Automation for Proof Engineering: Machine-Checked Proofs At Scale

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Automation for Proof Engineering

Machine-Checked Proofs At Scale

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Formal proofs, interactively developed and machine-checked, are a means to achieve the highest level of assurance in the correctness of software. In larger verification projects, with multi-year timelines and hundreds of thousands of lines of proof text, the emerging discipline of proof engineering plays a critical role in minimizing both the cost and effort of developing formal proofs. The work presented in this thesis targets the scalability challenges present in such projects. In a systematic analysis of several large software verification projects in the interactive proof assistant Isabelle, we demonstrate that in these projects, as the size of a formal specification increases, the required effort for its corresponding proof grows quadratically. Proof engineering encompasses both authoring proofs, and developing the necessary infrastructure to make those proofs tractable, scalable and robust against specification changes. Proof automation plays a key role here. However, in the context of Isabelle, many advanced features, such as developing custom automated reasoning procedures, are outside the standard repertoire of the majority of proof authors. To address this problem, we present Eisbach: an extension to Isabelle's formal proof document language Isar. Eisbach allows proof authors to write automated reasoning procedures, known as proof methods, at the familiar level of abstraction provided by Isar. Additionally, Eisbach is extensible through specialized methods that act as general language constructs, providing high-level access to advanced features of Isabelle, such as subgoal matching. We show how Eisbach provides a framework for extending Isar with more automation than was previously possible, by allowing proof methods to be treated as first-class language elements. Today, Eisbach is already used in many Isabelle proof developments. We further demonstrate its effectiveness by implementing several language extensions, together with a collection of proof methods for performing program refinement proofs. By applying these to proofs from the L4.verified project, the one of the largest formal proof projects in history, we show that effective use of Eisbach results in a reduction in the overall proof size and required effort for a given specification.

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Abstract

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The work presented in this thesis targets the scalability challenges present in such projects. In a systematic analysis of several large software verification projects in the interactive proof assistant *Isabelle*, we demonstrate that in these projects, as the size of a formal specification increases, the required effort for its corresponding proof grows quadratically.

Proof engineering encompasses both authoring proofs, and developing the necessary infrastructure to make those proofs tractable, scalable and robust against specification changes. Proof automation plays a key role here. However, in the context of Isabelle, many advanced features, such as developing custom automated reasoning procedures, are outside the standard repertoire of the majority of proof authors. To address this problem, we present *Eisbach*: an extension to Isabelle's formal proof document language *Isar*. Eisbach allows proof authors to write automated reasoning procedures, known as *proof methods*, at the familiar level of abstraction provided by Isar.

Additionally, Eisbach is extensible through specialized methods that act as general language constructs, providing high-level access to advanced features of Isabelle, such as subgoal matching. We show how Eisbach provides a framework for extending Isar with more automation than was previously possible, by allowing proof methods to be treated as first-class language elements. Today, Eisbach is already used in many Isabelle proof developments. We further demonstrate its effectiveness by implementing several language extensions, together with a collection of proof methods for performing program refinement proofs. By applying these to proofs from the L4.verified project [41], the one of the largest formal proof projects in history, we show that effective use of Eisbach results in a reduction in the overall proof size and required effort for a given specification.

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Contents

Chapter 1

Introduction

A proof is a justification for the truth of a given statement. In most applications these justifications are a mix of natural language and mathematical symbols, with the ultimate purpose of convincing a reader that the argument is sound. Such proofs are usually considered to be *informal*, as they rely on the intuition and intelligence of the reader. Errors or omissions in informal proofs are therefore only caught once inspected by a sufficiently diligent reviewer. Occasionally these oversights are shown to be more interesting than originally thought, either requiring additional proof or invalidating the entire result.

In contrast, formal logic requires that proofs are given in a precise, unambiguous syntax which can be checked algorithmically. This rigorous treatment of every detail is both the greatest benefit and most significant hurdle of formal logic. Proofs are trustworthy and robust, but the required level of precision quickly becomes intractable for real-world applications without assistance from a computer.

Computer-based reasoning tools have recently sparked a resurgence in the practical scope of formal logic, with the hope of automated theorem provers eventually abolishing the tedium of writing formal proofs. The role of automated reasoning in these systems varies widely, from simple proof checkers to fully automatic proof discovery. Interactive theorem provers (ITPs) have emerged as a bridge between these two extremes: high level proof strategies are given by human authors, while formal details and mechanical search are handled by the computer system. This admits the ability to develop formal proofs that would otherwise be too long or detailed for a human to produce, but too intricate for a computer to discover unaided. The result has been a rich ecosystem of formalized results in mathematics, philosophy and computer science [43] [4] [2].

Demand for *formal software verification* has been steadily increasing [28], as traditional software development techniques continue to leave systems vulnerable to attack or unexpected failure. In mission-critical software, such as in cars [20] or pacemakers [35], these vulnerabilities can result in significant loss of life or property. With formal verification, the security, safety or reliability of software is instead *proven* correct using formal logic, requiring only that the proof's assumptions are validated through testing. In ITPs, these proofs dwarf the size and complexity of the programs they verify, creating significant upfront cost to developing formally verified and trustworthy systems.

Large formal proofs quickly become difficult to develop and maintain, requiring diligence in order to avoid duplication or wasted effort. Comparable to software engineering, the emerging field of *proof engineering* is the practice of writing machine-checked proofs at scale. Every proof development exists in an ecosystem including: the theorem prover itself; other developments that the proof depends on; external tools for automating proofs or generating specifications; potentially a real-world artefact that the proof discusses, such as verified code or hardware. Changes to a proof's ecosystem will incur a corresponding change to the proof itself. A completed proof may also need to be generalised or extended in order for its results to be used in other projects. Proofs therefore exist in a life-cycle, whereby they are continuously updated, re-checked, and improved. A well-engineered proof acknowledges this in its design, minimising the required effort for both up-front and on-going development.

Despite recent successes in large-scale proof development, there has been relatively little research into proof engineering itself. With few projects to draw data from, and a wide range of proof systems used, it has so far been difficult to establish best practices or cost estimation techniques. One of the largest verification projects to date is L4.verified [42] [41], which proves that the C implementation of the seL4 micokernel correctly adheres to its functional specification. In a retrospective analysis of the proofs from this project [64], Staples et al. were able to draw a linear correlation between proof size (measured in source lines) and required effort (measured in person-weeks). Although this result is hardly definitive for proofs in all domains, it indicates that longer proofs are more difficult and costly to produce.

In ITPs, the source text of a proof development is a combination of specification and proof. Authors first define constants, functions and types, followed by hypothetical statements about them, and finally give formal proofs of those statements. Many ITPs support tactic-style reasoning: tools are invoked in a read-eval-print loop to transform a proof state. These tools, known as tactics or proof methods, justify these transformations internally by appealing to axioms of the underlying logic. Tactics range from extremely primitive (e.g. applying a single rule), to general purpose (e.g. invoking a first-order solver), to domain-specific (e.g. calculating a verification condition for a program specification).

The set of available tactics applicable to a given proof state imposes an upper bound on the expressive power of a single line of proof. Most ITPs have a facility for adding user-defined tactics, either through a specialized *tactic language* or in the implementation language of the prover itself. However, this presents a significant barrier-to-entry for many proof engineers, as either approach is most likely a jarring divergence from the language of simply writing proofs. This can result in excessively verbose proofs; authors hit the edge of the available automation, resorting instead to long sequences of tedious primitive proof operations rather than diving into the unfamiliar territory of writing custom tactics. Such proofs are costly to develop and are likely to be the first to break during subsequent proof maintenance iterations.

1.1 Thesis objectives and contributions

In this thesis, we provide a foundation for understanding some of the basic scalability issues in proof engineering through empirical analysis. By systematically analysing several large verification projects, we identify a key relationship between proofs and specifications, motivating improvements that can be made to the state-of-the-art in order to develop proofs at scale.

There have been few empirical studies of formal proofs to date, and even fewer which aim to provide insight into the scalability issues present in formal verification. As a result, it remains difficult today to answer even basic project management questions for both new and existing verification endeavours. Questions such as: "How long will it take to finish this proof?" or "What will be the verification impact of adding this feature?" currently cannot be answered without significant guesswork.

Through an analysis of both the L4.verified proofs and several proofs from Isabelle's AFP [43], we show that proof size grows quadratically with respect to the complexity of a given formal statement. Expanding on previous work by Staples et al. [64], which establishes a linear relationship between proof size and effort, this suggests that required proof effort will scale quadratically with specification size.

We hypothesize that this relationship is heavily influenced by the level of automation available to proof authors. Traditionally, the manual effort spent to write a particular proof is difficult to effectively re-use. If that effort could instead be used to write an automated proof procedure that solves a class of problems, subsequent instances of similar problems would instead become trivial applications of an existing strategy.

Motivated by this hypothesis, we designed and developed *Eisbach*, a language extension for Isar, the primary proof language of Isabelle. In Isar, *proof methods* are a syntactic interface into Isabelle's tactics and are traditionally written in Isabelle's implementation language of Standard ML. Eisbach's signature feature is a new command: **method**, which allows authors to write general-purpose proof methods without the need for ML. Instead, new methods are written by combining existing proof methods with Isar's *method combinators*, along with facilities for abstracting over arguments to be supplied at run-time.

Eisbach can be naturally extended by writing new proof methods in ML, which serve as general language constructs to be used in defining Eisbach methods. Chief among these is the included match method, which allows users to explicitly bind elements of a proof goal and provide control flow.

These form the basis of a *proof method language*, and have been included in the Isabelle distribution since the release of Isabelle-2015. Eisbach has since been widely adopted by Isabelle users as a means to express both simple and non-trivial proof methods.

We then present a comprehensive evaluation of Eisbach, both in its extensibility and efficacy. Our aim is to establish Eisbach as a platform for developing automated reasoning tools that are easily accessible to proof authors, and ultimately allow proof development to scale more effectively with larger and more complex problems. This evaluation includes a suite of tools built with the Eisbach framework, and an analysis of their efficacy when applied to a large proof development from the open source L4.verified project [41].

1.1.1 Summary of thesis contributions

The primary contributions of this thesis are as follows:

- We perform an empirical analysis of several large formal verification developments in Isabelle. In this study we precisely define metrics for the size of a particular proof, as well as the size of its statement, and show that the former grows quadratically with the latter. Building on previous work, this suggests that proof effort will grow quadratically with the size of the source code of a program.

- We present the core of *Eisbach*, an extension to Isabelle/Isar that provides a framework for writing proof automation at a high level of abstraction. With the included **method** command and match method, users can easily write expressive, custom proof methods.

- We present a debugging environment for writing automation, both for authoring proofs in Isar and proof methods with Eisbach.

- We extend the use of Eisbach beyond writing proof methods to additionally support automatically generating new facts.

- We show how Eisbach can be easily utilised to provide high-level access to functionality previously only available from Isabelle/ML.

- Finally, we evaluate the above by implementing several tools to automate refinement proofs from L4.verified. We refactor several existing proofs to use these tools, resulting in a reduction in both their size and complexity, demonstrating the capabilities and benefits of Eisbach.

1.2 Document Overview

Related Work In Chapter 2 we describe existing work on expressing domain-specific automated reasoning in interactive theorem provers, as well as existing proof engineering tools and empirical studies.

Background In Chapter 3 we explain general Isabelle concepts: the basic elements of an Isar proof, foundations of the meta-logic and term language, and proof methods with method combinators. We also introduce L4.verified in more detail, including several of its sub-projects.

Empirical Analysis of Proof Effort Scalability In Chapter 4 we motivate the need for domain-specific automated reasoning by investigating the relationship between a formal statement and the size of its corresponding proof. In a large-scale analysis of several proofs from both L4.verified and Isabelle's AFP, we demonstrate that proof size grows quadratically with the complexity of its corresponding statement. Building on previous work, this suggests that, in the context of formal software verification, proof *effort* grows quadratically with source code size.

Eisbach In Chapter 5 we present the core features of Eisbach: the **method** command and match proof method. We demonstrate how the **method** command can combine multiple existing proof methods into a new domain-specific method. The match method is presented as a way to manage control flow during method evaluation, as well as provide structure and document the method's intent.

Advanced Eisbach In Chapter 6 we demonstrate the use of Eisbach beyond its core features to establish it as an automation framework. We present a proof method debugging framework, that can be used for both writing proofs and developing proof methods. Additionally we demonstrate how a proof method can be lifted into an Isar *attribute*, allowing users to easily write automated tools that transform existing theorems. Finally, we show how proof methods can be written as language elements for Eisbach, easily extending what can be expressed in both method definitions and proof steps.

Case Study: L4.verified In Chapter 7 we evaluate Eisbach by applying it to an existing proof from L4.verified. We show how multiple proof methods can be rapidly prototyped, tested, and iterated in order to significantly reduce the size of proofs.

Chapter 2

Related Work

In the past few decades a large space of interactive theorem provers has emerged, each with its own flavour of logical foundations, proof presentation, and ideology. Differing cultural traditions of what is a “good” proof, or even what *is* a proof, have resulted in many approaches to proof development and engineering. In this chapter, we will present some of the important theorem provers that are in use today, with a focus on the end-user experience of constructing a formal proof. In particular, we consider their scalability of proof effort and the accessibility of developing domain-specific automated reasoning procedures. We then discuss some of the existing work on proof engineering, covering both systematic approaches to developing scalable proofs and previous empirical studies on large proof artefacts.

Chapter Outline

- **Proof Systems.** Section 2.1 gives a brief summary of some major proof systems and their approaches to proof automation.
- **Proof Engineering.** Section 2.3 provides some background on proof engineering.
- **Conclusion.** Section 2.4 summarizes the given related work.

Acknowledgements

Some content in Section 2.1 is based on the related work sections of previous publications [50][48] which were originally written in collaboration with Makarius Wenzel and Toby Murray. Additionally, some content in Section 2.3 is based on the related work section of a previous publication [47], originally written collaboration with Toby Murray, June Andronick, Ross Jeffery, Gerwin Klein and Mark Staples.

2.1 Proof Systems

2.1.1 Mizar

The Mizar system [58][75, §2] was introduced in 1978, comprising both the Mizar Language and the Mizar Checker. The intent was to stay close to standard mathematical vernacular, so that the translation from textbook proofs would be straightforward. Still active today, it boasts one of the largest collections of formalized mathematics in its Mizar Mathematical Library (MML) [12]. Figure 2.1 gives a typical Mizar proof. Here we can see the verbose-declarative style of Mizar scripts, as each individual proof step is explicitly stated, as well as a syntax which mimics a natural language pen-and-paper proof.

Proof Automation

Mizar closely resembles Isabelle's proof document language, Isar [73], with the notable exception that there are no proof tools explicitly referenced in the proof text. Indeed, the only two tools available to a Mizar proof author are justifications via `by` (simple automated reasoning) or `from` (consequence of rule application). If a reasoning step fails, the only recourse is to expand that step into sufficiently "obvious" steps for the Mizar system.

The rationale of this design choice is to ensure that Mizar proofs can be understood independently of the Mizar Checker. The disadvantage of this approach is that there is no capacity for the author to provide proof strategies in order to extend the scope of obvious proof steps. The resulting proofs are therefore extremely verbose and require significant manual effort to complete. The MML demonstrates that this is not a insurmountable hurdle for formalized mathematics, however it is unlikely to be effective in large-scale software verification.

2.1.2 Nqthm and ACL2

Nqthm [18] originates in 1973 with a focus on having a high level of automation at the cost of limited expressivity: formulas often need to be rewritten or "encoded" in the system [75]. It distinguishes itself from fully automatic theorem provers by allowing users to guide the automation with intermediate lemma suggestions. It is succeeded by ACL2 [40] which currently supports an impressive range of formalizations from hardware verification to the fundamental theorem of calculus.

One criticism of ACL2 is that it fails to satisfy the 'de Bruijn criterion': there is no *small* trusted checker that independently validates proofs. Although Mizar fails this criterion as well, the extensive use of automated decision procedures in ACL2 places much more trust in the correctness of the system.

Proof Automation

ACL2 and Nqthm are both implemented as variants of Common Lisp, where a first-order logic is embedded as a term-rewriting system. The semantics of the language are encoded as axioms in its logic, identifying the process of both writing and verifying programs. All logical formulas and definitions are encoded as Lisp functions, with the intention that

```
theorem
  sqrt 2 is irrational
proof
 assume sqrt 2 is rational;
 then consider i being Integer, n being Nat such that
 W1: n<>0 and
 W2: sqrt 2=i/n and
 W3: for i1 being Integer, n1 being Nat st n1<>0 & sqrt 2=i1/n1
         holds n<=n1
         by RAT_1:25;
 A5: i=sqrt 2*n by W1,XCMPLX_1:88,W2;
 C: sqrt 2>=0 & n>0 by W1,NAT_1:19,SQUARE_1:93;
  then i>=0 by A5,REAL_2:121;
  then reconsider m = i as Nat by INT_1:16;
 A6: m*m = n*n*(sqrt 2*sqrt 2) by A5
   .= n*n*(sqrt 2)^2 by SQUARE_1:def 3
   .= 2*(n*n) by SQUARE_1:def 4;
  then 2 divides m*m by NAT_1:def 3;
  then 2 divides m by INT_2:44,NEWTON:98;
  then consider m1 being Nat such that
 W4: m=2*m1 by NAT_1:def 3;
  m1*m1*2*2 = m1*(m1*2)*2
     .= 2*(n*n) by W4,A6,XCMPLX_1:4;
  then 2*(m1*m1) = n*n by XCMPLX_1:5;
  then 2 divides n*n by NAT_1:def 3;
  then 2 divides n by INT_2:44,NEWTON:98;
  then consider n1 being Nat such that
 W5: n=2*n1 by NAT_1:def 3;
 A10: m1/n1 = sqrt 2 by W4,W5,XCMPLX_1:92,W2;
 A11: n1>0 by W5,C,REAL_2:123;
  then 2*n1>1*n1 by REAL_2:199;
 hence contradiction by A10,W5,A11,W3;
end;
```

Figure 2.1: Mizar proof that $\sqrt{2}$ is irrational. Credit to Freek Wiedijk[75].

the proof of a given statement can be automatically discovered by the proof system. A failed proof attempt requires the user to modify the set of available lemmas, to guide the automated proof search to a valid proof.

New automated reasoning procedures are given as transformation functions to the simplifier, requiring that they produce a proof of equivalence to maintain logical soundness. These procedures can be arbitrarily complex, as they are simply Lisp functions, and are automatically applied during the proof search.

2.1.3 LCF

The so-called 'de Bruijn criterion' for theorem provers [13] requires that the soundness of the system only depend on a small, trusted core. Often this is referred to as the *proof kernel* of the system: a module that is solely responsible for performing logical justifications.

The original **LCF** (*Logic of Computable Functions*) proof assistant [32] was one of the earliest provers that satisfies this criterion, while simultaneously providing the ability to use automated reasoning procedures. LCF pioneered the notion of proof *tactics* and *tacticals* (i.e. operators on tactics) that can be still seen in its descendants today. A tactic can perform arbitrarily complex computations on the proof state. A tactical proof is generally more interactive than either Mizar or ACL2: each tactic manipulates the proof state, transforming it and presenting the result to the user. The proof is complete once the given series of tactic invocations successfully reduces the state to zero open proof obligations.

ML was invented for LCF as the *Meta Language* to implement tactics and other tools around the core logical engine. It is a fully-featured functional programming language, serving as both the implementation and proof language of LCF, with no formal distinction between proofs, tool implementations or theory specifications. LCF inspired many of the proof systems in use today. These LCF-style provers maintain logical soundness with a small, trusted kernel, while providing facilities for using and developing tactics for automated reasoning.

2.1.4 HOL

The **HOL** (higher-order logic) family [75, §1] continues the LCF tradition with ML as the main integrating platform for all activities of theory and proof tool development (using Standard ML or OCaml), however it replaces LCF's Logic of Computable Functions with simply-typed classical set theory. HOL provers provide only a bare-bones interface to the ML top level by default, although different interface languages have been developed. This has been done as various "Mizar modes" to imitate the mathematical proof language of Mizar (see Section 2.1.1), or as "SSReflect for HOL Light" that has emerged in the Flyspeck project, inspired by SSReflect for Coq [30].

Proof Automation

As in LCF, HOL tactics and tacticals are implemented in ML, which transform a given subgoal state to build up a series of justifications for an initial formal hypothesis. The full power of ML is always available to build advanced automated proof procedures as re-usable tools. HOL tactics work in the opposite direction to inferences in the core logic, which only allow deriving new facts from old ones.

This duality of backward reasoning from goals versus forward reasoning from facts is reconciled by *tactic justifications*: a tactic both performs the goal reduction and records an inference for the inverse direction. At the end of a tactical proof, all justifications are composed with the proof kernel to produce the final theorem. Programming errors in the tactic implementations could cause this final step to fail, i.e. if a proof step was taken that could not be logically justified.

Users of HOL are nearly always operating directly in ML, making the transition from only writing proofs to also developing proof tactics a natural progression. However this could also be seen as a disadvantage of the system: users are always exposed to its implementation details, with few abstractions or conveniences when simply writing proofs.

2.1.5 Coq

Coq [75, §4] started as another branch of the LCF family in 1985, but with different answers to old questions regarding the relationship between proofs and programs. Coq implements the *Calculus of Inductive Constructions* type theory as its core logic, where a logical statement is a *type* and its corresponding proof is some *term* of that type. A statement is therefore known to be true if its type is inhabited.

Similar to HOL, Coq provides a rich library of built-in proof tactics and tacticals for developing proofs. A series of tactics in a Coq proof ultimately construct a final *proof term*, which is formally justified by type checking. This type checking algorithm therefore also serves as Coq's proof kernel.

In contrast to HOL, the OCaml environment of Coq is not readily accessible for regular users. Implementing a new *Coq plug-in* in OCaml requires that it be compiled and linked separately. If using the bytecode compiler, users may `drop` into an adhoc interactive OCaml shell during a Coq session, however this is not accessible when using the default native compiler. Coq users are therefore generally limited to its high level specification and proof language. Custom proof automation is made available through several *tactic languages*, each designed for different applications and styles of proof development.

Proof Automation

Ltac is the untyped tactic scripting language for Coq [25], and has been successfully applied in large Coq theory developments [21]. It has familiar functional language elements, such as higher-order functions and let-bindings. However, it contains imperative elements as well, namely the implicit passing of the *proof goal* as global state. The main functionality of Ltac is provided by a `match` construct for performing both goal and term analysis. Matching performs *proof search* through implicit backtracking across matches, attempting multiple unifications and falling through to other patterns upon failure. Although syntactically similar to the `match` keyword in the term language of Coq, Ltac tactics have a different formal status than Coq functions. Although this serves to distinguish logical function application from on-line computation, it can result in obscure type errors that happen dynamically at run-time.

Mtac is a recently developed *typed tactic language* for Coq [76]. It follows an approach of dependently-typed functional programming: the behaviour of *Mtactics* may be characterized within the logical language of the prover. Mtac is notable by taking the existing language and type-system of Coq (including type-inference), and merely adds a minimal collection of monadic operations to represent impure aspects of tactical programming as first-class citizens: unbounded search, exceptions, and matching against logical syntax. Thus the formerly separate aspect of tactical programming in Ltac is incorporated into the logical language of Coq, which is made even more expressive to provide a uniform basis for all developments of theories, proofs, and proof tools. Thanks to strong static typing, Mtac avoids the dynamic type errors of Ltac. More recent work combines Mtac with SSReflect [31], to internalize a generic proof programming language into Coq, in analogy to the well-known type-class approach of Haskell.

This uniform proof language approach is quite elegant for Coq, but it relies on the inherent qualities of the Coq logic and its built-in computational approach. In contrast, the greater LCF family has always embraced multiple languages that serve different purposes: classic LCF-style systems are more relaxed about separating logical foundations from computation outside of it; potentially with access to external programs or network services.

SSReflect [30] is the common label for various tools and techniques for proof engineering in Coq that have emerged from large verification projects by Gonthier. This includes a sophisticated *proof scripting language* that provides fine-grained control over operations within the logical subgoal structure, and nested contexts for single-step equational reasoning. Actual *small-scale reflection* refers to implementation techniques within Coq, for propositional manipulations that could be done in HOL-based systems by more elementary means; the experimental SSReflect for HOL-Light re-uses the proof scripting language and its name, but without doing any reflection (this is neither possible nor required in HOL).

SSReflect emphasizes concrete proof scripts for particular problems, not general proof automation. Scripts written by an expert of SSReflect can be understood by the same, without stepping through the sequence of goal states in the proof assistant. General tools may be implemented nonetheless, by going into the Coq logic. The SSReflect toolbox includes specific support for generic theory development based on *canonical structures*.

Canonical Structures were later introduced by Gonthier [31] as a method of instrumenting automation within the logic while avoiding the use of explicit tactic invocations. He notes that tactics, being essentially untyped, operate as arbitrary proof state transformers. This makes it difficult to determine when a change will affect the execution of a tactic. There is no possibility to specify the intended behaviour of a tactic, let alone verify it. By using canonical structures, the unification/type inference algorithm of Coq is coerced into executing decision procedures while generating types, effectively lifting the automation as a first class object. The "execution" of the procedure, however, is intrinsically linked with the implementation of the unification algorithm, which is not specified formally and performs poorly when used for this application. Moreover, the automation is constrained to logic-style programming, which limits its utility.

2.1.6 The Lean Theorem Prover

Lean is a more recent entry into the space, with a strong focus on automation and interoperability with other systems [24]. Similar to Coq, Lean implements the dependent type theory of Calculus of Inductive Constructions to unify both proof and type checking. Proofs can be constructed as a mix of both tactics and declarative Mizar-style keywords.

Lean offers the ability to specify a *trust level* when invoked, allowing for relaxed correctness checking by invoking *macros* that sidestep the proof kernel. A high trust level may use heavily optimized macros in order to support rapid prototyping and proof development, while a trust level of zero forces macro expansion to ensure that all proofs are sound.

2.1.7 Isabelle

Isabelle was originally introduced as another *Logical Framework* by Paulson [56], to allow rapid prototyping of implementations of inference systems, especially versions of Martin-Löf type theory. This was provided by establishing a minimal meta-logic, known as Isabelle/Pure, and allowing end-users to establish their own object-logics. Many have emerged since: e.g. ZF set theory, FOL and HOLCF, however most applications today are done in the object-logic Isabelle/HOL.

Isabelle is written in Standard ML, descended from ML which was originally developed for use in LCF. Isabelle's Pure logic descends from the LCF-style of interactive theorem proving, limiting logical inferences to a small trusted core. Before Isar, proofs in Isabelle were simply ML programs, whose type signature guaranteed that they were limited to performing valid logical inferences according to Isabelle/Pure.

Isar was first seen in 1999 [70] and was introduced as a "declarative" language for proof text. Its design was heavily influenced by Mizar, with an aim to overcome its lack of extensibility and scalability. Isar's support for unstructured proofs (proof scripts) provided a convenient launching point for porting existing proofs from ML. Today nearly all Isabelle proofs are written in Isar, while ML is used to add functionality by defining new keywords or proof methods.

Proof Automation

As in other LCF-style systems, proofs in Isabelle can be represented a series of tactics that reduce a proof state to having zero open proof obligations. Similar to HOL, the soundness of these proofs is guaranteed by representing theorems as an abstract datatype that can only be transformed by a small proof kernel. In contrast to HOL, however, the proof state itself is represented as a theorem, avoiding the need for tactic justifications after the proof is completed.

Isar introduced the concept of *proof methods* (see Section 3.3) as an interface for invoking Isabelle's tactics in structured proofs. Each method defines its own syntax, which may be arbitrarily complex, for controlling the underlying tactics and dynamically extending Isabelle's *proof context* (see Section 3.4.4) to implicitly provide hints and facts.

Proof Automation

Tactics in Lean can be used both in proof blocks or invoked by the *elaborator* in order to compute missing subterms required to make a term type-correct. As in other LCF systems, Lean tactics are arbitrarily complex programs that ultimately appeal to the underlying proof kernel to guarantee soundness.

Lean supports defining *meta*-programs outside the axiomatic foundations of the system. Specifically a meta-definition may recurse infinitely, and have access to a set of meta-constants for interacting with Lean itself. Similar to Coq's Mtac, meta-programs in Lean can be used to define new proof tactics by representing a tactic as a state monad over proof subgoals.

2.2 Relationship to Isabelle/Eisbach

Tactics in Isabelle are written in ML, and proof methods have also historically been exclusively written in ML. In this thesis, we present *Eisbach* (see Chapter 5): a high level proof method language for Isabelle/Isar. Of the languages presented in this chapter, Eisbach most closely resembles Coq's Ltac: it is a untyped scripting language for writing automated reasoning procedures by combining existing tools and providing control flow through matching. This is no accident, as the accessibility and flexibility of Ltac has made it a popular choice for many Coq proof authors and thus serves as a good reference point for a successful tactic language.

Eisbach distinguishes itself from Ltac in two important ways: convenient access to advanced functionality via ML and ad-hoc extensibility through context data. Eisbach benefits greatly from Isar's ML integration, allowing for language extensions to be implemented as needed without requiring modifications to Eisbach or Isar. In contrast to Ltac, Eisbach's `match` (see Section 5.2.4) enjoys no special status in the language: it is implemented in ML as a standalone proof method. Although it is considered a core part of Eisbach as it provides critical functionality, `match` also demonstrates what is possible for end-users to implement in their own proofs. Additionally, in Section 5.2.7 we give an example of ad-hoc integration with ML, where a fact-producing ML function is used as a `match` target to expose previously-inaccessible functionality to Eisbach methods. Similar functionality is not readily available in Coq, as interfacing with OCaml directly requires separately compiling custom Coq plug-ins.

In Coq, the `Hint` command allows users to define sets of hint databases to be automatically applied by the `auto` tactic. Eisbach's integration with Isar's *named theorems* makes similar functionality easily available to *any* custom proof method, allowing post-hoc method extensions through managed collections of facts in the proof context. This is motivated by the knowledge management issues that become apparent in large scale proof engineering, where multiple extensive databases of theorems need to be effectively used by proof authors. Eisbach allows custom proof methods to easily provide structured access to user-defined named theorems, automatically using them where applicable rather than requiring users to manually search through thousands of irrelevant theorems. In Chapter 7 we show how this functionality can be used to define a core algorithm for a non-trivial proof method, while allowing large numbers of additional calculational and terminal rules to be incrementally proven and provided later.

2.3 Proof Engineering

In the previous sections, we saw how different proof systems approached both their logical foundations and availability of custom proof automation. Using these systems effectively to build scalable and maintainable proofs is the core challenge of *proof engineering*. Proof engineering requires both having effective tools that minimize the cost and effort of building proofs, while also collecting data on these proof artefacts to better predict and understand how effort will scale to further proofs.

Differences between proof systems, and differences in proof styles within those systems, have made it difficult to establish best practices in this field. Although there are many analogous problems in software engineering, they have their own unique challenges in the context of proof engineering.

2.3.1 Proof Maintenance

Proofs in a machine-checked system must be consistently maintained, where maintenance cycle can be triggered in response to changes to the proof system itself (e.g. the semantics or syntax of a command), a dependant proof or definition, or some external artefact (e.g. verified source code).

Bourke et al. report [17] on challenges faced in the maintenance of large scale proofs in the L4.Verified (see Section 3.5) and Verisoft [7] projects. They note the proof activity for a single module in L4.Verified, showing bursts of proof development followed by long periods of maintenance. A large portion of this maintenance is performing manual refactoring tasks on existing proofs, most frequently necessitated as a result of failing to avoid duplication during the initial proof development. The Levity tool was developed during the scope of the project to automatically reorganize lemmas.

Proof Refactoring

Restructuring proof is similar in many ways to code refactoring. However, in cases where untyped automated reasoning methods are applied as part of a proof script, it becomes nearly impossible to assign static semantics to the proof. Additionally in systems such as Isabelle, proofs are processed with respect to some *context* which implicitly affects the behaviour of automated methods without being represented syntactically (see Section 3.4.4).

Iain Whiteside gives a rigorous treatment of proof refactorings with respect to the formal proof language HiProofs and tactic language HiTac [74]. A small set of refactorings are considered, with a focus on reorganizing lemmas and unfolding automatic procedures within proofs. The significant drawback with this approach is that the semantics of existing proof systems need to be applied to HiProofs/HiTac for the proposed refactorings to have any use.

Alma et al. present two methods for dependency extraction in Coq and Mizar [6] with the motivation of fast refactoring of large mathematical libraries and instrumenting automated reasoning techniques. The significant difficulty cited is removing *unnecessary* dependencies, where the theorem prover has used more than is necessary to complete a given proof.

Ringer et al. use automation to compute reusable patches to proofs and specifications [57]. They note the inherent brittleness of formal proofs, as seemingly minor changes often have wide-reaching effects, and advocate for increased tool support to shift the burden of change away from proof authors.

Eisbach can be effectively used to reduce the maintenance overhead of proofs by providing a convenient interface for expressing duplicated reasoning. A significant aspect of proof maintenance is repairing proofs after a small specification or proof change has caused a particular manual proof step to fail that is repeated in multiple separate proofs. With Eisbach, this step can be easily factored into a re-usable proof method.

2.3.2 Proof Generalization

Modularization of code is a significant part of proper software engineering, grouping concerns into consistent sub-components and managing their interaction through abstract interfaces. The motivation is clear from a development and maintenance perspective: a distinct logical module can easily be re-used when its functionality is needed by another component, reducing duplication and maintenance overhead. Often this is supported by a *generalizing* refactoring, in which a piece of code is parametrized to make it more globally useful [29].

Similarly, large proof developments must be organized to reduce duplicated reasoning and lemmas are generalized to support more uses. Isabelle's locales [10] allow for definitions, proofs and other tools to be generalized over terms and assumptions. The module system in Coq [22] provides similar functionality to produced parametrized theories.

Felty and Howe implemented a prototype proof system designed to generalize tactic-based proofs to instrument proof-by-analogy [27]. Their approach operates on the explicit proof trees generated by the system, rather than the proof text given by the user. Grov and Maclean [34], in a more recent work, propose a generalization technique based on *Proof Strategies*, which leverages techniques from artificial intelligence. They develop a notion of *goal types* which capture goal features, matching against them to apply relevant tactics.

Eisbach is a natural extension to Grov and Maclean's proof strategies, as they ultimately require some native proof tool in order to manipulate the goal state of the underlying prover. Providing a suite of specialized Eisbach methods can therefore enhance the granularity of these strategies.

2.3.3 Empirical Evaluation of Proof Artefacts

Although the space of large-scale proof development and engineering is growing, there have been correspondingly few empirical investigations. In formal verification, research to date has concentrated on measuring formal specification of programs and the relationship between these measures and system implementations.

Formal Specifications

In Olszewska and Sere [55] the authors report on their use of Halstead's software science model [36] for the measurement of Event-B specifications. They used this framework to measure the "size of a specification, the difficulty of modelling it, as well as the effort". The specification metrics developed were seen as useful descriptors of the specifications studied when applied in the DEPLOY project [3].

Some research has also been carried out on other specification metrics. In 1987 Samson et al. [59] investigated metrics that might aid in cost prediction for software developed. They use McCabe's cyclomatic complexity metric [51] and lines of code to measure the implementation of a small system and measures of operators and equations to measure the HOPE formal specification. Although their sample size was small, they found a relationship between their measures of the specification and implementation. Tabareh's masters thesis [67] contained an investigation of relationships between specification and implementation measures. A number of specification metrics were defined for Z specifications. These were size-based metrics such as lines of code and conceptual complexity; structure based metrics such as logical complexity; and semantic based metrics such as slice-based coupling, cohesion and overlap. In a more recent paper Bollin [16] evaluated the use of specification metrics of complexity and quality in a case study comprising more than 65,000 lines of Z specification text. Staples et al. [65] analysed the relationship between the specification and implementation sizes for API functions in L4.verified, finding that the formal specification sizes correlated much more strongly than COSMIC Function Point count [38].

In Chapter 4 we define a general metric for measuring the size of a formal statement by analysing its dependency graph, and present a novel *idealisation* of this metric, useful in the presence of over-specified statements, by considering the content of its proof to prune the graph of unnecessary dependencies. We find that this idealised size correlates much more strongly with the size of the eventual proof of the statement, and therefore is likely a better estimation of its complexity.

Proofs and Formal Verification

Andronick et al. [8] provide a comprehensive retrospective analysis of the L4.verified project, citing the "middle-out" development process as a key factor in its success. A descriptive process model is given, highlighting the major phases of the project and the iterative nature of its development, along with the evolution of the size of each major artefact. Despite the data collected, the authors admit that "*We do not yet have a good understanding of what to measure in formal verification projects.*" They note that the lessons learned from this analysis were qualitative and not immediately useful for developing "decision-making models to inform project management judgments".

Later, Staples et al. [64] extracted size and effort data from L4.verified to understand their relationship. They found a "strong linear relationship" between the effort spent on a given proof (in person-weeks) and its eventual proof size (in lines of proof script), for both projects and individuals.

Jeffery et al. [39] propose a research agenda for "empirical study of productivity in software projects using formal methods". With support from prior literature, they identify over thirty research questions that require additional data. They note that although high-level concepts are shared between software engineering and formal methods, such as "size, effort" and "rework", the nature of their measurement in proof engineering will be significantly different. In this agenda, they identify following question:

> *How are characteristics of formal specifications, properties, or code related to effort in formal proofs?*

The study in Chapter 4 attempts to address this by investigating several verification projects to establish the relationship between the size of a formal statement and its formal proof. Finding a quadratic relationship between statement and proof size, we hypothesize that the effort required to verify software with current techniques will increase quadratically with the size of its code.

This study, and another question identified by Jeffery et al. motivates the development of Eisbach:

> *How can we best combine interactive proof and proof automation to achieve high proof productivity during initial proof development and subsequent proof maintenance?*

In Chapter 7 we address this question in the context of Eisbach by implementing a set of proof methods to automate previously-manual proofs in L4.verified.

2.4 Conclusion

In this chapter, we gave a brief overview of different proof systems and their approaches to proof automation. Some systems, like Mizar, expose very little or no functionality for providing additional automated reasoning procedures. Proofs are therefore extremely verbose and manual, but can be independently understood by a human reader. Other systems, like ACL2, are almost entirely automated. Users may extend the space of automated strategies available, but their application occurs as an extension to its standard proof search rather than as an interactive process. Neither Mizar nor ACL2 satisfy the 'de Bruijn criteron': they lack a small, trusted proof checker. In ACL2 this is of particular concern, given its high reliance on automated reasoning.

LCF-style provers like Coq, HOL and Isabelle provide a small proof kernel that all manual and automated proofs must use. This allows for powerful automated proof tools, called *tactics*, to be developed without compromising trust in the proofs that use them. In HOL, most tactics are developed in its implementation language ML, while Coq provides several high-level *tactic languages*. These languages have different advantages with respect to usability, expressibility and static guarantees, however the majority of custom Coq tactics use Ltac, its untyped scripting language. Eisbach provides similar functionality to Isabelle/Isar, focusing on large scale proof engineering and extensibility.

