Limitations of dynamic programming approach: singularity and time inconsistency

Author:
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Publication Date:
2016

DOI:
https://doi.org/10.26190/unsworks/2975

Limitations of dynamic programming approach: singularity and time inconsistency

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A thesis in fulfillment of the requirements for the degree of

Doctor of Philosophy

UNSW
AUSTRALIA

School of Mathematics and Statistics,

Faculty of Science.

March 2016

PLEASE TYPE

THE UNIVERSITY OF NEW SOUTH WALES
Thesis/Dissertation Sheet

Surname or Family name: **Wu**

First name: **Wei** Other name/s:

Abbreviation for degree as given in the University calendar: **PhD**

School: **School of Mathematics and Statistics** Faculty: **Faculty of Science**

Title: **Limitations of dynamic programming approach: singularity and time inconsistency**

Abstract 350 words maximum: (PLEASE TYPE)

Two failures of the dynamic programming (DP) approach to the stochastic optimal control problem are investigated. The first failure arises when we wish to solve a class of certain singular stochastic control problems in continuous time. It has been shown by Lasry and Lions (2000) that this difficulty can be overcome by introducing equivalent standard stochastic control problems. To solve this class of singular stochastic control problems, it remains to solve the equivalent standard stochastic control problems. Since standard stochastic control problems can be solved by applying the DP approach, this then solves the first failure. In the first part of the thesis, we clarify the idea of Lasry and Lions and extend their work to the case of controlled processes with jumps. This is particularly important in financial modelling where such processes are widely applied. For the purpose of application, we applied our result to an optimal trade execution problem studied by Lasry and Lions (2007b). The second failure of the DP approach arises when we wish to solve a multiperiod portfolio selection problem in which a mean-standard-deviation type criterion (a non-separable criterion) is used. We formulate such a problem as a discrete time stochastic control problem. By adapting a pseudo dynamic programming principle, we obtain a closed form optimal strategy for investors whose risk tolerances are larger than a lower bound. As a consequence, we develop a multiperiod portfolio selection scheme. The analysis is performed in the market of risky assets only, however, we allow both market transitions and intermediate cash injections and offtakes. This work provides a good basis for future studies of portfolio selection problems with selection criteria chosen from the class of translation-invariant and positive-homogeneous risk measures.

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Contents

Abstract

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Acknowledgements

Firstly, I would like to express my deepest gratitude to my supervisor, Professor Ben Goldys, and my co-supervisor, Associate Professor Spiro Penev. They have provided valuable suggestions to guide me through my research and study. There were a lot of obstacles during this long learning journey, it would be impossible to cope with these difficulties without their guidelines.

Secondly, I would like to say thank you to the School of Mathematics and Statistics at UNSW. They have offered excellent facilities and resources. In particular, I would like to thank our student service manager, Julie Hebblewhite, who takes care of all the administration issues. I really appreciate her assistance. In addition, I would also like to say thank you to the School of Mathematics and Statistics at the University of Sydney (USYD). They had kindly provided a friendly environment and sufficient resources for me to meet my supervisor, since the movement of my supervisor to USYD.

Special thanks go to my parents, Changyin Wu and Xiaoying Gao. Without their support and encouragement, I could not accomplish my degree.

Furthermore, I am very thankful to the Australian Mathematical Science Institute (AMSI) for their fantastic internship program. Thanks Cate Ballard, the National Program Manager of AMSI, who organized this internship. Thanks Optimo Financial, the industry partner, who offered adequate support during this internship. Finally, I am grateful to Hugh Bannister, the Principle of Optimo Financial, who taught me a lot and shared many practical experiences with me.

List of Symbols for Part I

$\mathbb{N}$	set of natural numbers
$\mathbb{R}^d$	d-dimensional Euclidean space
$\mathbb{R}^d_0$	$\mathbb{R}^d \backslash \{0\}$
$\mathbb{R}_+$	non-negative half line, i.e., $[0, \infty)$
$\bar{\mathbb{R}}_+$	extended non-negative half line, i.e., $[0, \infty]$
$< \cdot, \cdot >$	inner product on $\mathbb{R}^d$
$B(x_0, \epsilon)$	open ball of center $x_0 \in \mathbb{R}^d$ and radius $\epsilon > 0$, i.e., $\{x \in \mathbb{R}^d : \lvert x - x_0\rvert < \epsilon\}$
$\overline{B(x_0, \epsilon)}$	closed ball of center $x_0 \in \mathbb{R}^d$ and radius $\epsilon > 0$, i.e., $\{x \in \mathbb{R}^d : \lvert x - x_0\rvert \leq \epsilon\}$
$\mathcal{B}(\mathbb{R}^d)$	Borel σ-algebra of $\mathbb{R}^d$ in the metric topology
$\mathcal{B}(E)$	Borel σ-algebra of $E \subset \mathbb{R}^d$ in the metric topology induced from $\mathbb{R}^d$
$\bar{E}$	the closure of $E \subset \mathbb{R}^d$ in some topology
E^c	the complement of a set E
$\lvert x\rvert$	length of a vector $x \in \mathbb{R}^d$
$\lvert G\rvert$	Frobenius norm of a matrix G, i.e., if $G = (G_{ij})$, then $\lvert G\rvert = \sqrt{\sum_i \sum_j \lvert G_{ij}\rvert^2}$
$tr(G)$	trace of a matrix G
G^T	transpose of a matrix G
$\mathbb{S}^d$	space of d by d symmetric real valued matrices
I	identity matrix

$C([0,T];\mathbb{R}^d)$	space of continuous functions from $[0,T]$ to $\mathbb{R}^d$
$SC([0,T]\times\mathbb{R}^d;\mathbb{R})$	space of semicontinuous functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}$
$USC([0,T]\times\mathbb{R}^d;\mathbb{R})$	space of upper semicontinuous functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}$
$USC_b([0,T]\times\mathbb{R}^d;\mathbb{R})$	space of bounded, upper semicontinuous functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}$
$LSC([0,T]\times\mathbb{R}^d;\mathbb{R})$	space of lower semicontinuous functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}$
$LSC_b([0,T]\times\mathbb{R}^d;\mathbb{R})$	space of bounded, lower semicontinuous functions from $[0,T]\times\mathbb{R}^d$ to $\mathbb{R}$
$C^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$	space of functions from $(0,T]\times\mathbb{R}^d$ to $\mathbb{R}$ which are continuously differentiable with respect to the first argument and twice continuously differentiable with respect to the second argument
$C_b^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$	space of bounded functions from $(0,T]\times\mathbb{R}^d$ to $\mathbb{R}$ which are continuously differentiable with respect to the first argument and twice continuously differentiable with respect to the second argument
$C^{2,2,2}((0,T]\times\mathbb{R}^d\times\mathbb{R}^d;\mathbb{R})$	space of functions from $(0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ to $\mathbb{R}$ which are twice continuously differentiable with respect to the first argument, the second argument, and the third argument.
ΔL_t	the jump of the function L at time t, i.e., $\Delta L_t = L_t - L_{t-}$.
D_n^k	differential operator which denotes the k th derivative of a function with respect to the nth argument, with the understanding that $D_n^1 = D_n$, and if there is only one argument, the subscript n will be omitted.

Note: in this thesis, all constants are generic and may differ from one line to the other; subscripts may be added to emphasize the dependence of constants on particular parameters.

