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Securitization of Longevity Risk in Reverse Mortgages

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Abstract

Reverse mortgages are becoming remarkably popular in the last few years in Australia, and although they have been around a lot longer in the United States, they are receiving renewed interest among the elderly. Increase in life expectancies and decrease in the real income at retirement because of inflation continue to worry the retirees. Financial products that help alleviate the "risk of living longer" therefore continue to be attractive among the retirees.

Reverse mortgages involve various risks from the provider's perspective which may hinder the further development of these financial products. In this thesis, we address one method of transferring and financing the risks associated with these products through the form of securitization. Securitization is becoming an attractive alternative form of risk transfer of insurance liabilities. We therefore construct a securitization structure for reverse mortgages similar to the one applied in insurance risks. In particular, we investigate the merits of developing survivor bonds and survivor swaps for reverse mortgage products. In the case of survivor bonds, for example, we are able to develop premiums, both analytically and numerically through simulations, and to examine how the longevity risk is transferred to the investors. Our numerical calculations provide an indication of the economic benefits derived from developing survivor bonds to securitize the "longevity risk component" of reverse mortgage products. Some sensitivity analysis of these economic benefits also indicates that the survivor bond can be a promising tool for investment diversification.

Chapter 1

Introduction

In many countries today life expectancy is over 80 years; while life expectancy has increased in the last a few years, according to Bateman, Kingston and Piggott (2001, [5]), the labour market participation rates for males aged 60-64 have fallen from 70-90 percent, down to 20-50 percent in recent years. Living longer and retiring earlier exacerbate the risk that the elderly may not be able to fund their retirement in old age. This risk is known as "longevity risk".

Research has found that retired people often have plenty of equity locked in their homes but few liquid assets to support their daily lives. For example, in Australia home equity takes a proportion of 50 percent of all household assets in 1996 (1998, [3]). This is a common situation among retired people in many western countries called "house rich and cash poor". One option available to retirees in this position is a reverse mortgage (RM) product. A reverse mortgage is a non-recourse first lien loan that allows retirees, usually 62 years or older, to consume part or all of their home equity but still remain in their homes until they die or sell their homes. With home ownership and the elderly population steadily increasing in many developed countries, reverse mortgages are a viable way for house-rich, cash-strapped seniors to supplement their income and afford long-term care while staying in their homes.

In the USA, the reverse mortgage loan has become a very popular product in the financial market during the last decade with the most significant and rapid growth occurring since the beginning of the millennium. The various payment options for borrowers include: an up-front lump sum payment; a fixed monthly annuity payment; a line of credit with an upper limit and various combinations of these three types. Reverse mortgages were introduced in Australia in the early 1990's. Although the market has not yet proven viable, the product has a lot of potential according to Reed and Gibler (2003, [49]). Since major players in the financial market such as the Commonwealth bank started offering the product recently, the market has improved and there is indication that it will expand further in the near future.

Although a reverse mortgage sounds like an attractive product to the retirees, the product involves various risks from the provider's perspective. Most of the loan amount accrues interest at a variable rate, usually adjusted monthly or yearly based on an index rate, and is repaid only when the borrower dies or sells or permanently leaves the home. In most cases, the loan balance accumulates faster than the home equity value rises, so that in time it will exceed the value of the home equity. If the outstanding loan balance exceeds the home equity value before the loan is due, the lender will incur a loss. This "cross-over" risk is the crucial risk in reverse mortgages and is determined mainly by three underlying variables: mortality rates, interest rates and house prices. Improvements in the mortality rate, a high interest rate environment and a depressed real estate market can exacerbate the cross-over risk. Of the three variables, the mortality rate is believed to be the most important one in the pricing and risk management of reverse mortgages.

A traditional method of dealing with the risks in reverse mortgages is insurance. The Home Equity Conversion Mortgages (HECM) program in the United States is an example of one such scheme. This thesis suggests securitization as a possible means of managing the longevity risk in reverse mortgages, as mortality securitization is regarded as a flexible way to transfer the unwanted mortality risk from the insurers to the capital market, allowing the risks to be more efficiently distributed.

Securitization is a financial innovation emerged in the 1970's in the United States. Since then, the product has shown its incredibly fast growth and has expanded from the finance to the insurance industry. An economic justification of securitization of insurance risk is provided by Cox in 2000 (2000, [17]). Once insurance risk-based securities such as catastrophe bond proved popular, the idea of securitizing mortality risk was introduced in 2001 (2001, [6]). Several models of mortality-based securities were proposed including mortality bonds and swaps by academia. Shortly after the concepts were introduced, the insurance industry put them into practice. For examples, Swiss Re and European Investment Bank issued different types of mortality-based securities respectively in 2003 and 2004. While Swiss Re offered a brevity risk bond to hedge the risk of adverse changes in mortality rates, European Investment Bank issued a survivor bond to address the improvements in mortality rates. This thesis proposes a method to hedge the longevity risk inherent in reverse mortgage products. Following a similar approach used by Lin and Cox (2004, [40]), several examples of longevity securitization are given, including two types of survivor bonds and a survivor swap. The survivor bonds are priced and a sensitivity analysis is applied to each bond to assess the effectiveness in hedging the longevity risk. The results show that securitization can provide an efficient and economical way to hedge the longevity risk in reverse mortgages.

The thesis consists of six chapters. Chapter 2 discusses and reviews the historical development and the current market state of securitization in the relevant literature. An economic justification of securitization products of mortality risk is provided using Cox's approach. Different categories of securitization of mortality risk are examined and several recent examples are provided. Chapter 3, examines the history of reverse mortgage products in the US and discusses the risks involved. In Chapter 4, the risks are analyzed in detail and a model is provided to price the reverse mortgage. The results are calculated using a simulation technique. In Chapter 5, a model is developed to securitize the longevity risk in reverse mortgages. Several example schemes are given, including two types of survivor bonds and a survivor swap. The survivor bonds are then priced both analytically and numerically and the effect of risk hedging is examined at the end of this chapter. The thesis concludes with a summary of the findings and a recommendation for securitization as a method to manage the longevity risk.

Chapter 2

Securitization

2.1 History and current development in securitization

The first securitized transactions can be traced back to 1970 in the United States when the Government National Mortgage Association ("Ginnie Mae") began to sell guaranteed mortgage pass-through certificates (1996, [32]). In the late 1970s, private securitized residential mortgage transactions began to emerge, in respond to a funding shortfall in the U.S. home mortgage market. At that time, demand of homeowners and potential homeowners for mortgage loans exceeded the lenders capital ability to supply, leading the financial markets to find a more efficient way to transfer funds from investors in the capital markets to the mortgage demanders (2002, [19]).

Since then, the general financial markets have seen securitization grow whip and spur resulting in the creation of a numbers of new securities. Two factors have contributed to the blossoming of the securitization market over the last twenty years. The first was changes to the tax code. In 1986, new legislation was introduced in the U.S. to simplify tax structuring of some complex mortgage-backed securitization (1996, [32]). The second factor was the development in pricing technology and computing power, since to proper pricing of certain securities requires modern economic and financial techniques and extensive computation (1999, [29]).

In the USA, since the introduction of mortgage-backed securities in the 1970s the securitization volume has increased dramatically. By 2002, the newly issued volume of mortgage-backed securities (MBS) and asset-backed securities (ABS) reached \$1.5 trillion and \$450 billion respectively (2003, [18]). By the second quarter of 2003 a total of \$6.6 trillion worth of securities had been issued. Recently, securitization has begun to expand into other industries including the insurance industry.

In Europe also, the securitization market has shown strong growth. Figure (2.1) [1]illustrate the rapid growth of European securitization in the last decade. The issuance volume has dramatically increased by over 700%, from less than €40 billion in 1998 to a new historical record €319.6 billion in 2005.

Figure (2.2) illustrates the rapid growth of European securitization in the last year. Securitization volume has increased by 31.1%, from the previous record €241.2 billion in 2004 to €319.6 billion in 2005. The issuance in the fourth quarter of 2005 also set a new quarterly record of €135.7 billion. According to the latest ESF survey rapid growth is expected to continue in 2006, with securitized volume expected to increase 15.0 percent this year.

2.2 Classification of securitization

Securitization products can be classified as either asset securitization (or asset-backed securitization) or risk securitization based on whether a backing-asset is involved. The asset backed securities market enables the originator to move the asset off the balance sheet through securitization. Essentially, asset securitization repackages the asset and sell it in the market to either transfer the risk or refinance. ABS, MBS, CMBS (Commercial Mortgage Backed Securities) and CDO (Collateralized Debt Obligations) fall into this category.

Risk securitization is, on the other hand, more like reinsurance. Indeed in a broader sense, reinsurance can be considered a form of securitization. The sole aim of the transaction is to transfer the risk, either to a counterpart or a capital market; the key feature is that backing-assets are not involved in this type of securitization. Catastrophe bonds, mortality/survivor bonds or swaps all belong

[1]The statistics and figures are quoted from the European Securitization Forum (ESF) Securitisation Data Report. These quotations have been approved by ESF.

to this group. Compared to asset securitization, risk securitization is expensive due to the additional costs including the cost of measuring the risks and educating investors. These costs, however will decline as investors become more familiar with the risks in time.

Another classification approach is suggested by Gorvett (1999, [29]) based on the goal of the transaction. Based on his method, securitization products can be divided into those that transfer risk and those that provide contingent funding. The former group include reinsurance products which transfer risks to other companies within the insurance industry; swaps which exchange risk with a counter-party, or transfer risks to other insurers or the capital markets; catastrophe bonds which transfer risks to the capital markets; and exchange-traded derivatives which also transfer risks to the capital markets. The latter group includes line of credit (the right to borrow fund); contingent surplus notes which offer an option to borrow contingent upon the occurrence of an adverse event; and catastrophe equity puts which offer the option to sell equity (usually preferred shares) on redetermined terms, contingent upon an event.

2.3 Securitization of mortality risk

Cummins (2004, [19]) analyzed various traditional securitization models and discussed the emerging securitization classes in the insurance industry. He concluded that while securitization of insurance risk had great potential although growth could be impeded by its complexity and opacity. But Cummins deemed mortality risk one of the drivers of demand for insurance securitizations. Demand for new securities always arises when new risks appear and when existing risks become more significant. The appearance of mortality securitization is a result of the increasing importance of mortality risk.

A mortality-based security, as its name, is a particular type of securitization transferring mortality risk. They are usually long duration and high capacity. Examples of mortality-based securities include: annuity futures, where prices are linked to a specified future market annuity rate; mortality options, which include a range of contracts with option characteristics where payoff depends on an underlying mortality table at the payment date; and short-dated, mortality-linked securities which are market-traded securities where payments are linked to a mortality index. Some life insurance securitization deals since 1996 are listed in Appendix (D).

2.3.1 Survivor bonds

Mortality-based securities are very recent subject in the literature. The first paper on the topic of mortality-based securities was published by Blake and Burrows in 2001 (2001, [6]). In this paper the authors first introduced the concept of survivor bonds. "The idea behind the survivor bonds is to make the coupon payments contingent on mortality rate of a certain age; it pays a coupon proportional to the number of survivors in a cohort. The original example was an annuity bond with coupon payments tied to a survivor index published periodically by certain authorities." Blake and Burrows suggested that survivor bonds have considerable potential as mortality-hedging instruments for insurance companies.

Lin and Cox (2004, [40]) provide a detailed method to securitize the longevity risks in annuity products. The authors use Wang transform (1996, [60]) and (2000, [61]) to price a survivor bond with coupon payments linked to the survivorship of a pooled annuity portfolio. The thresholds of survivorship in each period are projected using Renshaw's GLM mortality model (1996, [50]) based on the U.S. mortality experience (1963, 1973, 1983 and 1996 US individual annuity mortality tables). However in their model, the investors are exposed to "basis risk" or "cohort risk" which is the risk that the mortality experience of the linked annuity pool could deteriorate significantly more than that of the mortality tables. Denuit, Dhaene, Le Bailly de Tilleghem and Teghem (2005, [22]) also undertook a pricing of survivor bond by projecting the future mortality rates using the Lee-Carter model (1992, [38]).

2.3.2 Survivor swaps

Dowd, Blake, Cairns and Dawson (2006, [25]) recommend another form of mortality-based security: survivor swaps. The authors define a survivor swap as "an agreement to exchange cash flows in the future based on the outcome of at least one survivor index" and list several advantages that survivor swaps have over survivor bonds. This include cost effectiveness and flexibility. Survivor swaps can be arranged at lower transaction costs than a bond issue and are more easily cancelled. They are more flexible and they can be tailor-made to suit diverse circumstances. Furthermore, they do not require the existence of a liquid market, just the willingness of counterparts to exploit their comparative advantages or trade views on the development of mortality over time.

Survivor swaps were discussed briefly by Blake (2003, [7]) and Dowd (2003, [24]), and in more detail by Dawson (2002, [21]) and Lin and Cox in (2004, [40]) and (2005, [39]). In (2005, [39]), Lin and Cox designed a specific mortality swap scheme between a life insurer and an annuity insurer, which they called "natural hedging". They used the same method to price the product as they price a survivor bond using Wang transform. Other than the benefits mentioned above, the authors show that the mortality swap can avoid basis risk problem since there is no need to project the future mortality thresholds. However, the method only holds based on the assumptions that investors accept the same transformed distribution and independence assumption for pricing mortality swaps.

The following are two recent examples illustrating how mortality-based securities are applied to hedge adverse mortality risk and longevity risk.

Example 1 *Swiss Re brevity risk bond*

It is based on the news that in December 2003, Swiss Re issued a bond linking principal payment to adverse mortality risk scenarios. The bond is designed to hedge the brevity risk in its life book of business, i.e. (the dramatic impact that premature death has on mortality rates) the excessive mortality changes of premature death. Brevity risk can be managed with the standard tools as long as there are no correlated mortality surprises.

To facilitate this transaction, Swiss Re set up a special purpose vehicle (SPV), which raised $400 million from investors. This issue was the first floating rate bond which links the return of principal solely to a "mortality index". The maturity of the bond is 4 years, and investors receive a floating coupon rate of US LIBOR plus 135 basis points. This coupon rate is higher than other straight bonds, however the principal payment is at risk if "the weighted average of general population mortality across five reference countries (US, UK, France, Italy and Switzerland) exceeds 130% of the 2002 level". Since mortality is generally improving over time, the probability of such high mortality are very low. so investors obtain a high coupon rate in return for assuming the risk.

The Swiss Re transaction is the first securitization that focus only and directly on the mortality risk. The transaction ignored all the other risks and cash flows in the life insurance business lines. In some sense, it is the first "pure" mortality bond. What is more, linking the return solely to a "mortality index" avoid the moral hazard from the issuer and can diversify the risk as great as possible. The disadvantage of the index linking is that it could introduce basis risk to the insurer, which is discussed in detail in Section (4.1.2).

Example 2 *EIB survivor bond*

In November 2004, the European Investment Bank (EIB) issued a longevity bond. This bond involves "time t coupon payments that are tied to an initial annuity payment of £50 million indexed to the time t survivor rates of English and Welsh males aged 65 in 2003". Unlike the Swiss Re brevity risk bond dealing with the extreme short-term adverse mortality risk, the EIB bond can be used to hedge against the long-term longevity risk since coupon payments are tied to a survivor index. It was reported that the total value of the issue was $540 million, and was primarily intended for purchase by UK pension funds.

Although the EIB survivor bond successfully realized the concept of survivor bond, which had been discussed for several years in the academia, it has the same problem as Swiss Re brevity risk bond. As the coupon rate is linked to the survivor index and there is mismatch between the index experience and the annuity issuers' individual mortality exposure, it does not provide a perfect hedge for the annuity issuers.

2.4 Economic justification of securitization of mortality risk

The benefits of mortality risk securitization has been extensively covered by Blake, Burrows, Dawson and Dowd ([6]), ([7]), ([21]), ([24]) and ([25]). The basic idea is that through mortality securitization, mortality risks can be distributed more efficiently over the whole economy. Cox (2000, [17]) identifies other benefits as well, “Securitization provides access to larger amounts of coverage. Counter-party risk is eliminated with securitization. Securitization may provide more favorable tax treatment. The special purpose reinsurer is usually located in a jurisdiction which allows favorable tax treatment of reserves.”

Cox (2000, [17]) also provides an economic justification of risk securitization based on Markowitz mean-variance portfolio theory. The Markowitz portfolio theory has been extensively covered in the academic literature and applied in the finance industry since the 1960's. An excellent introduction of the Markowitz model can be found in Luenberger (1998, [41]). In this section, following Cox's approach I show that the efficiency of the financial market is improved by introducing a mortality-based security to the financial market. First, a brief explanation of the Markowitz model (1959, [42]) is provided below.

Suppose a risky asset has a random rate of return r over period (t, T). The expected return of this asset is denoted as $E(r)$ and its variance is $Var(r)$. Now consider a portfolio P containing n risky assets, with percentage w_i invested in the i-th asset. The return of the i-th asset is r_i respectively. If the covariance of any of two assets r_i and r_j is $\sigma_{i,j}$, then the variance of the whole portfolio can be expressed as a n by n symmetric matrix denoted by $\sum = [\sigma_{i,j}]$. The diagonal elements of $\sum$ are the variances of each return σ_i^2. Given the weight vector of the portfolio P

$$W^T = [w_1, w_2, ..., w_n],$$

the portfolio return is calculated as:

$$r_W = \sum_{i=1}^{n} w_i r_i.$$

Taking expectations on both sides, the expected portfolio return $\mu_W = E(r_W)$ can be calculated as follows:

$$\mu_W = \sum_{i=1}^{n} w_i E(r_i).$$

The variance $\sigma_W = Var(r_W)$ is a product of weight vector W^T and the covariance matrix $\sum = [\sigma_{i,j}]$:

$$\sigma_W^2 = \sum_{i=1}^{n} w_i \sum_{i=1}^{n} w_j Cov(r_i, r_j)$$

or

$$\sigma_W^2 = W^T \sum W.$$

2.4.1 Efficient portfolios

"InvestorWord" defines Efficient Portfolio as: "A portfolio that provides the greatest expected return for a given level of risk, or equivalently, the lowest risk for a given expected return. Also called optimal portfolio." Figure (2.3) illustrates the concept of efficiency and portfolio dominance.

Note that portfolio A dominates portfolio B since it offers the same variance but has a higher expected return. Similarly, portfolio A dominates portfolio C because it offers the same expected return but a lower variance.

The Markowitz portfolio problem attempts to find the "efficient portfolio", that is the portfolio with the maximum return for a given portfolio variance or the portfolio with the minimum variance for a given portfolio return. Take the above portfolio P for example. If a required portfolio expected return μ is given, then the aim is to minimize the variance $\sigma_W^2 = W^T \sum W$ subject to the two constraints:

$$\sum_{i=1}^{n} w_i = 1$$

and

$$\sum_{i=1}^{n} w_i E(r_i) = \mu.$$

The first constraint requires that the whole portfolio P is invested in risky assets and the second set the expected return of P as required. This problem can be solved by using the Lagrange multiplier,

$$L = W^T \sum W + \lambda_1 \left[\mu - W^T E(r_i)\right] + \lambda_2 \left(1 - \sum_{i=1}^{n} w_i\right).$$

Differentiating L to get the first order conditions:

$$\sum_{j=1}^{n} \sigma_{i,j} w_i - \lambda_1 E(r_i) - \lambda_2 = 0 \text{ for } 0 \le i \le n$$

$$\sum_{i=1}^{n} w_i E(r_i) = \mu$$

$$\sum_{i=1}^{n} w_i = 1.$$

This system of linear equations can be expressed as a single matrix equation:

$$\sum{}^{*} [W, \lambda_1, \lambda_2]^T = [0, ..., \mu, 1]^T$$

where

$$\sum{}^{*} = \begin{bmatrix} \sigma_{1,1} & \sigma_{1,2} & ... & \sigma_{1,n} & -E(r_1) & -1 \\ \sigma_{2,1} & \sigma_2^2 & ... & \sigma_{2,n} & -E(r_2) & -1 \\ & & ... & & & \\ \sigma_{n,1} & \sigma_{n,2} & ... & \sigma_n^2 & -E(r_n) & -1 \\ E(r_1) & E(r_2) & ... & E(r_n) & 0 & 0 \\ 1 & 1 & ... & 1 & 0 & 0 \end{bmatrix}$$

This system consists of $n + 2$ equations for $n + 2$ unknowns. The solution of these equations will produce a weight vector for an efficient portfolio with the required expected return.

2.4.2 Efficient frontier and Two-fund theorem

An efficient frontier is simply the line comprised of n efficient portfolios. Given n expected returns μ, using the Lagrangian method, a corresponding set of n minimum σ can be solved. An efficient frontier can be plotted with the set of (σ_W, μ). According to the two-fund theorem given in Luenberger (1998, [41]), "Two efficient funds (portfolios) can be established so that any efficient portfolio can be duplicated, in terms of mean and variance, as a combination of these two. In other words, all investors seeking efficient portfolios need only invest in combinations of these two funds." A simple proof is provided below.

Consider a bond portfolio P^1 and a stock portfolio P^2, both of which are efficient. P^1 has expected return and volatility (μ_1, σ_1) and weight vector W^1. P^2 has expected return and volatility (μ_2, σ_2) and weight vector W^2. $\mu_2 > \mu_1$, $\sigma_2 > \sigma_1$.Given any required return μ with $\mu \geq \mu_1$, a portfolio P^3 can be formed with weights W^3, where W^3 satisfy Equation (2.1).

$$\left[W^3, \lambda_1^3, \lambda_2^3\right]^T = a\left[W^1, \lambda_1^1, \lambda_2^1\right]^T + (1-a)\left[W^2, \lambda_1^2, \lambda_2^2\right]^T \qquad (2.1)$$

where

$$a = \frac{\mu - \mu_1}{\mu_2 - \mu_1}.$$

It is easy to verify P^3 to be efficient. Because P^1 and P^2 are efficient, first order conditions (2.2) and (2.3) hold.

$$\sum^{a} \left[W^1, \lambda_1^1, \lambda_2^1\right]^T = [0, ..., 0, \mu_1, 1]^T, \tag{2.2}$$

and

$$\sum^{a} \left[W^2, \lambda_1^2, \lambda_2^2\right]^T = [0, ..., 0, \mu_2, 1]^T. \tag{2.3}$$

It is straightforward to obtain (2.4) from (2.2) and (2.3),

$$\sum^{a} \left[W^3, \lambda_{1,}^3 \lambda_2^3\right]^T = [0, ..., 0, \mu, 1]^T. \tag{2.4}$$

Thus (2.4) shows P^3 is a efficient portfolio. The return of P^3

$$r^3 = ar^1 + (1-a)r^2,$$

has an expected value $\mu = E[r^3]$, and a minimum variance $\sigma^2 = Var(r^3)$. Therefore given any required return μ, a new efficient portfolio can be formed from P^1 and P^2 with weight a. In other words, every efficient portfolio can be formed as a weighted average of the two other portfolios. Figure (2.4) illustrates the efficient frontier established from P^1 and P^2.

2.4.3 One-fund theorem and CML

After introducing a risk-free security R with deterministic return r_f and 0 variance to the market, although the investor's opportunity set is expanded, the shape of the efficient frontier is simplified. The efficient frontier is a line constructed as a combination of the risk-free asset and an efficient portfolio F of risky assets with weights $W^T = [w_1, ..., w_n]$. This line, which is also called the "capital market line" (CML), can be expressed mathematically as

$$\mu = r_f + \frac{r_F - r_f}{\sigma_F}\sigma.$$

The CML can be obtained by investing the proportion

$$a = \frac{(\sigma_F - \sigma)}{\sigma_F^{-1}}$$

in the risk-free asset R and $(1 - a)$ in F. The CML is illustrated in Figure (2.5).

Figure (2.5) shows that the capital market line is tangent to the original efficient frontier. With the exception of the tangent point F (also called "market portfolio"), all the portfolios in the CML are more efficient than those on the original efficient frontier. According to Luenberger (1998, [41]), the one-fund theorem states: "There is a single fund F of risky assets such that any efficient portfolio can be constructed as a combination of the fund F and the risk-free asset." It is optimal for mean-variance optimizing investors to hold a portfolio formed as a weighted average of F and R, for a certain weight a. This optimized portfolio lies on the CML and gives a return as $ar_f + (1-a)r_F$. The one-fund theorem is the final conclusion (the optimal application) of Markowitz mean-variance portfolio theory. Based on the Markowitz model, I will show that introducing mortality-based securities will improve market efficiency.

2.4.4 A more efficient market with mortality-based securities

Now we introduce a mortality-based security M, for example a bond with a higher expected return μ_M and higher variance σ_M^2 than a straight bond. Since mortality rates have nothing to do with the performance of the capital market and vice versa, this mortality bond has no correlation with other risky assets, that is $Cov(r_M, r_F)$ $= 0$.

Consider a portfolio K formed as a linear combination of market portfolio F and mortality-base security M with a return

$$r_K = ar_F + (1 - a)r_M.$$

The expected return of portfolio K is

$$\mu_K = a\mu_F + (1 - a)\mu_M,$$

and because $Cov(r_F, r_M) = 0$, the variance of portfolio K is simply

$$\sigma_K^2 = a^2\sigma_F^2 + (1-a)^2\sigma_M^2. \tag{2.5}$$

Suppose that $\sigma_M > \sigma_F$ and $\mu_M > \mu_F$. Equation (2.5) is quadratic and is minimized at

$$a = \frac{\sigma_M^2}{\sigma_F^2 + \sigma_M^2}.$$

The minimized σ_K^2 is

$$\sigma_K^{2*} = \frac{\sigma_M^2\sigma_F^2}{\sigma_F^2 + \sigma_M^2} \leq \sigma_F^2.$$

It is straightforward to show that the expected return μ_F

$$\mu_F \leq \mu_K.$$

Hence the new market portfolio $K^*(\mu_K, \sigma_K^*)$ formed by the mortality bond M and F dominates the original market portfolio F (σ_F, μ_F). The slope of the new CML is at least $\frac{\mu_K - r_f}{\sigma_K^*}$ since it must dominate all the feasible portfolios. Also it is clear that

$$\frac{\mu_K - r_f}{\sigma_K^*} \geq \frac{r_F - r_f}{\sigma_F},$$

Thus the slope of the new CML must be greater than the slope of the original one through F. A similar argument applies if $\sigma_M < \sigma_F$ and $\mu_M < \mu_F$. Thus the mean-variance portfolio theory shows that the introduction of the mortality bond results in better opportunity set for investors, and justifies the demand for the mortality bond. The two CMLs are illustrated in Figure (2.6).