Proof engineering is the discipline of developing proofs in these systems to be scalable and maintainable, while also providing empirical analysis of the artefacts and processes involved. We showed existing work on tools for maintaining, refactoring and generalizing proofs. We also presented several studies on the relationship between formal specifications and code size, as well as some larger retrospective studies on the successful L4.verified project. We find, as Jeffery et al. [39] note in their research agenda, that a lack of empirical analysis in proof engineering has made it difficult to build good models for proof effort.

In the next chapter we focus on Isabelle, its high-level proof language, Isar, and the structure of the L4.verified project. These form the basis for the later chapters, where we investigate the L4.verified proofs in an empirical analysis of proof size vs. specification size in Chapter 4, design Eisbach as an extension to Isar in Chapter 5, and ultimately use Eisbach in Chapter 7 to implement a set of proof methods that automate previously-manual proofs from L4.verified.

Chapter 3

Background

In the previous chapter we have seen a wide array of interactive proof systems, each with a different emphasis on the accessibility of writing custom proof tools. In this chapter we shift our focus to Isabelle, discussing foundations and the state of the art in automating its proofs. Isabelle is implemented in Standard ML, along with a rich ecosystem of powerful automation tools. Isar, Isabelle's proof document language, provides access to this proof automation at a high level of abstraction, allowing end users to focus on the logic of proofs rather than implementation details of the system. However, it is also designed to be extensible: users may define their own Isar commands in ML for both generating specifications and automating proof procedures. The result has been an ever-growing suite of Isabelle functionality that is easily accessible via Isar.

Readers familiar with Isabelle/Isar can skip to Section 3.5, which introduces the L4.verified project.

Chapter Outline

- **Introduction to Isar.** Section 3.1 gives a brief overview of the core concepts of Isabelle/Isar.
- **Isabelle/Pure.** Section 3.2 covers Isabelle's meta logic and proof kernel
- **Proof Methods.** Section 3.3 is an overview of Isabelle's existing proof methods and method combinators
- **Isar Revisited.** Section 3.4 presents a more in-depth discussion on relevant Isar concepts for this thesis.
- **L4.verified.** Section 3.5 provides an introduction to the L4.verified project
- **The Archive of Formal Proofs.** Section 3.6 briefly introduces Isabelle's Archive of Formal Proofs
- **Formatting Remarks.** Section 3.7 discusses the formatting of Isar input and output in this thesis

3.1 Introduction to Isar

Isar stands for *Intelligible, semi-automated reasoning* [71]. It provides a high-level interface to Isabelle's proof engine with a suite of commands for driving the proof process while simultaneously documenting the intuition of the proof author. As Isar itself is devoid of computation, *proof methods* (see Section 3.3) are used to interact with Isabelle's proof state in order to formally discharge proof obligations. Internally, proof methods use Isabelle's proof kernel (see Section 3.2.3) to ensure soundness, and most built-in methods simply provide a syntactic interface to some Isabelle *tactic*.

3.1.1 A Simple Proof

In Isar, a proof begins when some command (i.e. **lemma**) initiates a proof block, stating the assumptions and conclusion of a desired fact.

```
lemma IsarSimpleExample:
  assumes asm1: A ⟹ B
      and asm2: A
  shows B
```

In this example, A and B are arbitrary boolean terms that are fixed for the duration of the proof. The assumptions are given with the **assumes** keyword *asm1* ($A \Longrightarrow B$) and *asm2* (A). These are local facts that can be used in the scope of this proof. The conclusion, given with **shows**, is the goal of the proof.

The proof then proceeds as a series of interactive commands to show that the conclusion is true under the stated assumptions. The **apply** command can be given to evaluate *proof method* expression on the current proof state. This transforms the goal, either discharging it or introducing new subgoals. One of the simplest methods is rule that resolves the conclusion of the current goal against the conclusion of a given rule.

In this case, we can use rule twice to apply both of our assumptions to the goal and complete the proof.

```
  apply (rule asm1)
  apply (rule asm2)
```

Once the proof is complete and no proof obligations (subgoals) remain, we can use the **done** command to conclude.

```
  done
```

Upon successful completion of the proof, the fixed terms in the lemma are generalized into *schematics* (see Section 3.2.2) and the stated assumptions are inserted as antecedents to the conclusion. In this case the resulting IsarSimpleExample lemma is $(?A \Longrightarrow ?B) \Longrightarrow ?A \Longrightarrow ?B$ and is now generally available in further proofs.

3.1.2 The Languages of Isabelle

The previous example showed the *modus-ponens* rule of Isabelle's meta-logic, demonstrating the three main sub-languages of Isabelle/Isar:

- **The Isar command language**. Proof commands, such as **apply**, drive the proof state and interactively present the user with information about it.
- **The term language of Isabelle/Pure**. Terms are parsed as arguments to Isar commands as the formal basis of the lemma statement and proof. Here A and B are free variables in the term language, and $\Longrightarrow$ is a constant, representing meta-implication from the meta-logic of Isabelle/Pure.
- **Isar's proof method expressions**. Proof methods discharge the formal obligations of the proofs, appealing to the meta-logic of Isabelle/Pure.

In the next section, we present the core concepts of Isabelle/Pure as they relate to proof and proof method development in Isabelle, and in Section 3.3 we present a more comprehensive overview of proof methods.

3.2 Isabelle/Pure

3.2.1 Meta-logic Connectives

Isabelle/Pure is a higher-order logic, serving as a framework for performing Natural Deduction. Pure is based on simply-typed λ-calculus (modulo $\alpha\beta\eta$-conversions), with a special type prop defined as the type of *propositions.* Every logical statement in Pure is a prop and is constructed with the two meta-logical connectives, $\bigwedge$(universal quantification) and $\Longrightarrow$ (implication). Additional meta-connectives are derived from these primitives, such as $\equiv$ (meta-equivalence) and &&& meta-conjunction. Pure connectives have a low syntactic precedence, compared to connectives from object logics, such as $\forall$ or $\wedge$ from HOL (see Section 3.2.4). The Pure connectives outline inference rules declaratively, for example:

- conjunction introduction, traditionally $\dfrac{A \quad B}{A \wedge B}$, is: $A \Longrightarrow B \Longrightarrow A \wedge B$
- well-founded induction is: $\mathsf{wf}\ r \Longrightarrow (\bigwedge x.\ (\bigwedge y.\ (y,\ x) \in r \Longrightarrow P\ y) \Longrightarrow P\ x) \Longrightarrow P\ a$

3.2.2 Terms

A term in Isabelle/Pure is constructed from the following primitives:

Bound Variables in Lambda Abstractions. A lambda-abstraction represents a function in Isabelle, where bound variables refer to function arguments that are provided postfix, e.g. $(\lambda x\ y.\ x \Longrightarrow y)\ A\ B$ evaluates to $A \Longrightarrow B$. Universal quantification $\bigwedge$ is syntactic sugar for the higher-order function $\mathsf{Pure.all}$, where $\bigwedge x.\ P\ x$ is equivalent to $\mathsf{Pure.all}$ $(\lambda x.\ P\ x)$.

Constants. A constant is a term with a globally-fixed (potentially polymorphic) type, usually provided with one or more definitional axioms (i.e. $\mathsf{my\_equiv} \equiv (\lambda x\ y.\ (\bigwedge P.\ P\ x \Longrightarrow P\ y)))$. Object-logics may define their own interfaces for ensuring introduced definitional axioms are sound, and provide advanced functionality for constructing nontrivial definitions (e.g. for recursive functions).

Schematic Variables. A schematic variable is logically equivalent to a variable that is outermost meta-universally quantified, and is represented with a *?* prefix. Isabelle/Pure provides lifting functions to convert between these equivalent forms, for example the standard form of $\bigwedge x.\ (\bigwedge y.\ P\ x\ y) \Longrightarrow Q\ x$ is $(\bigwedge y.\ P\ ?x\ y) \Longrightarrow Q\ ?x$.

When a given rule is resolved against the current goal, Isabelle's unifier is used to first calculate valid instantiations for schematics appearing in both the rule and the goal in order for them to match.

Free Variables. A free variable is arbitrary-but-fixed within a given *context* (see Section 3.4.4), with optional type restrictions. Outside the context (e.g. when a proof is completed), a free variable can be generalized into a schematic. By default, most Isar commands will interpret any unknown terms (i.e. not constant or bound) as free variables, and proof commands (i.e **lemma**) will declare them as fixed for the duration of the proof.

Types

Terms in Isabelle/Pure have an associated type, where an explicit type constraint is represented with infix notation: *term* :: *type*. The syntax of functions is infix, i.e. $\_ \Rightarrow \_$. Logical statements have the special type prop, with meta-connectives used as prop-producing functions (i.e. $\Longrightarrow$ :: prop $\Rightarrow$ prop $\Rightarrow$ prop). Types are constructed from the following primitives:

Type Constructor. Similar to a constant term, a type constructor is globally defined and takes zero or more type arguments. For example, the function type ($\_ \Rightarrow \_$) takes two type arguments: the domain and range types, while prop takes zero arguments.

Schematic Type Variable. Similar to a schematic term variable, a schematic type variable represents a polymorphic type that can be instantiated. It is syntactically represented with a *?'* prefix. For example, meta-universal quantification is polymorphic in the quantified variable, and has the type ($?'a \Rightarrow$ prop) $\Rightarrow$ prop.

Free Type Variable. Similar to a free term variable, a free type variable represents an arbitrary-but-fixed type for some context, syntactically represented with a *'* prefix. Outside this context, free type variables can be generalized into schematics. For example, if we start a proof with **lemma** $\bigwedge x.\ P\ x \Longrightarrow P\ x$, the bound variable x is assigned the fresh, free type $'a$. Once the proof is complete, x is generalized into a schematic term (i.e. $P\ ?x \Longrightarrow P\ ?x$) and $'a$ is generalized into a schematic type (i.e. $?'a$).

Type variables (free and schematic) can be additionally declared as a particular *sort*, which restricts how they may be instantiated.

3.2.3 Proof Kernel

Isabelle's inference kernel follows the LCF tradition of providing an abstract ML type (thm) that represents the type of "true" statements, with respect to the underlying meta-logic, object-logic and any axioms used. A thm can only be created by ultimately appealing to the primitive interface of the kernel, although powerful tools can be built using this interface. The core internal data member of a thm is its underlying cterm: another abstract ML type representing a term that has been type-checked by the kernel. A thm can therefore be thought of as a term with an implicit certificate that it is a type-correct prop which has been derived from the primitive inference rules of Pure.

Rather than maintain an auxiliary data structure for the proof goal state in ML, as in the original LCF system, in Isabelle it is simply represented as a single thm. A proof of some hypothetical statement C begins with the trivial fact $C \Longrightarrow C$, provided as a primitive from Pure. This proof proceeds by manipulating the subgoal structure of this thm (i.e. its antecedents) and eventually reducing it to having zero subgoals, i.e. $\Longrightarrow C$, or simply C. Administrative goal operations, e.g. shuffling of subgoals or restricted subgoal views, work by elementary inferences involving $\Longrightarrow$ in Isabelle/Pure.

An intermediate goal state with n open subgoals has the form $H_1 \Longrightarrow ... \ H_n \Longrightarrow C$, each with its own substructure $H = (\bigwedge x.\ A\ x \Longrightarrow B\ x)$, for zero or more *goal parameters* (here just x) and *goal premises* (here just $A\ x$). Following [56], this local context is implicitly taken into account when natural deduction rules are composed by *lifting*, *higher-order unification*, and *backward chaining*. Often, long chains of meta-implication (as usually seen when pretty-printing subgoals) will be presented with a condensed syntax: $(A \Longrightarrow B \Longrightarrow C \Longrightarrow D) \equiv (\llbracket A;\ B;\ C \rrbracket \Longrightarrow D)$.

Isabelle tactics due to [56] follow the idea behind LCF tactics, but implement the backwards refinement more directly in the logical framework, without the replay tactic justifications (which is still seen in HOL or Coq today). A tactic is therefore simply an ML function from thm ⇒ thm list, which transforms an intermediate proof state thm into an unbounded lazy list of results. Tactic failure returns an empty set of results, while multiple results represents different backtracking options. Simple tactics include: backwards reasoning by applying a rule to the conclusion of a subgoal, forwards reasoning from a premise of a subgoal, and solving a subgoal by matching its conclusion with one of its premises. LCF-style *tacticals* combine and transform tactics, some common tacticals are: sequential composition, alternative composition, repeated application, optional application, and applying a given tactic to all subgoals. A traditional tactic-style proof script chains together a series of tactics to reduce a proof state to having no open subgoals.

Although the framework of Isabelle has changed significantly over the past few decades, the proof kernel has remained largely undisturbed. The Isar proof document language provides an alternative view on these primitive concepts, managing most of the administrative details behind-the-scenes while ultimately appealing to this small trusted core.

3.2.4 Isabelle/HOL

Isabelle's Pure logic is designed as a minimal implementation of an inference system, allowing users to establish their own object logics within it. Among the many logics that have emerged, Isabelle/HOL hosts the majority of the current Isabelle applications. Although Eisbach (see Chapter 5) and most of the tools implemented in Chapter 6 are compatible with other object logics, we present them in this thesis only in the context of Isabelle/HOL.

Table 3.1: Comparison of Isabelle's Pure and HOL connectives.

Connective Name	Pure Syntax	HOL Syntax
Universal Quantification	$\bigwedge x.\ P\ x$	$\forall\, x.\ P\ x$
Implication	$P \Longrightarrow Q$	$P \longrightarrow Q$
Equivalence	$P \equiv Q$	$P = Q$
Conjunction	$P\ \&\&\&\ Q$	$P \wedge Q$
Disjunction	-	$P \vee Q$
Existential Quantification	-	$\exists\, x.\ P\ x$
Negation	-	$\neg\ P$

HOL formulas are of type bool, where the special purpose function Trueprop :: bool ⇒ prop is used to embed them into the Pure logic. This is known as an object-logic *judgement*, and is implicitly inserted/hidden by the term parser/pretty printer when operating within HOL. Each Pure connective has an equivalent HOL interpretation (shown in Table 3.1), however HOL additionally defines existential quantification, disjunction and negation. Furthermore, HOL defines the constants True and False and axiomatizes the law of excluded middle (i.e. $\forall\, P.\ P = \mathsf{True} \vee P = \mathsf{False}$).

3.3 Proof Methods

As seen in Section 3.1.1, proof methods can be invoked during an Isar proof with the **apply** keyword to transform the goal state. An **apply** does not insist that the proof be completed, and outputs the resulting proof state to the user after successfully evaluating the given method. Further **apply** commands may follow to continue the proof, until it is eventually concluded by the command **done**. This chain of method invocations is often referred to as a *proof script*, where determining its logical content requires inspecting each intermediate goal state.

Isar additionally supports producing and presenting, human-readable formal proofs. These *structured* proofs explicitly state the assumptions and intermediate conclusions of each step, referring to proof methods to discharge intermediate proof obligations. Concluding a structured proof block a method is usually done with the **by** command, which evaluates one or two given proof methods, requiring that they successfully solve the goal.

The majority of the proofs presented in this thesis are largely unstructured, with some structured Isar elements (e.g. explicit lemma **assumes** or proofs concluding with **by**).

We can consider two rough classifications of methods: *basic* methods that perform a single reasoning step or technical adjustment, and *automated* methods that use heuristics and search to both simplify and solve goals. Here we present a small selection of proof methods, seen in the later chapters of this thesis.

3.3.1 Basic Proof Methods

- rule $fact_1$ ... $fact_n$ Backward reasoning from the conclusion. Resolves the conclusion of the current subgoal with the conclusion of each theorem in the given facts, backtracking over each result. The assumptions of the applied theorem are then introduced as subgoals. Example: use with conjI ($\llbracket ?P;\ ?Q \rrbracket \Longrightarrow ?P \wedge ?Q$)

- erule $fact_1$ ... $fact_n$ Forward reasoning by eliminating an assumption. Resolves the conclusion of the current subgoal with the conclusion of each theorem in the given facts, as well as resolving the first assumption of the theorem with an assumption in the current subgoal, backtracking over each result. The remaining assumptions of the applied theorem are then introduced as subgoals. Example: use with conjE ($\llbracket ?P \wedge ?Q;\ \llbracket ?P;\ ?Q \rrbracket \Longrightarrow ?R \rrbracket \Longrightarrow ?R$)

- drule $fact_1$ ... $fact_n$ Forward reasoning from an assumption. Resolves an assumption of the current subgoal with the first assumption of each theorem in the given facts, backtracking over each result. The remaining assumptions of the applied theorem are then introduced as subgoals. Example: use with conjunct1 ($?P \wedge ?Q \Longrightarrow ?P$)

- assumption Discharge the current subgoal by resolving it with one of its assumptions, backtracking over each result.

3.3.2 Automated Proof Methods

- simp *add:* $fact_1$ ... $fact_n$ Simplify the current subgoal through rewriting, potentially solving it. A large database of facts and processes are implicitly retrieved from the context, which can be managed through various declaration attributes (e.g. *rule*[simp]) or optionally provided in-place after *add:*.

- auto *simp:* $fact_1$ ... $fact_n$ *intro:* $fact_1$ ... $fact_n$ *elim:* $fact_1$ ... $fact_n$ Simplify *all* subgoals, additionally applying known introduction (backwards) and elimination (forward) rules from the context and attempting classical reasoning solvers.

3.3.3 Method Combinators and Backtracking

In general, **apply** and **by** accept proof method *expressions*, basic proof methods combined using method *combinators*. Unlike tacticals, there are only a minimal set of combinators provided. These support backtracking over method results, where the end of each **apply** is the first successful result of all combined methods. The standard proof method combinators are as follows:

1. Sequential composition of two methods with implicit backtracking: the expression "$method_1$, $method_2$" applies $method_1$, which may produce a set of possible results (new proof goals), before applying $method_2$ to all possible goal states produced by $method_1$. Effectively this produces all results in which the application of $method_1$ followed by $method_2$ is successful.

2. Alternative composition: "$method_1$ | $method_2$" tries $method_1$ and falls through to $method_2$ when $method_1$ fails (yields no results).

3. Suppression of failure: "*method?*" turns failure of *method* into an identity step on the goal state.

4. Repeated method application: “*method*+” repeatedly applies *method* (at least once) until it fails.

5. Goal restriction: “*method*[n]” restricts the view of *method* to the first n subgoals.

The subsequent example illustrates a proof method expression with combinators:

lemma $P \wedge Q \longrightarrow P$

by ((rule impI, (erule conjE)?) | assumption)+

Informally, this says: “Apply the implication introduction rule, followed by optionally eliminating any conjunctions in the assumptions. If this fails, solve the goal with an assumption. Repeat this action until it fails.”

This method expression is applicable to more than one proof, and indeed will solve a class of propositional logic problems involving implication and conjunction. In Chapter 5 we will see how Eisbach can be used to define a new method using this method expression, and how it can be refined into a general-purpose propositional logic solver.

3.4 Isar Revisited

In this section we present a small selection of additional Isar functionality, relevant to the following chapters of this thesis.

3.4.1 Facts and Theorems

Although Isabelle’s proof kernel operates on theorems (i.e. a thm), Isar operates on lists of theorems called *facts*. Individual members of a fact may be selected by an explicit index (i.e. *my_fact*(n) for selecting the nth theorem of *my_fact*).

For example, a lemma may prove multiple theorems and store them in a single fact:

lemma my_theorems: A B C

thm my_theorems — A, B, C

thm my_theorems(1) — A

thm my_theorems(2) — B

thm my_theorems(3) — C

3.4.2 Definitions

In Isabelle, new constants can be declared with optional type restrictions, and then definitional axioms can be added to allow the constant to be rewritten as its body. Isabelle’s primitive definition package enforces the logical consistency of any declared definition (i.e. avoiding cyclical dependencies).

The most straightforward access to this functionality is via the **definition** Isar command, which simultaneously declares a new constant and its definitional axiom. e.g.:

definition is_nonzero :: int ⇒ bool

where is_nonzero $x \equiv x \neq 0$

This declares a new constant is_nonzero with the type restriction int ⇒ bool and simultaneously produces a new fact named is_nonzero_def (is_nonzero $?x \equiv ?x \neq 0$).

3.4.3 Records

The **record** command, provided by Isabelle/HOL, allows for defining record types, with automatically generated field-update syntax and corresponding lemmas. e.g.:

```
record machine_state =
  program_counter :: int
  memory :: int ⇒ int

lemma
  (⦇program_counter = 1, memory = X⦈ :: machine_state)⦇program_counter := 0⦈ =
   ⦇program_counter = 0, memory = X⦈
  by simp
```

3.4.4 Proof Context

The structured proof context provides general administrative structure to complement primitive thm values from the kernel. The context tracks the scope of fixed terms, fixed types, and structured assumptions.

Revisiting our first example from Section 3.1.1, at the start of the proof, the context is locally augmented with the assumptions *asm1* and *asm2* as well as the fixed variables A and B.

```
lemma IsarSimpleExample:
  assumes asm1: A ⟹ B
      and asm2: A
  shows B
```

Once this proof concludes, proving B, the inner (proof) context is compared to the outer context to determine how to properly finalize the result. Since *asm1* and *asm2* are only declared in the inner context, they are added as antecedents to the conclusion (i.e. $[\![A \Longrightarrow B;\ A]\!] \Longrightarrow B$). Additionally, since A and B are only fixed in the inner context, they can be generalized into schematics to produce the final theorem: $[\![?A \Longrightarrow ?B;\ ?A]\!] \Longrightarrow ?B$). The inner context is then discarded, and the resulting rule is added to the set of known facts in the current (outer) context with the name IsarSimpleExample.

```
thm IsarSimpleExample — ⟦?A ⟹ ?B; ?A⟧ ⟹ ?B
```

Declaration Attributes

The proof context may also be augmented with arbitrary tool data, providing both global and local information. This is generally managed through *declaration attributes*, which add or remove rules from tool-specific collections (i.e. *rule*[simp] to add Simplifier rules to the context).

For example, we can declare local assumptions as simp so they are picked up by the simp method during a proof.

```
lemma
  assumes A: A
      and B: B
  shows A ∧ B by (simp add: A B)

lemma
  assumes A[simp]: A
      and B[simp]: B
  shows A ∧ B by simp
```

Once a proof concludes, the effects of any locally-applied declaration attributes are discarded.

Named collections of dynamic facts can be declared with the **named_theorems** command, with corresponding attributes to add or delete entries (see Section 5.2.1 for more details.)

```
named_theorems my_simps

lemma
  assumes A[my_simps]: A
      and B[my_simps]: B
  shows A ∧ B by (simp add: my_simps)
```

So far these attributes have all had local effects, persisting only for the duration of a single proof. A declaration can instead be made global (with respect to the outer context) by appending it to the name of a lemma.

```
lemma my_AB[my_simps]: A ⟹ B ⟹ A ∧ B by simp
```

Additionally, the **declare** command allows for applying declaration attributes to existing facts.

```
declare my_AB[my_simps]
```

Alternatively, the **lemmas** can declare multiple lemmas simultaneously.

```
lemmas [my_simps] = my_AB TrueI
```

3.4.5 Rule Attributes

Previously (see Section 3.4.4) we saw declaration attributes used to modify Isabelle's proof context. In this section we will cover *rule attributes*, which have the same syntax as declaration attributes, but instead perform in-place rule transformations.

The where and of attributes

These attributes are used to instantiate schematic variables in existing rules. With where each variable instantiation is named, while of takes an list of terms and instantiates schematics by order of their appearance in the given rule.

```
lemma ⟦A; B⟧ ⟹ A ∧ B
thm conjI — ⟦?P; ?Q⟧ ⟹ ?P ∧ ?Q

thm conjI[where P=A and Q=B] — ⟦A; B⟧ ⟹ A ∧ B
thm conjI[of A B] — ⟦A; B⟧ ⟹ A ∧ B

apply (rule conjI[of A B])
 apply assumption
apply assumption
done
```

The THEN and OF attributes

These attributes are used to perform in-place composition of existing rules. With OF the given facts are unified against the assumptions of the source rule. Conversely, THEN unifies the source rule against the first assumption of the given rule.

```
lemma
  assumes A: A
          and B: B
  shows A ∧ B
thm conjI — ⟦?P; ?Q⟧ ⟹ ?P ∧ ?Q

thm conjI[OF A] — ?Q ⟹ A ∧ ?Q
thm A[THEN conjI] — ?Q ⟹ A ∧ ?Q

thm conjI[OF A B] — A ∧ B

apply (rule conjI[OF A B])
done
```

3.4.6 Isabelle/ML

Isar supports embedding ML as a sub-language, where logical entities (i.e. terms and theorems) can be referred to via *antiquotations* [72]. The **ML** command can be used to execute arbitrary ML, optionally defining new variables or functions.

For example, we can define an ML tactic that attempts to solve the goal by resolving it with TrueI (i.e. the goal is trivially True).

```
ML ‹val trivial_tac = resolve0_tac [@{thm TrueI}] 1›
```

Here the thm antiquotation has retrieved the fact TrueI from the current context and embedded it in the tactic. We can then use this directly with the special-purpose tactic proof method, to invoke an ML tactic as a proof method.

```
lemma True by (tactic ‹trivial_tac›)
```

Additionally, various Isar "setup" commands take an ML function as an argument in order to define new Isar tools (e.g. attributes or methods, see Section 3.4.5 and Section 3.3). We can use **method_setup** to define a new proof method from this tactic.

```
method_setup trivially_True = ‹Scan.succeed (fn ctxt => SIMPLE_METHOD trivial_tac)›

lemma True by trivially_True
```

This natural integration with ML allows Isar to enjoy a significant amount of ad-hoc extensibility. However it also allows for many more opportunities for unexpected behaviour.

```
lemma False
  apply (tactic ‹fn _ => Seq.single @{thm TrueI}›)
```

```
No subgoals!
```

In this example it appears that we've successfully proven False. Fortunately an error is raised when we try to finish the proof.

```
done
```

```
Lost goal structure:
True
```

In this example we've destroyed the internal subgoal structure (see Section 3.2.3) of the proof by replacing it with simply True. Although this now appears to be a completed proof, with zero open subgoals, it does not prove the intended goal of False. Isar therefore throws an error when attempting to finalise the proof. This particular pitfall can be avoided by using only built-in tactics that maintain Isabelle's subgoal structure; however this demonstrates that, with ML, much more care must be taken to manage concepts that are otherwise abstracted away by Isar.

3.5 L4.verified

The L4.verified project [42] produced a formal, machine checked proof of the full functional correctness of the seL4 microkernel, down to the binary level, followed by proofs of security properties about the kernel. seL4 is a small operating system kernel, designed with explicit goals of high performance, formal verification, and secure access control. The kernel itself comprises $10,000$ lines of C code, while its corresponding proofs have grown to over $500,000$ lines of Isabelle.

The L4.verified proofs feature heavily in this thesis, as they are to-date among the largest proofs in both Isabelle and interactive theorem proving at large. In Chapter 4 we use several of its sub-projects in an empirical analysis analysing proof effort scalability, and in Chapter 7 we present case study where Eisbach is used to build proof methods that automate previous-manual proofs from L4.verified.

In L4.verified, the primary original result was *refinement* between the C implementation and its abstract specification of seL4. Roughly, a specification A is said to refine specification C if all possible observable behaviours of C are subsumed by the possible behaviours of A, given the same inputs (see Chapter 7 for more details). For seL4, this implies that the C implementation will always adhere to the abstract specification (under some assumptions). Desirable properties of the abstract specification, such as avoiding invalid memory accesses and maintaining invariant properties, are then easily shown to hold of the C implementation via refinement.

More recent work has extended this refinement stack in both directions; from showing that the final compiled binary is a faithful representation of its C source [62], to building more abstract specifications for reasoning about initialized systems on top of seL4 [19]. This allows for both greater trust in the verification result, and extends the scope of what kinds of properties can be proven about the kernel.

3.5.1 L4.verified specifications

Rather than directly prove refinement between the C and abstract specification, the project was instead divided into three sub-specifications:

The Abstract Specification (ASpec) The abstract specification is manually written in Isabelle/HOL as a functional specification for the seL4 microkernel. It is formalized as a *non-deterministic state monad* (see Section 7.1.2) in order to abstract away implementation details. For example, the concrete scheduling algorithm is abstracted into non-deterministically choosing a runnable thread. Additionally some kernel data structures, such as the *capability derivation tree*, are abstracted into pure Isabelle/HOL functions.

The Executable Specification (ExecSpec) In the early design phases of L4.verified, an initial prototype for seL4 was built in Haskell. This prototype was designed to be binary compatible with the C implementation by modelling hardware interactions as stateful transformations of a *state monad*(see Section 7.1.2). The model is sufficiently accurate such that it can be executed on the QEMU emulator [14], which allowed for design decisions to be rapidly tested in the early days of development. This Haskell kernel serves as a natural intermediate specification between the C and abstract. Rather than formalize the semantics of Haskell in Isabelle/HOL, however, it was more straightforward to generate monadic functions directly from the Haskell source. This resulted in a *shallow* embedding of the specification, which is better supported by the automation in Isabelle/HOL and is significantly easier to reason about.

The C Implementation and Specification (CSpec) The seL4 microkernel is written in a slightly restricted subset of C in order to facilitate translation into Isabelle/HOL and verification. To provide this C source with a formal semantics, it is preprocessed and given to Norrish's *C-to-Isabelle* parser [69]. This parser performs a conservative translation of C semantics into SIMPL [60], a minimalistic imperative language defined in Isabelle/HOL. The resulting SIMPL program (often referred to as the C specification) then serves as the primary verification artefact for L4.verified within Isabelle.

3.5.2 L4.verified refinement stack

Abstract invariants (AInvs) This proof contains the definition and proof of a collection of invariant properties about the abstract specification (ASpec). These were initially developed to ultimately support the executable-to-abstract (Refine) proof, and later used during the Access and InfoFlow proofs. It is supported by the `wp` method, developed to perform weakest-precondition style reasoning over monadic specifications (see Section 7.1).

Executable-to-abstract refinement (Refine) This proves that the executable specification (ExecSpec) is a refinement of the abstract specification (ASpec). It is combined with the result of CRefine to ultimately prove that the C implementation (CSpec) satisfies the abstract specification. Its refinement calculus previously lacked an automated proof method, prompting the case study presented in Chapter 7.

C-to-executable refinement (CRefine) This proves that the embedded C implementation (CSpec) is a refinement of the executable specification (ExecSpec). It is supported by the semi-automated `ctac` method, used for transforming proof states in the refinement calculus and optionally generating verification conditions for refinement.

3.5.3 Additional L4.verified proofs

Access Control (Access) The proof of functional correctness was followed by a proof that seL4 enforces integrity [63]. Phrased as an invariant property, integrity places an upper bound on the possible effects of each kernel invocation across security domains.

Information Flow (InfoFlow) Building on the Access proof, InfoFlow proves that seL4 enforces confidentiality [52]. This is proven by reasoning about multiple program traces, placing an upper bound on what information is read during a kernel invocation and ensuring that security domains can avoid leaking confidential information.

3.6 The Archive of Formal Proofs

The Archive of Formal Proofs (AFP) [43] is a collection of proofs in Isabelle, aimed at fostering the development of formal proofs and providing a place for archiving proof developments to be referred to in publications. The AFP counts over 400 entries. Two of the largest software verification proofs from the AFP are *JinjaThreads* [44] and *SATSolverVerification* [45], used in the empirical study presented in the following chapter.

JinjaThreads *Jinja* is a Java-like programming language formalised in Isabelle, with a formal semantics designed to exhibit core features of the Java language architecture, and formal proof of properties such as type safety and compiler correctness. *JinjaThreads* extends this development with Java-style arrays and threads, and shows preservation of the core properties.

SATSolverVerification This is a proof of correctness of several modern SAT solvers, including termination proofs as well as a large number of lemmas about propositional logic and CNF-formulae.

3.7 Formatting Remarks

3.7.1 Fonts

When presenting Isar input and Isabelle's output in this thesis, we use three fonts to indicate the status of the given text.

- For Isar commands and minor keywords we use **bold serif**. Example commands are **lemma**, **term** and **done**. Example minor keywords are **assumes**, **shows** and **uses**.
- When formatting both terms and inner Isar syntax, we use *italics* to indicate that content is either locally scoped (e.g. a bound or arbitrary-but-fixed variable) or that it is some other minor content (e.g. a flag or text argument).
- We use sans serif to indicate that content is some fixed constant, e.g a proven lemma, or a defined term or method.

In the example from Section 3.1.1, the formatting of the line **apply** (rule *asm1*) indicates that **apply** is an Isar command, rule is a defined method (as opposed to a *method parameter* as in Section 5.2.3), and that *asm1* is a locally-scoped assumption.

For readability (seen primarily in Chapter 7), Isar keywords will occasionally be omitted from the text when the meaning is clear.

3.7.2 Interactive Proof State

The proof of IsarSimpleExample in Section 3.1.1 does not show Isabelle's output during the proof process. Each command (i.e. **lemma**, **apply**, and **done**) produces interactive feedback to inform the author of Isabelle's proof state. Additionally, diagnostic commands (i.e. **term** and **thm**) allow the user to inspect logical elements of the current state.

To show Isabelle's output interactively in this thesis, we present the output from a given command between two horizontal lines. Revisiting our previous example, we may choose to show the proof state at each step.

```
lemma IsarSimpleExample:
  assumes asm1: A ⟹ B
      and asm2: A
  shows B
```

```
1. B
```

Here we see that the current proof state has a single subgoal, to prove *B*. We apply *asm1* with rule to perform backwards reasoning on this goal.

```
apply (rule asm1)
```

```
1. A
```

This has resolved the conclusion *asm1* with the current goal conclusion B, and now requires that we show A. At any point, we can use the diagnostic command **thm** to inspect the content of facts.

```
thm asm1
```

```
A ⟹ B
```

```
thm asm2
```

```
A
```

We can apply *asm2* to solve the current goal and finish the proof.

```
apply (rule asm2)
```

```
No subgoals!
```

At this point the proof is complete and we conclude with **done**.

```
done
```

For the sake of brevity, we may instead give Isabelle's output as inline comments in the Isar text (prefixed by —).

```
lemma IsarSimpleExample:
  assumes asm1: A ⟹ B
      and asm2: A
  shows B — 1. B
  apply (rule asm1) — 1. A

  thm asm1 — A ⟹ B
  thm asm2 — A

  apply (rule asm2) — No subgoals!
  done
```

Chapter 4

Empirical Analysis of Proof Effort Scalability

In this chapter, we investigate the relationship between the *formal specification* of a software verification proof development and the size of its eventual *formal proof.* We present an empirical analysis of proofs from both L4.verified (see Section 3.5) and Isabelle's Archive of Formal Proofs (see Section 3.6), finding a *consistent quadratic relationship* in these proof developments, between the size of the formal statement of a property, and the final size of its formal proof.

Combined with previous work by Staples et al. [64], which shows a linear relationship between the effort required to complete a proof and its final size, this suggests a quadratic relationship between the formal specification of a program and the effort required for its proof.

Informally this quadratic relationship can be justified by a common structure seen in software verification proofs: where a desired set of n properties must be verified over k lines of code, requiring $n \times k$ lines of proof. This presents a clear scalability challenge for verifying larger systems: as the number of lines grows, reducing the required effort of verifying each line becomes critical.

Later, in Chapter 5 we present Eisbach, a high-level extensible proof method language for Isabelle/Isar that allows proof authors to reduce duplicated reasoning and manual proof effort. In Chapter 7 we use Eisbach in a case study to develop a set of proof methods for performing automated reasoning in the refinement calculus of L4.verified [23]. We demonstrate the effectiveness of these methods by applying them to several existing L4.verified proofs to drastically reduce their size and complexity. By eliminating a significant manual component of these proofs, the resulting infrastructure becomes much more capable of scaling to both new, larger programs and subsequent iterations of L4.verified.

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to this paper. Specifically, his contributions were: the development of a proof metric collection framework for Isabelle, the collection of all the presented proof data, and the idealised metric detailed in Section 4.3.

Chapter Outline

- **Motivation and Summary.** Section 4.1 presents the need for empirical analysis of proof artefacts for proof engineering.
- **Approach and Measures.** Section 4.2 describes the precise measures used in this investigation.
- **Measures in Isabelle.** Section 4.3 specializes the measures from Section 4.2 to Isabelle.
- **Data Collected.** Section 4.4 presents the raw data from the investigation and its analysis.
- **Results and Discussion.** Section 4.5 reflects on the data and the metrics used.
- **Conclusion.** Section 4.6 summarizes and puts the discovered quadratic relationship of *specification size* and *proof size* into the context of this thesis.

4.1 Motivation and Summary

Despite a number of recent successes in software verification, a major hurdle has been a general lack of experience in large scale proof development. This makes it difficult to draw general conclusions about how proof effort scales, a critical question in *proof engineering.*

This suggests that empirical analysis of successful verification projects is needed in order to understand the scalability of existing approaches. Earlier work [65] has showed the need for a better understanding of how to measure artefacts in formal methods, to inform costs and estimation models. An analysis of the existing literature [39] revealed a shortage of empirical studies to provide industry with validated measures and models for the management and estimation of formal methods projects. As Stidolph and Whitehead [66] state:

> *experienced formal methodologists insist that cost and schedule estimation techniques are unsatisfactory and will remain so until a large body of experience becomes available.*

In other words, the absence of many successful large-scale verification projects makes it impossible to build a generally-applicable predictive model for required effort. We can, however, build explanatory models from the few existing suitable projects in order to understand existing scalability challenges. This paves the way for building predictive models as more experience becomes available, while informing the development of new proof techniques and technologies.

Previously [64], Staples et al. analysed proof productivity, and revealed a strong linear relationship between effort (in person-weeks) and proof size (in lines of proof script), for projects and for individuals. In this investigation we examine the inputs to software verification projects: formal statements of the properties to be proved about programs, and formal specifications of the programs. Our goal is to identify measures of these formal statements and specifications that relate to the final size of their interactive proofs.

We present results of an empirical analysis of a large number of proofs written in Isabelle. We measure the size of each lemma, in terms of the total number of concepts needed to state it, and compare that to the total number of lines used to prove it. We analysed four large sub-projects from L4.verified (see Section 3.5) as well as two proofs from the Archive of Formal Proofs (AFP) (see Section 3.6). From these 6 projects, we analysed a total of 15,018 lemma statements and associated proofs, covering a total of more than 215,000 lines of proof.

We find a consistent *quadratic relationship between statement size and proof size*, with the coefficient of determination (R^2) for the quadratic regressions varying from 0.154 to 0.845. One of the four sub-projects from L4.verified stands out with a lower R^2 and a significant collection of outliers, with proof sizes much smaller than would be expected given the statement size. Investigation revealed that these outliers were caused by *over-specified* lemma statements (see Section 4.2.4), with large constants mentioned unnecessarily, effectively inflating their statement size. To test this hypothesis, we defined an idealised measure for statement size that is an approximation of its minimum size. Using this measure greatly strengthens the relationship between statement size and proof size across all the projects, with R^2 between 0.73 and 0.937. This implies that there is a very strong quadratic relationship between statement size and proof size, when statements are not unnecessarily over-specified.

