Coherence for a closed functor

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COHERENCE FOR A CLOSED FUNCTOR

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Declaration of originality

All the work in this thesis is the original work of G. Lewis, with the exception of the concept of "club" which provides the setting for a precise statement of the results.

That concept was developed by the supervisor G.M. Kelly and expounded in the series of papers [4], [5], [6] and [7].

Historically, however, it was inspired by the first results of Lewis on the monoidal case of the present problem. When it became clear that even here "not all diagrams commute", as in (1.1) below, Kelly was at first prepared to settle for a partial result: those diagrams commute whose codomain is of the form αT. Lewis convinced him that one could do better, and gave necessary and sufficient conditions for commutativity.

These conditions, however, did not make precise sense while the morphisms were, as in Kelly-Mac Lane [8], actual natural transformations. It was this that led Kelly to replace them by their formal descriptions in the theory, and so to invent clubs and to investigate their nature. Kelly wishes here to acknowledge this debt to Lewis.

This setting apart, all the details of the covariant case and all the extension to the mixed-variance case is due to Lewis.

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G. Lewis

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G.M. Kelly

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I particularly want to thank my thesis supervisor, Professor G. Max Kelly, for his help throughout the four years of preparation and writing of this thesis. It was he who introduced me to category theory and in particular the present problem. He was also of great assistance in helping me express my ideas in readable language.

My thanks are also due to my parents, Shirley and Allan, for continually giving me encouragement to complete this thesis.

I would also like to express my appreciation to Cathy for her excellent typing.

Geoffrey Lewis.

February, 1974.

Abstract of the thesis

The theory of closed categories was greatly simplified by the coherence result of Kelly and Mac Lane [8], which showed that a large class of diagrams, writable in a generic closed category, commute in any particular one. In this thesis we carry this simplification further by considering diagrams writable in the context of a closed functor α: $A \to A'$ between two closed categories.

Like Kelly-Mac Lane we begin with a coherence result for the simpler problem where the functor and categories are not closed but only symmetric monoidal, and use the cut-elimination technique to pass to the closed case. Unlike Kelly-Mac Lane we find that it is not the case that "all diagrams commute" in the simpler case. Nevertheless, we have been able to determine in the symmetric monoidal case precisely which diagrams do commute.

It was already recognized in [8] that when dealing with a fragment f: $T \to S$ of a diagram, T and S had to be abstract descriptions of functors rather than their actual realizations in particular categories. We find that the morphism f must also be represented by an abstract description rather than as an actual natural transformation. This necessity, arising from this very problem, has led my thesis

supervisor, Professor G.M. Kelly, to examine these categories of "formal functors and formal natural transformations" associated to a coherence problem. The appropriate setting is found in his notion of club.

We define functors Γ_1, Γ_2 and Γ_3 whose domain categories may be either $\mathcal{P}'$, the club for a symmetric monoidal functor between symmetric monoidal categories, or $\mathcal{C}'$, the club for a closed functor between closed categories. We see that Γ_1 summarizes those parts of $\mathcal{P}'$ or $\mathcal{C}'$ which involve the first or domain category, Γ_2 summarizes those parts involving the second category, and Γ_3 the formal occurrences of the connecting symmetric monoidal or closed functor.

For the symmetric monoidal case we show that for morphisms $f,g\colon\ T \to S$ of $\mathcal{P}'$, their realizations are equal in any particular model if and only if $\Gamma_1 f = \Gamma_1 g$, $\Gamma_2 f = \Gamma_2 g$ and $\Gamma_3 f = \Gamma_3 g$.

An object T of $\mathcal{C}'$ is called proper if in its formation there is no use of any $[X,Y]$, for which $\Gamma_i Y = 0$ (Y is constant with respect to the appropriate invariant), yet $\Gamma_i X \neq 0$, for one of $i = 1$, 2, or 3.

For the closed case, we show that for any morphisms $f,g\colon\ T \to S$ of $\mathcal{C}'$ for which T and S are proper, their realizations are equal in any particular model if and only if $\Gamma_1 f = \Gamma_1 g$, $\Gamma_2 f = \Gamma_2 g$ and $\Gamma_3 f = \Gamma_3 g$.

1. Description of the problem

1.1 The theory of symmetric monoidal closed categories, which we shall call closed categories for short, was greatly simplified by the coherence result of Kelly and Mac Lane [8], which showed that a large class of diagrams, writable in a generic closed category, commute in any particular one. There are two obvious directions in which this simplification might be carried further. The first is to consider diagrams writable in the context of a closed category $\mathcal{A}$, together with a pair $\mathcal{B},\mathcal{C}$ of $\mathcal{A}$-categories, a pair T,S: $\mathcal{B} \to \mathcal{C}$ of $\mathcal{A}$-functors, and an $\mathcal{A}$-natural transformation η: $T \to S$. The second is to consider diagrams writable in the context of two closed categories $\mathcal{A},\mathcal{A}'$ together with a closed functor $\mathcal{A} \to \mathcal{A}'$. Here a closed functor means the same thing as a symmetric monoidal functor: a functor α: $\mathcal{A} \to \mathcal{A}'$ together with a natural transformation $\tilde{\alpha}$: $\alpha A \otimes' \alpha B \to \alpha(A \otimes B)$ and a morphism α^{o}: $I' \to \alpha I$, subject to the well-known axioms (see [2] pp 473 and 513).

Both of these extensions were considered in the recent volume on coherence problems [15]; the first in the paper of Kelly-Mac Lane [9], and the second in my own paper [12]. This last paper, written in a hurry while my work was still in a comparatively primitive form, contained some inaccuracies and some infelicities, as well as proofs more complicated than necessary. The purpose of the present thesis is to give an improved version of the results of that paper.

1.2 Our problem, then, is the "coherence problem" for the structure consisting of two closed categories A,A' and a closed functor $\alpha = (\alpha,\tilde{\alpha},\alpha^{o})$: $A \to A'$. As in the corresponding problem for a single closed category, studied by Kelly-Mac Lane in [8], we begin with a coherence result for the simpler problem in which A,A' are not closed but only symmetric monoidal. We also borrow the cut-elimination technique from Gentzen and Lambek ([10] and other papers) to pass to the closed case. There is, however, a significant difference: for a single symmetric monoidal category A, Mac Lane had proved in [14] the classical coherence result "all diagrams commute". For two symmetric monoidal categories joined by a symmetric monoidal functor α: $A \to A'$, there is no coherence result already available in the literature. We must prove our own, and it turns out that not all diagrams commute. (It is true that Epstein [3] has proved an "all diagrams commute" result in a related case: but the "tensor products" $\otimes,\otimes'$ in his A,A' had lacked identities I,I' and it is precisely these that cause non-commutativity. In this context see also Mac Donald [13];

and recall that it was the presence of the identity I that forced the Kelly-Mac Lane result in [8] to fall short of "all diagrams commute". It seems to always be the <u>constants</u> that cause trouble.)

<u>1.3</u> Nevertheless, we have been able in the symmetric monoidal case to determine precisely which diagrams <u>do</u> commute. Think of the two edges of a diagram as morphisms $f,g\colon\ T \to S$ in a suitable category. To a first approximation we can conceive of T,S as functors (of many variables and, in general, of mixed variances), and of f,g as natural transformations (of the generalized kind introduced by Eilenberg-Kelly [1]). An object such as T has a <u>type</u> ΓT, specifying its arity, the category from which the i-th argument is to be drawn, and the variance of this argument. A morphism such as f also has a <u>type</u> Γf, specifying the arguments of T and of S that it is to pair off; this was called its <u>graph</u> in [1] and then in [8], but we shall avoid this over-used word.

The generic components of f and of g form a closed diagram only when $\Gamma f = \Gamma g$; this then is a necessary condition for the diagram to be <u>writable</u> at all. When in a coherence problem one says "all diagrams commute" one means that every writable diagram does so; that is, one means that $\Gamma f = \Gamma g$ implies $f = g$; or in short that Γ is faithful. What Kelly-Mac Lane proved in [8] was a partial coherence result of the form "for <u>suitably restricted</u> T and S, and for $f,g\colon T \to S$, it is the case that $\Gamma f = \Gamma g$ implies $f = g$".

Our coherence result in the symmetric monoidal case $\alpha\colon \mathcal{A} \to \mathcal{A}'$, where, as we have said, not all diagrams commute, is a more complete one. We have said that Γ is not faithful; we could indeed prove a partial result as above that $\Gamma f = \Gamma g$ implies $f = g$ for suitably restricted T and S; but we do more. We assign to $f\colon T \to S$ a second invariant $\Delta f\colon \Delta T \to \Delta S$, and prove without restriction on T,S that "$\Gamma f = \Gamma g$ and $\Delta f = \Delta g$ imply $f = g$" - that is, that Γ and Δ are <u>jointly faithful</u>.

When we then pass to the closed case of $\alpha\colon \mathcal{A} \to \mathcal{A}'$ we get a result which, while like that of Kelly-Mac Lane [8] it is incomplete, is still more complete than if we looked at Γ alone. It takes the form: "Suppose T and S are Γ-proper (essentially the condition imposed by Kelly-Mac Lane in [8]) and also Δ-proper. Then $f,g\colon T \to S$ coincide if and only if $\Gamma f = \Gamma g$ and $\Delta f = \Delta g$." This is our main theorem, and it must of course be formulated with precision. Before passing to the discussion of the necessary precision we illustrate by examples two of our remarks above.

First, even in a single symmetric monoidal category, nobody expects to have $c = 1\colon\ A \otimes A \to A \otimes A$ where c is the symmetry. The generic components of c and of 1 look like

$$\begin{tikzcd} & B \otimes A \\ A \otimes B \arrow[ur, "c"] \arrow[dr, "1"'] & \\ & A \otimes B, \end{tikzcd}$$

and do not form a closed diagram. Equivalently, $c\colon\ \otimes \to \otimes$ and $1\colon\ \otimes \to \otimes$ are natural transformations of different types; $\Gamma\otimes = 2$, Γc is the non-identity permutation of 2 and $\Gamma 1$ is the identity permutation. Coherence is concerned with the natural transformations, and not with particular components such as $c_{AA}\colon\ A \otimes A \to A \otimes A$.

Secondly, an example of a writable but non-commuting diagram in the symmetric-monoidal $\alpha\colon\ \mathcal{A} \to \mathcal{A}'$ case. It is

(1.1)
$$\begin{tikzcd} \alpha I \arrow[r, "\cong"] \arrow[d, "\cong"'] & I' \otimes' \alpha I \arrow[d, "\alpha^{\circ} \otimes' 1"] \\ \alpha I \otimes' I' \arrow[r, "1 \otimes' \alpha^{\circ}"] & \alpha I \otimes' \alpha I \end{tikzcd}$$

Here the Γ of every vertex is 0 - all are constants - and the Γ of each edge is the identity permutation $0 \to 0$. Yet the diagram fails to commute even when α is the forgetful closed functor from Abelian Groups to Sets, the two legs sending $n \in \alpha I = \underline{\underline{Z}}$ to $(n,1)$ and to $(1,n)$. But the second invariant Δ looks at the occurrences of α; $\Delta(\alpha I) = 1$, $\Delta(\alpha I \otimes' \alpha I) = 2$, and the Δ's of the two legs are the two possible functions $1 \to 2$.

<u>1.4</u> Already in [8] it was recognized that the vertices T,S of the diagrams must not be actual functions in the model but their abstract descriptions in the theory - otherwise one would have unwanted composites of $f\colon\ T \to S$ and $g\colon\ S' \to R$, where S and S' although formally different had identical realizations in a particular model. Still, the morphisms f,g in [8] were actual natural transformations. Since each natural transformation f had a definite type Γf, this served well enough for the kind of result given in [8]. But it will not serve for us: the edges of (1.1) may coincide in a particular model, but are to be assigned different images under Δ. For us, not only the vertices of the diagrams but also the edges must be abstract descriptions in terms of the theory.

This necessity, arising originally from this very problem, where precise sense had to be made of my "second invariant" Δ, has led my thesis supervisor, Professor G.M. Kelly, to examine these categories of "formal functors and formal natural transformations" associated to a coherence problem. The appropriate setting is found in his notion of <u>club</u>, the first ideas on which were expounded in [4], [5] and [6], and a definitive, generalized treatment of which is to appear in [7]. We now turn to a discussion of clubs which, while merely an outline, should be sufficient for the reader wishing to understand the present thesis.

2. The idea of a club

2.1 Consider first a structure of the following kind, to be borne by a single category $\mathcal{A}$; examples would be a symmetric monoidal structure, or a strict monoidal structure.

We are first to be given, as part of the structure functors $|B|: \mathcal{A}^n \to \mathcal{A}$, indexed by the elements B of a set $\mathcal{B}$, and each associated with an arity or type $n \in \underline{\underline{N}}$, depending on B and written as $n = \Gamma B$. Note our careful distinction between the abstract B of the theory and its realization $|B|$ in the model $\mathcal{A}$. Add to $\mathcal{B}$ a formal identity $\underline{\underline{1}}$ with $\Gamma\underline{\underline{1}} = 1$, and then close $\mathcal{B}$ under the operation formal substitution; that is, from operations T of type n and $S_1, \ldots, S_n$ of types $m_1, \ldots, m_n$ we form $T(S_1, \ldots, S_n)$ of type $m_1 + \ldots + m_n$; we also write $n(m_1, \ldots, m_n)$ for $m_1 + \ldots + m_n$. Each T in this closure $\overline{\mathcal{B}}$ of $\mathcal{B}$ has itself an obvious realization $|T|: \mathcal{A}^n \to \mathcal{A}$, where $n = \Gamma T$. In the examples above, $\mathcal{B}$ consists of $\otimes$ and the unit I for $\otimes$, with $\Gamma\otimes = 2$, $\Gamma I = 0$.

We are next to be given axioms of the form $|T| = |S|$ for certain pairs $T, S \in \overline{\mathcal{B}}$ with $\Gamma T = \Gamma S$; for example in the strict monoidal case we have the axioms $|\otimes(\otimes,\underline{\underline{1}})| = |\otimes(\underline{\underline{1}},\otimes)|$, $|\otimes(I,\underline{\underline{1}})| = |\underline{\underline{1}}| = |\otimes(\underline{\underline{1}},I)|$, while in the non-strict monoidal case the list of such axioms is vacuous. The set of objects ob$\mathcal{K}$ of the corresponding club $\mathcal{K}$ is the quotient set of $\bar{\mathcal{B}}$ by the substitution-congruence generated by these axioms. Clearly each $T \in$ ob$\mathcal{K}$ again has a realization $|T|$: $A^n \to A$ when the axioms are satisfied in the model A.

We are then to be given a set $\mathcal{D}$ of formal natural transformations, each with a domain and codomain in ob$\mathcal{K}$, a typical one being then d: $T \to S$. These are to be generalized natural transformations in the sense of Eilenberg-Kelly [1]; but since T and S are covariant, we must have $\Gamma T = \Gamma S = n$ say, and type Γd of d is only a permutation of n. Each such d is to have a realization $|d|$: $|T| \to |S|$, an actual generalized natural transformation in A of type Γd. For example, in the symmetric monoidal case, $\mathcal{D}$ consists of a: $\otimes(\otimes,\underline{\underline{1}}) \to \otimes(\underline{\underline{1}},\otimes)$, ℓ: $\otimes(I,\underline{\underline{1}}) \to \underline{\underline{1}}$, r: $\otimes(\underline{\underline{1}},I) \to \underline{\underline{1}}$, c: $\otimes \to \otimes$, of respective types $\Gamma a = 1$, $\Gamma\ell = 1$, $\Gamma r = 1$, Γc = the non-identity permutation of 2; together with formal inverses $\bar{a}$: $\otimes(\underline{\underline{1}},\otimes) \to \otimes(\otimes,\underline{\underline{1}})$, $\bar{\ell}$, $\bar{r}$; we need no $\bar{c}$ since it is to be the same as c.

We close $\mathcal{D}$ to get $\bar{\mathcal{D}}$ = Exp Inst $\mathcal{D}$, the set of expanded instances of the $d \in \mathcal{D}$. An instance of d is a formal natural transformation

$e = d(R_1, \ldots, R_n)\colon\ T(R_1, \ldots, R_n) \to S(R_1, \ldots, R_n)$

where $R_i \in \mathrm{ob}K$; it has an obvious type $\Gamma e = \Gamma d(\Gamma R_1, \ldots, \Gamma R_n)$, and an obvious realization $|e|$ in the model A. An expansion of the instance $e\colon\ P \to Q$ is a formal natural transformation $h = T(1, \ldots, 1, e, 1, \ldots, 1)\colon\ T(S_1 \ldots, P \ldots S_m)$ $\to\ T(S_1 \ldots\ Q \ldots\ S_m)$, again with an obvious type $\Gamma h = \Gamma T(1, \ldots, 1, \Gamma e, 1, \ldots, 1)$ and an obvious realization $|h|$ in the model A.

The objects $\mathrm{ob}K$ and the axioms $\bar{D}$ form a <u>graph</u>, in the classical sense of the word; and Γ is a map of graphs $(\mathrm{ob}K, \bar{D}) \to \underline{\underline{P}}$, where $\underline{\underline{P}}$ is the category with $\underline{\underline{N}}$ as its set of objects and with permutations $\xi\colon\ n \to n$ as its only morphisms. We pass from the graph $(\mathrm{ob}K, \bar{D})$ to the category L it generates freely; then Γ extends to a functor $\Gamma\colon L \to \underline{\underline{P}}$. Clearly every $f\colon\ T \to S$ in L has a realization $|f|\colon\ |T| \to |S|$ which is a generalized natural transformation of type Γf.

Finally we are given a second set of axioms; a typical one is given by a pair $f, g\colon\ T \to S$ of morphisms of L, with $\Gamma f = \Gamma g$; and the axiom to be satisfied by A is that $|f| = |g|$. For example, in the monoidal case, we have the coherence axioms such as the "pentagon axiom" for a, as well as axioms asserting $a\bar{a} = 1$, $\bar{a}a = 1$, and so on. We close these axioms under the process of taking expanded instances; this gives us a set Δ_1 of pairs $f,g\colon\ T \to S$ in L. We add to these a second set Δ_2 of such pairs, asserting things like $|\otimes(1,f).\ \otimes(g,1)| = |\otimes(g,1).\ \otimes(1,f)|$, which follow from the functoriality of $|\otimes|$; or more generally of course of $|T|$. We add to these a third set Δ_3 of such pairs, formally asserting the naturality of $|d|$ for $d \in \mathcal{D}$; and we define the <u>club</u> K corresponding to the given structure as L/Δ where $\Delta = \Delta_1 + \Delta_2 + \Delta_3$.

We now have the following situation. We have a category K with a functor $\Gamma\colon\ K \to \underline{\underline{P}}$ (called its <u>augmentation</u>); that is an object of the 2-category $\underline{\underline{\mathrm{Cat}}}/\underline{\underline{P}}$. Next, K admits an operation of <u>substitution</u> $T(S_1, \ldots, S_n)$ for its objects, and a corresponding operation $f(g_1, \ldots, g_n)$ for its morphisms, with the special cases $T(g_1,\ldots,g_n)$ and $f(S_1,\ldots,S_n)$ where for example T denotes 1_T. Kelly [4], [5] has shown that $\underline{\underline{\mathrm{Cat}}}/\underline{\underline{P}}$ is a monoidal (indeed a closed) category with "tensor product" denoted by $\circ$; and that this substitution operation is a <u>multiplication</u> $\mu\colon\ K \circ K \to K$ in $\underline{\underline{\mathrm{Cat}}}/\underline{\underline{P}}$. Finally K has among its objects the formal identity $\underline{\underline{1}}$, and this corresponds to a <u>unit map</u> $\eta\colon\ J \to K$, where J is the identity for $\circ$. It turns out that μ and η satisfy the associative and identity axioms, so that K is a o-monoid in Cat/P.

2.2 We now make the definition: a club is a o-monoid in Cat/P. Then what we have shown above is that every structure of the kind considered gives rise to a club: its basic operations B,d and its two sets of axioms may be said to be generators and relations for K. A diagram in K is a pair f,g: T → S in K; it commutes if f = g; this cannot be so unless $\Gamma f = \Gamma g$, i.e. unless the diagram is writable (in terms of components); we may say "all diagrams commute" if Γ is faithful; the coherence problem is that of deciding which diagrams commute; it is essentially the problem of determining K explicitly, starting from its generators and relations.

We must say what we mean by a model A of a general club K; we call such an A a K-category. We embed Cat fully in Cat/P by giving to $A \in$ Cat the trivial augmentation Γ: $A \to$ P which is the constant functor at 0. Then $K \circ A \in$ Cat if $A \in$ Cat. A K-category is a category A together with an action θ: $K \circ A \to A$ satisfying the usual associativity and identity axioms. When K is constructed as above from basic operations and axioms, it is easy to see that a K-category A in the above sense is precisely a model for the structure in question.

This description of a K-category A exhibits it as an algebra for the monad $K\circ-$: $\underline{\underline{\mathrm{Cat}}} \to \underline{\underline{\mathrm{Cat}}}$. A morphism of ($K\circ-$)-algebras is a <u>strict morphism</u> of K-categories, or a <u>strict</u> K-<u>functor</u>; a functor $A \to B$ between K-categories, preserving all the structure on the nose. (In this <u>covariant</u> case, but not in <u>mixed-variance</u> cases like that of closed categories, the monad $K\circ-$ is actually a 2-<u>monad</u>, or <u>equational doctrine</u> in the sense of Lawvere [11].)

So the forgetful functor from the category of K-categories and strict K-functors, to the category $\underline{\underline{\mathrm{Cat}}}$ of categories, has a left adjoint sending A to $K\circ A$; thus $K\circ A$ is the <u>free</u> K-<u>category</u> <u>on</u> A. If 1 denotes the unit category with one object * and one morphism, we have $K\circ 1 \cong K$; so K itself is the free K-category on the object $\underline{\underline{1}} \in K$; given a K-category A and $A \in A$, there is a unique strict K-functor $K \to A$ sending $\underline{\underline{1}}$ to A.