List of Symbols for Part II

Let $\mathbb{R}^d$ denotes the Euclidean space of dimension d. We use a bold letter to distinguish a vector $\boldsymbol{v} \in \mathbb{R}^d$ from a scalar $v \in \mathbb{R}$. All vectors are column vectors. Moreover,

- for any matrix $\boldsymbol{B}$, $\boldsymbol{B}^T$ denotes its transpose, $\bar{\boldsymbol{B}}(i)$ denotes the sum of the elements of its ith row; if $\boldsymbol{B}$ is square, $\boldsymbol{B}^m$ denotes its mth power, where $m \geq 0$, and $\boldsymbol{B}^0 = \boldsymbol{I}$ (identity matrix);
- for any vector $\boldsymbol{v}$, we denote by $\boldsymbol{v}^i$ its ith component, and by $diag(\boldsymbol{v})$ a diagonal matrix whose diagonal elements $diag(\boldsymbol{v})_{ii} = \boldsymbol{v}^i$ for all i;
- for any matrix $\boldsymbol{B}$, and a vector $\boldsymbol{v}$, we define $\boldsymbol{B}_{\boldsymbol{v}}$ as the matrix product of $\boldsymbol{B}$ and $diag(\boldsymbol{v})$;
- for any sequence of matrices $(\boldsymbol{B}_n)_{n>0}$, and $m < l$, we put $\displaystyle\sum_{n=\ell}^{m} \boldsymbol{B}_n = 0$, and $\displaystyle\prod_{n=\ell}^{m} \boldsymbol{B}_n = I$.

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Introduction

Motivation and Background

This thesis is divided into two parts, each of which is devoted to investigate a failure of dynamic programming (DP) approach.

In the first part, we study the failure of the DP approach which arises when we wish to solve a class of singular stochastic control problems driven by Lévy noise in continuous time.

Now, let us consider the following stochastic control problem. The (controlled) state process $(X_t)_{t\geq s}$ is assumed to follow the stochastic differential equation (SDE):

$$\begin{cases} dX_t = a(X_{t-})dt + b(X_{t-})u_t dt + \sigma(X_{t-})dW_t + \displaystyle\int_{0<|\eta|<1} \gamma(X_{t-},\eta)\tilde{N}(dt,d\eta) \\ \qquad\quad + \displaystyle\int_{|\eta|\geq 1} \gamma(X_{t-},\eta)N(dt,d\eta) \\ \\ X_s = x \in \mathbb{R}^d, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \leq s \leq t \leq T, \end{cases}$$

where u is an admissible control process. The value function is defined as

$$V(s,x) = \sup_{u\in\mathcal{A}_s} V^u(s,x),$$

where $\mathcal{A}_s$ is a given admissible control set at time s, and

$$V^u(s,x) \;=\; \mathbb{E}\Big(\int_s^T f(X_t^u)dt + h(X_T^u)\Big) \tag{1}$$

is the revenue functional. Here, we add the superscript u to $(X_t)_{t\geq s}$ to emphasize its dependence on the control u.

From classical stochastic control theory, we know that the associated Integro-Hamilton-Jacobi-Bellman (henceforth HJB) equation takes the form

$$
\begin{aligned}
&-D_1V(s,x)+\sup_{u\in A}\Big(<-D_2V(s,x),a(x)+b(x)u>\Big)\\
&+\frac{1}{2}tr\big(-D_2^2V(s,x)\sigma(x)\sigma(x)^T\big)-\int_{0<|\eta|<\infty}\Big(V(s,x+\gamma(x,\eta))\\
&-V(s,x)-<D_2V(s,x),\gamma(x,\eta)>1_{\{0<|\eta|<1\}}\Big)\nu(d\eta)-f(x)=0,
\end{aligned}\tag{2}
$$

where $A\subseteq\mathbb{R}$. Here the solution is understood in the viscosity sense (we will clarify the meaning of this in Chapter 3).

When A is not bounded, the expression inside the supremum may be infinite and this yields a singularity. The singularity leads to the failure of the DP approach, since the HJB equation does not make sense any more. In stochastic control theory, this often happens when control enters into the state in a linear fashion (see for example Pham (2005) for more discussions).

There are a few ways to deal with this issue. We may reformulate the definition of viscosity solution to avoid the use of supremum (see for example Da Lio and Ley (2008)). Alternatively, we may consider to use variational inequalities instead of (2) (see Pham (2005) for more details). In recent years, a popular approach to deal with this issue is to formulate an equivalence result, see for example, Dufour and Miller (2002); Motta and Sartori (2007, 2010, 2011) and the references therein. By interpreting control problems in the weak sense (i.e., by considering the underlying probability space as part of the control), it has been shown that any member of this class of singular stochastic control problems is equivalent to the corresponding combined optimal stopping and stochastic control problem (with controls taking values in compact set). The equivalence is in the sense that the value functions of the two control problems are equal and the main approach is based on a time transformation technique.

Unlike the aforementioned works, the early work of Lasry and Lions (2000) proved a different equivalence result. They have shown that, within this class, the value function of any singular stochastic control problem is invariant under a flow associated to the drift coefficient of the corresponding state process. As a consequence, by interpreting control problems in the strong sense (i.e., by fixing the underlying probability space in advance), they have shown that, within this class, the value function of any singular stochastic control problem and the value function of the corresponding standard stochastic control problem are equal. Since controls of the corresponding standard stochastic control problem take values in a compact set, this then can be solved via classical argument (i.e., by using the DP approach). Thus, one can solve this class of singular stochastic control problems by solving the corresponding standard stochastic control problems.

Lasry and Lions (2000) proved their results for state processes driven by Brownian noise.

It is interesting to see if similar result holds when we consider state processes with general Lévy noise. Also, we note that the theory of Lasry and Lions provides a basis for their later studies of the impact of trading and hedging on the dynamic of an asset (see Lasry and Lions (2006, 2007b,a)). Since asset prices are better modeled by processes with jumps, (i.e., general Lévy noise; see for example Di Nunno et al. (2006) and the references therein), for the purpose of application, it is important to extend their theory to the case of general Lévy noise.

The primary aim of the first part of the thesis is to extend the work of Lasry and Lions (2000) to allow general Lévy noise. We will show that, with general Lévy noise, the invariance property outlined in their paper still holds, and as a consequence the equivalence to the corresponding standard stochastic optimal control problem is preserved. The main difficulty to extend the work of Lasry and Lions is the fact that their approach requires that the state process possess some finite moments. This is certainly true for Brownian noise with appropriate assumptions. However, for general Lévy noise, this may not hold. To overcome this difficulty, we use an approximation which is used in construction of solution of SDE with Lévy noise (see for example Theorem 6.2.9 on p374 in Applebaum (2009) or pp354-355 in Kunita (2004)). We close the first part of the thesis by applying the extended theory to an optimal trade execution problem studied by Lasry and Lions (2007b). The recent work of Kato (2014) studied a closely related problem with Brownian noise, and their work is generalized by Ishitani and Kato (2012) to allow jumps. We have independently obtained (in some sense) a more general result than Ishitani and Kato (2012). The work of Ishitani and Kato (2012) does not prove the equivalence result which is required to solve the optimal trade execution problem. Instead, they assume the value function is invariant under the same flow as in the case of Brownian noise, although they allow the jump term to depend on the control. Moreover, they have a stronger restriction on the Lévy measure to allow the existence of moments. In the current work, we remove this assumption.

The second part of the thesis studies a multiperiod time consistent portfolio selection problem which often encounters in personal wealth planning and management.

Due to the practical popularity, portfolio selection problems have been of a great interest by both academics and practitioners. There are various selection criteria available. Some examples include the classical mean-variance (MV) criterion introduced by Markowitz (1952), the safety-first criterion proposed by Roy (1952), and the criterion which targets a particular wealth level used by Skaf and Boyd (2009). In second part of the thesis, we choose a mean-standard-deviation (MSD) criterion which (in the single period case) has the form:

$$J_x(\boldsymbol{u}) \quad = \quad \mathbb{E}_x(W^{\boldsymbol{u}}) - \kappa\sqrt{Var_x(W^{\boldsymbol{u}})},$$

where $W^{\boldsymbol{u}}$ denotes investor's wealth at the end of the investment horizon, which depends on investor's initial wealth x, his investment strategy $\boldsymbol{u}$, and a parameter $\kappa > 0$ which characterizes investor's risk tolerance. All terms will be defined in a more precise way later. There are several reasons to choose this criterion. The most significant one is the fact that it provides a partial understanding on how to choose a dynamic portfolio for the class of translation-invariant and positive-homogeneous (TIPH) risk measures. The TIPH risk measure class contains many interesting examples such as the well-known Value at Risk, and the Conditional Value at Risk. In a single period portfolio selection model, it has been shown (see for example Landsman and Makov (2011)) that if the asset returns follow a (joint) elliptical distribution, optimizing a risk measure from the TIPH class is equivalent to optimizing the MSD criterion.