The readers should be aware that the Markowitz model is based on strict assumptions: First, there is no collinearity. That is, among the set of n risky assets, no single asset is a linear combination of the others. This restraint ensures that the n risky assets have an invertible covariance matrix. Second, the market is perfect: there are zero transaction costs; information is instantaneously revealed; there are no taxes and so on.

Cox (2000, [17]) also identified some benefits of securitization of insurance risk including elimination of counterpart risk, alleviation of capital strains, more favorable tax treatment and the capacity to handle very large losses. These benefits can certainly be applied to mortality securities. Many mortality risk deals provide longer term coverage compared to traditional reinsurance and mortality securitization also brings more capital to cover risks that would not be covered otherwise.

Mortality based securities pushed the market to a better equilibrium. Through mortality-based securities, investors can obtain the benefit of investment risk diversification and duration immunization. As mortality risk is independent of "market risk", a mutual fund could hold some mortality-based securities to diversify the market risk of its portfolio. Life insurance companies might also be interested in holding such securities to hedge against the adverse mortality risks. Through mortality-based securities, risks are distributed more efficiently throughout the financial market. The reverse mortgage lender can transfer the longevity risks to the capital market.

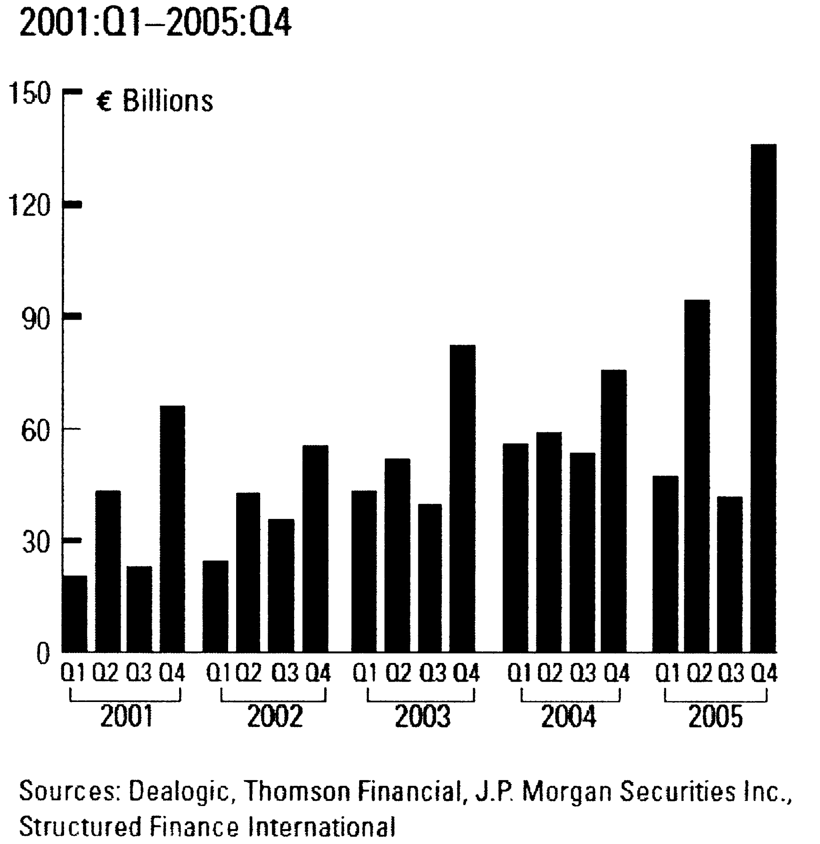

Figure 2.1: European securitization volume in the last decade

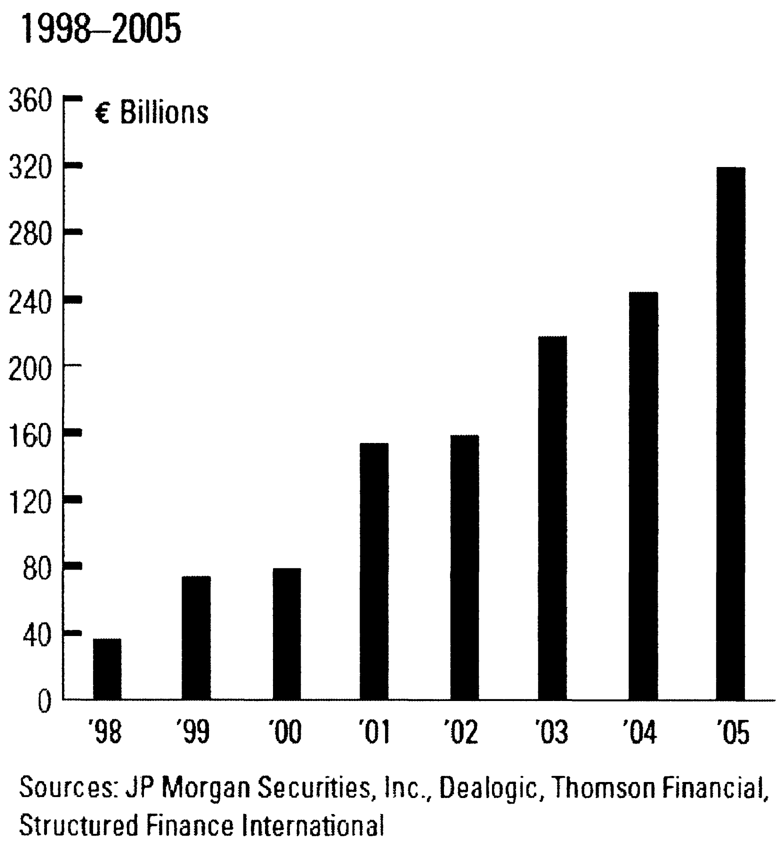

Figure 2.2: European securitization volume in 2005

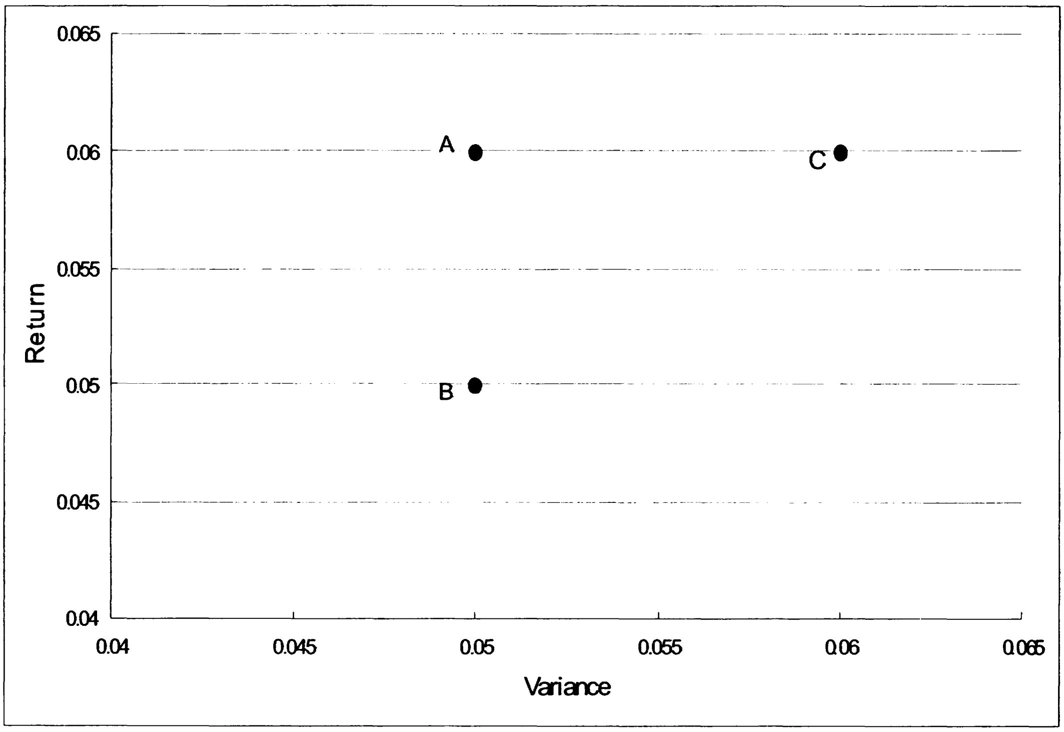

Figure 2.3: Portfolio dominance.

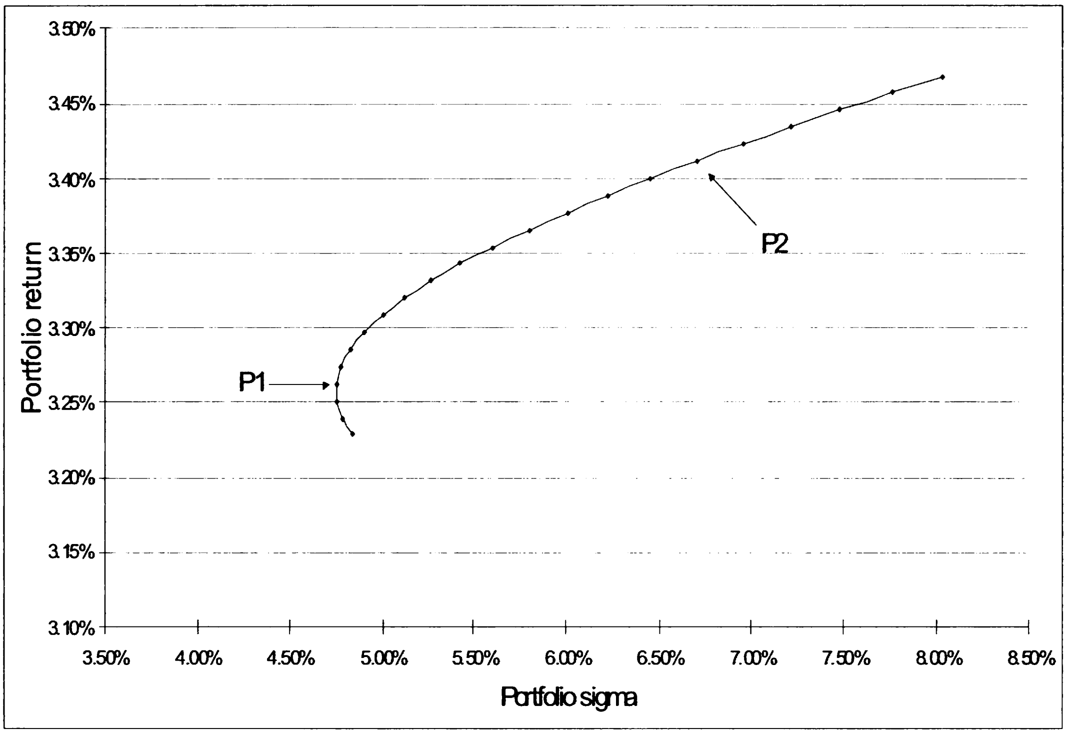

Figure 2.4: An efficient frontier

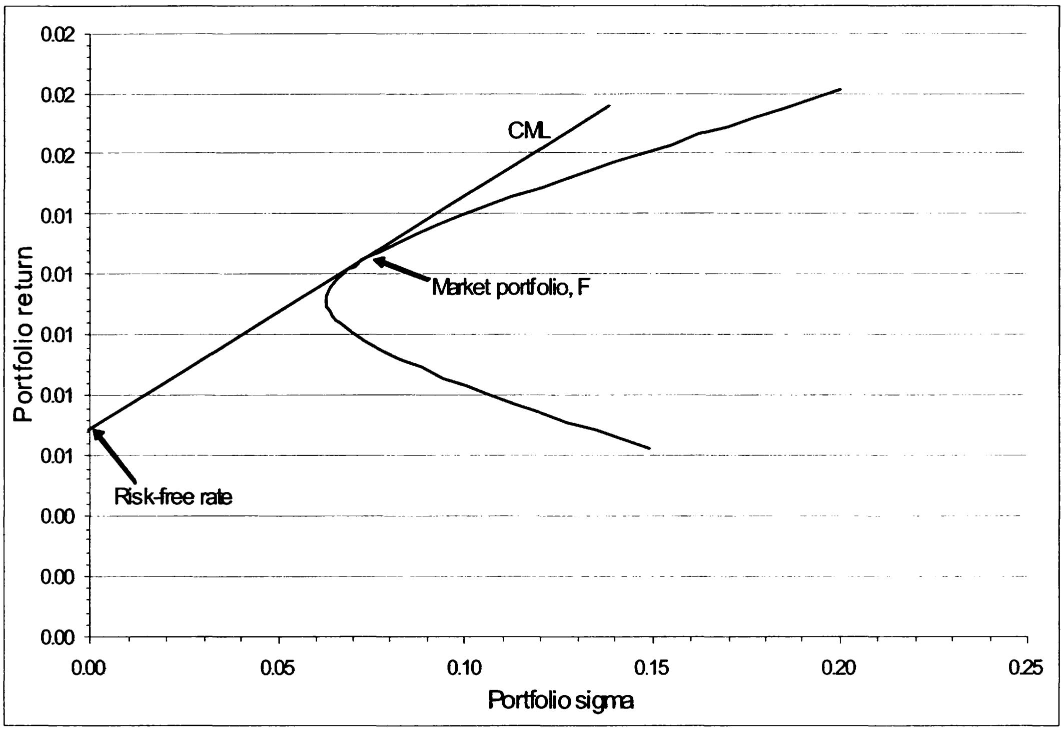

Figure 2.5: An efficient frontier with a CML

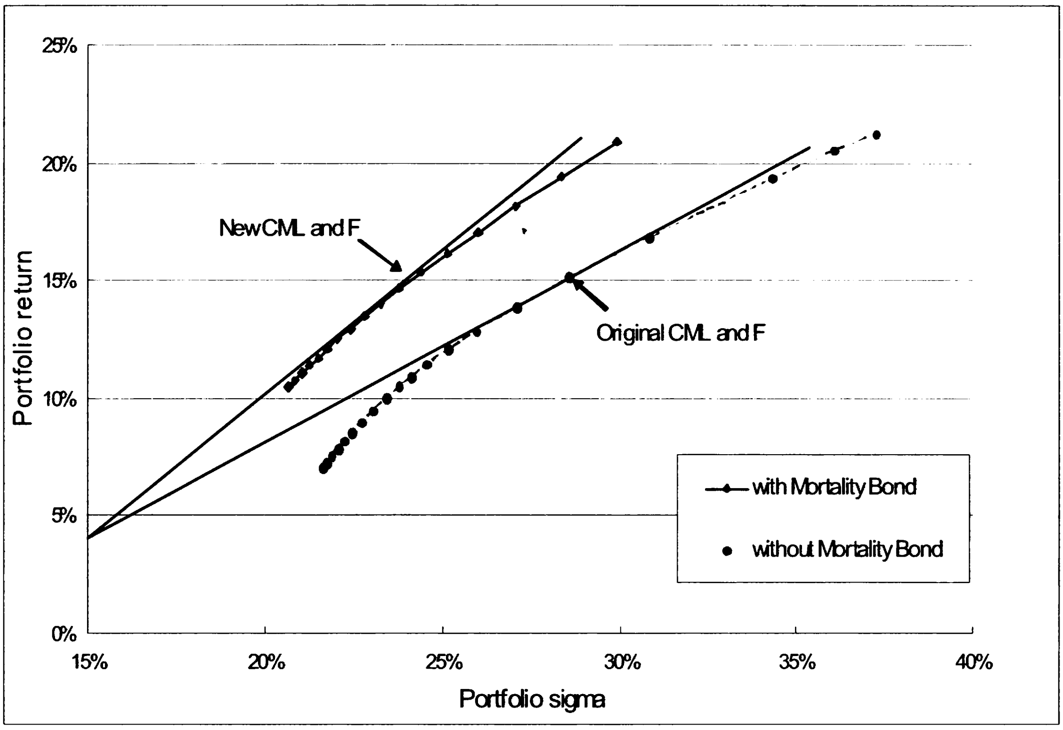

Figure 2.6: Comparing the two efficient frontiers

Chapter 3

Reverse mortgage products

3.1 Introduction of reverse mortgages

3.1.1 Ageing population and longevity risk

Today, while life expectancy is over 80 years in many countries, labour market participation rates for males aged 60-64 have fallen from between 70 and 90 per cent in the 1970s to between 20 and 50 percent (2001, [5]). According to Australian government actuaries, the life expectancy of a male aged 62 has almost doubled during the last 100 years from 13 years in the period 1881-1890 to 20 years in the period 2000-2002. (See Figure (3.1) .) The tendency towards increasing longevity has grown stronger since 1970 with life expectancy increasing 50 percent in the last twenty years alone.

While life expectancy has been increasing, decreasing real retirement income due to inflation and increasing health care costs has made it increasingly difficult for the elderly to maintain adequate standards of living during retirement. Consequently, longevity risk, the risk that the elderly may outlive their financial assets has become the subject of increasing attention from financial practitioners and academics. The traditional way to deal with longevity risk has been the annuity product; however, despite the apparent utility gain of the annuity product, the private annuity market has remained weak due to obstacles such as adverse selection. The weak annuity market has puzzled economists for decades. One reason put forward by the researchers is that retirees usually hold a most of their wealth in their home equity and have few liquid assets in hand. In some developed countries such as the United States, Japan and Australia, residential real estate represents more than half the value of private wealth of the elderly population (2003, [45]). With home ownership and the elderly population steadily expanding in many western countries, reverse mortgages represent a viable way for these house-rich, cash-strapped seniors to supplement their income and afford long-term care while remaining in their homes.

A reverse mortgage is a non-recourse first lien loan that enables retirees, usually 62 years or older, to convert part of their home equity into tax-free income while remaining in their homes until they die or sell their homes. Payment options to borrowers include an up-front lump sum payment; a fixed monthly annuity payment; a line of credit with an upper limit and various combinations of these three types. The loan amount accrues interest at a variable interest rate, usually adjusted monthly or yearly based on an index rate, and is repaid only when the borrower dies or sells or permanently leaves the home. During the course of the loan, the title of the residence is retained by the borrower; no repayments are made to the lender; and no assets other than the home may be attached to debt repayment. When the loan is due, the repayment amount is capped by the value of the home and if the borrower chooses to sell the home, she keeps any excess sales proceeds (2000, [1]).

3.1.2 History of reverse mortgages

The first reverse mortgage loan was issued by Nelson Haynes of Deering Savings & Loan (Portland, ME) in the United States in 1961. Only a few banks offered reverse mortgages until the product was approved for sale by the Federal Home Loan Bank Board in 1979 (1994, [14]). In 1989, after the Federal Housing Administration (FHA) introduced of the Home Equity Conversion Mortgages (HECM) program, reverse mortgages covered by public insurance became widely available in the United States.

Despite the great potential of reverse mortgages, the market grew very slowly until the beginning of the century. In seeking to explain this slow growth, Eschtruth and Tran (2001, [26]) listed several factors that depressed both demand and supply. On the demand side, the most important factor was that the up-front costs were too high. In 1999, the total up-front costs could add up to $10,000 to a median HECM loan ranging from $52,500 to $63,000, which is nearly 20% of the loan amount! Another major factor was that the limits on the size of HECM loans set by the FHA were too restrictive. Other reasons include that fear of debt (2000, [12]), concerns about future medical expenses, and the bequest motive. The bequest motive also explains why the demand for annuity reverse mortgages is lower than that for other forms of reverse mortgage loans (2003, [45]).

The low demand in turn discouraged supply. The expenses associated with the product are quite considerable when demand is low. Many lenders couldn't generate enough loans to covered costs of educating consumers, developing the market and maintaining trained staff. As a result, they had to exit the market. Legal issues were also a concern for the suppliers. According to Reed and Gibler (2003, [49]), regulations and laws governing the market were inadequate during the early stage of the market. For example, the wording of some of the requirements for the lender was often confusing. What is more, according to Boehm and Ehrhardt (1994, [8]), that required accounting techniques also made the loans financially unattractive. Last but not least, in the infancy of the market, the risk involved in a reverse mortgage was not insurable. According to Reed and Gibler (2003, [49]), the lenders also had to hold the products in their portfolios since there was no secondary market for liquidation.

However, many of these obstacles could be overcome if demand for reverse mortgages were strong. For example, higher demand would allow lenders to reduce their costs and would create a constituency which could lobby for the elimination of the remaining regulatory barriers and clarification of tax issues. After a decade of slow growth, the reverse mortgage market started to boom in the 21st century. According to the National Reverse Mortgages Lenders Association, nearly 40,000 new HECM loans originated in the years between 2000 to 2003. During 2004, a total number of 37,829 of HECM loans were approved, representing a 109% jump over the previous year, and nearly 500%growth since 2001. The growth continued in 2005 with a total number of 43,131 HECM reverse mortgages, which increased 14% from the year 2004. The number of RM lenders has tripled to 191 (2004, [57]). As HECMs are the most popular of the three reverse mortgage products currently available, accounting for about 90% of all reverse mortgages in the US today, the above statistics well represents the condition of the reverse mortgage market. The boom has been attributed to greater consumer awareness of and comfort with the product; and the increasingly tight budgets of America's seniors; the increase in home ownership and the growth of the elderly population. While the market has shown considerable growth, as of in 2000, only approximately 0.25% of elderly homeowners had entered into reverse mortgages contracts, indicating far more potential for growth in the market.

3.2 The merits of reverse mortgages

The most notable merit of a reverse mortgage is that the borrower is not required to repay the loan until he/she dies or vacates his/her house. As compared to normal asset-backed loans, reverse mortgage loans provide the elderly with a means of hedging their longevity risk. Another merit of a reverse mortgage is the "non recourse" feature. When the loan is terminated, the borrower only needs to repay the loan amount or proceeds from the sale of the house price: whichever is the lesser sum.

In a broader sense, allowing the elderly to consume the equity in their homes could help to stimulate the national economy. For instance, the unlocked equity could be spent on a holiday or a car, thus helping to maintain the consumption level of the economy (2003, [49]). Reverse mortgages may provide many other indirect benefits for society, although these are often difficult to predict and even harder to measure.

3.3 Various risks in reverse mortgages

While reverse mortgage loans provide many attractive benefits to the borrower, they involve many risks for the provider (lender). The most crucial risk in reverse mortgage loans is the "cross-over" risk. As the loan balance accumulates faster than the value of the borrower's home equity, the loan balance will in time exceed the value of the home equity. If the cross-over occurs before the loan is due, the lender will incur a loss. The cross-over risk is a combination of three underlying risks: the longevity risk, the interest risk and the house price risk. Improved mortality rates, a high interest rate, and a depressed real estate market will all exacerbate the cross-over risk. The three variables are the major inputs for the pricing model and will be analyzed in detail in Chapter 4.

In addition to the three major risks, other important risks include maintenance risk and expenses risk. Maintenance risk is sometimes called "moral hazard". Maintenance risk arises when the reverse mortgage borrowers fail to make the necessary repairs to maintain the value of their homes because they know that the lender bears the risk of the declining home resale value. Maintenance risk is discussed by Miceli and Sirmans (1994, [43]). The authors studied the optimal maintenance decision of the borrowers by maximizing a two-period utility model in a competitive market. Their model shows that the loan amount and expected state of the real estate market are the key factors affecting the maintenance decision. Their model also indicates that the best way to ensure the outcome for the lender results when either a limit of lending is set or a extra premium is charged to cover the risk. Shiller and Weiss (1998, [54]) presented another calibrated model for assessing this moral hazard risk. They argue that the risk might be reduced if the contracts are redesigned so that the settlement is determined by real estate price indices, rather than in terms of the sale price of the home itself.

One important risk which does not attract much attention is the expense risk. A good discussion on this topic can be found in Piggott, Mitchell, Valdez and Detzel (2003, [48]). The cost to enter the market is usually very large for a new comer. Considerable expenses may be associated with the marketing of the product and to educate potential buyers. After the loan is assessed and issued, there are costs associated with administrating the loan and complying with regulatory requirements. These include the ongoing monitoring expenses to ensure the borrower pays property taxes and insurance fees. When the homeowner dies or moves, there arise costs of inspection, evaluation and sale of the property. These expenses could be recovered by charging an up-front fee and ongoing administrating fees. Inaccurate assessing of the future expenses could result in fail to recover all the expenses incurred in product administration.

The subject of reverse mortgages has been covered extensively in the literature. Bartel, Daly, and Wrage (1980, [4]) published one of the first articles on reverse mortgages, which described the features of early reverse mortgage product. In 1987, the U.S. Housing Authorization Act was passed which contained provision for the establishment of a guarantee fund to cover losses in a reverse mortgage portfolio. Following the instruction of the US Housing Authorization Act, Weinrobe (1988, [62]) investigated the concept and operation of the guarantee fund, and calculated the cost of providing the guarantee fund. In the early 1990's two papers regarding the pricing and financial modeling of reverse mortgages appeared in the actuarial literature. Using Monte-Carlo simulation method, DiVenti and Herzog (1991, [23]) investigated the actuarial perspectives of the product, by estimating the amount of level payment in a life annuity reverse mortgage. In addition to quantitatively assessing the risks involved in a reverse mortgage, Phillips and Gwin (1992, [47]) introduced the idea of asset reserving to protect the solvency of a reverse mortgage lender.

Several papers on the topic of risks of reverse mortgages can be found in the special issue published by the Journal of the American Real Estate and Urban Economics Association in 1994. Boehm and Erhardt (1994, [8]) analyzed the interest risk in reverse mortgages. The authors calculated Mcaulay's duration and elasticity of a bond, a conventional mortgage and a fixed-rate annuity reverse mortgage. The comparison shows a reverse mortgage is much more sensitive to interest rate changes than the other products, which explains why reverse mortgages mostly involve variable interest. Szymanoski (1994, [56]) developed a general loss model to determine the fair premium that the HECM program should charge. The pricing model shows that the present value of the expected loss should never exceed the present value of expected insurance premiums. Using a simulation technique, he calculates the highest loan-to-value ratio of a reverse mortgage, which depends on age, interest and house price appreciation assumptions. Klein and Sirmans (1994, [36]) undertook an empirical research to determine the key drivers for the termination of the loans. They analyzed data offered from the reverse mortgage program by the Connecticut Housing Finance Authority, which provides term annuity reverse mortgage loans to low-income residents in Connecticut aged 68 years or older. The authors regressed the loan repayment rate on borrower and loan characteristic variables, and found that a borrower's age and martial status played the most important role in the termination of a reverse mortgage loan.