For the details of all this the reader may refer to Kelly's papers [4] and [5]. What he needs to know of clubs for the purposes of this thesis is the following:

(a) The fact above that K is the free K-category on its object $\underline{\underline{1}}$;

(b) The manner of constructing ob$K$ form the basic functorial operations;

(c) The fact that the morphisms of K are composites of expanded instances of the basic natural transformation equations.

There are a few further points we should notice. One is the matter of notation: an object of $K \circ A$ has the form $T[A_1, \ldots, A_n]$ where $T \in K$, $A_i \in A$, and $n = \Gamma T$. We write its image under an action θ: $K \circ A \to A$ as $T(A_1, \ldots, A_n)$ (which is therefore the same as $|T|(A_1, \ldots, A_n)$). Similarly we write $T(S_1, \ldots, S_n)$ for the image of $T[S_1, .., S_n]$ under the multiplication μ: $K \circ K \to K$, which is itself an action of K on K. Similarly too for morphisms: note that if f: $T \to T'$ has $\Gamma f = \xi$, a permutation of n, and if g_i: $A_i \to A_i'$, then $f(g_1, \ldots, g_n)$: $T(A_{\xi 1}, \ldots, A_{\xi n}) \to T'(A_1', \ldots, A_n')$.

Then there is the question of a <u>map of clubs</u>, that is, of a map ϕ: $K \to L$ of $\circ$-monoids in $\underline{\underline{Cat}}/\underline{\underline{P}}$. It is clear that any L-category A with an action θ: $L \circ A \to A$ is then also a K-category with action

$$K \circ A \xrightarrow{\phi \circ A} L \circ A \xrightarrow{\theta} A.$$

In particular $\underline{\underline{P}}$ itself is a club (with augmentation $1: \underline{\underline{P}} \to \underline{\underline{P}}$); and for any club K the augmentation $\Gamma: K \to \underline{\underline{P}}$ is a map of clubs. A $\underline{\underline{P}}$-category is, by Mac Lane's original coherence theorem [14], just a strict symmetric monoidal category. By the same theorem, if P denotes the club whose algebras are (non-strict) symmetric monoidal categories, the augmentation $\Gamma: P \to \underline{\underline{P}}$ is an equivalence of categories. Since we know the objects of P, as iterates of $\otimes$, I, and $\underline{\underline{1}}$, this information suffices to describe P completely. Again the discrete category $\underline{\underline{N}}$, with inclusion augmentation $\underline{\underline{N}} \to \underline{\underline{P}}$, is a club, whose algebras are the strict monoidal categories. If N is the club for (non-strict) monoidal categories, then N has the same objects as P, and by Mac Lane's results its augmentation $\Gamma: N \to \underline{\underline{P}}$ is an equivalence of N with its image under Γ, which is $\underline{\underline{N}}$.

$\underline{\underline{2.3}}$ There is little to change in the above when we allow the structure on A to involve functors of mixed variance, such as $A \times A \times A^{op} \to A$, and the most general natural transformations of Eilenberg-Kelly [1]. Such a structure is that of a closed category, which is a symmetric monoidal category with an extra basic functor $[\ ,\ ]: A^{op} \times A \to A$, and extra basic natural transformations $e: [A,B] \otimes A \to B$, $d: A \to [B, A \otimes B]$, satisfying as extra axioms the triangular axioms which make them the counit and unit of an adjunction $A(A \otimes B,C) \cong A(A,[B,C])$. Again we refer to Kelly's papers [4], [5] for details, just giving enough here to make this thesis readable.

The type ΓT of a functorial operation T is no longer just its arity n, but a string ν of + and - signs, such as ++-+--, indicating the variances of the arguments in T. The type of a natural transformation f: $T \to S$, where $\Gamma T = \nu$ and $\Gamma S = \mu$, is a pairing-off or linking of the arguments of T and of S taken together, as in the example of d and e above. Two arguments that are paired are to have the same variance if one is in T and one in S, and opposite variances if both are in T or both in S. This is best said by defining $\{\nu, -\mu\}$ to be the string obtained by first writing ν, and then writing μ with all its signs changed; then the type of f: $T \to S$ is a bijection between the + signs and the - signs in the type $\{\nu, -\mu\}$. We write this type of f as Γf: $\Gamma T \to \Gamma S$, or as say ξ: $\nu \to \mu$.

The difficulty is that generalized natural transformations f: $T \to S$, g: $S \to R$ can be composed only when Γf and Γg are compatible in the sense of [1] - we also discuss this in detail in §5.2 below. When ξ: $\nu \to \mu$

and η: $\mu \to \tau$ are compatible, we define their composite as in [1], and then $\eta\xi$ is the type of gf. When they are not, there is no natural transformation gf. We elect in this case to define the composite $\eta\xi$ to be a special "zero map" $*$: $\nu \to \tau$.

So we replace $\underline{\underline{P}}$, as the category of types, by a category T whose objects are strings ν, μ, τ, ... as above. A morphism from ν to μ is either a bijection of the +'s with the -'s in $\{\nu, -\mu\}$, called a non-trivial morphism, or the trivial morphism $*_{\nu\mu} = *$. Composition in T of non-trivial morphisms $\eta\xi$ is the Eilenberg-Kelly composite if they are compatible and is $*$ otherwise; the composite $\eta\xi$ is also $*$ if either $\eta = *$ or $\xi = *$.

Now in the mixed-variance case, starting from basic functors and natural transformations and their axioms, we again get as in §2.1 a category K of their formal iterates, this time with an augmentation functor Γ: $K \to T$. We can define $K \circ A$ if, for each f: $T \to S$ in K, $\Gamma f \neq *$: that is, if no incompatibilities arise. Kelly has shown in [6] that incompatibilities never arise if we start from a purely covariant situation (where they are impossible) and then add some right adjoints, such as $[-,-]$ in the closed category situation. This will cover all the cases we deal with.

So the "tensor product" $\circ$ is only partially defined on $\underline{\underline{\mathrm{Cat}}}/T$, but we shall not run outside its domain of definition. The K we get for such a mixed-variance structure will again be a $\circ$-monoid, that is, a <u>mixed-variance club</u>; a K-category A is a category A with an action θ: $K \circ A \to A$; the free such on A is $K \circ A$; and K itself is the free K-category on $\underline{\underline{1}} \in K$.

In these terms the coherence result of Kelly-Mac Lane [8] for closed categories is as follows. Let C be the club whose algebras are closed categories. Call an object T of C <u>proper</u> if in its construction by iteration from $\underline{\underline{1}}$, I, $\otimes$, [,], we never form [P,Q] where $\Gamma P \neq 0$ and $\Gamma Q = 0$ (0 is the empty string; in general we identify a string of n + signs with $n \in \underline{\underline{P}}$). They then show that, if f,g: T $\to$ S in C where T and S are <u>proper</u>, the condition $\Gamma f = \Gamma g$ (i.e. writability of the diagram) implies f = g. The club C for this case has never been fully determined: partial results have been obtained by Voreadu in her thesis. It is certainly <u>not</u> the case that all diagrams commute.

<u>2.4</u> Again, there is not much to change if the structure in question is borne not by a single category A but by a <u>family</u> of categories $(A_\lambda)_{\lambda \in \Lambda}$. Regarding the set Λ as a

discrete category, such a family may be taken as an object $\mathcal{A}$ of $\underline{\underline{\mathrm{Cat}}}/\Lambda$. The type ΓT of T now has to prescribe not only the arity n of T and the variance + or - of its i-th argument, but the index $\lambda \in \Lambda$ of the category $\mathcal{A}_\lambda$ from which the argument is drawn. The type Γf of f is still a pairing-off, but can only pair arguments from the same category $\mathcal{A}_\lambda$. This gives in place of $\mathcal{T}$ a new category $\mathcal{T}_\Lambda$, or $\underline{\underline{P}}_\Lambda$ in the purely covariant case: and the augmentation as a functor Γ: $\mathcal{K} \to \mathcal{T}_\Lambda$ (or $\underline{\underline{P}}_\Lambda$). The only other point of difference is that $\mathcal{K}$ now has among its objects "identities" $\underline{\underline{1}}_\lambda$ for each λ; the forgetful functor from $\mathcal{K}$-algebras (families $(\mathcal{A}_\lambda)$ with the given structure and strict functors) now lands not in $\underline{\underline{\mathrm{Cat}}}$ but in $\underline{\underline{\mathrm{Cat}}}/\Lambda$; and $\mathcal{K}$ itself is the free $\mathcal{K}$-algebra not on $\mathcal{I}$ but on Λ.

For our purposes it is simpler to write $\mathcal{K}$ as a family $(\mathcal{K}_\lambda)$ and to recognize the object $\underline{\underline{1}}_\lambda \in \mathcal{K}_\lambda$. Then the fact that $\mathcal{K}$ is free on Λ is expressed thus: the family $(\mathcal{K}_\lambda)$ bears the structure in question; and if $(\mathcal{A}_\lambda)$ does too, and if we choose $A_\lambda \in \mathcal{A}_\lambda$ for each $\lambda \in \Lambda$, then there are unique functors ϕ_λ: $\mathcal{K}_\lambda \to \mathcal{A}_\lambda$, sending $\underline{\underline{1}}_\lambda$ to A_λ, and constituting a strict morphism of the structure: everything preserved on the nose.

3. Our problem in terms of clubs

3.1 Let K be a covariant club of the single-category kind, as in §2.1 and §2.2. Then $K\circ -$: Cat $\to$ Cat is not only a functor but a 2-functor, as shown by Kelly [4], [5]; so it is not merely a monad but a 2-monad or doctrine. (This is false in the mixed-variance case.) This leads to the possibility of a lax or non-strict morphism of K-categories; that is, of a K-functor, as distinct from a strict K-functor. See Kelly [5], §7.

Let A, A' be K-categories with actions θ, θ'. A K-functor $A \to A'$ consists of a functor α: $A \to A'$, not required to preserve anything at all, together with a natural transformation $\bar{\alpha}$ as in

$$\begin{tikzcd} K\circ A \arrow[r, "\theta"] \arrow[d, "K\circ\alpha"'] \arrow[dr, phantom, "\Nearrow\bar{\alpha}"] & A \arrow[d, "\alpha"] \\ K\circ A' \arrow[r, "\theta'"'] & A' \end{tikzcd}$$

subject to the following axioms:

(3.1) The composite

$$\begin{tikzcd} KoKoA \arrow[r, "\mu oA"] \arrow[d, "KoKo\alpha"'] & KoA \arrow[r, "\theta"] \arrow[d, "Ko\alpha"] \arrow[dr, phantom, "\overset{\bar{\alpha}}{\Rightarrow}"] & A \arrow[d, "\alpha"] \\ KoKoA' \arrow[r, "\mu oA'"'] & KoA' \arrow[r, "\theta'"'] & A' \end{tikzcd}$$

coincides with the composite

$$\begin{tikzcd} KoKoA \arrow[r, "Ko\theta"] \arrow[d, "KoKo\alpha"'] \arrow[dr, phantom, "\overset{Ko\bar{\alpha}}{\Rightarrow}"] & KoA \arrow[r, "\theta"] \arrow[d, "Ko\alpha"] \arrow[dr, phantom, "\overset{\bar{\alpha}}{\Rightarrow}"] & A \arrow[d, "\alpha"] \\ KoKoA' \arrow[r, "Ko\theta'"'] & KoA' \arrow[r, "\theta'"'] & A' \end{tikzcd};$$

(3.2) The composite

$$\begin{tikzcd} A \arrow[r, "\eta oA"] \arrow[d, "\alpha"'] & KoA \arrow[r, "\theta"] \arrow[d, "Ko\alpha"] \arrow[dr, phantom, "\overset{\bar{\alpha}}{\Rightarrow}"] & A \arrow[d, "\alpha"] \\ A' \arrow[r, "\eta oA'"'] & KoA' \arrow[r, "\theta'"'] & A' \end{tikzcd}$$

is the identity.

It is clear how to compose such K-functors: they form a category, indeed a 2-category. The natural transformation $\bar{\alpha}$ has components

$$\bar{\alpha}_{T[A_1 \ldots A_n]}: \quad T(\alpha A_1 \ldots \alpha A_n) \to \alpha T(A_1 \ldots A_n).$$

When K is given by basic operations, it is easy to see that the above component need only be given for <u>basic</u> T, and its naturality in T only for <u>basic</u> d: $T \to T'$ (or for identifications $T = T'$ occurring in the axioms). Thus in the care where K corresponds to monoidal or to symmetric monoidal categories, strict or not, $\bar{\alpha}$ is determined by its components

$$\tilde{\alpha} = \bar{\alpha}_{\otimes(A,B)}: \quad \alpha A \otimes' \alpha B \to \alpha(A \otimes B)$$

$$\alpha^{o} = \bar{\alpha}_{I(-)}: \quad I' \to \alpha I.$$

The naturality with respect to a, ℓ, r when $K = N$ gives the usual definition of <u>monoidal functor</u>; in the case $K = \underline{\underline{N}}$ we have identifications in place of a, ℓ, r but we still get a monoidal functor, now between <u>strict</u> monoidal categories (but not itself strict). In the cases $K = P$ or $\underline{\underline{P}}$, we get the usual definition of <u>symmetric monoidal functor</u> (of Eilenberg-Kelly [2], pp 473 and 513). Note that a strict K-functor is a K-functor in which $\bar{\alpha} = 1$.

Now consider, for such a club K, the following structure borne by a pair of categories A, A' - or rather for the moment, by a pair A_1, A_2. The category A_1 is to have the structure of a K-category; so is the category A_2; and there is to be also a K-functor $(\alpha,\bar{\alpha})$: $A_1 \to A_2$. This is a structure (covariant) of the kind considered in §2.4 with $\Lambda = 2$; and it corresponds to a club $\hat{K}$ in $\underline{\underline{\mathrm{Cat}}}/\underline{\underline{\mathrm{P}}}_2$, with components say K_1 and K_2. Since $\hat{K}$ is itself a model, K_1 and K_2 are both K-categories, and there is a K-functor $(\kappa,\bar{\kappa})$: $K_1 \to K_2$. The fact that $\hat{K}$ is the free model on $\underline{\underline{1}}_1 \in K_1$ and $\underline{\underline{1}}_2 \in K_2$ means that, given any model $(\alpha,\bar{\alpha})$: $A_1 \to A_2$, and given $A_1 \in A_1$ and $A_2 \in A_2$, there are unique <u>strict</u> K-functors ϕ_1: $K_1 \to A_1$, ϕ_2: $K_2 \to A_2$, rendering commutative

(3.3)
$$\begin{tikzcd} K_1 \arrow[r, "\phi_1"] \arrow[d, "(\kappa,\bar{\kappa})"'] & A_1 \arrow[d, "(\alpha,\bar{\alpha})"] \\ K_2 \arrow[r, "\phi_2"'] & A_2 \end{tikzcd},$$

and such that $\phi_1(\underline{\underline{1}}_1) = A_1$, $\phi_2(\underline{\underline{1}}_2) = A_2$.

Now take A_2 to be the unit category I, which is trivially a K-category for any K; and take $(\alpha,\bar{\alpha})$ to be the only thing it can, which is trivially a K-functor. We deduce that there is a unique strict K-functor $\phi_1\colon\ K_1 \to A_1$ sending $\underline{\underline{1}}_1$ to A_1; that is, that K_1 is the free K-category on one object, namely K itself. This could also have been seen directly from the mode of construction of $\hat{K}$ in §2.1; the K_1 part of $\hat{K}$ must be K itself, since κ goes <u>from</u> K_1 to K_2, $\bar{\kappa}$ lives in K_2, and the K-structure on K_2 remains within K_2.

We therefore change notation, replacing (3.3) by

(3.4) $$\begin{tikzcd} K \arrow[r, "\phi"] \arrow[d, "(\kappa,\bar{\kappa})"'] & A \arrow[d, "(\alpha,\bar{\alpha})"] \\ K' \arrow[r, "\phi'"'] & A' \end{tikzcd} \quad ;$$

and $\underline{\underline{1}}_1$, $\underline{\underline{1}}_2$ by $\underline{\underline{1}}$ and $\underline{\underline{1}}'$; so that the determination of $\hat{K}$ reduces to that of K', κ and $\bar{\kappa}$.

3.2 The covariant part of our problem is such a determination in the case where $\mathcal{K} = \mathcal{P}$. We can solve this case but in the four cases $\mathcal{K} = \mathcal{P}$, $\underline{\underline{P}}$, $\mathcal{N}$, $\underline{\underline{N}}$ (see the end of §2.2 for the meanings of these); that is, when $\mathcal{K}$ corresponds to monoidal or symmetric monoidal categories, strict or not.

Going back to a general $\mathcal{K}$, recall that we have an augmentation $\hat{\Gamma}$: $\hat{\mathcal{K}} \to \underline{\underline{P}}_2$. We are of course interested only in its restriction $\hat{\Gamma}$: $\mathcal{K}' \to \underline{\underline{P}}_2$, since we know all about $\mathcal{K}$. The writable diagrams f,g: $T \to S$ in $\mathcal{K}'$ are those for which $\hat{\Gamma}f = \hat{\Gamma}g$. We can simplify this criterion of writability by splitting up $\hat{\Gamma}$ into its two parts Γ_1, Γ_2: $\mathcal{K}' \to \underline{\underline{P}}$; the first looks only at the arguments drawn from $\mathcal{K}$, and the second at those drawn from $\mathcal{K}'$; recall that a natural transformation may only link arguments drawn from the same category. It is clear that Γ_1 and Γ_2 are the unique strict $\mathcal{K}$-functors given by the following cases of (3.4) (recall that, as at the end of §2.2, $\underline{\underline{P}}$ is a $\mathcal{K}$-category by virtue of the club map Γ: $\mathcal{K} \to \underline{\underline{P}}$, with $T(m_1,\ldots,m_n) = \Gamma T(m_1,\ldots,m_n) = m_1 + \ldots + m_n$; that $\mathcal{I}$ is a $\underline{\underline{P}}$-category and hence a $\mathcal{K}$-category for any $\mathcal{K}$; and that a $\underline{\underline{P}}$-functor is automatically a $\mathcal{K}$-functor):

(3.5)
$$\begin{tikzcd}
K \arrow[r, "\Gamma"] \arrow[d, "(\kappa,\bar{\kappa})"'] & \underline{\underline{P}} \arrow[d, "1"] \\
K' \arrow[r, "\Gamma_1"'] & \underline{\underline{P}}
\end{tikzcd}$$

where $\Gamma(\underline{\underline{1}}) = 1$ and $\Gamma_1(\underline{\underline{1}}') = 0$;

(3.6)
$$\begin{tikzcd}
K \arrow[r] \arrow[d, "(\kappa,\bar{\kappa})"'] & I \arrow[d, "(\alpha,\bar{\alpha})"] \\
K' \arrow[r, "\Gamma_2"'] & \underline{\underline{P}}
\end{tikzcd}$$

where $K \to I$ is the unique functor, sending $\underline{\underline{1}}$ to $*$, where Γ_2 sends $\underline{\underline{1}}'$ to 1, and where $\alpha(*) = 0$, while the component

$$\bar{\alpha}_{T[*,\ \ldots,\ *]}\colon\ T(\alpha*,\ \ldots,\ \alpha*) \to \alpha T(*,\ \ldots,\ *)$$

or $T(0, \ldots, 0) \to 0$, is the identity map $0 \to 0$ (for $T(0, \ldots, 0) = \Gamma T(0, \ldots, 0) = 0$, when $\underline{\underline{P}}$ is regarded as a K-category via the club-map $\Gamma\colon\ K \to \underline{\underline{P}}$).

So the writable diagrams $f, g\colon\ T \to S$ in K' are those for which $\Gamma_1 f = \Gamma_1 g$ and $\Gamma_2 f = \Gamma_2 g$. As we said in

§1.2 and §1.3, <u>not</u> all writable diagrams commute.

We spoke in §1.3 of another invariant Δ alongside $\hat{\Gamma}$, i.e. alongside Γ_1 and Γ_2. We now write Δ as Γ_3. It is given as follows.

Denote by $\underline{\underline{S}}$ the skeletal category of finite sets, with objects $n \in \underline{\underline{N}}$ and with functions $n \to m$ as morphisms. It is a strict symmetric monoidal category if we take $m + n$ as its tensor product $m \otimes n$, with 0 as the identity for $\otimes$; hence it is a K-category. There is a symmetric monoidal functor $(\beta, \bar{\beta})$: $I \to \underline{\underline{S}}$, and hence a K-functor, where $\beta(*) = 1$, and where

$$\bar{\beta}_{T[*, \ldots, *]}: \quad T(\beta*, \ldots, \beta*) \to \beta T(*, \ldots, *)$$

is the unique map $n \to 1$, n being ΓT. We therefore get a case of (3.4), to wit

(3.7)
$$\begin{tikzcd} K \arrow[r] \arrow[d, "(\kappa, \bar{\kappa})"'] & I \arrow[d, "(\beta, \bar{\beta})"] \\ K' \arrow[r, "\Gamma_3"'] & \underline{\underline{S}} \end{tikzcd}$$

where $K \to I$ is the unique map sending $\underline{\underline{1}}$ to $*$, and Γ_3 sends $\underline{\underline{1}}'$ to 0. It is clear that Γ_3 "looks at the occurrences of κ'', and does what is claimed for it in §1.3, distinguishing for instance the two legs of (1.1).

Our first main result Theorem 4.5, to be proved in §4.4 below, is the assertion that Γ_1, Γ_2 and Γ_3 are <u>jointly faithful</u> in the cases $K = P, \underline{\underline{P}}, N, \underline{\underline{N}}$. Actually we first prove in Proposition 4.3 that, for any K, the morphisms of K' have a certain form; and we then prove in Theorem 4.5 the joint fidelity of $\Gamma_1, \Gamma_2, \Gamma_3$ under conditions satisfied by each of $P, \underline{\underline{P}}, N, \underline{\underline{N}}$.