There has been extensive research in the past regarding single period portfolio selection by using MSD criterion. For example, Landsman (2008) found a closed form solution by using matrix partitions. Owadally (2012) proposed two alternative ways in which the obtained solutions are more efficient computationally. The first approach is based on the relationship between optimizing the MSD criterion and optimizing the MV criterion which is close to a precommitment approach. For portfolio selection by precommitment approach, we refer to Li and Ng (2000); Çakmak and Özekici (2006). The second approach utilizes the standard Lagrange argument together with some facts from linear algebra. One may note that both Landsman (2008) and Owadally (2012) consider a market of risky assets. Later on, a risk free asset is added to the model in Landsman and Makov (2012), however only a trivial solution is obtained (when a budget constraint only is imposed).

Just like in the second method given by Owadally (2012) we follow a standard Lagrange method to solve the single period problem. However, the main interest of this work, is to extend the single period framework to a multiperiod model. In doing so, we note that the MSD and the MV criterion face the same difficulty due to the presence of the variance term in their formulation. This is known as non-separability, see for example Li and Ng (2000), which causes the failure of the DP approach since we can no longer apply the standard dynamic programming principle (DPP). In recent years, it is quite popular to use the time consistency concept to establish a pseudo DPP. This concept has been widely applied in the multiperiod portfolio selection problem with the MV criterion. We mention a few references here: Björk and Murgoci (2010); Wu (2013); Chen et al. (2013); Bensoussan et al. (2014) for discrete time, and Björk et al. (2014); Bensoussan et al. (2014) for continuous time setting. There are different definitions of time consistency. Here, we concentrate on the time consistency of optimal strategy with respect to a multiperiod selection criterion. To formulate a pseudo DPP, it has been argued that a rational investor should choose his strategy consistently through time. In other words, the investors only choose among strategies which they are going to follow in the future (see Strotz (1955-1956)). Thus, in discrete time, by utilizing this time consistency approach one can select an optimal strategy through a period-wise optimization and backward recursion. A meaningful explanation is given through a game theory point of view, and such a strategy has been called an equilibrium control (henceforth referred to as a weakly time consistent optimal strategy). It inherits the equilibrium concept that arises in game theory. We refer to Björk and Murgoci

(2010); Wu (2013); Bensoussan et al. (2014) and the references therein for more details. With a rather strong form of time consistency as proposed, for example by Kang and Filar (2006), an extra property of a time consistent optimal strategy is required. This property states that any sub-strategy of a weakly time consistent optimal strategy is also optimal for the corresponding subsequent periods. This is essentially satisfied for an optimal strategy that can be obtained through the standard DPP. Inspired by the work of Kovacevic and Pflug (2009), Chen et al. (2013) constructed a multiperiod separable selection criterion. With respect to this criterion, they proved that the optimal strategy obtained through the pseudo DPP satisfies the extra property of strong time consistency. They obtained a closed form optimal strategy with a multiperiod separable selection criterion of MV type. Later on, their work has been extended by Chen et al. (2014) to allow market transitions.

To the authors' best knowledge, the multiperiod portfolio selection problem in which a MSD type criterion is used as a selection criterion is only briefly mentioned in Kronborg and Steffensen (2015). However, the authors consider a model with two assets only, where one of the assets is supposed to be risk free. Within their setting, a trivial result (a special case of Landsman and Makov (2012)) only is obtained. In essence, the outcome is that is the reward is large enough, it would be advisable to invest as much as possible in the risky asset, whereas when the reward is too little in comparison to the investor's risk tolerance, the strategy is to invest in the risk free asset only. A similar result is obtained in the corresponding continuous time problem (see Kryger and Steffensen (2010); Kronborg and Steffensen (2015)). Thus, in this work we consider a market of risky assets only. We take the single period MSD criterion and formulate a separable multiperiod selection criteria of MSD type (similar to Chen et al. (2013) for the MV case). By applying the aforementioned pseudo DPP, we obtain a closed form optimal strategy. As a consequence, we develop a multiperiod portfolio selection scheme. In doing so, we allow for market transitions, and also for intermediate cash injections and offtakes. Thus, the wealth process of the investor is no longer self-financing in our setting. As far as we are aware, for multiperiod portfolio selection problem, the only work in which intermediate cash injections and offtakes are allowed and closed form solution is obtained, is by Wu and Li (2012). However unlike our work, the authors consider the multiperiod MV criterion, and follow a precommitment approach.

Outline and Contributions

The outline and the main contributions of this thesis are summarized below.

The first part of the thesis contains Chapter 1 to Chapter 4. In Chapter 1, we present some basic concepts and theories. These include the integral flow associated to a vector field, Poisson random measures, Lévy processes, and SDEs with random coefficients and Lévy noise.

In order to accomplish our main task of the first part of this thesis, we make some preparations in Chapter 2 and Chapter 3. In Chapter 2, we prove the DPP for standard stochastic control problems with general Lévy noise (i.e., Theorem 2.3.1). We stress that when we say the general Lévy noise, we mean that there is no assumptions imposed on the associated Lévy measure. This is the first contribution of this thesis. It is worth to note that our proof of the DPP does not require any moments assumption of the state process. In Chapter 3, we give an overview of the theory of viscosity solutions of HJB equations. We prove that the (relevant) value function is a viscosity solution of the associated HJB equation. We also present a rigorous derivation of a comparison theorem for semi-continuous bounded viscosity solutions of the HJB equations (i.e., Theorem 3.3.3). However, there are not really new contributions in this chapter.

In Chapter 4, we prove our main result of the first part of this thesis (i.e., Theorem 4.2.3). This extends the result of Lasry and Lions (2000) to the case of general Lévy noise. To prove this result, at the beginning of Chapter 4, we include some auxiliary results. Part of these results come from various references. We have, in some sense, partially extended some of these auxiliary results in various directions. Then, together with these auxiliary results and by following Lasry and Lions (2000) we prove Theorem 4.2.3 in Section 4.4. A financial application of this theory is presented in the last section of this chapter where the main result is summarized in Theorem 4.5.2. Finally, we include a brief outline of some possible future research directions. This closes the first part of this thesis.

The second part of the thesis is included in Chapter 5. At the beginning of this chapter, we set up the market model, outline our assumptions, and formulate our multiperiod portfolio selection problem as a discrete time stochastic control problem. We then briefly outline the properties of the single period MSD criterion and discuss the issue of the presence of the risk free asset. By using pseudo DPP and backward recursion, we obtain a closed form optimal strategy (i.e., Theorem 5.3.1). We also derive the optimal conditional expectation and conditional variance of the terminal wealth (i.e., Section 5.3.3). Moreover, we develop a multiperiod portfolio selection scheme (i.e., Algorithm 5.3.1) which is the main contribution of second part of this thesis. In addition, we perform some numerical illustrations and comparisons. A closing remark which includes some possible future research directions is presented at the end.

Part I

Limitation of Dynamic Programming Approach: Singularity

Chapter 1

Some Basic Concepts and Theories

In this chapter, we recall some basic concepts and theories which are related to the first part of the thesis.

1.1 Integral Flows Associated to Vector Fields

The concept of the integral flow (henceforth just flow) and its properties can be found in many standard textbooks. Here, we follow Teschl (2012); Barreira and Valls (2012) to outline some basic facts.

