UNIVERSITY OF NEW SOUTH WALES

Double Operator Integration with applications to Quantised Calculus.

by

E. McDonald

A thesis submitted in partial fulfillment for the degree of Doctor of Philosophy

in the
Faculty of Science School of Mathematics and Statistics

December 2019

Thesis/Dissertation Sheet

Surname/Family Name	:	**McDonald**
Given Name/s	:	**Edward**
Abbreviation for degree as give in the University calendar	:	**PhD**
Faculty	:	**Science**
School	:	**Mathematics and Statistics**
Thesis Title	:	**Double Operator Integration with applications to Quantised Calculus.**

Abstract 350 words maximum:

The theory of double operator integration provides a powerful set of tools for the study of spectral asymptotics of compact operators. We give a self-contained overview of the theory from its foundations, including a complete proof of the fundamental Peller's theorem. The theory is developed with the goal of proving a formula for the difference of complex powers of self-adjoint operators, which has recently been applied to problems in Connes' quantised calculus. The final two chapters give applications to the Conformal Trace Theorem for the Hausdorff measure of Julia sets of quadratic polynomials and to the characterisation of quantum differentiability on noncommutative Euclidean spaces.

ORIGINALITY STATEMENT

‘I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, or substantial proportions of material which have been accepted for the award of any other degree or diploma at UNSW or any other educational institution, except where due acknowledgement is made in the thesis. Any contribution made to the research by others, with whom I have worked at UNSW or elsewhere, is explicitly acknowledged in the thesis. I also declare that the intellectual content of this thesis is the product of my own work, except to the extent that assistance from others in the project's design and conception or in style, presentation and linguistic expression is acknowledged.’

Signed ……………………………………………….............

Date ……………………………………………….............

INCLUSION OF PUBLICATIONS STATEMENT

UNSW is supportive of candidates publishing their research results during their candidature as detailed in the UNSW Thesis Examination Procedure.

Publications can be used in their thesis in lieu of a Chapter if:

- The candidate contributed greater than 50% of the content in the publication and is the "primary author", ie. the candidate was responsible primarily for the planning, execution and preparation of the work for publication
- The candidate has approval to include the publication in their thesis in lieu of a Chapter from their supervisor and Postgraduate Coordinator.
- The publication is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in the thesis

Please indicate whether this thesis contains published material or not:

☐ This thesis contains no publications, either published or submitted for publication

☐ Some of the work described in this thesis has been published and it has been documented in the relevant Chapters with acknowledgement

☐ This thesis has publications (either published or submitted for publication) incorporated into it in lieu of a chapter and the details are presented below

CANDIDATE'S DECLARATION
I declare that:

- I have complied with the UNSW Thesis Examination Procedure
- where I have used a publication in lieu of a Chapter, the listed publication(s) below meet(s) the requirements to be included in the thesis.

Candidate's Name	Signature	Date (dd/mm/yy)

COPYRIGHT STATEMENT

Signed ……………………………………………..........................

Date ……………………………………………..............................

AUTHENTICITY STATEMENT

‘I certify that the Library deposit digital copy is a direct equivalent of the final officially approved version of my thesis.’

Signed ……………………………………………..........................

Date ……………………………………………..............................

"...a different choice of foundations can lead to a different way of thinking about the subject, and thus to ask a different set of questions and to discover a different set of proofs and solutions. Thus it is often of value to understand multiple foundational perspectives at once, to get a truly stereoscopic view of the subject."

Terence Tao

Abstract

The theory of double operator integration provides a powerful set of tools for the study of spectral asymptotics of compact operators. We give a self-contained overview of the theory from its foundations, including a complete proof of the fundamental Peller's theorem. The theory is developed with the goal of proving a formula for the difference of complex powers of self-adjoint operators, which has recently been applied to problems in Connes' quantised calculus. The final two chapters give applications to the Conformal Trace Theorem for the Hausdorff measure of Julia sets of quadratic polynomials and to the characterisation of quantum differentiability on noncommutative Euclidean spaces.

Acknowledgements

With sincerest thanks, admiration and appreciation to my supervisors: Fedor Sukochev and Dmitriy Zanin.

With acknowledgements to the School and those who helped and supported me: to Galina Levitina, Anna McGann and Xiao Xiong. Gratitude is also due to Anna Tomskova, who volunteered to read this thesis and made many helpful suggestions. Thanks also to Nigel Higson, who offered invaluable comments concerning index theory.

With special thanks to Magnus Goffeng and Marius Junge, who invited and supported me overseas. With additional thanks to Magnus Goffeng and Matthias Lesch, for careful reading and commenting on this manuscript.

With love and gratitude to my family: my mother, father, brother and grandfather who provided unfailing support.

To Angela; without whom I would never have made it so far.

Contents

Chapter 1

Introduction

The goal of this thesis is to illustrate the applications of the theory of double operator integration to a family of interesting and difficult problems in harmonic analysis concerning the singular values of the commutators of multiplication operators and Riesz transforms. The primary motivation for studying these questions comes from the work of A. Connes in the 1980s in his *quantised calculus*.

1.1 Quantised calculus

Connes' quantised calculus arose from considerations in noncommutative geometry, and although the original motivations for quantised calculus are largely not relevant for the present work, it is worthwhile and instructive to discuss Connes' reasoning.

1.1.1 Infinitesimals

Connes has promoted the idea that one can use the terminology of infinitesimals in quantised calculus [29]. Many of the results discussed in the present text can be helpfully recast as statements about infinitesimals.

An infinitesimal (from Latin *infinitesimus*) is a non-zero positive quantity which is smaller than $\frac{1}{n}$ for every natural number $n \geq 1$. Historically in mathematics, the notion of a "continuum" was not always clearly understood, and mathematicians and philosophers struggled with the apparent contradiction stemming from the fact that a line is made up of points, and yet a point has no length. Numerous solutions to this paradox were proposed, but one of the most enduring was the idea that a point does in fact have length – but this length is so small that when added to itself a finite number of times it is impossible to obtain the length of an interval. In algebraic terms, the "length" of a point must therefore be a quantity $\varepsilon > 0$ such that $n \cdot \varepsilon < 1$ for all natural numbers $n \geq 1$. Infinitesimals found use in the foundations of calculus: for example, one can say that a function f is continuous if $f(t)$ is infinitesimally close to $f(s)$ whenever t is infinitesimally close to s.

It is very important to note that there is great difficulty in interpreting historical mathematics from a modern perspective. Prior to the late 19th century, notions such as "function" and "set" did not exist in their modern form, and it is impossible to understand the works of Newton, Leibniz, Euler etc. without appreciating that those authors exist in their historical context. The above definition of an infinitesimal may not faithfully represent how all historical authors thought about infinitesimals and at least some authors may have had in mind a completely different notion. I do not count myself a historian and make no claims to authority on such subtle matters, the interested reader is directed to the extensive bibliography in [10].

In the 19th century, the perceived vagueness of the notion of an infinitesimal was troubling to many and there were numerous attempts to put infinitesimals on a secure footing or to develop calculus without them. Today we credit Bolzano, Dedekind, Weierstrass and many others with developing a convincing formulation of real analysis without infinitesimals.

Nonetheless, the idea of an infinitesimal remains appealing. It is an interesting problem to determine exactly why historical mathematicians had such success using them and whether they can be interpreted in a modern context. Indeed, there are several modern theories of real analysis which include infinitesimals. Notably, Robinson's theory of non-standard analysis has been promoted as a faithful recreation of the infinitesimals of Leibniz [86, 112, 126]. Smooth infinitesimal analysis is another approach based instead around nilpotent infinitesimals [9], and Conway's surreal numbers are based on an extension of the Dedekind completeness property of $\mathbb{R}$. A fascinating recent article with a survey of different approaches to infinitesimals is [48].

Connes' infinitesimals stand apart from many other approaches in that there has been little focus on the reinterpretation of classical real analysis. Instead, Connes' primary interest has been in new settings.

Let us return to the (debatably historically faithful) idea that an infinitesimal is supposed to be an object ε such that:

$$0 < \varepsilon < \frac{1}{n}, \qquad \text{for all } n \geq 1. \tag{1.1.1}$$

Of course, (1.1.1) is not satisfied for any real number ε. In non-standard analysis, surreal analysis and some other approaches, one considers an embedding of $\mathbb{R}$ into a larger ordered field which contains an element ε satisfying (1.1.1).

Connes, however, finds inspiration in quantum mechanics. Let us think of ε as being not a real number, but instead as a quantum observable. This opens up the possibility that we can instead interpret (1.1.1) as being a *superposition* of the statements "$0 < \varepsilon < \frac{1}{n}$" as n ranges over $1, 2, 3, \ldots$.

To be more precise, suppose that ε is, rather than a number, actually a positive compact operator on a Hilbert space, which has a sequence of distinct and positive eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ with corresponding orthonormal eigenvectors $\{v_j\}_{j=0}^{\infty}$. Consider the state:

$$\psi = \alpha_0 v_0 + \alpha_1 v_1 + \alpha_2 v_2 + \cdots = \sum_{j=0}^{\infty} \alpha_j v_j$$

where $\{\alpha_j\}_{j=0}^{\infty}$ is a sequence of complex numbers satisfying $\sum_{j=0}^{\infty} |\alpha_j|^2 = 1$. That is, ψ is a superposition of the eigenstates of ε. If we decide to measure ε, then the outcome will be one of $\{\lambda_j\}_{j=0}^{\infty}$ with corresponding probabilities $\{|\alpha_j|^2\}_{j=0}^{\infty}$.

The probability that the measured value of ε will be less than $\frac{1}{n}$ is:

$$\sum_{|\lambda_j|<\frac{1}{n}} |\alpha_j|^2.$$

So that if infinitely many $\{\alpha_j\}_{j=0}^{\infty}$ are nonzero, then for every $n \geq 1$ the outcome "$0 < \varepsilon < \frac{1}{n}$" will be observed with positive probability. This is an unconventional but not-unreasonable interpretation of (1.1.1).

We can remove the dependence on the choice of state ψ with the following definition: Say that a linear operator T on a Hilbert space H is *infinitesimal* if for all $n \geq 1$, there exists a finite dimensional subspace $E \subseteq H$ such that the norm of T restricted to E is less than or equal to $\frac{1}{n}$.

Thus, an operator T is infinitesimal if it can be approximated in the operator norm by finite rank operators, and so we come to the realisation that T is infinitesimal in the above sense if and only if T is compact.

Let us examine a simple case of Connes' infinitesimal arithmetic. Consider the algebra $\mathbb{C}1+\mathcal{K}(H)$, where H is a Hilbert space, $\mathcal{K}(H)$ denotes the algebra of compact operators and 1 denotes the identity operator. By mapping $z \in \mathbb{C}$ to $z1$, there is a natural embedding $\mathbb{C} \hookrightarrow \mathbb{C}1+\mathcal{K}(H)$. The above discussion is an attempt to motivate the idea that $\mathbb{C}1+\mathcal{K}(H)$ can be regarded as an "infinitesimal extension" of $\mathbb{C}$.

For the purposes of this example, adopt the following language: The "infinitesimal neighbourhood" of $Z \in \mathbb{C}1+\mathcal{K}(H)$ is the set $\{Z+T \ : \ T \in \mathcal{K}(H)\}$, and say that two operators $Z, W \in \mathbb{C}1+\mathcal{K}(H)$ are infinitesimally close if $Z-W$ is infinitesimal. That is, if Z and W have the same infinitesimal neighbourhood.

The self-adjoint subspace of $\mathbb{C}1+\mathcal{K}(H)$ is $\mathbb{R}1+\mathcal{K}_{sa}(H)$, where $\mathcal{K}_{sa}(H)$ denotes the space of self-adjoint compact operators. If $f : \mathbb{R} \to \mathbb{C}$ is a bounded Borel function, we may define $f(Z)$ for $Z \in \mathbb{R}1+\mathcal{K}_{sa}(H)$ by functional calculus. We then have the following (not particularly deep) theorem:

Theorem. *A bounded Borel function $f : \mathbb{R} \to \mathbb{C}$ is continuous at $t \in \mathbb{R}$ if and only if $f(t+\varepsilon)$ is infinitesimally close to $f(t)$ for all self-adjoint infinitesimals ε.*

Equivalently, f is continuous at t if f maps the self-adjoint part of the infinitesimal neighbourhood of t into the infinitesimal neighbourhood of $f(t)$. With sufficient care, one can also discuss continuity of general Borel functions $f : \mathbb{C} \to \mathbb{C}$ using the functional calculus of normal operators.

The above description of continuity is simply a restatement of the fact that f is continuous at t if and only if $f(t+\lambda_n) \to f(t)$ for all sequences $\lambda_n \to 0$. If this were the only product of Connes' infinitesimal arithmetic, then it would be nothing but a curiosity. However, one interesting feature of $\mathbb{C}1+\mathcal{K}(H)$ is that there is a natural way to make sense of the "size" or "order" of an infinitesimal $\varepsilon \in \mathcal{K}(H)$.

For $n \geq 0$, the nth singular value of $T \in \mathcal{K}(H)$ is defined as:

$$\mu(n,T) := \inf\{\|T-R\| \ : \ \mathrm{rank}(R) \leq n\}.$$

One may also characterise $\{\mu(n,T)\}_{n=0}^{\infty}$ as being the sequence of eigenvalues of the absolute value $|T|$ arranged in non-increasing order with multiplicities. The sequence

$\mu(T) := \{\mu(n,T)\}_{n=0}^\infty$ is a sequence of non-negative numbers vanishing towards zero, and it is reasonable to think of the rate of decay of $\mu(T)$ as measuring the size of the infinitesimal T. As justification, consider the case when T is positive. In this case, $\mu(T)$ is simply the sequence of eigenvalues of T arranged in non-increasing order with multiplicities. Denote the corresponding eigenvectors as $\{v_n\}_{n=0}^\infty$. When in a state $\psi \in H$ with $\|\psi\|_H = 1$, the probability that T is observed having value less than $\frac{1}{n}$ is:

$$\sum_{k\,:\,\mu(k,T)<\frac{1}{n}} |\langle \psi, v_k\rangle|^2.$$

This relates the rate of decay of $\mu(T)$ to the "observed size" of T.

Following Connes, an infinitesimal of order 1 is an operator $T \in \mathcal{K}(H)$ such that:

$$\mu(n,T) = O(\frac{1}{n}), \quad n \to \infty.$$

More generally, an infinitesimal of order $p > 0$ has singular value asymptotics:

$$\mu(n,T) = O(\frac{1}{n^{1/p}}), \quad n \to \infty.$$

In the language of operator theory, an infinitesimal of order p is thus identical to a compact operator in the weak Schatten ideal $\mathcal{L}_{p,\infty}$ (as defined in Subsection 1.5.2).

Much of the work of this thesis is devoted to the problem of determining when some compact operator is in an ideal $\mathcal{L}_{p,\infty}$. It can, in many cases, be enlightening to instead use this language of infinitesimals to describe our results.

1.1.2 Index Theory

Connes' quantised calculus was primarily motivated by his work in noncommutative geometry, and a large part of this particular branch of noncommutative geometry was motivated by index theory. Therefore, it is appropriate to say at least something about that topic.

The announcement in 1963 of the Atiyah–Singer index theorem [6] and the later dissemination of complete proofs [98] led to rapid developments in algebraic topology and index theory. Besides Atiyah and Singer, numerous authors including (but certainly not limited to) Kasparov [80], Bott and Patodi [4] and Getzler [56] began a thorough dissection of the original proofs of the index theorem. The literature on index theory is vast and a full account of the history of the subject is beyond the scope of the present text, further historical details may be found in [58, Chapter 5].

After substantial work by many authors, notably Kasparov and the important work of Brown, Douglas and Fillmore [22], it was eventually realised that an essential insight of the Atiyah–Singer index theorem is that an elliptic differential operator D on a manifold M defines a class $[D]$ in the K-homology of M, K-homology being a topological invariant which is dual to K-theory. After close examination it became clear that the Atiyah-Singer index theorem could be proved via an analytic construction of K-homology [5, 7].

The fundamental notion in what is now known as analytic K-homology is a Fredholm module. Subtly varying definitions are available in the literature, however typically a Fredholm module for a C^*-algebra A is a triple (π, H, F), where H is a Hilbert space, F is a bounded operator on H and π is a representation of A on the bounded operators of H such that for all $a \in A$ we have:

(i) $\pi(a)(F^2 - 1) \in \mathcal{K}(H)$

(ii) $\pi(a)(F - F^*) \in \mathcal{K}(H)$

(iii) $[F, \pi(a)] \in \mathcal{K}(H)$.

(c.f. [75, Chapter 8].) The K-homology of a C^*-algebra A is defined to be the set of Fredholm modules of A modulo a certain equivalence relation. Often, Fredholm modules are augmented with additional structure such as a $\mathbb{Z}_2$-grading γ, where γ is a self-adjoint idempotent on H which commutes with $\pi(A)$ and anticommutes with F. One can think that A is an algebra of functions of a "space", H is a space of sections of a "bundle" on that space, and F is an "order zero elliptic operator" on H.

Putting aside the technical details, the K-homological perspective on index theory begins by associating to an elliptic differential operator D on a compact manifold M a certain Fredholm module for $C(M)$ (the algebra of continuous functions on M).

The K-homology and K-theory of a manifold M are linked with its de Rham cohomology and homology by a functor called the Chern character. Within the framework of Fredholm modules, the Chern character is described as follows. One says that a Fredholm module (π, H, F) with grading γ is p-summable if $[F, \pi(a)] \in \mathcal{L}_{p+1}(H)$ (where the Schatten $\mathcal{L}_{p+1}$ classes are defined below in Section 1.5.2) for all a in a dense subalgebra $\mathcal{A} \subset A$. Given a p-summable Fredholm module and an integer $n > p$, the nth component of the Chern character is the multilinear functional on $\mathcal{A}$ given by:

$$\mathrm{ch}_n(a_0, a_1, a_2, \ldots, a_n) = c_n \mathrm{Tr}(\gamma F[F, \pi(a_0)][F, \pi(a_1)] \cdots [F, \pi(a_n)]), \quad a_0, a_1, \ldots, a_n \in \mathcal{A},$$

where c_n is a certain constant (strictly speaking, the Chern character is defined as a certain class in periodic cyclic cohomology rather than merely a multilinear functional, and it is essential that we assume $F^2 = 1$ in order that ch_n to be a cyclic cocycle). The above formula for the K-homological Chern character was the original motivation for the definition of cyclic cohomology, and moreover for quantised calculus. Connes [26] initiated a program of developing a formal analogy between expressions involving traces of products of commutators $[F, \pi(a)]$ and integrals of differential forms.

There is a certain formal similarity between the Chern character on K-homology and multilinear functionals of the form:

$$(f_0, f_1, \ldots, f_d) \mapsto \int_M f_0\, df_1 df_2 \cdots df_d, \quad f_0, f_1, \ldots, f_d \in C^\infty(M).$$

This is more than a visual resemblance: there is a close link between cyclic homology and de Rham cohomology, and under mild regularity assumptions an explicit identification between the two is possible [85, Section 2.3]. It is not the case that $[F, \pi(a)]$ could literally be interpreted as a differential form, but cyclic homology is a close analogy of de Rham cohomology, and $[F, \pi(a)]$ can play the same role as a differential df. It was this analogy that led Connes to postulate that $i[F,\pi(a)]$ should be interpreted as a "quantised differential" of a [28, Chapter 4].

For noncommutative geometry, this is one motivation to understand the singular values of operators of the form $[F,\pi(a)]$, and Chapters 5 and 7 are devoted to analysing the singular value asymptotics of $[F,\pi(a)]$ for certain very special examples.

For further details on the relationship to noncommutative geometry, see the survey [24] or [64, Chapter 8]. In a related vein to the preceding discussion of index theory, this thesis fits into the general context of *noncommutative geometry* as developed by Connes. Indeed, one of the features of the analytic perspective on K-homology is that to define Fredholm module there is no need for an underlying "space". Noncommutative geometry (especially in the sense related to the present work) is often attributed to Connes, due to his substantial contributions to and promotion of this philosophy [28, 30, 31]. Connes' contributions include (with H. Moscovici) a far-reaching extension of the Atiyah-Singer index theorem to a purely algebraic setting [34, 74]. Many of the techniques used in this thesis can be traced to that line of research.

1.2 Double operator integration

The notion of a double operator integral originates with the pioneering work of Yu. L. Daletskii and S. G. Krein [38, 39], and was later further developed and extended by M. S. Birman and M. Z. Solomyak [15–17] and others in the same research group, including S. Ju. Rotfel'd [114] and L. S. Koplienko [13].

Let H be a (complex, separable) Hilbert space, and let A and B be (potentially unbounded) self-adjoint linear operators on H, with spectral resolutions E^A and E^B respectively. Let $f:\mathbb{R}\to\mathbb{C}$ be a Borel function. The theory of double operator integration was developed to analyse expressions of the form $f(A)-f(B)$, where $f(A)$ and $f(B)$ are determined by functional calculus:

$$f(A) = \int_{\mathrm{Spec}(A)} f(\lambda)\,dE^A(\lambda),\ f(B) = \int_{\mathrm{Spec}(B)} f(\mu)\,dE^B(\mu).$$

The following questions are relevant to operator theory:

(i) if $A-B$ is bounded, is $f(A)-f(B)$ bounded?

(ii) Similarly, if $A-B$ is trace class, or in the Schatten-von Neumann ideal $\mathcal{L}_p$, is the same true of $f(A)-f(B)$?

(iii) When is the function $t\mapsto f(A+tB)$ differentiable?

A 1968 result of Yu. B. Farfarovskaya [51] states that not all Lipschitz functions on $\mathbb{R}$ are Lipschitz in the operator norm: a result which implies that there exist Lipschitz functions $f:\mathbb{R}\to\mathbb{R}$ and self-adjoint operators A and B with $A-B$ bounded but $f(A)-f(B)$ not bounded. It was later established by E. B. Davies that even the absolute value function $f(t)=|t|$ is not operator Lipschitz [40].

How can we study the difference $f(A)-f(B)$? Formally, we may compute $f(A)-f(B)$ as:

$$\begin{aligned}
f(A)-f(B) &= \int_{\mathrm{Spec}(A)} f(\lambda)\,dE^A(\lambda) - \int_{\mathrm{Spec}(B)} f(\mu)\,dE^B(\mu)\\
&= \iint_{\mathrm{Spec}(A)\times\mathrm{Spec}(B)} f(\lambda)-f(\mu) dE^A(\lambda)dE^B(\mu)\\
&= \iint_{\mathrm{Spec}(A)\times\mathrm{Spec}(B)} \frac{f(\lambda)-f(\mu)}{\lambda-\mu}(\lambda-\mu)dE^A(\lambda)dE^B(\mu)\\
&= \iint_{\mathrm{Spec}(A)\times\mathrm{Spec}(B)} \frac{f(\lambda)-f(\mu)}{\lambda-\mu}dE^A(\lambda)(A-B)dE^B(\mu).
\end{aligned}$$

In the final step, we have used the (formal) identities:

$$\begin{aligned}
&\lambda dE^A(\lambda)dE^B(\mu) = AdE^A(\lambda)dE^B(\mu) = dE^A(\lambda)AdE^B(\mu),\\
&\mu dE^A(\lambda)dE^B(\mu) = dE^A(\lambda)BdE^B(\mu).
\end{aligned}$$

A “double operator integral” refers to a formal expression:

$$\mathcal{T}_\phi^{A,B}(X) = \iint_{\mathrm{Spec}(A)\times\mathrm{Spec}(B)} \phi(\lambda,\mu)dE^A(\lambda)XdE^B(\mu). \tag{1.2.1}$$

Here, ϕ is a bounded measurable function on the product of the spectra $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$, and X is a linear operator on H. It is, in general, a difficult technical problem to give rigorous meaning to the above expression, and there is no universally accepted definition which can make sense of the above integral for all bounded operators X and for all ϕ. Nonetheless, there are compelling reasons to study expressions of the above form.

Daletskii and Krein noticed that if one can justify the preceding formal computations, then we have the following striking identity for a Lipschitz continuous function f:

$$f(A)-f(B) = \mathcal{T}_\phi^{A,B}(A-B). \tag{1.2.2}$$

where ϕ is the function $\phi(\lambda,\mu) = \frac{f(\lambda)-f(\mu)}{\lambda-\mu}$. Similar identities were already known to K. Löwner in 1934 [90], at least in the finite dimensional setting.

Thus if sense can be made of a double operator integral, then we have the possibility to “convert non-linear problems into linear problems”, in the sense that the non-linear relationship between $A-B$ and $f(A)-f(B)$ can be related to the linear map $X\mapsto \mathcal{T}_\phi^{A,B}(X)$.

A substantial proportion of the research into double operator integrals has been motivated by attempting to make sense of the formal expression (1.2.1) and verifying (1.2.2) for various classes of operators A and B and functions f. The potential rewards of such an endeavour are high: for example, if one can characterise the set of functions ϕ for which (1.2.1) makes sense and such that $X\mapsto \mathcal{T}_\phi^{A,B}(X)$ is a bounded linear operator in the operator norm, then one in principle has an analytical test to determine which functions f satisfy a Lipschitz estimate in the operator norm:

$$\|f(A)-f(B)\|_\infty \le C_f\|A-B\|_\infty$$

for some constant C_f.

A similar formal computation yields:

$$\frac{f(A+tB)-f(A)}{t} = \mathcal{T}_\phi^{A+tB,A}(B).$$

Suggesting that:

$$\frac{d}{dt}f(A+tB)|_{t=0} = \lim_{t\to 0}\mathcal{T}_\phi^{A+tB,A}(B) = \mathcal{T}_\phi^{A,A}(B). \tag{1.2.3}$$

The above formula is called the Daletskii-S. Krein formula, and it is a highly nontrivial matter to make rigorous sense of the latter equality, as one needs to understand the continuity of the mapping $(A,B)\mapsto \mathcal{T}_\phi^{A,B}$. Nonetheless, if (1.2.3) can be made rigorous then one has an explicit formula for the Gâteaux derivative of the function $A\mapsto f(A)$.

It is thanks to the far-reaching work of V. V. Peller [99] that we have what is arguably the central pillar of double operator integral theory: Peller's theorem. Peller's theorem precisely characterises the class of functions ϕ such that the double operator integral $\mathcal{T}_\phi^{A,B}$ defines a bounded linear operator on $\mathcal{L}_\infty$ (or equivalently, on $\mathcal{L}_1$). We shall explore Peller's theorem in Chapter 3. In his more recent publications and in joint work with A. B. Aleksandrov and F. L. Nazarov, Peller has contributed to the extension of the theory to more advanced problems: such as those involving multiple operator integrals and non-selfadjoint and even non-normal operators [1, 2].

Double operator integration and its applications remain an active area of research. Survey articles devoted to this topic include a 2003 survey of Birman and Solomyak [14] and the more recent expository work of Peller [101, 102].

1.2.1 Schur products

Some light can be shed on the formal identity (1.2.1) if one focuses initially on the finite dimensional case. If $H=\mathbb{C}^N$, then we should consider A,B and X as $N\times N$ matrices. The spectral resolutions of A and B are now discrete:

$$A=\sum_{j=0}^{N-1}\lambda(j,A)v_jv_j^*,\quad B=\sum_{j=0}^{N-1}\lambda(j,B)u_ju_j^*$$

where $\{\lambda(j,A)\}_{j=0}^{N-1}$, $\{\lambda(j,B)\}_{j=0}^{N-1}$ are the eigenvalues of A and B, and $\{v_j\}_{j=0}^{N-1}$ and $\{u_j\}_{j=0}^{N-1}$ are corresponding orthonormal bases of eigenvectors. In this case, the double integral (1.2.1) becomes a double sum:

$$\mathcal{T}_\phi^{A,B}(X)=\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\phi(\lambda(j,A),\lambda(k,B))u_jv_k^*\langle u_j,Xv_k\rangle.$$

If one writes $e_{j,k}=u_jv_k^*$, and $X_{j,k}=\langle u_j,Xv_k\rangle$, then we have:

$$\mathcal{T}_\phi^{A,B}(\sum_{j,k=0}^{N-1}X_{j,k}e_{j,k})=\sum_{j,k=0}^{N-1}\phi(\lambda(j,A),\lambda(k,B))X_{j,k}e_{j,k}$$

From the perspective of linear algebra, it now becomes clear that $\mathcal{T}_\phi^{A,B}$ is what is known as a Schur (or Hadamard) product. The Schur product of two matrices A, B of the same size (say, $n \times m$) is defined as the entrywise product:

$$A \circ B := \{A_{j,k}B_{j,k}\}.$$

The importance of the Schur product in linear algebra was already appreciated long before Daletskii and Krein, having appeared in I. Schur's 1911 paper [121].

A double operator integral may therefore be reasonably termed a "measurable Schur product" and indeed this terminology sometimes appears in the literature [133]. We will have more to say concerning the relationship between Schur products and double operator integrals in Chapter 3.

1.2.2 Functional calculus

Another perspective on the formal double operator integral (1.2.1) is from functional calculus. Let $p(x, y) = \sum_{j,k} a_{j,k}x^jy^k$ be a polynomial in two variables, and let A and B be bounded linear operators on the Hilbert space H. If A and B commute, then we can make unambiguous sense of the expression:

$$p(A, B) = \sum_{j,k} a_{j,k}A^jB^k.$$

From a more algebraic point of view, the assignment $p(x, y) \mapsto p(A, B)$ represents an algebra homomorphism from the space of polynomials in two variables to the space of bounded linear operators, and indeed is the unique algebra homomorphism which maps x to A and y to B.

If A and B do not commute, then the expression $p(A, B)$ is ambiguous; xy and yx represent identical polynomials but AB and BA in general are distinct operators. One can resolve this ambiguity with the use of various operator-ordering conventions (such as declaring that x^jy^k maps to A^jB^k) but there is in general no algebra homomorphism from the space of polynomials in two variables to the space of bounded linear operators which maps x to A and y to B.

The functional calculus perspective on double operator integration begins with the observation that A and B act not only on H, but also on the Hilbert-Schmidt space $\mathcal{L}_2(H)$. The Hilbert-Schmidt space is a two sided ideal of the algebra of bounded linear operators, and we can define the operators of "left multiplication" and "right multiplication":

$$L_AX = AX, \quad R_BX = XB, \quad X \in \mathcal{L}_2(H).$$

While A and B do not necessarily commute on H, L_A and R_B do commute on $\mathcal{L}_2(H)$, since:

$$A(XB) = (AX)B, \quad X \in \mathcal{L}_2(H).$$

So while $p(A, B)$ is an ambiguous expression, $p(L_A, R_B)$ may be unambiguously defined as a bounded linear operator on $\mathcal{L}_2(H)$:

$$p(L_A, R_B)X := \sum_{j,k} a_{j,k}A^jXB^k.$$

The mapping $p \mapsto p(L_A, R_B)$ inherits all of the desirable properties of functional calculus of two commuting variables (in particular, it is an algebra homomorphism).

Let us consider the relationship with the formal double operator integral (1.2.1). If A and B are self-adjoint, then they may be reconstructed as spectral resolutions:

$$A = \int_{\mathrm{Spec}(A)} \lambda\, dE^A(\lambda), \quad B = \int_{\mathrm{Spec}(B)} \mu\, dE^B(\mu).$$

For $X \in \mathcal{L}_2(H)$, we have:

$$A^j X B^k = \int_{\mathrm{Spec}(A)} \lambda^j\, dE^A(\lambda) X \int_{\mathrm{Spec}(B)} \mu^k\, dE^B(\mu).$$

Then formally applying Fubini's theorem,

$$A^j X B^k = \int_{\mathrm{Spec}(A)\times\mathrm{Spec}(B)} \lambda^j \mu^k\, dE^A(\lambda) X dE^B(\mu).$$

So for a general polynomial p,

$$p(L_A, R_B)X = \iint_{\mathrm{Spec}(A)\times\mathrm{Spec}(B)} p(\lambda,\mu)\, dE^A(\lambda) X dE^B(\mu).$$

While the application of Fubini's theorem was only heuristic, the left hand side $p(L_A, R_B)X$ is a rigorously defined object. The approach to double operator integration pursued by Birman, Solomyak [15–17] and Peller [99, 101, 102] is essentially to *define* the double operator integral as $p(L_A, R_B)X$,

$$T_p^{A,B}(X) := p(L_A, R_B)X.$$

This approach has the advantage that there is no need to restrict attention to polynomial functions or bounded operators A and B, since one may appeal to the existence of a functional calculus for arbitrary Borel functions of pairs of commuting self-adjoint operators. This approach does have the disadvantage that (1.2.1) is then only defined *a priori* for $X \in \mathcal{L}_2(H)$, although in some cases it is possible to extend the definition to wider classes of X using duality or density arguments. This point of view shall be pursued in Chapter 2.

1.2.3 Double operator integrals as operator-valued integrals

A third perspective on (1.2.1) views the double operator integral $T_\phi^{A,B}(X)$ as being a certain operator-valued integral.

Let us make the *ansatz* that one can separate the variables of the function ϕ in the sense that there is some measure space (Ω, μ) such that:

$$\phi(t,s) = \int_\Omega \alpha(t,\omega)\beta(s,\omega)\, d\mu(\omega), \quad t,s \in \mathbb{R} \tag{1.2.4}$$

where α and β are measurable functions on $\mathbb{R}\times\Omega$.

Formally applying Fubini's theorem now yields:

$$\begin{aligned}\mathcal{T}_\phi^{A,B}(X) &= \int_\Omega \int_{\mathbb{R}^2} \alpha(t,\omega)\beta(s,\omega) dE^A(t) X dE^B(s)\, d\mu(\omega)\\ &= \int_\Omega \alpha(A,\omega) X \beta(B,\omega)\, d\mu(\omega).\end{aligned}$$

While the preceding computation was only formal, the expression on the right-hand-side can be defined for large classes of X, α and β, and we will explore this definition in Section 4.1.1. This is a method which makes sense of (1.2.1) for wide classes of operators X, but is essentially limited to functions which can be decomposed as in (1.2.4).

1.2.4 Further perspectives

At present, there is no universally accepted definition of a double operator integral which applies equally well in all situations. If one wishes to consider a wide class of functions ϕ in (1.2.1), then the perspective in Subsection 1.2.2 is appropriate. If instead the focus is on considering a wider class of operators X, then the perspective of Subsection 1.2.3 is advantageous. In some circumstances, even more exotic interpretations of (1.2.1) are needed. For example, in [106] a definition was developed based on taking limits of discrete Schur multipliers and in [41] the theory was based on integration of functions with respect to finitely additive spectral measures on Banach spaces.

The perspective of this thesis is to take a "stereoscopic" approach to double operator integrals: rather than promote a single definition, instead two competing definitions will be presented on an equal footing. These definitions are consistent whenever they are both meaningful, however for certain applications it can be helpful to adopt one or the other point of view.

1.3 Julia sets

One of the most impressive applications of Connes' quantised calculus has been to the apparently unrelated area of holomorphic dynamics. This line of inquiry originates with Connes and D. Sullivan, and the first results in this direction were announced in [28]. Holomorphic dynamics is a well-established research area, and several monographs on the topic exist [8, 25, 95].

1.3.1 Background on holomorphic dynamics

The primary focus of study in holomorphic dynamics is the iteration of functions. Let $\phi : \mathbb{C} \to \mathbb{C}$ be a holomorphic function, and let $z_0 \in \mathbb{C}$. Define a sequence $\{z_n\}_{n\geq 1}$ by the rule $z_{n+1} = \phi(z_n)$ for $n \geq 0$. In other words, $z_n = \phi^n(z_0)$. It is usually hopeless to derive a closed form for z_n in terms of n and z_0; instead the main focus of the theory of holomorphic dynamics is to describe the qualitative features of the sequence $\{z_n\}_{n\geq 0}$. Of special interest is the relationship between the choice of z_0 and the asymptotic behaviour of $\{z_n\}_{n\geq 0}$ as $n \to \infty$.

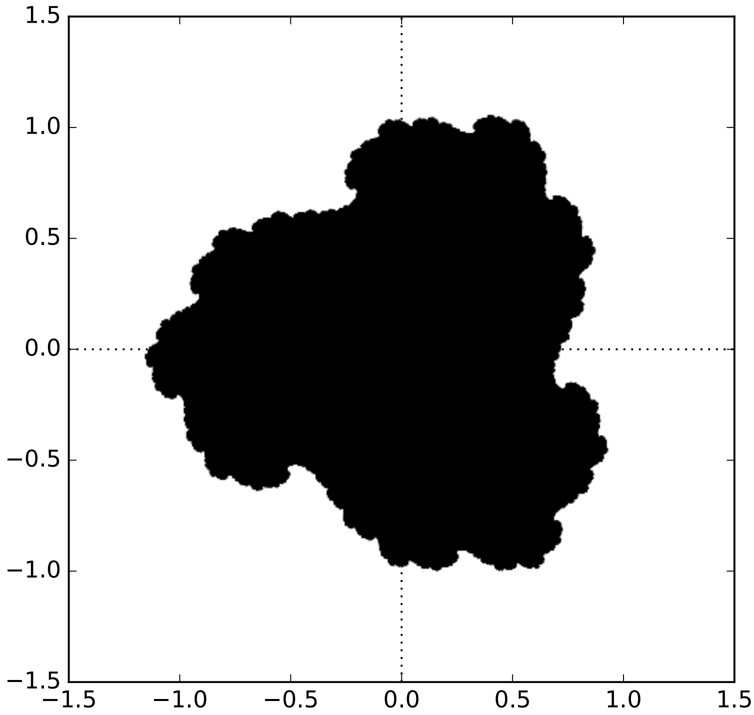


FIGURE 1.1: $K(\phi)$ for $\phi(z) = z^3 + 0.4 - 0.1i$

For example, if ϕ is a polynomial of degree $d > 1$, then when z is sufficiently large $\phi(z)$ is dominated by the degree d term, so there is the following dichotomy: either $|z_n| \to \infty$ as $n \to \infty$ or $|z_n|$ remains uniformly bounded in n.

Define the *filled in Julia set*, $K(\phi)$, of a polynomial ϕ to be the set of z_0 such that the sequence $z_n = \phi^n(z_0)$ is uniformly bounded as $n \to \infty$ [25, Section III.4]. The set $K(\phi)$ is necessarily bounded, and it can also be proved to be closed [25, Section II.1]. Since the equation $\phi(z) = z$ always has at least one solution, $K(\phi)$ is also necessarily non-empty. The *Julia set* $J(\phi)$ may be defined in this context to be the boundary of $K(\phi)$.

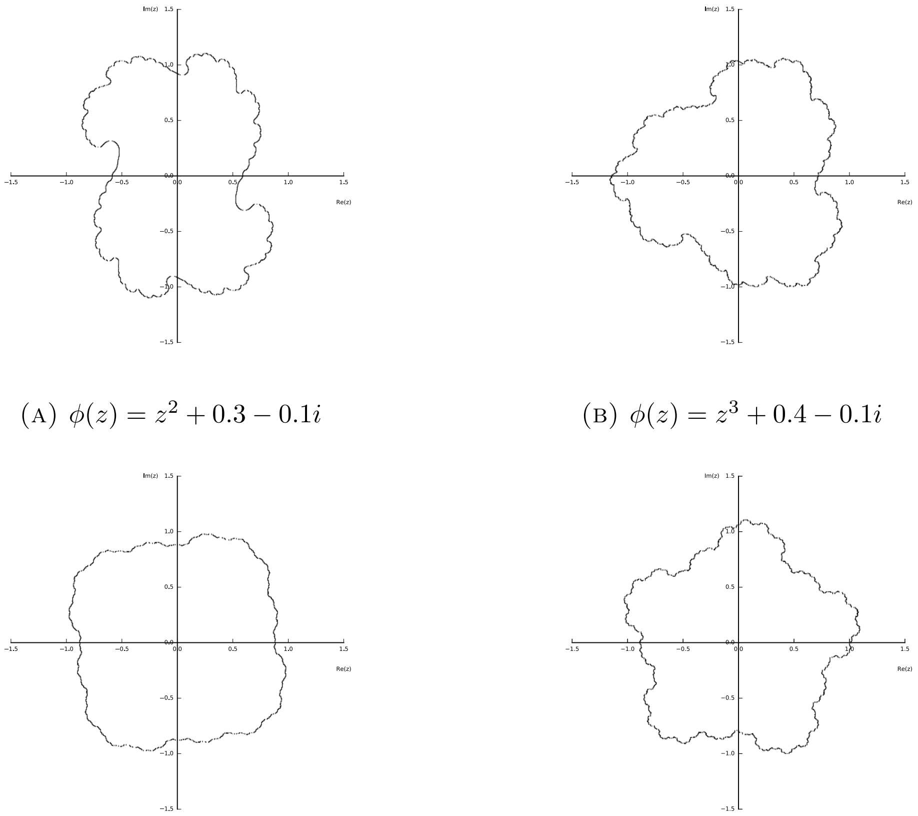
One can approximate $K(\phi)$ on a computer. Several algorithms exist [95, Appendix H], but the simplest one is described as follows: consider a large $(2N+1)\times(2N+1)$ grid of complex numbers, say $\{n\varepsilon + im\varepsilon\}_{n,m=-N}^{N}$ for some grid spacing $\varepsilon > 0$. Assign to each number in the grid a pixel, initially coloured white. Select appropriate constants J and K, and colour the (n, m)th pixel black if $|\phi^J(n\varepsilon + im\varepsilon)| < K$.

For example, Figure 1.1 shows $K(\phi)$ for $\phi(z) = z^3 + 0.4 - 0.1i$.

The Julia set $J(\phi)$ can be approximated from $K(\phi)$ by applying an edge-detection algorithm. Figure 1.2 was produced with such an algorithm.

1.3.2 The Conformal Trace Theorem

In Chapter 1.2, we give an indication of the utility of quantised calculus in the concrete example of the so-called *Conformal Trace Theorem* for Julia sets.



(A) $\phi(z) = z^2 + 0.3 - 0.1i$

(B) $\phi(z) = z^3 + 0.4 - 0.1i$

(C) $\phi(z) = z^4 + 0.2 - 0.1i$

(D) $\phi(z) = z^5 - 0.29 + 0.5i$

FIGURE 1.2: Numerical estimation of of $J(\phi)$ for various ϕ.

An important feature of Julia sets in general is that they are *fully invariant*: that is, z belongs to the Julia set of a polynomial ϕ if and only if $\phi(z)$ also belongs to the Julia set of ϕ [95, Lemma 4.3]. We will be concerned with examples where ϕ is *hyperbolic* on $J(\phi)$, which roughly means that $|\phi'(z)| > 1$ for all $z \in J(\phi)$ (see Definition 5.1.1). This hyperbolic self-similarity accounts for the rough appearance of the Julia sets in Figure 1.2 and in the cases that interest us, $J(\phi)$ will have Hausdorff dimension strictly between 1 and 2.

Let us briefly discuss the motivation behind the conformal trace theorem. Suppose that one has a simple closed curve $\mathcal{C}$ in the plane $\mathbb{R}^2$, parametrised by a function $s : [0, 1) \to \mathcal{C}$. If s is of bounded variation, then one can recover the arc-length measure on $\mathcal{C}$ by integration: the integral of a continuous function f on $\mathcal{C}$ with respect to the arc-length measure is given by the Riemann-Stieltjes integral:

$$\int_0^1 f(s(t))|ds|(t).$$

On the other hand, typical Julia sets cannot be parametrised by a function of bounded variation, although in place of an arc-length measure they do have a Hausdorff measure. The conformal trace theorem states that we may recover the integral of a function f on

J with respect to the p-dimensional Hausdorff measure by the formula:

$$⨍ f(Z)|đZ|^p$$

where Z is an appropriate parametrisation of J, and now $đZ$ denotes a quantised differential and $⨍$ is a singular trace.

In general it is a nontrivial matter to determine if the Julia set of a polynomial is a simple closed curve. In [33], the decision was made to restrict attention to the heavily studied class of examples of the form $\phi(z) = z^2+c$ for c sufficiently small and nonzero. In Chapter 5, instead we work with a larger class of polynomials which we term *admissible* (see Definition 5.1.1). The Julia sets of admissible polynomials are simple closed curves, and certain basic polynomials such as $\phi(z) = z^d + c$ for $d \geq 2$ and c sufficiently small and nonzero are admissible.

The precise version of the Conformal Trace Theorem to be proved in Chapter 5 is as follows. The relevant notations will be reviewed in Chapter 5.

Theorem. *Let $p \in (1,2)$ be the Hausdorff dimension of the Julia set J of an admissible polynomial ϕ. Let m_p be the p-dimensional Hausdorff measure on J. Then,*

(a) $[F, M_Z] \in \mathcal{L}_{p,\infty}$.

(b) *For every continuous Hermitian trace φ on $\mathcal{L}_{1,\infty}$, there exists a constant $K(\varphi,\phi)$ such that for every $f \in C(J)$ we have:*

$$\varphi(M_{f\circ Z}|[F, M_Z]|^p) = K(\varphi,\phi)\int_J f\,dm_p.$$

(c) *If ω is a dilation invariant extended limit on $L_\infty(0,\infty)$ such that $\omega \circ \log$ is also dilation invariant, then $K(\mathrm{tr}_\omega,\phi) > 0$. Here, tr_ω is a Dixmier trace corresponding to the extended limit ω.*

This version of the theorem has not previously appeared in writing, and the proofs given in Chapter 5 are original to this thesis.

1.4 Plan of the thesis

In the next section of this chapter we will review some background material concerning von Neumann algebras (Subsection 1.5.1) and operator ideals (Subsection 1.5.2).

Chapter 2 continues exposition of background material with an exhaustive account of the theory of integration with respect to spectral measures. This material is included primarily in service of Chapter 3, and so readers familiar with this material could move very quickly through Chapter 2. In Chapter 3, we include a complete proof of Peller's theorem concerning necessary and sufficient conditions for the boundedness of double operator integrals on the trace class ideal corresponding to a separable Hilbert space.

In Chapter 4, we include an alternative exposition of double operator integration theory based instead on the concept of a weak* or Gel'fand integral. This is developed for the purposes of proving a theorem relating the difference of complex powers of operators.

Chapter 5 details a proof of the Conformal Trace Theorem. It is primarily based on the paper [33].

Chapters 6 and 7 are concerned with the problem of characterising the singular value asymptotics of quantised differentials associated to noncommutative Euclidean spaces. These chapters are based on the published paper [94] and the submitted paper [92] respectively.

1.5 Preliminary material

1.5.1 Von Neumann algebras

Let us briefly recall the basics of von Neumann algebra theory. Further details may be found in, for example, the books [43, 77, 118, 131]. A von Neumann algebra $\mathcal{M}$ is a unital $*$-subalgebra of the algebra of bounded linear operators on a Hilbert space H which is closed in the weak operator topology (or equivalently in the strong operator topology).

If $X \subseteq \mathcal{B}(H)$, let $X' \subseteq \mathcal{B}(H)$ denote the set of operators which commute with every element of X. Von Neumann's celebrated bicommutant theorem states that a $*$-subalgebra $\mathcal{M}$ of $\mathcal{B}(H)$ is a von Neumann algebra if and only if $\mathcal{M} = \mathcal{M}''$.

The algebra $\mathcal{M}$ inherits a partial ordering from its representation on H: for self-adjoint $a, b \in \mathcal{M}$ we say that $a \leq b$ if for all $\xi \in H$ we have that $\langle \xi, a\xi \rangle \leq \langle \xi, b\xi \rangle$. We say that $a \in \mathcal{M}$ is positive if $a \geq 0$. Equivalently, an element a is positive if $a = b^*b$ for some $b \in \mathcal{M}$. Let $\mathcal{M}^+$ denote the set of positive elements of $\mathcal{M}$. The cone $\mathcal{M}^+$ satisfies the following Dedekind completeness property: if $\{x_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{M}^+$ is a net where $(\Lambda, \preccurlyeq)$ is a directed set, we say that $\{x_\lambda\}_{\lambda\in\Lambda}$ is monotone increasing if $\lambda \preccurlyeq \mu$ implies that $x_\lambda \leq x_\mu$. Given a monotone net $\{x_\lambda\}_{\lambda\in\Lambda}$ which is bounded above in the sense that there is some $y \in \mathcal{M}^+$ with $x_\lambda \leq y$ for all $\lambda \in \Lambda$, there exists a least upper bound $\sup_{\lambda\in\Lambda} x_\lambda$ in $\mathcal{M}^+$. Note that in $\mathcal{M}^+$ we can only prove the existence of the supremum of a monotone net: for $A, B \in \mathcal{B}(H)$ a least upper bound $A \vee B$ exists if and only if either $A \leq B$ or $B \leq A$.

A self-adjoint idempotent of $\mathcal{M}$ is called a projection. Many of the properties of the lattice of projections of $\mathcal{B}(H)$ transfer to $\mathcal{M}$. For example, if $p, q \in \mathcal{M}$ are projections, and if $p \vee q$ denotes the projection onto the closed subspace $pH + qH$ then $p \vee q \in \mathcal{M}$ due to the bicommutant theorem. We also have that $p \vee q$ is the minimal projection such that $p \leq p \vee q$ and $q \leq p \vee q$. We denote $\mathcal{P}(\mathcal{M})$ for the lattice of projections in $\mathcal{M}$.

A distinguishing feature of von Neumann algebras is that a von Neumann algebra $\mathcal{M}$ has a Banach space pre-dual $\mathcal{M}_*$. This introduces a new notion of convergence in the setting of von Neumann algebras, weak* convergence: say that a net $\{x_\alpha\}_{\alpha\in A}$ (indexed by a directed set A) in $\mathcal{M}$ converges to $x \in \mathcal{M}$ in the weak*-sense if $x_\alpha(\omega) \to x(\omega)$ for all $\omega \in \mathcal{M}_*$.

It will later be important to give sufficient conditions for weak*-convergence in terms of the representation of $\mathcal{M}$ on H, and for this purpose we use the σ-weak topology. Consider subsets of H of the form $\{\xi_k\}_{k=0}^\infty$ and $\{\eta_k\}_{k=0}^\infty$ where ξ_j is orthogonal to ξ_k for

$j \neq k$, η_j is orthogonal to η_k for $j \neq k$, and

$$\sum_{k=0}^{\infty} \|\xi_k\|_H^2 < \infty, \quad \sum_{k=0}^{\infty} \|\eta_k\|_H^2 < \infty.$$

Define the following seminorm on $\mathcal{M}$:

$$\rho_{\{\xi_k\},\{\eta_k\}}(x) = \left|\sum_{k=0}^{\infty} \langle \xi_k, x\eta_k \rangle\right|, \quad x \in \mathcal{M}. \tag{1.5.1}$$

Then the σ-weak topology of $\mathcal{M}$ is defined to be the topology generated by the family of seminorms $\rho_{\{\xi_k\},\{\eta_k\}}$ as $\{\xi_k\}$ and $\{\eta_k\}$ vary over all possible choices satisfying the stated conditions (c.f. [131, Definition II.2.1]). The predual $\mathcal{M}_*$ can be identified with the set of σ-weakly continuous functionals on $\mathcal{M}$ [131, Theorem II.2.6(iii)] and hence if $\{x_\alpha\}_{\alpha \in A}$ is a net which converges to x in the σ-weak sense then $\{x_\alpha\}_{\alpha \in A}$ converges to x in the weak*-topology of $\mathcal{M}$.

A faithful normal trace τ on $\mathcal{M}^+$ is an additive mapping from $\mathcal{M}^+$ to $[0, \infty]$ which satisfies the following properties:

(i) Faithfulness: $\tau(x) = 0$ if and only if $x = 0$.

(ii) Normality: if $\{x_\lambda\}_{\lambda \in \Lambda}$ is a monotone increasing net of operators which is bounded above, then $\tau(\sup_{\lambda \in \Lambda} x_\lambda) = \sup_{\lambda \in \Lambda} \tau(x_\lambda)$.

(iii) Traciality: If $u \in \mathcal{M}$ is unitary, then $\tau(uxu^*) = \tau(x)$ for all $x \in \mathcal{M}^+$.

There exist von Neumann algebras where every trace is infinite on every nonzero element. For this reason, the notion of a semifinite von Neumann algebra is introduced. Say that a projection $p \in \mathcal{P}(\mathcal{M})$ is τ-finite if $\tau(p) < \infty$. A pair $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra if the identity projection $1 \in \mathcal{P}(\mathcal{M})$ is a monotone limit of τ-finite projections. If $(\mathcal{M}, \tau)$ is semifinite, then every $x \in \mathcal{M}^+$ can be obtained as the supremum of a monotone net $\{x_\alpha\}_{\alpha \in A}$ such that $\tau(x_\alpha)$ is finite for each α.

In the semifinite case, an explicit description of $\mathcal{M}_*$ is available. Let $p \geq 1$ and consider the set

$$N_p = \{x \in \mathcal{M} \ : \ \tau(|x|^p) < \infty\}$$

then the quantity $\|x\|_p := \tau(|x|^p)^{1/p}$ is a norm on N_p, and the completion of N_p with respect to this norm is called the L_p space associated to $(\mathcal{M}, \tau)$, denoted $L_p(\tau)$ for brevity. One can identify $L_1(\tau)$ with $\mathcal{M}_*$ with the duality pairing:

$$(x, z) \mapsto \tau(xz), \quad x \in \mathcal{M}, z \in L_1(\tau).$$

The particular case $p = 2$ is a Hilbert space, with inner product $\langle y, x \rangle_{L_2} := \tau(y^*x)$.

A von Neumann algebra $\mathcal{M}$ is said to be σ-finite if $\mathcal{M}$ admits at most countably many pairwise orthogonal projections ([131, Definition 3.18]). The σ-finiteness of $\mathcal{M}$ is equivalent to the assumption that $\mathcal{M}$ admits a faithful representation on a separable Hilbert space [131, Proposition 3.19] and further implies that every $x \in \mathcal{M}$ can be obtained as a weak*-limit of a sequence in $L_1(\tau) \cap \mathcal{M}$.

1.5.2 Singular traces and operator ideals

This section introduces notation and terminology concerning singular traces (in particular Dixmier traces) and operator ideals. Let H be a complex separable Hilbert space with orthonormal basis $\{e_n\}_{n=0}^{\infty}$. Denote by $\mathcal{B}(H)$ the $*$-algebra of bounded linear operators on H and denote by $\mathcal{K}(H)$ the set of compact operators. Given an operator $T \in \mathcal{K}(H)$, the singular value function $s \mapsto \mu(s, T)$ is defined to be the distance of T to the set of all operators of rank at most s:

$$\mu(s, T) := \inf\{\|T - R\| \ : \ \mathrm{rank}(R) \leq s\}, \quad s \geq 0.$$

Equivalently, $\{\mu(k, T)\}_{k=0}^{\infty}$ is the sequence of eigenvalues of $|T|$ arranged in non-increasing order with multiplicities.

Given $p \in (0, \infty]$, the p-Schatten class $\mathcal{L}_p$ is defined to be the set of operators $T \in \mathcal{B}(H)$ such that $\{\mu(n, T)\}_{n=0}^{\infty}$ is in the sequence space ℓ_p. The weak Schatten class $\mathcal{L}_{p,\infty}$ is the set of operators $T \in \mathcal{B}(H)$ such that $\mu(n, T) = O(n^{-1/p})$. The Schatten p-class $\mathcal{L}_p$ (resp. the weak Schatten class $\mathcal{L}_{p,\infty}$) is equipped with the norm (resp. quasi-norm) given by $\|T\|_p := \|\{\mu(n, T)\}_{n=0}^{\infty}\|_{\ell_p}$ (resp. $\|T\|_{p,\infty} := \sup_{n\geq 0} n^{1/p}\mu(n, T)$). The closure of the set of finite rank operators in $\mathcal{L}_{p,\infty}$ shall be denoted $(\mathcal{L}_{p,\infty})_0$.

When $p \geq 1$, $\mathcal{L}_p$ is a Banach space, and an ideal of $\mathcal{B}(H)$. Although the weak Schatten quasinorm $\|\cdot\|_{p,\infty}$ is not a norm, when $p > 1$ there is an equivalent norm for $\mathcal{L}_{p,\infty}$, given by:

$$\|T\|'_{p,\infty} = \sup_{n\geq 0} (n+1)^{1/p-1} \sum_{k=0}^{n} \mu(k, T).$$

As with the $\mathcal{L}_p$ spaces, $\mathcal{L}_{p,\infty}$ is an ideal of $\mathcal{B}(H)$. We also have the following form of Hölder's inequality,

$$\|TS\|_{r,\infty} \leq c_{p,q}\|T\|_{p,\infty}\|S\|_{q,\infty}$$

where $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, for some constant $c_{p,q}$. For $0 < r < \infty$, the closure of the set of finite rank operators in the $\mathcal{L}_{r,\infty}$ quasinorm is denoted $(\mathcal{L}_{r,\infty})_0$. It is straightforward to check that $(\mathcal{L}_{r,\infty})_0$ is again an ideal of $\mathcal{B}(H)$.

An operator theoretic result which will be useful is the Araki-Lieb-Thirring inequality [3, Page 169] (see also [81, Theorem 2]) which states that if A and B are bounded operators and $r \geq 1$, then:

$$|AB|^r \prec\prec_{\log} |A|^r|B|^r$$

where $\prec\prec_{\log}$ denotes logarithmic submajorisation. In particular this implies the following inequality for the $\mathcal{L}_{r,\infty}$ quasinorm, when $r \geq 1$:

$$\|AB\|_{r,\infty} \leq e\||A|^r|B|^r\|_{1,\infty} \leq e\|A\|_{\infty}^{r-1}\|A|B|^r\|_{1,\infty}. \tag{1.5.2}$$

For $q \in [1, \infty)$, we also consider the ideal $\mathcal{L}_{q,1}$, defined as the set of bounded operators T on H satisfying:

$$\|T\|_{\mathcal{L}_{q,1}} := \sum_{n\geq 0} \frac{\mu(n, T)}{(n+1)^{1-\frac{1}{q}}} < \infty.$$

We have the following Hölder-type inequality, if $\frac{1}{p} + \frac{1}{q} = 1$ then:

$$\|TS\|_1 \leq \|T\|_{p,\infty}\|S\|_{q,1}. \tag{1.5.3}$$

A functional $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is called a continuous trace if it is continuous in the $\mathcal{L}_{1,\infty}$ quasinorm and for all $A \in \mathcal{L}_{1,\infty}$ and $B \in \mathcal{B}(H)$, we have $\varphi(BA) = \varphi(AB)$. A trace φ is hermitian if $\varphi(A^*) = \overline{\varphi(A)}$ for all $A \in \mathcal{L}_{1,\infty}$.

An important fact about traces is that any trace φ on $\mathcal{L}_{1,\infty}$ vanishes on $\mathcal{L}_1$ [89, Theorem 5.7.8]. It is known that not all traces on $\mathcal{L}_{1,\infty}$ are continuous [88, Remark 3.1(3)]. Within the class of continuous traces on $\mathcal{L}_{1,\infty}$ there are the well-known Dixmier traces [89, Chapter 6], which we discuss below.

There is a bijective correspondence between traces on $\mathcal{L}_{1,\infty}$ and certain functionals on ℓ_∞ which we will describe here for later use. A continuous linear functional $\theta \in \ell_\infty^*$ is called translation-invariant if it is invariant under translations in the sense that

$$\theta(x_0, x_1, \ldots) = \theta(0, x_0, x_1, \ldots), \qquad \text{for all } (x_0, x_1, \ldots) \in \ell_\infty.$$

Additionally, a functional θ is Hermitian if $\theta(x^*) = \overline{\theta(x)}$ for all $x \in \ell_\infty$

The following result is a combination of [123, Theorem 4.1, Theorem 4.9]. Note that in [123] the implicit assumption is made that all functionals are Hermitian.

Theorem 1.5.1. *For every continuous Hermitian trace φ on $\mathcal{L}_{1,\infty}$ there exists a unique translation-invariant Hermitian functional $\theta \in \ell_\infty^*$ such that for all $A \geq 0$ in $\mathcal{L}_{1,\infty}$ we have:*

$$\varphi(A) = \theta\left(\frac{1}{\log 2}\left\{\sum_{k=2^n-1}^{2^{n+1}-2} \mu(k, A)\right\}_{n\geq 0}\right). \tag{1.5.4}$$

Moreover, for every translation invariant $\theta \in \ell_\infty^*$ the right hand side of (1.5.4) defines a trace on $\mathcal{L}_{1,\infty}$.

Corollary 1.5.2. *Every continuous Hermitian trace φ on $\mathcal{L}_{1,\infty}$ can be written as a difference $\varphi = \varphi_+ - \varphi_-$ where φ_- and φ_+ are positive continuous traces.*

Proof. Due to Theorem 1.5.1, the result will follow from the assertion that for any translation invariant Hermitian linear functional θ on ℓ_∞ that there are positive translation-invariant linear functionals θ_+, θ_- such that $\theta = \theta_+ - \theta_-$. This fact is established in [123, Lemma 4.8], thus completing the proof. □

In Section 5.6 we also refer to the specific subclass of traces on $\mathcal{L}_{1,\infty}$ of Dixmier traces. A linear positive linear functional ω on the von Neumann algebra $L_\infty(0,\infty)$ is called an extended limit if ω vanishes on all functions of bounded support and $\omega(1) = 1$. The dilation semigroup $\{\sigma_s\}_{s>0}$ on $L_\infty(0,\infty)$ is defined by:

$$(\sigma_s f)(t) = f(t/s).$$

A dilation invariant extended limit is defined to be an extended limit ω such that $\omega \circ \sigma_s = \omega$ for all $s > 0$.

Given a dilation invariant extended limit ω, the Dixmier trace tr_ω is defined on $0 \leq A \in \mathcal{L}_{1,\infty}$ by

$$\mathrm{tr}_\omega(A) = \omega\left(t \mapsto \frac{1}{\log(1+t)} \int_0^t \mu(s, A)\, ds\right).$$

It is proved in [89, Theorem 6.3.6] that tr_ω extends by linearity to a continuous trace on $\mathcal{L}_{1,\infty}$.

Chapter 2

Spectral integration

The theory of double operator integration on the Hilbert-Schmidt class rests on the standard theory of spectral integration.

As mentioned in Subsection 1.2.2 of the introduction, the definition of a double operator integral $\mathcal{T}_\phi^{A,B}(X)$ when $X \in \mathcal{L}_2(H)$ (the Hilbert-Schmidt space of a Hilbert space H) can be seen as an application of the functional calculus of two commuting operators of "left multiplication" $L_A X := AX$ and "right multiplication" $R_B X = XB$. Many of the elementary properties of $\mathcal{T}_\phi^{A,B}$ can be seen as consequences of well-known facts about functional calculus of commuting self-adjoint operators [120, Chapters 4 and 5], [18, Chapter 6].

We will now attempt a thorough overview of this theory. Much of the following material is standard, but there appears to be no single reference which adequately covers the material in sufficient generality for our purposes.

An important distinction occurs between the notion of spectral measure valued in the projection lattice of a Hilbert space and spectral measure valued in the projection lattice of a von Neumann algebra. In the former theory, many of the desirable properties of double operator integrals follow from the specific tensor product structure of the Hilbert-Schmidt class. On the other hand, when working in the setting of general semifinite von Neumann algebras, we must instead take greater care to ensure that product measures are well-defined. We will review the problem of defining products in Subsection 2.2.2.

Ultimately, we will be able to define double operator integrals for spectral measures valued in the projections of a von Neumann algebra provided that the underlying measurable space is a σ-compact standard Borel space. This is more than sufficiently general to cover all known applications of double operator integration theory.

Sections 2.1 and 2.2 cover the elementary theory of spectral measures on a Hilbert space. Readers familiar with spectral theory can skip to Subsection 2.2.2, where we discuss the more subtle issues involving in defining products of spectral measures. Section 2.3 begins the discussion on spectral measures on von Neumann algebras. Double operator integrals in the functional-calculus sense are defined in Section 2.4.

2.1 Finitely additive measures on algebras

Recall that a family $\mathcal{A}$ of subsets of a set X is called an algebra (sometimes a Boolean algebra) if $\mathcal{A}$ is non-empty and closed under the set operations of finite union, finite intersection and complementation. A σ-algebra is an algebra which is further closed under countable union.

Denote the σ-algebra generated by an algebra $\mathcal{A}$ as $\sigma(\mathcal{A})$.

A *finitely additive measure* μ on an algebra $\mathcal{A}$ is a mapping from $\mathcal{A}$ to $[0, \infty]$ which is:

(i) Finitely additive: if $A, B \in \mathcal{A}$ are disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$.

(ii) Nontrivial: $\mu(\emptyset) = 0$ (equivalently, at least one set has finite measure).

A finitely additive measure μ on an algebra $\mathcal{A}$ is called σ-additive on $\mathcal{A}$ if it satisfies the following additional property: if $A \in \mathcal{A}$ is a countable disjoint union $A = \bigcup_{n=0}^{\infty} A_n$ where each $A_n \in \mathcal{A}$, then $\mu(A) = \sum_{n=0}^{\infty} \mu(A_n)$.

A σ-additive measure on a σ-algebra is simply called a *measure*. A central problem in elementary measure theory is to determine when a finitely additive measure μ on an algebra $\mathcal{A}$ extends to a measure on a σ-algebra containing $\mathcal{A}$. Certainly, it is necessary that μ be σ-additive on $\mathcal{A}$. The classical Hahn-Kolmogorov theorem (see e.g. [135, Theorem 11.20], [47, Section III.5, Theorem 8] or [70, Theorem 13A] for the σ-finite case) states that σ-additivity on $\mathcal{A}$ is also sufficient:

Theorem 2.1.1 (Hahn-Kolmogorov)**.** *A finitely additive measure μ on an algebra $\mathcal{A}$ of subsets of a set X admits an extension to a measure on $\sigma(\mathcal{A})$ if and only if μ is σ-additive on the algebra $\mathcal{A}$.*

Moreover, the extension can be described precisely by:

$$\mu^*(Z) = \inf\{\sum_{j=0}^{\infty} \mu(A_j) \ : \ Z \subseteq \bigcup_{j=0}^{\infty} A_j,\, A_j \in \mathcal{A}\}$$

and μ^* defines a measure on the σ-algebra of sets C which satisfy:

$$\mu^*(Z) = \mu^*(Z \cap C) + \mu^*(Z \setminus C)$$

for all $Z \subseteq X$.

If X is in fact a separable complete metric space, and $\mathcal{A}$ consists of Borel subsets of X, then we have another tool at our disposal to prove that a finitely additive measure μ on $\mathcal{A}$ extends to a measure on $\sigma(\mathcal{A})$ [21, Theorem 7.1.7, Theorem 7.3.11]:

Theorem 2.1.2. *Let X be a complete separable metric space, and let $\mathcal{A}$ be an algebra whose elements are Borel subsets of X. A finitely additive measure μ on $\mathcal{A}$ is σ-additive if and only if it satisfies the following inner regularity condition:*

$$\mu(A) = \sup\{\mu(K) \ : \ K \in \mathcal{A}, \overline{K} \subseteq A \textit{ and } \overline{K} \textit{ is compact}\}$$

where $\overline{K}$ denotes the closure of a set K.

Combining the Hahn-Kolmogorov theorem and Theorem 2.1.2 yields the following: if μ is a finitely additive measure on an algebra of Borel sets on a complete separable metric space, and μ satisfies the inner regularity condition of Theorem 2.1.2, then μ is the restriction to $\mathcal{A}$ of a measure on $\sigma(\mathcal{A})$.

The above theorem turns out to be crucial in the theory of double operator integrals in the semifinite setting. Without it, we would be unable to even state the basic definitions.

Ultimately a double operator integral on the L_2-space of a semifinite von Neumann algebra will be defined as an integral with respect to a so-called *spectral measure*. However in order to prove that the claimed spectral measure even exists, we must review the theory of finitely additive spectral measures.

