ldots, u_m$ and the remaining vertices by $v_1, \ldots, v_m, v_{m+1}$. Note that if any of the v_i's is connected to two consecutive vertices in C_m in $\overline{L}$, then we will have a C_{m+1} in $\overline{L}$ and we are done. Suppose there is no such v_i. If all vertices $v_1, \ldots, v_{n+1}$ are adjacent to each other in $\overline{L}$ and form K_{n+1}, then there is a C_{m+1} in $\overline{L}$ and we are done. Suppose that is not the case. Then two distinct v_i, say v_1 and v_2, are adjacent in L. Now, there are three cases to be considered.

Case 1. First, assume that there is a vertex from v_is other than v_1 and v_2, say v_3, such that v_1 and v_3 are joined to a vertex u_i of C_m in L, and v_2 and v_3 are joined to a vertex u_j of C_m in L. If $i \neq j$, then $(v_1, u_i, v_3, u_j, v_2, v_1)$ forms a C_5 in L. If $i = j$, then v_1, v_2 and v_3 are adjacent to a vertex u_i, without loss of generality, say u_1 in L. Note that at least one of u_2 and u_3, say u_2, must be adjacent to v_3 in L. Similarly, at least one of u_m and u_{m-1}, say u_m, must be adjacent to v_3 in L. Then, both u_2 and u_m must be adjacent to v_1 and v_2 in $\overline{L}$ or else L will contain C_5. This will force both u_3 and u_{m-1} to be adjacent to v_1 and v_2 in L. Now, consider the edge $\{u_1, u_3\}$. If it is in L, then $(u_1, u_3, v_2, v_1, u_1)$ will form C_5 in L. If it is in $\overline{L}$, then $(u_m, v_1, u_2, u_1, u_3, \ldots, u_m)$ will form C_m in $\overline{L}$.

Case 2. Next, assume that the first case does not hold, and there is some vertex from v_is, say v_3, that is not adjacent in L to the vertex of C_m whichever is joined to

v_1 or v_2 in L. Note that one of the u_1 and u_2, say u_1, must be adjacent to v_1 in L. Then, u_1 must be adjacent to v_3 in $\overline{L}$ and, hence, both u_m and u_2 must be adjacent to v_3 in L. Based on our assumption, both u_m and u_2 must be adjacent to v_1 and v_2 in $\overline{L}$. Then, u_1 must be adjacent to v_2 in L and both u_{m-1} and u_3 must be adjacent to v_1 and v_2 in L. By our assumption, both u_{m-1} and u_3 must be adjacent to v_3 in $\overline{L}$. Similarly, consider the edge $\{u_1, u_3\}$. If it is in L, then $(u_1, u_3, v_2, v_1, u_1)$ will form C_5 in L. If it is in $\overline{L}$, then $(u_m, v_1, u_2, u_1, u_3, \ldots, u_m)$ will form C_m in $\overline{L}$.

Case 3. Lastly, assume that the previous two cases do not hold. Then, for each vertex from v_i, $i \neq 1, 2$, whenever the edges $\{v_1, u_j\}$ and $\{v_i, u_j\}$ are in L, the edge $\{v_2, u_j\}$ is in $\overline{L}$, or whenever the edges $\{v_2, u_j\}$ and $\{v_i, u_j\}$ are in L, the edge $\{v_1, u_j\}$ is in $\overline{L}$. For simplicity, we look at the vertex v_3. Since *Case 2* does not hold, there is at least one vertex from C_m, say u_1, that is adjacent to both v_3 and one of v_1 and v_2, say v_1, in L. Then, the edge $\{u_1, v_2\}$ must be in $\overline{L}$, based on our assumption. This will force both u_m and u_2 to be adjacent to v_2 in L. Now, since *Case 1* does not hold, both u_2 and u_m must be adjacent to v_3 in $\overline{L}$. Then, both u_3 and u_{m-1} must be adjacent to v_3 in L and v_2 in $\overline{L}$. Continuing this argument, v_2 will be adjacent to u_i in L for all even i and v_3 will be adjacent to u_i in L for all odd i. Therefore, m must be even for this case. Now, consider the vertices $v_4, \ldots, v_{m+1}$. If any of them are adjacent to u_i in L for even i, then v_1 must be adjacent to u_i in L for odd i and in $\overline{L}$ for even i. Then, the edge $\{v_1, v_3\}$ must be in L, or else we will get a C_{m+1} in $\overline{L}$. Suppose that all of them are adjacent to u_i in L for odd i. If all the edges $\{v_1, v_k\}$ are in $\overline{L}$, where $k \neq 1, 2, 3$, then we have $(u_m, u_1, v_4, v_1, v_5, u_2, \ldots, u_m)$ as a C_{m+1} in $\overline{L}$. However, if there is an edge $\{v_1, v_k\}$, $k \neq 1, 2, 3$, that is in L, then we will get $(v_3, u_1, v_1, v_k, u_3, v_3)$ as a C_5 in L. Therefore, this subcase cannot happen, and we must have the edge $\{v_1, v_3\}$ in L. Relabel the vertex v_2 as v_3 and v_3 as v_2. Then, we will reach the condition in the second case where the result holds.

In each of these cases, the result holds. By induction, we have $R(C_5, C_m) \leq 2m-1$ for $m \geq 5$.

Now, note that there is complete bipartite graph $K_{m-1,m-1}$ as a subgraph of K_{2m-2}. Colour the edges of this subgraph in colour c_1 and the remaining edges in colour c_2. In this way, we will get a c_1-coloured $K_{m-1,m-1}$ and c_2-coloured $K_{m-1} \cup K_{m-1}$. Hence, there is neither a c_1-coloured C_5 nor a c_2-coloured C_m in this colouring. Thus, we have $R(C_5, C_m) \geq 2m-1 > 2m-2$ for $m \geq 5$.

Therefore, $R(C_5, C_m) = 2m-1$ for $m \geq 5$. Thus, the theorem is valid. □

The complete theorem on $R(C_{m_1}, C_{m_2})$ was given by Faudree and Schelp [35] and Rosta [85] independently. However, these proofs are complicated. In 2001, a simpler proof was provided by Károlyi and Rosta [61]. The theorem is mentioned below but the interested reader is referred to the publications cited above for detailed proofs.

Theorem 3.16. [35, 61, 85]

$$R(C_{m_1}, C_{m_2}) = \begin{cases} 6 & \textit{if } m_1 = m_2 = 3 \textit{ or } 4; \\ 2m_2 - 1 & \textit{if } 3 \le m_1 \le m_2,\ m_1 \textit{ is odd} \\ & \textit{and } (m_1, m_2) \ne (3,3); \\ m_2 - 1 + \frac{m_1}{2} & \textit{if } 4 \le m_1 \le m_2, (m_1, m_2) \ne (4,4) \\ & \textit{and } m_1 \textit{ and } m_2 \textit{ are both even}; \\ \max\{m_2 - 1 + \frac{m_1}{2}, 2m_2 - 1\} & \textit{if } 4 \le m_1 < m_2 \\ & \textit{and } m_1 \textit{ is even and } m_2 \textit{ is odd.} \end{cases}$$

3.4 Ramsey-type Results for Bipartite Graphs

In this section, we present some Ramsey results on the bipartite graphs. We first study the existence of the monochromatic bipartite graph in the edge-colouring of a complete graph. Then, we will introduce the bipartite Ramsey theorem.

Definition 3.17. *We define $R(K_{p_1,p_2}, K_{q_1,q_2})$ as the least N such that, for every edge-colouring of a complete graph K_N with the colours c_1 and c_2, we can get either a monochromatic c_1-coloured K_{p_1,p_2} or a monochromatic c_2-coloured K_{q_1,q_2}.*

Now, we will present some results on this type of Ramsey number.

Theorem 3.18. [51]

$$R(K_{1,m_1}, K_{1,m_2}) = \begin{cases} m_1 + m_2, & \textit{if } m_1 \textit{ or } m_2 \textit{ is odd.} \\ m_1 + m_2 - 1, & \textit{if both } m_1 \textit{ and } m_2 \textit{ are even.} \end{cases}$$

Proof. First, suppose that m_1 or m_2 is odd. Consider any edge-colouring of $K_{m_1+m_2}$ with the colours c_1 and c_2. For each vertex, there are $m_1 + m_2 - 1$ edges incident to it. If at least m_1 of them are c_1-coloured, then we have a c_1-coloured K_{1,m_1}. Suppose that this is not the case. Then there are at least m_2 c_2-coloured edges, forming a c_2-coloured K_{1,m_2}. Hence, we have $R(K_{1,m_1}, K_{1,m_2}) \le m_1 + m_2$. Now, we need to show that $R(K_{1,m_1}, K_{1,m_2}) \ge m_1 + m_2$. Without loss of generality, suppose m_1 is odd. Then $m_1 - 1$ must be even. Hence, in the complete graph $K_{m_1+m_2-1}$, there must exist a regular subgraph of degree $m_1 - 1$. We call it subgraph G. Note that the complement of G, $\overline{G}$ is a regular graph of degree $m_2 - 1$. We colour the complete graph $K_{m_1+m_2-1}$ in such a way that the edges in G has colour c_1 and the edges in $\overline{G}$ have colour c_2. In this way, we have neither a c_1-coloured K_{1,m_1} nor a c_2-coloured K_{1,m_2}. Hence, we have $R(K_{1,m_1}, K_{1,m_2}) \ge m_1 + m_2$. Therefore, $R(K_{1,m_1}, K_{1,m_2}) = m_1 + m_2$ if m_1 or m_2 is odd.

Now, suppose that m_1 and m_2 are both even. Consider any edge-colouring of $K_{m_1+m_2-1}$ with the colours c_1 and c_2. For each vertex, there are $m_1 + m_2 - 2$ edges incident to it. Note that we cannot have exactly $m_1 - 1$ of these edges coloured c_1 for all vertices, because if that were the case, then we would have an odd number of c_1-coloured edges in a graph with an odd number of vertices, which is impossible. Hence, there is at least one vertex v in $K_{m_1+m_2-1}$ for which one of the following conditions holds.

(1) There are at least m_1 c_1-coloured edges incident to v. In this case, we have a c_1-coloured K_{1,m_1}.

(2) There are at most $m_1 - 2$ c_1-coloured edges incident to v. Then, we have more than m_2 c_2-coloured edges incident to v, and so we have a c_2-coloured K_{1,m_2}.

Thus, we have $R(K_{1,m_1}, K_{1,m_2}) \leq m_1 + m_2 - 1$. However in the complete graph $K_{m_1+m_2-2}$, we can find a regular subgraph of degree $m_1 - 1$ and its complementary graph is a regular graph of degree $m_2 - 2$. Thus, we have $R(K_{1,m_1}, K_{1,m_2}) \geq m_1 + m_2 - 1$. Therefore, $R(K_{1,m_1}, K_{1,m_2}) = m_1 + m_2 - 1$ if both m_1 and m_2 are even. □

We have generalised the theorem above as follows.

Theorem 3.19.

$$R(K_{1,m_1}, \ldots, K_{1,m_k}) \leq \begin{cases} m_1 + \cdots + m_k - (k-1) + 1 & \textit{if any of } m_i \textit{ or } k \textit{ is odd;} \\ m_1 + \cdots + m_k - (k-1) & \textit{if all } m_i \textit{ and } k \textit{ are even.} \end{cases}$$

Proof. Suppose that any of the integers $m_1, \ldots, m_k, k$ is odd. Consider any edge-colouring of $K_{m_1+m_2+\cdots+m_k-(k-1)+1}$ with colours $c_1, c_2, \ldots, c_k$. For each vertex, there are $m_1 + m_2 + \cdots + m_k - (k-1)$ edges incident to it. By the Pigeonhole Principle, at least m_i of these edges are c_i-coloured. Then we have a c_i-coloured K_{1,m_i} and we are done. Hence, we have $R(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_k}) \leq m_1 + m_2 + \cdots + m_k - (k-1) + 1$ if m_i or k is odd.

Now, suppose that all of the integers $m_1, \ldots, m_k, k$ are even. Consider any edge-colouring of $K_{m_1+m_2+\cdots+m_k-(k-1)}$ with colours $c_1, c_2, \ldots, c_k$. For each vertex, there are $m_1 + m_2 + \cdots + m_k - k$ edges incident to it. Note that we cannot have exactly $m_1 - 1$ of these edges coloured c_1 for all vertices, because if that is the case, then we have an odd number of c_1-coloured edges in a graph with an odd number of vertices, which is impossible. Hence, for there is at least one vertex v in $K_{m_1+m_2+\cdots+m_k-(k-1)}$ for which one of the following conditions holds.