This supports a hypothesis made by Cock et al. regarding the scalability of their approach in L4.verified, stating "We hypothesise that the effort for the whole verification so far was quadratic in the size of the kernel." [23] Microkernel code is highly non-modular by nature, and so verification is dominated by proving invariants. Each invariant needs to be preserved by each feature, which in turn relies on and modifies data structures used by other features. Our current work provides empirical evidence for this hypothesis, by correlating proof size to statement size.

4.1.1 Limitations

In general, it is not possible to create a universally applicable model that relates a formal statement to its proof size or required proof effort. For example, Fermat's Last Theorem can be trivially stated, and yet over 350 years passed after its initial conjecture before a proof was found[1].

A universal model limited to only software verification is impossible as well. We could encode Fermat's last theorem as a verification problem (i.e. a program that asserts $a^n + b^n \neq c^n$ for any inputs where $n > 2$). If the integer sizes are unbounded, verifying this program would be at least as difficult as proving the original theorem.

[1] Aside from Fermat's hypothetical proof, of course.

Conversely, we could construct a degenerate 100,000 line program with extremely complex source code, but where the main entry point immediately returns without calling any other functions. This would be trivial to verify (assuming this is the intended behaviour), but would likely appear to require significant effort by any metric.

With that said, we posit that most software to be verified will not contain such edge cases, and indeed must be well-formed in order for a human to be intuitively convinced of its correctness. The bulk of the work will therefore not be in solving any hard problems, but rather iterating each desired property over the entire program's structure. From this, we hypothesize that by gathering data on existing, successful verification projects, we can gain insight into how real-world proof effort will scale.

4.2 Approach and Measures

We can consider two primary artefacts when discussing a proof development in an interactive theorem prover: a specification S and its corresponding proof P. For example, in formal verification S comprises the formalized program code (with corresponding semantics) along with desired correctness properties for the software. To prove S in an ITP, we must provide a proof P in the proof language of the ITP, most likely appealing to results from existing libraries or intermediate lemmas. The set of all definitions, intermediate lemmas, and the final proof P of S form the *proof development* that ultimately establishes the property expressed in S. This S is then referred to as the top-level statement or property of the proof development.

In order to successfully develop a proof of S it must be decomposed into subcomponents that can be stated and proven as independent lemmas. Each of these lemmas therefore contains a fragment of the overall specification as its *formal statement*. The final proof P is the culmination of these lemmas, which ultimately describe all of the components of S.

In this section we describe a general measure for the size of a *formal statement* of a lemma, as well as the size of its *formal proof*.

4.2.1 Proofs and Specifications

A proof development in an ITP is comprised of *definitions*, lemma *statements* and *proofs* of those lemmas. Definitions are used to introduce new *constants* and give them semantics, such as function specifications, program invariants or data structures. These definitions are given in the *term language* of the theorem prover, which has a precise syntax and semantics. Lemma statements relate constants, positing a fact that is then proved. Similar to definitions, lemma statements are written in the term language. Term and proof languages vary significantly between theorem provers, however they all share similar traits. A term can express logical statements, with connectives (e.g. conjunction, implication) and quantifiers (e.g. universal, existential). Proof languages have syntax for appealing to automated reasoning tools and previously proven results. A proof, in this context, is a sequence of appeals which eventually demonstrate that a lemma statement is true.

As a running example, consider the following definitions of two new constants C and

E, which mention some propositions A and B from a previous proof development. They also depend on the exclusive-OR operator $\oplus$ and the usual logical connectives.

$$\mathsf{C} \equiv (\neg\mathsf{A}) \vee (\neg\mathsf{B})$$
$$\mathsf{E} \equiv \mathsf{B} \oplus \mathsf{C}$$

The constants implicitly form a *dependency graph*. A constant c directly depends on another constant c' if c' appears in the definition of c. This represents an edge in the constant dependency graph from node c to node c'. In our example, E directly depends on B, C, and the $\oplus$ connective. We say that c depends on c'' if c'' is reachable from c in the constant dependency graph. In other words, c'' must be defined at some point in order to define c. In our example E depends on all of A, B, C, and the three connectives $\oplus$, $\vee$ and $\neg$.

Similarly lemmas also implicitly form a *dependency graph*: a lemma l directly depends on another lemma l' if l' is used to justify some step in the proof of l. Lemma l is then said to depend on l'' if at any point l'' had to be proved for the proof of l to hold. In our example, one can state the following lemmas to be proved (where the $\longrightarrow$ operator is logical implication).

$$\mathsf{E} \longrightarrow (\mathsf{A} \vee \neg\mathsf{B}) \tag{4.1}$$
$$\mathsf{E} \wedge \mathsf{B} \longrightarrow \mathsf{A} \tag{4.2}$$

The truth of Formula 4.1 and Formula 4.2 simply relies on the definitions of E and C and standard propositional logic. Their Isabelle proofs might look like the following:

```
lemma Eq1: E ⟶ (A ∨ ¬B)
  unfolding E_def C_def
  apply (rule HOL.impI)
  apply (elim xorE HOL.disjE HOL.conjE)
    apply (subst (asm) HOL.de_Morgan_disj)
    apply (subst (asm) HOL.not_not)
    apply (rule HOL.disjI1)
    apply (elim HOL.conjE)
   apply assumption
  apply (rule HOL.disjI2,assumption)+
  done

lemma Eq2: E ∧ B ⟶ A
  apply (metis Eq1)
done
```

It is not necessary to understand these proofs. We observe, however, that the proof of the lemma Eq1 refers only to facts from the **HOL** proof development, which defines the Higher Order Logic of Isabelle (see Section 3.2.4), as well as the facts that capture the definitions of C and E, C_def and E_def respectively, plus the fact xorE that in this example has already been proved in an existing proof development on which it builds.

4.2.2 Proof Size

In our analysis we consider a given lemma and relate its statement size to its proof size. Formally we define the size of the proof of a lemma l to be:

> **Proof size of lemma l:** the total number of source lines used to state and prove l, excluding definitions.

Note that this includes the proofs of all lemmas that l depends on. We exclude definition declarations (such as C_def) because, although they are part of the proof source, we are interested in the total number of *new* source lines written to complete stages (2) and (3) of a proof development; definition declarations are all written in stage (1).

We refine this notion of size slightly by only counting source lines from a particular proof development D. We call this the proof size of l with respect to D, defined as follows.

> **Proof size of lemma l with respect to proof development D:** the total number of source lines required to state and prove l, excluding definitions, as well as lemmas and proofs outside of D.

The reason that we contextualise proof size this way is because proof development is cumulative, and new proofs often build on old ones. For example, a proof development D, proving the correctness of Dijkstra's algorithm, would appeal to lemmas and definitions from an existing proof development G, a formalisation of graph theory. G would provide a definition of a graph, edges, paths, and would have lemmas proved about reachability. When measuring the size of the proof of a lemma from D in order to gauge its effort, one would consider the lemmas in G to have come at zero cost, and not take their size into account. More precisely, in general we measure the size of any proof of a lemma from a proof development D *with respect to* D, in order to exclude from its size any pre-existing lemmas on which it depends.

In our example, all the facts used in the proof of Eq1 come from an pre-existing proof development except C_def and D_def that are the definitions of C and D respectively. Thus none of the direct dependencies of the proof of Eq1 are counted when computing its size. The size of the proof of Formula 4.1 therefore is just its immediate size (i.e. 11 lines), while the proof of Formula 4.2 would be measured as its immediate size summed with the size of Formula 4.1 (i.e. $3 + 11 = 14$ lines).

4.2.3 Raw Statement Size

Here we define a measure for the statement of l that we found correlates well with the proof size of l as defined above:

> **Raw statement size for lemma l:** the total number of unique constants required to write the statement for l, including all of its dependencies, recursively.

"Unique" specifies that each constant is counted at most once per statement. This measure, importantly, is computable after stage (1) in the proof development as it only requires definitions to have been written, and does not depend on proofs. We refer to this as a statement's *raw* size. This is distinguished from the statement's *idealised* size, introduced later.

Similarly to proof size, it often makes sense to measure statement size with respect to some proof development D. Doing so excludes all constants that fall outside of D. However, while we measure the size of a *proof* in proof development D with respect to D itself, it often makes more sense to measure *statements* in D with respect to a larger proof development D' that includes D. In the example of the previous section, when measuring the size of Formula 4.1, we might choose to count the sizes of A and B, even though these constants have been defined in a pre-existing proof development, and even though proofs about A and B in this pre-existing development will not be counted in the size of the proof of Formula 4.1. The reason is that the effort of proving a new fact about a constant c (here e.g. A) might be highly impacted by the size of c even though it has been defined in a pre-existing development. In the context of the seL4 proofs we observed this effect to be extremely strong, where most statements referring to seL4's abstract specification (defined in a separate proof development)would have proofs which follow the structure of the abstract specification and carry the complexity of reasoning about it.

In the example of the previous section, assume we choose to measure the size of Formula 4.1 and Formula 4.2 with respect to the entire proof development down to the axioms of the logic, i.e. including the definitions of A and B and all the operators. Assume that A and B are complex constants with sizes 100 and 200 respectively. For simplicity we assume their dependencies are disjoint, so C would have a size of $1 + 100 + 200 + 1 + 1 = 303$, where we add 1 for C itself, the size of A, the size of B and then the size of $\neg$ and $\vee$ (we assume for simplicity that they are defined axiomatically). We can then calculate the size of E as $1 + 1 + 303 = 305$, by adding 1 for E itself, 1 for $\oplus$ (also assumed to be defined axiomatically) and then the size of C. Note that we do not add the size of B, as it has been considered already when calculating the size of C, and we are only counting unique constant dependencies. Then Formula 4.1 would have a size of $305 + 1 = 306$, where we count the size of E and the size of $\longrightarrow$ (the other constants being already considered in the size of E). Similarly Formula 4.2 would have a size of 306.

Note that both proof size and statement size are inherently recursive; the proof size of l includes the size of its dependent lemmas, and the statement size of l is based on its dependent definitions. No attempt is made to measure the immediate size of any lemma, as this is far too susceptible to fluctuations in individual proof style. In our example, Formula 4.2 has a small immediate proof size (3 lines), but would be given the same statement size as Formula 4.1. By considering each in terms of all of its dependencies we get a much more robust measure.

4.2.4 Idealised Statement Size

Most lemmas make stronger assumptions than are actually necessary. In particular, a lemma might have a concrete term where an abstract one will suffice: the statement "1 is odd" is less general than the statement "$2n + 1$ is odd", which has an abstract term "$2n + 1$" in place of the concrete term "1". It is the job of the proof engineer to decide the appropriate level of generality for a lemma, based on its intended use. In cases where a constant with a large definition is included unnecessarily, we observe a large discrepancy between statement and proof size.

In our example, consider the statement $\neg\mathsf{C} \vee \mathsf{C}$. Suppose it was proved in one step by appealing directly to the *law of excluded middle*, an axiom of HOL in Isabelle. The raw size of this statement is $303 + 1 + 1 = 305$. This is an over-estimation of the statement's complexity because the statement mentions C unnecessarily — C could be abstracted without affecting the proof. Doing so (by replacing the constant C by a variable x) would yield a statement with size of just 2 (1 for each logical connective), a much better indication of its proof complexity.

In practice, over-specificity can save effort in provers like Isabelle, as it can aid automated reasoning by simplifying *higher order unification*, a primitive procedure in Isabelle. Additionally it is not often worth the effort to generalise a lemma that will only be used once. As a result, it is common to see many over-specific lemmas in large proof developments.

To address this, we introduce *idealised statement size.* The idealised size for the statement of some lemma l is the size it would have been given had it been stated in its most general terms. More precisely, it is defined as follows.

> **Idealised statement size for lemma l:** the raw size of the statement of l had it been abstracted over all possible constants such that l's proof remains valid.

Note that we apply this recursively, conceptually removing mentions of redundant constants in all dependant constants of l's statement. The idealised statement size of l is always smaller or equal to the raw size of l, as it may only remove unnecessary constants from measurement. Unfortunately, computing this size is undecidable as it would require a precise analysis of why l is true. We show how it can be approximated in Section 4.3.3.

4.3 Measures in Isabelle

The definitions given in the previous section abstracted away from any specific theorem prover. Here we explain how we compute these measures for Isabelle.

4.3.1 Measuring Proof Size

Previously, in Section 3.1 we covered the structure of a proof in Isabelle/Isar. Each Isar proof begins with a proof command (e.g. **lemma**) and ends with a terminal command (e.g. **done**) once the proof is complete. The *lemma statement* as described in Section 4.2.2 is both the stated conclusion and any assumptions.

To measure proof size, we distinguish Isabelle *fact* (see Section 3.4.1) from the more specific Isabelle *lemmas.* Although Isabelle makes no formal distinction between these two, we consider a *lemma* to be a fact which has been explicitly stated by a proof engineer, with a provided Isabelle/Isar proof. A *fact* is more general: it is a statement that has been proved by any means. This includes lemmas, but also all the statements automatically generated and internally proved by Isabelle. For example, defining a recursive function f requires a proof of its termination. In many cases this termination proof can be done automatically with no manual invocations of tools. This fact is implicitly used in any lemma l that reasons about f, but it does not have a proof size that can be measured in such a way that would correspond to the effort required to prove it (since it is automatic). Therefore, when computing the size of the proof of a lemma l, we will only count the proof sizes of used *lemmas*. Lemmas are the only facts whose proofs require substantial human effort, therefore they are the only ones relevant to our investigation.

For a given lemma l in Isabelle we say that the lines between the beginning and ending keywords (inclusive) constitute the proof of l. The immediate size of l is therefore simply its line count. Then we compute all the lemmas which l recursively depends on and add their sizes to get l's total size. Here we only count unique lemmas: if multiple dependencies of l depend on some l', we only count the size of l' once. We compute lemma dependencies by examining proof terms produced by Isabelle [15], where a proof term is the internal formal representation of a proof. The total size of l with respect to some proof development D considers all lemmas outside of D to have size 0. Simply put, for each proof development, lemmas not from that development (e.g. pre-existing library lemmas), do not contribute to the measured size of proofs.

This measure approximates the proof size of l as described in Section 4.2.2. It is incomplete, however, as it does not include source lines which must exist for l to be valid, but for which this dependency relationship is not easily found. For example, the proof of l may depend on certain syntax existing, declared with one line using the **notation** keyword. This is not factored into the total size of l. The impact of this on the validity of the measured proof size is assumed to be minimal, as the size of Isabelle proof developments is dominated by proof text.

4.3.2 Measuring Raw Statement Size

In Section 3.4.2 we saw how new constants could be introduced to Isabelle with Isar. Internally these definitional commands create simple definitions based on the user specification, proving canonical facts for interpreting them. In our running example, the definition of C introduces the new name C with body $(\neg\mathsf{A}) \lor (\neg\mathsf{B})$, and a new fact $\mathsf{C\_def}$ for the statement $\mathsf{C} = (\neg\mathsf{A}) \lor (\neg\mathsf{B})$.

To measure the size of a given lemma statement S as described in Section 4.2.2 we recursively inspect all definitions mentioned in S. The number of unique definitions traversed will be the size of S according to this measure.

4.3.3 Approximating Idealised Statement Size

In Section 4.2.4 we introduced idealised statement size as a refinement of raw statement size. Although computing the idealised size of S is, in general, undecidable, it can be approximated by examining the finished proof of S.

Intuitively, if S refers to C but the defining fact $\mathsf{C\_def}$ of the constant C does not appear in the dependency graph of the proof of S, this means that the truth of S does not depend on the definition of C, and that S could be rewritten by replacing the constant C by a variable x. We therefore exclude the size of C when computing the approximate idealised size of S. For instance, the approximate idealised statement size of the statement

$\neg \mathsf{C} \lor \mathsf{C}$, proved by appealing directly to the law of excluded middle without using C_def, is 2.

More generally, for each constant c, we have a set of *defining equations*. These are Isabelle facts which are the canonical interpretation of c. In the case of simple definitions, this is just the equation that was given when c was defined (such as C_def). To compute the idealised size of a statement S, we exclude all the constants whose defining equations are never used in the proof of S. This will always be an over-approximation of the idealised size of S, but is at worst the original statement size.

This approximation of idealised statement size cannot be a leading predictor of proof size, as it requires stage (3) of the proof to be complete. However, it is useful when trying to build an explanatory model for understanding the relationship between statement size and proof size. The implications of using this measure are discussed in Section 4.5

4.4 Data Collected

We applied our exploratory analysis to six projects: (1) four top-level statements from L4.verified, and (2) two proof developments in the Archive of Formal Proofs.

4.4.1 L4.verified Proofs

Previously (see Section 3.5) we outlined the structure of L4.verified. We applied these measures to the AInvs, Refine, Access and Infoflow proof developments, taken from the public release of the seL4 proofs [68]. The statements measured for each proof development are all the dependencies of its top-level statement, computed with respect to that development. The statement sizes for these proofs are computed with respect to the whole seL4 verification development, where the large kernel specifications are defined.

AInvs: The top-level statement in this proof depends on both the abstract specification for seL4 (*ASpec*) and the abstract invariants (invs). We measured 2,790 lemmas from *AInvs*, including the top-level statement with a raw size of 1,292, ideal size of 949 and proof size of 32,214 lines (measured as described in Section 4.3).

Refine: The top-level statement depends on the abstract specification (*ASpec*) and the executable specification (*ExecSpec*), as well as a corresponding global invariant for each of them. *Refine* builds on *AInvs*, but we compute the proof sizes for *Refine* with respect to itself. That is, proofs from *AInvs* do not contribute to the size of proofs from *Refine*. We measured 4,143 lemmas from *Refine*, including the top-level statement with a raw size of 2,398, ideal size of 1,746 and proof size of 67,856 lines.

Access: The top-level statement depends on *ASpec*, the definition of integrity (see Section 3.5.3), as well as invs. Similar to *Refine*, we only measure *Access* proofs with respect to itself, taking proofs from *Refine* and *AInvs* for granted. We measured 724 lemmas from *Access*, including the top-level statement with a raw size of 1,395, ideal size of 1,083 and proof size of 8,116 lines.

InfoFlow: The top-level statement depends on ASpec, the definition of integrity, and invs. Additionally it includes a more involved discussion of program execution traces. As previously done, proof sizes from *InfoFlow* are measured with respect to itself. We

Table 4.1: R^2 and equations for Figure 4.1 and Figure 4.2

Proof Name	Size Used	R^2	Equation $f(x) = a \cdot x^2 + b \cdot x + c$		
			a	b	c
AInvs	**raw**	**0.845**	**0.02579**	**-8.181**	**394.9**
	idealised	*0.937*	*0.04325*	*-4.399*	*100.4*
Refine	**raw**	**0.724**	**0.01198**	**-5.355**	**519.1**
	idealised	*0.799*	*0.01737*	*2.365*	*<2.2E-16*
Access	**raw**	**0.735**	**0.0032**	**-1.112**	**86.45**
	idealised	*0.889*	*0.006039*	*-0.915*	*57.36*
InfoFlow	**raw**	**0.154**	**-0.0003736**	**1.743**	**<2.2E-16**
	idealised	*0.73*	*0.007893*	*-3.652*	*260.4*
JinjaThreads	**raw**	**0.457**	**0.05631**	**-16.46**	**472.9**
	idealised	*0.694*	*0.1166*	*-16.04*	*281*
SATSolver Verification	**raw**	**0.798**	**1.711**	**-65.43**	**375.7**
	idealised	*0.802*	*4.123*	*-77.52*	*223.7*

measured 1,665 lemmas from *InfoFlow*, including the top-level statement with a raw size of 2,029, ideal size of 1,323 and proof size of 19,579 lines.

4.4.2 Proofs from the AFP

The Archive of Formal Proofs (see Section 3.6) is a public collection of proofs in Isabelle. We applied these measures to two of its largest software verification proof developments.

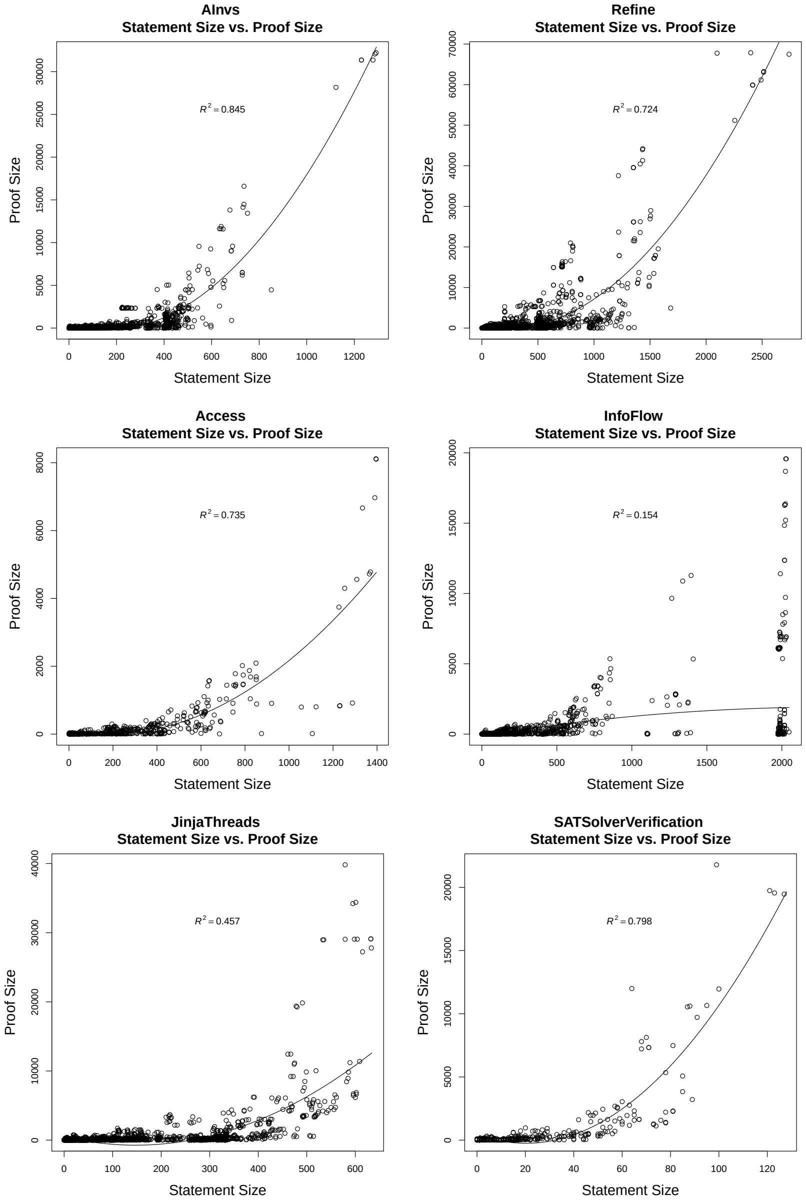
JinjaThreads: We measured 5,215 lemmas from *JinjaThreads*, including the top-level statement with a raw size of 579, ideal size of 453 and proof size of 39,821 lines.

SATSolverVerification: We measure the proof with respect to the *FunctionalImplementation* theory, an implementation of a SAT solver within Isabelle's HOL. We measured 481 lemmas from *SATSolverVerification*, including the top-level statement with a raw size of 99, ideal size of 57 and proof size of 21,788 lines.

4.5 Results and Discussion

For each of the six projects analysed, we computed the statement size and proof size of all of its lemmas, using the measures described in Section 4.3. As explained, we used two variants for the statement size: a raw measure and an idealised measure, where the latter represents what the size of the statement would be if stated in its most general form.

The results are given in Figure 4.1 for the raw measure and in Figure 4.2 for the idealised measure. The analysis demonstrates a *consistent quadratic relationship* between statement size and proof size across all the projects, with a stronger relationship when using the idealised measure. The respective R^2 and equations of the regression lines are given in Table 4.1. We now discuss the results in detail.



AInvs
Statement Size vs. Proof Size

Refine
Statement Size vs. Proof Size

Access
Statement Size vs. Proof Size

InfoFlow
Statement Size vs. Proof Size

JinjaThreads
Statement Size vs. Proof Size

SATSolverVerification
Statement Size vs. Proof Size

Figure 4.1: Relation between raw statement size and proof size (measured as described in Section 4.3) for the six projects analysed.

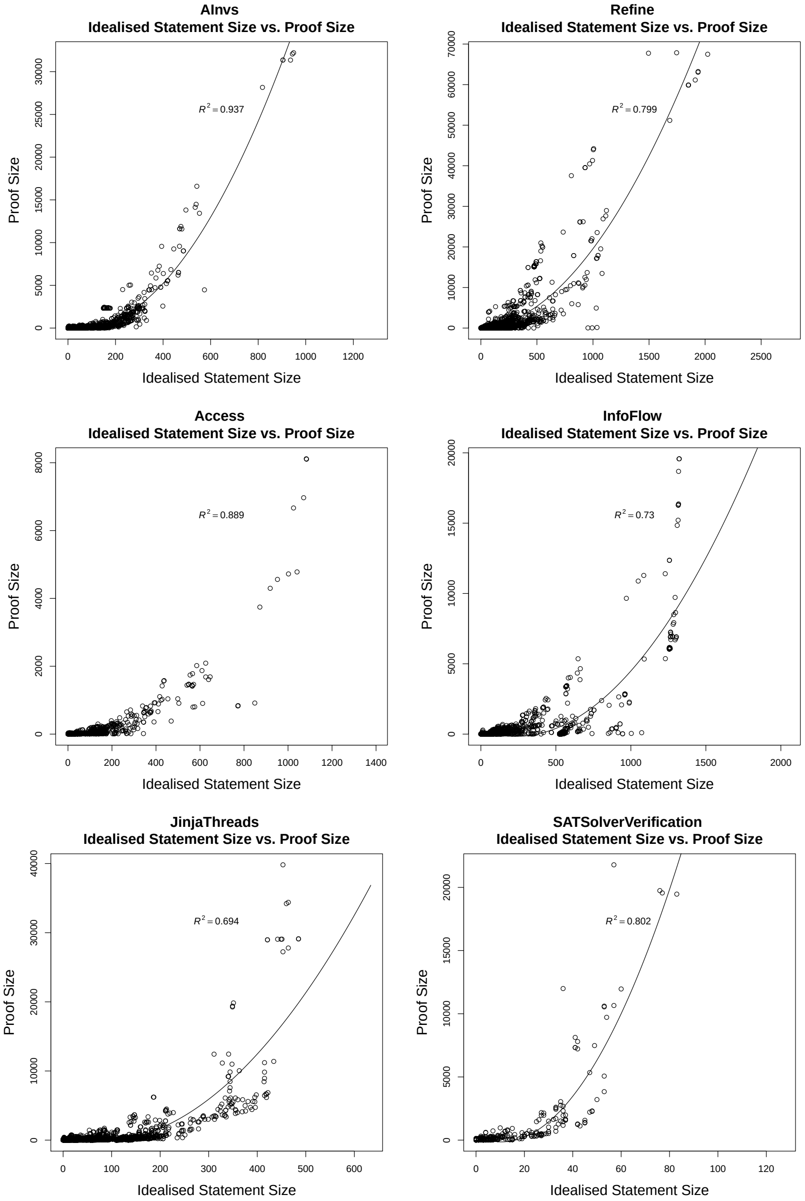


Figure 4.2: Relation between idealised statement size and proof size (measured as described in Section 4.3) for the six projects analysed.

4.5.1 Results using the Raw Measure

For the first three results the regression for the raw size fits the data nicely, as we can see in the plots and confirmed by the R^2 results. However, even in *Access* some outliers can be seen in the lower right corner of the graph. In *InfoFlow* this effect is even stronger: a cluster of points with a large statement size ($\sim 2,000$) and a negligible proof size. This has the effect of flattening the regression line and obfuscating the relationship. After a thorough examination we determined that these are statements that have been over-specified (see Section 4.2.4), resulting in a larger statement size than expected.

There likely exist many other over-specified lemmas, but they have a less apparent effect on the overall shape of the graph. The lemmas identified in this case were pathological: they mentioned the entire abstract specification but were stating a general property. The abstract specification is one of the largest constants in this development, so including it in a statement makes its size completely dominated by the size of the abstract specification. All of the lemmas in *AInvs* are similarly over-specified, but more subtly so. Indeed the presence of this over-specification was only made clear after performing the idealised size analysis. This over-specification is a result of abstract specification being *extensible* [46], embedding an optional deterministic implementation of certain operations, which can be used in place of non-deterministic ones when necessary. However, no proof in *AInvs* appeals to this deterministic implementation. This is simply because, by design, the standard invariants do not discuss the program state required to resolve this determinism. As a result, these deterministic operations unnecessarily inflate the statement size as measured in *AInvs*.

4.5.2 Effectiveness of the Idealised Measure

The idealised size, introduced in Section 4.2.4, is meant to capture the idea that redundant constants do not contribute to the difficulty of a proof. Analysis using the idealised size (shown in Figure 4.2 and Table 4.1) show improved results across all projects. In particular, the *InfoFlow* results are now much more aligned and consistent with the others. R^2 now varies from 0.694 to 0.937. We can also see from the graphs that all of the statement sizes shrunk, reflecting the fact that the idealised size is always smaller or equal to the raw size.

We made no attempt to investigate the outliers in the proofs from the AFP, simply because we do not know their proofs well enough to perform the same level of analysis. Despite this, applying the idealised size measure had significant improvement on the clarity of the data in *JinjaThreads*, with a lesser effect in *SATSolverVerification*, which was already quite clear.

Previous work [65] showed a linear relationship between the size of formal specifications (as measured in source lines of Isabelle) and the number of lines of source code. Although not measured, we argue that the specification size (as described in Section 4.2.3) will scale linearly with source lines. Combined with the result above, this indicates a quadratic relationship between code size, property size, and eventual proof size. This confirms our intuition based on our experience working with the seL4 proofs: the correctness of each property depends on its interaction with the entire program, resulting in the observed quadratic relationship. Despite this intuition, and the apparent shape of the data, we investigated other regressions in the course of this study. Specifically we performed linear, cubic and exponential regressions against the measured proof size and ideal/raw specification sizes. Linear and exponential regressions were less compelling than quadratic, with extremely low R^2 values and clearly not fitting the data. A cubic regression yielded a marginal increase in R^2, but with an extremely small leading coefficient, indicating that the relationship is indeed quadratic.

4.6 Conclusion

The goal of this investigation was to gain insight into how the complexity of the inputs to large-scale verification projects (i.e. the program source and specification) ultimately affect their required proof effort. The relatively low number of successful large-scale applications of formal verification, and corresponding lack of empirical studies, have made it difficult to understand how proof effort will scale when verifying even larger programs. Earlier work by Staples et al. [64] established a linear relationship between proof size and proof effort. Expanding on this, we investigated the relationship between the size of a *formal statement* and its corresponding *formal proof*.

We have established two measures for statement size, *raw* and *idealised*, and use them in an analysis of six Isabelle proof developments. Raw size is the number of unique constants required to write a statement, recursively including all dependencies. This measure was shown to be highly susceptible to over-specified statements having inflated sizes. This prompted the introduction of idealised size, a refinement of raw size, which removes redundant constants in order to reduce the impact of over-specified statements. In total we examined the size of 15,018 statements and compared them against their proof size.

Our analysis shows a quadratic relationship between statement and proof size, and that our idealised measure strengthens this correlation. Combined with previous work from Staples et al. [65], this result suggests a quadratic relationship between verified code and its corresponding proof size. We speculate that this relationship will hold most consistently when verifying invariant properties over non-modular code.

This poses a significant scaling challenge for formal verification. The source of seL4 is only on the order of 10,000 lines of code, and yet has resulted in over 500,000 lines of formal proof across all of its related developments. For the next grand verification challenges, with 100,000 lines of code, a quadratic scaling factor would likely bring the corresponding lines of proof into the tens of millions.

Motivated by this, in the following chapters we will see how Eisbach provides high-level access to the automated reasoning infrastructure of Isabelle. In Chapter 7 we use Eisbach to develop a collection of proof methods for automating refinement proofs. We apply these methods to existing proofs from *Refine* in L4.verified, achieving a significant reduction in proof size and complexity. This demonstrates that by empowering proof engineers to easily develop domain-specific automated reasoning tools, we can increase the efficiency of manual proof effort and reduce the work required to verify each line of code.

Chapter 5

Eisbach

Motivated by the proof effort scalability issues presented in the previous chapter, in this chapter we present *Eisbach*, a proof method language for Isabelle. Isabelle's primary proof language, Isar (see Section 3.1), has previously lacked a means to write automated proof procedures. In many proofs, the result has been a significant duplication of effort, exacerbating both the initial development and ongoing maintenance costs in large proof developments.

Isar is a convenient and effective interface for new Isabelle users, as it abstracts away many of Isabelle's technical and formal details. It also provides advanced features for experts, including the ability to embed ML functions in order to interact with Isabelle directly (see Section 3.4.6). The vast majority of Isabelle proof developments today, however, almost exclusively use Isar, with little to no ML source. In particular, very few custom proof methods are developed[1], instead relying exclusively on Isabelle's built-in automated methods.

Eisbach incorporates language elements from Isar to allow users to write proof methods at a familiar level of abstraction, while supporting more advanced behaviour through the included match method. In this chapter we describe the language itself and the design principles on which it was developed.

Later, in Chapter 6, we will see how this core functionality can be extended to increase the scope of developing proof methods with Eisbach.

Acknowledgements

Eisbach first appeared at the International Conference on Interactive Theorem Proving [50], and later in the Journal of Automated Reasoning [48], in collaboration with Makarius Wenzel and Toby Murray. It has been included in the main Isabelle release since Isabelle-2015, with a system manual [49] since Isabelle-2016-1. The content in this chapter is based on these publications.

The initial design and prototype implementation of Eisbach was completed independently by the author. Subsequent versions were designed in collaboration with Makarius

[1] With the exception of proof developments that now take advantage of Eisbach since its inclusion in Isabelle.

Wenzel, although the majority of the implementation was by the author. Makarius Wenzel, however, has continued to update the Eisbach sources since its initial release in conjunction with improvements to Isar. Some functionality has since been moved from Eisbach's module to Isar's proof method module.

Thanks to Thomas Sewell for suggesting the name "Eisbach", referring to the tributary of the Isar river in Munich.

Chapter Outline

- **Motivation.** Section 5.1 motivates Eisbach as a natural extension of Isar.
- **Eisbach.** Section 5.2 demonstrates the functionality of Eisbach by incrementally building a simple first-order logic solver.
- **Design and Implementation.** Section 5.3 presents the design principles of Eisbach and some implementation details.
- **Conclusion.** Section 5.4 summarizes the primary benefits of Eisbach.

5.1 Motivation

In the previous chapter, we investigated the relationship between the size of a formal specification and the size of its eventual proof. We found that, for the software verification projects considered, as the size of a formal statement increases, the size of its corresponding proof grows quadratically. Building on previous work by Staples et al. [64][65], this suggests that a quadratic increase in proof effort will be required when applying existing techniques to verify larger programs.

Machine-checked proofs in many domains have been steadily growing in size, with verification projects accounting for the largest among these. Isabelle's Archive of Formal Proofs (see Section 3.6) now comprises over 1.8 million lines, having over doubled in size in the past 5 years [1] and continuing to grow. As ever-larger proof developments are attempted, the scalability of proof effort becomes paramount, as does the maintainability of the produced artefacts.

In Section 3.1 we gave a brief overview of Isar, Isabelle's primary proof language. Isar provides a suite of keywords and commands for structuring proofs, but does not perform any computation or reasoning directly. Arbitrarily complex proof tools called *proof methods* (see Section 3.3) operate on the proof state. Proof methods are traditionally written in Isabelle/ML: Standard ML that is embedded into the logical context of Isar (see Section 3.4.6). With Isabelle/ML, the full power of ML is always available in proofs and proof methods, however the vast majority of Isabelle theories are written solely in Isar.

In this chapter, we present a proof method language for Isabelle, called *Eisbach*, that allows writing proof procedures by appealing to existing proof tools with their usual syntax. The new Isar command **method** allows proof methods to be combined, named, and abstracted over terms, facts and other methods. Eisbach is inspired by Coq's Ltac (see Section 2.1.5), and includes similar features such as matching on facts and the current goal. However, Eisbach's matching behaves differently from Ltac's, especially with respect to backtracking (see Section 5.2.4). Eisbach continues the Isabelle philosophy of exposing carefully designed features to the user while leaving more sophisticated functionality to Isabelle/ML. Eisbach benefits from general Isabelle concepts, while easing their exposure to users: pervasive backtracking (see Section 3.3), the structured proof context with named facts, and attributes to declare hints for proof tools (see Section 3.4.4).

As a quick motivating example, consider the following lemma which proves a simple property of lists, by induction on the argument list *xs*, and application of the auto proof method with an explicit simplification rule passed as its argument.

```
lemma length (xs @ ys) = length xs + length ys
  by (induct xs ; auto simp: append_Nil)
```

Indeed, as anyone who has worked through the first examples in the Isabelle/HOL tutorial [54] can attest, many simple properties of lists are proved using exactly this same procedure, perhaps varying only on the extra simplification rules to be applied by auto.

The following simple usage of Eisbach defines a new proof method which generalises this procedure. The method defined identifies a list in the conclusion of the current subgoal and applies induction to it; all newly emerging subgoals are solved with auto, with additional simplification rules given as argument.

```
method induct_list uses simps =
  (match conclusion in ?P (x :: 'a list) for x ⇒
    ‹induct x ; auto simp: simps›)
```

Now induct_list can be called as a proof method to prove simple properties about lists such as the one above.

```
lemma length (xs @ ys) = length xs + length ys
  by (induct_list simps: append_Nil)
```

The term *xs* @ *ys* is now selected from the goal implicitly via match, and the proof succeeds as before.

The primary goal of Eisbach is to make writing proofs more productive, increasing the scalability of proof effort by avoiding duplication. Its design principles are:

- To be easy to use for beginners and experts.
- To expose limited functionality, leaving complex functionality to Isabelle/ML.
- To be extensible by end-users.
- Seamless integration with other Isabelle languages.
- To continue Isar's principle of readable proofs, creating *readable proof procedures*.

5.2 Eisbach

The core functionality of Eisbach is exposed via the **method** command. This allows compound proof methods, combined with method combinators (see Section 3.3.3), to be named and re-used. Method definitions may abstract over parameters: terms, facts or other methods. The provided match method exposes expressive matching facilities when defining new methods, used to manage control flow and perform goal analysis via unification.

Recall the following example from Section 3.3:

lemma $P \wedge Q \longrightarrow P$

  by ((rule impI, (erule conjE)*?*) | assumption)+

In this example, the rules impI and conjE are repeatedly applied to the goal until the proof is complete. As well as the above lemma, this invocation will prove the correctness of a small class of propositional logic tautologies. With the **method** command we can define a proof method that makes the above functionality available generally.

method prop_solver$_1$ =

  ((rule impI, (erule conjE)*?*) | assumption)+

lemma $P \wedge Q \wedge R \longrightarrow P$

  by prop_solver$_1$

5.2.1 Fact Abstraction

We can generalize prop_solver$_1$ by abstracting it over the introduction and elimination rules it currently applies. In the previous example, the facts impI and conjE are static. They are evaluated once when the method is defined and cannot be changed later. This makes the method stable in the sense of *static scoping*: naming another fact impI in a later context won't affect the behaviour of prop_solver$_1$. To instead pass these facts to the method when it is invoked, we can declare some fact-parameters with the **uses** keyword.

method prop_solver$_2$ **uses** *intros elims* =

  ((rule *intros*, (erule *elims*)*?*) | assumption)+

lemma $P \wedge Q \wedge R \longrightarrow P \wedge (Q \longrightarrow R)$

  by (prop_solver$_2$ *intros*: impI conjI *elims*: conjE)

In this particular example, however, providing these rules on each invocation of prop_solver$_2$ is cumbersome. In the following section we will see how we can create an Eisbach method that is extensible, but also has a database of fact hints that are implicitly used.