Definition 2.1.3. *Let $\mathcal{A}$ be an algebra of subsets of a set X, and let H be a Hilbert space. A finitely additive spectral measure ν is a mapping from $\mathcal{A}$ to the lattice $\mathcal{P}(H)$ of projections in H which satisfies the following two properties:*

(i) $\nu(X) = 1$ *(here, 1 is the identity operator on H)*

(ii) *Finite additivity: if $A, B \in \mathcal{A}$ are disjoint, then $\nu(A \cup B) = \nu(A) + \nu(B)$.*

Wherever necessary, the inner product and norm on H will be denoted $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively.

Although these assumptions of Definition 2.1.3 appear quite mild the algebraic properties of $\mathcal{P}(H)$ are sufficiently powerful that we can prove the following:

Lemma 2.1.4. *Let ν be a finitely additive spectral measure on an algebra $\mathcal{A}$. If $B_1, B_2 \in \mathcal{A}$ are disjoint, then $\nu(B_1)$ and $\nu(B_2)$ are orthogonal.*

Proof. To see this, one simply need note that if p and q are two projections on H whose sum is again a projection, then p and q are orthogonal. For the sake of completeness, we include the argument here. Since $(p+q)^2 = p+q$, we have $pq + qp = 0$.

Therefore, $pqp + qp = 0$, and since pqp is self-adjoint, we therefore have that $pqp = -\frac{1}{2}(qp + pq) = 0$. Thus, $pq = ppq = -pqp = 0$, and similarly $qp = 0$.

To conclude, since $\nu(B_1 \cup B_2)$, $\nu(B_1)$, $\nu(B_2)$ are all projections and $\nu(B_1 \cup B_2) = \nu(B_1) + \nu(B_2)$, it follows that $\nu(B_1)$ and $\nu(B_2)$ must be orthogonal as claimed. $\square$

Remark 2.1.5. *Due to Lemma 2.1.4, it follows immediately that:*

$$\nu(X \setminus A) = \nu(A)^{\perp} = 1 - \nu(A).$$

where $\perp$ denotes the orthogonal complement.

In fact, a finitely additive spectral measure induces a lattice homomorphism from the algebra $\mathcal{A}$ to the lattice of projections $\mathcal{P}(H)$. The key to this fact is that the image of a spectral measure ν in fact consists of pairwise commuting projections, as the following lemma shows:

Lemma 2.1.6. *Let ν be a finitely additive spectral measure on an algebra $\mathcal{A}$. Then for any $A_1, A_2 \in \mathcal{A}$, we have:*

$$\nu(A_1 \cap A_2) = \nu(A_1)\nu(A_2) = \nu(A_2)\nu(A_1).$$

Proof. Let $\Delta = A_1 \cap A_2$. Since $\mathcal{A}$ is an algebra, we of course have $\Delta \in \mathcal{A}$. Decompose A_1 and A_2 as:

$$A_1 = A_1 \setminus \Delta \cup \Delta, \quad A_2 = A_2 \setminus \Delta \cup \Delta.$$

Since ν is finitely additive:

$$\nu(A_1) = \nu(A_1 \setminus \Delta) + \nu(\Delta), \quad \nu(A_2) = \nu(A_2 \setminus \Delta) + \nu(\Delta).$$

Since the family of sets $\{A_1 \setminus \Delta, A_2 \setminus \Delta, \Delta\}$ is pairwise disjoint, it follows from Lemma 2.1.4 that the family of projections $\{\nu(A_1 \setminus \Delta), \nu(A_2 \setminus \Delta), \nu(\Delta)\}$ is pairwise orthogonal, and thus:

$$\begin{aligned}
\nu(A_1)\nu(A_2) &= (\nu(A_1 \setminus \Delta) + \nu(\Delta))(\nu(A_2 \setminus \Delta) + \nu(\Delta)) \\
&= \nu(\Delta)^2 \\
&= \nu(\Delta) \\
&= \nu(A_1 \cap A_2).
\end{aligned}$$

By symmetry, we also have $\nu(A_2)\nu(A_1) = \nu(A_1 \cap A_2)$. □

Since $\nu(A_1)$ and $\nu(A_2)$ commute for all $A_1, A_2 \in \mathcal{A}$, it follows that $\nu(A_1)\nu(A_2) = \nu(A_1) \wedge \nu(A_2)$, and hence:

$$\nu(A_1 \cap A_2) = \nu(A_1) \wedge \nu(A_2).$$

By taking complements, it follows immediately that:

$$\nu(A_1 \cup A_2) = \nu(A_1) \vee \nu(A_2).$$

In algebraic terms, a finitely additive spectral measure ν on an algebra $\mathcal{A}$ is a lattice homomorphism from $\mathcal{A}$ to $\mathcal{P}(H)$.

2.2 Spectral measures on a Hilbert space

While there does exist a theory of integration relative to finitely additive spectral measures, we will instead develop the theory of countably additive spectral measures.

Definition 2.2.1. *Let (X, Σ) be a measurable space, and let H be a Hilbert space. A spectral measure ν is a mapping from Σ to the lattice of projections $\mathcal{P}(H)$ in H satisfying the following two properties:*

(i) *Completeness: $\nu(X) = 1$ (again, 1 is the identity operator on H).*

(ii) *Weak σ-additivity: If $\{A_j\}_{j=0}^{\infty}$ is a countable family of pairwise disjoint sets in Σ, then:*

$$\nu\left(\bigcup_{j=0}^{\infty} A_j\right) = \lim_{n\to\infty} \sum_{j=0}^{n} \nu(A_j)$$

where the limit is taken in the sense of the weak operator topology of $\mathcal{B}(H)$.

Call the tuple (X, Σ, H, ν) a spectral measure space.

It is perhaps not immediately obvious that a spectral measure is automatically a finitely additive spectral measure. One way to see that this is true is to note that the infinite family $\{\emptyset\}_{n=0}^{\infty}$ is pairwise disjoint, so we can apply (ii) to conclude that $\nu(\emptyset) = 0$, and from there one can see that (ii) implies finite additivity.

Remark 2.2.2. *Within the literature there are some variations in the way that the definition of a spectral measure is stated. For example, some authors (such as [18, Section 5.1.1]) assume that the limit in Definition 2.2.1.(ii) is taken in the strong operator topology.*

Since a spectral measure ν is in particular a finitely additive spectral measure, if a family $\{A_j\}_{j=0}^{\infty} \subseteq \Sigma$ is pairwise disjoint then:

$$p_n := \nu(\bigcup_{j=0}^{\infty} A_j) - \sum_{j=0}^{n} \nu(A_j) = \nu(\bigcup_{j=n+1}^{\infty} A_j)$$

defines a sequence of projections converging in the weak operator topology to zero as $n \to \infty$. Thus for all $x \in H$, $\langle x, p_n x\rangle = \|p_n x\|^2$ converges to zero, and so p_n also converges strongly to zero. Thus in Definition 2.2.1.(ii), we could have equivalently assumed that the convergence is in the strong operator topology.

Other authors (such as [110, Section VIII.3]) assume as part of the definition that if $A, B \in \Sigma$ then $\nu(A \cap B) = \nu(A)\nu(B)$. However this follows from Lemma 2.1.6.

Almost immediately from Definition 2.2.1, we get the following:

Theorem 2.2.3. *Let (X, Σ, H, ν) be a spectral measure space and let $x, y \in H$. Then the mapping:*

$$\nu^{x,y}(A) := \langle \nu(A)x, y\rangle, \quad A \in \Sigma$$

is a (complex) measure on (X, Σ).

In particular, taking $x = y$, the mapping:

$$\nu^{x,x}(A) = \|\nu(A)x\|^2, \quad A \in \Sigma$$

is a (non-negative) measure on Σ.

Moreover if $\mathcal{A}$ is an algebra, and ν is a finitely additive spectral measure on $\mathcal{A}$, then $\nu^{x,x}$ defines a finitely additive non-negative measure on $\mathcal{A}$.

As with scalar-valued measures, a central question in the theory of spectral measures concerns the problem of extending a finitely additive spectral measure on an algebra $\mathcal{A}$

to a spectral measure on $\sigma(\mathcal{A})$. In parallel to the Hahn-Kolmogorov theorem, we have the following:

Theorem 2.2.4 (Spectral Hahn-Kolmogorov)**.** *Let $\mathcal{A}$ be an algebra of subsets of a set X, and let $\nu : \mathcal{A} \to \mathcal{P}(H)$ be a finitely additive spectral measure.*

Then ν extends to a spectral measure on $\sigma(\mathcal{A})$ if and only if for all $x \in H$, the finitely additive measure $\nu^{x,x}$ is σ-additive on $\mathcal{A}$.

Proof. One direction of the implication is clear: if ν is the restriction to $\mathcal{A}$ of a spectral measure, then each $\nu^{x,x}$ is the restriction to $\mathcal{A}$ of a measure (Theorem 2.2.3) and hence must be σ-additive on $\mathcal{A}$. We now focus attention on the reverse implication.

Recall from the precise statement of the Hahn-Kolmogorov theorem that the finitely additive measure $\nu^{x,x}$ can be extended to a measure in the following way:

For each fixed x, consider the following mapping μ_x^* on a subset $Z \subseteq X$:

$$\mu_x^*(Z) = \inf\{\sum_{j=0}^{\infty} \nu^{x,x}(A_j) \ : \ Z \subseteq \bigcup_{j=0}^{\infty} A_j, A_j \in \mathcal{A}\}.$$

The family of sets $\mathcal{C}(x)$ defined by:

$$\mathcal{C}(x) = \{C \subseteq X \ : \ \mu_x^*(Y) = \mu_x^*(Y \cap C) + \mu_x^*(Y \setminus C), \text{ for all } Y \subseteq X\}$$

is a σ-algebra, and μ_x^* defines a measure on $\mathcal{C}(x)$. According to the Hahn-Kolmogorov theorem, $\mathcal{A} \subseteq \mathcal{C}(x)$ and $\mu_x^*(A) = \nu^{x,x}(A)$ for all $A \in \mathcal{A}$.

We now define a "spectral outer measure" on X from ν as follows. If $Z \subseteq X$, define:

$$\nu^*(Z) := \inf\{s - \lim_{n\to\infty} \sum_{j=0}^{n} \nu(A_j) \ : \ A_j \in \mathcal{A}, A_j \cap A_k = \emptyset \text{ for } j \neq k, Z \subseteq \bigcup_{j=0}^{\infty} A_j\}.$$

Here, $s - \lim$ denotes the limit in the strong operator topology. It should be noted that the strong limit of a monotone family of projections is again a projection, and thus for each pairwise disjoint family $\{A_j\}_{j=0}^{\infty}$ the limit $s - \lim_{n\to\infty} \sum_{j=0}^{n} \nu(A_j)$ is a projection. Hence, $\nu^*(Z)$ is well defined as the infimum of a family of projections, and in particular is a projection.

It is not hard to see that for each $x \in H$ and an arbitrary subset $Z \subseteq X$ that we have:

$$\langle x, \nu^*(Z)x\rangle = \mu_x^*(Z). \tag{2.2.1}$$

Let $\mathcal{C}(\nu)$ denote the family of subsets V of X such that:

$$\nu^*(Y) = \nu^*(V \cap Y) + \nu^*(Y \setminus V)$$

for all $Y \subseteq X$. If $C \in \mathcal{C}(x)$, then by the definition of $\mathcal{C}(x)$ for all subsets $Y \subseteq X$ we have:

$$\mu_x^*(Y) = \mu_x^*(Y \cap C) + \mu_x^*(Y \setminus C).$$

Using (2.2.1), it follows that:

$$\langle x, \nu^*(Y)x\rangle = \langle x, \nu^*(Y \cap C)x\rangle + \langle x, \nu^*(Y \setminus C)x\rangle.$$

So if $Z \in \mathcal{C}(x)$ for every $x \in H$, then by an application of the polarisation identity we have:

$$\langle x, \nu^*(Y)y \rangle = \langle x, \nu^*(Y \cap C)y \rangle + \langle x, \nu^*(Y \setminus C)y \rangle$$

for all $x, y \in H$, and therefore $C \in \mathcal{C}(\nu)$. Conversely, if $C \in \mathcal{C}(\nu)$ then immediately $C \in \mathcal{C}(x)$ for every $x \in H$.

Therefore $\mathcal{C}(\nu) = \bigcap_{x \in H} \mathcal{C}(x)$ and so $\mathcal{C}(\nu)$ is a σ-algebra.

Since every μ_x^* is σ-additive on $\mathcal{C}(x)$, μ_x^* is in particular σ-additive on $\mathcal{C}(\nu)$. Applying the polarisation identity, it follows that $\langle x, \nu^*(\cdot)y \rangle$ is σ-additive on $\mathcal{C}(\nu)$ and therefore ν^* is a spectral measure on $\mathcal{C}(\nu)$. ☐

Now that we have necessary and sufficient conditions for it to be possible to extend a finitely additive measure to a spectral measure, we should also discuss the question of uniqueness.

Lemma 2.2.5. *Let $\nu : \mathcal{A} \to \mathcal{P}(H)$ be a finitely additive spectral measure on an algebra $\mathcal{A}$ of subsets of a set X. Then there is at most one spectral measure $\widetilde{\nu}$ on $\sigma(\mathcal{A})$ which extends ν.*

Proof. Suppose that there are two extensions, μ_0 and μ_1 of ν to $\sigma(\mathcal{A})$. That is,

$$\mu_0(A) = \mu_1(A) = \nu(A) \quad A \in \mathcal{A}$$

and μ_0 and μ_1 are spectral measures on $\sigma(\mathcal{A})$. Let $\mathcal{F} \subset \sigma(\mathcal{A})$ denote the family of subsets where μ_0 and μ_1 agree. That is,

$$\mathcal{F} = \{A \in \sigma(\mathcal{A}) \; : \; \mu_0(A) = \mu_1(A)\}.$$

By assumption we have $\mathcal{A} \subseteq \mathcal{F}$. Let us show that $\mathcal{F}$ is a σ-algebra. First, thanks to Lemma 2.1.6, it is clear that $\mathcal{F}$ is closed under finite unions and intersections, and from Remark 2.1.5 $\mathcal{F}$ is also closed under complementations. Hence $\mathcal{F}$ is an algebra.

To complete the proof, it suffices to show that $\mathcal{F}$ is closed under countable increasing unions. Thus let $\{A_n\}_{n \geq 0}$ be an upward-nested family of sets, where each $A_n \in \mathcal{F}$. Then,

$$\mu_0(A_n) = \mu_1(A_n), \quad n \geq 0.$$

Since μ_0 and μ_1 are spectral measures, we therefore have:

$$\mu_0\left(\bigcup_{n \geq 0} A_n\right) = \lim_{n \to \infty} \mu_0(A_n) = \lim_{n \to \infty} \mu_1(A_n) = \mu_1\left(\bigcup_{n \geq 0} A_n\right)$$

where the limits are in the weak operator topology. Thus $\mathcal{F}$ is a σ-algebra containing $\mathcal{A}$, and so in particular $\mathcal{F}$ contains $\sigma(\mathcal{A})$. ☐

2.2.1 Integration with respect to spectral measures

Let (X, Σ, H, ν) be a spectral measure space, and let $x, y \in H$. Recall that $\nu^{x,y}$ denotes the (scalar) measure:

$$\nu^{x,y} = \langle x, \nu(\cdot)y \rangle : \Sigma \to \mathbb{C}$$

In fact $\nu^{x,y}$ is of finite total variation.

Lemma 2.2.6. *For each $x, y \in H$, the measure $\nu^{x,y}$ defined above has finite total variation, and:*

$$|\nu^{x,y}|(X) \leq \|x\|\|y\|.$$

Proof. Let $\{A_j\}_{j=0}^{\infty}$ be a family of pairwise disjoint sets in Σ. Then:

$$\sum_{j=0}^{\infty} |\nu^{x,y}(A_j)| = \sum_{j=0}^{\infty} |\langle x, \nu(A_j)y\rangle| = \sum_{j=0}^{\infty} |\langle \nu(A_j)x, \nu(A_j)y\rangle|.$$

Thus by the Cauchy-Schwarz inequality:

$$\sum_{j=0}^{\infty} |\nu^{x,y}(A_j)| \leq \sum_{j=0}^{\infty} \|\nu(A_j)x\|\|\nu(A_j)y\| \leq \left(\sum_{j=0}^{\infty} \|\nu(A_j)x\|^2\right)^{1/2} \left(\sum_{j=0}^{\infty} \|\nu(A_j)y\|^2\right)^{1/2}.$$

Since the projections $\{\nu(A_j)\}_{j=0}^{\infty}$ are pairwise orthogonal (Lemma 2.1.4), we can apply Bessel's inequality to arrive at:

$$\sum_{j=0}^{\infty} |\nu^{x,y}(A_j)| \leq \|x\|\|y\|.$$

Taking the supremum over all countable families $\{A_j\}_{j=0}^{\infty}$ of pairwise disjoint sets, we can thus bound the total variation of $\nu^{x,y}$ above by $\|x\|\|y\|$. □

Lemma 2.2.6 implies that if ϕ is a bounded measurable function on X, for each $x, y \in H$ we may define:

$$T_{x,y} := \int_X \phi \, d\nu^{x,y}.$$

It is reasonable to think that there should be an operator $T \in \mathcal{B}(H)$ such that:

$$\langle x, Ty\rangle = T_{x,y}.$$

As we will demonstrate in the next proposition, this is indeed the case, and moreover T satisfies:

$$\|T\| \leq \|\phi\|_\infty.$$

From now on, denote the class of bounded measurable functions on (X, Σ) as $B(X)$. We will develop an integration theory for functions ϕ in $B(X)$, although we remark that many authors instead prefer to work with almost-everywhere equivalence classes of functions, where a set $A \in \Sigma$ is declared to be a null set if $\nu(A)$ is the zero projection. The decision to work with bounded functions rather than pointwise-almost-everywhere equivalence classes has been made since we will later need to integrate the same function with respect to different measures on the same measurable space.

The uniform norm $\|\phi\|_\infty$ for $\phi \in B(X)$ is defined in the usual way as $\sup_{x \in X} |\phi(x)|$.

Proposition 2.2.7. *Let (X, Σ, H, μ) be a spectral measure space, and let $\phi \in B(X)$. Then there exists a unique operator $T \in \mathcal{B}(H)$ such that for all $x, y \in H$ we have:*

$$\langle x, Ty \rangle = \int_X \phi \, d\nu^{x,y}$$

and:

$$\|T\| \leq \|\phi\|_\infty.$$

Proof. Since $\nu^{x,y}$ is a scalar measure of bounded total variation, the integral $\int_X \phi \, d\nu^{x,y}$ is indeed well-defined for all $x, y \in H$.

Since $\nu^{x,y}(A) := \langle x, \nu(A)y \rangle$, it follows from the sesquilinearity of the inner product that we have:

$$\nu^{x,y+z} = \nu^{x,y} + \nu^{x,z}, \; \nu^{x+y,z} = \nu^{x,z} + \nu^{y,z}$$

and if $\alpha \in \mathbb{C}$,

$$\nu^{\alpha x,y} = \overline{\alpha}\nu^{x,y}, \; \nu^{x,\alpha y} = \alpha\nu^{x,y}.$$

Therefore the mapping:

$$B(x, y) = \int_X \phi \, d\nu^{x,y}.$$

is a sesquilinear map on H. Thanks to Lemma 2.2.6,

$$|B(x, y)| \leq \|\phi\|_\infty \|x\| \|y\|.$$

Hence from the Riesz theorem, there exists a unique bounded operator T on H such that:

$$B(x, y) = \langle x, Ty \rangle$$

and $\|T\| \leq \|\phi\|_\infty$. □

Proposition 2.2.7 permits the following definition:

Definition 2.2.8. *Let (X, Σ, H, ν) be a spectral measure space, and let $\phi \in B(X)$. Define the integral $\int_X \phi \, d\nu$ as the unique bounded linear operator on H such that:*

$$\langle x, \int_X \phi \, d\nu \, y \rangle = \int_X \phi \, d\nu^{x,y}, \quad x, y \in H.$$

One could also define the spectral integral as a continuous extension of the spectral integrals of simple functions in an appropriate sense [120, Section 4.3.1].

The second part of Proposition 2.2.7 can now be restated as:

$$\left\| \int_X \phi \, d\nu \right\| \leq \|\phi\|_\infty, \quad \phi \in B(X). \tag{2.2.2}$$

Moreover since each $\nu^{x,y}$ for $x, y \in H$ is a measure, it is immediate that if $\phi, \psi \in B(X)$ and $\alpha \in \mathbb{C}$, then:

$$\int_X \alpha\phi + \psi \, d\nu = \alpha \int_X \phi \, d\nu + \int_X \psi \, d\nu.$$

Since $\nu^{x,y} = \overline{\nu^{y,x}}$, we also have:

$$\left(\int_X \phi \, d\nu\right)^* = \int_X \overline{\phi} \, d\nu.$$

Moreover, it is easy to see that if $A \in \Sigma$ then:

$$\int_X \chi_A \, d\nu = \nu(A).$$

It is an immediate consequence of (2.2.2) and the linearity of the integral that uniform convergence of the integrand implies norm convergence of the integral:

Lemma 2.2.9. *Let (X, Σ, H, ν) be a spectral measure space. Let $\{\phi_j\}_{j=0}^{\infty} \subseteq B(X)$ converge uniformly to some ϕ:*

$$\lim_{n\to\infty} \|\phi_n - \phi\|_\infty = 0.$$

Then,

$$\lim_{n\to\infty} \int_X \phi_n \, d\nu = \int_X \phi \, d\nu$$

in the norm topology.

An arbitrary bounded measurable function can be uniformly approximated by simple functions. Indeed, if $\phi \in B(X)$ is real-valued for each $n \geq 0$, define:

$$A_{k,n} := \phi^{-1}([\|\phi\|_\infty \frac{k}{2^n}, \|\phi\|_\infty \frac{k+1}{2^n})), \quad k = -2^n, -2^n + 1, \ldots, 2^n.$$

Then the function:

$$\phi_n := \sum_{k=-2^n}^{2^n} \chi_{A_{k,n}} \frac{k}{2^n}$$

is within $\|\phi\|_\infty 2^{-n}$ of ϕ in the uniform norm, so $\|\phi_n - \phi\|_\infty$ converges to zero. We can similarly approximate bounded complex valued functions arbitrarily well by a sequence of simple functions in the uniform norm.

There is a property of integration with respect to spectral measures which is of great use in double operator integral theory, and which does not have any obvious analogy in the scalar-valued case: integration with respect to a spectral measure defines a functional calculus. The following theorem, in essence, states that integration is an algebra homomorphism from $B(X)$ to the algebra of bounded linear operators on H.

Theorem 2.2.10. *Let (X, Σ, H, ν) be a spectral measure space, and let ϕ and ψ be bounded measurable functions. Then:*

$$\int_X \phi\psi \, d\nu = \int_X \phi \, d\nu \int_X \psi \, d\nu.$$

Proof. Suppose initially that ϕ and ψ are characteristic functions of measurable sets. That is, suppose that $\phi = \chi_{A_1}$ and $\psi = \chi_{A_2}$, where $A_1, A_2 \in \Sigma$. Then:

$$\phi\psi = \chi_{A_1 \cap A_2}.$$

Thus by Lemma 2.1.6,

$$\int_X \phi\psi \, d\nu = \nu(A_1 \cap A_2) = \nu(A_1)\nu(A_2).$$

Thus the theorem is true when ϕ and ψ are characteristic functions of measurable sets. Due to linearity of the integral, the result extends straightforwardly to simple functions.

Now if ϕ and ψ are arbitrary bounded measurable functions, we can select sequences $\{\phi_n\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$ of simple functions which approximate ϕ and ψ respectively in the uniform norm. Lemma 2.2.9 then implies:

$$\lim_{n\to\infty} \int_X \phi_n \, d\nu = \int_X \phi \, d\nu, \quad \lim_{n\to\infty} \int_X \psi_n \, d\nu = \int_X \psi \, d\nu$$

where the limits are in the operator norm.

Moreover, $\phi_n\psi_n$ approximates $\phi\psi$ in the uniform norm. Thus using Lemma 2.2.9, we have the norm topology convergence:

$$\begin{aligned}
\int_X \phi\psi \, d\nu &= \lim_{n\to\infty} \int_X \phi_n\psi_n \, d\nu \\
&= \lim_{n\to\infty} (\int_X \phi_n \, d\nu \int_X \psi_n \, d\nu) \\
&= (\lim_{n\to\infty} \int_X \phi_n \, d\nu)(\lim_{n\to\infty} \int_X \psi_n \, d\nu) \\
&= \int_X \phi \, d\nu \int_X \psi \, d\nu.
\end{aligned}$$

□

With the dominated convergence theorem for scalar valued functions on measure spaces, we also have the following result in the same spirit as Lemma 2.2.9, where instead we deal with weaker notions of convergence.

Corollary 2.2.11. *Suppose that $\{\phi_n\}_{n=0}^{\infty}$ is a sequence of bounded functions on X which is uniformly bounded:*

$$\sup_{n\geq 0} \|\phi_n\|_\infty < \infty$$

and which converges pointwisely to some bounded function ϕ on X. Then,

$$\lim_{n\to\infty} \int_X \phi_n \, d\nu = \int_X \phi \, d\nu$$

in the strong operator topology.

Proof. Let $x \in H$, and consider:

$$\left\| \left(\int_X \phi_n - \phi \, d\nu \right) x \right\|^2.$$

In terms of the inner product on H, this is:

$$\left\langle x, \left(\int_X \phi_n - \phi\, d\nu\right)^* \left(\int_X \phi_n - \phi\, d\nu\right) x \right\rangle.$$

Now using Theorem 2.2.10, this reduces to the scalar integral:

$$\int_X |\phi_n - \phi|^2\, d\nu_{x,x}$$

which by the dominated convergence theorem converges to zero as $n \to \infty$.

Hence,

$$\lim_{n\to\infty} \left\| \int_X \phi\, d\nu \cdot x - \int_X \phi_n\, d\nu \cdot x \right\| = 0$$

and this is precisely the desired claim. □

An important tool for double operator integration theory is the change-of-variables formula.

Theorem 2.2.12. *Let (X, Σ, H, ν) be a spectral measure space, and let (Y, Ω) be a measurable space. If $h : X \to Y$ is measurable, we can define the pushforward spectral measure $h_*\nu$ by:*

$$h_*\nu(A) = \nu(h^{-1}(A)) \quad A \in \Omega.$$

Then for all $\phi \in B(Y)$:

$$\int_X \phi \circ h\, d\nu = \int_Y \phi\, d(h_*\nu).$$

Proof. Observe that for $x, y \in H$ and $A \in \Sigma$ we have:

$$\langle x, (h_*\nu)(A)y\rangle = \langle x, \nu(h^{-1}(A))y\rangle = \nu^{x,y}(h^{-1}(A)).$$

So that for each $x, y \in H$, the mapping $A \mapsto \langle x, h_*\nu(A)y\rangle$ is the pushforward of the (scalar-valued) measure $\nu^{x,y}$. It then follows that $A \mapsto \langle x, h_*\nu(A)y\rangle$ is a measure for each $x, y \in H$. Since $A \mapsto h_*\nu(A)$ is projection valued, this completes the proof of the claim that $h_*\nu$ is a spectral measure. We have also proved that:

$$(h_*\nu)^{x,y} = h_*(\nu^{x,y}),$$

where the pushforward on the left is the pushforward of the spectral measure ν, and the pushforward on the right is the pushforward of the scalar-valued measure $\nu^{x,y}$.

By definition, we have:

$$\langle x, \int_X \phi \circ h\, d\nu\, y\rangle = \int_X \phi \circ h\, d\nu^{x,y}.$$

The integral on the right may be computed using the (scalar-valued) change of variables formula:

$$\int_X \phi \circ h\, d\nu^{x,y} = \int_Y \phi\, dh_*(\nu^{x,y})$$

However $h_*(\nu^{x,y}) = (h_*\nu)^{x,y}$, and thus:

$$\langle x, \int_X \phi \circ h \, d\nu \, y\rangle = \int_Y \phi \, d(h_*\nu)^{x,y}.$$

□

We conclude with a mention of the spectral theorem for self-adjoint operators. See e.g. [116, Theorem 13.30], [110, Theorem VIII.6] or [18, Chapter 6, Theorem 1.1].

Theorem 2.2.13 (Spectral theorem)**.** *Let $T : \mathrm{dom}(T) \subseteq H \to H$ be a (possibly unbounded) self-adjoint operator on H. Then there is a unique spectral measure E_T on $\mathbb{R}$ such that for all $x \in H$ and $y \in \mathrm{dom}(T)$ we have:*

$$\langle x, Ty\rangle = \int_{\mathbb{R}} t \, dE_T^{x,y}(t).$$

Moreover $\mathrm{dom}(T)$ is precisely the set of $x \in H$ such that $\int_{\mathbb{R}} t^2 \, dE_T^{x,x}(t) < \infty$.

If f is a bounded Borel function on the real line and T is a self-adjoint operator on H with spectral measure E_T, then by definition ([110, Section VIII.3]) $f(T)$ is the unique operator such that:

$$\langle x, f(T)y\rangle = \int_{\mathbb{R}} f \, dE_T^{x,y}, \quad x, y \in H.$$

In this way the theory of integration with respect to a spectral measure is compatible with Borel functional calculus, as we have:

$$f(T) = \int_{\mathbb{R}} f \, dE_T, \quad f \in B(\mathbb{R}).$$

Combining the Spectral theorem and Theorem 2.2.12 yields a useful identity: if $f \in B(\mathbb{R})$ and T is a self-adjoint operator, and $h : \mathbb{R} \to \mathbb{R}$ is Borel, then:

$$f(h(T)) = \int_{\mathbb{R}} f \circ h \, dE_T = \int_{\mathbb{R}} f \, dh_*E_T$$

Moreover,

$$E_{h(T)} = h_*E_T.$$

2.2.2 Products of spectral measures

Given two measurable spaces (X_1, Σ_1), (X_2, Σ_2), one conventionally defines the product $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ by defining $\Sigma_1 \otimes \Sigma_2$ to be the σ-algebra generated by $\{E \times F \ : \ E \in \Sigma_1, \ F \in \Sigma_2\}$. If μ_1 and μ_2 are σ-finite measures on X_1 and X_2 respectively, then there is a unique product measure $\mu_1 \times \mu_2$ on $X_1 \times X_2$, specified by the “product measure” property that $(\mu_1 \times \mu_2)(E \times F) = \mu_1(E)\mu_2(F)$ for all $E \in \Sigma_1$ and $F \in \Sigma_2$ (see [70, Chapter VII]).

It is natural to ask whether the same can be said for spectral measures defined on the same Hilbert space. Birman, Solomyak and Vershik [20] considered the following question: given two spectral measure spaces $(X_j, \Sigma_j, H, \nu_j)$, $j = 1, 2$ defined on the same Hilbert space H which commute in the sense that:

$$\nu_1(E)\nu_2(F) = \nu_2(F)\nu_1(E), \quad E \in \Sigma_1,\ F \in \Sigma_2 \tag{2.2.3}$$

does there exist a spectral measure ν on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ such that:

$$\nu(E \times F) = \nu_1(E)\nu_2(F), \quad E \in \Sigma_1,\ F \in \Sigma_2?$$

Birman, Solomyak and Vershik resolved this question in the negative, by providing the counterexample given in the proof of the next theorem. See [18, Section 5.5]. It is interesting to note that this impossibility result can also be obtained from older works concerning non-direct products of measures due to Marczewski and Ryll-Nardezewski [91], and the counterexample provided in [91] is virtually the same as that of Birman, Solomyak and Vershik. The same counterexample appears to have been rediscovered a third time by Berg, Christensen and Ressel [12, Exercise 1.31, Chapter 2] and was later applied by Karni and Merzbach [79], in a very similar context.

Theorem 2.2.14 (Birman, Solomyak and Vershik)**.** *There exist spectral measure spaces* $(X_j, \Sigma_j, H, \nu_j)$, $j = 1, 2$ *on the same Hilbert space* H *which commute in the sense of* (2.2.3) *such that there is no spectral measure* ν *on* $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_j)$ *such that* $\nu(E \times F) = \nu_1(E)\nu_2(F)$ *where* $E \in \Sigma_1$ *and* $F \in \Sigma_2$*.*

Proof. Let $\mathfrak{B}([0, 1])$ denote the Borel σ-algebra of $[0, 1]$, and let λ denote the Lebesgue measure on $[0, 1]$ restricted to $\mathfrak{B}([0, 1])$. Denote λ^* for the outer-Lebesgue measure.

Select two disjoint nonmeasurable subsets N_1 and N_2 of $[0, 1]$ such that $N_1 \cup N_2 = [0, 1]$ and $\lambda^*(N_1) = \lambda^*(N_2) = 1$ and define the following two families of sets:

$$\Sigma_1 = \{N_1 \cap \Delta \ :\ \Delta \in \mathcal{B}([0, 1])\},$$
$$\Sigma_2 = \{N_2 \cap \Delta \ :\ \Delta \in \mathcal{B}([0, 1])\}.$$

Evidently, Σ_1 and Σ_2 are σ-algebras for N_1 and N_2 respectively.

Our choice of measurable spaces will be (N_j, Σ_j), $j = 1, 2$ and the Hilbert space will be $L_2([0, 1], \lambda)$.

For a set $A \in \Sigma_j$, with $j = 1, 2$ fixed, choose $\widetilde{A} \in \mathfrak{B}([0, 1])$ such that $\widetilde{A} \cap N_j = A$. We shall define:

$$\nu_j(A) := M_{\chi_{\widetilde{A}}} \in \mathcal{P}(L_2([0, 1], \lambda)).$$

where $M_{\chi_{\widetilde{A}}}$ denotes the operator on $L_2([0, 1], \lambda)$ of pointwise multiplication by the characteristic function of $\widetilde{A}$. To show that this is a well-defined spectral measure on Σ_j, we must first show that $\widetilde{A}$ is unique up to λ-null sets.

Suppose that $Y_1, Y_2 \in \mathfrak{B}([0, 1])$ are such that $Y_1 \cap N_j = Y_2 \cap N_j = A$, and let $Y_1 \Delta Y_2$ denote the symmetric difference. Since $Y_1 \Delta Y_2$ is (in particular) Lebesgue measurable, the Lebesgue outer measure λ^* of N_j decomposes as:

$$\lambda^*(N_j) = \lambda^*(N_j \cap (Y_1 \Delta Y_2)) + \lambda^*(N_j \setminus (Y_1 \Delta Y_2)) \tag{2.2.4}$$

and the Lebesgue outer measure of $N_j \cup (Y_1 \Delta Y_2)$ decomposes as:

$$\lambda^*(N_j \cup (Y_1 \Delta Y_2)) = \lambda^*(Y_1 \Delta Y_2) + \lambda^*(N_j \setminus (Y_1 \Delta Y_2)). \tag{2.2.5}$$

Since $\lambda^*(N_j) = \lambda^*(N_j \cup (Y_1 \Delta Y_2)) = 1$, it follows by subtracting (2.2.4) from (2.2.5) that:

$$\lambda^*(Y_1 \Delta Y_2) = \lambda^*(N_j \cap (Y_1 \Delta Y_2)).$$

However $N_j \cap Y_1 = N_j \cap Y_2 = A$ and so in fact $N_j \cap (Y_1 \Delta Y_2) = \emptyset$. Thus:

$$\lambda^*(Y_1 \Delta Y_2) = 0.$$

Thus Y_1 and Y_2 differ only by a Lebesgue null set, and therefore the mapping $\nu_j(A) = M_{\chi_{\widetilde{A}}}$ is well defined. It then follows easily that ν_j is in fact a spectral measure on (N_j, Σ_j), $j = 1, 2$.

Now we show that there is no product measure $\nu_1 \times \nu_2$ on $N_1 \times N_2$. Let $n \geq 1$ and consider the following subset of $N_1 \times N_2$:

$$\delta_n = \bigcup_{k=0}^{2^n-1} ([k \cdot 2^{-n}, (k+1) \cdot 2^{-n}) \cap N_1) \times ([k \cdot 2^{-n}, (k+1) \cdot 2^{-n}) \cap N_2)$$

If there were a product measure $\nu_1 \times \nu_2$, then we would have:

$$(\nu_1 \times \nu_2)(\delta_n) = \sum_{k=0}^{2^n-1} M_{\chi_{[k2^{-n},(k+1)2^{-n})}} = 1.$$

Note that if $(x_1, x_2) \in \delta_n$, then $|x_1 - x_2| \leq 2^{-n}$. Thus if $(x_1, x_2) \in \bigcap_{n \geq 1} \delta_n$, then $x_1 = x_2$. But $x_1 \in N_1$ and $x_2 \in N_2$ and by assumption $N_1 \cap N_2 = \emptyset$. Therefore:

$$\bigcap_{n \geq 1} \delta_n = \emptyset.$$

That is, we have a nested family of subsets $\{\delta_n\}_{n \geq 1}$ which has empty intersection, but $(\nu_1 \times \nu_2)(\delta_n)$ does not converge weakly to zero. This is in contradiction to σ-additivity. □

Nonetheless, there are circumstances where the product of two commuting spectral measures is well defined. In particular, if $(X_j, \Sigma_j, H_j, \nu_j)$, $j = 1, 2$ are two spectral measure spaces, then one can define a spectral measure ν on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2, H_1 \otimes H_2)$, where $H_1 \otimes H_2$ is the Hilbert space tensor product, such that $\nu(E \times F) = \nu_1(E) \otimes \nu_2(F)$, $E \in \Sigma_1$, $F \in \Sigma_2$ [19]. This construction was then used by Birman and Solomyak to prove that if $(X_j, \Sigma_j, H, \nu_j)$, $j = 1, 2$ are two spectral measure spaces, then there is a unique measure ν on $X_1 \times X_2$ valued in projections on the Hilbert-Schmidt space $\mathcal{L}_2(H)$ such that:

$$\nu(E \times F)X = \nu_1(E) X \nu_2(F), \quad X \in \mathcal{L}_2(H), E \in \Sigma_1, F \in \Sigma_2.$$

(Recall that $\mathcal{L}_2(H)$ is isometrically isomorphic to the tensor product $H \otimes H$). This construction is sufficient to develop the theory of double operator integrals in the setting of bounded linear operators on a Hilbert space.

However, for our purposes this will not be sufficient, and we seek alternative conditions under which the product of commuting spectral measures defines a spectral measure. The following was originally obtained (in fact in even more generality) by H. Schaefer [119], and can also be found as [18, Theorem V.2.6]. Recall that a measurable space (X, Σ) is called a *standard Borel space* if (X, Σ) is isomorphic to the Borel σ-algebra of a complete separable metric space. Standard Borel spaces include $\mathbb{R}$, open intervals in $\mathbb{R}$ and separable Banach spaces.

It is important to note that if (X_1, Σ_1) and (X_2, Σ_2) are standard Borel spaces, then so is $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ [21, Lemma 6.4.2].

Theorem 2.2.15 (Schaefer)**.** *Let (X_1, Σ_1) and (X_2, Σ_2) be standard Borel spaces, and let H be a Hilbert space. If $(X_j, \Sigma_j, H, \nu_j)$, $j = 1, 2$ are spectral measure spaces such that ν_1 and ν_2 commute:*

$$\nu_1(E)\nu_2(F) = \nu_2(F)\nu_1(E), \quad E \in \Sigma_1, F \in \Sigma_2$$

then there exists a unique spectral measure $\nu_1 \times \nu_2$ on $(X_1 \times X_2, \Sigma_1 \otimes \Sigma_2)$ such that:

$$(\nu_1 \times \nu_2)(E \times F) = \nu_1(E)\nu_2(F), \quad E \in \Sigma_1, F \in \Sigma_2.$$

Proof. This proof is based on a combination of Theorem 2.2.4 and Theorem 2.1.2.

Let $\mathcal{A}$ denote the algebra generated by $\Sigma_1 \times \Sigma_2$. That is, $\mathcal{A}$ is the algebra formed by taking finite unions of sets of the form $A_1 \times A_2$, where $A_1 \in \Sigma_1$ and $A_2 \in \Sigma_2$. On $\mathcal{A}$ we can define the finitely additive spectral measure $\nu_1 \times \nu_2$ by extension of the identity:

$$(\nu_1 \times \nu_2)(A_1 \times A_2) = \nu_1(A_1)\nu_2(A_2), \quad A_j \in \Sigma_j,\ j = 1, 2.$$

According to Theorem 2.2.4, to prove that $\nu_1 \times \nu_2$ extends to a spectral measure on $\Sigma_1 \otimes \Sigma_2 = \sigma(\mathcal{A})$, it suffices to show that for all $x \in H$, the finitely additive measure $\langle x, (\nu_1 \times \nu_2)(\cdot)x\rangle$ is σ-additive on $\mathcal{A}$. Theorem 2.2.5 implies that this extension will be unique if it exists.

On the other hand, since $\Sigma_1 \otimes \Sigma_2$ is Borel the σ-algebra of $X_1 \times X_2$, we can also appeal to Theorem 2.1.2.

Let us now show that the finitely additive measure $\mu_x(\cdot) := \langle x, \nu_1 \times \nu_2(\cdot)x\rangle$ satisfies the "inner regularity" condition of Theorem 2.1.2. Since $\nu_1^{x,x}$ and $\nu_2^{x,x}$ are Borel measures, they are inner regular due to the "necessity" component of Theorem 2.1.2. Let $\varepsilon > 0$. Suppose that $A = A_1 \times A_2 \in \mathcal{A}$, and let $K_1 \subset A_1$ and $K_2 \subset A_2$ be compact and chosen such that:

$$\nu_j^{x,x}(A_j \setminus K_j) < \varepsilon, \quad j = 1, 2.$$

But then since μ_x is finitely additive (and using Lemma 2.2.6)

$$\mu_x(A \backslash (K_1 \times K_2)) \leq \mu_x((A_1 \backslash K_1) \times K_2) + \mu_x(A_1 \times (A_2 \backslash K_2)) \leq \varepsilon(\nu_1^{x,x}(A_1) + \nu_2^{x,x}(A_2)) \leq 2\varepsilon\|x\|^2.$$

Since $K_1 \times K_2$ is compact and contained within A, it follows that the finitely additive measure μ_x satisfies the condition of Theorem 2.1.2 and thus is σ-additive. Now applying Theorem 2.2.4 it follows that $\nu_1 \times \nu_2$ extends to a measure on $\Sigma_1 \otimes \Sigma_2$. $\square$

The most famous result which can be seen as a corollary of Theorem 2.2.15 is the existence of a bounded functional calculus for commuting self-adjoint operators [18, Chapter 6, Section 5]:

Theorem 2.2.16 (Functional calculus for commuting operators)**.** *Let A_1 and A_2 be two (possibly unbounded) self-adjoint operators on the same Hilbert space which commute in the sense that their respective spectral measures commute,*

$$E_{A_1}(\Delta_1)E_{A_2}(\Delta_2) = E_{A_2}(\Delta_2)E_{A_1}(\Delta_1) \quad \Delta_1, \Delta_2 \in \mathfrak{B}(\mathbb{R}).$$

Then there exists an algebra homomorphism from $B(\mathbb{R}^2)$ to $\mathcal{B}(H)$ given by:

$$f \in B(\mathbb{R}^2) \mapsto \int_{\mathbb{R}^2} f\, d(E_{A_1} \times E_{A_2})$$

which extends the usual Borel functional calculus in the sense that if f depends only on the first variable, then the functional calculus for A_1 is recovered.

Moreover, under this algebra homomorphism pointwise convergent uniformly bounded sequences are mapped to sequences which converge in the strong operator topology.

Corollary 2.2.17. *Let (X_j, Σ_j), $j = 1, 2$ be standard Borel spaces with commuting spectral measures ν_1 and ν_2, and let f be a bounded function on $X_1 \times X_2$ which depends only on the first variable. Then:*

$$\int_{X_1 \times X_2} f\, d(\nu_1 \times \nu_2) = \int_{X_1} f\, d\nu_1.$$

Proof. If f is the characteristic function of an element of Σ_1, then the assertion follows from the definition of the product measure. By linearity, the assertion extends to simple functions and then Lemma 2.2.9 implies that the result holds for arbitrary bounded measurable functions. □

An immediate application of Theorem 2.2.12 yields:

Corollary 2.2.18. *Let X_1, X_2, Y_1 and Y_2 be standard Borel spaces, and $(X_j, \Sigma_j, H, \nu_j)$, $j = 1, 2$ are commuting spectral measures. If $h_1 : X_1 \to Y_1$ and $h_2 : X_2 \to Y_2$ are measurable functions and $\phi \in B(Y_1 \times Y_2)$ then:*

$$\int_{X_1 \times X_2} \phi(h_1, h_2)\, d(\nu_1 \times \nu_2) = \int_{Y_1 \times Y_2} \phi\, d((h_1)_*\nu_1 \times (h_2)_*\nu_2).$$

2.2.3 Convergence of spectral measures

Suppose that (X, Σ) is a measurable space, H is a Hilbert space and we have a family $\{\nu_j\}_{j=0}^{\infty}$ of spectral measures on (X, Σ) with values as projections on H.

Of the various notions of convergence of the sequence of spectral measures $\{\nu_j\}_{j=0}^{\infty}$, the one which is of most importance to us is weak convergence:

Definition 2.2.19. *Let (X, Σ) be a standard Borel space, and write $C_b(X)$ for the algebra of functions continuous on X and bounded. A sequence of spectral measures*

$\{\nu_j\}_{j=0}^{\infty}$ on the same Hilbert space H is said to converge weakly to a spectral measure ν if for all $x \in H$ and $h \in C_b(X)$ we have:

$$\left(\int_X h\, d\nu_j\right) x \to \left(\int_X h\, d\nu\right) x$$

That is, the integrals $\int_X h\, d\nu_j$ converge in the strong operator topology to $\int_X h\, d\nu$.

The importance of weak convergence of spectral measures is the following, which is essentially [110, Theorem VIII.20]. Recall that a sequence of (possibly unbounded) self-adjoint operators $\{A_n\}_{n=0}^{\infty}$ is said to converge to A in the strong resolvent sense if for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the sequence $\{(\lambda - A_n)^{-1}\}_{n=0}^{\infty}$ of the resolvents of $\{A_n\}_{n=0}^{\infty}$ converges in the strong operator topology to $(\lambda - A)^{-1}$.

Theorem 2.2.20. *A sequence of self-adjoint operators $\{A_n\}_{n=0}^{\infty}$ converges in the strong resolvent sense to a self-adjoint operator A if and only if the corresponding spectral measures $\{E_{A_n}\}_{n=0}^{\infty}$ on $\mathbb{R}$ converge weakly to the spectral measure of A on $\mathbb{R}$.*

The following theorem is based on the elementary fact that if $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ are sequences of bounded operators which converge strongly to A and B respectively, and moreover $\sup_{n\geq 0} \|A_n\| < \infty$, then A_nB_n converges strongly to AB.

Recall that a topological space X is called σ-compact if X is a union of at most countably many compact subspaces. In particular, this implies that every $h \in C_b(X)$ can be obtained as a pointwise limit of a sequence of compactly supported continuous functions.

Theorem 2.2.21. *Let (X, Σ), (Y, Ω) be σ-compact standard Borel spaces, and let H be a Hilbert space. Suppose that $\{\nu_n\}_{n=0}^{\infty}$ and $\{\mu_n\}_{n=0}^{\infty}$ are sequences of spectral measures valued in projections on H and defined on X and Y respectively.*

Suppose that there are spectral measures ν and μ on X and Y respectively so that $\nu_n \to \nu$ and $\mu_n \to \mu$ weakly.

If, for all $n \geq 0$, the measures ν_n and μ_n commute, then ν and μ commute and we have:

$$\nu_n \times \mu_n \to \nu \times \mu$$

in the weak sense.

Proof. First let us show that ν and μ commute, so that the product measure $\nu \times \mu$ is defined.

Let U and V be open sets in X and Y respectively, and select uniformly bounded sequences $\{f_k\}_{k\geq 0} \subseteq C_b(X)$ and $\{g_k\}_{k\geq 0} \subset C_b(Y)$, supported in U and V respectively, such that f_k converges pointwisely to the characteristic function of U and g_k converges pointwisely to the characteristic function of V. Then for each fixed k and n we have:

$$\int_X f_k\, d\nu_n \int_Y g_k\, d\mu_n = \int_Y g_k\, d\mu_n \int_X f_k\, d\nu_n.$$

Passing $n \to \infty$ and using the fact stated before the theorem, we have that:

$$\int_X f_k\, d\nu \int_Y g_k\, d\mu = \int_Y g_k\, \mu \int_X f_k\, d\nu.$$

Now taking $k \to \infty$, and appling Corollary 2.2.11 yields:

$$\nu(U)\mu(V) = \mu(V)\nu(U)$$

for all open sets $U \subseteq X$ and $V \subseteq Y$. Let $\mathcal{F}_V$ denote the family of measurable subsets $A \in \Sigma$ which satisfy:

$$\nu(A)\mu(V) = \mu(V)\nu(A).$$

Since ν is a spectral measure, it is easy to see that $\mathcal{F}_V$ is a σ-algebra, and since $\mathcal{F}_V$ contains all open subsets of X, it follows that $\mathcal{F}_V = \Sigma$. Similarly, for any $A \in \Sigma$ let $\mathcal{G}_A$ denote the set of $B \in \Omega$ such that $\nu(A)\mu(B) = \mu(B)\nu(A)$. Again it is easy to see that $\mathcal{G}_A$ is a σ-algebra containing all open subsets of Y, and hence $\mathcal{G}_A = \Omega$. Finally we have established that:

$$\nu(A)\mu(B) = \mu(B)\nu(A) \quad A \in \Sigma, B \in \Omega$$

and so ν and μ commute as claimed.

Now we prove the claimed convergence $\nu_n \times \mu_n \to \nu \times \mu$. First, we show that for functions of the form $f(x, y) = g(x)h(y)$, where $x \in X$, $y \in Y$, $g \in C_b(X)$ and $h \in C_b(Y)$ that we have:

$$\int_{X\times Y} f \, d(\nu_n \times \mu_n) \to \int_{X\times Y} f \, d(\nu \times \mu) \tag{2.2.6}$$

in the strong operator topology. This follows from Theorem 2.2.10 and Corollary 2.2.17:

$$\int_{X\times Y} f \, d(\nu_n\times\mu_n) = \int_{X\times Y} g(x) \, d(\nu_n\times\mu_n)(x, y) \int_{X\times Y} h(y) \, d(\nu_n\times\mu_n)(x, y) = \int_X g \, d\nu_n \int_Y h \, d\mu_n.$$

Now passing $n \to \infty$ and using the assertion stated before the theorem yields (2.2.6) for functions f of the form $f(x, y) = g(x)h(y)$.

Since the linear span of such functions is dense in $C_0(X \times Y)$[1] in the uniform norm, it then follows from Lemma 2.2.9 that (2.2.6) holds for all $f \in C_0(X \times Y)$.

To complete the proof, we must use the σ-compactness of $X \times Y$. Select a uniformly bounded sequence $\{g_k\}_{k\geq 0} \subset C_0(X \times Y)$ which converges pointwisely to the identity function. Then using Corollary 2.2.11, the sequence of integrals $\int_{X\times Y} g_k \, d(\nu \times \mu)$ converges strongly to the identity operator as $k \to \infty$.

[1]Here, $C_0(X \times Y)$ denotes the subalgebra of functions vanishing at infinity

Let $x \in H$. Then using Theorem 2.2.10 we have:

$$\begin{aligned}
&\left\| \int_{X \times Y} f \, d(\nu_n \times \mu_n) x - \int_{X \times Y} f \, d(\nu \times \mu) x \right\| \\
&\leq \left\| \int_{X \times Y} (f - f g_k) \, d(\nu_n \times \mu_n) x \right\| \\
&\quad + \left\| \int_{X \times Y} f g_k \, d(\nu_n \times \mu_n) x - \int_{X \times Y} f g_k \, d(\nu \times \mu) x \right\| \\
&\quad + \left\| \int_{X \times Y} f g_k \, d(\nu \times \mu) x - \int_{X \times Y} f \, d(\nu \times \mu) x \right\| \\
&\leq \left\| \int_{X \times Y} f \, d(\nu_n \times \mu_n) \right\| \left\| x - \int_{X \times Y} g_k \, d(\nu_n \times \mu_n) x \right\| \\
&\quad + \left\| \int_{X \times Y} f g_k \, d(\nu_n \times \mu_n) x - \int_{X \times Y} f g_k \, d(\nu \times \mu) x \right\| \\
&\quad + \left\| \int_{X \times Y} f \, d(\nu \times \mu) \right\| \left\| x - \int_{X \times Y} g_k \, d(\nu \times \mu) x \right\|.
\end{aligned}$$

Taking $n \to \infty$, we have for arbitrary fixed k, since $g_k \in C_0(X \times Y)$:

$$\lim_{n \to \infty} \left\| \int_{X \times Y} f \, d(\nu_n \times \mu_n) x - \int_{X \times Y} f \, d(\nu \times \mu) x \right\| \leq 2 \|f\|_\infty \left\| x - \int_{X \times Y} g_k \, d(\nu \times \mu) x \right\|.$$

Now taking $k \to \infty$ yields the result. □

To conclude our discussion of convergence of measures, we discuss an important example where convergence occurs in practice:

Theorem 2.2.22. *Let (X, Σ, H, ν) be a standard Borel measure space, and let (Y, Ω) be another standard Borel space. Suppose that $\{h_n\}_{n=0}^\infty$ is a sequence of measurable functions from X to Y which converges pointwisely to a measurable function $h : X \to Y$. Then we have:*

$$(h_n)_* \nu \to h_* \nu$$

in the weak sense.

Proof. Let $f \in C_b(Y)$. Then due to Theorem 2.2.12, for each $n \geq 0$ we have:

$$\int_Y f \, d(h_n)_* \nu = \int_X f \circ h_n \, d\nu.$$

The sequence $\{f \circ h_n\}_{n \geq 0} \subseteq B(X)$ is uniformly bounded, and since f is continuous we have the pointwise limit $f \circ h_n \to f \circ h$. Thus from Corollary 2.2.11, for all $x \in H$ we have:

$$\left(\int_X f \circ h_n \, d\nu \right) x \to \left(\int_X f \circ h \, d\nu \right) x$$

That is,

$$\left(\int_Y f \, d(h_n)_* \nu \right) x \to \left(\int_Y f \, dh_* \nu \right) x$$

for all $f \in C_b(Y)$. □

Note that we do not need to assume in the above theorem that each h_n is continuous.

2.3 Spectral measures on a von Neumann algebra

Before proceeding to the definition of a double operator integral, a remark is necessary concerning spectral measure spaces (X, Σ, H, ν) when ν is considered as being valued in the lattice of projections of a von Neumann algebra.

Recall (from Section 1.5.1) that a von Neumann algebra is equipped with weak*-topology, and convergence in the weak* topology of a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ is in general stronger than convergence in the weak operator topology of H. Therefore in principle if one replaced the weak operator topology convergence in Definition 2.2.1 with weak* convergence then one would have a different notion of spectral measure.

Fortunately this is not the case, and although weak* convergence is a stricter condition than weak operator toplogy convergence, both notions of convergence define the same notion of spectral measure, as the next lemma shows.

Lemma 2.3.1. *Let H be a Hilbert space, and let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a von Neumann algebra with pre-dual $\mathcal{M}_*$. Let (X, Σ, H, ν) be a spectral measure such that $\nu(E) \in \mathcal{P}(\mathcal{M})$ for all $E \in \Sigma$. Let $\{A_j\}_{j=0}^\infty$ be a pairwise disjoint family in Σ.*

Then for all $\omega \in \mathcal{M}_*$ we have:

$$\nu(\bigcup_{j=0}^\infty A_j)(\omega) = \lim_{n\to\infty} \sum_{j=0}^n \nu(A_j)(\omega).$$

Proof. As discussed in Remark 2.2.2, the sequence given by

$$p_n := \nu(\bigcup_{j=0}^\infty A_j) - \sum_{j=0}^n \nu(A_j), \quad n \geq 0$$

is a sequence of projections in $\mathcal{B}(H)$ which converges to zero in the strong operator topology. Let us show that p_n converges to zero in the weak*-topology of $\mathcal{M}$.

As discussed in Section 1.5.1, it suffices to show that $\rho_{\{\xi_k\},\{\eta_k\}}(p_n)$ converges to zero for all pairwise orthogonal families $\{\xi_k\}_{k=0}^\infty$ and $\{\eta_k\}_{k=0}^\infty$ such that $\sum_{k=0}^\infty \|\xi_k\|^2 + \|\eta_k\|^2 < \infty$, where $\rho_{\{\xi_k\},\{\eta_k\}}$ is the seminorm defined in (1.5.1).

Using the Cauchy-Schwarz inequality:

$$\begin{aligned}
\rho_{\{\xi_k\},\{\eta_k\}}(p_n) &= \left|\sum_{k=0}^{\infty}\langle \xi_k, p_n\eta_k\rangle\right| \\
&\leq \sum_{k=0}^{\infty} |\langle \xi_k, p_n\eta_k\rangle| \\
&\leq \sum_{k=0}^{\infty} \|\xi_k\|_H \|p_n\eta_k\|_H \\
&\leq \left(\sum_{k=0}^{\infty} \|\xi_k\|_H^2\right)^{1/2} \left(\sum_{k=0}^{\infty} \|p_n\eta_k\|_H^2\right)^{1/2}.
\end{aligned}$$

Since p_n converges strongly to zero, it follows that for each $k \geq 0$ we have $\lim_{n\to\infty} \|p_n\eta_k\|_H = 0$. Since,

$$\sum_{k=0}^{\infty} \|\xi_k\|_H^2 < \infty, \quad \sum_{k=0}^{\infty} \|\eta_k\|_H^2 < \infty$$

we may apply the dominated convergence theorem to conclude that:

$$\lim_{n\to\infty} \rho_{\{\xi_k\},\{\eta_k\}}(p_n) = 0.$$

Therefore p_n converges to zero in the weak*-topology, and this is precisely the desired result. □

Our primary interest in the von Neumann algebraic setting is semifinite algebras. For a semifinite von Neumann algebra $(\mathcal{M}, \tau)$, a spectral measure valued in $\mathcal{P}(\mathcal{M})$ determines two distinct spectral measures on the Hilbert space $L_2(\tau)$, defined by left and right multiplication.

Corollary 2.3.2. *Let (X, Σ, H, ν) be a spectral measure space such that ν is valued in the projections of a von Neumann algebra $\mathcal{M} \subseteq \mathcal{B}(H)$ which has a semifinite trace τ. Then ν defines spectral measures on the Hilbert space $L_2(\tau)$ by "left multiplication" and "right multiplication". Namely, if we define:*

$$L_\nu(E)x = \nu(E)x, \quad R_\nu(E)x = x\nu(E), \quad E \in \Sigma,\ x \in L_2(\tau)$$

then L_ν and R_ν are spectral measures on the Hilbert space $L_2(\tau)$.

Proof. The inner product for $L_2(\tau)$ is:

$$\langle y, x\rangle_{L_2} = \tau(y^*x), \quad x, y \in L_2(\tau).$$

So,

$$\langle y, L_\nu(E)x\rangle_{L_2} = \tau(y^*\nu(E)x)$$

and

$$\langle y, R_\nu(E)x\rangle_{L_2} = \tau(y^*x\nu(E)) = \tau(x\nu(E)y^*).$$

Since $\nu(E)$ is a projection on H, it follows immediately that $L_\nu(E)$ and $R_\nu(E)$ are projections for $L_2(\tau)$ for all $E \in \Sigma$.

To complete the proof, it suffices to check that,

$$E \mapsto \tau(x\nu(E)y)$$

is a σ-additive measure on Σ for all $x, y \in L_2(\tau)$. Since $\tau(x\nu(E)y) = \tau(yx\nu(E))$ and $yx \in L_1(\tau)$, it suffices to show instead that:

$$E \mapsto \tau(z\nu(E)), \quad E \in \Sigma, z \in L_1(\tau)$$

is σ-additive. According to our identification of $L_1(\tau)$ with $\mathcal{M}_*$, the σ-additivity of $E \mapsto \tau(z\nu(E))$ is provided by Lemma 2.3.1. $\square$

2.4 Double operator integrals as spectral integrals

Now we can put the machinery of spectral integration theory to use to define double operator integrals.

A double operator integral on a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ is conventionally defined in terms of two spectral measures E and F on $\mathbb{R}$ with values in the lattice of projections of $\mathcal{M}$, $\mathcal{P}(\mathcal{M})$. Corollary 2.3.2 states that the mappings:

$$L_E(\Delta)x = E(\Delta)x, \quad R_F(\Delta)x = xF(\Delta), \quad x \in L_2(\tau), \quad \Delta \in \mathfrak{B}(\mathbb{R})$$

are spectral measures on $\mathbb{R}$ for the Hilbert space $L_2(\tau)$. In particular, these two measures commute, since:

$$L_E(\Delta_1)R_F(\Delta_2)x = E(\Delta_1)(xF(\Delta_2)) = (E(\Delta_1)x)F(\Delta_2) = R_F(\Delta_2)L_E(\Delta_1)x, \quad x \in L_2(\tau).$$

It follows from Theorem 2.2.15 that a product measure $L_E \times R_F$ exists on $\mathbb{R}^2$. This permits the definition of a double operator integral:

Definition 2.4.1 (Double operator integral)**.** *Let E and F be spectral measures on the Borel σ-algebra $\mathfrak{B}(\mathbb{R})$ valued in $\mathcal{P}(\mathcal{M})$. Let $E \otimes F$ denote the spectral measure on the Hilbert space $L_2(\tau)$ determined uniquely by:*

$$(E \otimes F)(\Delta_1 \times \Delta_2)(x) = E(\Delta_1)xF(\Delta_2), \quad x \in L_2(\tau), \Delta_1, \Delta_2 \in \mathfrak{B}(\mathbb{R}).$$

For a bounded Borel function ϕ on $\mathbb{R}^2$, the double operator integral is defined to be the spectral integral:

$$\mathcal{T}_\phi^{E,F} := \int_{\mathbb{R}^2} \phi \, d(E \otimes F) \in \mathcal{B}(L_2(\tau)).$$

Since $\mathbb{R}$ is in particular a σ-compact standard Borel space, Theorem 2.2.15 implies that $E \otimes F$ is indeed a uniquely defined spectral measure. The already established properties of the spectral integral (see Subsection 2.2.1) transfer immediately to the double operator integral. In particular, we have the following:

Theorem 2.4.2. *Let ϕ, ψ be bounded functions on $\mathbb{R}^2$ and $\alpha \in \mathbb{C}$, then:*

(i) $\mathcal{T}_{\alpha\phi}^{E,F} = \alpha \mathcal{T}_\phi^{E,F}$

(ii) $\mathcal{T}_{\phi+\psi}^{E,F} = \mathcal{T}_\phi^{E,F} + \mathcal{T}_\psi^{E,F}$

(iii) $\left\| \mathcal{T}_{\phi}^{E,F} \right\| \leq \|\phi\|_{\infty}$

(iv) $\mathcal{T}_{\phi\psi}^{E,F} = \mathcal{T}_{\phi}^{E,F} \mathcal{T}_{\psi}^{E,F}$

(v) *If* $\{\phi_n\}_{n=0}^{\infty}$ *converges uniformly to* ϕ*, then* $\mathcal{T}_{\phi_n}^{E,F}$ *converges to* $\mathcal{T}_{\phi}^{E,F}$ *in the norm topology*

(vi) *If* $\sup_{n\geq 0} \|\phi_n\|_{\infty} < \infty$ *and* ϕ_n *converges pointwisely to a bounded function* ϕ*, then* $\mathcal{T}_{\phi_n}^{E,F}$ *converges to* $\mathcal{T}_{\phi}^{E,F}$ *in the strong operator topology.*

(vii) *If* $\phi \in C_b(\mathbb{R}^2)$*, and* $\{E_n\}_{n\geq 0}$ *and* $\{F_n\}_{n\geq 0}$ *converge in the weak sense to spectral measures* E *and* F*, then* $\mathcal{T}_{\phi}^{E_n,F_n}$ *converges in the strong operator topology to* $\mathcal{T}_{\phi}^{E,F}$*. The same conclusion may not hold for discontinuous* ϕ*.*

(viii) *If* ϕ *depends only on the first variable, then* $\mathcal{T}_{\phi}^{E,F} = \int_{\mathbb{R}} \phi\, dE$*, and similarly if* ϕ *depends only on the second variable then* $\mathcal{T}_{\phi}^{E,F} = \int_{\mathbb{R}} \phi\, dF$*.*

Having defined $\mathcal{T}_{\phi}^{E,F}$ for functions ϕ on $\mathbb{R}^2$, it is worth remaking that the role of $\mathbb{R}$ can be replaced by any σ-compact standard Borel space. In particular, there is an essentially identical theory for functions on the unit circle $\mathbb{T}$.

Recall that a (possibly unbounded) self-adjoint operator A is called *affiliated* with a von Neumann algebra $\mathcal{M}$ if $XA \subseteq AX$ for all $X \in \mathcal{M}'$. In particular, this implies that the spectral measure of A is $\mathcal{P}(\mathcal{M})$-valued.

Definition 2.4.3. *If* A *and* B *are self-adjoint operators affiliated with* $\mathcal{M}$ *and with spectral measures* E_A *and* E_B*, we shall adopt the notation:*

$$\mathcal{T}_{\phi}^{A,B} = \mathcal{T}_{\phi}^{E_A,E_B}.$$

Theorem 2.2.20, combined with the Theorem 2.2.21 yields the following:

Theorem 2.4.4. *If* $\{A_n\}_{n=0}^{\infty}$ *and* $\{B_n\}_{n=0}^{\infty}$ *are sequences of self-adjoint operators affiliated with* $\mathcal{M}$ *which converge to self-adjoint operators* A *and* B *respectively in the strong resolvent sense and* $\phi \in C_b(\mathbb{R}^2)$*, then:*

$$\mathcal{T}_{\phi}^{A_n,B_n} \to \mathcal{T}_{\phi}^{A,B}$$

in the strong operator topology.

Theorem 2.4.5. *Suppose that* $\phi \in C_b(\mathbb{R}^2)$*, and* A *and* B *are self-adjoint operators affiliated with* $\mathcal{M}$*. If* $\{h_n\}_{n\geq 0}$ *and* $\{g_n\}_{n\geq 0}$ *are sequences of measurable functions from* $\mathbb{R}$ *to* $\mathbb{R}$ *which converge pointwisely to* h *and* g *respectively, then we have:*

$$\mathcal{T}_{\phi}^{h_n(A),g_n(B)} = \mathcal{T}_{\phi(h_n,g_n)}^{A,B} \to \mathcal{T}_{\phi}^{h(A),h(B)}$$

in the strong operator topology.

Chapter 3

Peller's theorem

We have so far discussed double operator integrals on the Hilbert-Schmidt class $L_2(\tau)$. Consider f, a Lipschitz function on $\mathbb{R}$ and let $f^{[1]}(t,s)$ denote the divided difference $\frac{f(t)-f(s)}{t-s}$ (set to an arbitrary value on the diagonal $t = s$). If A and B are two self-adjoint operators affiliated with $\mathcal{M}$ such that $A - B \in L_2(\tau)$, then Theorem 2.4.2.(iv) yields the formula:

$$f(A) - f(B) = \mathcal{T}^{A,B}_{f^{[1]}}(A - B).$$

A problem of great interest is to determine when f is operator Lipschitz. That is, does there exist a constant C_f such that:

$$\|f(A) - f(B)\|_{\mathcal{M}} \leq C_f \|A - B\|_{\mathcal{M}}?$$

Via the theory of double operator integration, the problem of characterising operator Lipschitz functions is reduced to determining the class of $\phi \in B(\mathbb{R}^2)$ such that $\mathcal{T}^{A,B}_{\phi}$ extends to a bounded linear operator from $\mathcal{M}$ to $\mathcal{M}$ (or from $L_1(\tau)$ to $L_1(\tau)$).