(1) There are at least m_1 c_1-coloured edges incident to v. In this case, we have a c_1-coloured K_{1,m_1}.

(2) There are at most $m_1 - 2$ c_1-coloured edges incident to v. Then we have at least $m_2 + m_3 + \cdots + m_k - (k-2)$ remaining edges incident to v. By the Pigeonhole Principle, m_i of them must be c_i-coloured, for $i = 2, 3, \ldots, k$. Hence, we have a c_i-coloured K_{1,m_i}.

Thus, we have $R(K_{1,m_1}, K_{1,m_2}, \ldots, K_{1,m_k}) \leq m_1 + m_2 + \cdots + m_k - (k-1)$ if all m_i and k are even. □

Theorem 3.20. $R(K_{1,t}, K_{m_1,m_2}) \leq m_1 + m_2 + t - 1$.

Proof. Consider the complete graph $K_{m_1+m_2+t-1}$. Suppose that we colour all of the edges so that there is no c_1-coloured $K_{1,t}$. Then, for every vertex in $K_{m_1+m_2+t-1}$, there are at most $t-1$ c_1-coloured edges incident to it. Hence, we can find m_1 vertices of $K_{m_1+m_2+t-1}$ such that there are at most $t-1$ of the remaining vertices that are adjacent via a c_1-coloured edge to any one of the m_1 vertices. These m_1 vertices, together with the remaining m_2 vertices, form a c_2-coloured K_{m_1,m_2}. Therefore, $R(K_{1,t}, K_{m_1,m_2}) \leq m_1 + m_2 + t - 1$. □

Theorem 3.21. [53] $R(K_{1,3}, K_{m_1,m_2}) = m_1 + m_2 + 2$.

Proof. From Theorem 3.20, we get $R(K_{1,3}, K_{m_1,m_2}) \leq m_1+m_2+3-1 = m_1+m_2+2$.

On the other hand, in the complete graph $K_{m_1+m_2+1}$ there exists a cycle graph $C_{m_1+m_2+1}$ as the subgraph. Suppose that we colour the complete graph $K_{m_1+m_2+1}$ in such a way that the edges in $C_{m_1+m_2+1}$ are c_1-coloured and the remaining edges are c_2-coloured. In this way, we have neither a c_1-coloured $K_{1,3}$ nor a c_2-coloured K_{m_1,m_2}. Therefore, we have $R(K_{1,3}, K_{m_1,m_2}) \geq m_1 + m_2 + 2$.

Hence, $R(K_{1,3}, K_{m_1,m_2}) = m_1 + m_2 + 2$. □

Previously, we were looking at the results on the existence of monochromatic, complete bipartite graph in the edge-colouring of a complete graph. Now, we will present a Ramsey-type result for edge-colourings of a complete bipartite graph. However, we first introduce the theorem below.

Theorem 3.22. [45] *Let $m \in \mathbb{N}$ and $0 < \varepsilon \leq 1$. There exists a sufficiently large n such that if G is a subgraph of $K_{n,n}$ with at least εn^2 edges, then G has $K_{m,m}$ as a subgraph.*

Proof. The idea of the proof is from [45]; we have added missing proof details here.

We can take any n satisfying $n\binom{\varepsilon n}{m} \geq m\binom{n}{m}$. Let U and V be disjoint sets of n vertices in $K_{n,n}$. Let G be any subgraph of $K_{n,n}$ with at least εn^2 edges. For each $i \in U$, we set $D_i = \{j \in V : \{i,j\} \in G\}$ and $d_i = |D_i|$. Thus, $\sum\limits_{i \in U} d_i \geq \varepsilon n^2$. Set $S = \{(i, X) : X \subset V, |X| = m, X \subset D_i\}$. For each $i \in U$, there are precisely $\binom{d_i}{m}$ X's such that $(i, X) \in S$. Therefore, we have

$$|S| = \sum_{i \in U} \binom{d_i}{m} \geq n \binom{\frac{1}{n}\sum\limits_{i \in U} d_i}{m} \geq n \binom{\frac{\varepsilon n^2}{n}}{m} = n \binom{\varepsilon n}{m} \geq m \binom{n}{m}.$$

For $X \subseteq V, |X| = m$, we set $T_X = \{i \in U : (i, X) \in S\}$. Then, we have $|S| = \sum |T_X|$. Hence, there are $\binom{n}{m}$ summands X such that $|T_X| \geq \frac{|S|}{\binom{n}{m}} \geq \frac{m\binom{n}{m}}{\binom{n}{m}} \geq m$. Let $T_X^* \subseteq T_X$ with $|T_X^*| = m$. We have that $T_X^* \cup X$, which is a subgraph of G, is a complete bipartite graph $K_{m,m}$. □

Theorem 3.23 (Ramsey's Theorem for Bipartite Graphs)**.** *Let $r, m \in \mathbb{N}$. If n is sufficiently large, then each r-colouring of the edges of $K_{n,n}$ gives a complete monochromatic subgraph $K_{m,m}$. The least of such n is known as the bipartite Ramsey Number, $BR(K_{m,m}; r)$.*

Proof. Note that there are n^2 edges in the complete bipartite graph $K_{n,n}$. By the Pigeonhole Principle, in any r-colouring of the edges of $K_{n,n}$, there is at least one colour, say c_i $(1 \leq i \leq r)$, such that at least $\frac{n^2}{r}$ of the edges are c_i-coloured. Now, consider Theorem 3.22. Let $\varepsilon = \frac{1}{r}$ and G be the subgraph of $K_{n,n}$ which consists of all the c_i-coloured edges. Then, there are at least $\frac{n^2}{r} = \varepsilon n^2$ edges in the subgraph G. By Theorem 3.22, G has some c_i-coloured $K_{m,m}$ - which is monochromatic. □

Theorem 3.24. [87] $BR(K_{m_1,m_2};r) \geq (m_1!m_2!r^{m_1m_2-1})^{\frac{1}{m_1+m_2}}$.

Proof. The outline of this proof is given in [87]; we have added some missing details.

Consider a r-colourings of the edges of $K_{n,n}$. First, note that there are r^{n^2} ways to colour the edges of $K_{n,n}$. Further note that there are $r^{n^2-(m_1m_2-1)}$ ways to colour K_{m_1,m_2} in order to obtain a monochromatic K_{m_1,m_2}. It is clear that there are $\binom{n}{m_1}\binom{n}{m_2}$ copies of K_{m_1,m_2} in a complete graph $K_{n,n}$. Hence, there are at most $\binom{n}{m_1}\binom{n}{m_2}r^{n^2-(m_1m_2-1)}$ colourings of $K_{n,n}$ containing some monochromatic K_{m_1,m_2}. Therefore, if we have $\binom{n}{m_1}\binom{n}{m_2}r^{n^2-(m_1m_2-1)} < r^{n^2}$, then there is some r-colouring of the edges of $K_{n,n}$ which has no monochromatic K_{m_1,m_2}.

Now, suppose that we choose our n such that $r^{m_1m_2-1} > (\frac{n^{m_1}}{m_1!})(\frac{n^{m_2}}{m_2!})$ or, equivalently, $n < (m_1!m_2!r^{m_1m_2-1})^{\frac{1}{m_1+m_2}}$. Then we have

$$r^{m_1m_2-1} > \left(\frac{n^{m_1}}{m_1!}\right)\left(\frac{n^{m_2}}{m_2!}\right) > \binom{n}{m_1}\binom{n}{m_2}$$

and thus

$$r^{n^2} > \binom{n}{m_1}\binom{n}{m_2}r^{n^2-(m_1m_2-1)}.$$

Therefore, $BR(K_{m_1,m_2};r) \geq (m_1!m_2!r^{m_1m_2-1})^{\frac{1}{m_1+m_2}}$. □

Chapter 4
Van der Waerden's Theorem

In the previous chapters, we were mainly discussing Ramsey-type results on colourings of the edges of graphs. Starting from this chapter, we will look into the Ramsey-type results on colourings of the set of integers.

In this chapter, we focus on Ramsey-type results guaranteeing the existence of monochromatic arithmetic progressions in colourings of integers. In Section 4.1, we introduce Van der Waerden's Theorem. In Section 4.2, we construct a proof of the theorem. In Section 4.3, we also present the polynomial version of Van der Waerden's Theorem. Then, in Section 4.4, we discuss some bounds on the Van der Waerden numbers.

4.1 Van der Waerden's Theorem

In this section, we present Van der Waerden's Theorem. Before doing so, we first introduce some terminology.

Definition 4.1 (Arithmetic Progression)**.** *An* arithmetic progression *is a sequence of numbers such that the differences between the consecutive terms is constant. An arithmetic progression* $\{a, a+d, \ldots, a+(k-1)d\}$ *is said to be projected from term* a *with common difference* d *and length* k*.*

Example 4.2. $\{3, 7, 11, 15\}$ is an arithmetic progression projected from $a = 3$ and with the common difference $d = 4$ and length $k = 4$. $\{1, 4, 7, 9\}$ is not an arithmetic progression since the difference is not constant as $7 - 4 = 3$ but $9 - 7 = 2$.

Theorem 4.3 (Van der Waerden's Theorem)**.** [102] *Let* $k, r \in \mathbb{N}$*. For sufficiently large* n*, each* r*-colouring of* $[n]$ *gives a monochromatic arithmetic progression of length* k*. The least of such* n *is called the Van der Waerden number, denoted by* $W(k, r)$*.*

Example 4.4. Consider a 2-colouring of [9] in the following way: 1, 4, 5 and 8 are c_1-coloured and 2, 3, 6, 7 and 9 are c_2-coloured. Note that the c_2-coloured 3, 6 and 9 form a monochromatic arithmetic progression of length 3 with the common difference of 3. In fact, $W(3, 2) = 9$; a detailed proof will be given in Section 4.4.

Before we proceed to the proof of Van der Waerden's Theorem, we want to introduce the density version of the theorem, which is also known as Szemerédi's Theorem.

Theorem 4.5 (Szemerédi's Theorem)**.** [100] *Let $k \in \mathbb{N}$ and $S \subset \mathbb{N}$. Suppose S has positive upper density, which means*

$$\limsup_{n\to\infty} \frac{|S \cup [n]|}{n} > 0,$$

then S contains infinitely many arithmetic progression of length k.

Szemerédi's Theorem was first conjectured by Erdős and Turán in 1936 [26], and proven by Szemerédi in 1975 [100]. We are not going to discuss the proof here, interested reader is referred to [100].

4.2 Proof of Van der Waerden's Theorem

In this section, we construct a proof of Van der Waerden's Theorem. We first introduce a lemma to help us.

Lemma 4.6. [37] *Let $k, r \in \mathbb{N}$. Suppose that the Van der Waerden number $W(k-1, r)$ exists for all r. Then for all c, there exists a number $U(k-1, r, c)$ such that if $[U(k-1, r, c)]$ is r-coloured, then there exists $a \in \mathbb{N}$ for which one of the following conditions hold.*

(1) *There are c monochromatic arithmetic progressions of length $k-1$, all with projected first term a, all of different colours, and different from a.*
(2) *There is a monochromatic arithmetic progression of length k.*

Proof. This proof is from [37]. We prove it by induction on c. For $c = 1$, we can take $U(k-1, r, 1) = 2W(k-1, r)$. Let χ be any r-colouring of $[U(k-1, r, 1)]$. Consider the colouring of the last half of $[U(k-1, r, 1)]$, which is of the size of $[W(k-1, r)]$. By the definition of $W(k-1, r)$, there exists a monochromatic progression of length $k-1$ in the r-colouring of $[W(k-1, r)+1, 2W(k-1, r)]$, say $\{a, a+d, \ldots, a+(k-2)d\}$. Now, let $a' = a - d$ and $d' = d$. Then we will get $\{a'+d', a'+2d', \ldots, a'+(k-1)d' \in [W(k-1, r)+1, 2W(k-1, r)]\}$ such that $\chi(a'+d') = \chi(a'+2d') = \cdots = \chi(a'+(k-1)d')$. Now, note that $a' \in [W(k-1, r] \in [U(k, r, 1)]$. If $\chi(a') \neq \chi(a'+d')$, then the first condition in Lemma 4.6 holds; otherwise, we have the second condition.