This is all, strictly speaking, that we need know about the covariant case for our applications to the mixed-variance one of two closed categories and a closed functor; but because we can go further here, we do so. We can give K', κ, $\bar{\kappa}$ explicitly in the four cases $P, \underline{\underline{P}}, N, \underline{\underline{N}}$, and we do so; our method is to "guess" the result, and then to use the above Theorem 4.5 to prove it.

<u>3.3</u> We are now in a position to formulate our main result, in terms of a mixed-variance analogue of §3.2 above. Let C be the mixed-variance club for closed categories; there is of course a club-map $P \to C$. (Note that every covariant club $K \to \underline{\underline{P}}$ can be considered as a mixed-variance club $K \to \underline{\underline{P}} \to T$, with the obvious embedding

$\underline{\underline{P}} \to T$ sending n to a string of n + signs.) Now there is no such thing as a "non-strict map of C-algebras", as such; for a mixed-variance club such as C, the functor Co- is no longer a 2-functor, and a diagram such as (3.1) makes no sense when $K = C$, there being no "$K \circ \bar{\alpha}$". What has always been meant by a <u>closed functor</u> $\alpha\colon\ A \to A'$ between closed categories is just a symmetric monoidal functor $(\alpha, \tilde{\alpha}, \alpha^{\circ})$; these are what occur in nature. Such a functor <u>induces</u> a natural transformation $\hat{\alpha}\colon\ \alpha[A,B] \to [\alpha A, \alpha B]'$; but this is no independent datum.

So we consider the structure, borne by a pair of categories A, A', consisting of closed structures on each and a closed, i.e. a symmetric monoidal, functor $(\alpha, \tilde{\alpha}, \alpha^{\circ})\colon\ A \to A'$; or in the more general notation, $(\alpha, \bar{\alpha})\colon\ A \to A'$. The basic functors and natural transformations are those which generate the club $\hat{P}$ in $\underline{\underline{\mathrm{Cat}}}/\underline{\underline{P}}_2$, together with $[\ ,\ ]\colon\ A^{op} \times A \to A$, $[\ ,\ ]'\colon\ A'^{op} \times A' \to A'$ and natural transformations $e\colon\ [A,B] \otimes A \to B$, $d\colon\ A \to [B, A \otimes B]$, $e'\colon\ [X,Y]' \otimes' X \to Y$, $d'\colon\ X \to [Y, X \otimes' Y]'$, satisfying the extra axioms asserting that d,e provide an adjunction $A(A \otimes B, C) \cong A(A, [B,C])$ and that d', e' provide an adjunction $A'(X \otimes' Y, Z) \cong A'(X, [Y,Z]')$. By Kelly's result in [6], since we do no more than add right adjoints to certain of the functors in a club $\hat{P}$, this structure is given by a club $\hat{C}$ in $\underline{\underline{\mathrm{Cat}}}/T_2$.

As in §3.1, $\hat{C}$ is itself the free such structure on $\underline{\underline{1}}$ and $\underline{\underline{1}}'$, and may be written as $(\gamma,\bar{\gamma})$: $C \to C'$ the domain part being C itself for the same reasons as in §3.1. Our task is to determine C' as far as we are able: we cannot at the moment determine it completely, any more than C is known completely at this time - see our comments in §2.3.

We certainly know the objects of C'; they are those iterates of the basic functors whose codomain is C'; it is immediately clear that each such is uniquely writable in the form $T(X_1, \ldots, X_n)$ where $T \in C$ and where each X_i is either $\underline{\underline{1}}'$ or γS_i for $S_i \in C$. We also know the generators for the morphisms of C' - namely the expanded instances of a', ℓ', r', c', d', e', γa, $\gamma\ell$, γr, γc, γd, γe, $\tilde{\gamma}$ and γ^{o}, and the (formal) inverses of a', ℓ', r', γa, $\gamma\ell$ and γr. Our partial determination of C' consists in determining which writable diagrams f,g: $T \to S$ commute for <u>restricted</u> T,S - as in the Kelly-Mac Lane result [8] for C itself.

As in §3.2, we break up the functor Γ, which determines "writability", into two functors Γ_1 and Γ_2; and we add a third functor Δ or Γ_3 as a further invariant. Γ_1 and Γ_2 are determined as in (3.5) and (3.6), except that $(\kappa,\bar{\kappa})$: $K \to K'$ is replaced by $(\gamma,\bar{\gamma})$: $C \to C'$, and $\underline{\underline{P}}$

is replaced by T. Γ_3 is determined as in (3.7), with C etc. in place of K etc., and with a suitable replacement G for $\underline{\underline{S}}$.

Just as $\underline{\underline{P}}$ is a subcategory of $\underline{\underline{S}}$, with the same objects, so T is a subcategory of G, with the same objects. Whereas a non-trivial morphism $\nu \to \mu$ in T is a <u>bijection</u> from the + signs to the - signs in $\{\nu, -\mu\}$, a non-trivial morphism in G is a <u>function</u> from the + signs to the - signs; there is still the trivial morphism *, and the matter of compatibility. It is the case that both T and G are <u>closed categories</u>, so that we do indeed get the analogues of (3.5) - (3.7); since $T \subset G$, we may if we like consider Γ_i as a functor $C \to G$ for i = 1, 2, 3. These Γ_i are of course extensions of the Γ_i of §3.2 in the case $K = P$.

We call an object T of C' <u>proper</u> if, in its construction from the basic functors, one never forms $[A,B]$ or $[A,B]'$ where, for some i, $\Gamma_i A \neq 0$ and $\Gamma_i B = 0$. Our main result then becomes:

Let $T, S \in C'$ be proper, and let $f, g\colon T \to S$. Then $f = g$ if and only if $\Gamma_i f = \Gamma_i g$ for i = 1, 2, 3. Note that the third invariant Γ_3 is necessary here; mere writability, given by $\Gamma_i f = \Gamma_i g$ for i = 1, 2, does not imply $f = g$ even for proper T and S.

The method of proof is parallel to that of Kelly-Mac Lane in [8]. We first prove a cut-elimination result, Lemma 5.8 below, providing an inductive construction of the morphisms of C' from those of lower "rank"; we then use induction on rank T + rank S to prove the main theorem, using as a starting point the corresponding result for P' obtained from Corollary 4.6.

4. Coherence in the covariant case

4.1 Let $\mathcal{K}$ be a covariant club of the single-category kind, as in §2.1 and §2.2. We begin with a description of $\mathcal{K}'$, the $\mathcal{K}$-category mentioned at the conclusion of §3.1.

The objects of $\mathcal{K}'$ are generated as a $\mathcal{K}$-category by $\underline{1}'$ and κB for all $B \in \mathcal{K}$. Thus if A is an object of $\mathcal{K}$ with $\Gamma A = n$, and $Z_1, \ldots, Z_n$ are objects of $\mathcal{K}'$, then there is an object

$$(4.1) \qquad A(Z_1, \ldots, Z_n)$$

of $\mathcal{K}'$. If all the Z_i are either $\underline{1}'$ or κB_i, we say that the object (4.1) is in its prime factorization and that the Z_i are the prime factors of (4.1). All objects of $\mathcal{K}'$ have prime factorizations.

All morphisms of $\mathcal{K}'$ are composites of expansions of morphisms of the following forms:

$$(4.2) \qquad a(Z_1, \ldots, Z_n)\colon \; A(Z_{\xi 1}, \ldots Z_{\xi n}) \to B(Z_1 \ldots Z_n)$$

where $a\colon \; A \to B$ is a morphism of $\mathcal{K}$ with $\Gamma a = \xi$, and each $Z_i \in \mathcal{K}'$;

$$(4.3) \qquad \kappa a\colon \; \kappa A \to \kappa B$$

for $a\colon \; A \to B$ in $\mathcal{K}$; and

$$(4.4) \qquad \bar{\kappa}(A; B_1, \ldots, B_n)\colon \; A(\kappa B_1 \ldots \kappa B_n) \to \kappa A(B_1 \ldots B_n)$$

where $A; B_1, \ldots, B_n$ are objects of $\mathcal{K}$.

We have the following two relations:

(4.5) The composite

$$\begin{array}{l}
A(B_1(\kappa C_1 \ldots) \;\ldots\; B_n(\ldots \kappa C_m) \\
\Big\downarrow \; A(\bar{\kappa}(B_1;C_1\ldots) \;\ldots\; \bar{\kappa}(B_n;\ldots C_m)) \\
A(\kappa B(C_1 \ldots) \;\ldots \kappa B_n(\ldots C_m)) \\
\Big\downarrow \; \bar{\kappa}(A;B(C_1\ldots) \;\ldots B_n(\ldots C_m)) \\
\kappa A(B_1(C_1 \ldots) \;\ldots B_n(\ldots C_m))
\end{array}$$

is equal to

$$\begin{array}{l}
A(B_1 \ldots B_n)(\kappa C_1 \ldots \kappa C_m) \\
\Big\downarrow \; \bar{\kappa}(A(B_1\ldots B_n);C_1\ldots C_m) \\
\kappa A(B_1 \ldots B_n)(C_1 \ldots C_m);
\end{array}$$

(4.6) $\bar{\kappa}(\underline{\underline{1}};A)\colon \ \underline{\underline{1}}(\kappa A) \to \kappa(\underline{\underline{1}}(A))$ is the identity morphism; because $(\kappa,\bar{\kappa})\colon \ K \to K'$ is a K-functor.

<u>4.2</u> Let the <u>central morphisms</u> of K' be those morphisms (4.2) for which all Z_i are prime, i.e. $\underline{\underline{1}}'$ or κA_i. Clearly the central morphisms are closed under composition,

and each identity morphism is central. We observe that if $z: Z \to Y$ is central then Z and Y have the same prime factors to within order.

Lemma 4.1: If $a: A \to B$ in K has $\Gamma a = \xi$, then

$$(4.7) \qquad a(Z_1,\ldots Z_n): A(Z_{\xi 1}\ldots Z_{\xi n}) \to B(Z_1\ldots Z_n)$$

is central for all $Z_i \in K'$.

Proof: Let the prime factorizations of the Z_i be $Z_1 = C_1(Y_1,\ldots), \ldots, Z_n = C_n(\ldots, Y_m)$. Then (4.7) is $a(C_1\ldots C_n)(Y_1\ldots Y_m)$ which is central because $a(C_1\ldots C_n) : A(C_{\xi 1}\ldots C_{\xi n}) \to B(C_1\ldots C_n)$ is a morphism of K. □

Lemma 4.2: If $z_i: Z_i \to Y_i$ is central for each i, and if $A \in K$, then the following morphism is central

$$(4.8) \qquad A(z_1\ldots z_n): A(Z_1\ldots Z_n) \to A(Y_1\ldots Y_n).$$

Proof: Let z_1 be

$$a_1(X_1,\ldots): B_1(X_{\xi 1},\ldots) \to C_1(X_1,\ldots),$$

etc, where $a_i: B_i \to C_i$ is in K and the X_i are prime. Then (4.8) is $A(a_1\ldots a_n)(X_1\ldots X_m)$ which is central since

$$A(a_1\ldots a_n): A(B_1\ldots B_n) \to A(C_1\ldots C_n)$$

is a morphism of K. □

<u>4.3</u> In this section we show that each morphism of $\mathcal{K}'$ can be expressed as the composite of a central morphism, an expansion of instances of $\bar{\kappa}$, and an expansion of instances of κa, <u>in that order</u>.

Let a morphism $z\colon Z \to Y$ of $\mathcal{K}'$ be called <u>decomposable</u> if z can be written as

(4.9) $$Z \xrightarrow{t} W \xrightarrow{x} X \xrightarrow{y} Y$$

where

(4.10) The prime factorization of Y is $A(V_1 \ldots V_n)$ and of X is $A(U_1 \ldots U_n)$.

(4.11) The morphism y is $A(w_1 \ldots w_n)$ where for each $1 \leqslant i \leqslant n$ either

$$V_i = U_i = \underline{\underline{1}}' \text{ and } w_i = 1; \text{ or}$$

$$V_i = \kappa B_i,\ U_i = \kappa C_i \text{ and } W_i = \kappa a_i \text{ where}$$

$$a_i\colon\ C_i \to B_i \text{ in } \mathcal{K}.$$

(4.12) One factorization of W (not necessarily prime) is $A(T_1 \ldots T_n)$.

(4.13) The morphism x is $A(v_1 \ldots v_n)$ where for each $1 \leq i \leq n$ either

$$U_i = T_i = \underline{\underline{1}}' \text{ and } v_i = 1; \text{ or}$$

$$U_i = \kappa E_i(\ldots D_j \ldots),\ T_i = E_i(\ldots \kappa D_j \ldots) \text{ and}$$

v_i is

$$\bar{\kappa}(E_i;\ldots D_j\ldots):\ E_i(\ldots\kappa D_j\ldots) \to \kappa E_i(\ldots D_j \ldots).$$

(4.14) t is central.

Proposition 4.3: All morphisms of K' are decomposable.

Proof: Since $\kappa 1$: $\kappa A \to \kappa A$ and
$\bar{\kappa}(\underline{\underline{1}};A)$: $\underline{\underline{1}}(\kappa A) \to \kappa(\underline{\underline{1}}(A))$

are both equal to 1: $\kappa A \to \kappa A$, it readily follows that 1: $Z \to Z$ satisfies both (4.11) and (4.13). Thus any central morphism z: $Z \to Y$ is decomposable being

$$Z \xrightarrow{z} Y \xrightarrow{1} Y \xrightarrow{1} Y.$$

The morphism (4.3) is decomposable as

$$\kappa A \xrightarrow{1} \kappa A \xrightarrow{1} \underline{\underline{1}}(\kappa A) \xrightarrow{\underline{\underline{1}}(\kappa a)} \underline{\underline{1}}(\kappa B).$$

The morphism (4.4) is decomposable as

$$A(\ldots\kappa B_i\ldots) \xrightarrow{1} \underline{\underline{1}}(A(\ldots\kappa B_i\ldots)) \xrightarrow{\underline{\underline{1}}(\bar{\kappa})} \underline{\underline{1}}(\kappa A(\ldots B_i\ldots)$$
$$\xrightarrow{1} \underline{\underline{1}}(\kappa A(\ldots B_i \ldots)).$$

Suppose z_k: $Z_k \to Y_k$ are decomposable for $1 \leqslant k \leqslant p$, and that each z_k is

$$(4.15) \qquad Z_k \xrightarrow{t_k} W_k \xrightarrow{x_k} X_k \xrightarrow{y_k} Y_k$$

as in (4.9). For $F \in K$ with $\Gamma F = p$, it is necessary to show that $F(z_1 \ldots z_p)$ is decomposable.

But $F(z_1 \ldots z_p)$ is

$$F(Z_1 \ldots Z_p) \xrightarrow{F(t_1 \ldots t_p)} F(A_1 \ldots A_p)(\ldots T_i \ldots)$$
$$\xrightarrow{F(A_1 \ldots A_p)(\ldots V_i \ldots)} F(A_1 \ldots A_p)(\ldots U_i \ldots)$$
$$\xrightarrow{F(A_1 \ldots A_p)(\ldots W_i \ldots)} F(A_1 \ldots A_p)(\ldots V_i \ldots)$$

which is decomposable.

The proof of the proposition will be completed with the following lemma:

<u>Lemma 4.4</u>: <u>If z: $Z \to Y$ and u: $Y \to S$ are decomposable so is the composite uz: $Z \to S$.</u>

<u>Proof</u>: Let z be (4.9). It is sufficient to show the truth of the lemma when

(i) u is central;

(ii) u is an expansion of instances of $\bar{\kappa}$;

(iii) u is an expansion of instances of κa.

Case (i): Let u be

(4.16) $\quad b(V_{\xi 1}\ldots V_{\xi n})\colon \; A(V_1\ldots V_n) \to F(V_{\xi 1}\ldots V_{\xi n})$

for $b\colon \; A \to F$ in K with $\Gamma b = \xi^{-1}$.

But uz is now

$$Z \xrightarrow{t} A(T_1\ldots T_n) \xrightarrow{b(T_{\xi 1}\ldots T_{\xi n})} F(T_{\xi 1}\ldots T_{\xi n})$$

$$\xrightarrow{F(v_{\xi 1}\ldots v_{\xi n})} F(U_{\xi 1}\ldots U_{\xi n}) \xrightarrow{F(w_{\xi 1}\ldots w_{\xi n})}$$

$$F(V_{\xi 1}\ldots V_{\xi n})$$

which is decomposable by the centrality of

$$b(T_{\xi 1}\ldots T_{\xi n}).t.$$

Case (ii): Let u be

(4.17) $\quad A(V_1\ldots V_n) = G(F_1(V_1\ldots)\ldots F_m(\ldots V_n)) \xrightarrow{G_1(s_1\ldots s_m)}$

$$G(R_1\ldots R_m)$$

where $R_1,\ldots,R_m$ are prime, and if

(a) $\quad R_i$ is $\underline{\underline{1}}'$, then $F_i(\ldots V_j\ldots) = \underline{\underline{1}}(\underline{\underline{1}}')$ and $s_i = 1$;

(b) $\quad R_i$ is κJ_i, then $F_i(\ldots V_j\ldots) = F_i(\ldots \kappa H_j\ldots)$,

$J_i = F_i(\ldots H_j\ldots)$, and $s_i = \bar{\kappa}(F_i;\ldots H_j\ldots)$.

Since $A = G(F_1 \ldots F_m)$, uz is the composite

$$(4.18) \quad Z \xrightarrow{t} G(\ldots F_i(\ldots T_j \ldots)\ldots) \xrightarrow{F(\ldots F_i(\ldots w_j v_j \ldots)\ldots)}$$

$$G(\ldots F_i(\ldots V_j \ldots)\ldots) \xrightarrow{G(\ldots s_i \ldots)} G(\ldots R_i \ldots),$$

This is decomposable if each $s_i . F_i(\ldots w_j v_j \ldots)$ is, because as we have seen in the proof of Proposition 4.3, an expansion of decomposables is decomposable.

If R_i is $\underline{\underline{1}}'$, then $s_i = 1$, $F_i(\ldots V_j \ldots) = \underline{\underline{1}}'$, $w_j = 1$, $v_j = 1$, $F_i(\ldots T_j \ldots) = \underline{\underline{1}}'$, so that $s_i . F_i(\ldots w_j v_j \ldots)$ is $1 \colon \underline{\underline{1}}' \to \underline{\underline{1}}'$, and so decomposable.

Suppose R_i is κJ_i. Then $s_i . F_i(\ldots w_j v_j \ldots)$ is

$$(4.19) \quad F_i(\ldots E_j(\ldots \kappa D_k \ldots)\ldots) \xrightarrow{F_i(\ldots \bar{\kappa} \ldots)}$$

$$F_i(\ldots \kappa E_j(\ldots D_k \ldots)\ldots) \xrightarrow{F_i(\ldots \kappa a_j \ldots)}$$

$$F_i(\ldots \kappa H_j \ldots) \xrightarrow{\bar{\kappa}} \kappa F_i(\ldots H_j \ldots)$$

But by the naturality of $\bar{\kappa}$, (4.19) is

$$F_i(\ldots E_j(\ldots \kappa D_k \ldots)\ldots) \xrightarrow{F_i(\ldots \bar{\kappa} \ldots)}$$

$$F_i(\ldots \kappa E_j(\ldots D_k \ldots)\ldots) \xrightarrow{\bar{\kappa}}$$

$$\kappa F_i(\ldots E_j(\ldots D_k \ldots)\ldots) \xrightarrow{\kappa F_i(\ldots a_j \ldots)} \kappa F_i(\ldots H_j \ldots).$$

However this is decomposable since $\bar{\kappa}.\ F_i(\ldots\bar{\kappa}\ldots)$ equals

$$F_i(\ldots E_j\ldots)(\ldots\kappa D_k\ldots) \xrightarrow{\bar{\kappa}} \kappa F_i(\ldots E_j\ldots)(\ldots D_k\ldots)$$

by (4.5).

Case (iii): Let u be

(4.20) $\quad A(s_1\ldots s_n)\colon\ A(V_1\ldots V_n) \to A(R_1\ldots R_n)$

where if

(a) V_i is $\underline{\underline{1}}'$, so is R_i, and $s_i = 1\colon\ \underline{\underline{1}}' \to \underline{\underline{1}}'$.

(b) V_i is κB_i, then R_i is κF_i and s_i is κb_i for $b_i\colon\ B_i \to F_i$

But uy is

$$A(s_1w_1\ldots s_nw_n)\colon\ A(U_1\ldots U_n) \to A(R_1\ldots R_n)$$

where s_iw_i is either $1\colon\ \underline{\underline{1}}' \to \underline{\underline{1}}'$, or $\kappa(b_ia_i)\colon\ \kappa C_i \to \kappa F_i$.

Thus uz is decomposable. □

4.4 We proceed to use Proposition 4.3 in order to prove a theorem concerning certain sufficient conditions upon K ensuring the joint faithfulness of Γ_1, Γ_2 and Γ_3.

Theorem 4.5: Suppose that every $a\colon\ C(A_1\ldots A_n) \to C(B_1\ldots B_n)$ in K with $\Gamma a = n(\xi_1\ldots\xi_n)$ for some $\xi_i\colon\ \Gamma A_i \to \Gamma B_i$ is of the

<u>form $a = C(b_1 \ldots b_n)$ for some $b_i\colon\ A_i \to B_i$ in K with $\Gamma b_i = \xi_i$. Then if every map in K is an isomorphism, Γ_1, Γ_2 and Γ_3 are jointly faithful.</u>

<u>Proof</u>: Let $z, z'\colon\ Z \to Y$ in K' be such that $\Gamma_i z = \Gamma_i z'$ for $i = 1, 2, 3$. We must show that $z = z'$.