A condition that is sufficient is that ϕ is in the so-called Birman-Solomyak class. That is, there exists a probability space (Ω, σ) and bounded measurable functions $\alpha, \beta : \mathbb{R} \times \Omega \to \mathbb{C}$ such that:

$$\phi(t,s) = \int_{\Omega} \alpha(t,\omega)\beta(s,\omega)\, d\sigma(\omega), \quad t,s \in \mathbb{R}.$$

If ϕ has the above form, then for $X, Y \in L_2(\tau)$ we have:

$$\tau(Y^* \mathcal{T}^{A,B}_{\phi}(X)) = \int_{\Omega} \tau(Y^* \alpha(A,\omega) X \beta(B,\omega))\, d\sigma(\omega)$$

from which it (not entirely trivially) follows that $\mathcal{T}^{A,B}_{\phi}|_{L_1(\tau)\cap L_2(\tau)}$ extends to a bounded linear map from $L_1(\tau)$ to $L_1(\tau)$ (see Theorem 3.4.5).

Peller's theorem (named for V.V. Peller due to his foundational paper [99]) essentially states that this "Birman-Solomyak" condition is also necessary. Peller's original proof (which is similar to the proof given by Hiai and Kosaki [72, Section 2.1]) is based on the theory of operator ideals.

The proof given here is intended to be more elementary, and a particular goal was to highlight the role of Grothendieck's inequality (stated below as Theorem 3.2.1). While many of the steps in the following proof exist in various places throughout the literature, to the best of our knowledge they have never been assembled together with the goal of proving Peller's theorem.

Before stating the theorem, it is useful to define the notion of a null-set and essential boundedness in the context of a spectral measure (X, Σ, ν, H). Say that a set $A \in \Sigma$ is null if $\nu(A)$ is the zero projection. One can define the essential supremum of a measurable function ϕ on X similarly to the case of scalar-valued measures. Similarly the notion of "almost everywhere equivalence" is meaningful in the setting of spectral measures.

The theorem states the following:

Theorem 3.0.1 (Peller's theorem)**.** *Let E and F be two spectral measures on $\mathbb{R}$, valued in the projections of a separable Hilbert space H. Let ϕ be a measurable function on $\mathbb{R}^2$, essentially bounded with respect to $E \otimes F$. The following two statements are equivalent:*

(i) *The double operator integral $\mathcal{T}_\phi^{E,F}$ is bounded from the trace class $\mathcal{L}_1(H)$ to $\mathcal{L}_1(H)$.*

(ii) *The function ϕ admits a* Birman-Solomyak decomposition *relative to E and F. That is, there exists a σ-finite measure space (Ω, σ) such that*

$$\phi(t, s) = \int_\Omega \alpha(t, \omega)\beta(s, \omega)d\sigma(\omega),$$

for almost every $t, s \in \mathbb{R}$, where α and β are essentially bounded measurable functions such that:

$$\int_\Omega \operatorname*{esssup}_{t\in\mathbb{R}}|\alpha(t, \omega)|\operatorname*{esssup}_{s\in\mathbb{R}}|\beta(s, \omega)|\, d|\sigma|(\omega) < \infty. \tag{3.0.1}$$

Moreover, there is a universal constant K_G such that:

$$\frac{1}{K_G}\|\phi\|_{\mathfrak{BS}(E\times F)} \leq \|\mathcal{T}_\phi^{E,F}\|_{\mathcal{L}_1\to\mathcal{L}_1} \leq \|\phi\|_{\mathfrak{BS}(E\times F)}$$

where $\|\phi\|_{\mathfrak{BS}(E\times F)}$ is the infimum of (3.0.1) *over all such representations of ϕ.*

Remark 3.0.2. *Let us make a few observations about Peller's theorem.*

- *The meaning of "essential supremum" in* (3.0.1) *is taken with respect to the measures E and F.*
- *It is important to emphasise that the Birman-Solomyak representation is only taken pointwise almost everywhere with respect to the measures E and F. Indeed, we may modify ϕ on a null set relative to $E \otimes F$ without changing the spectral integral defining $\mathcal{T}_\phi^{E,F}$. Moreover, the choice of the measure space (Ω, σ) depends on the spectral type of the spectral measures E and F.*
- *We have stated the result for spectral measures on $\mathbb{R}$ only for the sake of simplicity. A similar statement holds for spectral measures on $\mathbb{R}^d$ on $\mathbb{T}^d$ or on any σ-compact standard Borel space.*
- *The constant K_G is known as Grothendieck's constant and it is known to be strictly between 1 and 2. Further details are in Remark 3.2.2.*

- *The assumption that H is separable is needed to ensure that H admits elements of maximal spectral type relative to E and F, as we will see in Lemma 3.4.7.*
- *Finally, the measure space (Ω, σ) appearing in the theorem may seem quite abstract but in fact the following proof of the theorem gives an explicit choice of Ω, depending only on the spectral types of E and F. In particular, Ω may always be taken to be a compact Hausdorff space with its Borel σ-algebra, the measure σ may always be chosen to be a finite Borel regular measure.*

3.1 The injective tensor product

Peller's theorem is often stated in terms of tensor product norms for Banach spaces. Arguably the most important such norm in this context is the injective tensor product, which we now describe.

For Banach spaces X and Y, let B_{X^*} and B_{Y^*} denote the closed unit balls of the duals X^* and Y^* respectively.

Definition 3.1.1. *Let X and Y be Banach spaces, and let $X \odot Y$ denote their algebraic tensor product. For linear functionals $\alpha \in X^*$ and $\beta \in Y^*$, let $\alpha \otimes \beta$ denote the linear functional on $X \odot Y$ given by the linear extension of the mapping:*

$$(\alpha \otimes \beta)(x \otimes y) = \alpha(x)\beta(y), \quad x \in X,\, y \in Y.$$

The injective tensor norm, $\|\cdot\|_{X\otimes_\varepsilon Y}$, is the norm on $T \in X \odot Y$ given by:

$$\|T\|_{X\otimes_\varepsilon Y} := \sup_{\alpha\in B_{X^*},\, \beta\in B_{Y^*}} |(\alpha\otimes\beta)(T)|.$$

We will denote the completion of $X \odot Y$ with the norm $\|\cdot\|_{X\otimes_\varepsilon Y}$ as $X \otimes_\varepsilon Y$.

Recall that the linear dual of $X \odot Y$ can be identified with the space of bilinear mappings from $X \times Y$ to $\mathbb{C}$. The continuous dual of $X \otimes_\varepsilon Y$ can therefore be identified with the space of bilinear mappings $X \times Y \to \mathbb{C}$ which are continuous in the injective tensor product norm.

The following theorem describes those bilinear maps $X \times Y \to \mathbb{C}$ which are continuous for the injective tensor product, and is (a minor modification of) [42, Chapter VIII, Section 1, Theorem 5]. Recall that due to the Banach-Alaoglu theorem, the unit balls B_{X^*} and B_{Y^*} are compact Hausdorff spaces when equipped with the weak*-topology.

Theorem 3.1.2. *Let X and Y be Banach spaces, and let X_0 and Y_0 be (possibly not closed) subspaces of X and Y respectively. Let $\Psi : X_0 \odot Y_0 \to \mathbb{C}$ be a bilinear map. Suppose that Ψ is continuous in the $X \otimes_\varepsilon Y$ norm. That is, assume that there is a constant C_Ψ such that for all $T \in X_0 \odot Y_0$ we have:*

$$|\Psi(T)| \leq C_\Psi \|T\|_{X\otimes_\varepsilon Y} = C_\Psi \sup_{\alpha\in B_{X^*},\, \beta\in B_{Y^*}} |(\alpha\otimes\beta)(T)|.$$

Then there is a regular Borel measure σ on the compact Hausdorff space $B_{X^*} \times B_{Y^*}$ such that for all $x \in X_0$ and $y \in Y_0$:

$$\Psi(x \otimes y) = \int_{B_{X^*}\times B_{Y^*}} \alpha(x)\beta(y)\, d\sigma(\alpha, \beta)$$

and moreover, the total variation norm of σ is no greater than C_Ψ.

Proof. Consider the embedding map J from $X_0 \odot Y_0$ into the Banach space $C(B_{X^*} \times B_{Y^*})$ defined as:

$$J(x \otimes y) = ((\alpha, \beta) \mapsto \alpha(x)\beta(y)), \quad \alpha \in B_{X^*},\, \beta \in B_{Y^*}.$$

Here, $C(B_{X^*} \times B_{Y^*})$ is equipped with the uniform norm, and so it follows almost immediately from the definition of the injective tensor product norm that J is an isometric embedding when $X_0 \odot Y_0$ is equipped with the norm $\|\cdot\|_{X \otimes_\varepsilon Y}$. Indeed, if $T \in X_0 \odot Y_0$ then:

$$\|J(T)\|_{C(B_{X^*} \times B_{Y^*})} = \sup_{(\alpha,\beta) \in B_{X^*} \times B_{Y^*}} |(\alpha \otimes \beta)(T)| = \|T\|_{X \otimes_\varepsilon Y}.$$

Thus $\Psi \circ J^{-1}$ defines a bounded linear functional on the (possibly not closed) subspace $J(X_0 \odot Y_0)$ of $C(B_{X^*} \times B_{Y^*})$, with norm at most C_Ψ. By the Hahn-Banach extension theorem, there is a continuous linear functional W on $C(B_{X^*} \times B_{Y*})$, with norm at most C_Ψ such that:

$$W(J(x \otimes y)) = \Psi(x \otimes y), \quad x \in X_0,\, y \in Y_0.$$

Thanks to the Riesz representation theorem for linear functionals on the space of continuous functions on compact Hausdorff spaces, there exists a complex Borel regular measure σ on $B_{X^*} \times B_{Y^*}$ with total variation norm at most C_Ψ such that:

$$W(J(x \otimes y)) = \int_{B_{X^*} \times B_{Y^*}} J(x \otimes y)\, d\sigma, \quad x \in X_0,\, y \in Y_0.$$

By the definition of J, for a given $(\alpha, \beta) \in B_{X^*} \times B_{Y^*}$ we have:

$$J(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y), \quad x \in X_0,\, y \in Y_0$$

and therefore:

$$W(J(x \otimes y)) = \int_{B_{X^*} \times B_{Y^*}} \alpha(x)\beta(y)\, d\sigma(\alpha, \beta), \quad x \in X_0,\, y \in Y_0.$$

Since $W(J(x \otimes y)) = \Psi(x \otimes y)$, the proof is complete. $\square$

3.2 Consequences of Grothendieck's theorem

For $n, p \geq 1$, we will use the notation ℓ_p^n to denote the Banach space on $\mathbb{C}^n$ with norm $\|x\|_{\ell_p^n} := (\sum_{k=1}^n |x_k|^p)^{1/p}$. Recall the classical result that the dual of ℓ_1^n is isometric to ℓ_∞^n. Let $\{e_j\}_{j=1}^n$ denote the canonical basis of $\mathbb{C}^n$, and we regard the space $M_{n,m}(\mathbb{C})$ of $n \times m$ matrices as being identical to $\mathbb{C}^n \otimes \mathbb{C}^m$, according to the isomorphism $\{A_{j,k}\} \mapsto \sum_{j,k} A_{j,k} e_j \otimes e_k$. By the definition of the injective tensor product, for a matrix $A \in M_{n,m}(\mathbb{C})$ we have:

$$\|A\|_{\ell_1^n \otimes_\varepsilon \ell_1^m} = \sup \left\{ \left| \sum_{j=1}^n \sum_{k=1}^m A_{j,k} t_j s_k \right| \;:\; \max_{1 \leq j \leq n} |t_j| \leq 1,\; \max_{1 \leq k \leq m} |s_k| \leq 1 \right\}. \tag{3.2.1}$$

Grothendieck's celebrated inequality, (see e.g. [104, Theorem 1.1], [103, Corollary 5.7] and [84, Section 2]), states the following:

Theorem 3.2.1. *Let $n,m \geq 1$, and let $A \in M_{n,m}(\mathbb{C})$. Then there is a universal constant K_G such that for all complex Hilbert spaces $(H,\langle\cdot,\cdot\rangle)$ and finite sets $\{x_j\}_{j=1}^n$, and $\{y_k\}_{k=1}^m$ in H we have:*

$$\left|\sum_{j=1}^n\sum_{k=1}^m A_{j,k}\langle x_j,y_k\rangle\right| \leq K_G\|A\|_{\ell_1^n\otimes_\varepsilon \ell_1^m}\max_{1\leq j\leq n}\|x_j\|_H\max_{1\leq k\leq m}\|y_k\|_H.$$

Remark 3.2.2. *To be precise, Theorem 1.1 of [104] is stated for the case where $n=m$. However there is no additional generality from considering the $n\neq m$ case, since one may consider an $n\times m$ matrix as being the top left submatrix of the square matrix $A\oplus 0\in M_{n+m,n+m}(\mathbb{C})$.*

The optimal value of K_G is unknown, but it is known the optimal value satisfies:

$$\frac{4}{\pi}\leq K_G < \frac{\pi}{2\log(1+\sqrt{2})}$$

(in fact better results are known, see [104, Section 4]).

The following theorem is in fact an equivalent form of Grothendieck's inequality. However, following [104, Section 2], we prove it as a corollary of Theorem 3.2.1.

Theorem 3.2.3. *Let $A\in M_{n,m}(\mathbb{C})$, and let $\{X_r\}_{r=1}^d$ and $\{Y_r\}_{r=1}^d$ be sequences in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, with components denoted $X_r=(X_r^{(1)},X_r^{(2)},\ldots,X_r^{(n)})$, etc. Then:*

$$\left|\sum_{r=1}^d\sum_{j=1}^n\sum_{k=1}^m A_{j,k}X_r^{(j)}Y_r^{(k)}\right| \leq K_G\|A\|_{\ell_1^n\otimes_\varepsilon\ell_1^m}\max_{1\leq j\leq n}\left(\sum_{r=1}^d|X_r^{(j)}|^2\right)^{1/2}\max_{1\leq k\leq m}\left(\sum_{r=1}^d|Y_r^{(k)}|^2\right)^{1/2}$$

Proof. Exchanging the order of summation, we have:

$$\sum_{r=1}^d\sum_{j=1}^n\sum_{k=1}^m A_{j,k}X_r^{(j)}Y_r^{(k)} = \sum_{j=1}^n\sum_{k=1}^m A_{j,k}\left(\sum_{r=1}^d X_r^{(j)}Y_r^{(k)}\right).$$

Let H be the Hilbert space ℓ_2^d, and let x_j and y_k for $1\leq j\leq n$ and $1\leq k\leq m$ be the vectors:

$$x_j=\sum_{r=1}^d\overline{X_r^{(j)}}e_r,\quad y_k=\sum_{r=1}^d Y_r^{(k)}e_r.$$

Then for each j and k,

$$\|x_j\|_H=\left(\sum_{r=1}^d|X_r^{(j)}|^2\right)^{1/2},\ \|y_k\|_H=\left(\sum_{r=1}^d|Y_r^{(k)}|^2\right)^{1/2}$$

and:

$$\langle x_j,y_k\rangle=\sum_{r=1}^d X_r^{(j)}Y_r^{(k)}.$$

So applying Theorem 3.2.1:

$$\begin{aligned}
&\left|\sum_{j=1}^{n}\sum_{k=1}^{m} A_{j,k}\left(\sum_{r=1}^{d} X_r^{(j)}Y_r^{(k)}\right)\right| \\
&= \sum_{j=1}^{n}\sum_{k=1}^{m} A_{j,k}\langle x_j, y_k\rangle \\
&\leq K_G\|A\|_{\ell_1^n\otimes_\varepsilon \ell_1^m} \max_{1\leq j\leq n}\|x_j\|_H \max_{1\leq k\leq m}\|y_k\|_H \\
&= K_G\|A\|_{\ell_1^n\otimes_\varepsilon \ell_1^m} \max_{1\leq j\leq n}\left(\sum_{r=1}^{d}|X_r^{(j)}|^2\right)^{1/2} \max_{1\leq k\leq m}\left(\sum_{r=1}^{d}|Y_r^{(k)}|^2\right)^{1/2}.
\end{aligned}$$

□

The most technical component of this section is the following corollary of Theorem 3.2.3. This is an argument which is largely inspired by Pisier's [104, Proposition 23.3]. The argument rests on a version of the Hahn-Banach theorem which is sometimes called the Hahn-Banach separation theorem, or simply the hyperplane separation theorem. That result states that if V is a real Banach space, and $\mathcal{E}$ and $\mathcal{G}$ are two non-empty disjoint convex sets in V (and $\mathcal{G}$ is open), then there is a continuous linear functional $\omega \in V^*$ and a real number c such that $\omega(x) \geq c$ for all $x \in \mathcal{E}$ and $\omega(x) < c$ for all $x \in \mathcal{G}$. See e.g. [116, Theorem 3.4].

Corollary 3.2.4. *Let $A \in M_{n,m}(\mathbb{C})$. There exist vectors of non-negative numbers $\{\lambda_j\}_{j=1}^n$, $\{\mu_k\}_{k=1}^m$ with $\sum_{j=1}^n \lambda_j = \sum_{k=1}^m \mu_k = 1$ such that for all $x \in \mathbb{C}^n$ and $y \in \mathbb{C}^m$ we have:*

$$\left|\sum_{j=1}^{n}\sum_{k=1}^{m} \overline{x_j}A_{j,k}y_k\right| \leq K_G\|A\|_{\ell_1^n\otimes_\varepsilon \ell_1^m}\left(\sum_{j=1}^{n}\lambda_j|x_j|^2\right)^{1/2}\left(\sum_{k=1}^{m}\mu_k|y_k|^2\right)^{1/2}.$$

Proof. In order to lighten the notation, rescale A so that $K_G\|A\|_{\ell_1^n\otimes_\varepsilon \ell_1^m} = 1$ (unless A is zero, in which case the result is trivial).

Let Δ^n denote the $n-1$-dimensional simplex:

$$\Delta^n = \{(\lambda_1, \dots, \lambda_n) \in [0,1]^n \ : \ \sum_{j=1}^{n}\lambda_j = 1\}.$$

Similarly Δ^m denotes the $m-1$-dimensional simplex.

Let $\{X_r\}_{r=1}^d$, $\{Y_r\}_{r=1}^d$ be sets of vectors in $\mathbb{C}^n$ and $\mathbb{C}^m$ respectively, with components denoted $X_r = (X_r^{(1)}, X_r^{(2)}, \cdots, X_r^{(n)})$, and similarly with Y_r. Let $t > 0$. By the arithmetic-geometric mean inequality, we have:

$$\max_{1\leq j\leq n}\left(\sum_{r=1}^{d}|X_r^{(j)}|^2\right)^{1/2}\max_{1\leq k\leq m}\left(\sum_{r=1}^{d}|Y_r^{(k)}|^2\right)^{1/2} \leq \frac{1}{2}\left(t\max_{1\leq j\leq n}\sum_{r=1}^{d}|X_r^{(j)}|^2 + (1/t)\max_{1\leq k\leq m}\sum_{r=1}^{d}|Y_r^{(k)}|^2\right).$$

Replacing the maximum with the supremum over $\Delta^n \times \Delta^m$, Theorem 3.2.3 yields:

$$\left|\sum_{r=1}^{d}\sum_{j=1}^{n}\sum_{k=1}^{m} A_{j,k} X_r^{(j)} Y_r^{(k)}\right| \leq \frac{1}{2} \sup_{(\lambda,\mu)\in\Delta^n\times\Delta^m} (t\sum_{j=1}^{n}\lambda_j \sum_{r=1}^{d} |X_r^{(j)}|^2 + (1/t)\sum_{k=1}^{m}\mu_k \sum_{r=1}^{d}|Y_r^{(k)}|^2). \tag{3.2.2}$$

Note that the right hand side of (3.2.2) does not change if we multiply each X_r or Y_r by some z_r with $|z_r| = 1$, so we can replace the left hand side with:

$$\sum_{r=1}^{d}\left|\sum_{j=1}^{n}\sum_{k=1}^{m} A_{j,k} X_r^{(j)} Y_r^{(k)}\right|.$$

By rearranging the right hand side of (3.2.2), we arrive at:

$$\sum_{r=1}^{d}\left|\sum_{j=1}^{n}\sum_{k=1}^{m} A_{j,k} X_r^{(j)} Y_r^{(k)}\right| \leq \frac{1}{2} \sup_{(\lambda,\mu)\in\Delta^n\times\Delta^m} \sum_{r=1}^{d}\left(t\sum_{j=1}^{n}\lambda_j |X_r^{(j)}|^2 + (1/t)\sum_{k=1}^{m}\mu_k |Y_r^{(k)}|^2\right). \tag{3.2.3}$$

Consider the following real-valued function on $\Delta^n \times \Delta^m$, defined for each $d \geq 1$ and each choice of $X = \{X_r\}_{r=1}^d \subset \mathbb{C}^n$ and $Y = \{Y_r\}_{r=1}^d \subset \mathbb{C}^m$ as:

$$F_{X,Y}(\lambda,\mu) = \sum_{r=1}^{d}\left(\frac{1}{2}\left(t\sum_{j=1}^{n}\lambda_j |X_r^{(j)}|^2 + (1/t)\sum_{k=1}^{m}\mu_k |Y_r^{(k)}|^2\right) - \left|\sum_{j=1}^{n}\sum_{k=1}^{m} A_{j,k} X_r^{(j)} Y_r^{(k)}\right|\right).$$

For each X and Y, (3.2.3) states that there exists a point $(\lambda,\mu) \in \Delta^n \times \Delta^m$ such that $F_{X,Y}(\lambda,\mu) \geq 0$.

Let us prove that we can choose a fixed $(\lambda^*,\mu^*) \in \Delta^n \times \Delta^m$ such that $F_{X,Y}(\lambda^*,\mu^*) \geq 0$ for all X and Y.

Note that each $F_{X,Y}$ is continuous (in fact, affine linear) on $\Delta^n \times \Delta^m$, and let $\mathcal{F}$ denote the convex cone formed by all positive linear combinations of the functions $F_{X,Y}$ over all X and Y. Let $\mathcal{N}$ denote the convex cone of all real-valued continuous functions on $\Delta^n \times \Delta^m$ which are strictly negative. That is, define:

$$\mathcal{N} := \{f \in C(\Delta^n \times \Delta^m, \mathbb{R}) \ : \ \sup_{(\lambda,\mu)\in\Delta^n\times\Delta^m} f(\lambda,\mu) < 0\}.$$

Note that a positive linear combination of functions of the form $F_{X,Y}$ is again of the form $F_{X,Y}$. To see this, note that if $X_1, X_2 \subset \mathbb{C}^n$ and $Y_1, Y_2 \subset \mathbb{C}^m$ are sets with $|X_j| = |Y_j| = d_j$, $j = 1, 2$, then $F_{X_1,Y_1} + F_{X_2,Y_2} = F_{X_3,Y_3}$, where X_3 and Y_3 are the disjoint unions of X_1, X_2 and Y_1, Y_2 respectively. Thus each $f \in \mathcal{F}$ is non-negative at at least one point, and so we have:

$$\mathcal{F} \cap \mathcal{N} = \emptyset.$$

Since $\mathcal{N}$ is clearly open, we can apply the Hahn-Banach separation theorem to $\mathcal{F}$ and $\mathcal{N}$. Thus there exists a nonzero bounded linear functional $\nu \in C(\Delta^n \times \Delta^m, \mathbb{R})^*$ and $c \in \mathbb{R}$ such that:

$$\int_{\Delta^n\times\Delta^m} f\,d\nu \geq c, \quad f \in \mathcal{F} \tag{3.2.4}$$

and

$$\int_{\Delta^n\times\Delta^m} g\,d\nu<c, \quad g\in\mathcal{N}. \tag{3.2.5}$$

Let us show that c is necessarily zero. To see this, simply note that if $f\in\mathcal{F}$ and $g\in\mathcal{N}$ then $\lambda f\in\mathcal{F}$ and $\lambda g\in\mathcal{N}$ for all $\lambda>0$. Then (3.2.4) and (3.2.5) remain valid with c replaced by c/λ, and since λ is arbitrarily large it follows that $c=0$. With $c=0$, (3.2.5) implies that ν is a positive (in particular, nonzero) measure on $\Delta^n\times\Delta^m$. By rescaling ν if necessary, we may further assume that ν is a probability measure. Define:

$$\lambda_j^* := \int_{\Delta^n\times\Delta^m} t_j d\nu(t_j,s_k), \quad \mu_k^* := \int_{\Delta^n\times\Delta^m} s_k\,d\nu(t_j,s_k), \quad 1\le j\le n,\ 1\le k\le m.$$

Since ν is positive, λ_j^* and μ_j^* are non-negative and since ν is a probability measure, $\lambda^*\in\Delta^n$ and $\mu^*\in\Delta^m$.

Now for each choice of X and Y, since $F_{X,Y}$ is affine linear in λ and μ we have:

$$F_{X,Y}(\lambda^*,\mu^*) = \int_{\Delta^n\times\Delta^m} F_{X,Y}(t,s)\,d\nu(t,s)$$

and this is non-negative, due to (3.2.4).

Consider the special case where $d=1$, and $X=\{x\}$ and $Y=\{y\}$. From the definition of $F_{X,Y}$, we have proved that:

$$\left|\sum_{j=1}^n\sum_{k=1}^m A_{j,k}x_jy_k\right| \le \frac{1}{2}\left(t\sum_{j=1}^n \lambda_j^*|x_j|^2+(1/t)\sum_{k=1}^m\mu_k^*|y_k|^2\right).$$

Using the numerical identity:

$$(ab)^{1/2} = \inf_{t>0}\frac{1}{2}(ta+(1/t)b), \quad a,b\ge 0$$

we can take the infimum over t to arrive at:

$$\left|\sum_{j=1}^n\sum_{k=1}^m A_{j,k}x_jy_k\right| \le \left(\sum_{j=1}^n\lambda_j^*|x_j|^2\right)^{1/2}\left(\sum_{k=1}^m\mu_k^*|y_k|^2\right)^{1/2}.$$

Finally, replacing x_j with $\overline{x_j}$ yields the result. $\square$

Remark 3.2.5. *Note that by choosing the vectors $x=e_j$ and $y=e_k$ for $1\le j\le n$ and $1\le k\le m$ in Corollary 3.2.4, it follows that:*

$$|A_{j,k}|\le K_G\|A\|_{\ell_1^n\otimes_\varepsilon\ell_1^m}\lambda_j\mu_k.$$

Thus, if $\lambda_j=0$ or $\mu_k=0$, then $A_{j,k}=0$

We now arrive at the main result of this section, which is again a reformulation of Grothendieck's inequality (see [104, Theorem 2.1]).

Theorem 3.2.6. *Let $n,m\ge 1$. Then for all $A=\{A_{j,k}\}\in M_{n,m}(\mathbb{C})$, there exist unit vectors $\xi=\{\xi_j\}_{j=1}^n\in\ell_2^n$, $\eta=\{\eta_k\}_{k=1}^m\in\ell_2^m$ and a matrix $T=\{T_{j,k}\}_{j=1}^n{}_{k=1}^m\in M_{n,m}(\mathbb{C})$*

such that:

$$A_{j,k} = \xi_j T_{j,k} \eta_k, \quad 1 \leq j \leq n,\, 1 \leq k \leq m$$

and

$$\|T\|_{\mathcal{B}(\ell_2^m, \ell_2^n)} \leq K_G \|A\|_{\ell_1^n \otimes_\varepsilon \ell_1^m}.$$

Proof. Select $\lambda \in \Delta^n$ and $\mu \in \Delta^m$ so that Corollary 3.2.4 holds for A. Let:

$$\xi_j = \lambda_j^{1/2}, \quad \eta_k = \mu_k^{1/2}, \quad 1 \leq j \leq n,\, 1 \leq k \leq m.$$

Then $\xi \in \ell_2^n$ and $\eta \in \ell_2^m$ are unit vectors. Let T be the matrix with entries:

$$T_{j,k} = \begin{cases} \frac{A_{j,k}}{\xi_j \eta_k} & \xi_j \neq 0 \text{ and } \eta_k \neq 0. \\ 0, \text{ otherwise.} \end{cases}$$

From Remark 3.2.5, we have that if $\xi_j = 0$ or $\eta_k = 0$ then $A_{j,k} = 0$, thus for all $1 \leq j \leq n$ and $1 \leq k \leq m$ we have:

$$T_{j,k} \xi_j \eta_k = A_{j,k}$$

Corollary 3.2.4 yields that for all $\alpha \in \mathbb{C}^n$ and $\beta \in \mathbb{C}^m$ we have:

$$\left| \sum_{j=1}^n \sum_{k=1}^m \overline{\alpha_j} T_{j,k} \beta_k \right| \leq K_G \|A\|_{\ell_1^n \otimes_\varepsilon \ell_1^m} \|\alpha\|_{\ell_2^n} \|\beta\|_{\ell_2^m}.$$

Hence,

$$\|T\|_{\mathcal{B}(\ell_2^m, \ell_2^n)} \leq K_G \|A\|_{\ell_1^n \otimes \ell_1^m}$$

as required. $\square$

Before continuing, it is worth making some remarks concerning the meaning of Theorem 3.2.6. It is not hard to prove from (3.2.1) that there is an isometric isomorphism:

$$\mathcal{B}(\ell_\infty^m, \ell_1^n) \cong \ell_1^n \otimes_\varepsilon \ell_1^m.$$

So that we can view the $\ell_1^n \otimes_\varepsilon \ell_1^m$ norm of $A \in M_{n,m}(\mathbb{C})$ as being exactly its norm as a map from ℓ_∞^m to ℓ_1^m. Theorem 3.2.6 then states the following: for all linear maps $A : \ell_\infty^m \to \ell_1^n$ there exist unit vectors $\eta \in \ell_2^m$ and $\xi \in \ell_2^n$ and a linear map $T : \ell_2^m \to \ell_2^n$ such that the following diagram commutes:

$$\begin{tikzcd} \ell_\infty^m \arrow[r, "A"'] \arrow[d, "M_\xi"'] & \ell_1^n \\ \ell_2^m \arrow[r, "T"'] & \ell_2^m \arrow[u, "M_\eta"'] \end{tikzcd}$$

where M_ξ and M_η denote pointwise multiplication, and

$$\|T\|_{\mathcal{B}(\ell_2^m, \ell_2^n)} \leq K_G \|A\|_{\mathcal{B}(\ell_\infty^m, \ell_1^n)}.$$

3.3 Matrix Schur multipliers

For $n, m \geq 1$, let $\mathcal{L}_1(\ell_2^m, \ell_2^n)$ denote the Banach space on $M_{n,m}(\mathbb{C})$ with norm given by:

$$\|T\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)} := \sup_{\|A\|_{\mathcal{B}(\ell_2^m,\ell_2^n)}\leq 1} |\mathrm{tr}(A^*T)|.$$

The matrix Schur product $A \circ B$ of two $n \times m$ matrices is given by:

$$A \circ B = \sum_{j=1}^{n} \sum_{k=1}^{m} A_{j,k} B_{j,k} e_j \otimes e_k.$$

The $\mathcal{L}_1$-Schur multiplier norm, $\|\cdot\|_{\mathrm{m}}$, of a matrix $A \in M_{n,m}(\mathbb{C})$, is defined as:

$$\|A\|_{\mathrm{m}} := \sup_{\|B\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)}\leq 1} \|A \circ B\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)}. \tag{3.3.1}$$

We will call this simply the Schur norm of A.

Lemma 3.3.1. *Let $A, B \in M_{n,m}(\mathbb{C})$, and $\xi \in \mathbb{C}^n$ and $\eta \in \mathbb{C}^m$. Then:*

$$|\mathrm{tr}(B^*(A \circ (\xi \otimes \eta)))| \leq \|A\|_{\mathrm{m}} \|B\|_{\mathcal{B}(\ell_2^m,\ell_2^n)} \|\xi\|_{\ell_2^n} \|\eta\|_{\ell_2^m}.$$

Proof. By the definition of the $\mathcal{L}_1(\ell_2^m, \ell_2^n)$ norm and the Schur multiplier norm, we have:

$$\begin{aligned} |\mathrm{tr}(B^*(A \circ (\xi \otimes \eta)))| &\leq \|B\|_{\mathcal{B}(\ell_2^m,\ell_2^n)} \|A \circ (\xi \otimes \eta)\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)} \\ &\leq \|B\|_{\mathcal{B}(\ell_2^m,\ell_2^n)} \|A\|_{\mathrm{m}} \|\xi \otimes \eta\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)}. \end{aligned}$$

However it is easily seen that:

$$\|\xi \otimes \eta\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)} = \|\xi\|_{\ell_2^n} \|\eta\|_{\ell_2^m}$$

and this completes the proof. □

The following result is of crucial importance in our proof of Peller's theorem. It allows us, via Theorem 3.2.6, to relate the Schur norm (given by (3.3.1)) to (the dual of) the injective tensor product norm.

Corollary 3.3.2. *Let $A, B \in M_{n,m}(\mathbb{C})$. Then:*

$$|\mathrm{tr}(B^*A)| \leq K_G \|A\|_{\mathrm{m}} \|B\|_{\ell_1^n \otimes_\varepsilon \ell_1^m}.$$

Proof. Choosing a factorisation as in Theorem 3.2.6, we have:

$$B_{j,k} = \xi_j T_{j,k} \eta_k$$

where ξ and η are unit vectors in ℓ_2^n and ℓ_2^m respectively, and $\|T\|_{\mathcal{B}(\ell_2^m,\ell_2^n)} \leq K_G \|B\|_{\ell_1^n \otimes_\varepsilon \ell_1^m}$.

Then simply computing $\mathrm{tr}(B^*A)$, we have:

$$\begin{aligned}\mathrm{tr}(B^*A) = \sum_{j=1}^{n}\sum_{k=1}^{m} \overline{B_{j,k}} A_{j,k} &= \sum_{j=1}^{n}\sum_{k=1}^{m} \overline{\xi_j T_{j,k} \eta_k} A_{j,k} \\ &= \sum_{j=1}^{n}\sum_{k=1}^{m} \overline{T_{j,k}} \cdot \overline{\xi_j}\,\overline{\eta_k} A_{j,k} \\ &= \mathrm{tr}(T^*(A \circ (\overline{\xi} \otimes \overline{\eta}))).\end{aligned}$$

Thus by Lemma 3.3.1, we have:

$$|\mathrm{tr}(B^*A)| \leq \|A\|_{\mathrm{m}} \|T\|_{\mathcal{B}(\ell_2^m, \ell_2^n)} \|\xi\|_{\ell_2^n} \|\eta\|_{\ell_2^m} \leq K_G \|A\|_{\mathrm{m}} \|B\|_{\ell_1^n \otimes_\varepsilon \ell_1^m}.$$

□

Before moving onto the completion of the proof of Peller's theorem, at this point it should be remarked that we can also obtain a "finite dimensional" Peller's theorem from the above. Corollary 3.3.2 states that:

$$\|A\|_{(\ell_1^n \otimes_\varepsilon \ell_1^m)^*} \leq K_G \|A\|_{\mathrm{m}}.$$

If we appeal to some outside knowledge concerning tensor products, there is in fact an isometric isomorphism between the dual of $\ell_1^n \otimes_\varepsilon \ell_1^m$ and $\ell_\infty^n \otimes_\pi \ell_\infty^m$, where π is the projective tensor product. We can then arrive at:

$$\frac{1}{K_G} \|A\|_{\ell_\infty^n \otimes_\pi \ell_\infty^m} \leq \|A\|_{\mathrm{m}} \leq \|A\|_{\ell_\infty^n \otimes_\pi \ell_\infty^m}.$$

That is, the Schur norm $\|A\|_{\mathrm{m}}$ of an $n \times m$ matrix is equivalent to the norm of A in $\ell_\infty^n \otimes_\pi \ell_\infty^m$, where the constants are independent of n and m. For further details see [103, Chapter 5].

3.4 Measurable Schur multipliers

Let (L, λ) and (M, μ) be two σ-finite measure spaces. To lighten notation, we will write $L_p(\lambda)$ for $L_p(L, \lambda)$, and similarly $L_p(\mu)$ for $L_p(M, \mu)$ and $L_p(\lambda \times \mu)$ for $L_p(L \times M, \lambda \times \mu)$.

Definition 3.4.1. *Let $\phi \in L_\infty(\lambda \times \mu)$. Then ϕ defines a bilinear map $B(\phi) : L_1(\lambda) \odot L_1(\mu) \to \mathbb{C}$ given by:*

$$B(\phi)(f \otimes g) = \int_{L \times M} \phi(t, s) f(t) g(s)\, d\lambda(t) d\mu(s), \quad f \in L_1(\lambda),\ g \in L_1(\mu).$$

Definition 3.4.2. *Suppose that $\mathcal{I} = \{I_j\}_{j=1}^n$ and $\mathcal{J} = \{J_k\}_{k=1}^m$ are two families of pairwise-disjoint measurable sets in L and M respectively, such that each I_j and J_k have finite and nonzero measure, (that is, $0 < \lambda(I_j) < \infty$, $0 < \mu(J_k) < \infty$). Let $\mathcal{I} \times \mathcal{J}$ denote the family of sets:*

$$\mathcal{I} \times \mathcal{J} = \{I_j \times J_k \ : \ 1 \leq j \leq n,\ 1 \leq k \leq m\}.$$

Define $[\phi]_{\mathcal{I}\times\mathcal{J}}$ as the $n \times m$ matrix with (j,k)th entry given by:

$$\frac{1}{\lambda(I_j)\mu(J_k)} \int_{I_j \times J_k} \phi \, d(\mu \times \lambda).$$

We are now ready to prove what is essentially Peller's theorem, stated in terms of Schur multipliers.

Theorem 3.4.3. *Suppose that $\phi \in L_\infty(\lambda \times \mu)$ is such that:*

$$\sup_{\mathcal{I},\mathcal{J}} \|[\phi]_{\mathcal{I}\times\mathcal{J}}\|_{\mathrm{m}} = C_\phi < \infty$$

where the supremum is over all $\mathcal{I}$ and $\mathcal{J}$ as in Definition 3.4.2.

Then there is a compact Hausdorff space Ω, a Borel regular measure σ on Ω with total variation norm at most $K_G C_\phi$, and there are measurable functions $\alpha \in L_\infty(L \times \Omega)$ and $\beta \in L_\infty(M \times \Omega)$ with $\|\alpha\|_\infty \leq 1$ and $\|\beta\|_\infty \leq 1$ such that:

$$\phi(t,s) = \int_\Omega \alpha(t,\omega)\beta(s,\omega)\, d\sigma(\omega)$$

for pointwise almost-all $(t,s) \in L \times M$.

Proof. Let $X := L_1(\lambda)$ and $Y := L_1(\mu)$. Let X_0 and Y_0 be the subspaces of X and Y consisting of simple functions.

Recall that since λ and μ are σ-finite, we have an isometric identification:

$$X^* = L_\infty(\lambda), \quad Y^* = L_\infty(\mu).$$

Let $T \in X_0 \odot Y_0$. Then $T = \sum_{r=1}^N x_r \otimes y_r$, where each x_r and y_r are simple functions, i.e., finite linear combinations of indicator functions. Choose familes of measurable subsets $\mathcal{I} = \{I_j\}_{j=1}^n$ and $\mathcal{J} = \{J_k\}_{k=1}^m$ of L and M respectively so that each x_r (resp. y_r) is a finite linear combination of indicator functions of sets in $\mathcal{I}$ (resp. $\mathcal{J}$).

Therefore T can be written as a linear combination of indicator functions of $\mathcal{I} \times \mathcal{J}$. Hence,

$$T := \sum_{j=1}^n \sum_{k=1}^m \frac{\overline{a_{j,k}}}{\lambda(I_j)\mu(J_k)} \chi_{I_j} \otimes \chi_{J_k}$$

for some matrix $a \in M_{n,m}(\mathbb{C})$.

In fact the norm of T in $L_1(\lambda) \otimes_\varepsilon L_1(\mu)$ is exactly $\|a\|_{\ell_1^n \otimes_\varepsilon \ell_1^m}$. To see this, note that by definition we have:[1]

$$\begin{aligned} \|T\|_{L_1(\lambda)\otimes_\varepsilon L_1(\mu)} &= \sup_{f\in B_{X^*}, g\in B_{Y^*}} \left| \sum_{j=1}^n \sum_{k=1}^m \overline{a_{j,k}} \frac{1}{\lambda(I_j)} \int_{I_j} f\, d\lambda \cdot \frac{1}{\mu(J_k)} \int_{J_k} g\, d\mu \right| \\ &= \sup_{|\alpha_j|\le 1, |\beta_k|\le 1} \left| \sum_{j=1}^n \sum_{k=1}^m a_{j,k}\alpha_j\beta_k \right| \\ &= \|a\|_{\ell_1^n \otimes_\varepsilon \ell_1^m} \end{aligned} \tag{3.4.1}$$

Then by the definition of $B(\phi)$ (Definition 3.4.1) we have:,

$$\begin{aligned} B(\phi)(T) &= \int_{L\times M} \phi(t,s) \sum_{j=1}^n \sum_{k=1}^m \overline{a_{j,k}} \frac{\chi_{I_j}(t)\chi_{J_k}(s)}{\lambda(I_j)\mu(J_k)}\, d\lambda(t) d\mu(s) \\ &= \sum_{j=1}^n \sum_{k=1}^m \frac{\overline{a_{j,k}}}{\lambda(I_j)\mu(J_k)} \int_{I_j\times J_k} \phi\, d(\lambda\times\mu) \\ &= \operatorname{tr}(a^*[\phi]_{\mathcal{I}\times\mathcal{J}}). \end{aligned}$$

Therefore, from Corollary 3.3.2 and (3.4.1), we have:

$$\begin{aligned} |B(\phi)(T)| = |\operatorname{tr}(a^*[\phi]_{\mathcal{I}\times\mathcal{J}})| &\le K_G \|[\phi]_{\mathcal{I}\times\mathcal{J}}\|_{\mathrm{m}} \|a\|_{\ell_1^n\otimes_\varepsilon \ell_1^m} \\ &\le K_G C_\phi \|T\|_{L_1(\lambda)\otimes_\varepsilon L_1(\mu)}. \end{aligned}$$

Hence $B(\phi)$ satisfies the assumptions of Theorem 3.1.2. Thus there is a regular Borel measure on $B_{X^*} \times B_{Y^*}$ with total variation norm at most $K_G C_\phi$ and such that for all sets of finite measure, $I \subset L$ and $J \subset M$ we have:

$$B(\phi)(\chi_I \otimes \chi_J) = \int_{B_{X^*}\times B_{Y^*}} \omega_0(\chi_I)\omega_1(\chi_J)\, d\sigma(\omega_0,\omega_1) \tag{3.4.2}$$

For each $\omega_0 \in B_{X^*}$ and $\omega_1 \in B_{Y^*}$, we can identify ω_0 and ω_1 with elements of $L_\infty(\lambda)$ and $L_\infty(\mu)$ respectively. Write $\omega = (\omega_0, \omega_1)$, and consider the functions α and β defined by $\alpha(t,\omega) := \omega_0(t)$ and $\beta(s,\omega) = \omega_1(s)$. We can use α and β to write the pairing of ω_0 and ω_1 with χ_I and χ_J as integrals:

$$\omega_0(\chi_I) = \int_I \alpha(t,\omega)\, d\lambda(t), \quad \omega_1(\chi_J) = \int_J \beta(s,\omega)\, d\mu(s).$$

Inserting this into (3.4.2) and using Fubini's theorem, (which is valid due to the σ-finiteness of λ and μ), we have:

$$\begin{aligned} B(\phi)(\chi_I\otimes\chi_J) &= \int_{B_{X^*}\times B_{Y^*}} \int_I \alpha(t,\omega)\, d\lambda(t) \int_J \beta(s,\omega)\, d\mu(s) d\sigma(\omega) \\ &= \int_{I\times J} \int_{B_{X^*}\times B_{Y^*}} \alpha(t,\omega)\beta(s,\omega)\, d\sigma(\omega) d(\lambda\times\mu)(t,s) \\ &= \int_{L\times M} \left(\int_{B_{X^*}\times B_{Y^*}} \alpha(t,\omega)\beta(s,\omega)\, d\sigma(\omega) \right) \chi_I(t)\chi_J(s)\, d(\lambda\times\mu)(t,s). \end{aligned}$$

[1] Note that $\|\overline{a}\|_{\ell_1^n\otimes_\varepsilon \ell_1^m} = \|a^*\|_{\ell_1^m\otimes_\varepsilon \ell_1^n} = \|a\|_{\ell_1^n\otimes_\varepsilon \ell_1^m}$.

From the definition of $B(\phi)$ (Definition 3.4.1) we have:

$$\int_{L\times M}\phi(t,s)\chi_I(t)\chi_J(s)\,d\lambda(t)d\mu(s) = \int_{L\times M}\left(\int_{B_{X^*}\times B_{Y^*}}\alpha(t,\omega)\beta(s,\omega)\,d\sigma(\omega)\right)\chi_I(t)\chi_J(s)\,d(\lambda\times\mu)(t,s).$$

Since I and J are arbitrary, it follows that for almost all t and s we have:

$$\phi(t,s) = \int_{B_{X^*}\times B_{Y^*}}\alpha(t,\omega)\beta(s,\omega)\,d\sigma(\omega).$$

This gives precisely the desired result. □

Definition 3.4.4. *The* Birman-Solomyak *class of functions $\phi \in L_\infty(\lambda\times\mu)$, denoted $\mathfrak{BS}(\lambda\times\mu)$, is defined to be the class of functions which can be expressed as:*

$$\phi(t,s) = \int_\Omega \alpha(t,\omega)\beta(s,\omega)d\sigma(\omega), \quad a.e. t\in L, s\in M.$$

for some σ-finite measure space (Ω,σ) and measurable functions α and β such that:

$$\int_\Omega \mathrm{esssup}_{t\in L}|\alpha(t,\omega)|\mathrm{esssup}_{s\in M}|\beta(s,\omega)|\,d|\sigma|(\omega) < \infty. \tag{3.4.3}$$

Such a representation is called a Birman-Solomyak decomposition. The infimum of (3.4.3) over all Birman-Solomyak decompositions of ϕ is called the Birman-Solomyak norm, and denoted $\|\phi\|_{\mathfrak{BS}}$.

If ν_0 and ν_1 are spectral measures, then $\mathfrak{BS}(\nu_0\times\nu_1)$ is defined in precisely the same way (given that the product exists), where instead the representation is assumed to hold pointwise almost everywhere with respect to $\nu_0\times\nu_1$.

With this definition, the content of Theorem 3.4.3 could be stated as: if $\|[\phi]_{\mathcal{I}\times\mathcal{J}}\|_{\mathrm{m}} \leq C$ over all $\mathcal{I}$ and $\mathcal{J}$, then $\phi\in\mathfrak{BS}(\lambda\times\mu)$ and $\|\phi\|_{\mathfrak{BS}} \leq K_G C$, where K_G is Grothendieck's constant.

Recall that $(\mathcal{M},\tau)$ is a semifinite von Neumann algebra, and $B(\mathbb{R}^2)$ denotes the algebra of bounded Borel measurable functions on $\mathbb{R}^2$.

We now deal with the easiest direction of Peller's theorem: the sufficiency of the Birman-Solomyak condition for a double operator integral to map $\mathcal{L}_1(H)$ into itself continuously. Actually, for the sake of future use in Chapter 4, we will state and prove the following theorem in slightly more generality. Recall that $(\mathcal{M},\tau)$ denotes a semifinite von Neumann algebra.

Theorem 3.4.5. *Let (X_0,Σ_0,H,ν_0), (X_1,Σ_1,H,ν_1) be two $\mathcal{P}(\mathcal{M})$-valued spectral measure spaces on σ-compact standard Borel spaces (X_0,Σ_0) and (X_1,Σ_1). Let $x\in L_1(\tau)\cap\mathcal{M}$, and $\|\phi\|_{\mathfrak{BS}(\nu_0\times\nu_1)} < \infty$. Then $\mathcal{T}_\phi^{\nu_0,\nu_1}(x)\in L_1(\tau)\cap L_2(\tau)$, and:*

$$\|\mathcal{T}_\phi^{\nu_0,\nu_1}(x)\|_1 \leq \|\phi\|_{\mathfrak{BS}}\|x\|_1.$$

Proof. Note that since $L_1(\tau)\cap\mathcal{M}$ is a subset of $L_2(\tau)$, the double operator integral $\mathcal{T}_\phi^{\nu_0,\nu_1}(x)$ is certainly defined.

Let $x, y \in L_1(\tau) \cap \mathcal{M}$, and recall that $\nu_0 \otimes \nu_1$ denotes the spectral measure on $L_2(\tau)$ given as in Definition 2.4.1. We have:

$$\tau(y^* \mathcal{T}_\phi^{\nu_0,\nu_1}(x)) = \int_{\mathbb{R}^2} \phi \, d(\nu_0 \otimes \nu_1)^{x,y}.$$

Since $(\nu_0 \otimes \nu_1)^{x,y}$ is a measure of bounded variation (Lemma 2.2.6), we may write ϕ in terms of its Birman-Solomyak decomposition and use Fubini's theorem:

$$\begin{aligned} \tau(y^* \mathcal{T}_\phi^{\nu_0,\nu_1}(x)) &= \int_{X_0 \times X_1} \int_\Omega \alpha(t,\omega)\beta(s,\omega) \, d\sigma(\omega) d(\nu_0 \otimes \nu_1)^{x,y}(t,s) \\ &= \int_\Omega \int_{X_0 \times X_1} \alpha(t,\omega)\beta(s,\omega) d(\nu_0 \otimes \nu_1)^{x,y}(t,s) d\sigma(\omega). \end{aligned} \tag{3.4.4}$$

Let us examine the inner integral. This is the function (of σ) given by:

$$\int_{X_0 \times X_1} \alpha(t,\omega)\beta(s,\omega) d(\nu_0 \otimes \nu_1)^{x,y}(t,s).$$

We may recognise this as being precisely a double operator integral:

$$\int_{X_0 \times X_1} \alpha(t,\omega)\beta(s,\omega) d(\nu_0 \otimes \nu_1)^{x,y}(t,s) = \tau(y^* \mathcal{T}^{\nu_0,\nu_1}_{\alpha(\cdot,\omega)\beta(\cdot,\omega)}(x)).$$

Using Corollary 2.2.17, this is:

$$\tau(y^* \mathcal{T}^{\nu_0,\nu_1}_{\alpha(\cdot,\omega)\beta(\cdot,\sigma)}(x)) = \tau(y^* \int_{X_0} \alpha(t,\omega) \, d\nu_0(t) x \int_{X_1} \beta(s,\omega) \, d\nu_1(s)).$$

So that pointwisely for σ-almost all σ, we have:

$$\left| \int_{X_0 \times X_1} \alpha(t,\omega)\beta(s,\omega) d(\nu_0 \otimes \nu_1)^{x,y}(t,s) \right| \leq \|y\|_\mathcal{M} \|x\|_1 \mathrm{esssup}_{t \in X_0} |\alpha(t,\omega)| \mathrm{esssup}_{s \in X_1} |\beta(s,\omega)|.$$

Applying this upper bound to (3.4.4) yields:

$$|\tau(y^* \mathcal{T}_\phi^{\nu_0,\nu_1}(x))| \leq \|x\|_1 \|y\|_\mathcal{M} \|\phi\|_{\mathfrak{BS}}, \quad x, y \in L_1(\tau) \cap \mathcal{M}. \tag{3.4.5}$$

Let p be a τ-finite projection in $\mathcal{M}$, and let u be a partial isometry defining a polar decomposition for $\mathcal{T}_\phi^{\nu_0,\nu_1}(x)$:

$$u \mathcal{T}_\phi^{\nu_0,\nu_1}(x) = \left| \mathcal{T}_\phi^{\nu_0,\nu_1}(x) \right|$$

Let $y^* = pu$ Then (3.4.5) yields:

$$\tau(p \left| \mathcal{T}_\phi^{\nu_0,\nu_1}(x) \right| p) \leq \|x\|_1 \|\phi\|_{\mathfrak{BS}}$$

for all τ-finite projections p. Since τ is semifinite, we may take the supremum over a monotone family $\{p_\alpha\}_\alpha$ with $p_\alpha \uparrow 1$ to arrive at:

$$\|\mathcal{T}_\phi^{\nu_0,\nu_1}(x)\|_1 \leq \|x\|_1 \|\phi\|_{\mathfrak{BS}}.$$

□

To complete the proof of Peller's theorem, we explain how the assumption of Theorem 3.4.3 is related to the boundedness of double operator integrals $\mathcal{L}_1(H)$ for a separable Hilbert space H.

The following lemma will be important to relate double operator integrals with scalar integrals.

Lemma 3.4.6. *Let $(X_j, \Sigma_j, H, \nu_j)$, $j = 1, 2$ be spectral measures on a σ-compact standard Borel space, and let $\nu_0 \otimes \nu_1$ be the corresponding measure on the Hilbert-Schmidt class $\mathcal{L}_2(H)$ by left and right multiplication.*

If $\xi, \eta \in H$ are vectors, let $\xi\eta^*$ denote the rank one operator $v \mapsto \xi\langle \eta, v\rangle$. Then:

$$\langle \xi, \mathcal{T}_\phi^{\nu_0,\nu_1}(\xi\eta^*)\eta\rangle = \int_{X_0\times X_1} \phi d(\nu_0^{\xi,\xi} \times \nu_1^{\eta,\eta}).$$

Proof. This simply amounts to the claim that we have an equality of scalar measures:

$$(\nu_0 \otimes \nu_1)^{\xi\otimes\eta^*,\xi\otimes\eta^*} = \nu_0^{\xi,\xi} \times \nu_1^{\eta,\eta}.$$

This is easily verified on measurable sets $\Delta_0 \times \Delta_1 \in \Sigma_0 \otimes \Sigma_1$, and hence the coincidence of the above measures follows from the monotone class theorem. □

Consider a pair of spectral measure spaces $(X_j, \Sigma_j, H, \nu_j)$, $j = 1, 2$ defined on σ-compact standard Borel spaces (X_j, Σ_j), $j = 1, 2$, and H is separable. To complete the proof, we need to select (scalar) measures μ_0 and μ_1 which are equicontinuous with respect to ν_0 and ν_1 respectively.

Here, we say that a measure μ is equicontinuous with a spectral measure ν if $\nu(\Delta) = 0$ if and only if $\mu(\Delta) = 0$.

It is that this point we use the assumption that H is separable: if E is a spectral measure on a separable Hilbert space H, there exists $x \in H$ such that the scalar measure $E^{x,x}$, defined by $E^{x,x}(\Delta) = \|E(\Delta)x\|^2$, is equicontinuous with respect to E. An element $x \in H$ satisfying this property is called an element of maximal spectral type. A proof that elements of maximal spectral type exist for spectral measures on separable Hilbert spaces may be found as [18, Theorem 4, Chapter 7].

Theorem 3.4.7. *Suppose that $(X_j, \Sigma_j, H, \nu_j)$, $j = 1, 2$ are two spectral measure spaces on σ-compact standard Borel spaces and on a separable Hilbert space H.*

Let $x, y \in H$ be chosen such that the scalar-valued measures:

$$\mu_0 = \nu_0^{x,x}, \quad \mu_1 = \nu_1^{y,y}.$$

are equicontinuous with respect to ν_0 and ν_1. Assume that $\phi \in L_\infty(\mu_0 \times \mu_1)$ satisfies:

$$\sup_{\|T\|_{\mathcal{L}_1(H)}\leq 1} \|\mathcal{T}_\phi^{\nu_0,\nu_1}(T)\|_{\mathcal{L}_1(H)} = K < \infty.$$

Then

$$\sup_{\mathcal{I},\mathcal{J}} \|[\phi]_{\mathcal{I}\times\mathcal{J}}\|_{\mathrm{m}} \leq K$$

where the supremum is taken over all finite pairwise disjoint families of sets $\mathcal{I} = \{I_j\}$ and $\mathcal{J} = \{J_k\}$ in $\mathbb{R}$, and the entries of the matrix $[\phi]_{\mathcal{I}\times\mathcal{J}}$ are computed with the measures μ_0 and μ_1, defined by x and y respectively.

Proof. Let $\mathcal{I} = \{I_j\}_{j=1}^n \subset \Sigma_0$ and $\mathcal{J} = \{J_k\}_{k=1}^m \subset \Sigma_1$ be two finite families of disjoint measurable sets, with finite and nonzero μ_0 and μ_1 measure respectively.

For each $1 \leq j \leq n$ and $1 \leq k \leq m$, let ξ_j and η_k denote the following elements of $L_2(\tau)$:

$$\xi_j = \frac{\nu_0(I_j)x}{\|\nu_0(I_j)x\|}, \quad \eta_k = \frac{\nu_1(J_k)y}{\|\nu_1(J_k)y\|}.$$

Since the sets $\{I_j\}$ and $\{J_k\}$ are assumed to be pairwise disjoint, the sets of vectors $\{\xi_j\}$ and $\{\eta_k\}$ are pairwise orthogonal.

We define the following two linear maps, $\Phi : H \to \ell_2^n$ and $\Psi : H \to \ell_2^m$,

$$\Phi(v) = \sum_{j=1}^n e_j\langle \xi_j, v\rangle, \quad \Psi(v) = \sum_{k=1}^m e_k\langle \eta_k, v\rangle.$$

The adjoints of Φ and Ψ map ℓ_2^n and ℓ_2^m into H respectively,

$$\Phi^*(w) = \sum_{j=1}^n \xi_j w_j, \quad \Psi^*(z) = \sum_{k=1}^m \eta_k z_k, \quad w \in \ell_2^n, z \in \ell_2^m.$$

Since the sets $\{\xi_j\}$ and $\{\eta_k\}$ are pairwise orthonormal, Φ and Ψ are partial isometries, and we have:

$$\Phi\Phi^* = 1_{\ell_2^n}, \quad \Psi\Psi^* = 1_{\ell_2^m}.$$

The operators $\Phi^*\Phi$ and $\Psi^*\Psi$ are the orthogonal projections onto the linear spans of $\{\xi_j\}$ and $\{\eta_k\}$ respectively.

For a matrix $a \in M_{n,m}(\mathbb{C})$, considered as a linear map from ℓ_2^m to ℓ_2^n, we may consider:

$$\Phi^* a\Psi : H \to H.$$

Similarly, a linear operator $T : H \to H$ descends to a linear map from ℓ_2^m to ℓ_2^n given by:

$$\Phi T\Psi^* : \ell_2^m \to \ell_2^n.$$

Since Φ and Ψ are partial isometries, it is easy to see that:

$$\|\Phi^* a\Psi\|_{\mathcal{L}_1(H)} = \|a\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)}, \quad a \in M_{n,m}(\mathbb{C})$$

and

$$\|\Phi T\Psi^*\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)} \leq \|T\|_{\mathcal{L}_1(H)}, \quad T \in \mathcal{L}_1(H).$$

Let $a \in M_{n,m}(\mathbb{C})$. We shall prove that:

$$\Phi\mathcal{T}_\phi^{\nu_0,\nu_1}(\Phi^* a\Psi)\Psi^* = [\phi]_{\mathcal{I}\times\mathcal{J}} \circ a. \tag{3.4.6}$$

It will be helpful to write x^* for the linear functional defined by $v \mapsto \langle x, v\rangle$, and similarly with y. We have:

$$\begin{aligned}
\Phi \mathcal{T}_\phi^{\nu_0,\nu_1}(\Phi^* a \Psi)\Psi^* &= \sum_{j=1}^n \sum_{k=1}^m a_{j,k} \Phi \mathcal{T}_\phi^{\nu_0,\nu_1}(\Phi^*(e_j \otimes e_k)\Psi)\Psi^* \\
&= \sum_{j=1}^n \sum_{k=1}^m a_{j,k} \frac{1}{\sqrt{\mu_0(I_j)\mu_1(J_k)}} \Phi \mathcal{T}_\phi^{\nu_0,\nu_1}(\nu_0(I_j) x y^* \nu_1(J_k))\Psi^* \\
&= \sum_{j=1}^n \sum_{k=1}^m a_{j,k} \frac{1}{\sqrt{\mu_0(I_j)\mu_1(J_k)}} \Phi \nu_0(I_j) \mathcal{T}_\phi^{\nu_0,\nu_1}(x y^*) \nu_1(J_k)\Psi^* \\
&= \sum_{j=1}^n \sum_{k=1}^m a_{j,k} \frac{1}{\sqrt{\mu_0(I_j)\mu_1(J_k)}} e_j \otimes e_k \langle \xi_j, \mathcal{T}_{\phi\chi_{I_j\times J_k}}^{\nu_0,\nu_1}(xy^*)\eta_k\rangle \\
&= \sum_{j=1}^n \sum_{k=1}^m a_{j,k} \frac{1}{\mu_0(I_j)\mu_1(J_k)} e_j \otimes e_k \langle x, \mathcal{T}_{\phi\chi_{I_j\times J_k}}^{\nu_0,\nu_1}(xy^*)y\rangle.
\end{aligned}$$

Now Lemma 3.4.6 implies:

$$\langle x, \mathcal{T}_{\phi\chi_{I_j\times J_k}}^{\nu_0,\nu_1}(xy^*)y\rangle = \iint_{I_j\times J_k} \phi\, d(\mu_0 \times \mu_1).$$

So we have proved (3.4.6).

Finally,

$$\begin{aligned}
\|[\phi]_{\mathcal{I}\times\mathcal{J}} \circ a\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)} &= \|\Phi \mathcal{T}_\phi^{\nu_0,\nu_1}(\Phi^* a\Psi)\Psi^*\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)} \\
&\leq \|\mathcal{T}_\phi^{\nu_0,\nu_1}(\Phi^* a \Psi)\|_{\mathcal{L}_1(H)} \\
&\leq K\|\Phi^* a\Psi\|_{\mathcal{L}_1(H)} \\
&= K\|a\|_{\mathcal{L}_1(\ell_2^m,\ell_2^n)}.
\end{aligned}$$

So indeed $\|[\phi]_{\mathcal{I}\times\mathcal{J}}\|_{\mathrm{m}} \leq K$.

□

Proof of Peller's theorem. Specialising Lemma 3.4.5 to the case $\mathcal{M} = \mathcal{B}(H)$, we have that $\phi \in \mathfrak{BS}(E\times F)$ is sufficient for $\mathcal{T}_\phi^{E,F}$ to be bounded from $\mathcal{L}_1(H)$ to $\mathcal{L}_1(H)$, and the norm of $\mathcal{T}_\phi^{E,F}$ is no more than $\|\phi\|_{\mathfrak{BS}(E\times F)}$

For the necessity of the Birman-Solomyak condition, if $\mathcal{T}_\phi^{E,F}$ is bounded from $\mathcal{L}_1$ to $\mathcal{L}_1$, then Theorem 3.4.7 implies that $\sup_{\mathcal{I},\mathcal{J}} \|[\phi]_{\mathcal{I}\times\mathcal{J}}\|_{\mathrm{m}} < \infty$, Finally, Lemma 3.4.3 yields the conclusion that $\phi \in \mathfrak{BS}(E\times F)$, with the appropriate norm bound. □

3.5 Final comments

Note that in the proof of Theorem 3.4.3 we have used the σ-finiteness of λ and μ in a number of places (at the very least, σ-finiteness is used to ensure that $\lambda \times \mu$ is well defined). However there is good reason to try and work in a slightly more general situation. For example we can also prove that if $\{\phi(t,s)\}_{t\in T, s\in S}$ is a bounded function, where T and S are arbitrary sets, and

$$\sup_{|T_0|<\infty, |S_0|<\infty} \|\{\phi(t,s)\}_{t\in T_0, s\in S_0}\|_{\mathrm{m}} < \infty$$

then $\phi \in \mathfrak{BS}(S\times T)$. This follows from a version of Theorem 3.4.3 stated for S and T considered as discrete measures. If S and T are uncountable then these measures are not σ-finite, however the proof of Theorem 3.4.3 can easily be reworded to suit this situation.

It is worth remarking that Peller's theorem does not in fact give necessary conditions for $\phi \in B(\mathbb{R}^2)$ such that $\mathcal{T}_\phi^{A,B}$ is bounded from $\mathcal{L}_1$ to $\mathcal{L}_1$ for *all* affiliated operators A and B. If $\phi \in B(\mathbb{R}^2)$ is such that $\mathcal{T}_\phi^{A,B}$ is bounded from $\mathcal{L}_1$ to $\mathcal{L}_1$ for all affiliated operators A and B, then Peller's theorem states that for each pair of measures λ_A and λ_B equicontinuous with respect to the spectral measures of A and B, we have that $\phi \in \mathfrak{BS}(\lambda_A \times \lambda_B)$. Note however that the Birman-Solomyak decomposition necessarily depends on the choice of measures λ_A and λ_B.

In the following chapter, we will develop a theory of double operator integration based on the *a priori* assumption that the symbol is in the Birman-Solomyak class.

Chapter 4

Weak*-integration and the difference of powers formula

This section is devoted to a proof of a useful integral representation for the difference of complex powers of two non-negative operators. For lack of a better name, let us call it the "difference of powers" formula (Theorem 4.2.1).

The difference of powers formula originally appeared in a particular special case as [35, Lemma 5.2], and was later strengthened in [128]. In Chapter 5, we will put the formula to use to prove the conformal trace theorem for Julia sets, and then in Chapter 7 we will highlight the application to quantum differentiability.

We begin with a section on certain technical preliminaries concerning so-called weak*-integration (Section 4.1). This includes the definition of $\mathcal{T}^{A,B}_{\phi,\mathcal{M}}$, an alternative definition of a double operator integral transformer (Definition 4.1.9). Section 4.2 contains the proof of the difference of powers formula. Finally we give two applications of the formula in Section 4.3.

4.1 Preliminaries

In Chapter 2, we discussed at length the definition of a double operator integral as a spectral integral on the Hilbert space $L_2(\tau)$ corresponding to a von Neumann algebra $\mathcal{M}$ with semifinite normal trace τ. This is a definition that is particularly well-suited to the study of double operator integrals on ideals whose intersection with $L_2(\tau)$ is dense: for example on the ideal $L_1(\tau)$. It is possible to extend to other ideals using duality, interpolation or weak*-density, however it is preferable to have a direct definition.

In this chapter we present an alternative definition of a double operator integral better suited to study double operator integrals on $\mathcal{M}$, which we denote $\mathcal{T}^{A,B}_{\phi,\mathcal{M}}$. While this definition is particularly well suited for later applications, it has a limitation in the fact that it presently only applies to those von Neumann algebras $\mathcal{M}$ which are σ-finite. This is an observation which has at times gone unobserved in the literature, for example it was apparently unnoticed in [107]. The recently submitted paper [45] was an inspiration for this section, although the proofs provided here are different and have not previously appeared in writing.