Now, assume that $U(k-1, r, c)$ exists, we want to show the existence of $U(k-1, r, c+1)$. Take $U(k-1, r, c+1) = 2U(k-1, r, c)W(k-1, r^{U(k-1,r,c)})$. Let χ be any r-colouring of $[U(k-1, r, c+1)]$. Now, we divide $[U(k-1, r, c+1)]$ into $U(k-1, r, c)W(k-1, r^{U(k-1,r,c)})$ numbers, followed by $W(k-1, r^{U(k-1,r,c)})$ blocks of size $U(k-1, r, c)$. We denote these blocks by $B_1, B_2, \ldots, B_{W(k-1, r^{U(k-1,r,c)})}$. By the definition of $W(k-1, r^{U(k-1,r,c)})$, there exists a monochromatic arithmetic progression of length $k-1$ of blocks, say $B_A, B_{A+D}, \ldots, B_{A+(k-2)D}$, which means that these blocks are identically coloured in the r-colouring of $[U(k-1, r, c+1)]$. Consider the block B_A. Note that there are $U(k-1, r, c)$ terms in the block. If the second condition of Lemma 4.6 holds, then we are done. Otherwise, we will have c arithmetic progressions of length $k-1$, all with projected first term a, all different colours, and different from a. We need to find one more monochromatic set of an arithmetic progression. Since $B_A, B_{A+D}, \ldots, B_{A+(k-2)D}$ are identically coloured, the terms $a+D, a+2D, \ldots, a+(k-2)D$ must be monochromatic. Note that this monochromatic progression of length $k-1$ is different from the previous c arithmetic progressions as all these arithmetic progressions are coloured differently with a. Hence, we have $c+1$ arithmetic progressions of length $k-1$, all with projected first term a, and all of different colours.

Thus, by induction, the lemma holds. $\square$

Now we prove the Van der Waerden's Theorem (Theorem 4.3).

Proof. Note that to prove the theorem, we only need to show the existence of $W(k,r)$ for all $k \geq 1$. We prove it by induction on k. First, notice that the case $k=1$ is trivial and that $W(1,r)=r$ since we can take any term from r-coloured $[r]$ and we can get a monochromatic arithmetic progression of length 1. Now, suppose that $W(k-1,r)$ exists. We wish to show that $W(k,r)$ exists. Since Lemma 4.6 is valid for all c, we consider the case when $c=r$. Hence, there exists $n=U(k-1,r,r)$ such that if $[n]$ is r-coloured, then there is a monochromatic arithmetic progression of length k or r arithmetic progressions of length $k-1$, all of different colours, with projected term a whose colour differs from all of them. Now, note that there are only r colours, so the latter case cannot happen. Therefore, we must have a monochromatic arithmetic progression of length k. Hence, $W(k,r)$ exists. Thus, by induction, $W(k,r)$ exists for all $k \geq 1$ and the theorem is proven. $\square$

4.3 Polynomial Van der Waerden's Theorem

In this section, we are going to introduce the polynomial Van der Waerden's Theorem. First, we introduce the following definition.

Definition 4.7 (Polynomial)**.** *A polynomial $P(x)$ is defined as an expression built from constants and variables by the means of addition, multiplication and exponential to a non-negative power.*

$$P(x) = \sum_{k=0}^{n} a_k x^n = a_n x_n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \,,$$

where a_i are constants and x is the variable.

Theorem 4.8 (Polynomial Van der Waerden's Theorem)**.** [6]
Let $k, r \in \mathbb{N}$ and $p_1, \ldots, p_k \in \mathbb{Z}[x]$ with $p_i(0)=0$. If n is sufficiently large, then any r-colouring of $[n]$ will give a monochromatic $a, a+p_1(d), \ldots, a+p_k(d)$.

We will not prove the theorem here. The interested reader is referred to [6, 105] for combinatorial proofs.

4.4 Van der Waerden Numbers

In this section, we will present some results on the Van der Waerden numbers.

Theorem 4.9.

(1) $W(k,1)=k$
(2) $W(2,r)=r+1$.

Proof.

(1) Note that if we colour the set $[k]$ with a single colour, then the set $[k]$ itself will form a monochromatic arithmetic progression of length k with common difference 1.
(2) Consider any r-colouring of $[r+1]$. By the Pigeonhole Principle, at least two of them are coloured with the same colour. These two same coloured terms will form a monochromatic arithmetic progression of length 2.

□

Theorem 4.10. *$W(3,2) = 9$.*

Proof. First, we need to show that $W(3,2) \leq 9$. We divide the 2-colouring of $[9]$ into two cases.

Case 1: Both 1 and 2 are coloured with the same colour.
Without loss of generality, let both 1 and 2 be c_1-coloured. To avoid having monochromatic arithmetic progression of length 3, 3 must be c_2-coloured. Suppose that 4 is c_1-coloured. Then, both 6 and 7 must be c_2-coloured, or else $\{1,4,7\}$ and $\{2,4,6\}$ will form c_1-coloured arithmetic progressions of length 3. Then 5 must be c_1-coloured, or $\{5,6,7\}$ will form a c_2-coloured arithmetic progression. Now, if 8 is c_1-coloured, then $\{2,5,8\}$ will form a c_1-coloured arithmetic progression. If 8 is c_2-coloured, then $\{6,7,8\}$ will form a c_2-coloured arithmetic progression. On the other hand, suppose that 4 is c_2-coloured. Then 5 must be c_1-coloured or $\{3,4,5\}$ will form a c_2-coloured arithmetic progression. Then both 8 and 9 must be c_2-coloured, or else $\{1,5,9\}$ and $\{2,5,8\}$ will form a c_1-coloured arithmetic progressions of length 3. Then 7 will then be forced to be c_1-coloured to avoid having monochromatic progression of length 3. Now if 6 is c_1-coloured, then $\{5,6,7\}$ will form a c_1-coloured arithmetic progression. If 6 is c_2-coloured, then $\{4,6,8\}$ will form a c_2-coloured arithmetic progression. Hence, in this case, we will get a monochromatic arithmetic progression of length 3 no matter how we colour the set $[9]$.

Case 2: 1 and 2 are coloured with different colours.
Without loss of generality, we let 1 be c_1-coloured and 2 be c_2-coloured. Suppose that 3 is c_1-coloured. Since 1 and 3 are both c_1-coloured, 5 must be c_2-coloured. Then 8 will be forced to be c_1-coloured or else $\{2,5,8\}$ will form a c_2-coloured arithmetic progression. If 4 is c_1-coloured, then 7 must be c_2-coloured or else $\{1,4,7\}$ will form a c_1-coloured arithmetic progression. In this way, if 6 is c_1-coloured, we will get $\{4,6,8\}$ as a c_1-coloured arithmetic progression and if 6 is c_2-coloured, then we will get $\{5,6,7\}$ as a c_2-coloured arithmetic progression. Now, if 4 is c_2-coloured, then 6 must be c_1-coloured or else $\{2,4,6\}$ will form a c_2-coloured arithmetic progression. This will force 7 to be c_2-coloured or $\{6,7,8\}$ will form a c_1-coloured arithmetic progression. In this way, if 9 is c_1-coloured, then we will get $\{3,6,9\}$ as a c_1-coloured arithmetic progression. If 9 is c_2-coloured, then we will get $\{5,7,9\}$ as a c_2-coloured arithmetic progression. On the other hand, suppose that 3 be c_2-coloured. This will force 4 to be c_1-coloured or $\{2,3,4\}$ will form a c_2-coloured arithmetic progression. Since 1 and 4 are both c_1-coloured, 7 must be c_2-coloured. Then, 5 must be c_1-coloured, or else $\{3,5,7\}$ will form a c_2-coloured arithmetic progression. This will

force 6 to be c_2-coloured to avoid getting c_1-coloured arithmetic progression $\{4, 5, 6\}$. Since both 6 and 7 are c_2-coloured, we must colour 8 with c_1. In this way, if 9 is c_1-coloured, $\{1, 5, 9\}$ will form a c_1-coloured arithmetic progression. If 9 is c_2-coloured, $\{3, 6, 9\}$ will form a c_2-coloured arithmetic progression. Hence, in this case, we will get a monochromatic arithmetic progression of length 3 no matter how we colour the set $[9]$.

Thus, in both cases, we will get a monochromatic arithmetic progression of length 3. Hence, $W(3, 2) \leq 9$.

Now, we need to show that $W(3, 2) \geq 9$. Consider the colouring of $[8]$ in following way: 1,2,5,6 are c_1-coloured and 3,4,7,8 are c_2-coloured. In this way, we have no monochromatic arithmetic progression of length 3. Hence, $W(3, 2) \geq 9 > 8$.

Therefore, we have $W(3, 2) = 9$. $\square$

Here, we state some famous results on the upper bound of the Van der Waerden number by Gowers [41] with the proofs omitted.

Theorem 4.11. [41] *For $k \geq 2$, $W(k, 2) \leq 2^{2^{2^{2^{2^{k+9}}}}}$.*

Theorem 4.12. [41] *Let $f(k, r) = r^{2^{2^{k+9}}}$. Then $W(k, r) \leq 2^{2^{f(k,r)}}$.*

In [57], Huang and Yang had claimed that the following theorem holds. The interested reader is referred to [57] for the proof.

Theorem 4.13. [57] *Let $r > 5$. Then $W(3, r) < (\frac{r}{4})^{3^r}$.*

We now extend the concept of Van der Waerden Number $W(k, r)$ and define $W(k_1, \ldots, k_r; r)$ to be the least n such that in each r-colouring of $[n]$, there is always a c_i-coloured arithmetic progression of length k_i, for some $1 \leq i \leq r$. Note that the existence of $W(k_1, \ldots, k_r; r)$ is guaranteed as $W(k_1, \ldots, k_r; r) \leq W(\max(k_1, \ldots, k_r), r)$ because a monochromatic arithmetic progression of length $\max(k_1, \ldots, k_r)$ contains an arithmetic progression of length k_i, for $1 \leq i \leq r$.

Theorem 4.14.

(1) *$W(1, k; 2) = k$.*
(2) *$W(2, k; 2) = 2k$, if k is odd.*
(3) *$W(2, k; 2) = 2k - 1$, if k is even.*

Proof.

(1) In any 2-colouring of $[k]$, we have that either all terms are c_2-coloured, forming an arithmetic progression of length k with common difference of 1, or at least one c_1-coloured term, forming an arithmetic progression of length 1.
(2) Consider any 2-colouring of $[2k]$. If there is none or at least 2 c_1-coloured terms, then we are done. Suppose there is only one c_1-coloured term, say a. Now we partition the remaining $2k-1$ c_2-coloured terms into two classes: those less than a and those more than a. By the Pigeonhole Principle, one of the partitions will have at least k terms, forming a c_2-coloured arithmetic progression. Hence, $W(2, k; 2) \leq 2k$. Now, if we colour the $[2k - 1]$ in such a way that k is c_1-coloured and the rest of the elements in $[2k - 1]$ are c_2-coloured, we will have neither a c_1-coloured arithmetic progression of length 2 nor a c_2-coloured

arithmetic progression of length k. Therefore, $W(2,k;2) \geq 2k > 2k-1$. Thus, we have $W(2,k;2) = 2k$.

(3) Consider any 2-colouring of $[2k-1]$. If there is none or at least 2 c_1-coloured terms, then we are done. Suppose that there is only one c_1-coloured term, say a. Now we partition the remaining $2k-2$ c_2-coloured terms into two classes, those less than a and those more than a. If either one of the partitions have at least k terms, then we will get a c_2-coloured arithmetic progression of length k. Or else, a must be even and equal to k and both of the partitions must contain $k-1$ terms. In this way, $1, 3, 5, \ldots, 2k-1$ will form a c_2-coloured arithmetic progression of length k. Hence, $W(2,k;2) \leq 2k$. Now, if we colour the $[2k-2]$ in such a way that k is c_1-coloured and the rest are c_2-coloured, then we will have neither a c_1-coloured arithmetic progression of length 2 nor a c_2-coloured arithmetic progression of length k. Therefore, $W(2,k;2) \geq 2k-1 > 2k-2$. Thus, we have $W(2,k;2) = 2k-1$.