Named Theorems

A *named theorem* is a fact whose contents are produced dynamically within the current proof context. The Isar command **named_theorems** provides simple access to this concept: it declares a dynamic fact with corresponding *attribute* (see Section 3.4.4) for managing this particular data slot in the context.

```
named_theorems intros
```

So far intros refers to the empty fact. Using the Isar command **declare** we may apply declaration attributes to the context. Below we declare both conjI and impI as intros, adding them to the named theorem slot.

```
declare conjI [intros] and impI [intros]
```

We can refer to named theorems as dynamic facts within a particular proof context, which are evaluated whenever the method is invoked. Instead of explicitly providing these arguments to prop_solver$_2$ on each invocation, we can instead refer to these named theorems.

```
named_theorems elims
declare conjE [elims]

method prop_solver_3 =
  ((rule intros, (erule elims)?) | assumption)+

lemma P ∧ Q ⟶ P
  by prop_solver_3
```

Often these named theorems need to be augmented on the spot, when a method is invoked. The **declares** keyword in the signature of **method** adds the common method syntax *method decl*: *facts* for each named theorem *decl*.

```
method prop_solver_4 declares intros elims =
  ((rule intros, (erule elims)?) | assumption)+

lemma P ∧ (P ⟶ Q) ⟶ Q ∧ P
  by (prop_solver_4 elims: impE intros: conjI)
```

5.2.2 Term Abstraction

Methods can also abstract over terms using the **for** keyword, optionally providing type constraints. For instance, the following proof method intro_ex takes a term y of any type, which it uses to instantiate the x-variable of exI (existential introduction) before applying the result as a rule. The instantiation is performed here by Isar's where attribute. If the current subgoal is to find a witness for the given predicate Q, then this has the effect of committing to y.

```
method intro_ex for Q :: 'a ⇒ bool and y :: 'a =
  (rule exI [where P = Q and x = y])
```

The term parameters y and Q can be used arbitrarily inside the method body, as part of attribute applications or arguments to other methods. The expression is type-checked as far as possible when the method is defined, however dynamic type errors can still occur when it is invoked (e.g. when terms are instantiated in a parameterized fact). Actual term arguments are supplied positionally, in the same order as in the method definition.

```
lemma P a ⟹ ∃x. P x
  by (intro_ex P a)
```

5.2.3 Custom Combinators

The original proof method combinators (see Section 3.3.3) were chosen as a minimal subset of Isabelle's standard tacticals. Additionally, methods intentionally do not have subgoal addressing, as they are either implicitly applied to the first subgoal or apply to all subgoals simultaneously. This quickly proves to be too restrictive when writing proof methods in Eisbach.

The new method combinator for *structured concatenation* was introduced to Isar with initial release of Eisbach. Structured concatenation ($method_1$; $method_2$) is similar to sequential composition ($method_1$, $method_2$), except that $method_2$ is invoked on *all* subgoals that have newly emerged from $method_1$. This is useful to handle cases where the number of subgoals produced by a method is determined dynamically at run-time (e.g. when defining an Eisbach method). This is analogous to the `THEN_ALL_NEW` tactical available in ML.

```
method conj_with uses my_rule =
  (intro conjI ; intro my_rule)

lemma
  assumes A: P
  shows P ∧ P ∧ P
  by (conj_with my_rule: A)
```

Moreover, Eisbach method definitions may take other methods as arguments, and thus implement method combinators with prefix syntax. For example, to more usefully exploit Isabelle's backtracking, it can often be useful to require a method to solve all produced subgoals. This can easily be written as a *higher-order method* using ";". The **methods** keyword denotes method parameters that are other proof methods to be invoked by the method being defined.

```
method solves methods m = (m ; fail)
```

Given some method-argument m, *solve* ‹m› applies the method m and then fails whenever m produces any new unsolved subgoals — i.e. when m fails to completely discharge the goal it was applied to.

With these simple features we are ready to write our first non-trivial proof method. Returning to the first-order logic example, the following method definition applies various rules with their canonical methods.

```
named_theorems subst

method prop_solver declares intros elims subst =
  (assumption |
    rule intros | erule elims |
    subst subst | subst (asm) subst |
    (erule notE ; solves ‹prop_solver›))+
```

The only non-trivial part above is the final alternative (erule notE ; solve ‹prop_solver›).

Here, in the case that all other alternatives fail, the method takes one of the assumptions $\neg\ P$ of the current goal and eliminates it with the rule notE, causing the goal to be proved to become P. The method then recursively invokes itself on the remaining goals. The job of the recursive call is to demonstrate that there is a contradiction in the original assumptions (i.e. that P can be derived from them). Note that this recursive invocation is applied with the solves method combinator to ensure that a contradiction will indeed be shown. In the case where a contradiction cannot be found, backtracking will occur and a different assumption $\neg\ Q$ will be chosen for elimination.

Note that the recursive call to prop_solver does not have any parameters passed to it. Recall that fact parameters, e.g. intros, elims, and subst, are managed by declarations in the current proof context. They will therefore be passed to any recursive call to prop_solver and, more generally, any invocation of a method which declares these named theorems.

After declaring some standard rules to the context, the prop_solver becomes capable of solving non-trivial propositional tautologies.

```
lemmas [intros] =
  conjI  — ⟦?P; ?Q⟧ ⟹ ?P ∧ ?Q
  impI  — (?P ⟹ ?Q) ⟹ ?P ⟶ ?Q
  disjCI  — (¬ ?Q ⟹ ?P) ⟹ ?P ∨ ?Q
  iffI  — ⟦?P ⟹ ?Q; ?Q ⟹ ?P⟧ ⟹ ?P = ?Q
  notI  — (?P ⟹ False) ⟹ ¬ ?P

lemmas [elims] =
  impCE  — ⟦?P ⟶ ?Q; ¬ ?P ⟹ ?R; ?Q ⟹ ?R⟧ ⟹ ?R
  conjE  — ⟦?P ∧ ?Q; ⟦?P; ?Q⟧ ⟹ ?R⟧ ⟹ ?R
  disjE  — ⟦?P ∨ ?Q; ?P ⟹ ?R; ?Q ⟹ ?R⟧ ⟹ ?R

lemma (A ∨ B) ∧ (A ⟶ C) ∧ (B ⟶ C) ⟶ C
  by prop_solver
```

5.2.4 Matching

So far we have seen methods defined as simple combinations of other methods. Some familiar programming language concepts have been introduced (i.e. abstraction and recursion). The only control flow has been implicitly the result of backtracking. When designing more sophisticated proof methods this proves too restrictive and too difficult to manage conceptually.

We therefore introduce the match method, which provides more direct access to the higher-order matching facility at the core of Isabelle. It is implemented as a separate proof method (in Isabelle/ML), and thus can be directly applied to proofs. However, it is most useful when applied in the context of writing Eisbach method definitions.

Matching allows methods to introspect the goal state, and to implement more explicit control flow. In the basic case, a term or fact *ts* is given to match against as a *match target*, along with a collection of pattern-method pairs $(p,\ m)$: roughly speaking, when the pattern p matches any member of *ts*, the *inner* method m will be executed.

Consider the following example:

lemma
fixes P
assumes X:
$Q \longrightarrow P$
Q
shows P
by (match X **in** I: $Q \longrightarrow P$ **and** I': $Q \Rightarrow$ ‹rule mp [OF I I']›)

Here we have a structured Isar proof, with the named assumption X and a conclusion P. With the match method we can find the local facts $Q \longrightarrow P$ and Q, binding them separately as I and I'. We then specialize the modus-ponens rule $Q \longrightarrow P \Longrightarrow Q \Longrightarrow P$ to these facts to solve the goal.

Here we were able to match against an assumption out of the Isar proof state. In general, however, proof subgoals can be *unstructured*, with goal parameters and premises arising from rule application. For example, an unstructured version of the previous proof state would be the Pure implication $\bigwedge P.\ Q \longrightarrow P \Longrightarrow Q \Longrightarrow P$. Here the premises $Q \longrightarrow P$ and Q are unnamed and P is a quantified variable (or goal parameter) rather than a fixed term.

To handle unstructured subgoals, match uses *subgoal focusing* (see also Section 5.3.5) to produce structured goals out of unstructured ones.

In place of fact or term, we may give the keyword **premises** as the match target. This causes a subgoal focus on the first subgoal, lifting local goal parameters to fixed term variables and premises into hypothetical theorems. The match is performed against these theorems, naming them and binding them as appropriate. Similarly giving the keyword **conclusion** matches against the conclusion of the first subgoal.

An unstructured version of the previous example can then be similarly solved through focusing.

lemma $\bigwedge P.\ Q \longrightarrow P \Longrightarrow Q \Longrightarrow P$
by (match **premises in** I: $Q \longrightarrow ?A$ **and** I': $Q \Rightarrow$ ‹rule mp [OF I I']›)

In this example the goal parameter P is first fixed as an anonymous internal term (e.g. P__, where an underscore suffix indicates that the term cannot be referenced directly), and then the premises $Q \longrightarrow P$__ and Q are assumed as hypothetical theorems. In the first pattern, the schematic variable $?A$ acts as a wildcard, and thus the pattern $Q \longrightarrow ?A$ matches the first premise by matching $?A$ to the newly-fixed P__. The second pattern then matches the second premise Q, binding it to I'. Finally the inner method is executed, similar to the previous example, and successfully solves the goal.

Match variables may be specified by giving a list of **for**-fixes after the pattern description. These variables are then considered wildcards, similar to schematics. In contrast to schematic variables, however, **for**-fixed terms are bound to the result of the match, and may be referred to inside of the inner method body. In the previous example we could not give $Q \longrightarrow P$__ as a match pattern, because P__ cannot be referred to directly. If we want to refer to P__ directly we must first bind it with a **for**-fix in a pattern.

lemma $\bigwedge P.\ Q \longrightarrow P \Longrightarrow Q \Longrightarrow P$
by (match **premises in** I: $Q \longrightarrow A$ **and** I': Q **for** $A \Rightarrow$
‹match **conclusion in** $A \Rightarrow$ ‹rule mp [OF I I']››)

In this example A is a match variable which is effectively bound to the goal parameter P upon a successful match. The inner match then matches the now-bound A (bound to P) against the conclusion (also P), finally applying the specialized rule to solve the goal.

In the following example we extract the predicate of an existentially quantified conclusion in the current subgoal and search the current premises for a matching fact. If both matches are successful, we then instantiate the existential introduction rule with both the witness and predicate, solving with the matched premise.

```
method solve_ex =
  (match conclusion in ∃ x. Q x for Q ⇒
    ‹match premises in U: Q y for y ⇒
      ‹rule exI [where P = Q and x = y, OF U]››)
```

The first match matches the pattern $\exists\, x.\ Q\ x$ against the current conclusion, binding the term Q in the inner match. Next the pattern $Q\ y$ is matched against all premises of the current subgoal. In this case Q is fixed and y may be instantiated. Once a match is found, the local fact U is bound to the matching premise and the variable y is bound to the matching witness. The existential introduction rule exI: $P\ x \Longrightarrow \exists\, x.\ P\ x$ is then instantiated with y as the witness and Q as the predicate, with its proof obligation solved by the local fact U (using the Isar attribute OF). The following example is a trivial use of this method.

```
lemma halts p ⟹ ∃ x. halts x
  by solve_ex
```

Within a match pattern for a fact, each outermost meta-universally quantified variable specifies the requirement that a matching fact must have a schematic variable at that point. This gives a corresponding name to this "slot" for the purposes of forming a static closure, allowing the where attribute to perform an instantiation at run-time.

```
lemma
  assumes A: Q ⟹ False
  shows ¬ Q
  by (match intros in X: ⋀R. (R ⟹ False) ⟹ ¬ R ⇒
        ‹rule X [where R = Q, OF A]›)
```

In this example, the match expression successfully matches notI (($?P \Longrightarrow$ False) $\Longrightarrow \neg\ ?P$) from intros and renames its schematic $?P$ to $?R$ in the local fact X, which can then be instantiated with where. This pattern statically guarantees that, given a successful match, such an instantiation will be possible.

Subgoal focusing (see Section 5.3.5) converts the outermost meta-universally quantified variables of premises into schematics when lifting them to hypothetical facts. This allows us to instantiate them with where when using an appropriate match pattern.

```
lemma (⋀x :: 'a. A x ⟹ B x) ⟹ A y ⟹ B y
  by (match premises in I: ⋀z :: 'a. ?P z ⟹ ?Q z ⇒
        ‹rule I [where z = y]›)
```

Here, the first premise of the goal has been lifted into the local fact $A\ ?x \Longrightarrow B\ ?x$, and matched successfully against the given pattern. This produces a named local fact I, where $?x$ has been renamed to $?z$, allowing the inner where instantiation to successfully produce the fact $A\ y \Longrightarrow B\ y$.

Match Backtracking

Multiple pattern-method pairs can be given to match, separated by a "|". These patterns are considered top-down, executing the inner method m of the first pattern which is satisfied by the current match target. By default, matching performs extensive backtracking by attempting all valid variable and fact bindings according to the given pattern. In particular, all unifiers for a given pattern will be explored, as well as each matching fact. The inner method m will be re-executed for each different variable/fact binding during backtracking. A successful match is considered a cut-point for backtracking. Specifically, once a match is made no other pattern-method pairs will be considered.

The method foo below fails for all goals that are conjunctions. Any such goal will match the first pattern, causing the second pattern (that would otherwise match all goals) to never be considered. If multiple unifiers exist for the pattern $?P \wedge ?Q$ against the current goal, then the failing method *fail* will be (uselessly) tried for all of them.

```
method foo =
  (match conclusion in ?P ∧ ?Q ⇒ ‹fail› | ?R ⇒ ‹prop_solver›)
```

This behaviour is in direct contrast to the backtracking done by Coq's Ltac [25], which will attempt all patterns in a match before failing. This means that the failure of an inner method that is executed after a successful match does not, in Ltac, cause the entire match to fail, whereas it does in Eisbach. In Eisbach the distinction is important due to the pervasive use of backtracking. When a method is used in a combinator chain, its failure becomes significant because it signals previously applied methods to move to the next result. Therefore, it is necessary for match to not mask such failure. In contrast to supplying multiple pattern-method pairs to a single match, we can combine multiple invocations of match with the "| combinator. This allows inner methods to instead "fall through" upon failure. The following proof method, for example, always invokes prop_solver for all goals because its first alternative either never matches or (if it does match) always fails.

```
method foo₁ =
  (match conclusion in ?P ∧ ?Q ⇒ ‹fail›)
 | (match conclusion in ?R ⇒ ‹prop_solver›)
```

Backtracking may be controlled more precisely by marking individual patterns as *cut*. This causes backtracking to not progress beyond this pattern: once a match is found no others will be considered.

```
method foo₂ =
  (match premises in I: P ∧ Q (cut) and I′: P ⟶ ?U for P Q ⇒
    ‹rule mp [OF I′ I [THEN conjunct1]]›)
```

In this example, once a conjunction is found ($P \wedge Q$), all possible implications of P in the premises are considered, evaluating the inner rule with each consequent. No other conjunctions will be considered, with method failure occurring once all implications of the form $P \longrightarrow ?U$ have been explored. Here the left-right processing of individual patterns is important, as all patterns after of the cut will maintain their usual backtracking behaviour.

lemma $[\![A \wedge B;\ A \longrightarrow D;\ A \longrightarrow C]\!] \Longrightarrow C$
  by foo_2

lemma $[\![C \wedge D;\ A \wedge B;\ A \longrightarrow C]\!] \Longrightarrow C$
  apply $(\mathsf{foo}_2)?$
  oops

In this example, the first lemma is solved by foo_2, by first picking $A \longrightarrow D$ for I', then backtracking and ultimately succeeding after picking $A \longrightarrow C$. In the second lemma, however, $C \wedge D$ is matched first, the second pattern in the match cannot be found and so the method fails[2].

5.2.5 Premises within a Subgoal Focus

Subgoal focusing provides a structured form of a subgoal, allowing for more expressive introspection of the goal state. This requires some consideration in order to be used effectively. When the keyword **premises** is given as the match target, the premises of the subgoal are lifted into hypothetical theorems, which can be found and named via match patterns. Additionally these premises are stripped from the subgoal, leaving only the conclusion. This renders them inaccessible to standard proof methods which operate on the premises, such as frule (forward reasoning from premises) or erule (eliminating/decomposing premises). Naive usage of these methods within a match will most likely not function as the method author intended.

method my_allE_bad **for** $y :: 'a$ =
  (match **premises in** I: $\forall x :: 'a.\ ?Q\ x \Rightarrow$
    ‹erule allE [where $x = y$]›)

Here we take a single parameter y and specialize the universal elimination rule ($\forall x.\ P\ x \Longrightarrow (P\ x \Longrightarrow R) \Longrightarrow R$) to it, then attempt to apply this specialized rule with erule. The method erule will attempt to unify with a universal quantifier in the premises that matches the type of y. Suppose we tried to use my_allE_bad to prove a trivial lemma the following:

lemma $\forall x.\ P\ x \Longrightarrow P\ a$
  apply (my_allE_bad a)$?$
  oops

When my_allE_bad is invoked, since **premises** causes a focus, the premise $\forall x.\ P\ x$ is nowhere to be found, and thus my_allE_bad will always fail. If focusing instead left the premises in place, using methods like erule would lead to unintended behaviour, specifically during backtracking. In my_allE_bad, erule could choose an alternate premise while backtracking, while leaving I bound to the original match. In the case of more complex inner methods, where either I or bound terms are used, this would almost certainly not be the intended behaviour.

An alternative implementation would be to specialize the elimination rule to the bound term and apply it with rule instead of erule.

[2] This is why we need to use the $?$ combinator in this example and the **oops** keyword to terminate an unfinished proof.

```
method my_allE_almost for y :: 'a =
  (match premises in I: ∀ x :: 'a. ?Q x ⇒
    ‹rule allE [where x = y, OF I]›)
```

This method will insert a specialized duplicate of a universally quantified premise. Although this will successfully apply in the presence of such a premise, it is not likely the intended behaviour. To understand why, consider the following example:

```
lemma ∀ x. P x ⟹ P a
 apply (my_allE_almost a)
 apply (my_allE_almost a)
 by assumption
```

Here, after applying my_allE_almost, the goal state is: $\llbracket \forall x.\ P\ x;\ P\ a \rrbracket \Longrightarrow P\ a$. Observe that the premise $P\ a$ has been inserted as intended, but that the original premise $\forall x.\ P\ x$ still remains. A second application of my_allE_almost therefore succeeds, yielding a goal state of $\llbracket \forall x.\ P\ x;\ P\ a;\ P\ a \rrbracket \Longrightarrow P\ a$. Repeated application of my_allE_almost would thus produce an infinite stream of duplicate specialized premises, due to the original premise never being removed. To address this, matched premises may be declared with the thin attribute. This will hide the premise from subsequent inner matches, and remove it from the list of premises after the inner method has finished. It can be considered analogous to the old-style thin_tac, used for removing goal premises that match a given pattern.

To complete our example, the correct implementation of the method will thin the premise from the match and then apply it to the specialized elimination rule.

```
method my_allE for y :: 'a =
  (match premises in I [thin]: ∀ x :: 'a. ?Q x ⇒
    ‹rule allE [where x = y, OF I]›)

lemma ∀ x. P x ⟹ ∀ x. Q x ⟹ P y ∧ Q y
  by (my_allE y)+ (rule conjI)
```

Other attributes may also be applied to matched facts. This is most applicable when focusing, in order to inform methods which would otherwise use premises implicitly.

```
lemma A = B ⟹ (A ⟶ B) ∧ (B ⟶ A)
  by (match premises in I [subst]: ?P ⟷ ?Q ⇒ ‹prop_solver›)
```

In this example, the pattern $?P \longleftrightarrow ?Q$ matches against the premise $A \longleftrightarrow B$ and binds it to the local fact I. Additionally it declares this fact as a subst rule, adding it to the subst named theorem for the duration of the match. This is then implicitly used by prop_solver to solve the goal.

5.2.6 Example

We complete our tour of the features of Eisbach by extending the propositional logic solver presented earlier to first-order logic. The following method instantiates universally quantified assumptions by simple guessing, relying on backtracking to find the correct instantiation. Specifically, it instantiates assumptions of the form $\forall x.\ ?P\ x$ by finding some type-correct term y by matching other assumptions against $?H\ y$, using type annotations to ensure that the types match correctly. The matched universal quantifier is marked as *thin* to remove it from the premises, while using the universal elimination rule *allE* to specialize U to y. The same matching is also performed against the conclusion to find possible instantiations there as well.

```
method guess_all =
  (match premises in U [thin]: ∀x. P (x :: 'a) for P ⇒
    ‹(match premises in ?H (y :: 'a) for y ⇒
      ‹rule allE [where x = y, OF U]›)?,
     (match conclusion in ?H (y :: 'a) for y ⇒
      ‹rule allE [where x = y, OF U]›)?›)
```

The pattern *?H y* is used to find arbitrary subterms y within the premises or conclusion of the current goal. It makes use of Isabelle/Pure's workhorse of higher-order unification (although matching involves pattern-matching only). While such a pattern-match need not bind all variables to be valid, to avoid trivial matches, *match* considers only those matches that bind all **for**-fixed variables mentioned in the pattern.

The inner match must be duplicated over both the premises and conclusion because they are logically different entities: the premises are *facts*, in that they are (assumed) true; the conclusion is not and must be proved, and so is a *term*. This might look strange to users of Coq's Ltac, where these notions are identified; however, it does not limit the expressivity of Eisbach.

The thin attribute is necessary here in order to guarantee termination (see Section 5.2.4). However, since the premise is "consumed", care must be taken to ensure that this does not render the goal unsolvable (i.e. in the case where the premise needs to be specialized multiple times). Here we assume that this is handled by a previous application of prop_solver, which decomposes the goal into sufficiently small subgoals such that only a single instantiation is required.

Similar to our previous solve_ex method, we introduce a method which attempts to guess at an appropriate witness for an existential proof. In this case, however, the method simply guesses the witness based on terms found in the current premises, again using higher-order matching as in the guess_all method above.

```
method guess_ex =
  (match conclusion in
    ∃x. P (x :: 'a) for P ⇒
      ‹match premises in ?H (x :: 'a) for x ⇒
        ‹rule exI [where x = x and P = P]››)
```

These methods can now be combined into a surprisingly powerful first-order solver.

```
method fol_solver =
  ((prop_solver | guess_ex | guess_all) ; solves ‹fol_solver›)
```

The use of solves above ensures that the recursive subgoals are solved. Without it, the recursive call could terminate prematurely and leave the goal in an unsolvable state (due to an incorrect guess for a quantifier instantiation).

After declaring some standard rules in the context, this method is capable of solving various example problems.

```
declare
  allI [intros] — (⋀x. ?P x) ⟹ ∀x. ?P x
  exE [elims] — ⟦∃x. ?P x; ⋀x. ?P x ⟹ ?Q⟧ ⟹ ?Q
  ex_simps [subst]
  all_simps [subst]

lemma (∀x. P x) ∧ (∀x. Q x) ⟹ (∀x. P x ∧ Q x)
  and ∃x. P x ⟶ (∀x. P x)
  and (∃x. ∀y. R x y) ⟶ (∀y. ∃x. R x y)
  by fol_solver+
```

5.2.7 Integration with ML

Although Isar's built-in methods and attributes are sufficient for most purposes, often proof methods will require functionality that is simply not accessible from Isar. In many cases this can be isolated into a reusable piece of functionality, to be implemented as a generally-available attribute or proof method.

For example, consider the following proof of a simple case analysis statement on lists:

```
lemma (case l of (x # xs) ⇒ length l = length xs + 1 | [] ⇒ length l = 0)
  by (rule list.split[THEN iffD2, of l], simp)
```

This proof has applied the *split* rule for lists to decompose the goal into a form that simp can solve. The split rule for lists is as follows:

```
P (case list of [] ⇒ f1 | x · xa ⇒ f2 x xa) =
((list = [] ⟶ P f1) ∧ (∀x21 x22. list = x21 · x22 ⟶ P (f2 x21 x22)))
```

This shows how a predicate P of a case statement over some list l can be rewritten into a set of conjuncts over P. Specializing this rule with THEN iffD2 ($[\![P = Q;\ Q]\!] \Longrightarrow P$) rewrites it into a backwards reasoning rule.

```
(list = [] ⟶ P f1) ∧ (∀x21 x22. list = x21 · x22 ⟶ P (f2 x21 x22)) ⟹
P (case list of [] ⇒ f1 | x · xa ⇒ f2 x xa)
```

Once this rule is specialized to the given l and applied to the goal, the case statement no longer appears and simp can solve it.

If we want to generalize this pattern to perform case analysis on arbitrary datatypes, we need to be able to retrieve the split rule for a given term at run-time. As there is no facility in Isar to do this directly, we need to use ML. We can write a simple function that retrieves the split rule for a given term based on the name of its type.

```
ML ‹
fun get_split_rule ctxt term =
let
  val typ = Term.fastype_of term;
  val typeNm = fst (dest_Type typ);
in Proof_Context.get_thm ctxt (typeNm ^ .split) end;›
```

With this function, we can then define a rule attribute.

```
attribute_setup split_rule_of =
  ‹Args.term >> (fn t =>
    Thm.rule_attribute [] (fn context => fn _ =>
      (case try (get_split_rule (Context.proof_of context)) t of
        SOME thm => thm
      | NONE => Drule.dummy_thm)))›
```

This attribute evaluates to the split rule for the given term *t*, if possible, otherwise resulting in a dummy theorem. We use *try* to wrap *get_split_rule* as a partial function, where any raised errors will result in *NONE*, and a successful result will be wrapped in *SOME*. We can then use this attribute as the target of a match in the definition of an Eisbach method. The double square bracket syntax in this method is standard Isar notation, used to evaluate the given rule attribute against a dummy fact (which is ignored by split_rule_of).

```
method apply_split for t :: 'a =
  (match [[split_rule_of t]] in U: (?x :: bool) = ?y ⇒
    ‹match U[THEN iffD2] in U': ⋀t'. ?A ⟹ ?P (t' :: 'c) ⇒
      ‹match (t) in t' :: 'c for t' ⇒
        ‹rule U'[where t'=t']›››)
```

This method first retrieves the split rule for the given term t and ensures it is in the expected form $?x = ?y$. If the attribute produces a dummy theorem (i.e. due to t not being a datatype) then this pattern fails to match and the method evaluation fails. We then transform the split rule with THEN iffD2 as before, and match on the resulting rule. Here we identify some schematic subterm of the rule's conclusion and bind its type to $'c$. In the final match, we require that t match some term t' of type $'c$. This ensures that $'c$ can be unified to $'a$ (backtracking the previous match if this is not the case) before instantiating the rule with the given term.

Our original proof can then be solved as before, but with our new method abstracting away the splitting logic.

```
lemma (case l of (x # xs) ⇒ length l = length xs + 1 | [] ⇒ length l = 0)
  by (apply_split l, simp)
```

Additionally, since this method fails gracefully on an invalid term, we could write a method to speculatively apply it to different subterms in the goal.

```
method guess_split =
  (match conclusion in ?P x for x ⇒ ‹apply_split x›)
```

This method guesses which subterm may be applicable for splitting, backtracking on choices that cause apply_split to fail.

```
lemma (case n of (Suc k) ⇒ n = k + 1 | 0 ⇒ ∀k. k ≥ n)
  by (guess_split, simp)
```

In general, there will always exist functionality that is inaccessible directly to Eisbach, requiring a bridge to some ML interface. However, as more bridges are developed, and as Isabelle package developers begin to include more method-level programmatic interfaces, the extent of automation expressible in pure Eisbach will continue to expand.

5.3 Design and Implementation

A core design goal of Eisbach is a seamless integration with other Isabelle languages, notably Isar, ML, and object-logics. The primary motivation clearly being to make it accessible to existing Isabelle/Isar users, with a secondary objective of both forward and backward compatibility.

5.3.1 Readable Proof Methods

In Isar there is a clear distinction between a structured and an unstructured proof. The former makes use of the rich reasoning framework provided by Isar, while the latter relies more heavily on the implicit behaviour of proof methods. An unstructured proof cannot be understood without checking the proof in Isabelle and inspecting the subgoal state at each stage of the proof. This can create significant issues during proof maintenance phases, where a proof needs to be updated in response to an update to a specification or to Isabelle itself. The original intention of a proof cannot be easily extracted from the unstructured proof script, and a now-failing proof may require significant time and effort to perform the necessary archaeological exploration of its history to recover some insight.

One of the aims of Eisbach is to address this by providing a means to describe reasoning procedures. A proof method designed to solve a particular class of problems serves as a better record of the author's intent than an ad-hoc series of general tools. Arguably this simply shifts the problem of *proof maintenance* to that of *proof method maintenance*, and indeed this is a well-known concern in Ltac today. This indicates the necessity for writing proof methods that are *readable* and thus also maintainable.

In Eisbach, match can be considered as a structured language element, and is meant to serve both as implementation and documentation. Many methods shown here could have been implemented without using match, but would have been significantly more difficult to understand, and may happen to work in unintended cases. In practice, a match pattern is a much more explicit description of the expected goal state than, for example, the expectation that erule successfully finds an appropriate premise for the given rule. As with Isar, Eisbach method authors are free to use as much structure as they consider necessary for their specific application.

5.3.2 Design Goals and Comparison to Ltac

Here we explicitly revisit the design goals presented in Section 5.1, discussing each in the context of both Eisbach and Ltac.

To be easy to use for beginners and experts. Both Eisbach and Ltac methods/tactics can be constructed from tools that users naturally become familiar with when writing proofs, thus providing a low barrier-to-entry. In both languages, matching allows experts to explicitly introspect the goal state when writing complex tools.

To expose limited functionality, leaving complex functionality to Isabelle/ML. At its core, Eisbach provides the ability to name a method expression and abstract it over parameters. Many Eisbach methods can then be written by simply combining Isabelle/Isar's built-in methods. This is similar to the functionality provided by Ltac, however, as opposed to Ltac's match keyword, Eisbach's match method is implemented as a stand-alone tool.

Isar's integration with ML (see Section 5.2.7) makes this a natural design choice, and demonstrates how similar language extensions can be written by end-users. In contrast, Coq discourages ad-hoc use of OCaml by requiring separate compilation of custom plugins.

To be extensible by end-users. In addition to providing new language features by writing custom ML methods, Eisbach methods can be made extensible through Isar's named theorems (see Section 5.2.1), allowing databases of facts to be managed as context data. This can be seen as a generalization of the hint databases used by Coq's `auto` tactic.

Seamless integration with other Isabelle languages. By forming static closures of Isar's existing method syntax (see Section 5.3.4), Eisbach immediately benefits from existing methods and Isabelle languages. This underlying functionality can be used to easily include method expressions as arguments to both (higher-order) methods and other tools. For example, later in Section 6.2 we use this functionality to implement a pair of rule attributes for transforming proven facts with method expressions.

Although Coq tools can similarly be implemented to take tactics as arguments, this functionality is more immediately expressive in Isabelle/Isar, as methods can be defined to have arbitrarily complex syntax (e.g. match).

To continue Isar's principle of readable proofs, creating *readable proof procedures* In an Isar proof, authors are free to choose the extent of structured proof elements to use. Highly structured proofs are generally more easily understood, at the cost of being more verbose. Similarly, Eisbach's match can be used as documentation for the intent of a particular method.

5.3.3 Method Correctness and Types

In contrast to Coq's Mtac (see Section 2.1.5), Eisbach and Ltac are *untyped* languages. For Eisbach, this was a necessary design choice in order to integrate with Isabelle's existing untyped tactics and methods. This may seem ironic, given that both ML and Isabelle's Pure logic are strongly typed. Indeed this discussion dates back to the tactical proofs that first appeared in the original LCF proof assistant (see Section 2.1.3).

It is important to clarify that both Ltac and Eisbach are subject to the correctness constraints of their respective proof systems. In the case of Eisbach, each defined method eventually appeals to the primitive inferences of Isabelle's Pure logic using the proof kernel (see Section 3.2.3). Eisbach methods, like all Isabelle proof tools, can therefore at most fail to produce a valid proof, without concerns that programming errors will allow for unsound reasoning.

In tactical proving there is therefore less of an immediate need for the static guarantees that would be provided by a type system. Instead, we can focus on providing run-time checks, meaningful error messages and interactive debugging facilities (see Section 6.1). In Eisbach, intermediate match clauses could explicitly check that the subgoal state has some expected form before invoking a tool that makes this assumption implicitly. In general, method semantics can be encoded as run-time pre and postcondition checks. For example, a verification-condition generator (see Section 7.1) might have a precondition check that the initial subgoal is an annotated program, and a postcondition check that no produced subgoals discuss the semantics of the original program.

5.3.4 Static Closure of Concrete Syntax

Isabelle provides a rich selection of powerful proof methods, each with its own concrete syntax, which is implemented by parser combinators in ML. Additionally, Isabelle's theorem attributes, which perform context and fact transformations, have their own parsers. Rather than re-write all tools from the libraries to support Eisbach, we build on existing features of the Isabelle parsing framework whereby tokens have values (types, terms, facts etc.) assigned to them implicitly during parsing.

This syntax/value assignment mechanism was originally introduced to support *locale expressions* in the sense of [10]. Thus expressions over facts and attributes became transformable by morphisms, to move them from an abstract locale context to a concrete application context.[3]

The same principle of syntax closure and interpretation is now the main workhorse of Eisbach. After some modifications, it works for method expressions as well, including their embedded facts and attribute expressions. For example, the basic method "(simp *add*: *foo* [OF *bar*])" is wrapped up as static closure, where the embedded fact expression "*foo* [OF *bar*]" is treated like a pre-evaluated constant.

Eisbach then simply serves as an interpretation environment for the carefully prepared method syntax tokens. When a proof method is applied, Eisbach instantiates these token values appropriately (via some morphism), based on the supplied arguments to the method or results of matching, and then executes the resulting method body. Although this presents some technical challenges and required various modifications of the Isar implementation itself, this proves to be a very effective solution to performing this kind of language extension.

As a result of this, the inner methods that appear in Eisbach method definitions may have arbitrarily complex internal syntax. For example, Eisbach's match is implemented as a standard proof method, rather than requiring any special status in the language. This opens the door for users to develop their own advanced proof methods to serve as language extensions for method development in Eisbach.

5.3.5 Subgoal Focusing

In Isabelle/Pure there is a logical distinction between universally quantified parameters (like x in $\bigwedge x.\ P\ x$) and arbitrary-but-fixed terms (like a in $P\ a$). A subgoal in the former form does not allow the x to be explicitly referenced, because it is hidden within

[3] See also [11] for a recent exposition of the possibilities of locales and locale interpretations via morphisms in Isabelle.

a closed formula; for example, *my_fact* [where $y = x$] does not produce a valid theorem. Historically, some special tactics were provided to descend into the sub-goal structure and provide ad-hoc access to its local parameters: these are available in Isar via so-called *improper methods* (like rule_tac).

Likewise, premises within a subgoal are not yet local facts. In a structured Isar proof, assumptions are stated explicitly in the text via **assumes** or **assume** and are accessible to attributes etc. In contrast, the local prefix $A \Longrightarrow$ ⌑ of a subgoal is not accessible to structured reasoning and cannot be provided to standard methods as explicit arguments.

Isabelle/ML provides systematic support for *subgoal focusing*, and Isar provides access to it via the **subgoal** command. Focusing creates a new goal out of a given subgoal, but with its parameters turned into fixed variables (actual terms), and premises into local assumptions (actual facts). For example:

lemma $\bigwedge A\ B\ C.\ [\![(\bigwedge x.\ A\ x \Longrightarrow B);\ B \Longrightarrow C;\ A\ y]\!] \Longrightarrow C$
  subgoal premises *prems* **for** $A\ B\ C$

---

1. C

At this point the goal has been *focused*, where the goal parameters (A, B and C) are converted into fixed terms, and the subgoal becomes focused on the conclusion (C). The goal premises have been lifted into proper assumptions, lifting meta-universally quantified variables into schematics, and then stored in the local fact *prems*.

thm *prems* — $A\ ?x \Longrightarrow B$, $B \Longrightarrow C$, $A\ y$

We can then complete this proof by referring to *prems* directly. As *prems* contains three individual theorems (one for each premise in the original goal), we can refer to each directly with the standard syntax *prems*(n) for selecting the nth theorem of *prems*.

  apply (rule *prems*(2))
  apply (rule *prems*(1)[where $x = y$])
  apply (rule *prems*(3))
  done
done

Here the first **done** ends the subgoal focus, having successfully discharged the subgoal and the second **done** concludes the proof.

For Eisbach, we incorporated this into the language with some concrete syntax, to allow the user to write methods that can operate within the local subgoal structure as required. This allows for uniform treatment of the goal state when matching and parameter passing. In Eisbach's match method, focusing is accessed by specifying the keyword **conclusion** or **premises** as the match target (see Section 5.2.4).

5.4 Conclusion

In this chapter we have presented Eisbach, a high-level language for writing proof methods in Isabelle/Isar. It supports familiar Isar language elements, such as method combinators and rule attributes, as well as being compatible with existing Isabelle proof methods.

An expressive match method enables the use of higher-order matching against facts and subgoals to provide control flow.

One of Eisbach's greatest virtues is that it provides a framework for thinking of proof methods like programming language elements. This applies to methods written in Eisbach, like solves, but also to those written in Isabelle/ML. A good example of the latter is the match method which, while its implementation is entirely independent of the **method** command, became necessary to implement only in the presence of Eisbach. The method language of Isar was already arbitrarily expressive, as methods define their own syntax. Eisbach now opens up this space allowing users to write methods that serve as elements of a high-level *method programming language*, rather than one-off proof tools. Thus methods that may have had little use in proof scripts now become useful and, in the case of match, powerful, as elements of Eisbach-defined proof methods.

In the next chapter, we will see how Isar and Eisbach can be extended into a rich ecosystem for proof method development. Later, in Chapter 7, we show how Eisbach can be used to implement a set of real-world proof methods in L4.verified, and demonstrate their impact on existing proofs.

Chapter 6

Advanced Eisbach

With Eisbach we can rapidly prototype and develop new proof methods by using Isar's familiar method expressions, use named theorems to manage and implicitly use collections of facts as proof context data, and perform matching on proof states to manage the control flow of methods. Although this is sufficient for many use cases, the core infastructure of Eisbach can be used for many advanced applications that were previously not immediately possible in Isar.

In particular, Eisbach provides a facility for forming a static closure from a method expression (see Section 5.3.4). From this, methods can be treated as first class language elements, to be easily provided to existing tools as standalone arguments.