The definition of $\mathcal{T}^{A,B}_{\phi,\mathcal{M}}$ is inspired by Peller's Theorem. In Chapter 3 (specifically, Theorems 3.4.3 and 3.4.5) we proved that a double operator integral $\mathcal{T}^{A,B}_{\phi}$ is continuous on $\mathcal{L}_1(H)$ in the $\mathcal{L}_1$-norm if and only if its symbol ϕ admits a representation in terms of a σ-finite measure space (Ω, μ) of the form:

$$\phi(t,s) = \int_\Omega \alpha(t,\omega)\beta(s,\omega)\,d\mu(\omega), \quad t,s \in \mathbb{R}$$

pointwise almost everywhere relative to the spectral measures of A and B, and where

$$\int_\Omega \operatorname*{esssup}_{t\in\mathbb{R}}|\alpha(t,\omega)|\operatorname*{esssup}_{s\in\mathbb{R}}|\beta(s,\omega)|\,d\mu(\omega) < \infty.$$

To define $\mathcal{T}^{A,B}_{\phi,\mathcal{M}}$, we exclusively consider symbols of the above form. We shall define $\mathcal{T}^{A,B}_{\phi,\mathcal{M}}$ on $x \in \mathcal{M}$ as

$$\mathcal{T}^{A,B}_{\phi,\mathcal{M}}(x) := \int_\Omega \alpha(A,\omega)x\beta(B,\omega)\,d\mu(\omega). \tag{4.1.1}$$

For this to be a meaningful definition, we must discuss the sense in which the above right hand side converges. Since von Neumann algebras are typically non-separable, the theory of Bochner integration turns out to be inadequate to develop the theory in full generality. It is better to understand the integral in a weak sense, and for us the theory of Gel'fand or weak*-integration is the most suitable.

4.1.1 Weak*-integration

Let X be a Banach space, and let (Ω, Σ, μ) be a measure space, where there is no need to assume that μ is positive. The theory of weak*-integration concerns defining integrals of X^*-valued functions $f : \Omega \to X^*$ which converge in the weak*-topology. According to at least some authors, this is called a Gel'fand integral [42, Chapter II, Section 3].

We shall denote the dual pairing between X and X^* as:

$$(z, x) := z(x), \quad z \in X^*, x \in X.$$

Definition 4.1.1. *If X is a Banach space, and (Ω, Σ, μ) a σ-finite measure space, then a function $f : \Omega \to X^*$ is called weak*-measurable if for all $x \in X$, the scalar-valued function $(f, x) := \omega \mapsto (f(\omega), x)$ is measurable. Furthermore we say that f is weak*-integrable if $(f, x) \in L_1(\Omega, \mu)$ for all $x \in X$.*

The following is adapted from [42, Chapter II, Section 3, Lemma 1]:

Lemma 4.1.2. *Let $f : \Omega \to X^*$ be weak*-integrable, where (Ω, Σ, μ) is a measure space with $\mu(\Omega) < \infty$. There exists a unique $I_f \in X^*$ such that for all $x \in X$ we have:*

$$I_f(x) = \int_\Omega (f(\omega), x)\,d\mu(\omega).$$

Proof. Consider the linear mapping $T : X \to L_1(\Omega, \mu)$ defined by:

$$T(x) = (f, x).$$

The mapping T is indeed linear and well defined by the assumption that f is weak*-integrable. We will show that T is in fact bounded using the closed graph theorem. Since T is everywhere defined, it suffices to show that T is closed. Suppose that $\{x_n\}_{n\geq 0}$ is a sequence converging in X to $x \in X$ and such that $T(x_n)$ converges in $L_1(\Omega, \mu)$ to some $g \in L_1(\Omega, \mu)$,

$$x_n \to x, \quad T(x_n) \to g.$$

Then since $T(x_n) \to g$ in $L_1(\Omega, \mu)$ and μ is finite, there is a subsequence $\{T(x_{n_k})\}_{k\geq 0}$ which converges pointwisely μ-almost everywhere to g. But for fixed $\omega \in \Omega$, we have:

$$\lim_{n\to\infty} (f(\omega), x_n) = (f(\omega), x)$$

since $f(\omega) \in X^*$ is a continuous linear functional. Hence $T(x_n)$ converges everywhere to the function (f, x) and has a subsequence which converges pointwise almost everywhere to g, and thus $(f, x) = g$ pointwise μ-almost everywhere. Thus T is closed and everywhere defined, and hence bounded. Hence,

$$\left| \int_\Omega (f, x)\, d\mu \right| \leq \|(f, x)\|_{L_1(\Omega)} \leq \|T\|_{X\to L_1} \|x\|_X.$$

Thus the functional I_f on X defined by

$$I_f(x) = \int_\Omega (f, x)\, d\mu$$

is continuous, and thus an element of X^*. □

Note that the finiteness of $\mu(\Omega)$ was used in an essential way in the above proof in order to extract a pointwise-almost everywhere convergent subsequence of $\{T(x_n)\}_{n\geq 0}$.

We now define the weak*-integral.

Definition 4.1.3. *Let $f : \Omega \to X^*$ be a weak* integrable function. If there exists an element $I_f \in X^*$ such that for all $x \in X$ we have:*

$$(I_f, x) = \int_\Omega (f(\omega), x)\, d\mu(\omega).$$

then called I_f the weak-integral of f, and denote:*

$$I_f := \int_\Omega f\, d\mu.$$

It is clear that I_f is necessarily unique. Lemma 4.1.2 states that weak*-integrable functions on spaces of finite measure have weak*-integrals.

Let (Ω, Σ, μ) be a measure space where μ is a positive measure, and let $h : \Omega \to [0, \infty]$ be an arbitrary (possibly non-measurable) function. Recall that the lower Lebesgue integral of h is defined as the extended real number:

$$\int_\Omega h\, \underline{d\mu} := \sup\{\int_\Omega g\, d\mu \ : \ 0 \leq g \leq h,\ g \text{ is measurable }\}.$$

The lower Lebesgue integral is not necessarily additive, but it is easily seen to be monotone and positively homogeneous.

Instead of assuming weak*-integrability, we can show existence of the weak*-integral under the assumption that $\int_\Omega \|f(\omega)\|_{X^*} \,\underline{d|\mu|(\omega)}$ is finite. Note that in contrast to Lemma 4.1.2, the following lemma does not require $\mu(\Omega) < \infty$.

Lemma 4.1.4. *Suppose that (Ω, Σ, μ) is σ-finite, and let $f : \Omega \to X^*$ be weak*-measurable. If:*

$$\int_\Omega \|f(\omega)\|_{X^*} \,\underline{d|\mu|(\omega)} < \infty$$

then f is weak-integrable and the weak*-integral satisfies:*

$$\left\| \int_\Omega f \, d\mu \right\|_{X^*} \leq \int_\Omega \|f(\omega)\|_{X^*} \,\underline{d|\mu|(\omega)}.$$

Proof. Let $x \in X$. Then since

$$|(f(\omega), x)| \leq \|x\|_X \|f(\omega)\|_{X^*}$$

and $(f(\omega), x)$ is measurable by assumption, the monotonicity of the lower Lebesgue integral implies:

$$\left| \int_\Omega (f(\omega), x) \, d\mu(\omega) \right| \leq \int_\Omega |(f(\omega), x)| \, d|\mu|(\omega) \leq \int_\Omega \|x\|_X \|f(\omega)\|_{X^*} \,\underline{d|\mu|(\omega)}. \tag{4.1.2}$$

Thus f is weak*-integrable. Then same is true for $f\chi_{\Omega'}$ for any measurable subset $\Omega' \subseteq \Omega$, and Lemma 4.1.2 implies that the weak* integral $\int_{\Omega'} f \, d\mu$ is uniquely defined for each $\Omega' \subseteq \Omega$ with $\mu(\Omega') < \infty$.

Since the lower Lebesgue integral is positively homogeneous, we may take out $\|x\|_X$ in (4.1.2) to obtain the following norm bound for any subset $\Omega' \subseteq \Omega$ of finite μ-measure:

$$\left| \int_{\Omega'} (f(\omega), x) \, d\mu(\omega) \right| \leq \|x\|_X \int_{\Omega'} \|f(\omega)\|_{X^*} \,\underline{d|\mu|(\omega)}.$$

By the definition of the weak*-integral, the above inequality can be restated as:

$$\left| \left(\int_{\Omega'} f \, d\mu, x \right) \right| \leq \|x\|_X \int_{\Omega'} \|f(\omega)\|_{X^*} \,\underline{d|\mu|(\omega)}.$$

Taking the supremum over $x \in X$ with $\|x\|_X \leq 1$ yields the norm bound:

$$\left\| \int_{\Omega'} f \, d\mu \right\|_{X^*} \leq \int_{\Omega'} \|f(\omega)\|_{X^*} \,\underline{d|\mu|(\omega)} \leq \int_\Omega \|f(\omega)\|_{X^*} \,\underline{d|\mu|(\omega)}. \tag{4.1.3}$$

Since (Ω, Σ, μ) is σ-finite, there exists a sequence $\Omega_0 \subseteq \Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega$ such that $\bigcup_{n \geq 0} \Omega_n = \Omega$ and each Ω_n has finite μ-measure. Then (4.1.3) implies that the sequence:

$$I_n := \int_{\Omega_n} f \, d\mu \in X^*$$

is uniformly bounded in the norm of X^*. From the Banach-Alaoglu theorem, it follows that the sequence $\{I_n\}_{n \geq 0}$ has a limit point I in the weak*-topology of X^*. However since (f, x) is integrable for each $x \in X$, the dominated convergence theorem implies that for all $x \in X$ we have:

$$\lim_{n\to\infty} (I_n, x) = \lim_{n\to\infty} \int_{\Omega_n} (f, x)\, d\mu = \int_{\Omega} (f, x)\, d\mu.$$

Hence the weak*-limit point I satisfies:

$$(I, x) = \int_{\Omega} (f, x)\, d\mu, \qquad \text{for all } x \in X.$$

Thus I is the integral of f, and satisfies the stated norm bound.

□

4.1.2 Weak*-integration in von Neumann algebras

Let $\mathcal{M}$ be a von Neumann algebra, with semifinite faithful normal trace τ. We will apply the theory of weak* integration developed in the preceding subsection to the pair (X, X^*), where we take $X = L_1(\mathcal{M}, \tau)$ and identify $\mathcal{M}$ with the dual of $L_1(\mathcal{M}, \tau)$ according to the dual pairing:

$$(z, x) := \tau(zx), \quad z \in \mathcal{M}, x \in L_1(\tau).$$

In this setting, for a function $f : \Omega \to \mathcal{M}$ to be weak* measurable means that for all $x \in L_1(\mathcal{M}, \tau)$, the function $\omega \mapsto \tau(xf(\omega))$ is measurable in (Ω, Σ, μ), and similarly to be weak*-integrable means that $\omega \mapsto \tau(xf(\omega))$ is in $L_1(\Omega, \mu)$ for all $x \in L_1(\mathcal{M}, \tau)$. We assume in the sequel that (Ω, Σ, μ) is σ-finite, since it will be necessary to apply Fubini's theorem.

According to Lemma 4.1.4, the assumption that $\int_\Omega \|f\|_{\mathcal{M}} \underline{d|\mu|} < \infty$ is sufficient for there to exist a weak*-integral $\int_\Omega f\, d\mu \in \mathcal{M}$ and such that for all $x \in L_1(\mathcal{M}, \tau)$ we have:

$$\tau(x \int_\Omega f\, d\mu) = \int_\Omega \tau(xf(\omega))\, d\mu(\omega).$$

The following lemma is routine, and so the proof is omitted.

Lemma 4.1.5. *Let $f : \Omega \to \mathcal{M}$, and let $x \in \mathcal{M}$.*

(i) *If f is weak* measurable (resp. integrable), then $\omega \mapsto f(\omega)^*$ is weak*-measurable (resp. integrable),*

(ii) *If f is weak* measurable (resp. integrable), then $\omega \mapsto xf(\omega)$ and $\omega \mapsto f(\omega)x$ are weak*-measurable (resp. integrable),*

(iii) *If f is weak* measurable, and $g : \Omega \to \mathbb{C}$ is a measurable scalar valued function, then $\omega \mapsto f(\omega)g(\omega)$ is weak*-measurable.*

Let us now verify that the integrand appearing in (4.1.1) is weak*-measurable.

Lemma 4.1.6. *Let (Ω,Σ,μ) be a measure space, and let α,β be bounded measurable functions on $\mathbb{R}\times\Omega$ such that $\sup_{t\in\mathbb{R}}|\alpha(t,\omega)|<\infty$ and $\sup_{t\in\mathbb{R}}|\beta(t,\omega)|<\infty$ for each $\omega\in\Omega$.*

Let $x\in\mathcal{M}$ and A,B be two self-adjoint operators affiliated with $\mathcal{M}$. We define,

$$\alpha(A,\omega):=\int_{\mathbb{R}}\alpha(t,\omega)\,dE_A(t),\quad \beta(B,\omega)=\int_{\mathbb{R}}\beta(s,\omega)\,dE_B(s).$$

Then the $\mathcal{M}$-valued function on Ω

$$\omega\mapsto\alpha(A,\omega)x\beta(B,\omega)$$

is weak-measurable.*

Proof. Let us first remark that if α and β are functions of the form,

$$\alpha(t,\omega)=\chi_{A_1}(t)\chi_{A_2}(\omega),\quad \beta(t,\omega)=\chi_{B_1}(t)\chi_{B_2}(\omega) \tag{4.1.4}$$

for some measurable sets $A_1,B_1\subseteq\mathbb{R}$ and $A_2,B_2\subseteq\Omega$, then

$$\alpha(A,\omega)x\beta(B,\omega)=\chi_{A_1}(A)x\chi_{B_1}(B)\chi_{A_2\cap B_2}(\omega)$$

and so the weak*-measurability of $\alpha(A,\omega)x\beta(B,\omega)$ follows in this case from parts (ii) and (iii) of Lemma 4.1.5. Similarly, if α and β are linear combinations of functions of the form (4.1.4) then the measurability follows.

Suppose that $u,v\in L_2(\tau)$. Recall (from Lemma 2.3.2) that A and B define operators L_A and L_B of left multiplication on $L_2(\tau)$, and it can be easily seen that:

$$\alpha(A,\omega)u=\alpha(L_A,\omega)u,\quad \beta(B,\omega)v=\beta(L_B,\omega)v$$

where $\alpha(L_A,\omega)$ and $\beta(L_B,\omega)$ may be defined in terms of the spectral measures of L_A and L_B on $L_2(\tau)$. So we may write $\tau(u^*\alpha(A,\omega)x\beta(B,\omega)v)$ as an inner product in the space $L_2(\tau)$,

$$\tau(u^*\alpha(A,\omega)x\beta(B,\omega)v)=\langle\overline{\alpha(L_A,\omega)}u,x\beta(L_B,\omega)v\rangle.$$

Every measurable function on the product space $\mathbb{R}\times\Omega$ may be obtained as a pointwise limit of linear combinations of functions of the form (4.1.4), so suppose that $\{\alpha_n\}_{n\geq1}$ and $\{\beta_n\}_{n\geq1}$ are linear combinations of functions of the form (4.1.4) such that $\alpha_n\to\alpha$ and $\beta_n\to\beta$ pointwisely. We may assume that $\sup_{n\geq0}\sup_{t\in\mathbb{R}}|\alpha_n(t,\omega)|<\infty$ for all ω, and similarly with $\{\beta_n\}_{n=0}^\infty$. Appealing to Corollary 2.2.11, it follows that $\overline{\alpha_n}(L_A,\omega)u\to\overline{\alpha}(L_A,\omega)u$ and $\beta_n(L_B,\omega)v\to\beta(L_B,\omega)v$ in $L_2(\tau)$. Thus,

$$\lim_{n\to\infty}\tau(u^*\alpha_n(A,\omega)x\beta_n(B,\omega)v)=\tau(u^*\alpha(A,\omega)x\beta(B,\omega)v).$$

Therefore $\omega\mapsto\tau(u^*\alpha(A,\omega)x\beta(B,\omega)v)$ is a pointwise limit of measurable functions, and is thus measurable. Since every $z\in L_1(\tau)$ can be factorised as $z=vu^*$ for some $u,v\in L_2(\tau)$, it follows that $\omega\mapsto\tau(z\alpha(A,\omega)x\beta(B,\omega))$ is measurable for all $z\in L_1(\tau)$. □

Now we may discuss the connection to double operator integration theory. Recall Theorem 3.4.5: if a function ϕ is in the Birman-Solomyak class, then the double operator integral transformer $\mathcal{T}_\phi^{A,B}$ admits an extension to a bounded linear mapping from $L_1(\tau)$ to $L_1(\tau)$. With the theory of weak*-integration, we can shed additional light on this result.

Theorem 4.1.7. *Let $x, y \in L_1(\tau) \cap \mathcal{M}$, and let A, B be self-adjoint operators affiliated with $\mathcal{M}$. If ϕ is a function admitting the representation:*

$$\phi(t,s) = \int_\Omega \alpha(t,\omega)\beta(s,\omega)\,d\mu(\omega)$$

for almost all $t, s \in \mathbb{R}$ relative to the joint spectral measures of A and B and a σ-finite measure space (Ω, μ), and we have

$$\int_\Omega \operatorname*{esssup}_{t\in\mathbb{R}} |\alpha(t,\omega)| \operatorname*{esssup}_{s\in\mathbb{R}} |\beta(s,\omega)|\,d|\mu|(\omega) < \infty$$

then

$$\tau(y^*\mathcal{T}_\phi^{A,B}(x)) = \int_\Omega \tau(y^*\alpha(A,\omega)x\beta(B,\omega))\,d\mu(\omega) = \tau(y^* \int_\Omega \alpha(A,\omega)x\beta(B,\omega)\,d\mu(\omega))$$

where the latter integral is a weak integral.*

Proof. The assumption that:

$$\int_\Omega \sup_{t\in\mathbb{R}} |\alpha(t,\omega)| \sup_{s\in\mathbb{R}} |\beta(s,\omega)|\,d|\mu|(\omega) < \infty$$

implies:

$$\int_\Omega \|\alpha(A,\omega)\|_\mathcal{M} \|\beta(B,\omega)\|_\mathcal{M}\,\underline{d|\mu|(\omega)} < \infty$$

and hence Lemma 4.1.4 implies that $\omega \mapsto \alpha(A,\omega)x\beta(B,\omega)$ is weak*-integrable, and so the weak*-integral indeed exists.

By definition of the weak*-integral, we then have:

$$\int_\Omega \tau(y^*\alpha(A,\omega)x\beta(B,\omega))\,d\mu(\omega) = \tau(y^* \int_\Omega \alpha(A,\omega)x\beta(B,\omega)\,d\mu(\omega)).$$

As was already noted in the proof of Theorem 3.4.5, we also have:

$$\tau(y^*\mathcal{T}_\phi^{A,B}(x)) = \int_\Omega \tau(y^*\alpha(A,\omega)x\beta(B,\omega))\,d\mu(\omega)$$

and this completes the proof. □

One of the more subtle consequences of Theorem 4.1.7 is that when $x \in L_1(\tau) \cap \mathcal{M}$, the value of the weak*-integral

$$\int_\Omega \alpha(A,\omega)x\beta(B,\omega)\,d\mu(\omega)$$

does not depend on α, β and μ specifically, but instead depends only on ϕ, A, B and x.

It is tempting to think that the same will hold not only for $x \in L_1(\tau) \cap \mathcal{M}$, but for all $x \in \mathcal{M}$. It is possible to prove that this is the case – provided that the von Neumann algebra $\mathcal{M}$ is σ-finite.

Corollary 4.1.8. *Assume that $(\mathcal{M}, \tau)$ is σ-finite. Let α and β and ϕ satisfy the same conditions as in Theorem 4.1.7 and let $x \in \mathcal{M}$. Then the weak*-integral*

$$\int_\Omega \alpha(A,\omega)x\beta(B,\omega)\,d\mu(\omega)$$

exists, and depends only on ϕ, A, B and x, and not the representation of ϕ in terms of α, β and μ.

Proof. The existence of the weak*-integral follows from the assumption that

$$\|\phi\|_{\mathfrak{B}\mathfrak{S}(E_A\times E_B)} = \int_\Omega \operatorname*{esssup}_{t\in\mathbb{R}}|\alpha(t,\omega)|\operatorname*{esssup}_{s\in\mathbb{R}}|\beta(s,\omega)|\,d|\mu|(\omega) < \infty$$

combined with Lemma 4.1.4.

Let T denote the linear map from $\mathcal{M}$ to $\mathcal{M}$,

$$T(x) := \int_\Omega \alpha(A,\omega)x\beta(B,\omega)\,d\mu(\omega)$$

Lemma 4.1.4 then implies that T is bounded in the norm topology of $\mathcal{M}$ with norm at most $\|\phi\|_{\mathfrak{B}\mathfrak{S}}$.

Since we have made the extra assumption that $\mathcal{M}$ is σ-finite, there exists a sequence $\{x_n\}_{n\geq 0} \subset L_1(\tau)\cap\mathcal{M}$ such that $x_n \to x$ in the weak*-topology and $\sup_{n\geq 0}\|x_n\|_{\mathcal{M}} < \infty$. Let us show that $T(x_n) \to T(x)$ in the weak*-topology. By definition, for all $z \in L_1(\tau)$ we have:

$$\tau(zT(x_n)) = \int_\Omega \tau(z\alpha(A,\omega)x_n\beta(B,\omega))\,d\omega.$$

Since $x_n \to x$ in the weak*-sense, for each $\omega \in \Omega$ we have:

$$\lim_{n\to\infty}\tau(z\alpha(A,\omega)x_n\beta(B,\omega)) = \tau(z\alpha(A,\omega)x\beta(B,\omega)).$$

Moreover for each ω the sequence $\{\tau(z\alpha(A,\omega)x_n\beta(B,\omega))\}_{n\geq 0}$ is uniformly bounded by the function

$$\sup_{t\in\mathbb{R}}|\alpha(t,\omega)|\sup_{s\in\mathbb{R}}|\beta(s,\omega)|$$

which is integrable by assumption.

Thus by Lebesgue's dominated convergence theorem:

$$\begin{aligned}
\lim_{n\to\infty}\tau(zT(x_n)) &= \lim_{n\to\infty}\int_\Omega \tau(z\alpha(A,\omega)x_n\beta(B,\omega))\,d\mu(\omega)\\
&= \int_\Omega \lim_{n\to\infty}\tau(z\alpha(A,\omega)x_n\beta(B,\omega))\,d\mu(\omega)\\
&= \int_\Omega \tau(z\alpha(A,\omega)x\beta(B,\omega))\,d\mu(\omega)\\
&= \tau(zT(x)).
\end{aligned}$$

That is, $T(x_n) \to T(x)$ in the weak*-sense.

Let $\widetilde{\phi}$ be the function obtained by exchanging the variables of ϕ, that is, define $\widetilde{\phi}(t,s) = \phi(s,t)$. Note that $\|\widetilde{\phi}\|_{\mathfrak{BS}} = \|\phi\|_{\mathfrak{BS}}$.

Let $y \in L_1(\tau) \cap \mathcal{M}$. Theorem 4.1.7 yields:

$$\tau(y^*T(x_n)) = \tau(y^* \int_\Omega \alpha(A,\omega)x_n\beta(B,\omega)\,d\mu(\omega)) = \tau(y^*\mathcal{T}_\phi^{A,B}(x_n)) = \tau(\mathcal{T}_{\widetilde{\phi}}^{B,A}(y^*)x_n).$$

Taking the limit as $n \to \infty$,

$$\tau(y^*T(x)) = \tau(\mathcal{T}_{\widetilde{\phi}}^{B,A}(y^*)x), \quad y \in L_1(\tau) \cap \mathcal{M}.$$

Since $L_1(\tau) \cap \mathcal{M}$ is dense in $L_1(\tau)$, it follows that $T(x)$ is uniquely determined by the mapping $y \mapsto \tau(y^*T(x))$ for $y \in L_1(\tau) \cap \mathcal{M}$. However the above right hand side depends only on ϕ, A, B, y and x, and does not depend on the choice of representation of ϕ. $\square$

Since the value of $\int_\Omega \alpha(A,\omega)x\beta(B,\omega)\,d\omega$ does not depend on the specific choice of representation of the function ϕ, we are now permitted to give the following definition:

Definition 4.1.9. *Suppose that $(\mathcal{M},\tau)$ is σ-finite, and let $\phi \in \mathfrak{BS}(\mathbb{R}^2)$ admit a representation as in Theorem 4.1.7. Define $\mathcal{T}_{\phi,\mathcal{M}}^{A,B}$ on $x \in \mathcal{M}$ as the weak*-integral:*

$$\mathcal{T}_{\phi,\mathcal{M}}^{A,B}(x) = \int_\Omega \alpha(A,\omega)x\beta(B,\omega)\,d\mu(\omega).$$

Note that unlike Definition 2.4.1, we have stated the definition in terms of two self-adjoint operators A and B rather than two spectral measures. There is no substantial technical difference between the two ways of writing the definition, however here it is notationally slightly more convenient to define the double operator integral transformer in terms of operators rather than spectral measures.

The following properties of $\mathcal{T}_{\phi,\mathcal{M}}^{A,B}$ follow in a straightforward manner from the definition, Corollary 4.1.8 and Theorem 3.4.5.

Theorem 4.1.10. *Let $\phi, \psi \in \mathfrak{BS}(E_A \times E_B)$. Then we have:*

1. $\mathcal{T}_{\phi\psi,\mathcal{M}}^{A,B} = \mathcal{T}_{\phi,\mathcal{M}}^{A,B}\mathcal{T}_{\psi,\mathcal{M}}^{A,B}$*,*
2. $\mathcal{T}_{\psi+\phi,\mathcal{M}}^{A,B} = \mathcal{T}_{\phi,\mathcal{M}}^{A,B} + \mathcal{T}_{\psi,\mathcal{M}}^{A,B}$*.*
3. $\mathcal{T}_{\phi,\mathcal{M}}^{A,B}$ *is a bounded linear map from $\mathcal{M}$ to $\mathcal{M}$, and can be extended to a bounded linear map from $L_1(\tau)$ to $L_1(\tau)$, in both cases having norm at most $\|\phi\|_{\mathfrak{BS}}$.*

Proof. We assume that ϕ and ψ have Birman-Solomyak representations:

$$\phi(t,s) = \int_{\Omega_0} \alpha_0(t,\omega)\beta_0(s,\omega)\,d\sigma_0(\omega), \quad \psi(t,s) = \int_{\Omega_1} \alpha_1(t,\omega)\beta_1(s,\omega)\,d\sigma_1(\omega).$$

To prove (1), we recall that since the measures σ_0 and σ_1 are σ-finite, Fubini's theorem is applicable and we have the following Birman-Solomyak representation of $\phi\psi$:

$$(\phi\psi)(t,s) = \int_{\Omega_0\times\Omega_1} \alpha_0(t,\omega_0)\alpha_1(t,\omega_1)\beta_0(s,\omega_0)\beta_1(s,\omega_1)\,d\sigma(\omega_0,\omega_1).$$

where $\sigma = \sigma_0 \times \sigma_1$ is the product measure. Thus for $x \in \mathcal{M}$ and $z \in L_1(\tau)$, we have:

$$\tau(z\mathcal{T}^{A,B}_{\phi\psi,\mathcal{M}}(x)) = \int_{\Omega_0\times\Omega_1} \tau(z\alpha_0(A,\omega_0)\alpha_1(A,\omega_1)x\beta_0(B,\omega_0)\beta_1(B,\omega_1))\,d\sigma(\omega_0,\omega_1).$$

On the other hand,

$$\begin{aligned}
\tau(z\mathcal{T}^{A,B}_{\phi,\mathcal{M}}(\mathcal{T}^{A,B}_{\psi,\mathcal{M}}(x))) &= \int_{\Omega_0} \tau(z\alpha_0(A,\omega_0)\mathcal{T}^{A,B}_{\psi,\mathcal{M}}(x)\beta_0(B,\omega_0))\,d\sigma_0(\omega_0)\\
&= \int_{\Omega_0} \tau(\beta_0(B,\omega_0)z\alpha_0(A,\omega_0)\mathcal{T}^{A,B}_{\psi,\mathcal{M}}(x))\,d\sigma_0(\omega_0)\\
&= \int_{\Omega_0}\int_{\Omega_1} \tau(\beta_0(B,\omega_0)z\alpha_0(A,\omega_0)\alpha_1(A,\omega_1)x\beta_1(B,\omega_1))\,d\sigma_1(\omega_1)d\sigma_0(\omega_0).
\end{aligned}$$

Thus (1) follows from Fubini's theorem.

To prove (2), one can give $\phi+\psi$ a Birman-Solomyak representation on the disjoint union measure space $\Omega_0 \oplus \Omega_1$. The details are elementary and so we omit this proof.

To prove the $\mathcal{M} \to \mathcal{M}$ boundedness component of (3), this follows from (4.1.4) and the definition of the Birman-Solomyak norm (we also noted this in the proof of Corollary 4.1.8). Finally, the $L_1(\tau)$-component of (3) is essentially the result of Theorem 3.4.5, since $L_1(\tau)\cap\mathcal{M} \subseteq L_2(\tau)\cap L_1(\tau)$, and on $L_2(\tau)\cap L_1(\tau)$ the transformer $\mathcal{T}^{A,B}_{\phi,\mathcal{M}}$ coincides with $\mathcal{T}^{A,B}_{\phi}$, thanks to Theorem 4.1.7. □

4.2 Proof of the difference of powers formula

Let us assume henceforth that $(\mathcal{M},\tau)$ is a σ-finite von Neumann algebra.

For an integrable function h on $\mathbb{R}$, we let $\widehat{h}$ denote the (rescaled) Fourier transform:

$$\widehat{h}(s) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-ist}h(t)\,dt.$$

With this particular convention, the Fourier inversion theorem takes the form:

$$h(t) = \int_{-\infty}^{\infty} \exp(ist)\widehat{h}(s)\,ds$$

(under suitable assumptions on h).

In what follows, A and B will be non-negative elements of $\mathcal{M}$, and we adopt the (unusual) convention that $0^{is} = 0$ for all $s \in \mathbb{R}$, including $s = 0$.

For a self-adjoint element $B \in \mathcal{M}$, we denote by $\mathrm{supp}(B)$ the support projection of B, that is the maximal projection $p \in \mathcal{M}$ such that $Bp = B$. Equivalently, $\mathrm{supp}(B) =$

$\chi_{\mathbb{R}\setminus\{0\}}(B)$. Due to our convention that $0^z = 0$ for all $\Re(z) \geq 0$, the operator B^z is given by:

$$B^z := \int_{\mathrm{Spec}(B)\setminus\{0\}} \lambda^z dE_B(\lambda), \quad \Re(z) \geq 0.$$

where E_B is the $\mathcal{P}(\mathcal{M})$-valued spectral measure of B. This ensures that $0^z = 0$ as required.

Theorem 4.2.1 (Difference of powers formula)**.** *Let $A, B \in \mathcal{M}$ be positive, and let*

$$C := A^{1/2}BA^{1/2}.$$

Let $z \in \mathbb{C}$ be in the half-plane $\Re(z) > 1$, and define the mapping $T_z : \mathbb{R} \to \mathcal{M}$ as:

$$T_z(0) := B^{z-1}[BA^{1/2}, A^{z-1/2}] + [BA^{1/2}, A^{1/2}]C^{z-1},$$
$$T_z(s) := B^{z-1+is}[BA^{1/2}, A^{z-1/2+is}]C^{-is} + B^{is}[BA^{1/2}, A^{1/2+is}]C^{z-1-is}, \quad s \neq 0.$$

Define the function $g_z : \mathbb{R} \to \mathbb{C}$ by

$$g_z(t) := 1 - \frac{e^{zt/2} - e^{-zt/2}}{(e^{t/2} - e^{-t/2})(e^{(z-1)t/2} + e^{-(z-1)t/2})}, \quad t \neq 0$$

with $g_z(0) := 1 - \frac{z}{2}$. Then:

(i) *For each z with $\Re(z) > 1$, the mapping $T_z : \mathbb{R} \to \mathcal{M}$ is continuous in the weak*-sense.*

(ii) *We have:*

$$B^zA^z - C^z = T_z(0) - \int_{-\infty}^{\infty} T_z(s)\widehat{g}_z(s)\,ds$$

where the integral is a weak-integral in $\mathcal{M}$.*

Theorem 4.2.1 originated in [35] in the special case where z is real and B is compact. Theorem 4.2.1 was proved in the above form as [128, Theorem 5.2.1]. In this section we will give an overview of the proof, which follows similar lines to [128] but admits certain simplifications due to our version of double operator integration theory.

The following is [128, Remark 5.22]. The proof is more tedious than insightful, and therefore is omitted.

Lemma 4.2.2. *For $\Re(z) > 1$, the function g_z is in the Schwartz class.*

Theorem 4.2.3. *Let $0 \leq X, Y \in \mathcal{M}$, $z \in \mathbb{C}$ and define $V_z := X^{z-1}(X - Y) + (X - Y)Y^{z-1}$. Then we have a weak*-integral representation:*

$$X^z - Y^z = V_z - \int_{-\infty}^{\infty} X^{is}V_zY^{-is}\widehat{g}_z(s)\,ds.$$

Proof. Let $\phi_{1,z}$ denote the function:

$$\phi_{1,z}(\lambda, \mu) = 1 - \frac{\lambda^z - \mu^z}{(\lambda - \mu)(\lambda^{z-1} + \mu^{z-1})}, \quad \lambda \neq \mu > 0.$$

Also define $\phi_{1,z}(\lambda,0) = \phi_{1,z}(0,\mu) = 0$ when $\lambda,\mu \geq 0$. When $\lambda,\mu > 0$, then $\phi_{1,z}(\lambda,\mu) = g_z(\log(\lambda/\mu))$.

Let $t = \log(\lambda/\mu)$. Then by the Fourier inversion theorem:

$$\phi_{1,z}(\lambda,\mu) = \int_{-\infty}^{\infty} \widehat{g}_z(s)e^{its}\,ds = \int_{-\infty}^{\infty} \widehat{g}_z(s)\lambda^{is}\mu^{-is}\,ds, \quad \lambda \neq \mu \geq 0.$$

The above identity is also easily verified when either λ or μ is zero due to our standing assumption that $0^{is} = 0$.

Since $\widehat{g}_z \in L_1(\mathbb{R})$, it follows that the measure $\widehat{g}_z dt$ has finite total variation, and so the above furnishes a Birman-Solomyak representation of $\phi_{1,z}$. That is, $\phi_{1,z} \in \mathfrak{BS}(E^X \times E^Y)$, and therefore for all $T \in \mathcal{M}$ we have the weak*-integral representation:

$$\mathcal{T}^{X,Y}_{\phi_{1,z},\mathcal{M}}(T) = \int_{-\infty}^{\infty} \widehat{g}_z(s)X^{is}TY^{-is}\,ds \in \mathcal{M}. \tag{4.2.1}$$

Now define:

$$\phi_{2,z}(\lambda,\mu) := (\lambda^{z-1}+\mu^{z-1})(\lambda-\mu), \quad \lambda,\mu \geq 0$$

so that

$$\mathcal{T}^{X,Y}_{\phi_{2,z},\mathcal{M}}(1) = X^{z-1}(X-Y)+(X-Y)Y^{z-1} = V_z.$$

However:

$$\phi_{1,z}(\lambda,\mu)\phi_{2,z}(\lambda,\mu) = \phi_{2,z}(\lambda,\mu) - (\lambda^z - \mu^z).$$

and thus Theorem 4.1.10 implies

$$\mathcal{T}^{X,Y}_{\phi_{1,z},\mathcal{M}}(\mathcal{T}^{X,Y}_{\phi_{2,z},\mathcal{M}}(1)) = \mathcal{T}^{X,Y}_{\phi_{2,z},\mathcal{M}}(1) - (X^z - Y^z)$$

That is,

$$V_z - (X^z - Y^z) = \mathcal{T}^{X,Y}_{\phi_{1,z},\mathcal{M}}(V_z).$$

Putting $T = V_z$ in (4.2.1) yields the result.

□

We will now explain how to prove Theorem 4.2.1 in the special case that one of the operators has finite spectrum. Specifically, we will assume that $B = \sum_{j=1}^n \lambda_j p_j$, where $\{\lambda_1,\ldots,\lambda_n\} = \sigma(B)$ and $\{p_j\}_{j=1}^n$ are pairwise orthogonal projections in $\mathcal{M}$. After this is established, the general result will follow from an approximation argument.

Lemma 4.2.4. *Theorem 4.2.1 holds under the assumption that there exists some $\varepsilon > 0$ such that $B \geq \varepsilon$ and that the spectrum of B is finite.*

Proof. Since B is non-negative, if $\sigma(B)$ is finite then there exist pairwise orthogonal projections $\{p_j\}_{j=1}^n$ with $\sum_{j=1}^n p_j = 1$ and non-negative real numbers $\{\lambda_j\}_{j=1}^n$ such that:

$$B = \sum_{j=1}^n \lambda_j p_j.$$

Let $z \in \mathbb{C}$ have real part strictly greater than 1. Then:

$$B^z = \sum_{j=1}^{n} \lambda_j^z p_j.$$

Note that the above is consistent with our assertion that $0^{is} = 0$ for all $s \in \mathbb{R}$.

Since the sum $\sum_{j=1}^{n} \lambda_j^z p_j$ is finite and $\sum_{j=1}^{n} p_j = 1$, we have:

$$B^z A^z - C^z = \sum_{j=1}^{n} p_j(\lambda_j^z A^z - C^z).$$

We proceed by applying Theorem 4.2.3 to each summand, with $X = \lambda_j A$ and $Y = C$ for the jth summand. Let $v_{j,z} := (\lambda_j A)^{z-1}(\lambda_j A - C) + (\lambda_j A - C)C^{z-1}$. Then Theorem 4.2.3 implies that for each j we have:

$$\lambda_j^z A^z - C^z = v_{j,z} - \int_{-\infty}^{\infty} (\lambda_j A)^{is} v_{j,z} C^{-is} \widehat{g}_z(s)\, ds.$$

Then multiplying by p_j and summing over j:

$$B^z A^z - C^z = \sum_{j=1}^{n} p_j(\lambda_j^z A^z - C^z) = \sum_{j=1}^{n} p_j v_{j,z} - \int_{-\infty}^{\infty} \sum_{j=1}^{n} p_j (\lambda_j A)^{is} v_{j,z} C^{-is} \widehat{g}_z(s)\, ds. \tag{4.2.2}$$

For the first sum on the right hand side, we have:

$$\begin{aligned}
\sum_{j=1}^{n} p_j v_{j,z} &= \sum_{j=1}^{n} \left(p_j((\lambda_j A)^z - (\lambda_j A)^{z-1}C) + p_j(\lambda_j A - C)C^{z-1}\right)\\
&= B^z A^z - B^{z-1}A^{z-1}C + (BA - C)C^{z-1}\\
&= B^{z-1}(BA^z - A^{z-1}C) + (BA - C)C^{z-1}.
\end{aligned}$$

Now recalling that $C = A^{1/2}BA^{1/2}$, we have:

$$\begin{aligned}
\sum_{j=1}^{n} p_j v_{j,z} &= B^{z-1}(BA^z - A^{z-1/2}BA^{1/2}) + (BA - A^{1/2}BA^{1/2})C^{z-1}\\
&= B^{z-1}[BA^{1/2}, A^{z-1/2}] + [BA^{1/2}, A^{1/2}]C^{z-1}\\
&= T_z(0).
\end{aligned}$$

For the second sum on the right hand side of (4.2.2), we have:

$$\begin{aligned}
\sum_{j=1}^{n} p_j(\lambda_j A)^{is} v_{j,k} &= \sum_{j=1}^{n} p_j((\lambda_j A)^{z+is} - (\lambda_j A)^{z-1+is}C) + p_j((\lambda_j A)^{1+is} - (\lambda_j A)^{is}C)C^{z-1}\\
&= B^{z+is}A^{z+is} - B^{z-1+is}A^{z-1+is}C + (B^{1+is}A^{1+is} - B^{is}A^{is}C)C^{z-1}\\
&= B^{z-1+is}[BA^{1/2}, A^{z-1/2+is}] + B^{is}[BA^{1/2}, A^{1/2+is}]C^{z-1}.
\end{aligned}$$

Substituting this into (4.2.2) yields:

$$\begin{aligned} B^z A^z - C^z &= T_z(0) - \int_{-\infty}^{\infty} \widehat{g}_z(s) B^{z-1+is}[BA^{1/2}, A^{z-1/2+is}]C^{-is} + B^{is}[BA^{1/2}, A^{1/2+is}]C^{z-1-is}\, ds \\ &= T_z(0) - \int_{-\infty}^{\infty} \widehat{g}_z(s) T_z(s)\, ds \end{aligned}$$

exactly as needed. $\square$

Now we remove the assumption that the spectrum of B is finite, using an approximation argument and the continuity properties of functional calculus.

Proof of Theorem 4.2.1. Let $n \geq 0$, and let F_n denote the function:

$$F_n(t) = \sum_{k=0}^{\infty} \frac{k+1}{2^n} \chi_{(k2^{-n},(k+1)2^{-n}]}(t), \quad t \in [0, \|B\|].$$

Then,

$$B_n := F_n(B) = \sum_{k=0}^{\infty} \frac{k+1}{2^n} \chi_{(k2^{-n},(k+1)2^{-n}]}(B).$$

Since B is bounded, this sum is actually finite, and $F_n(B)$ (being a finite linear combination of orthogonal projections) has spectrum given by a finite subset of $\{2^{-n}, 2 \cdot 2^{-n}, 3 \cdot 2^{-n}, \ldots\}$. The support projection of B_n is the projection:

$$\mathrm{supp}(B_n) = \sum_{k \geq 0} \chi_{(k2^{-n},(k+1)2^{-n}]}(B) = \chi_{(0,\infty)}(B) = \mathrm{supp}(B).$$

Moreover,

$$B_n - B = \sum_{k=0}^{\infty} (\frac{k+1}{2^n} - B) \chi_{[k2^{-n},(k+1)2^{-n})}(B).$$

Since F_n converges uniformly to the identity function on $[0, \|B\|]$, the spectral theory of self-adjoint operators (or Lemma 2.2.9) implies that B_n converges to B in the operator norm topology, or equivalently in the topology of $\mathcal{M}$. Hence, the spectral measure of B_n converges weakly to that of B (Theorem 2.2.20).

Since the function $\lambda \mapsto \lambda^z$ is continuous and bounded on $[0, \|B\|]$ for $\Re(z) > 1$, the definition of weak convergence of spectral measures (Definition 2.2.19) implies that in particular we have $B_n^z \mathrm{supp}(B) \to B^z \mathrm{supp}(B)$ in the strong operator topology, and since $\mathrm{supp}(B_n) = \mathrm{supp}(B)$, we have:

$$B_n^z \mathrm{supp}(B_n) \to B^z \mathrm{supp}(B)$$

in the strong operator topology. Since strong convergence implies weak*-convergence in $\mathcal{M}$ (see Section 2.3), it follows that $B_n^z \mathrm{supp}(B_n) \to B^z \mathrm{supp}(B)$ in the weak*-topology of $\mathcal{M}$.

For each $n \geq 0$, define $C_n = A^{1/2}B_nA^{1/2}$ and:

$$T_{n,z}(0) := B_n^{z-1}[B_nA^{1/2}, A^{z-1/2}] + [B_nA^{1/2}, A^{1/2}]C_n^{z-1},$$
$$T_{n,z}(s) := B_n^{z-1+is}[B_nA^{1/2}, A^{z-1/2+is}]C_n^{-is} + B_n^{is}[B_nA^{1/2}, A^{1/2+is}]C_n^{z-1-is}, \quad s \neq 0.$$

That is, $T_{n,z} : \mathbb{R} \to \mathcal{M}$ is defined identically to T_z in Theorem 4.2.1, but with B_n and C_n in place of B and C respectively.

Note that since $B_n^z \to B^z$ in the weak*-topology, it follows easily that $T_{n,z}(s) \to T_z(s)$ in the weak*-topology.

Since B_n has finite spectrum, Lemma 4.2.4 implies that:

$$B_n^zA^z - C_n^z = T_{n,z}(0) - \int_{-\infty}^{\infty} T_{n,z}(s)\widehat{g}_z(s)\,ds. \tag{4.2.3}$$

Taking the limit in the weak*-topology as $n \to \infty$, the above left hand side converges to $B^zA^z - C^z$. As for the right hand side, since the integral is in the weak*-sense, for all $\omega \in \mathcal{M}_*$ we have:

$$\omega\left(\int_{-\infty}^{\infty} T_{n,z}(s)\widehat{g}_z(s)\,ds\right) = \int_{-\infty}^{\infty} \omega(T_{n,z}(s))\widehat{g}_z(s)\,ds.$$

We have established that $T_{n,z}(s) \to T_z(s)$ in the weak*-topology, and since evidently $\|T_{n,z}(s)\|_{\mathcal{M}}$ is uniformly bounded in s and $\widehat{g}_z$ is Schwartz class, it follows from the Lebesgue dominated convergence theorem that:

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \omega(T_{n,z}(s))\widehat{g}_z(s)\,ds = \int_{-\infty}^{\infty} \omega(T_z(s))\widehat{g}_z(s)\,ds = \omega\left(\int_{-\infty}^{\infty} T_z(s)\widehat{g}_z(s)\,ds\right).$$

so that the right hand side of (4.2.3) converges in the weak*-topology to

$$T_z(0) - \int_{-\infty}^{\infty} \widehat{g}_z(s)T_z(s)\,ds.$$

□

4.3 Applications of the difference of powers formula

The main utility of the difference of powers formula is to give sufficient conditions on A, B and $1 < r < \infty$ such that $B^rA^r - (A^{1/2}BA^{1/2})^r$ is in a desired operator ideal. Since our primary applications are to ideals of $\mathcal{B}(H)$ for a Hilbert space H, we will now restrict attention to that setting, and consider the Schatten-von Neumann ideals $\mathcal{L}_p$.

One technical result we use is the following, which is essentially [108, Equation (14)]. Suppose that $\mathcal{E}$ is an interpolation space between $\mathcal{L}_p$ and $\mathcal{L}_q$ for some $1 < p < q < \infty$. If X and Y are positive operators such that $[X, Y] \in \mathcal{E}$, and f is a Lipschitz function on $\mathbb{R}$ then

$$\|[X, f(Y)]\|_{\mathcal{E}} \leq c_{\mathcal{E}}\|f'\|_{L_\infty(\mathbb{R})}\|[X, Y]\|_{\mathcal{E}}.$$

In particular, if $f(t) = t^{1+2is}$ then f is a Lipschitz function on $\mathbb{R}$, with $|f'(t)| = |1+2is|$ for $t \in \mathbb{R}$, and therefore:

$$\|[X, Y^{1+2is}]\|_{\mathcal{E}} \leq 2c_{\mathcal{E}}(1+|s|)\|[X,Y]\|_{\mathcal{E}}. \tag{4.3.1}$$

We will use $\mathcal{E} = \mathcal{L}_{r,1}$ and $\mathcal{E} = \mathcal{L}_{r,\infty}$ where $1 < r < \infty$.

4.3.1 Sufficient conditions for a difference of powers to be trace class

The result of this subsection concerns conditions on positive bounded operators A and B so that the difference $B^rA^r - (A^{1/2}BA^{1/2})^r$ is trace class, and originally appeared in [32, Appendix B]. We will use this result in Chapter 7.

Before proving the theorem, it is worth remarking on $\mathcal{L}_1$-valued integrals. Suppose that $T : \mathbb{R} \to \mathcal{B}(H)$ is a weak*-measurable function such that for all $s \in \mathbb{R}$, the value of T at s is in $\mathcal{L}_1$. If $\int_{\mathbb{R}} \|T(s)\|_1\, ds < \infty$, then the weak*-integral is in $\mathcal{L}_1$.

To see this, note that the assumption implies that $\int_{-\infty}^{\infty} \|T(s)\|_\infty\, ds < \infty$, and therefore Theorem 4.1.4 implies that the weak*-integral exists. Therefore, there exists a polar decomposition:

$$\left|\int_{-\infty}^{\infty} T(s)\, ds\right| = u\int_{-\infty}^{\infty} T(s)\, ds$$

for some partial isometry u. Since u is bounded, it may be moved inside the weak*-integral. Then letting $X \in \mathcal{L}_1$ the definition of the weak*-integral implies that:

$$\operatorname{tr}\left(X\left|\int_{-\infty}^{\infty} T(s)\, ds\right|\right) = \int_{-\infty}^{\infty} \operatorname{tr}(XuT(s))\, ds \leq \int_{-\infty}^{\infty} \|X\|_\infty \|T(s)\|_1\, ds.$$

If one then takes the supremum over all $X \in \mathcal{L}_1$ with $\|X\|_\infty \leq 1$, it follows that the absolute value of the weak*-integral $\int_{-\infty}^{\infty} T(s)\, ds$ has finite trace, and so the weak* integral is in $\mathcal{L}_1$.

Theorem 4.3.1. *Let A and B be two positive bounded operators, and let $r > 1$. If the following four conditions hold:*

(i) $B^{r-1}A^{r-1} \in \mathcal{L}_{\frac{r}{r-1},\infty}$,

(ii) $A^{1/2}BA^{1/2} \in \mathcal{L}_{r,\infty}$,

(iii) $[BA^{1/2}, A^{1/2}] \in \mathcal{L}_{r,1}$,

(iv) $B^{r-1}[B, A^{r-1}]A \in \mathcal{L}_1$.

Then

$$B^rA^r - (A^{1/2}BA^{1/2})^r \in \mathcal{L}_1.$$

Proof. Taking $z = r$ and observing that $\Re(z) = r > 1$ allows us to apply Theorem 4.2.1 to get:

$$B^rA^r - (A^{1/2}BA^{1/2})^r = T_r(0) - \int_{\mathbb{R}} T_r(s)\widehat{g}_r(s)\, ds.$$

We now focus on proving that $T_r(0) \in \mathcal{L}_1$ and

$$\int_{\mathbb{R}} \|T_r(s)\|_1 |\widehat{g}_r(s)|\, ds < \infty.$$

Let $s \in \mathbb{R}$. As the function $t \mapsto t^{is}$ on $\mathbb{R}$ takes values in $\{z \in \mathbb{C} \,:\, |z| = 0, 1\}$, the operator $C^{is} = (A^{1/2}BA^{1/2})^{is}$ is a partial isometry. So we have by the triangle inequality:

$$\|T_r(s)\|_1 \le \|B^{r-1}[BA^{1/2}, A^{r-1/2+is}]\|_1 + \|[BA^{1/2}, A^{1/2+is}]C^{r-1}\|_1.$$

Note that this holds even in the $s = 0$ case. By the Leibniz rule:

$$\begin{aligned}
[BA^{1/2}, A^{r-\frac{1}{2}+is}] &= [BA^{1/2}, A^{r-1}A^{1/2+is}] \\
&= [BA^{1/2}, A^{r-1}]A^{1/2+is} + A^{r-1}[BA^{1/2}, A^{1/2+is}] \\
&= [B, A^{r-1}]A^{1+is} + A^{r-1}[BA^{1/2}, A^{1/2+is}].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|T_r(s)\|_1 \le \|B^{r-1}[B, A^{r-1}]A\|_1 &+ \|B^{r-1}A^{r-1}[BA^{1/2}, A^{1/2+is}]\|_1 \\
&+ \|[BA^{1/2}, A^{1/2+is}]C^{r-1}\|_1.
\end{aligned}$$

Using the Hölder-type inequality (1.5.3) we have:

$$\begin{aligned}
\|T_r(s)\|_1 \le \|B^{r-1}[B, A^{r-1}]A\|_1 &+ \|B^{r-1}A^{r-1}\|_{\frac{r}{r-1},\infty}\|[BA^{1/2}, A^{1/2+is}]\|_{r,1} \\
&+ \|[BA^{1/2}, A^{1/2+is}]\|_{r,1}\|C^{r-1}\|_{\frac{r}{r-1},\infty}.
\end{aligned}$$

By assumption (iv), the first norm $\|B^{r-1}[B, A^{r-1}]A\|_1$ is finite, and by (i) the norm $\|B^{r-1}A^{r-1}\|_{\frac{r}{r-1},\infty}$ is finite. Finally by (ii), we have $C^{r-1} \in \mathcal{L}_{\frac{r}{r-1},\infty}$ and so $\|C^{r-1}\|_{\frac{r}{r-1},\infty}$ is finite.

So there are constants c_1 and c_2 such that:

$$\|T_r(s)\|_1 \le c_1 + c_2\|[BA^{1/2}, A^{1/2+is}]\|_{r,1}.$$

If $s = 0$, then by assumption (iii) the latter norm is finite, so we have proved that $\|T_r(0)\|_1 < \infty$.

Since by (iii) we have that $[BA^{1/2}, A^{1/2}] \in \mathcal{L}_{r,1}$, we can apply (4.3.1) with $X = BA^{1/2}$, $Y = A^{1/2}$ and $\mathcal{E} = \mathcal{L}_{r,1}$ to get:

$$\|T_r(s)\|_1 \le c_1 + 2c_2c_r(1 + |s|)$$

with $c_r := \|[BA^{1/2}, A^{1/2}]\|_{r,1}$. Since g_r is a Schwartz-class function, the Fourier transform $\widehat{g}_r$ is also in the Schwartz-class, and therefore,

$$\int_{\mathbb{R}} |\widehat{g}_r(s)|\|T_r(s)\|_1\, ds \le c_1 \int_{\mathbb{R}} |\widehat{g}_r(s)|\, ds + 2c_2c_r \int_{\mathbb{R}} |\widehat{g}_r(s)|(1 + |s|)\, ds < \infty.$$

The argument preceding the theorem then implies that $\int_{\mathbb{R}} \widehat{g}_r(s)T_r(s)\, ds \in \mathcal{L}_1$, and so $B^rA^r - (A^{1/2}BA^{1/2})^r \in \mathcal{L}_1$. □

4.3.2 Sufficient conditions for a difference of powers to be in $(\mathcal{L}_{1,\infty})_0$

In this subsection we provide sufficient conditions for a difference of powers of operators to be in the quasi-Banach ideal $(\mathcal{L}_{1,\infty})_0$. We will use this in Chapter 5.

We begin with a simple observation. Recall that if $1 < r < \infty$ then $(\mathcal{L}_{r,\infty})_0$ is a separable ideal of $\mathcal{B}(H)$ defined as the closure of the set of all finite rank operators in the quasinorm $\|\cdot\|_{r,\infty}$. Since $r > 1$, there is a norm equivalent to the defining quasinorm $\|\cdot\|_{r,\infty}$.

Lemma 4.3.2. *Let f be a continuous function from $\mathbb{R}$ to $(\mathcal{L}_{r,\infty})_0$ where $1 < r < \infty$. If*

$$\int_{\mathbb{R}} \|f(s)\|_{r,\infty}\, \underline{ds} < \infty$$

then the weak-integral $\int_{\mathbb{R}} f(s)\,ds$ is in $(\mathcal{L}_{r,\infty})_0$.*

Proof. The proof relies on the fact that $(\mathcal{L}_{r,\infty})_0$ can be given an equivalent Banach norm, and the assumption implies that f is integrable in the $(\mathcal{L}_{r,\infty})_0$-valued Bochner sense.

Indeed, since f is continuous, and $\|f(\cdot)\|_{r,\infty}$ is integrable, it follows that f is integrable in the $(\mathcal{L}_{r,\infty})$-valued Bochner sense. So the integral:

$$\int_{-\infty}^{\infty} f(s)\,ds$$

taken in the Bochner sense, is an element of the space $(\mathcal{L}_{r,\infty})_0$. Let us show that the Bochner integral must coincide precisely with the weak*-integral.

Since the $\mathcal{L}_{r,\infty}$ topology is finer than the norm topology, f is also continuous in the operator norm, and hence in particular is weak*-measurable. Since $\|f(s)\|_\infty \le \|f(s)\|_{r,\infty}$ for all $s \in \mathbb{R}$, it follows from Lemma 4.1.4 that f is weak*-integrable.

For all $X \in \mathcal{L}_1$ and $T \in (\mathcal{L}_{r,\infty})_0$, we have:

$$|\mathrm{Tr}(XT)| \le \|X\|_1\|T\|_\infty \le \|X\|_1\|T\|_{r,\infty}.$$

Therefore the map $T \mapsto \mathrm{Tr}(XT)$ is a continuous linear functional on $(\mathcal{L}_{r,\infty})_0$. Since continuous linear functionals can be moved inside a Bochner integral, we have:

$$\mathrm{Tr}\left(X\int_{-\infty}^{\infty} f(s)\,ds\right) = \int_{-\infty} \mathrm{Tr}(Xf(s))\,ds$$

where the integral on the left is a Bochner integral. However the right hand side is precisely the definition of:

$$\mathrm{Tr}\left(X\int_{-\infty}^{\infty} f(s)\,ds\right)$$

where now the integral is a $\mathcal{B}(H)$-valued weak*-integral. Thus, for all $X \in \mathcal{L}_1$ we have:

$$\mathrm{Tr}\left(X\int_{-\infty}^{\infty} f(s)\,ds\right) = \mathrm{Tr}\left(X\int_{-\infty}^{\infty} f(s)\,ds\right)$$

where the integral on the left is an $(\mathcal{L}_{r,\infty})_0$-valued Bochner integral, and the integral on the right is a $\mathcal{B}(H)$-valued weak*-integral. Since this holds for all $X \in \mathcal{L}_1$, it follows that these two integrals are identical. $\square$

The following result first appeared in the literature as [28, Lemma 3.β.11] without proof. A proof was later provided in [35, Lemma 5.3].

Lemma 4.3.3. *Let A and B be non-negative bounded operators, and $1 < r < \infty$. If $B \in \mathcal{L}_{r,\infty}$ and $[A^{1/2}, B] \in (\mathcal{L}_{r,\infty})_0$, then*

$$B^r A^r - (A^{1/2} B A^{1/2})^r \in (\mathcal{L}_{1,\infty})_0.$$

Proof. Since $[A^{1/2}, B] \in (\mathcal{L}_{r,\infty})_0$, it follows that for all $k \geq 0$,

$$[A^{k/2}, B] = \sum_{l=0}^{k-1} A^{(k-l-1)/2}[A^{1/2}, B]A^{l/2} \in (\mathcal{L}_{r,\infty})_0.$$

By linearity, for all polynomials p we have:

$$[p(A^{1/2}), B] \in (\mathcal{L}_{r,\infty})_0.$$

Since $B \in \mathcal{L}_{r,\infty}$, we also have the trivial bound for all continuous functions f on $[0, \|A\|_\infty]$:

$$\|[f(A^{1/2}), B]\|_{r,\infty} \leq \|f\|_{L_\infty([0,\|A\|_\infty])}\|B\|_{r,\infty}.$$

Now let f be an arbitrary continuous function on the interval $[0, \|A\|_\infty]$, and select a sequence $\{p_n\}_{n=0}^\infty$ such that $p_n \to f$ uniformly on $[0, \|A\|_\infty]$. Then,

$$\|[f(A^{1/2}), B] - [p_n(A^{1/2}), B]\|_{r,\infty} \leq \|f - p_n\|_{L_\infty([0,\|A\|_\infty])}\|B\|_{r,\infty} \to 0$$

as $n \to \infty$. Since each $[p_n(A^{1/2}), B]$ is in $(\mathcal{L}_{r,\infty})_0$, and this ideal is by definition closed in the $\mathcal{L}_{r,\infty}$ quasinorm, it follows that:

$$[f(A^{1/2}), B] \in (\mathcal{L}_{r,\infty})_0.$$

In particular, if we take $f(t) = t^{2r-1+2is}$ for $s \in \mathbb{R}$ then

$$[A^{r-1/2+is}, B] \in (\mathcal{L}_{r,\infty})_0.$$

Since $A^{1/2}$ is bounded, we have immediately that

$$[BA^{1/2}, A^{r-1/2+is}] \in (\mathcal{L}_{r,\infty})_0. \tag{4.3.2}$$

for all $s \in \mathbb{R}$. If we use (4.3.1), with $X = BA^{1/2}$, $Y = A^{r-1/2}$ then we have the estimate:

$$\|[BA^{1/2}, A^{r-1/2+is(2r-1)}]\|_{r,\infty} \leq 2c_r(1 + |s|)\|[BA^{1/2}, A^{r-1/2}]\|_{r,\infty}$$

which is finite by (4.3.2) with $s = 0$. Rescaling s by a factor of $\frac{1}{2r-1}$ yields:

$$\|[BA^{1/2}, A^{r-1/2+is}]\|_{r,\infty} \leq 2c_r(1 + \frac{|s|}{2r-1})\|[BA^{1/2}, A^{r-1/2}]\|_{r,\infty}. \tag{4.3.3}$$

Similarly, $[BA^{1/2}, A^{1/2+is}] \in (\mathcal{L}_{r,\infty})_0$ and

$$\|[BA^{1/2}, A^{1/2+is}]\|_{r,\infty} \leq 2c_r(1 + |s|)\|[BA^{1/2}, A^{1/2}]\|_{r,\infty}. \tag{4.3.4}$$

We write the integral formula in Theorem 4.2.1 as follows. We have:

$$B^rA^r-(A^{1/2}BA^{1/2})^r = T_r(0) - B^{r-1}\int_{-\infty}^{\infty}\widehat{g}_r(s)B^{is}[BA^{1/2},A^{r-1/2+is}]C^{-is}ds \\ + \int_{-\infty}^{\infty}\widehat{g}_r(s)B^{is}[BA^{1/2},A^{1/2+is}]C^{-is}\,ds\cdot C^{r-1}. \tag{4.3.5}$$

Since $T_r(0) = B^{r-1}[BA^{1/2},A^{r-1/2}]+[BA^{1/2},A^{1/2}]C^{r-1}$, we have:

$$T_r(0)\in\mathcal{L}_{\frac{r}{r-1},\infty}\cdot(\mathcal{L}_{r,\infty})_0+(\mathcal{L}_{r,\infty})_0\cdot\mathcal{L}_{\frac{r}{r-1},\infty}$$

so by the Hölder inequality, $T_r(0)\in\mathcal{L}_{1,\infty}$, and since $(\mathcal{L}_{r,\infty})_0$ is an ideal, we have further that $T_r(0)\in(\mathcal{L}_{r,\infty})_0$.

Finally, using the fact that $\widehat{g}_r$ is in the Schwartz class and the estimates (4.3.3) and (4.3.4), Lemma 4.3.2 applied to (4.3.5) yields:

$$B^rA^r-(A^{1/2}BA^{1/2})^r\in(\mathcal{L}_{1,\infty})_0+\mathcal{L}_{\frac{r}{r-1},\infty}\cdot(\mathcal{L}_{r,\infty})_0+(\mathcal{L}_{r,\infty})_0\cdot\mathcal{L}_{\frac{r}{r-1},\infty}.$$

From the Hölder inequality, it follows that $B^rA^r-(A^{1/2}BA^{1/2})^r\in(\mathcal{L}_{1,\infty})_0$ as required.

□

Chapter 5

Application to Julia sets

5.1 Introduction

Using the machinery developed in Chapter 4, we are able to finally move to one of the most surprising applications of quantised calculus: the Conformal Trace Theorem (Theorem 5.1.3). This result (in the specific setting of Julia sets of quadratic polynomials) was first announced by Connes in [28, Page 23], and later in [30] and [31] but the detailed proofs were not given. Eventually, a complete proof for the quadratic case was published in [33] which consisted of joint work of the author with A. Connes, F. Sukochev and D. Zanin. The content of this chapter largely reproduces [33], however a number of edits have been made.

The most substantial change made here compared to the published version of [33] is that we consider a much larger class of polynomials. In [33], for the sake of simplicity the authors restricted attention to Julia sets of quadratic polynomials $z \mapsto z^2 + c$ where c is in the set $\{w(1-w)\ :\ 0 < |w| < 1/2\}$. Here we consider a larger class which contains polynomials of arbitrarily high degree (in particular, we can consider $z \mapsto z^d + c$ for $d \geq 2$ and $c \neq 0$ is sufficiently small). This has necessitated changes to some of the proofs: most notably, Lemma 5.5.5 has a new and arguably simpler proof than the published version of [33].

We recall the definition of Julia sets of polynomials, as outlined in [25, Chapter III]. Let ϕ be a polynomial. For $k \geq 1$ we denote ϕ^k for the k-fold iteration of ϕ. The Julia set $J(\phi)$ of ϕ may be defined to be the boundary of the set of points $z \in \mathbb{C}$ such that $\phi^k(z)$ is bounded as $k \to \infty$.

The results of this chapter will be applicable to a class of polynomials which we call "admissible".

Definition 5.1.1. *We call a polynomial ϕ of degree $d \geq 2$* admissible *if the following two conditions hold:*

(i) *The Julia set J of ϕ is a Jordan curve of Hausdorff dimension $p > 1$ (we recall the definition of Hausdorff dimension in Section 5.4).*

(ii) *ϕ is hyperbolic on J (i.e, there exists an $n \geq 0$ such that for every $z \in J$ we have $|(\phi^n)'(z)| > 1$).*

The second assumption (that ϕ is hyperbolic) implies that J has Hausdorff dimension strictly less than 2 (see Theorem 5.4.2).

It is in general not trivial to determine if a given polynomial is admissible, but there exist examples of admissible polynomials of arbitrary degree greater than 1. The following lemma demonstrates the most important class of examples.

Lemma 5.1.2. *Let $d \geq 2$ and consider $\phi(z) := z^d + c$. If c is sufficiently small and nonzero, then ϕ is admissible.*

Proof. Recall that an attracting fixed point of a polynomial ϕ is a point $z \in \mathbb{C}$ such that $\phi(z) = z$ and $|\phi'(z)| < 1$. The fixed points of ϕ correspond to solutions to to $z^d - z + c = 0$, and a fixed point z_0 is attracting when $|\phi'(z_0)| < 1$. That is, when $|z_0|^{d-1} < d^{-1}$. One can then see directly that ϕ has an attracting fixed point if and only if $c \in \{w - w^d \,:\, 0 < |w|^{d-1} < d^{-1}\}$.

Since the function $w \mapsto w - w^d$ is one-to-one on the open disc $|w|^{d-1} < d^{-1}$, it follows that for each $c \in M_0$ there exists a unique $0 < |w|^{d-1} < d^{-1}$ such that $c = w - w^d$. Thus, ϕ always has a unique attracting fixed point which we denote z_0.

A critical point of ϕ is a solution to $\phi'(z) = 0$. According to [25, Theorem III.2.2], for every attracting fixed point of a polynomial there is at least one critical point z such that $\lim_{k\to\infty} \phi^k(z) = z_0$. In this case, ϕ has precisely one critical point at zero and hence $\lim_{k\to\infty} \phi^k(0) = z_0$.

Since z_0 is attractive, there is a bounded neighbourhood U of z_0 such that $\phi^k(U) \subseteq U$ for all $k \geq 0$ (see [25, Section II.2]). Thus $z_0 \notin J$. Moreover, as $\lim_{k\to\infty} \phi^k(0) = z_0$, it follows that:

$$\overline{\bigcup_{k=1}^{\infty} \{\phi^k(0)\}} \subseteq \bigcup_{k=1}^{\infty} \phi^{-k}(U)$$

the latter set is an open neighbourhood of z_0, and thus is disjoint from J. Therefore

$$\overline{\bigcup_{k=1}^{\infty} \{\phi^k(0)\}} \cap J = \emptyset.$$

According to a well-known characterisation of hyperbolicity (see [25, Lemma V.2.1] and [95, Theorem 19.1]) a general polynomial ϕ is hyperbolic on its Julia set J if and only if

$$\overline{\bigcup_{k=1}^{\infty} \phi^k(\mathrm{CP})} \cap J = \emptyset$$

where CP is the set of all critical points of ϕ. Since 0 is the only critical point of ϕ, we have therefore demonstrated that ϕ is hyperbolic on J.

Let us now show that J is a Jordan curve. Since we know that ϕ is hyperbolic on J, it suffices to show that the attracting basin of ∞ is simply connected [8, Lemma 9.9.1]. Indeed, this follows from the fact that ϕ has no critical points which are iterated to infinity, see the discussion in [25, Section III.4].

Finally, the fact that J has Hausdorff dimension strictly greater than 1 follows from a computation of Ruelle [117, Appendix 2], which shows that the Hausdorff dimension of

J is $1 + \frac{|c|^2}{4\log(d)} + O(|c|^3)$, so that when c is sufficiently small and nonzero then J has Hausdorff dimension strictly greater than 1.