3.4 Transferring the mortality risk in reverse mortgages

A traditional method to deal with the risks in reverse mortgages is insurance. Recently, securitization has been applied to reverse mortgages. In March 2001, the first European reverse mortgage securitization transaction worth GBP 222.5 million was launched by Citibank N.A. in Europe. The offer was given a preliminary rating of AAA by Standard&Poor (2001, [55]). In October 2004, the United States amended the Internal Revenue Code of 1986 to eliminate financial asset securitization investment trusts (FASITs) and to expand the real estate mortgage investment conduit (REMIC) rules to allow securitization of certain mortgages with increasing balances and certain government-originated loans. The REMIC amendments are designed to facilitate the securitization of reverse mortgages. These transactions are similar to mortgage-backed securitizations, which sell the consolidated asset to release the capital. This thesis propose a risk securitization model only focusing on transferring the longevity risk in reverse mortgages.

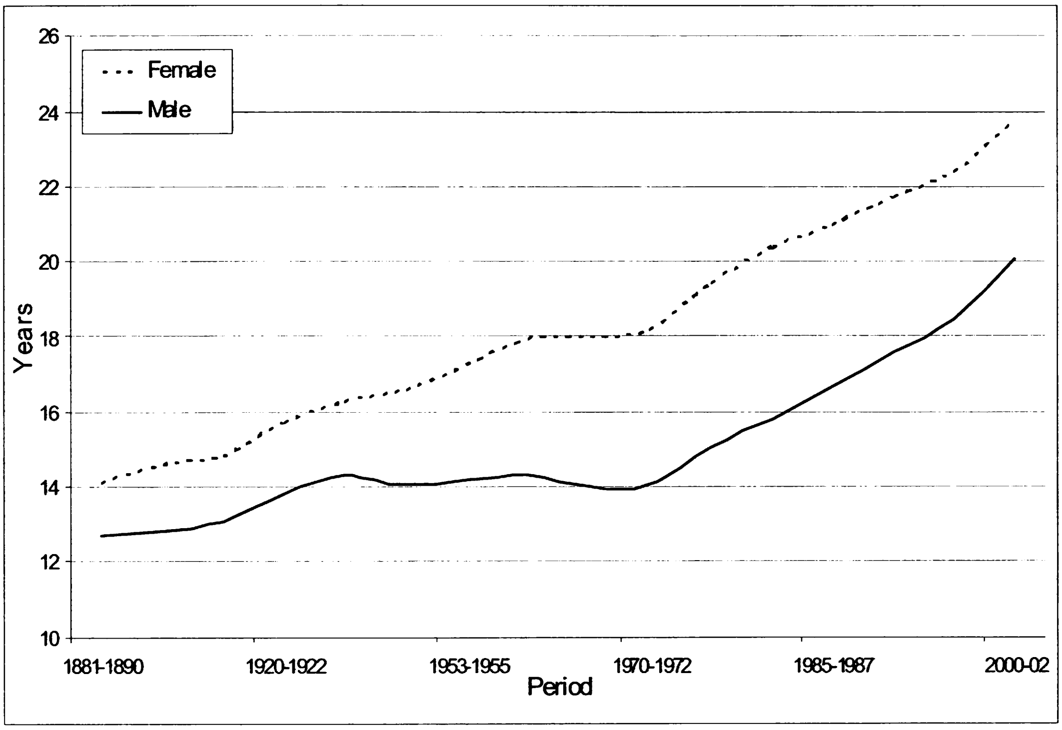

Figure 3.1: Life expectancies for Australian aged 62

Chapter 4

Pricing reverse mortgages

4.1 Risk analysis for reverse mortgage providers

4.1.1 The cross-over risk

When a reverse mortgage loan is repaid, the borrower only needs to repay the proceeds of the sale of the house or the balance of the loan amount: whichever is the lesser amount. In the case that the loan value has outgrown the collateral house value, the borrower only repays the proceeds of sale the house. Any access is considered a loss to the lender. Since the interest rate is usually higher than the house price appreciation rate, the loan value will definitely exceed the house value at some future point. The question is whether the cross-over happens before the loan is terminated or after. If the cross-over occurs before the loan termination, the lender will incur a loss. However, if the loan is terminated before the cross-over, any excess proceeds from the sale will go to the borrower or their heir, rather than becoming the lender's gain. This characteristic makes reverse mortgages more similar to options rather than futures.

From the reverse mortgage provider's perspective, avoiding loss is basically avoiding cross-over before the loan is repaid. The timing of the cross-over is mainly affected by three crucial variables: mortality rate, interest rate and house price. To illustrate their effects on the loss to the lender, a simple example is provided below. In this scenario it is assumed that the interest rate is flat at 6% annually, and the house price appreciation rate is 3% per year. The loan-to-value ratio is

0.5, allowing a 62-year old male to borrow a lump sum amount of $\$200,000$ against his house which is currently valued at $\$400,000$.

Figure (4.1) illustrates the cross-over risk. If the loan is repaid before the cross-over point, there is no loss to the lender. However, if the loan is repaid after the cross-over point, the difference between the balance of the loan and the house price is the loss to the lender for that particular year. Assuming the lender finances the capital at an interest rate of 4% per annum, Figure (4.2) illustrates the net cash flow to the lender in each year. After the cross-over point, the net cash flow becomes negative.

In some sense, to price a financial product is just to price its risks. In the following section, the three major risks — longevity risk, interest rate risk, and house price risk — are thoroughly examined for the purpose of pricing.

4.1.2 Longevity risk and occupancy risk

The occupancy risk is the risk that the borrower could live in the house too long so that the loan value accumulates to a point where it exceeds the house value. The occupancy risk is believed to be independent of the other two major risks, as life expectancy in most cases does not interact with the market variables of interest rates and house prices. The occupancy time is a function of both mortality rate and the mobility rate.

Although the decision to move and repay the loan may be affected by the condition of the real estate market and the interest rate, the real attractiveness of the reverse mortgage loan is that the product allows the borrower, (who is usually elderly low-income and doesn't want to move), to stay in their home until they die. Some US researchers, like DiVenti and Herzog (1991, [23]), assume that the mobility rate is approximately proportional to the mortality rate. The same assumption is employed in this thesis in the absence of available data. Thus the duration of the loan is mainly determined by the mortality rate. Due to the dramatic improvement in the mortality rate since the 1970's, the longevity risk has become the most crucial risk in reverse mortgage product.

Because of the great mortality improvement in the last decade, projecting future mortality has become the subject of increasing attention from academia and practitioners. However, to predict the human being's future mortality is very difficult. The simple belief that life expectancy will continue to increase indefinitely may be too optimistic. As we can see from Figure (4.3), although the force of mortality generally improves over time, sometimes it deteriorates. Lin and Cox (2004, [40]) summarized the different opinions about mortality trends and discussed the difficulties of predicting future mortality changes. A summary is provided below.

First of all, although mortality rates have been observed to improve for more than a decade throughout the world, not all academics believe that human mortality rates will continually improve. For one thing, Lin and Cox suspect that the recent improvement in mortality could be exaggerated. The dramatic improvements for certain age groups in the U.S. may be in part the result of the adjustments to previous mortality decline. What is more, the mortality rates for different age groups have not all improved. For example, Goss, Wade, and Bell (1998, [30]) show that the age-adjusted annual death rates for ages 85 and over in the United States actually deteriorated between 1990-1994. Secondly, medical and technological advances have not eliminated all the severe threats to life and of course there is always the threat of new unforeseen diseases. SARS in 2003 and avian flu in 2004 are examples of infectious diseases which could severely impact on mortality rates. According to Rogers (2002, [52]), it is possible that HIV may expand or develop variants. There is always the chance that new viruses may emerge due to the abuse of drugs and medicines.

Even within the group believing that future mortality will continually improve, researchers can not agree on the extent to which mortality will improve. First, they can not agree on whether there exists a natural limit for human life expectancy. Some researchers predict that there will be no limit to the age a human can live based on the great progress of the Human Gnome Project and medical research into anti-ageing drugs. Some others, like Hayflick (2002, [31]), believe that there is a limit to human life expectancy and that improvements in the mortality rate will slow down as the limit is approached. Lin and Cox called this "Life table entropy". Buettner (2002, [10]) provides a review and summary of these different opinions.

Lin and Cox conclude "that there is no agreement among experts on the future of mortality. Steady improvement is the trend, but changes in either direction are feasible."

Traditionally, actuaries project future mortality rates using life tables — a kind of parametric model. Based on the Law of Large Number, the longevity risk is believed to be well diversified given a large enough pool. However, solely using the life tables has some shortcomings. The first problem is the "table risk" or "model risk". Most actuarial tables are based on the population census several years prior. The paces at which actuaries and demographers update their life tables may never keep pace with the current mortality rate. Especially in recent years, rapid improvements of health and innovations in medical technology have exacerbated the model risk.

Another problem of the life table approach is the "basis risk": the mismatch of the table experience and the product experience. In the case of reverse mortgages, the major target customers for the product are retirees on low-incomes, many of whom are short of cash for health care and have a shorter life expectancy than that of the average population. Obviously, there is no such experience in the life table. If we use an annuitant table the projected mortality rates could be underestimated, since people who purchase annuities are mostly wealthier and have a relatively longer life expectancy than the general population. Those RM lenders who price the product with annuitant tables could over-charge the borrowers and lose competence in the market. To sum up, if a mortality table has to be used, some appropriate adjustments need to be made to match the customer experience.

Actuaries and statisticians have proposed many ways to solve the problems with the life table. For example, Frees, Carriere and Valdez found that the Gompertz model fits the force of mortality very well in the range of old ages (60 above), and can be used to generate the future force of mortality (1996, [27]). Their model are widely used in the actuarial literature and adopted in economics literature for optimal annuity asset allocation and insurance product pricing; see Milevsky and Young (2003, [44]), for example. The force of mortality in their model is shown only as a function of ages x, not time t, which does not represent the systematic mortality improvement. However, the model could be extended to be stochastic after adding some extra components. For example, Charupat and Milevsky (2001, [13]) attached an time-dependent random shock to the Gompertz model and applied to determine the optimal annuitization polices.

Some other models also treat mortality rate as a function of both age x and time t. Lee and Carter (1992, [38]) introduced a simple model for central mortality rates $m(t, x)$ which involves age-dependent and time-dependent terms. Renshaw, Haberman and Hatzoupoulos (1996, [50]) suggested a Generalized Linear Model for the force of mortality similar to Lee and Carter's approach. Cairns et al. (2004, [11]) provided an excellent review for the various mortality rate models and a theoretical framework to pricing mortality derivatives.

4.1.3 Interest rate and house price risk

Today in the U.S., most reverse mortgage loans carry a variable interest rate. In a high interest rate environment, the loan amount will accumulate at a faster rate than it would in a normal interest rate environment, resulting in the interest becoming a larger part of the repayment. This obviously makes the loan amount more likely to exceed the future house value. Since the loan repayment is capped by the future house price, the excess interest may not be fully recovered from the borrower, resulting in the lender possibly incurring a loss. Figure (4.4) illustrate the effect of interest rates on the lender's loss holding the other variables unchanged. Using the simple example in the last section, if the interest rate raised 50 basis point, the cross-over will occur about 5 years earlier than it would have in a normal interest rate environment. The extent of the loss in each year after the cross-over is also larger if the interest rate rises. Figure (4.5) shows that in a high interest rate scenario, the negative cash flows come earlier and are of a larger magnitude.

Since the interest rate is a nationwide economical variable, the interest risk can only be hedged, not diversified. Interest rate modeling is extensively covered in the finance and economics literature. There are plenty of models, from simple to complex, depending on the situation adopted. One of the most used models is the Vasicek model, which features a mean reversion stochastic interest rate. The Vasicek model is used in the simulation task and discussed in next section.

House prices also play an important part in whether the lender will incur a loss. Since the loan repayment is capped by the house price, the loan repayment could be heavily reduced if the condition of the real estate market is adverse during the course or at the termination of the loan. Figure (4.6)shows the effect of the house price appreciation rate on the lender's loss based on the same example above. Holding the other variables constant, if the house price appreciation rate is 50 basis points lower, the cross-over will be about 5 years earlier than expected, and the amount of loss in each year after the cross-over will also be larger than expected. Figure (4.7) illustrates that in a depressed real estate market, the negative cash flows to the lender come earlier and are on a larger scale than expected.

The house price risk can be partially diversified. The systematic part of the house price is affected only by the fluctuation of the national economy, and consequently is not diversifiable. Such economical variables as interest rates or inflation rates and such catastrophic events as earthquakes or hurricanes can result in changes to the systematic part of the house price. The idiosyncratic part of the house price is affected by the location or maintenance of the house, and therefore can be diversified by a large pool of properties throughout the country. While a "practical" house price model could be very complicated, a simple stochastic house price model is adopted in this thesis to focus on the longevity risk. The house price is assumed to be a geometric Brownian motion, which means the price has no memory and is only affected by random shocks. Although sometimes the house price process presents certain mean reversion feature similar to interest rates, Gau (1987, [28]) found the auto-correlation is statistically insignificant in the long run based on the U.S. experience and argued that the no-memory property should be preserved.

Many researchers believe interest rates and house prices are negatively correlated, at least in a lagged sense. The possible economic explanation is that high mortgage interest rates (adjusted based on the index interest rate) may result in suppressed the demand in the real estate market, thus lowering the house price. If this is true, the reverse mortgages lender can expect more fluctuation in the loss experience. In a high interest rate situation, the house price appreciation will be lower and the loan amount will grow faster. This will lead to a greater loss since the loan repayment is capped by the house price. On the other hand, a low interest rate and a strong real estate market can reduce the possibility of loss. Thus a negative correlation between the interest rate and the house price increases the variance of the lender's loss distribution, which makes the risk management more difficult. The opposite is true for a positive correlation. So it is necessary to test the correlation between the two variables. An empirical analysis is performed in the next section based on Australian 25 years' experience (1980-2005). Quarterly 10-year Australian government bond yields and the average of quarterly median house prices of eight Australian capital cities are used for the test. The result shows that the long-term real (after inflation) interest rate is positively correlated to 1-year lagged house prices and negatively correlated to 2-year lag prices. Both results are statistically significant. The detailed results are provided in the section 4.3.

4.2 Pricing equation

In this section, a lump sum reverse mortgage is priced following an approach adopted by Piggott et. al (2003, [48]). Suppose a 62-year old male takes out a loan to the value of Q_0 dollars, against his house currently valued at H_0 dollars. If at time t the loan amount is Q_t, the house price is H_t, and the cost of the capital is M_t, then by definition the value of the reverse mortgage loan is

$$V_t = Min\left(Q_t, H_t\right),$$

and the loss to the lender L_t is

$$L_t = M_t - V_t$$

$$L_t = M_t - Min\left(Q_t, H_t\right)$$

If the loan amount Q_t accumulates at a risk free interest rate r_t plus a risk premium λ, the house price H_t appreciates at a rate of δ_t and the cost of the capital M_t accumulates at a interest rate of η, then the loan value process is

$$Q_t = Q_0 e^{\int_0^t (r_t+\lambda)dt},$$

the house price process is

$$H_t = H_0 e^{\int_0^t \delta_t dt},$$

and the process of cost of capital is

$$M_t = Q_0 e^{\int_0^t \eta_t dt}.$$

Since the value of the loan repayment V_t is the smaller of the house price and accumulated loan amount,

$$V_t = Min\left[Q_0 e^{\int_0^t (r_t+\lambda)dt}, H_0 e^{\int_0^t \delta_t dt}\right].$$

The loss to the lender L_t at time t thus becomes

$$L_t = Q_0 e^{\int_0^t \eta_t dt} - Min\left[Q_0 e^{\int_0^t (r_t+\lambda)dt}, H_0 e^{\int_0^t \delta_t dt}\right].$$

Suppose the future life span of the loan is a random variable T, then when the loan is repaid the loss L_T is

$$L_T = Q_0 e^{\int_0^T \eta_t dt} - Min\left[Q_0 e^{\int_0^T (r_t+\lambda)dt}, H_0 e^{\int_0^T \delta_t dt}\right] \tag{4.1}$$

According to the actuarial equivalence principle, the present value of total expected loss should be equal to zero,

$$E\left(e^{-rT} L_T\right) = 0. \tag{4.2}$$

Substituting (4.1) into (4.2), the pricing equation of the reverse mortgage is obtained:

$$E\left(e_0^{-rT} Q e^{\int_0^T \eta_t dt}\right) = E\left[e^{-rT} Min\left(Q_0 e^{\int_0^T (r_t+\lambda)dt}, H_0 e^{\int_0^T \delta_t dt}\right)\right] \tag{4.3}$$

With the pricing equation (4.3), given a certain risk premium λ charged by the lender, the maximal safe loan amount Q_0 can be calculated. Or given a certain initial loan amount Q_0, the actuarially fair risk premium λ that the lender should charge can be determined. A numerical example is provided in the next section to illustrate the pricing of reverse mortgages.

4.3 A numerical example

4.3.1 Projecting the mortality rate

In this section, a Gompertz mortality model used by Frees et. al (1996, [27]) is adopted to project the future force of mortality μ_x. In the case of Gompertz mortality, the force of mortality

$$\mu_x = \frac{1}{\sigma} e^{(x-m)/\sigma},$$

where σ is the scale parameter and m is the location parameter. Then the survival probability can be expressed as:

$$_t p_x = e^{-\int_x^{x+t} \mu_s ds}$$

$$ {}_tp_x = \exp\left[-\int_x^{x+t} \frac{1}{\sigma} e^{(s-m)/\sigma} ds\right] $$

$$ {}_tp_x = \exp\left[\sigma\mu_x\left(1 - e^{t/\sigma}\right)\right]. $$

The probability distribution function (PDF) of future life time T of aged x is

$$ f(t) = \frac{d}{dt}({}_tp_x) = -\mu_x \exp\left[\frac{t}{\sigma} + \sigma\mu_x\left(1 - e^{t/\sigma}\right)\right] $$

The repayment rate of the contract is the sum of mortality rate q_{x+t} and mobility rate m_{x+t}. Following the example set by DiVenti and Herzog (1991, [23]), the mobility rate is assumed to be 30% of the mortality rate, where the overall repayment rate is denoted as q^*_{x+t}, then

$$ q^*_{x+t} = q_{x+t} + m_{x+t} $$

$$ q^*_{x+t} = q_{x+t}(1 + 30\%) $$

4.3.2 Projecting interest rates and house prices

In this example the well-known one-factor stochastic Vasicek model is adopted to project the interest rate. The key feature of Vasicek interest model is mean reversion (1977, [59]). Unlike alternative models like the CIR model proposed by Cox, Ingersoll, and Ross (1985, [16]), in the Vasicek model interest rate may actually become negative. The Vasicek model describes the short rate's Q dynamics by the following SDE:

$$ dr_t = \alpha\left(\beta - r_{t-1}\right) dt + \sigma_r dZ_t, \tag{4.4} $$

where Z_t is a standard Brownian motion. A discrete approximation can be expressed as

$$ r_t - r_{t-\Delta t} = \alpha\left(\beta - r_{t-\Delta t}\right)\Delta t + \sigma_r \varepsilon_{r,t}, $$

where $\varepsilon_{r,t} = \left(Z_{r,t} - Z_{r,_{t-\Delta t}}\right)$, is normally distributed as normal with mean 0 and variance Δt. After fitting the model, the future interest rate can be projected as

$$ r_{t_k} = \Delta t\widehat{\alpha}\widehat{\beta} + \left(1 - \Delta t\widehat{\alpha}\right) r_{t_k-\Delta t} + \widehat{\sigma}_r\varepsilon_{r,t}, $$

where $\widehat{\alpha}$, $\widehat{\beta}$, and $\widehat{\sigma}_r$ are fitted parameters.

For the house price model, to distinguish between the diversifiable idiosyncratic price risk and the non-diversifiable systematic price risk, a specific and a general house price appreciation rate are introduced. The basic structure of the house price model is a geometric Brownian motion:

$$d\ln(H_t) = \mu_H dt + \sigma_H dZ_H.$$

where μ_H is the drift and σ_H is the volatility parameter. Using Ito's lemma, this stochastic differential equation can be solved as

$$H_t = H_{t-1}\exp(\mu_H + \sigma_H Z_H),$$

where $Z_H \sim N(0,1)$. A discrete time approximation is

$$H_t = H_{t-\Delta t}\exp\left(\mu_H \Delta t + \sigma_H \varepsilon_{H,t}\sqrt{\Delta t}\right),$$

where $\varepsilon_{H,t}$ are iid (identical independent distributed) standard normal variable $N(0,1)$. With the projected house price, the general house appreciation rate c_t is calculated to represent the systematic part of price risk:

$$c_t = \frac{H_t}{H_{t-1}} - 1.$$

As discussed above, the idiosyncratic part of the house price is greatly affected by the borrower's maintenance behavior and regional economic fluctuation. To capture this characteristic, a random shock is attached to the general appreciation rate to determine the specific house appreciation rate c_t^s for each house:

$$c_t^s = c_t + \sigma_s Z_s,$$

where $Z_s \sim N(0,1)$. It is obvious that c_t^s is a normal variable with a mean of general house appreciation rate c_t and a variance σ_s^2. Finally each house has its own price process expressed as

$$H_t = H_{t-1}(1 + c_t^s)$$

4.3.3 Calibration of the models

Australian Life Tables 2000-02 for male in the age range of 62-100 are used to calibrate the Gompertz mortality model. Using iterative trials in MS *Excel*, the

Parameters	Values
Mean	4.16%
α	0.5757
β	4.8825
σ_r	4.7891

Table 4.1: Fitted parameters for the Vasicek model

Parameters	Values
μ_H	0.034
σ_H	0.1003
σ_s	0.08

Table 4.2: Fitted parameters for the house price model

location parameter m is found to be 82.119 and the scalar σ is 9.786. The life expectancy for the table experience is 17.3582, and the modelled life expectancy is 17.3238. The model fits very well except for some very old ages in the neighborhood of 100. In this age range, the Gompertz model seems to underestimate the survival rate, and as a result gives a slightly lower life expectancy. Figure (4.8) shows the fitness of the Gompertz model.

The Australian 10-year government bond yield (1980-2005) is used to fit the Vasicek interest model. Using Ordinary Least Squares (OLS) estimation, the sum of the squares of the difference between the real interest rate and the modeled interest rate is minimized. A brief introduction to OLS is provided in Appendix A. The model is fitted by iterations in MS *Excel*. The fitted results are listed in Table (4.1).

The house price model is calibrated to the quarterly median house prices from eight capital cities in Australia. Similarly using OLS estimation, the parameters for the house price model are calculated in Table (4.2).

The correlation between interest rates and house prices is calculated in Table (4.3). Although the results are statistically significant for both lags, it is hard to conclude how the two series are correlated overall. So in the following simulation process, the two series are generated independently with zero lag and zero correlation.

	Statistic	Values
lag 1	ρ	0.144
	t value	0.935
lag 2	ρ	−0.661
	t value	0.495
	95% percentile	1.9665

Table 4.3: The correlation between interest rates and house prices

Large business indicator lending rate	9.10
Bank standard housing loans rate	7.30
10-year Australian government bond yield	**5.35**
Credit premium for RMs capital financing	3.75
Mortgage risk premium	1.95

Table 4.4: Calculation of risk premiums

4.3.4 The risk premiums

Besides the above parameter calibrations, it is also assumed that the total risk premium for the lender is 7% to lend the money and 3.75% to finance the capital. Following are some of the bases of this assumption. 10-year Australian government bond yields are used as a base rate. The difference between the base rate and the other rates is calculated as an approximation of the market premium for the risks. For example, the credit premium for reverse mortgage lender capital financing is the difference between the "large business indicator lending rate" and the base rate. The mortgage risk (approximately for the sum of interest rate risk and house price risk) is the difference between "bank standard housing loans rate" and the base rate.

The premium for mortality risk can be calculated based on the annuity price data and the mortality table covering the same period. Using Wang transformation (1996, [60]) and (2000, [61]), the transformed distribution of future life time $F^*(t)$ for aged x can be expressed as

$$F^*(t) = g_\lambda [F(t)] = \Phi \left[\Phi^{-1}({}_tq_x) - \lambda\right], \tag{4.5}$$

where λ is the risk premium. Using Elliptical transformation (2005, [58]), the transformed distribution of future life time $F^*(t)$ for aged x can be expressed as

$$F^*(t) = F_Z\left[\Phi^{-1}({}_tq_x) - \lambda^*\right], \tag{4.6}$$

where λ^* is the risk premium, $F_Z(\cdot)$ is the distribution function of a spherical random variable with density generator. If g_Z is chosen to be the density generator of a Normal distribution, then the Elliptical transformation can be simplified as Wang transformation. Given the table mortality rate ${}_tq_x$ (the original distribution) and the annuity price $F^*(t)$ (the transformed distribution), solving Equation (4.5) or (4.6) numerically can lead us to the risk premium or market price of mortality risk. However without the actual annuity price data, we just assume the risk premium for mortality risk is 1.8% in the following simulations. Thus the sum of the risk premiums that the lender charges is approximately 7%.

Notice that the total premium for all the risks may not be equal to the sum of the premium for each different risk. Besides, the risk premiums may change over time. The most straightforward way to estimate the total risk premium is applying Wang transformation or Elliptical transformation to real reverse mortgage price data. In the absence of reverse mortgage data, I have to make simplified assumptions.

4.3.5 The simulation results

Two portfolios are considered, containing 50 and 1000 loans respectively. For each portfolio, 1000 trials are calculated in *Matlab*. In the 50-loan portfolio for example, in each trial an interest rate process, 50 different house price processes and 50 life spans of loans are generated. For each loan, a repayment and a loss are calculated. Then a maximal safe loan amount is calculated by trial and error using the relationship in Equation (4.3). The outcomes of the of 50 trials are averaged to arrive at the final result.