By Proposition 4.3, z and z' are decomposable so may be written as in (4.9) as

$$Z \xrightarrow{t} A(T_1 \ldots T_n) \xrightarrow{A(v_1 \ldots v_n)} A(U_1 \ldots U_n) \xrightarrow{A(w_1 \ldots w_n)} A(V_1 \ldots V_n)$$

and

$$Z \xrightarrow{t'} A(T'_1 \ldots T'_n) \xrightarrow{A(v'_1 \ldots v'_n)} A(U'_1 \ldots U'_n)$$

$$\xrightarrow{A(w'_1 \ldots w'_n)} A(V_1 \ldots V_n).$$

If $s = t'.\ t^{-1}$ we must show the commutativity of

(4.21)
$$\begin{tikzcd}
A(T_1 \ldots T_n) \arrow[rr, "s"] \arrow[d, "A(v_1 \ldots v_n)"] & & A(T'_1 \ldots T'_n) \arrow[d, "A(v'_1 \ldots v'_n)"] \\
A(U_1 \ldots U_n) \arrow[dr, "A(w_1 \ldots w_n)"'] & & A(U'_1 \ldots U'_n) \arrow[dl, "A(w'_1 \ldots w'_n)"] \\
& A(V_1 \ldots V_n) &
\end{tikzcd}$$

given that $\Gamma_i(4.21)$ commutes for $i = 1, 2, 3$.

We assert that s associates each prime factor of T_i with a prime factor of T_i'. Otherwise Γ_2 or Γ_3 of (4.21) fails to commute in view of the evident character of Γ_2 and Γ_3 of the left and right legs of (4.21).

Let the prime factorizations of the T_i' be

$T_1' = B_1'(Y_1 \ldots), \ldots, T_n' = B_n'(\ldots Y_m)$, and of the T_i be $T_1 = B_1(Y_{\eta 1} \ldots), \ldots, T_n = B_n(\ldots Y_{\eta m})$. Since T has the same factors as T_i', $\Gamma B_i = \Gamma B_i'$, and s is

$$a(Y_1 \ldots Y_m)\colon\ A(B_1 \ldots B_n)(Y_{\eta 1} \ldots Y_{\eta m}) \to A(B_1' \ldots B_n')(Y_1 \ldots Y_m)$$

where a: $A(B_1 \ldots B_n) \to A(B_1' \ldots B_n')$, $\Gamma a = \eta = A(\xi_1 \ldots \xi_n)$ for $\xi_i\colon \Gamma B_i \to \Gamma B_i'$. By the property of K mentioned in the statement of the theorem there exist $b_i\colon\ B_i \to B_i'$ for which $\Gamma b_i = \xi_i$ and $a = A(b_1 \ldots b_n)$. Denote the central morphisms $b_1(Y_1 \ldots)\colon\ B_1(Y_{\eta 1} \ldots) \to B_1'(Y_1 \ldots), \ldots,$ $b_n(\ldots Y_m)\colon\ B_n(\ldots Y_{\eta m}) \to B_n'(\ldots Y_m)$ by $s_1, \ldots, s_n$. Then $s = A(s_1 \ldots s_n)$.

By the strictness of each Γ_i and the commutativity of Γ_i of (4.21), it follows that Γ_i of

$$(4.22) \qquad \begin{tikzcd}
T_j \arrow[rr, "s_j"] \arrow[d, "v_j"'] & & T'_j \arrow[d, "v'_j"] \\
U_j \arrow[dr, "w_j"'] & & U'_j \arrow[dl, "w'_j"] \\
& V_j &
\end{tikzcd}$$

commutes.

If $V_j = \underline{\underline{1}}'$, then (4.22) reduces to

$$\begin{tikzcd}
\underline{\underline{1}}' \arrow[r, bend left, "1"] \arrow[r, bend right, "1"'] & \underline{\underline{1}}'
\end{tikzcd}$$

which obviously commutes.

If V_j is κC for some $C \in K$, then (4.22) is

$$(4.23) \qquad \begin{tikzcd}
D(\kappa E_{\zeta 1} \ldots E_{\zeta p}) \arrow[rr, "s_j"] \arrow[d, "\bar{\kappa}"'] & & D'(\kappa E_1 \ldots \kappa E_p) \arrow[d, "\bar{\kappa}"] \\
\kappa D(E_{\zeta 1} \ldots E_{\zeta p}) \arrow[dr, "\kappa f"'] & & \kappa D'(E_1 \ldots E_p) \arrow[dl, "\kappa f'"] \\
& \kappa C &
\end{tikzcd}$$

for various $D, D', E_1, \ldots, E_p, f, f'$. But s_j is central and so may be written as

$$d(\kappa E_1 \ldots \kappa E_p): \quad D(\kappa E_{\zeta 1} \ldots \kappa E_{\zeta p}) \to D'(\kappa E_1 \ldots \kappa E_p)$$

for $d: D \to D'$ with $\Gamma d = \zeta$. By the naturality of $\bar{\kappa}$ the right leg of (4.23) equals

$$(4.24) \quad D(\kappa E_{\zeta 1} \ldots \kappa E_{\zeta p}) \xrightarrow{\bar{\kappa}} \kappa D(E_{\zeta 1} \ldots E_{\zeta p}) \xrightarrow{\kappa d(E_1 \ldots E_p)} \kappa D'(E_1 \ldots E_p) \xrightarrow{\kappa f'} \kappa C.$$

The commutativity of $\Gamma_1(4.23)$ means that

$$\begin{tikzcd} D(E_{\zeta 1} \ldots E_{\zeta p}) \arrow[rr, "d(E_1 \ldots E_p)"] \arrow[dr, "f"'] & & D'(E_1 \ldots E_p) \arrow[dl, "f'"] \\ & C & \end{tikzcd}$$

commutes. By this result and the expansion (4.24) we see that (4.23) commutes. Thus (4.22) commutes for any prime V_j. Consequently (4.21) commutes, so that $z = z'$.

This completes the proof of Theorem 4.5. $\square$

<u>Corollary 4.6:</u> <u>If $\Gamma: K \to \underline{\underline{P}}$ is full and faithful, then Γ_1, Γ_2 and Γ_3 are jointly faithful.</u>

Proof: Let $a\colon\ C(A_1\ldots A_n) \to C(B_1\ldots B_n)$ in K be such that $\Gamma a = n(\xi_1\ldots\xi_n)$ and $\xi_i\colon\ \Gamma A_i \to \Gamma B_i$. Since Γ is full and faithful there exist unique $b_i\colon\ A_i \to B_i$ with $\Gamma b_i = \xi_i$. Since $\Gamma a = \Gamma C(b_1\ldots b_n)$ and Γ is faithful $a = C(b_1\ldots b_n)$.

We now want to show that every morphism $a\colon\ A \to B$ of K is an isomorphism. There is an inverse $\eta\colon\ \Gamma B \to \Gamma A$ of Γa in $\underline{\underline{P}}$. Since Γ is full and faithful there exists a unique $b\colon\ B \to A$ with $\Gamma b = \eta$. But $\Gamma(ba) = \Gamma(1_A)$ and $\Gamma(ab) = \Gamma(1_B)$, so $ba = 1_A$ and $ab = 1_B$, thus a is an isomorphism. □

4.5 We now turn our attention to the specific cases $K = \underline{\underline{P}}$, P, $\underline{\underline{N}}$ and N. In this section we shall define and consider a $\underline{\underline{P}}$-category E, an $\underline{\underline{N}}$-category B, a $\underline{\underline{P}}$-functor $(\varepsilon,\bar{\varepsilon})\colon\ \underline{\underline{P}} \to E$, and an $\underline{\underline{N}}$-functor $(\beta,\bar{\beta})\colon\ \underline{\underline{N}} \to B$. In §4.6 we shall show that $\hat{\underline{\underline{P}}}$ is isomorphic to $(\varepsilon,\bar{\varepsilon})\colon\ \underline{\underline{P}} \to E$, and that $\hat{\underline{\underline{N}}}$ is isomorphic to $(\beta,\bar{\beta})\colon\ \underline{\underline{N}} \to B$; and in §4.7 we show that $\hat{P}$ is equivalent to $\hat{\underline{\underline{P}}}$, and $\hat{N}$ is equivalent to $\hat{\underline{\underline{N}}}$.

The objects of E are

$$(f;\ n \to p,\ u,\ \phi) \tag{4.25}$$

where n,p,u are non-negative integers, f is an increasing map, and ϕ is a (p,u)-shuffle. A (p,u)-shuffle is a permutation ϕ of $p+u$ for which $\phi 1 < \phi 2 < \ldots < \phi p$, and $\phi(p+1) < \phi(p+2) < \ldots < \phi(p+u)$

A morphism of $\mathcal{E}$ from $(f\colon\ n \to p,u,\phi)$ to $(f'\colon\ n' \to p',u',\phi')$ exists only when $n' = n$ and $u' = u$ It is written

$$(4.26)\qquad (\xi,h,\theta)$$

where $\xi\colon\ n \to n$ and $\theta\colon\ u \to u$ are permutations, and $h\colon\ p \to p'$ is any map such that the following diagram commutes

$$\begin{tikzcd} n \arrow[r, "f"] \arrow[d, "\xi"'] & p \arrow[d, "h"] \\ n \arrow[r, "f'"'] & p' \end{tikzcd}.$$

The composite of (ξ,h,θ) and (ξ',h',θ') is $(\xi'\xi,h'h,\theta'\theta)$. The morphism $(1,1,1)$ is the identity for any object.

The $\underline{\underline{P}}$-action on $\mathcal{E}$ is given by

$$(f\colon\ n \to p,u,\phi) \otimes (f'\colon\ n' \to p',u',\phi') =$$
$$(f+f'\colon\ n+n' \to p+p',\ u+u',\ \phi+\phi')$$

and

$$I = (0 \to 0,\ 0,\ 1)$$

where $f+f'$, for example, is the tensor product of f and f' in $\underline{\underline{S}}$.

The functor $\varepsilon\colon\ \underline{\underline{P}} \to \mathcal{E}$ is given by

$$\varepsilon n = (\Omega\colon\ n \to 1,\ 0,\ 1)$$

where Ω is the unique morphism $n \to 1$ in $\underline{\underline{S}}$; and

$$\varepsilon(\xi\colon\ n \to n) = (\xi, 1, 1).$$

The $\underline{\underline{P}}$-functor structure on ε is defined by

$$\bar{\varepsilon}(\otimes; n, m) = \tilde{\varepsilon} = \begin{tikzcd} n+m \arrow[r, "\Omega+\Omega"] \arrow[d, "1"'] & 1+1 \arrow[d, "\Omega"] & , & 0 \arrow[d] & , & 1 \\ n+m \arrow[r, "\Omega"'] & 1 & , & 0 & , & 1 \end{tikzcd}$$

$$\bar{\varepsilon}(I;-) = \varepsilon^{o} = \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] & , & 0 \arrow[d] & , & 1 \\ 0 \arrow[r] & 1 & , & 0 & , & 1. \end{tikzcd}$$

The objects of $\mathcal{B}$ are the same as those of $\mathcal{E}$. The morphisms of $\mathcal{B}$ are those of $\mathcal{E}$ for which ξ and θ are 1, h is increasing, and the following condition holds. For each $i \leqslant p$, and $0 < j, k \leqslant u$ such that $\phi(p+j) < \phi i$, $\phi i < \phi(p+k)$, ϕ' is such that $\phi'(q+j) \leqslant \phi' hi$ and $\phi' hi < \phi'(q+k)$. The $\underline{\underline{N}}$-action on $\mathcal{B}$ is the restriction of the $\underline{\underline{P}}$-action on $\mathcal{E}$; and β and $\bar{\beta}$ are the restrictions of ε and $\bar{\varepsilon}$.

Lemma 4.7: The morphisms of E are generated by

(4.27)
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] & , & u \arrow[d, "\theta"] & , & 1 \\ 0 \arrow[r] & 0 & , & u & , & 1 & , \end{tikzcd}$$

(4.28)
$$\begin{tikzcd} n \arrow[r, "f"] \arrow[d, "1"'] & p \arrow[d, "1"] & , & u \arrow[d, "1"] & , & \phi \\ n \arrow[r, "f"'] & p & , & u & , & \phi' & , \end{tikzcd}$$

(4.29)
$$\begin{tikzcd} p(n_{\eta 1}\ldots n_{\eta p}) \arrow[r, "p(\Omega\ldots\Omega)"] \arrow[d, "\eta(n_1\ldots n_p)"'] & p \arrow[d, "\eta"] & , & 0 \arrow[d] & , & 1 \\ p(n_1\ldots n_p) \arrow[r, "p(\Omega\ldots\Omega)"'] & p & , & 0 & , & 1 & , \end{tikzcd}$$

(4.30)
$$\begin{tikzcd} p(n_1\ldots n_p) \arrow[r, "p(\Omega\ldots\Omega)"] \arrow[d, "1"'] & p \arrow[d] & , & 0 \arrow[d] & , & 1 \\ p(n_1\ldots n_p) \arrow[r, "\Omega"'] & 1 & , & 0 & , & 1 & \text{and} \end{tikzcd}$$

(4.31)
$$\begin{tikzcd} n \arrow[r, "\Omega"] \arrow[d, "\xi"'] & 1 \arrow[d] & , & 0 \arrow[d] & , & 1 \\ n \arrow[r, "\Omega"'] & 1 & , & 0 & , & 1 \ . \end{tikzcd}$$

Proof: A typical morphism of $\mathcal{E}$

(4.32)
$$\begin{tikzcd} n \arrow[r, "f"] \arrow[d, "\xi"'] & p \arrow[d, "h"] & , & u \arrow[d, "\theta"] & , & \phi \\ n \arrow[r, "g"'] & q & , & u & , & \phi' \end{tikzcd}$$

can be written as the composite

(4.33)
$$\begin{tikzcd} n \arrow[r, "f"] \arrow[d, "1"'] & p \arrow[d, "1"] & , & u \arrow[d, "1"] & , & \phi \\ n \arrow[r, "f"] \arrow[d, "1"'] & p \arrow[d, "1"] & , & u \arrow[d, "\theta"] & , & 1 \\ n \arrow[r, "f"] \arrow[d, "\xi"'] & p \arrow[d, "h"] & , & u \arrow[d, "1"] & , & 1 \\ n \arrow[r, "g"] \arrow[d, "1"'] & q \arrow[d, "1"] & , & u \arrow[d, "1"] & , & 1 \\ n \arrow[r, "g"] & q & , & u & , & \phi' \ . \end{tikzcd}$$

The first and fourth of the four factors of (4.33) are instances of (4.28). But the second factor is the tensor product

$$\left(\begin{tikzcd} n \arrow[r, "f"] \arrow[d, "1"'] & p \arrow[d, "1"] & , & 0 \arrow[d] & , & 1 \\ n \arrow[r, "f"'] & p & , & 0 & , & 1 \end{tikzcd} \right) \otimes \left(\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] & , & u \arrow[d, "\theta"] & , & 1 \\ 0 \arrow[r] & 0 & , & u & , & 1 \end{tikzcd} \right)$$

$$= 1 \otimes \left(\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] & , & u \arrow[d, "\theta"] & , & 1 \\ 0 \arrow[r] & 0 & , & u & , & 1 \end{tikzcd} \right) \quad ,$$

an expansion of an instance of (4.27).

The third factor is the tensor product

$$\left(\begin{tikzcd} n \arrow[r, "f"] \arrow[d, "\xi"'] & p \arrow[d, "h"] & , & 0 \arrow[d] & , & 1 \\ n \arrow[r, "g"'] & q & , & 0 & , & 1 \end{tikzcd} \right) \otimes \left(\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] & , & u \arrow[d, "1"] & , & 1 \\ 0 \arrow[r] & 0 & , & u & , & 1 \end{tikzcd} \right)$$

i.e. an expansion of

$$(4.34)\qquad \begin{array}{ccccc} n & \xrightarrow{f} & p & , & 0 & , & 1 \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle h} & & \downarrow & & \\ n & \xrightarrow[g]{} & q & , & 0 & , & 1 \end{array} \quad .$$

We will show that (4.34) is generated by (4.29), (4.30) and (4.31). We will only consider the square diagrams, for example (4.35) instead of (4.34)

$$(4.35)\qquad \begin{array}{ccc} n & \xrightarrow{f} & p \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle h} \\ n & \xrightarrow[g]{} & q \end{array} \quad .$$

Every increasing map $f\colon\ n \to p$ can be written as

$$(4.36)\qquad p(\Omega,\ldots\Omega)\colon\ n = p(n_1\ldots n_p) \to p(1\ldots 1) = p$$

for a unique selection $n_1,\ldots,n_p$. Also __any__ map $h\colon\ p \to q$ may be written

$$(4.37)\qquad p \overset{\eta}{\to} p = q(p_1\ldots p_q) \xrightarrow{q(\Omega\ldots\Omega)} q$$

for a unique selection $p_1,\ldots,p_q$, and a not necessarily unique permutation η.

The following diagram in $\underline{\underline{S}}$ commutes

$$
\text{(4.38)} \qquad
\begin{tikzcd}
n = p(n_{\eta 1}\ldots n_{\eta p}) \arrow[r, "p(\Omega\ldots\Omega)"] \arrow[d, "\eta(n_1\ldots n_p)"'] & p \arrow[d, "\eta"] \\
p(n_1\ldots n_p) \arrow[r, "p(\Omega\ldots\Omega)"] \arrow[d, "="'] & p = q(p_1\ldots p_q) \arrow[d, "q(\Omega\ldots\Omega)"] \\
q(p_1(n_1\ldots)\ldots p_q(\ldots n_p)) \arrow[r, "q(\Omega\ldots\Omega)"] \arrow[d, "q(\xi_1\ \ldots\ \xi_q)"'] & q \arrow[d] \\
n = q(p_1(n_1\ldots)\ldots p_q(\ldots n_p)) \arrow[r, "q(\Omega\ldots\Omega)"] & q
\end{tikzcd}
$$

for any permutations ξ_i. But by (4.36) and (4.37) the right leg of (4.38) is hf. By (4.35) $hf = g\xi$, so that $g = q(\Omega\ldots\Omega)$ by (4.37). For any $x \in n$ suppose $hfx = i$. By (4.38) $\eta(n_1\ldots n_p)x \in p_i(\ldots n_j\ldots)$. But $g\xi x = i$, so that $\xi x \in p_i(\ldots n_j\ldots)$ also. Thus there exist permutations ξ_i of $p_i(\ldots n_j\ldots)$ so that $\xi = q(\xi_1\ldots\xi_q) \,.\, \eta(n_1\ldots n_p)$. Thus diagram (4.35) is the same as diagram (4.38).

It is now sufficient to show that each of the three factors of (4.38) is generated by (4.29), (4.30) and (4.31).

The first factor is already an instance of (4.29). The second factor is tensor product of the morphisms $1 \leq i \leq q$,

$$\begin{tikzcd}
p_i(\ldots n_j \ldots) \arrow[r, "p_i(\Omega\ldots\Omega)"] \arrow[d, "1"'] & p_i \arrow[d, "\Omega"] \\
p_i(\ldots n_j \ldots) \arrow[r, "\Omega"'] & 1
\end{tikzcd} ,$$

all instances of (4.30). The third factor is a tensor product of the morphisms $1 \leq i \leq q$

$$\begin{tikzcd}
p_i(\ldots n_j \ldots) \arrow[r, "\Omega"] \arrow[d, "\xi_i"'] & 1 \arrow[d, "1"] \\
p_i(\ldots n_j \ldots) \arrow[r, "\Omega"'] & 1
\end{tikzcd} ,$$

all instances of (4.31).

This completes the proof of Lemma 4.7. $\square$

<u>Lemma 4.8</u>: <u>The morphisms of $\mathcal{B}$ are generated by</u>

(4.39) $$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd} , \quad \begin{tikzcd}
1 \arrow[d, "1"] \\
1
\end{tikzcd} , \quad \begin{matrix} 1 \\ \\ 1 \end{matrix}$$ <u>and</u>

(4.40)
$$\begin{tikzcd} p(n_1\ldots n_p) \arrow[r, "p(\Omega\ldots\Omega)"] \arrow[d, "1"'] & p \arrow[d, "\Omega"] & , & 0 \arrow[d] & , & 1 \\ p(n_1\ldots n_p) \arrow[r, "\Omega"'] & 1 & , & 0 & , & 1 \end{tikzcd} .$$

Proof: Consider a morphism of $\mathcal{B}$ of the form

(4.41)
$$\begin{tikzcd} n \arrow[r, "f"] \arrow[d, "1"'] & p \arrow[d, "h"] & , & 0 \arrow[d] & , & 1 \\ n \arrow[r, "g"'] & q & , & 0 & , & 1 \end{tikzcd} ,$$

By (4.36) this may be written

$$\begin{tikzcd} q(p_1(n_1\ldots)\ldots p_q(\ldots n_p)) \arrow[r, "q(p_1(\Omega\ldots)\ldots p_q(\ldots\Omega))"] \arrow[d, "1"'] & q(p_1\ldots p_q) \arrow[d, "q(\Omega\ldots\Omega)"] & ,0 \arrow[d] & , 1 \\ q(p_1(n_1\ldots)\ldots p_q(\ldots n_p)) \arrow[r, "q(\Omega\ldots\Omega)"'] & q & ,0 & , 1 \end{tikzcd}$$

which is a tensor product of the morphisms $1 \leqslant i \leqslant q$

$$\begin{tikzcd} p_i(\ldots n_j\ldots) \arrow[r, "p_i(\ldots\Omega\ldots)"] \arrow[d, "1"'] & p_i \arrow[d, "\Omega"] & , & 0 \arrow[d] & , & 1 \\ p_i(\ldots n_j\ldots) \arrow[r, "\Omega"'] & 1 & , & 0 & , & 1 \end{tikzcd}$$

all instances of (4.40).

Because of the condition relating ϕ, ϕ' and h, a general morphism of $\mathcal{B}$

$$\begin{tikzcd} n \arrow[r, "f"] \arrow[d, "1"'] & p \arrow[d, "h"] & , & u \arrow[d] & , & \phi \\ n \arrow[r, "g"'] & q & , & u & , & \phi' \end{tikzcd}$$

may be readily "disentangled" as the tensor product of morphisms like (4.41) (with the "u" part = 0), and

$$(4.42)\qquad \begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] & , & u_i \arrow[d, "1"] & , & 1 \\ 0 \arrow[r] & 0 & , & u_i & , & 1 \end{tikzcd} \; .$$

But (4.42) is the tensor product of u_i of (4.39).