□

More specifically, [25, Theorem V.2.1] implies the stronger result that J is a *quasicircle*, but for our present purposes it suffices to know that J is a Jordan curve.

Let $\mathbb{T}$ denote the unit circle in the complex plane, equipped with its standard Haar measure. Let $F : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ be the Hilbert transform, defined on exponential basis functions $e_n(z) := z^n$, $n \in \mathbb{Z}$ by $Fe_n = \mathrm{sgn}(n)e_n$. Given an essentially bounded function f on the unit circle $\mathbb{T}$, the symbol M_f denotes the operator on $L_2(\mathbb{T})$ of pointwise multiplication by f. Since J is a Jordan curve, due to the Riemann mapping theorem there is a conformal mapping Z from the exterior of the unit disc $\{z \in \mathbb{C} : |z| > 1\}$ to the unbounded component of $\mathbb{C} \setminus J$. By the Carathèodory theorem on continuous extensions of conformal maps, Z extends to a continuous bijection $Z : \mathbb{T} \to J$. We may therefore consider Z as a function on the circle, and we write M_Z as the corresponding linear operator on $L_2(\mathbb{T})$. It is known that Z may be chosen such that for all $z \in \mathbb{T}$ we have $Z(z^d) = \phi(Z(z))$ (see Subsection 5.4.3).

The main result of this chapter is the following theorem (all heretofore unexplained symbols and notions will be defined in Section 1.5.2).

Theorem 5.1.3 (The Conformal Trace Theorem)**.** *Let $p \in (1, 2)$ be the Hausdorff dimension of the Julia set J of an admissible polynomial ϕ. Let m_p be the p-dimensional Hausdorff measure on J. Then,*

(a) $[F, M_Z] \in \mathcal{L}_{p,\infty}$.

(b) *For every continuous Hermitian trace φ on $\mathcal{L}_{1,\infty}$, there exists a constant $K(\varphi, \phi)$ such that for every $f \in C(J)$ we have:*

$$\varphi(M_{f\circ Z}|[F, M_Z]|^p) = K(\varphi, \phi) \int_J f\, dm_p.$$

(c) *If ω is a dilation invariant extended limit on $L_\infty(0, \infty)$ such that $\omega \circ \log$ is also dilation invariant, then $K(\mathrm{tr}_\omega, \phi) > 0$. Here, tr_ω is a Dixmier trace corresponding to the extended limit ω.*

Theorem 5.1.3 should be compared with [35, Theorem 1.1] which concerns geometric measures on limit sets of finitely generated quasi-Fuchsian groups. The statement of the result is very similar, however it should be noted that the methods of proof used in this text are completely different to those used in [35]. We follow a proof outlined by the Connes in [28, Chapter 4], which proceeds by identifying the functional $f \mapsto \varphi(M_{f\circ Z}|[F, M_Z]|^p)$ on the space $C(J)$ with the (essentially unique) p-conformal measure with respect to ϕ on J (as defined by Sullivan [130, Theorem 3]). Another theorem of Sullivan [130, Theorem 4] identifies this p-conformal measure with the p-dimensional Hausdorff measure.

The pair $(L_2(\mathbb{T}), F)$ is a Fredholm module for $C(\mathbb{T})$ in the sense discussed in Section 1.1.2. It was the analysis of this particular Fredholm module by Connes and Sullivan which ultimately led to Theorem 5.1.3. To provide some intuition for the appearance of

F in a formula for the Hausdorff measure, it is worth noting that F is invariant under endomorphisms of $\mathbb{T}$ which are the restrictions of conformal maps. This is the essential property of F which leads to the identification of the measure $f \mapsto \varphi(M_{f\circ Z}|[F, M_Z]|^p)$ with the conformal measure of the Julia set.

This chapter is broken up as follows:

1. Section 5.2 collects necessary results concerning commutators of multiplication operators and the Hilbert transform. Many of the proofs relevant to this section are provided in Appendix A.
2. Section 5.3 proves that if $\mathcal{C}$ is any Jordan curve in the complex plane with finite upper s-dimensional Minkowski content, and ζ is a conformal equivalence between the exterior of the unit disc $\mathbb{D}$ and the exterior of $\mathcal{C}$, then $[F, M_\zeta]$ (where M_ζ is considered as an operator on $L_2(\mathbb{T})$) is in the weak Schatten ideal $\mathcal{L}_{s,\infty}$.
3. Section 5.4 collects properties of Julia sets of admissible polynomials, and demonstrates that the Julia set of an admissible polynomial is a Jordan curve with Hausdorff dimension $p \in (1, 2)$ and with finite upper and strictly positive lower p-Minkowski content. Combined with the results of Section 5.3, this immediately yields Theorem 5.1.3.(a).
4. Section 5.5 then completes the proof of Theorem 5.1.3.(b), by showing that the functional $f \mapsto \varphi(M_{f\circ Z}|[F, M_Z]|^p)$ is p-conformal with respect to ϕ (in the sense of Subsection 5.4.2).
5. Section 5.6 then provides a proof of Theorem 5.1.3.(c) by referring to known results on the relationship between Dixmier traces and zeta-function residues.

Sections 5.3 and 5.6 are in a sense self-contained in that no reference is made to Julia sets. Instead, we work with arbitrary Jordan curves with finite upper and positive lower s-dimensional Minkowski content. We have opted to work at this level of generality because it is anticipated that in future work we may be able to work with more general conformally self-similar Jordan curves.

We give thanks to Professors Smirnov and Sullivan for useful discussions and Professor Bishop for communicating to us the idea of the proof of Proposition 5.4.3.

5.2 Commutators of multiplication operators and the Hilbert transform

Denote by $\mathbb{D}$ the open unit disc in the complex plane. Given $f \in L_1(\mathbb{T})$[1], let $\hat{f}(n)$ be the nth Fourier coefficient of f. It is well known that f can be identified with the non-tangential boundary values of a holomorphic function in the interior of the unit disc if and only if $\hat{f}(n) = 0$ for all $n < 0$. In this case we identify f with its holomorphic extension. The Hilbert transform $F : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ is defined on functions $f \in L_2(\mathbb{T})$ by

$$(Ff)(z) = \sum_{n\in\mathbb{Z}} \mathrm{sgn}(n)\hat{f}(n)z^n, z \in \mathbb{T}.$$

[1]Spaces $L_p(\mathbb{T})$ on the circle are always defined with respect to the Haar measure

Given $f \in L_\infty(\mathbb{T})$, the symbol M_f stands for the operator on $L_2(\mathbb{T})$ given by pointwise multiplication by f. We are concerned with conditions on f which are necessary and sufficient for the commutator $[F, M_f]$ to be in the Schatten p-class $\mathcal{L}_p$. The following result is a restatement of a result due to Peller [100, Chapter 6] and a full proof is included in Appendix A. Recall that $dzd\bar{z}$ denotes the Lebesgue measure on $\mathbb{C}$.

Theorem 5.2.1. *Let f be a function on $\mathbb{T}$ with holomorphic extension to $\mathbb{D}$, and let $p_0 > 1$. There exist constants $k, K > 0$ (depending on p_0) such that for all $p \in (p_0, 2)$ we have*

$$k\left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}\,dzd\bar{z}\right)^{1/p} \leq \|[F,f]\|_p \leq K\left(\int_{\mathbb{D}} |f'(z)|^p(1-|z|^2)^{p-2}\,dzd\bar{z}\right)^{1/p}.$$

Note that f' denotes the derivative of the holomorphic extension of f to the interior of $\mathbb{D}$, not the holomorphic extension of the derivative of f. We also utilise the following one-sided result, giving sufficient conditions for $[F, M_f]$ to be in the weak p-Schatten class and which is identical to [35, Lemma 3.5].

Theorem 5.2.2. *Let $p > 1$, and let f be a function on $\mathbb{T}$ with holomorphic extension to $\mathbb{D}$. Define $h(z) := f'(z)(1-|z|^2)$ and let ν be the measure on $\mathbb{D}$ given by $d\nu = \frac{dzd\bar{z}}{(1-|z|^2)^2}$. Then there exists a constant $c_p > 0$ such that*

$$\|[F, M_f]\|_{p,\infty} \leq c_p\|h\|_{L_{p,\infty}(\mathbb{D},\nu)}.$$

The proof of Theorem 5.2.2 amounts to a combination of Theorem 5.2.1 and an interpolation argument.

5.3 Jordan curves with finite upper Minkowski content

Let A be an arbitrary subset of $\mathbb{R}^d$ (we will ultimately be concerned with the case $d = 2$). The δ-neighbourhood of A is the set,

$$S_\delta(A) = \{x \in \mathbb{R}^d \ : \ \mathrm{dist}(x, A) < \delta\}.$$

Let $|S_\delta(A)|$ denote the Lebesgue measure of $S_\delta(A)$ and let $0 \leq s \leq d$. The upper s-dimensional Minkowski content of A is defined by:

$$M^s(A) := \limsup_{\delta\to 0} \delta^{s-d}|S_\delta(A)|.$$

By definition, $M^s(A)$ is finite if and only if $|S_\delta(A)| = O(\delta^{d-s})$ as $\delta \to 0$.

The lower s-dimensional Minkowski content is defined as,

$$M_s(A) := \liminf_{\delta\to 0} \delta^{s-d}|S_\delta(A)|.$$

The above given definitions of upper and lower Minkowski content follow [52, Definition 3.2.37].

Let $\mathcal{C}$ be a Jordan curve in the plane, and let $\Omega \subset \mathbb{C}$ be the bounded component of $\mathbb{C} \setminus \mathcal{C}$, so that $\mathcal{C} = \partial\Omega$. By the Riemann mapping theorem, there is a conformal mapping $\xi : \mathbb{D} \to \Omega$ which by the Carathèodory theorem extends to a continuous function $\xi : \mathbb{T} \to \mathcal{C}$. This section is devoted to the proof of the fact that if $\mathcal{C}$ has finite upper s-dimensional Minkowski content then the commutator $[F, M_\xi]$ (considered as an operator on $L_2(\mathbb{T})$) is in the weak Schatten s-class $\mathcal{L}_{s,\infty}$.

The following Lemma appears as [28, Equation (4.21)], and we supply a detailed proof for convenience.

Lemma 5.3.1. *Let Ω be a domain in $\mathbb{C}$ whose boundary $\partial\Omega$ is a Jordan curve, and let $\xi : \mathbb{D} \to \Omega$ be a conformal map. Then for all $|z| < 1$,*

$$\frac{1}{4}(1-|z|^2)|\xi'(z)| \leq \mathrm{dist}(\xi(z), \partial\Omega) \leq (1-|z|^2)|\xi'(z)|$$

Proof. Let h be any conformal mapping $h : \mathbb{D} \to \Omega$. Since h is conformal, it is bijective and we have $h'(0) \neq 0$. Hence we may define a function k by

$$k(z) = \begin{cases} \frac{z}{h(z)-h(0)}, & z \neq 0 \\ \frac{1}{h'(0)}, & z = 0. \end{cases}$$

Since h is holomorphic, k is also holomorphic. Since $\partial\Omega$ is a Jordan curve, by the Carathèodory theorem [54, Theorem 3.1], h extends to a continuous function on the circle $\mathbb{T}$. Since $h(0)$ is in the interior of the curve $\partial\Omega$, we have $\inf_{z\in\mathbb{T}} |h(z) - h(0)| > 0$, so it follows that k also extends continuously to $\mathbb{T}$. By the maximum modulus principle, since k is holomorphic in the open unit disc,

$$|k(0)| \leq \sup_{|z|=1} |k(z)|$$

Equivalently,

$$\frac{1}{|h'(0)|} \leq \sup_{|z|=1} \frac{|z|}{|h(z)-h(0)|}.$$

Since $|z| = 1$, we then obtain

$$\inf_{|z|=1} |h(z) - h(0)| \leq |h'(0)|$$

When $|z| = 1$, the point $h(z)$ lies in the boundary $\partial\Omega$, so immediately:

$$\mathrm{dist}(h(0), \partial\Omega) \leq |h'(0)| \tag{5.3.1}$$

We now refer to the Koebe 1/4-theorem, [115, Theorem 14.14], which states that if h is a conformal mapping from $\mathbb{D}$ to a simply connected domain Ω, then Ω contains the disc centred at $h(0)$ with radius $\frac{|h'(0)|}{4}$. Equivalently, $\mathrm{dist}(h(0), \partial\Omega)$ is not less than $\frac{1}{4}|h'(0)|$, so:

$$\frac{1}{4}|h'(0)| \leq \mathrm{dist}(h(0), \partial\Omega). \tag{5.3.2}$$

Combining (5.3.1) and (5.3.2),

$$\frac{1}{4}|h'(0)| \leq \mathrm{dist}(h(0), \partial\Omega) \leq |h'(0)|. \tag{5.3.3}$$

Let $|z| < 1$. Consider the function

$$h(w) := \xi\left(\frac{z-w}{1-\overline{z}w}\right), \quad |w| < 1.$$

Note that the map $w \mapsto \frac{z-w}{1-\overline{z}w}$ is a conformal automorphism of the unit disc, so the image of the unit disc under h is the same as the image under ξ. Thus, h is a conformal mapping from the unit disc to Ω. We can then simply compute:

$$h(0) = \xi(z), \quad h'(0) = -\xi'(z)(1-|z|^2).$$

So immediately from (5.3.3)

$$\frac{1}{4}(1-|z|^2)|\xi'(z)| \leq \operatorname{dist}(\xi(z), \partial\Omega) \leq (1-|z|^2)|\xi'(z)|.$$

□

The next result shows how we can use Lemma 5.3.1 to reduce the question of whether $(1-|z|^2)|\xi'(z)| \in L_{s,\infty}(\mathbb{D}, \frac{dzd\overline{z}}{(1-|z|^2)^2})$ to a purely geometric question concerning $\mathcal{C}$.

Proposition 5.3.2. *Let $\mathcal{C}$ be a Jordan curve in the plane with interior Ω, and let $\xi : \mathbb{D} \to \Omega$ be a conformal map. Let h be the function on $\mathbb{D}$ given by $h(z) = |\xi'(z)|(1-|z|^2)$. Let D be the function on Ω defined by $D(z) = \operatorname{dist}(z, \partial\Omega) = \operatorname{dist}(z, \mathcal{C})$. Then for all $s > 0$,*

$$h \in L_{s,\infty}(\mathbb{D}, \frac{dzd\overline{z}}{(1-|z|^2)^2}) \iff D \in L_{s,\infty}(\Omega, \frac{dwd\overline{w}}{\operatorname{dist}(w,\partial\Omega)^2}).$$

Proof. Restating Lemma 5.3.1, we have:

$$\frac{1}{4}h(z) \leq D(\xi(z)) \leq h(z). \tag{5.3.4}$$

Rearranging the result of Lemma 5.3.1 yields

$$\frac{1}{(1-|z|^2)^2} \leq \frac{|\xi'(z)|^2}{\operatorname{dist}(\xi(z),\partial\Omega)^2} \leq \frac{16}{(1-|z|^2)^2}, \quad z \in \mathbb{D}. \tag{5.3.5}$$

The function ξ maps $\mathbb{D}$ conformally into Ω, so in particular it is injective. If $w = \xi(z)$, then $dwd\overline{w} = |\xi'(z)|^2 dzd\overline{z}$, so from (5.3.5) for any Borel set $A \subseteq \mathbb{D}$ we have

$$\int_A \frac{dzd\overline{z}}{(1-|z|^2)^2} \leq \int_{\xi(A)} \frac{dwd\overline{w}}{\operatorname{dist}(w,\partial\Omega)^2} \leq 16\int_A \frac{dzd\overline{z}}{(1-|z|^2)^2}.$$

Thus, the images of the measure $\frac{dzd\overline{z}}{(1-|z|^2)^2}$ under ξ is equivalent to the measure $\frac{dwd\overline{w}}{\operatorname{dist}(w,\partial\Omega)^2}$. Combining (5.3.4) and (5.3.5) yields the equivalence that $h \in L_{s,\infty}(\mathbb{D}, \frac{dzd\overline{z}}{(1-|z|^2)^2})$ if and only if $D \in L_{s,\infty}(\Omega, \frac{dwd\overline{w}}{\operatorname{dist}(w,\partial\Omega)^2})$. □

The following is the key result which yields $[F, M_\xi] \in \mathcal{L}_{s,\infty}$ if $\mathcal{C}$ has finite upper s-dimensional Minkowski content.

Proposition 5.3.3. *If $\partial\Omega$ has finite upper s-dimensional Minkowski content, then:*

$$z \mapsto \mathrm{dist}(z, \partial\Omega) \in L_{s,\infty}\left(\Omega, \frac{dz d\bar{z}}{\mathrm{dist}(z,\partial\Omega)^2}\right).$$

Proof. We partition the region Ω into countably many regions, $\{A_k\}_{k\geq 0}$ defined by:

$$A_k := \{z \in \Omega \ : \ \mathrm{dist}(z,\partial\Omega) \in [2^{1-k}, 2^{-k})\}$$

and define $A_{-1} := \{z \in \Omega \ : \ \mathrm{dist}(z,\partial\Omega) > 2\}$. Then Ω is a disjoint union:

$$\Omega = \bigcup_{k=-1}^{\infty} A_k.$$

Let μ be the measure $d\mu = \frac{dzd\bar{z}}{\mathrm{dist}(z,\partial\Omega)^2}$. Then for all $n \geq 0$,

$$\mu(\{z \in \Omega \ : \ \mathrm{dist}(z,\partial\Omega) \geq 2^{-n}) = \sum_{k=-1}^{n} \mu(A_k).$$

Inside the region A_k, the function $z \mapsto \frac{1}{\mathrm{dist}(z,\partial\Omega)^2}$ is bounded from above by 2^{2k}. So for $k \geq 0$,

$$\begin{aligned}
\mu(A_k) &\leq 2^{2k}|A_k| \\
&= 2^{2k}(|S_{2^{1-k}}(\partial\Omega) \cap \Omega| - |S_{2^{-k}}(\partial\Omega) \cap \Omega|) \\
&\leq 2^{2k}|S_{2^{1-k}}(\partial\Omega)|.
\end{aligned}$$

By the assumption that the s-dimensional Minkowski content is finite there exists $C > 0$ such that for all k,

$$\begin{aligned}
|S_{2^{1-k}}(\Omega)| &\leq C \cdot 2^{(1-k)(2-s)} \\
&= C \cdot 2^{2-s} \cdot 2^{-2k} \cdot 2^{ks}.
\end{aligned}$$

Letting $K = C2^{2-s}$, we obtain that for all $k \geq 0$ we have $\mu(A_k) \leq K2^{ks}$. So,

$$\begin{aligned}
\mu(\{z \in \Omega \ : \ \mathrm{dist}(z,\partial\Omega) \geq 2^{-n}) &\leq \mu(A_{-1}) + K\sum_{k=0}^{n} 2^{ks} \\
&= O(2^{ns}).
\end{aligned}$$

Thus, $\mu(\{z \in \Omega \ : \ \mathrm{dist}(z,\partial\Omega) \geq t\}) = O(t^{-s})$ as $t \to 0$. □

We obtain our main result concerning conformal maps from the unit disc to the interior of a Jordan curve.

Theorem 5.3.4. *Let $\mathcal{C}$ be a Jordan curve in the plane with finite s-dimensional upper Minkowski content, and let ξ be a conformal map from the interior of the unit disc to the interior of $\mathcal{C}$. Then the extension of ξ to the boundary, considered as a function on the circle $\mathbb{T}$, satisfies*

$$[F, M_\xi] \in \mathcal{L}_{s,\infty}.$$

Proof. Let Ω denote the interior of $\mathcal{C}$ and let D be the function on Ω given by $D(w) := \mathrm{dist}(w, \mathcal{C})$. From Proposition 5.3.3, we have $D \in L_{s,\infty}(\Omega, \frac{dwd\overline{w}}{\mathrm{dist}(w,\mathcal{C})^2})$. Applying Proposition 5.3.2, it follows that the function $h(z) := (1-|z|^2)|\xi'(z)|$ is in $L_{s,\infty}(\mathbb{D}, \frac{dzd\overline{z}}{(1-|z|^2)^2})$.

Due to Theorem 5.2.2, if $h \in L_{s,\infty}(\mathbb{D}, \frac{dzd\overline{z}}{(1-|z|^2)^2})$ then $[F, M_\xi] \in \mathcal{L}_{s,\infty}$. □

Theorem 5.3.4 concerns conformal equivalences between the open unit disc and the *interior* of a Jordan curve. In fact, similar results hold for equivalences between the *exterior* of the unit disc and the exterior of a Jordan curve.

Theorem 5.3.5. *Let $\mathcal{C}$ be a Jordan curve in the plane with finite s-dimensional upper Minkowski content, and let ζ be a conformal map from the exterior of the unit disc, $\{z \in \mathbb{C} : |z| > 1\}$ to the exterior of $\mathcal{C}$. Then the extension of ζ to the $\mathbb{T}$, considered as a function on the circle $\mathbb{T}$, satisfies*

$$[F, M_\zeta] \in \mathcal{L}_{s,\infty}.$$

Proof. Without loss of generality we may assume that the point 0 is in the interior of $\mathcal{C}$, and also ζ may be chosen such that as $|z| \to \infty$ we have $|\zeta(z)| \to \infty$. Define the function η on $\mathbb{D} \setminus \{0\}$ by

$$\eta(z) := \zeta(z^{-1})^{-1}.$$

Since 0 is in the interior of $\mathcal{C}$, the range of $\zeta(z^{-1})$ is bounded away from zero, so η is bounded in any punctured neighbourhood of zero and so has holomorphic extension to $\mathbb{D}$, and by our assumption is extended to $\mathbb{D}$ by defining $\eta(0) = 0$. Since ζ is injective, η is also injective and hence is a conformal equivalence onto its image. Since $0 \notin \mathcal{C}$, the image $\mathcal{C}^{-1}$ is also a Jordan curve. Hence, η is a conformal equivalence between $\mathbb{D}$ and the interior of the Jordan curve $\mathcal{C}^{-1}$.

For all $\delta > 0$, by definition we have

$$S_\delta(\mathcal{C}^{-1}) = \bigcup_{z \in \mathcal{C}} B(z^{-1}, \delta).$$

Since the function $z \mapsto z^{-1}$ is Lipschitz when restricted to the complement of any ball containing 0, then for any $\varepsilon > 0$ there exists a constant $C > 0$ such that for all $|z| > \varepsilon$ and all $\delta < \varepsilon/2$ we have,

$$B(z^{-1}, \delta) \subseteq B(z, C\delta)^{-1}.$$

Hence for δ sufficiently small the inclusion

$$S_\delta(\mathcal{C}^{-1}) \subseteq S_{C\delta}(\mathcal{C})^{-1}$$

holds.

The Jacobian of the function $z \mapsto z^{-1}$ is uniformly bounded on compact subsets of $\mathbb{C} \setminus 0$. Hence, there is a constant $K > 0$ (depending on $\mathcal{C}$) such that for δ sufficiently small,

$$|S_{C\delta}(\mathcal{C})^{-1}| \leq K|S_{C\delta}(\mathcal{C})| = O(\delta^{2-s}).$$

So finally $|S_\delta(\mathcal{C}^{-1})| = O(\delta^{2-s})$. Hence $\mathcal{C}^{-1}$ has finite s-dimensional upper Minkowski content.

Let W be the unitary map on $L_2(\mathbb{T})$ which maps the basis function z^n to z^{-n} for all $n \in \mathbb{Z}$. Then $W^*FW = -F + R$, where R is a rank one map, and $M_\zeta = (WM_\eta W^*)^{-1}$. Thus,

$$\begin{aligned}[F, M_\zeta] &= -(WM_\eta W^*)^{-1}[F, WM_\eta W^*](WM_\eta W^*)^{-1} \\ &= -(WM_\eta W^*)^{-1}W[-F + R, M_\eta]W^*(WM_\eta W^*)^{-1}.\end{aligned}$$

So finally, $[F, M_\zeta] \in \mathcal{L}_{s,\infty}$. □

Theorem 5.3.5 will yield Theorem 5.1.3.(a) once it is shown that the Julia sets of admissible polynomials ϕ are Jordan curves with finite p-dimensional upper Minkowski content.

5.4 Julia sets

We now specialise to Jordan curves which arise as Julia sets of admissible polynomials. We use the concepts of Hausdorff measure and Hausdorff dimension, conventionally defined as follows (see e.g. [50, Section 2.4], [52, Section 2.10.2]).

Let S be a Borel subset of $\mathbb{R}^d$, and let $A \subseteq S$ be Borel. Let $s, \delta > 0$, and define:

$$\mathcal{H}^s_\delta(A) := \inf\{\sum_{j=1}^\infty r_j^s; : \text{ there is a covering of } A \text{ with open sets with diameters } r_j < \delta\}.$$

The Hausdorff measure $m_s(A)$ is defined to be $\sup_{\delta>0} \mathcal{H}^s_\delta(A)$. The assignment $A \mapsto m_s(A)$ is then a Borel measure on S, and the Hausdorff dimension of S is defined to be the infimum of the set of all s such that $m_s(S)$ is positive.

Let $d \geq 2$, and fix an admissible polynomial ϕ of degree d. As usual, we denote the k-fold iteration of the function ϕ with itself by ϕ^k.

Definition 5.4.1. *The Julia set J of ϕ is the boundary of the set of points $z \in \mathbb{C}$ such that $\phi^k(z)$ remains bounded as $k \to \infty$ (see [25, Chapter III] and [95, Lemma 9.4]). Let p denote the Hausdorff dimension of J.*

It is well known that J is invariant under ϕ, and also $\phi^{-1}(J) = J$ (see [25, Theorem III.1.3] and [95, Lemma 4.3]).

Recall that we have assumed that an admissible polynomial ϕ is hyperbolic on J. That is, there exists $n \geq 1$ such that

$$\inf\{|(\phi^n)'(z)| \ : \ z \in J\} > 1.$$

This condition is important for characterising the Hausdorff measure on J.

From [130, Theorem 4], the Julia set of a hyperbolic map has Hausdorff dimension strictly between 0 and 2. So immediately it follows that:

Lemma 5.4.2. *If ϕ is an admissible polynomial, then the Hausdorff dimension p of J satisfies $1 < p < 2$.*

The hyperbolicity of ϕ also implies that the Hausdorff measure $m_p(J)$ of J is finite [134, Theorem 2.3].

5.4.1 Minkowski content of the Julia set

The following proposition allows us to apply the results of Section 5.3 to the Julia set J.

Proposition 5.4.3. *If ϕ is an admissible polynomial, then the Julia set J has finite upper p-dimensional Minkowski content, and positive lower p-dimensional Minkowski content.*

Proof. Let $\delta > 0$. The set $S_\delta(J)$ can be written as a union of balls of radius δ,

$$S_\delta(J) = \bigcup_{z \in J} B(z, \delta).$$

By the Vitali covering lemma, there is a disjoint finite subset $\{B(z_j, \delta)\}_{j=1}^{K(\delta)}$ such that

$$\bigcup_{j=1}^{K(\delta)} B(z_j, \delta) \subseteq S_\delta(J) \subseteq \bigcup_{j=1}^{K(\delta)} B(z_j, 5\delta). \tag{5.4.1}$$

Since the finite set $\{B(z_j, \delta)\}_{j=1}^{K(\delta)}$ is disjoint, applying the Lebesgue measure to (5.4.1):

$$\sum_{j=1}^{K(\delta)} |B(z_j, \delta)| \leq |S_\delta(J)| \leq \sum_{j=1}^{K(\delta)} |B(z_j, 5\delta)|.$$

So,

$$K(\delta)\pi\delta^2 \leq |S_\delta(J)| \leq 25K(\delta)\pi\delta^2. \tag{5.4.2}$$

Let m_p denote the p-dimensional Hausdorff measure on J. Now applying m_p to (5.4.1):

$$\sum_{j=1}^{K(\delta)} m_p(B(z_j, \delta) \cap J) \leq m_p(J) \leq \sum_{j=1}^{K(\delta)} m_p(B(z_j, 5\delta) \cap J).$$

Now we refer to [134, Theorem 2.3], where it is stated (as a consequence of the hyperbolicity of ϕ) that there exist constants $\alpha, \beta > 0$ such that for all $r \in (0, 1)$ and $z \in J$ we have,

$$\alpha r^p \leq m_p(B(z, r) \cap J) \leq \beta r^p.$$

So,

$$K(\delta)\alpha\delta^p \leq m_p(J) \leq K(\delta)\beta 5^p \delta^p. \tag{5.4.3}$$

Rearranging the inequalities (5.4.3) we obtain

$$\frac{m_p(J)}{5^p \beta}\delta^{-p} \leq K(\delta) \leq \frac{m_p(J)}{\alpha}\delta^{-p}. \tag{5.4.4}$$

Combining (5.4.2) and (5.4.4) yields,

$$\frac{m_p(J)\pi}{5^p\beta}\delta^{2-p} \leq |S_\delta(J)| \leq \frac{25\pi m_p(J)}{\alpha}\delta^{2-p}.$$

Since $\frac{m_p(J)\pi}{5^p\beta} > 0$, the lower p-dimensional Minkowski content is positive, and since $\frac{25m_p(J)\pi}{\alpha} < \infty$, the upper p-dimensional Minkowski content is finite. $\square$

Remark 5.4.4. *Proposition 5.4.3 also shows that the Minkowski dimension of J is equal to the Hausdorff dimension p.*

5.4.2 Conformal measures on the Julia set

Let $q \in (0,\infty)$, and let ν be a Borel measure on J. The measure ν is said to be q-conformal with respect to ϕ (in the sense of Sullivan [130, Theorem 3]) if for any measurable set $A \subseteq J$ such that $\phi|A$ is injective we have

$$\nu(\phi(A)) = \int_A |\phi'(z)|^q \, d\nu(z).$$

Conditions which guarantee the uniqueness of a q-conformal measure with respect to ϕ have been previously studied, of particular interest is the case where $q = p$, the Hausdorff dimension of J. We refer to [130, Theorem 4], where it is proved that there is (up to a scaling factor) a unique p-conformal measure for ϕ when ϕ is a hyperbolic map. Moreover, [130, Theorem 4] states that this essentially unique measure coincides with the Hausdorff measure on J.

5.4.3 Conformal equivalence of the exterior of J with the exterior of the unit disc

By the Riemann mapping theorem, we can choose a conformal map Z from the exterior of the unit disc to the exterior of J. By the Carathèodory theorem ([54, Theorem 3.1]), Z extends continuously to the boundary, $Z : \mathbb{T} \to J$. It is known as a special case of [25, Chapter 2, Theorem 4.1] that Z can be chosen such that:

$$Z(z^d) = \phi(Z(z)), \text{ for all } |z| \geq 1. \tag{5.4.5}$$

The above equation is due to L. Böttcher, and implies that the map Z provides a conjugacy between the endomorphism $\phi : J \to J$ and the dth power map $z \mapsto z^d$ on the unit circle.

A combination of Corollary 5.4.3 and Theorem 5.3.5 immediately yields Theorem 5.1.3.(a). That is, that $[F, M_Z] \in \mathcal{L}_{p,\infty}$.

The reason for considering Z as a mapping from the exterior of the unit disc to the exterior of J is precisely so that (5.4.5) holds. Indeed, [25, Theorem II.4.1] shows that there is a conformal map Z such that $Z(z^d) = \phi(Z(z))$ defined for all z in a neighbourhood of a superattracting fixed point of the extension of ϕ to the Riemann sphere. The extension of the map $z \mapsto \phi(z)$ has a superattracting fixed point on the Riemann sphere at ∞ (see the Example at the end of page 34 in [25]).

For the remainder of this text, we assume that Z satisfies (5.4.5).

5.5 The Conformal Trace Formula

As in the previous section, we assume that ϕ is an admissible polynomial, so that the Julia set J of ϕ is a Jordan curve of Hausdorff dimension $1 < p < 2$, with finite upper and positive lower p-dimensional Minkowski content, and moreover that there is a unique (up to scaling) p-conformal measure on J with respect to ϕ. Everywhere in this section, Z is a fixed conformal map from the exterior of the unit disc $\mathbb{D}$ to the exterior of J, identified with its continuous extension to $\mathbb{T}$, and satisfying (5.4.5).

Due to Theorem 5.3.5 we have $|[F, M_Z]|^p \in \mathcal{L}_{1,\infty}$ so the following functional is well defined and bounded on $C(J)$.

Definition 5.5.1. *Let φ be a continuous trace on $\mathcal{L}_{1,\infty}$. Due to Theorem 5.3.5, we may consider the linear functional l_φ on $C(J)$ given by:*

$$l_\varphi(f) := \varphi(M_{f\circ Z}|[F, M_Z]|^p), \quad f \in C(J).$$

Remark 5.5.2. *Suppose φ in definition 5.5.1 is positive. By the Riesz theorem there is a unique regular non-negative Borel measure ν_φ on J with the normalisation $\nu_\varphi(J) = 1$ such that*

$$\varphi(M_{f\circ Z}|[F, M_Z]|^p) = K(\varphi, \phi) \int_J f \, d\nu_\varphi$$

where $K(\varphi, \phi)$ is a constant.

The first part of the following proposition appears as [28, Chapter 4, Section 3.β, Theorem 8(a)].

Proposition 5.5.3. *Let $f \in C(\mathbb{T})$. Then,*

(i) $[M_f, [F, M_Z]] \in (\mathcal{L}_{p,\infty})_0$*, and*

(ii) $[M_f, |[F, M_Z]|^p] \in (\mathcal{L}_{1,\infty})_0$.

Proof. For both part (i) and part (ii), it suffices to prove the result for $f(z) = e_n(z) = z^n$, $n \in \mathbb{Z}$, due to linearity and continuity.

First we prove part (i). Since M_f commutes with M_Z,

$$M_{e_n}[F, M_Z]M_{e_n}^* = [M_{e_n}FM_{e_n}^*, M_Z]$$

However it can be computed that,

$$M_{e_n}FM_{e_n}^*e_k = \operatorname{sgn}(k-n)e_k, \quad \text{for all } k \in \mathbb{Z}.$$

Hence, $M_{e_n}FM_{e_n}^* - F$ is a finite rank operator, and in particular is in $(\mathcal{L}_{p,\infty})_0$. Thus,

$$[M_{e_n}FM_{e_n}^*, M_Z] - [F, M_Z] = M_{e_n}[F, M_Z]M_{e_n}^* - [F, M_Z] \in (\mathcal{L}_{p,\infty})_0. \tag{5.5.1}$$

Multiplying (5.5.1) on the right by M_{e_n} yields $[M_{e_n}, [F, M_z]] \in (\mathcal{L}_{p,\infty})_0$, thus completing the proof of part (i).

Now we prove part (ii). Applying [28, Chapter 4, Section 3.β, Proposition 10] to (5.5.1) yields,

$$|M_{e_n}[F, M_Z]M_{e_n}^*|^p - |[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0.$$

Since M_{e_n} is unitary, it follows that $|M_{e_n}[F, M_Z]M_{e_n}^*|^p = M_{e_n}|[F, M_Z]|^p M_{e_n}^*$, so

$$M_{e_n}|[F, M_Z]|^p M_{e_n}^* - |[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0.$$

As in part (i), multiplying on the right by M_{e_n} yields $[M_{e_n}, |[F, M_Z]|^p] \in (\mathcal{L}_{1,\infty})_0$, thus completing the proof.

□

The following theorem consists of special cases of parts (b) and (c) of [28, Chapter 4, Section 3.β, Theorem 8], however the proof of part (c) in that reference was not included and so for the convenience of the reader we supply a self-contained proof.

Note that we make repeated use of the following fact: if X and Y are bounded operators with $X - Y \in (\mathcal{L}_{p,\infty})_0$, then $|X|^p - |Y|^p \in (\mathcal{L}_{1,\infty})_0$. This may be found as [28, Chapter 4, Section 3.β].

Theorem 5.5.4. *Let f be a complex polynomial. Then,*

(i) $[F, f(M_Z)] - f'(M_Z)[F, M_Z] \in (\mathcal{L}_{p,\infty})_0$

(ii) $|[F, f(M_Z)]|^p - |f'(M_Z)|^p |[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0$.

Proof. First we prove part (i). Due to linearity, it suffices to prove (i) for $f(z) = z^n$, $n \geq 0$. By the Leibniz rule,

$$[F, M_Z^n] = \sum_{k=1}^{n} M_Z^{n-k}[F, M_Z]M_Z^{k-1}.$$

From Lemma 5.5.3.(i), $[[F, M_Z], M_Z^{k-1}] \in (\mathcal{L}_{p,\infty})_0$ for all $k \geq 1$, so

$$[F, M_Z^n] - nM_Z^{n-1}[F, M_Z] \in (\mathcal{L}_{p,\infty})_0.$$

Since $f'(z) = nz^{n-1}$, this completes the proof of part (i).

Now we prove part (ii). Firstly, we apply [28, Chapter 4, Section 3.β, Proposition 10] to the difference $[F, f(M_Z)] - f'(M_Z)[F, M_Z]$, which gives us

$$|[F, f(M_Z)]|^p - |f'(M_Z)[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0.$$

Note that[2],

$$|f'(M_Z)[F, M_Z]| = \big||f'(M_Z)|[F, M_Z]\big|$$

Hence,

$$|[F, f(M_Z)]|^p - \big||f'(M_Z)|[F, M_Z]\big|^p \in (\mathcal{L}_{1,\infty})_0 \tag{5.5.2}$$

[2] For any operators A and B, we have $||A|B|^2 = B^*A^*AB = |AB|^2$, so $||A|B| = |AB|$

From Proposition 5.5.3.(i), since the function $|f' \circ Z|$ is continuous on $\mathbb{T}$, we have:

$$[|f'(M_Z)|, [F, M_Z]] \in (\mathcal{L}_{p,\infty})_0.$$

Again applying [28, Chapter 4, Section 3.β, Proposition 10], it follows that:

$$\big||f'(M_Z)|[F, M_Z]\big|^p - \big|[F, M_Z]|f'(M_Z)|\big|^p \in (\mathcal{L}_{1,\infty})_0. \tag{5.5.3}$$

Subtracting (5.5.3) from (5.5.2) yields

$$|[F, f(M_Z)]|^p - \big|[F, M_Z]|f'(M_Z)|\big|^p \in (\mathcal{L}_{1,\infty})_0. \tag{5.5.4}$$

Hence, since $|[F, M_Z]|f'(M_Z)|| = ||[F, M_Z]||f'(M_Z)||$

$$|[F, f(M_Z)]|^p - \big||[F, M_Z]| \cdot |f'(M_Z)|\big|^p \in (\mathcal{L}_{1,\infty})_0.$$

Since the function $|f' \circ Z|^{1/2}$ is continuous on $\mathbb{T}$, from Proposition 5.5.3.(i) (with M_f in that proposition given as $M_{|f' \circ Z|^{1/2}}$) the double commutator $[|f'(M_Z)|^{1/2}, [F, M_Z]]$ is in $(\mathcal{L}_{p,\infty})_0$. Taking the adjoint, we also have that $[|f'(M_Z)|^{1/2}, [F, M_Z]^*] \in (\mathcal{L}_{p,\infty})_0$. Thus from [35, Lemma 6.2],

$$[|f'(M_Z)|^{1/2}, |[F, M_Z]|] \in (\mathcal{L}_{p,\infty})_0. \tag{5.5.5}$$

Multiplying (5.5.5) on the right by $|f'(M_Z)|^{1/2}$, it follows that

$$|f'(M_Z)|^{1/2}|[F, M_Z]||f'(M_Z)|^{1/2} - |[F, M_Z]| \cdot |f'(M_Z)| \in (\mathcal{L}_{p,\infty})_0.$$

Applying [28, Chapter 4, Section 3.β, Proposition 10], it follows that

$$(|f'(M_Z)|^{1/2} \cdot |[F, M_Z]| \cdot |f'(M_Z)|^{1/2})^p - \big||[F, M_Z]| \cdot |f'(M_Z)|\big|^p \in (\mathcal{L}_{1,\infty})_0. \tag{5.5.6}$$

Subtracting (5.5.6) from (5.5.4) yields

$$|[F, f(M_Z)]|^p - (|f'(M_Z)|^{1/2} \cdot |[F, M_Z]| \cdot |f'(M_Z)|^{1/2})^p \in (\mathcal{L}_{1,\infty})_0. \tag{5.5.7}$$

From (5.5.5), $[|f'(M_Z)|^{1/2}, |[F, M_Z]|] \in (\mathcal{L}_{p,\infty})_0$, so we may apply Lemma 4.3.3 to get:

$$|[F, M_Z]|^p|f'(M_Z)|^p - (|f'(M_Z)|^{1/2}|[F, M_Z]||f'(M_Z)|^{1/2})^p \in (\mathcal{L}_{1,\infty})_0. \tag{5.5.8}$$

Subtracting (5.5.8) from (5.5.7) yields

$$|[F, f(M_Z)]|^p - |[F, M_Z]|^p \cdot |f'(M_Z)|^p \in (\mathcal{L}_{1,\infty})_0.$$

Taking the adjoint, we arrive at

$$|[F, f(M_Z)]|^p - |f'(M_Z)|^p|[F, M_Z]|^p \in (\mathcal{L}_{1,\infty})_0.$$

□

We wish to show that ν_φ from Remark 5.5.2 is p-conformal with respect to ϕ, thus identifying it as the unique such measure on J (up to a constant). Let U be the linear map on $L_2(\mathbb{T})$ defined by $(Uh)(z) = h(z^d)$. By the definition of Z, we have:

$$UM_Z = M_{\phi \circ Z}U.$$

More generally, if g is any Borel function on J then:

$$UM_{g\circ Z} = M_{g\circ\phi\circ Z}U.$$

The following lemma contains the details required to prove that ν_φ is p-conformal with respect to ϕ.

Lemma 5.5.5. *Let φ be a positive continuous trace on $\mathcal{L}_{1,\infty}$, and let ν_φ be the corresponding measure from Remark 5.5.2. Suppose that A is an open subset of J such that $\phi|A$ is injective. Then ν_φ satisfies the following transformation property: for all $g \in C(J)$ supported in $\phi(A)$, we have*

$$\int_{\phi(A)} g\, d\nu_\varphi = \int_A (g\circ\phi)\cdot|\phi'|^p\, d\nu_\varphi.$$

Proof. Let $q(z) = z^d$, so that (5.4.5) may be restated as

$$\phi\circ Z = Z\circ q. \tag{5.5.9}$$

Recall that U is defined as the linear operator on $L_2(\mathbb{T})$ given by $(Uh)(z) = h(z^d) = (h\circ q)(z)$. Since by assumption ϕ is injective on A, it follows that q is injective on $Z^{-1}(A)$. Select a branch cut u of the function $z\mapsto z^{1/d}$ such that $u\circ q|_{Z^{-1}(A)} = \mathrm{id}|_{Z^{-1}(A)}$, and define the operator V on $L_2(\mathbb{T})$ given by:

$$(Vh)(z) = h(u(z)), \quad z\in\mathbb{T}.$$

Then,

$$VU = 1, \quad UVM_{\chi_{Z^{-1}(A)}} = M_{\chi_{Z^{-1}(A)}}.$$

Here, $\chi_{Z^{-1}(A)}$ is the indicator function of the set $Z^{-1}(A)\subset\mathbb{T}$, and so $M_{\chi_{Z^{-1}(A)}}$ is a projection on $L_2(\mathbb{T})$ and since g is supported in $\phi(A)$,

$$M_{g\circ\phi\circ Z} = M_{g\circ\phi\circ Z}M_{\chi_{Z^{-1}(A)}}.$$

As $VU = 1$, we have:

$$\varphi(M_{g\circ Z}|[F,M_Z]|^p) = \varphi(VUM_{g\circ Z}|[F,M_Z]|^p).$$

Since $UM_h = M_{h\circ q}U$ for all $h\in L_2(\mathbb{T})$, it follows that,

$$\varphi(M_{g\circ Z}|[F,M_Z]|^p) = \varphi(VM_{g\circ Z\circ q}U|[F,M_Z]|^p).$$

From (5.5.9), we get:

$$\varphi(M_{g\circ Z}|[F,M_Z]|^p) = \varphi(VM_{g\circ\phi\circ Z}U|[F,M_Z]|^p). \tag{5.5.10}$$

Since U commutes with F,

$$\begin{aligned} U[F,M_Z] &= [F,UM_Z] \\ &= [F,M_{Z\circ q}U] \\ &= [F,M_{Z\circ q}]U. \end{aligned}$$

The same argument yields

$$U[F, M_Z]^* = [F, M_{Z\circ q}]^* U.$$

So,

$$\begin{aligned} U|[F, M_Z]|^2 &= [F, M_{Z\circ q}]^* U[F, M_Z] \\ &= |[F, M_{Z\circ q}]|^2 U. \end{aligned}$$

By induction, for every $n \geq 1$,

$$U|[F, M_Z]|^{2n} = |[F, M_{Z\circ q}]|^{2n} U.$$

Hence for any polynomial r we have $Ur(|[F, M_Z]|^2) = r(|[F, M_{Z\circ q}]|^2)U$. Applying the continuous functional calculus with the function $r(t) = |t|^{p/2}$, it then follows that:

$$U|[F, M_Z]|^p = |[F, M_{Z\circ q}]|^p U.$$

We now have:

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \varphi(VM_{g\circ\phi\circ Z}|[F, M_{Z\circ q}]|^p U). \tag{5.5.11}$$

Applying (5.5.9), it follows that

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \varphi(VM_{g\circ\phi\circ Z}|[F, M_{\phi\circ Z}]|^p U).$$

However ϕ is a polynomial, so we can apply Theorem 5.5.4.(2) to the right hand side of the above to obtain:

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \varphi(VM_{g\circ\phi\circ Z}|\phi'(M_Z)|^p|[F, M_Z]|^p U).$$

Now using the cyclicity of the trace,

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \varphi(UVM_{(g\circ\phi Z)\cdot|\phi'\circ Z|^p}|[F, M_Z]|^p).$$

Now we use the fact that g is supported in $\phi(A)$. So we can multiply by the indicator function of $Z^{-1}(A)$:

$$M_{g\circ Z} = M_{\chi_{Z^{-1}(A)}} M_{g\circ Z}.$$

Since we have chosen V such that $UVM_{\chi_{Z^{-1}(A)}} = M_{\chi_{Z^{-1}(A)}}$, it follows that:

$$\varphi(M_{g\circ Z}|[F, M_Z]|^p) = \varphi(M_{(g\circ\phi\circ Z)\cdot|\phi'\circ Z|^p}|[F, M_Z]|^p).$$

and this is the desired result. □

The following proposition is the main result of this section.

Proposition 5.5.6. *The measure ν_φ from Remark 5.5.2 corresponding to a positive continuous trace φ is p-conformal with respect to the map ϕ.*

Proof. Let U be an open subset of J such that $\phi|U$ is injective and let g be a continuous function on J supported in $\phi(U)$. Since ϕ is injective when restricted to U, it is easy to see that $\phi(U)$ is also open. Since g is supported on $\phi(U)$, Lemma 5.5.5 states that:

$$\int_{\phi(U)} g\, d\nu_\varphi = \int_U (g \circ \phi)|\phi'|^p\, d\nu_\varphi.$$

Since $\phi|U$ is injective, as g varies over all continuous functions supported in $\phi(U)$, $(g \circ \phi)|U$ varies over all continuous functions supported in U. So,

$$\sup_{\operatorname{supp}(g)\subseteq\phi(U),\|g\|_\infty\leq 1} \int_{\phi(U)} g\, d\nu_\varphi = \sup_{\operatorname{supp}(h)\subseteq U,\|h\|_\infty\leq 1} \int_U h|\phi'|^p\, d\nu_\varphi.$$

Since ν_φ is positive, it follows from the Riesz theorem that we have an equality of measures,

$$\nu_\varphi(\phi(U)) = \int_U |\phi'|^p\, d\nu_\varphi$$

for all open subsets U such that $\phi|_U$ is injective. Due to the regularity of the measure ν_φ it follows that for all Borel subsets A such that $\phi|A$ is injective that:

$$\nu_\varphi(\phi(A)) = \int_A |\phi'|^p\, d\nu_\varphi.$$

This is precisely the desired result. □

We may now finally complete the proof of Theorem 5.1.3.(b).

Corollary 5.5.7. *Let φ be a continuous Hermitian (not necessarily positive) trace on $\mathcal{L}_{1,\infty}$. There is a constant $K(\varphi,\phi)$ such that for all $f \in C(J)$*

$$\varphi(M_{f\circ Z}|[F, M_Z]|^p) = K(\varphi,\phi)\int_J f\, d\nu \tag{5.5.12}$$

where ν is the (essentially unique) p-conformal measure on J with respect to ϕ.

Proof. If φ is positive, then this is simply a restatement of Proposition 5.5.6. For general traces φ, we may use Corollary 1.5.2 to write $\varphi = \varphi_+ - \varphi_-$ for positive traces φ_+ and φ_-. Then,

$$\varphi(M_{f\circ Z}|[F, M_z]|^p) = K(\varphi_+,\phi)\int_J f\, d\nu_{\varphi_+} - K(\varphi_-,\phi)\int_J f\, d\nu_{\varphi_-}.$$

Then applying Proposition 5.5.6 to the positive traces φ_+ and φ_- individually, we have that ν_{φ_+} and ν_{φ_-} is p-conformal. Hence the measure $K(\varphi_+,\phi)\nu_{\varphi_+} - K(\varphi_-,\phi)\nu_{\varphi_-}$ is p-conformal, so there is a constant $K(\varphi,\phi)$ such that

$$K(\varphi_+,\phi)\nu_{\varphi_+} - K(\varphi_-,\phi)\nu_{\varphi_-} = K(\varphi,\phi)\nu$$

where ν is the essentially unique p-conformal measure on J with respect to ϕ. □

Remark 5.5.8. *Since ϕ is hyperbolic on J by assumption, the p-conformal measure on J is identical to the p-dimensional Hausdorff measure on J by [130, Theorem 4], so Corollary 5.5.7 could also be stated with ν denoting the Hausdorff measure m_p.*

5.6 Non-triviality of the conformal trace formula

The remaining task is to show that the formula (5.5.12) is nontrivial: that is, that there is φ such that $K(\varphi, \phi) > 0$. We show that indeed such a φ does exist, and is given by a Dixmier trace tr_ω where ω is a dilation invariant extended limit such that $\omega \circ \log$ is also dilation invariant.[3] We achieve this using [89, Theorem 8.6.8], which states that if ω is a dilation invariant extended limit on $L_\infty(0,\infty)$ such that $\omega \circ \log$ is still dilation invariant, then the Dixmier trace tr_ω is equal to the following ζ-function residue:

$$\mathrm{tr}_\omega(T) = (\omega \circ \log)(t \mapsto \frac{1}{t}\mathrm{tr}(T^{1+1/t})), \quad 0 \leq T \in \mathcal{L}_{1,\infty}.$$

Hence to show that $\mathrm{tr}_\omega(|[F,Z]|^p) > 0$, it suffices to show that

$$\liminf_{s\to 0} s \cdot \mathrm{tr}(|[F,Z]|^{p+s}) > 0.$$

The crucial result is the following, which is stated as [28, Chapter 4, Section 3.α, Proposition 7]:

Proposition 5.6.1. *Let $\mathcal{C}$ be a Jordan curve with interior Ω, and let ξ be a conformal map $\xi : \mathbb{D} \to \Omega$. Since ξ extends continuously to $\mathbb{T}$, we may consider ξ as a function on $\mathbb{T}$. Let $p_0 > 1$. Then then there are positive constants C_{p_0} and c_{p_0} such that:*

$$c_{p_0}\int_\Omega \mathrm{dist}(z,\partial\Omega)^{p-2}dzd\overline{z} \leq \mathrm{tr}(|[F,M_\xi]|^p) \leq C_{p_0}\int_\Omega \mathrm{dist}(z,\partial\Omega)^{p-2}dzd\overline{z}$$

for all $p > p_0$.

Proof. This result is an immediate consequence of Lemma 5.3.1 and Theorem 5.2.1. □

Proposition 5.6.2. *Let $\mathcal{C}$ be a Jordan curve with finite upper p-dimensional Minkowski content and positive lower p-dimensional Minkowski content. Let Ω be the interior of $\mathcal{C}$ so that $\partial\Omega = \mathcal{C}$. Then,*

$$\liminf_{s\to 0} s \cdot \int_\Omega \mathrm{dist}(z,\mathcal{C})^{p+s-2}\, dzd\overline{z} > 0.$$

Proof. By the assumption that $\mathcal{C}$ has positive lower p-Minkowski content and finite upper p-Minkowski content, there are constants $b, B > 0$ such that

$$b\delta^{2-p} \leq |S_\delta(C)\cap\Omega| \leq B\delta^{2-p}, \quad \forall \delta > 0.$$

Let $\lambda > 0$. Define $A_k \subseteq \Omega$, $k \geq 1$ by

$$A_k = \{z \in \Omega \;:\; \mathrm{dist}(z,\mathcal{C}) \in [\lambda^{-k}, \lambda^{1-k})\}.$$

[3] For an extended limit $\omega \in L_\infty(0,\infty)^*$, the notation $\omega \circ \log$ denotes the extended limit defined as $f \mapsto \omega(f \circ \max\{\log, 0\})$.

So,

$$\begin{aligned}|A_k| &= |S_{\lambda^{1-k}}(\mathcal{C})\cap\Omega| - |S_{\lambda^{-k}}(\mathcal{C})\cap\Omega|\\ &\geq (b\lambda^{(1-k)(2-p)} - B\lambda^{-k(2-p)})\\ &= (b\lambda^{2-p}-B)\lambda^{-k(2-p)}.\end{aligned}$$

Now fix $\lambda > 1$ such that $b_0 := b\lambda^{2-p} - B > 0$. Then,

$$\begin{aligned}\int_\Omega \mathrm{dist}(z,\mathcal{C})^{p+s-2}\,dzd\overline{z} &\geq \sum_{k=0}^\infty b_0\lambda^{-kp-ks+kp}\lambda^{-2k+kp}\\ &= \sum_{k=0}^\infty b_0\lambda^{-ks}\\ &= \frac{b_0}{1-\lambda^{-s}}.\end{aligned}$$

From the l'Hôpital rule, the limit as $s\to 0$ of $\frac{s}{1-\lambda^{-s}}$ is $\frac{1}{\log(\lambda)}$. Hence $\liminf_{s\to 0}\frac{b_0 s}{1-\lambda^{-s}} = \frac{b_0}{\log(\lambda)} > 0$. □

Due to Lemma 5.4.3 we can apply the above proposition to immediately obtain:

Corollary 5.6.3. *Let ω be a dilation invariant extended limit on $L_\infty(0,\infty)$ such that $\omega\circ\log$ is still dilation invariant. Then*

$$\mathrm{tr}_\omega(|[F,M_Z]|^p) > 0.$$

5.7 Final comments

A reasonable conjecture is that $K(\varphi,\phi)$ does not depend on φ up to normalisation, however this problem is still open.

It is noteworthy that Sections 5.2, 5.3, 5.6 and the proof of Proposition 5.5.3 make no explicit reference to Julia sets. In fact, much of the mathematics in this chapter applies equally well to arbitrary Jordan curves with finite upper and positive lower p-Minkowski content.

Indeed, with an almost verbatim repetition of the proofs in this chapter one can prove the following: if $\mathcal{C}$ is a Jordan curve with finite upper and positive lower p-Minkowski content, where $1 < p < 2$, and Z is a conformal equivalence between the exterior of $\mathbb{D}$ and the exterior of $\mathcal{C}$, extended continuously to the boundary, then $[F,M_Z]\in\mathcal{L}_{p,\infty}$ and the functional:

$$g\mapsto \varphi(M_{g\circ Z}|[F,M_Z]|^p),\quad g\in C(\mathcal{C})$$

represents a measure on $\mathcal{C}$ for any continuous trace φ. If φ is a Dixmier trace satisfying the conditions of Section 5.6, then the measure is non-trivial. It would be of great interest to identify this measure for Jordan curves other than limit sets of quasi-Fuchsian groups or Julia sets of admissible polynomials.

Before concluding this chapter, let us make one more aside about the curious place of the Conformal Trace Theorem in noncommutative geometry. Despite making essential use of noncommutative tools, such as singular traces and the difference of powers formula, it is not yet clear how the theorem fits into the broader noncommutative paradigm. There has been significant research on the application of operator algebraic and noncommutative geometric ideas to dynamical systems and in particular to Julia sets (such as the recent work of Kaminker, Putnam and Whittaker in the geometry of Smale spaces [78]), however the relationship to the conformal trace theorem remains unclear to the author.

Chapter 6

Connes' trace theorem for noncommutative Euclidean spaces

The following chapter is primarily based on the published paper [94], a joint work of the author with F. Sukochev and D. Zanin. The presentation here is slightly simplified from the published version of [94], since we restrict attention to the example of quantum Euclidean spaces. Nonetheless, some of the proofs remain essentially unchanged.

The purpose of [94] was to prove an analogy for Connes' trace theorem for noncommutative planes and noncommutative tori. Connes' trace theorem [27, Theorem 1] concerns classical pseudodifferential operators on compact manifolds. Suppose that (X, g) is a compact n-dimensional manifold, and T is a classical pseudodifferential operator of order $-n$ on X. If we consider T as an operator on the space $L_2(X, g)$, then in fact T is in the ideal $\mathcal{L}_{1,\infty}$. If we evaluate a Dixmier trace tr_ω (c.f. Section 1.5.2) on T, then the result is the *Wodzicki residue* of T, which may be computed as the integral of the degree $-n$ homogeneous component of the symbol of T over the cosphere bundle of X, with respect to the measure induced by g. Similar statements are also possible with certain non-compact manifolds (such as $\mathbb{R}^d$) and with wider classes of traces [89, Chapter 11].

Here we take a "simple minded" approach to Connes' trace theorem. Rather than develop a theory of pseudodifferential operators, we consider a C^*-algebra Π generated by homogeneous Fourier multipliers and left multiplication operators. The algebra Π is our substitute for the algebra of pseudodifferential operators of order zero. Connes' trace theorem in this setting is ultimately a consequence of the structure of tensor products of C^*-algebras. The essential idea is to associate the principal symbol map (strictly speaking, the zeroth order symbol map) with the quotient map with respect to the ideal of compact operators. This is an idea which originates with and which was a central feature of the work of H. O. Cordes [37]. Generally speaking, the identification of the symbol mapping as a Calkin quotient map is essential to the K-homological viewpoint on index theory, briefly alluded to in Section 1.1.2. See for example the pioneering work of Brown, Douglas and Fillmore [22] for this approach.

Our original motivation for developing such an operator algebraic approach was to handle operators with non-smooth symbols. Indeed, the C^*-algebraic machinery in the following sections permits the use of operators with low regularity with relative ease. The cost of this is a certain lack of flexibility: the theory developed in [94] relies on the triviality of the tangent bundles of the manifolds under consideration. This is a severe disadvantage compared to the standard theory of pseudodifferential operators. In addition, we make no attempt to study operators of order greater than zero, so in particular differential operators are excluded. This theory is certainly no substitute for a full pseudodifferential calculus, and is intended to be complementary to the standard theory rather than a replacement.

Despite these weaknesses, the C^*-algebraic perspective yields previously unknown extensions of Connes' trace theorem and it does so with clarity and simplicity. The trace theorem for quantum tori has since been used to obtain a very general characterisation of quantum differentiability in that setting [93]. In Chapter 7, we will use the machinery developed here to study quantum differentiability on noncommutative Euclidean spaces.

Quantum Euclidean spaces were first introduced by a number of authors, including Groenewold [65] and Moyal [96], for the study of quantum mechanics in phase space. The constructions of Groenewold and Moyal were later abstracted into more general canonical commutation relation (CCR) algebras, and have since become fundamental in mathematical physics. Under the names Moyal planes or Moyal-Groenewold planes, these algebras play the role of a central and motivating example in noncommutative geometry [23, 55]. As geometrical spaces with noncommuting spatial coordinates, noncommutative Euclidean spaces have appeared frequently in the mathematical physics literature [46], in the contexts of string theory [122] and noncommutative field theory [97].

Quantum Euclidean spaces have also been studied as an interesting noncommutative setting for classical and harmonic analysis, and for this we refer the reader to recent work such as [62, 83, 94, 129].

6.1 Algebraic preliminaries

Before discussing Connes' trace theorem, it is helpful and insightful to take a broader view and consider an abstract setting.

6.1.1 C^*-norms on tensor products of C^*-algebras

Given two C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, we denote the algebraic tensor product as $\mathcal{A}_1 \odot \mathcal{A}_2$. The following results are taken from [118] (see Theorem 1.22.6, Propositions 1.22.5 and 1.22.3 there).

Theorem 6.1.1. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital C^*-algebras. There are pre-C^*-norms on the algebraic tensor product $\mathcal{A}_1 \odot \mathcal{A}_2$, and there exists a norm which is minimal.*

The completion of $\mathcal{A}_1 \odot \mathcal{A}_2$ with respect to the minimal C^*-norm is denoted by $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$.

Theorem 6.1.2. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital C^*-algebras. If $\mathcal{A}_2$ is commutative, then there exists a unique pre-C^*-norm on $\mathcal{A}_1 \odot \mathcal{A}_2$ (which we may take to be the minimal one).*

The above theorem is a essentially a statement of the fact that commutative algebras are nuclear.

Theorem 6.1.3. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital C^*-algebras. If $\mathcal{A}_2$ is commutative (read $\mathcal{A}_2 = C(X)$ for some compact Hausdorff space X), then $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ is isometrically $*$-isomorphic to $C(X, \mathcal{A}_1)$.*

Theorem 6.1.3 is an immediate consequence of Theorem 6.1.2, since the embedding of $\mathcal{A}_1 \odot C(X)$ into $C(X, \mathcal{A}_1)$ induces a C^* norm on $\mathcal{A}_1 \odot \mathcal{A}_2$.

We also have,

Theorem 6.1.4. *Let $\psi_1 \in \mathcal{A}_1^*$ and $\psi_2 \in \mathcal{A}_2^*$. Then the tensor product $\psi_1 \otimes \psi_2$ extends continuously to $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$.*

Proof. First we may normalise $\|\psi_1\|_{\mathcal{A}_1^*} = \|\psi_2\|_{\mathcal{A}_2^*} = 1$. Then for $T \in \mathcal{A}_1 \odot \mathcal{A}_2$,

$$|(\psi_1 \otimes \psi_2)(T)| \leq \sup_{\alpha \otimes \beta \in \mathcal{A}_1^* \odot \mathcal{A}_2^*, \|\alpha\| = \|\beta\| = 1} |(\alpha \otimes \beta)(T)| = \|T\|_{\mathcal{A}_1 \otimes_\varepsilon \mathcal{A}_2}.$$

(Recall the injective tensor product norm $\|\cdot\|_{\mathcal{A}_1 \otimes_\varepsilon \mathcal{A}_2}$ from Definition 3.1.1). From [118, Proposition 1.22.2], we have that $\|T\|_{\mathcal{A}_1 \otimes_\varepsilon \mathcal{A}_2} \leq \|T\|_{\min}$. □

6.2 Noncommutative Euclidean space

For this section, let θ be a $d \times d$ real antisymmetric matrix with trivial kernel[1]. Our approach is to define the Noncommutative Euclidean space (also known as the Moyal plane) in terms of a certain family of unitary operators $\{U(t)\}_{t \in \mathbb{R}^d}$.

Definition 6.2.1. *Let $t \in \mathbb{R}^d$. We define the following linear operator on $L_2(\mathbb{R}^d)$,*

$$(U(t)\xi)(u) = e^{-\frac{i}{2}(t, \theta u)} \xi(u - t).$$

The family $\{U(t)\}_{t \in \mathbb{R}^d}$ then consists of unitary operators satisfying

$$U(t)U(s) = e^{\frac{i}{2}(t, \theta s)} U(t + s). \tag{6.2.1}$$

The algebra $L_\infty(\mathbb{R}^d_\theta)$ is then defined to be the von Neumann algebra generated by $\{U(t)\}_{t \in \mathbb{R}^d}$.

Denote the representation of $L_\infty(\mathbb{R}^d_\theta)$ on $L_2(\mathbb{R}^d)$ as π_1[2].

It is known (see [129]) that there is an isometric $*$-isomorphism from $\mathcal{B}(L_2(\mathbb{R}^{d/2})) \to L_\infty(\mathbb{R}^d_\theta)$. Denote the image of the compact operators $\mathcal{K}(L_2(\mathbb{R}^{d/2}))$ under this isomorphism $C_0(\mathbb{R}^d_\theta)$. The standard trace Tr on $\mathcal{B}(L_2(\mathbb{R}^{d/2}))$ then induces a semifinite trace on the algebra $L_\infty(\mathbb{R}^d_\theta)$, which we denote as τ_θ.

We define $L_2(\mathbb{R}^d_\theta)$ to be the GNS-space for $L_\infty(\mathbb{R}^d_\theta)$ defined by τ_θ.

Remark 6.2.2. *We also note that if we formally take $\theta = 0$ in Definition 6.2.1 we recover the commutative algebra $L_\infty(\mathbb{R}^d)$. However our definitions of τ_θ and $C_0(\mathbb{R}^d_\theta)$ rely on the non-degeneracy of θ.*

[1] this implies that d is automatically even
[2] π_1 is simply the identity mapping

Definition 6.2.3. *For $k = 1, \ldots, n$, let ∂_k denote the multiplication operators on $L_2(\mathbb{R}^d)$,*

$$D_k\xi(t) = t_k\xi(t).$$

We define the operators $\partial_k x$, $k = 1, \ldots, d$ by

$$\partial_k x := i[D_k, x].$$

There exists a dense subspace $\mathcal{D} \subset L_2(\mathbb{R}^d_\theta)$ such that the operators $\partial_k, k = 1, \ldots, d$ may be considered as self-adjoint operators on $L_2(\mathbb{R}^d_\theta)$ with common core $\mathcal{D}$. We denote $\nabla = (\partial_1, \partial_2, \ldots, \partial_d)$, considered as a self-adjoint linear operator from $L_2(\mathbb{R}^d_\theta)$ to $L_2(\mathbb{R}^d_\theta) \otimes \mathbb{C}^d$. For a multi-index α, define

$$\partial^\alpha := \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_d^{\alpha_d}$$

which is also considered as a self-adjoint operator on $L^2(\mathbb{R}^d_\theta)$.

Definition 6.2.4. *With τ_θ we can define L_p-spaces associated to $L_\infty(\mathbb{R}^d_\theta)$ with the norm:*

$$\|x\|_p := \tau_\theta(|x|^p)^{1/p}, \quad x \in L_\infty(\mathbb{R}^d_\theta).$$

Note that this is consistent with our definition of $L_2(\mathbb{R}^d_\theta)$ as a GNS-space.

The corresponding Sobolev space, $W^k_p(\mathbb{R}^d_\theta)$ is defined to be the set of $x \in L_p(\mathbb{R}^d_\theta)$ with $\partial^\alpha x \in L_p(\mathbb{R}^d_\theta)$ for all $|\alpha| \leq k$. The W^k_p norm is the sum of the L_p norms of $\nabla^\alpha x$ for all $0 \leq |\alpha| \leq k$.