For a house currently valued at \$100,000, the maximal safe loan amount for a lump sum reverse mortgage is \$38,128 for the portfolio of 50 loans and \$39,222 for the portfolio of 1000 loans. If the reverse mortgage takes the form of annuity, the maximal safe annual payment is \$4,271 and \$4,447 for the portfolio of 50 and 1000 loans respectively. Figures (4.9) and (4.10) show a single simulation path for each portfolio. (Both the figures are for the lump sum case).

Notice the portfolio of 1000 loans gives a larger maximal safe amount of loan. This result is reasonable, because the risks are better diversified in the larger portfolio. Since the larger portfolio actually involves less risk, it can afford a larger maximal safe loan amount. Figure (4.11) shows the loss (present value) distribution of a single contract. It is clear the loss is approximately normally distributed with a little long right tail.

The distribution of the number of loan repayments in each year is illustrated in Figure (4.12). The figure shows that in most years, the number of repayments is normally distributed. This makes sense. In each year the number of repaid loans d_{x+t} is binomially distributed at the repayment rate q^*_{x+t}. If the number of loans is large, say more than 30, then the number of loan repayments is approximately distributed as normal according to the Central Limit Theorem (CLM). The situation is more complicated for the distribution of the aggregate loss amount.

The distribution of the aggregate loss amount in each year is illustrated in Figure (4.13). Some of the histograms look like normal distribution. In fact, all the distributions are asymptotically normal if the number of loan repayments approaches infinity. In the early years, most of the repayments turn out to be negative losses (or gains). Especially in the first 2 years, all the losses are negative. This is to be expected because the cross-over happens very rarely in the early period. It would only happen in extreme scenarios where house prices collapse as a result of earthquake for example. In the early course of the loan, the distributions are skewed to the right. But this gradually changes to the opposite over time. In about the 20th year, the distribution of aggregate loss begins to become symmetric around 0. (In this period, the number of repayments is the larger, so the aggregate loss is more closer to normal distribution.) After that, the distribution starts to skew to the left because the losses start to outnumber the gains. In the last few years, nearly all the repayments turn out to be losses. This is consistent with our analysis in chapter 3.

The average number of losses and average aggregate loss amount in each year are also calculated. Figure (4.14) shows that the number of repayments continues to increase until year 20, after that the number of repayments gradually decreases to 0 at year 48. The average aggregate loss amount is negative in the early period, but after year 18, all the losses become positive. In year 28, although the number of losses is not the greatest, the average aggregate loss is the largest. From then on, the aggregate loss amount decreases rapidly, because the increase in the size of the loss of a single contract is offset by the decrease in the number of repayments. In the last few years, although the loss amount of a single contract may be very large as shown in Figure (4.14), the aggregate loss amount need not be a big concern for the lender. According to the above analysis, the most financially stressful period for the lender is the several years in the period year 28-33, not the last few years.

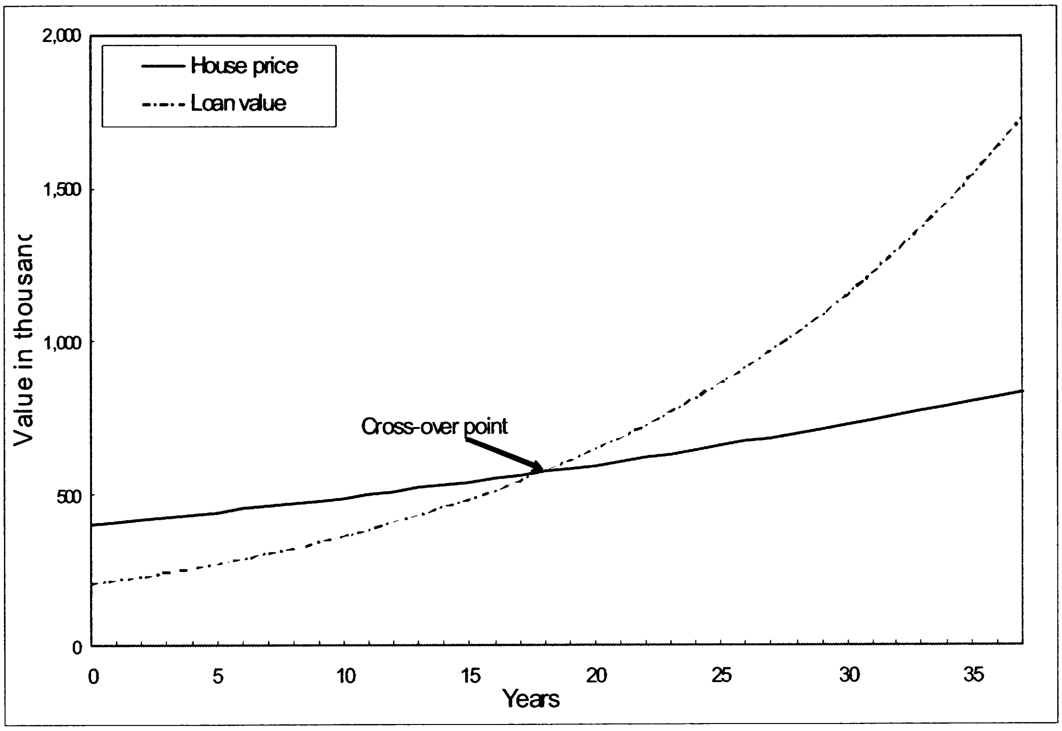

Figure 4.1: The cross-over risk

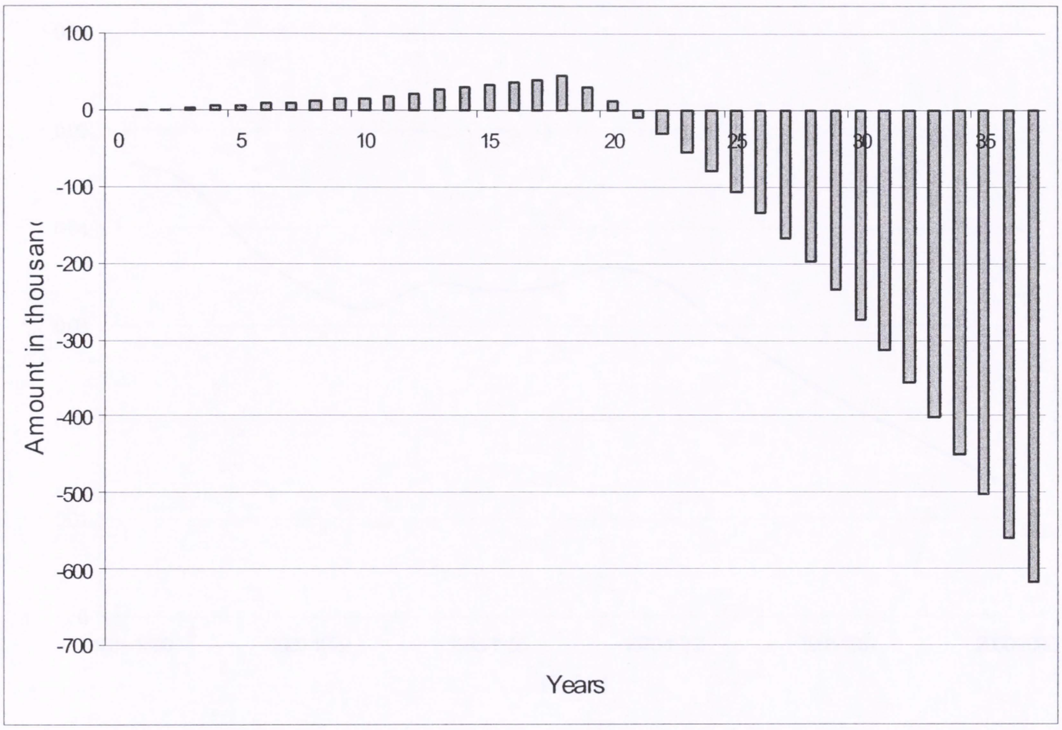

Figure 4.2: Cash flow analysis for a reverse mortgage lender

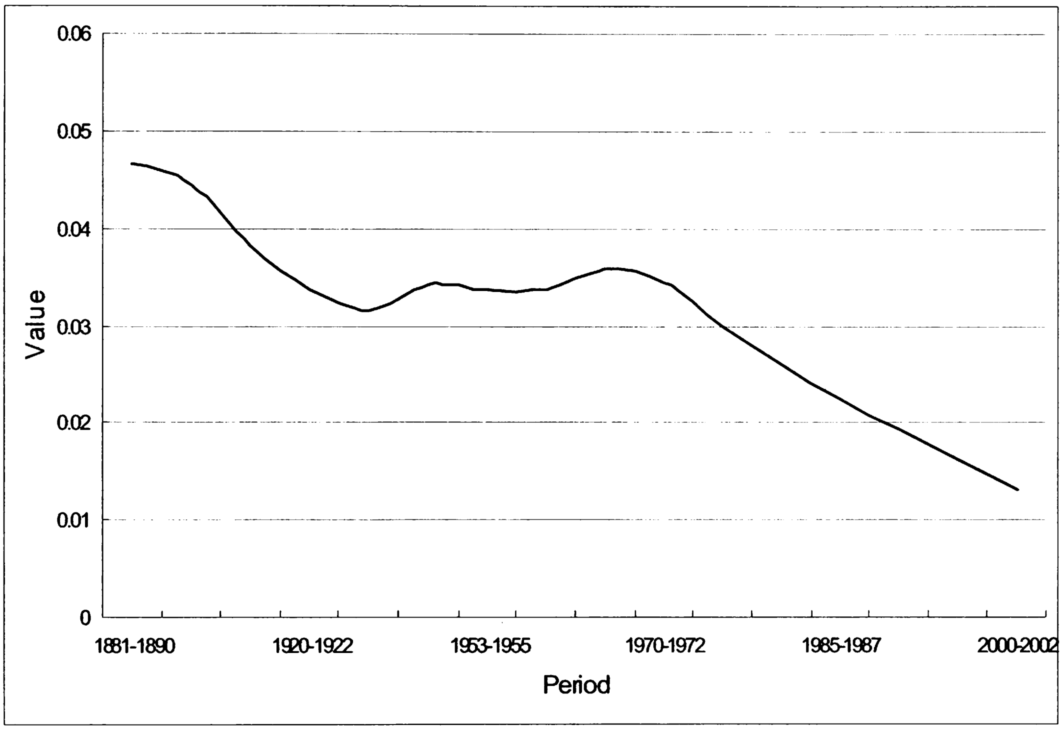

Figure 4.3: Force of mortality of Australian aged 62

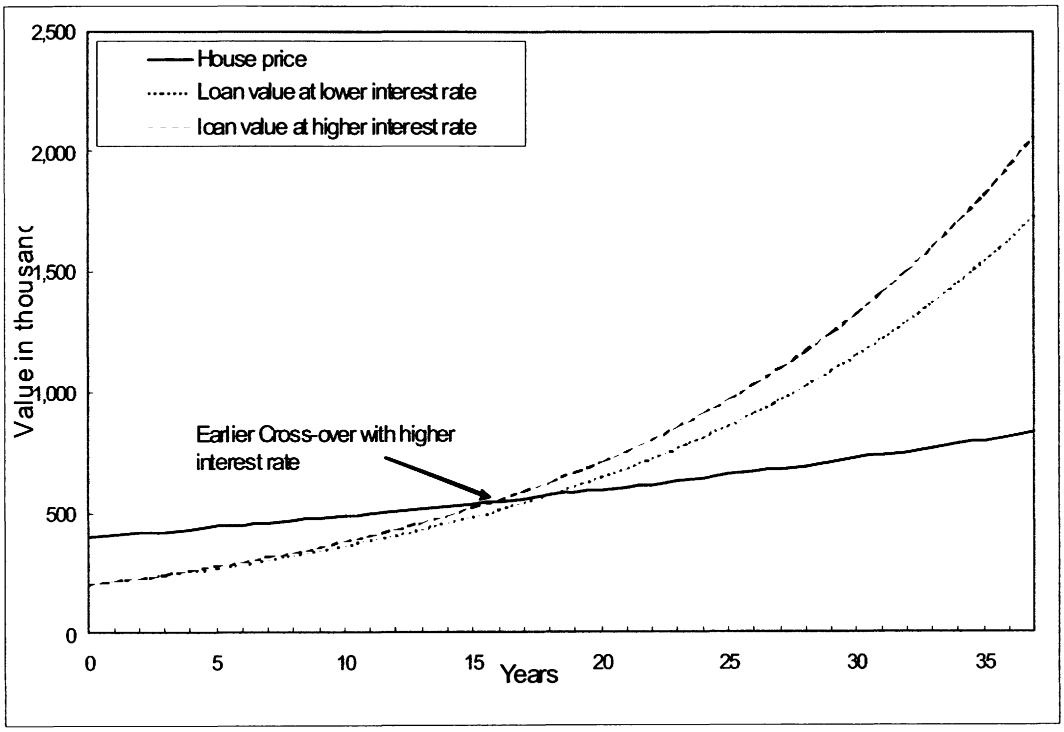

Figure 4.4: The interest rate risk

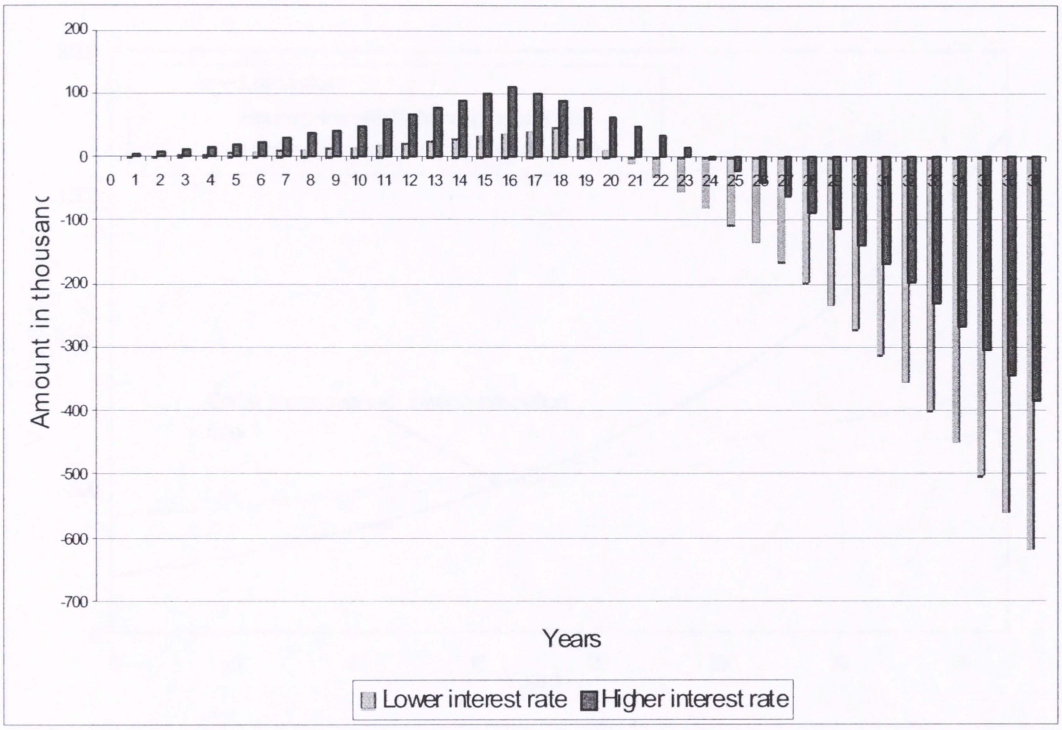

Figure 4.5: The lender's cash flows with different interest rates

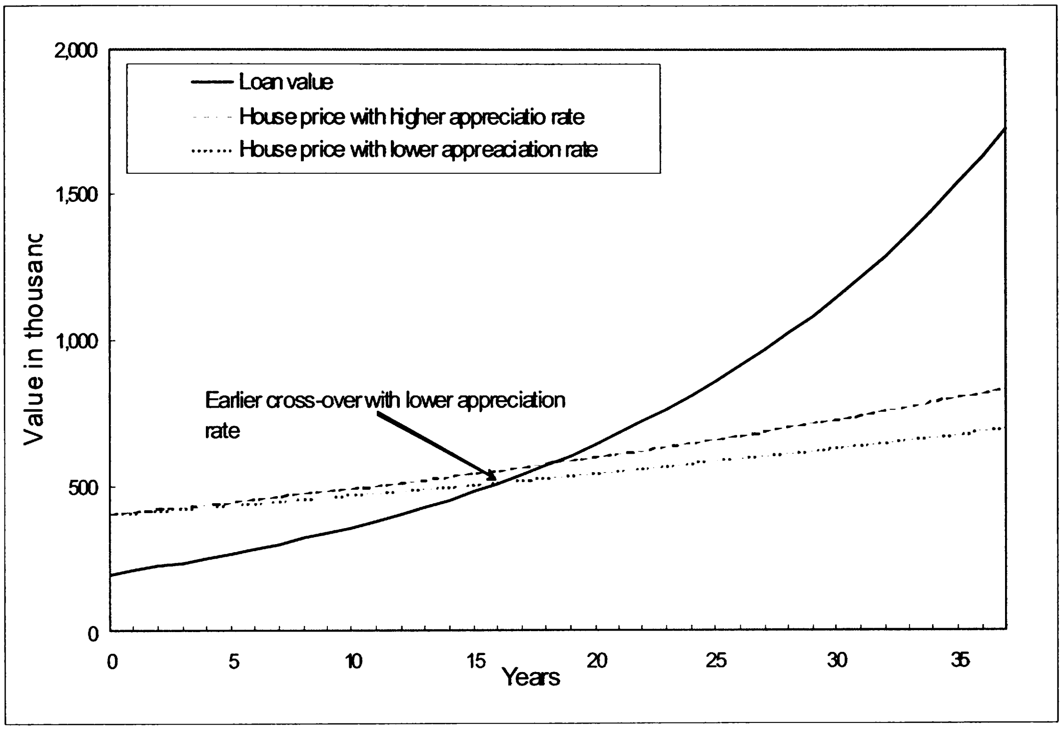

Figure 4.6: The house price risk

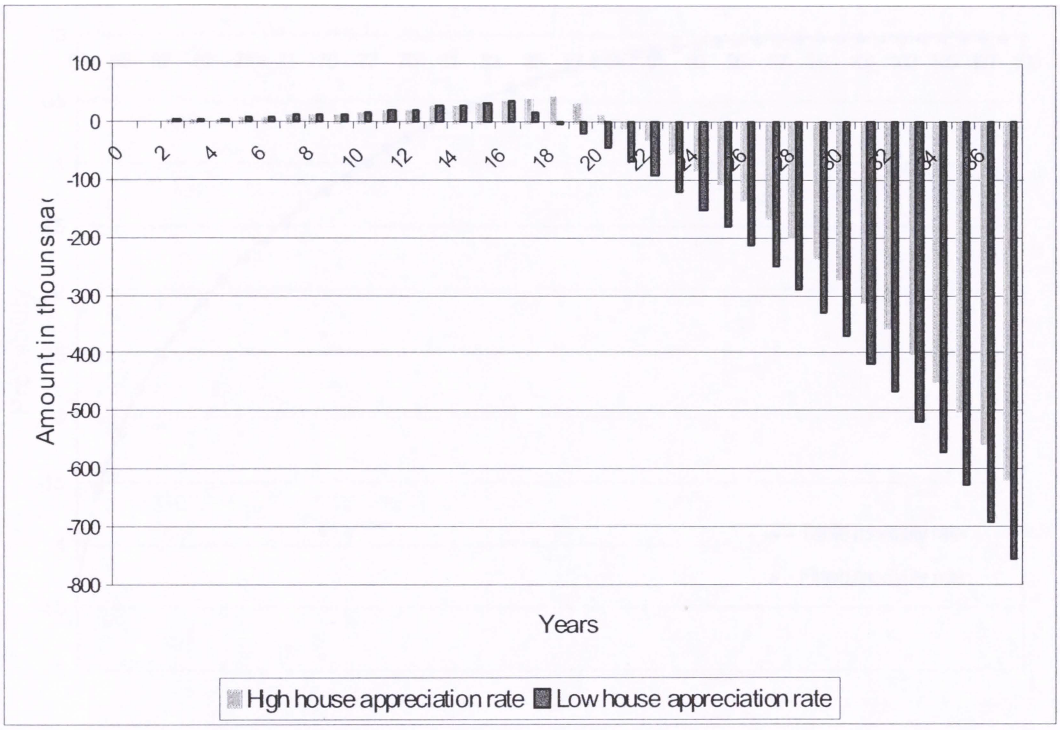

Figure 4.7: The lender's cash flows with different house appreciation rates

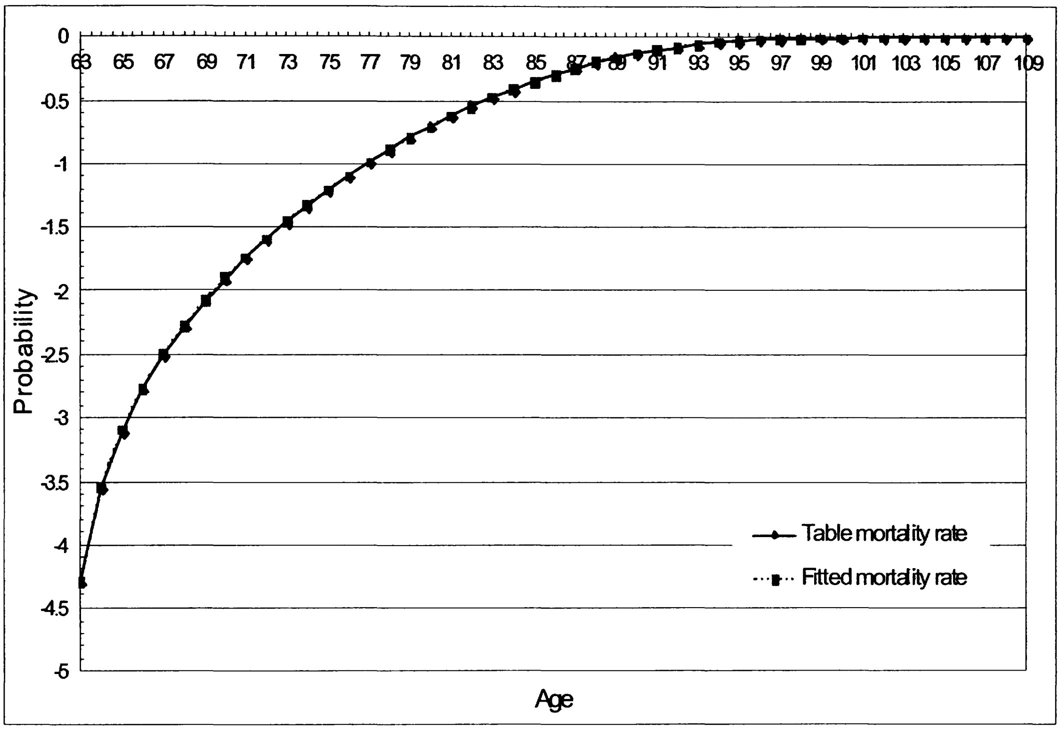

Figure 4.8: The Gompertz for Australian male aged 62

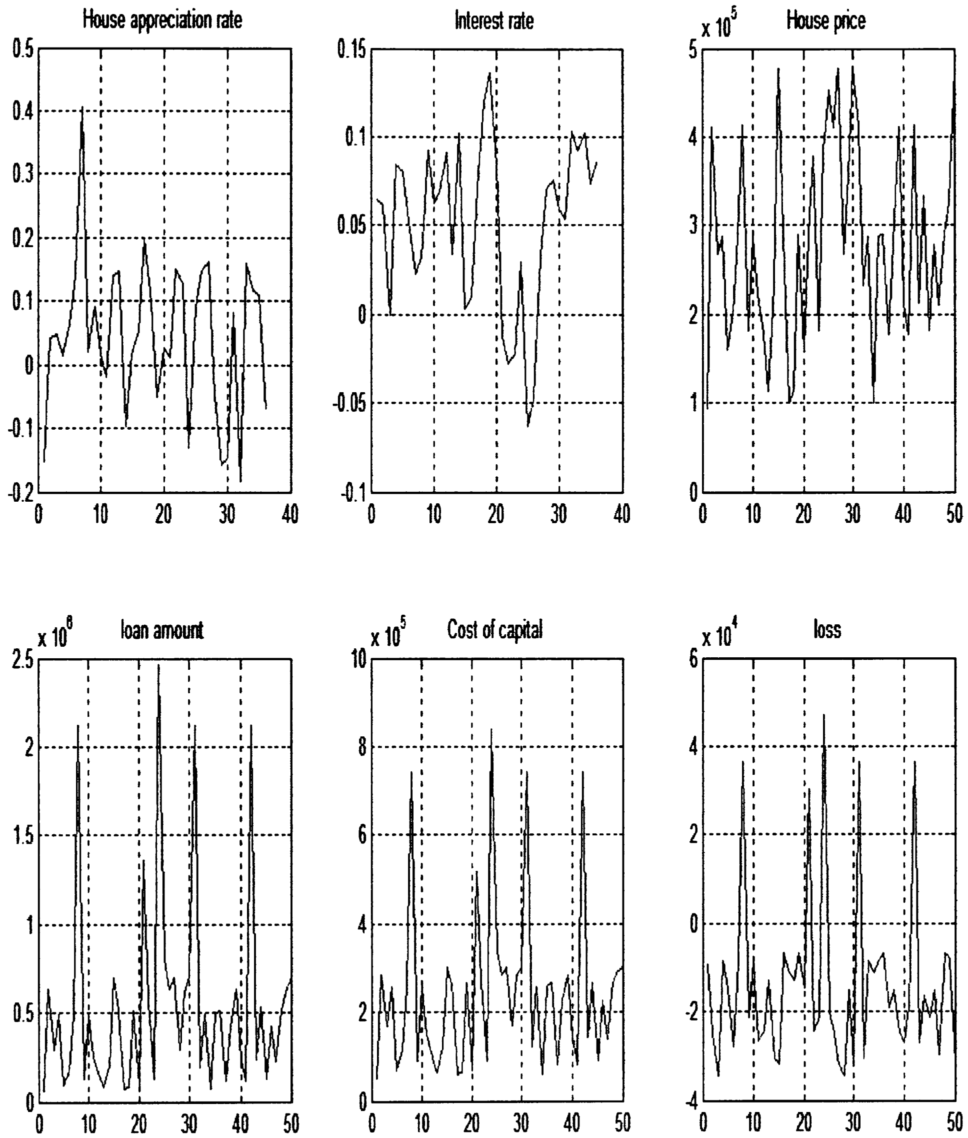

Figure 4.9: One simulation path for 50-loan portfolio.