This completes the proof of Lemma 4.8. □

<u>4.6</u> Define the strict symmetric monoidal functors $\Gamma_{1,\mathcal{E}}$: $\mathcal{E} \to \underline{\underline{P}}$, $\Gamma_{2,\mathcal{E}}$: $\mathcal{E} \to \underline{\underline{P}}$ and $\Gamma_{3,\mathcal{E}}$: $\mathcal{E} \to \underline{\underline{S}}$ on (4.25) by n, u and p respectively, and on (4.26) by ξ, θ and h respectively. Let the strict monoidal functors $\Gamma_{1,\mathcal{B}}$: $\mathcal{B} \to \underline{\underline{P}}$, $\Gamma_{2,\mathcal{B}}$: $\mathcal{B} \to \underline{\underline{P}}$ and $\Gamma_{3,\mathcal{B}}$: $\mathcal{B} \to \underline{\underline{S}}$ be the restrictions of the $\Gamma_{i,\mathcal{E}}$ to $\mathcal{B}$.

Let us write $\hat{\underline{\underline{P}}}$ as $(\pi,\bar{\pi})$: $\underline{\underline{P}} \to \underline{\underline{P}}'$, and $\hat{\underline{\underline{N}}}$ as $(\nu,\bar{\nu})$: $\underline{\underline{N}} \to \underline{\underline{N}}'$. By (3.3) there are <u>strict</u> symmetric monoidal functors ρ' and ρ, and <u>strict</u> monoidal functors σ' and σ rendering commutative

(4.43)
$$\begin{tikzcd} \underline{\underline{P}} \arrow[r, "\rho'"] \arrow[d, "(\pi{,}\bar{\pi})"'] & \underline{\underline{P}} \arrow[d, "(\varepsilon{,}\bar{\varepsilon})"] \\ \underline{\underline{P}}' \arrow[r, "\rho"'] & \mathcal{E} \end{tikzcd}$$

(4.44)
$$\begin{tikzcd} \underline{\underline{N}} \arrow[r, "\sigma'"] \arrow[d, "(\nu{,}\bar{\nu})"'] & \underline{\underline{N}} \arrow[d, "(\beta{,}\bar{\beta})"] \\ \underline{\underline{N}}' \arrow[r, "\sigma"'] & \mathcal{B} \end{tikzcd}$$

and such that $\rho'(\underline{\underline{1}}) = 1$, $\sigma'(\underline{\underline{1}}) = 1$, and $\rho(\underline{\underline{1}}') = \sigma(\underline{\underline{1}}') = (0 \to 0, 1, 1)$. But ρ' and σ' are the identity functors. In the remainder of this section we shall show that ρ and σ are isomorphisms.

<u>Lemma 4.9</u>: <u>Both ρ and σ are bijections on objects.</u>

<u>Proof</u>: Since ρ and σ have the same underlying object function, we only consider ρ.

Any object of $\mathcal{E}$ can be written uniquely as

$$(4.45) \qquad p(\Omega\ldots\Omega):\ p(n_1\ldots n_p) \to p,\ \ u,\ \phi$$

which is of the form

$$(4.46) \qquad (p+u)(E_{\phi 1}\ldots E_{\phi(p+u)})$$

where $E_{\phi i} = (0 \to 0,\ 1,\ 1)$ if $i > p$,

and $E_{\phi i} = (\Omega:\ n_i \to 1,\ 0,\ 1)$ if $i \leqslant p$.

But $\rho(\underline{\underline{1}}') = (0 \to 0,\ 1,\ 1)$ and $\rho(\pi n) = (\Omega:\ n \to 1,\ 0,\ 1)$. Thus ρ maps the generators of $\underline{\underline{P}}'$ to the generators of $\mathcal{E}$. Therefore ρ is bijective on objects. □

Lemma 4.10: Both ρ and σ are surjective on morphisms.

Proof: We know the generators of the morphisms of $\mathcal{E}$ and of $\mathcal{B}$ by Lemma 4.7 and Lemma 4.8. We just need to check that each such generator is the image of a morphism of $\underline{\underline{P}}'$ and $\underline{\underline{N}}'$ respectively:

$$(4.27) \quad \text{is} \quad \rho\left(\begin{array}{c} u(\underline{\underline{1}}'\ldots\underline{\underline{1}}') \\ \big\downarrow \theta(\underline{\underline{1}}'\ldots\underline{\underline{1}}') \\ u(\underline{\underline{1}}'\ldots\underline{\underline{1}}') \end{array}\right);$$

$$(4.28) \quad \text{is} \quad \rho\left(\begin{array}{c} (p+u)(E_{\phi 1}\ldots E_{\phi(p+u)}) \\ \Big\downarrow \phi(\phi')^{-1}(E_{\phi' 1}\ldots E_{\phi'(p+u)}) \\ (p+u)(E_{\phi' 1}\ldots E_{\phi'(p+u)}) \end{array}\right)$$

using the notation of (4.46);

(4.29) is $\rho \left(\begin{array}{c} p(\pi n_{\eta 1} \ldots \pi n_{\eta p}) \\ \Big\downarrow \eta(\pi n_1 \ldots \pi n_p) \\ p(\pi n_1 \ldots \pi n_p) \end{array} \right)$;

(4.30) is $\rho \left(\begin{array}{c} p(\pi n_1 \ldots \pi n_p) \\ \Big\downarrow \bar{\pi}(p;\ n_1 \ldots n_p) \\ \pi(n_1 + \ldots + n_p) \end{array} \right)$;

(4.31) is $\rho(\pi\xi\colon\ \pi n \to \pi n)$;

(4.39) is $\sigma(1\colon\ \underline{\underline{1}}' \to \underline{\underline{1}}')$; and

(4.40) is $\sigma \left(\begin{array}{c} p(\nu n_1 \ldots \nu n_p) \\ \Big\downarrow \bar{\nu}(p;\ n_1 \ldots n_p) \\ \nu(n_1 + \ldots + n_p) \end{array} \right)$. □

Lemma 4.11: Both ρ and σ are faithful.

Proof: We now use the $\Gamma_{i,E}$ and $\Gamma_{i,B}$ constructed at the beginning of this section.

We know that $\Gamma_{i,\underline{\underline{P}}}(\underline{\underline{1}}')$ are 0, 1 and 0 respectively. But $\Gamma_{i,\mathcal{E}} \cdot \rho$ applied to $\underline{\underline{1}}'$ also yield 0, 1, 0. From the uniqueness of $\Gamma_{i,\underline{\underline{P}}}$, it follows that $\Gamma_{i,\mathcal{E}} \cdot \rho = \Gamma_{i,\underline{\underline{P}}}$ for $i = 1, 2, 3$. Similarly $\Gamma_{i,\mathcal{B}} \cdot \sigma = \Gamma_{i,\underline{\underline{N}}}$ for $i = 1, 2, 3$.

Suppose $z, y\colon\ Z \to Y$ are morphisms of $\underline{\underline{P}}'$ such that $\rho z = \rho y$. Then $\Gamma_{i,\mathcal{E}}(\rho z) = \Gamma_{i,\mathcal{E}}(\rho y)$, that is, $\Gamma_{i,\underline{\underline{P}}}(z) = \Gamma_{i,\underline{\underline{P}}}(y)$ for $i = 1, 2, 3$. But $\Gamma\colon\ \underline{\underline{P}} \to \underline{\underline{P}}$ is the identity so Corollary 4.6 applies. Consequently $z = y$ and ρ is faithful.

It follows similarly for any morphisms $f, g\colon\ T \to S$ of $\underline{\underline{N}}'$ with $\sigma f = \sigma g$, that $\Gamma_{i,\underline{\underline{N}}}(f) = \Gamma_{i,N}(g)$ for $i = 1, 2, 3$. However $\underline{\underline{N}}$ readily satisfies the conditions of Theorem 4.5, $\Gamma\colon\ \underline{\underline{N}} \to \underline{\underline{P}}$ being the inclusion. Thus the $\Gamma_{i,\underline{\underline{N}}}$ are jointly faithful so $f = g$ and σ is faithful. □

Theorem 4.12: *$\hat{\underline{\underline{P}}}$ is isomorphic to $(\varepsilon, \bar{\varepsilon})\colon\ \underline{\underline{P}} \to \mathcal{E}$, and $\hat{\underline{\underline{N}}}$ is isomorphic to $(\beta, \bar{\beta})\colon\ \underline{\underline{N}} \to \mathcal{B}$.*

Proof: By Lemmas 4.9, 4.10 and 4.11, ρ and σ are isomorphisms. In diagrams (4.43) and (4.44), ρ' and σ' are known to be identity functors. □

4.7 Let us write $\hat{P}$ as $(P, \bar{P})\colon\ P \to P'$ and $\hat{N}$ as $(N, \bar{N})\colon\ N \to N'$

We know that $\underline{\underline{P}}$ and $\underline{\underline{P}}'$ are $\mathcal{P}$-categories (symmetric monoidal categories) and that $(\pi,\bar{\pi})$ is a $\mathcal{P}$-functor (symmetric monoidal functor). Thus if we let $\mathcal{K} = \mathcal{P}$ in (3.3) we know that there exist <u>unique</u> <u>strict</u> symmetric monoidal functors U,V rendering commutative

$$\begin{tikzcd} \mathcal{P} \arrow[r, "U"] \arrow[d, "{(P,\bar{P})}"'] & \underline{\underline{P}} \arrow[d, "{(\pi,\bar{\pi})}"] \\ \mathcal{P}' \arrow[r, "V"'] & \underline{\underline{P}}' \end{tikzcd}$$

such that $U(\underline{\underline{1}}) = \underline{\underline{1}}$ and $V(\underline{\underline{1}}') = \underline{\underline{1}}'$.

Similarly by considering $\mathcal{K} = \mathcal{N}$, there exist <u>unique</u> <u>strict</u> monoidal functors W, X rendering commutative

$$\begin{tikzcd} \mathcal{N} \arrow[r, "W"] \arrow[d, "{(N,\bar{N})}"'] & \underline{\underline{N}} \arrow[d, "{(\nu,\bar{\nu})}"] \\ \mathcal{N}' \arrow[r, "X"'] & \underline{\underline{N}}' \end{tikzcd}$$

such that $W(\underline{\underline{1}}) = \underline{\underline{1}}$ and $X(\underline{\underline{1}}') = \underline{\underline{1}}'$.

<u>Theorem 4.13</u>: <u>$\hat{\mathcal{P}}$ is equivalent to $\hat{\underline{\underline{P}}}$, and $\hat{\mathcal{N}}$ is equivalent to $\hat{\underline{\underline{N}}}$.</u>

Proof: We will show that U, V, W, X are equivalences. But we already know that U and W are equivalences by the work of Mac Lane [14].

It is clear that both V and X are surjective on both objects and morphisms. It remains to show that V and X are faithful.

For i = 1, 2, 3, $\Gamma_{i,\underline{\underline{P}}} \cdot V(\underline{\underline{1}}') = \Gamma_{i,\mathcal{P}}(\underline{\underline{1}}')$. But by the uniqueness of $\Gamma_{i,\mathcal{P}}$ it follows that $\Gamma_{i,\mathcal{P}} = \Gamma_{i,\underline{\underline{P}}} \cdot V$. If the morphisms $z, y\colon\ Z \to Y$ in $\mathcal{P}'$ have $Vz = Vy$, then $\Gamma_{i,\mathcal{P}}(z) = \Gamma_{i,\underline{\underline{P}}}(Vz) = \Gamma_{i,\underline{\underline{P}}}(Vy) = \Gamma_{i,\mathcal{P}}(y)$. But $\Gamma\colon\ \mathcal{P} \to \underline{\underline{P}}$ is full and faithful, so by Corollary 4.6, the $\Gamma_{i,\mathcal{P}}$ are jointly faithful. Therefore $z = y$ and V is faithful.

For morphisms $f, g\colon\ T \to S$ in $\mathcal{N}'$ with $Xf = Xg$, we obtain in the same way that $\Gamma_{i,\mathcal{N}}(f) = \Gamma_{i,\mathcal{N}}(g)$ for i = 1, 2, 3. But $\mathcal{N}$ easily satisfies the conditions of Theorem 4.5 so the $\Gamma_{i,\mathcal{N}}$ are jointly faithful. Consequently X is faithful. $\square$

5. The closed categories G and C'

5.1 We shall confine our study of mixed-variance clubs to the case $K = C$. Since our main Theorem 6.11 involves functors Γ_1: $C' \to T$, Γ_2: $C' \to T$, Γ_3: $C' \to G$ it is necessary that we investigate the closed categories C', T and G.

5.2 We begin with a study of G.

Let the objects of G be the finite lists of the signs + and −. Of course the empty list, which we shall write 0, is an object of G. For any list μ, let $-\mu$ be the list with all signs changed. Let $\{\mu,\nu\}$ be the list consisting of the elements of μ followed by those of ν. Let μ_+ (respectively μ_-) be the set of + elements (respectively −) of μ. A non-trivial morphism from μ to ν is a function from $\{\mu,-\nu\}_+$ to $\{\mu,-\nu\}_-$. For every pair of objects μ,ν let there be a morphism $*$: $\mu \to \nu$ called the trivial morphism.

We say that the non-trivial morphisms f: $\mu \to \nu$ and g: $\nu \to \pi$ are incompatible (written $f \nmid g$) if there is a subset

$$(5.1) \qquad \nu_1, \nu_2, \ldots, \nu_{2n} \qquad\qquad n \geqslant 1$$

of the elements of ν, such that f maps ν_i to ν_{i+1} for i odd, and g maps ν_i to ν_{i+1} for i even ($g(\nu_{2n}) = \nu_1$). Otherwise f and g are compatible (written $f \sim g$).

If $f \sim g$ we define gf: $\mu \to \pi$. Consider the sequence

$$(5.2) \qquad x_0, x_1, \ldots, x_n \qquad\qquad n \geq 1$$

where $x_0 \in \mu_+$ or π_-; and x_{i+1} is $f(x_i)$ if $x_i \in \mu_+$ or ν_-, and x_{i+1} is $g(x_i)$ if $x_i \in \nu_+$ or π_-; and x_n is the first x_i in π_+ or μ_-. We define the composite gf to be the map which sends each x_0 to x_n as in (5.2). If $f \nmid g$ or f or g is trivial, let gf be the trivial morphism.

Lemma 5.1: Suppose f: $\mu \to \nu$, g: $\nu \to \pi$ and h: $\pi \to \rho$ are non-trivial morphisms of G. Then $f \sim g$ and $gf \sim h$, if and only if, $g \sim h$ and $f \sim hg$. (We write this as $f \sim g \sim h$.)

Proof: Suppose $f \nmid g$. Then there exists a sequence (5.1). But if $g \sim h$, hg maps ν_i to ν_{i+1} for i even. Thus $f \nmid hg$.

Suppose $f \sim g$, but $gf \nmid h$. Then there exist

$$\pi_1, \ldots, \pi_{2n} \qquad\qquad n \geq 1$$

elements of π, such that gf maps π_i to π_{i+1}(i odd) and h maps π_i to π_{i+1} (i even). Since gf maps π_{2i-1} to π_{2i}, there exists a sequence (possibly empty)

$$\nu_{i1}, \nu_{i2}, \ldots, \nu_{ir_i} \qquad\qquad r_i \text{ even}$$

of elements of ν, such that $g(\pi_{2i-1}) = \nu_{i1}$, $g(\nu_{ir_i}) = \pi_{2i}$, $g(\nu_{ij}) = \nu_{i,j+1}$ if j is even; and $f(\nu_{ij}) = \nu_{i,j+1}$ if j is odd. If $r_i = 0$ for all i then $g \dagger h$. Suppose r_i is not always 0. Then the sequence

$$\nu_{11}, \ldots \nu_{1r_1}, \nu_{21}, \ldots, \nu_{2r_2}, \ldots, \nu_{nr_n}$$

shows that $f \dagger hg$. The converse is proved in exactly the same way. $\square$

<u>Lemma 5.2</u>: <u>Suppose $f: \mu \to \nu$, $g: \nu \to \pi$ and $h: \pi \to \rho$ are non-trivial morphisms of $\mathcal{G}$ such that $f \sim g \sim h$. Then $h(gf) = (hg)f$.</u>

<u>Proof</u>: Consider the sequence

$$x_0, x_1, \ldots x_n \qquad\qquad n \geqslant 1$$

where $x_0 \in \mu_+$ or ρ_-; $x_{i+1} = f(x_i)$ if $x_i \in \mu_+$ or ν_-; $x_{i+1} = g(x_i)$ if $x_i \in \nu_+$ or π_-; $x_{i+1} = h(x_i)$ if $x_i \in \pi_+$ or ρ_-; and x_n is the first x_i in ρ_+ or μ_-.

Let

$$x_0 = y_0, y_1, \ldots, y_m = x_n$$

be those x_i which are in μ, ν or ρ. But $y_{i+1} = f(y_i)$ if $y_i \in \mu_+$ or ν_-; and $y_{i+1} = hg(y_i)$ if $y_i \in \nu_+$ or ρ_-. Thus $(hg)f(x_0) = x_n$. Similarly by considering the x_i in μ, π or ρ we find that $h(gf)(x_0) = x_n$. Consequently $(hg)f = h(gf)$. $\square$

For any object μ of G we define the identity morphism $1: \mu \to \mu$. It is a function from $\{\mu,-\mu\}_+$ to $\{\mu,-\mu\}_-$, i.e. from $\mu_+ \cup (-\mu)_+$ to $\mu_- \cup (-\mu)_-$. Let it be the function comprising the identity maps from μ_+ to $(-\mu)_-$ and from $(-\mu)_+$ to μ_-.

Given objects μ,ν of G, define $\mu\otimes\nu$ to be $\{\mu,\nu\}$, and $[\mu,\nu]$ to be $\{-\mu,\nu\}$. Given non-trivial morphisms $f: \mu \to \pi$ and $g: \nu \to \rho$ we want to define $f\otimes g: \mu\otimes\nu \to \pi\otimes\rho$ and $[f,g]: [\pi,\nu] \to [\mu,\rho]$. Now $f\otimes g$ will be a map from $\{\mu,\nu,-\pi,-\rho\}_+$ to $\{\mu,\nu,-\pi,-\rho\}_-$, i.e. from $\{\mu,-\pi\}_+ \cup \{\nu,-\rho\}_+$ to $\{\mu,-\pi\} \cup \{\nu,-\rho\}_-$. We define $f\otimes g$ to be the morphism which acts as f on $\{\mu,-\pi\}_+$ and as g on $\{\nu,-\rho\}_+$. We can similarly define $[f,g]$. If one or both of f and g is trivial we define $f\otimes g$ and $[f,g]$ as the respective trivial maps.

If $f\sim h$ and $g\sim k$ then $f\otimes g\sim h\otimes k$ and $h\otimes k.f\otimes g = hf\otimes kg$. Thus $\otimes$ is a functor. Similarly $[\ ,\ ]$ is a functor.

<u>Proposition 5.3</u>: <u>G is a closed category.</u>

<u>Proof</u>: We have already defined composition of morphisms, identity morphisms, tensor product and internal hom functors. Let the identity object of G be the empty list 0.

Lemma 5.2 states that composition is associative when $f\sim g\sim h$. Consider the cases when we do not have $f\sim g\sim h$. If any of f,g and h is $*$, then so are both $(hg)f$ and $h(gf)$.

So suppose that at least one of $f \nmid g$, $gf \nmid h$, $g \nmid h$ and $f \nmid hg$. But by Lemma 4.1 we deduce that $(hg)f = h(gf) = *$. Thus composition is always associative. Since the identity morphisms readily satisfy the category axioms, G is a category.

Since $(\mu \otimes \nu) \otimes \rho = \mu \otimes (\nu \otimes \rho)$ the associativity isomorphism is the identity morphism. Also since $\mu \otimes 0 = \mu = 0 \otimes \mu$, the left and right identity isomorphisms are also the identity morphisms. The commutativity isomorphism $c: \ \mu \otimes \nu \to \nu \otimes \mu$ is a map from $\{\mu, \nu, -\nu, -\mu\}_+$ to $\{\mu, \nu, -\nu, -\mu\}_-$, and comprises the identity maps $\{\mu, -\mu\}_+$ to $\{\mu, -\mu\}_-$, and from $\{\nu, -\nu\}_+$ to $\{\nu, -\nu\}_-$.

The maps $d: \ \mu \to [\nu, \mu \otimes \nu]$ and $e: \ [\mu, \nu] \otimes \mu \to \nu$ are respectively the maps from the + elements to the − elements of respectively $\{\mu, \nu, -\mu, -\nu\}$ and $\{-\mu, \nu, \mu, -\nu\}$, induced from the evident identity maps.

It is easy to check that the relevant axiom-diagrams [see Eilenberg-Kelly [2]] commute so G is a closed category. □

We can see that T is a closed subcategory of G with the same objects. The non-trivial morphisms of T are those morphisms of G whose underlying functions are <u>bijections</u>.

Let a <u>central morphism</u> $f: \ \mu \to \nu$ of G be a morphism of T which maps μ_+ bijectively to ν_+, and ν_- bijectively to μ_-.

It is easy to see that if $g\colon \nu \to \pi$ is a non-trivial morphism of G, and $f\colon \mu \to \nu$ and $h\colon \pi \to \rho$ are central, then $f \sim g \sim h$.

Note that the centrals are the smallest set of the morphisms of G containing $1, a, c, \ell, r$ and closed under $\otimes$ and $[\ ,\ ]$.

<u>5.3</u> Define the closed functor $(\sigma,\bar{\sigma})\colon I \to G$ by: $\sigma(*) = +$; $\tilde{\sigma}(\otimes;*,*)\colon +,+ \to +$ is the unique such morphism in G; and $\sigma^{0}\colon 0 \to +$ is also the unique such morphism in G. Let $(\beta,\bar{\beta})\colon I \to G$ be the closed functor with $\beta(*) = 0$.