6.2.1 Cwikel-type estimates for Noncommutative Euclidean Space

The following is [83, Proposition 6.15(v)],

Lemma 6.2.5. *$W^m_2(\mathbb{R}^d_\theta)$ is a norm-dense subset of $C_0(\mathbb{R}^d_\theta)$ for every $m \geq 0$.*

We also require the following Theorem, which is a special case of [83, Theorem 7.2]:

Theorem 6.2.6. *Let $p \in [2, \infty)$. If $x \in L_p(\mathbb{R}^d_\theta)$ and $g \in L_p(\mathbb{R}^d)$, then*

$$\|\pi_1(x)g(\nabla)\|_{\mathcal{L}_p} \leq C(p, d, \theta)\|x\|_{L_p(\mathbb{R}^d_\theta)}\|g\|_{L_p(\mathbb{R}^d)}.$$

If $p \in (2, \infty)$, and $g \in L_{p,\infty}(\mathbb{R}^d)$, then:

$$\|\pi_1(x)g(\nabla)\|_{\mathcal{L}_{p,\infty}} \leq C(p, d, \theta)\|x\|_{L_p(\mathbb{R}^d_\theta)}\|g\|_{L_{p,\infty}(\mathbb{R}^d)}.$$

The space $\ell_{1,\infty}(L_\infty)(\mathbb{R}^d)$ is defined as the set of $g \in L_\infty(\mathbb{R}^d)$ such that:

$$\left\{\operatorname*{esssup}_{t \in n+[0,1]^d} |g(t)|\right\}_{n \in \mathbb{Z}^d} \in \ell_{1,\infty}(\mathbb{Z}^d).$$

The space $\ell_1(L_\infty)(\mathbb{R}^d)$ is defined similarly, with ℓ_1 in place of $\ell_{1,\infty}$.

The following is a special case of [83, Theorem 7.6]:

Theorem 6.2.7. *For every $x \in W^d_1(\mathbb{R}^d_\theta)$ and $g \in \ell_{1,\infty}(L_\infty(\mathbb{R}^d))$ we have that $\pi_1(x)g(\nabla) \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$ and $\|\pi_1(x)g(\nabla)\|_{1,\infty} \leq C_{d,\theta}\|x\|_{W^d_1}\|g\|_{\ell_{1,\infty}(L_\infty)}$.*

Applying Theorem 6.2.7 to the function $g(t) = (1 + |t|^2)^{-d/2}$, we obtain a corollary,

Corollary 6.2.8. *If* $x \in W_1^d(\mathbb{R}_\theta^d)$, *then*

$$\|\pi_1(x)(1-\Delta)^{-d/2}\|_{1,\infty} \le C_d \|x\|_{W_1^d}.$$

The following is an $\mathcal{L}_1$ Cwikel estimate, proved in [83, Theorem 7.7].

Lemma 6.2.9. *If* $g \in \ell^1(L_\infty)(\mathbb{R}^d)$ *and* $x \in W_1^d(\mathbb{R}_\theta^d)$, *then* $\pi_1(x)g(\nabla) \in \mathcal{L}_1$.

6.3 Main construction

We now proceed to define the simple algebraic construction which underlies our version of Connes' trace theorem.

Lemma 6.3.1. *Let* $\mathcal{A}_1$, $\mathcal{A}_2$ *and* $\mathcal{B}$ *be* C^*-*algebras and let* $\rho_1 : \mathcal{A}_1 \to \mathcal{B}$ *and* $\rho_2 : \mathcal{A}_2 \to \mathcal{B}$ *be* C^*-*homomorphisms. Suppose that*

1. *$\rho_1(x)$ commutes with $\rho_2(y)$ for all $x \in \mathcal{A}_1$, $y \in \mathcal{A}_2$.*
2. *The mapping $\theta : \mathcal{A}_1 \odot \mathcal{A}_2 \to \mathcal{B}$ defined by the formula*

$$\theta(a_1 \otimes a_2) = \rho_1(a_1)\rho_2(a_2), \quad a_1 \in \mathcal{A}_1,\ a_2 \in \mathcal{A}_2,$$

   is injective.
3. *$\mathcal{A}_1, \mathcal{A}_2$ are unital and $\mathcal{A}_2$ is abelian.*
4. *$\mathcal{B}$ is generated by $\rho_1(\mathcal{A}_1)$ and $\rho_2(\mathcal{A}_2)$.*

Under these conditions, θ extends to a C^*-algebra isomorphism

$$\theta : \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \to \mathcal{B}.$$

Proof. Condition (1) is required to ensure that that θ is a $*-$homomorphism. Condition (2) states that θ is an injection on the algebraic tensor product.

At this stage, we have an injective $*$-homomorphism from $\mathcal{A}_1 \odot \mathcal{A}_2$ to $\mathcal{B}$. This allows us to define a pre-C^*-norm on $\mathcal{A}_1 \odot \mathcal{A}_2$ by setting

$$\|T\| = \|\theta(T)\|_{\mathcal{B}}, \quad T \in \mathcal{A}_1 \odot \mathcal{A}_2.$$

By condition (3) and Theorem 6.1.2, the latter norm coincides with the minimal pre-C^*-norm on $\mathcal{A}_1 \odot \mathcal{A}_2$. Thus, $\theta : \mathcal{A}_1 \odot \mathcal{A}_2 \to \mathcal{B}$ is an isometric embedding of the algebra $\mathcal{A}_1 \odot \mathcal{A}_2$ equipped with the minimal C^*-norm into $\mathcal{B}$. Since $\mathcal{A}_1 \odot \mathcal{A}_2$ is dense in $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$, the surjectivity of θ and the conclusion of the lemma follow from the condition (4).

□

Remark 6.3.2. *Lemma 6.3.1 uses the fact that there is a unique pre-C^*-norm on $\mathcal{A}_1 \odot \mathcal{A}_2$. It is enough to assume that one of the factors is nuclear, instead of abelian. For the remainder of this text we restrict to the case where one factor is abelian.*

Let $\mathcal{Q}(H)$ be the Calkin algebra and let $q : \mathcal{L}(H) \to \mathcal{Q}(H)$ be the quotient mapping.

Theorem 6.3.3. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be C^*-algebras and let $\pi_1 : \mathcal{A}_1 \to \mathcal{L}(H)$ and $\pi_2 : \mathcal{A}_2 \to \mathcal{L}(H)$ be representations. Let $\Pi(\mathcal{A}_1, \mathcal{A}_2)$ be the C^*-algebra generated by $\pi_1(\mathcal{A}_1)$ and $\pi_2(\mathcal{A}_2)$. Suppose that*

1. *$\mathcal{A}_1, \mathcal{A}_2$ are unital and $\mathcal{A}_2$ is abelian.*
2. *The representations π_1 and π_2 "commute modulo compact operators" i.e., for all $a_1 \in \mathcal{A}_1$ and $a_2 \in \mathcal{A}_2$ the commutator $[\pi_1(a_1), \pi_2(a_2)]$ is compact.*
3. *If $x_k \in \mathcal{A}_1$, $y_k \in \mathcal{A}_2$, $1 \le k \le n$, then*
$$\sum_{k=1}^n \pi_1(x_k)\pi_2(y_k) \in \mathcal{K}(H) \Longrightarrow \sum_{k=1}^n x_k \otimes y_k = 0.$$

There exists a unique continuous $*-$homomorphism $\mathrm{sym} : \Pi(\mathcal{A}_1, \mathcal{A}_2) \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ *such that*

$$\mathrm{sym}(\pi_1(x)) = x \otimes 1, \quad \mathrm{sym}(\pi_2(y)) = 1 \otimes y, \quad x \in \mathcal{A}_1, y \in \mathcal{A}_2.$$

Proof. This is a special case of Lemma 6.3.1 with $\mathcal{B} = q(\Pi(\mathcal{A}_1, \mathcal{A}_2))$, and $\rho_j = q \circ \pi_j$, $j = 1, 2$. We verify each of the required conditions. Condition 6.3.1(4) is satisfied since by definition $\mathcal{B} = q(\Pi(\mathcal{A}_1, \mathcal{A}_2))$ is generated by $\rho_1(\mathcal{A}_1)$ and $\rho_2(\mathcal{A}_2)$. Condition 6.3.1(1) follows from (2). Condition 6.3.1(3) is automatic, due to (1).

Finally, condition 6.3.1(2) is a consequence of (3).

Thus, Lemma 6.3.1 states that
$$\theta := \rho_1 \otimes \rho_2$$
defines an isometric $*$-isomorphism $\theta : \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2 \to \mathcal{B}$.

Define
$$\mathrm{sym} := \theta^{-1} \circ q.$$

By construction $\mathrm{sym} : \Pi(\mathcal{A}_1, \mathcal{A}_2) \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ is a continuous $*$-algebra homomorphism. Let $x \in \mathcal{A}_1$. Then $\mathrm{sym}(\pi_1(x)) = \theta^{-1}(q(\pi_1(x)))$, and since $\theta(x \otimes 1) = \rho_1(x) = q(\pi_1(x))$, we get that $\mathrm{sym}(\pi_1(x)) = x \otimes 1$. Similarly, if $y \in \mathcal{A}_2$ then $\mathrm{sym}(\pi_2(y)) = 1 \otimes y$.

As $\Pi(\mathcal{A}_1, \mathcal{A}_2)$ is generated by $\pi_1(\mathcal{A}_1)$ and $\pi_2(\mathcal{A}_2)$, and sym is continuous, it follows that sym is uniquely determined by its restriction to $\pi_1(\mathcal{A}_1)$ and $\pi_2(\mathcal{A}_2)$. $\square$

Lemma 6.3.4. *If $T \in \mathcal{K}(H)$ and if $\{p_k\}_{k \ge 0} \subset \mathcal{L}(H)$ is a sequence of pairwise orthogonal projections, then $\|Tp_k\|_\infty \to 0$ as $k \to \infty$.*

Proof. Let $\varepsilon > 0$, and let $T = T_1 + T_2$, where T_1 is finite rank and $\|T_2\|_\infty < \varepsilon$. Since,

$$\|Tp_k\|_\infty \leq \|T_1 p_k\|_\infty + \varepsilon$$

it suffices to show that $\|T_1 p_k\|_\infty \to 0$.

Note that $\|T_1 p_k\|_\infty \leq \|T_1 p_k\|_2$. As each p_k is pairwise orthogonal and $T_1 \in \mathcal{L}_2$,

$$\sum_{k=0}^{\infty} \|T_1 p_k\|_2^2 = \left\| T_1 \sum_{k=0}^{\infty} p_k \right\|_2^2 < \infty.$$

Thus $\lim_{k\to\infty} \|T_1 p_k\|_2 = 0$. □

Lemma 6.3.5. *Let $x_k \in C_0(\mathbb{R}^d_\theta)$ and $y_k \in C(S^{d-1})$, $1 \leq k \leq n$. If*

$$\sum_{k=1}^{n} \pi_1(x_k)\pi_2(y_k) \in \mathcal{K}(L_2(\mathbb{R}^d)),$$

then

$$\sum_{k=1}^{n} x_k \otimes y_k = 0.$$

Proof. Fix $s \in S^{d-1}$ and choose a sequence $\{m_j\}_{j\geq 0} \subset \mathbb{Z}^d$ such that $\frac{m_j}{|m_j|} \to s$ and $|m_j| \to \infty$ as $j \to \infty$. It follows that

$$\sup_{t\in m_j+[0,1]^d} \left| \frac{t}{|t|} - s \right| \to 0, \quad j \to \infty.$$

By continuity, we have

$$\sup_{t\in m_j+[0,1]^d} \left| y_k(\frac{t}{|t|}) - y_k(s) \right| \to 0, \quad j \to \infty.$$

By the spectral theorem, we have

$$\pi_2(y_k)\chi_{m_j+[0,1]^d}(\nabla) - y_k(s)\chi_{m_j+[0,1]^d}(\nabla) \to 0$$

in the uniform norm as $j \to \infty$.

By Lemma 6.3.4, we have that

$$\sum_{k=1}^{n} \pi_1(x_k)\pi_2(y_k)\chi_{m_j+[0,1]^d}(\nabla) \to 0$$

in the uniform norm as $j \to \infty$. By the preceding paragraph, we have

$$\sum_{k=1}^{n} \pi_1(x_k)y_k(s)\chi_{m_j+[0,1]^d}(\nabla) \to 0$$

in the uniform norm as $j\to\infty$. By Lemma [83, Lemma 7.5], there exists a unitary operator $V_j\in\mathcal{L}(L_2(\mathbb{R}^d))$ which commutes with $L_\infty(\mathbb{R}^d_\theta)$ and such that

$$V_j\chi_{m_j+[0,1]^d}(\nabla)V_j^{-1}=\chi_{[0,1]^d}(\nabla).$$

Thus,

$$\sum_{k=1}^n\pi_1(x_k)y_k(s)\chi_{[0,1]^d}(\nabla)=V\cdot(\sum_{k=1}^n\pi_1(x_k)y_k(s)\chi_{m_j+[0,1]^d}(\nabla))\cdot V^{-1}\to 0$$

in the uniform norm as $j\to\infty$. The left hand side does not depend on j and, therefore,

$$\sum_{k=1}^n\pi_1(x_k)y_k(s)\chi_{[0,1]^d}(\nabla)=0.$$

Since each $y_k(s)$ is a scalar, we have

$$\pi_1(\sum_{k=1}^n x_k\cdot y_k(s))\cdot\chi_{[0,1]^d}(\nabla)=0.$$

Appealing once again to [83, Lemma 7.5], there exists a family of unitaries $\{V_n\}_{n\in\mathbb{Z}^d}$ on $L_2(\mathbb{R}^d)$, each of which commutes with $L_\infty(\mathbb{R}^d_\theta)$ and such that $V_n\chi_{[0,1]^d}(\nabla)V_n^{-1}=\chi_{n+[0,1]^d}(\nabla)$. Conjugating by V_n and summing over n yields:

$$\pi_1\left(\sum_{k=1}^n x_k\cdot y_k(s)\right)\cdot\sum_{n\in\mathbb{Z}^d}\chi_{n+[0,1]^d}(\nabla)=0$$

where the sum converges in the strong operator topology. However $\sum_{n\in\mathbb{Z}^d}\chi_{n+[0,1]^d}(\nabla)=1$. Hence,

$$\sum_{k=1}^n x_k\cdot y_k(s)=0.$$

Since $s\in S^{d-1}$ is arbitrary, the assertion follows.

$\square$

The preceding Lemma applies for $\mathbb{R}^d_\theta$, where as always the assumption is made that $\det(\theta)\neq 0$. We also record the following, which applies for the commutative case $\mathbb{R}^d$:

Lemma 6.3.6. *Let $x_k\in L_\infty(\mathbb{R}^d)$ and $y_k\in C(S^{d-1})$, $1\le k\le n$. If*

$$\sum_{k=1}^n\pi_1(x_k)\pi_2(y_k)\in\mathcal{K}(L_2(\mathbb{R}^d)),$$

then

$$\sum_{k=1}^n x_k\otimes y_k=0.$$

Proof. The argument of Lemma 6.3.5 works *mutatis mutandi* for this case, by taking instead $(V_j\xi)(t):=e^{-i(m_j,t)}\xi(t)$ rather than referring to [83, Lemma 7.5].

□

6.4 Verification of the commutator condition

In the published version of [94], a self-contained proof of the following lemma was included. Here, such a proof is redundant since in the next chapter we will prove a much more general result (Theorem 7.1.6). Therefore the proof of the following will be deferred to the next chapter.

Lemma 6.4.1. *There is a norm dense subspace $\mathcal{S}(\mathbb{R}^d_\theta) \subset C_0(\mathbb{R}^d_\theta)$ such that for all $x \in \mathcal{S}(\mathbb{R}^d_\theta)$, we have:*

$$[(1-\Delta)^{\frac{1}{2}}, \pi_1(x)](1-\Delta)^{-\frac{1}{2}} \in \mathcal{K}(L_2(\mathbb{R}^d)). \tag{6.4.1}$$

The subspace $\mathcal{S}(\mathbb{R}^d_\theta)$ will be explicitly defined in (7.1.6), and its density in $C_0(\mathbb{R}^d_\theta)$ is an immediate consequence of Proposition 7.1.9. For now, we do not need the details of the definition and only need to know that $\mathcal{S}(\mathbb{R}^d_\theta)$ is norm-dense in $C_0(\mathbb{R}^d_\theta)$.

The operators $\frac{D_k}{\sqrt{-\Delta}}$, $k = 1, \ldots, d$ are the noncommutative equivalent of the Riesz transforms R_k. The following Lemma can be viewed as a noncommutative variant of the classical result that if $f \in C_0(\mathbb{R}^d)$, then the commutators $[M_f, R_k]$ are compact.

Lemma 6.4.2. *If $x \in \mathcal{S}(\mathbb{R}^d_\theta)$, then*

$$[\pi_1(x), \frac{D_k}{(-\Delta)^{\frac{1}{2}}}] \in \mathcal{K}(L_2(\mathbb{R}^d)), \quad k = 1, \ldots, d \tag{6.4.2}$$

Proof. Firstly, we consider the commutator

$$[\pi_1(x), \frac{D_k}{(1-\Delta)^{\frac{1}{2}}}] = -[D_k, \pi_1(x)](1-\Delta)^{-\frac{1}{2}} + \frac{D_k}{(1-\Delta)^{\frac{1}{2}}} \cdot [(1-\Delta)^{\frac{1}{2}}, \pi_1(x)](1-\Delta)^{-\frac{1}{2}}.$$

Using Theorem 6.2.7 for the first summand and Lemma 6.4.1 for the second summand, we infer that

$$[\pi_1(x), \frac{D_k}{(1-\Delta)^{\frac{1}{2}}}] \in \mathcal{K}(L_2(\mathbb{R}^d)).$$

Define a function h_k on $\mathbb{R}^d$ by setting

$$h_k(t) = \frac{t_k}{|t|} - \frac{t_k}{(1+|t|^2)^{\frac{1}{2}}} = \frac{t_k}{|t|} \cdot \frac{1}{(1+|t|^2)^{\frac{1}{2}} \cdot ((1+|t|^2)^{\frac{1}{2}} + |t|)}, \quad t \in \mathbb{R}^d.$$

It follows from Theorem 6.2.7 that

$$[\pi_1(x), h_k(\nabla)] = \pi_1(x)h_k(\nabla) - h_k(\nabla)\pi_1(x) \in \mathcal{L}_{d+1}(\mathbb{R}^d_\theta).$$

Thus,

$$[\pi_1(x), \frac{D_k}{(-\Delta)^{\frac{1}{2}}}] = [\pi_1(x), \frac{D_k}{(1-\Delta)^{\frac{1}{2}}}] + [\pi_1(x), h_k(\nabla)] \in \mathcal{K}(L_2(\mathbb{R}^d)).$$

□

Now we may complete the verifications of the condition 6.3.3(2) for $\mathbb{R}^d_\theta$,

Theorem 6.4.3. *If $x \in C_0(\mathbb{R}^d_\theta)$ and if $y \in C(S^{d-1})$, then $[\pi_1(x), \pi_2(y)] \in \mathcal{K}(L_2(\mathbb{R}^d))$.*

Proof. Lemma 6.4.2 shows that $[\pi_1(x), \pi_2(y)] \in \mathcal{K}(L_2(\mathbb{R}^d))$ when $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ and $y(t) = \frac{t_k}{|t|}$. Since and the compact operators are closed in the norm topology, the result follows for arbitrary $x \in C_0(\mathbb{R}^d_\theta)$ and $y(t) = \frac{t_k}{|t|}$.

We may now extend the result to all y given as a polynomial in the variables $\frac{t_k}{|t|}$ using the Leibniz rule. Finally by the Stone-Weierstrass theorem, we may approximate arbitrary $y \in C(S^{d-1})$ by polynomials in the uniform norm. Hence again using the fact that $\mathcal{K}(L_2(\mathbb{R}^d))$ is norm-closed, this completes the proof. $\square$

At this point we should compare the present approach to that of [127], where many similar results were proved for functions in $L_\infty(\mathbb{R}^d)$ without assuming any continuity. The restriction to $C_0(\mathbb{R}^d_\theta)$ and $C(S^{d-1})$ is most likely necessary for the present approach however, since Lemma 6.4.3 is untrue in the commutative case if we do not assume that x is at least continuous.

6.5 Connes' Trace Formula

We now proceed to establish a variant of Connes' trace theorem which applies to non-commutative Euclidean space. Let H be a separable Hilbert space. We recall that a linear functional $\varphi : \mathcal{L}_{1,\infty}(H) \to \mathbb{C}$ is called a continuous trace if $\varphi([A, B]) = 0$ for all $A \in \mathcal{B}(H)$ and $B \in \mathcal{L}_{1,\infty}(H)$ and $|\varphi(B)| \lesssim \|B\|_{1,\infty}$. We will call φ normalised if

$$\varphi\left(\operatorname{diag}\left\{\frac{1}{n+1}\right\}_{n=0}^{\infty}\right) = 1.$$

If φ is a normalised trace, then note also that

$$\varphi\left(\operatorname{diag}\left\{\frac{1}{(1+|n|^2)^{d/2}}\right\}_{n\in\mathbb{Z}^d}\right) = \frac{\operatorname{Vol}(S^{d-1})}{d}.$$

It is known (see [89, Corollary 5.7.7]) that any continuous trace φ on $\mathcal{L}_{1,\infty}(H)$ vanishes on $\mathcal{K}(H) \cdot \mathcal{L}_{1,\infty}(H)$.

We establish that for any continuous normalised trace φ on $\mathcal{L}_{1,\infty}$, $T \in \Pi(C_0(\mathbb{R}^d_\theta) + \mathbb{C}, C(S^{d-1}))$ and $z \in W^d_1(\mathbb{R}^d_\theta)$,

$$\varphi(T\pi_1(z)(1-\Delta)^{-d/2}) = c_{d,\theta}\left(\tau_\theta \otimes \int_{S^{d-1}}\right)(z\operatorname{sym}(T)). \tag{6.5.1}$$

To prove (6.5.1), we use the following two results. Lemma 6.5.1 follows immediately from the fact that any continuous normalised trace on $\mathcal{L}_{1,\infty}(H)$ vanishes on $\mathcal{K}(H) \cdot \mathcal{L}_{1,\infty}(H)$

Lemma 6.5.1. *Let $V \in \mathcal{L}_{1,\infty}(H)$, and let φ be a continuous trace on $\mathcal{L}_{1,\infty}(H)$. Then,*

$$T \mapsto \varphi(TV)$$

is a continuous linear functional on $\mathcal{B}(H)$ which vanishes on $\mathcal{K}(H)$.

Proof. All traces on the ideal $\mathcal{L}_{1,\infty}$ are singular (i.e., vanishing on finite rank operators) [89, Corollary 5.7.7].

Thus, if T is finite rank then TV is finite rank and:

$$\varphi(TV) = 0.$$

Since φ is assumed to be continuous, we have:

$$|\varphi(TV)| \leq \|\varphi\|_{(\mathcal{L}_{1,\infty})^*}\|TV\|_{1,\infty} \leq \|\varphi\|_{(\mathcal{L}_{1,\infty})^*}\|T\|_\infty\|V\|_{1,\infty}.$$

So the functional $T \mapsto \varphi(TV)$ is continuous in the operator norm and vanishes on finite rank operators. Immediately it follows that $\varphi(TV) = 0$ whenever T is compact. $\square$

Lemma 6.5.2. *Let $\mathcal{A}_1, \mathcal{A}_2$ and H be as in Theorem 6.3.3. Suppose that ω is a continuous linear functional on $\Pi(\mathcal{A}_1,\mathcal{A}_2)$ which vanishes $\Pi(\mathcal{A}_1,\mathcal{A}_2)\cap\mathcal{K}(H)$. Then there exists a unique linear functional ρ on $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$ such that*

$$\omega(T) = \rho(\mathrm{sym}(T))$$

for all $T \in \Pi(\mathcal{A}_1,\mathcal{A}_2)$.

If, in addition, we have $\psi_1 \in \mathcal{A}_1^*$ and $\psi_2 \in \mathcal{A}_2^*$, and

$$\omega(\pi_1(a)\pi_2(b)) = \psi_1(a)\psi_2(b)$$

for all $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$, then

$$\omega(T) = (\psi_1 \otimes \psi_2)(\mathrm{sym}(T))$$

or in other words, $\rho = \psi_1 \otimes \psi_2$.

Proof. Since ω vanishes on $\Pi(\mathcal{A}_1,\mathcal{A}_2)\cap\mathcal{K}(H)$, ω descends to a linear functional $\tilde{\omega}$ on $\Pi(\mathcal{A}_1,\mathcal{A}_2)/(\Pi(\mathcal{A}_1,\mathcal{A}_2)\cap\mathcal{K}(H))$, which is simply $q(\Pi(\mathcal{A}_1,\mathcal{A}_2)$. Theorem 6.3.3 gives an isometric $*$-isomorphism $j : q(\Pi(\mathcal{A}_1,\mathcal{A}_2) \to \mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$. Defining $\rho = \tilde{\omega}\circ j^{-1}$ gives the required linear functional.

Now to prove that $\rho = \psi_1 \otimes \psi_2$, first we note that it follows from Theorem 6.1.4 that $\psi_1 \otimes \psi_2$ is well defined on $\mathcal{A}_1 \otimes_{\min} \mathcal{A}_2$. Since by assumption ψ_1 and ψ_2 are continuous, $\psi_1 \otimes \psi_2$ is determined by its values on the algebraic tensor product $\mathcal{A}_1 \odot \mathcal{A}_2$. Hence, the linear functional $\psi_1 \otimes \psi_2$ is uniquely characterised by

$$(\psi_1 \otimes \psi_2)(a \otimes b) = \psi_1(a)\psi_2(b) \quad a \in \mathcal{A}_1,\, b \in \mathcal{A}_2.$$

Since by assumption $\rho(a \otimes b) = \omega(\pi_1(a)\pi_2(b)) = \psi_1(a)\psi_2(b)$, it follows that $\rho = \psi_1 \otimes \psi_2$. $\square$

We fix $z \in W_1^d(\mathbb{R}_\theta^d)$, and consider the functional

$$\omega(T) := \varphi(T\pi_1(z)(1-\Delta)^{-d/2})$$

and we must prove that for all $x \in C_0(\mathbb{R}^d_\theta) + \mathbb{C}$ and $g \in C(S^{d-1})$,

$$\omega(\pi_1(x)\pi_2(g)) = \tau_\theta(xz) \int_{S^{d-1}} g(t)\,dt \tag{6.5.2}$$

(at least up to some constant). From Lemma 6.5.2 it will that follows that $\omega(T) = c_{d,\theta}\left(\tau_\theta \otimes \int_{S^{d-1}}\right)(\mathrm{sym}(T)(z\otimes 1))$ for an appropriate constant $c_{d,\theta}$.

6.5.1 Connes' Trace formula on noncommutative Euclidean space

The following assertion is proved in [129].

Theorem 6.5.3. *If $x \in W^d_1(\mathbb{R}^d_\theta)$, then $x(1-\Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}$ and there is a constant $C(d,\theta) > 0$ such that*

$$\varphi(x(1-\Delta)^{-\frac{d}{2}}) = C(d,\theta)\tau_\theta(x)$$

for every normalised continuous trace on $\mathcal{L}_{1,\infty}$.

We also need a pair of important intermediate results from [129]. Firstly,

Lemma 6.5.4. *If F is a continuous functional on $W^d_1(\mathbb{R}^d_\theta)$ such that*

$$F(x) = F(U(-t)xU(t)),\quad x \in W^d_1(\mathbb{R}^d_\theta),\quad t\in\mathbb{R}^d,$$

then $F = \tau_\theta$ (up to a constant factor).

Let $M_d(\mathbb{R})$ be the space of $d\times d$ real matrices. We define

$$\mathrm{Sp}(\theta,d) := \Big\{g \in M_d(\mathbb{R}):\ g^*\theta g = \theta\Big\}.$$

As we are working under the assumption that $\det(\theta)\neq 0$, it follows that $\mathrm{Sp}(\theta,d)$ is a group under usual matrix multiplication.

By our assumption that $\det(\theta)\neq 0$, it follows that if $g\in\mathrm{Sp}(\theta,d)$ then $|\det(g)| = 1$.

The second result from [129] we require is:

Lemma 6.5.5. *Let $g \in \mathrm{Sp}(\theta,d)$. We define an action $g\mapsto W_g$ on $L_2(\mathbb{R}^d)$ by*

$$(W_g\xi) = \xi\circ g^{-1}.$$

The operator W_g is unitary on $L_2(\mathbb{R}^d)$, and conjugation by W_g defines a trace-preserving group of automorphisms of $L_\infty(\mathbb{R}^d_\theta)$.

Note that the assumption that $g\in\mathrm{Sp}(\theta,d)$ in Lemma 6.5.5 is crucial: otherwise we do not necessarily have that $W_gxW_g^* \in L_\infty(\mathbb{R}^d_\theta)$ when $x\in L_\infty(\mathbb{R}^d_\theta)$. Let Ω be the antisymmetric matrix $\Omega := \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}^{\oplus d/2}$. Then $\mathrm{Sp}(\Omega,d)$ is the usual symplectic group.

Let $g\in\mathrm{GL}(d,\mathbb{R})$. Referring to Appendix A, consider the operator V_g on $C(S^{d-1})$ defined by

$$(V_gf)(t) = \frac{1}{|gt|^d} f\left(\frac{gt}{|gt|}\right).$$

It can be easily verified that $g \mapsto V_g$ is an "opposite group action" in the sense that it satisfies the rule $V_{gh} = V_h \circ V_g$ for all $g, h \in \mathrm{GL}(d, \mathbb{R})$. Lemma 6.6.1 proves that the rotation-invariant integration functional m on $C(S^{d-1})$ transforms under V_g by $m \circ V_g = (\det(g))^{-1} m$.

Lemma 6.5.6. *Let $l \in C(S^{d-1})^*$. If $l \circ V_g = l$ for every $g \in \mathrm{Sp}(\theta, d)$, then $l = \alpha m$ for some $\alpha \in \mathbb{C}$.*

Proof. The following result of linear algebra is well known, and follows easily from [124, Section 9.44]. There exists a real invertible matrix β with $\beta\beta^* = \beta^*\beta = |\det(\theta)|^{-1}$ such that

$$\beta^* \theta \beta = \Omega. \tag{6.5.3}$$

Hence, if $g \in \mathrm{Sp}(\Omega, d)$ is arbitrary, then:

$$\begin{aligned}(\beta g \beta^{-1})^* \theta (\beta g \beta^{-1}) &= (\beta^*)^{-1} g^* \beta^* \theta \beta g \beta^{-1} \\ &= (\beta^*)^{-1} \Omega \beta^{-1} \\ &= \theta,\end{aligned}$$

so $\beta g \beta^{-1} \in \mathrm{Sp}(\theta, d)$. Since by assumption, $l \circ V_h = l$ for all $h \in \mathrm{Sp}(\theta, d)$, we have:

$$l \circ V_\beta^{-1} \circ V_g \circ V_\beta = l.$$

Therefore for arbitrary $g \in \mathrm{Sp}(\theta, d)$,

$$(l \circ V_{\beta^{-1}}) \circ V_g = l \circ V_{\beta^{-1}}, \qquad \text{for all } g \in \mathrm{Sp}(\Omega, d).$$

So by Theorem 6.6.2, there is a constant C such that $l \circ V_{\beta^{-1}} = Cm$. Hence $l = Cm \circ V_\beta$. By Lemma 6.6.1, $m \circ V_\beta = \det(\beta)^{-1} m$. Let $\alpha = C \det(\beta)^{-1}$, so that $l = \alpha m$. $\square$

Lemma 6.5.7. *Let φ be a continuous trace on $\mathcal{L}_{1,\infty}$. There is a continuous functional $l \in C(S^{d-1})^*$ such that for all $x \in W_1^d(\mathbb{R}^d_\theta)$ and all $b \in C(S^{d-1})$ we have*

$$\varphi(\pi_1(x)\pi_2(b)(1-\Delta)^{-\frac{d}{2}}) = \tau_\theta(x) \cdot l(b).$$

Proof. Since φ is unitarily invariant, it follows that

$$\varphi(\pi_1(x)\pi_2(b)(1-\Delta)^{-\frac{d}{2}}) = \varphi(e^{i\langle \theta t, \nabla\rangle}\pi_1(x)\pi_2(b)(1-\Delta)^{-\frac{d}{2}} e^{-i\langle \theta t, \nabla\rangle})$$

However, ∇ commutes with Δ and with $\pi_2(b)$. Thus,

$$\varphi(\pi_1(x)\pi_2(b)(1-\Delta)^{-\frac{d}{2}}) = \varphi(e^{i\langle \theta t, \nabla\rangle}\pi_1(x) e^{-i\langle \theta t, \nabla\rangle}\pi_2(b)(1-\Delta)^{-\frac{d}{2}}).$$

Note that if $\xi \in L_2(\mathbb{R}^d)$,

$$\begin{aligned}e^{i\langle \theta t, \nabla\rangle} U(s) e^{-i\langle \theta t, \nabla\rangle}\xi(r) &= e^{i\langle \theta t, \nabla\rangle} U(s) e^{-i(\theta t, r)}\xi(r) \\ &= e^{i\langle \theta t, \nabla\rangle} e^{\frac{i}{2}(s, \theta r) - i(\theta t, r-s)}\xi(r-s) \\ &= e^{i(\theta t, r) + \frac{i}{2}(s,\theta r) - i(\theta t, r) + i(\theta t, s)}\xi(r-s) \\ &= e^{i(\theta t, s)}(U(s)\xi)(r)\end{aligned}$$

On the other hand from (6.2.1),

$$U(-t)U(s)U(t) = e^{i\langle \theta t, s\rangle}U(s).$$

Since the family $\{U(t)\}_{t\in\mathbb{R}^d}$ generates $L_\infty(\mathbb{R}^d_\theta)$, it follows that for all $x \in L_\infty(\mathbb{R}^d_\theta)$ we have:

$$e^{i\langle \theta t,\nabla\rangle}xe^{-i\langle \theta t,\nabla\rangle} = U(-t)xU(t).$$

Since π_1 is actually the identity function, this is equivalent to

$$e^{i\langle \theta t,\nabla\rangle}\pi_1(x)e^{-i\langle \theta t,\nabla\rangle} = \pi_1(U(-t)xU(t)).$$

Hence,

$$\varphi(\pi_1(x)\pi_2(b)(1-\Delta)^{-\frac{d}{2}}) = \varphi(\pi_1(U(-t)xU(t))\pi_2(b)(1-\Delta)^{-d/2}).$$

Consider now the linear functional on $W^d_1(\mathbb{R}^d_\theta)$,

$$F(x) = \varphi(\pi_1(x)\pi_2(b)(1-\Delta)^{-d/2})$$

From Corollary 6.2.8, F is continuous in the W^d_1-norm. We have proved that $F(U(-t)xU(t)) = F(x)$, and so from Lemma 6.5.4 we can conclude that $F(x)$ is a scalar multiple of $\tau_\theta(x)$. So,

$$\varphi(\pi_1(x)\pi_2(b)(1-\Delta)^{-\frac{d}{2}}) = \tau_\theta(x)\cdot l(b), \tag{6.5.4}$$

for some functional l on $C(S^{d-1})$. Since φ is continuous,

$$|l(b)| \leq C\|b\|_\infty$$

for some $C \geq 0$. So l is continuous. □

Lemma 6.5.8. *Let $x \in W^d_1(\mathbb{R}^d_\theta)$ and $b \in C(S^{d-1})$, then for any continuous normalised trace φ on $\mathcal{L}_{1,\infty}$.*

$$\varphi(\pi_1(x)\pi_2(b)(1-\Delta)^{-\frac{d}{2}}) = \frac{C(d,\theta)}{\mathrm{Vol}(S^{d-1})}\tau_\theta(x)\int_{S^{d-1}} b(t)\,dt.$$

where $C(d,\theta)$ is the same constant as in Theorem 6.5.3.

Proof. Let l be the linear functional from Lemma 6.5.7. It is required to show that we have:

$$l(b) = \frac{C(d,\theta)}{\mathrm{Vol}(S^{d-1})}\int_{S^{d-1}} b(t)\,dt.$$

From Lemma 6.5.6, it suffices to show that $l\circ V_g = l$ for all $g \in \mathrm{Sp}(d,\theta)$, and we will be able to recover the constant by substituting $b = 1$.

Now let $g \in \mathrm{Sp}(\theta, d)$. Since the operator W_g from Lemma 6.5.5 is unitary, it follows that:

$$\tau_\theta(x)l(b) = \varphi(W_g^*\pi_1(x)\pi_2(b)(1-\Delta)^{-d/2}W_g) \tag{6.5.5}$$

$$= \varphi(\pi_1(W_g^*xW_g)W_g^*\pi_2(b)(1-\Delta)^{-d/2}W_g). \tag{6.5.6}$$

We now show that for all $y \in W_1^d(\mathbb{R}^d_\theta)$,

$$\pi_1(y)W_g^*\pi_2(b)(1-\Delta)^{-d/2}W_g - \pi_1(y)\pi_2(V_g b)(1-\Delta)^{-d/2} \in \mathcal{L}_1. \tag{6.5.7}$$

Let $\xi \in L_2(\mathbb{R}^d)$, then

$$\begin{aligned}(W_g^*\pi_2(b)(1-\Delta)^{-d/2}W_g\xi)(t) &= W_g^*(b\left(\frac{t}{|t|}\right)(1+|t|^2)^{-d/2}\xi(g^{-1}t))\\ &= b\left(\frac{gt}{|gt|}\right)(1+|gt|^2)^{-d/2}\xi(t)\\ &= b\left(\frac{gt}{|gt|}\right)\frac{|t|^d}{|gt|^d}\frac{|gt|^d}{|t|^d}(1+|gt|^2)^{-d/2}\xi(t).\end{aligned}$$

The above computation shows that:

$$W_g^*\pi_2(b)(1-\Delta)^{-d/2}W_g = \pi_2(V_g b)\frac{|g\nabla|^d}{|\nabla|^d}(1+|g\nabla|^2)^{-d/2}.$$

Hence,

$$\begin{aligned}&\pi_1(y)W_g^*\pi_2(b)(1-\Delta)^{-d/2}W_g - \pi_1(y)\pi_2(V_g b)(1-\Delta)^{-d/2}\\ &\qquad= \pi_1(y)\pi_2(V_g(b))\left(\frac{|g\nabla|^d}{|\nabla|^d}(1+|g\nabla|^2)^{-d/2} - (1-\Delta)^{-d/2}\right)\\ &\qquad= \pi_1(y)\left(\frac{|g\nabla|^d}{|\nabla|^d}(1+|g\nabla|^2)^{-d/2} - (1-\Delta)^{-d/2}\right)\pi_2(V_g(b)).\end{aligned}$$

Due to Lemma 6.2.9, to prove (6.5.7), it suffices to show that:

$$h(t) := \frac{|gt|^d}{|t|^d}(1+|gt|^2)^{-d/2} - (1+|t|^2)^{-d/2}.$$

is in $\ell_1(L_\infty(\mathbb{R}^d))$. It is clear that h is bounded in the ball $\{|t| \le 1\}$. Supposing $|t| > 1$, we rewrite h as,

$$h(t) = |t|^{-d}\left(\frac{|gt|^d}{(1+|gt|^2)^{d/2}} - \frac{|t|^d}{(1+|t|^2)^{d/2}}\right)$$

Since $\frac{|gt|^2}{1+|gt|^2}$ and $\frac{|t|^2}{1+|t|^2}$ are bounded above by 1, we may use the numerical inequality:

$$|\alpha^{d/2} - \beta^{d/2}| \le \frac{d}{2}|\alpha-\beta|, \quad |\alpha|,|\beta| \le 1$$

to obtain,

$$|h(t)| \le \frac{d}{2}|t|^{-d}\left|\frac{|gt|^2}{1+|gt|^2} - \frac{|t|^2}{1+|t|^2}\right|$$

However,

$$\begin{aligned}\frac{|gt|^2}{1+|gt|^2} - \frac{|t|^2}{1+|t|^2} &= (1+|t|^2)^{-1}\frac{|gt|^2-|t|^2}{1+|gt|^2}\\ &= O((1+|t|^2)^{-1}), \quad |t| \to \infty.\end{aligned}$$

Hence, $|h(t)| = O(|t|^{-d-2})$ as $|t|\to\infty$. From there it is easy to see that $h\in\ell_1(L_\infty)(\mathbb{R}^d)$. This completes the proof of (6.5.7).

As φ vanishes on $\mathcal{L}_1$, we may use (6.5.7) with $y = W_g^*xW_g$ to obtain in (6.5.6) to obtain,

$$\tau_\theta(x)l(b) = \varphi(\pi_1(W_g^*xW_g)\pi_2(V_gb)(1-\Delta)^{-d/2})$$

so by Lemma 6.5.7:

$$\tau_\theta(x)l(b) = \tau_\theta(W_g^*xW_g)l(V_gb).$$

From Lemma 6.5.5 we have $\tau_\theta(W_g^*xW_g) = \tau_\theta(x)$, so now

$$\tau_\theta(x)l(b) = \tau_\theta(x)l(V_gb).$$

Since $x\in W_1^d(\mathbb{R}^d_\theta)$ is arbitrary, it follows that $l(b) = l(V_gb)$. So from Lemma 6.5.6, $l(b) = \alpha\int_{S^{d-1}} b(t)\,dt$ for some constant α. By substituting $b=1$ and using Theorem 6.5.3, we recover the constant α. □

Theorem 6.5.9. *Let $z\in W_1^d(\mathbb{R}^d_\theta)$. Then for every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$, and every $T\in\Pi(C_0(\mathbb{R}^d_\theta)+\mathbb{C}, C(S^{d-1}))$,*

$$\varphi(T\pi_1(z)(1-\Delta)^{-d/2}) = \frac{C(d,\theta)}{\mathrm{Vol}(S^{d-1})}\left(\tau_\theta\otimes\int_{S^{d-1}}\right)(\mathrm{sym}(T)(z\otimes 1)).$$

In particular, if $T = T\pi_1(z)$, then

$$\varphi(T(1-\Delta)^{-d/2}) = \frac{C(d,\theta)}{\mathrm{Vol}(S^{d-1})}\left(\tau_\theta\otimes\int_{S^{d-1}}\right)(\mathrm{sym}(T)).$$

Once again, $C(d,\theta)$ is the same constant as in Theorem 6.5.3.

Proof. We apply Lemma 6.5.2 to the functional

$$\omega(T) = \varphi(T\pi_1(z)(1-\Delta)^{-d/2}).$$

Since $\pi_1(z)(1-\Delta)^{-d/2}\in\mathcal{L}_{1,\infty}$, it follows from Lemma 6.2.7 that this functional is well defined and vanishes on compact operators. Consider the functionals $\psi_1(x) := C(d,\theta)\tau_\theta(xz)$ and $\psi_2(b) = \frac{1}{\mathrm{Vol}(S^{d-1})}\int_{S^{d-1}} b(t)\,dt$ on $\mathbb{C}+C_0(\mathbb{R}^d_\theta)$ and $C(S^{d-1})$ respectively. From Lemma 6.5.2, to show that $\omega(T) = (\psi_1\otimes\psi_2)(\mathrm{sym}(T))$ it suffices to prove:

$$\omega(\pi_1(x)\pi_2(b)) = \psi_1(x)\psi_2(b).$$

To this end, we compute $\omega(\pi_1(x)\pi_2(b))$. Since $[\pi_1(x),\pi_2(b)]$ is compact,

$$\omega(\pi_1(x)\pi_2(b)) = \omega(\pi_2(b)\pi_1(x)).$$

Hence

$$\begin{aligned}\omega(\pi_1(x)\pi_2(b)) &= \varphi(\pi_2(b)\pi_1(x)\pi_1(z)(1-\Delta)^{-d/2})\\ &= \varphi(\pi_2(b)\pi_1(xz)(1-\Delta)^{-d/2}).\end{aligned}$$

Using the cyclicity of the trace φ, and that $\pi_2(b)$ commutes with Δ,

$$\omega(\pi_1(x)\pi_2(b)) = \varphi(\pi_1(xz)\pi_2(b)(1-\Delta)^{-d/2}).$$

The right hand side may be computed using Lemma 6.5.8,

$$\begin{aligned}\varphi(\pi_1(xz)\pi_2(b)(1-\Delta)^{-d/2}) &= \frac{C(d,\theta)}{\mathrm{Vol}(S^{d-1})}\tau_\theta(xz)\int_{S^{d-1}} b(t)\,dt \\ &= \psi_1(x)\psi_2(b).\end{aligned}$$

So finally, we have $\omega(\pi_1(x)\pi_2(b)) = \psi_1(x)\psi_2(b)$. So from Lemma 6.5.2, we immediately obtain $\omega = \psi_1 \otimes \psi_2$, and this completes the proof. □

6.6 Measures invariant under the action of symplectic groups

For $g \in \mathrm{GL}(d,\mathbb{R})$, we define an action V_g on $C(S^{d-1})$ as follows:

$$(V_g b)(t) = \frac{1}{|gt|^d} b\left(\frac{gt}{|gt|}\right), \quad t \in S^{d-1}.$$

This is indeed an (opposite) action: we have

$$V_{g_1} \circ V_{g_2} = V_{g_2 g_1}, \quad g_1, g_2 \in \mathrm{GL}(d,\mathbb{R}).$$

Lemma 6.6.1. *If m is a rotation-invariant measure on $C(S^{d-1})$, then $m \circ V_g = \det(g^{-1}) \cdot m$.*

Proof. By converting to polar coordinates, for every $b \in C(S^{d-1})$ we have the formula,

$$m(b) = \frac{1}{\Gamma(d)}\int_{\mathbb{R}^d} b\left(\frac{t}{|t|}\right) e^{-|t|}\,dt$$

So,

$$m(V_g b) = \frac{1}{\Gamma(d)}\int_{\mathbb{R}^d} b\left(\frac{gt}{|gt|}\right)\frac{|t|^d}{|gt|^d} e^{-|t|}\,dt.$$

Applying the linear transformation $s = gt$, we get,

$$\begin{aligned} m(V_g b) &= \frac{1}{\Gamma(d)}\int_{\mathbb{R}^d} b\left(\frac{s}{|s|}\right)\frac{|g^{-1}s|^d}{|s|^d} e^{-|g^{-1}s|}\,d(g^{-1}s) \\ &= \frac{\det(g^{-1})}{\Gamma(d)}\int_{\mathbb{R}^d} b\left(\frac{s}{|s|}\right)\frac{|g^{-1}s|^d}{|s|^d} e^{-|g^{-1}s|}\,ds.\end{aligned}$$

Now using polar coordinates,

$$m(V_g b) = \frac{\det(g)^{-1}}{\Gamma(d)}\int_{S^{d-1}} b(s)\int_0^\infty |g^{-1}s|^d e^{-r|g^{-1}s|} r^{d-1}\,dr ds.$$

Applying the formula $\Gamma(d) = \alpha^d \int_0^\infty r^{d-1}e^{-\alpha r}\,dr$, we get

$$\begin{aligned} m(V_g b) &= \frac{\det(g^{-1})}{\Gamma(d)}\Gamma(d)\int_{S^{d-1}} b(s)\,ds \\ &= \det(g^{-1}) m(b).\end{aligned}$$

□

Recall that d is even. The symplectic group $\mathrm{Sp}(d,\mathbb{R})$ is the subgroup of $\mathrm{GL}(d,\mathbb{R})$ defined as follows:

$$\mathrm{Sp}(d,\mathbb{R}) = \Big\{T \in M_d(\mathbb{R}) :\ T^*\Omega T = \Omega\Big\}, \quad \Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\oplus \frac{d}{2}}.$$

In the published version of [94], the next theorem was proved by a lengthy induction argument. It was suggested to us by M. Goffeng that a much simpler proof is possible by instead restricting $g \mapsto V_g$ to orthogonal matrices. Even though the restriction of V to $\mathrm{SO}(d)$ uses only a fraction of the available symmetries, it is still sufficient for our purposes. We supply this simplified argument below.

The key component of the argument is that if μ is a measure on the unit sphere $\{(z_1,\ldots,z_d) \in \mathbb{C}^k \ :\ |z_1|^2+|z_2|^2+\cdots+|z_d|^2 = 1\}$ which is invariant under the action of the unitary group $\mathrm{U}(k)$, then μ is the (up to a scalar factor) the usual rotation-invariant measure on S^{2k-1}. This follows from the fact that $\mathrm{U}(k)$ acts transitively on the sphere in $\mathbb{C}^k$. One way to see this implication is that for all $U \in \mathrm{U}(k)$ and $f \in C(S^{2k-1})$ we have (by assumption)

$$\int_{S^{2k-1}} f\,d\mu = \int_{S^{2k-1}} f\circ U\,d\mu$$

Now if we integrate over $U \in \mathrm{U}(k)$ with respect to the Haar measure dU and use Fubini's theorem, we have:

$$\int_{S^{2k-1}} f\,d\mu = \int_{S^{2k-1}} \left(\int_{\mathrm{U}(k)} f\circ U\,dU\right)\,d\mu.$$

Since $\mathrm{U}(k)$ acts transitively on S^{2k-1}, the integral $\int_{\mathrm{U}(k)} f\circ U\,dU$ is a scalar and hence:

$$\int_{S^{2k-1}} f\,d\mu = \mu(S^{2k-1})\int_{\mathrm{U}(k)} f(Uz)\,dU$$

where $z \in S^{2k-1}$ is arbitrary. It follows that all $\mathrm{U}(k)$-invariant measures on S^{2k-1} are proportional to the pushforward of the Haar measure on $\mathrm{U}(k)$ to S^{2k-1}, so in particular are proportional to each other.

Theorem 6.6.2. *If $l \in C(S^{d-1})^*$ is such that $l\circ V_g = l$, for all $g \in \mathrm{Sp}(d,\mathbb{R})$, then l is invariant under rotations, and hence up to rescaling is the unique rotation-invariant measure on S^{d-1}.*

Proof. Let K be the compact subgroup of $\mathrm{Sp}(d,\mathbb{R})$ given by:

$$K = \mathrm{SO}(d)\cap \mathrm{Sp}(d,\mathbb{R}).$$

We will prove that if $l\circ V_g = l$ for all $g \in K$, then necessarily $l\circ V_R = l$ for all $R \in \mathrm{SO}(d)$. Since $V|_{\mathrm{SO}(d)}$ is exactly the action of $\mathrm{SO}(d)$ on S^{d-1} by rotations, this is the required result.

Recall that d is even, and let $k \geq 1$ be such that $d = 2k$. The unit sphere S^{d-1} in $\mathbb{R}^d$ can be identified with the unit sphere $S^k_{\mathbb{C}} := \{(z_1,\ldots,z_k) \in \mathbb{C}^k \ :\ \sum_{j=1}^k |z_k|^2 = 1\}$ in k-dimensional complex space $\mathbb{C}^k$. We claim that the action of K on S^{d-1} can be identified precisely with the action of $\mathrm{U}(k)$ on $S^k_{\mathbb{C}}$, and this yields the uniqueness of the functional l due to the discussion preceding the theorem.

Write $T \in K$ in block-matrix form:

$$T = \begin{pmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{pmatrix}$$

where $T_{j,k} \in M_{k,k}(\mathbb{R})$. The condition that $T\Omega T^* = \Omega$ implies that $T\Omega = \Omega T$. It follows that we have $T_{1,1} = T_{2,2}$ and $T_{2,1} = -T_{1,2}$. Rewriting $A = T_{1,1}$ and $B = T_{2,1}$, we have:

$$T = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

We can identify T with the $k \times k$ complex matrix $U = A + iB$. In fact the mapping $T \mapsto U$ is a group isomorphism. To see this, note that since T is orthogonal we have:

$$T^*T = \begin{pmatrix} A^*A + B^*B & -A^*B + B^*A \\ -B^*A + A^*B & B^*B + A^*A \end{pmatrix} = \begin{pmatrix} I_{k,k} & 0 \\ 0 & I_{k,k} \end{pmatrix}$$

and similarly, $AA^* + BB^* = I$ and $BA^* = AB^*$. These relations imply that $U = A + iB$ is unitary, and moreover if U is a unitary matrix, we can consider the matrix

$$\begin{pmatrix} \Re(U) & -\Im(U) \\ \Im(U) & \Re(U) \end{pmatrix}$$

where $\Re(U)$ and $\Im(U)$ are the real and imaginary parts of U respectively[3]. It is easily checked that the above matrix belongs to K. This $T \mapsto U$ is a bijection between K and $\mathrm{U}(k)$, and it is straightforward to check that it is indeed a group homomorphism.

Consider the following ($\mathbb{R}$-linear) isomorphism from $\mathbb{R}^d$ to $\mathbb{C}^k$:

$$\iota(x_1, x_2, \ldots, x_{2k}) = (x_1 + ix_{k+1}, x_2 + ix_{k+2}, \ldots, x_k + ix_{2k}), \quad x \in \mathbb{R}^d.$$

That is, the point $(x, y) \in \mathbb{R}^d$ is mapped to $x + iy \in \mathbb{C}^k$. Note that $x \in S^{d-1}$ if and only if $\iota(x) \in S^k_{\mathbb{C}}$. It is readily verified that for all $x \in \mathbb{R}^d$, we have:

$$\iota(Tx) = U\iota(x).$$

It follows that ι intertwines the action $V|_K$ on S^{d-1} with the standard action of $\mathrm{U}(k)$ on $S^k_{\mathbb{C}}$, and this gives the desired identification of $V|_K$ with the action of $\mathrm{U}(k)$ on $S^k_{\mathbb{C}}$.

□

[3]By "real part", we do not mean $\frac{1}{2}(U + U^*)$, but instead the matrix formed by taking the real parts of the entries of U

Chapter 7

Quantum differentiability on noncommutative Euclidean spaces

This chapter is based on the research paper [92], which consists of joint work of the author with F. Sukochev and X. Xiong.

The topic of the paper concerns quantum differentiability conditions for so-called Moyal or Groenewold Euclidean spaces. This is a line of research which follows on from [87] and [93] which proved the analogous results for classical Euclidean spaces and quantum tori respectively.

Some changes have been made here to simplify the presentation: in particular, one of the most difficult parts of [92] was in redeveloping the general theory of noncommutative Euclidean spaces to apply simultaneously to the commutative and noncommutative cases. Here, we restrict attention to the exclusively noncommutative case.

7.1 Introduction

Following [26], quantised calculus may be defined defined in terms of a Fredholm module. A Fredholm module can be defined with the following data: a separable Hilbert space H, a unitary self-adjoint operator F on H and a C^*-algebra $\mathcal{A}$ represented on H such that the commutator $[F,a]$ is a compact operator on H for all a in A. The quantised differential of $a \in \mathcal{A}$ is then defined to be the operator $đa = i[F,a]$.

A problem of particular interest in quantised calculus is to precisely quantify the asymptotics of the sequence $\{\mu(n, đa)\}_{n=0}^\infty$ in terms of a. In operator theoretic language, we seek conditions under which the operator $đa$ is in some ideal of the algebra of bounded operators on H. Of the greatest importance are Schatten-von Neumann $\mathcal{L}_p$ ideals, the Schatten-Lorentz $\mathcal{L}_{p,\infty}$ spaces and the Macaev-Dixmier ideals $\mathcal{M}_{1,\infty}$.

The link between quantised calculus and geometry is discussed by Connes in [27]. A model example for quantised calculus is to take a compact d-dimensional Riemannian spin manifold M (with $d \geq 2$) with Dirac operator D, and define H to be the Hilbert space of pointwise almost-everywhere equivalence classes of square integrable sections of the spinor bundle. The algebra $\mathcal{A} = C(M)$ of continuous functions on M acts by pointwise multiplication on H, and one defines F as a difference of spectral projections:

$$F := \chi_{[0,\infty)}(D) - \chi_{(-\infty,0)}(D).$$

One then has $đf = i[F, M_f]$, where M_f is the operator on H of pointwise multiplication by $f \in C(M)$. In quantised calculus the immediate question is to determine the relationship between the degree of differentiability of $f \in C(M)$ and the rate of decay of the singular values of $đf$. This is the problem which we term "characterising quantum differentiability". In general, we have the following inclusion [27, Theorem 3.1]:

$$f \in C^\infty(M) \Rightarrow |đf|^d \in \mathcal{M}_{1,\infty}.$$

This corresponds to the implication:

$$f \in C^\infty(M) \Rightarrow \sup_{n\geq 0} \frac{1}{\log(2+n)} \sum_{j=0}^{n} \mu(j, đf)^d < \infty.$$

It is possible to specify even more precise details about the asymptotics of $\{\mu(j, đf)\}_{j\geq 0}$. Suppose that ω is an extended limit (a continuous linear functional on the space of bounded sequences $\ell_\infty(\mathbb{N})$ which extends the limit functional). If ω is invariant under dilations (in the sense of [89, Definition 6.2.4]) then [27, Theorem 3.3] states that:

$$\omega\left(\left\{\frac{1}{\log(2+n)} \sum_{j=0}^{n} \mu(j, đf)^d\right\}_{n=0}^{\infty}\right) = c_d \int_M |\mathrm{d}f \wedge \star \mathrm{d}f|^{d/2} \tag{7.1.1}$$

where c_d is a known constant, d is the exterior differential and $\star$ denotes the Hodge star operator associated to the orientation of M. The quantity on the left hand side of (7.1.1) is precisely the Dixmier trace $\mathrm{tr}_\omega(|đf|^d)$. According to Connes, this formula "shows how to pass from quantised 1-forms to ordinary forms, not by a classical limit, but by a direct application of the Dixmier trace" [27, Page 676].

When working with particular manifolds, rather than general compact manifolds, it is possible to specify with even greater precision the relationship between f and the singular values of $đf$. In the one dimensional cases of the circle and the line, the appropriate choice for F turns out to be the Hilbert transform (see [28, Chapter 4, Section 3.α]) and the commutators of pointwise multiplication operators and the Hilbert transform are very well understood. If f is a function on either the line $\mathbb{R}$ or the circle $\mathbb{T}$, necessary and sufficient conditions on f for $đf$ to be in virtually any named operator ideal are known (see e.g. [57]).

In higher dimensions (in particular $\mathbb{T}^d$ and $\mathbb{R}^d$ for $d \geq 2$), an appropriate choice for F is given by a linear combination of Riesz transforms [36, 87]. Commutators of pointwise multiplication operators and Riesz transforms are well studied in classical harmonic analysis, and Janson and Wolff [76] determined necessary and sufficient conditions for such a commutator to be in $\mathcal{L}_p$ for all $p \in (0, \infty)$. For the case of Euclidean space $\mathbb{R}^d$ an even more precise characterisation including a complete if and only if characterisation for $đf$ to be in a wide class of operator ideals was obtained by Rochberg and Semmes [113].

If $f \in C^\infty(\mathbb{T}^d)$, let $\nabla f = (\partial_1 f, \partial_2 f, \ldots, \partial_d f)$ be the gradient vector of f, and let $\|\nabla f\|_2 = \left(\sum_{j=1}^d |\partial_j f|^2\right)^{\frac{1}{2}}$. Then as a special case of (7.1.1), we have the following:

$$\mathrm{tr}_\omega(|đf|^d) = k_d \int_{\mathbb{T}^d} \|\nabla f(t)\|_2^d dm(t), \tag{7.1.2}$$

where $k_d > 0$ is a known constant, and m denotes the flat (Haar) measure on $\mathbb{T}^d$. A similar integral formula can also be obtained in the non-compact setting of $\mathbb{R}^d$ [87, Theorem 2]. Quantum differentiability in the commutative setting is a topic of active research interest, with many results known outside the classical examples of Euclidean spaces and tori. For example, Goffeng and Gimperlein have explored the deviations from classical spectral asymptotics for non-smooth functions [59, 60]. A number of results are also known in the strictly noncommutative setting, especially as regards sufficient conditions. For example, Goffeng and Mesland have explored the quantum differentiability relating to spectral triples on Cuntz-Krieger algebras [61] and Emerson and Nica provided sufficient conditions for finite summability of Fredholm modules relating to hyperbolic groups [49].

Recently the author, F. Sukochev and X. Xiong have established a characterisation of the $\mathcal{L}_{d,\infty}$-ideal membership of quantised differentials for noncommutative tori [93]. The primary result of [93] is as follows. Let θ be an antisymmetric real $d\times d$ matrix with $d > 2$, and consider the noncommutative tori $\mathbb{T}^d_\theta$. In this setting, there is a conventional choice of Fredholm module and an associated quantised calculus [64, Section 12.3]. An element $x \in L_2(\mathbb{T}^d_\theta)$ belongs to the (noncommutative) homogeneous Sobolev space $\dot{W}^1_d(\mathbb{T}^d_\theta)$ if and only if its quantised differential $đx$ has bounded extension in $\mathcal{L}_{d,\infty}$. The quantum torus analogue of (7.1.2) is also obtained as [93, Theorem 1.2]: for $x \in \dot{W}^1_d(\mathbb{T}^d_\theta)$, there is a certain constant c_d such that for any continuous normalised trace φ on $\mathcal{L}_{1,\infty}$ we have

$$\varphi(|đx|^d) = c_d \int_{S^{d-1}} \tau\left(\left(\sum_{j=1}^d |\partial_j x - s_j \sum_{k=1}^d s_k \partial_k x|^2\right)^{\frac{d}{2}}\right) ds, \tag{7.1.3}$$

where τ is the standard trace on the algebra $L_\infty(\mathbb{T}^d_\theta)$, and the integral is over the $(d-1)$-sphere S^{d-1} with respect to its rotation invariant measure ds. To the best of our knowledge, these results were the first concerning quantum differentiability in the strictly noncommutative setting.

The primary task of this paper is to determine similar results for noncommutative Euclidean spaces. A number of major obstacles make this task far more difficult than for noncommutative tori. In particular, the methods of [93] were facilitated by a well-developed theory of pseudodifferential operators on noncommutative tori [66, 67]. However, despite recent advances [62, 82, 94], the theory of pseudodifferential operators for noncommutative Euclidean spaces is still in its infancy and it is not clear how to directly adapt the existing theory to this problem. It has therefore been necessary for us to develop new arguments, based on a very simple form of pseudodifferential calculus (see Section 7.3).

Another difficulty with $\mathbb{R}^d_\theta$ compared to $\mathbb{T}^d_\theta$ is that the nature of the required analysis changes dramatically with θ. For example, the range of the canonical trace τ on the algebra $L_\infty(\mathbb{T}^d_\theta)$ on projections is $[0,1]$, while for the canonical trace on $L_\infty(\mathbb{R}^d_\theta)$ the range of the trace on projections is either $[0,\infty]$ if $\det(\theta) = 0$ or instead ranges over integral multiples of $(2\pi)^{d/2}|\det(\theta)|^{1/2}$ if $\det(\theta) \neq 0$.

On a related issue, in the non-degenerate case the C^*-algebra $C_0(\mathbb{R}^d_\theta)$ is isomorphic to the algebra of compact linear operators on $L_2(\mathbb{R}^d)$, and $L_\infty(\mathbb{R}^d_\theta)$ is isomorphic to all of $\mathcal{B}(L_2(\mathbb{R}^d))$. This has some counterintuitive consequences for both the analysis and geometry of $\mathbb{R}^d_\theta$. Firstly, $C_0(\mathbb{R}^d_\theta)$ becomes an ideal of $L_\infty(\mathbb{R}^d_\theta)$ so that "continuous times bounded is continuous". Secondly, $C_0(\mathbb{R}^d_\theta)$ is generated by projections as a C^*-algebra. This is quite unlike Euclidean space, where there are no nontrivial continuous projections.

From the point of view of K-theory and the algebraic-topological point of view on noncommutative geometry, $C_0(\mathbb{R}^d_\theta)$ is strongly Morita equivalent to a point. From that point of view, the "space" $\mathbb{R}^d_\theta$ is trivial. This situation is analogous to classical K-theory: Euclidean space $\mathbb{R}^d$ is contractable, and so is homotopically indistinguishable from a point.

A noteworthy side effect of the self-contained approach is that we obtain in an abstract manner the following commutator estimates for quantum Euclidean spaces: Let Δ be the Laplace operator associated to the noncommutative Euclidean space $\mathbb{R}^d_\theta$ (see Section 6.2 for complete definitions). For an appropriate class of smooth elements $x \in L_\infty(\mathbb{R}^d_\theta)$, if $\alpha,\beta \in \mathbb{R}$ are such that $\alpha < \beta + 1$, then we have

$$[(1-\Delta)^{\alpha/2}, x](1-\Delta)^{-\beta/2} \in \mathcal{L}_{\frac{d}{\beta-\alpha+1},\infty}.$$

In the classical (commutative) case, this estimate follows almost immediately from the calculus and mapping properties of pseudodifferential operators (see [87, Lemma 13]).

7.1.1 Main results on quantum differentiability

In this section we state the main results of this chapter.

Let θ be an antisymmetric real $d \times d$ matrix, where $d \geq 2$. As in Section 6.2, we will exclude the degenerate case when $\det(\theta) = 0$. In the published version of [92] the degenerate case $\det(\theta) = 0$ was included, however this came at the cost of substantially increasing the length and complexity of the paper. For the sake of clarity, we have opted to exclude the degenerate case here.

Our first main result provides sufficient conditions for $đx \in \mathcal{L}_{d,\infty}$:

Theorem 7.1.1. *Assume that* $\det(\theta) \neq 0$. *If* $x \in L_p(\mathbb{R}^d_\theta) \cap \dot{W}^1_d(\mathbb{R}^d_\theta)$ *for some* $d \leq p < \infty$, *then* $đx$ *has bounded extension, and the extension is in* $\mathcal{L}_{d,\infty}$.

The space $\dot{W}^1_d(\mathbb{R}^d_\theta)$ is a noncommutative homogeneous Sobolev space for $\mathbb{R}^d_\theta$. This is distinct from the Sobolev spaces defined in Section 6.2, and will be introduced in Definition 7.1.11.

The *a priori* assumption $x \in L_p(\mathbb{R}^d_\theta)$ for some $d \leq p < \infty$ may not be necessary, however we have been unable to remove it. One reason for this difficulty is that there is no clear replacement for the use of the Poincaré inequality in the noncommutative situation. See Proposition 7.1.17.

With Theorem 7.1.1, we can prove our second main result, the following trace formula:

Theorem 7.1.2. *Assume that* $\det(\theta) \neq 0$. *Let* $x \in L_p(\mathbb{R}^d_\theta) \cap \dot{W}^1_d(\mathbb{R}^d_\theta)$ *for some* $d \leq p < \infty$. *Then there is a constant* c_d *depending only on the dimension* d *such that for any continuous normalised trace* φ *on* $\mathcal{L}_{1,\infty}$ *we have:*

$$\varphi(|đx|^d) = c_d \int_{\mathbb{S}^{d-1}} \tau_\theta\left(\Big(\sum_{j=1}^d |\partial_j x - s_j \sum_{k=1}^d s_k \partial_k x|^2\Big)^{\frac{d}{2}}\right) ds.$$

Here, the integral over $\mathbb{S}^{d-1}$ *is taken with respect to the rotation-invariant measure* ds *on* $\mathbb{S}^{d-1}$, *and* $s = (s_1, \ldots, s_d)$.

Recall that τ_θ is the canonical trace on the algebra $L_\infty(\mathbb{R}^d_\theta)$ (see Section 6.2). Although the above integral formula is identical in appearance to (7.1.3), the proof involves different techniques.

The next corollary is a direct application of Theorem 7.1.2. The proof is the same as [93, Corollary 1.3], so we omit the details.

Corollary 7.1.3. *Assume that* $\det(\theta) \neq 0$. *Let* $x \in L_p(\mathbb{R}^d_\theta) \cap \dot{W}^1_d(\mathbb{R}^d_\theta)$ *for some* $d \leq p < \infty$. *Then there are constants* c_d *and* C_d *depending only on* d *such that for any continuous normalised trace* φ *on* $\mathcal{L}_{1,\infty}$ *we have*

$$c_d \|x\|^d_{\dot{W}^1_d} \leq \varphi(|đx|^d) \leq C_d \|x\|^d_{\dot{W}^1_d}.$$

Since φ vanishes on the trace class $\mathcal{L}_1$, Corollary 7.1.3 immediately yields the following noncommutative version of the $p \leq d$ component of [76, Theorem 1]:

Corollary 7.1.4. *Assume that* $\det(\theta) \neq 0$. *If* $x \in L_p(\mathbb{R}^d_\theta) + \mathbb{C}$ *for some* $d \leq p < \infty$ *and* $đx$ *has bounded extension in* $\mathcal{L}_p$ *for* $p \leq d$, *then* x *is a constant.*

As a converse to Theorem 7.1.1, we prove our third main result: the necessity of the condition $x \in \dot{W}^1_d(\mathbb{R}^d_\theta)$ for $đx \in \mathcal{L}_{d,\infty}$.