Definition 1.1.1. *For all $\kappa_1, \kappa_2 \in \mathbb{R}$, and $z_0 \in \mathbb{R}^d$, the mapping $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfies the following properties:*

***(FP1)**: $\varphi(0, z_0) = z_0$;*

***(FP2)**: $\varphi(\kappa_2 + \kappa_1, z_0) = \varphi(\kappa_2, \varphi(\kappa_1, z_0))$;*

is called a flow.

Under suitable conditions, the solution of a first order autonomous ordinary differentiation equation (ODE) defines a flow in the sense of Definition 1.1.1. This is summarized in the following proposition.

Proposition 1.1.2. *Consider the ODE:*

$$\begin{cases} \dfrac{dZ(\kappa)}{d\kappa} = \beta(Z(\kappa)), \\ \\ Z(0) = z_0 \in \mathbb{R}^d, \end{cases} \tag{1.1}$$

where $\kappa \in \mathbb{R}$. Assume $\beta : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz, i.e., there exists a constant $C > 0$ such that for all $z_1, z_2 \in \mathbb{R}^d$ we have

$$|\beta(z_1) - \beta(z_2)| \leq C|z_1 - z_2|.$$

Then there exists a unique solution Z to (1.1). Moreover, $\phi(\kappa, z_0) = Z_\kappa^{z_0}$ defines a flow (associated to the vector field β).

Proof. The proof follows from section 2.2 on p36 in Teschl (2012), and Proposition 1.13 on p8 in Barreira and Valls (2012). □

Now, we make the following assumption.

Assumption 1.1. $D\beta : \mathbb{R}^d \to \mathbb{R}^d$ is bounded, twice continuously differentiable functions with bounded derivatives up to second order.

Then we obtain some nice properties of the flow. These are outlined in the following sequence of results.

Theorem 1.1.3. *The flow ϕ defined by the solution of (1.1) is Lipschitz, i.e., there exists a constant $C > 0$ such that for all $z_1, z_2 \in \mathbb{R}^d$, and $\kappa_1, \kappa_2 \in \Gamma$, where $\Upsilon \in \mathbb{R}$ is compact such that*

$$|\phi(\kappa_1, z_1) - \phi(\kappa_2, z_2)| \leq C|z_1 - z_2| + C|t - s|.$$

Proof. The proof follows from the proof of Theorem 2.9 in Teschl (2012) (see p44 of Teschl (2012)). □

Theorem 1.1.4. *The flow ϕ defined by the solution of (1.1) belongs to $C^3(\mathbb{R} \times \mathbb{R}^d; \mathbb{R}^d)$.*

Proof. The proof follows from the proof of Theorem 2.10 in Teschl (2012) (see p46 of Teschl (2012)). □

Lemma 1.1.5. *There exists a constant $C > 0$ such that for all $z, z_1, z_2 \in \mathbb{R}^d$, $\kappa, \kappa_1, \kappa_2 \in \Gamma$, where $\Upsilon \subset \mathbb{R}$ is compact, we have*

$$|D_2\phi(\kappa, z)| \leq C, \quad |D_2\phi(\kappa_1, z_1) - D_2\phi(\kappa_2, z_2)| \leq C|z_1 - z_2| + C|\kappa_1 - \kappa_2|, \tag{1.2}$$
$$|D_2^2\phi(\kappa, z)| \leq C, \quad |D_2^2\phi(\kappa_1, z_1) - D_2^2\phi(\kappa_2, z_2)| \leq C|z_1 - z_2| + C|\kappa_1 - \kappa_2|. \tag{1.3}$$

Proof. Let

$$A(\kappa, x) = D_2\varphi(\kappa, x),$$

by the argument on p46 of Teschl (2012), we see that $A(\kappa, x)$ solves

$$A(\kappa, x) = I + \int_0^\kappa D\beta(\varphi(r, x))A(r, x)dr.$$

First, it is easy to see that

$$|A(\kappa, x)| \quad \leq \quad C + C\int_0^\kappa |A(r, x)|dr,$$

The first claim of (1.2) then follows from Gronwall's inequality.

Next, let us show the second claim of (1.2). For all $x_1, x_2 \in \mathbb{R}^d$, and $\kappa_1, \kappa_2 \in \Gamma$ (without loss of generality we assume $\kappa_2 > \kappa_1 > 0$) we see that

$$|A(\kappa_1, x_1) - A(\kappa_2, x_2)| \leq |A(\kappa_1, x_1) - A(\kappa_1, x_2)| + |A(\kappa_1, x_2) - A(\kappa_2, x_2)|.$$

The first term, after an application of Gronwall's inequality, yields

$$|A(\kappa_1, x_1) - A(\kappa_1, x_2)| \leq C|x_1 - x_2|,$$

and second term implies

$$|A(\kappa_1, x_2) - A(\kappa_2, x_2)| \leq C|\kappa_2 - \kappa_1|.$$

Thus, we have

$$|A(\kappa_1, x_1) - A(\kappa_2, x_2)| \leq C\Big(|x_1 - x_2| + |\kappa_2 - \kappa_1|\Big).$$

This completes the proof of (1.2). The proof of (1.3) follows a similar argument. □

1.2 Poisson Random Measures and Lévy Processes

There are many good sources which contain a detailed discussion on Poisson random measures and Lévy process. Here, we mainly follow Applebaum (2009); Çinlar (2011) to present some basic theories on this topic.

Throughout this section, we assume that all random quantities are defined on given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We first define the random measure.

Definition 1.2.1. *Given a measurable space $(\mathbb{R}^d_0, \mathcal{B}(\mathbb{R}^d_0))$, a random measure is a mapping $N : \Omega \times \mathcal{B}(\mathbb{R}^d_0) \to \bar{\mathbb{R}}_+$ such that*

- *for every $E \in \mathcal{B}(\mathbb{R}^d_0)$, $\omega \to N(\omega, E)$ is a random variable;*
- *for every $\omega \in \Omega$, $E \to N(\omega, E)$ is a measure on $(\mathbb{R}^d_0, \mathcal{B}(\mathbb{R}^d_0))$.*

Here, we have defined the random measure on the measurable space $(\mathbb{R}^d_0, \mathcal{B}(\mathbb{R}^d_0))$. This is actually not essential, we can define a random measure on a more general measurable space. However, for the purpose of this thesis, this would be enough for us. To define the Poisson random measure, we need an intensity measure, it is convenient to use the Lévy measure.

Definition 1.2.2. *A Borel measure ν defined on $(\mathbb{R}^d_0, \mathcal{B}(\mathbb{R}^d_0))$ is called a Lévy measure, if*

$$\int_{\mathbb{R}^d_0} (|\eta|^2 \wedge 1)\nu(d\eta) < \infty. \tag{1.4}$$

On $(\mathbb{R}^d_0, \mathcal{B}(\mathbb{R}^d_0))$, every Lévy measure is σ-finite.

Next, we define the Poisson random measure.

Definition 1.2.3. *A Poisson random measure $N : \Omega \times \mathcal{B}(\mathbb{R}^d_0) \to \bar{\mathbb{R}}_+$ is a random measure such that*

- *for every $E \in \mathcal{B}(\mathbb{R}^d_0)$, $N(\omega, E)$ has a Poisson distribution with mean $\nu(E)$;*
- *for every disjoint sets $E_1, ..., E_n \in \mathcal{B}(\mathbb{R}^d_0)$, where $n \geq 2$, $N(\omega, E_1), ..., N(\omega, E_n)$ are independent.*

Let $E = (0, t] \times A$. If $A \in \mathcal{B}(\mathbb{R}^d_0)$ is bounded below, i.e., $0 \notin \bar{A}$, by Lemma 2.3.4 in Applebaum (2009) (see p101 in Applebaum (2009)), the Poisson random measure N is finite a.s. For the sake of notations, we will sometimes omit the ω and simply write $N(\omega, (0, t] \times A)$ as $N(t, A)$. For every $t \geq 0$, and $A \in \mathcal{B}(\mathbb{R}^d_0)$ bounded below we define the compensated Poisson random measure $\tilde{N}(t, A)$ (associated to N) by

$$\tilde{N}(t, A) = N(t, A) - t\nu(A).$$

A Lévy process can be characterized by the Poisson random measure and the compensated Poisson random measure through the Lévy-Ito decomposition. Before we present this decomposition, let us define the Levy process.