We know that there exist unique strict symmetric monoidal functors Λ_1, Λ_2, Λ_3, Γ_1, Γ_2 and Γ_3 rendering commutative

$$\begin{array}{ccc} C & \xrightarrow{\Lambda_1} & G \\ {\scriptstyle (\gamma,\bar{\gamma})}\downarrow & & \downarrow{\scriptstyle (1,1)} \\ C' & \xrightarrow[\Gamma_1]{} & G' \end{array} ,$$

$$\begin{array}{ccc} C & \xrightarrow{\Lambda_2} & I \\ {\scriptstyle (\gamma,\bar{\gamma})}\downarrow & & \downarrow{\scriptstyle (\beta,\bar{\beta})} \\ C' & \xrightarrow[\Gamma_2]{} & G \end{array} ,$$

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Lambda_3"] \arrow[d, "{(\gamma,\bar{\gamma})}"'] & \mathcal{I} \arrow[d, "{(\sigma,\bar{\sigma})}"] \\
\mathcal{C}' \arrow[r, "\Gamma_3"'] & \mathcal{G}
\end{tikzcd}$$

and such that $\Lambda_1(\underline{\underline{1}}) = 1$, $\Lambda_2(\underline{\underline{1}}) = *$, $\Lambda_3(\underline{\underline{1}}) = *$, $\Gamma_1(\underline{\underline{1}}') = 0$, $\Gamma_2(\underline{\underline{1}}') = +$ and $\Gamma_3(\underline{\underline{1}}') = 0$. We know that Λ_1: $\mathcal{C} \to \mathcal{G}$ is the graph functor of Kelly-Mac Lane [8].

<u>5.4</u> We now turn our attention to the category $\mathcal{C}'$. We shall define the constructible morphisms of $\mathcal{C}'$. Our aim is to show that all morphisms of $\mathcal{C}'$ satisfy this definition. In Chapter 6, we shall use this characterization of the morphisms of $\mathcal{C}'$.

The objects of $\mathcal{C}'$ may be considered to be $T\{X_1,\ldots,X_m\}$ where $T \in \mathcal{C}$ and each X_i is $\underline{\underline{1}}'$ or γA_i for $A_i \in \mathcal{C}$. Mentions of $\otimes$ and $[\ ,\ ]$ in T correspond to $\otimes'$ and $[\ ,\ ]'$ in $T(X_1,..,X_m)$. Having been inspired by this correspondence, we shall frequently abbreviate $\otimes'$ and $[\ ,\ ]'$ to $\otimes$ and $[\ ,\ ]$.

However, we find it more useful to factorize the objects of $\mathcal{C}'$ as

(5.3) $P(Z_1,\ldots,Z_n)$

where $P \in \mathcal{P}$ and each Z_i is either $\underline{\underline{1}}'$; γA_i for $A_i \in \mathcal{C}$; or $[X_i,Y_i]'$ for $X_i,Y_i \in \mathcal{C}'$. We call such Z_i the <u>prime factors</u> of (5.3). We write I' for the identity object $\underline{I}(-)$ of $\mathcal{C}'$.

If f: $A \otimes B \to C$ is a morphism of $\mathcal{C}$, and z: $Z \otimes Y \to X$ is a morphism of $\mathcal{C}'$ denote by πf and πz the respective composites

$$A \xrightarrow{d} [B, A\otimes B] \xrightarrow{[1,f]} [B,C],$$

$$Z \xrightarrow{d'} [Y, Z\otimes Y] \xrightarrow{[1,z]} [Y,X].$$

If g: $A \to B$ and y: $Z \to Y$ are morphisms of $\mathcal{C}$ and $\mathcal{C}'$ respectively, anc C and X are objects of $\mathcal{C}$ and $\mathcal{C}'$, denote by $\langle g\rangle_C$ and $\langle y\rangle_X$ (usually abbreviated to $\langle g\rangle$ and $\langle y\rangle$) the respective composites

$$[B,C]\otimes A \xrightarrow{1\otimes g} [B,C]\otimes B \xrightarrow{e} C,$$

$$[Y,X]\otimes Z \xrightarrow{1\otimes y} [Y,X]\otimes Y \xrightarrow{e'} X.$$

We define the <u>central morphisms</u> of $\mathcal{C}'$ to be those of the form

(5.4) $\quad p(Z_1 \ldots Z_n)$: $\quad P(Z_{\xi 1} \ldots Z_{\xi n}) \to Q(Z_1 \ldots Z_n)$

where $P,Q \in \mathcal{P}$, p: $P \to Q$ is a morphism of $\mathcal{P}$ with $\Gamma p = \xi$ and the Z_i are prime.

<u>Lemma 5.4</u>: <u>Let f be the central morphism (5.4) of $\mathcal{C}'$. For $i = 1,2,3$, $\Gamma_i(f)$ is central in G.</u>

Proof: Because the Γ_i are strict $\mathcal{P}$-functors, $\Gamma_i(f) = p(\Gamma_i(Z_1)\ldots\Gamma_i(Z_n))$. These morphisms are clearly central in $\mathcal{G}$. □

For each object Z of $\mathcal{C}$ or $\mathcal{C}'$ we define its rank written r(Z) or rZ. Let

$$r(I) = r(I') = 0,$$

$$r(\underline{\underline{1}}) = r(\underline{\underline{1}}') = 1,$$

$$r(Z\otimes Y) = rZ + rY,$$

$$r([Z,Y]) = rZ + rY + 1,$$

$$r(\gamma Z) = rZ + 1.$$

For each morphism z: $Z \to Y$ of $\mathcal{C}'$, let its rank rz be $rZ + rY$. Note that if z is central $rZ = rY$.

<u>5.5</u> We define the <u>constructible morphisms</u> of $\mathcal{C}'$ to be the smallest class of morphisms of $\mathcal{C}'$ satisfying the following conditions:

(5.5) Every central morphism is in the class;

(5.6) If x: $X \to V$ and y: $W \to U$ are in the class with $r(x) > 0$, $r(y) > 0$, then so is

$$Z \xrightarrow{a} X\otimes W \xrightarrow{x\otimes y} V\otimes U \xrightarrow{b} Y$$

where a and b are central;

(5.7) If y: $Z\otimes X \to W$ is in the class, then so is

$$Z \xrightarrow{\pi y} [X,W] \xrightarrow{b} Y$$

where b is central;

(5.8) If $y\colon X \to W$ and $x\colon V\otimes U \to Y$ are in the class then so is

$$Z \xrightarrow{a} ([W,V]\otimes X)\otimes U \xrightarrow{\langle y\rangle\otimes 1} V\otimes U \xrightarrow{x} Y$$

where a is central;

(5.9) If $P \in \mathcal{P}$ with $\Gamma P = n$ and $A_1,\ldots,A_n$ are objects of $\mathcal{C}$ and $f\colon P(A_1\ldots A_n) \to B$ is a morphism of $\mathcal{C}$, then the following morphism is in the class

$$Z \xrightarrow{a} P(\gamma A_1\ldots\gamma A_n) \xrightarrow{\bar{\gamma}(P;A_1\ldots A_n)} \gamma P(A_1\ldots A_n)$$

$$\xrightarrow{\gamma f} \gamma B \xrightarrow{b} Y$$

where a and b are central.

Suppose $z\colon Z \to Y$ with $r(z) = 0$ is constructible. By elimination z can only be central.

<u>Lemma 5.5</u>: <u>Suppose $z\colon Z \to Y$ is constructible and $a'\colon Z' \to Z$ and $b'\colon Y \to Y'$ are central. Then $b'za'$ is constructible. For $i = 1,2,3$, $\Gamma_i a' \sim \Gamma_i z \sim \Gamma_i b'$.</u>

<u>Proof</u>: The latter part of the lemma follows directly from Lemma 5.4 and the second-last paragraph of §5.2.

It is readily seen from the definition of central morphisms of $\mathcal{C}'$ that the composite of two central morphisms is central. Thus we need only consider the cases:

(i) za' where z is defined by (5.7); and

(ii) $b'z$ where z is defined by (5.8).

We use induction on $r(z)$ assuming that $b'za'$ is constructible for all z with smaller rank.

Case (i):

$$Z' \xrightarrow{a'} Z \xrightarrow{\pi y} [X,W] \xrightarrow{b'b} Y'$$

$$= Z' \xrightarrow{a'} Z \xrightarrow{d'} [X,Z\otimes X] \xrightarrow{[1,y]} [X,W] \xrightarrow{b'b} Y'$$

$$= Z' \xrightarrow{d'} [X,Z'\otimes X] \xrightarrow{[1,a'\otimes 1]} [X,Z\otimes X] \xrightarrow{[1,y]} [X,W] \xrightarrow{b'b} Y'$$

$$= Z' \xrightarrow{\pi(y.a'\otimes 1)} [X,W] \xrightarrow{b'b} Y'$$

which is constructible by (5.7) if $y.a'\otimes 1$ is. But this is so by the induction assumption, $r(y)$ being less than $r(z)$.

Case (ii): $b'za'$

$$= b'x \;.\; \langle y\rangle\otimes 1 \;.\; aa'$$

which is constructible by (5.8) if $b'x$ is. But this is so because $r(x) < r(z)$. □

Lemma 5.6: If $x\colon\ X \to V$ and $y\colon\ W \to U$ are constructible, so is $x\otimes y\colon\ X\otimes W \to V\otimes U$.

Proof: By (5.6) we need only consider the cases where at least one of $r(x)$ and $r(y)$ is 0. But then at least one of x and y is central. Thus at lease one of

$x\otimes 1$: $X\otimes W \to V\otimes W$ and $1\otimes y$: $V\otimes W \to V\otimes U$ is central. But $x\otimes y = 1\otimes y$. $x\otimes 1$ must be constructible by Lemma 5.5. □

Lemma 5.7: If for $i = 1,2,3$, $\Gamma_i(x)$, $\Gamma_i(y)$ and $\Gamma(f)$ are non-trivial, then Γ_i (5.6), Γ_i (5.7), Γ_i (5.8) and Γ_i (5.9) are non-trivial.

Proof: The morphisms can be checked to be non-trivial by their straightforward but tedious evaluation. □

5.6

Lemma 5.8 (Cut-elimination): If z: $Z \to Y$ and w: $Y\otimes X \to W$ are constructible so is

(5.10) $Z\otimes X \xrightarrow{z\otimes 1} Y\otimes X \xrightarrow{w} W$

Proof: Write $\sigma = rZ + rY + rX + rW$, and $\sigma_o = rZ + rY$. The proof is by double induction; we suppose the lemma to be true for all situations with lower σ, or the same σ and lower σ_o. If either z or w is central, the lemma is a case of Lemma 5.5.

For non-central z and w we consider cases according to whether z or w be defined by (5.6) - (5.9). Clearly we may omit central factors occurring at the beginning or end of z, and at the end of w. In the proof any numbered arrow represents the evident central morphism.

Case 1: z is defined by (5.6)

Let z be $f\otimes g$: $A\otimes B \to C\otimes D$. Then (5.10)

$$= (A\otimes B)\otimes X \xrightarrow{(f\otimes g)\otimes 1} (C\otimes D)\otimes X \xrightarrow{w} W$$

$$= (A\otimes B)\otimes X \xrightarrow{2} B\otimes(A\otimes X) \xrightarrow{g\otimes 1} D\otimes(A\otimes X) \xrightarrow{3} A\otimes(D\otimes X) \xrightarrow{f\otimes 1}$$

$$C\otimes(D\otimes X) \xrightarrow{4} (C\otimes D)\otimes X \xrightarrow{w} W.$$

Now $w4.f\otimes 1$ is constructible by the induction hypothesis. So too is $w4(f\otimes 1)3.g\otimes 1$. Thus $w.z\otimes 1$ is constructible.

Case 2: z is defined by (5.8).

Let z be

$$([A,B]\otimes C)\otimes D \xrightarrow{<f>\otimes 1} B\otimes D \xrightarrow{g} Y.$$

Then $w.z\otimes 1$

$$= (([A,B]\otimes C)\otimes D)\otimes X \xrightarrow{(<f>\otimes 1)\otimes 1} (B\otimes D)\otimes X \xrightarrow{g\otimes 1} Y\otimes X \xrightarrow{w} W$$

$$= (([A,B]\otimes C)\otimes D)\otimes X \xrightarrow{5} ([A,B]\otimes C) \otimes (D\otimes X) \xrightarrow{<f>\otimes 1} B\otimes(D\otimes X)$$

$$\xrightarrow{6} (B\otimes D)\otimes X \xrightarrow{g\otimes 1} Y\otimes X \xrightarrow{w} W$$

which we shall call (5.11). But $w.g\otimes 1$ is constructible by the induction hypothesis. Thus (5.11) is constructible.

We next consider the remaining cases where z is defined by either (5.7) or (5.9). In either case Y is a prime factor.

<u>Case 3</u>: w is defined by (5.6).

Let w be

$$Y\otimes X \xrightarrow{7} A\otimes B \xrightarrow{f\otimes g} C\otimes D.$$

Without loss of generality assume 7 associates the prime factor Y with a prime factor of A. Let E be a tensor product of those prime factors of X associated via 7 with prime factors of A. Then $w.g\otimes 1$

$$= Z\otimes X \xrightarrow{z\otimes 1} Y\otimes X \xrightarrow{8} (Y\otimes E)\otimes B \xrightarrow{9\otimes 1} A\otimes B \xrightarrow{f\otimes g} C\otimes D$$

$$= Z\otimes X \xrightarrow{10} (Z\otimes E)\otimes B \xrightarrow{(z\otimes 1)\otimes 1} (Y\otimes E)\otimes B \xrightarrow{9\otimes 1} A\otimes B \xrightarrow{f\otimes g} C\otimes D.$$

This is constructible by Lemma 5.5 and Lemma 5.6 if

$$(5.12) \qquad Z\otimes E \xrightarrow{z\otimes 1} Y\otimes E \xrightarrow{9} A \xrightarrow{f} C$$

is constructible. But $\sigma(5.12) = rZ + rY + rE + rC$, while $\sigma(5.10) = rZ + rY + rX + rC + rD$. So the induction assumption applies unless $rE \geqslant rX + rD$, i.e. $rB + rD \leqslant 0$. But since w is formed by (5.6), $r(g) = rB + rD > 0$.

<u>Case 4</u>: w is defined by (5.7).

Let w be πf: $Y\otimes X \to [A,B]$. Then (5.10) is

$$Z\otimes X \xrightarrow{z\otimes 1} Y\otimes X \xrightarrow{\pi f} [A,B]$$

which is π applied to

$$(5.13)\quad (Z\otimes X)\otimes A \xrightarrow{(z\otimes 1)\otimes 1} (Y\otimes X)\otimes A \overset{f}{\to} B.$$

So we must show that (5.13) is constructible. But (5.13) is

$$Z\otimes(X\otimes A) \xrightarrow{z\otimes 1} Y\otimes(X\otimes A) \xrightarrow{11} (Y\otimes X)\otimes A \overset{f}{\to} B$$

The result follows by induction since $rZ + rY + r(X\otimes A) + rB < rZ + rY + rX + r[A,B]$.

<u>Case 5</u>: w is defined by (5.9).

Let w be

$$Y\otimes X \xrightarrow{12} P(\gamma A_1\ldots\gamma A_n) \overset{\bar{\gamma}}{\to} \gamma P(A_1\ldots A_n) \xrightarrow{\gamma f} \gamma B$$

where $P \in \mathcal{P}$. The prime factor Y must be one of the γA's, say γA_k, so z must be defined by (5.9) not (5.7). Let z be

$$Q(\gamma C_1\ldots\gamma C_m) \overset{\bar{\gamma}}{\to} \gamma Q(C_1\ldots C_m) \xrightarrow{\gamma g} \gamma A_k$$

Then (5.10)

$$= Q(\gamma C_1\ldots\gamma C_m)\otimes X \xrightarrow{\bar{\gamma}\otimes 1} \gamma Q(C_1\ldots C_m)\otimes X \xrightarrow{\gamma g\otimes 1} \gamma A_k\otimes X$$

$$\xrightarrow{12} P(\gamma A_1\ldots\gamma A_n) \overset{\bar{\gamma}}{\to} \gamma P(A_1\ldots A_n) \xrightarrow{\gamma f} \gamma B$$

$$= Z\otimes X \overset{13}{—} P(\gamma A_1\ldots Q(\gamma C_1\ldots\gamma C_m)\ldots\gamma A_n) \xrightarrow{P(1\ldots\bar{\gamma}\ldots 1)}$$

$$P(\gamma A_1\ldots\gamma Q(C_1\ldots C_m)\ldots\gamma A_n) \overset{\bar{\gamma}}{\to}$$

$$\gamma P(A_1\ldots Q(C_1\ldots C_m)\ldots A_n) \xrightarrow{\gamma P(1\ldots g\ldots 1)}$$

$$\gamma P(A_1\ldots A_k\ldots A_n) \xrightarrow{\gamma f} \gamma B$$

$$= Z\otimes X \xrightarrow{13} P(\underline{\underline{1}}\ldots Q\ldots\underline{\underline{1}})(\gamma A_1\ldots\gamma C_1\ldots\gamma C_m\ldots\gamma A_n) \xrightarrow{\bar{\gamma}}$$

$$\gamma P(\underline{\underline{1}}\ldots Q\ldots\underline{\underline{1}})(A_1\ldots C_1\ldots C_m\ldots A_n)$$

$$\xrightarrow{\gamma(f\ .\ P(1\ldots g\ldots 1))} \gamma B.$$

This last morphism is constructible by (5.9) since $f.P(1\ldots g\ldots 1)$ is a morphism of $\mathcal{C}$.

<u>Case 6</u>: w is defined by (5.8)

Let w be

$$Y\otimes X \xrightarrow{14} ([A,B]\otimes C)\otimes D \xrightarrow{\langle f\rangle\otimes 1} B\otimes D \xrightarrow{g} W.$$

<u>Subcase 1</u>: The prime factor Y is associated via 14 with the prime factor $[A,B]$.

Here z cannot be defined by (5.9) so must be defined by (5.7). Also Y must be $[A,B]$. Let z be

πh: $Z \to [A,B]$. Thus (5.10) is

$$Z\otimes X \xrightarrow{\pi h\otimes 1} [A,B]\otimes X \xrightarrow{14} ([A,B]\otimes C)\otimes D \xrightarrow{\langle f\rangle\otimes 1} B\otimes D \xrightarrow{g} W$$

$$= Z\otimes X \xrightarrow{15} (Z\otimes C)\otimes D \xrightarrow{(\pi h\otimes 1)\otimes 1} ([A,B]\otimes C)\otimes D \xrightarrow{\langle f\rangle\otimes 1} B\otimes D \xrightarrow{g} W$$

This is constructible, by induction if

$$(5.14)\quad Z\otimes C \xrightarrow{\pi h\otimes 1} [A,B]\otimes C \xrightarrow{\langle f\rangle} B$$

is constructible.

However (5.14) is

$$Z\otimes C \xrightarrow{d'\otimes 1} [A, Z\otimes A]\otimes C \xrightarrow{[1,h]\otimes 1} [A,B]\otimes C \xrightarrow{1\otimes f} [A,B]\otimes A \xrightarrow{e'} B$$

$$= \quad Z\otimes C \xrightarrow{1\otimes f} Z\otimes A \xrightarrow{d'\otimes 1} [A, Z\otimes A]\otimes A \xrightarrow{e'} Z\otimes A \xrightarrow{h} B$$

$$= \quad Z\otimes C \xrightarrow{1\otimes f} Z\otimes A \xrightarrow{h} B$$

$$(5.15) \qquad = Z\otimes C \xrightarrow{16} C\otimes Z \xrightarrow{f\otimes 1} A\otimes Z \xrightarrow{17} Z\otimes A \xrightarrow{h} B$$

using the adjunction axiom that $e'.d'\otimes 1 = 1$.

But (5.15) is constructible since $rC + rA + rZ + rB < rZ + r[A,B] + rX + rW$.

<u>Subcase 2</u>: 14 associates Y with a prime factor of C.

Let E be a tensor product of the prime factors of X which are associated via 14 with prime factors of C. Then (5.10) is

$$Z\otimes X \xrightarrow{18} ([A,B]\otimes(Z\otimes E))\otimes D \xrightarrow{(1\otimes(z\otimes 1))\otimes 1} ([A,B]\otimes(Y\otimes E))\otimes D$$

$$\xrightarrow{(1\otimes 19)\otimes 1} ([A,B]\otimes C)\otimes D \xrightarrow{\langle f\rangle\otimes 1} B\otimes D \xrightarrow{g} W$$

$$(5.16) \qquad = Z\otimes X \xrightarrow{18} ([A,B]\otimes(Z\otimes E))\otimes D \xrightarrow{\langle(5.17)\rangle\otimes 1} B\otimes D \xrightarrow{g} W$$

where (5.17) is

$$(5.17) \qquad Z\otimes E \xrightarrow{z\otimes 1} Y\otimes E \xrightarrow{19} C \xrightarrow{f} A.$$

By definition (5.8), (5.16) is constructible if (5.17) is. But this is so because $rZ + rY + rE + r\,A < rZ + rY + r[A,B] + rE + rD + rW$.

Subcase 3: 14 associates Y with a prime factor of D.

Let E be a tensor product of the prime factors of X which are associated via 14 with prime factors of D. Then (5.10) is

$$Z\otimes X \xrightarrow{20} ([A,B]\otimes C)\otimes(Z\otimes E) \xrightarrow{1\otimes(z\otimes 1)} ([A,B]\otimes C)\otimes(Y\otimes E)$$
$$\xrightarrow{1\otimes 21} ([A,B]\otimes C)\otimes D \xrightarrow{<f>\otimes 1} B\otimes D \xrightarrow{g} W$$

$$(5.18) \quad = \quad Z\otimes X \xrightarrow{20} ([A,B]\otimes C)\otimes(Z\otimes E) \xrightarrow{<f>\otimes 1} B\otimes(Z\otimes E) \xrightarrow{(5.19)} W$$

where

$$(5.19) \quad = \quad Z\otimes E \xrightarrow{z\otimes 1} Y\otimes E \xrightarrow{21} D \xrightarrow{g} W$$

But (5.19) is constructible by induction, hence (5.18) is constructible by Definition (5.8).