Theorem 7.1.5. *Assume that* $\det(\theta) \neq 0$. *Let* $x \in L_p(\mathbb{R}^d_\theta)$ *for some* $d \leq p < \infty$. *If* $đx$ *has bounded extension in* $\mathcal{L}_{d,\infty}$, *then* $x \in \dot{W}^1_d(\mathbb{R}^d_\theta)$, *and there is a constant* $c_d > 0$ *depending only on* d *such that*

$$c_d \|x\|_{\dot{W}^1_d} \leq \|đx\|_{\mathcal{L}_{d,\infty}}.$$

It is worth noting that one may consider the commutative ($\theta = 0$) case in Theorems 7.1.1, 7.1.2 and 7.1.5 and in this case the results obtained are very similar to those of [87]. The only difference being in the integrability assumptions: in [87], boundedness was assumed, and here we assume p-integrability for some $d \leq p < \infty$. Nonetheless the proofs we give here are independent to those of [87]. It would be of great interest to extend the results here to include $p = \infty$, however technical obstacles have so far prevented this. See the discussion preceding Proposition 7.1.17.

7.1.2 Main commutator estimate

As a byproduct of the proof of Theorem 7.1.2, we obtain a commutator estimate on quantum Euclidean spaces. We shall define the noncommutative Schwartz space $\mathcal{S}(\mathbb{R}^d_\theta)$ in Subsection 7.1.3.

Theorem 7.1.6. *Let $\alpha, \beta \in \mathbb{R}$, and let $x \in \mathcal{S}(\mathbb{R}^d_\theta)$. Then if $\alpha < \beta + 1$:*

$$[(1-\Delta)^{\alpha/2}, x](1-\Delta)^{-\beta/2} \in \mathcal{L}_{\frac{d}{\beta-\alpha+1},\infty}.$$

On the other hand if $\alpha = \beta + 1$, then the operator

$$[(1-\Delta)^{\alpha/2}, x](1-\Delta)^{-\beta/2}$$

has bounded extension.

This estimate is to be compared with the Cwikel type estimates provided in [83]. Using the latter estimates, one can deduce that $(1-\Delta)^{\alpha/2}x(1-\Delta)^{-\beta/2} \in \mathcal{L}_{\frac{d}{\beta-\alpha},\infty}$ and $x(1-\Delta)^{(\alpha-\beta)/2} \in \mathcal{L}_{\frac{d}{\beta-\alpha},\infty}$, however showing that the difference of these two operators is in the smaller ideal $\mathcal{L}_{\frac{d}{\beta-\alpha+1},\infty}$ requires additional argument.

If we consider the classical (commutative) setting, the result of Theorem 7.1.6 would follow from a standard application of pseudodifferential operator calculus: x is viewed as an order 0 pseudo-differential operator, while $(1-\Delta)^{\alpha/2}$ is of order α. It follows that the commutator $[(1-\Delta)^{\alpha/2}, x]$ is of order $\alpha - 1$, and thus $[(1-\Delta)^{\alpha/2}, x](1-\Delta)^{-\beta/2}$ is of order $\alpha - \beta - 1$. From there, a short argument which essentially uses the fact that x decays at infinity can be used to show that the result of Theorem 7.1.6 holds (an argument of precisely this nature was used in [87, Lemma 13]). It likely is possible to carry out a similar argument in the noncommutative setting using the quantum pseudodifferential operator theory of [62], however we have found the direct argument to be insightful.

The layout of this chapter is the following. In the following section we introduce notation, terminology and required background material concerning operator ideals and analysis on quantum Euclidean spaces, and we also recount some elementary properties such as the dilation action and Cwikel type estimates. Section 7.2 is devoted to the proof of Theorem 7.1.1. Section 7.3 concerns our proof of Theorem 7.1.6, and is the most technical component of the paper. The final section, Section 7.4, completes the proofs of Theorems 7.1.2 and 7.1.5.

7.1.3 Weyl quantisation

Let $f \in L_1(\mathbb{R}^d)$. We will define $U(f) \in L_\infty(\mathbb{R}^d_\theta)$ as the operator given by the $L_\infty(\mathbb{R}^d_\theta)$-valued weak*-integral:

$$U(f) = \int_{\mathbb{R}^d} f(t)U(t)\,dt \in L_\infty(\mathbb{R}^d_\theta).$$

Since the family $\{U(t)\}_{t\in\mathbb{R}^d}$ is strongly continuous, the weak* measurability of integrand is immediate. Since each $U(t)$ is unitary, Lemma 4.1.4 implies that:

$$\|U(f)\|_\infty \leq \|f\|_1. \tag{7.1.4}$$

We will denote $U = U_\theta$ when there is a need to refer to the dependence on θ. The map U has other names and notations in the literature: for example composing U with the Fourier transform determines a mapping $\mathcal{S}(\mathbb{R}^d) \to \mathcal{B}(L_2(\mathbb{R}^{d/2}))$ which is also known as the Weyl quantisation map [69, Section 13.3]. In the $\det(\theta) \neq 0$ case, the map U is also essentially the same as the so-called Weyl transform [132, Page 138]. In [62], the map denoted there λ_θ is very similar to U, the only difference being that $U(t_1e_1)U(t_2e_2)\cdots U(t_de_d)$ is used in place of $U(t)$.

Remark 7.1.1. *In [92], the Weyl transform was defined by giving $U(f)\xi$ as an $L_2(\mathbb{R}^d)$-valued Bochner integral for $\xi \in L_2(\mathbb{R}^d)$. Since we have already discussed weak*-integrals at length in Section 4.1.1, it is more convenient to instead define $U(f)$ as a weak*-integral here.*

Assume now that $f \in \mathcal{S}(\mathbb{R}^d)$. For $\xi \in \mathcal{S}(\mathbb{R}^d)$, by the definition of $U(t)$ we have:

$$(U(f)\xi)(s) = \int_{\mathbb{R}^d} f(t)e^{-\frac{i}{2}(t,\theta s)}\xi(s-t)\,dt. \tag{7.1.5}$$

Since ξ is continuous, it is easy to see that $(U(f)\xi)(s)$ is continuous as a function of s. Evaluating $U(f)\xi(s)$ at $s = 0$ yields:

$$(U(f)\xi)(0) = \int_{\mathbb{R}^d} f(t)\xi(-t)\,dt.$$

Hence, if $U(f) = U(g)$ for $f, g \in L_1(\mathbb{R}^d)$, it follows that:

$$\int_{\mathbb{R}^d} (f(t) - g(t))\xi(-t)\,dt = 0$$

for all $\xi \in \mathcal{S}(\mathbb{R}^d)$, and thus $f = g$ pointwise almost everywhere. It follows that U is injective.

The class of Schwartz functions on $\mathbb{R}^d_\theta$ is defined as the image of $\mathcal{S}(\mathbb{R}^d)$ under U. That is,

$$\mathcal{S}(\mathbb{R}^d_\theta) := \{x \in L_\infty(\mathbb{R}^d_\theta) : x = \int_{\mathbb{R}^d} f(s)U(s)ds, \text{ for some } f \in \mathcal{S}(\mathbb{R}^d)\}, \tag{7.1.6}$$

The Schwartz space $\mathcal{S}(\mathbb{R}^d_\theta)$ is equipped with the topology induced by the isomorphism $U : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d_\theta)$, where $\mathcal{S}(\mathbb{R}^d)$ is equipped with its canonical Fréchet topology. It is important to note that the Fréchet topology of $\mathcal{S}(\mathbb{R}^d_\theta)$ is finer than the $L_p(\mathbb{R}^d_\theta)$ topology for every $1 \le p \le \infty$. This follows, for example, from Proposition 7.1.10 below.

It is worth emphasising that in the non-degenerate case ($\det(\theta) \neq 0$), the non-commutativity of $L_\infty(\mathbb{R}^d_\theta)$ implies that $\mathcal{S}(\mathbb{R}^d_\theta)$ has a number of properties quite unlike the classical Schwartz space $\mathcal{S}(\mathbb{R}^d)$ (for example, see Theorem 7.1.7 below). In terms of the isomorphism of $L_\infty(\mathbb{R}^d_\theta)$ with $\mathcal{B}(L_2(\mathbb{R}^{d/2}))$, it is possible to select a specific basis such that $\mathcal{S}(\mathbb{R}^d_\theta)$ is an algebra of infinite matrices whose entries have rapid decay ([63, Theorem 6] and [109, Theorem 6.11]). While we will not need the specific details of the matrix description, we do make use of the following result, which is [55, Lemma 2.4].

Theorem 7.1.7. *There exists a sequence $\{p_n\}_{n\ge 0} \subset \mathcal{S}(\mathbb{R}^d_\theta)$ such that:*

(i) *Each p_n is a projection of rank n (considered as an operator on $L_2(\mathbb{R}^{d/2})$).*

(ii) *We have that* $p_n \uparrow 1$, *where* 1 *is the identity operator in* $L_\infty(\mathbb{R}^d_\theta)$.

(iii) $\bigcup_{n\geq 0} p_n L_\infty(\mathbb{R}^d_\theta) p_n$ *is dense in* $\mathcal{S}(\mathbb{R}^d_\theta)$ *in its Fréchet topology.*

The presence of smooth projections is a feature of analysis on quantum Euclidean spaces in the $\det(\theta) \neq 0$ case entirely distinct from analysis on Euclidean space. For our purposes we do not need to know the precise form of the sequence $\{p_n\}_{n\geq 0}$, however a description using the map U may be found in [55, Section 2]. The following is a property of $\mathcal{S}(\mathbb{R}^d_\theta)$ which we use several times. It has a short proof which can be found in [63, pg. 877].

Proposition 7.1.8. *The Schwartz class* $\mathcal{S}(\mathbb{R}^d_\theta)$ *has the following factorisation property: every* $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ *is the product* $x = yz$ *of two elements* $y, z \in \mathcal{S}(\mathbb{R}^d_\theta)$.

An important property of the Weyl transform is that $f \mapsto U(f)$ effects an isometry from $L_2(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d_\theta)$ [132, Chapter 2, Lemma 3.1].

Proposition 7.1.9. *Let* $f \in \mathcal{S}(\mathbb{R}^d)$. *Then we have*

$$\|U(f)\|_2 = \|f\|_2 .$$

Proposition 7.1.9 permits us to extend the domain of U from $L_1(\mathbb{R}^d)$ to $L_1(\mathbb{R}^d)+L_2(\mathbb{R}^d)$. The following inequality may be thought of as the quantum Euclidean analogue of the Hausdorff-Young inequality.

Proposition 7.1.10. *Let* $1 \leq p \leq 2$ *with* $\frac{1}{p}+\frac{1}{q} = 1$. *Then for every* $f \in L_p(\mathbb{R}^d)\cap L_1(\mathbb{R}^d)$, *we have* $U(f) \in L_q(\mathbb{R}^d_\theta)$, *and*

$$\|U(f)\|_q \leq \|f\|_p$$

and hence U *has continuous extension from* $L_p(\mathbb{R}^d)$ *to* $L_q(\mathbb{R}^d_\theta)$.

Proof. First consider the case $p = 1$ and $q = \infty$. According to (7.1.4), we have:

$$\|U(f)\|_\infty \leq \|f\|_1, \quad f \in L_1(\mathbb{R}^d_\theta).$$

The case $p = 2$ is provided by Proposition 7.1.9:

$$\|U(f)\|_2 = \|f\|_2.$$

We may deduce the result for all $1 \leq p \leq 2$ by using complex interpolation for the couples $(L_1(\mathbb{R}^d), L_2(\mathbb{R}^d))$ and $(L_\infty(\mathbb{R}^d_\theta), L_2(\mathbb{R}^d_\theta))$. The complex interpolation method for the latter couple is covered by the standard theory of interpolation of noncommutative L_p-spaces (see e.g. [105]). □

We now define the space $\mathcal{S}'(\mathbb{R}^d_\theta)$ of tempered distributions, and the associated operations.

Definition 7.1.2. *Let* $\mathcal{S}'(\mathbb{R}^d_\theta)$ *be the space of continuous linear functionals on* $\mathcal{S}(\mathbb{R}^d_\theta)$, *which may be called the space of quantum tempered distributions.*

As in the classical case, denote the pairing of $T \in \mathcal{S}'(\mathbb{R}^d_\theta)$ *with* ϕ *in* $\mathcal{S}(\mathbb{R}^d_\theta)$ *by* (T, ϕ), *and* $L_\infty(\mathbb{R}^d_\theta)$ *is embedded into* $\mathcal{S}(\mathbb{R}^d_\theta)$ *by:*

$$(x, \phi) := \tau_\theta(x\phi), \quad x \in L_\infty(\mathbb{R}^d_\theta),\ \phi \in \mathcal{S}(\mathbb{R}^d_\theta).$$

For a multi-index $\alpha \in \mathbb{N}_0^d$ and $T \in \mathcal{S}(\mathbb{R}^d_\theta)$, define $\partial^\alpha T$ as the distribution $(\partial^\alpha T, \phi) = (-1)^{|\alpha|}(T, \partial^\alpha \phi)$.

It is not hard to verify that ∂^α on distributions extends ∂^α on $L_\infty(\mathbb{R}^d_\theta)$ as defined in Section 6.2, so there is no conflict of notation.

In terms of the isomorphism $U : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d_\theta)$, we can compute derivatives easily:

$$\partial^\alpha U(\phi) = U(t_1^{\alpha_1} \cdots t_d^{\alpha_d} \phi(t)). \tag{7.1.7}$$

By duality, we can extend the derivatives D_k to operators on $\mathcal{S}'(\mathbb{R}^d_\theta)$. With these generalised derivatives, we are able to reintroduce the homogeneous Sobolev spaces $\dot{W}^m_p(\mathbb{R}^d_\theta)$ associated to noncommutative Euclidean space.

Definition 7.1.11. *The homogeneous Sobolev space $\dot{W}^m_p(\mathbb{R}^d_\theta)$ consists of those $x \in \mathcal{S}'(\mathbb{R}^d_\theta)$ such that every partial derivative of x of order m is in $L_p(\mathbb{R}^d_\theta)$, equipped with the norm:*

$$\|x\|_{\dot{W}^m_p} = \sum_{|\alpha|=m} \|\partial^\alpha x\|_p .$$

We will have frequent need to refer to the operator $(1-\Delta)^{1/2}$, which we abbreviate as J,

$$J := (1-\Delta)^{1/2}.$$

That is, J is the operator on $L_2(\mathbb{R}^d)$ of pointwise multiplication by $(1+|t|^2)^{1/2}$, with domain $L_2(\mathbb{R}^d, (1+|t|^2)dt)$. Classically, the operator J is called the Bessel potential.

Definition 7.1.12. *Let $N = 2^{\lfloor d/2 \rfloor}$ and $\{\gamma_j\}_{1\leq j\leq d}$ be self-adjoint $N\times N$ matrices satisfying $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{j,k}$. The Dirac operator D associated with $L_\infty(\mathbb{R}^d_\theta)$ is the operator on $\mathbb{C}^N \otimes L_2(\mathbb{R}^d)$ defined by*

$$D := \sum_{j=1}^d \gamma_j \otimes D_j.$$

7.1.4 The dilation map

We now describe the "dilation" action of $\mathbb{R}^+$ on a quantum Euclidean space. A peculiarity of the noncommutative situation is that the natural dilation semigroup does not define an automorphism of $L_\infty(\mathbb{R}^d_\theta)$ to itself, but instead the value of θ varies.

The heuristic motivation for the dilation mapping is as follows. Recall that we consider $\mathbb{R}^d_\theta$ as being generated by elements $\{x_1, \ldots, x_d\}$ satisfying the commutation relation

$$[x_j, x_k] = i\theta_{j,k}.$$

However this relation is not invariant under rescaling. That is, if we let $\lambda > 0$ then the family $\{\lambda x_1, \ldots, \lambda x_d\}$ satisfies the relation:

$$[\lambda x_j, \lambda x_k] = i\lambda^2\theta_{j,k}.$$

It therefore becomes clear that if we wish to define a "dilation by λ" map on $\mathbb{R}^d_\theta$, we should instead consider dilation as mapping between two different noncommutative spaces. That is, from $\mathbb{R}^d_\theta$ to $\mathbb{R}^d_{\lambda^2\theta}$.

The following rigorous definition of the "dilation by λ" map follows [62]. Given $\lambda > 0$, define the map Ψ_λ from $L_\infty(\mathbb{R}^d_\theta)$ to $L_\infty(\mathbb{R}^d_{\lambda^2\theta})$ as

$$\Psi_\lambda : U_\theta(s) \mapsto U_{\lambda^2\theta}(\frac{s}{\lambda}). \tag{7.1.8}$$

Recall that we include a subscript θ (or $\lambda^2\theta$) to indicate the dependence on the matrix.

Denote by σ_λ the usual L_2-norm preserving dilation on Euclidean space:

$$\sigma_\lambda\xi(t) = \lambda^{d/2}\xi(\lambda t), \quad \xi \in L_2(\mathbb{R}^d).$$

We have $\sigma_\lambda^* = \sigma_{\lambda^{-1}}$. It is standard to verify that

$$U_\theta(s) = \sigma_\lambda^* U_{\lambda^2\theta}(\frac{s}{\lambda})\,\sigma_\lambda. \tag{7.1.9}$$

Moreover, by (7.1.9), it is evident that for every $\lambda > 0$, Ψ_λ is a $*$-isomorphism from $L_\infty(\mathbb{R}^d_\theta)$ to $L_\infty(\mathbb{R}^d_{\lambda^2\theta})$.

The following proposition shows how the dilation Ψ_λ affects the L_p norms for quantum Euclidean spaces.

Proposition 7.1.13. *Let $\lambda > 0$ and $x \in L_p(\mathbb{R}^d_\theta)$, and denote $\xi = \lambda^2\theta$. Then for all $2 \le p < \infty$, we have:*

$$\|\Psi_\lambda x\|_{L_p(\mathbb{R}^d_\xi)} \le \lambda^{d/p}\|x\|_{L_p(\mathbb{R}^d_\theta)}$$

and Ψ_λ is an isometry from $L_\infty(\mathbb{R}^d_\theta)$ to $L_\infty(\mathbb{R}^d_\xi)$.

If in addition $x \in W^1_p(\mathbb{R}^d_\theta)$, then:

$$\|\partial^j\Psi_\lambda(x)\|_{L_p(\mathbb{R}^d_\xi)} \le \lambda^{d/p-1}\|\partial_j x\|_{L_p(\mathbb{R}^d_\theta)}, \quad j = 1, \cdots, d. \tag{7.1.10}$$

Proof. As was already mentioned, Ψ_λ is a $*$-isomorphism between $L_\infty(\mathbb{R}^d_\theta)$ and $L_\infty(\mathbb{R}^d_\xi)$, and since a $*$-isomorphism of C^*-algebras is an isometry, it follows immediately that $\Psi_\lambda : L_\infty(\mathbb{R}^d_\theta) \to L_\infty(\mathbb{R}^d_\xi)$ is an isometry.

For $p = 2$, recall from Proposition 7.1.9 that the mapping $(2\pi)^{-d/2}U_\theta$ (resp. $(2\pi)^{-d/2}U_\xi$) defines an isometry from $L_2(\mathbb{R}^d_\theta)$ (resp. $L_2(\mathbb{R}^d_\xi)$) to $L_2(\mathbb{R}^d)$. Also note that:

$$\Psi_\lambda \circ U_\theta = U_\xi \circ d_\lambda, \quad \lambda > 0.$$

where d_λ is the dilation by λ map $f \mapsto f(\cdot/\lambda)$. Hence Ψ_λ has the same norm between $L_2(\mathbb{R}^d_\theta)$ and $L_2(\mathbb{R}^d_\xi)$ as d_λ does on $L_2(\mathbb{R}^d)$. This is easily computed to be $\lambda^{d/2}$.

Finally, the result for $2 < p < \infty$ follows from complex interpolation of the couples $(L_2(\mathbb{R}^d_\theta), L_\infty(\mathbb{R}^d_\theta))$ and $(L_2(\mathbb{R}^d_\xi), L_\infty(\mathbb{R}^d_\xi))$.

We recall that the complex interpolation space $(L_2(\mathbb{R}^d_\theta), L_\infty(\mathbb{R}^d_\theta))_\eta$ is $L_{2/\eta}(\mathbb{R}^d_\theta)$, where $\eta \in (0,1)$, and that we have:

$$\|\Psi_\lambda\|_{L_{2/\eta}\to L_{2/\eta}} \leq \|\Psi_\lambda\|^{\eta}_{L_2\to L_2}\|\Psi_\lambda\|^{1-\eta}_{L_\infty\to L_\infty} \leq \lambda^{d\eta/2}.$$

Taking $\eta = \frac{2}{p}$ yields the desired norm bound.

The second claim follows from the easily-verified identity:

$$\partial_j(\Psi_\lambda(x)) = \lambda^{-1}\Psi_\lambda\partial_j(x).$$

□

7.1.5 Density of $\mathcal{S}(\mathbb{R}^d_\theta)$ in Sobolev spaces

Let us discuss the density of $\mathcal{S}(\mathbb{R}^d_\theta)$ in the Sobolev spaces $W^m_p(\mathbb{R}^d_\theta)$ and $\dot{W}^m_p(\mathbb{R}^d_\theta)$. Proving the density of $\mathcal{S}(\mathbb{R}^d_\theta)$ in the homogeneous Sobolev space $\dot{W}^m_p(\mathbb{R}^d_\theta)$ presents certain difficulties and we have been unable to achieve this for the full range of indices (m,p).

For this section, we fix $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}\psi(s)\,ds = 1$. We do not assume that ψ is positive or has compact support. For $\varepsilon > 0$, define:

$$\psi_\varepsilon(t) = \varepsilon^{-d}\psi(\frac{t}{\varepsilon}). \tag{7.1.11}$$

The following theorem provides us a means of "approximation in the spatial variables".

The proof is straightforward and shall be omitted here, but full details may be found in [92, Theorem 3.8].

Theorem 7.1.14. *Let $1 \leq p < \infty$. For all $x \in L_p(\mathbb{R}^d_\theta)$, we have that $U(\psi_\varepsilon)x \to x$ in the $L_p(\mathbb{R}^d_\theta)$ norm as $\varepsilon \to 0$.*

The $p = 2$ component of Theorem 7.1.14 may be equivalently, stated as $U(\psi_\varepsilon) \to 1$ in the strong operator topology of $L_\infty(\mathbb{R}^d_\theta)$ in its representation on $L_2(\mathbb{R}^d_\theta)$.

We note one further property of $U(\psi_\varepsilon)$:

Lemma 7.1.15. *Let $1 \leq j \leq d$. Then for all $2 \leq p \leq \infty$, we have:*

$$\|\partial_j U(\psi_\varepsilon)\|_p \leq \varepsilon^{1-\frac{d}{p}}\|\psi_1\|_q.$$

where q satisfies $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. Recall (from (7.1.7)) that:

$$\partial_j U(\psi_\varepsilon) = U(t_j\psi_\varepsilon(t))$$

so that we may apply Proposition 7.1.10 to bound $\|\partial_j U(\psi_\varepsilon)\|_p$ by:

$$\left(\int_{\mathbb{R}^d} t_j^q\varepsilon^{-dq}|\psi(\frac{t}{\varepsilon})|^q dt\right)^{1/q}$$

where q is Hölder conjugate to p.

Applying the change of variable $s = \frac{t}{\varepsilon}$, we get the norm bound:

$$\|\partial_j U(\psi_\varepsilon)\|_p \leq \varepsilon^{1-d+\frac{d}{q}} \|\psi_1\|_q. \qquad \square$$

Proposition 7.1.16. *Let $m \geq 0$ and $1 \leq p < \infty$, and $x \in W_p^m(\mathbb{R}_\theta^d)$. Then:*

$$\lim_{\varepsilon \to 0} \|U(\phi_\varepsilon)x - x\|_{W_p^m} = 0.$$

In particular, $\mathcal{S}(\mathbb{R}_\theta^d)$ is norm-dense in $W_p^m(\mathbb{R}_\theta^d)$.

At the time of this writing, we are unable to prove that the inclusion $\mathcal{S}(\mathbb{R}_\theta^d) \subset \dot{W}_p^m(\mathbb{R}_\theta^d)$ is dense. In the classical (commutative) setting or on quantum tori, this can be achieved by an application of a Poincaré inequality (see, e.g., [68, Theorem 7]). To the best of our knowledge, no adequate replacement is known in the noncommutative setting. In the following proposition, to obtain the desired convergence in $\dot{W}_d^1(\mathbb{R}_\theta^d)$ norm, we have to assume additionally that $x \in L_p(\mathbb{R}_\theta^d)$ for some $d \leq p < \infty$. This is the ultimate cause of the *a priori* assumption in the statements of Theorems 7.1.1, 7.1.2 and 7.1.5 that $x \in L_p(\mathbb{R}_\theta^d)$ for some $d \leq p < \infty$.

Proposition 7.1.17. *If $x \in \dot{W}_d^1(\mathbb{R}_\theta^d) \cap L_p(\mathbb{R}_\theta^d)$ for some $d \leq p < \infty$, then the sequence $U(\phi_\varepsilon)x$ converges to x in $\dot{W}_d^1$-seminorm when $\varepsilon \to 0^+$.*

7.2 Proof of Theorem 7.1.1

This section is devoted to the proof of Theorem 7.1.1, that is, that the condition $x \in \bigcup_{d \leq p < \infty} L_p(\mathbb{R}_\theta^d) \cap W_d^1(\mathbb{R}_\theta^d)$ is sufficient for $đx \in \mathcal{L}_{d,\infty}$, and with an explicit norm bound:

$$\|đx\|_{d,\infty} \lesssim_d \|x\|_{\dot{W}_d^1(\mathbb{R}_\theta^d)}.$$

The proof given here is similar to the corresponding result on quantum tori [93], relying heavily on the Cwikel type estimate stated in the last section.

Lemma 7.2.1. *Suppose that $p > \frac{d}{2}$ and $x \in L_p(\mathbb{R}_\theta^d)$. If $p \geq 2$, then:*

$$\left\| \left[\operatorname{sgn}(D) - \frac{D}{\sqrt{1+D^2}}, 1 \otimes x \right] \right\|_{\mathcal{L}_p} \lesssim_{p,d} \|x\|_p.$$

Proof. Let $1 \leq j \leq d$, and for $\xi \in \mathbb{R}^d$ define

$$h_j(\xi) := \frac{\xi_j}{|\xi|} - \frac{\xi_j}{(1+|\xi|^2)^{\frac{1}{2}}}.$$

Thus,

$$M_{h_j} = h_j(i\nabla) = \frac{D_j}{\sqrt{-\Delta}} - \frac{D_j}{(1-\Delta)^{\frac{1}{2}}}$$

Note that there is no ambiguity in writing $\frac{D_j}{\sqrt{-\Delta}}$, as this is simply M_g for $g(\xi) = \frac{\xi_j}{|\xi|}$. and so,

$$\operatorname{sgn}(D) - \frac{D}{\sqrt{1+D^2}} = \sum_{j=1}^{d} \gamma_j \otimes \Big(\frac{D_j}{\sqrt{-\Delta}} - \frac{D_j}{(1-\Delta)^{\frac{1}{2}}}\Big) = \sum_{j=1}^{d} \gamma_j \otimes M_{h_j}.$$

One can easily check that $h_j \in L_p(\mathbb{R}^d)$ as $p > \frac{d}{2}$. Expanding out the commutator,

$$\left[\operatorname{sgn}(D) - \frac{D}{\sqrt{1+D^2}}, 1\otimes x\right] = \left[\sum_{j=1}^{d} \gamma_j\otimes M_{h_j}, 1\otimes x\right] = \sum_{j=1}^{d} \gamma_j \otimes [M_{h_j}, x].$$

Hence,

$$\begin{aligned}
\left\|\left[\operatorname{sgn}(D) - \frac{D}{\sqrt{1+D^2}}, 1\otimes x\right]\right\|_{\mathcal{L}_p} &\leq d \max_{1\leq j\leq d} \|[M_{h_j}, x]\|_{\mathcal{L}_p} \\
&\leq d \max_{1\leq j\leq d} \Big(\|M_{h_j} x\|_{\mathcal{L}_p} + \|x M_{h_j}\|_{\mathcal{L}_p}\Big) \\
&= d \max_{1\leq j\leq d} \big(\|x^* M_{h_j}\|_{\mathcal{L}_p} + \|x M_{h_j}\|_{\mathcal{L}_p}\big).
\end{aligned}$$

The desired conclusion follows then from Theorem 6.2.6.(i). □

The proof of the next lemma is modelled on that of [93, Lemma 4.2] and [87, Lemma 10], via the technique of double operator integrals.

Lemma 7.2.2. *The function ψ on $\mathbb{R}^2$ given by:*

$$\psi(\lambda,\mu) = \frac{(1+\lambda^2)^{1/4}(1+\mu^2)^{1/4}}{(1+\lambda^2)^{1/2} + (1+\mu^2)^{1/2}}$$

is in the Birman-Solomyak class $\mathfrak{BS}(\nu_0 \times \nu_1)$ for any pair of spectral measures ν_0 and ν_1.

Proof. Let $t := \frac{1}{4}(\log(1+\lambda^2) - \log(1+\mu^2))$. Then,

$$\psi(\lambda,\mu) = \frac{1}{e^t + e^{-t}} = \frac{1}{2}\operatorname{sech}(t).$$

The function $t \mapsto \frac{1}{2}\operatorname{sech}(t)$ is Schwartz class, and so has Schwartz-class Fourier transform F, so by the Fourier inversion theorem:

$$\psi(\lambda,\mu) = \int_{-\infty}^{\infty} e^{ist} F(s)\, ds = \int_{-\infty}^{\infty} (1+\lambda^2)^{is/4}(1+\mu^2)^{-is/4} F(s)\, ds.$$

This is a Birman-Solomyak decomposition, since F is in particular integrable. □

Lemma 7.2.3. *Let $x \in \dot{W}^1_d(\mathbb{R}^d_\theta) \cap L_p(\mathbb{R}^d_\theta)$ for some $d \leq p < \infty$. Then*

$$\Big\|\Big[\frac{D}{\sqrt{1+D^2}}, 1\otimes x\Big]\Big\|_{\mathcal{L}_{d,\infty}} \lesssim_d \|x\|_{\dot{W}^1_d}.$$

Proof. Set $g(t) = t(1+t^2)^{-\frac{1}{2}}$ for $t \in \mathbb{R}$. Thanks to Theorem 4.1.10, we have:

$$[g(D), 1 \otimes x] = \mathcal{T}^{\mathcal{D},\mathcal{D}}_{g(\lambda)-g(\mu),\mathcal{B}(L_2)}(x) = \mathcal{T}^{D,D}_{g^{[1]},\mathcal{B}(L_2)}([D, 1\otimes x]), \tag{7.2.1}$$

where $g^{[1]}(\lambda,\mu) := \frac{g(\lambda)-g(\mu)}{\lambda-\mu} = \psi_1(\lambda,\mu)\psi_2(\lambda,\mu)\psi_3(\lambda,\mu)$, with

$$\psi_1 = 1 + \frac{1-\lambda\mu}{(1+\lambda^2)^{\frac{1}{2}}(1+\mu^2)^{\frac{1}{2}}}, \quad \psi_2 = \frac{(1+\lambda^2)^{\frac{1}{4}}(1+\mu^2)^{\frac{1}{4}}}{(1+\lambda^2)^{\frac{1}{2}}+(1+\mu^2)^{\frac{1}{2}}}, \quad \psi_3 = \frac{1}{(1+\lambda^2)^{\frac{1}{4}}(1+\mu^2)^{\frac{1}{4}}}.$$

For $k = 1, 3$, the function ψ_k can be written as a linear combination of products of bounded functions of λ and μ individually, so $\psi_1, \psi_3 \in \mathfrak{BS}(E^D \times E^D)$. From Lemma 7.2.2, we also have that $\psi_2 \in \mathfrak{BS}(E^D \times E^D)$.

Applying Theorem 4.1.10, we have:

$$\mathcal{T}^{D,D}_{g^{[1]},\mathcal{B}(L_2)} = \mathcal{T}^{D,D}_{\psi_1,\mathcal{B}(L_2)}\mathcal{T}^{D,D}_{\psi_2,\mathcal{B}(L_2)}\mathcal{T}^{D,D}_{\psi_3,\mathcal{B}(L_2)}. \tag{7.2.2}$$

Theorem 4.1.10 yields the boundedness of $\mathcal{T}^{D,D}_{\psi_k}$ on $\mathcal{L}_1$ and $\mathcal{B}(L_2(\mathbb{R}^d))$ for $k = 1, 2, 3$. Then by real interpolation of $(\mathcal{L}_1, \mathcal{L}_\infty)$ (see [44]), the transformers $T^{D,D}_{\psi_k}$ with $k = 1, 2, 3$ are bounded linear transformations from $\mathcal{L}_{d,\infty}$ to $\mathcal{L}_{d,\infty}$. Using (7.2.1) and the product representation of g in (7.2.2), we have

$$\begin{aligned}\|[g(D), 1\otimes x]\|_{\mathcal{L}_{d,\infty}} &\leq \|\mathcal{T}^{D,D}_{\psi_1,\mathcal{B}(L_2)}\|_{\mathcal{L}_{d,\infty}\to\mathcal{L}_{d,\infty}}\|\mathcal{T}^{D,D}_{\psi_2,\mathcal{B}(L_2)}\|_{\mathcal{L}_{d,\infty}\to\mathcal{L}_{d,\infty}}\|\mathcal{T}^{D,D}_{\psi_3,\mathcal{B}(L_2)}([D, 1\otimes x])\|_{\mathcal{L}_{d,\infty}} \\ &\lesssim_d \|\mathcal{T}^{D,D}_{\psi_3,\mathcal{B}(L_2)}([D, 1\otimes x])\|_{\mathcal{L}_{d,\infty}}.\end{aligned}$$

Since $\psi_3(\lambda,\mu) = (1+\lambda^2)^{-1/4}(1+\mu^2)^{-1/4}$, Theorem 4.1.10 implies that:

$$\mathcal{T}^{D,D}_{\psi_3,\mathcal{B}(L_2)}([D, 1\otimes x]) = (1+D^2)^{-1/4}[D, 1\otimes x](1+D^2)^{-1/4}.$$

Recalling that $D = \sum_{j=1}^d \gamma_j \otimes D_j$,

$$\begin{aligned}\|[g(D), 1\otimes x]\|_{\mathcal{L}_{d,\infty}} &\lesssim_d \|(1+D^2)^{-1/4}[D, 1\otimes x](1+D^2)^{-1/4}\|_{\mathcal{L}_{d,\infty}} \\ &\lesssim_d \sum_{j=1}^d \|(1+D^2)^{-1/4}[\gamma_j\otimes D_j, 1\otimes x](1+D^2)^{-1/4}\|_{\mathcal{L}_{d,\infty}}.\end{aligned}$$

But by definition, $[\gamma_j \otimes D_j, 1\otimes x] = \gamma_j \otimes \partial_j x$, thus we obtain

$$\|(1+D^2)^{-1/4}[\gamma_j\otimes D_j, 1\otimes x](1+D^2)^{-1/4}\|_{\mathcal{L}_{d,\infty}} = \|J^{-1/2}\,\partial_j x\, J^{-1/2}\|_{\mathcal{L}_{d,\infty}}.$$

Here the first norm $\|\cdot\|_{\mathcal{L}_{d,\infty}}$ is the norm of $\mathcal{L}_{d,\infty}(\mathbb{C}^N \otimes L_2(\mathbb{R}^d))$, and the second one is the norm of $\mathcal{L}_{d,\infty}(L_2(\mathbb{R}^d))$, and $J = (1-\Delta)^{1/2}$. We are reduced to estimating the quantity $\|J^{-1/2}\,\partial_j x\, J^{-1/2}\|_{\mathcal{L}_{d,\infty}}$. By polar decomposition, for every j, there is a partial isometry V_j on $L_2(\mathbb{R}^d)$ such that

$$\partial_j x = V_j|\partial_j x| = V_j|\partial_j x|^{\frac{1}{2}}|\partial_j x|^{\frac{1}{2}}.$$

Recalling that x is such that $\|V_j|\partial_j x|^{\frac{1}{2}}\|_{2d} \le \|\,|\partial_j x|^{\frac{1}{2}}\|_{2d} = \|\partial_j x\|_d^{\frac{1}{2}} < \infty$, we apply Theorem 6.2.8 to get

$$\||\partial_j x|^{\frac{1}{2}} J^{-1/2}\|_{\mathcal{L}_{2d,\infty}} = \|J^{-1/2}|\partial_j x|^{\frac{1}{2}}\|_{\mathcal{L}_{2d,\infty}} \lesssim_d \|\,|\partial_j x|^{\frac{1}{2}}\|_{2d}$$

and

$$\|J^{-1/2}V_j|\partial_j x|^{\frac{1}{2}}\|_{\mathcal{L}_{2d,\infty}} \lesssim_d \|V_j|\partial_j x|^{\frac{1}{2}}\|_{2d} \lesssim_d \|\,|\partial_j x|^{\frac{1}{2}}\|_{2d}.$$

Thus, by Hölder's inequality for weak Schatten classes,

$$\|J^{-1/2}\partial_j x\, J^{-1/2}\|_{\mathcal{L}_{d,\infty}} \lesssim_d \|\,|\partial_j x|^{\frac{1}{2}}\|_{2d}^2 \lesssim_d \|\partial_j x\|_d.$$

Combining the preceding estimates, we arrive at

$$\|[g(D), 1\otimes x]\|_{\mathcal{L}_{d,\infty}} \lesssim_d \sum_{j=1}^{d} \|\partial_j x\|_d \lesssim_d \|x\|_{\dot{W}_d^1},$$

which completes the proof. □

Now we are able to complete the proof of Theorem 7.1.1.

Proof of Theorem 7.1.1. Lemmas 7.2.1 and 7.2.3 already yield

$$\|{\rm đ}x\|_{\mathcal{L}_{d,\infty}} \lesssim_d \|x\|_d + \|x\|_{\dot{W}_d^1}, \tag{7.2.3}$$

for all $x \in W_d^1(\mathbb{R}_\theta^d)$, and with constants independent of θ. We are going to get rid of the dependence on $\|x\|_d$ by a dilation argument as follows. Let $\lambda > 0$ and $\Psi_\lambda : L_\infty(\mathbb{R}_\theta^d) \to L_\infty(\mathbb{R}_{\lambda^2\theta}^d)$ be the $*$-isomorphism defined in (7.1.8). By (7.1.9), for $x \in L_\infty(\mathbb{R}_\theta^d)$, we have $\Psi_\lambda(x) = \sigma_\lambda x \sigma_\lambda^*$. Since the operator $\frac{D_j}{\sqrt{-\Delta}}$, viewed as a Fourier multiplier on $\mathbb{R}^d$, commutes with σ_λ (and σ_λ^*), we have

$$\begin{aligned}{\rm đ}\big(\Psi_\lambda(x)\big) &= i[\mathrm{sgn}(D), 1\otimes\Psi_\lambda(x)] = i[\mathrm{sgn}(D), 1\otimes\sigma_\lambda x\sigma_\lambda^*]\\ &= i\sigma_\lambda[\mathrm{sgn}(D), 1\otimes x]\sigma_\lambda^* = \sigma_\lambda {\rm đ}x\, \sigma_\lambda^*.\end{aligned}$$

Whence, $\|{\rm đ}\big(\Psi_\lambda(x)\big)\|_{\mathcal{L}_{d,\infty}} = \|{\rm đ}x\|_{\mathcal{L}_{d,\infty}}$. Applying (7.2.3) to $\Psi_\lambda(x) \in L_\infty(\mathbb{R}_{\lambda^2\theta}^d)$, we obtain

$$\|{\rm đ}\big(\Psi_\lambda(x)\big)\|_{\mathcal{L}_{d,\infty}} \lesssim_d \|\Psi_\lambda(x)\|_d + B_d\|\Psi_\lambda(x)\|_{\dot{W}_d^1}.$$

By virtue of Proposition 7.1.13, we return back to $x \in L_\infty(\mathbb{R}_\theta^d)$:

$$\|{\rm đ}x\|_{\mathcal{L}_{d,\infty}} = \|{\rm đ}\big(\Psi_\lambda(x)\big)\|_{\mathcal{L}_{d,\infty}} \lesssim_d \lambda\|x\|_d + \|x\|_{\dot{W}_d^1}.$$

Letting $\lambda \to 0$ completes the proof of Theorem 7.1.1 for $x \in \mathcal{S}(\mathbb{R}_\theta^d)$.

The general case $x \in \dot{W}_d^1(\mathbb{R}_\theta^d) \cap \bigcup_{d\le p<\infty} L_p(\mathbb{R}_\theta^d)$ is achieved by approximation. By Corollary 7.1.17, select a sequence $\{x_n\}$ in $\mathcal{S}(\mathbb{R}_\theta^d)$ such that $x_n \to x$ in $\dot{W}_d^1$ seminorm. Proposition 7.1.16 implies that we can choose this sequence such that we also have that $x_n \to x$ in the $L_p(\mathbb{R}_\theta^d)$-sense. For these Schwartz elements x_n, we have $\|{\rm đ}x_m - {\rm đ}x_n\|_{\mathcal{L}_{d,\infty}} \lesssim_d \|x_m - x_n\|_{\dot{W}_d^1}$, so $\{{\rm đ}x_n\}$ is Cauchy in $\mathcal{L}_{d,\infty}$, and thus converges to some limit (say, L) in the $\mathcal{L}_{d,\infty}$ quasinorm.

Let $\eta \in L_2(\mathbb{R}^d)$ be compactly supported, and let $K \subset \mathbb{R}^d$ be a compact set containing the support of η. Then $(x_n - x)\eta = (x_n - x)M_{\chi_K}\eta$. We have:

$$\|(x_n - x)\eta\|_2 = \|(x_n - x)\chi_K\eta\|_2 \le \|(x_n - x)M_{\chi_K}\|_\infty\|\eta\|_2 \le \|(x_n - x)\chi_K\|_{\mathcal{L}_p}\|\eta\|_2.$$

Theorem 6.2.6 implies that $\|(x_n - x)M_{\chi_K}\|_{\mathcal{L}_p} \lesssim_{p,K} \|x_n - x\|_p$, and since we have selected the sequence to converge in the $L_p(\mathbb{R}^d_\theta)$ sense:

$$\lim_{n\to\infty} \|(x_n - x)\eta\|_2 = 0. \tag{7.2.4}$$

Similarly, if $\xi \in \mathbb{C}^N \otimes L_2(\mathbb{R}^d)$ is compactly supported, then $\mathrm{sgn}(D)\xi$ is still compactly supported and we have:

$$\lim_{n\to\infty} \|1 \otimes (x_n - x)\,\mathrm{sgn}(D)\xi\|_2 = 0. \tag{7.2.5}$$

Combining (7.2.4) and (7.2.5) implies that $(\text{đ}x_n)\xi \to (\text{đ}x)\xi$ for all compactly supported $\xi \in \mathbb{C}^N \otimes L_2(\mathbb{R}^d)$. Since we know that $\text{đ}x_n \to L$ in the $\mathcal{L}_{d,\infty}$ topology, it follows that $\text{đ}x = L$, and therefore $\text{đ}x \in \mathcal{L}_{d,\infty}$.

To complete the proof, we use the quasinorm triangle inequality:

$$\|\text{đ}x\|_{\mathcal{L}_{d,\infty}} \lesssim_d \|\text{đ}x - \text{đ}x_n\|_{\mathcal{L}_{d,\infty}} + \|\text{đ}x_n\|_{\mathcal{L}_{d,\infty}} \lesssim_d \|x - x_n\|_{\dot{W}^1_d} + \|x_n\|_{\dot{W}^1_d}.$$

Upon taking the limit $n\to\infty$ we arrive at:

$$\|\text{đ}x\|_{\mathcal{L}_{d,\infty}} \lesssim_d \|x\|_{\dot{W}^1_d}.$$

□

7.3 Commutator estimates for $\mathbb{R}^d_\theta$

This section is devoted to a proof of Theorem 7.1.6, which is an essential ingredient for our proof of Theorem 7.1.2 i.e., the computation of $\varphi(|\text{đ}x|^d)$ when $x \in L_\infty(\mathbb{R}^d_\theta) \cap \dot{W}^1_d(\mathbb{R}^d_\theta)$ and φ is a continuous normalised trace on $\mathcal{L}_{1,\infty}$. One powerful tool used in [93] for quantum tori is the theory of noncommutative pseudodifferential operators. The proof in [93] proceeds by viewing the quantised differential $\text{đ}x = i[\mathrm{sgn}(D), 1\otimes x]$ as a pseudodifferential operator, then determining its (principal) symbol and order, and finally appealing to Connes' trace formula as obtained in [94].

For potential future utility we will prove Theorem 7.1.6 for the full range of parameters (α, β), although ultimately we will only need certain specific choices of α and β.

7.3.1 The pseudodifferential calculus

Our method of proof for Theorem 7.1.6 is to develop a very basic pseudodifferential calculus for $\mathbb{R}^d_\theta$. It is possible to provide a fully developed symbol calculus, as in [62] and [82, Section 3], however for our purposes only a much weaker framework is necessary. The calculus developed here is essentially a special case of the abstract pseudodifferential calculus developed by Connes and Moscovici [34] and Higson [74]. The ideas discussed here have also been greatly extended by L. Gao, M. Junge and the author to noncommutative Euclidean spaces with non-commuting derivatives [53].

Recall that J denotes the Bessel potential operator $J = (1-\Delta)^{1/2}$. The following definition is essentially the same as [73, Definition 1.1], adapted to $\mathbb{R}^d_\theta$.

Definition 7.3.1. *For $s \in \mathbb{R}$, a linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is said to have* analytic order s *if for all $r \in \mathbb{R}$, the operator:*

$$J^r T J^{-r-s}$$

extends to a bounded linear operator on $L_2(\mathbb{R}^d)$. If T has analytic order s for some s, say that T is an operator of finite analytic order.

Obviously, if T has analytic order t and S has analytic order s, then TS and ST have analytic order $t+s$, and $T+S$ has analytic order $\max\{t,s\}$. In [53], the term "asymptotic degree" was used for essentially the same concept as analytic order.

According to the above definition, it T has analytic order s and $t > s$, then T also has analytic order t. It is tempting to define the "minimal order" of an operator T as the infimum of all s such that T has analytic order s. However this can be misleading, since an operator can be analytic of every order $t > 0$ but not analytic of order zero (e.g., the operator $\log(J)$).

In practice it is not necessary to check that $J^r T J^{-r-s}$ is bounded for every $r \in \mathbb{R}$, due to an interpolation argument. Recall that the operator J is defined as being simply the multiplication operator $(1+|t|^2)^{1/2}$ on $L_2(\mathbb{R}^d, (1+|t|^2)^{1/2}dt)$.

Definition 7.3.2. *For $s \in \mathbb{R}$, let $L_2^s(\mathbb{R}^d)$ denote the space:*

$$L_2^s(\mathbb{R}^d) := L_2(\mathbb{R}^d, (1+|t|^2)^{s/2}\, dt).$$

Equivalently, $L_2^s(\mathbb{R}^d)$ is the domain of the self-adjoint operator J^s.

Evidently, $L_2^s(\mathbb{R}^d)$ is the image under the Fourier transform of the Bessel potential Sobolev space $W_2^s(\mathbb{R}^d)$ on $\mathbb{R}^d$.

Proposition 7.3.3. *Let $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ be linear, and let $s \in \mathbb{R}$. The following are equivalent:*

(i) *T has analytic order s.*

(ii) *For every $k \in 2\mathbb{Z}$, the map $J^k T J^{-k-s}$ has bounded extension.*

(iii) *T extends to a bounded linear map from $L_2^{k+s}(\mathbb{R}^d)$ to $L_2^k(\mathbb{R}^d)$ for every $k \in 2\mathbb{Z}$.*

(iv) *T extends to a bounded linear map from $L_2^{r+s}(\mathbb{R}^d)$ to $L_2^r(\mathbb{R}^d)$ for every $r \in \mathbb{R}$.*

Proof. That (i) implies (ii) is trivial. Since J^k maps $L_2^k(\mathbb{R}^d)$ to $L_2(\mathbb{R}^d)$ continuously, that (ii) implies (iii) is self-evident from the definitions. Similarly, (iv) implies (i) for the same reason.

The only somewhat nontrivial implication is that (iii) implies (iv). For this, we use the fact that the family $\{L_2^s(\mathbb{R}^d)\}_{s\in\mathbb{R}}$ is a complex interpolation scale. This is well-known (see [11, Chapter 4, Theorem 3.6]), since $L_2^s(\mathbb{R}^d)$ is simply a weighted L_2 space. □

As one would expect, the analytic order of an operator of multiplication by $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ is zero.

Lemma 7.3.4. *Every $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ has analytic order zero.*

Proof. Note that $x : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, this can be seen directly from (7.1.5).

Due to Proposition 7.3.3.(ii), it suffices to check that $J^{2k}xJ^{-2k}$ has bounded extension for every $k \in \mathbb{Z}$. For $k \geq 0$, the proof can be achieved by induction on k, with the case $k = 0$ being immediate. Recall that $J^2 = 1+\sum_{j=1}^d D_j^2$, and that $[D_j^2, x] = 2\partial_j x D_j + \partial_j^2 x$.

Assuming that $k \geq 0$ and $J^{2k}xJ^{-2k}$ has bounded extension for every $x \in \mathcal{S}(\mathbb{R}^d_\theta)$, we have (on $\mathcal{S}(\mathbb{R}^d)$)

$$\begin{aligned} J^{2k+2}xJ^{-2k-2} &= J^{2k}xJ^{-2k} + J^{2k}[J^2, x]J^{-2k-2} \\ &= J^{2k}xJ^{-2k} - \sum_{j=1}^d 2iJ^{2k}\partial_j x D_j J^{-2k-2} + J^{2k}\partial_j^2 x J^{-2k}. \end{aligned}$$

Since $\partial_j x$ and $\partial_j^2 x$ are in $\mathcal{S}(\mathbb{R}^d_\theta)$ for all $j = 1, \ldots, d$, and since each $D_j J^{-2}$ is bounded, it follows that the operator $J^{2k+2}xJ^{-2k-2}$ coincides on $\mathcal{S}(\mathbb{R}^d)$ with a sum of operators with bounded extension to $L_2(\mathbb{R}^d)$. Hence, $J^{2k+2}xJ^{-2k-2}$ has bounded extension, and thus by induction $J^{2k}xJ^{-2k}$ has bounded extension for every $k \geq 0$.

One can handle the case $k < 0$ using the identity:

$$\langle J^{2k}xJ^{-2k}\xi, \eta\rangle_{L_2(\mathbb{R}^d)} = \langle \xi, J^{-2k}x^*J^{2k}\eta\rangle_{L_2(\mathbb{R}^d)}, \quad \xi, \eta \in \mathcal{S}(\mathbb{R}^d).$$

That is, on $\mathcal{S}(\mathbb{R}^d)$ the operator $J^{2k}xJ^{-2k}$ coincides with the adjoint of the bounded extension of $J^{-2k}x^*J^{2k}$, so $J^{2k}xJ^{-2k}$ itself has bounded extension. □

The following Lemma strengthens the Cwikel-type estimate given in Lemma 6.2.8, given additional smoothness assumptions on x.

Lemma 7.3.5. *For every $\beta > 0$ and $x \in \mathcal{S}(\mathbb{R}^d_\theta)$, the operator $xJ^{-\beta}$ is in the ideal $\mathcal{L}_{d/\beta,\infty}$.*

Proof. For $\beta = d$, this is already the result of Lemma 6.2.8 since $\mathcal{S}(\mathbb{R}^d_\theta) \subset W_1^d(\mathbb{R}^d)$.

We can extend the result to $0 < \beta \leq d$ using the Araki-Lieb-Thirring inequality (1.5.2), with $r = d/\beta$, $A = x$ and $B = J^{-\beta}$ to obtain:

$$xJ^{-\beta} \in \mathcal{L}_{d/\beta,\infty}$$

We may complete the proof by showing that if the result holds for some $\beta > 0$, then it continues to hold for $\beta + 2$. Since we know the result is true for $0 < \beta \leq 2 \leq d$, this suffices to complete the proof.

Assume that the result holds for some $\beta > 0$. That is, assume that $\beta > 0$ is such that for all $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ we have $xJ^{-\beta} \in \mathcal{L}_{d/\beta,\infty}$. Since every $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ can be factorised as $x = yz$ for some $y, z \in \mathcal{S}(\mathbb{R}^d_\theta)$ (Proposition 7.1.8), we have that:

$$J^{-2}xJ^{-\beta} = J^{-2}yzJ^{-\beta} = (y^*J^{-2})^*(xJ^{-\beta})$$

and this product is in $\mathcal{L}_{\frac{d}{\beta+2},\infty}$, due to Hölder's inequality. Hence

$$J^{-2}xJ^{-\beta} \in \mathcal{L}_{\frac{d}{\beta+2},\infty} \tag{7.3.1}$$

for all $x \in \mathcal{S}(\mathbb{R}^d_\theta)$.

Now write $xJ^{-\beta-2}$ as follows:

$$\begin{aligned} xJ^{-\beta-2} &= J^{-2}xJ^{-\beta} + [x, J^{-2}]J^{-\beta} \\ &= J^{-2}xJ^{-\beta} + J^{-2}[J^2, x]J^{-\beta-2}. \end{aligned}$$

Using $J^2 = 1 + \sum_{j=1}^d D_j^2$ and $[D_j^2, x] = -2i\partial_j x D_j - \partial_j^2 x$, we have:

$$xJ^{-\beta-2} = J^{-2}xJ^{-\beta} - 2i\sum_{j=1}^d J^{-2}\partial_j x D_j J^{-\beta-2} - \sum_{j=1}^d J^{-2}\partial_j^2 x J^{-\beta-2}.$$

Since each $\partial_j x$ and $\partial_j^2 x$ is still in $\mathcal{S}(\mathbb{R}^d_\theta)$ and each $D_j J^{-2}$ is bounded, it now follows from (7.3.1) that $xJ^{-\beta-2}$ is in $\mathcal{L}_{\frac{d}{\beta+2},\infty}$. □

Immediately from Lemma 7.3.5, we have the following:

Corollary 7.3.6. *If T has analytic order $-\beta < 0$ and $x \in \mathcal{S}(\mathbb{R}^d_\theta)$, then $xT \in \mathcal{L}_{d/\beta,\infty}$.*

Proof. Since $J^\beta T$ has analytic order zero, $J^\beta T$ is in particular bounded on $L_2(\mathbb{R}^d)$. Hence by Lemma 7.3.5, the product $xT = xJ^{-\beta}J^\beta T$ is in the ideal $\mathcal{L}_{d/\beta,\infty}$. □

7.3.2 Commutator identities

The following integral formula will be useful: let $\zeta < 1$ and $\eta > 1-\zeta$. Then for all $t > 0$ we have

$$\int_0^\infty \frac{1}{\lambda^\zeta (t+\lambda)^\eta}\, d\lambda = t^{1-\zeta-\eta}\,\mathrm{B}(\eta+\zeta-1, 1-\zeta). \tag{7.3.2}$$

where $\mathrm{B}(\cdot,\cdot)$ is the Beta function.

For a linear operator $T : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$, let $L(T) := J^{-1}[J^2, T]$, and define $\delta(T) := [J, T]$ similarly. Inductively, for $k \in \mathbb{N}$ we define $L^k(T) = L(L^{k-1}(T))$ and $\delta^k(T) = \delta(\delta^{k-1}(T))$. We also make the convention that $L^0(T) = T$ and $\delta^0(T) = T$. Note that $L(T)J^{-1} = L(TJ^{-1})$.

In conventional pseudodifferential calculus, the order of a commutator of two operators with commuting symbols is one less than the sum of their orders. In our very basic calculus, the following theorem is a suitable substitute for our purposes when one of the operators is a power of J.

The essential idea behind the proof of the following goes back to Connes and Moscovici [34, Appendix B].

Theorem 7.3.1. *Let T be an operator with finite analytic order and let $\alpha \in \mathbb{R}$. Suppose that $L(T)$ and $L^2(T)$ have analytic order s. Then $[J^\alpha, T]$ has analytic order $s - 1 + \alpha$.*

Proof. Initially consider the case where $0 < \alpha < 2$. Using (7.3.2) with the parameters $\eta = 1$ and $\zeta = 1 - \frac{\alpha}{2}$, we have:

$$(J^2)^{\alpha/2-1} = \frac{1}{\mathrm{B}(1-\alpha/2,\alpha/2)}\int_0^\infty \lambda^{\alpha/2-1}\frac{1}{J^2+\lambda}\,d\lambda.$$

Since $J^2+\lambda \geq 1+\lambda$, the integrand is bounded in norm by $\lambda^{\alpha/2-1}(1+\lambda)^{-1}$ and so the above integral exists in the weak* sense thanks to Lemma 4.1.4. The coincidence of the integral and the operator $J^{\alpha-2} = (J^2)^{\alpha/2-1}$ follows from the spectral theorem. Multiplying through by J^2 yields the identity:

$$J^\alpha = \frac{1}{\mathrm{B}(1-\alpha/2,\alpha/2)}\int_0^\infty \lambda^{\alpha/2-1}\frac{J^2}{J^2+\lambda}\,d\lambda.$$

One should take caution about about the interpretation of this identity, since the integral is not absolutely convergent in the operator norm (indeed, the integral is J^α which is unbounded). A valid interpretation is that if $\xi,\eta\in\mathcal{S}(\mathbb{R}^d)$, then we have:

$$\langle J^\alpha\xi,\eta\rangle = \langle J^{\alpha-2}\cdot J^2\xi,\eta\rangle = \frac{1}{\mathrm{B}(1-\alpha/2,\alpha/2)}\int_0^\infty \lambda^{\alpha/2-1}\langle\frac{1}{J^2+\lambda}\cdot J^2\xi,\eta\rangle\,d\lambda$$

by the definition of the weak* integral, since the map $X\mapsto\langle XJ^2\xi,\eta\rangle$ is in the predual of $\mathcal{B}(L_2(\mathbb{R}^d))$.

The above interpretation justifies the identity:

$$[J^\alpha,T] = \frac{1}{\mathrm{B}(1-\alpha/2,\alpha/2)}\int_0^\infty \lambda^{\alpha/2-1}\left[\frac{J^2}{J^2+\lambda},T\right]\,d\lambda.$$

Using the identity:

$$\left[\frac{J^2}{J^2+\lambda},T\right] = \lambda(J^2+\lambda)^{-1}[J^2,T](J^2+\lambda)^{-1},$$

we have:

$$[J^\alpha,T] = \frac{1}{\mathrm{B}(1-\alpha/2,\alpha/2)}\int_0^\infty \lambda^{\alpha/2}\frac{J}{\lambda+J^2}L(T)(J^2+\lambda)^{-1}\,d\lambda.$$

Now we commute the resolvent $(\lambda+J^2)^{-1}$ with $L(T)$ as follows:

$$L(T)(\lambda+J^2)^{-1} = (\lambda+J^2)^{-1}L(T) + \frac{J}{\lambda+J^2}L^2(T)(\lambda+J^2)^{-1}$$

Substituting this into the integral, we have:

$$\begin{aligned}[J^\alpha,T] &= \frac{1}{\mathrm{B}(1-\alpha/2,\alpha/2)}\int_0^\infty \lambda^{\alpha/2}\frac{J}{(\lambda+J^2)^2}\,d\lambda\cdot L(T)\\ &\quad+\frac{1}{\mathrm{B}(1-\alpha/2,\alpha/2)}\int_0^\infty \lambda^{\alpha/2}\frac{J^2}{(\lambda+J^2)^2}L^2(T)(\lambda+J^2)^{-1}\,d\lambda.\end{aligned}$$

Using (7.3.2) with the parameters $\zeta = -\alpha/2$ and $\eta = 2$, we arrive at:

$$\begin{aligned}[J^\alpha, T] &= \frac{\mathrm{B}(1-\alpha/2, 1+\alpha/2)}{\mathrm{B}(1-\alpha/2,\alpha/2)} J^{\alpha-1}L(T) \\ &+ \frac{1}{\mathrm{B}(1-\alpha/2,\alpha/2)}\int_0^\infty \lambda^{\alpha/2}\frac{J^2}{(\lambda+J^2)^2}L^2(T)(\lambda+J^2)^{-1}\,d\lambda.\end{aligned}$$

To complete the proof of the $0 < \alpha < 2$ case, we need to show that the integral term has analytic order $s-1+\alpha$. Since $L^2(T)$ has analytic order s, by definition for every $r \in \mathbb{R}$ we have:

$$\|J^{r-s}L^2(T)J^{-r}\|_\infty < \infty.$$

Let $\beta, r \in \mathbb{R}$. For all $\lambda > 0$, it follows that we have the bound:

$$\begin{aligned}&\|\lambda^{\alpha/2}J^{r-s-\beta}\frac{J^2}{(\lambda+J^2)^2}L^2(T)(\lambda+J^2)^{-1}J^{-r}\| \\ &\qquad\le \lambda^{\alpha/2}\|J^{2-\beta}(\lambda+J^2)^{-2}\|_\infty\|J^{r-s}L^2(T)J^{-r}\|_\infty\|(\lambda+J^2)^{-1}\|_\infty.\end{aligned}$$

By functional calculus,

$$\|J^{2-\beta}(\lambda+J^2)^{-2}\|_\infty \le \sup_{t\ge 0} t^{2-\beta}(\lambda+t^2)^{-2} \lesssim_\beta \lambda^{-\beta/2-1}$$

and similarly $\|(\lambda+J^2)^{-1}\| \le (\lambda+1)^{-1}$. Thus we have the upper bound:

$$\|\lambda^{\alpha/2}J^{r-s-\beta}\frac{J^2}{(\lambda+J^2)^2}L^2(T)(\lambda+J^2)^{-1}J^{-r}\| \lesssim_{\beta,r,s_2} \lambda^{(\alpha-\beta)/2-1}(\lambda+1)^{-1}.$$

If $\frac{\alpha-\beta}{2} - 1 \in (-1,0)$, the above function is integrable over $[0,\infty)$ and hence the integral:

$$\int_0^\infty \lambda^{\alpha/2}\frac{J^2}{(\lambda+J^2)^2}L^2(T)(\lambda+J^2)^{-1}\,d\lambda$$

converges in the operator norm from $L_2^r(\mathbb{R}^d)$ to $L_2^{r-s+\beta}(\mathbb{R}^d)$. Therefore, $[J^\alpha, T]$ has order at most $\max\{s+\alpha-1, s+\beta\}$ for all $\beta \in (\alpha-2, 0)$. Taking $\beta = \alpha - 1$ completes the proof of the $0 < \alpha < 2$ case.

Note that the cases $\alpha = 0$ and $\alpha = 2$ are trivial, so the result holds for $\alpha \in [0,2]$. Using the identities:

$$[J^{\alpha+2}, T] = J^\alpha[J^2, T] + J^2[J^\alpha, T], \quad [J^{-\alpha}, T] = -J^{-\alpha}[J^\alpha, T]J^{-\alpha}$$

the result is easily extended to all $\alpha \in \mathbb{R}$. □

Remark 7.3.7. *Note that if we continued taking iterated commutators with $(\lambda+J^2)^{-1}$ in the proof of Theorem 7.3.1, we would have for each n the identity:*

$$L(T)(\lambda+J^2)^{-1} = \sum_{k=1}^{n-1} J^k(\lambda+J^2)^{-k}L^k(T) + J^{n+1}(\lambda+J^2)^{-n}L^{n+1}(T)(\lambda+J^2)^{-1}.$$

This expansion yields the following formula for $[J^\alpha, T]$, when $\alpha \in (0,2)$:

$$[J^\alpha, T] = \sum_{k=1}^{n-1} \frac{\mathrm{B}(k-\alpha/2, 1+\alpha/2)}{\mathrm{B}(1-\alpha/2,\alpha/2)} J^{\alpha-k} L^k(T) \\ + \frac{1}{\mathrm{B}(1-\alpha/2,\alpha/2)} \int_0^\infty \lambda^{\alpha/2} \frac{J^{n+1}}{(\lambda+J^2)^{n+1}} L^{n+1}(T)(\lambda+J^2)^{-1}\,d\lambda.$$

In some circumstances (e.g., $T = x \in \mathcal{S}(\mathbb{R}^d_\theta)$ as discussed below), this is an asymptotic expansion in the sense of Higson [74, Definition 4.18].

Corollary 7.3.8. *If T is an operator such that $L^k(T)$ has analytic order s for all $k \geq 0$, then $\delta^k(T)$ has analytic order s for all $k \geq 0$.*

Proof. Let us show that $\delta^k(T)$ has analytic order s for every k by proving that for every l and k, the operator:

$$\delta^l(L^k(T)) \tag{7.3.3}$$

has analytic order s, by induction on l. For $l = 0$, this is simply the claim that $L^k(T)$ has analytic order s for every k, which is assumed.

Supposing that the operator (7.3.3) has analytic order s for some $l \geq 0$, let us show that $\delta^{l+1}(L^k(T))$ has analytic order s. Since L and δ commute, we have:

$$L(\delta^l(L^k(T)) = \delta^l(L^{k+1}(T)), \quad L^2(\delta^l(L^k(T))) = \delta^l(L^{k+2}(T)).$$

So by the inductive hypothesis, both $L(\delta^l(L^k(T)))$ and $L^2(\delta^l(L^k(T)))$ have analytic order s. Theorem 7.3.1 with $\alpha = 1$ then proves that $\delta^{l+1}(L^k(T))$ has analytic order s, and thus by induction the operator (7.3.3) has analytic order s for every $l \geq 0$ and $k \geq 0$. □

The converse of Corollary 7.3.8 also holds: if $\delta^k(T)$ has analytic order s for every k, then $L^k(T)$ has analytic order s for all $k \geq 0$, and the proof follows from the simple identity:

$$L(T) = 2\delta(T) - J^{-1}\delta^2(T).$$

This converse was already known to Connes and Moscovici [34, Appendix B].

Lemma 7.3.9. *For every $k \geq 0$ and $x \in \mathcal{S}(\mathbb{R}^d_\theta)$, the operators $L^k(x)$ and $\delta^k(x)$ have analytic order zero.*

Proof. Thanks to Corollary 7.3.8, it suffices to prove that $L^k(x)$ has analytic order zero for every $k \geq 0$.

We first show that $[J^2, x]$ has analytic order 1.

Since $J^2 = 1 + \sum_{j=1}^d D_j^2$ and $[D_j^2, x] = 2\partial_j x D_j + \partial_j^2 x$, it follows that $[J^2, x]$ can be expanded as:

$$[J^2, x] = \sum_{j=1}^d -2i\partial_j x D_j - \partial_j^2 x,$$

Since each D_j has order 1 and each $\partial_j x$ and $\partial_j^2 x$ has order zero (Lemma 7.3.4), it follows that $D(x) = [J^2, x]$ has order 1, and therefore $L(x)$ has analytic order zero.