□

Theorem 4.15.

$$W(k_1,k_2;2) \leq k_2 W(k_1,k_1,\ldots,k_1;2^{k_2}) = k_2 W(k_1,2^{k_2}) \leq k_2 2^{2^{2^{k_2}2^{2^{k_1+9}}}}.$$

Proof. Consider any 2-colouring of $[k_2 W(k_1,2^{k_2})]$. Now partition these $k_2 W(k_1,2^{k_2})$ terms into $W(k_1,2^{k_2})$ blocks $B_1, B_2, \ldots, B_{W(k_1,2^{k_2})}$ where

$$B_i = \{(i-1)k_2+1, (i-1)k_2+2, \ldots, (i-1)k_2+k_2\}.$$

By the definition of $W(k_1,2^{k_2})$, we have that $B_j, B_{j+d}, B_{j+2d}, \ldots, B_{j+(k_1-1)d}$ have the same colour. If any term in the block B_j is c_1-coloured, say a, then $a+k_2d \in B_{j+d}$, $a+2k_2d \in B_{j+2d}, \ldots, a+(k_1-1)k_2d \in B_{j+(k_1-1)d}$ will also be c_1-coloured and these will form a c_1-coloured arithmetic progression of length k_1. Or else, all the k_2 terms in the block B_j are c_2-coloured and these will form a c_2-coloured arithmetic progression of length k_2. Thus, we have $W(k_1,k_2;2) \leq k_2 W(k_1,k_1,\ldots,k_1;2^{k_2}) = k_2 W(k_1,2^{k_2})$. By Theorem 4.12, we have $W(k_1,2^{k_2}) \leq 2^{2^{2^{k_2}2^{2^{k_1+9}}}}$. Therefore, $W(k_1,k_2;2) \leq k_2 2^{2^{2^{k_2}2^{2^{k_1+9}}}}$. □

Theorem 4.16. *Let $k \geq 3$. Then $W(3,k;2) < k(2^{3^{2^k}(k-2)})$.*

Proof. By Theorem 4.15, we have $W(3,k;2) \leq kW(3,2^k)$. Now for $k \geq 3$, we have $2^k > 5$. Hence, by Theorem 4.13, we have $W(3,2^k) < (\frac{2^k}{4})^{3^{2^k}} = 2^{3^{2^k}(k-2)}$. Thus, we have $W(3,k;2) \leq kW(3,2^k) < k(2^{3^{2^k}(k-2)})$. □

To end this chapter, we tabulate some known exact Van der Waerden numbers in the following tables, after compiling from various reference as cited respectively.

(k,r)	1	2	3	4
1	1	2	3	4
2	2	3	4	5
3	3	9	27[14]	76[5]
4	4	35[14]	293[66]	
5	5	178[96]		
6	6	1132[67]		

Table 4.1: Van der Waerden number $W(k,r)$.

(k_1,k_2)	3	4	5	6	7
3	9	18[14]	22[14]	32[14]	46[14]
4	18	35	55[14]	73[5]	109[4]
5	22	55	178	206[65]	260[1]
6	32	73	206	1132	
7	46	109	260		

Table 4.2: Van der Waerden number $W(k_1,k_2,2)$.

Chapter 5
Schur's Theorem

In this chapter, we introduce Schur's Theorem, one of the Ramsey-type results concerning equations. In Section 5.1, we present and prove Schur's Theorem. Next, in Section 5.2, we look at some results on Schur numbers. In Section 5.3, we discuss generalisations of Schur's Theorem, specifically looking at Rado's Theorem and Folkman's Theorem.

5.1 Schur's Theorem

In this section, we present and prove Schur's Theorem. Schur's Theorem was given and proven by Issai Schur in his publication in 1916 [92].

Theorem 5.1 (Schur's Theorem)**.** [92] *Let $N, r \in \mathbb{N}$. If $[N]$ is r-coloured, then there is some same-coloured $a, b, c \in [N]$, such that $a + b = c$, where a and b are not necessarily distinct. The least of such N is called the* Schur number *and is denoted by $S(r)$; furthermore, such a, b and c are called a* Schur triple*.*

Proof. By Ramsey's Theorem (Theorem 2.7), there exists $N+1 = R(k_1, k_2, \ldots, k_r) = R(3; r)$, where $k_1 = k_2 = \cdots = k_r = 3$ such that for any r-colouring K_{N+1}, there exists a monochromatic subgraph K_3. Now, consider any r-colouring of $[N]$ and let K_{N+1} be a complete graph with $N + 1$ vertices. Label each vertex of K_N from 1 to $N + 1$. Colour each edge with the colour corresponding to the positive difference of the connecting vertices in the r-colouring of $[N]$. For instance, colour the edge connecting the vertices labelled 2 and 3 with colour 1 in the r-colouring of $[N]$. By the definition of $N + 1$, there is a monochromatic triangle in K_{N+1}, with three labelled vertices, say i, j and k, for $i < j < k$. Since the edges $\{i, j\}$, $\{j, k\}$ and $\{i, k\}$ are of the same colour, $j - i$, $k - j$, and $k - i$ are of the same colour in the r-colouring of $[N]$. Let $a = j - i$, $b = k - j$ and $c = k - i$, and note that a, b and c are same-coloured and that $a + b = (j - i) + (k - j) = k - i = c$. Then, we have proven that the theorem is valid. □

Example 5.2. Let $r = 2$. Consider a 2-colouring of [5] in the following way: 1, 4 and 5 are c_1-coloured and 2 and 3 are c_2-coloured. Note that 1, 4 and $1 + 4 = 5$ are all c_1-coloured. In fact, $S(2) = 5$; a detailed proof will be given in Section 5.2.

5.2 Schur's Numbers

In this section, we present some results on Schur numbers, the first of which is, as far as we can tell, new. We also independently prove that $S(1) = 2$, $S(2) = 5$ and $S(3) = 14$ (see Theorems 5.4 and 5.7). These results are without doubt not new; however, we could not find neither proofs nor original references to them in the literature.

Theorem 5.3. *For $r \geq 1$, $S(r) \leq R(3;r) - 1 \leq r!(\frac{e-e^{-1}+3}{2}) \approx 2.68r!$.*

Proof. It follows from the proof of Theorem 5.1 that there is a monochromatic Schur triple in the r-colouring of $[N]$, where $N+1 = R(3;r)$. Hence, $S(r) \leq R(3;r) - 1$. Now, by Theorem 2.24, we have $S(r) \leq R(3;r) - 1 \leq r!(\frac{e-e^{-1}+3}{2}) \approx 2.68r!$. □

Theorem 5.4.

(1) $S(1) = 2$
(2) $S(2) = 5$.

Proof.

(1) It is clear that there is no monochromatic Schur triple in any colouring of [1], since there is only one element in [1] and $1 + 1 \neq 1$. Therefore, $S(1) \geq 2 > 1$. Note that in any single-coloured [2], we can find a monochromatic Schur triple, $a + b = c$ in the colouring where $a = b = 1$ and $c = 2$. Hence, $S(1) \leq 2$. Thus, we have $S(1) = 2$.
(2) Consider a colouring of [4] in such a way that 1 and 4 are c_1-coloured and 2 and 3 are c_2-coloured. Note that there is no monochromatic Schur triple in this colouring. Therefore, $S(2) \geq 5 > 4$. Now, by Theorem 5.3, $S(2) \leq R(3;2) - 1 = 6 - 1 = 5$. Thus, we have $S(2) = 5$.

□

Now, we will look at some lower bounds on Schur numbers.

Theorem 5.5. [92] $S(r+1) \geq 3S(r) - 1 > 3S(r) - 2$.

Proof. The main idea of this proof is given in [92]; here, we have added some details.

Let $S(r) = n$. Then there is a r-colouring, say χ, of $[n-1]$ that does not contain the monochromatic Schur triple. Now, let χ' be a $(r+1)$-colouring of $[3n-2]$ in such a way that

$$\chi'(x) = \begin{cases} \chi(x) & \text{for } x \in [1, n-1]\,; \\ c_{r+1} & \text{for } x \in [n, 2n-1]\,; \\ \chi(x - (2n-1)) & \text{otherwise}\,. \end{cases}$$

We claim that there is no monochromatic Schur triple in the χ'-colouring of $[3n-2]$. Suppose to the contrary that there is one, say a, b, c, with $a \leq b < c$. Consider the colour c_{r+1}. Since $c = a + b \geq 2a \geq 2n \notin [n, 2n-1]$, the Schur triple cannot be c_{r+1}-coloured. Now, consider the other colours. Since there is no monochromatic triple in $[1, n-1]$ as $\chi' = \chi$ for $[1, n-1]$ and $[n-1+1, 2(n-1)] = [n, 2n-2]$ is c_{r+1}-coloured, we see that a and b of the Schur triple cannot both be from the interval $[1, n-1]$. Similarly, it is also impossible for both a and b to be from the interval $[2n, 3n-2]$ because if that is the case, then $c = a + b \geq 2a \geq 2(2n) > 3n - 2 \notin [2n, 3n-2]$. Hence, we must have $a \in [1, n-1]$, $b \in [2n, 3n-2]$ and $a + b = c \in [2n, 3n-2]$ with $\chi'(a) = \chi'(b) = \chi'(c) = \chi(a)$. Now, let $b' = b - (2n-1) \in [1, n-1]$ and $c' = c - (2n-1) \in [1, n-1]$. Note that $\chi'(b) = \chi(b') = \chi(a), \chi'(c) = \chi(c') = \chi(a)$ and $a + b' = a + b - (2n-1) = c - (2n-1) = c'$: we have a monochromatic Schur triple in χ, a contradiction. Hence, there is no monochromatic Schur triple in χ'. Thus, $S(r+1) > 3n - 2 = 3S(r) - 2$. Therefore, we have $S(r+1) \geq 3S(r) - 1$. □

Theorem 5.6. [92] *For $r \geq 1$, $S(r) \geq \frac{3^r+1}{2}$.*

Proof. We prove the theorem by induction on r. First, note that, by Theorem 5.4, $S(1) = 2 \geq \frac{3^1+1}{2}$. Now, assume that $S(r) \geq \frac{3^r+1}{2}$. We need to show that $S(r+1) \geq \frac{3^{r+1}+1}{2}$. By Theorem 5.5, $S(r+1) \geq 3S(r) - 1 \geq 3(\frac{3^r+1}{2}) - 1 = \frac{3^{r+1}+1}{2}$. Hence by induction, we have $S(r) \geq \frac{3^r+1}{2}$ for $r \geq 1$. □

Theorem 5.7. $S(3) = 14$.

Proof. By Theorem 5.6, $S(3) \geq \frac{3^3+1}{2} = 14$. Now, we need to show that $S(3) \leq 14$. Suppose to the contrary that there is a 3-colouring of [14] that does not contain any monochromatic Schur triple. Without loss of generality, we assume that 1 is c_1-coloured. Since $1+1 = 2$, 2 cannot be c_1-coloured. Again, without loss of generality, we let 2 be c_2-coloured. Now, there are several cases to be considered.

Case A: 3 is c_1-coloured.

Note that in this case, 4 must be c_3-coloured. Now consider the colour of 5.

Case A1: 5 is c_1-coloured.

Then, 8 must be c_2-coloured. This will force 6 to be c_3-coloured. Now, note that in this case, no matter how 10 be coloured, we will get a monochromatic Schur triple, hence a contradiction.

Case A2: 5 is c_2-coloured.

Note that in this case, 6 can only be either c_2-coloured or c_3-coloured. We first consider the case that 6 is c_2-coloured. In this case, 8 must be c_1-coloured. This will force 7 to be c_3-coloured. Now, no matter how 11 be coloured, we will obtain a monochromatic Schur triple, thus a contradiction. On the other hand, if 6 is c_3-coloured, 10 must be c_1-coloured. To avoid having monochromatic Schur triple, 7 must be c_3-coloured, and hence 11 must be c_2-coloured. Now, note that no matter how 13 is coloured, we will get a monochromatic Schur triple, a contradiction.

Case A3: 5 is c_3-coloured.

Similar to Case A2, 6 can only be either c_2-coloured or c_3-coloured. Suppose that 6 is c_2-coloured. 8 must be c_1-coloured. Then, 9 must be c_2-coloured and this will force 11 to be c_3-coloured. Now, consider the colour of 7. We will get a monochromatic Schur triple no matter what colour is used, hence a contradiction. Now, suppose that 6 is c_3-coloured. Since 8 cannot be c_3-coloured, there are two subcases to be considered. First, 8 is c_1-coloured. This will force 11 to be c_2-coloured. Then, no matter how 9 be coloured, we will get a monochromatic Schur triple, hence a contradiction. Now, if 8 is c_2-coloured, then 10 need to be c_1-coloured. This will cause 11 to be c_2-coloured, and hence 13 must be c_3-coloured. In this case, any colouring of 9 will grant us a monochromatic Schur triple, thus a contradiction.

Case B: 3 is c_2-coloured.

In this case, 4 cannot be c_2-coloured. We consider the colour of 4.

Case B1: 4 is c_1-coloured.