This completes the proof of Lemma 5.8, all cases having been dealt with. □

5.7

Proposition 5.9: Every morphism of C' is constructible.

Proof: All the generators of C' are constructible. If z and y are constructible, z⊗y is constructible by Lemma 5.6.

Suppose $z\colon\ Z \to Y$ and $y\colon\ Y \to X$ are constructible. Then $z\otimes 1\colon\ Z\otimes I' \to Y\otimes I'$ is constructible by Lemma 5.6. The composite

$$(5.20)\qquad Z \xrightarrow{22} Z\otimes I' \xrightarrow{z\otimes 1} Y\otimes I' \xrightarrow{23} Y \xrightarrow{y} X$$

is constructible by Lemma 5.5 and Lemma 5.8. But (5.20) is yz.

Suppose $z\colon\ Z \to Y$ is constructible. Then so are

$$(5.21)\qquad [X,Z]\otimes X \xrightarrow{e'} Z \xrightarrow{z} Y$$

and $\pi(5.21)$

$$= [X,Z] \xrightarrow{d'} [X,[X,Z]\otimes X] \xrightarrow{[1,e']} [X,Z] \xrightarrow{[1,z]} [X,Y]$$

$$= [1,z]\colon\ [X,Z] \to [X,Y].$$

Also

$$(5.22)\qquad [Y,X]\otimes Z \xrightarrow{1\otimes z} [Y,X]\otimes Y \xrightarrow{e'} X$$

is constructible as is $\pi(5.22)$

$$= [Y,X] \xrightarrow{d'} [Z,[Y,X]\otimes Z] \xrightarrow{[1,1\otimes z]} [Z,[Y,X]\otimes Y] \xrightarrow{[1,e']} [Z,X]$$

$$= [Y,X] \xrightarrow{d'} [Y,[Y,X]\otimes Y] \xrightarrow{[z,1]} [Z,[Y,X]\otimes Y] \xrightarrow{[1,e']} [Z,X]$$

(by the naturality of d')

$$= [Y,X] \xrightarrow{d'} [Y,[Y,X]\otimes Y] \xrightarrow{[1,e']} [Y,X] \xrightarrow{[z,1]} [Z,X]$$

$$= [z,1]: \quad [Y,X] \to [Z,X].$$

If f: $A \to B$ is a morphism of C, the following is constructible

$$\underline{\underline{1}}(\gamma A) \xrightarrow{\bar{\gamma}} \gamma A \xrightarrow{\gamma f} \gamma B.$$

But this is γf since $\bar{\gamma}(\underline{\underline{1}};A)$ is the identity morphism.

Thus the category with objects the same as C', and with the constructibles as morphisms, is a closed category containing the generators of the morphisms of C'. Consequently, all morphisms of C' are constructible. □

Theorem 5.10: No incompatibilities arise in C'. That is, if z: $Z \to Y$ and z': $Y \to X$ are morphisms of C', then $\Gamma_i(z) \sim \Gamma_i(z')$ for $i = 1,2,3$.

Proof: We know that $z'z$ is constructible, so it is only necessary to show that each Γ_i of every morphism in C' is non-trivial.

This is certainly true if the morphism is central. If the morphism is defined by (5.6), (5.7) or (5.8), we need only consider, by Lemma 5.7, the relevant x and y. But in each case $r(x)$ and $r(y)$ are less than the rank of our morphism, so the theorem follows by induction. Suppose z is defined by

(5.9). Then $\Gamma_2(z)$ is the map $0 \to 0$, and $\Gamma_3(z)$ the map $n \to 1$. But $\Gamma_1(z)$ is

$$\Gamma f\colon \quad \Gamma P(A_1 \ldots A_n) \to \Gamma B$$

for $f \in \mathcal{C}$. But we know from Kelly-Mac Lane [8], that f is a constructible morphism of $\mathcal{C}$, so the graph Γf is allowable, i.e. Γf is a non-trivial morphism of $\mathcal{T}$. $\square$

6. Coherence for a closed functor.

6.1 In this chapter we prove our main theorem, using methods of proof based heavily on those used by Kelly and Mac Lane in §7 of [8].

Let the reduced objects of $\mathcal{P}$ be I, and any object of $\mathcal{P}$ formed by iterates of $\underline{\underline{1}}$ and $\otimes$. Let the reduced objects of $\mathcal{C}'$ be $P(Z_1,\ldots,Z_n)$ where $P \in \mathcal{P}$ is reduced and the Z_i are prime.

If $Z_1,\ldots Z_n$ are prime objects of $\mathcal{C}'$, a tensor product of $Z_1,\ldots,Z_n$ is $P(Z_1,\ldots,Z_n)$ where P is a reduced object of $\mathcal{P}$ with $\Gamma P = n$.

An object T of $\mathcal{C}$ is constant if $\Gamma T = 0$, and an object Z of $\mathcal{C}'$ is i-constant if $\Gamma_i Z = 0$, where i = 1,2 or 3. A constant object of $\mathcal{C}'$ is one which is 1-constant, 2-constant and 3-constant.

The proper objects of $\mathcal{C}$ are those satisfying the following rules:

$\underline{\underline{1}}$ and I are proper;

If T and S are proper, so is $T\otimes S$; and

If T and S are proper, so is $[T,S]$, unless S is constant and T is not constant.

For i = 1,2 or 3, let the <u>i-proper objects of $\mathcal{C}'$</u> be those satisfying the following rules:

For each i, $\underline{\underline{1}}'$ and I' are i-proper;

If $T \in \mathcal{C}$, γT is 2-proper and 3-proper;

If $T \in \mathcal{C}$ is proper, γT is 1-proper;

For each i, if Z and Y are i-proper so is $[Z,Y]$ unless Y is i-constant and Z is not i-constant;

If $P \in \mathcal{P}$, and $Z_1,\ldots,Z_n$ are i-proper, then so is $P(Z_1,\ldots,Z_n)$.

A <u>proper object of $\mathcal{C}'$</u> is an object that is 1-proper, 2-proper and 3-proper.

6.2

<u>Lemma 6.1</u>: <u>For any object Z of $\mathcal{C}'$ there exist a reduced object Y of $\mathcal{C}'$, and a central morphism z: $Z \to Y$.</u>

<u>Proof:</u> Let the prime factorization of Z be $P(X_1\ldots X_n)$. From our knowledge of $\mathcal{P}$ we know that there is a reduced object Q of $\mathcal{P}$ with $\Gamma Q = n$, and a morphism y: $P \to Q$ of $\mathcal{P}$ with Γy the identity permutation. Let Y be $Q(X_1,\ldots,X_n)$ and z be $y(X_1\ldots X_n)$ □

Consequently:

<u>Lemma 6.2</u>: <u>In Definition (5.6) we may assume that X,W,V and U</u>

are reduced. In (5.8) we may assume that X and U are reduced. In (5.9) we may assume that P is reduced. □

We make some observations about i-proper objects. If Z is i-constant then Z is i-proper. If $[Z,Y]$ is i-proper, so are both Z and Y; and $Z \otimes Y$ is i-proper if and only if Z and Y are i-proper, whence Z is i-proper if and only if each prime factor of Z is i-proper.

Lemma 6.3: Let $z\colon\ Z \to Y$ be a central morphism of C'. If either Z or Y is i-proper so is the other one. If either Z or Y is i-constant so is the other.

Proof: Z and Y have the same prime factors. □

Proposition 6.4: Let $z\colon\ Z \to Y$ be a morphism of C' with Y i-constant and Z i-proper, where i is 1,2 or 3. Then Z is i-constant.

Proof: Suppose inductively that the proposition is true for all smaller values, if any, of $r(z)$. If Z is central, Z is i-constant by Lemma 6.3. Also by Lemma 6.3 we may ignore central factors occurring at the beginning and end of z. Since z is constructible we consider cases according to whether z is defined by (5.6) - (5.9).

If z is defined by (5.6), let z be $f \otimes g$: $A \otimes B \to C \otimes D$. Then C and D are i-constant and A and B are i-proper. Since $r(f) < r(z)$ and $r(g) < r(z)$, A and B are i-constant by induction.

If z is defined by (5.7) let z be πf: $A \to [B,C]$. Since $[B,C]$ is i-constant so are both B and C. Indeed they are i-proper. Thus $A \otimes B$ is i-proper. But f: $A \otimes B \to C$ satisfies the conditions of the proposition and $r(f) < r(z)$. Thus $A \otimes B$ is i-constant so A is i-constant.

If z is defined by (5.8), let z be

$$([A,B] \otimes C) \otimes D \xrightarrow{\langle f \rangle \otimes 1} B \otimes D \xrightarrow{g} Y.$$

Since $([A,B] \otimes C) \otimes D$ is i-proper, so are $[A,B]$, C, D, A, B and $B \otimes D$. Bur $r(g) < r(z)$ so $B \otimes D$ is i-constant, as are B and D. But $[A,B]$ is i-proper, so A must be i-constant. But f: $C \to A$ satisfies the conditions of the proposition and $r(f) < r(z)$, so C is i-constant. Thus $([A,B] \otimes C) \otimes D$ is i=constant.

If z is defined by (5.9), let z be

$$P(\gamma A_1 \ldots \gamma A_n) \xrightarrow{\bar{\gamma}} \gamma P(A_1 \ldots A_n) \xrightarrow{\gamma f} \gamma B.$$

But γB is not 3-constant, so this case does not exist for $i = 3$. Also $P(\gamma A_1 \ldots \gamma A_n)$ is 2-constant, so we only have to consider $i = 1$. If $P(\gamma A_1 \ldots \gamma A_n)$ is 1-proper, each A_j and hence $P(A_1 \ldots A_n)$ is proper. By Proposition 7.4 of Kelly-Mac Lane [8], $P(A_1 \ldots A_n)$ is constant. Thus $P(\gamma A_1 \ldots \gamma A_n)$ is 1-constant. □

Our next lemma concerns the elimination of constant prime factors, i.e. $[T,S]'$ where T and S are constant.

<u>Lemma 6.5</u>: <u>For any object Z of $\mathcal{C}'$, there exist an object Y with $r(Y) \leqslant r(Z)$, and an isomorphism $z\colon\ Z \to Y$ in $\mathcal{C}'$ such that</u>

(i) <u>Y is reduced;</u>

(ii) <u>Y has no constant prime factors, its prime factors being precisely the non-constant prime factors of Z;</u>

(iii) <u>If Z is i-proper, so is Y;</u>

(iv) <u>There is a constant object X of $\mathcal{C}'$ and a central morphism $y\colon\ Z \to Y \otimes X$ with $\Gamma_i z = \Gamma_i y$ for $i = 1,2,3$.</u>

<u>Proof</u>: Let $Y_1, \ldots, Y_n$ be the non-constant prime factors of Z, and $X_1, \ldots, X_m$ the constant prime factors of Z, both lists keeping the factors <u>in the same order</u> as they occur in Z. Let Y be a tensor product of the $Y_1, \ldots, Y_n$, and X a product of $X_1, \ldots, X_m$. There exists a central morphism $y\colon\ Z \to Y \otimes X$. It is now sufficient to find an isomorphism $z\colon\ Z \to Y$.

We show that if W is constant, there exists an isomorphism k_W: $W \to I'$ in C'. Let the numbered morphisms denote the obvious central morphisms.

If $W = [I',I']$ let $k_W = h$ be

$$[I',I'] \xrightarrow{2} [I',I'] \otimes I' \xrightarrow{e'} I'.$$

The inverse of h is

$$I' \xrightarrow{d'} [I',I' \otimes I'] \xrightarrow{[1,3]} [I',I'].$$

We now define k_W inductively by setting $k_{I'} = 1$; by taking $k_{U \otimes V}$ to be the composite

$$U \otimes V \xrightarrow{k_U \otimes k_V} I' \otimes I' \xrightarrow{3} I';$$

and $k_{[U,V]}$ to be the composite

$$[U,V] \xrightarrow{[k_U{}^{-1}, k_V]} [I',I'] \xrightarrow{h} I'.$$

We let z be the composite

$$Z \xrightarrow{y} Y \otimes X \xrightarrow{1 \otimes k_X} Y \otimes I' \xrightarrow{4} Y. \qquad \square$$

<u>Lemma 6.6</u>: <u>If Z is an i-proper object of C' for which there are no + elements of $\Gamma_i Z$, then Z is i-constant.</u>

<u>Proof</u>: By Lemma 6.5 we may assume that Z is reduced with no constant prime factors. Consider the class of objects X of C' which are i-proper, reduced, have no i-constant prime

factors, and for which $\Gamma_i X_+ = 0$, $\Gamma_i X_- \neq 0$. We shall show that this class is empty. Suppose Y is a member of this class with least rank.

Clearly $Y \neq \underline{\underline{1}}'$ or I'. If $Y = W \otimes V$ then both W and V are in the class (remember Y is reduced) and $rW < rY$, $rV < rY$, which contradicts the hypothesis that Y has minimum rank.

If Y is γA for $A \in \mathcal{C}$, $\Gamma_3 Y_+ \neq 0$ and $\Gamma_2 Y_- = 0$. Thus we only consider the case $i = 1$. Clearly Y is not $\gamma\underline{\underline{1}}$ or γI. If $A = B \otimes C$ then both γB and γC are in the class, and $r\gamma B < rY$, $r\gamma C < rY$, again contradicting the minimum rank hypothesis. If $A = [B,C]$ then $\Gamma_1 \gamma C_+ = 0$, $\Gamma_1 \gamma B_- = 0$. If $\Gamma_1 \gamma C_- \neq 0$, there exists by Lemma 6.5 an object W, with $rW \leqslant r\,C$ which is in the class. But $r\gamma C < rY$ so again a contradiction. Since $\Gamma_1 \gamma Y_- \neq 0$, $\Gamma_1 \gamma B_+$ must not be 0. But then $[B,C]$ is not a proper object of $\mathcal{C}$, so $\gamma[B,C]$ is not 1-proper.

Suppose $Y = [W,V]$. Then $\Gamma_i W_- = 0$, $\Gamma_i V_+ = 0$. If $\Gamma_i V_- \neq 0$, there exists by Lemma 6.5 an object U with $rU \leqslant rV$ which is in the class. But $rV < rY$ so again a contradiction. This leaves $\Gamma_i V_- = 0$ and $\Gamma_i W_+ \neq 0$. But then Y is not i-proper. Consequently Y does not exist, so that the class is empty and the lemma is proved. $\square$

6.3

<u>Proposition 6.7</u>: <u>Let $z: Z\otimes Y \to X\otimes W$ be a morphism of C' where Z,Y,X,W are proper. Suppose that each $\Gamma_i z = \xi_i\otimes\eta_i$ for $\xi_i: \Gamma_i Z \to \Gamma_i X$ and $\eta_i: \Gamma_i Y \to \Gamma_i W$. Then there are morphisms $x: Z \to X$ and $y: Y \to W$ such that $z = x\otimes y$, $\Gamma_i x = \xi_i$ and $\Gamma_i y = \eta_i$.</u>

<u>Proof</u>: Suppose inductively that the proposition is true for all smaller values, if any, of $r(z)$. By Lemma 6.5 we may suppose each of Z,Y,X,W to be reduced with no constant prime factors. Since z is constructible we shall consider cases according to the Definitions (5.5) - (5.9). A numbered arrow will indicate the appropriate central morphism.

Suppose z is central. By Lemma 5.4, each $\Gamma_i z$ is central, so by the forms $\Gamma_i z = \xi_i\otimes\eta_i$ there are one-to-one correspondences set up between the prime factors of Z and X, and Y and W. Hence there exist central x and y with the desired properties.

If z is defined by (5.6), let z be

$$Z\otimes Y \xrightarrow{2} A\otimes B \xrightarrow{f\otimes g} C\otimes D \xrightarrow{3} X\otimes W.$$

Let a tensor product of those factors of Z associated via 2 with a prime factor of A (respectively B) be E(respectively F)

Let a tensor product of those factors of Y associated via 3 with a prime factor of A (respectively B) be G (respectively H). In the same way let E',F',G',H' be tensor products of the prime factors "common" to X anc C, X and D, W and C, W and D respectively. Define ρ_i: $\Gamma_i E \to \Gamma_i E'$ as the restrictions of $\Gamma_i z$ to $\{\Gamma_i E, \Gamma_i E'\}$. These are indeed morphisms of $\mathcal{G}$ because $\Gamma_i z$ is of the form $\xi_i \otimes \eta_i$.
Similarly define σ_i: $\Gamma_i F \to \Gamma_i F'$, τ_i: $\Gamma_i G \to \Gamma_i G'$,

κ_i: $\Gamma_i H \to \Gamma_i H'$. For the morphisms

$$(6.1) \qquad E\otimes G \xrightarrow{4} A \xrightarrow{f} C \xrightarrow{5} E'\otimes G'$$

$$(6.2) \qquad F\otimes H \xrightarrow{6} B \xrightarrow{g} D \xrightarrow{7} F'\otimes H',$$

$\Gamma_i(6.1) = \rho_i \otimes \tau_i$, $\Gamma_i(6.2) = \sigma_i \otimes \kappa_i$. By the inductive hypothesis we conclude that (6.1) and (6.2) are respectively $r\otimes t$ and $s\otimes k$, where $\Gamma_i r = \rho_i$, $\Gamma_i s = \sigma_i$, $\Gamma_i t = \tau_i$, $\Gamma_i k = \kappa_i$. Define x and y to be the composites

$$Z \xrightarrow{8} E\otimes F \xrightarrow{r\otimes s} E'\otimes F' \xrightarrow{9} X$$

$$Y \xrightarrow{10} G\otimes H \xrightarrow{t\otimes k} G'\otimes H' \xrightarrow{11} W$$

But $z = x\otimes y$ and $\Gamma_i x = \xi_i$, $\Gamma_i y = \eta_i$.

If z is defined by (5.7) or (5.9) let z be

$$Z\otimes Y \xrightarrow{W} V \xrightarrow{12} X\otimes W$$

where V is prime. Then either X = V and W = I', or X = I' and

$W = V$. Without loss of generality we assume the former. Then z is the composite

$$Z\otimes Y \xrightarrow{W} V \xrightarrow{13} V\otimes I'$$

If z is defined by (5.7) let w be πf: $Z\otimes Y \to [A,B]$. Since $\Gamma_i z = \xi_i \otimes \eta_i$, it follows that Γ_i of

$$(6.3) \qquad (Z\otimes A)\otimes Y \xrightarrow{14} (Z\otimes Y)\otimes A \xrightarrow{f} B \xrightarrow{15} B\otimes I$$

is $\nu_i \otimes \eta_i$ where $\pi(\nu_i) = \xi_i$. Therefore by induction, (6.3) is $v\otimes y$ where $\Gamma_i(v\colon \; Z\otimes A \to B) = \nu_i$ and $\Gamma_i(y\colon \; Y \to I) = \eta_i$. Let x: $Z \to [A,B]$ be πu. Then $\Gamma_i x = \xi_i$ and $z = x\otimes y$.

If z is defined by (5.9) let z be

$$(6.4) \qquad Z\otimes Y \xrightarrow{16} P(\gamma A_1 \ldots \gamma A_n) \xrightarrow{\bar{\gamma}} \gamma P(A_1 \ldots A_n) \xrightarrow{\gamma f} \gamma B \xrightarrow{17} \gamma B\otimes I'.$$

By the form (6.4) each prime factor γA is mapped by Γ_3(6.4) to γB. Thus by the form $\xi_i \otimes \eta_i$ of $\Gamma_i(z)$ each prime factor γA is associated via 16 with Z, so $Y = I'$.

Thus $y = 1$: $I' \to I'$, and x is

$$Z \xrightarrow{18} P(\gamma A_1 \ldots \gamma A_n) \xrightarrow{\bar{\gamma}} \gamma P(A_1 \ldots A_n) \xrightarrow{\gamma f} \gamma B.$$

Then $\Gamma_i x = \xi_i$ and $z = x\otimes y$.

If z is defined by (5.8) let z be

$$(6.5) \qquad Z\otimes Y \xrightarrow{19} ([A,B]\otimes C)\otimes D \xrightarrow{\langle f\rangle \otimes 1} B\otimes D \xrightarrow{g} X\otimes W.$$

Assume that 19 associates $[A,B]$ with a prime factor of Z. Let a tensor product of those prime factors of C associated via 19 with a prime factor of Z (resp Y) be E (resp F). The image under $\Gamma_i z$ of an element of $\Gamma_i F_+$ is in $\Gamma_i A$ or $\Gamma_i B$ by the form (6.5) of z, but is in $\Gamma_i Y$ or $\Gamma_i W$ by the hypothesis that $\Gamma_i z = \xi_i \otimes \eta_i$. It must therefore be in $\Gamma_i F_-$. Thus Γ_i of

$$(6.6) \qquad E\otimes F \xrightarrow{20} C \xrightarrow{f} A \xrightarrow{21} A\otimes I'$$

is $\rho_i \otimes \sigma_i$ where ρ_i: $\Gamma_i E \to \Gamma_i A$ and σ_i: $\Gamma_i F \to \Gamma_i I'$. By the inductive hypothesis (6.6) is $r\otimes s$ where $\Gamma_i(r\colon\ E \to A) = \rho_i$ and $\Gamma_i(s\colon\ F \to I') = \sigma_i$. But by Proposition 6.4, F is constant, so must be I'. This means that all prime factors of C are associated via 19 with Z.

Let a tensor product of those prime factors of D associated via 19 with Z be G. But Γ_i of

$$(6.7) \qquad (B\otimes G)\otimes Y \xrightarrow{22} B\otimes D \xrightarrow{g} X\otimes W$$

is $\zeta_i \otimes \eta_i$ where ζ_i: $\Gamma_i(B\otimes G) \to \Gamma_i X$ is the restriction of ξ_i to $\{\Gamma_i B, \Gamma_i G, \Gamma_i X\}$. By induction (6.7) is $w\otimes y$ where $\Gamma_i(w\colon\ B\otimes G \to X) = \zeta_i$ and $\Gamma_i(y\colon\ Y \to W) = \eta_i$. Let x be the composite

$$Z \xrightarrow{23} ([A,B]\otimes C)\otimes G \xrightarrow{\langle f\rangle\otimes 1} B\otimes G \xrightarrow{w} X.$$

Then $z = x\otimes y$ and $\Gamma_i x = \xi_i$.