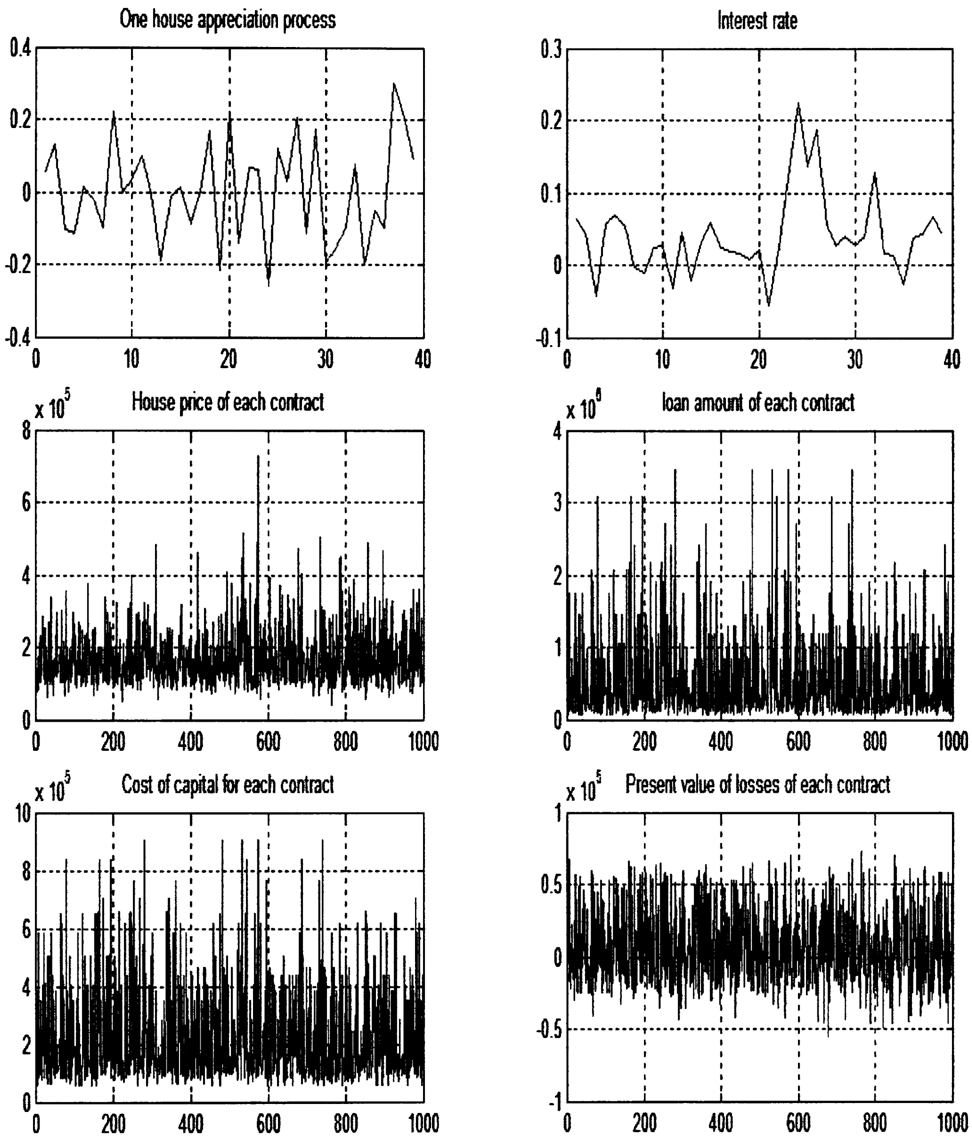

Figure 4.10: One simulation path for 1000-loan portfolio

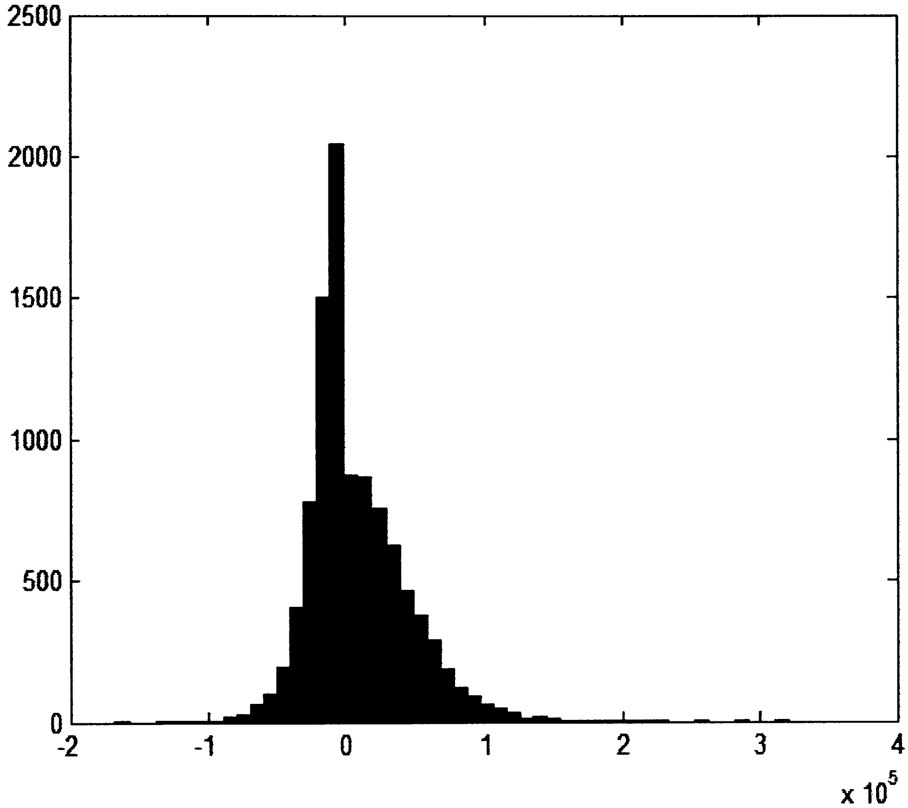

Figure 4.11: The loss distribution of one contract

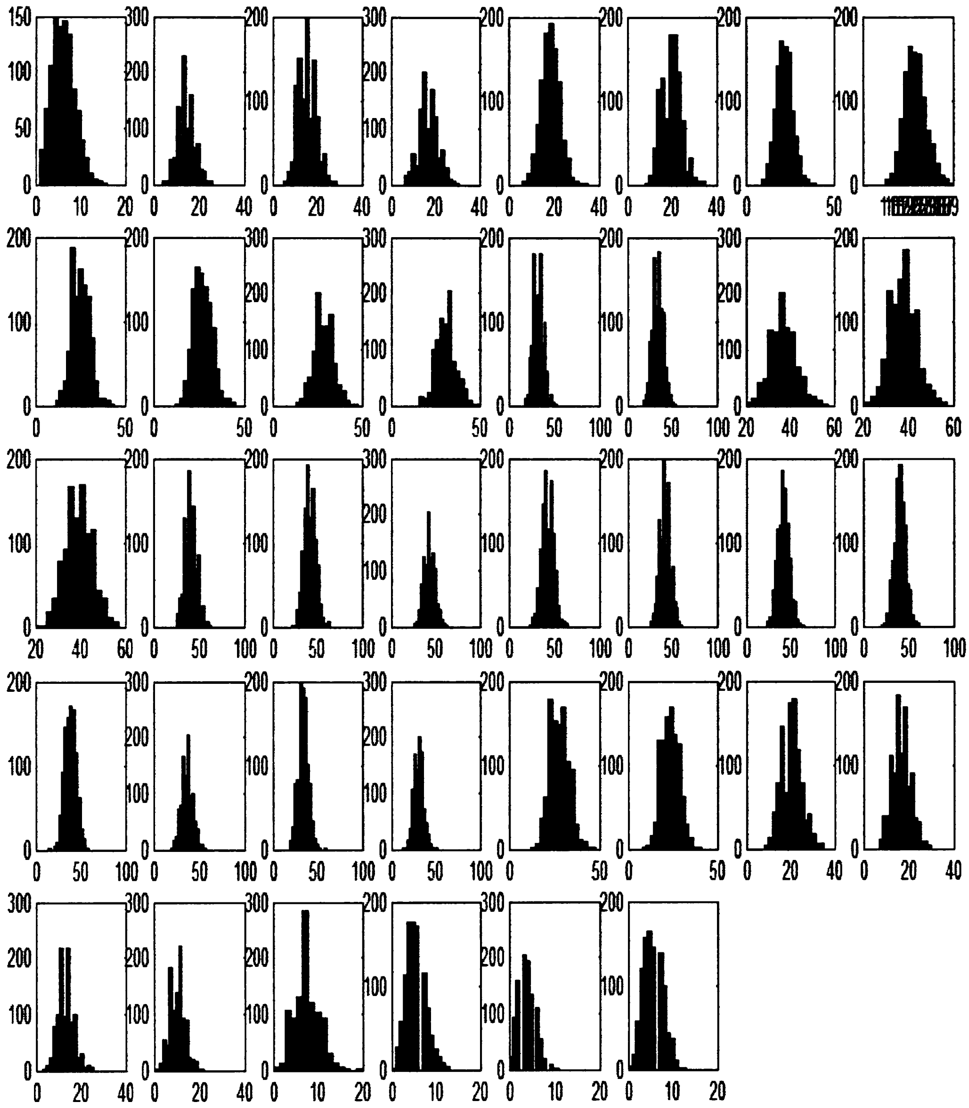

Figure 4.12: The number of repayments in each year

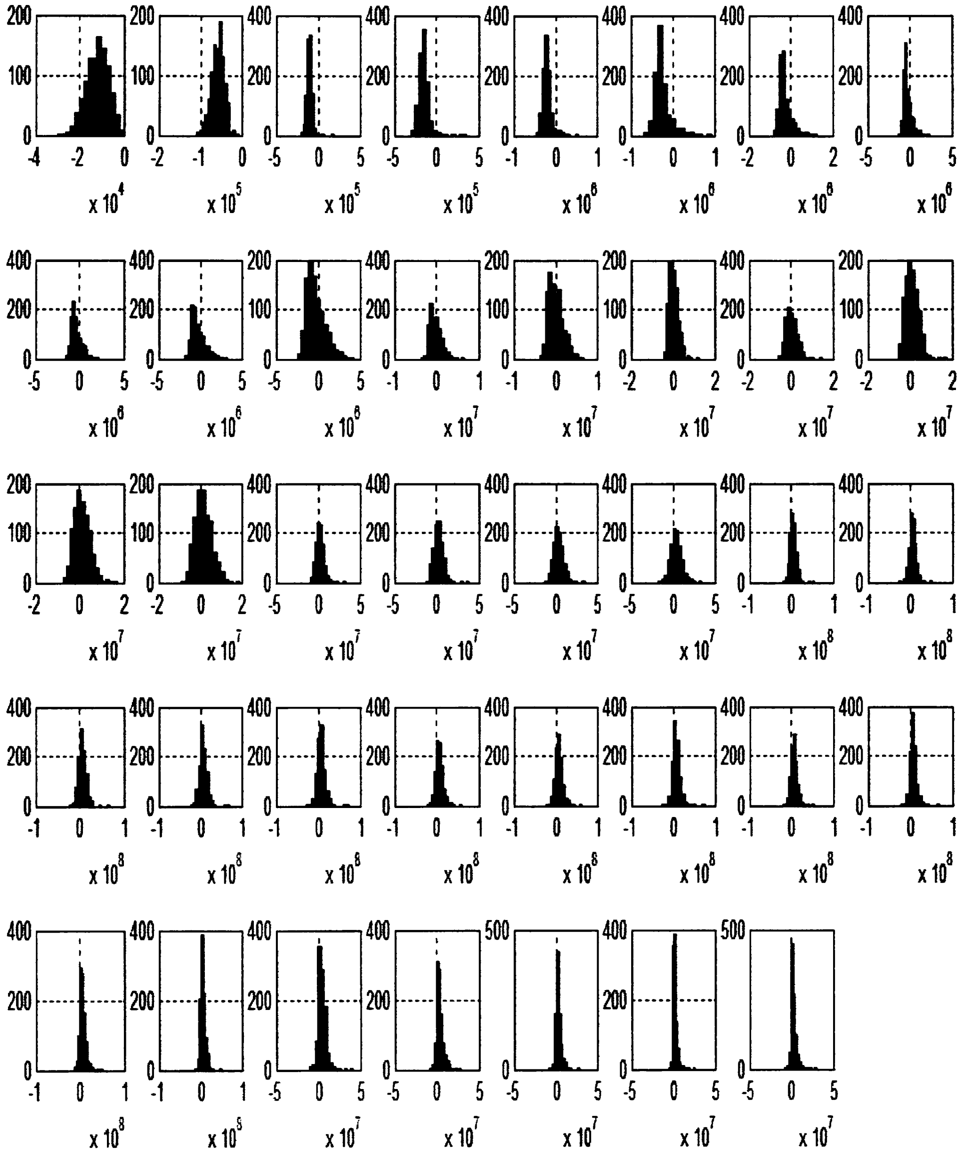

Figure 4.13: Aggregate loss amount in each year

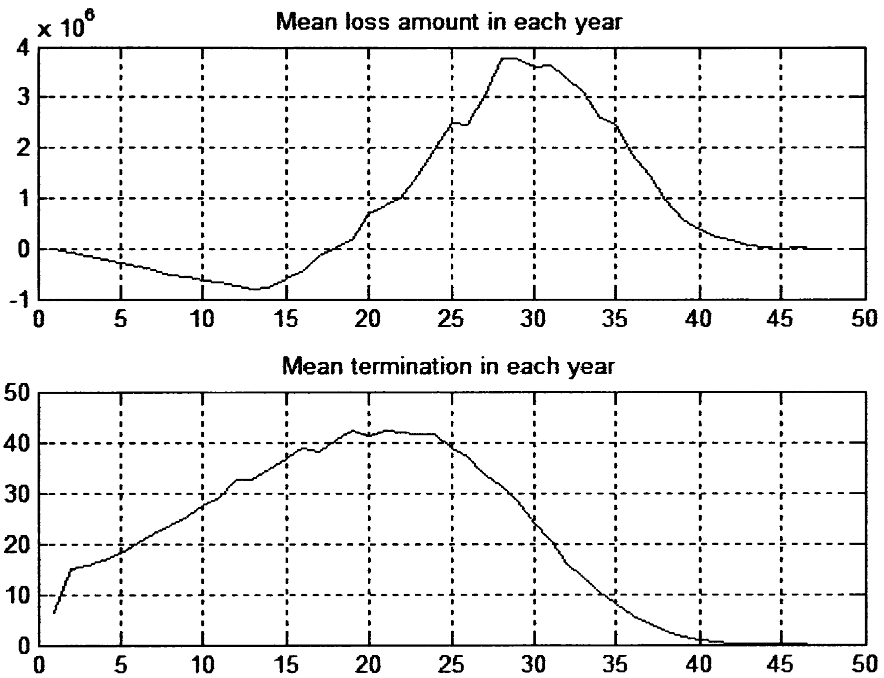

Figure 4.14: The average loss amount and number of repayments in each year

Chapter 5

Securitization with reverse mortgages

5.1 Structure of securitization

Cox (2000, [17]) pointed out that a common structure for the asset and liability securitizations involves four entities: retail customers, a retail contract issuer, a special purpose company, and investors. The authors then illustrate the process using the examples of several recent catastrophe bonds including USAA hurricane bonds, Winterthur Windstorm Bonds, and Swiss Re California Earthquake Bonds.

In this section, the general structure is applied to a lump sum case of a reverse mortgage product. In the case of reverse mortgage securitization, the process should involve at least five components:

- Borrower (Homeowner)
- Loan originator (Retailer)
- Special Purpose Company (or Special Purpose Vehicle)
- Lender (Investment bank)
- Investors (Capital markets)

Figure (5.1) illustrates the general structure of reverse mortgage securitization and all the cash flows involved in the process.

The transaction starts from the RMs retailer. The retailer is the front office that makes contact with the reverse mortgage loan borrower and negotiates the loan. After the retailer initiates the loan, it collects the lump sum from the reverse mortgages lender and then pays the amount to the borrower. To protect itself from the risk of not being able to fully recover the accumulated loan amount, the lender enters into an insurance contract with the Special Purpose Company (SPC). The insurance contract sets up a schedule of fixed trigger levels such that if the loss amount exceeds the triggers, the SPC will pay the lender a certain amount of benefit up to an upper limit. In return, the SPC collects a premium from the lender up front. The SPC issues a survivor bond on the market. The bond is sold at a price lower than the normal market price, because in the event that the loss of the lender exceeds the trigger, part or all of the coupon could be defaulted to the bond holders and transferred to the insured — the lender.

Readers should notice that the above model only provides a very basic structure for the securitization process. In the real world, the process usually involves many other components which serve various purposes. For example, to protect the bond investors from the default risk from the SPC, the process can involve some forms of credit enhancement, from institutional rating agencies.

5.1.1 Cash flow analysis for each component

For the retailer

As an independent servicing institution, the retailer provides service to the customers, monitors their repayments of the loans, and maintains the integrity of the cash flows and payment process. In each period after the loan starts, the retailer collects the repaid loan amount from the borrowers and transfers the amount to the lenders. From the perspective of the retailer, the cash inflows from the borrowers are exactly the same as the cash outflows to the lenders and therefore there is no risk of loss at all.

For the lender

The lender predicts the number of survivors of the loans and the loss amount in each period by analyzing the past mortality improvement, interest rate fluctuation and the real estate market conditions before the loans start. In each period after the loans commence, the lender's cash inflows are the loan repayments collected from the retailer and the cash outflows are the accumulated cost of capital. If the net of the two is less than the scheduled triggers, no insurance benefit is claimed from the SPC. Otherwise, the lender can collect a benefit from the SPC to cover the loss.

For the SPC

The SPC is a passive entity that only exists to securitize the mortality risk and sell the security on the market. For this purpose, it collects premiums from the lender and issues a survivor bond. The premium and the capital from selling the bond are invested at a risk-free interest rate. In each period after the loans commence, the SPC's only cash inflow is the risk-free investment proceeds. The SPC's cash outflows are the claims from the retailer with high priority and the coupons paid to the bond holders. At the end of the term of the bond, the SPC repays the principal to the investors. The net cash flow should be always zero for the SPC.

For the investors

The survivor bond investors purchase the survivor bond at a lower price than a straight bond, but bear the risk of losing some of the future coupons. In each period after the loans commence, the cash inflows for the bond holders are the random coupons from the SPC. At the end of the term, the investor collects the full principal. To illustrate the securitization process, several examples of securitization schemes are provided, including two types of survivor bonds and one survivor swap.

Example 1: Reverse mortgage survivor bond type 1

In this case, to illustrate the effect of longevity securitization, the interest rate and house appreciation rate are both assumed to be constant. Suppose the lender holds a portfolio of l_0 loans. At time 0, all the borrowers are aged 62 and each borrow a lump sum of Q_0 against their home property currently valued at H_0. To hedge the longevity risk, the lender purchases insurance from the SPC at a lump sum premium P. Under the contract, in each period after the cross-over, the SPC will pay the lender a benefit of $A_t\left(l_t - \widehat{l_t}\right)$, up to a ceiling amount C, if the number of survived loans l_t exceeds the predetermined trigger $\widehat{l_t}$. In period t the loss amount for each loan i is $L_{i,t}$, and since the interest rate and house appreciation rate are constant, $L_{i,t} = L_t$ for all i and all t. A_t is determined as the 1-period discounted difference between the expected loss in t and $t+1$.

$$A_t = \frac{L_{t+1}}{1+r} - L_t.$$

If the risk-free interest rate is r, the house appreciation rate is c, the risk premium the lender charges is λ_1 and the premium that the lender is charged for capital finance is λ_2, then

$$L_t = Q_0\left(1+r+\lambda_2\right)^t - \min\left[Q_0\left(1+r+\lambda_1\right)^t, H_0\left(1+c\right)^t\right].$$

For example, if $r = 6.5\%$, $c = 3\%$, $\lambda_1 = 3\%$, $\lambda_2 = 1.5\%$, $Q_0 = 50000$ and $H_0 = 100000$, the L_t and A_t in each period are calculated and graphed as below in Figure (5.2) and (5.3).

In Figure (5.2), L_t is always increasing with t after the cross-over. This means after the cross-over the loss amount increases over time. In Figure (5.3), A_t is always positive and increasing with t after cross-over. This means the 1-period discounted loss L_{t+1} is larger than the current period loss L_t. This finding implies that after the cross-over, the lender is always better off incurring loss at the current time than incurring loss later. The actual and expected number of repaid loans in each period t are denoted as d_t and $\hat{d}_t$, and the first period that the lender claims benefit is j. It is straightforward to show

$$\sum_{t=j}^{T}\left(l_t - \widehat{l_t}\right) = \sum_{t=j}^{T}\left(d_t - \widehat{d_t}\right). \tag{5.1}$$

Equation (5.1) indicates that less loss incurred in the current period means more loss will be incurred in the future, because the excessive survived loans ultimately will need to be repaid in some time in the future. When these excessive loans are repaid later, the present value of the these losses will be greater than if they had been incurred in present time. The difference between any two losses L_j and L_k

$(j \leq k)$ in two different periods after the cross-over can be expressed as

$$L_k - L_j = \sum_{t=j}^{k} A_t s_{t-j\rceil r}, \tag{5.2}$$

where $s_{t-j\rceil r}$ denotes the accumulation of 1 from time j to k at risk free interest rate r. Protected by the insurance contract, after the cross-over the lender can claim payments from the SPC to construct a reserve to cover the unexpected future loss in the event that $l_t > \widehat{l_t}$. The benefit payments B_t of each period are determined in (5.3). After collection, the benefits are invested at a risk free interest rate r and accumulate until the excessive survived loans are repaid.

$$B_t = \begin{cases} 0 & \text{if } l_t \leq \widehat{l_t} \\ A_t\left(l_t - \widehat{l_t}\right) & \text{if } \widehat{l_t} < l_t < \frac{C}{A_t} \\ C & \text{if } l_t > \frac{C}{A_t} \end{cases} \tag{5.3}$$

Let the first period that the lender claims benefit B_t be j, then the reserve R_t the lender accumulates continually with B_t during the loan process is

$$R_t = \sum_{t=j}^{t} B_t s_{t-j\rceil r}.$$

Denote the actual and expected aggregate loss in each period t as $\underline{L}_t$ and $\hat{\underline{L}}_t$. By definition, the actual aggregate loss is

$$\hat{\underline{L}}_t = \hat{d}_t L_t,$$

and expected aggregate loss is

$$\underline{L}_t = d_t L_t.$$

In the case of $\underline{L}_t > \hat{\underline{L}}_t$, the lender will have an unexpected loss of

$$\underline{L}_t - \hat{\underline{L}}_t = \left(d_t - \widehat{d_t}\right) L_t.$$

This unexpected loss in period t could be partly or fully covered by the reserve R_t. Notice that

$$R_t = \sum_{t=j}^{T} B_t s_{t-j\rceil r} \leq \sum_{t=j}^{T} \left(\underline{L}_t - \hat{\underline{L}}_t\right), \tag{5.4}$$

which means the accumulated unexpected loss is the upper limit of the reserve R_t. This avoids the problem of over-insurance. The accumulated unexpected loss is fully covered only if

$$R_t = \sum_{t=j}^{T} \left(\underline{L}_t - \hat{\underline{L}}_t \right),$$

or

$$R_t = \sum_{t=j}^{t} A_t s_{t-j\rceil r},$$

in which case all the benefits collected from the insurer are less than the ceiling C.

The SPC issues a bond with a face value F and survivorship-contingent coupon payments C_t at a price of $V \leq F$. The coupon payments are deducted whenever $\widehat{l}_t < l_t$ and the net coupon payments that the bond holders receive in each period are

$$C_t = \begin{cases} C & \text{if } l_t \leq \widehat{l}_t \\ C - A_t \left(l_t - \widehat{l}_t \right) & \text{if } \widehat{l}_t < l_t < \frac{C}{A_t} \\ 0 & \text{if } l_t > \frac{C}{A_t} \end{cases} . \tag{5.5}$$

In each period, the SPC needs to pay the coupon C_t and in the last period, the SPC needs to pay back the principal as well. To cover the cash outflow, the SPC invests the premium P and the survivor bond price V in a straight risk-free bond with a face value F sold at a price of W. If $v = \frac{1}{1+r}$ is the one period discount factor, as long as

$$P + V \geq W = Fv^T + \sum_{k=1}^{T} v^k C,$$

the SPC can collect the amount C in each period and will be able to fulfill both his insurance and bond contract. To avoid any arbitrage, we should have $P + V = W$.