Definition 1.2.4. *A stochastic process $(L_t)_{t\geq 0}$ taking values in $\mathbb{R}^d$ is a Lévy process if*

- *$L_0 = 0$ ($\mathbb{P}$-a.s.);*
- *L has independent increments, i.e., for all $s < t < r$, $L_r - L_t$ and $L_t - L_s$ are independent;*
- *L has stationary increments, i.e., for all $s < t < r$, $L_r - L_t$ and $L_t - L_s$ have the same distribution;*
- *L is stochastically continuous, i.e., for all $\epsilon > 0$, and $s \geq 0$*

$$\lim_{t\to s} \mathbb{P}(|L_t - L_s| > \epsilon) = 0.$$

Now, we ready to present the Ito-Lévy decomposition.

Theorem 1.2.5. *Let $(L_t)_{t\geq 0}$ be a Lévy process. There exists $b \in \mathbb{R}^d$, a Brownian motion W with covariance matrix D and a independent Poisson random measure N such that for every $t \geq 0$ we have*

$$L_t = bt + W_t + \int_{0<|\eta|<1} \eta \tilde{N}(t, d\eta) + \int_{|\eta|\geq 1} \eta N(t, d\eta). \tag{1.5}$$

Proof. see the proof of Theorem 2.4.16 on p126 in Applebaum (2009) ☐

We may classify a jump as a large jump if $|\eta| > 1$, and a jump as a small jump if $0 < |\eta| < 1$ (for this classification, see p364 of Applebaum (2009)) The large jumps part give rise to a compound Poisson process $(P_t)_{t\geq 0}$, where

$$P_t = \int_{|\eta|\geq 1} \eta N(t, d\eta). \tag{1.6}$$

Given a general Lévy process $(L_t)_{t\geq 0}$, we do not expect that it has any finite moments at all. It is the large jumps (or the compound Poisson process given in (1.6)) cause the failure of the existence of finite moments. This result is summarized in the following theorem.

Theorem 1.2.6. *Let $(L_t)_{t\geq 0}$ be a Lévy process. For every $n \in \mathbb{N}$ and $t \geq 0$, $\mathbb{E}(|L_t|^n) < \infty$ if and only if $\int_{|\eta|\geq 1} |\eta|^n \nu(d\eta) < \infty$.*

Proof. see the proof of Theorem 2.5.2 on p132 in Applebaum (2009). ☐

1.3 Stochastic Differential Equation for Jump Diffusion with Random Coefficients

The theory of stochastic differential equations (SDEs) for jump diffusions with random coefficients plays an important role. The basic theory of SDE are well known and can be found, for example, in Applebaum (2009); Kunita (2004); Menaldi (2014). The existence and uniqueness of solutions of SDE are often obtained under the Lipschitz (or local Lipschitz) and growth conditions. However, these assumptions can be weaken (see, for example, the recent work of Xu et al. (2015) and Kulinich and Kushnirenko (2014)). In this section, we will review some of the the basic theories of SDE driven by Lévy noise and random coefficients. Our main references are Kunita (2004); Applebaum (2009); Menaldi (2014); Xu et al. (2015).

We will work on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ on which an m-dimension standard Brownian motion W and a Poisson random measure N are defined. We shall assume that W and N are independent, and the filtration satisfies usual conditions (i.e., $\mathcal{F}_0$ contains all the $\mathbb{P}$ null sets, and $(\mathcal{F}_t)_{t\geq 0}$ is right continuous).

A predictable set is a subset of $[0, T] \times \Omega$ which has a form of $(s, t] \times A$ for some $A \in \mathcal{F}_s$. Sometimes, the set $\{0\} \times \mathcal{F}_0$ is also considered as a predictable set. The sigma algebra generated by all predictable sets associated to the filtration $(\mathcal{F}_t)_{t\geq 0}$ is denoted as $\mathcal{P}$. A process is called predictable if it is measurable with respect to $\mathcal{P}$.

For a fixed $T > 0$ and $s \in [0, T]$, consider the SDE

$$\begin{cases} dX_t = b_0(t, \omega, X_{t-})dt + \gamma_0(t, \omega, X_{t-})dL_t, \\ \\ X_s = x \in \mathbb{R}^d, \qquad\qquad 0 \leq s \leq t \leq T, \end{cases} \tag{1.7}$$

where $(L_t)_{t\geq 0}$ is a Lévy process, X_{t-} is the left limit of X_t, $b_0 : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$, and $\gamma_0 : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$ are $\mathcal{P}\times\mathcal{B}(\mathbb{R}^d)$ measurable mappings.

For $b \in \mathbb{R}^d$, define the mapping $b_1 : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$ such that

$$b_1(t,\omega,x) = b_0(t,\omega,x) + b\gamma_0(t,\omega,x).$$

By Theorem 1.2.5, we may replace the Lévy process by its Ito-Lévy decomposition. Thus, (1.7) becomes

$$\begin{cases} dX_t = b_1(t,\omega,X_{t-})dt + \gamma_0(t,\omega,X_{t-})dW_t + \int_{0<|\eta|<1} \gamma_0(t,\omega,X_{t-})\eta\tilde{N}(dt,d\eta) \\ \qquad + \int_{|\eta|\geq 1} \gamma_0(t,\omega,X_{t-})\eta N(dt,d\eta) \\ X_s = x \in \mathbb{R}^d. \qquad\qquad 0 \leq s \leq t \leq T. \end{cases} \tag{1.8}$$

In what follows we will consider a more general SDE, i.e.,

$$\begin{cases} dX_t = \hat{b}(t,\omega,X_{t-})dt + \hat{\sigma}(t,\omega,X_{t-})dW_t + \int_{0<|\eta|<1} \hat{\gamma}(t,\omega,X_{t-},\eta)\tilde{N}(dt,d\eta) \\ \qquad + \int_{|\eta|\geq 1} \hat{\gamma}(t,\omega,X_{t-},\eta)N(dt,d\eta) \\ X_s = x \in \mathbb{R}^d, \qquad\qquad 0 \leq s \leq t \leq T, \end{cases} \tag{1.9}$$

where $\hat{b} : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}^d$, $\hat{\sigma} : [0,T]\times\Omega\times\mathbb{R}^d \to \mathbb{R}^{d\times m}$ are $\mathcal{P}\times\mathcal{B}(\mathbb{R}^d)$ measurable mappings and $\hat{\gamma} : [0,T]\times\Omega\times\mathbb{R}^d\times\mathbb{R}^q_0 \to \mathbb{R}^d$ is a $\mathcal{P}\times\mathcal{B}(\mathbb{R}^d)\times\mathcal{B}(\mathbb{R}^q_0)$ measurable mapping.

We shall assume (local) Lipschitz and growth conditions which will be the key assumptions in future chapters.