Similarly, if we assume that $L^k(x)$ has analytic order k for every $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ then since J commutes with each D_j and $\partial_j x = [D_j, x]$, we have:

$$L^{k+1}(x) = J^{-1}[J^2, L^k(x)] = \sum_{j=1}^d -2iJ^{-1}L^k(\partial_j x)D_j - J^{-1}L^k(\partial_j^2 x)$$

so it follows that $L^{k+1}(x)$ has analytic order zero for every $x \in \mathcal{S}(\mathbb{R}^d_\theta)$, and thus the result follows by induction. $\square$

Corollary 7.3.10. *Let T be an operator which is analytic of order $-\beta < 0$. Then for every $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ and $k \geq 0$, we have $\delta^k(x)T \in \mathcal{L}_{d/\beta,\infty}$.*

Proof. By Lemma 7.3.9, every $\delta^k(x)$ has analytic order zero.

We prove the result by induction on k, with the $k = 0$ case being the result of Corollary 7.3.6. Supposing the result is true for $0 \leq l < k$, we prove it for k by factorising x as $x = yz$ for $y, z \in \mathcal{S}(\mathbb{R}^d_\theta)$ (Proposition 7.1.8). Then by the kth order Leibniz rule we have

$$\delta^k(yz)T = \sum_{l=0}^k \binom{k}{l}\delta^l(y)\delta^{k-l}(z)T = \delta^k(y)zT + y\delta^k(z)T + \sum_{l=1}^{k-1}\binom{k}{l}\delta^l(y)\delta^{k-l}(z)T.$$

Since every $\delta^l(y)$ and $\delta^{k-l}(z)$ have order zero, the result now follows from the inductive hypothesis and Corollary 7.3.6. $\square$

Using Theorem 7.3.1, we obtain the key theoretical tool behind the proof of Theorem 7.1.6.

Corollary 7.3.11. *For every $\alpha, \beta \in \mathbb{R}$, $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ and $k \geq 0$, the operator $[J^\alpha, \delta^k(x)]J^{-\beta}$ has analytic order $\alpha - \beta - 1$.*

Proof. From Lemma 7.3.9, in particular $L(x)$ and $L^2(x)$ have analytic order zero for every $x \in \mathcal{S}(\mathbb{R}^d_\theta)$. Theorem 7.3.1 then implies that $[J^\alpha, x]$ has analytic order $\alpha - 1$. Similarly, since every $L^{k+1}(x)$ and $L^{k+2}(x)$ have analytic order zero, it follows that $[J^\alpha, L^k(x)]$ has analytic order $\alpha - 1$.

Since we have:

$$[J^\alpha, L^k(x)] = L^k([J^\alpha, x])$$

it follows from Corollary 7.3.8 that $\delta^k([J^\alpha, x])$ has analytic order $\alpha - 1$ for every k, and hence $[J^\alpha, \delta^k(x)]$ has analytic order $\alpha - 1$.

Multiplying by $J^{-\beta}$ reduces the order by β, so the operator $[J^\alpha, x]J^{-\beta}$ has analytic order $\alpha - \beta - 1$. $\square$

Next, we prove Theorem 7.1.6 for the cases where $\alpha \leq 1$ and $\beta > 0$ by a factorisation argument.

Lemma 7.3.12. *Let $x \in \mathcal{S}(\mathbb{R}^d_\theta)$. For every $\alpha \leq 1$, $\beta > \max\{\alpha - 1, 0\}$, $\gamma \geq 0$ and $k \geq 0$, the operator $J^{-\gamma}[J^\alpha, \delta^k(x)]J^{-\beta}$ is in the ideal $\mathcal{L}_{d/(\beta+\gamma-\alpha+1),\infty}$.*

Proof. For simplicity, consider the case $\gamma = 0$. Since the Schwartz class $\mathcal{S}(\mathbb{R}^d_\theta)$ can be factorised (Proposition 7.1.8), write $x = yz$ for $y, z \in \mathcal{S}(\mathbb{R}^d_\theta)$. Then:

$$\delta^k(yz) = \sum_{l=0}^{k} \binom{k}{l} \delta^{k-l}(y)\delta^l(z).$$

Thus it suffices to prove the result for $\delta^{k-l}(y)\delta^l(z)$ in place of $\delta^k(x)$. By the Leibniz rule, we have:

$$[J^\alpha, \delta^{k-l}(y)\delta^l(z)]J^{-\beta} = \delta^{k-l}(y)[J^\alpha, \delta^l(z)]J^{-\beta} + [J^\alpha, \delta^{k-l}(y)]\delta^{k-l}(z)J^{-\beta}. \tag{7.3.4}$$

Corollary 7.3.11 shows that the first term on the right hand side of (7.3.4) has order $\alpha - \beta - 1 < 0$. This combines with Corollary 7.3.10 to deduce that the first term is in $\mathcal{L}_{d/(\beta-\alpha+1),\infty}$.

As for the second term in (7.3.4), we factorise z into $z = wv$ for $w, v \in \mathcal{S}(\mathbb{R}^d_\theta)$, and write $\delta^{k-l}(z)$ as a linear combination of terms of the form $\delta^{l_0}(u)\delta^{l_1}(v)$. If $\alpha < 1$, the operators $[J^\alpha, \delta^{k-l}(y)]\delta^{l_0}(u)$ and $\delta^{l_1}(v)J^{-\beta}$ are in $\mathcal{L}_{d/(1-\alpha),\infty}$ and $\mathcal{L}_{d/\beta,\infty}$ respectively due to Lemma 7.3.10, so by Hölder's inequality it follows that $[J^\alpha, y]\delta^{k-l}(z)J^{-\beta}$ is in $\mathcal{L}_{d/(\beta-\alpha+1),\infty}$. On the other hand, if $\alpha = 1$ then Corollary 7.3.11 implies that $[J^\alpha, y]$ is order zero, so in particular $[J^\alpha, y]$ is bounded, and since $\delta^{k-l}(z)J^{-\beta} \in \mathcal{L}_{d/\beta,\infty}$, it follows that $[J^\alpha, \delta^l(y)]\delta^{k-l}(z)J^{-\beta} \in \mathcal{L}_{d/\beta,\infty}$.

It is possible to prove the cases where $\gamma > 0$ by a similar factorisation argument, so this proof is omitted. ☐

7.3.3 Proof of Theorem 7.1.6

In Lemma 7.3.12 we have already proved Theorem 7.1.6 when $\beta > 0$ and $\alpha \leq 1$. Moreover, the cases where $\beta - \alpha + 1 = 0$ are covered by Corollary 7.3.11, since operators of analytic order zero are in particular bounded on $L_2(\mathbb{R}^d)$.

We now complete the proof for the full range of parameters $\{(\alpha, \beta) \,:\, \beta - \alpha + 1 \geq 0\}$. First we note that since $[J^2, \delta^k(x)]$ is computed as:

$$[J^2, \delta^k(x)] = -\sum_{j=1}^{d} 2i\delta^k(\partial_j x)D_j + \delta^k(\partial_j^2 x).$$

It then follows from Lemma 7.3.10 that if $\beta - \alpha + 1 > 0$, then $[J^2, \delta^k(x)]J^{\alpha-\beta-2} \in \mathcal{L}_{d/(\beta-\alpha+1),\infty}$.

Now we consider the cases where $\alpha > 1$ and $\beta > 0$. By the Leibniz rule:

$$\begin{aligned}
[J^\alpha, \delta^k(x)]J^{-\beta} &= [J^2 \cdot J^{\alpha-2}, \delta^k(x)]J^{-\beta} \\
&= [J^2, \delta^k(x)]J^{\alpha-\beta-2} + J^2[J^{\alpha-2}, \delta^k(x)]J^{-\beta} \\
&= [J^2, \delta^k(x)]J^{\alpha-\beta-2} - J^{-\alpha}[J^{2-\alpha}, \delta^k(x)]J^{\alpha-\beta-2}.
\end{aligned}$$

The first summand is in $\mathcal{L}_{d/(\beta-\alpha+1),\infty}$. Since $\alpha > 1$, it follows that $2 - \alpha < 1$ and since $\beta - \alpha + 1 > 0$ it follows that $\alpha - \beta - 2 < -1 < 0$, so Lemma 7.3.12 is applicable to the second summand, which is hence in $\mathcal{L}_{d/(\beta-\alpha+1),\infty}$.

Thus, the result remains true for the parameter range $\{(\alpha,\beta)\ :\ \beta-\alpha+1\geq 0, \beta>0\}$.

To complete the proof, we show that if $[J^\alpha,\delta^k(x)]J^{-\beta}\in\mathcal{L}_{\frac{d}{\beta-\alpha+1},\infty}$ for every k then also $[J^{\alpha-1},\delta^k(x)]J^{1-\beta}\in\mathcal{L}_{\frac{d}{\beta-\alpha+1},\infty}$ for every k. By the Leibniz rule, we have:

$$\begin{aligned}
[J^{\alpha-1},\delta^k(x)]J^{1-\beta} &= [J^\alpha,\delta^k(x)]J^{-\beta} + J^\alpha[J^{-1},\delta^k(x)]J^{1-\beta}\\
&= [J^\alpha,\delta^k(x)]J^{-\beta} - J^{\alpha-1}\delta^{k+1}(x)J^{-\beta}\\
&= [J^\alpha,\delta^k(x)]J^{-\beta} - J^{-1}[J^\alpha,\delta^{k+1}(x)]J^{-\beta}\\
&\quad - J^{-1}\delta^{k+1}(x)J^{\alpha-\beta}\\
&= [J^\alpha,\delta^k(x)]J^{-\beta} - J^{-1}[J^\alpha,\delta^{k+1}(x)]J^{-\beta}\\
&\quad - [J^{-1},\delta^{k+1}(x)]J^{\alpha-\beta} - \delta^{k+1}(x)J^{\alpha-\beta-1}\\
&= [J^\alpha,\delta^k(x)]J^{-\beta} - J^{-1}[J^\alpha,\delta^{k+1}(x)]J^{-\beta}\\
&\quad + J^{-1}\delta^{k+2}(x)J^{\alpha-\beta-1} - \delta^{k+1}(x)J^{\alpha-\beta-1}.
\end{aligned}$$

It now follows from Lemma 7.3.10 that each of the above four summands is in $\mathcal{L}_{\frac{d}{\beta-\alpha+1},\infty}$ if $\alpha<\beta+1$.

Therefore the result holds for the parameters $(\alpha-1,\beta-1)$ whenever it holds for (α,β), and this suffices to handle every possible case.

7.4 Proofs of Theorems 7.1.2 and 7.1.5

Using Theorem 7.1.6 and the commutator estimates developed in Section 7.3, we are able to establish the trace formula in Theorem 7.1.2, and finally prove Theorem 7.1.5. This will be done by showing that for all $x\in\mathcal{S}(\mathbb{R}^d_\theta)$,

$$|đx|^d - |A|^d(1+D^2)^{-d/2}\in\mathcal{L}_1$$

for a certain bounded operator A on $\mathbb{C}^N\otimes L_2(\mathbb{R}^d)$ (depending on x), and then applying the trace formula given by [94, Theorem 6.15] to $|A|^d(1+D^2)^{-d/2}$.

7.4.1 Operator difference estimates

We begin with the construction of the above mentioned operator A. For $1\leq j,k\leq d$, denote $g_{j,k}(t)=\frac{t_jt_k}{|t|^2}$ on $\mathbb{R}^d$. Let $x\in\mathcal{S}(\mathbb{R}^d_\theta)$. Define the operator A_j on $L_2(\mathbb{R}^d)$ as

$$A_j\xi := (\partial_j x)\xi - \sum_{k=1}^d (M_{g_{j,k}}\partial_k x)\xi = (\partial_j x)\xi - \sum_{k=1}^d g_{j,k}(D_1,\cdots,D_d)(\partial_k x)\xi,\quad \xi\in L_2(\mathbb{R}^d) \tag{7.4.1}$$

and define the operator A on $\mathbb{C}^N\otimes L_2(\mathbb{R}^d)$

$$A := \sum_{j=1}^d \gamma_j\otimes A_j,$$

where N and γ_j are the same as in Definition 7.1.12.

The main result in this subsection is the following theorem:

Theorem 7.4.1. *Let $x \in \mathcal{S}(\mathbb{R}^d_\theta)$. Then we have:*

$$|\text{đ}x|^d - |A|^d(1+D^2)^{-d/2} \in \mathcal{L}_1.$$

Recall that $D = \sum_{j=1}^d \gamma_j \otimes D_j$, and $\text{đ}x = i[\mathrm{sgn}(D), 1\otimes x]$. Let $g(t) = t(1+t^2)^{-1/2}$ and write

$$\text{đ}x = i[\mathrm{sgn}(D) - g(D), 1\otimes x] + i\sum_{j=1}^d \gamma_j \otimes [D_j J^{-1}, x].$$

By Lemma 7.2.1, $[\mathrm{sgn}(D) - g(D), 1\otimes x]$ belongs to $\mathcal{L}_p$ when $p > \frac{d}{2}$. Define the auxiliary operator $\widetilde{A}_j$ for $1 \le j \le d$ on $L_2(\mathbb{R}^d)$ as

$$\widetilde{A}_j := \partial_j x - \sum_{k=1}^d D_j D_k J^{-2} \partial_k x\,. \tag{7.4.2}$$

The following proposition connects the commutator $[D_j J^{-1}, x]$ with $\widetilde{A}_j$.

Proposition 7.4.2. *Let $1 \le j \le d$, and $x \in \mathcal{S}(\mathbb{R}^d_\theta)$. Then,*

$$i[D_j J^{-1}, x] - \widetilde{A}_j J^{-1} \in \mathcal{L}_{\frac{d}{2},\infty}.$$

Proof. From the Leibniz rule, we have

$$i[D_j J^{-1}, x] = \partial_j x J^{-1} + iD_j[J^{-1}, x] = \partial_j x J^{-1} - iD_j J^{-1}\delta(x)J^{-1}.$$

We have the following algebraic identities:

$$\begin{aligned}
J^{-1}\delta(x)J^{-1} &= \frac{1}{2}J^{-1}([J^2 \cdot J^{-1}, x] + [J^{-1}\cdot J^2, x])J^{-1}\\
&= \frac{1}{2}J^{-1}(J^2[J^{-1}, x] + [J^2, x]J^{-1} + J^{-1}[J^2, x] + [J^{-1}, x]J^2)J^{-1}\\
&= \frac{1}{2}J^{-2}[J^2, x]J^{-1} + \frac{1}{2}J^{-1}([J^2, x]J^{-1} - J[J, x]J^{-1} - J^{-1}[J, x]J)J^{-1}\\
&= \frac{1}{2}J^{-1}L(x)J^{-1} + \frac{1}{2}J^{-1}(\delta(x) - J^{-1}\delta(x)J)J^{-1}\\
&= \frac{1}{2}J^{-1}L(x)J^{-1} + \frac{1}{2}J^{-2}\delta^2(x)J^{-1}.
\end{aligned}$$

From Corollary 7.3.10, we have that that $D_j J^{-2}\delta^2(x)J^{-1}$ is in $\mathcal{L}_{d/2,\infty}$.

So we obtain:

$$i[D_j J^{-1}, x] \in \partial_j x J^{-1} - \frac{i}{2}D_j J^{-1}L(x)J^{-1} + \mathcal{L}_{\frac{d}{2},\infty}. \tag{7.4.3}$$

Examining the second term, we have:

$$\begin{aligned}
D_jJ^{-2}[J^2,x]J^{-1} &= D_jJ^{-2}[J^2,x]J^{-1}\\
&= D_jJ^{-2}\sum_{k=1}^{d}[D_k^2,x]J^{-1}\\
&= \sum_{k=1}^{d} D_jJ^{-2}(-iD_k\partial_kx - i\partial_kx\, D_k)J^{-1}\\
&= \sum_{k=1}^{d} D_jJ^{-2}(-2iD_k\partial_kx + i\partial_k^2x)J^{-1}
\end{aligned}$$

Due to the factorisation property of $\mathcal{S}(\mathbb{R}^d_\theta)$ and Lemma 7.3.5, we have $D_jJ^{-2}\partial_k^2xJ^{-1}\in\mathcal{L}_{d/2,\infty}$, and therefore

$$D_jJ^{-1}L(x)J_\theta^{-1} \in -2i\sum_{k=1}^{d} D_jD_kJ^{-2}\partial_kxJ^{-1} + \mathcal{L}_{d/2,\infty}. \tag{7.4.4}$$

Combining (7.4.3) and (7.4.4) yields:

$$i[D_jJ^{-1},x] \in \partial_jxJ^{-1} - \sum_{k=1}^{d} D_jD_kJ^{-2}\partial_kxJ^{-1} + \mathcal{L}_{d/2,\infty} = \widetilde{A}_jJ^{-1} + \mathcal{L}_{d/2,\infty}$$

as was claimed. □

Let us also compare $\widetilde{A}_jJ^{-1}$ with A_jJ^{-1}.

Proposition 7.4.3. *Let $1\le j\le d$, and $x\in\mathcal{S}(\mathbb{R}^d_\theta)$. Then,*

$$A_jJ^{-1} - \widetilde{A}_jJ^{-1} \in \mathcal{L}_{\frac{d}{2},\infty}.$$

Proof. By definition, $A_j = \sum_{k=1}^d M_{g_{j,k}}\partial_kx$ and $\widetilde{A}_j = \sum_{k=1}^d M_{\widetilde{g}_{j,k}}\partial_kx$ with $\widetilde{g}_{j,k}(t) = \frac{t_jt_k}{1+|t|^2}$. So we are reduced to estimating $M_{g_{j,k}}\partial_kxJ^{-1} - M_{\widetilde{g}_{j,k}}\partial_kxJ^{-1}$ for every k. Using the factorisation of $x=yz\in\mathcal{S}(\mathbb{R}^d_\theta)$ (Proposition 7.1.8) and the Leibniz rule, we have

$$M_{g_{j,k}}\partial_kxJ^{-1} - M_{\widetilde{g}_{j,k}}\partial_kxJ^{-1} = (M_{g_{j,k}} - M_{\widetilde{g}_{j,k}})\partial_ky\,zJ^{-1} + (M_{g_{j,k}} - M_{\widetilde{g}_{j,k}})y\,\partial_kzJ^{-1}.$$

From Lemma 7.3.5, both zJ^{-1} and ∂_kzJ^{-1} belong to $\mathcal{L}_{d,\infty}$. On the other hand, one can easily check that $g_{j,k}-\widetilde{g}_{j,k}\in L_p(\mathbb{R}^d)$ as $p>\frac{d}{2}$, which yields by Theorem 6.2.6 that

$$(M_{g_{j,k}} - M_{\widetilde{g}_{j,k}})y \in \mathcal{L}_p\subset\mathcal{L}_{d,\infty},\quad (M_{g_{j,k}} - M_{\widetilde{g}_{j,k}})\partial_ky \in \mathcal{L}_p\subset\mathcal{L}_{d,\infty}.$$

Thus it follows from the Hölder inequality that

$$M_{g_{j,k}}\partial_kxJ^{-1} - M_{\widetilde{g}_{j,k}}\partial_kxJ^{-1} \in \mathcal{L}_{d/2,\infty},$$

whence the proposition. □

For $g(t) = t(1+t^2)^{-1/2}$ on $\mathbb{R}$, Propositions 7.4.2 and 7.4.3 imply that

$$i[g(D), 1\otimes x] - A(1+D^2)^{-1/2} \in \mathcal{L}_{\frac{d}{2},\infty}. \tag{7.4.5}$$

This – combined with Lemma 7.2.1 – yields:

$$đx - A(1+D^2)^{-1/2} \in \mathcal{L}_{\frac{d}{2},\infty}$$

for all $x \in \mathcal{S}(\mathbb{R}^d_\theta)$.

Lemma 7.4.4. *Let $x \in \mathcal{S}(\mathbb{R}^d_\theta)$. We have*

$$|đx|^d - ((1+D^2)^{-1/2}|A|^2(1+D^2)^{-1/2})^{d/2} \in \mathcal{L}_1.$$

Proof. We already know from Lemma 7.2.1 that $i[g(D), 1\otimes x] - đx \in \mathcal{L}_{\frac{d}{2}}$, which together with (7.4.5) ensures that

$$đx - A(1+D^2)^{-1/2} \in \mathcal{L}_{\frac{d}{2},\infty}.$$

Taking the adjoint:

$$đx^* - (1+D^2)^{-1/2}A^* \in \mathcal{L}_{\frac{d}{2},\infty}.$$

Recall that $đx \in \mathcal{L}_{d,\infty}$ by Theorem 7.1.1 (as has been proved in Section 7.2), so it follows that $A(1+D^2)^{-1/2} \in \mathcal{L}_{d,\infty}$. Using the Hölder inequality, we have

$$\begin{aligned} |đx|^2 - (1+D^2)^{-1/2}|A|^2(1+D^2)^{-1/2} &= đx^*\big(đx - A(1+D^2)^{-1/2}\big) \\ &\quad + \big(đx^* - (1+D^2)^{-1/2}A^*\big)A(1+D^2)^{-1/2} \\ &\in \mathcal{L}_{\frac{d}{3},\infty} \subset \mathcal{L}_{\frac{5d}{12}}. \end{aligned}$$

If $d=2$, then we are done.

Now assume that $d>2$. We appeal to a recent result from E. Ricard [111, Theorem 3.4], which says that we can take a power $1/2$ to each term of the preceding inclusion to get

$$|đx| - \Big((1+D^2)^{-1/2}|A|^2(1+D^2)^{-1/2}\Big)^{1/2} \in \mathcal{L}_{\frac{5d}{6},\infty}.$$

Next we introduce a power d:

$$\begin{aligned} &|đx|^d - \Big((1+D^2)^{-1/2}|A|^2(1+D^2)^{-1/2}\Big)^{d/2} \\ &= \sum_{k=0}^{d-1} |đx|^{d-k-1}\Big(|đx| - ((1+D^2)^{-1/2}|A|^2(1+D^2)^{-1/2})^{1/2}\Big)\Big((1+D^2)^{-1/2}|A|^2(1+D^2)^{-1/2}\Big)^{\frac{k}{2}} \\ &\in \sum_{k=0}^{d-1} \mathcal{L}_{\frac{d}{d-k-1},\infty} \cdot \mathcal{L}_{\frac{5d}{6}} \cdot \mathcal{L}_{\frac{d}{k},\infty} \subset \mathcal{L}_{\frac{5d}{5d+1},\infty} \subset \mathcal{L}_1. \end{aligned}$$

□

Writing $|A|^2 = A^*A$, it follows from Theorem 7.1.6 that

$$[|A|^2, (1+D^2)^{\alpha/2}](1+D^2)^{-\beta/2} \in \mathcal{L}_{\frac{d}{\beta-\alpha+1},\infty} \tag{7.4.6}$$

for all $\beta > 0$ and $\alpha < 1$. Therefore, if $d = 2$, letting $\alpha = -1$ and $\beta = 1$ in (7.4.6), we have

$$[|A|^2, (1+D^2)^{-1/2}](1+D^2)^{-1/2} \in \mathcal{L}_{2/3,\infty} \subset \mathcal{L}_1$$

This inclusion can be combined with Lemma 7.4.4 to arrive at

$$|đx|^2 - |A|^2(1+D^2)^{-1} \in \mathcal{L}_1$$

which completes the proof of Theorem 7.4.1 for the $d = 2$ case.

For $d > 2$, we need

Proposition 7.4.5. *Let $d > 2$. Then*

$$|A|^d(1+D^2)^{-d/2} - ((1+D^2)^{-1/2}|A|^2(1+D^2)^{-1/2})^{d/2} \in \mathcal{L}_1.$$

Proof. From Theorem 4.3.1 it suffices to show the following four conditions:

(i) $|A|^{d-2}(1+D^2)^{1-\frac{d}{2}} \in \mathcal{L}_{\frac{d}{d-2},\infty}$.

(ii) $(1+D^2)^{-1/2}|A|^2(1+D^2)^{-1/2} \in \mathcal{L}_{\frac{d}{2},\infty}$.

(iii) $[|A|^2(1+D^2)^{-1/2}, (1+D^2)^{-1/2}] \in \mathcal{L}_{\frac{d}{2},1}$.

(iv) $|A|^{d-2}[|A|^2, (1+D^2)^{1-\frac{d}{2}}](1+D^2)^{-1} \in \mathcal{L}_1$.

Since $d > 2$, we have that $|A|^{d-2} = |A|^{d-3}\,\mathrm{sgn}(A)A$, so (i) follows immediately from Lemma 7.3.5. Similarly using $|A|^2 = A^*A$, we get also get (ii) immediately from the Hölder inequality and the fact that $A(1+D^2)^{-1/2}$ and its adjoint operator belong to $\mathcal{L}_{d,\infty}$.

For (iii), we write:

$$[|A|^2(1+D^2)^{-1/2}, (1+D^2)^{-1/2}] = [|A|^2, (1+D^2)^{-1/2}](1+D^2)^{-1/2}$$

which is in $\mathcal{L}_{\frac{2d}{5},\infty}$ due to (7.4.6) (with $\alpha = -1$ and $\beta = 1$). Since $\frac{2d}{5} < \frac{d}{2}$, it follows that $\mathcal{L}_{2d/5,\infty} \subset \mathcal{L}_{d/2,1}$ and this proves (iii). Finally, (iv) immediately follows from (7.4.6) with $\alpha = 2 - d$ and $\beta = 2$. □

Lemma 7.4.4 and Proposition 7.4.5 yield Theorem 7.4.1 for the case $d > 2$, and thus we have completed the proof of Theorem 7.4.1.

7.4.2 Proof of Theorem 7.1.2

Now we are able to give the proof of Theorem 7.1.2, via Theorem 6.5.9

Proof of Theorem 7.1.2. We will assume initially that $x \in \mathcal{S}(\mathbb{R}^d_\theta)$. For a continuous normalised trace φ on $\mathcal{L}_{1,\infty}$, Theorem 7.4.1 ensures that

$$\varphi(|đx|^d) = \varphi\big(|A|^d(1+D^2)^{-d/2}\big).$$

But since $A = \sum_j \gamma_j \otimes A_j$ self-adjoint unitary matrices γ_j, the only part that contributes to the trace on the right hand side above is $(1 \otimes \sum_j A_j^* A_j)^{d/2}(1+D^2)^{-d/2}$. Hence,

$$\varphi(|đx|^d) = \varphi\big((\sum_j A_j^* A_j)^{d/2}(1-\Delta)^{-d/2}\big).$$

However, note that each A_j is a linear combination of operators of multiplication by a function $x \in \mathcal{S}(\mathbb{R}^d_\theta)$ and Fourier multiplication by a function $g \in C(\mathbb{S}^{d-1})$, and so is in the algebra $\Pi(C_0(\mathbb{R}^d_\theta)+\mathbb{C}, C(\mathbb{S}^{d-1}))$, with symbol:

$$\mathrm{sym}(A_j) = \partial_j x \otimes 1 - \sum_{k=1}^d s_j s_k \otimes \partial_k x.$$

Since sym is a norm-continuous $*$-homomorphism, we have

$$\mathrm{sym}(\sum_j A_j^* A_j)^{d/2} = \Big(\sum_{j=1}^d \big|\partial_j x - s_j \sum_{k=1}^d s_k \partial_k x\big|^2\Big)^{d/2}.$$

Since d is necessarily even, this is indeed an operator of the form Tz for $T \in \Pi(C_0(\mathbb{R}^d_\theta)+\mathbb{C}, C(\mathbb{S}^{d-1}))$ and $z \in W^d_1(\mathbb{R}^d_\theta)$. Hence Theorem 6.5.9 is applicable, and thus we have proved Theorem 7.1.2 for $x \in \mathcal{S}(\mathbb{R}^d_\theta)$.

By virtue of Proposition 7.1.17, the general case of Theorem 7.1.2 is done via an approximation argument, identically to the proof of [93, Theorem 1.2]. $\square$

7.4.3 Proof of Theorem 7.1.5

Finally, we prove Theorem 7.1.5.

Proof of Theorem 7.1.5. Assume that $x \in L_p(\mathbb{R}^d_\theta)$ for some $d \le p < \infty$ and that $đx \in \mathcal{L}_{d,\infty}$.

Let $\{\phi_\varepsilon\}_{\varepsilon>0}$ be a family of Schwartz class functions as in Theorem 7.1.14. Then $U(\phi_\varepsilon)x \in \mathcal{S}(\mathbb{R}^d_\theta)$, and Theorem 7.1.1 implies:

$$\|đ(U(\phi_\varepsilon)x)\|_{d,\infty} \lesssim_d \|U(\phi_\varepsilon)\|_\infty \|đx\|_{d,\infty} + \|U(\phi_\varepsilon)\|_{W^1_d}\|x\|_\infty.$$

Since $\|U(\phi_\varepsilon)\|_\infty \le \|\phi_\varepsilon\|_1$ is uniformly bounded in ε, and $\|U(\phi_\varepsilon)\|_{W^1_d}$ is uniformly bounded in ε by Lemma 7.1.15, it follows that $\{U(\phi_\varepsilon)x\}_{\varepsilon>0}$ is uniformly bounded in $\dot{W}^1_d(\mathbb{R}^d_\theta)$, so for every $1 \le j \le d$, $\{\partial_j(U(\phi_\varepsilon)x)\}_{\varepsilon>0}$ is uniformly bounded in $L_d(\mathbb{R}^d_\theta)$. Since $d \ge 2$, the space $L_d(\mathbb{R}^d_\theta)$ is reflexive and therefore $\{\partial_j(U(\phi_\varepsilon)x)\}_{\varepsilon>0}$ has a weak limit point in $L_d(\mathbb{R}^d_\theta)$. But we know from Theorem 7.1.14 that $U(\phi_\varepsilon)x \to x$ in the L_p sense. In particular, in the distributional sense, and therefore $\partial_j(U(\phi_\varepsilon)x) \to \partial_j x$ in the distributional sense.

Therefore, $\partial_j x \in L_d(\mathbb{R}^d_\theta)$ for every $1 \le j \le d$. That is, $x \in W^1_d(\mathbb{R}^d_\theta)$.

Finally, we obtain the bound on the norm using Corollary 7.1.3. That result implies that there exists a constant $c_d > 0$ such that for all continuous normalised traces φ on

$\mathcal{L}_{1,\infty}$,
$$\|x\|_{\dot{W}^1_d} \lesssim_d \varphi(|\,\text{đ}x|^d)^{\frac{1}{d}}.$$
Since φ is continuous,
$$\|x\|_{\dot{W}^1_d} \lesssim_d \|\varphi\|_{(\mathcal{L}_{1,\infty})^*}\|\,\text{đ}x\|_{d,\infty}.$$
Selecting a continuous normalised trace φ of norm 1 completes the proof.

$\square$

Appendix A

Function spaces and commutators with the Hilbert transform

This section of the appendix is devoted to a self-contained proof of Theorem 5.2.1. The content here is largely unaltered from the published version in the appendix of [33].

A.1 Bergman and Analytic Besov spaces

Definition A.1.1. *Let μ be the measure on $\mathbb{D}$ defined by*

$$d\mu(z) = \frac{dzd\overline{z}}{(1-|z|^2)^2}$$

(i.e., the Poincaré disc model volume form). For $p \in (0,\infty]$ the space $A_p^{1/p}$ is defined to be the set of functions f holomorphic in the unit disc satisfying $z \mapsto (1-|z|^2)^2|f''(z)| \in L_p(\mathbb{D},\mu)$ with the seminorm $\|f\|_{A_p^{1/p}} := \|(1-|z|^2)^2|f''(z)|\|_{L_p(\mathbb{D},\mu)}$.

We also define the space $C_p^{1/p}$ of functions f holomorphic in the interior of the unit disc satisfying $(1-|z|^2)|f'(z)| \in L_p(\mathbb{D},\mu)$ with corresponding seminorm $\|f\|_{C_p^{1/p}} = \|(1-|z|^2)|f'(z)|\|_{L_p(\mathbb{D},\mu)}$.

The following result shows that for $p \in (1,2)$ the spaces $A_p^{1/p}$ and $C_p^{1/p}$ coincide, and in fact there is an equivalence of (semi)-norms with constants that are uniform for $p \in (1+\varepsilon, 2)$ for all $\varepsilon > 0$.

Theorem A.1.2. *Let $p_0 > 1$, and let f be a function holomorphic in the unit disc satisfying $f'(0) = 0$. There exist constants $k, K > 0$ (depending only on p_0) such that for all $p_0 < p < 2$ we have*

$$k\|f\|_{A_p^{1/p}} \le \|f\|_{C_p^{1/p}} \le K\|f\|_{A_p^{1/p}}.$$

Proof. Let $h(z) := f'(z)(1-|z|^2)$ and $g(z) := f''(z)(1-|z|^2)^2$, and fix $p_0 > 1$. The assertion of this theorem is equivalent to saying that there are positive constants $k, K > 0$

such that for all $p \in [p_0, 2]$,

$$k\|h\|_{L_p(\mathbb{D},\mu)} \leq \|g\|_{L_p(\mathbb{D},\mu)} \leq K\|h\|_{L_p(\mathbb{D},\mu)}. \tag{A.1.1}$$

By applying the method of complex interpolation as in [35, Lemma 3.4], it suffices to prove (A.1.1) for $p = p_0$ and $p = 2$. Firstly, for $p = 2$, we note that the spaces $A_2^{1/2}$ and $C_2^{1/2}$ are Hilbert spaces, and that the functions $e_n(z) = z^n$, $n \geq 0$ are orthogonal in both spaces with dense linear span. Hence for $p = 2$ it suffices to prove (A.1.1) for $f(z) = z^n$, $n \geq 0$. When $f(z) = z^n$, we have $h(z) = nz^{n-1}(1-|z|^2)$ and $g(z) = n(n-1)z^{n-2}(1-|z|^2)^2$. Then,

$$\begin{aligned}
\|h\|^2_{L_2(\mathbb{D},\mu)} &= \int_{\mathbb{D}} n^2|z|^{2n-2}dzd\overline{z} \\
&= \frac{2\pi n^2}{2n} \\
&= \pi n
\end{aligned}$$

and furthermore,

$$\begin{aligned}
\|g\|^2_{L_2(\mathbb{D},\mu)} &= \int_{\mathbb{D}} n^2(n-1)^2|z|^{2n-4}(1-|z|^2)^2dzd\overline{z} \\
&= 2\pi n^2(n-1)^2\int_0^1 r^{2n-3}(1-r^2)^2dr \\
&= \pi n^2(n-1)^2\frac{2}{n(n^2-1)} \\
&\leq Kn
\end{aligned}$$

for some $K > 0$, hence proving (A.1.1) for $p = 2$.

The case $p = p_0$ is more subtle. We refer to [71, Proposition 1.11], a special case of which states that $f \in C_{p_0}^{1/p_0}$ if and only if $f \in A_{p_0}^{1/p_0}$. We explain how it is possible to modify the proof of [71, Proposition 1.11] to obtain the left hand inequality of (A.1.1) for $p = p_0$. The right hand side inequality of (A.1.1) will then follow from the open mapping theorem, since [71, Proposition 1.11] establishes that there is a bijective correspondence between $A_{p_0}^{1/p_0}$ and $C_{p_0}^{1/p_0}$.

We refer to the following formula, given in the proof of [71, Proposition 1.11]. If $\beta > \alpha > -1$ and $\xi \in L_{p_0}(\mathbb{D}, (1-|z|^2)^\alpha dzd\overline{z})$ is holomorphic and n is a positive integer, then there is a universal constant C such that

$$(1-|z|^2)^n\xi^{(n)}(z) = C(1-|z|^2)^n\int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{(1-z\overline{w})^{2+n+\beta}}\overline{w}^n\xi(w)dwd\overline{w}.$$

We apply this result with $\xi = f'$, $n = 1$ and $\alpha = p_0 - 2$, which is possible since $p_0 > 1$ so $\alpha > -1$. Fix any $\beta > \alpha$ Thus,

$$(1-|z|^2)f''(z) = C(1-|z|^2)\int_{\mathbb{D}} \frac{(1-|w|^2)^\beta}{(1-z\overline{w})^{3+\beta}}\overline{w}f'(w)dwd\overline{w}.$$

Restating this in terms of h, g and μ,

$$g(z) = C(1-|z|^2)^2 \int_{\mathbb{D}} \frac{(1-|w|^2)^{\beta+1}}{(1-z\overline{w})^{\beta+3}} \overline{w} h(w)\, d\mu(w).$$

It is established in [71, Theorem 1.9] that the right hand side (considered as a function of h) is an integral operator which maps $L_{p_0}(\mu)$ to $L_{p_0}(\mu)$. Hence, there is a constant k such that for all $p > p_0$

$$k\|g\|_{L_{p_0}(\mathbb{D},\mu)} \leq \|h\|_{L_{p_0}(\mathbb{D},\mu)}.$$

□

A.2 Peller's theorem on commutators

The following two theorems express the equivalence of the $\mathcal{L}_p$ norm of the commutator $[F, M_f]$ with the $A_p^{1/p}$ norm of f. This result is originally due to Peller [100, Chapter 6]. We include a proof here since the results in [100, Chapter 3] are not stated with bounds on the norms.

Theorem A.2.1. *Let f be a function holomorphic in the unit disc such that $f'(0) = 0$. There exists a universal constant $K > 0$ such that for all $p \in [1, 2]$:*

$$\|[F, M_f]\|_p \leq K\|f\|_{A_p^{1/p}}. \tag{A.2.1}$$

Proof. The required inequality (A.2.1) for $p \in [1, 2]$ can be obtained from the $p = 1$ and $p = 2$ cases by the method of complex interpolation as discussed in [35, Lemma 3.4].

The case $p = 2$ is omitted since it can be verified by computing $\|[F, M_f]\|_p$ and $\|f\|_{A_2^{1/2}}$ for the exponential basis functions $f(z) = z^n$, $n \geq 0$. For the case $p = 1$, we use [136, Lemma 2], which implies that if $f \in L_1(\mathbb{D}, dzd\overline{z})$ is holomorphic and $f(0) = f'(0) = \cdots = f^{(4)}(0) = 0$, then

$$f(z) = \frac{1}{2} \int_{\mathbb{D}} \frac{(1-|w|^2)^2 f''(w)}{\overline{w}^2 (1-z\overline{w})^2}\, dwd\overline{w}.$$

Hence,

$$[F, f(z)] = \frac{1}{2} \int_{\mathbb{D}} \frac{(1-|w|^2)^2 f''(w)}{\overline{w}^2} [F, \frac{1}{(1-z\overline{w})^2}]\, dwd\overline{w}. \tag{A.2.2}$$

However,

$$[F, \frac{1}{(1-z\overline{w})^2}] = -(1-z\overline{w})^{-2}[F, (1-z\overline{w})^2](1-z\overline{w})^{-2}.$$

Since $(1-z\overline{w})^2 = 1 - 2z\overline{w} + z^2\overline{w}^2$, the commutator $[F, (1-z\overline{w})^2]$ is finite rank, and in fact rank at most 5. So there is a constant C such that

$$\|[F, (1-z\overline{w})^{-2}]\|_1 \leq C\|(1-z\overline{w})^{-2}\|_\infty.$$

If $w \neq 0$, then the function on $\mathbb{T}$ $z \mapsto |1-z\overline{w}|^{-2}$ is maximised when z is as close to $\frac{1}{\overline{w}} = \frac{w}{|w|^2}$ as possible, which is at the point $\frac{w}{|w|}$. Hence $|1-z\overline{w}|^{-2} \leq (1-|w|)^{-2}$. Applying the $\mathcal{L}_1$-triangle inequality to (A.2.2) it follows that

$$\|[F, M_f]\|_1 \leq \frac{C}{2} \int_{\mathbb{D}} \frac{|f''(w)|(1-|w|^2)^2}{|w|^2(1-|w|)^2}\, dw d\overline{w}.$$

Since $1 \leq (1+|w|) \leq 2$ we may simplify this to

$$\|[F, M_f]\|_1 \leq C \int_{\mathbb{D}} \frac{|f''(w)|}{|w|^2}\, dw d\overline{w}.$$

We now split the integral over $\mathbb{D}$ into the two regions $R_1 := \{z \in \mathbb{D} \ : \ |w| \leq 1/2\}$ and $R_2 := \{z \in \mathbb{D} \ : \ |w| > 1/2\}$. For $w \in R_2$, we have

$$\frac{|f''(w)|}{|w|^2} \leq 4|f''(w)|$$

so

$$\int_{R_2} \frac{|f''(w)|}{|w|^2}\, dw d\overline{w} \leq 4 \int_{R_2} |f''(w)|\, dw d\overline{w}.$$

Since f is holomorphic and $f(0) = f'(0) = \cdots = f^{(4)}(0) = 0$, we have the power series expansion:

$$f(w) = \sum_{n>4} \hat{f}(n) w^n.$$

Substituting the series expansion into the integral,

$$\begin{aligned}
\int_{R_1} \frac{|f''(w)|}{|w|^2}\, dw d\overline{w} &\leq \sum_{n>4} \int_{R_1} n(n-1)|\hat{f}(n)||w|^{n-4} dw d\overline{w} \\
&= \sum_{n>4} 2\pi n(n-1)|\hat{f}(n)| \int_0^{1/2} r^{n-3} dr \\
&= \sum_{n>4} 2\pi \frac{n(n-1)}{n-2} |\hat{f}(n)| 2^{2-n}.
\end{aligned}$$

Hence,

$$\int_{R_1} \frac{|f''(w)|}{|w|^2}\, dw d\overline{w} \leq \sum_{n>4} 2\pi \frac{n(n-1)}{n-2} |\hat{f}(n)| 2^{2-n}. \tag{A.2.3}$$

Let ξ be a function holomorphic in $\mathbb{D}$, then by the Cauchy integral formula, for every $r > 0$,

$$\hat{\xi}(n) = \frac{1}{2\pi i} \int_0^1 \frac{\xi(re^{2\pi it})}{r^{n+1} e^{2\pi i(n+1)t}} \cdot 2\pi i e^{2\pi it} r dt,$$

Hence,

$$|\hat{\xi}(n)| \leq \int_0^1 |\xi(re^{2\pi it})| r^{-n} dt.$$

So in fact,

$$r^{n+1}|\hat{\xi}(n)| \leq \int_0^1 |\xi(re^{2\pi it})| r dt$$

So integrating over $0 \leq r \leq 1$,

$$\frac{1}{n+2}|\hat{\xi}(n)| \leq \|\xi\|_{L_1(\mathbb{D})}.$$

Applying this result to $\xi = f''$, we have $\hat{\xi}(n) = (n+2)(n+1)\hat{f}(n+2)$, so

$$(n+1)|\hat{f}(n+2)| \le \|f''\|_{L_1(\mathbb{D})}. \tag{A.2.4}$$

Hence, for all $n > 1$,

$$(n-1)|\hat{f}(n)| \le \|f''\|_{L_1(\mathbb{D})} \text{ for all } n \ge 0.$$

Applying this to (A.2.3), it follows that there exists $C > 0$ such that,

$$\begin{aligned}\int_{R_1} \frac{|f''(w)|}{|w|^2}\, dw d\overline{w} &\le 2\pi \left(\sum_{n>4} \frac{n}{n-2} 2^{2-n} \right) \|f''\|_{L_1(\mathbb{D})} \\ &\le C\|f\|_{A_1^1}.\end{aligned}$$

This proves the desired result when $f(0) = \cdots = f^{(4)}(0) = 0$. For arbitrary functions f holomorphic in $\mathbb{D}$, consider the function g given by

$$g(z) = f(z) - \sum_{k=0}^{4} \hat{f}(k) z^k.$$

By the preceding argument, there is a constant $K > 0$ such that $\|[F, M_g]\|_1 \le K\|g\|_{A_1^1}$. Thus, there is a constant $C > 0$ such that

$$\|[F, M_f]\|_1 \le C(\|f\|_{A_1^1} + |\hat{f}(1)| + |\hat{f}(2)| + |\hat{f}(3)| + |\hat{f}(4)).$$

Applying (A.2.4) with $n = 0, 1, 2$, there is a constant $K > 0$ such that:

$$\|[F, M_f]\|_1 \le C\|f\|_{A_1^1}.$$

$\square$

Theorem A.2.2. *There is a universal constant $k > 0$ such that for all $p \in [1, 2]$,*

$$k\|f\|_{A_p^{1/p}} \le \|[F, M_f]\|_p.$$

Proof. We first prove the result for $p = 1$ and $p = 2$, with the general case following from interpolation. This result essentially follows from [100, Theorem 6.1.1], where it is proved that there is a constant $k > 0$ such that

$$k\|f\|_{B_{1,1}^1} \le \|[F, M_f]\|_1$$

where $B_{1,1}^1$ is the Besov space on the circle.

However, following [125, Chapter 5, Proposition 7], $\|f\|_{A_1^1} \le \alpha\|f\|_{B_{1,1}^1}$. This prove the result for $p = 1$, and for $p = 2$ it is enough to check the inequality on exponential basis functions $f(z) = z^n$, for which it is easily verified. $\square$

Combining Theorems A.1.2, A.2.1 and A.2.2 we obtain the following corollary,

Corollary A.2.3. *Let $p_0 > 1$. There exist constants $b_{p_0}, B_{p_0} > 0$ (depending on p_0) such that for all $p \in (p_0, 2)$ we have*

$$b_{p_0}\|f\|_{C_p^{1/p}} \leq \|[F, M_f]\|_p \leq B_{p_0}\|f\|_{C_p^{1/p}}.$$

Bibliography

[1] Aleksandrov, A. B., Nazarov, F. L., and Peller, V. V. (2016). Functions of non-commuting self-adjoint operators under perturbation and estimates of triple operator integrals. *Adv. Math.*, 295:1–52.

[2] Aleksandrov, A. B. and Peller, V. V. (2019). Dissipative operators and operator Lipschitz functions. *Proc. Amer. Math. Soc.*, 147(5):2081–2093.

[3] Araki, H. (1990). On an inequality of Lieb and Thirring. *Lett. Math. Phys.*, 19(2):167–170.

[4] Atiyah, M., Bott, R., and Patodi, V. K. (1973). On the heat equation and the index theorem. *Invent. Math.*, 19:279–330.

[5] Atiyah, M. F. (1970). Global theory of elliptic operators. In *Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969)*, pages 21–30. Univ. of Tokyo Press, Tokyo.

[6] Atiyah, M. F. and Singer, I. M. (1963). The index of elliptic operators on compact manifolds. *Bull. Amer. Math. Soc.*, 69:422–433.

[7] Baum, P. and Douglas, R. G. (1982). K homology and index theory. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 117–173. Amer. Math. Soc., Providence, R.I.

[8] Beardon, A. F. (1991). *Iteration of rational functions*, volume 132 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. Complex analytic dynamical systems.

[9] Bell, J. L. (1998). *A primer of infinitesimal analysis*. Cambridge University Press, Cambridge.

[10] Bell, J. L. (2017). Continuity and infinitesimals. In Zalta, E. N., editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2017 edition.

[11] Bennett, C. and Sharpley, R. (1988). *Interpolation of operators*, volume 129 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA.

[12] Berg, C., Christensen, J. P. R., and Ressel, P. (1984). *Harmonic analysis on semi-groups*, volume 100 of *Graduate Texts in Mathematics*. Springer-Verlag, New York. Theory of positive definite and related functions.

[13] Birman, M. S., Koplienko, L. S., and Solomyak, M. Z. (1975). Estimates of the spectrum of a difference of fractional powers of selfadjoint operators. *Izv. Vysš. Učebn. Zaved. Matematika*, (3(154)):3–10.

[14] Birman, M. S. and Solomyak, M. (2003). Double operator integrals in a Hilbert space. *Integral Equations Operator Theory*, 47(2):131–168.

[15] Birman, M. S. and Solomyak, M. Z. (1966). Double Stieltjes operator integrals. In *Probl. Math. Phys., No. I, Spectral Theory and Wave Processes (Russian)*, pages 33–67. Izdat. Leningrad. Univ., Leningrad.

[16] Birman, M. S. and Solomyak, M. Z. (1967). Double Stieltjes operator integrals. II. In *Problems of Mathematical Physics, No. 2, Spectral Theory, Diffraction Problems (Russian)*, pages 26–60. Izdat. Leningrad. Univ., Leningrad.

[17] Birman, M. S. and Solomyak, M. Z. (1973). Double Stieltjes operator integrals. III. pages 27–53.

[18] Birman, M. S. and Solomyak, M. Z. (1987). *Spectral theory of selfadjoint operators in Hilbert space*. Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht. Translated from the 1980 Russian original by S. Khrushchëv and V. Peller.

[19] Birman, M. S. and Solomyak, M. Z. (1996). Tensor product of a finite number of spectral measures is always a spectral measure. *Integral Equations Operator Theory*, 24(2):179–187.

[20] Birman, M. S., Veršik, A. M., and Solomyak, M. Z. (1979). The product of commuting spectral measures may fail to be countably additive. *Funktsional. Anal. i Prilozhen.*, 13(1):61–62.

[21] Bogachev, V. I. (2007). *Measure theory. Vol. I, II.* Springer-Verlag, Berlin.

[22] Brown, L. G., Douglas, R. G., and Fillmore, P. A. (1977). Extensions of C^*-algebras and K-homology. *Ann. of Math. (2)*, 105(2):265–324.

[23] Carey, A. L., Gayral, V., Rennie, A., and Sukochev, F. A. (2014). Index theory for locally compact noncommutative geometries. *Mem. Amer. Math. Soc.*, 231(1085):vi+130.

[24] Carey, A. L., Phillips, J., and Rennie, A. (2011). Spectral triples: examples and index theory. In *Noncommutative geometry and physics: renormalisation, motives, index theory*, ESI Lect. Math. Phys., pages 175–265. Eur. Math. Soc., Zürich.

[25] Carleson, L. and Gamelin, T. W. (1993). *Complex dynamics*. Universitext: Tracts in Mathematics. Springer-Verlag, New York.

[26] Connes, A. (1985). Noncommutative differential geometry. *Inst. Hautes Études Sci. Publ. Math.*, (62):257–360.

[27] Connes, A. (1988). The action functional in noncommutative geometry. *Comm. Math. Phys.*, 117(4):673–683.

[28] Connes, A. (1994). *Noncommutative geometry*. Academic Press, Inc., San Diego, CA.

[29] Connes, A. (1995). Noncommutative geometry and reality. *J. Math. Phys.*, 36(11):6194–6231.

[30] Connes, A. (1997a). Brisure de symétrie spontanée et géométrie du point de vue spectral. *Astérisque*, (241):Exp. No. 816, 5, 313–349. Séminaire Bourbaki, Vol. 1995/96.

[31] Connes, A. (1997b). Noncommutative differential geometry and the structure of space time. In *Deformation theory and symplectic geometry (Ascona, 1996)*, volume 20 of *Math. Phys. Stud.*, pages 1–33. Kluwer Acad. Publ., Dordrecht.

[32] Connes, A., Levitina, G., McDonald, E., Sukochev, F., and Zanin, D. (2019a). Noncommutative geometry for symmetric non-self-adjoint operators. *J. Funct. Anal.*, 277(3):889–936.

[33] Connes, A., McDonald, E., Sukochev, F., and Zanin, D. (2019b). Conformal trace theorem for Julia sets of quadratic polynomials. *Ergodic Theory Dynam. Systems*, 39(9):2481–2506.

[34] Connes, A. and Moscovici, H. (1995). The local index formula in noncommutative geometry. *Geom. Funct. Anal.*, 5(2):174–243.

[35] Connes, A., Sukochev, F. A., and Zanin, D. V. (2017). Trace theorem for quasi-Fuchsian groups. *Mat. Sb.*, 208(10):59–90.

[36] Connes, A., Sullivan, D., and Teleman, N. (1994). Quasiconformal mappings, operators on Hilbert space, and local formulae for characteristic classes. *Topology*, 33(4):663–681.

[37] Cordes, H. O. (1979). *Elliptic pseudodifferential operators—an abstract theory*, volume 756 of *Lecture Notes in Mathematics*. Springer, Berlin.

[38] Daleckiĭ, Y. L. and Kreĭn, S. G. (1951). Formulas of differentiation according to a parameter of functions of Hermitian operators. *Doklady Akad. Nauk SSSR (N.S.)*, 76:13–16.

[39] Daleckiĭ, Y. L. and Kreĭn, S. G. (1956). Integration and differentiation of functions of Hermitian operators and applications to the theory of perturbations. *Voronež. Gos. Univ. Trudy Sem. Funkcional. Anal.*, 1956(1):81–105.

[40] Davies, E. B. (1988). Lipschitz continuity of functions of operators in the Schatten classes. *J. London Math. Soc. (2)*, 37(1):148–157.

[41] de Pagter, B., Witvliet, H., and Sukochev, F. A. (2002). Double operator integrals. *J. Funct. Anal.*, 192(1):52–111.

[42] Diestel, J. and Uhl, Jr., J. J. (1977). *Vector measures*. American Mathematical Society, Providence, R.I. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[43] Dixmier, J. (1981). *von Neumann algebras*, volume 27 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York. With a preface by E. C. Lance, Translated from the second French edition by F. Jellett.

[44] Dodds, P. G., Dodds, T. K., and de Pagter, B. (1992). Fully symmetric operator spaces. *Integral Equations Operator Theory*, 15(6):942–972.

[45] Dodds, P. G., Dodds, T. K., Sukochev, F. A., and Zanin, D. (2019). Arithmetic-geometric mean and related submajorisation and norm inequalities for τ-measurable operators. *(submitted manuscript).*

[46] Douglas, M. R. and Nekrasov, N. A. (2001). Noncommutative field theory. *Rev. Modern Phys.*, 73(4):977–1029.

[47] Dunford, N. and Schwartz, J. T. (1958). *Linear Operators. I. General Theory.* With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York; Interscience Publishers, Ltd., London.

[48] Ehrlich, P. (2012). The absolute arithmetic continuum and the unification of all numbers great and small. *Bull. Symbolic Logic*, 18(1):1–45.

[49] Emerson, H. and Nica, B. (2018). K-homological finiteness and hyperbolic groups. *J. Reine Angew. Math.*, 745:189–229.

[50] Falconer, K. (2003). *Fractal geometry.* John Wiley & Sons, Inc., Hoboken, NJ, second edition. Mathematical foundations and applications.

[51] Farforovskaja, J. B. (1968). The connection of the Kantorovič-Rubinšteĭn metric for spectral resolutions of selfadjoint operators with functions of operators. *Vestnik Leningrad. Univ.*, 23(19):94–97.

[52] Federer, H. (1969). *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York.

[53] Gao, L., Junge, M., and McDonald, E. Quantum euclidean spaces with noncommutative covariant derivatives. (in preparation).

[54] Garnett, J. B. and Marshall, D. E. (2005). *Harmonic measure*, volume 2 of *New Mathematical Monographs.* Cambridge University Press, Cambridge.

[55] Gayral, V., Gracia-Bondí a, J. M., Iochum, B., Schücker, T., and Várilly, J. C. (2004). Moyal planes are spectral triples. *Comm. Math. Phys.*, 246(3):569–623.

[56] Getzler, E. (1986). A short proof of the local Atiyah-Singer index theorem. *Topology*, 25(1):111–117.

[57] Gheorghe, L. G. (2001). Hankel operators in Schatten ideals. *Ann. Mat. Pura Appl. (4)*, 180(2):203–210.

[58] Gilkey, P. B. (1995). *Invariance theory, the heat equation, and the Atiyah-Singer index theorem.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, second edition.

[59] Gimperlein, H. and Goffeng, M. (2017). Nonclassical spectral asymptotics and Dixmier traces: from circles to contact manifolds. *Forum Math. Sigma*, 5:e3, 57.

[60] Gimperlein, H. and Goffeng, M. (2019). Commutator estimates on contact manifolds and applications. *J. Noncommut. Geom.*, 13(1):363–406.

[61] Goffeng, M. and Mesland, B. (2015). Spectral triples and finite summability on Cuntz-Krieger algebras. *Doc. Math.*, 20:89–170.

[62] González-Pérez, A. M., Junge, M., and Parcet, J. (2017). Singular integrals in quantum euclidean spaces.

[63] Gracia-Bondía, J. M. and Várilly, J. C. (1988). Algebras of distributions suitable for phase-space quantum mechanics. I. *J. Math. Phys.*, 29(4):869–879.

[64] Gracia-Bondía, J. M., Várilly, J. C., and Figueroa, H. (2001). *Elements of noncommutative geometry*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA.

[65] Groenewold, H. J. (1946). On the principles of elementary quantum mechanics. *Physica*, 12:405–460.

[66] Ha, H., Lee, G., and Ponge, R. (2018a). Pseudodifferential calculus on noncommutative tori, i. oscillating integrals.

[67] Ha, H., Lee, G., and Ponge, R. (2018b). Pseudodifferential calculus on noncommutative tori, ii. main properties.

[68] Hajłasz, P. and Kałamajska, A. (1995). Polynomial asymptotics and approximation of Sobolev functions. *Studia Math.*, 113(1):55–64.

[69] Hall, B. C. (2013). *Quantum theory for mathematicians*, volume 267 of *Graduate Texts in Mathematics*. Springer, New York.

[70] Halmos, P. R. (1950). *Measure Theory*. D. Van Nostrand Company, Inc., New York, N. Y.

[71] Hedenmalm, H., Korenblum, B., and Zhu, K. (2000). *Theory of Bergman spaces*, volume 199 of *Graduate Texts in Mathematics*. Springer-Verlag, New York.

[72] Hiai, F. and Kosaki, H. (2003). *Means of Hilbert space operators*, volume 1820 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin.

[73] Higson, N. (2004). Meromorphic continuation of zeta functions associated to elliptic operators. In *Operator algebras, quantization, and noncommutative geometry*, volume 365 of *Contemp. Math.*, pages 129–142. Amer. Math. Soc., Providence, RI.

[74] Higson, N. (2006). The residue index theorem of Connes and Moscovici. In *Surveys in noncommutative geometry*, volume 6 of *Clay Math. Proc.*, pages 71–126. Amer. Math. Soc., Providence, RI.

[75] Higson, N. and Roe, J. (2000). *Analytic K-homology*. Oxford Mathematical Monographs. Oxford University Press, Oxford. Oxford Science Publications.

[76] Janson, S. and Wolff, T. H. (1982). Schatten classes and commutators of singular integral operators. *Ark. Mat.*, 20(2):301–310.

[77] Kadison, R. V. and Ringrose, J. R. (1983). *Fundamentals of the theory of operator algebras. Vol. I*, volume 100 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York. Elementary theory.

[78] Kaminker, J., Putnam, I. F., and Whittaker, M. F. (2017). K-theoretic duality for hyperbolic dynamical systems. *J. Reine Angew. Math.*, 730:263–299.

[79] Karni, S. and Merzbach, E. (1990). On the extension of bimeasures. *J. Analyse Math.*, 55:1–16.

[80] Kasparov, G. G. (1975). Topological invariants of elliptic operators. I. K-homology. *Izv. Akad. Nauk SSSR Ser. Mat.*, 39(4):796–838.

[81] Kosaki, H. (1992). An inequality of Araki-Lieb-Thirring (von Neumann algebra case). *Proc. Amer. Math. Soc.*, 114(2):477–481.

[82] Lesch, M. and Moscovici, H. (2016). Modular curvature and Morita equivalence. *Geom. Funct. Anal.*, 26(3):818–873.

[83] Levitina, G., Sukochev, F., and Zanin, D. (2017). Cwikel estimates revisited (submitted manuscript).

[84] Lindenstrauss, J. and Peł czyński, A. (1968). Absolutely summing operators in L_p-spaces and their applications. *Studia Math.*, 29:275–326.

[85] Loday, J.-L. (1998). *Cyclic homology*, volume 301 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[86] Loeb, P. A. and Wolff, M. P. H., editors (2015). *Nonstandard analysis for the working mathematician.* Springer, Dordrecht, second edition.

[87] Lord, S., McDonald, E., Sukochev, F., and Zanin, D. (2017). Quantum differentiability of essentially bounded functions on Euclidean space. *J. Funct. Anal.*, 273(7):2353–2387.

[88] Lord, S., Sukochev, F., and Zanin, D. Advances in dixmier traces and applications. *(Submitted manuscript).*

[89] Lord, S., Sukochev, F., and Zanin, D. (2013). *Singular traces*, volume 46 of *De Gruyter Studies in Mathematics.* De Gruyter, Berlin. Theory and applications.

[90] Löwner, K. (1934). Über monotone Matrixfunktionen. *Math. Z.*, 38(1):177–216.

[91] Marczewski, E. and Ryll-Nardzewski, C. (1953). Remarks on the compactness and non direct products of measures. *Fund. Math.*, 40:165–170.

[92] McDonald, E., Sukochev, F., and Xiong, X. (2019a). Quantum differentiability on quantum planes. *(submitted manuscript).*

[93] McDonald, E., Sukochev, F., and Xiong, X. (2019b). Quantum differentiability on quantum tori. *Commun. Math. Phys.*

[94] McDonald, E., Sukochev, F., and Zanin, D. (2019c). A C^*-algebraic approach to the principal symbol ii. *D. Math. Ann.*

[95] Milnor, J. (2006). *Dynamics in one complex variable*, volume 160 of *Annals of Mathematics Studies.* Princeton University Press, Princeton, NJ, third edition.

[96] Moyal, J. E. (1949). Quantum mechanics as a statistical theory. *Proc. Cambridge Philos. Soc.*, 45:99–124.

[97] Nekrasov, N. and Schwarz, A. (1998). Instantons on noncommutative $\mathbf{R}^4$, and $(2,0)$ superconformal six-dimensional theory. *Comm. Math. Phys.*, 198(3):689–703.

[98] Palais, R. S. (1965). *Seminar on the Atiyah-Singer index theorem.* With contributions by M. F. Atiyah, A. Borel, E. E. Floyd, R. T. Seeley, W. Shih and R. Solovay. Annals of Mathematics Studies, No. 57. Princeton University Press, Princeton, N.J.

[99] Peller, V. V. (1985). Hankel operators in the theory of perturbations of unitary and selfadjoint operators. *Funktsional. Anal. i Prilozhen.*, 19(2):37–51, 96.

[100] Peller, V. V. (2003). *Hankel operators and their applications.* Springer Monographs in Mathematics. Springer-Verlag, New York.

[101] Peller, V. V. (2010). The behavior of functions of operators under perturbations. In *A glimpse at Hilbert space operators*, volume 207 of *Oper. Theory Adv. Appl.*, pages 287–324. Birkhäuser Verlag, Basel.

[102] Peller, V. V. (2016). Multiple operator integrals in perturbation theory. *Bull. Math. Sci.*, 6(1):15–88.

[103] Pisier, G. (2001). *Similarity problems and completely bounded maps*, volume 1618 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, expanded edition. Includes the solution to "The Halmos problem".

[104] Pisier, G. (2012). Grothendieck's theorem, past and present. *Bull. Amer. Math. Soc. (N.S.)*, 49(2):237–323.

[105] Pisier, G. and Xu, Q. (2003). Non-commutative L^p-spaces. In *Handbook of the geometry of Banach spaces, Vol. 2*, pages 1459–1517. North-Holland, Amsterdam.

[106] Potapov, D., Skripka, A., and Sukochev, F. (2013). Spectral shift function of higher order. *Invent. Math.*, 193(3):501–538.

[107] Potapov, D. and Sukochev, F. (2010). Double operator integrals and submajorization. *Math. Model. Nat. Phenom.*, 5(4):317–339.

[108] Potapov, D. and Sukochev, F. (2011). Operator-Lipschitz functions in Schatten-von Neumann classes. *Acta Math.*, 207(2):375–389.

[109] Potapov, D., Sukochev, F., Vella, D., and Zanin, D. (2019). A residue formula for locally compact noncommutative manifolds. In *Positivity and Noncommutative Analysis*, Trends in Mathematics. Birkhäuser Basel.

[110] Reed, M. and Simon, B. (1980). *Methods of modern mathematical physics. I.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York, second edition. Functional analysis.

[111] Ricard, E. (2018). Fractional powers on noncommutative L_p for $p < 1$. *Adv. Math.*, 333:194–211.

[112] Robinson, A. (1996). *Non-standard analysis.* Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ. Reprint of the second (1974) edition, With a foreword by Wilhelmus A. J. Luxemburg.

[113] Rochberg, R. and Semmes, S. (1989). Nearly weakly orthonormal sequences, singular value estimates, and Calderon-Zygmund operators. *J. Funct. Anal.*, 86(2):237–306.

[114] Rotfel'd, S. J. (1977). Asymptotic behavior of the spectrum of abstract integral operators. *Trudy Moskov. Mat. Obšč.*, 34:105–128.

[115] Rudin, W. (1987). *Real and complex analysis.* McGraw-Hill Book Co., New York, third edition.

[116] Rudin, W. (1991). *Functional analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition.

[117] Ruelle, D. (1982). Repellers for real analytic maps. *Ergodic Theory Dynamical Systems*, 2(1):99–107.

[118] Sakai, S. (1971). *C^*-algebras and W^*-algebras.* Springer-Verlag, New York-Heidelberg. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 60.

[119] Schaefer, H. H. (1962). Spectral measures in locally convex algebras. *Acta Math.*, 107:125–173.

[120] Schmüdgen, K. (2012). *Unbounded self-adjoint operators on Hilbert space*, volume 265 of *Graduate Texts in Mathematics.* Springer, Dordrecht.

[121] Schur, J. (1911). Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen. *J. Reine Angew. Math.*, 140:1–28.

[122] Seiberg, N. and Witten, E. (1999). String theory and noncommutative geometry. *J. High Energy Phys.*, (9):Paper 32, 93.

[123] Semenov, E., Sukochev, F., Usachev, A., and Zanin, D. (2015). Banach limits and traces on $\mathcal{L}_{1,\infty}$. *Adv. Math.*, 285:568–628.

[124] Shilov, G. E. (1977). *Linear algebra.* Dover Publications, Inc., New York, english edition. Translated from the Russian and edited by Richard A. Silverman.

[125] Stein, E. M. (1970). *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J.

[126] Stroyan, K. D. and Luxemburg, W. A. J. (1976). *Introduction to the theory of infinitesimals.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London. Pure and Applied Mathematics, No. 72.

[127] Sukochev, F. and Zanin, D. (2018a). A C^*-algebraic approach to the principal symbol. I. *J. Operator Theory*, 80(2):481–522.

[128] Sukochev, F. and Zanin, D. (2018b). The Connes character formula for locally compact spectral triples.

[129] Sukochev, F. and Zanin, D. (2018c). Connes integration formula for the noncommutative plane. *Comm. Math. Phys.*, 359(2):449–466.

[130] Sullivan, D. (1983). Conformal dynamical systems. In *Geometric dynamics (Rio de Janeiro, 1981)*, volume 1007 of *Lecture Notes in Math.*, pages 725–752. Springer, Berlin.

[131] Takesaki, M. (2002). *Theory of operator algebras. I*, volume 124 of *Encyclopaedia of Mathematical Sciences.* Springer-Verlag, Berlin. Reprint of the first (1979) edition, Operator Algebras and Non-commutative Geometry, 5.

[132] Takhtajan, L. A. (2008). *Quantum mechanics for mathematicians*, volume 95 of *Graduate Studies in Mathematics.* American Mathematical Society, Providence, RI.

[133] Todorov, I. G. and Turowska, L. (2010). Schur and operator multipliers. In *Banach algebras 2009*, volume 91 of *Banach Center Publ.*, pages 385–410. Polish Acad. Sci. Inst. Math., Warsaw.

[134] Urbański, M. (2003). Measures and dimensions in conformal dynamics. *Bull. Amer. Math. Soc. (N.S.)*, 40(3):281–321.

[135] Wheeden, R. L. and Zygmund, A. (2015). *Measure and integral*. Pure and Applied Mathematics (Boca Raton). CRC Press, Boca Raton, FL, second edition. An introduction to real analysis.

[136] Zhu, K. H. (1991). Analytic Besov spaces. *J. Math. Anal. Appl.*, 157(2):318–336.