Example 2: Reverse mortgage survivor bond type 2

In this case, the interest rate and house appreciation rate are no longer assumed to be deterministic, but stochastic. Suppose the lender holds the same portfolio of l_0 loans as in survivor bond type 1. At time 0, all the borrowers are aged 62 and each borrow a lump sum of Q_0 against a property currently valued at H_0. If the survivor bond type 1 is applied, there is a chance that the lender is over-insured, that is that the reserve R_t may sometimes exceed what the lender actually needs:

$$R_t > \sum_{t=j}^{T} \left(\underline{L}_t - \hat{\underline{L}}_t \right),$$

where $\underline{L}_t$ and $\hat{\underline{L}}_t$ stand for the aggregate actual loss and the preset trigger amount. This is because L_t may not be necessarily increasing all the time, and thus A_t may not be positive all the time. To avoid this problem, the lender may purchase another type of insurance contract from the SPC at a lump sum premium P. Under this insurance contract, in each period after the cross-over, the SPC will cover the lender's aggregate loss up to a ceiling amount C if the actual total amount of loss $\underline{L}_t$ exceeds the trigger amount $\hat{\underline{L}}_t$, for example 95% percentile of the distribution of $\underline{L}_t$. Under this arrangement, the benefit paid to the lender in period t is

$$B_t = \begin{cases} 0 & \text{if } \underline{L}_t \leq \hat{\underline{L}}_t \\ \underline{L}_t - \hat{\underline{L}}_t & \text{if } \hat{\underline{L}}_t < \underline{L}_t \leq C \\ C & \text{if } \underline{L}_t > C \end{cases} . \tag{5.6}$$

The lender's net loss after the benefit in period t is

$$\underline{L}_t - B_t = \begin{cases} \underline{L}_t - 0 & \text{if } \underline{L}_t \leq \hat{\underline{L}}_t \\ \underline{L}_t - \left(\underline{L}_t - \hat{\underline{L}}_t \right) & \text{if } \hat{\underline{L}}_t < \underline{L}_t \leq C \\ \underline{L}_t - C & \text{if } \underline{L}_t > C \end{cases}$$

$$\underline{L}_t - B_t = \begin{cases} \underline{L}_t & \text{if } \underline{L}_t \leq \hat{\underline{L}}_t \\ \hat{\underline{L}}_t & \text{if } \hat{\underline{L}}_t < \underline{L}_t \leq C \\ \underline{L}_t - C & \text{if } \underline{L}_t > C \end{cases} . \tag{5.7}$$

Similar to the above case, the survivor bond has a face value F and random coupon payments of C_t sold at a price of $V \leq F$. But in this case, the coupon payments are linked to the lender's aggregate loss, not the number of survived loans. The coupons for the bond holders in period t are

$$C_t = \begin{cases} C & \text{if } \underline{L}_t \leq \hat{\underline{L}}_t \\ C - \left(\underline{L}_t - \hat{\underline{L}}_t \right) & \text{if } \hat{\underline{L}}_t < \underline{L}_t \leq C \\ 0 & \text{if } \underline{L}_t > C \end{cases} . \tag{5.8}$$

As in type 1, the SPC invests the premium P and the survivor bond price V in a straight risk-free bond with a face value of F sold at a price of W. If v is the one period discount factor, as long as

$$P + V \geq W = Fv^T + \sum_{k=1}^{T} v^k C,$$

the SPC can collect amount C in each period and fulfill both his insurance and bond contract. For the securitization to be actuarially fair, we should have $P+V = W$.

Example 3: Reverse mortgage survivor swap

Another possible securitization structure is a reverse mortgage swap. In the survivor swap transaction, there is no principal payment at time T. At one side, the SPC pays the same cash flows B_t to the insurer, $t = 1, 2, ..., T$. In exchange for the floating benefit B_t, the lender pays a fixed annual premium x to the SPC instead of paying a lump sum premium P. Eventually, we have

$$P = x a_{\overline{T}|}.$$

At the other side, the SPC pays C_t to the bond holders. The investors pay the SPC a fixed amount y each year in order to receive the same coupons C_t, instead of paying V for the survivor bond. So we have

$$y a_{\overline{T}|} = \sum_{k=1}^{T} v^k E\left(C_t\right).$$

As in the RMs bond example, the SPC has cash flows of B_t to the lender and C_t to the investors. Assuming there is no counter–party risk, in each year the SPC gets $x + y$, exactly enough to finance its obligation $B_t + C_t$. One advantage of swaps over issuing survivor bonds is the lower transaction costs, but the trade–off is that swaps could introduce default risk. As part of the solution, the swap might be provided by a broker or investment banker to reduce the default risk.

5.2 Pricing a reverse mortgage survivor bond

Basically to price a bond is just to discount all the expected future cash flows at the appropriate discount rate. The general bond pricing equation is

$$V = Fv^T + \sum_{k=1}^{T} v^k E(C_t) \tag{5.9}$$

where the notation follows the examples in the last section. The two types of survivor bonds are priced below.

5.2.1 Case 1: Survivor bond type 1

Following the example in the above section, it is assumed that the interest rate and house appreciation rate are constant. The only difference between survivor bond type 1 and a straight bond is that the former's coupon payments are linked to the survivorship of the loans and is thus uncertain. Suppose a series of $\widehat{l}_t$ are determined as the triggers, for a portfolio of l_0 loans borrowed by persons aged x with the identical house value H_0, the bond holders will receive coupons in each period t

$$C_t = \begin{cases} C & \text{if } l_t \le \widehat{l}_t \\ C - A_t\left(l_t - \widehat{l}_t\right) & \text{if } \widehat{l}_t < l_t < \frac{C}{A_t} \\ 0 & \text{if } l_t > \frac{C}{A_t} \end{cases} .$$

This is equivalent to

$$C_t = C - \left[A_t\left(l_t - \widehat{l}_t\right), 0\right]_+ + \left[A_t\left(l_t - \widehat{l}_t\right) - C, 0\right]_+ .$$

Taking the expectation on both sides,

$$E(C_t) = C - E\left[A_t\left(l_t - \widehat{l}_t\right), 0\right]_+ + E\left[A_t\left(l_t - \widehat{l}_t\right) - C, 0\right]_+ . \tag{5.10}$$

The pricing equation of the survivor bond type 1 can be obtained by substituting equation (5.10) into (5.9):

$$V = Fv^T + \sum_{k=1}^{T} v^k \left\{C - E\left[A_t\left(l_t - \widehat{l}_t\right), 0\right]_+ + E\left[A_t\left(l_t - \widehat{l}_t\right) - C, 0\right]_+\right\} . \tag{5.11}$$

As mentioned in Chapter 4, the number of survived loans l_t follows binomial distribution at the same repayment rate q^*_{x+t}. If the loan number is large, for example more than 30, according to the Central Limit Theorem, l_t is approximately distributed as normal with mean $\mu_t = l_t \left(1 - q^*_{x+t}\right)$ and variance $\sigma_t^2 = l_t \left(1 - q^*_{x+t}\right) q^*_{x+t}$. In equation (5.10), we can rewrite the expectations term as

$$E\left[A_t\left(l_t - \widehat{l_t}\right), 0\right]_+ = A_t E\left[(l_t - \mu_t) - \left(\widehat{l_t} - \mu_t\right), 0\right]_+$$

$$E\left[A_t\left(l_t - \widehat{l_t}\right), 0\right]_+ = A_t\sigma_t E\left[\frac{(l_t - \mu_t)}{\sigma_t} - \frac{\left(\widehat{l_t} - \mu_t\right)}{\sigma_t}, 0\right]_+ .$$

Let $E(X-k)_+ = \Psi(k)$ [1], $k_t = \frac{\left(\widehat{l_t} - \mu_t\right)}{\sigma_t}$, we have

$$E\left[A_t\left(l_t - \widehat{l_t}\right), 0\right]_+ = A_t\sigma_t\Psi(k_t).$$

Similarly,

$$E\left[A_t\left(l_t - \widehat{l_t}\right) - C, 0\right]_+ = A_t\sigma_t\Psi\left(k_t + \frac{C}{\sigma_t}\right).$$

Thus succinctly, equation (5.10) can be rewritten as

$$E(C_t) = C - A_t\sigma_t\Psi(k_t) + A_t\sigma_t\Psi\left(k_t + \frac{C}{\sigma_t}\right). \tag{5.12}$$

Substituting (5.12) in (5.9), we have the approximated pricing equation of the reverse mortgage survivor bond type 1,

$$V = Fv^T + \sum_{k=1}^{T} v^k \left[C - A_t\sigma_t\Psi(k_t) + A_t\sigma_t\Psi\left(k_t + \frac{C}{\sigma_t}\right)\right]. \tag{5.13}$$

From equation (5.13), the survivor bond price can be easily calculated. From the derivation in Appendix B, we know

$$\Psi(k) = E(X-k)_+ = \phi(k) - k\left[1 - \Phi(k)\right], \tag{5.14}$$

where $\phi(x)$ and $\Phi(x)$ are PDF and CDF of a standard normal random variable X. After obtaining the bond price, we can calculate the internal return rate (IRR) of the survivor bond.

[1] $E(X-k)_+ = E\left[(X-k), 0\right]_+$

Period	Loss L_t	Period	Loss L_t	Period	Loss L_t
1	−588.33	14	−3590.22	27	16738.21
2	−691.29	15	−4059.30	28	18400.75
3	−807.79	16	2500.51	29	20206.01
4	−939.43	17	5935.67	30	22165.68
5	−1087.97	18	6651.23	31	24292.41
6	−1255.36	19	7431.25	32	26599.90
7	−1443.77	20	8281.12	33	29102.90
8	−1655.62	21	9206.63	34	31817.40
9	−1893.57	22	10214.08	32	34760.64
10	−2160.57	23	11310.25	36	37951.27
11	−2459.91	24	12502.48	37	41409.46
12	−2795.21	25	13798.72	38	45156.95
13	−3170.50	26	15207.54		

Table 5.1: Single loss in each period

A numerical example

Following the above example, let the annual interest rate r be 6.5%, annual house price appreciation c be 3%, the risk premium the lender charges λ_1 and is charged λ_2 be 3% and 1.5% respectively, the initial loan amount Q_0 be \$50,000 and the house price H_0 be \$100,000, then the L_t and A_t in each period are calculated in Table (5.1) and (5.2).

To get the projected number of survivors, the future force of mortality is projected as a function of both age x and time t. As discussed in Chapter 4, many projection methods have been proposed in the literature. This thesis adopts the Generalized Linear Model (GLM) suggested by Renshaw et al. (1996, [50]). According to the Renshaw model, the force of mortality $\mu_{x,t}$ is a log-linear function of age x and time t. Based on the Australian life table for 1881-2002, the model is calibrated as:

$$\mu_{x,t} = \exp\left(\beta_0 + \sum_{j=1}^{3}\beta_j L_j(x) + \sum_{i=1}^{2}\alpha_i t^i + \sum_{i=1}^{2}\sum_{j=1}^{3}\gamma_{i,j} L_j(x)\, t^i\right).$$

where $L_j(x)$ is the Legendre polynomial. In S-plus, a GLM regression is performed using the mortality rates from the Australian Life Table for 1881-2002. The regression results are listed in Table (5.3).

Period	Appreciation A_t	Period	Appreciation A_t	Period	Appreciation A_t
1	−552.42	14	−2092.50	27	11173.30
2	−613.19	15	−2313.83	28	11712.80
3	−680.39	16	4093.37	29	12284.82
4	−754.69	17	7166.25	30	12891.65
5	−836.83	18	7475.87	31	13535.76
6	−927.60	19	7802.34	32	14219.77
7	−1027.90	20	8146.79	33	14946.54
8	−1138.70	21	8510.39	34	15719.13
9	−1261.07	22	8894.45	32	16540.83
10	−1396.21	23	9300.32	36	17415.19
11	−1545.42	24	9729.49	37	18346.04
12	−1710.12	25	10183.56	38	19337.48
13	−1891.91	26	10664.21		

Table 5.2: Appreciation of each loss in each period

Parameters	Value	Standard Error	t value
β_0	−2.08911420	0.2313983	−9.02821598
β_1	1.62026486	0.4471106	3.62385658
β_2	−0.06886849	0.5358593	−0.12851972
β_3	−0.01089401	0.4734314	−0.02301074
α_1	−0.31284970	0.2433404	−1.28564656
α_2	−0.31091752	0.4554777	−0.68261856
$\gamma_{1,1}$	0.09956416	0.4690509	0.21226730
$\gamma_{1,2}$	−0.05981924	0.5609072	−0.10664731
$\gamma_{1,3}$	−0.06188307	0.5008452	−0.12355728
$\gamma_{2,1}$	−0.02977038	0.8790215	−0.03386763
$\gamma_{2,2}$	−0.04404868	1.0510683	−0.04190849
$\gamma_{2,3}$	−0.08197422	0.9356919	−0.08760813
Residual Deviance	0.4061248		

Table 5.3: Fitted parameters in the GLM model

Age range	GLM $\hat{\mu}_{x,t}$ (2000-02)	Current table $\hat{\mu}_{x,t}$	Projected improvement	Improvement rate
62 – 66	0.0096	0.016603	0.007013	0.00140
67 – 71	0.0152	0.027621	0.012461	0.00109
72 – 76	0.0304	0.046858	0.016447	0.00080
77 – 81	0.0656	0.075844	0.010215	−0.00125
82 – 86	0.1171	0.130129	0.012998	0.00056
87 – 91	0.1189	0.189262	0.070341	0.01147
92 – 96	0.0417	0.242077	0.200356	0.02600
97 – 101	0.0027	0.297514	0.294837	0.01890

Table 5.4: Projected improvement of force of mortality

The goodness-of-fit are illustrated in Figures (5.4), (5.5) and (5.6).

Future improvements in the force of mortality are calculated on a 5 year interval basis, as the mortality tables are mostly published every 5 years. In each 5 year interval, the mortality improvement is assumed to be linear. The projected improvements in the force of mortality are listed in Table (5.4).

With the above improvement factors, the corresponding trigger $\widehat{l_t}$ for each period can be easily obtained. The trigger values are calculated with formula (5.15).

$$\widehat{l_t} = \begin{cases} l_0\left({}_tp_x\right)e^{0.0014t} & \text{for } 0 < t \leq 5 \\ l_0\left({}_tp_x\right)e^{0.007+0.00109(t-5)} & \text{for } 5 < t \leq 10 \\ l_0\left({}_tp_x\right)e^{0.0125+0.0008(t-10)} & \text{for } 10 < t \leq 15 \\ l_0\left({}_tp_x\right)e^{0.0164-0.00125(t-15)} & \text{for } 15 < t \leq 20 \\ l_0\left({}_tp_x\right)e^{0.0102+0.00056(t-20)} & \text{for } 20 < t \leq 25 \\ l_0\left({}_tp_x\right)e^{0.0130+0.01147(t-25)} & \text{for } 25 < t \leq 30 \\ l_0\left({}_tp_x\right)e^{0.0703+0.026(t-30)} & \text{for } 30 < t \leq 35 \\ l_0\left({}_tp_x\right)e^{0.2004+0.0189(t-35)} & \text{for } 35 < t \leq 40 \end{cases}. \tag{5.15}$$

The results are illustrated in Figure (5.7) and listed in Table (5.5).

The survivor bond price is calculated with 1000-run simulation. Some of the results are listed in Table (5.6).

The results show prices for survivor bonds for a group of 1000 loans with loan amount of \$50,000 per person and the trigger levels calculated in Table (5.5). The

Period	Trigger $\widehat{l_t}$	Period	Trigger $\widehat{l_t}$	Period	Trigger $\widehat{l_t}$
1	986	14	677	27	152
2	973	15	639	28	124
3	958	16	599	29	99
4	942	17	557	30	79
5	924	18	514	31	61
6	904	19	470	32	48
7	883	20	427	33	37
8	859	21	384	34	28
9	834	22	342	32	21
10	807	23	300	36	15
11	778	24	259	37	11
12	746	25	220	38	8
13	712	26	183		

Table 5.5: Projected trigger values in each period

Number of loans	1000
Initial house value	$100, 000
Lump sum borrowed	$50, 000
Face value of straight bond	$100, 000, 000
Face value of survivor bond	$100, 000, 000
Coupon rate for both bonds	6.5% p.a.
Annual aggregate cash flow out of SPC	$6, 500, 000
Straight bond price	$100, 000, 000
Survivor bond price	$99, 902, 898
Premium paid to SPC	$97, 102

Table 5.6: Calculation of the mortality bond price (Type 1)

annual aggregate cash flow out of the SPC is \$6,500,000 and the coupon rate for both the straight bond and the survivor bond is 6.5%. The price of the survivor bond is \$999.02 per \$1000 of face value. The total premium is actually quite small relative to the total amount of loans: only $97,102/(1000 \times 50,000)$ which equals 0.2%. For as long as 38 years' protection, this price is very cheap.

Sensitivity testing

To examine how sensitive the value of a survivor bond is to the mortality change, a random shock is attached to the projected force of mortality $\widehat{\mu}_{x,t}$. Suppose the distribution of mortality shocks ε_t at time t is a beta distribution with parameters a and b. The mortality improvement shock ε_t is expressed as a percentage of the force of mortality $\widehat{\mu}_{x,t}$, so it ranges from 0 to 1, that is, $0 < \varepsilon_t < 1$ with probability 1. Without the shock, the projected survival probability $\hat{p}_{x,t} = e^{\left(-\widehat{\mu}_{x,t}\right)}$. With the shock, the new survival probability can be expressed as:

$$\hat{p}'_{x,t} = \exp\left(-\widehat{\mu}_{x,t}\right)^{1-\varepsilon_t} = \left(\hat{p}_{x,t}\right)^{1-\varepsilon_t}.$$

It is clear that

$$\hat{p}'_{x,t} \leq \left(\hat{p}_{x,t}\right)^{1-\varepsilon_t}.$$

After 10,000 simulation trials, the impact of various mortality shocks is summarized in the Table (5.7). The table lists how many survivors will remain in the portfolio after 20 years, and how much of the total value of the coupons and principal the investors will lose after the shocks. For example, when $a = 1.38$, $b = 26.30$, $E[\varepsilon_t] = 0.05$, on average the investors will lose only 0.11% of the total value of the coupons and principal. The maximal loss is 0.13%.

The results show that the impact of mortality shocks is very limited in terms of the entire investment. Even in a scenario of a 50% mortality surprise, the investors lose $99,902,898 - 99,663,909$ which equals \$238,989 on average, which is less than 3.7% of the total value of expected coupons and 1% of the total value of expected coupons and principal. Therefore there is very little chance of investors losing large amounts of coupons.

Shock ε_t	Statistic	l_{20}	PV of coupons and principal	Percentage change
1%	Min	429	99,790,111	−0.11%
	5% percentile	430	99,793,980	−0.11%
	95% percentile	432	99,795,946	−0.11%
	Max	435	99,796,337	−0.11%
	Mean	431	99,795,199	−0.11%
	Stdev	1	635	
5%	Min	435	99,768,435	−0.13%
	5% percentile	440	99,785,248	−0.12%
	95% percentile	453	99,795,809	−0.11%
	Max	465	99,797,493	−0.11%
	Mean	446	99,791,922	−0.11%
	Stdev	4	3,369	
10%	Min	444	99,744,368	−0.16%
	5% percentile	453	99,773,260	−0.13%
	95% percentile	481	99,798,092	−0.11%
	Max	504	99,786,999	−0.10%
	Mean	465	99,786,940	−0.12%
	Stdev	8	6,946	
25%	Min	465	99,663,820	−0.24%
	5% percentile	491	99,714,999	−0.19%
	95% percentile	569	99,783,484	−0.12%
	Max	641	99,793,361	−0.11%
	Mean	528	99,758,254	−0.14%
	Stdev	24	21,573	
50%	Min	411	99,482,588	−0.42%
	5% percentile	483	99,557,089	−0.35%
	95% percentile	699	99,751,549	−0.15%
	Max	825	99,774,291	−0.13%
	Mean	584	99,663,909	−0.24%
	Stdev	65	69,833	

Table 5.7: Sensitivity Testing (Type 1)

5.2.2 Case 2: Survivor bond type 2

In this case, the future coupon payments of the survivor bond are a function of the lender's aggregate loss amount in each period, which is affected by both the number of survivors and the single loss amount of each repaid loan. In this example, the values of trigger $\hat{\underline{L}}_t$ are set to be the 95% percentile of the aggregate loss distribution in each period. To take account of the randomness of mortality improvement, a 1% shock ε_t is attached to the projected force of mortality $\widehat{\mu}_{x,t}$. Following the example in the last section of a portfolio of l_0 loans borrowed by persons aged x with the identical house value H_0, in each period t, the bond holder will receive coupon

$$C_t = \begin{cases} C & \text{if } \underline{L}_t \leq \hat{\underline{L}}_t \\ C - \left(\underline{L}_t - \hat{\underline{L}}_t\right) & \text{if } \hat{\underline{L}}_t < \underline{L}_t \leq C \\ 0 & \text{if } \underline{L}_t > C \end{cases} .$$

This is equivalent to

$$C_t = C - \left[\left(\underline{L}_t - \hat{\underline{L}}_t\right), 0\right]_+ + \left[\left(\underline{L}_t - \hat{\underline{L}}_t\right) - C, 0\right]_+ .$$

Taking the expectation on both sides,

$$E\left(C_t\right) = C - E\left[\left(\underline{L}_t - \hat{\underline{L}}_t\right), 0\right]_+ + E\left[\left(\underline{L}_t - \hat{\underline{L}}_t\right) - C, 0\right]_+ . \tag{5.16}$$

Substituting Equation (5.16) into (5.9), the pricing equation of the survivor bond type 2 is

$$V = Fv^T + \sum_{k=1}^{T} v^k \left\{C - E\left[\left(\underline{L}_t - \hat{\underline{L}}_t\right), 0\right]_+ + E\left[\left(\underline{L}_t - \hat{\underline{L}}_t\right) - C, 0\right]_+\right\} \tag{5.17}$$

Since it is impossible to find out the distribution of $\underline{L}_t$, the simulation techniques are used. A numerical example follows.

A numerical example

Using the algorithm in Chapter 2, the trigger values $\hat{\underline{L}}_t$ in each year are calculated as the average of 1000 simulation trials. The results are listed in Table (5.8).

The price of the survivor bond is calculated in Table (5.9).

Period	Trigger $\hat{L}_t$	Period	Trigger $\hat{L}_t$	Period	Trigger $\hat{L}_t$
1	0	14	2,281,434	27	17,132,581
2	0	15	2,907,624	28	17,435,844
3	0	16	3,681,598	29	17,452,028
4	0	17	4,446,271	30	17,168,945
5	0	18	5,419,246	31	15,990,877
6	0	19	6,587,557	32	15,343,864
7	0	20	7,955,241	33	13,715,978
8	116,142	21	9,333,113	34	11,531,379
9	278,462	22	11,068,592	32	10,051,137
10	446,561	23	12,229,021	36	8,538,117
11	837,594	24	13,476,656	37	6,651,498
12	1,182,790	25	15,116,196	38	4,828,433
13	1,641,429	26	16,262,309		

Table 5.8: The projected triggers with simulation

Number of loans	1000
Initial house value	\$100,000
Lump sum borrowed	\$50,000
Face value of straight bond	\$100,000,000
Face value of survivor bond	\$100,000,000
Coupon rate for both bonds	6.5% p.a.
Annual aggregate cash flow out of SPC	\$6,500,000
Straight bond price	\$100,000,000
Survivor bond price	\$98,504,875
Premium paid to SPC	\$1,495,125

Table 5.9: Calculation of the bond price (Type 2)

The results show prices for survivor bonds for a group of $1,000$ loans with loan amount of $\$50,000$ per person and the trigger levels calculated above. The annual aggregate cash outflow of the SPC is $\$6,500,000$ and the coupon rate for both the straight bond and the survivor bond is 6.5% . The price of the survivor bond is \$985.04 per $\$1,000$ of face value. As we can see, the premium here is much larger than that of survivor bond type one, since much greater risks are involved in this case. But the total premium is still very small relative to the total amount of loans: $1,495,125/\,(1000 \times 50,000)$ which equals 2.99%.

Sensitivity testing

To examine the impact of mortality improvement in some extreme scenarios, the value of the shock parameter ε_t is increased. The results of sensitivity testing are summarized in Table (5.10).

The results show that the impact of mortality shock is very limited in terms of the whole investment. Since in this case other risk variables are not controlled, it is difficult to tell exactly how much the change can be attributed to mere mortality improvement. But the results show no matter how great the mortality shock is, the present value of the bond does not change much, which means the present value is not sensitive to mortality improvement.

Shock ε_t	Statistic	PV of coupons and principal	Percentage change
5%	Min	59,169,754	−39.93%
	5% percentile	88,218,673	−10.44%
	95% percentile	100,000,000	1.52%
	Max	100,000,000	1.52%
	Mean	98,250,787	−0.26%
	Stdev	5,346,482	
10%	Min	59,927,863	−39.16%
	5% percentile	86,700,844	−11.98%
	95% percentile	100,000,000	1.52%
	Max	100,000,000	1.52%
	Mean	98,196,112	−0.31%
	Stdev	1,803,888	
25%	Min	61,486,654	−37.58%
	5% percentile	86,745,006	−11.94%
	95% percentile	100,000,000	1.52%
	Max	100,000,000	1.52%
	Mean	98,070,043	−0.44%
	Stdev	5,708,053	
50%	Min	61,880,827	−37.18%
	5% percentile	87,868,901	−10.80%
	95% percentile	100,000,000	1.52%
	Max	100,000,000	1.52%
	Mean	98,267,554	−0.24%
	Stdev	1,732,446	

Table 5.10: Sensitivity Testing (Type 2)

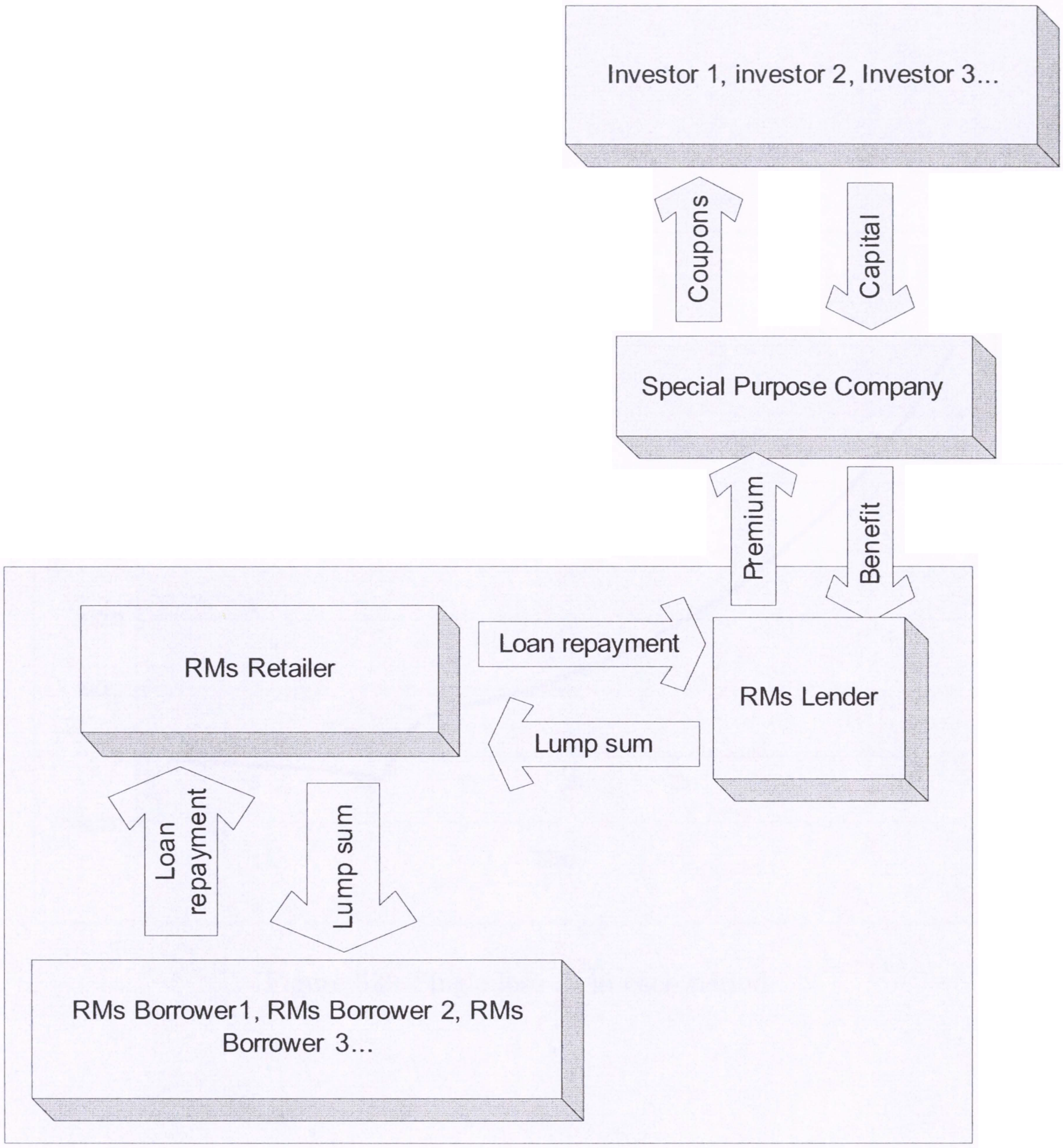

Figure 5.1: The process of reverse mortgage securitization.