In this chapter, we explore both the consequences of this new ability, and demonstrate some of the advanced capabilities of Eisbach. Implementations of the tools presented here are available in the L4.verified repository [5].

Chapter Outline

- **Method Expression Debugging.** In Section 6.1 we present a new Isar command, **apply_debug**, and specialized proof method, #break, that allow for interactively stepping through the execution of a method expression.

- **Rule Attributes from Methods.** In Section 6.2 we present a pair of new rule attributes that allow proof methods to be used as fact transformers.

- **Advanced Methods and Combinators.** In Section 6.3 we present a suite of specialized methods for performing technical operations in proof methods.

6.1 Method Expression Debugging

Although Isabelle's proof kernel precludes writing proof methods which produce logically unsound results, while under development they will often have unexpected or unintended behaviour. When a method loops indefinitely, or fails to make progress on a goal, a proof engineer requires tools to discover the source of the problem.

General method debugging capabilities in Isabelle/Isar are limited. Usually a proof author is forced to experimentally remove or add rules from different rulesets to determine their effect on various automated methods (e.g. removing a rule from the simpset to see if it is causing the simp method to loop). This process is time-consuming and often frustrating, as it requires a deep knowledge of both the particular proof domain (to know what rules may be causing issues), and of how the rules are used by the proof method.

Some methods have built-in facilities for tracing their internal behaviour (e.g. simplifier tracing [37]). However, when debugging a general method expression, either in-place or in the body of an Eisbach method, users are often concerned with understanding its control flow and backtracking behaviour.

In this section we introduce the method debugging command **apply_debug** and its corresponding special-purpose #break method. This command allows the evaluation of a method expression to be interactively inspected and manipulated at set breakpoints, so that users may trace the execution of proof methods and experiment with alternate proof strategies.

6.1.1 Example Debugging Session

Consider the following example, where a method expression fails to solve the goal.

lemma $A \Longrightarrow (A \wedge B) \vee A$
apply ((rule conjI disjI1 disjI2)+, assumption)

---

1. $A \Longrightarrow B$

In this example, the author may have expected the following semantics: “apply conjunction and disjunction introduction rules, backtracking on the chosen disjunct until the goal is solved by assumption.” However, the resulting subgoal is instead $A \Longrightarrow B$, which is unsolvable. To debug a method expression, the most common strategy is to dismantle the it, copy-pasting its components as a proof script in order to step through its execution.

To debug our example, we would take the expression fragment under the + combinator and apply it as single proof step.

lemma $A \Longrightarrow (A \wedge B) \vee A$
apply (rule conjI disjI1 disjI2)

---

1. $A \Longrightarrow A \wedge B$

Now we can see that the first disjunct has been introducted with disjI1. Applying this fragment again, we see that conjI is used and introduces two subgoals.

apply (rule conjI disjI1 disjI2)

---

1. $A \Longrightarrow A$
2. $A \Longrightarrow B$

Our first subgoal can be solved trivially by assumption.

```
apply assumption
```

```
1. A ⟹ B
```

Now we see the final proof state that was encountered after evaluating the original expression.

The issue in this example is that the first subgoal produced by applying conjI is indeed solvable by assumption, which terminates the method evaluation and leaves $A \Longrightarrow B$ to be solved. Since the method has successfully evaluated, no backtracking results are attempted, and the intended strategy of initially applying disjI2 is not explored.

Without additional tool support, debugging Eisbach methods requires a similar strategy: copy-pasting sections of method definitions into Isar proofs in order to interactively trace their execution.

Consider the following modification of the previous example, where the method expression has been generalized into an Eisbach method. The proof fails as expected, however isolating the cause of the error is now much more involved.

```
named_theorems my_solver_rules
method my_solver declares my_solver_rules =
  ((rule my_solver_rules)+, assumption)

lemma A ⟹ (A ∧ B) ∨ A
  apply (my_solver my_solver_rules: conjI disjI1 disjI2)
  oops
```

This attempt fails, as in our initial example. However re-creating the original trace requires first populating the my_solver_rules with Isar's **supply** command.

```
lemma A ⟹ (A ∧ B) ∨ A
  supply [my_solver_rules] = conjI disjI1 disjI2
  apply (rule my_solver_rules)
  apply (rule my_solver_rules)
   apply assumption
  oops
```

This style of method expression tracing proves to be time-consuming, error-prone, and extremely difficult in non-trivial proofs and method expressions. In larger projects, with many inter-dependent proof methods, manually constructing a proof trace with this strategy becomes nearly impossible.

6.1.2 The apply_debug command

When debugging a proof method we are often faced with two important questions: "What is the execution path that lead to the resulting proof state?" and "What alternate strategies could be applied in order to make progress?". Answering the former question allows us to identify *when* a method may have made an unexpected or incorrect reasoning step, and the latter gives insight into *how* a method could be modified in order to resolve the issue.

In this section, we introduce the new Isar command **apply_debug**. This command takes a method expression and evaluates it against the current proof state, identical to how the **apply** command functions. However, it is aware of a special-purpose method #break, which acts as a breakpoint for method evaluation. When a #break is encountered during the invocation of **apply_debug**, the method evaluation halts and the resulting proof state is the *intermediate* proof state that was seen by the breakpoint. The **continue** command resumes execution, until either the next #break is encountered or the method expression terminates.

Returning to our previous example, we can debug our method expression by placing a #break inside the loop.

```
lemma A ⟹ (A ∧ B) ∨ A
  apply_debug ((rule conjI disjI1 disjI2, #break)+, assumption)
```

```
1. A ⟹ A ∧ B
```

Here we see the goal state after **apply_debug** is not the final state, but rather the goal after only applying disjI1 (i.e. the method has committed to solving the first disjunct). If we execute **continue** we can see the result after applying conjI.

```
continue
```

```
1. A ⟹ A
2. A ⟹ B
```

Finally, if we **continue** again, method evaluation terminates without encountering any additional breakpoints and we are presented with the expected goal.

```
continue
```

```
1. A ⟹ B

Final Result.
```

The output "*Final Result.*" indicates that the **apply_debug** evaluation has finished and it is not possible to **continue** any more. If the expression is evaluated with **apply** instead, as part of a proof rather than a debugging session, the #break is ignored and the proof continues as usual, resulting in the same final proof state.

The same functionality is available when #break is used in the definition of an Eisbach method.

```
method my_solver declares my_solver_rules =
  ((rule my_solver_rules, #break)+, assumption)

lemma A ⟹ (A ∧ B) ∨ A
  apply_debug (my_solver my_solver_rules: conjI disjI1 disjI2)
  continue — 1. A ⟹ A  2. A ⟹ B
  continue — 1. A ⟹ B
  oops
```

In this example, the breakpoint inside of my_solver is hit twice, as in the previous example. In both cases, we were readily able to trace the execution of the expression without needing to copy-paste portions of it into the proof script.

Tagging

Without any additional arguments, a #break will be triggered *unconditionally*, where it will halt execution during any invocation of **apply_debug**. In general, however, this can result in hitting many breakpoints that are unrelated to the current debugging task. Indeed if a user is investigating a complex method expression in a proof script, they likely are not interested in the intermediate states of each of its constituent methods.

It is important, therefore, that the debugging detail level be configurable, so users can inspect and interact with proof states that are relevant to solving a particular issue without becoming overwhelmed with irrelevant technical details.

Revisiting our example, we may want to debug a method expression containing my_solver, but not my_solver itself. Its implementation, however, contains an unconditional breakpoint, and so we encounter it unintentionally.

lemma $A \longrightarrow (A \wedge B) \vee A$
  apply_debug (rule impI, #break,
    my_solver *my_solver_rules*: conjI disjI1 disjI2, #break)

1. $A \Longrightarrow A \wedge B \vee A$

As expected, our first breakpoint is hit after impI is applied and the implication has been introduced. However, subsequent invocations of **continue** see the trace of my_solver as well.

continue — *1.* $A \Longrightarrow A \wedge B$
continue — *1.* $A \Longrightarrow A$ *2.* $A \Longrightarrow B$
continue — *1.* $A \Longrightarrow B$
continue — *1.* $A \Longrightarrow B$ *Final Result.*
oops

If we were only interested in seeing the proof state before and after the execution of my_solver then this is much more detail than is needed, indeed likely obscuring the error being investigated.

To address this, we can use *tagged* breakpoints: the #break method takes a single optional string argument which, when present, tags it to instead be *conditionally* active. When **apply_debug** is invoked, a comma-delimited list of active tags can be given, such that any breakpoints with matching tags will be hit (in addition to unconditional breakpoints). In the absence of a matching tag for a breakpoint during a given **apply_debug** invocation, the #break is skipped.

method my_solver **declares** my_solver_rules =
  ((rule my_solver_rules, #break *my_solver_tag*)+, assumption)

This modified my_solver now conditionally breaks on the presence of "my_solver_tag", given as an argument to **apply_debug**.

lemma $A \Longrightarrow (A \wedge B) \vee A$
apply_debug (**tags** *my_solver_tag*)
(my_solver *my_solver_rules*: conjI disjI1 disjI2) — *1.* $A \Longrightarrow A \wedge B$
continue — *1.* $A \Longrightarrow A$ *2.* $A \Longrightarrow B$
continue — *1.* $A \Longrightarrow B$ *Final Result.*

We can now debug a method expression that contains my_solver without tracing its execution. By giving no tags to **apply_debug**, only unconditional breakpoints are encountered.

lemma $A \longrightarrow (A \wedge B) \vee A$
apply_debug (rule impI, #break,
my_solver *my_solver_rules*: conjI disjI1 disjI2,
#break) — *1.* $A \Longrightarrow A \wedge B \vee A$
continue — *1.* $A \Longrightarrow B$
continue — *1.* $A \Longrightarrow B$ *Final Result.*

6.1.3 Proof state interaction

So far we have only used **apply_debug** to observe the goal state during the execution of a method expression evaluation. Importantly, however, it also grants the ability to interact directly with the run-time proof state: the proof context can be examined with standard Isar diagnostic commands (e.g. **thm** and **term**), and the goal state can be modified ad-hoc with proof methods. Each #break provides an opportunity for the user to interact with the intermediate state in a proof method evaluation as if it were an intermediate state in a proof script.

Returning to our example, we can use **thm** to examine the contents of the **named_theorem** my_solver_rules during the execution of my_solver.

lemma $A \Longrightarrow (A \wedge B) \vee A$
apply_debug (**tags** *my_solver_tag*)
(my_solver *my_solver_rules*: conjI disjI1 disjI2)
continue

1. $A \Longrightarrow A$
2. $A \Longrightarrow B$

thm my_solver_rules

$[\![?P;\ ?Q]\!] \Longrightarrow ?P \wedge ?Q$
$?P \Longrightarrow ?P \vee ?Q$
$?Q \Longrightarrow ?P \vee ?Q$

Additionally, we may apply methods to modify the run-time state. In this state, for example, if we manually apply assumption, we can trigger the desired backtracking behaviour from the expression.

apply assumption

---

1. $A \Longrightarrow B$

Since the first subgoal is now solved, the assumption from the body of my_solver will fail on the new head subgoal, causing the method to explore other backtracking branches.

continue

---

1. $A \Longrightarrow A$

The method has backtracked and applied disjI2 instead, breaking afterwards. Now if we **continue**, the assumption from my_solver successfully solves the goal and the method execution finishes.

continue

No subgoals!

Final Result.

Our debugging session has now altered the outcome of the method evaluation, resulting in a successful proof.

With this, we have both identified the source of the error, and a potential solution: by requiring that *all* emerging subgoals be solvable by assumption, we force the backtracking behaviour of the method to explore all possible combinations of the given rules. This can be accomplished by replacing sequential composition (,) with structured concatenation (;). Implementing this change in an updated my_solver successfully solves the original goal.

method my_solver_fixed **declares** my_solver_rules =
  ((rule my_solver_rules)+; assumption)

lemma $A \Longrightarrow (A \wedge B) \vee A$
  apply (my_solver_fixed *my_solver_rules*: conjI disjI1 disjI2)
  done

Methods as Breakpoint Filters

In addition to using tagged breakpoints, we can further control the breakpoints that are encountered by providing **continue** with a method as an optional argument. This method acts as a filter against the proof state at each breakpoint, and only halts the evaluation of the debugged method expression if the given method successfully applies to the proof state.

Consider the following example, where structured concatenation has been used to break on each subgoal produced by intro conjI.

```
lemma A ∧ (∀x. P x) ∧ B ∧ C
  apply_debug (#break, (intro conjI; #break)) — 1. A ∧ (∀x. P x) ∧ B ∧ C
  continue — 1. C
  continue — 1. B
  continue — 1. ∀x. P x
  continue — 1. A
  continue — 1. A  2. ∀x. P x  3. B  4. C
```

The second #break is executed on a restricted view of the subgoal multiple times, once per conjunct. Now, consider the case where we are interested in investigating the proof state prior to encountering a particular conjunct, e.g. $\forall x.\ P\ x$. If there are a large number of subgoals, it is impractical to manually **continue** over each one. One option to address this is to conditionally break by using Eisbach's match method.

```
lemma A ∧ (∀x. P x) ∧ B ∧ C
  apply_debug (#break,
    (intro conjI; match conclusion in ∀x. _ ⇒ ‹#break› | _ ⇒ ‹−›))
  continue — 1. ∀x. P x
  continue — 1. A  2. ∀x. P x  3. B  4. C Final Result.
```

Although this strategy is effective in isolating the desired subgoal, it requires modifying the method expression itself to do so. To address this, the **continue** command can instead be given a single method as an argument, to be used as filter on the proof states encountered by each break point. The breakpoint is hit if the evaluation of the given method is successful. If the method fails to apply, the breakpoint is skipped.

In our example, we can provide a match as a filter to **continue** which succeeds only when the conclusion of the goal is a universal quantifier.

```
lemma A ∧ (∀x. P x) ∧ B ∧ C
  apply_debug (#break, (intro conjI; #break))
  continue (match conclusion in ∀x. _ ⇒ ‹−›)
```

```
1. ∀x. P x
```

In this session, the first two encounters of #break are skipped since the match fails, and the **continue** halts on the intended subgoal $\forall x.\ P\ x$. Further uses of **continue**, however, will hit the remaining breakpoints as expected.

```
  continue — 1. A
  continue — 1. A  2. ∀x. P x  3. B  4. C Final Result.
```

Rather than using match, we could instead apply a rule that distinguishes the target breakpoint. Note, however, that the effect of the given method is preserved in the resulting proof state. In our example, we could use allI $((\bigwedge x.\ ?P\ x) \Longrightarrow \forall x.\ ?P\ x)$ to halt on $\forall x.\ P\ x$ and introduce its variable.

```
lemma A ∧ (∀x. P x) ∧ B ∧ C
  apply_debug (#break, (intro conjI; #break))
  continue (rule allI) — 1. ⋀x. P x
  continue — 1. A
  continue — 1. A  2. ⋀x. P x  3. B  4. C Final Result.
```

The command **finish** is an alias for **continue** fail, which will simply skip over all remaining breakpoints to finish the evaluation of the method and end the debugging session.

6.1.4 Document-based debugging

Isabelle's PIDE (Prover IDE) approach to interactive theorem proving advances the command-line interfaces of early IDEs by providing a document-centric approach to the formal proof text. The philosophy is to allow users to edit and develop the text of a given formalization without much consideration for the state of the underlying prover. As the document evolves, the system continuously checks it and provides markup to the given commands, identifying any errors or outputting the proof state. This is in contrast to the traditional approach of a read-eval-print loop in ITPs, where users explicitly step forwards or backwards over each proof command.

Since **apply_debug** is implemented as an Isar command, and not as a separate user interface tool (i.e. as a panel in Isabelle/jEdit), it must support this PIDE model of interaction. A method debugging session therefore behaves as a proof, where users are free to arbitrarily modify any part of the debugging script and expect the entire session to update accordingly. The user should not be concerned with the state of the thread executing the debugged proof method, therefore **apply_debug** performs sufficient bookkeeping to manage this automatically.

Method Traces and Threads

In the following example, assume that the debugging session is a checked as a PIDE document. Each command has received markup according to the output produced by Isabelle, and **finish** has successfully evaluated to the final result of the method expression.

```
lemma assumes A B C D E shows (A ∧ B) ∧ (C ∧ D ∧ E)
  apply_debug (rule conjI, #break)+  — 1. A ∧ B  2. C ∧ D ∧ E
  continue  — 1. A  2. B  3. C ∧ D ∧ E
  finish  — 1. A  2. B  3. C ∧ D ∧ E Final Result.
```

If we change a line of this document, PIDE re-executes any following lines in order to produce updated markup. For example, if we add a new method after **apply_debug**, the following two commands must be re-evaluated.

```
  apply_debug (rule conjI, #break)+  — 1. A ∧ B  2. C ∧ D ∧ E
    apply (rule conjI; fact)  — 1. C ∧ D ∧ E
  continue
  finish ⇒
```

We use ⇒ to show the current state of the thread managing the evaluation of the method: it points to the command corresponding to the breakpoint that the thread is currently blocked on. Commands which still need to be processed are shown in grey. Here we see that immediately after adding the **apply**, the remaining commands become unprocessed and lose their markup. The thread that was executing the method is currently terminated (represented by ⇒ after **finish**), therefore it will be automatically restarted and re-played up to this breakpoint in order to produce the new state after **continue**.

```
⇒ apply_debug (rule conjI, #break)+   — 1. A ∧ B  2. C ∧ D ∧ E
     apply (rule conjI; fact) — 1. C ∧ D ∧ E
   continue
   finish
```

When evaluated in the updated document, the **continue** command detects that the executing thread has passed it. This thread is therefore restarted, and the method is re-evaluated, starting with the the initial proof state.

```
   apply_debug (rule conjI, #break)+   — 1. A ∧ B  2. C ∧ D ∧ E
     apply (rule conjI; fact) — 1. C ∧ D ∧ E
⇒ continue
   finish
```

Once the first breakpoint is hit, the current proof state (just after having applied (rule conjI; fact)) is given as its result state. The method execution then continues from this modified state, and **continue** evaluates to state of the next breakpoint.

```
   apply_debug (rule conjI, #break)+   — 1. A ∧ B  2. C ∧ D ∧ E
     apply (rule conjI; fact) — 1. C ∧ D ∧ E
   continue — 1. C  2. D ∧ E
⇒ finish
```

At this point the executing thread has caught up to the current document state, so the **finish** proceeds as normal: completing the evaluation of the method and terminating the thread.

```
   apply_debug (rule conjI, #break)+   — 1. A ∧ B  2. C ∧ D ∧ E
     apply (rule conjI; fact) — 1. C ∧ D ∧ E
   continue — 1. C  2. D ∧ E
   finish ⇒ — 1. C  2. D ∧ E Final Result.
```

In this example, restarting the executing thread was relatively straightforward since we had only modified the result of the first breakpoint. In general, modifying the document after several debugging commands will require skipping over each of their corresponding breakpoints. Additionally, any ad-hoc changes to the proof state that were made in the debugging script will need to be preserved.

To accomplish this, each debug command records its initial proof state and the order that it appeared during the method's execution. When the executing thread is restarted, the resulting proof state after each skipped #break is its corresponding previously-stored proof state, preserving the effect of any ad-hoc changes that were made in the document.

For example, if we now add an **apply** after the **continue**, the restarted thread needs to recreate the result of (rule conjI; fact) in order to match the document state.

```
⇒ apply_debug (rule conjI, #break)+ — 1. A ∧ B  2. C ∧ D ∧ E
     apply (rule conjI; fact) — 1. C ∧ D ∧ E
   continue — 1. C  2. D ∧ E
     apply fact — 1. D ∧ E
   finish
```

As before, the executing thread is restarted and the method is evaluated starting with the initial proof state. When the first breakpoint is encountered, its starting state is *1.* $A \wedge B$ *2.* $C \wedge D \wedge E$. The document source that produced the resulting state of this

#break, however, has already been processed (i.e. **apply** (rule conjI; fact)). Rather than attempting to re-evaluate this Isar source, its resulting state from the previous execution (i.e. *1.* $C \wedge D \wedge E$) is retrieved from the trace and provided as the result of #break. The method evaluation then continues from there.

```
   apply_debug (rule conjI, #break)+  — 1. A ∧ B  2. C ∧ D ∧ E
     apply (rule conjI; fact) — 1. C ∧ D ∧ E
⇒ continue — 1. C  2. D ∧ E
     apply fact — 1. D ∧ E
   finish
```

After skipping the first breakpoint, the method evaluation continues until the second breakpoint, where the current proof state (after applying fact) is given as its result.

```
   apply_debug (rule conjI, #break)+  — 1. A ∧ B  2. C ∧ D ∧ E
     apply (rule conjI; fact)— 1. C ∧ D ∧ E
   continue — 1. C  2. D ∧ E
     apply fact — 1. D ∧ E
⇒ finish
```

At this point, the thread has caught up to the document state and **finish** can be successfully evaluated by resuming the method evaluation as usual.

```
   apply_debug (rule conjI, #break)+  — 1. A ∧ B  2. C ∧ D ∧ E
     apply (rule conjI; fact) — 1. C ∧ D ∧ E
   continue — 1. C  2. D ∧ E
     apply fact — 1. D ∧ E
   finish ⇒  — 1. D  2. E
```

6.2 Rule Attributes from Methods

Isar's rule attributes (see Section 3.4.5) allow for in-place transformations of facts. For example, the built-in attributes where and of are used to instantiate schematic variables of a fact with given arguments, while OF and THEN compose facts together. These attributes perform precise technical adjustments, in contrast to, for example, the simplified attribute, which rewrites all sub-terms in the fact according to the current simpset (or any provided rules). This enables some automation in fact transformations, but is not sufficient for many applications.

Similar to proof methods, it is often desirable to have domain-specific rule attributes for automating common rule transformations. However, in comparison to method expressions, Isar's attribute language is relatively primitive.

For example, an existing lemma may be proven in its *uncurried* form (i.e. $A \wedge B \longrightarrow C$) while a further proof may require it to be in its *curried* form (i.e. $A \longrightarrow B \longrightarrow C$). Rather than explicitly stating the curried form and proving it as a consequence of the existing lemma, it is more robust and convenient to instead convert a lemma in-place using a rule attribute.

Using the built-in THEN attribute and sequential composition of attributes (i.e. $[attribute_1,attribute_2]$), we can explicitly apply the curry rule ($?P \wedge ?Q \longrightarrow ?R \Longrightarrow ?P \longrightarrow ?Q \longrightarrow ?R$) a number of times to curry a given fact.

lemma ABCD: $((A \wedge B) \wedge C) \longrightarrow D$

thm ABCD[THEN curry] — $A \wedge B \longrightarrow C \longrightarrow D$

After applying curry once to ABCD, C is pulled out of the conjunction and into the implication. Applying curry again yields the desired rule in its fully uncurried form.

thm ABCD[THEN curry, THEN curry] — $A \longrightarrow B \longrightarrow C \longrightarrow D$

In this example, the transformation from the uncurried to curried form of a specific rule was straightforward. In general, however, these transformations can quickly become non-trivial as lemmas grow larger and more complex. Similar to proof methods, rule attributes are traditionally written in Isabelle/ML, thus presenting a significant barrier-to-entry for many proof authors.

The simplified rule attribute is an existing example of exposing the functionality of proof methods to rule attributes. Both the attribute and simp method utilize the same underlying functionality, however each requires a separate Isabelle/ML implementation, parser and syntax.

lemma ABB: $A \wedge B \wedge B$

thm ABB[simplified] — $A \wedge B$

lemma $A \wedge B \wedge B \Longrightarrow A \wedge B$ **by** simp

In this example, both simplified and simp implicitly apply the rule conj_absborb ($(?A \wedge ?A) = ?A$).

Generalizing this concept, we present a pair of rule attributes that allow proof method expressions to be used for performing rule transformations. This exposes the power of Isabelle's built-in proof automation to rule attributes, and effectively allows Eisbach to be used as a rule attribute language, as well as a proof method language.

Using methods to transform facts

We can define a specialized rule attribute @ that takes a single method expression as a parameter and evaluates it against the provided rule. With this we can, for example, replicate the functionality of simplified by instead invoking the simp method.

thm ABB[@ ‹simp›] — $A \wedge B$

Internally (see Section 6.2) this is accomplished by inserting the given fact as the lone premise of a dummy proof state. The given method expression then performs *forward* reasoning on this proof state, taking the final resulting premise as the produced fact. The drule method, which replaces a goal premise by resolving it with the first assumption of a given fact, is therefore equivalent to the THEN attribute.

In our conjunction currying example we needed to apply the THEN attribute a fixed number of times in order to convert the lemma into its fully curried form. In contrast, the drule method can be combined with + to apply the given rule as many times as possible.

lemma *dummy_ conclusion*
  apply (insert ABCD)

---

1. $(A \wedge B) \wedge C \longrightarrow D \Longrightarrow$ *dummy_ conclusion*

---

  apply (drule curry)+

---

1. $A \longrightarrow B \longrightarrow C \longrightarrow D \Longrightarrow$ *dummy_ conclusion*

In this dummy proof, we can see that the given method expression has transformed the inserted fact as desired. However this goal is not solvable, thus this transformation cannot be exported in a usable form. With the @ attribute we can retain this transformation, so the resulting fact can be used.

thm ABCD[@ ‹(drule curry)+›] — $A \longrightarrow B \longrightarrow C \longrightarrow D$

Although already an improvement over manually applying THEN, this simple method expression is not sufficient if we wish to have a general-purpose attribute for rule currying. If the antecedent is not left-associated, or if the implication is partially curried, the resulting rule will not be in the desired form.

lemma *ABCDEFG*: $(A \wedge B) \wedge (C \wedge D) \longrightarrow E \wedge F \longrightarrow G$

thm *ABCDEFG*[@ ‹(drule curry)+›] — $A \longrightarrow B \longrightarrow C \wedge D \longrightarrow E \wedge F \longrightarrow G$

In this example, the method expression successfully curries the first conjunct, but then the evaluation terminates since curry is no longer directly applicable via drule. To address this, we can use a combination of the uncurry rule ($?P \longrightarrow ?Q \longrightarrow ?R \Longrightarrow ?P \wedge ?Q \longrightarrow ?R$) and imp_conj_assoc to re-associate conjunctions in antecedents.

lemma imp_conj_assoc: $P \wedge Q \wedge R \longrightarrow S \Longrightarrow ((P \wedge Q) \wedge R) \longrightarrow S$

To complete our transformation, we want to first fully uncurry the rule so we have a single implication. After this, we repeatedly re-associate the antecedent until it exposes a single conjunct and then pull out this conjunct with curry.

thm *ABCDEFG*[@ ‹((drule uncurry)+)?, (drule imp_conj_assoc curry)+›]
  — $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow G$

Since this method expression performs a general purpose transformation, we may decide to wrap it up in an Eisbach method for easy re-use.

method curry_rule = ((drule uncurry)+)?, (drule imp_conj_assoc curry)+

thm *ABCDEFG*[@ ‹curry_rule›] — $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow G$

Transforming assumptions in facts

Previously, we have seen the @ attribute used to transform the conclusion of a given fact, by inserting it as a premise to a dummy goal and performing forward reasoning to transform it. Instead, if we wish to transform a fact assumption, we have to perform backwards reasoning.

For example, if we instead wanted to curry the implication in the assumption of a rule, we need reverse our reasoning with the OF attribute.

lemma ABCD_E: $((A \wedge B) \wedge C) \longrightarrow D \Longrightarrow E$

thm ABCD_E[OF uncurry,OF uncurry] — $A \longrightarrow B \longrightarrow C \longrightarrow D \Longrightarrow E$

This transformation is not possible with the @ attribute. Instead, the # attribute takes multiple method expressions and applies them in order to the assumptions of the given fact. In contrast to the @ attribute, the dummy proof state used by # has the assumption inserted as the conclusion, and thus can be manipulated through backwards reasoning (e.g. the rule method).

thm ABCD_E[# ‹(rule uncurry)+›] — $A \longrightarrow B \longrightarrow C \longrightarrow D \Longrightarrow E$

Similar to the previous example, additional consideration would be required to handle non-trivial implications. We can simply reverse all of the rules to achieve the same transformation for rule assumptions.

lemma imp_conj_assoc_rev: $((P \wedge Q) \wedge R) \longrightarrow S \Longrightarrow P \wedge Q \wedge R \longrightarrow S$

method curry_rule_asm = ((rule curry)+)*?*, (rule imp_conj_assoc_rev uncurry)+

This new method curry_rule_asm can now be used to transform rule. *assumptions* with #.

lemma ABCDEFG_H: $(A \wedge B) \wedge (C \wedge D) \longrightarrow E \wedge F \longrightarrow G \Longrightarrow H$

thm ABCDEFG_H[# ‹curry_rule_asm›]
— $A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow F \longrightarrow G \Longrightarrow H$

If a rule contains multiple assumptions, each method given to # is evaluated against each assumption in the order given.

lemma ABC_DD_E: $\llbracket A \wedge B \wedge C \longrightarrow D;\ D \wedge D \rrbracket \Longrightarrow E$

thm ABC_DD_E[# ‹curry_rule_asm› ‹simp›]
— $\llbracket A \longrightarrow B \longrightarrow C \longrightarrow D;\ D \rrbracket \Longrightarrow E$

Of course it is not always straightforward or even possible to perform a given transformation on both the assumptions and conclusion of a given fact. Specifically, the @ attribute may only weaken its conclusion, while # must strengthen its assumptions.

lemma *AB_CD*: $A \vee B \Longrightarrow C \wedge D$

thm *AB_CD*[# ‹rule disjI1›, @ ‹drule conjunct1›] — $A \Longrightarrow C$
thm *AB_CD*[# ‹rule disjI2›, @ ‹drule conjunct2›] — $B \Longrightarrow D$

Internal Subgoal Structure

Internally, a rule attribute is already very similar to a proof method: both take a thm (see Section 3.2.3) to another thm and are therefore guaranteed by Isabelle's proof kernel. Isabelle's LCF tradition embraces the identification of proofs and fact transformations, where a proof is simply a series of fact transformations, with conventions to maintain its subgoal structure.

In @ and #, the provided method is evaluated against a specially-constructed internal subgoal, such that the result of the evaluation can be used as a new fact. Using **apply_debug** we can inspect this subgoal during the method's evaluation.

Beginning with a dummy proof state, we can trace the evaluation of curry_rule in @ by evaluating it as part of a method expression given to **apply_debug**.

lemma True
  thm ABCD — $(A \wedge B) \wedge C \longrightarrow D$
  apply_debug (insert ABCD[@ ‹#break, curry_rule, #break›])

---

1. $(A \wedge B) \wedge C \longrightarrow D \Longrightarrow$ *thesis*

When evaluating insert, **apply_debug** encounters the #break inside of the method given to @ and halts on its internal goal state. To establish this state, @ first introduces a fixed variable *thesis* and creates a trivial thm: *thesis* $\Longrightarrow$ *thesis*. Treated as a proof state, this has a single subgoal *thesis* and overall conclusion *thesis*.

In this state, the original fact ABCD is inserted into the head subgoal as a goal premise, resulting in internal state $((A \wedge B) \wedge C \longrightarrow D \Longrightarrow \mathit{thesis}) \Longrightarrow (\mathit{thesis})$. The given method is then evaluated on this subgoal, as seen at this breakpoint. We can then **continue** to evaluate curry_rule.

continue

---

1. $A \longrightarrow B \longrightarrow C \longrightarrow D \Longrightarrow$ *thesis*

The internal state is now $(A \longrightarrow B \longrightarrow C \longrightarrow D \Longrightarrow \mathit{thesis}) \Longrightarrow (\mathit{thesis})$, which @ will convert into the resulting fact. After the method is completely evaluated, *thesis* is generalized into a schematic *?thesis* and then the first subgoal is solved by assumption, instantiating *?thesis* to our desired fact. The resulting thm is then simply $A \longrightarrow B \longrightarrow C \longrightarrow D$.

continue

---

1. $A \longrightarrow B \longrightarrow C \longrightarrow D \Longrightarrow$ True

Final Result.

The initial insert has inserted the final result from @ into the outer proof state. We can trace the evaluation of the # attribute similarly.

lemma True
  thm ABCD_E — $(A \wedge B) \wedge C \longrightarrow D \Longrightarrow E$
  apply_debug (insert ABCD_E[# ‹#break, curry_rule_asm, #break›])

---

1. $(A \wedge B) \wedge C \longrightarrow D$

To establish this subgoal state, similar to as in @, a fixed variable *thesis* is introduced and initialized in the trivial proof state *thesis* $\Longrightarrow$ *thesis*. The original fact ABCD is then inserted as a premise to the head subgoal, so the internal state is $(((A \wedge B) \wedge C \longrightarrow D \Longrightarrow E) \Longrightarrow \textit{thesis}) \Longrightarrow \textit{thesis}$. This thm is then re-associated to produce the desired subgoal structure $(A \wedge B) \wedge C \longrightarrow D \Longrightarrow ((E \Longrightarrow \textit{thesis}) \Longrightarrow \textit{thesis})$. The head subgoal can now be transformed with curry_rule_asm.

continue

---

1. $A \longrightarrow B \longrightarrow C \longrightarrow D$

After the method has been evaluated, the internal thm is re-associated to $((A \longrightarrow B \longrightarrow C \longrightarrow D \Longrightarrow E) \Longrightarrow \textit{thesis}) \Longrightarrow \textit{thesis}$. The fixed *thesis* is then generalized to a schematic variable, and the resulting head subgoal is solved by assumption as before. The resulting underlying thm is the final fact $A \longrightarrow B \longrightarrow C \longrightarrow D \Longrightarrow E$.

continue

---

1. $(A \longrightarrow B \longrightarrow C \longrightarrow D \Longrightarrow E) \Longrightarrow \mathsf{True}$

Final Result.

Additional Arguments

By default, the schematic variables in the given fact are fixed in the internal goal state to prevent the given method from performing unintended instantiations. To leave them schematic, and able to be instantiated, the flag (*schematic*) can be given.

thm conjI — $[\![?P;\ ?Q]\!] \Longrightarrow ?P \wedge ?Q$
thm allI — $(\bigwedge x.\ ?P\ x) \Longrightarrow \forall x.\ ?P\ x$

thm conjI[# ‹rule allI›] — *Error: Failed to apply method to rule assumption.*
thm conjI[# (*schematic*) ‹rule allI›] — $[\![\bigwedge x.\ ?P\ x;\ ?Q]\!] \Longrightarrow (\forall x.\ ?P\ x) \wedge ?Q$

However care must be taken when using this feature, since many methods will aggressively instantiate schematic variables in order to "solve" the internal subgoal.

thm conjI[# (*schematic*) ‹simp›] — $?Q \Longrightarrow \mathsf{True} \wedge ?Q$

Additionally, by default, the internal *thesis* is an object-level judgement (i.e. bool in HOL) in order to maintain compatibility with most methods and rules. This necessitates an extra step performed by these attributes that was not previously discussed, where the rule must be converted to an object-level judgement (e.g. $\forall x.\ P\ x \longrightarrow Q\ x$) prior to the final instantiation of *thesis*, and then have its meta-connectives restored in the final fact (e.g. $\bigwedge x.\ P\ x \Longrightarrow Q\ x$).

Since converting a given fact into an object logic judgement is not always possible (i.e. if the rule contains meta-logic judgements), the flag (*raw_prop*) can be given to instead have *thesis* be of type prop.

thm conjunctionD1 — *PROP ?A &&& PROP ?B* $\Longrightarrow$ *PROP ?A*

thm conjunctionD1[@ ‹–›] — *Error: Failed to fully atomize result.*
thm conjunctionD1[@ (*raw_prop*) ‹–›] — *PROP ?A &&& PROP ?B* $\Longrightarrow$ *PROP ?A*

This, however, can cause incompatibilities with existing rules and methods (e.g. elimination rules with erule, where the conclusion is a object logic judgement). In this example, the schematic *?R* in allE is a bool, so it cannot be applied when the conclusion of the goal is a prop.

```
thm allE — ⟦∀x. ?P x; ?P ?x ⟹ ?R⟧ ⟹ ?R

thm allI[@ ‹erule allE›] — (⋀x. ?P x) ⟹ ?P ?x
thm allI[@ (raw_prop) ‹erule allE›] — Error: Failed to apply method to rule assumption.
```

Most often these flags are not necessary, but exist to support specific use cases.

In Chapter 7 we show a nontrivial application of these attributes, where a large library of lemmas, stated in the original refinement calculus of L4.verified [23], can be automatically transformed into the calculus of a new proof method.

6.3 Advanced Methods and Combinators

With Eisbach it is straightforward to lift existing tacticals into method combinators, since methods can be easily taken as arguments to new (higher-order) methods. For example, the common tactical *ALLGOALS* can be made available to methods with a few lines of ML. This takes a method and repeatedly evaluates it against the goal, once with each subgoal rotated to be the head goal.

```
ML
‹fun method_evaluate text ctxt facts =
   Method.NO_CONTEXT_TACTIC ctxt
     (Method.evaluate_runtime text ctxt facts)›

method_setup all =
 ‹Method.text_closure >> (fn m => fn ctxt => fn facts =>
   let
     fun tac i st' =
       Goal.restrict i 1 st'
       |> method_evaluate m ctxt facts
       |> Seq.map (Goal.unrestrict i)
   in SIMPLE_METHOD (ALLGOALS tac) facts end)›
```

This method takes a method as an argument with the *Method.text_closure* parser. This forms a *static closure* of the method syntax (see Section 5.3.4) which is then re-evaluated by *Method.evaluate_runtime* on each subgoal. These functions were introduced to Isar's *Method* module as of Isabelle-2016 in an effort to standardize this aspect of Eisbach.

```
lemma ⟦A; B; C⟧ ⟹ A ∧ B ∧ C
  apply (intro conjI)
```

```
 1. ⟦A; B; C⟧ ⟹ A
 2. ⟦A; B; C⟧ ⟹ B
 3. ⟦A; B; C⟧ ⟹ C
```

Each of these three subgoals is now solvable by assumption.

```
  apply (all ‹assumption›)
  done
```

Some new methods which have no analogous tactical can be extremely useful for writing methods in Eisbach. For example, we can define a succeeds method combinator which leaves the goal unchanged, but fails if the method would not have applied.

```
method_setup succeeds =
‹Method.text_closure >> (fn m => fn ctxt => fn facts =>
  let
    fun can_tac st =
      (case SINGLE (method_evaluate m ctxt facts) st of
        SOME _ => Seq.single st
      | NONE => Seq.empty)
  in SIMPLE_METHOD can_tac facts end)›
```

This allows for writing methods that function as tests, when more complex control flow is required.

Many general-purpose combinators can be written without requiring ML. For example, we can write a combinator to hide the conclusion of a subgoal from a given method by using only Eisbach's match.

```
lemma A ∧ A ⟹ B ∧ B
  apply (only_asm ‹simp›)
```

```
1. A ⟹ B ∧ B
```

This new method only_asm (defined below) wraps the subgoal such that simp can only see the premise $A \wedge A$, thus leaving the conclusion untouched.