This completes the proof of Proposition 6.7. □

Proposition 6.8: Let $z\colon\ Z \otimes Y \to X$ be a morphism of C' where Z,Y,X are proper. Suppose that each $\Gamma_i z$ sends each element of $\Gamma_i Y_+$ to an element of $\Gamma_i Y_-$. Then Y is constant.

Proof: Each Γ_i of

$$Z \otimes Y \xrightarrow{z} X \xrightarrow{24} X \otimes I'$$

is of the form $\xi_i \otimes \eta_i$. Therefore by Proposition 6.7 there is a morphism $y\colon\ Y \to I'$. But by Proposition 6.4, Y must be constant. □

Proposition 6.9: Let $z\colon\ ([Z,Y] \otimes X) \otimes W \to V$ be a morphism of C' between proper shapes, with $[Z,Y]$ not constant. Suppose, for each i, $\Gamma_i z = \eta_i(\langle \xi_i \rangle \otimes 1)$ for $\xi_i\colon\ \Gamma_i X \to \Gamma_i Z$ and $\eta_i\colon\ \Gamma_i(Y \otimes W) \to \Gamma_i V$. Suppose finally that there do not exist U,T,S,R, such that for each i, ξ_i can be written

$$(6.8) \qquad \Gamma_i X \xrightarrow{\omega_i} \Gamma_i(([U,T] \otimes R) \otimes S) \xrightarrow{\rho_i(\langle \sigma_i \rangle \otimes 1)} \Gamma_i Z$$

where ω_i is a central morphism of G. Then there exist $x\colon\ X \to Z$ and $y\colon\ Y \otimes W \to V$ such that $z = y(\langle x \rangle \otimes 1)$, $\Gamma_i x = \xi_i$ and $\Gamma_i y = \eta_i$.

Proof: Suppose inductively that the proposition is true for all smaller values, if any, of $r(z)$. By Lemma 6.5 we may suppose each of X,W,V is reduced with no constant prime factors

Since z is constructible we consider cases according to the Definitions (5.5) - (5.9). Note that once we have $z = y(\langle x\rangle\otimes 1)$ it is automatic that $\Gamma_i x = \xi_i$, $\Gamma_i y = \eta_i$.

Suppose that z is central. Then the image under $\Gamma_i z$ of any element of $\Gamma_i X$ or $\Gamma_i Z$ is in $\Gamma_i V$. But by the form $\xi_i(\langle\eta_i\rangle\otimes 1)$ of $\Gamma_i z$, any element of $\Gamma_i X_+$ or $\Gamma_i Z_-$ is mapped to $\Gamma_i X_-$ or $\Gamma_i Z_+$. Hence for each i, there are no elements of $\Gamma_i X_+$ or $\Gamma_i Z_-$. By Lemma 6.6, $X = I'$. Since z is central, V has a prime factor $[Z,Y]$. But any element of $\Gamma_i Z_+$ in $\Gamma_i V_-$ is mapped to an element of $\Gamma_i Z_+$ in $\Gamma_i(([Z,Y]\otimes I')\otimes W)_-$ by the centrality of z; and to $\Gamma_i V$, $\Gamma_i W$ or $\Gamma_i Y$ by the form $\xi_i(\langle\eta_i\rangle\otimes 1)$ of $\Gamma_i z$. Hence $\Gamma_i Z_+$ is empty so Z is constant too.

Let x be k_Z^{-1} where k_Z is defined in the proof of Lemma 6.5. Let y be the composite

$$Y\otimes W \xrightarrow{d'\otimes 1} ([Z,Y]\otimes Z)\otimes W \xrightarrow{(1\otimes k_Z)\otimes 1} ([Z,Y]\otimes I')\otimes W \xrightarrow{z} V.$$

But upon simplification we find that $y(\langle x\rangle\otimes 1) = z$.

If z is defined by (5.6) let z be

$$(6.9) \qquad ([Z,Y]\otimes X)\otimes W \xrightarrow{2} A\otimes B \xrightarrow{f\otimes g} C\otimes D \xrightarrow{3} V.$$

We may suppose that $[Z,Y]$ is associated via 2 with a prime factor of A. Let E be a tensor product of those prime factors of X associated via 2 with a prime factor of B. Each element of $\Gamma_i E_+$ is mapped by $\Gamma_i z$ to an element of $\Gamma_i Z$ or $\Gamma_i X$ by the form $\eta_i(<\xi_i>\otimes 1)$ of $\Gamma_i z$, and to an element of $\Gamma_i B$ or $\Gamma_i D$ by the form (6.9). Thus each element of $\Gamma_i E_+$ is mapped to an element of $\Gamma_i E_-$. By Proposition 6.8, E must be constant. Thus all prime factors of X are associated via 2 with prime factors of A.

Let F be a tensor product of the prime factors of W, associated via 2 with prime factors of A. Then Γ_i of

$$(6.10) \qquad ([Z,Y]\otimes X)\otimes F \xrightarrow{4} A \xrightarrow{f} C$$

is $\zeta_i(<\xi_i>\otimes 1)$ where ζ_i is the restriction of η_i to $\{\Gamma_i Y, \Gamma_i F, \Gamma_i C\}$. By induction (6.10) = $w(<x>\otimes 1)$ where $\Gamma_i(x\colon\ X \to Z) = \xi_i$ and $\Gamma_i(w\colon\ Y\otimes F \to C) = \zeta_i$. Let y be

$$Y\otimes W \xrightarrow{5} (Y\otimes F)\otimes B \xrightarrow{w\otimes g} C\otimes D \xrightarrow{3} V.$$

Then $z = y(<x>\otimes 1)$.

If z is defined by (5.7) let z be $\pi f\colon\ ([Z,Y]\otimes X)\otimes W \to [A,B]$. But $\Gamma_i f$ is

$$([\Gamma_i Z,\Gamma_i Y]\otimes\Gamma_i X)\otimes\Gamma_i(W\otimes A) \xrightarrow{<\xi_i>\otimes 1} \Gamma_i Y\otimes\Gamma_i(W\otimes A) \xrightarrow{\zeta_i} \Gamma_i B$$

where $\pi\zeta_i = \eta_i$. By induction there exist $x\colon\ X \to Z$ and $w\colon\ Y\otimes(W\otimes A) \to B$ with $\Gamma_i x = \xi_i$ and $\Gamma_i w = \zeta_i$, such that f is

$$(([Z,Y]\otimes X)\otimes W)\otimes A \xrightarrow{6} ([Z,Y]\otimes X)\otimes(W\otimes A) \xrightarrow{<x>\otimes 1} Y\otimes(W\otimes A) \xrightarrow{w} B.$$

But then πf is

$$([Z,Y]\otimes X)\otimes W \xrightarrow{<x>\otimes 1} Y\otimes W \xrightarrow{\pi w} [A,B].$$

If z is defined by (5.9) let z be

$$([Z,Y]\otimes X)\otimes W \xrightarrow{7} P(\gamma A_1\ldots\gamma A_n) \xrightarrow{\bar{\gamma}} \gamma P(A_1\ldots A_n) \xrightarrow{\gamma f} \gamma B \xrightarrow{8} V.$$

But $[Z,Y]$ cannot be associated via 7 with any prime factor γA. Consequently z cannot be defined by (5.9).

If z is defined by (5.8) let z be

$$(6.11) \qquad ([Z,Y]\otimes X)\otimes W \xrightarrow{9} ([A,B]\otimes C)\otimes D \xrightarrow{<f>\otimes 1} B\otimes D \xrightarrow{g} V.$$

We consider three cases, namely $[Z,Y]$ is associated via 9 with (1) $[A,B]$, (2) a prime factor of C, or (3) a prime factor of D.

<u>Case 1</u>: Here A = Z and B = Y. Let E be a tensor product of the prime factors of X associated via 9 with prime factors of D. Each element of $\Gamma_i E_+$ is mapped by $\Gamma_i z$ to an element of $\Gamma_i Y$ or $\Gamma_i D$ or $\Gamma_i V$ by the form (6.11) of z; and to an element of $\Gamma_i X$ or $\Gamma_i Z$ by the form $\eta_i(<\xi_i>\otimes 1)$ of $\Gamma_i z$. Consequently each element of $\Gamma_i E_+$ is mapped to an element of $\Gamma_i E_-$. By Proposition 6.8, E is constant so all prime factors of X are associated via 9 with prime factors of C. A similar argument

shows that all prime factors of W are associated via 9 with prime factors of D.

Thus z is

$$([Z,Y]\otimes X)\otimes W \xrightarrow{(1\otimes 10)\otimes 11} ([Z,Y]\otimes C)\otimes D \xrightarrow{\langle f\rangle\otimes 1} Y\otimes D \xrightarrow{g} V.$$

Let $x = f.10$ and $y = g(1\otimes 11)$.

<u>Case 2</u>: $[Z,Y]$ is associated via 9 with a prime factor of C.

Suppose, if possible, that $[A,B]$ was associated via 9 with a prime factor of X. Let E be a tensor product of those prime factors of $([Z,Y]\otimes X)\otimes W$, that either are prime factors of X, or else are associated via 9 with prime factors of C. Each element of $\Gamma_i C_+$ is mapped by $\Gamma_i z$ to an element of $\Gamma_i A$ or $\Gamma_i C$ by the form (6.11) of z. Each element of $\Gamma_i X_+$ is mapped by $\Gamma_i z$ toan element of $\Gamma_i X$ or $\Gamma_i Z$ by the hypothesis that $\Gamma_i z = \eta_i(\langle\xi_i\rangle\otimes 1)$. Thus each element of $\Gamma_i E_+$ is mapped by $\Gamma_i z$ to an element of $\Gamma_i E$. Thus by Proposition 6.8 E is constant, which contradicts the hypothesis that $[Z,Y]$ is not constant.

Thus $[A,B]$ must be associated via 9 with a prime factor of W. Let F be a tensor product of those prime factors of X associated via 9 with a prime factor of D. By considering the two forms of $\Gamma_i z$ we see that $\Gamma_i z$ maps each element of

$\Gamma_i F_+$ to $\Gamma_i F$, so F is constant. Thus each prime factor of X is associated via 9 with a prime factor of C.

Let G be a tensor product of the prime factors of W associated via 9 with C. Then Γ_i of

$$([Z,Y]\otimes X)\otimes G \xrightarrow{12} C \xrightarrow{f} A \tag{6.12}$$

is $\zeta_i(<\xi_i>\otimes 1)$ where ζ_i: $\Gamma_i(Y\otimes G) \to \Gamma_i A$ is the restriction of η_i to $\{\Gamma_i Y, \Gamma_i G, \Gamma_i A\}$. By induction (6.12) is $w(<x>\otimes 1)$ where $\Gamma_i(x\colon\ X \to Z) = \xi_i$ and $\Gamma_i(w\colon\ Y\otimes G \to A) = \zeta_i$. Let y be

$$Y\otimes W \xrightarrow{13} ([A,B]\otimes(Y\otimes G))\otimes D \xrightarrow{<w>\otimes 1} B\otimes D \xrightarrow{g} V.$$

Then $z = y(<x>\otimes 1)$.

<u>Case 3</u>: $[Z,Y]$ is associated via 9 with a prime factor of D.

Suppose if possible that $[A,B]$ was associated via 9 with a prime factor of X. Let E be a tensor product of those prime factors of C associated via 9 with prime factors of W. By considering the two forms of $\Gamma_i z$, we see that $\Gamma_i z$ maps each element of $\Gamma_i E_+$ to an element of $\Gamma_i E$. Thus E is constant so every prime factor of C is associated via 9 with X. This implies that each ξ_i is of the form

$$\Gamma_i X \xrightarrow{\omega i} (\Gamma_i[A,B] \otimes \Gamma_i C) \otimes \Gamma_i F \xrightarrow{<\Gamma_i f> \otimes 1} \Gamma_i B \otimes \Gamma_i F \xrightarrow{\rho_i} \Gamma_i Z$$

for central ω_i. But this is excluded by hypothesis so $[A,B]$ must be associated via 9 with a prime factor of W.

Let G be a tensor product of those prime factors of X associated via 9 with prime factors of C. By the two forms of $\Gamma_i z$ we see that $\Gamma_i z$ maps each element of $\Gamma_i G_+$ to an element of $\Gamma_i G$. Thus G is constant, so each prime factor of X is associated via 9 with D.

Let H be a tensor product of the prime factors of W associated via 9 with prime factors of D.

Then Γ_i of

$$([Z,Y] \otimes X) \otimes (B \otimes H) \xrightarrow{14} B \otimes D \xrightarrow{g} V$$

is $\zeta_i(<\xi_i> \otimes 1)$ where ζ_i: $\Gamma_i(Y \otimes (B \otimes H)) \to \Gamma_i V$ is the restriction of η_i to $\{\Gamma_i Y, \Gamma_i B, \Gamma_i H, \Gamma_i V\}$. By induction (6.13) = $w(<x> \otimes 1)$ where $\Gamma_i(x\colon\ X \to Z) = \xi_i$ and $\Gamma_i(w\colon\ Y \otimes (B \otimes H) \to V) = \zeta_i$. Let y be

$$Y \otimes W \xrightarrow{15} Y \otimes (([A,B] \otimes C) \otimes H) \xrightarrow{1 \otimes (<f> \otimes 1)} Y \otimes (B \otimes H) \xrightarrow{w} V.$$

Then $z = y\ (<x> \otimes 1)$. □

Proposition 6.10: Let $z\colon\ P(\gamma A_1 \ldots \gamma A_n) \to \gamma B$ be a morphism of C' between proper objects, P being an object of P. Then z may be written

$$P(\gamma A_1 \ldots \gamma A_n) \xrightarrow{\bar{\gamma}} \gamma P(A_1 \ldots A_n) \xrightarrow{\gamma y} \gamma B.$$

Proof: By Lemma 5.5 we may assume that P is reduced. Obviously z cannot be defined by (5.7) or (5.8).

If z is defined by (5.6) let z be

$$P(\gamma A_1 \ldots \gamma A_n) \xrightarrow{2} C \otimes D \xrightarrow{f \otimes g} E \otimes F \xrightarrow{3} \gamma B.$$

We may suppose that γB is associated via 3 with a prime factor of E. But then F is constant. However D is proper, so by Proposition 6.4 D is constant. Thus $r(g) = 0$, so z cannot be defined by (5.6).

If z is defined by (5.9) let z be

$$(6.14)\qquad P(\gamma A_1 \ldots \gamma A_n) \xrightarrow{4} Q(\gamma C_1 \ldots \gamma C_n) \xrightarrow{\bar{\gamma}} \gamma Q(C_1 \ldots C_n) \xrightarrow{\gamma x} \gamma B.$$

Since 4 is central, 4 may be written as

$$w(\gamma C_1 \ldots \gamma C_n)\colon\ P(\gamma C_{\xi 1} \ldots C_{\xi n}) \to Q(\gamma C_1 \ldots \gamma C_n)$$

where $w\colon\ P \to Q$ is a morphism of P with $\Gamma w = \xi$, and $C_{\xi i} = A_i$. Thus (6.14) equals

$$P(\gamma A_1 \ldots \gamma A_n) \xrightarrow{\bar{\gamma}} \gamma P(A_1 \ldots A_n) \xrightarrow{\gamma w(C_1 \ldots C_n)} \gamma Q(C_1 \ldots C_n) \xrightarrow{\gamma x} \gamma B$$

which is of the desired form, with $y = x.w(C_1 \ldots C_n)$.

If z is central, n must be 1, and A_1 must be B. But P is a reduced object of $\mathcal{P}$ with $\Gamma P = 1$, so P must be $\underline{\underline{1}}$. Thus $z\colon \gamma B \to \gamma B$ must be the identity morphism, which can be written in the desired form as

$$\underline{\underline{1}}(\gamma B) \xrightarrow{\bar{\gamma}} \gamma(\underline{\underline{1}}B) \xrightarrow{\gamma 1} \gamma B. \qquad \square$$

6.4

Theorem 6.11: Let $z, z'\colon\ Z \to Y$ be morphisms of C' between proper objects, such that $\Gamma_i z = \Gamma_i z'$ for $i = 1,2,3$. Then $z = z'$.

Proof: Suppose inductively that the theorem is true for all smaller value, if any, of $r(z) = r(z')$. By Lemma 6.5 we may suppose that Z and Y are reduced with no constant prime factors. If both z and z' are central, $z = z'$, so we may suppose that z is not central.

If z is defined by (5.7) let z be

$$Z \xrightarrow{\pi f} [A,B] \xrightarrow{2} Y.$$

But then $2^{-1}.z = \pi f$, so $2^{-1}.z' = \pi(f')$ where $f' = \pi^{-1}(2^{-1}.z')$. But $\Gamma_i f = \Gamma_i f'$ because $\Gamma_i z = \Gamma_i z'$. Hence $f = f'$ by induction, whence $z = z'$.

If z is defined by (5.6) let z be

$$Z \xrightarrow{3} A\otimes B \xrightarrow{f\otimes g} C\otimes D \xrightarrow{4} Y$$

Then $4^{-1}.z.3^{-1} = f\otimes g$: $A\otimes B \to C\otimes D$. Then

$\Gamma_i(4^{-1}.z'.3^{-1}) = \Gamma_i(4^{-1}.z.3^{-1}) = \Gamma_i f\otimes\Gamma_i g$.

By Proposition 6.7, $4^{-1}.z.3^{-1} = f'\otimes g'$ where $\Gamma_i(f'\colon\ A \to C) = \Gamma_i f$ and $\Gamma_i(g'\colon\ B \to D) = \Gamma_i g$, whence $f = f'$, $g = g'$ by induction, so that $z = z'$.

If z is defined by (5.9) let z be

$$Z \xrightarrow{5} P(\gamma A_1\ldots\gamma A_n) \xrightarrow{\bar{\gamma}} \gamma P(A_1\ldots A_n) \xrightarrow{\gamma f} \gamma B \xrightarrow{6} Y.$$

Then $6^{-1}.z'.5^{-1} = 6^{-1}.z.5^{-1}$: $P(\gamma A_1\ldots\gamma A_n) \to \gamma B$.

By Proposition 6.10, $6^{-1}.z'.5^{-1}$ equals

$$P(\gamma A_1\ldots\gamma A_n) \xrightarrow{\bar{\gamma}} \gamma P(A_1\ldots A_n) \xrightarrow{\gamma f'} \gamma B.$$

But $\Gamma_1(6^{-1}.z'.5^{-1}) = \Gamma f'$ and $\Gamma_1(6^{-1}.z.5^{-1}) = \Gamma f$.

Therefore by Theorem 2.4 of Kelly-Mac Lane [8], $f = f'$, whence $z = z'$.

Finally, if z is defined by (5.8) let z be

$$Z \xrightarrow{7} ([X,W]\otimes V)\otimes U \xrightarrow{\langle y\rangle\otimes 1} W\otimes U \xrightarrow{x} Y.$$

Then it may be the case that there exist A,B,C,D such that for each i, $\Gamma_i y$ is

$$\Gamma_i V \xrightarrow{\omega_i} ([\Gamma_i A,\ \Gamma_i B]\otimes\Gamma_i C)\otimes\Gamma_i D \xrightarrow{\langle\sigma_i\rangle\otimes 1} \Gamma_i B\otimes\Gamma_i D \xrightarrow{\rho_i} \Gamma_i X$$

for central ω_i in G. In this case $\Gamma_i z = \Gamma_i x(\langle\Gamma_i y\rangle\otimes 1)\Gamma_i 7$. But $\langle\Gamma_i y\rangle = \langle\rho_i\rangle(1\otimes(\langle\sigma_i\rangle\otimes 1))(1\otimes\omega_i)$. But now $\Gamma_i z =$ $\tau_i(\langle\sigma_i\rangle\otimes 1)\psi_i$ for some τ_i and central ψ_i. But perhaps there exist E,F,G,H such that for each i, σ_i is of the form

$$\Gamma_i C \xrightarrow{\phi i} ([\Gamma_i E,\Gamma_i F]\otimes\Gamma_i G)\otimes\Gamma_i H \xrightarrow{\langle\lambda_i\rangle\otimes 1} \Gamma_i F\otimes\Gamma_i H \xrightarrow{\kappa_i} \Gamma_i A$$

for central ϕ_i. But C has strictly fewer prime factors than V, since $[X,W]$ is a prime factor of V but not of C; G has strictly fewer prime factors than C; and so on. Thus this process terminates, and ultimately we have an expression for $\Gamma_i z$ of the form

$$\Gamma_i Z \xrightarrow{\mu_i} ([\Gamma_i Q,\Gamma_i M]\otimes\Gamma_i P)\otimes\Gamma_i N \xrightarrow{\langle\xi_i\rangle\otimes 1} \Gamma_i M\otimes\Gamma_i N \xrightarrow{\eta_i} \Gamma_i Y$$

for μ_i central and ξ_i not of the form (6.8). Moreover $[Q,M]$ is not constant since Z has no constant prime factors. There exists a central

$$8\colon\quad Z \to ([Q,M]\otimes P)\otimes N$$

with $\Gamma_i 8 = \mu_i$. From Proposition 6.9 applied to $z.8^{-1}$ and $z'.8^{-1}$ we conclude that $z.8^{-1} = g(\langle f\rangle\otimes 1)$ and $z'.8^{-1} = g'(\langle f'\rangle\otimes 1)$ for $f,f'\colon\ P \to Q$, and $g,g'\colon\ M\otimes N \to Y$ with $\Gamma_i f = \Gamma_i f'$ and $\Gamma_i g = \Gamma_i g'$. By the inductive hypothesis $f = f'$ and $g = g'$ so that $z = z'$.

This completes the proof of Theorem 6.11. $\square$

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