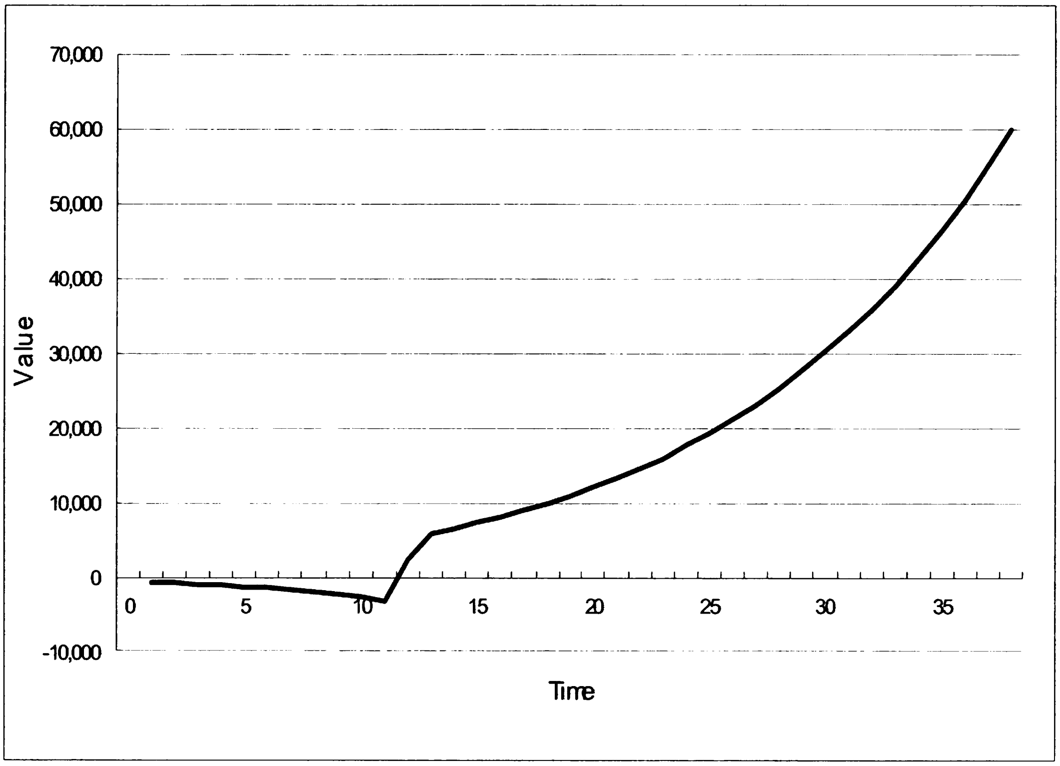

Figure 5.2: Single loss L_t in each period

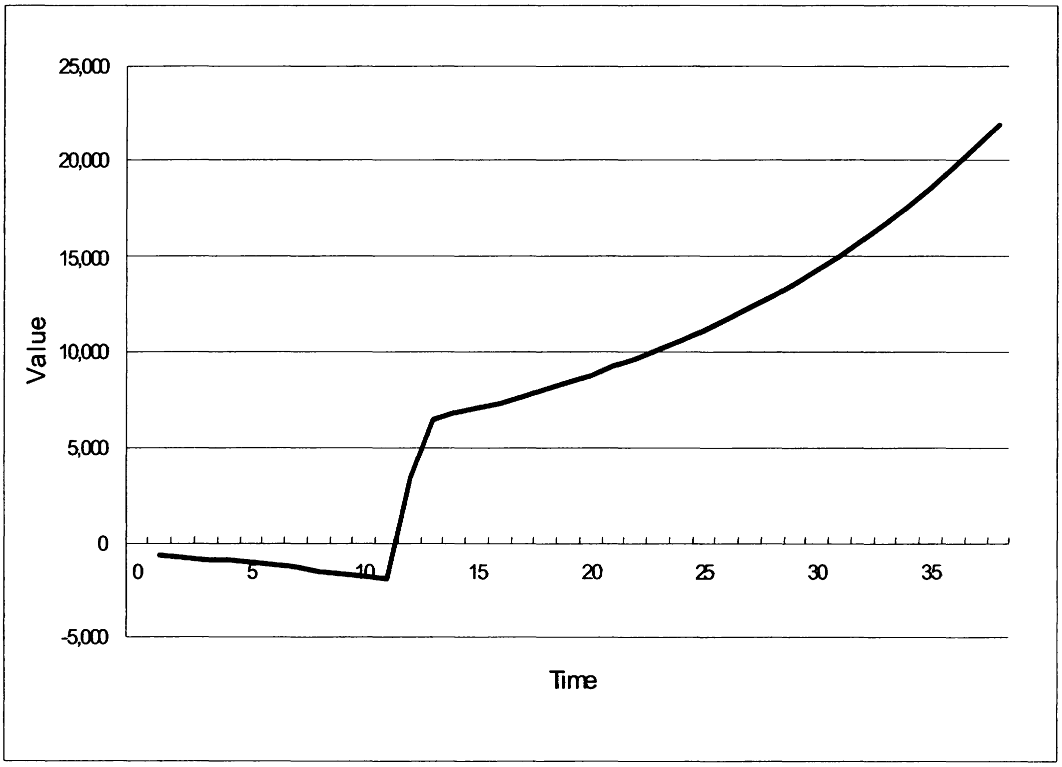

Figure 5.3: Appreciation A_t of each loss in each period

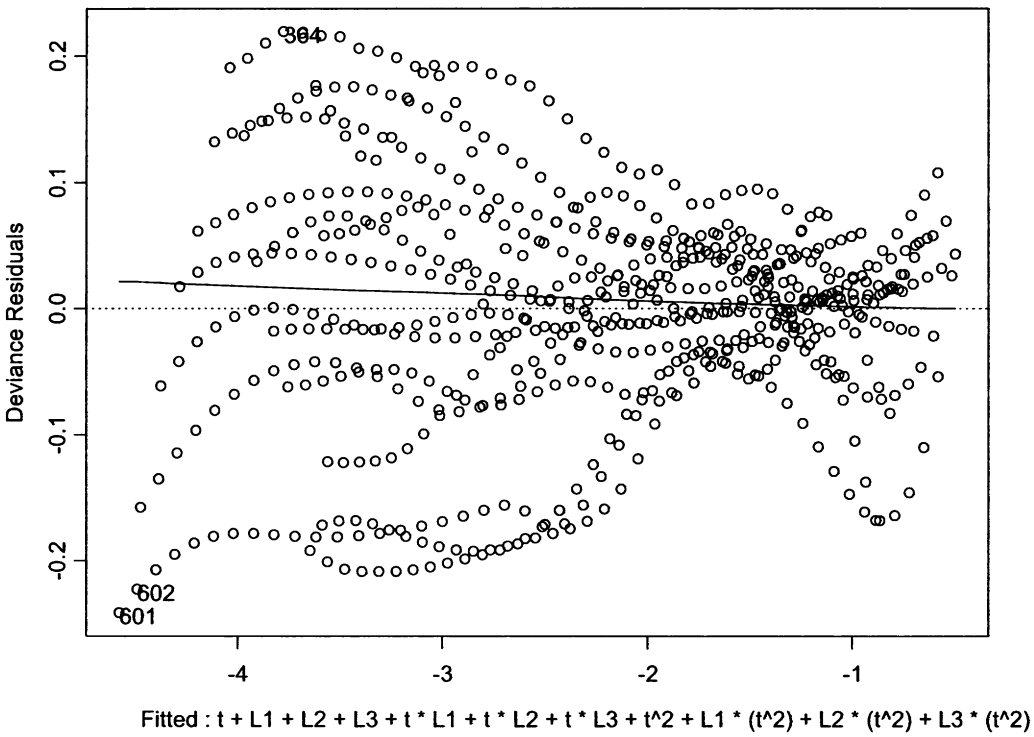

Figure 5.4: Residuals vs Fit

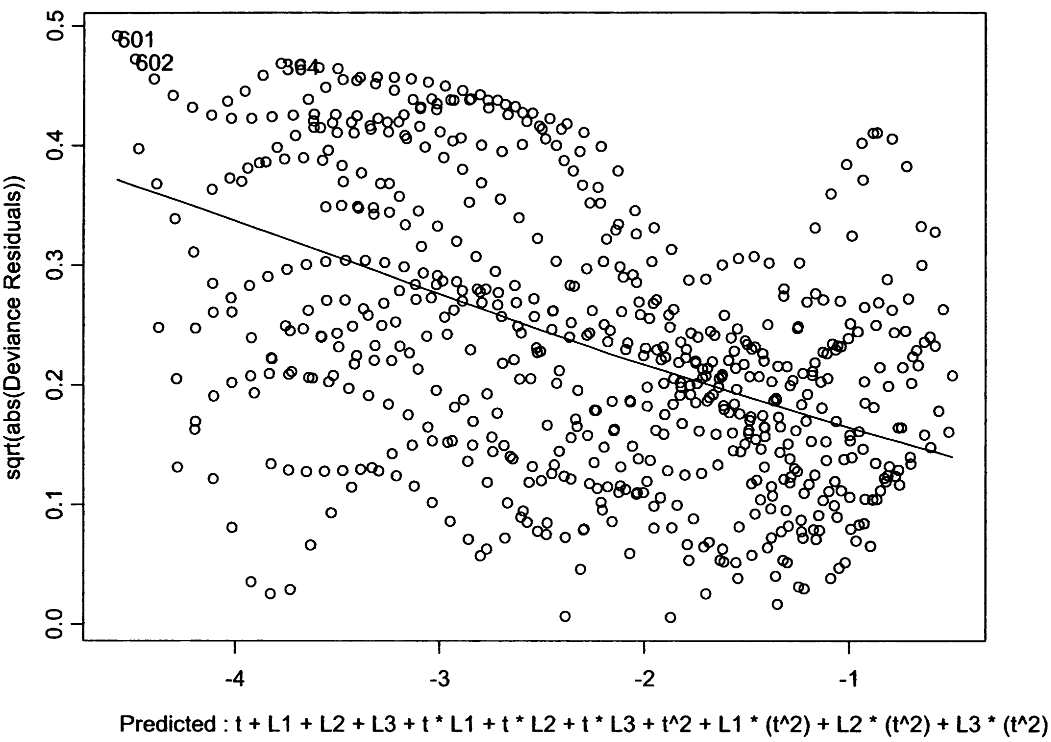

Figure 5.5: Sqrt Abs Residuals vs Fit

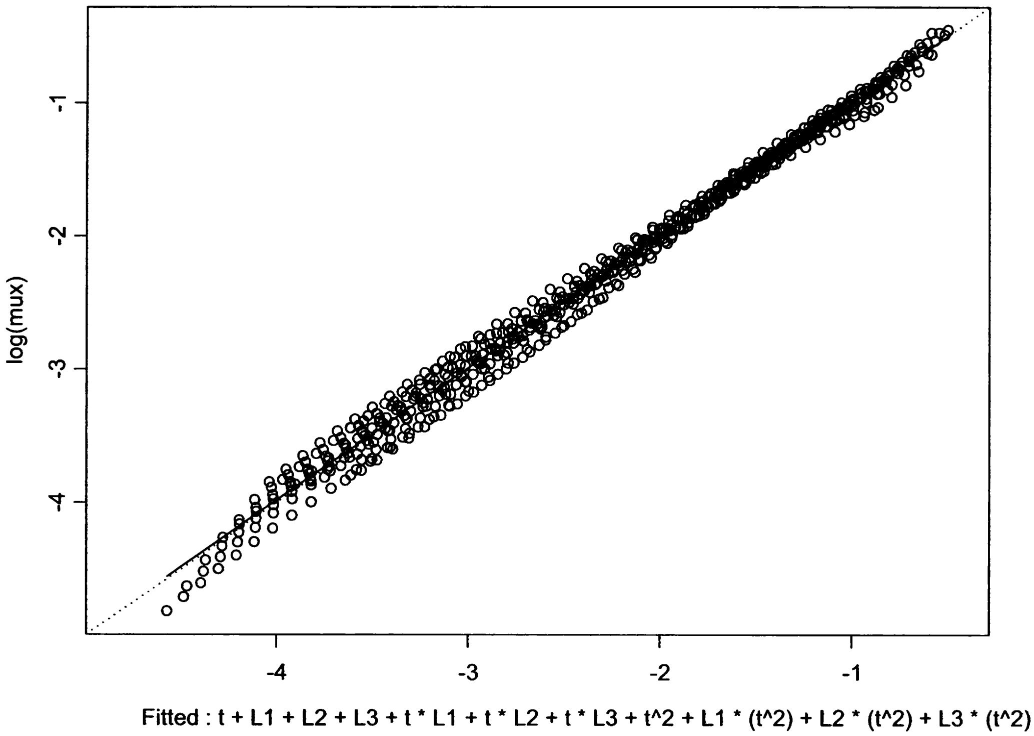

Figure 5.6: Residuals Normal QQ

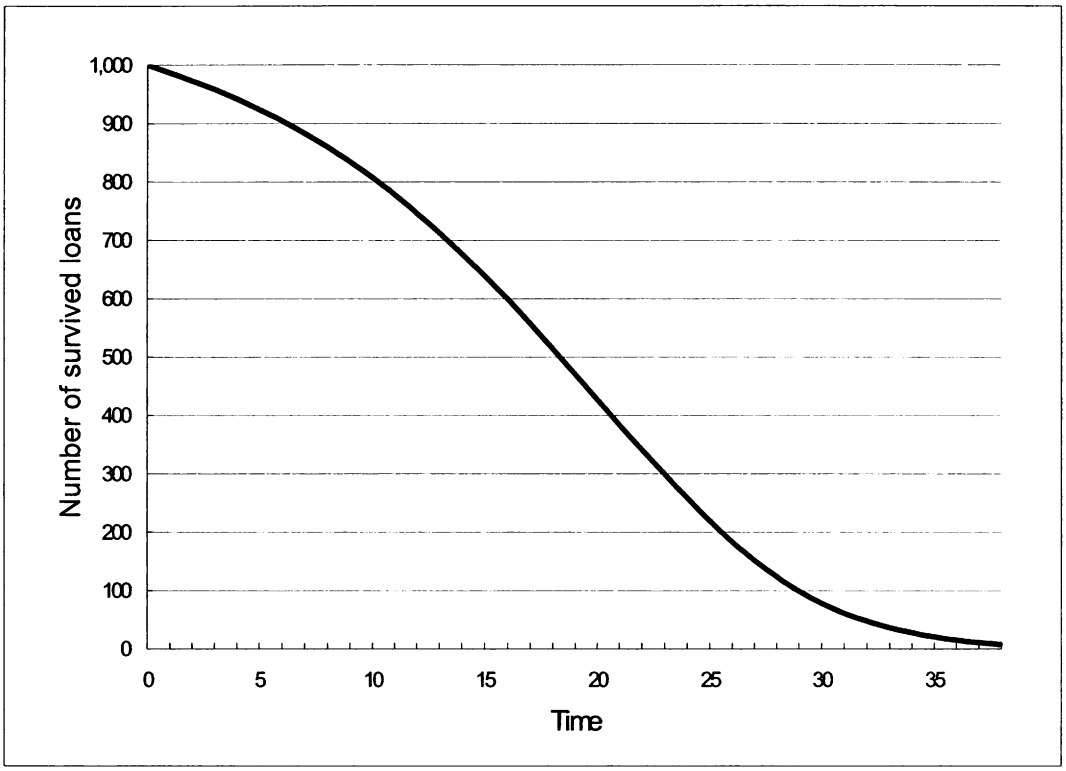

Figure 5.7: Projected trigger values

Chapter 6

Conclusion

6.1 Summary of results

The reverse mortgage is a promising product with many benefits, which has matured and become increasingly popular in recent years. Due to the various risks involved in reverse mortgages, especially the longevity risk component, the development of the product has to some extent been stunted. The risks are compulsorily insured in the United States to protect the loan borrowers. In this thesis, I suggest a more flexible method, namely securitization, to deal with the risks to the lender, particularly the longevity risk.

In Chapter 2, a review of securitization is provided. Securitization as a form of financial engineering has existed for little more than 30 years, but it has had a significant impact on the financial industry, and most recently the insurance industry. Compared to traditional insurance, securitization has certain advantages. These advantages include: greater coverage of risks; the elimination of counterparty risk; and more favorable tax treatment. Securitization of mortality risk, is one of the major developments in the securitization of insurance risks. It has been the subject of academic interest since the rapid improvement in mortality rates. Several securitization products have been proposed and discussed, and even put into practice in the insurance industry recently. Following Cox's approach (2000, [17]), an economic justification of securitization of mortality risk is provided in this chapter based on Markowitz mean-variance portfolio theory.

In Chapter 3, the concept of the reverse mortgage product is introduced and its development is reviewed. I find that although the product experienced slow growth in its infancy, it has been understood and accepted more widely, which has resulted in rapid growth in recent years. The advantages of the reverse mortgage over other ordinary collateralized loans is discussed. The unique features of the product not only allow the borrower to convert their home equity into a liquid asset, but also to hedge their longevity risk, which is an important financial risk for retirees. These benefits to the borrower, however, pose a huge risk to the lender. A brief discussion of the risks inherent in the product is provided in this chapter. The three major risk variables are interest rates, house prices and mortality rates. Other risks include the maintenance risk and the expense risk. At the end of this Chapter, the trend towards transferring the mortality risk in reverse mortgages through securitization is examined.

In Chapter 4, to acquaint the readers with the reverse mortgage and its risks at a higher level, a pricing model for the product adopted by Piggott et. al (2003, [48])is introduced. The four input variables of the pricing model are interest rates, house price, mortality rates and the risk premium that the lender charges. How these variables affect the price are analyzed in detail. A numerical example to illustrate the pricing of the product is then provided. In the example, the well-known Vasicek model is adopted to project the interest rates. For simplicity, the geometric Brownian motion is used to model the systematic part of the house price. To capture the impact of the borrower's maintenance behavior and regional economic fluctuation, a random shock is attached to represent the idiosyncratic part of the house price. The correlation between the interest rate and the house price are discussed in this chapter and tested later. To project the most important variable — mortality rate — the Gompertz mortality model adopted by Frees et. al (1996, [27]) is used.

The model is then calibrated to the Australian dataset including the ten year government bond yield (1980-2005), the median house price of eight capital cities (1980-2005) and the Australian male life tables (2002). Certain models to calculate the risk premiums are also suggested. The simulation results of two different sized portfolios are discussed at the end of the chapter. The results show that the larger portfolio, the greater the maximal safe loan amount it allows. The results also reveal that the average loss to the lender does not necessarily increase with the length of the loan. Instead, in our example the most financially stressful period for the lender is the several years between years 28-33 rather than the last few years.

In Chapter 5, the securitization technique is applied to the reverse mortgage product. Following a similar approach used by Lin and Cox (2004, [40]), a model is provided to securitize the longevity risk in a reverse mortgage. The cash flows for each component of the transaction are analyzed in detail. Several examples of longevity securities based on the model are provided, including two types of survivor bonds and a survivor swap. The formulae for pricing the securities are provided subsequently. A numerical example for pricing each type of survivor bond is also provided.

For survivor bond type 1, the interest rate and house appreciation rate are assumed to be constant to emphasize the effect of longevity securitization. To grasp the stochastic feature of the mortality improvement rates, the GLM model suggested by Renshaw et al. (1996, [50]) is used. The model is calibrated to the Australian male life tables (1881-2002). The number of parameters are chosen based on both accuracy and efficiency. Using the fitted model, the triggers for the survived loans are calculated and the survivor bond prices obtained. The results show that for the lender, the insurance premium is cheap relative to the total value of the portfolio and the period of protection. To test its value to the investor, a sensitivity analysis is undertaken to show how much the bond value will change in responds to mortality shocks. If mortality rates improve by 5%, the investor will lose 0.12% of the total value of expected coupons. Even a dramatic mortality improvement shock of 50% will only result in the investors losing less than 3.7% of the total value of expected coupons. Therefore there is very little chance of investors losing large amounts of coupons.

For survivor bond type 2, the randomness of interest rates and house prices is taken into account. The same interest rate and house price models adopted in pricing reverse mortgages are used to project the inputs. A simulation approach is applied because it is very difficult to solve the equation of bond prices analytically. Since more risks are involved in this case, the insurance premiums are more expensive than in the case of survivor bond type 1. But compared to the total loan amount of the portfolio, it still only amounts to 2.99%. On the other hand, the investor can expect a maximal loss of 2.01% of the total expected coupons even in the worst case scenario where a combination of shocks occur. Since none of the three risk variables are controlled, it is impossible to tell exactly how much the change can be attributed to each shock in this case, but the investor does not lose a large portion of their returns even in the most adverse situations.

This thesis shows that the securitization model is a good method to control the longevity risk in reverse mortgages. Examples are given including two types of survivor bond and a survivor swap. The survivor bonds are priced using analytical and simulation methods. The results show that through these securitization schemes, long term protection for the reverse mortgage lender can be obtained at relatively cheap prices. The impact of mortality improvement shocks on the bond prices are examined using sensitivity testing. The test results reveal that even in the situation where mortality dramatically improves, there is little chance that survivor bond investors will lose large portion of the investment. Given the many benefits of mortality securitization, I believe it can help the development of reverse mortgage products.

6.2 Limitation of the results

The readers of this thesis should be aware that there are simplifying assumptions in this analysis. For example, in the case of the survivor bond type 1, the interest rates and the house appreciation rate are controlled to be constant. In reality, these rates are never constant. In addition, there are other limitations due to model choice and data limitation.

6.2.1 Model choice

In choosing the house price model, for simplicity the geometric Brownian motion is used to model the systematic part of the house price. In the real world, the house price model could be extremely complicated when the effects of many economic variables are taken into account. Further, when applying this model to calculate the correlation between the interest rate and the house price, our results show that although the two are correlated in the lagged sense, the correlation is not simply positive or negative. I choose to ignore the correlation in the further analysis considering that a linear correlation test may not be convincing due to the simplified house price model. And this simplification may affect the price of the reverse mortgage as well as the prices of the survivor bonds.

6.2.2 Data limitation

In calculating the premiums that the lender charges, the most straightforward way is to apply the Distortion methods (Wang Transformation or Elliptical Transformation) to the real world reverse mortgage price data. Without the reverse mortgage data, I can only suggest some approximate models to estimate the total risk premiums, which still require certain annuity price data. However, unable to get the appropriate annuity price data, for this analysis, I have had to choose the total risk premium subjectively and conservatively. Therefore the risk premium may be lower than that in the industry, and hence the numerical results of the maximal safe loan amount derived could be overstated.

When pricing the reverse mortgage using simulation, the number of survived loans is only generated to 38 years after the loan starts. The reason is that the Australian life tables used to calibrate the Gompertz model are truncated to age 100. Although the Gompertz model is parametric and can be used to generate death numbers for older ages, it does not fit very well in very old ages. For this reason and simplicity, any loss beyond the 38th year is forced to occur in the 38th year. This approximation will overstate the numbers and the aggregate amount of losses in the last year. Subsequently, the numerical results of the maximal safe loan amount could be understated since the losses could grow greater if they were not brought forward to the 38th year.

What is more, since table mortality data is used rather than the real mortality experiences obtained from reverse mortgage provider, the basis risk is introduced.

6.3 Further work

In this thesis, only pricing examples of survivor bonds are given. It has been argued that survivor swaps have advantages over survivor bonds and can be applied between the reverse mortgage lender and the life insurer. It would be interesting to see how a survivor swap is priced in reverse mortgages. Dawson (2002, [21]), Dowd (2006, [25])and Lin et. al (2004, [39]) are useful references on this topic.

A copula approach could also be applied to model the correlation between the interest rate and the house price. This would allow the researcher to investigate the effects of the correlation on the pricing and the risk management of reverse mortgages. Lastly, a calibration of our models to real data would reveal the actual effect of securitization of longevity risk in reverse mortgages. These topics are left for future research.