To implement only_asm, first we establish a new **context** in order to write **private** wrapper definitions and lemmas. This allows us to assume that the method passed to only_asm will be unable to unfold these definitions, or otherwise inadvertently make use of them, since they will necessarily be out of scope.

```
context
begin

private definition protect_concl x ≡ ¬ x
private definition protect_false ≡ False

private lemma protect_start: (protect_concl P ⟹ protect_false) ⟹ P
private lemma protect_end: protect_concl P ⟹ P ⟹ protect_false
```

Here we are simply wrapping negation and False, in order to carefully move the conclusion of a subgoal as a negated premise, as in the rule ccontr $((\neg\ ?P \Longrightarrow \mathsf{False}) \Longrightarrow ?P)$. This allows us to use the subgoal focusing provided by match to hide it.

```
method only_asm methods m =
  (match premises in H[thin]:_ (multi) ⇒
    ‹rule protect_start,
     match premises in H'[thin]:protect_concl _ ⇒
       ‹insert H, #break, (m, #break);rule protect_end[OF H']››)

end
```

In this method match is used to control the goal premises that m can see. The bound fact H holds the goal's initial premises (binding to *all* premises due to the *multi* flag), while *H'* holds the newly-introduced protect_concl premise after applying protect_start. By removing protect_concl from the subgoal state with thin (see Section 5.2.5) and binding it to *H'*, we ensure that m cannot inadvertently remove or manipulate it, and that the original goal can therefore be restored with protect_end. Additionally, since we are in a local **context**, our **private** lemmas and definitions are inaccessible after the **end**.

With **apply_debug**, we can inspect the goal state just before the given method is evaluated.

```
lemma A ∧ A ⟹ B ∧ B
  apply_debug (only_asm ‹simp›)
```

```
1. A ∧ A ⟹ protect_false
```

Here we can see that $B \wedge B$ has been replaced with protect_false, and that *H'* contains the wrapped conclusion.

```
thm H' — protect_concl (B ∧ B)

continue
```

```
1. A ⟹ protect_false
```

After simp is evaluated, the resulting state has is premise simplified but leaves protect_false untouched.

```
continue
```

```
1. A ⟹ B ∧ B

Final Result.
```

Finally, protect_false has been replaced with $B \wedge B$, the original conclusion.

This pattern of *window reasoning* turns out to be extremely powerful when combined with Eisbach, where a combinator focuses the effect of a given method on a sub-component of the subgoal. In the next section, we'll see how this can be generalized to write window reasoning combinators for different proof calculi.

6.3.1 A Hoare Logic Combinator

We have seen how Eisbach can be used to write a general-purpose method combinator only_asm that controls the perspective the provided method has on the goal state. This suggests a general strategy for extending this technique to other domains: wrap a subgoal to focus on a particular sub-component, execute the provided method, then unwrap to recover the original structure.

As an example, we can consider the monadic Hoare logic framework used in L4.verified [23]. In this framework, a Hoare triple $\{\!|P|\!\}\ f\ \{\!|Q|\!\}$ states that given the precondition P, the postcondition Q will hold after executing f.

$\{\!|P|\!\}\ f\ \{\!|Q|\!\} \equiv \forall\, s.\ P\ s \longrightarrow (\forall\,(r,\ s')\in \mathsf{fst}\ (f\ s).\ Q\ r\ s')$

A Hoare triple is usually encountered as the conclusion to a subgoal, where the precondition is a schematic variable, and the postcondition is a (potentially large) conjunction resulting from using a verification condition generator. These conjuncts need to be propagated into the precondition by applying appropriate rules from the calculus.

Often it is the case that a given Hoare triple requires a very specific or technical adjustment in order for a proof to proceed. If this cannot be accomplished with existing calculus rules, the user must resort to manually specifying the desired postcondition, then prove that it implies the target one. In large proofs, with dozens of individual conjuncts in a postcondition, this can result in excessively verbose and brittle proof scripts.

For example, consider the following scenario:

lemma assumes AA': $\bigwedge r\ s.\ A\ r\ s \Longrightarrow A'\ r\ s$
  shows $\{\!|P|\!\}\ f\ \{\!|\lambda r\ s.\ A\ r\ s \longrightarrow B\ r\ s|\!\}$
  apply (rule hoare_strengthen_post[where $Q{=}\lambda r\ s.\ A'\ r\ s \longrightarrow B\ r\ s$])

---

1. $\{\!|P|\!\}\ f\ \{\!|\lambda r\ s.\ A'\ r\ s \longrightarrow B\ r\ s|\!\}$
2. $\bigwedge r\ s.\ A'\ r\ s \longrightarrow B\ r\ s \Longrightarrow A\ r\ s \longrightarrow B\ r\ s$

By manually applying hoare_strengthen_post ($[\![\{\!|?P|\!\}\ ?a\ \{\!|?Q|\!\};\ \bigwedge r\ s.\ ?Q\ r\ s \Longrightarrow\ ?R\ r\ s]\!] \Longrightarrow \{\!|?P|\!\}\ ?a\ \{\!|?R|\!\}$) and instantiating $?Q$ to our desired postcondition, we have introduced two new subgoals: we now must show that we can establish $A'\ r\ s \longrightarrow B\ r\ s$ as the postcondition for f, and that it implies the original condition. The second goal is a trivial consequence of the assumption.

  defer
  apply (simp *add*: AA')

---

1. $\{\!|P|\!\}\ f\ \{\!|\lambda r\ s.\ A'\ r\ s \longrightarrow B\ r\ s|\!\}$

The problem, as stated earlier, is that quoting the postcondition in the proof script is often impractical. Instead, we would like to be able to use the assumption AA' directly to strengthen the subgoal. To accomplish this, we can write a specialized method combinator that allows us to treat the postcondition as if it were the conclusion of a subgoal.

Similar to as we did with the only_asm method, we start by opening a new context to create private definitions specific to our method implementation.

```
context begin

private definition packed_triple P f r s ≡ ∃si. P si ∧ (r, s) ∈ fst (f si)

private lemma packed_triple_start:
  (⋀r s. packed_triple P f r s ⟹ Q r s) ⟹ ⦃P⦄ f ⦃λr s. Q r s⦄

private lemma packed_tripleE:
  packed_triple P f r s ⟹ ⦃P⦄ f ⦃λr s. Q r s⦄ ⟹ Q r s
```

Here the new constant packed_triple wraps the first half of a Hoare triple, assuming the existence of an initial state *si* that satisfies the precondition P and f gives the return value/result state r and s after *si*. This allows a Hoare triple to be packed by applying packed_triple_start, leaving the postcondition as the conclusion. After the goal has been modified, it can be unpacked back into a Hoare triple with packed_tripleE. With Eisbach, this is a simple method to define.

```
method post_simple methods m =
  (rule packed_triple_start,
   match premises in H[thin]: packed_triple _ _ _ _ ⇒
    ‹#break, m, #break, (atomize (full))?, rule packed_tripleE[OF H]›)
```

Note the use of atomize after m is invoked. This is a built-in method that rewrites the subgoal into the current object logic (HOL in this case). Any assumptions are turned into HOL implications, and goal parameters are turned into universally quantified variables. Here this is necessary to ensure that the goal has been completely collapsed into a single term that can be packed with packed_tripleE.

With this method, we can easily modify the postcondition of our subgoal in-place.

```
lemma assumes AA': ⋀r s. A r s ⟹ A' r s
  shows ⦃P⦄ f ⦃λr s. A r s ⟶ B r s⦄
  apply_debug (post_simple ‹rule impI, drule AA'›)
```

```
1. A r s ⟶ B r s
```

The first breakpoint from post_simple shows the goal just before the method is invoked. We can see that the postcondition has indeed been lifted to the conclusion of the head subgoal. The local fact H contains the introduced packed_triple assumption.

```
thm H — packed_triple P f r s
```

Our second breakpoint, hit after the method has executed, shows the resulting goal just before it is atomized and packed.

```
continue
```

```
1. A' r s ⟹ B r s
```

The resulting Hoare triple is now in the expected form.

```
continue
```

```
1. ⋀r s. ⦃P⦄ f ⦃λr s. A′ r s ⟶ B r s⦄

Final Result.
```

Note that the introduced goal parameters r and s are not present in the subgoal itself, and will be ignored (or removed) by most methods.

A key issue with this implementation is the implicit assumption that the given method will produce a single subgoal. Indeed if multiple goals are produced, then our combinator will not behave as expected.

```
lemma assumes DA: ⋀r s. D r s ⟹ A r s
  shows ⦃P⦄ f ⦃λr s. C ⟶ A r s ∧ B r s⦄
  apply_debug (post_simple ‹rule impI, rule conjI, rule DA›)
  continue
```

```
1. C ⟹ D r s
2. C ⟹ B r s
```

After introducing the conjunct in the postcondition with conjI, the resulting proof state has one subgoal per conjunct. This additional subgoal causes the method to behave unexpectedly, since atomize and rule only affect the head subgoal.

```
continue
```

```
1. ⋀r s. ⦃P⦄ f ⦃λr s. C ⟶ D r s⦄
2. ⋀r s. C ⟹ B r s

Final Result.
```

Although the first subgoal has been correctly packed into the postcondition, the second has resulted an unsolvable subgoal.

Addressing this issue turns out to be non-trivial, and a common issue with method development in Eisbach, since there are no built-in methods that allow direct control over the subgoal structure. The solution is a new fold_subgoals method that will wrap all current subgoals into a single meta-conjunction (&&&).

6.3.2 Subgoal Folding

With a few exceptions, the majority of proof methods operate solely on the first (head) subgoal, where the only possible global effects are the instantiation of schematic variables. We can use method combinators, such as structured concatenation (;) or the custom all combinator, to lift the effect of a given method to multiple subgoals. However, precise control over the subgoal structure is still not immediately accessible.

When using Isabelle's built-in methods, subgoals are either introduced by applying a rule with multiple assumptions or discharged by applying a rule with no assumptions. Introducing a new subgoal is therefore a non-reversible effect.

For example, once we apply conjI to a subgoal, we now are obligated to solve each conjunct independently.

lemma $\forall x.\ A\ x \longrightarrow (\forall y.\ B\ y\ x) \wedge (C\ x \longrightarrow D\ x)$
apply (intro allI conjI impI)

---

1. $\bigwedge x\ y.\ A\ x \Longrightarrow B\ y\ x$
2. $\bigwedge x.\ \llbracket A\ x;\ C\ x\rrbracket \Longrightarrow D\ x$

At this point there is no built-in method that can return the proof state into its original form. Although this is logically sound (i.e. no information has been lost), it can pose an issue when writing methods in Eisbach.

The new method fold_subgoals, collapses the entire subgoal state into a single meta-conjunct.

apply fold_subgoals

---

1. $(\bigwedge x\ y.\ A\ x \Longrightarrow B\ y\ x)\ \&\&\&\ (\bigwedge x.\ \llbracket A\ x;\ C\ x\rrbracket \Longrightarrow D\ x)$

We can then use atomize to convert this into a HOL formula, rewriting meta-connectives to their HOL equivalents.

apply (atomize (*full*))

---

1. $(\forall x\ y.\ A\ x \longrightarrow B\ y\ x) \wedge$
$(\forall x.\ A\ x \longrightarrow C\ x \longrightarrow D\ x)$

The resulting single subgoal is now logically equivalent to the initial one, although with the antecedent $A\ x$ duplicated in each conjunct.

As seen in this example, the subgoals of a given proof state are likely to share a common *prefix* of goal parameters and assumptions. In our example, the initial assumption $A\ x$ was duplicated in each subgoal, and thus appeared in each resulting conjunct. To capture this common case, an optional argument (*prefix*) can be given to fold_subgoals to lift the longest common prefix of parameters and assumptions out of the meta-conjunction.

lemma $\forall x.\ A\ x \longrightarrow (\forall y.\ B\ y\ x) \wedge (C\ x \longrightarrow D\ x)$
apply (intro allI conjI impI)
apply (fold_subgoals (*prefix*))

---

1. $\bigwedge x.\ A\ x \Longrightarrow (\bigwedge y.\ B\ y\ x)\ \&\&\&\ (C\ x \Longrightarrow D\ x)$

Although this is still logically equivalent to the original use of fold_subgoals, the common assumption $A\ x$ now only appears once and the goal parameter x is common to both meta-conjuncts. Applying atomize to this subgoal now returns it to its original form.

apply (atomize (*full*))

---

1. $\forall x.\ A\ x \longrightarrow (\forall y.\ B\ y\ x) \wedge (C\ x \longrightarrow D\ x)$

Focusing while Folding

The effect of fold_subgoals is global to the proof state: all currently visible subgoals are folded into the resulting meta-conjunct. When performing a subgoal focus (as in a match), only the focused subgoal is visible to the inner method. Applying fold_subgoals in a match will therefore only fold subgoals visible in the current focus.

```
lemma assumes DA: D ⟹ A
  shows (A ∧ B) ∧ C
  apply (rule conjI)
```

```
1. A ∧ B
2. C
```

In a proof state with multiple subgoals, match **conclusion** will focus on the first one as the match target. The inner method therefore can only see this goal.

```
  apply_debug
    (match conclusion in _ ∧ _ ⇒
      ‹rule conjI, rule DA, #break, fold_subgoals, atomize (full), #break›)
```

```
1. D
2. B
```

After applying conjI and DA, in the focused goal, we can fold and atomize to recover a single subgoal. Note that the third subgoal C from the outer proof state is not visible.

```
continue
```

```
1. D ∧ B
```

```
continue
```

```
1. D ∧ B
2. C
```

Final Result.

After the match been evaluated, the focus ends and we see that the effect of fold_subgoals was restricted to the head goal, leaving the second subgoal untouched.

Hoare Logic Combinator Revisited

Returning to the Hoare logic example from the previous section, armed with fold_subgoals, it is straightforward to modify our method to simply fold all the resulting subgoals after applying m before atomizing.

```
method post methods m =
  (rule packed_triple_start,
   match premises in H[thin]: packed_triple _ _ _ _ ⇒
   ‹#break, m, #break,
      fold_subgoals (prefix),(atomize (full))?, #break,
      rule packed_tripleE[OF H]›)
```

This will now have the intended effect when m produces multiple subgoals.

```
lemma assumes DA: ⋀r s. D r s ⟹ A r s
  shows ⦃P⦄ f ⦃λr s. C ⟶ A r s ∧ B r s⦄
  apply_debug (post ‹rule impI, rule conjI, rule DA›)
  continue
```

```
1. C ⟹ D r s
2. C ⟹ B r s
```

After the given method is applied, there are multiple remaining open subgoals. They are then folded by fold_subgoals into a single goal and atomized.

```
continue
```

```
1. C ⟶ D r s ∧ B r s
```

This can now be packed into a single postcondition with packed_tripleE.

```
continue
```

```
1. ⋀r s. ⦃P⦄ f ⦃λr s. C ⟶ D r s ∧ B r s⦄

Final Result.
```

Finally, the resulting state is as expected, where the assumption *DA* has been used to transform one conjunct of the consequent in the postcondition.

6.4 Conclusion

In this chapter we have seen a variety of tools implemented directly in Eisbach (only_asm and post methods), implemented using Eisbach's method evaluation framework (@ and # attributes), and designed to support proof method development in Eisbach (**apply_debug** and the fold_subgoals method). These tools both demonstrate the capabilities of Eisbach and expand the scope of what is possible with it. In the next chapter, we evaluate this infrastructure in the context of large scale proof engineering by developing a collection of proof methods for real-world use in L4.verified.

Chapter 7

Case Study: L4.verified

In the previous chapters we have seen the features provided by the basic Eisbach framework and shown how it can be easily extended into a rich ecosystem for developing proof automation. In this chapter we outline the development of a non-trivial proof method using Eisbach. This proof method, corres, is actively being used in the L4.verified project to aid in its refinement proofs. We show how Eisbach simplifies much of the standard boilerplate for developing proof methods, and how it facilitates rapid prototyping. We apply the corres method to refactor several existing proofs from L4.verified, resulting in smaller, simpler and more maintainable proofs.

This chapter contains a large amount of technical detail that is not crucial to understanding the main result, but nonetheless is included for completeness and reproducibility. It assumes a greater degree of familiarity with Isabelle than previous chapters and contains many examples with non-trivial proof states. The casual reader can safely read the beginning of Section 7.1, followed by Section 7.4 and after.

Chapter Outline

- **L4.verified, VCGs, and Refinement.** Section 7.1 motivates the need for a new proof method in L4.verified for refinement proofs.

- **Corres.** Section 7.2 introduces the corres refinement calculus from L4.verified.

- **The Corres Proof Method.** Section 7.3 follows the process of incrementally building a proof method for automating corres proofs

- **Application to L4.verified.** Section 7.4 presents the results of applying this new method to L4.verified.

- **Conclusion.** Section 7.5 summarizes and puts this case study in the context of this thesis.

7.1 L4.verified, VCGs, and Refinement

The Hoare logic and refinement calculus used in L4.verified [23] were instrumental to its success. Importantly, the VCG (verification-condition generator) for solving Hoare triples is extensively used in the L4.verified proofs. This VCG, implemented in Isabelle/ML as the `wp` method, performs *weakest-precondition*[1] style reasoning [26] to prove invariant properties about the seL4 microkernel. These invariants are then used as a stepping stone for the major verification result: proving refinement between seL4's abstract functional specification and its C implementation. This proof provides guarantees about the implementation, such as the absence of invalid memory accesses, but also allows for additional properties to be proven against the abstract specification (such as integrity [61] and confidentiality [53]) and have them carried down to the implementation without significant additional effort.

The refinement calculus used in L4.verified is substantially more complicated than the Hoare logic. It is contextual refinement under preconditions, and therefore incorporates Hoare logic in order to solve and propagate additional refinement conditions. Rather than pay the initial cost of developing a VCG for refinement proofs, however, the original L4.verified authors instead chose to perform these proofs manually. This decision was made for a number of reasons:

- There was no previous work (in contrast to Hoare logic) in developing refinement VCGs.
- Early attempts at developing a refinement VCG were unsuccessful due to frequent changes to the refinement calculus.
- It was not clear that the initial effort to build a and maintain proof method in Isabelle/ML could be justified given the project's time-frame.

In the years since the completion of the original proof, these refinement proofs have undergone numerous maintenance iterations, and have grown in complexity as seL4 evolved. These highly manual proofs have proven to be among the most brittle in the L4.verified project, as even minor changes to specifications or lemma statements would then require manual intervention by a proof engineer.

We therefore decided to develop a new VCG for automating refinement proofs in L4.verified. The goal was to design a VCG that could accelerate the tediously manual process of locating and applying rules from the refinement calculus, while also automatically attempting some of the advanced proof techniques that had emerged over time. This VCG would need to be applicable for refactoring existing refinement proofs to make them more robust, lowering the cost of proof maintenance, and for accelerating the development of new refinement proofs.

The result was the `corres` proof method, which borrows heavily from weakest-precondition style reasoning for Hoare logic. This method is not necessarily specific to the L4.verified refinement proofs, and indeed is applicable in any refinement proof formalised in

[1] The preconditions calculated by `wp` are not necessarily the weakest possible. Occasionally these are referred to as the weakest *reasonable* preconditions.

the monadic refinement calculus from Cock, et al. [23] (e.g. an abstract specification built on the output of AutoCorres [33]). However, the L4.verified proofs remain the canonical application of this calculus to date.

7.1.1 Refinement

Here we present refinement as defined by Cock et al. in their previous work [23] on formalizing the Hoare logic and refinement calculus used in L4.verified. Refinement is a relationship between two processes: an abstract (A) and a concrete (C). We say that A is refined by C if all possible observable behaviours of C are subsumed by the possible behaviours of A, given the same set of inputs. Formally, we can define a process as having three functions: an initialization function (Init) which takes an external state into a set of internal states, a step function (Step) which takes an external event and gives a set of internal state transitions, and a finalize function (Fin), which takes an internal state and gives a corresponding external state.

record process =
  Init :: $'external \Rightarrow 'state$ set
  Step :: $'event \Rightarrow ('state \times 'state)$ set
  Fin :: $'state \Rightarrow 'external$

From this we can define an execution function, which evaluates a given process A on some list of events.

steps δ s $events$ $\equiv$ foldl ($\lambda states\ event.$ (δ $event$) `` $states$) s $events$
execution A s $events$ $\equiv$ (Fin A) ` (steps (Step A) (Init A s) $events$)

We can then say that A is refined by C if the execution result of C for some given list of the events is a subset of the execution of A.

$A \sqsubseteq C \equiv \forall s\ events.$ execution C s $events$ $\subseteq$ execution A s $events$

Importantly, refinement is transitive, which allows for multiple formal specifications at different levels of abstraction.

$\forall A\ B\ C.\ A \sqsubseteq B \longrightarrow B \sqsubseteq C \longrightarrow A \sqsubseteq C$

To prove refinement, it is often more convenient to instead show the equivalent property of *forward simulation*. To prove forward simulation between two processes, we first establish an explicit state relation (S) between the *internal* states of each one. We then show that each of the functions that the process comprises establish and maintain this relation.

$A \sqsubseteq_S C \equiv$
  ($\forall s.$ Init C s $\subseteq$ S `` Init A s) $\wedge$
  ($\forall event.$ (S ;; Step C $event$) $\subseteq$ (Step A $event$;; S)) $\wedge$
  ($\forall s\ s'.$ (s, s') $\in S \longrightarrow$ Fin C s' = Fin A s)

Roughly, the definition of $A \sqsubseteq_S C$ says: given a state relation S, a concrete process C and an abstract process A, A is a forward simulation of C given the following conditions:

- For any *external* state, the set of initial states for C is a subset of those for A, modulo the state relation S.
- For any single *event*, the result of stepping C is a subset of stepping A, modulo the state relation S.

- For any two *internal* states satisfying S, their *external* final states are equivalent for both C and A.

It is straightforward to prove that forward simulation implies refinement.

$$\forall A\ C.\ \exists S.\ A \sqsubseteq_S C \longrightarrow A \sqsubseteq C$$

Recall from Section 3.5 that L4.verified is divided into two separate refinement proofs, linked by transitivity: C-to-executable (CRefine) and executable-to-abstract (Refine). The focus of our case study is the latter proof, which comprises over 80,000 lines of Isabelle/Isar source to date. Our aim is to increase level of proof automation in the existing executable-to-abstract refinement proof, and to expedite similar proofs in the future.

7.1.2 The State Monad

Both the abstract and executable specifications of L4.verified are formalized as a *non-deterministic state monad* [23]. The monad state models the entire state of the hardware, including physical memory and machine registers, as well as abstract data structures such as the kernel heap or capability derivation tree [41]. Each function returns a set of result states (and return values), where multiple results represent non-deterministic choice (to abstract away implementation details). To indicate catastropic failure of a function (e.g. invalid memory access), a flag is added to the state which is tripped if *any* possible computation branch would fail.

$$('s,'a)\ \mathsf{nondet\_monad} = {'s} \Rightarrow ('a \times {'s})\ \mathsf{set} \times \mathsf{bool}$$

The basic monad constructors are return and bind (>>=)). The return constructor takes a single argument and immediately returns it, leaving the state unmodified and always succeeding (leaving the failure flag False).

$$\mathsf{return} :: {'a} \Rightarrow {'s} \Rightarrow {'a} \times {'s}\ \mathsf{set} \times \mathsf{bool}$$
$$\mathsf{return}\ a \equiv \lambda s.\ (\{(a,\ s)\},\ \mathsf{False})$$

Monadic bind takes two functions, f and g, and composes them in sequence. The input state s is evaluated on f, and the set of resulting state-return value pairs (s' and r) are given to g. The final result is the union of all the possible results of g.

$$\begin{aligned} >\!>= :: &('s,'a)\ \mathsf{nondet\_monad} \Rightarrow ('a \Rightarrow ('s,\ 'b)\ \mathsf{nondet\_monad}) \\ &\Rightarrow ('s,\ 'b)\ \mathsf{nondet\_monad} \end{aligned}$$

$$\begin{aligned} f >\!>= g \equiv \lambda s.\ (&\bigcup(\mathsf{fst}\ `\ (\lambda(s',r).\ g\ s'\ r)\ `\ \mathsf{fst}\ (f\ s)), \\ &\mathsf{True} \in \mathsf{snd}\ `\ (\lambda(s',r).\ g\ s\ r)\ `\ \mathsf{fst}\ (f\ s) \vee \mathsf{snd}\ (f\ s)) \end{aligned}$$

The failure flag is set to True if any branch of g or f failed. Failure is therefore prolific, and a non-failure result of some function guarantees that none of its non-deterministic branches will result in failure. The do syntax from Haskell is supported, to make longer chains of computations more concise and readable:

$$\mathsf{do}\ x \leftarrow f;\ g\ x\ \mathsf{od} \equiv f >\!>= g$$

The three canonical monadic laws for return are as follows:

$$\begin{aligned} \mathsf{return}\ x >\!\!>\!= f &= f\,x \\ m >\!\!>\!= \mathsf{return} &= m \\ (m >\!\!>\!= f) >\!\!>\!= g &= m >\!\!>\!= (\lambda x.\ f\,x >\!\!>\!= g) \end{aligned}$$

To interact with the state we require two primitives: gets and modify, which allow for retrieving and modifying respectively.

$\mathsf{gets} :: ('s \Rightarrow 'a) \Rightarrow ('s,\ 'a)\ \mathsf{nondet\_monad}$
$\mathsf{gets}\ f\ = (\lambda s.\ (\{(f\ s,\ s)\},\ \mathsf{False}))$

$\mathsf{modify} :: ('s \Rightarrow 's) \Rightarrow ('s,\ \mathsf{unit})\ \mathsf{nondet\_monad}$
$\mathsf{modify}\ f \equiv \lambda s.\ (\{((),\ f\ s)\},\ \mathsf{False})$

The monadic laws for gets and modify are similar to those for return.

7.1.3 Monadic Hoare Logic

In L4.verified, Hoare logic is used extensively to prove invariant properties about the abstract and executable specifications. These invariants are necessary conditions for refinement, and are used to discharge intermediate subgoals arising from the corres calculus (see Section 7.2).

The Hoare triple $\{\!|P|\!\}\ f\ \{\!|Q|\!\}$ (briefly introduced previously in Section 6.3.1) states that given the precondition P, the postcondition Q will hold after executing *f*.

$\{\!|P|\!\}\ f\ \{\!|Q|\!\} \equiv \forall\, s.\ P\ s \longrightarrow (\forall\, (r,\ s')\in\mathsf{fst}\ (f\ s).\ Q\ r\ s')$

Note that Q takes two parameters: the return value of *f* and the resulting state. The rule for return is straightforward:

return_wp: $\{\!|P\ x|\!\}\ \mathsf{return}\ x\ \{\!|P|\!\}$

Since return leaves the state unmodified, any predicate P will be preserved by it. Similar rules for gets and modify can be easily proven.

gets_wp: $\{\!|\lambda s.\ P\ (f\ s)\ s|\!\}\ \mathsf{gets}\ f\ \{\!|P|\!\}$
modify_wp: $\{\!|\lambda s.\ P\ ()\ (f\ s)|\!\}\ \mathsf{modify}\ f\ \{\!|P|\!\}$

The rule for bind however is complicated by the need for a fresh predicate.

seq: $[\![\bigwedge x.\ \{\!|B\ x|\!\}\ g\ x\ \{\!|C|\!\};\ \{\!|A|\!\}\ f\ \{\!|B|\!\}]\!] \Longrightarrow \{\!|A|\!\}\ f >\!\!>\!= g\ \{\!|C|\!\}$

In seq the intermediate predicate B serves as both the precondition for g and postcondition for *f*. Since B does not appear in the conclusion of the goal, we are free to choose any suitable condition, provided it is sufficient for g to establish C and can be established by *f*. Determining these intermediate conditions (and proving they satisfy the conditions of seq) is the main challenge in automating Hoare logic proofs. A verification-condition generator takes a Hoare triple and applies seq and similar rules from the Hoare logic calculus, calculating and connecting the intermediate conditions in order to produce a final verification condition, whose proof ultimately entails the validity of the Hoare triple.

Weakest Preconditions and Verification Conditions

The seq rule is used in Hoare logic in order to decompose triples over composite functions into smaller proof obligations. Further rules are required to handle more complex control flow (i.e. conditionals, loops, exceptions), with the goal of eventually decomposing the main goal into a sequence of Hoare triples over atomic functions. Ideally these atomic triples are trivially solved with previously proven results.

In practice, there are two strategies to calculate the intermediate condition (B) required by seq: strongest postcondition or weakest precondition. In strongest postcondition style reasoning, the second precondition is solved first ($\{\!|A|\!\}\ f\ \{\!|B|\!\}$) and thus B is calculated as the strongest postcondition that can be established after f assuming A. Conversely, in weakest precondition style reasoning, the other precondition is solved first ($\bigwedge x.\ \{\!|B\ x|\!\}\ g\ x\ \{\!|C|\!\}$), and B is calculated as the weakest precondition required to establish C after g.

In L4.verified, proofs are done in weakest-precondition style, supported by the wp method which acts as a verification condition generator. The goal of the wp method is to convert a Hoare triple (precondition, function, and postcondition) into a single *verification* condition, which implies that the Hoare triple is valid. Effectively this translates the proof obligation from Hoare logic into Isabelle's HOL logic, where Isabelle's built-in proof methods such as simp and auto can be effectively used.

Although the actual wp method is written in ML, we can give a simplified implementation with a few lines of Eisbach.

```
named_theorems wp and wp_comb and wp_split

method wp declares wp wp_comb wp_split =
((rule wp | (rule wp_comb, rule wp)) | rule wp_split)+
```

Example wp Proof

As a simple example, consider the following function:

```
record my_state =
  my_int :: int

definition my_add :: int ⇒ (my_state, unit) nondet_monad
  where my_add i ≡
    do i' ← gets my_int;
         modify (λs. (s ⦇my_int := i + i'⦈))
    od
```

We can phrase the behaviour of this function as a simple Hoare triple, and prove it with wp. Using **apply_debug** (see Section 6.1) we can trace the execution of wp by supplying the tag *wp* (the implementation of wp contains a single tagged breakpoint before every rule application).

```
lemma my_add_adds:
  ⦃λs. z = my_int s + y⦄ my_add y ⦃λ_ s. my_int s = z⦄
  unfolding my_add_def
  apply_debug (tags wp) wp
```

The first step in a wp proof to apply the following weakening rule hoare_pre.

hoare_pre: $[\![\{\!|P'|\!\}\ f\ \{\!|Q|\!\};\ \bigwedge s.\ P\ s \Longrightarrow P'\ s]\!] \Longrightarrow \{\!|P|\!\}\ f\ \{\!|Q|\!\}$

This replaces the concrete precondition in the current subgoal with a schematic one, ultimately to be established as the consequent in the verification condition.

```
 1. ⦃?Q⦄ do i′ ← gets my_int;
              modify (λs. s(|my_int := y + i′|))
           od
    ⦃λ_ s. my_int s = z⦄
 2. ⋀s. z = my_int s + y ⟹ ?Q s
```

Then seq is applied to introduce an intermediate condition and decompose the function body into two Hoare triples.

continue

```
 1. ⋀i′. ⦃?B3 i′⦄ modify (λs. s(|my_int := y + i′|))
          ⦃λ_ s. my_int s = z⦄
 2. ⦃?Q⦄ gets my_int ⦃?B3⦄
 3. ⋀s. z = my_int s + y ⟹ ?Q s
```

The first Hoare triple is solved with modify_wp.

continue

```
 1. ⦃?Q⦄ gets my_int ⦃λi′ s. my_int (s(|my_int := y + i′|)) = z⦄
 2. ⋀s. z = my_int s + y ⟹ ?Q s
```

The second is solved by gets_wp.

finish

```
 1. ⋀s. z = my_int s + y ⟹ my_int (s(|my_int := y + my_int s|)) = z

 Final Result.
```

At this point the method terminates, as the verification condition has been successfully generated. This goal is a trivial consequence of Isabelle's records, and can be solved by simp.

by simp

The wp attribute

The wp declaration attribute (see Section 3.4.4) can be used to manage a database of Hoare triples, to be automatically applied by the wp method. Each conjunct in a composite postcondition is matched against this database using Isabelle's usual rule resolution. The wp rule set also contains calculational rules for handling control flow, exception handling, etc. For example, if we declare my_add_adds as a wp rule we can then solve Hoare triples containing my_add with wp:

```
declare my_add_adds[wp]

lemma
  ⦃λs. my_int s = 0⦄
   do
     my_add 1;
     my_add 1
   od
  ⦃λ_ s. my_int s = 2⦄
  by (wp | simp)+
```

In the ideal case, solving a Hoare triple simply requires a single invocation of wp, followed by a manual proof of the calculated verification condition. If this fails, however, proof authors must then determine either what rules need to be marked with the wp attribute in order for the wp method to succeed, or if manual intervention is required (e.g. a loop invariant must be explicitly provided).

As more rules are added to this wp database, the wp method becomes capable of generating verification conditions with minimal manual intervention.

7.2 Corres

To prove forward simulation between the abstract and executable specifications of L4.verified, we need to decompose the problem in to manageable sub-problems. Here we present the corres calculus used in L4.verified, originally given by Cock et al. [23] and updated here. This calculus was designed to prove forward simulation for functions defined over a non-deterministic state monad.

Informally, corres_underlying *srel nf nf' rrel P P' a c* states: for any two states s and s' satisfying the state relation *srel*, if $P\ s$ and $P'\ s'$ hold, then every result state of $c\ s'$ has a corresponding result state in $a\ s$ which satisfies *srel* and the return-value relation *rrel*. The flags *nf* and *nf'* control how failure is treated: if *nf* is True then $a\ s$ must not fail (assuming $P\ s$), and if *nf'* is True then $c\ s'$ must not fail (assuming $P'\ s'$).

```
corres_underlying srel nf nf' rrel P P' a c ≡
∀(s, s')∈srel.
   P s ∧ P' s' ⟶
   (nf ⟶ ¬ snd (a s)) ⟶
   (∀(r', t')∈fst (c s').
       ∃(r, t)∈fst (a s).
          (t, t') ∈ srel ∧ rrel r r') ∧
   (nf' ⟶ ¬ snd (c s'))
```

The name corres_underlying is historical: originally this framework was only used for the abstract-to-executable refinement proof for L4.verified, so parameters *srel*, *nf* and *nf'* were concretely instantiated in the definition of the corres constant. The corres framework has since been abstracted for use in other projects (e.g. Greenaway's AutoCorres [33]).

To avoid rewriting the existing L4.verified proofs, the base constant was renamed to corres_underlying and corres was defined as a special case of it.

$\mathsf{corres}\ \mathit{rrel}\ P\ P'\ a\ c \equiv \mathsf{corres\_underlying}\ \mathsf{state\_relation}\ \mathsf{a\_nofail}\ \mathsf{c\_nofail}\ \mathit{rrel}\ P\ P'\ a\ c$

For the sake of brevity, and without loss of generality, in this chapter we will use corres to refer to corres_underlying $\mathit{srel}\ \mathit{nf}\ \mathit{nf}'$ with some arbitrary-but-fixed instantiation for srel (state_relation), nf (a_nofail) and nf' (c_nofail).

Similar to the seq rule for Hoare logic, we can prove a rule that can be used to distribute a corres obligation across bind.

corres_split:
$[\![\mathsf{corres}\ R'\ P\ P'\ a\ c;$
$\bigwedge rv\ rv'.\ R'\ rv\ rv' \Longrightarrow \mathsf{corres}\ R\ (S\ rv)\ (S'\ rv')\ (b\ rv)\ (d\ rv');$
$\{\!|Q|\!\}\ a\ \{\!|S|\!\};\ \{\!|Q'|\!\}\ c\ \{\!|S'|\!\}]\!] \Longrightarrow$
$\mathsf{corres}\ R\ (P\ \mathsf{and}\ Q)\ (P'\ \mathsf{and}\ Q')\ (a >>= (\lambda rv.\ b\ rv))\ (c >>= (\lambda rv'.\ d\ rv'))$

As in seq, the intermediate predicates S and S' serve as both preconditions and postconditions. They are calculated as the required preconditions in order to establish corres between b and d. The two Hoare triples then show that these conditions are established by a and c assuming Q and Q' respectively. The relation R' is required to hold between all return values of a and c by assumption A, therefore it can be assumed to hold between the quantified r and r' in assumption B.

Similar to the hoare_pre rule in the previous section, the start of a corres proof requires that the provided preconditions be replaced with schematic ones. The most straightforward rule simply weakens both preconditions.

corres_guard_imp:
$[\![\mathsf{corres}\ R\ Q\ Q'\ a\ c;\ \bigwedge s\ s'.\ [\![P\ s;\ P'\ s';\ (s,\ s') \in \mathsf{state\_relation}]\!] \Longrightarrow Q\ s \wedge Q'\ s']\!] \Longrightarrow$
$\mathsf{corres}\ R\ P\ P'\ a\ c$

For elementary functions, we can prove general rules which reduce a corres goal to one or more HOL obligations, which can be solved with standard proof methods.

corres_return: $R\ a\ b \Longrightarrow \mathsf{corres}\ R\ \top\ \top\ (\mathsf{return}\ a)\ (\mathsf{return}\ b)$

Where $\top$ is the universal predicate (i.e. $\top \equiv (\lambda \mathrm{s}.\ \mathsf{True})$). With this rule we can show corres between a pair of abstract and concrete return functions if we can prove that the given return value relation R is satisfied by the returned values.

Similarly, we can prove corres rules for get and put.

corres_gets:
$(\bigwedge s\ s'.\ [\![P\ s;\ P'\ s';\ (s,\ s') \in \mathsf{state\_relation}]\!] \Longrightarrow \mathit{rrel}\ (f\ s)\ (f'\ s')) \Longrightarrow$
$\mathsf{corres}\ \mathit{rrel}\ P\ P'\ (\mathsf{gets}\ f)\ (\mathsf{gets}\ f')$

Since gets leaves the state unmodified the resulting states will always correspond. The proof obligation is to therefore show that the return relation rrel is satisfied by the given state projections f and f', assuming the preconditions and state_relation hold.

corres_modify:
$(\bigwedge s\ s'.\ [\![P\ s;\ P'\ s';\ (s,\ s') \in \mathsf{state\_relation}]\!] \Longrightarrow (f\ s,\ f'\ s') \in \mathsf{state\_relation}) \Longrightarrow$
$\mathsf{corres}\ \mathsf{dc}\ P\ P'\ (\mathsf{modify}\ f)\ (\mathsf{modify}\ f')$

Here we see that two modify functions will correspond if the modified states that are being stored (i.e. $f\ s$ and $f'\ s'$) satisfy the state_relation. The return-value relation is the universal relation (dc $\equiv \lambda\_\ \_.$ True) since put always returns a unit.

To introduce nondeterminism into monadic specifications (as shown later in Section 7.2.1) we can use the select primitive, which nondeterministically returns a single element from a set and leaves the state unmodified.

select $A \equiv \lambda s.\ (A \times \{s\}$, False$)$

This can be seen as a generalization of return (i.e. return a = select $\{a\}$). The most straightforward corres rule for select mirrors the definition of corres.

corres_select:
$(\bigwedge s\ s'\ b.\ [\![P\ s;\ P'\ s';\ b \in B]\!] \Longrightarrow \exists\, a{\in}A.\ R\ a\ b) \Longrightarrow$
corres $R\ P\ P'$ (select A) (select B)

This can then be specialized to return.

corres_select_return:
$(\bigwedge s\ s'.\ [\![P\ s;\ P'\ s']\!] \Longrightarrow \exists\, a{\in}A.\ R\ a\ b) \Longrightarrow$ corres $R\ P\ P'$ (select A) (return b)

Note that corres_return, corres_select and corres_select_return do not mention state_relation as select and return do not read or modify the monadic state.