In this case, 5 must be c_3-coloured. Since 6 cannot be c_2-coloured, there are two subcases to be considered. Suppose that 6 is c_1-coloured. Then, 10 must be c_2-coloured. This will force 7 to be c_3-coloured. In this case, no matter how we colour 12, we will get a monochromatic Schur triple, hence a contradiction. Now, suppose that 6 is c_3-coloured. Note that 8 cannot be c_1-coloured. If 8 is c_2-coloured, then both 10 and 11 cannot be coloured with c_2 and c_3, but this will yield a c_1-coloured Schur triple, thus a contradiction. On the other hand, suppose that 8 is c_3-coloured. In this subcase, 10, 11, 12, 13 and 14 cannot be c_3-coloured. If 12 is c_1-coloured, then 11 and 13 must be c_2-coloured and we will get a c_2-coloured Schur triple. If 12 is c_2-coloured, then 10 and 14 must be c_1-coloured. This will give us a c_1-coloured Schur triple, thus a contradiction.

Case B2: 4 is c_3-coloured.

In this case, 5 cannot be c_2-coloured. There are two subcases to be considered for the colouring of 5. Suppose that 5 is c_1-coloured. Then, 6 must be c_3-coloured. This forces 10 to be c_2-coloured; hence, 8 must be c_1-coloured. Then, 13 must be c_3-coloured. Now, any colouring of 7 will give us a monochromatic Schur triple, hence leads us to a contradiction. Now, suppose 5 is c_3-coloured. Note that both 8 and 10 cannot be c_3-coloured. Since $2+8=10$, 8 and 10 cannot be both c_2-coloured and hence one of them must be c_1-coloured. Then, 9 must be c_2-coloured. Consider the colouring of 6. Since 3 is c_2-coloured, 6 cannot be c_2-coloured. If 6 is c_1-coloured, then 7 must be c_3-coloured. Then, no matter how 12 be coloured, we will obtain a monochromatic Schur triple, thus gives us a contradiction. If 6 is c_3-coloured, then 11 must be c_1-coloured and hence 10 must be c_2-coloured. Then, no matter how we colour 12, we will have a monochromatic Schur triple, hence a contradiction.

Case C: 3 is c_3-coloured.

Consider the colour of 4; it can only be c_1 or c_3.

Case C1: 4 is c_1-coloured.

In this case, 8 cannot be c_1-coloured. Therefore, there are two subcases to be considered on the colouring of 8. First, suppose 8 is c_2-coloured. Then, 6 must be c_1-coloured and hence 10 must be c_3-coloured. This will force 5 to be c_2-coloured and 13 to be c_1-coloured. Now, no matter how we colour 7, we will get a monochromatic Schur triple, hence leads us to a contradiction. Next, suppose 8 is c_3-coloured. 5 must be c_2-coloured. Note that 11 cannot be c_3-coloured. If 11 is c_1-coloured, then 7 must be c_3-coloured. Whichever colour we use to colour 10, we will obtain a monochromatic Schur triple, thus a contradiction. If 11 is c_3-coloured, then 6 must be c_1-coloured and 10 must be c_3-coloured. Then, any colouring of 7 will give us a monochromatic Schur triple, hence again a contradiction.

Case C2: 4 is c_3-coloured.

In this case, 6, 7 and 8 cannot be c_3-coloured. If 7 is c_1-coloured, then 6 and 8 must be c_2-coloured. However, 2, 6 and 8 would form a c_2-coloured Schur triple. Hence, 7 must be c_2-coloured. Now, consider the colour of 6. Suppose that 6 is c_1-coloured; then 5 must be c_3-coloured and so 9 must be c_1-coloured. This forces both 8 and 10 to be c_2-coloured but 2, 8 and 10 then form a c_2-coloured Schur triple, a contradiction. On the other hand, suppose that 6 is c_2-coloured. Then 8 must be c_1-coloured and 9 must be c_3-coloured. This causes 5 to be c_1-coloured. However we colour 13, we will have a monochromatic Schur triple, a contradiction.

All possible cases lead us to a contradiction. Therefore, in any colouring of [14], we will have a monochromatic Schur triple. Hence, $S(3) \leq 14$.

Thus, we have $S(3) = 14$. $\square$

To end this section, we list some known Schur numbers in the table below.

r	S(r)
1	2
2	5
3	14
4	45[40]

Table 5.1: Known Schur numbers $S(r)$

5.3 Generalisations of Schur's Theorem

In this section, we present some generalisations of Schur's Theorem. Previously, we have discussed the existence of monochromatic Schur triples $a + b = c$ in colourings of $[N]$ where $N \in \mathbb{N}$. Now, we will look into the case with four terms, which is $a + b + c = d$ (Theorem 5.8), and then the case with k terms, namely $x_1 + x_2 + \cdots + x_{k-1} = x_k$ (Theorem 5.9). We conceived of these results by looking at the proof of Schur's Theorem and wondering if Schur's Theorem could be generalised with respect to more terms, and it was possible to prove that it indeed was. Later, however, we found that these results already existed in the literature; see [69].

Similarly, we independently discovered and proved a bound on k-term Schur numbers $S(k; r)$ (Theorem 5.10), proved that $S(4; 2) = 11$ (Theorem 5.11) and proved an exact expression for $S(k; 2)$ (Theorem 5.12). These and more general results were also later found to already exist in the literature; see [69].

Theorem 5.8. *Let $N, r \in \mathbb{N}$. If $[N]$ is r-coloured, then there are same-coloured $a, b, c, d \in [N]$ such that $a + b + c = d$. We denote the least of such N by $S(4; r)$.*

Proof. By Ramsey's Theorem (Theorem 2.7), there exists $N+1 = R(k_1, k_2, \ldots, k_r) = R(4; r)$, where $k_1 = k_2 = \cdots = k_r = 4$ such that there is a monochromatic K_4 in any r-colouring of K_{N+1}, . Now, consider any r-colouring of $[N]$. Let K_{N+1} be a complete graph with $N + 1$ vertices. Label each of the vertices of K_N from 1 to $N + 1$. Colour each edge with the colour corresponding to the positive difference of the end vertices in the r-colouring of $[N]$. By the definition of $N + 1$, there is a monochromatic K_4 in K_{N+1}, with four labelled vertices, say i, j, k and l, for $i < j < k < l$. Since the edges $\{i, j\}$, $\{j, k\}$, $\{k, l\}$ and $\{i, l\}$ are of the same colour, it follows that $j - i$, $k - j$, $l - k$ and $l - i$ are of the same colour in the r-colouring of $[N]$. Let $a = j - i$, $b = k - j$, $c = l - k$ and $d = l - i$, and note that a, b, c and d are of the same colour and that $a + b + c = (j - i) + (k - j) + (l - k) = l - i = d$. Then, we have proven that the theorem is valid. □

Theorem 5.9. *Let $N, r \in \mathbb{N}$ and $k \geq 3 \in \mathbb{N}$. If $[N]$ is r-coloured, then there are some same-coloured $x_1, x_2, \ldots, x_k \in [N]$ such that $x_1 + x_2 + \cdots + x_{k-1} = x_k$. We denote the least of such N by $S(k; r)$ where $S(3; r)$ is the Schur number $S(r)$, as defined in Theorem 5.1.*

Proof. By Ramsey's Theorem (Theorem 2.7), there exists $N+1 = R(k_1, k_2, \ldots, k_r) = R(k; r)$, where $k_1 = k_2 = \cdots = k_r = k$ such that for any r-colouring K_{N+1}, there exists a monochromatic K_k, where $k \geq 3 \in \mathbb{N}$. Now, consider any r-colouring of $[N]$. Let K_{N+1} be a complete graph with $N+1$ vertices and label these vertices from 1 to $N+1$. Colour each edge with the colour corresponding to the positive difference of the end vertices in the r-colouring of $[N]$. By the definition of $N+1$, there is a monochromatic K_k in K_{N+1}, with k labelled vertices, say $v_1, v_2, \ldots, v_k$, where $v_1 < v_2 < \ldots < v_k$. Since the edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{k-1}, v_k\}$ and $\{v_1, v_k\}$ are of the same colour, it follows that $v_2 - v_1, v_3 - v_2, \ldots, v_k - v_{k-1}$ and $v_k - v_1$ are of the same colour in the r-colouring of $[N]$. Let $x_1 = v_2 - v_1, x_2 = v_3 - v_2, \ldots, x_{k-1} = v_k - v_{k-1}$ and $x_k = v_k - v_1$, and note that $x_1, x_2, \ldots, x_k$ are same-coloured and that we have $x_1 + x_2 + \cdots + x_{k-1} = (v_2 - v_1) + (v_3 - v_2) + \cdots + (v_k - v_{k-1}) = v_k - v_1 = x_k$. Thus, the theorem is valid. □

Theorem 5.10. *For $r \geq 1$ and $k \geq 3 \in \mathbb{N}$, $S(k; r) \leq R(k; r) - 1$.*

Proof. It follows from the proof of Theorem 5.9 that there are monochromatic $x_1, x_2, \ldots, x_k \in [N]$, where $x_1 + x_2 + \cdots + x_{k-1} = x_k$ in the r-colouring of $[N]$, in which $N + 1 = R(k; r)$. Hence, $S(k; r) \leq R(k; r) - 1$. □

Theorem 5.11. $S(4; 2) = 11$.

Proof. First, we need to show that $S(4; 2) \geq 11 > 10$. Consider the following colouring of $[10]$. Colour 1, 2, 9 and 10 with colour c_1 and 3, 4, 5, 6, 7 and 8 with colour c_2. In this colouring, we have no monochromatic set of four terms that satisfy $a + b + c = d$. Hence, $S(4; 2) \geq 11 > 10$. Next, we have to show that $S(4; 2) \leq 11$. Suppose to the contrary that there is no monochromatic set $\{a, b, c, d\}$ in the colouring of $[11]$ with 2 colours. Without loss of generality, we assume that 1 is c_1-coloured. Since $1 + 1 + 1 = 3$, 3 must be c_2-coloured. Then, 9 must be c_1-coloured as $3 + 3 + 3 = 9$. This will force 7 and 11 to be c_2-coloured because $1+1+7 = 9$ and $1+1+9 = 11$. Then, 5 must be c_1-coloured because $3+3+5 = 11$. Now, no matter how 2 is coloured, we will have either c_1-coloured $1 + 2 + 2 = 5$ or c_2-coloured $7 + 2 + 2 = 11$. Hence, $S(4; 2) \leq 11$. Thus, we have $S(4; 2) = 11$. □

Theorem 5.12. *For $k \geq 3$, $S(k; 2) = (k-1)^2 + (k-2)$.*

Proof. First, we need to show that $S(k; 2) \geq (k-1)^2 + (k-2)$. Let $N = (k-1)^2 + (k-2) - 1 = (k-1)^2 + (k-3)$. Let χ be the colouring of $[N]$ with two colours in the following way:

$$\chi(x) = \begin{cases} c_1, & \text{for } x \in [k-1, (k-1)^2 - 1]; \\ c_2, & \text{otherwise.} \end{cases}$$

Consider all c_1-coloured elements. Note that the smallest c_1-coloured element is $k-1$. Hence, the smallest possible sum of k c_1-coloured numbers is $(k-1) + \cdots + (k-1) = (k-1)(k-1) = (k-1)^2$ which is strictly greater than $(k-1)^2 - 1$ which is the largest c_1-coloured element. Thus, it is impossible to have monochromatically c_1-coloured $x_1, x_2, \ldots, x_k \in [N]$, where $x_1 + x_2 + \cdots + x_{k-1} = x_k$. Now, consider all c_2-coloured elements. Divide them into two partitions, $[1, k-2]$ and $[(k-1)^2, (k-1)^2+(k-3)]$. For the numbers in the first partition, all the possible sums are from $1+1+\cdots+1 = k-1$ to $(k-2)+\cdots+(k-2) = (k-1)(k-2)$, which are all c_1-coloured. Therefore, there is no monochromatic solution for $x_1+x_2+\cdots+x_{k-1} = x_k$ within the elements from the first partition. Similarly, in the second partition, the minimum possible sum is $(k-1)^2+\cdots+(k-1)^2 = (k-1)^3 > (k-1)^2+(k-3) = N$. Therefore, there is no monochromatic solution for $x_1+x_2+\cdots+x_{k-1} = x_k$ within the elements from the second partition. Now, consider summations involving both partitions. The minimum possible sum is

$$1+\cdots+1+(k-1)^2 = (k-1)^2+(k-2) > (k-1)^2+(k-3) = N\,.$$

Thus, there is impossible to have a monochromatically c_2-coloured solution to $x_1+x_2+\cdots+x_{k-1} = x_k$. Hence, there is no monochromatic solution to $x_1+x_2+\cdots+x_{k-1} = x_k$ in this 2-colouring of $[N]$. Therefore, we get $S(k;2) \geq (k-1)^2+(k-2)$.