Assumption 1.2. There exists a constant $C > 0$ such that for all $t \in [s,T]$, $|x_1|, |x_2| \leq N$, $0 < |\eta| < 1$, and $\omega \in \Omega$, the following hold.

$$\begin{aligned} |\hat{b}(t,\omega,x_1) - \hat{b}(t,\omega,x_2)| + |\hat{\sigma}(t,\omega,x_1) - \hat{\sigma}(t,\omega,x_2)| &\leq C|x_1 - x_2|; \\ |\hat{\gamma}(t,\omega,x_1,\eta) - \hat{\gamma}(t,\omega,x_2,\eta)| &\leq C|\eta||x_1 - x_2|; \end{aligned}$$

Assumption 1.3. There exists a constant $C > 0$ such that for all $t \in [s,T]$, $x \in \mathbb{R}^d$, $0 < |\eta| < 1$, and $\omega \in \Omega$, the following hold.

$$\begin{aligned} |\hat{b}(t,\omega,x)| + |\hat{\sigma}(t,\omega,x)| &\leq C(1+|x|); \\ |\hat{\gamma}(t,\omega,x,\eta)| &\leq C|\eta|(1+|x|). \end{aligned}$$

As a consequence of Assumption 1.2 and Assumption 1.3, we have the following theorem.

Theorem 1.3.1. *There exists a unique càdlàg and adapted solution $(X_t)_{t\geq s}$ of (1.9).*

Proof. By a construction of solution (see for example Theorem 6.2.9 on p374 in Applebaum (2009) or pp354-355 in Kunita (2004)), it is sufficient to consider the existence and uniqueness of the solution of the following SDE:

$$\begin{cases} dX_t = \hat{b}(t,\omega,X_{t-})dt + \hat{\sigma}(t,\omega,X_{t-})dW_t + \displaystyle\int_{0<|\eta|<1} \hat{\gamma}(t,\omega,X_{t-},\eta)\tilde{N}(dt,d\eta) \\ \\ X_s = x, \qquad\qquad\qquad\qquad 0 \le s \le t \le T. \end{cases} \tag{1.10}$$

The proof of such a result then follows from the argument in Section 5.1.1 of Menaldi (2014). □

Given a predictable process $(u_t)_{t\ge 0}$, we may view the following SDE:

$$\begin{cases} dX_t = b(t,X_{t-},u_t)dt + \sigma(t,X_{t-},u_t)dW_t + \displaystyle\int_{0<|\eta|<1} \gamma(t,X_{t-},u_t,\eta)\tilde{N}(dt,d\eta) \\ \qquad + \displaystyle\int_{|\eta|\ge 1} \gamma(t,X_{t-},u_t,\eta)N(dt,d\eta) \\ \\ X_s = x \in \mathbb{R}^d, \qquad\qquad\qquad\qquad 0 \le s \le t \le T, \end{cases} \tag{1.11}$$

in terms of (1.9). Indeed, we may write

$$\begin{aligned} b(t,X_t,u_t(\omega)) &= \hat{b}(t,\omega,X_t), \\ \sigma(t,X_t,u_t(\omega)) &= \hat{\sigma}(t,\omega,X_t), \\ \gamma(t,X_t,\eta,u_t(\omega)) &= \hat{\gamma}(t,\omega,X_t,\eta), \end{aligned}$$

then, by Theorem 1.3.1 there exists a unique càdlàg and adapted solution to (1.11).

Next, consider a special case of (1.11):

$$\begin{cases} dX_t = b(X_{t-})u_t dt + \sigma(X_{t-})dW_t + \displaystyle\int_{0<|\eta|<1} \gamma(X_{t-},\eta)\tilde{N}(dt,d\eta) \\ \qquad + \displaystyle\int_{|\eta|\ge 1} \gamma(X_{t-},\eta)N(dt,d\eta) \\ \\ X_s = x \in \mathbb{R}^d, \qquad\qquad\qquad\qquad 0 \le s \le t \le T, \end{cases} \tag{1.12}$$

Instead of assuming Assumption 1.2 and Assumption 1.3, we impose the following assumptions.

Assumption 1.4. There exists a constant $C > 0$ such that for all $t \in [s,T]$, $|x_1|, |x_2| \le N$, and $0 < |\eta| < 1$, the following hold.

$$\begin{aligned} |b(x_1) - b(x_2)| + |\sigma(x_1) - \sigma(x_2)| &\le C|x_1 - x_2|; \\ |\gamma(x_1,\eta) - \gamma(x_2,\eta)| &\le C|\eta||x_1 - x_2|. \end{aligned}$$

Assumption 1.5. There exists a constant $C > 0$ such that for all $t \in [s,T]$, $x \in \mathbb{R}^d$, $0 < |\eta| < 1$, and $\omega \in \Omega$, the following hold.

$$\begin{aligned} |b(x)| + |\sigma(x)| &\leq C(1+|x|); \\ |\gamma(x,\eta)| &\leq C|\eta|(1+|x|). \end{aligned}$$

Assumption 1.6. There exists a constant $C > 0$ such that

$$\int_0^T |u_t| dt < C \quad \mathbb{P} - \text{a.s.}.$$

Theorem 1.3.2. *Under Assumption 1.4 - Assumption 1.6, there exists a unique càdlàg and adapted solution $(X_t)_{t\geq s}$ of (1.11).*

Proof sketch. Again, it is sufficient to prove the existence and uniqueness of solution of

$$\begin{cases} dX_t = b(X_{t-})u_t dt + \sigma(X_{t-})dW_t + \displaystyle\int_{0<|\eta|<1} \gamma(X_{t-},\eta)\tilde{N}(dt,d\eta) \\ X_s = x, \qquad\qquad\qquad\qquad 0 \leq s \leq t \leq T, \end{cases} \tag{1.13}$$

The result then follows by adapting to the the argument in Section 5.1.1 of Menaldi (2014). □

Remark 1.3.3. *One may note Theorem 1.3.2 remains valid, if*

- *we replace s and x with a stopping time τ and a $\mathcal{F}_\tau$-measurable random variable ζ respectively;*
- *we replace the set $\{|\eta| \geq 1\}$ by $\{1 \leq |\eta| < M\}$ for $M \geq 1$.*

Chapter 2

Dynamic Programming Principle for Stochastic Control Problems driven by General Lévy Noise

The dynamic programming principle (DPP) is a well-known device in studying stochastic optimal control problems. For a standard control problem with finite horizon, it states that the value function for the control problem starting at time $s \in [0, T]$ from a position $X_s^u = x$ is given by the formula

$$V(s, x) = \sup_{u \in \mathcal{A}_s} \mathbb{E}\Big(\int_s^\tau f(r, X_r^u, u_r) dr + V(\tau, X_\tau^u) \Big), \tag{2.1}$$

where τ is some stopping time, u is an admissible control process, $\mathcal{A}_s$ is a given admissible control set at time s, and X^u is a controlled state process (where the superscript emphasize the dependence on the control). All terms will be defined in a more precise way later.

To complete the main task of the first part of this thesis, the DPP plays an important role. Thus, in this chapter, we extend the proof of the DPP for standard stochastic optimal control problems driven by general Lévy noise.

2.1 Some Comments on the Proof of DPP

There are many ways to prove the DPP. When the underlying probability space is fixed in advance, we say a stochastic control problem is under a strong formulation. In this case, one may use the theory of piecewise constant controls to construct appropriate supermartingales and show that the DPP holds through properties of supermartingales (see Krylov (2009) for the diffusion case, and Ishikawa (2004) for the jump case). Alternatively, we can prove the DPP by partitioning the state space, provided the value function satisfies certain regularity conditions or using its semicontinuous envelope (see for example Bouchard and Touzi (2011)). When a control problem is defined in the weak sense, that is the underlying probability space is taken to be part of the control, we can also apply this approach (see for example Azevedo et al. (2014); Yong and Zhou (1999)). Moreover, recently by interpreting controls in the weak sense, El Karoui and Tan (2013) proved the DPP by using a probabilistic approach.