Using these and similar rules, we can then proceed to prove corres results over intermediate functions in monadic specifications, and compose them to prove refinement of the top-level entry points.

7.2.1 Example

In this section we will develop an abstract and concrete implementation for a toy scheduler and prove corres between them.

Scheduler Specification

First we define the state for each scheduler.

record abstract_state =
cur_thread :: thread_id
threads :: thread_id set

record concrete_state =
cur_thread' :: thread_id
threads' :: thread_id list

In both states we store the currently running thread, and in the concrete state we maintain an ordered list of runnable threads while the abstract state leaves it as an unordered set. The relationship between these two states is straightforward to define.

definition state_relation $\equiv \{(s,s').$
cur_thread' s' = cur_thread s $\wedge$
set (threads' s') = threads $s\}$

Where set is the unordered set of elements in a given list, ie:

```
next_thread_abstract ≡
    do ts ← gets threads;
        select ts
    od

dequeue_thread_abstract t ≡
    modify (λs.
      s⦇threads := (threads s) − {t}⦈)

set_cur_thread_abstract t ≡
    modify(λs. (s⦇cur_thread := t⦈))
```

```
next_thread_concrete ≡
    do ts′ ← gets threads′;
        return (hd ts′)
    od

dequeue_thread_concrete t′ ≡
    modify (λs.
      s⦇threads′ := removeAll t′ (threads′ s)⦈)

set_cur_thread_concrete t′ ≡
    modify(λs. (s⦇cur_thread′ := t′⦈))
```

Figure 7.1: Intermediate functions for scheduler_abstract and scheduler_concrete

```
set :: 'a list ⇒ 'a set
set [] = ∅
set (x · l) = {x} ∪ set l
```

Next we define our abstract and concrete schedulers.

```
definition
  schedule_abstract :: (abstract_state, unit) nondet_monad
  where schedule_abstract ≡
    do t ← next_thread_abstract;
       dequeue_thread_abstract t;
       set_cur_thread_abstract t
    od

definition
  schedule_concrete :: (concrete_state, unit) nondet_monad
  where schedule_concrete ≡
    do t′ ← next_thread_concrete;
       dequeue_thread_concrete t′;
       set_cur_thread_concrete t′
    od
```

Each scheduler selects a new thread according to its scheduling algorithm (next_thread_*), removes it from the set of schedulable threads (dequeue_thread_*), then marks it as the current thread (set_cur_thread_*). The definitions for these functions are given in Figure 7.1.

In the concrete scheduler, next_thread_concrete will simply select the first element of the list with the partial function hd.

```
hd :: 'a list ⇒ 'a
hd (x · xs) = x
```

For the abstract scheduler, since our thread pool is unordered, our only option is to use non-determinism via the select function when defining next_thread_abstract.

To dequeue the selected thread in dequeue_thread_concrete, we remove it from the list of schedulable threads with removeAll.

```
removeAll :: 'a ⇒ 'a list ⇒ 'a list
removeAll x [] = []
removeAll x (y · l) = (if x = y then removeAll x l else y · removeAll x l)
```

Whereas in dequeue_thread_abstract we can use set difference.

```
op − :: 'a set ⇒ 'a set ⇒ 'a set
A − B = {x ∈ A | x ∉ B}
```

Scheduler Corres Proof

The most interesting functions in this example are the thread selectors (next_thread_abstract and next_thread_concrete). For these functions to correspond, we require that at least one thread can be scheduled (otherwise their behaviours diverge). To ensure this on both sides, it is sufficient to assume that the concrete list of threads is nonempty (i.e. $\lambda s'$. threads' $s' \neq []$) since this will imply that the abstract set of threads is nonempty via the state relation. We know that there is always a nondeterministic branch in next_thread_abstract where the selected thread is the same as next_thread_concrete, so we set our return relation to equivalence.

```
lemma corres_next_thread:
  corres (λt t'. t = t') ⊤ (λs'. threads' s' ≠ [])
    next_thread_abstract next_thread_concrete
  unfolding next_thread_abstract_def next_thread_concrete_def
```

After unfolding the definitions of next_thread_abstract and next_thread_concrete we start our proof by replacing the precondition with a schematic.

```
apply (rule corres_guard_imp)
```

```
 1. corres (λt t'. t = t') ?Q ?Q'
          (do ts ← gets threads;
              select ts
           od)
          (do ts' ← gets threads';
              return (hd ts')
           od)
 2. ⋀s s'.
       ⟦True; threads' s' ≠ []; (s, s') ∈ state_relation⟧
       ⟹ ?Q s ∧ ?Q' s'
```

We now have schematic variables for our corres preconditions in the head subgoal, and two additional subgoals showing that they are implied by the given preconditions.

```
apply (rule corres_split)
```

```
 1. corres ?R'3 ?P3 ?P'3 (gets threads) (gets threads')
 2. ⋀ts ts'.
       ?R'3 ts ts' ⟹
       corres (λt t'. t = t') (?S3 ts)
              (?S'3 ts') (select ts) (return (hd ts'))
 3. ⦃?Q3⦄ gets threads ⦃?S3⦄
 4. ⦃?Q'3⦄ gets threads' ⦃?S'3⦄
 5. ⋀s s'.
       ⟦True; threads' s' ≠ []; (s, s') ∈ state_relation⟧
       ⟹ (?P3 and ?Q3) s ∧ (?P'3 and ?Q'3) s'
```

After applying the split rule we get one subgoal for each pair of elementary functions. To solve the first goal we specialize corres_gets to have trivial preconditions, and set the return relation to state that each set of schedulable threads is equivalent.

```
apply (rule corres_gets[where P=⊤ and P'=⊤ and rrel=λts ts'. set ts' = ts])
```

```
 1. ⋀s s'. ⟦True; True; (s, s') ∈ state_relation⟧ ⟹ set (threads' s') = threads s
A total of 5 subgoals...
```

This goal is now a trivial consequence of state_relation and can be solved by unfolding its definition.

```
apply (simp add: state_relation_def)
```

```
 1. ⋀ts ts'.
       set ts' = ts ⟹
       corres (λt t'. t = t') (?S3 ts)
              (?S'3 ts') (select ts) (return (hd ts'))
 2. ⦃?Q3⦄ gets threads ⦃?S3⦄
 3. ⦃?Q'3⦄ gets threads' ⦃?S'3⦄
 4. ⋀s s'.
       ⟦True; threads' s' ≠ []; (s, s') ∈ state_relation⟧
       ⟹ ((λ_. True) and ?Q3) s ∧
           ((λ_. True) and ?Q'3) s'
```

The first corres subgoal is now solved and we are left with the second. However our return-value relation is now instantiated concretely in the goal assumptions (i.e. set $ts' = ts$). This goal can be solved with our previously shown corres_select_return rule, where we explicitly assume that the fetched thread set (now the *ts* bound goal parameter) is nonempty.

```
apply (rule_tac P=⊤ and P'=λ_. ts' ≠ []in corres_select_return)
```

```
 1. ⋀ts ts' s s'.
       ⟦set ts' = ts; True; ts' ≠ []⟧
       ⟹ ∃ a∈ts. a = hd ts'
A total of 4 subgoals...
```

This goal now only discusses sets and lists, therefore it can be solved by fastforce alone.

```
apply fastforce
```

```
 1. ⦃?Q3⦄ gets threads ⦃λts _. True⦄
 2. ⦃?Q'3⦄ gets threads' ⦃λts' _. ts' ≠ []⦄
 3. ⋀s s'.
       ⟦True; threads' s' ≠ []; (s, s') ∈ state_relation⟧
       ⟹ ((λ_. True) and ?Q3) s ∧
          ((λ_. True) and ?Q'3) s'
```

Now we are left with only Hoare triples, which are both easily discharged with *gets_wp* (or by applying the wp method).

```
apply (rule gets_wp)+
```

```
 1. ⋀s s'.
       ⟦True; threads' s' ≠ []; (s, s') ∈ state_relation⟧
       ⟹ ((λ_. True) and (λs. True)) s ∧
          ((λ_. True) and (λs. threads' s ≠ [])) s'
```

The rest of the goals are now trivial implications which can be solved by auto.

```
by auto
```

The proofs for dequeuing and setting the current thread are trivial consequences of corres_modify and the definition of state_relation.

corres_dequeue:

```
corres dc ⊤ ⊤
  (dequeue_thread_abstract t) (dequeue_thread_concrete t)
```

corres_set_cur_thread:

```
corres dc ⊤ ⊤
  (set_cur_thread_abstract t) (set_cur_thread_concrete t)
```

Finally, with these three lemmas we can prove our main result.

```
lemma corres_schedule:
  corres dc ⊤ (λs′. threads′ s′ ≠ [])
     (schedule_abstract) (schedule_concrete)
  unfolding schedule_abstract_def schedule_concrete_def
  apply (rule corres_guard_imp)
    apply (rule corres_split)
       apply (rule corres_next_thread)
      apply (rule corres_split)
```

```
 1. ⋀t t′.
       t = t′ ⟹
       corres (?R′8 t t′) (?P9 t) (?P′11 t′)
              (dequeue_thread_abstract t) (dequeue_thread_concrete t′)
A total of 7 subgoals...
```

Attempting to apply corres_dequeue here directly will fail, since the arguments to each dequeue function differ (t on the left and t' on the right). We first need to simplify the goal in order to rewrite the conclusion with the assumption ($t = t'$) so that our rule will be applicable (this assumption comes from the return-value relation of corres_next_thread).

```
  apply simp
```

```
 1. ⋀t t′.
       t = t′ ⟹
       corres (?R′8 t′ t′) (?P9 t′) (?P′11 t′)
              (dequeue_thread_abstract t′) (dequeue_thread_concrete t′)
A total of 7 subgoals...
```

The rest of the proof proceeds as expected, with a similar simplification step required for corres_set_cur_thread.

```
         apply (rule corres_dequeue)
        apply simp
        apply (rule corres_set_cur_thread)
       apply wp+
  by auto
```

7.3 The Corres Proof Method

In this section we will incrementally build a proof method for solving corres proofs using Eisbach. The objective is to demonstrate how a complex method can be iteratively constructed using a combination of Eisbach and Isabelle/Isar. To this end, many implementation details have been omitted that appear in the final methods used in L4.verified. The full source of the Eisbach methods, auxiliary Isabelle/ML tools, and proof calculus are available in the L4.verified repository [5].

7.3.1 First Steps and Limitations of Corres

The core algorithm of a corres proof is the same as a Hoare logic proof:

1. Apply weakening rule to replace precondition(s) with schematics.
2. Repeatedly apply split rules to distribute across binds.
3. Solve terminal goals with previously proven rules.
4. Solve the final verification conditions.

For our VCG we are concerned with the first three steps, ideally generating the verification condition with as little intervention from the end user as possible. A first attempt at encoding this as a proof method using Eisbach might look as follows (including breakpoints to trace its execution via **apply_debug** (see Section 6.1)):

```
method corres_simple uses corres_rules =
  (rule corres_guard_imp, #break,
    (rule corres_split corres_rules, #break)+)
```

This method weakens the preconditions in the goal, then attempts to repeatedly apply either the split rule or some given rule for solving terminal goals. A naive attempt at applying this to our thread selector proof, however, does not make significant progress.

```
lemma corres_next_thread:
  corres (λt t'. t = t') ⊤ (λs'. threads' s' ≠ [])
    next_thread_abstract next_thread_concrete
  unfolding next_thread_abstract_def next_thread_concrete_def
  apply_debug
   (corres_simple corres_rules:
          corres_gets[where P=⊤ and P'=⊤ and rrel=λts ts'. set ts' = ts]
          corres_select_return)
```

```
 1. corres (λt t'. t = t') ?Q ?Q'
            (do ts ← gets threads;
                select ts
             od)
            (do ts' ← gets threads';
                return (hd ts')
             od)
A total of 2 subgoals...
```

After applying corres_guard_imp, the preconditions in the goal have been replaced with schematics. From here, corres_split is then applied to decompose the subgoal.

continue

```
 1. corres ?R'3 ?P3 ?P'3 (gets threads) (gets threads')
 2. ⋀ts ts'.
       ?R'3 ts ts' ⟹
       corres (λt t'. t = t') (?S3 ts)
               (?S'3 ts') (select ts) (return (hd ts'))
A total of 5 subgoals...
```

Now that the head subgoal is an atomic function, corres_simple applies the given *corres_rules* to discharge it.

continue

```
 1. ⋀s s'. ⟦True; True; (s, s') ∈ state_relation⟧ ⟹ set (threads' s') = threads s
A total of 5 subgoals...
```

The method has now successfully applied corres_gets, introducing its assumption as a new subgoal.

finish

```
 1. ⋀s s'. ⟦True; True; (s, s') ∈ state_relation⟧ ⟹ set (threads' s') = threads s
 2. ⋀ts ts'.
       set ts' = ts ⟹
       corres (λt t'. t = t') (?S3 ts)
               (?S'3 ts') (select ts) (return (hd ts'))
 3. ⦃?Q3⦄ gets threads ⦃?S3⦄
 4. ⦃?Q'3⦄ gets threads' ⦃?S'3⦄
 5. ⋀s s'.
       ⟦True; threads' s' ≠ []; (s, s') ∈ state_relation⟧
       ⟹ ((λ_. True) and ?Q3) s ∧
           ((λ_. True) and ?Q'3) s'

Final Result.
```

The method has now finished evaluating, leaving the assumption from corres_gets to be solved. The issue here is that corres_gets has a non-corres assumption which needs to be proven. Standard resolution (via the rule method) puts this assumption as the head subgoal, where it blocks our simple VCG from making progress and the process terminates.

In general any rule of this form will be a problem for corres_simple. Since we are interested in generating a single verification condition to be solved interactively, while automatically processing all corres obligations, we need to defer any domain-specific reasoning (e.g. set or list reasoning) to the end of the proof.

One solution would be to instead move the rule assumptions into one of the corres preconditions, e.g.:

```
lemma corres_gets2:
  corres rrel
    (λs. ∀s'. (s,s') ∈ state_relation ⟶ P s ⟶ P' s' ⟶ rrel (f s) (f' s')) P'
    (gets f) (gets f')

lemma corres_select_return2:
  corres R (λs. P s ⟶ (∃a∈A. R a b)) P' (select A) (return b)

lemma corres_next_thread:
  corres (λt t'. t = t') ⊤ (λs'. threads' s' ≠ [])
    next_thread_abstract next_thread_concrete
  unfolding next_thread_abstract_def next_thread_concrete_def
  apply (corres_simple corres_rules:
          corres_gets2[where P=⊤ and P'=⊤ and rrel=λts ts'. set ts' = ts]
          corres_select_return2)
```

```
 1. ⋀ts ts'.
       set ts' = ts ⟹
       corres (λt t'. t = t') (?S3 ts)
              (?S'3 ts') (select ts) (return (hd ts'))
 2. ⦃?Q3⦄ gets threads ⦃?S3⦄
 3. ⦃?Q'3⦄ gets threads' ⦃?S'3⦄
 4. ⋀s s'.
       ⟦True; threads' s' ≠ []; (s, s') ∈ state_relation⟧
       ⟹ ((λs. ∀s'. (s, s') ∈ state_relation ⟶
True ⟶ True ⟶ set (threads' s') = threads s) and
             ?Q3)
             s ∧
            ((λ_. True) and ?Q'3) s'
```

This attempt has made more progress and successfully applied corres_gets2 (with its precondition now deferred to the 4th subgoal), however corres_select_return2 has not been applied as expected. Now the problem is the scope of the goal parameters *ts* and *ts'*. Note that the abstract schematic precondition in our head subgoal is a function of *ts* while the concrete precondition is a function of *ts'*. This is a result of the corres_split rule, restated here:

```
 corres_split:
  ⟦corres R' P P' a c;
   ⋀rv rv'. R' rv rv' ⟹ corres R (S rv) (S' rv') (b rv) (d rv');
   ⦃Q⦄ a ⦃S⦄; ⦃Q'⦄ c ⦃S'⦄⟧ ⟹
     corres R (P and Q) (P' and Q') (a >>= (λrv. b rv)) (c >>= (λrv'. d rv'))
```

In this rule, S and S' only depend on *rv* and *rv'* respectively. This is necessary because these parameters are placeholders for the return values of the respective functions a and c. In corres_select_return2, however, our new precondition contains terms from both sides ($\exists\, a{\in}A.\ R\ a\ b$ where A is the abstract set and b is the concrete return value). This prevents it from being stated as a precondition on either side of corres in a way that's compatible with corres_split.

The solution is to extend the corres constant to add an additional slot for these extra conditions. This gives us more control over the order that they are solved in, and ultimately how they are propagated through the subgoal structure.

7.3.2 CorresK

We define a simple extension to corres that adds a stateless precondition.

$$\mathsf{corresK}\ F\ r\ P\ P'\ f\ g \equiv F \longrightarrow \mathsf{corres}\ r\ P\ P'\ f\ g$$

Additionally, we define a weaker version of corres that exclusively handles return-value relations.

$$\begin{array}{l}
\mathsf{corres\_rv} :: \mathsf{bool} \Rightarrow ('a \Rightarrow 'b \Rightarrow \mathsf{bool}) \Rightarrow ('s \Rightarrow \mathsf{bool}) \Rightarrow ('t \Rightarrow \mathsf{bool}) \\
\qquad\qquad \Rightarrow ('s,\ 'a)\ \mathsf{nondet\_monad} \Rightarrow ('t,\ 'b)\ \mathsf{nondet\_monad} \Rightarrow \\
\qquad\qquad\quad ('a \Rightarrow 'b \Rightarrow \mathsf{bool}) \Rightarrow \mathsf{bool} \\
\mathsf{corres\_rv}\ F\ r\ P\ P'\ f\ f'\ R \equiv \\
\ F \longrightarrow (\forall s\ s'.\ P\ s \longrightarrow P'\ s' \longrightarrow \\
\quad (\forall sa\ rv.\ (rv,\ sa) \in \mathsf{fst}\ (f\ s) \longrightarrow \\
\quad (\forall sa'\ rv'.\ (rv',\ sa') \in \mathsf{fst}\ (f'\ s') \longrightarrow \\
\qquad r\ rv\ rv' \longrightarrow R\ rv\ rv')))
\end{array}$$

Where two functions f and f' satisfy corres_rv if their return values satisfy the given return value relation R, under the assumption F and preconditions P and P'. This is useful to write the split rule for corresK:

corresK_split:

$$\begin{array}{l}
[\![\mathsf{corresK}\ F\ r'\ P\ P'\ a\ c; \\
\bigwedge rv\ rv'.\ r'\ rv\ rv' \Longrightarrow \mathsf{corresK}\ (F'\ rv\ rv')\ r\ (R\ rv)\ (R'\ rv')\ (b\ rv)\ (d\ rv'); \\
\mathsf{corres\_rv}\ F''\ r'\ Q\ Q'\ a\ c\ F'; \\
\{\!|S|\!\}\ a\ \{\!|R|\!\};\ \{\!|S'|\!\}\ c\ \{\!|R'|\!\}]\!] \Longrightarrow \\
\quad \mathsf{corresK}\ (F \wedge F'')\ r\ (Q\ \mathsf{and}\ P\ \mathsf{and}\ S)\ (Q'\ \mathsf{and}\ P'\ \mathsf{and}\ S') \\
\qquad\qquad (a >>= (\lambda rv.\ b\ rv))\ (c >>= (\lambda rv'.\ d\ rv'))
\end{array}$$

This rule looks very similar to corres_split, however additional stateless preconditions F, F' and F'' have been added for corresK. Importantly, F' is a function of both return values in the second assumption. The third assumption is analogous to a Hoare triple: it states that under some stateless precondition F'', and additional preconditions Q and Q', the stateless precondition F', required for proving corres between b and d, is satisfied by a and c. These conditions are then all propagated back up through the stateless, abstract and concrete preconditions in the goal conclusion.

In comparison to the original corres_split rule, here we have additional stateless preconditions for handling rule assumptions that would otherwise create additional subgoals. The precondition weakening rule is straightforward to write:

corresK_guard_imp:

$$\begin{array}{l}
[\![\mathsf{corresK}\ F'\ R\ Q\ Q'\ a\ c; \\
\ \bigwedge s\ s'.\ [\![P\ s;\ P'\ s';\ F;\ (s,\ s') \in \mathsf{state\_relation}]\!] \Longrightarrow F' \wedge Q\ s \wedge Q'\ s']\!] \Longrightarrow \\
\ \mathsf{corresK}\ F\ R\ P\ P'\ a\ c
\end{array}$$

We can additionally rephrase our previous terminal rules over corresK, moving the assumption to the stateless precondition.

corresK_select_return:
$\mathsf{corresK}\ (\exists\, a{\in}A.\ R\ a\ b)\ R\ \top\ \top\ (\mathsf{select}\ A)\ (\mathsf{return}\ b)$

corresK_gets:
$\mathsf{corresK}$
$\quad(\forall\, s\ s'.\ (s{,}s') \in \mathsf{state\_relation} \longrightarrow P\ s \longrightarrow P'\ s' \longrightarrow rrel\ (f\ s)\ (f'\ s'))$
$\quad rrel\ P\ P'\ (\mathsf{gets}\ f)\ (\mathsf{gets}\ f')$

corresK_modify:
$\mathsf{corresK}$
$\quad(\forall\, s\ s'.\ P\ s \longrightarrow P'\ s' \longrightarrow (s,\ s') \in \mathsf{state\_relation} \longrightarrow (f\ s,\ f'\ s') \in \mathsf{state\_relation})$
$\quad\mathsf{dc}\ P\ P'\ (\mathsf{modify}\ f)\ (\mathsf{modify}\ f')$

We now also require rules for discharging corres_rv subgoals. The corresK_split rule uses corres_rv as an opportunity to introduce extra preconditions in order to prove the stateless precondition after the bind F'. When corres_rv is encountered during a proof, its preconditions F, P and P' will all be schematic, to later be instantiated appropriately. This can normally be handled through one of the three following rules:

corres_rv_wp_right:
$\lbrace\!|P'|\!\rbrace\ f'\ \lbrace\!|\lambda rv'\ s.\ \forall\, rv.\ r\ rv\ rv' \longrightarrow Q\ rv\ rv'|\!\rbrace \Longrightarrow \mathsf{corres\_rv}\ \mathsf{True}\ r\ \top\ P'\ f\ f'\ Q$

corres_rv_wp_left:
$\lbrace\!|P|\!\rbrace\ f\ \lbrace\!|\lambda rv\ s.\ \forall\, rv'.\ r\ rv\ rv' \longrightarrow Q\ rv\ rv'|\!\rbrace \Longrightarrow \mathsf{corres\_rv}\ \mathsf{True}\ r\ P\ \top\ f\ f'\ Q$

corres_rv_defer:
$\mathsf{corres\_rv}\ (\forall\, rv\ rv'.\ r\ rv\ rv' \longrightarrow Q\ rv\ rv')\ r\ \top\ \top\ f\ f'\ Q$

The rule corres_rv_wp_right will instantiate the stateless and abstract preconditions to simply True, and lift the schematic P' as the precondition to a Hoare triple with a postcondition Q, assuming the return-value relation r and a fixed, arbitrary return value for f. This effectively removes f from consideration, and assumes that Q can be established as a postcondition of f' in Hoare logic. Similarly corres_rv_wp_left discards f' and lifts the goal into a Hoare triple over f.

Finally, corres_rv_defer discards all information about f and f', instantiating P and P' to True and the stateless precondition to showing that Q is a direct consequence of the return-value relation r. Automatically applying these rules turns out to be non-trivial, so we will defer their discussion to a later section. For now we will manually apply them as appropriate.

We can now write a version of our corres_simple method, but for corresK. Additionally, we declare a new named theorem corresK to maintain a collection of rules in the context.

named_theorems corresK

method corresK_simple **declares** corresK =
  (rule corresK_guard_imp, (rule corresK_split corresK)+)

Rules that can be applied safely by this method can now simply be declared as correK rules.

declare corresK_select_return[corresK]

Example

Now we can revisit our scheduler example, armed with a new calculus and VCG.

```
lemma corresK_next_thread[corresK]:
  corresK True (λt t'. t = t') ⊤ (λs'. threads' s' ≠ [])
    next_thread_abstract next_thread_concrete
  unfolding next_thread_abstract_def next_thread_concrete_def
  apply (corresK_simple corresK:
        corresK_gets[where P=⊤ and P'=⊤ and rrel=λts ts'. set ts' = ts])
```

```
 1. corres_rv ?F''3 (λts ts'. set ts' = ts) ?Q3 ?Q'3
             (gets threads) (gets threads') (λts ts'. ∃ a∈ts. a = hd ts')
 2. ⦃?S3⦄ gets threads ⦃λts _. True⦄
 3. ⦃?S'3⦄ gets threads' ⦃λts' _. True⦄
 4. ⋀s s'.
       ⟦True; threads' s' ≠ []; True; (s, s') ∈ state_relation⟧
       ⟹ ((∀ s s'.
                (s, s') ∈ state_relation ⟶
                True ⟶ True ⟶ set (threads' s') = threads s) ∧
             ?F''3) ∧
           (?Q3 and (λ_. True) and ?S3) s ∧
           (?Q'3 and (λ_. True) and ?S'3) s'
```

The corresK subgoals have now been successfully discharged and we are left with a corres_rv goal. We can defer this postcondition to a Hoare triple on the concrete (right) side of corresK with corres_rv_wp_right.

```
apply (rule corres_rv_wp_right)
```

```
 1. ⦃?Q'3⦄ gets threads'
    ⦃λrv' s. ∀ rv. set rv' = rv ⟶ (∃ a∈rv. a = hd rv')⦄
 2. ⦃?S3⦄ gets threads ⦃λts _. True⦄
 3. ⦃?S'3⦄ gets threads' ⦃λts' _. True⦄
 4. ⋀s s'.
       ⟦True; threads' s' ≠ []; True; (s, s') ∈ state_relation⟧
       ⟹ ((∀ s s'.
                (s, s') ∈ state_relation ⟶ True ⟶ True ⟶ set (threads' s') = threads s) ∧
             True) ∧
           ((λ_. True) and (λ_. True) and ?S3) s ∧
           (?Q'3 and (λ_. True) and ?S'3) s'
```

The resulting postcondition is solvable, since it will follow from the top-level precondition threads' $s' \neq []$. The top 3 subgoals now are all Hoare triples and can be solved by wp, instantiating their preconditions in the verification condition.

```
apply wp+
```

```
 1. ⋀s s'.
       ⟦True; threads' s' ≠ []; True; (s, s') ∈ state_relation⟧
       ⟹ ((∀s s'.
                (s, s') ∈ state_relation ⟶ True ⟶ True ⟶ set (threads' s') = threads s) ∧
            True) ∧
           ((λ_. True) and (λ_. True) and (λs. True)) s ∧
           ((λs. ∀rv. set (threads' s) = rv ⟶
                        (∃a∈rv. a = hd (threads' s))) and
            (λ_. True) and
            (λs. True))
            s'
```

Finally we are left with a verification condition, solved by unfolding the definition of state_relation.

```
by (fastforce simp add: state_relation_def)
```

Our proof now has a more natural flow: solve corres goals, then solve Hoare logic goals, then finally solve the verification condition. The corresK calculus gives us fine-grained control over how additional subgoals are produced, ensuring that the head subgoal remains over corresK until all the open subgoals have been reduced to Hoare logic or verification conditions.

The corresK equivalents for dequeue and set_cur_thread are similarly straightforward consequences of corresK_modify.

```
corresK_dequeue[corresK]:
 corresK (t = t') dc ⊤ ⊤
   (dequeue_thread_abstract t) (dequeue_thread_concrete t')

corresK_set_cur_thread[corresK]:
 corresK (t = t') dc ⊤ ⊤
   (set_cur_thread_abstract t) (set_cur_thread_concrete t')
```

Note that in corres_dequeue and corres_set_cur_thread, the assumption $t = t'$ was implicit since only t was used. Now, by stating it explicitly as the stateless precondition for corresK, we will defer proving this to the verification condition.

With these added to the corresK named theorem, they will be automatically used by corresK_simple and the proof of corresK_schedule becomes straightforward.

```
lemma corresK_schedule:
  corresK True dc ⊤ (λs′. threads′ s′ ≠ [])
     (schedule_abstract) (schedule_concrete)
  unfolding schedule_abstract_def schedule_concrete_def
  apply (corresK_simple)
```

```
 1. ⋀t t′.
       t = t′ ⟹
       corres_rv (?F″9 t t′) dc (?Q10 t) (?Q′13 t′)
                  (dequeue_thread_abstract t) (dequeue_thread_concrete t′) (λy ya. t = t′)
 2. ⋀t t′. t = t′ ⟹ ⦃?S12 t⦄ dequeue_thread_abstract t ⦃λy _. True⦄
 3. ⋀t t′.
       t = t′ ⟹ ⦃?S′15 t′⦄ dequeue_thread_concrete t′ ⦃λya _. True⦄
 4. corres_rv ?F″3 (λt t′. t = t′) ?Q3 ?Q′3 next_thread_abstract next_thread_concrete
               (λt t′. t = t′ ∧ ?F″9 t t′)
 5. ⦃?S3⦄ next_thread_abstract ⦃λt. ?Q10 t and (λ_. True) and ?S12 t⦄
 6. ⦃?S′3⦄ next_thread_concrete
    ⦃λt′. ?Q′13 t′ and (λ_. True) and ?S′15 t′⦄
 7. ⋀s s′.
       ⟦True; threads' s′ ≠ []; True; (s, s′) ∈ state_relation⟧
       ⟹ (True ∧ ?F″3) ∧
           (?Q3 and (λ_. True) and ?S3) s ∧
           (?Q′3 and (λs′. threads' s′ ≠ []) and ?S′3) s′
```

All corresK subgoals have been discharged, leaving only corres_rv and Hoare triples. Both corres_rv goals are trivial consequences of the return-value relation established by corresK_next_thread, and so they can be deferred.

```
  apply (rule corres_rv_defer | wp)+
```

```
 1. ⋀s s′.
       ⟦True; threads' s′ ≠ []; True; (s, s′) ∈ state_relation⟧
       ⟹ (True ∧
            (∀rv rv′.
                rv = rv′ ⟶
                rv = rv′ ∧
                (∀rva rv′a. dc rva rv′a ⟶ rv = rv′))) ∧
           ((λ_. True) and (λ_. True) and (λs. True ∧ True ∧ True)) s ∧
           ((λ_. True) and (λs′. threads' s′ ≠ []) and (λs. True ∧ True ∧ True)) s′
```

Finally, the resulting verification condition is trivially solved by simp.

```
  by simp
```

7.3.3 Mismatched Functions

In our examples so far, there has been no structural difference between the abstract and concrete functions. In cases where there are an equal number of matching atomic steps on each side, the simple strategy employed by corresK_simple is sufficient to complete the proof.

```
add_thread_abstract t ≡
  modify (λs.
    s⦇threads := (threads s) ∪ {t}⦈)
```

```
add_thread_concrete t′ ≡
  modify (λs.
    s⦇threads′ := insort t′ (threads′ s)⦈)
```

Figure 7.2: Intermediate functions for enqueuing threads.

In general, however, there can be significant structural differences between abstract and concrete functions. A concrete function may, for example, require extra steps to maintain the consistency of a concrete data structure, which is unnecessary in its abstract version. These steps have no corresponding abstract function, and therefore must be handled differently.

In corres, there are a number of strategies that can be used for handling mismatched functions. In some cases, functions can be proven equivalent to alternate representations which do structurally match. Often, however, more advanced calculus rules must be applied to "step" only the left or right side of a corres subgoal, until they have synchronized.

Our naive VCG corresK_simple will generate unsolvable goals if functions are mismatched, since corresK_split is applied unconditionally. A safe and effective VCG needs to both detect that the functions are mismatched, and be able to handle it.

Returning to our scheduler example, we now add specifications for enqueueing new threads.

```
definition
  enqueue_thread_abstract :: thread_id ⇒ (abstract_state, bool) nondet_monad
  where enqueue_thread_abstract t ≡
    do add_thread_abstract t;
       select {True, False}
    od

definition
  enqueue_thread_concrete :: thread_id ⇒ (concrete_state, bool) nondet_monad
  where enqueue_thread_concrete t′ ≡
    do add_thread_concrete t′;
       cur_thread′ ← gets cur_thread′;
       return (t′ < cur_thread′)
    od
```

In this example, each enqueue function takes a thread identifier and returns a boolean indicating whether or not the scheduler needs to be re-run. The definitions for add_thread_abstract and add_thread_concrete are given in Figure 7.2. In enqueue_thread_abstract the thread is simply inserted into the unordered set of threads. In enqueue_thread_concrete, the given thread is inserted into its sorted position in the list of threads with insort.

```
insort :: 'a ⇒ 'a list ⇒ 'a list
insort x [] = [x]
insort x (y#ys) = (if x ≤ y then (x#y#ys) else y#(insort x ys))
```

To determine if the scheduler needs to be re-run, enqueue_thread_abstract can only non-deterministically return True or False since it has no concept of thread priority. In enqueue_thread_concrete, however, t' needs to be compared to the currently running thread.

Similar to corres_dequeue, the corresK lemma for add_thread is a straightforward consequence of corresK_modify.

```
corresK_add_thread[corresK]:
  corresK (t = t′) dc ⊤ ⊤ (add_thread_abstract t)(add_thread_concrete t′)
```

Although these functions indeed correspond, applying corresK_simple yields unexpected results.

```
lemma corresK_enqueue_thread:
  corresK (t = t′) (λb b′. b = b′) ⊤ ⊤
    (enqueue_thread_abstract t) (enqueue_thread_concrete t′)
  unfolding enqueue_thread_abstract_def enqueue_thread_concrete_def
  apply corresK_simple
```

```
 1. ⋀y ya.
       dc y ya ⟹
       corresK (?F′3 y ya) (λb b′. b = b′)
                (?R3 y) (?R′3 ya)
                 (select {True, False})
                 (do cur_thread′ ← gets cur_thread';
                     return (t′ < cur_thread′)
                  od)
 A total of 5 subgoals...
```

The proof is now stuck at an unsolvable goal, since select and gets cur_thread' certainly do not correspond.

Generating an unsolvable goal is a serious issue for a VCG, as it prevents exploring alternate strategies when used as part of a larger method expression.

To address this, we can update our method to instead recursively apply itself after a corresK_split. This then requires that at a rule from corresK eventually be successfully applied after applying any corresK_split. Additionally, we abstract this split rule into a named theorem corresK_calc, designed to hold other unsafe calculational rules.

```
named_theorems corresK_calc

declare corresK_split[corresK_calc]

method corresK_once declares corresK corresK_calc =
  (rule corresK | (rule corresK_calc, corresK_once))

method corresK declares corresK corresK_calc =
  (corresK_once | (rule corresK_guard_imp, corresK_once))+
```

Returning to our example, this new method will now successfully split the goal and solve the first obligation.

```
lemma corresK_enqueue_thread:
  corresK (t = t') (λb b'. b = b') ⊤ ⊤
          (enqueue_thread_abstract t) (enqueue_thread_concrete t')     unfolding enqueue_thread_abstract_def enqueue_thread_concrete_def
  apply corresK
```

```
 1. ⋀y ya.
       dc y ya ⟹
       corresK (?F'3 y ya) (λb b'. b = b')
               (?R3 y) (?R'3 ya)
                (select {True, False})
                (do cur_thread' ← gets cur_thread';
                    return (t' < cur_thread')
                 od)
A total of 5 subgoals...
```

To make progress on a subgoal like this, we require non-standard corresK rules that can handle mismatched function bodies. In this case, we need to step the concrete function body across gets cur_thread' in order to make progress.

```
 corresK_gets_bind:
 ⟦⋀rv. corresK (F rv) r P (Q' rv) a (c rv);
  corres_rv F' dc ⊤ P' (return ()) (gets f) (λ_ rv'. F rv')⟧ ⟹
    corresK F' r P (λs'. P' s' ∧ Q' (f s') s') a (gets f >>= c)

apply (rule corresK_gets_bind)
```

```
 1. ⋀y ya rv.
       dc y ya ⟹
       corresK (?F10 y ya rv) (λb b'. b = b')
               (?R3 y) (?Q'14 ya rv)
               (select {True, False}) (return (t' < rv))
A total of 6 subgoals...
```

Now the function bodies match on each side of corresK and the proof can proceed as usual.

```
apply corresK
```

```
 1. ⋀y ya.
       dc y ya ⟹
       corres_rv (?F'3 y ya) dc (λ_. True) (?P'15 ya)
                 (return ()) (gets cur_thread')
                 (λ_ rv. ∃ a∈{True, False}. a = (t' < rv))
 2. corres_rv ?F''3 dc ?Q3 ?Q'3 (add_thread_abstract t) (add_thread_concrete t') ?F'3
 3. ⦃?S3⦄ add_thread_abstract t ⦃λy _. True⦄
 4. ⦃?S'3⦄ add_thread_concrete t' ⦃λa b. ?P'15 a b ∧ True⦄
 5. ⋀s s'.
       ⟦True; True; t = t'; (s, s') ∈ state_relation⟧
       ⟹ (t = t' ∧ ?F''3) ∧
           (?Q3 and (λ_. True) and ?S3) s ∧
           (?Q'3 and (λ_. True) and ?S'3) s'
```

This corres_rv obligation comes from corresK_select_return and in this case it is trivially true. It can therefore be propagated to the final verification condition with corres_rv_defer. This final verification condition is then solved by simp.

```
apply (rule corres_rv_defer | wp)+
by simp
```

This proof is successful, and the corresK method now successfully avoids creating unsolvable goals.

Corres Search

In general, handling mismatched function bodies in a corresK proof requires applying irregular rules like corresK_gets_bind, however these rules should not be applied unconditionally or fully automatically. The goal, therefore, will be to maintain a collection of unsafe rules that can be speculatively applied until the subgoal reaches some expected form.

Specifically, we define a method corresK_search that recursively applies a set of rules until the given rule can be successfully applied.

```
named_theorems corresK_search

method corresK_search uses search declares corresK_search corresK_calc =
   (rule search)
  | (rule corresK_search corresK_calc, corresK_search search: search)

declare corresK_gets_bind[corresK_search]
```

This method will backtrack over all possible combinations of the unsafe corresK_search rules and calculational rules from corresK_calc until the given fact *search* successfully applies.

In our thread queuing example, we can see that the final matching functions are select and return, therefore by providing corresK_select_return as our *search* fact to corresK_search, we can safely apply corresK_gets_bind.

```
lemma corresK_enqueue_thread:
  corresK (t = t′) (λb b′. b = b′) ⊤ ⊤
    (enqueue_thread_abstract t) (enqueue_thread_concrete t′)
  unfolding enqueue_thread_abstract_def enqueue_thread_concrete_def
  apply (corresK | corresK_search search: corresK_select_return)+
  apply (rule corres_rv_defer | wp)+
  by simp
```

Critically, we have not specified how corresK_select_return should be applied, rather that its successful application will indicate that the search has been successful. This effectively allows the user to provide hints to the method indicating how the function bodies correspond, without necessarily knowing the required calculational rules to prove it.