Next, we want to show that $S(k;2) \leq (k-1)^2+(k-2)$. Let $M = (k-1)^2+(k-2)$. Suppose to the contrary that there is no monochromatic solution to $x_1+x_2+\cdots+x_{k-1} = x_k$ in some 2-colouring of $[M]$. Without loss of generality, we assume that 1 is c_1-coloured. Since $1+1+\cdots+1 = k-1$, $k-1$ must be c_2-coloured. Then $(k-1)^2$ must be c_1-coloured as $(k-1)+(k-1)+\cdots+(k-1) = (k-1)(k-1) = (k-1)^2$. Since $1+1+\cdots+1+(k-1)^2-(k-2) = (k-1)^2$ and $1+1+\cdots+1+(k-1)^2 = (k-1)^2+(k-2)$, both $(k-1)^2-(k-2)$ and $(k-1)^2+(k-2)$ must be c_2-coloured. This will force $(k-2)+(k-1)$ to be c_1-coloured because $(k-1)+(k-1)+\cdots+(k-1)+[(k-2)+(k-1)] = (k-1)^2+k-2$. Now, consider the colouring of $k-2$. If $k-2$ is c_1-coloured, then we have a monochromatic solution: $(k-2)+(k-2)+\cdots+(k-2)+[(k-2)+(k-1)] = (k-1)^2$. If $k-2$ is c_2-coloured, then we also have a monochromatic solution: $(k-2)+(k-2)+\cdots+(k-2)+(k-1) = (k-1)^2-(k-2)$. Either way, we have a contradiction. Hence, in any 2-colouring of $[(k-1)^2+(k-2)]$, there is a monochromatic solution to the equation $x_1+x_2+\cdots+x_{k-1} = x_k$. Therefore, $S(k;2) \leq (k-1)^2+(k-2)$. Thus, we have $S(k;2) = (k-1)^2+(k-2)$. □

Next, we look at the Rado's Theorem which was proved by a student of Schur's, Richard Rado, in the 1933 paper [81]. While Schur's Theorem concerns the equation $a+b-c = 0$, Rado's Theorem addresses the equation $a_1x_1+\cdots+a_kx_k = 0$.

Theorem 5.13 (Rado's Theorem)**.** [81] *Let $k \geq 2$ and $c_i \in \mathbb{Z}$ for $1 \leq i \leq k$. Then, for any finite colouring of $\mathbb{N}$, $c_1x_1+\cdots+c_kx_k = 0$ has a monochromatic solution $x_1, \ldots, x_k \in \mathbb{N}$ if and only*

$$\sum_{i\in I} a_i = 0 \qquad \textit{for some nonempty subset } I \subseteq [k].$$

Proof. The following proof follows the general outline of the proof in [63] but we have adapted it so as to provide better clarity of argument and notation.

We first prove that if there is any finite colouring of $\mathbb{N}$ such that $a_1x_1+\cdots+a_kx_k = 0$ has a monochromatic solution $x_1,\ldots,x_k \in \mathbb{N}$, then there exists a nonempty subset $I \subseteq [k]$ such that

$$\sum_{i\in I} a_i = 0\,.$$

We prove the contrapositive. Assume that there exist $a_1,\ldots,a_k$ such that no nonempty subset $I \subseteq [k]$ satisfies $\sum_{i\in I} a_i = 0$. We need to show that for some r, there is an r-colouring of $\mathbb{N}$ without monochromatic solutions to $a_1x_1+\cdots+a_kx_k = 0$. First, we choose a prime number p that does not divide $\sum_{j\in J} a_j$ for any $J \subseteq [k]$. Since there are only finitely many choices for J, we can always do so. Now, for $n \in \mathbb{N}$, let s be the largest integer such that $p^s|n$, so $n = p^s m$ where $m \not\equiv 0 \pmod{p}$. We define χ as a $(p-1)$-colouring of $\mathbb{N}$ in such a way that $\chi(n) = m \pmod{p}$ with the colours $c_1,\ldots,c_{p-1}$. We wish to show that in χ, there is no monochromatic solution to $a_1x_1+\cdots+a_kx_k = 0$. Suppose to the contrary that we have one. Let $\{y_1,\ldots,y_k\}$ be a monochromatic solution under χ with the colour c_b. Then, we have $1 \le b \le p-1$, and since $y_i \in \mathbb{N}$, for each y_i, there are numbers s_i and k_i such that $y_i = p^{s_i}(pk_i + b)$. Let $s = \min\{s_1,\ldots,s_k\}$. We have

$$0 = \sum_{i=1}^{k} a_i y_i = \sum_{i=1}^{k} a_i p^{s_i}(pk_i + b) = p^s \sum_{i=1}^{k} a_i p^{s_i - s}(pk_i + b)\,.$$

Note that $s = s_i$ and $s - s_i = 0$, for some i. Hence, modulo p, we will get

$$0 \equiv b \sum_{i=1}^{k} p^{s_i - s} a_i \pmod{p}\,.$$

Since p is prime and p does not divide b, we have that p divides $\sum_{i\in\{1,k\};s_i=s} a_i$. This gives a contradiction since p is chosen such that p does not divide $\sum_{j\in J} a_j$. Hence, the result holds.

Now we need to show that if there is a nonempty subset $I \subseteq [k]$ such that

$$\sum_{i\in I} a_i = 0\,,$$

then, for any r-colouring of $\mathbb{N}$, $a_1x_1+\cdots+a_kx_k = 0$ has a monochromatic solution $x_1,\ldots,x_k \in \mathbb{N}$. Suppose that $\sum_{i\in I} a_i = 0$. If $I = [k]$, then we have

$$\sum_{i=1}^{k} a_i = 0\,.$$

Then, choosing x_1 to be any integer and setting $x_i = x_1$ for all $2 \le i \le k$, we will have a monochromatic solution for $a_1x_1+\cdots+a_kx_k = 0$. Now, suppose that $I \subset [k]$ and assume without the loss of generality that $a_1 > 0$ and $I = [m]$ where $m < k$. Let $s = a_{m+1} + a_{m+2} + \cdots + a_k$. We take $x_2 = x_3 = \cdots = x_m$ and $x_{m+1} = x_{m+2} = \cdots = x_k$. Then, the equation $a_1x_1+\cdots+a_kx_k = 0$ will become

$a_1x_1+x_2(a_2+a_3+\cdots+a_m)+x_{m+1}(a_{m+1}+a_{m+2})+\cdots+a_k=0$. Since $a_1+\cdots+a_m=\sum_{i\in I}a_i=0$ and $s=a_{m+1}+a_{m+2}+\cdots+a_k$, we have $a_1(x_1-x_2)+sx_{m+1}=0$.

Now, we use induction on r. Suppose that $r=1$. Then we can choose x_1 and x_2 so that $x_2-x_1=s$ and $x_{m+1}=a_1$. Suppose that $r\geq 2$ and assume that the result holds for $r-1$. Let $b=\sum_{i=1}^{k}|a_i|$. Now, consider χ to be any r-colouring of $[W(n+1,r)b]$ where $W(n+1,r)$ is the Van der Waerden number defined in Theorem 4.3 and n is the least positive integer such that in any $(r-1)$-colouring of $[n]$, there is a monochromatic solution for $a_1(x_1-x_2)+sx_{m+1}=0$. Such n exists from the induction hypothesis. We want to show that, in χ, we will get a monochromatic solution for $a_1(x_1-x_2)+sx_{m+1}=0$. Since $0\neq s=a_{m+1}+a_{m+2}+\cdots+a_k$ and $b=\sum_{i=1}^{k}|a_i|$, we have $1\leq|s|<b$. For $1\leq l\leq b$, we define χ_l to be the colouring of [W(n+1,r)] so that $\chi_l(i)=\chi(li)$. Then for each l, we will have a χ_l-monochromatic set $\{a,a+d,\ldots,a+nd\}\subseteq[W(n+1,r)]$. Hence, under the colouring of χ, we have a monochromatic set $\{la,la+ld,\ldots,la+lnd\}\subseteq[W(n+1,r)l]\subseteq[W(n+1,r)b]$.

Now, we let $l=|s|$ and $a'=la$. Then we will have a monochromatic set $\{a',a'+|s|d,\ldots,a'+n|s|d\}\subseteq[W(n+1,r)b]$ for some $d\geq 1$ under the colouring of χ. Consider this subset $\{a_1d,2a_1d,\ldots,na_1d\}\subseteq[W(n+1,r)b]\subset\mathbb{N}$ under the colouring of χ.

If $\chi(ja_1d)=\chi(a')$ for some $j\in[1,n]$, then consider the following cases. First, if $s<0$, then we can take $x_2=a'$, $x_1=a'+jd|s|$ and $x_{m+1}=ja_1d$: we then get $a_1(x_1-x_2)+sx_{m+1}=a_1(a'+jd|s|-a')+sja_1d=0$. If $s\geq 0$, then we take $x_2=a'+jd|s|$, $x_1=a'$ and $x_{m+1}=ja_1d$: we thereby get $a_1(x_1-x_2)+sx_{m+1}=a_1(a'-a'jd|s|)+sja_1d=0$. In both cases, we have a monochromatic solution to $a_1(x_1-x_2)+sx_{m+1}=0$.

On the other hand, if $\chi(ja_1d)\neq\chi(a')$ for all $j\in[1,n]$, then the elements of the set $\{a_1d,2a_1d,\ldots,na_1d\}\subseteq[W(n+1,r)b]\subset\mathbb{N}$ are coloured with $r-1$ colours. By the induction hypothesis, we have a monochromatic solution for $a_1(x_1-x_2)+sx_{m+1}=0$.

Thus, in all cases, we have a monochromatic solution for $a_1(x_1-x_2)+sx_{m+1}=0$. Hence, by the induction, the result holds.

Therefore, the theorem is valid. $\square$

Example 5.14. Consider the equation

$$x_1+x_2-x_3=0\,.$$

In this equation, $a_1=1$, $a_2=1$ and $a_3=-1$. Note that $\{a_1,a_3\}\subset\{a_1,a_2,a_3\}$ and $a_1+a_3=0$. By Theorem 5.13, $x_1+x_2-x_3=0$ has monochromatic solution. This is affirmed by Schur's Theorem (Theorem 5.1) as $x_1+x_2-x_3=0$ can be rewritten as $x_1+x_2=x_3$.

We now turn out attention to others generalisation of Schur's Theorem, now involving sum sets and product sets. We first introduce these notions, as well as a useful lemma.

Definition 5.15 (Sum set)**.** *For any set $S\subseteq\mathbb{N}$, the* sum set*, denoted by $\sum(S)$, is the set of all finite sums of the elements of S.*

Definition 5.16 (Product set). *For any set $S \subseteq \mathbb{N}$, a* product set*, denoted by $\prod(S)$, is the set of all finite products of the elements of S.*

Example 5.17. Let $S = \{1, 2, 5, 8\}$. Then

$$\sum(S) = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 15, 16\}$$
$$\text{and} \quad \prod(S) = \{1, 2, 5, 8, 10, 16, 40, 80\}.$$

Lemma 5.18. *For all $k, r \geq 1$, there is an integer $N = N(k; r)$ such that for any r-colouring of $[N]$, there exists $x_1 < x_2 < \cdots < x_k \in [N]$ with $\sum_{i=1}^{k} x_i < N$ where*

$$S_t = \{\sum_{r \in R} x_r : R \subseteq [k], \max_{r \in R} r = t\}$$

is monochromatic for $t = 1, 2, \ldots, k$.

Proof. The following proof is provided in [69]. We prove it by induction on k. For $k = 1$, the result is immediate. Now, let r be arbitrary and assume that $N(k; r)$ exists. We wish to show that $N(k+1; r)$ exists and that $N(k+1; r) \leq 2W(N(k; r) + 2, r)$, where $W(N(k; r)+2, r)$ is the Van der Waerden number defined in Theorem 4.3. Let $m = 2W(N(k; r) + 2, r)$. Consider an arbitrary r-colouring of $[m]$. By the definition of $W(N(k; r) + 2, r)$, there is a monochromatic arithmetic progression

$$A = \{a, a + d, \ldots, a + (N(k; r) + 1)d\} \subseteq [\frac{m}{2}, m].$$

Now, consider the set $D = \{d, 2d, \ldots, N(k; r)d\}$. By the induction hypothesis, there exist $x_1 < x_2 < \cdots < x_k \in D \subseteq [m]$ such that the associated sets $S_1, S_2, \ldots S_k$ are each monochromatic. Now, we wish to find an x_{k+1} so that S_{k+1} is also monochromatic. Take $x_{k+1} = a + d$. Note that $a > \frac{m}{2}$ and $a + N(k; r)d < m$; hence $N(k; r)d < \frac{m}{2}$. Therefore, we have $x_{k+1} = a + d > N(k; r)d \geq x_k$. For S_{k+1}, note that $S_{k+1} \subseteq (a + d) + D \subseteq A$. Hence, S_{k+1} is monochromatic. Thus, $N(k + 1; r)$ exists.