To prove the DPP, in most cases, the state process is required to have finite second moments (see for example Azevedo et al. (2014); Bouchard and Touzi (2011); Ishikawa (2004); Krylov (2009); Yong and Zhou (1999)). A stochastic control problem is often formulated with the state process assumed to follow a certain stochastic differential equation (SDE). For SDE driven by Brownian noise, with appropriate assumptions on the coefficients of SDE, it is well known that the existence of finite second moments is assured. However, this does not hold in general case when the SDE is driven by a more general Lévy type noise. For example, let us consider the following state process:

$$\begin{cases} dX_t = b(t, X_{t-}, u_t)dt + \sigma(t, X_{t-}, u_t)dW_t + \displaystyle\int_{0<|\eta|<1} \gamma(t, X_{t-}, u_t, \eta)\tilde{N}(dt, d\eta) \\ \qquad\quad + \displaystyle\int_{|\eta|\geq 1} \gamma(t, X_{t-}, u_t, \eta)N(dt, d\eta) \\ \\ X_s = x \in \mathbb{R}^d, \qquad\qquad\qquad\qquad\qquad\qquad 0 \leq s \leq t \leq T, \end{cases}$$

where W is a Brownian motion, N is a Poisson random measure, and $\tilde{N}$ is the associated compensated Poisson random measure. In this case, we need further assumption on the measure ν, for example

$$\int_{|\eta|\geq 1} |\eta|^p \nu(d\eta)dt \quad < \quad \infty, \text{ for some } p \geq 2, \tag{2.2}$$

to assure that there exists a finite second moment for the state process. This would restrict us to only a subclass of Lévy type noise. However, in order to study controlled state processes with heavy tailed distributions, one needs to relax the moments assumption.

Zălinescu (2011) extended the proof of DPP to stable processes, which requires (2.2) to hold for a certain $p > 0$. He proved the DPP in the context of a combined control and optimal stopping problem in which a C^2-approximation of the state process is introduced. In contrast, the recent work of El Karoui and Tan (2013) formulated the stochastic control problems in terms of controlled martingale problems. Their proof assumes that (2.2) holds for $p = 1$.

In this chapter, we shall extend the proof of the DPP (under the strong formulation) by relaxing (2.2) in which no finite moments assumption are imposed. To this end, we use an approximation which is used in construction of solution of SDE with Lévy noise (see Section 1.3). The idea behind this is to define a new state process by cutting off the jumps if they are too large. Since large jumps cause the failure of the existence of finite moments (similar as in Theorem 1.2.6), by cutting of the large jumps we retain the nice property of existence of finite moments. We will follow Bouchard and Touzi (2011) and Zălinescu

(2011) to establish a DPP for the approximated state process. Then, by passing through the limit we obtain the desired result.

2.2 Problem Formulation

We will work on the Wiener-Poisson space. Let us recall the construction of such a space given in Bouchard and Touzi (2011); Ishikawa and Kunita (2006). To this end, we first recall the definitions of Wiener and Poisson spaces. Fix a $T > 0$. Let $\Omega_W = C([0,T];\mathbb{R}^d)$, and for $\omega_1 \in \Omega_W$, set $W_t(\omega_1) := \omega_1(t)$. Define $\mathbb{F}^W := (\mathcal{F}_t^W)_{t\geq 0}$ as the smallest filtration such that W_s is measurable with respect to $\mathcal{F}_t^W$ for all $s \in [0,t]$. On $(\Omega_W, \mathcal{F}^W)$, let $\mathbb{P}_W$ be the probability measure such that W is the m-dimensional standard Brownian motion, where $\mathcal{F}^W = \mathcal{F}_T^W$. Then, we obtain the Wiener space $(\Omega_W, \mathcal{F}^W, \mathbb{P}_W)$. Next, let Ω_N be the set of integer-valued measures on $[0,T]\times\mathbb{R}_0^q$, and for $\omega_2 \in \Omega_N$, set $N(\omega_2, I\times A) := \omega_2(I\times A)$, where $I \in \mathcal{B}([0,t])$, and $A \in \mathcal{B}(\mathbb{R}_0^q)$. Define $\mathbb{F}^N := (\mathcal{F}_t^N)_{t\geq 0}$ as the smallest filtration such that $N(\cdot, I\times A)$ is measurable with respect to $\mathcal{F}_t^N$ for all $I \in \mathcal{B}([0,t])$ and $A \in \mathcal{B}(\mathbb{R}_0^q)$. On $(\Omega_N, \mathcal{F}^N)$, let $\mathbb{P}_N$ be the probability measure such that N is the Poisson random measure with intensity ν, where $\mathcal{F}^N = \mathcal{F}_T^N$, and ν is the Lévy measure, i.e., it satisfies

$$\int_{\mathbb{R}_0^q} (|\eta|^2 \wedge 1)\nu(d\eta) < \infty.$$

Then, we obtain the Poisson space $(\Omega_N, \mathcal{F}^N, \mathbb{P}_N)$. Now, consider the product space $\Omega = \Omega_W \times \Omega_N$. For $\omega = (\omega_1, \omega_2) \in \Omega$, set $W_t(\omega) := W_t(\omega_1)$, and $N(\omega, I\times A) := N(\omega_2, I\times A)$. Let $\mathbb{P} := \mathbb{P}_W \otimes \mathbb{P}_N$ be the probability measure on $(\Omega, \mathcal{F})$, where $\mathcal{F}$ is the completion of $\mathcal{F}^W \otimes \mathcal{F}^N$. This then yields the Wiener-Poisson space $(\Omega, \mathcal{F}, \mathbb{P})$. Without of loss generality, we may assume that this space is complete. On this space, we may associate a filtration $(\mathcal{F}_t)_{t\geq 0}$ which is the right-continuous completed version of the filtration $(\mathcal{F}_t^W \otimes \mathcal{F}_t^N)_{t\geq 0}$.

Let $\mathcal{F}_t^{W,s}$ be the smallest σ-algebra such that $W_r - W_s$ is measurable with respect to $\mathcal{F}_t^{W,s}$ for all $r \in [s, t\vee s]$, and $\mathcal{F}_t^{N,s}$ be the smallest σ-algebra such that $N(\cdot, I_2\times A) - N(\cdot, I_1 \times A)$ is measurable with respect to $\mathcal{F}_t^{N,s}$ for all $I_1, I_2 \in \mathcal{B}([s, t\vee s])$, $A \in \mathcal{B}(\mathbb{R}_0^q)$, where $I_1 \subset I_2$. We define a commonly used filtration $(\mathcal{F}_t^s)_{t\geq s}$ which is the right-continuous completed version of $(\mathcal{F}_t^{W,s} \otimes \mathcal{F}_t^{N,s})_{t\geq s}$ (see for example Bouchard and Touzi (2011) for this filtration).

Next, we consider the following control problem. Fix $s \in [0,T)$, the (controlled) state process $(X_t^u)_{t\geq s}$ is assumed to follow the SDE:

$$\begin{cases} dX_t = b(t, X_{t-}, u_t)dt + \sigma(t, X_{t-}, u_t)dW_t + \displaystyle\int_{0<|\eta|<1} \gamma(t, X_{t-}, u_t, \eta)\tilde{N}(dt, d\eta) \\ \qquad\qquad + \displaystyle\int_{|\eta|\geq 1} \gamma(t, X_{t-}, u_t, \eta)N(dt, d\eta) \\ X_s = x \in \mathbb{R}^d, \qquad\qquad 0 \leq s \leq t \leq T, \end{cases} \tag{2.3}$$

where X_{t-} is the left limit of X_t, and $u : [0,T]\times\Omega \to \mathbb{R}^\ell$ is a predictable process which acts as a control. Moreover, $b : [0,T] \times \mathbb{R}^d \times \mathbb{R}^\ell \to \mathbb{R}^d$ is a continuous function, $\sigma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^\ell \to \mathbb{R}^{d\times m}$ is a continuous function, $\gamma : [0,T] \times \mathbb{R}^d \times \mathbb{R}^\ell \times \mathbb{R}^q_0 \to \mathbb{R}^d$ is a Borel measurable function, and γ is continuous in (t,x,u) for every $\eta \in \mathbb{R}^q_0$.