7.3.4 Automating Corres_rv

The corresK calculus introduced a new constant corres_rv in order to propagate stateless preconditions through the subgoal structure. In our examples, however, each corres_rv proof obligation required manual intervention in order to address it. Similar to corresK, we can build a simple VCG for discharging corres_rv subgoals.

A corres_rv statement is similar to a Hoare triple in that it has a single postcondition to be proven to hold as the result of some execution, except that it has multiple preconditions and multiple functions. We can decompose the postcondition across conjuncts in order to decide how to handle each one independently.

```
corres_rv_conj_lift:
⟦corres_rv F r P PP f g Q; corres_rv F′ r P′ PP′ f g Q′⟧ ⟹
  corres_rv (F ∧ F′) r (λs. P s ∧ P′ s) (λs′. PP s′ ∧ PP′ s′) f g
    (λrv rv′. Q rv rv′ ∧ Q′ rv rv′)
```

By repeatedly applying this rule, we can isolate each conjunct in a corres_rv goal.

Next, we can decide if an (atomic) postcondition should be propagated as a Hoare triple based on which goal parameters it discusses. If a postcondition only contains goal parameters that represent abstract return values, then it can be propagated as an abstract Hoare triple. Similarly if it only contains concrete parameters it can be propagated as a concrete Hoare triple.

We can decide this with a simple method that determines if two terms can be unified. Note that match will not work here, since information about the scope of goal parameters is lost inside of a subgoal focus.

```
definition can_unify P Q ≡ True

lemma can_unify_refl: can_unify P P
lemma can_unify_trivial: can_unify P Q

method can_unify = (succeeds ‹rule can_unify_refl›, rule can_unify_trivial)
```

With succeeds, can_unify checks if the two terms from can_unify could be unified with can_unify_refl, but discards the result.

corres_rv_wp_left_safe:
$[\![\mathsf{can\_unify}\ P\ (\lambda\_.\ \forall rv\ rv'.\ Q\ rv\ rv');\ \{\!|P|\!\}\ f\ \{\!|\lambda rv\ s.\ \forall rv'.\ r\ rv\ rv' \longrightarrow Q\ rv\ rv'|\!\}]\!] \Longrightarrow$
$\mathsf{corres\_rv}\ \mathsf{True}\ r\ P\ \top\ f\ f'\ Q$

corres_rv_wp_right_safe:
$[\![\mathsf{can\_unify}\ P'\ (\lambda\_.\ \forall rv\ rv'.\ Q\ rv\ rv');\ \{\!|P'|\!\}\ f'\ \{\!|\lambda rv'\ s.\ \forall rv.\ r\ rv\ rv' \longrightarrow Q\ rv\ rv'|\!\}]\!] \Longrightarrow$
$\mathsf{corres\_rv}\ \mathsf{True}\ r\ \top\ P'\ f\ f'\ Q$

In cases where the postcondition doesn't depend on the return values of f or f', it can be safely propagated through the stateless precondition of corres_rv.

corres_rv_defer_safe:
$\mathsf{corres\_rv}\ (\forall rv\ rv'.\ r\ rv\ rv' \longrightarrow F)\ r\ \top\ \top\ f\ f'\ (\lambda\_\ \_.\ F)$

Alternatively, if the function is a trivial consequence of the return-value relation then the goal can be discharged without propagating any preconditions.

corres_rv_trivial:
$\mathsf{corres\_rv}\ \mathsf{True}\ r\ \top\ \top\ f\ f'\ r$

We define two named theorems, one to maintain rules which emit a can_unify subgoal and one to maintain rules which can be safely applied unconditionally.

named_theorems corres_rv **and** corres_rv_unify

lemmas [corres_rv] = corres_rv_defer_safe corres_rv_trivial
lemmas [corres_rv_unify] = corres_rv_wp_left_safe corres_rv_wp_right_safe

From this it is straightforward to define a corres_rv method.

method corres_rv_once **declares** corres_rv corres_rv_unify =
(rule corres_rv) | (rule corres_rv_unify, can_unify)

method corres_rv **declares** corres_rv corres_rv_unify =
(corres_rv_once | (rule corres_rv_conj_lift, corres_rv_once))+

7.3.5 Integration with Corres

Although it is possible to convert an existing proof development that uses the corres calculus to instead use corresK, it is impractical if a large number of proofs already exist. Our final VCG therefore must be able to make use of existing corres rules, and be applicable to a corres subgoal.

With the @ attribute (see Section 6.2) converting corres rule into a corresK rule is straightforward.

lemma corresK_drop:
$\mathsf{corres}\ r\ Q\ Q'\ f\ g \Longrightarrow \mathsf{corresK}\ \mathsf{True}\ r\ Q\ Q'\ f\ g$

lemma corresK_convert:
$A \longrightarrow \mathsf{corres}\ r\ P\ Q\ f\ f' \Longrightarrow \mathsf{corresK}\ A\ r\ P\ Q\ f\ f'$

method corresK_convert = (((drule uncurry)+)?, drule corresK_convert corresK_drop)

Combined with the atomized attribute, which converts a rule into its object-logic equivalent, this will move any rule assumptions from a corres rule into the stateless precondition of corresK.

```
thm corres_return — ?R ?a ?b ⟹ corres ?R (λ_. True) (λ_. True) (return ?a) (return ?b)

thm corres_return[atomized, @corresK_convert]
  — corresK (?R ?a ?b) ?R (λ_. True) (λ_. True) (return ?a) (return ?b)
```

Recall, however, in our example the previously-implicit assumption that the function arguments were equivalent needed to be made explicit in the stateless precondition of corresK. To handle this case, we can use match.

```
lemma lift_corresK_args:
  corresK F r (P x) (P′ x) (f x) (f′ x) ⟹
   corresK (F ∧ x = x′) r (P x) (P′ x′) (f x) (f′ x′)

lemma corresK_drop_True:
  corresK (True ∧ F) r P P′ f f′ ⟹ corresK F r P P′ f f′

method lift_corresK_args =
  (drule corresK_drop_True |
   match premises in
    H[thin]: corresK _ _ (P x) (P′ x) (f x) (f′ x) (cut 5) for P P′ f f′ x ⇒
      ‹match ((f,f′)) in
        (_, λ_. g) for g ⇒ fail |
        (λ_. g, _) for g ⇒ fail | _ ⇒
          ‹cut_tac lift_corresK_args[where f=f and f′=f′ and P=P and P′=P′, OF H]››)+
```

This attempts to match the corres statement by determining if all of its elements can be abstracted over some x. Since all of these patterns are higher-order (i.e. both P and x are variables in the expression $P\ x$) the number of valid match results is unbounded. To avoid looping when no arguments exist to be lifted, we *cut* the number of backtracking results to 5. We then explicitly fail on any match results where f or f' discard the argument (i.e. $(\lambda\_.\ g)$ and x).

```
method corresK_lift_convert = (corresK_convert, lift_corresK_args)

thm corres_dequeue
  — corres dc ⊤ ⊤ (dequeue_thread_abstract ?t) (dequeue_thread_concrete ?t)

thm corres_dequeue[@corresK_lift_convert]
  — corresK (?t = ?x′) dc ⊤ ⊤ (dequeue_thread_abstract ?t) (dequeue_thread_concrete ?x′)
```

Finally, we can write a corres method and named theorem, that internally converts rules and subgoals into its native corresK calculus.

```
lemma corresK_start:
  corresK True r P P' a c ⟹ corres r P P' a c

named_theorems corres

method corres declares corresK corresK_calc corres =
  ((rule corresK_start)?, corresK corresK: corres[atomized, @corresK_lift_convert])
```

7.3.6 Coressimp

By decomposing the automation strategy for corres into multiple components, we have the ability to carefully apply these specialized methods in order to observe and control their individual behaviour on proofs, and integrate them into other tools. In many cases, however, the proof author will simply wish to attempt *all* known strategies in order to maximally leverage the available automation.

We can integrate this into a single corressimp method, that applies our corres methods along with simp and wp repeatedly to the first subgoal. Note that corresK_search is guarded with a match to ensure it is not attempted when no *search* fact is given.

```
method corressimp uses search wp simp declares corres corresK corresK_calc =
  ((corres
  | corres_rv
  | match search in _ ⇒ ‹corresK_search search: search›
  | wp add: wp
  | simp add: simp)+)[1]
```

7.4 Application to L4.verified

The methods developed in the previous section are a simplification of their final implementation in L4.verified. In practice, there are many side conditions that have to be considered in order to have a robust method that can integrate with existing proofs. The final implementation, including auxiliary methods and the corresK calculus, is approximately 922 lines of Isabelle source: 32 Eisbach methods spanning 121 lines, 36 lines of ML[2], 14 definitions spanning 32 lines, and 135 lemmas spanning 733 lines.

Development began by identifying several corres proofs from the L4.verified refinement proof that were excessively manual as a result of the corres calculus. The goal was to be able to refactor the proofs in-place to use an automated method, without requiring significant modifications to the surrounding proofs or rules.

The implementation progressed iteratively, with naive implementations for the core methods that gradually grew more complex as more side conditions were encountered when the corres proof method was used in more proofs. The majority of the complexity in the implementation resulted from handling mismatched function bodies and the corres_rv obligations that emerged in those proofs.

[2] This is a single ML function to retrieve splitting rules for case statements over inductive datatypes, and has been lifted into a standalone rule attribute.

To evaluate the method, 9 individual proofs from L4.verified were refactored to use the corres proof method. In Table 7.1 we show the number of lines of proof for each lemma before and after it is refactored. The total number of proof lines was reduced 45%, from 475 to 261, with an average reduction of 21 lines and 45% per proof. Some proofs had much more dramatic changes in relative proof size than others, for example set_ep was reduced by almost 80% after refactoring, while resolve_address_bits was only reduced by 24%. This is largely explained by differences in the complexity of the verification conditions that appear in the proofs. In resolve_address_bits, for example, the proof text is primarily dedicated to solving the verification condition, thus the benefit of the corres method is not significant when viewed as a percentage of the total proof size.

In addition to measuring the total reduction in proof size, we can also consider the number of corres lemmas that appear in the proof text itself. Each lemma that is explicitly used indicates a place where the proof author needed to stop to think about the underlying calculus, and potentially look up the relevant rule. Learning the library of corres rules has proven to be a major hurdle when on-boarding new proof engineers, and it is precisely this manual burden that the corres method aims to alleviate.

In Table 7.2 we show instead the number of corres lemmas that were explicitly used in the source proof text before and after each proof was refactored. Nearly all corres lemmas no longer appear in the proof text, and the few that remain are either simple terminal rules used to guide corres_search or induction rules used to manually instantiate loop invariants, and cannot be automatically applied.

There is little to no change in the overall proof processing time for the refactored proofs, as this is still largely dominated by solving the final verification condition.

7.4.1 Proof Example

To understand the effect of the corres method, and indeed proof automation in general, we show the original 38 line proof of set_asid_pool_corres in Figure 7.3. After the initial use of 4 corres rules, the rest of the proof is dedicated to solving the emerging verification conditions. At this point there are 15 open subgoals, with Hoare logic obligations interleaved with verification conditions, as seen by the combined used of wp and auto. Clearly there is some repeated reasoning, as many of the clarsimp, simp and auto lines look very similar, but the subgoal structure makes this extremely difficult to capture or exploit.

After the proof has been refactored, it is much more straightforward.

```
lemma set_asid_pool_corres:
  a = inv ASIDPool a′ o ucast ⟹
  corres dc (asid_pool_at p and valid_etcbs) (asid_pool_at′ p)
          (set_asid_pool p a) (setObject p a′)
  unfolding set_asid_pool_def
  apply (corressimp search: set_other_obj_corres[where P=λ_. True]
                        wp: get_object_ret get_object_wp)
  apply (clarsimp simp: obj_at_simps other_obj_relation_def asid_pool_relation_def )
  by (auto simp: obj_at_simps typ_at_to_obj_at_arches
           split: Structures_A.kernel_object.splits if_splits arch_kernel_obj.splits)
```

In this case, corressimp has successfully handled the corres proof obligation and emitted a

Table 7.1: Sizes of L4.verified corres proofs before and after refactoring, measured in lines of proof.

Function Name	Original	Refactored	#	%
	Size	Size	Reduction	Reduction
create_mapping_entries	36	17	19	53
set_asid_pool	38	10	28	74
lookup_pt_slot	24	13	11	46
copy_global_mappings	47	20	27	57
find_pd_for_asid	74	55	19	26
getSlotCap	14	11	3	21
resolve_address_bits	156	119	37	24
set_ep	33	7	26	79
set_ntfn	32	8	24	75
Total	**454**	**260**	**194**	**43**
Average	**50**	**29**	**21**	**51**

Table 7.2: Number of corres lemmas appearing in proof text of L4.verified corres proofs before and after refactoring.

Function Name	Original	Refactored	#	%
	# Lemmas	# Lemmas	Reduction	Reduction
create_mapping_entries	12	0	12	100
set_asid_pool	4	1	3	75
lookup_pt_slot	4	0	4	100
copy_global_mappings	9	1	8	89
find_pd_for_asid	7	2	5	71
getSlotCap	4	0	4	100
resolve_address_bits	6	1	5	83
set_ep	4	1	3	75
set_ntfn	3	1	2	67
Total	**53**	**7**	**46**	**87**
Average	**6**	**1**	**9**	**84**

```
lemma set_asid_pool_corres:
  a = inv ASIDPool a' o ucast ⟹
  corres dc (asid_pool_at p and valid_etcbs) (asid_pool_at' p)
            (set_asid_pool p a) (setObject p a')
  apply (simp add: set_asid_pool_def)
  apply (rule corres_symb_exec_l)
     apply (rule corres_symb_exec_l)
        apply (rule corres_guard_imp)
          apply (rule set_other_obj_corres [where P=λko::asidpool. True])
                apply simp
               apply (clarsimp simp: obj_at'_def projectKOs)
               apply (erule map_to_ctes_upd_other, simp, simp)
              apply (simp add: a_type_def is_other_obj_relation_type_def)
             apply (simp add: objBits_simps archObjSize_def)
            apply simp
           apply (simp add: objBits_simps archObjSize_def pageBits_def)
          apply (simp add: other_obj_relation_def asid_pool_relation_def)
         apply assumption
        apply (simp add: typ_at'_def obj_at'_def ko_wp_at'_def projectKOs)
        apply clarsimp
        apply (case_tac ko; simp)
        apply (rename_tac arch_kernel_object)
        apply (case_tac arch_kernel_object; simp)
       prefer 5
       apply (rule get_object_sp)
     apply (clarsimp simp: obj_at_def exs_valid_def assert_def a_type_def return_def fail_def)
      apply (auto split: Structures_A.kernel_object.split_asm arch_kernel_obj.split_asm if_split_asm)[1]
     apply wp
     apply (clarsimp simp: obj_at_def a_type_def)
     apply (auto split: Structures_A.kernel_object.split_asm arch_kernel_obj.split_asm if_split_asm)[1]
    apply (rule no_fail_pre, wp)
    apply (clarsimp simp: simp: obj_at_def a_type_def)
    apply (auto split: Structures_A.kernel_object.splits arch_kernel_obj.splits if_split_asm)[1]
   apply (clarsimp simp: obj_at_def exs_valid_def get_object_def exec_gets)
   apply (simp add: return_def)
  apply (rule no_fail_pre, wp)
  apply (clarsimp simp add: obj_at_def)
  done
```

Figure 7.3: The corres proof for set_asid_pool before refactoring.

single verification condition. Although the corressimp method has only directly replaced 4 uses of the corres rules, it has additionally produced a subgoal structure which is much more amenable to using Isabelle's built-in automation. Previously, verification conditions were interleaved with Hoare logic obligations, and many related and similar conditions had been spread across multiple subgoals.

In the refactored proof, the single verification condition can now be handled with one invocation of auto (and some help from clarsimp). This has effectively factored out a significant amount of duplicated reasoning from the original proof, while also abstracting away the details of the corres calculus.

As the development of L4.verified continues, the corres method has now started to appear in new proofs.

7.5 Conclusion

In this chapter we have outlined the design and implementation of a new proof method for handling refinement proofs using Eisbach. This was motivated by identifying a significant source of manual proof effort in L4.verified. Although the Hoare logic verification-condition generator (VCG) wp has been used extensively in the L4.verified proofs, no such VCG existed for automating refinement proofs.

Building on the corres calculus [23], we identified a key factor that made it unsuitable to effectively use automated reasoning. This motivated the development of a new corresK calculus, which enabled greater control over how verification conditions were propagated through Isabelle's subgoal structure. With this, we were able to build a corresK method for automating corresK proofs, and ultimately integrate this into the existing corres infrastructure of L4.verified in a corres method.

We refactored several existing refinement proofs from L4.verified to use this new proof method, achieving a significant reduction in overall proof size in many cases, and in every case eliminating most of the corres calculus rules that appear in the final proof text. The benefit of the corres method is therefore twofold: explicit use of corres rules is no longer required except when completely necessary, reducing the manual burden of generating a verification condition; and the resulting condition is now a single subgoal, enabling much more effective use of Isabelle's built-in automation (or domain-specific methods) and avoiding duplicated reasoning in the proof text.

This case study motivated and made use of the advanced features presented in Chapter 6. An initial design goal for this VCG was for the majority of the source to be Isar and Eisbach, with as little as possible ML. Where ML functionality was initially required, general-purpose tools were often built instead (i.e. the @ attribute and fold_subgoals method). The **apply_debug** command was designed to trace the backtracking behaviour of corresK_search, and was later updated to support proof state modifications in order to perform ad-hoc experiments with different corresK calculational rules. The final result of this study was both an effective corres proof method, and a rich library of tools for future method development.

Developing the corresK calculus was necessary due to the original focus of the corres calculus on manual, interactive proofs. It is a conservative extension of corres that provides a minimal extra layer in order to support the necessary formal bookkeeping for a VCG. Were Eisbach available during the initial development of corres, it is highly likely that a VCG would have been developed in tandem with the calculus itself and the results would have been much different. For example, annotated specifications could have been generated from the invariant proofs and avoided the need for Hoare logic reasoning in the middle of each corres proof.

Eisbach allows such experiments to be carried out easily, allowing for more informed decisions when embarking on new proof developments.

Chapter 8

Conclusion

The demand for software verification is growing [28]: as complex computer systems are entrusted with an ever-increasing range of mission-critical tasks, their correctness and resilience to attack is paramount. In the past few decades, interactive theorem provers (ITPs) have increased the practical scope of using formal logic to *prove* the correctness of software. Large-scale verification projects, such as L4.verified, are now becoming more common-place. L4.verified consists of over 500,000 lines of Isabelle proof text, verifying seL4's relatively small 10,000 line C implementation.

In Section 2.3 we introduced *proof engineering* as the practice of managing the scalability issues inherent to producing and maintaining large proofs in ITPs. Effectively engineering a proof requires productively using (or building) frameworks in order to minimize the cost of both its initial development and ongoing maintenance. A critical aspect of proof engineering is performing empirical analysis of proof projects and artefacts, in order to inform cost and effort models of formal proof development. We discussed a research agenda by Jeffery et al. [39], in which they identify over thirty open questions in proof engineering that require additional data. In this thesis, we noted this question from the agenda:

> *How are characteristics of formal specifications, properties, or code related to effort in formal proofs?*

We considered this question in Chapter 4 by building on previous work by Staples et al. [64][65], where they establish a linear relationship between proof effort and proof size, as well as between code size and formal specification size. In our study, we developed metrics to investigate the relationship between the size of a formal statement and the size of its proof, finding a quadratic relationship in the verification projects considered. This suggests, combined with the results from Staples et al., that with current methods the effort required to prove the correctness of software will increase quadratically with code size.

This quadratic scaling factor poses a significant challenge for formal verification of larger software systems. It indicates that hundreds of thousands of lines of verified code potentially results in tens of millions of lines of proof, requiring dozens of person-years to complete.

Motivated by this, we considered a second question from the research agenda:

> *How can we best combine interactive proof and proof automation to achieve high proof productivity during initial proof development and subsequent proof maintenance?*

The availability and use of automated reasoning is intrinsically linked with the scalability of proof effort. In an interactive theorem prover, each line of proof indicates some upper bound on the amount of reasoning that can be expressed with a single step. Most modern ITPs provide a *tactic language* for writing new proof tactics, either as a domain-specific language (e.g. Ltac in Coq), or a set of standard libraries in the implementation language of the prover (e.g. ML in HOL).

In Isabelle, powerful automated *proof methods* come built-in with Isar. This has allowed for significant results to be achieved without requiring a specialized language for developing new methods, as custom proof methods can be implemented in ML by expert users. However, larger proof developments have seen scalability and maintenance challenges arise from a lack of custom proof methods. We hypothesized that this was due to a lack of familiarity with ML within the Isabelle community. Although many proofs would benefit from custom proof methods, they can still be completed without them, at the cost of duplicated reasoning in the proof text or additional manual effort.

To address this, in Chapter 5 we presented *Eisbach*, a proof method language and extensible proof automation framework for Isabelle that leverages Isar's syntax and integration with ML. Eisbach's expressive match method uses Isabelle's unifier to give control-flow to inner method expressions, while using *subgoal focusing* to provide direct access to logical elements of Isabelle's subgoal state. Although match is a core aspect of Eisbach, it is implemented in ML as a standard proof method with no special status. This is a testament to the extensibility of Eisbach, as similarly expressive language extensions can easily be provided by end users. This is explored further in Chapter 6 where we introduced a suite of language extensions, including tools to treat proof methods as fact-producing functions, as well as an interactive debugger.

Eisbach's integration with *named theorems* additionally allows for ad-hoc extensibility, where proof methods can refer to named collections of facts that are dynamically extended later. This functionality is critical in the proof methods introduced in Chapter 7 to reduce the manual effort of proofs in L4.verified. Using this, as well as extensions presented in Chapter 6, we developed a set of proof methods for calculating verification conditions in refinement proofs. By applying these methods to several existing proofs from L4.verified, we demonstrated that their size and complexity could be drastically reduced. Across 9 lemmas we reduced proof size by 21-77%, with an average reduction of 51%. Additionally, we reduced explicit references to refinement calculus rules by 67-100%, with an average reduction of 84%.

These results are a promising indication that powerful automated methods can address the quadratic scaling factor identified in Chapter 4. By reducing the required proof effort for each individual instance of a particular problem, we hypothesize that we can reduce the coefficient on this quadratic relationship and, in some cases, reduce it to linear. In the presence of automation, however, even measuring proof effort becomes more complex as the role of the proof author shifts from providing individual proof steps to simply guiding automated methods. Each method will therefore likely require its own model for understanding how much effort is required for a proof author to determine the necessary input parameters.

Eisbach has been incorporated into many other proof developments since its initial 2015 release, confirming our experience that Eisbach can make proof development more productive. In his PhD thesis [9], Alasdair Armstrong remarks "*compared to the development of tactics in Isabelle/ML... Eisbach makes the development of such tactics available to even the novice Isabelle user.*"

Building trustworthy, verified systems is crucial for the increasing number of mission-critical applications of computers. As larger, more complex software systems are being formally verified in interactive theorem provers, the discipline of proof engineering is becoming more important. A proof engineer requires both the tools and knowledge to manage the challenges, costs and efforts inherent in developing and maintaining large-scale proofs.

In this thesis, we have presented Eisbach as a supporting framework for scalable proof engineering in Isabelle, motivated by an empirical investigation and evaluated with a case study. By increasing the accessibility of writing custom proof automation with Eisbach, we can achieve better scalability of proof efforts. This enables the field of proof engineering to tackle future grand verification challenges, expanding our capacity to verify larger mission-critical systems.

Bibliography

[1] Afp statistics. `https://www.isa-afp.org/statistics.html`. [Online; accessed 2018-03-23].

[2] Coq package index. `https://coq.inria.fr/opam/www/`. [Online; accessed 2018-03-23].

[3] Deploy: Industrial deployment of system engineering methods providing high dependability and productivity. `http://www.deploy-project.eu/`. [Online; accessed 2015-02-06].

[4] Mizar mathematical library. `http://mizar.org/library/`. [Online; accessed 2018-03-23].

[5] sel4/l4v: Proofs for sel4-8.0.0. `https://doi.org/10.5281/zenodo.1168016`, Feb. 2018. [Online; accessed 2018-04-15].

[6] Alama, J., Mamane, L., and Urban, J. Dependencies in formal mathematics: Applications and extraction for Coq and Mizar. In *Intelligent Computer Mathematics* (2012), pp. 1–16.

[7] Alkassar, E., Paul, W., Starostin, A., and Tsyban, A. Pervasive verification of an OS microkernel. In *Verified Software: Theories, Tools, Experiments* (2010), pp. 71–85.

[8] Andronick, J., Jeffery, R., Klein, G., Kolanski, R., Staples, M., Zhang, H. J., and Zhu, L. Large-scale formal verification in practice: A process perspective. In *International Conference on Software Engineering* (Zurich, Switzerland, June 2012), ACM, pp. 1002–1011.

[9] Armstrong, A. *Formal Analysis of Concurrent Programs.* PhD thesis, University of Sheffield, 2015.

[10] Ballarin, C. Locales and locale expressions in Isabelle/Isar. In *Types for Proofs and Programs (TYPES 2003)* (2003), S. Berardi, M. Coppo, and F. Damiani, Eds., vol. 3085 of *Lecture Notes in Computer Science*, Springer.

[11] Ballarin, C. Locales: A module system for mathematical theories. *Journal of Automated Reasoning 52*, 2 (2014), 123–153.

[12] Bancerek, G., and Rudnicki, P. Information retrieval in MML. In *International Conference on Mathematical Knowledge Management* (2003), Springer, pp. 119–132.

[13] Barendregt, H., and Wiedijk, F. The challenge of computer mathematics. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences 363*, 1835 (2005), 2351–2375.

[14] Bellard, F. Qemu, a fast and portable dynamic translator. In *Proceedings of the Annual Conference on USENIX Annual Technical Conference* (Berkeley, CA, USA, 2005), ATEC '05, USENIX Association, pp. 41–41.

[15] Berghofer, S., and Nipkow, T. Proof terms for simply typed higher order logic. In *Theorem Proving in Higher Order Logics*, M. Aagaard and J. Harrison, Eds., vol. 1869 of *Lecture Notes in Computer Science*. Springer Berlin Heidelberg, 2000, pp. 38–52.

[16] Bollin, A. Metrics for quantifying evolutionary changes in Z specifications. *Journal of Software: Evolution and Process 25*, 9 (2013), 1027–1059.

[17] Bourke, T., Daum, M., Klein, G., and Kolanski, R. Challenges and experiences in managing large-scale proofs. In *Intelligent Computer Mathematics* (2012), pp. 32–48.

[18] Boyer, R., and Moore, J. *A Computational Logic*. Academic Press, 1979.

[19] Boyton, A., Andronick, J., Bannister, C., Fernandez, M., Gao, X., Greenaway, D., Klein, G., Lewis, C., and Sewell, T. Formally verified system initialisation. In *Proceedings of the 15th International Conference on Formal Engineering Methods* (Queenstown, New Zealand, Oct. 2013), Lindsay Groves, Jing Sun, Ed., Springer, pp. 70–85.

[20] Checkoway, S., McCoy, D., Kantor, B., Anderson, D., Shacham, H., Savage, S., Koscher, K., Czeskis, A., Roesner, F., and Kohno, T. Comprehensive experimental analyses of automotive attack surfaces. In *Proceedings of the 20th USENIX Security Symposium* (2011).

[21] Chlipala, A. Mostly-automated verification of low-level programs in computational separation logic. *ACM SIGPLAN Notices 46*, 6 (Jun 2011), 234.

[22] Chrząszcz, J. Implementing modules in the coq system. In *International Conference on Theorem Proving in Higher Order Logics* (2003), Springer, pp. 270–286.

[23] Cock, D., Klein, G., and Sewell, T. Secure microkernels, state monads and scalable refinement. In *Proceedings of the 21st International Conference on Theorem Proving in Higher Order Logics* (Montreal, Canada, Aug. 2008), Otmane Ait Mohamed, César Muñoz, Sofiène Tahar, Ed., Springer, pp. 167–182.

[24] de Moura, L., Kong, S., Avigad, J., Van Doorn, F., and von Raumer, J. The lean theorem prover (system description). In *International Conference on Automated Deduction* (2015), Springer, pp. 378–388.

[25] Delahaye, D. A tactic language for the system Coq. In *International Conference on Logic for Programming, Artificial Intelligence and Reasoning* (Nov. 2000), vol. 1955 of *Lecture Notes in Computer Science*, Springer.

[26] Dijkstra, E. W. Guarded commands, nondeterminacy and formal derivation of programs. *Commun. ACM 18*, 8 (Aug. 1975), 453–457.

[27] Felty, A., and Howe, D. Generalization and reuse of tactic proofs. *Logic Programming and Automated Reasoning* (1994).

[28] Fisher, K. HACMS: High assurance cyber military systems. In *Proceedings of the 2012 ACM Conference on High Integrity Language Technology* (Boston, Massachusetts, USA, 2012), HILT '12, ACM, pp. 51–52.

[29] Fowler, M., Beck, K., Brant, J., Opdyke, W., and Roberts, D. *Refactoring: improving the design of existing code.* Addison-Wesley Professional, 1999.

[30] Gonthier, G., and Mahboubi, A. An introduction to small scale reflection in Coq. *J. Formalized Reasoning 3*, 2 (2010).

[31] Gonthier, G., Ziliani, B., Nanevski, A., and Dreyer, D. How to make ad hoc proof automation less ad hoc. *J. Funct. Program. 23*, 4 (2013), 357–401.

[32] Gordon, M. J. C., Milner, R., and Wadsworth, C. P. *Edinburgh LCF: A Mechanized Logic of Computation.* LNCS 78. Springer, 1979.

[33] Greenaway, D. *Automated Proof-Producing Abstraction of C Code.* PhD thesis, Sydney, Australia, jan 2015.

[34] Grov, G., and Maclean, E. Towards automated proof strategy generalisation. *CoRR abs/1303.2* (2013), 1–15.

[35] Halperin, D., Clark, S. S., Fu, K., Heydt-Benjamin, T. S., Defend, B., Kohno, T., Ransford, B., Morgan, W., and Maisel, W. H. Pacemakers and implantable cardiac defibrillators: Software radio attacks and zero-power defenses. In *Proceedings of the IEEE Symposium on Security and Privacy* (Oakland, CA, USA, May 2008), pp. 129–142.

[36] Halstead, M. H. *Elements of Software Science (Operating and Programming Systems Series).* Elsevier Science Inc., New York, NY, USA, 1977.

[37] Hupel, L. Interactive simplifier tracing and debugging in Isabelle. In *Intelligent Computer Mathematics.* Springer, 2014, pp. 328–343.

[38] ISO/IEC 19761:2011, *Software engineering – COSMIC: a functional size measurement method*, 2011.

[39] Jeffery, R., Staples, M., Andronick, J., Klein, G., and Murray, T. An empirical research agenda for understanding formal methods productivity. *Information and Software Technology 60* (2015), 102–112.

[40] Kaufmann, M., and Moore, J. S. ACL2: An industrial strength version of Nqthm. In *Computer Assurance* (1996).

[41] Klein, G., Andronick, J., Elphinstone, K., Murray, T., Sewell, T., Kolanski, R., and Heiser, G. Comprehensive formal verification of an OS microkernel. *ACM Transactions on Computer Systems 32*, 1 (Feb. 2014), 2:1–2:70.

[42] Klein, G., Elphinstone, K., Heiser, G., Andronick, J., Cock, D., Derrin, P., Elkaduwe, D., Engelhardt, K., Kolanski, R., Norrish, M., Sewell, T., Tuch, H., and Winwood, S. seL4: Formal verification of an OS kernel. In *ACM Symposium on Operating Systems Principles* (Big Sky, MT, USA, Oct. 2009), ACM, pp. 207–220.

[43] Klein, G., Nipkow, T., and Paulson, L. The archive of formal proofs. `http://afp.sf.net`. [Online; accessed 2018-03-28].

[44] Lochbihler, A. Jinja with threads. *Archive of Formal Proofs* (Dec. 2007). `http://afp.sf.net/entries/JinjaThreads.shtml`, Formal proof development [Online; accessed 2018-04-15].

[45] Maric, F. Formal verification of modern SAT solvers. *Archive of Formal Proofs* (July 2008). `http://afp.sf.net/entries/SATSolverVerification.shtml`, Formal proof development [Online; accessed 2018-04-15].

[46] Matichuk, D., and Murray, T. Extensible specifications for automatic re-use of specifications and proofs. In *10th International Conference on Software Engineering and Formal Methods* (Thessaloniki, Greece, Dec. 2012), p. 8.

[47] Matichuk, D., Murray, T., Andronick, J., Jeffery, R., Klein, G., and Staples, M. Empirical study towards a leading indicator for cost of formal software verification. In *Proceedings of the 37th International Conference on Software Engineering - Volume 1* (Piscataway, NJ, USA, 2015), ICSE '15, IEEE Press, pp. 722–732.

[48] Matichuk, D., Murray, T., and Wenzel, M. Eisbach: A proof method language for Isabelle. *J. Autom. Reason. 56*, 3 (Mar. 2016), 261–282.

[49] Matichuk, D., Wenzel, M., and Murray, T. The Eisbach user manual. `https://isabelle.in.tum.de/doc/eisbach.pdf`. [Online; accessed 2018-03-23].

[50] Matichuk, D., Wenzel, M., and Murray, T. An Isabelle proof method language. In *Interactive Theorem Proving (ITP)* (Vienna, Austria, July 2014), p. 16.

[51] McCabe, T. A complexity measure. *Software Engineering, IEEE Transactions on SE-2*, 4 (Dec 1976), 308–320.

[52] Murray, T., Matichuk, D., Brassil, M., Gammie, P., Bourke, T., Seefried, S., Lewis, C., Gao, X., and Klein, G. seL4: from general purpose to a proof of information flow enforcement. In *IEEE Symposium on Security and Privacy* (San Francisco, CA, May 2013), pp. 415–429.

[53] Murray, T., Matichuk, D., Brassil, M., Gammie, P., and Klein, G. Noninterference for operating system kernels. In *The Second International Conference on Certified Programs and Proofs* (Kyoto, Dec. 2012), Chris Hawblitzel and Dale Miller, Ed., Springer, pp. 126–142.

[54] Nipkow, T., Paulson, L., and Wenzel, M. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*, vol. 2283 of *Lecture Notes in Computer Science*. Springer, 2002.

[55] Olszewska (Pląska), M., and Sere, K. Specification metrics for Event-B developments. In *Proceedings of the CONQUEST 2010: "Software Quality Improvement"* (2010), I. Schieferdecker, R. Seidl, and S. Goericke, Eds., International Software Quality Institute, pp. 1–12.

[56] Paulson, L. C. Isabelle: the next 700 theorem provers. In *Logic and Computer Science*, P. Odifreddi, Ed. Academic Press, 1990.

[57] Ringer, T., Yazdani, N., Leo, J., and Grossman, D. Adapting proof automation to adapt proofs. In *CPP* (2018).

[58] Rudnicki, P. An overview of the Mizar project. In *Types for Proofs and Programs* (1992), pp. 1–22.

[59] Samson, W. B., Nevill, D. G., and Dugard, P. I. Predictive software metrics based on a formal specification. *Inf. Softw. Technol. 29*, 5 (June 1987), 242–248.

[60] Schirmer, N. *Verification of sequential imperative programs in Isabelle/HOL*. PhD thesis, Technische Universität München, 2006.

[61] Sewell, T. seL4 enforces integrity. *Interactive Theorem Proving 6898* (2011), 325–340.

[62] Sewell, T., Myreen, M., and Klein, G. Translation validation for a verified OS kernel. In *ACM SIGPLAN Conference on Programming Language Design and Implementation* (Seattle, Washington, USA, June 2013), ACM, pp. 471–481.

[63] Sewell, T., Winwood, S., Gammie, P., Murray, T., Andronick, J., and Klein, G. seL4 enforces integrity. In *Proceedings of the 2nd International Conference on Interactive Theorem Proving* (Nijmegen, The Netherlands, Aug. 2011), M. C. J. D. van Eekelen, H. Geuvers, J. Schmaltz, and F. Wiedijk, Eds., vol. 6898 of *Lecture Notes in Computer Science*, Springer, pp. 325–340.

[64] Staples, M., Jeffery, R., Andronick, J., Murray, T., Klein, G., and Kolanski, R. Productivity for proof engineering. In *Empirical Software Engineering and Measurement* (Turin, Italy, Sept. 2014).

[65] Staples, M., Kolanski, R., Klein, G., Lewis, Corey nd Andronick, J., Murray, T., Jeffery, R., and Bass, L. Formal specifications better than function points for code sizing. In *International Conference on Software Engineering* (San

Francisco, USA, May 2013), David Notkin, Betty H. C. Cheng, Klaus Pohl, Ed., IEEE, pp. 1257–1260.

[66] Stidolph, D. C., and Whitehead, J. Managerial issues for the consideration and use of formal methods. In *In Stefania Gnesi, Keijiro Araki, and Dino Mandrioli (eds.), FME 2003, International Symposium of Formal Methods Europe* (2003), pp. 8–14.

[67] Tabareh, A. Predictive software measures based on formal Z specifications. Master's thesis, University of Gothenburg - Department of Computer Science and Engineering, 2011.

[68] Trustworthy Systems Team. seL4 proofs for API 1.03, release 2014-08-10, Aug 2014.

[69] Tuch, H., Klein, G., and Norrish, M. Types, bytes, and separation logic. In *ACM SIGPLAN Notices* (2007), vol. 42, ACM, pp. 97–108.

[70] Wenzel, M. Isar - a generic interpretative approach to readable formal proof documents. In *Proceedings of the 12th International Conference on Theorem Proving in Higher Order Logics* (London, UK, UK, 1999), TPHOLs '99, Springer-Verlag, pp. 167–184.

[71] Wenzel, M. *Isabelle/Isar—a versatile environment for human-readable formal proof documents*. PhD thesis, Technische Universität München, 2002.

[72] Wenzel, M., and Chaieb, A. SML with antiquotations embedded into Isabelle/Isar. In *Workshop on Programming Languages for Mechanized Mathematics (PLMMS 2007). Hagenberg, Austria* (June 2007), J. Carette and F. Wiedijk, Eds.

[73] Wenzel, M., and Wiedijk, F. A comparison of Mizar and Isar. *Journal of Automated Reasoning 29*, 3-4 (2002), 389–411.

[74] Whiteside, I., Aspinall, D., Dixon, L., and Grov, G. Towards formal proof script refactoring. In *Intelligent Computer Mathematics* (2011), pp. 260–275.

[75] Wiedijk, F., Ed. *The Seventeen Provers of the World*, vol. 3600. 2006.

[76] Ziliani, B., Dreyer, D., Krishnaswami, N. R., Nanevski, A., and Vafeiadis, V. Mtac: a monad for typed tactic programming in Coq. In *ICFP* (2013), G. Morrisett and T. Uustalu, Eds., ACM.