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Appendix A

OLS method for model fitting

- In the Gompertz model, the sum of squares is

$$SSE = \sum_t \left[{}_tp_x^{table} - \exp\left(-\frac{B}{\ln c}c^x\left(c^t - 1\right)\right)\right]^2,$$

To minimize the SSE, we take derivatives of B and c to get the first order conditions (FOCs):

$$\frac{\partial SSE}{\partial B} = 2\sum_t \left[{}_tp_x^{table} - \exp\left(-\frac{B}{\ln c}c^x\left(c^t - 1\right)\right)\right] \frac{c^{65}\left(c^t - 1\right)}{\ln c} \exp\left(-\frac{B}{\ln c}c^x\left(c^t - 1\right)\right)$$

$$\begin{aligned} & \frac{\partial SSE}{\partial c} \\ = & -2\sum_{tt} \left[p_x^{table} - \exp\left(-\frac{B}{\ln c}c^x\left(c^t - 1\right)\right)\right] \left[\frac{Bc^{64}}{\ln c}\left(c^t - 1\right)\left(\left(c^t - 1\right)\left(\frac{1}{\ln c} - 65\right) - tc^t\right)\right] \end{aligned}$$

Equating the two FOCs to zero and solving them simultaneously give us the B and c that minimize the squared errors. Unfortunately, the system of equations can not be solved analytically, so we solve them in *Excel* numerically. The *Excel* macro is attached in Appendix E.

- In Vasicek interest model, we can find the parameters α and β through the following formula:

$$\widehat{\alpha} = \frac{\sum_{k=1}^{N} r_{t_k} \cdot \sum_{k=1}^{N} r_{t_k-\Delta t} - N \sum_{k=1}^{N} r_{t_k-\Delta t} r_{t_k} - \left(\sum_{k=1}^{N} r_{t_k-\Delta t} \right)^2 + N \sum_{k=1}^{N} r^2_{t_k-\Delta t}}{\Delta t \cdot \left[N \sum_{k=1}^{N} r^2_{t_k-\Delta t} - \left(\sum_{k=1}^{N} r_{t_k-\Delta t} \right)^2 \right]}$$

$$\widehat{\beta} = \frac{\sum_{k=1}^{N} r_{t_k} \cdot \sum_{k=1}^{N} r^2_{t_k-\Delta t} - \sum_{k=1}^{N} r_{t_k-\Delta t} \cdot \sum_{k=1}^{N} r_{t_k-\Delta t} r_{t_k}}{\sum_{k=1}^{N} r_{t_k} \cdot \sum_{k=1}^{N} r_{t_k-\Delta t} - N \sum_{k=1}^{N} r_{t_k-\Delta t} r_{t_k} - \left(\sum_{k=1}^{N} r_{t_k-\Delta t} \right)^2 + N \sum_{k=1}^{N} r^2_{t_k-\Delta t}}$$

where $t_k - \Delta t = t_{k-1}$and N is the number of observations without the starting value. The standard deviation is

$$\widehat{\sigma} = \sqrt{\frac{MSE\,(r_t)}{\Delta t \cdot (N-2)}},$$

with

$$MSE\,(r_t) = \sum_{k=1}^{N} \left[r_{t_k} - \left(\Delta t \widehat{\alpha} \widehat{\beta b} + (1 - \Delta t \widehat{\alpha})\, \widehat{r}_{t_k-\Delta t} \right) \right]^2$$

- In the geometric Brownian motion model for house price, μ_H and σ_H are estimated by

$$\widehat{\mu} = \frac{1}{\Delta t N} \sum_{k=1}^{N} \left(\ln \frac{H_{t_k}}{H_{t_k-\Delta t}} - \widehat{\mu} \right)^2$$

Appendix B

Integration by parts

For a random variable X with a Probability Distribution Function (PDF) $f_X(x)$,Cumulated Distribution Function (CDF) $F_X(x)$, if $E(X) < \infty$, by definition of expectation, we know that

$$E\left[(X-k)_+\right] = \int_k^\infty (x-k)\, f_X(x)\, dx. \tag{B.1}$$

Integrate the right-hand side by applying integration by parts:

$$u = -(x-k) \quad \text{and} \quad v = 1 - F_X(x).$$

so that $du = -dx$ and $dv = -f_X(x)$. Therefore

$$E\left[(X-k)_+\right] = -(x-k)\left[1-F(x)\right]|_k^\infty + \int_k^\infty \left[1-F(x)\right] d(x-k)$$

$$E\left[(X-k)_+\right] = -0 + 0 + \int_k^\infty \left[1-F(x)\right] dx$$

$$E\left[(X-k)_+\right] = \int_k^\infty \left[1-F(x)\right] dx. \tag{B.2}$$

For a standard normal random variable $X\tilde{}N\ (0,1)$ with a *PDF* $\phi(x)$ and a *CDF* $\Phi(x)$. $\phi(x)$ is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right).$$

Differentiating on both sides, we have

$$\phi'(x) = -x\phi(x). \tag{B.3}$$

From equation (B.2), we get

$$E\,(X-k)_+ = \int_k^\infty [1-\Phi(x)]\,dx.$$

Again we integrate the right-hand side by applying integration by parts:

$$u = x \quad \text{and} \quad v = 1-\Phi(x)$$

so that $du = dx$ and $dv = -f_X(x)$. Therefore

$$E\left[(X-k)_+\right] = x\,[1-F(x)]\,|_k^\infty - \int_k^\infty x d\,[1-F(x)]$$

$$E\left[(X-k)_+\right] = 0 - k\,[1-\Phi(x)] + \int_k^\infty x\phi(x)\,dx.$$

Using the result in equation (B.1), we have

$$E\left[(X-k)_+\right] = -\int_k^\infty \phi'(x)\,dx - k\,[1-\Phi(x)]$$

$$E\,(X-k)_+ = \phi(k) - k\,[1-\Phi(k)]\,.$$

Appendix C

GLM for projecting the future mortality

As discussed in Chapter 3, many projection methods of mortality rates have been proposed in the literature. In Chpater 5, we use the generalized linear model first suggested by Renshaw et al. (1996, [50]). According to the Renshaw model, the force of mortality is a log-linear function of age x and time t,

$$\mu_{(x,t)} = \exp\left(\beta_0 + \sum_{j=1}^{s} \beta_j L_j\left(x'\right) + \sum_{i=1}^{r} \alpha_i t'^i + \sum_{i=1}^{r}\sum_{j=1}^{s} \gamma_{i,j} L_j\left(x'\right) t'^i\right), \tag{C.1}$$

where r, s are the number of parameters and $L_j\left(x\right)$ is the Legendre polynomial defined below:

$$L_0\left(x\right) = 1$$
$$L_1\left(x\right) = x$$
$$L_2\left(x\right) = \frac{3x^2 - 1}{2}$$
$$L_3(x) = \frac{5x^3 - 3x}{2}$$
$$\cdot$$

$$\left(n+1\right) L_{n+1}\left(x\right) = (2n+1)xL_n\left(x\right) - nL_{n-1}\left(x\right)$$

Equation (C.1) can be expressed as

$$\mu_{(x,t)} = \exp\left(\sum_{j=0}^{s} \beta_j L_j\left(x'\right)\right) \exp\left(\sum_{i=1}^{r} \alpha_i t'^i + \sum_{i=1}^{r}\sum_{j=1}^{s} \gamma_{i,j} L_j\left(x'\right) t'^i\right) \tag{C.2}$$

Notice that the first term in the right-hand side is a special case of Gompertz–Makeham law of mortality:

$$\mu_x = \mathrm{GM}\,(r,s) = \sum_{i=1}^{r-1} \alpha_i x^i + \exp\left(\sum_{j=0}^{s} \beta_j \left(x^j\right)\right).$$

To estimate the unknown parameters α_i, β_j and $\gamma_{i,j}$, the actual number of deaths $a_{(x,t)}$ are modelled as independent Poisson response variables $A_{(x,t)}$ of a generalized linear model with mean and variance given by

$$E[A_{(x,t)}] = m_{(x,t)} = R^c_{(x,t)} \mu_{(x,t)}$$

$$var\left[A_{(x,t)}\right] = \phi m_{(x,t)}$$

$R^c_{(x,t)}$ is the central exposed-to-risk, and ϕ is a scale parameter.

$$\log m_{(x,t)} = \log R^c_{(x,t)} + \log \mu_{(x,t)} = \eta_{(x,t)}$$

$$\log m_{(x,t)} = \log R^c_{(x,t)} + \beta_0 + \sum_{j=1}^{s} \beta_j L_j\left(x'\right) + \sum_{i=1}^{r} \alpha_i t'^i + \sum_{i=1}^{r}\sum_{j=1}^{s} \gamma_{i,j} L_j\left(x'\right) t'^i$$

Estimation of the parameters is carried out using the quasi-log-likelihood approach, which, in this case, involves maximizing the expression

$$\frac{1}{\phi}\sum_{x,t}\left(-m_{(x,t)} + a_{(x,t)} \log m_{(x,t)}\right)$$

To determine the optimum values of r and s, the improvement in the scaled deviance for successive increases in the values of r and s is compared with a 2 random variable with 1 d.f. as an approximation. The optimum values chosen are the minimum values of r and s beyond which improvement in the deviance is not statistically significant.

The unscaled deviance corresponding to the predicted rates, $\widehat{\mu}_{(x,t)}$, is

$$D\,(c,f) = 2\sum_{x,t}\left[a_{(x,t)} \log\left(\frac{a_{(x,t)}}{\widehat{m}_{(x,t)}}\right) - \left(a_{(x,t)} - \widehat{m}_{(x,t)}\right)\right]$$

where

$$\widehat{m}_{(x,t)} = R^c_{(x,t)} \widehat{\mu}_{(x,t)},$$

Appendix D

Life insurance securitization deals since 1996

Life Insurance Securitization Deals Since 1996			
Purpose	Amount (m)	Date	Issuer
Liquidity/M&E Securitization	$900+	1996-1998	American Skandia
VIF Securitization	€731	1998-2002	Hannover Re
VIF Monetization	£260	Apr 1998	NPI
Closed Block Monetization	$1,750	Dec 2001	Prudential
Closed Block Monetization	$300	Apr 2002	MONY
Reg XXX Financing	$1,150	Jul 2003	Genworth I
VIF Monetization	$150	Jun 2004	Forethought
VIF Monetization	£400	Oct 2003	Barclays Life
Reg XXX Financing	$600	Nov 2004	Banner Life
Reg XXX Financing	$850	Dec 2004	Genworth II
VIF Monetization	£380	Dec 2004	Friends Provident
Source: Cowley and Cummins (2005)			

Appendix E

Program Codes

E.1 Excel macro for fitting the Gompertz

```
'fit Gompertz to the 2000-02 Australian mortality experience

Sub fit_2000()

Application.ScreenUpdating = False
test = Range("C5") + Range("C6")

Do Until test = 0
 differentials = Range("C5") + Range("C6")
 For i = 1 To 30
 Sheets("2000").Range("I65").GoalSeek Goal:=0,
  ChangingCell:=Sheets("2000").Range("C5")
 Sheets("2000").Range("J65").GoalSeek Goal:=0,
  ChangingCell:=Sheets("2000").Range("C6")
  Next i
 test = differentials - Range("C5") - Range("C6")
Loop

Application.ScreenUpdating = True

End Sub

'fit Gompertz to all the past Australian mortality experience

Sub fit_all()
```

```
Application.ScreenUpdating = False
For j = 0 To 15
For s = 0 To 38
Worksheets("male").Cells(45 + s, 3) =
Worksheets("male").Cells(2 + s, 2 + j)
Next s
test = Range("M47") + Range("M48")

Do Until test = 0
 differentials = Range("M47") + Range("M48")
 For i = 1 To 20
 Sheets("male").Range("H85").GoalSeek Goal:=0,
  ChangingCell:=Sheets("male").Range("M47")
 Sheets("male").Range("I85").GoalSeek Goal:=0,
  ChangingCell:=Sheets("male").Range("M48")
  Next i
 test = differentials - Range("M47") - Range("M48")
Loop

Worksheets("results").Cells(6 + j, 2)
  = Worksheets("male").Cells(50, 13)
Worksheets("results").Cells(6 + j, 3)
  = Worksheets("male").Cells(51, 13)
Next j

Application.ScreenUpdating = True
End Sub
```

E.2 Matlab code for simulations

```
% Generate random life time of each contract.                       %
% Simulate the stochastic process of interest and house price%
% Find the maximal safe loan amount given the risk premium       %
% Plot the loss distribution and other figures                      %
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
clear all;
p=10;  % number of runs
n=1000;  % number of loans in each portfolio
l=zeros(p,38); % number of survivors
```

```
d=zeros(p,38); % number of terminations
sumloss=zeros(p,38); % total loss in each year
meanloss=zeros(p,38); % average loss in each year
meanlossd=zeros(p,38); % 1-period discounted average loss in each year
interest=ones(p,38);
for s=1:p
    %%%%% Random life time  %%%%%%%
    % age=62; mode=82.118134; sigma=9.786905707;   % 2000 data fitted
    age=62; mode=82.3782; sigma=9.73327;   % projeted future mu fitted
    T=zeros(1,n); % life span for each loan
    for j=1:n  % Gompertz
        u1=rand(1);
        T(j) = ceil(log(1 - log(u1) / exp((age - mode) / sigma))
                   * sigma + 0.5);
        if T(j)>38
            T(j)=38;
        end;
    end
    %%% Interest rate and house price appreciation rate %%%
    rho=0; lag=0; % correlation parameters
    r0=6.5; a=0.5757; b=4.8825; sigmaR=4.7891; % Vasicek
    meanH=0.0354; sigmaH=0.1003; % general shock
    sigmaS=0.08; % individual shock
    general=ones(1,38);
    houseapp=ones(n,38);
    for i=1:38
        n1=randn(1); n2=randn(1);
        eI(i) = n1;
        if i >= lag
            eH(i) = rho * eI(i - lag) + sqrt(1 - rho ^2) * n2;
        end
        k = i - lag;
        if k <= 1
            interest(s,k) = r0 / 100;
            general(k) = exp(meanH + sigmaH * eH(i));
        elseif k > 1
            interest(s,k) =
(a * b + (1 - a) * interest(s,k - 1) * 100 + sigmaR * eI(i)) / 100;
            general(k) = exp(meanH + sigmaH * eH(i)) * general(k - 1);
        end
```

```
end
for j=1:n
    n3=randn(1);
    houseapp(j,1) = general(1) - 1 + sigmaS * n3;
    for i=2:38
        n4=randn(1);
        houseapp(j,i) = general(i) / general(i - 1) - 1 + sigmaS * n4;
    end
end
%%%%%Accumulation factors %%%%%%
accL=ones(1,38); % accmulation factor for loan
accCL=ones(1,38); % accmulation factor for cost of capital
accH=ones(n,38); % accmulation factor for house appreciation
premL=0.075; % total risk premium lender charges
premC=0.0375;  % total risk premium lender is charged
disc=0.065;  % discount rate
accL(1)=1+interest(s,1)+premL;
accCL(1)=1+interest(s,1)+premC;
% Lump sum and cost acc factor
for i=2:38
    accL(i)=(1+interest(s,i)+premL)*accL(i-1);
    accCL(i)=(1+interest(s,i)+premC)*accCL(i-1);
end
% House price acc factor
for j=1:n
    accH(j,1)=1+houseapp(j,1);
    for i=2:38
        accH(j,i)=(1+houseapp(j,i))*accH(j,i-1);
    end
end
%%%%%%%%% Loan amount, house price and losses %%%%
Q0=50000;   % starting value
H0=100000;  % initial house value
QL=ones(1,n);  % accumulated loan amount when repaid
HT=ones(1,n);  % accumulated house price when repaid
VL=ones(1,n);  % recoverable amount (minimum of the above two)
costL=ones(1,n);  % accumulated cost of capital when terminated
lossL=ones(1,n);  % the diff between cost and recoverable amount
% pvlossL=ones(1,n);
for j=1:n
```

```
        QL(j)=Q0*accL(T(j));
        HT(1,j)=H0*accH(j,T(j));
        VL(j)=min(QL(j),HT(j));
        costL(j)=Q0*accCL(T(j));
        lossL(j)=costL(j)-VL(j);
        %     pvlossL(j)=lossL(j)/(1+disc)^T(j);
    end
    %%%%% Minimization of the loss and take average %%%%
    [Q0,SL]=fminbnd(@pvL,1,H0,optimset('Display','off'),
                accL,HT,accCL,disc,T,n);
    iterL(s)=Q0; % find the lump sum minimizes the APV of loss

    [m]=find(T==1); % find the loans terminated at t=1
    d(s,1)=length(m); % number of terminations
    l(s,1)=n-length(m); % number of survivors
    sumloss(s,1)=sum(lossL(m)); % total loss
    if length(m)~=0
        meanloss(s,1)=mean(lossL(m)); % average loss
        meanlossd(s,1)=meanloss(s,1)/(1+interest(s,1));
    end
    for i=2:38
        [m]=find(T==i);  % find the loan terminated at time i
        d(s,i)=length(m); % number of terminations
        l(s,i)=l(s,i-1)-length(m); % number of survivors
        sumloss(s,i)=sum(lossL(m)); % total loss
        if length(m)~=0
            meanloss(s,i)=mean(lossL(m));
            meanlossd(s,i)=meanloss(s,i)/(1+interest(s,i));
        end
    end
end
meansumloss=zeros(1,38);
mmloss=zeros(1,38);
mmlossd=zeros(1,38);
accmeanloss=zeros(1,38);
meand=zeros(1,38);
meanl=zeros(1,38);
trigger=zeros(1,38);
for i=1:38
    meansumloss(1,i)=sum(sumloss(:,i))/p;
```

```
    mmloss(1,i)=sum(meanloss(:,i))/p;
    mmlossd(1,i)=sum(meanlossd(:,i))/p;
    meand(1,i)=sum(d(:,i))/p;
    meanl(1,i)=sum(l(:,i))/p;
    trigger(1,i)=quantile(sumloss(:,i),0.95);
end
accmeanloss=zeros(1,38);
for i=1:38
    accmeanloss(i+1)=mmlossd(i+1)-mmloss(i);
end
meanL=sum(iterL)/p;
meanL

subplot (2,2,1), plot(meanl),
title('average survivors in the portfolio each year')
grid on
subplot(2,2,2), plot(mmloss),
title('average loss of all the
            terminated loans each year')
grid on
subplot(2,2,3), plot(meansumloss),
title('average total loss of the portfolio each year')
grid on
subplot (2,2,4), plot(accmeanloss),
title('differenced average loss
           of all the terminated loans each year')
grid on


figure(2)
for i=1:38
 subplot(5,8,i), hist(l(:,i),15),title(' ');
 grid on
end
figure(3)
for i=1:38
 subplot(5,8,i), hist(lossamtA(:,i),15),title(' ');
 grid on
end
```

```
figure(4)
subplot(3,1,1), plot(meanlossamtA),
title('Average loss amount in each year')
grid on
subplot(3,1,2), plot(meand),
title('Average terminated loans in each year')
grid on
subplot(3,1,3), plot(meanl),
title('Average survived loans in each year')
grid on

%%%%%%Funtions used in the scripts %%%%%%

function pvA=pvA(x,accA,HT,accCA,disc,T,n)
for j=1:n
    f(j)=(min(x.*accA(T(j)),HT(j))-x.*accCA(T(j)))
              ./((1+disc).^T(j));
end
pvA=abs(sum(f));

function pvL=pvL(x,accL,HT,accCL,disc,T,n)
for j=1:n
    f(j)=(min(x.*accL(T(j)),HT(j))-x.*accCL(T(j)))
              ./((1+disc).^T(j));
end
pvL=abs(sum(f));

%%%%% Survivor Bond Type 2 Pricing  %%%%%%%

p=1000;  % number of runs
n=1000;  % number of loans in each portfolio
l=zeros(p,38); % number of survivors
d=zeros(p,38); % number of terminations
sumloss=zeros(p,38); % total loss in each year
meanloss=zeros(p,38); % average loss in each year
interest=ones(p,38);
shockp=zeros(p,38);
c=zeros(p,38);
pvmbc=zeros(p,38);
pvmb=zeros(1,p);
```

```
for s=1:p

    for i=1:38
        shock=betarnd(1.49,147.51);
        shockp(s,i)=px(i)^(1-shock);
    end
    l(s,1)=ceil(n*shockp(1));
    d(s,1)=n-l(s,1);
    for i=2:38
        l(s,i)=ceil(l(s,i-1)*shockp(s,i));
        d(s,i)=l(s,(i-1))-l(s,i);
    end

    rho=0; lag=0; % correlation parameters
r0=6.5; a=0.5757; b=4.8825; sigmaR=4.7891; % Vasicek
meanH=0.0354; sigmaH=0.1003; % general shock
sigmaS=0.08; % individual shock
general=ones(1,38);
houseapp=ones(n,38);
for i=1:38
    n1=randn(1); n2=randn(1);
    eI(i) = n1;
    if i >= lag
        eH(i) = rho * eI(i - lag) + sqrt(1 - rho ^2) * n2;
    end
    k = i - lag;
    if k <= 1
        interest(s,k) = r0 / 100;
        general(k) = exp(meanH + sigmaH * eH(i));
    elseif k > 1
        interest(s,k) =
(a * b + (1 - a) * interest(s,k - 1) * 100 + sigmaR * eI(i)) / 100;
        general(k) = exp(meanH + sigmaH * eH(i)) * general(k - 1);
    end
end

for j=1:n
    n3=randn(1);
    houseapp(j,1) = general(1) - 1 + sigmaS * n3;
```

```
    for i=2:38
        n4=randn(1);
        houseapp(j,i) = general(i) / general(i - 1) - 1 + sigmaS * n4;
    end
end

accL=ones(1,38); % accmulation factor for each loan
accCL=ones(1,38); % accmulation factor for cost of capital
accH=ones(n,38); % accmulation factor for house appreciation
premL=0.075; % total risk premium lender charges
premC=0.0375;  % total risk premium lender is charged
disc=0.065;  % discount rate

accL(1)=1+interest(s,1)+premL;
accCL(1)=1+interest(s,1)+premC;
for i=2:38
        accL(i)=(1+interest(s,i)+premL)*accL(i-1);
        accCL(i)=(1+interest(s,i)+premC)*accCL(i-1);
end
for j=1:n
       accH(j,1)=1+houseapp(j,1);
       for i=2:38
       accH(j,i)=(1+houseapp(j,i))*accH(j,i-1);
       end
end

Q0=39222;    % starting value
H0=100000;  % initial house value
QL=ones(1,n);  % accumulated loan amount when terminated
HT=ones(1,n);  % accumulated house price when terminated
VL=ones(1,n);  % recoverable amount (minimum of the above two)
costL=ones(1,n); % accumulated cost of capital when terminated
lossL=ones(1,n); % diffe between cost and recoverable amount

for j=1:d(s,1)
    QL(j)=Q0*accL(1);
    HT(1,j)=H0*accH(j,1);
    VL(j)=min(QL(j),HT(j));
    costL(j)=Q0*accCL(1);
    lossL(j)=costL(j)-VL(j);
```

```
    sumloss(s,1)=sum(lossL(1:d(s,1)));
end

for i=2:38
    for j=(n-l(s,(i-1))+1):(n-l(s,i))
        QL(j)=Q0*accL(i);
        HT(1,j)=H0*accH(j,i);
        VL(j)=min(QL(j),HT(j));
        costL(j)=Q0*accCL(i);
        lossL(j)=costL(j)-VL(j);
        sumloss(s,i)=sum(lossL((n-l(s,(i-1))+1):(n-l(s,i))));
    end
end

for i=1:38
    if sumloss(s,i)<trigger(i)
        c(s,i)=0;
    elseif sumloss(s,i)>6500000
        c(s,i)=6500000;
    else c(s,i)=sumloss(s,i);
    end
end

for i=1:38
    pvmbc(s,i)=(6500000-c(s,i))/(1+disc)^(i);
end
pvmb(s)=sum(pvmbc(s,:))+n*H0/(1+disc)^38;

end

mbprice=sum(pvmb(:))/p
premium=n*H0-mbprice

subplot(2,2,1), plot(lossL),title('')
grid on
subplot(2,2,2), plot(sumloss(p,:)),title('')
grid on
subplot(2,2,3), plot(c(p,:)),title('')
grid on
subplot(2,2,4), plot(trigger),title('value of trigger each year')
```

```
grid on

%%%%%senario analysis for goodness of hedge %%%%

p=10000;
n=1000;
T=38;
tpx=zeros(p,T);
l=zeros(p,T);
shockp=zeros(p,T);
mean=zeros(p,T);
sigma=zeros(p,T);
k1=zeros(p,T);
k2=zeros(p,T);
fi1=zeros(p,T);
fi2=zeros(p,T);
E=zeros(p,T);
PVEC=zeros(p,T);
V=zeros(p,1);
disc=0.065;
a=1.49; b=147.51; % 1% shock
% a=1.38; b=26.30;  % 5% shock
% a=1.26; b=11.37;  % 10% shock
% a=0.88; b=2.65;   % 25% shock
% a=0.25; b=0.25;   % 50% shock
C=6500000; F=100000000;

for s=1:p
    for i=1:T
        shock=betarnd(a,b);
        shockp(s,i)=px(i)^(1-shock);
    end
    l(s,1)=n*shockp(1);
    tpx(s,1)=shockp(s,1);
    for i=2:T
        l(s,i)=l(s,i-1)*shockp(s,i);
        tpx(s,i)=tpx(s,i-1)*shockp(s,i);
    end
    for i=1:T
        mean(s,i)=n*tpx(s,i);
```

```
        sigma(s,i)=(mean(s,i)*(1-tpx(s,i)))^0.5;
        k1(s,i)=(l(s,i)-mean(s,i))/sigma(s,i);
        k2(s,i)=C/sigma(s,i)+k1(s,i);
        fi1(s,i)=normpdf(k1(s,i),0,1)-k1(s,i)*(1-normcdf(k1(s,i),0,1));
        fi2(s,i)=normpdf(k2(s,i),0,1)-k2(s,i)*(1-normcdf(k2(s,i),0,1));
        E(s,i)=C-At(i)*sigma(s,i)*fi1(s,i)+At(i)*sigma(s,i)*fi2(s,i);
        PVEC(s,i)=E(s,i)/(1+disc)^i;
    end
    V(s)=sum(PVEC(s,:))+ F/(1+disc)^T;
end
mbprice=sum(V(:))/p
premium=F-mbprice
```