By induction, the result holds. □

Theorem 5.19 (Folkman's theorem). [90] *If $r, k \in \mathbb{N}$ and $M \in \mathbb{N}$ is sufficiently large, then for any r-colouring of $\mathbb{N}$, there is a k-subset $S \subseteq [M]$ with monochromatic $\sum(S)$.*

Proof. The proof generally follows that of [69]. We will prove that $M = N((k-1)r + 1; r)$ will satisfy the conditions of the theorem, where $N((k-1)r + 1; r)$ is defined in Lemma 5.18. Let $x_1 < x_2 < \cdots < x_{(k-1)r+1}$ satisfy Lemma 5.18 and consider the associated sets $S_1, S_2, \ldots, S_{(k-1)r+1}$. By the Pigeonhole Principle, k of them must be same-coloured, say $S_{i_1}, \ldots, S_{i_k}$. Let $S = \{i_1, \ldots, i_k\} \subseteq [M]$. Then, by Lemma 5.18, $\sum(S)$ is monochromatic. Hence, the theorem is valid. □

Example 5.20. Consider the case $k = 2$ in Folkman's Theorem (Theorem 5.19). We can take $M = S(r)$ where $S(r)$ is the Schur number as defined in Theorem 5.1. Then, for each r-colouring of $[M]$, there are integers a, b and c such that the set $\{a, b, c = a + b\}$ is monochromatic.

Theorem 5.21. *If $r, k \in \mathbb{N}$ and $M \in \mathbb{N}$ is sufficiently large, then for any r-colouring of $\mathbb{N}$, there is a k-subset $S \subseteq [M]$ with monochromatic $\prod(S)$.*

Proof. Note that by Theorem 5.19, there exists an M' such that for any r-colouring of $\mathbb{N}$, there is a k-subset $S' \subseteq [M']$ with monochromatic $\sum(S')$. Now, we take $M = 2^{M'}$. Let χ be any r-colouring of $[M]$ and χ' be the r-colouring of $[M']$ defined by $\chi'(i) = \chi(2^i)$ for $1 \leq i \leq M'$. By Theorem 5.19, there is a k-subset $S' \subseteq [M']$, say $\{s_1, \ldots, s_k\}$ with monochromatic $\sum(S')$. By the definition of χ', there is a set $S = \{2^{s_1}, \ldots, 2^{s_k}\}$ which is monochromatic under χ colouring. Note that

$$\prod_{r \in R} 2^{s_r} = 2^{\sum_{r \in R} s_r}$$

for any $R \subseteq S'$. Hence, $\prod(S)$ is monochromatic. $\square$

Chapter 6

The Hales-Jewett Theorem

In this chapter, we present another key theorem in Ramsey Theory, the Hales-Jewett Theorem. In Section 6.1, we present this theorem and prove it. We will also mention the density version of the theorem. In Section 6.2, we present the another proof of Van der Waerden's Theorem by using the Hales-Jewett Theorem.

6.1 The Hales-Jewett Theorem

In this section, we will present the Hales-Jewett Theorem and prove it. Hales-Jewett Theorem is a fundamental theorem in Ramsey Theory with a geometric focus, and was proven by Alfred W. Hales and Robert I. Jewett in 1963 [50]. Before looking into the theorem, we introduce some notation and definitions required.

Definition 6.1 (n-cube over k elements)**.** *We define the n-cube over k elements by*

$$C_k^n = \{(x_1, \ldots, x_n) : x_i \in [0, k-1]\}\,.$$

Definition 6.2 (Line)**.** *A line in C_k^n is a set of points $x_0, \ldots, x_{k-1}$, where $x_i = (x_{i1}, \ldots, x_{in})$ so that in each coordinate $j \in [n]$, either*

$$x_{0j} = \cdots = x_{k-1,j}$$

or

$$x_{sj} = s, \text{ where } s \in [0, k-1], \text{ for some } j\,.$$

Example 6.3. For $k = 5, n = 4$, $\{0130, 1131, 2132, 3133, 4134\}$ forms a line in C_k^n. For clarity purpose, the parentheses and commas may be omitted when k is small.

Definition 6.4 (Equivalence class)**.** *There is a collection of $n+1$ equivalence classes on $C_k^n = [0, k-1]^n$ in such a way that i-th equivalence class is the set of all points where $k-1$ appears in the i rightmost positions, for $0 \leq i \leq n$.*

Example 6.5. Consider $C_4^2 = [0, 3]^2$. There are 3 equivalence classes of C_4^2: the 0th equivalence class is $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2)\}$; the first equivalence class is $\{(0,3), (1,3), (2,3)\}$ and the second equivalence class is $\{(3,3)\}$.

Definition 6.6 (Layered c-dimensional subspace)**.** *A c-dimensional subspace of C_k^n is a c-dimensional cube. A c-dimensional subspace of C_k^n is said to be layered if there is a line where the first $k-1$ points are monochromatic. The mentioned line is also known as a layered line.*

Theorem 6.7 (Hales-Jewett Theorem)**.** [50] *Let $k, r \in \mathbb{N}$. If n is sufficiently large, then for any r-colouring of the cube $C_k^n = \{(x_1, \ldots, x_n) : x_i \in [0, k-1]\}$, there is a monochromatic line. The least of such n is known as the Hales-Jewett Number and is denoted by $HJ(r, k)$.*

Before we prove the theorem, we introduce the following lemma to help us.

Lemma 6.8. [45] *Let $k, r \in \mathbb{N}$. Suppose that $HJ(r, k)$ exists for all r. Then for all $c \in \mathbb{N}$, there exists a number $LHJ(r, k, c)$ so that for $n \geq LHJ(r, k, c)$, if C_{k+1}^n is r-coloured, then there exists a layered c-dimensional subspace.*

Proof. The proof is mainly from [45]. We use induction on c. Letting $c = 1$, we can take $LHJ(r, k, 1) = HJ(r, k)$. Consider any r-colouring of C_{k+1}^n for $n \geq LHJ(r, k, 1) = HJ(r, k)$. Note that there is C_k^n in C_{k+1}^n. By definition of $HJ(r, k)$, there is a monochromatic line in C_k^n and this line is also a layered line in the 1-dimensional subspace.

Now, suppose that $LHJ(r, k, c)$ exists. We need to show that $LHJ(r, k, c+1)$ also exists. Let $m = LHJ(r, k, c)$ and $s = r^{(k+1)^m}$. Since $LHJ(r, k, 1)$ exists for all r, $m' = LHJ(s, k, 1) = HJ(s, k)$ must exist. We intend to show that we can take $LHJ(r, k, c+1) = m' + m$.

Let χ be an r-colouring on $C_{k+1}^{m'+m}$. Now, consider $x \in C_{k+1}^{m'}$ and $y \in C_{k+1}^m$. We let $xy \in C_{k+1}^{m'+m} = C_{k+1}^{m'} \times C_{k+1}^m$ denote their concatenation. Consider χ' be a s-colouring of $C_{k+1}^{m'}$ in such a way that

$$\chi'(x) = \chi'(x') \text{ if and only if } \chi(xy) = \chi(x'y) \text{ for all } y \in C_{k+1}^m \,.$$

Since there are only s colours, there exists a layered line $x_0, x_1, \ldots, x_t \in C_{k+1}^{m'}$ under χ'. Now, we colour C_{k+1}^m by χ'', where

$$\chi''(y) = \chi(x_i y), \text{ for } 0 \leq i \leq k-1 \,,$$

in which the x_i's are the points in the layered line. Note that there are r colours in χ'' and that $m = LHJ(r, k, c)$, so there is a layered c-dimensional subspace, say $S \subseteq C_{k+1}^m$ under the colouring of χ''.

Now, let $T = \{x_i s : 0 \leq i \leq k, s \in S\} \subseteq C_{k+1}^{m'+m}$. Suppose that S has equivalence classes $S_0, \ldots, S_c$. Then T has equivalence classes $T_j = \{x_{is} : 0 \leq i \leq k, s \in S_j\}$, for $0 \leq j \leq c$, and T_{c+1} which consists of a single point beginning with x_k. Note that for $x_{is}, x_{is'} \in T_j$, where $0 \leq j \leq c$, we have

$$\chi(x_{is}) = \chi''(s) = \chi''(s') = \chi(x_{is'}) \,.$$

Hence, T is our layered $(c+1)$-dimensional subspace. Thus, $LHJ(r, k, c+1)$ exists. By induction, the result holds. □

Now, we proceed to the proof of Hales-Jewett Theorem (Theorem 6.7).

Proof. The proof mainly follows that given in [45]. We use induction on k. If $k = 1$, we can just take $n = 1$ and the result is trivial. Suppose that the theorem holds for k. We need to show that the result also holds for the case $k + 1$. Since $HJ(r, k)$ exists by the induction hypothesis, Lemma 6.8 that $LHJ(r,k,c)$ exists for all $c \in \mathbb{N}$. Take $c = r$ and $n = LHJ(r,k,r)$. By the definition of $LHJ(r,k,r)$, if C_{k+1}^n is r-coloured, then there exists a layered r-dimensional subspace. Now, let C_{k+1}^r be the layered subspace and consider these $r+1$ points x_i for $0 \leq i \leq r$:

$$x_i = (x_{i1}, \ldots, x_{ir}),\ x_{ij} = \begin{cases} k, & \text{if } j \leq i\,; \\ 0, & \text{if } j > i\,. \end{cases}$$

Since there are only r colours, the Pigeonhole Principle implies that x_u and x_v are of the same colour, say c_1, for some $u < v$. Then, the points $y_0, \ldots, y_k$ in which

$$y_s = (y_{s1}, \ldots, y_{sr}),\ y_{si} = \begin{cases} k, & \text{if } i \leq u\,; \\ s, & \text{if } u < i \leq v\,; \\ 0, & \text{if } v < i. \end{cases}$$

will also be c_1-coloured and form the monochromatic line. Hence by induction, the theorem holds. □

Now, we will introduce the density version of Hales-Jewett Theorem. In this strengthened version, instead of colouring the entire C_k^n with r colours, we colour an arbitrary subset, say $A \subset C_k^n$, with density $0 < \delta < 1$, where $\delta = \frac{|A|}{k^n}$.

Theorem 6.9 (Density version of Hales-Jewett Theorem)**.** [36]
Let $k, r \in \mathbb{N}$ and $0 < \delta < 1 \in \mathbb{R}$. If n is sufficiently large, then for any r-colouring of $A \subset C_k^n$ with density δ, there is a monochromatic line.

The proof of this theorem is rather technical so we are not going to prove it here. The interested reader is referred to [19, 36, 80].

6.2 Proof of Van der Waerden's Theorem by Hales-Jewett Theorem

In this section, we present another proof of Van der Waerden's Theorem (Theorem 4.3) by using the Hales-Jewett Theorem (Theorem 6.7). Van der Waerden's Theorem may indeed be proven as a corollary of the Hales-Jewett Theorem.

Let $k, r \in \mathbb{N}$. Recall that Van der Waerden's Theorem states that, for sufficiently large n, each r-colouring of $[n]$ gives a monochromatic arithmetic progression of length k. Recall that the Van der Waerden number $W(k,r)$ is the least such n.

Proof. This proof mainly follows that of [45]. We want to show that $W(k,r) \leq k^{HJ(r,k)}$ where $HJ(r,k)$ is defined as in the Hales-Jewett Theorem (Theorem 6.7). We represent each number $a \in [k^{HJ(r,k)}]$ as $HJ(r,k)$-tuples $(a_1, \ldots, a_{HJ(r,k)})$ by translating a into base-k expression $a = \sum_{HJ(r,k)}^{i=1} a_i k^{i-1}$, where $0 \leq a_i < k$. Now, note that the r-colouring of $[k^{HJ(r,k)}]$ induces an r-colouring of $C_k^{HJ(r,k)}$. By Theorem 6.7, there is a monochromatic line of length k. Note that, in this monochromatic line, the coordinate of each point is either constant or increasing by one each time. Hence, by translating back every point of the monochromatic line, we will get a monochromatic progression of length k, with the common difference in the form k^α where $\alpha \in \mathbb{N}$. Then, we are done. □

Chapter 7

Applications of Ramsey Theory

To highlight the significance of Ramsey Theory, we include some of the applications of Ramsey Theory in this chapter. In Section 7.1, we consider the application of Ramsey Theory to Graph Theory. Next, in Section 7.2, we will show a geometric application of Ramsey Theory. Finally, in Section 7.3, we will apply Ramsey Theory to Number Theory.

7.1 Applications to Graph Theory

In this section, we will present some applications of Ramsey Theory to Graph Theory. The first application is to prove Mantel's Theorem.

Theorem 7.1 (Mantel's Theorem)**.** [72] *Let G be a simple graph with $n \geq 3$ vertices. If the number of edges of G is $|E(G)| > \frac{n^2}{4}$, then G has at least one triangle.*

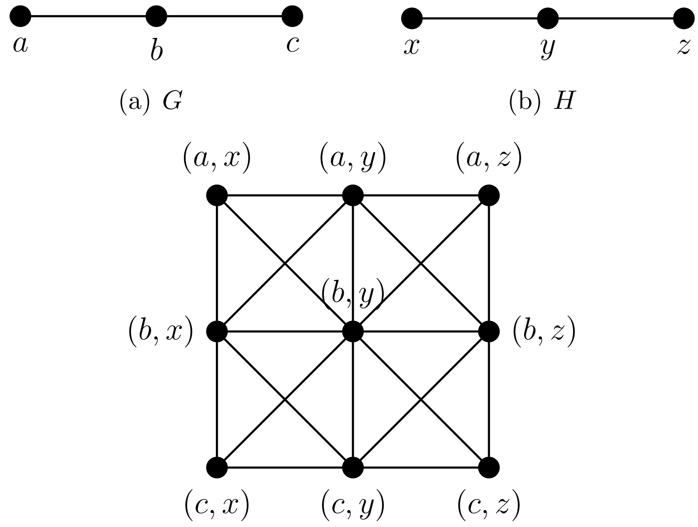
Proof. The proof mainly follows that of [103]. We use induction on n. Suppose that $n = 3$. If $|E(G)| > \frac{3^2}{4} = \frac{9}{4} \geq 3$, then $|E(G)| = 3$ and G is itself a triangle K_3. Now, suppose that the theorem holds for the case $n - 1$. We wish to show that the theorem is also true for n. Let G be a simple graph with $|E(G)| > \frac{n^2}{4}$ and let $\{u, v\}$ be an edge of G. Let H be the subgraph of G obtained by deleting both vertices u and v and each edge incident to them. If $|E(H)| > \frac{(n-2)^2}{4}$, then the subgraph H has at least one triangle and, hence, G must also contain a triangle. If $|E(H)| \leq \frac{(n-2)^2}{4}$, then the number of edges between H and the vertices u and v will be $|E(G) - \{u, v\}| - |E(H)| > \frac{n^2}{4} - 1 - \frac{(n-2)^2}{4} = n - 2$. Hence, there are at least $n - 1$ edges between H and the vertices u and v. Note that there are only $n - 2$ vertices in H. By the Pigeonhole Principle, some vertex in H must be joined to both u and v. Thus, G has a triangle. By induction, the theorem holds for all $n \geq 3$. $\square$

Before we go to the next application, there is a definition that we here introduce.

Definition 7.2 (Strong product of two graphs)**.** *The* strong product *of two graphs G and H, denoted by $G \boxtimes H$, is the graph whose vertex set is the Cartesian product $V(G) \times V(H)$ and whose edges are given as follows: $\{(a, b), (c, d)\}$ is an edge in $G \boxtimes H$ if and only if one of the following conditions holds:*

(1) *$\{a, c\} \in E(G)$ and $\{b, d\} \in E(H)$,*
(2) *$a = c$ and $\{b, d\} \in E(H)$,*
(3) *$b = d$ and $\{a, c\} \in E(G)$.*

Example 7.3. Figure 7.1 shows the graphs G and H and their normal product $G \boxtimes H$.



(a) G (b) H

(c) $G \boxtimes H$

Figure 7.1: The normal product of graph G and H, $G \boxtimes H$.

Definition 7.4. *A set of vertices in a graph is* independent *if no two of these vertices are adjacent. For each graph G, let $\alpha(G)$ be the largest size of an independent set in G.*

Theorem 7.5. [54] *If G and H are graphs, then $\alpha(G \boxtimes H) \leq R(\alpha(G)+1, \alpha(H)+1)-1$, where $R(\alpha(G)+1, \alpha(H)+1)$ is the Ramsey number as defined in Theorem 2.1.*

Proof. This proof is provided in [84]. Let $N = R(\alpha(G)+1, \alpha(H)+1)$. Suppose to the contrary that $\alpha(G \boxtimes H) \geq N$. Let I be an independent set of $G \boxtimes H$ with N vertices. Let (a, b) and (c, d) are two distinct vertices in I. Since I is independent, then one of the following conditions holds:

(1) $a \neq c$ and $\{a, c\} \notin E(G)$,

(2) $b \neq d$ and $\{b, d\} \notin E(H)$.

Now, consider a 2-colouring of the complete graph K_N. Label each of the vertex of K_N as in I. Colour the edge $\{(a, b), (c, d)\}$ with c_1 if (1) holds and c_2 otherwise. By the definition of N, there is either a c_1-coloured $K_{\alpha(G)+1}$ or a c_2-coloured $K_{\alpha(H)+1}$. Suppose that there is a c_1-coloured $K_{\alpha(G)+1}$. Since (1) holds for this subgraph, $\{a : a \in V(G) \text{ and } \{a, b\} \in K_{\alpha(G)+1} \text{ for some } b\}$ is an independent set of G with $\alpha(G)+1$ vertices, which is a contradiction since $\alpha(G)$ is the largest size of an independent set in G. On the other hand, suppose that there is some c_2-coloured $K_{\alpha(H)+1}$. Since (2) holds for this subgraph, $\{b : b \in V(H) \text{ and } \{a, b\} \in K_{\alpha(H)+1} \text{ for some } a\}$ is an independent set of H with $\alpha(H)+1$ vertices, which is a contradiction. Thus, we have $\alpha(G \boxtimes H) \leq N - 1 = R(\alpha(G)+1, \alpha(H)+1) - 1$. $\square$

7.2 Application to Geometry

In this section, we will present a geometric application of Ramsey Theory.

Theorem 7.6. [25] *Let $k \in \mathbb{N}$. If n is sufficiently large, then among any n points in the plane with no three points collinear, there are k of the points that form a convex polygon.*

Proof. The proof mainly follows that which is outlined in [25]. We will show that we can take $n = R_3(k,k)$, where $R_3(k,k)$ is the Ramsey number as defined in Theorem 2.7. Let K_n be a complete graph with n vertices. Label each of the n points with $1, 2, \ldots, n$ respectively in any order. Colour every triple $\{i, j, l\}$ with colour c_1 if $i < j < l$ is clockwise orientated and c_2 otherwise. By the definition of $R_3(k,k)$, there are k points whose triples are monochromatic. Then, each triangle among these k points is same-orientated, and hence these k points will form a convex polygon. $\square$

Many study on such a minimum integer n in Theorem 7.6, denoted by $ES(k)$, has been conducted over the past several decades. The most recent result is contributed by Suk, who proved that $SE(k) \leq 2^{k+6k^{\frac{2}{3}} \log_2 n}$ [98].

7.3 Applications to Number Theory

In this section, we present applications of Ramsey Theory to Number Theory. The first application to be presented is on the multiplicative representation of integers, proposed by Erdős in 1964.

Theorem 7.7. [21] *Let $A \subseteq \mathbb{N}$ where, for each $n \in \mathbb{N}$, there are $a, b \in A$ such that $n = ab$. For each $k \in \mathbb{N}$, there is some integer $n \in \mathbb{N}$ such that the equation $n = ab$ has at least k solutions with $a, b \in A$.*

Proof. The proof mainly follows that in [77] in the form in which it was reproduced and modified in [76]. Note that A must contain all prime numbers; hence it is sufficient for us to consider the integers n that are products of distinct primes only. Let $M(n)$ be the set of prime factors of n. Whenever we have any partition of $M(n) = M_1 \cup M_2$ for which $a = \prod M_1$, $b = \prod M_2$ and $n = ab$, the definition of A implies that $a, b \in A$. Now, by Ramsey's Theorem (Theorem 2.7), for a sufficiently large n and $|M|$, we can have one of the partitions, without loss of generality, say M_1, have at least k elements of M. Hence, there are at least k ways to partition M. Thus, $n = ab$ have at least k solutions in A. $\square$

Next, we are going to present an application of the Pigeonhole Principle to Number Theory, namely to prove Proizvolov's Identity, proposed by Vyacheslav Proizvolov after the 1985 All-Union Olympiad [91].

Theorem 7.8 (Proizvolov's Identity)**.** [91]
If $[2n]$ is biparted into sets $A = \{a_1 > \cdots > a_n\}$ and $B = \{b_1 < \cdots < b_n\}$, then

$$\sum_{i=1}^{n} |a_i - b_i| = n^2 .$$

Proof. This proof follows that provided in [7]. Consider the pairs a_i and b_i. We wish to show that one of them must be in $[n]$ and the other one must be in $[n+1, 2n]$. First, assume the contrary that both a_i and b_i are in $[n]$ for some i. Then, at least $n-i+1$ a_js and i b_js are in $[n]$. Therefore, at least $n-i+1+i = n+1$ of the a_j's and b_j's lie in $[n]$. By the Pigeonhole Principle, at least two of these are identical, which is a contradiction since $A \cap B = \emptyset$. Hence, a_i and b_i cannot be both in $[n]$. Next, we assume that both a_i and b_i are in $[n+1, 2n]$. Then, at least i a_js and $n-i+1$ b_js lie in $[n+1, 2n]$. This means that there are in total at least $i+n-i+1 = n+1$ a_js and b_js in $[n+1, 2n]$. Again, by the Pigeonhole Principle, at least two of them are identical, which is a contradiction. Hence, a_i and b_i cannot be both in $[n+1, 2n]$. Then, we have shown that one of them must be in $[n]$ and the other one must be in $[n+1, 2n]$. Thus, we have

$$\sum_{i=1}^{n} |a_i - b_i| = [(n+1) + \cdots + 2n] - (1 + \cdots + n) = n^2 .$$

□

There are many other applications of Ramsey Theory, particularly in Information Theory, information retrieval, design of packet switched networks, Games Theory and many other applications. The interested reader is referred to the overviews given by F.S. Roberts [84, 86] and V. Rosta [86].

Chapter 8
Conclusion

In conclusion, Ramsey Theory is a rapidly developing field of mathematics. In the thesis, we have studied several different types of Ramsey-type results, including results pertaining to edge-colourings of the complete graph, monochromatic arithmetic progressions and Schur triple $a + b = c$ in colourings of integers, monochromatic lines in cube colourings. We have also presented bounds on the various types of Ramsey numbers. However, there are still many Ramsey-type topics that have not been included in this thesis. We hope that through this thesis, the reader might have found interest and appreciation for Ramsey Theory, as Ramsey Theory has become an important and active area of research. More importantly, there is still room for advancement in this field of knowledge, such as the study of the various bounds and the relationships between Ramsey Theory and the other fields of study. We look forward to conducting more such research. Before ending our thesis, we list some of the interesting open problems and conjectures in the field.

Conjecture 8.1. [22] *Let $R(k,k)$ denote the k^{th} diagonal Ramsey number. Then,*

$$\lim_{k\to\infty} R(k,k)^{\frac{1}{k}}$$

exists.

Question 8.2. [22] *What is the limit in the Conjecture 8.1 if such a limit exists?*

Conjecture 8.3. [20] *Let $W(k;2)$ be the Van der Waerden number. Then,*

$$\lim_{k\to\infty} \frac{W(k;2)}{2^k} = \infty$$

and

$$\lim_{k\to\infty} W(k;2)^{\frac{1}{k}} = \infty.$$

Conjecture 8.4. [62] *Let $k \geq l > 2$. Then,*

$$W(k,l;2) \geq W(k+1,l-1) \geq W(k+2,l-2) \geq \cdots .$$

Question 8.5. [55] *Suppose that $r \in \mathbb{N}$ and $\mathbb{N}$ is r-coloured. Must there be a arbitrarily large finite set $S \subseteq \mathbb{N}$ such that $\sum(S) \cup \prod(S)$ is monochromatic?*

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