Fix a compact set $A \subset \mathbb{R}^\ell$. The set of admissible controls $(u_t)_{t\in[0,T]}$ is denoted by $\mathcal{A}_s$, where

$$\begin{aligned}\mathcal{A}_s \quad = \quad & \Big\{ u : [0,T] \times \Omega \to A \mid u \text{ is predictable with respect to } (\mathcal{F}^s_t)_{t\geq s} \\ & \text{and } u_r = 0 \text{ for all } r \in [0,s] \Big\}.\end{aligned}$$

In the rest of this chapter, we shall make the following assumption.

Assumption 2.1. There exist constants $C > 0$ and $C_M > 0$ such that for all $t \in [0,T]$, $u \in A$, $x_1, x_2 \in \mathbb{R}^d$, and $0 < |\eta| < M$, we have

$$\begin{aligned}\big|\sigma(t,x_1,u) - \sigma(t,x_2,u)\big| + \big|b(t,x_1,u) - b(t,x_2,u)\big| \quad &\leq \quad C|x_1 - x_2|, \\ \big|\gamma(t,x_1,u,\eta) - \gamma(t,x_2,u,\eta)\big| \quad &\leq \quad C_M|\eta||x_1 - x_2|, \\ \big|\gamma(t,x,\eta,u)\big| \quad &\leq \quad C_M|\eta|(1 + |x|).\end{aligned}$$

It is well known that under Assumption 2.1 and the compactness of A, there exists a constant $C > 0$ such that for all $t \in [s,T]$, $u \in A$, $x \in \mathbb{R}^d$, we have

$$\big|\sigma(t,x,u)\big| + \big|b(t,x,u)\big| \quad \leq \quad C(1 + |x|).$$

By Theorem 1.3.1 there exists a unique càdlàg and adapted solution solution of SDE (2.3). To emphasize dependence on initial conditions and the control, we may write X_t as $X^{u,s,x}_t$.

The revenue functional for a given $u \in \mathcal{A}_s$ is defined as

$$V^u(s,x) = \mathbb{E}\Big(\int_s^T f(t, X^{u,s,x}_t, u_t)dt + h(X^{u,s,x}_T) \Big), \tag{2.4}$$

where $f : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^\ell \to \mathbb{R}$ and $h : \mathbb{R}^d \to \mathbb{R}$ are continuous bounded functions. We will say that

$$V(s,x) = \sup_{u\in\mathcal{A}_s} V^u(s,x) \tag{2.5}$$

is the value function. If there exists a maximizer $u^*(s) := u^* \in \mathcal{A}_s$, then

$$V(s,x) \quad = \quad V^{u^*}(s,x).$$

2.3 The Dynamic Programming Principle

We start this section by stating the DPP. For $s \in [0,T]$, let $\mathcal{T}_{[s,T]}$ be the set of stopping times in $[s,T]$ adapted to $(\mathcal{F}^s_t)_{t\geq s}$. The DPP is then stated in the following Theorem.

Theorem 2.3.1. *(Dynamic Programming Principle): For every $\tau \in \mathcal{T}_{[s,T]}$ and all $x \in \mathbb{R}^d$,*

$$V(s,x) \quad = \quad \sup_{u \in \mathcal{A}_s} \mathbb{E}\Big(\int_s^\tau f(r, X_r^{u,s,x}, u_r)dr + V(\tau, X_\tau^{u,s,x}) \Big). \tag{2.6}$$

In order to prove the DPP, we need some preparations.

2.3.1 An Approximation of State Process

In this subsection, we present an approximation of the state process. Fix $s \in [0,T]$. Let $\tau_0 = s$, and for $k = 1, 2, ...$, let τ_k be the arrival time of kth jump of a compound Poisson process $(L_t)_{t \geq 0}$ after τ_0, where

$$L_t = \int_{|\eta| \geq 1} \eta N(t, d\eta).$$

Lemma 2.3.2. *For $M \geq 1$, let τ_M be a stopping time such that*

$$\tau_M = \inf\{t > s : \Delta L_t \in E_M\},$$

where $E_M = \{\eta \in \mathbb{R}_0^q : |\eta| \geq M\}$. As $M \to \infty$, we have $1_{\{\tau_M \leq T\}} \to 0$, $\mathbb{P}$-a.s. In particular, we have $1_{\{\tau_M \leq \tau\}} \to 0$ $\mathbb{P}$-a.s. for every $\tau \in \mathcal{T}_{[s,T]}$.

Proof. **Step 1:** We show that $1_{\{\tau_M \leq T\}} \to 0$ as $M \to \infty$ in probability, i.e.

$$\lim_{M \to \infty} \mathbb{P}\Big(1_{\{\tau_M \leq T\}} = 1\Big) = 0.$$

First, it is easy to see that

$$\{\tau_M \leq t\} = \Big\{ \sup_{r \in [s,t]} |\Delta L_r| \geq M \Big\}.$$

In particular if $t = T$, we have

$$\{\tau_M \leq \tau\} \subseteq \{\tau_M \leq T\} = \Big\{ \sup_{r \in [s,T]} |\Delta L_r| \geq M \Big\}$$

for every $\tau \in \mathcal{T}_{[s,T]}$.

Thus, to show

$$\lim_{M \to \infty} \mathbb{P}\Big(1_{\{\tau_M \leq T\}} = 1\Big) = 0,$$

it is enough to show

$$\lim_{M \to \infty} \mathbb{P}\Big(\Big\{ \sup_{r \in [s,T]} |\Delta L_r| \geq M \Big\}\Big) = 0.$$

Let $(M_n)_{n\geq 1}$ be an increasing sequence such that $M_n \to \infty$ as $n \to \infty$, and let

$$A_n = \Big\{ \sup_{r\in[s,T]} |\Delta L_r| \geq M_n \Big\}.$$

Since $A_{n+1} \subseteq A_n$, we have $\Big\{ \sup_{r\in[s,T]} |\Delta L_r| = \infty \Big\} = \bigcap_n A_n$.

Now, we see that

$$\begin{aligned}
\lim_{M\to\infty} \mathbb{P}\Big(\big\{ \sup_{r\in[s,T]} |\Delta L_r| \geq M \big\}\Big) &= \lim_{n\to\infty} \mathbb{P}(A_n) \\
&= \mathbb{P}\Big(\sup_{r\in[s,T]} |\Delta L_r| = \infty \Big) \\
&= 0,
\end{aligned}$$

where the last equality is due to the fact that the size of the jump can not be infinity in a finite time period.

Step 2: We next show that as $M \to \infty$, $1_{\{\tau_M \leq T\}} \to 0$ $\mathbb{P}$-a.s.

From **Step 1**, we know that there exists an increasing subsequence $(\hat{M}_n)_n$ (with $\hat{M}_n \to \infty$ as $n \to \infty$) such that $1_{\{\tau_{\hat{M}_n} \leq T\}} \to 0$ $\mathbb{P}$-a.s. as $n \to \infty$. Now, we observe that $1_{\{\tau_M \leq T\}}$ is bounded and non-increasing in M thus it converges to a limit a $\mathbb{P}$-a.s. and a must equal to 0, since we can't have two different limits. $\square$

Set $\zeta_0^M = x$, and for $k = 1, 2, ...,$ define

$$\zeta_k^M = X_{\tau_k}^{u,\tau_{k-1},\zeta_{k-1}^M} 1_{\{|\Delta L_{\tau_k}| < M\}} + X_{\tau_k -}^{u,\tau_{k-1},\zeta_{k-1}^M} 1_{\{|\Delta L_{\tau_k}| \geq M\}},$$

and

$$X_t^M = \sum_{k=0}^{\infty} X_t^{u,\tau_k,\zeta_k^M} 1_{[\tau_k,\tau_{k+1})}(t) 1_{[s,T]}(t). \tag{2.7}$$

By construction of solution, we see that $(X_t^M)_{t\geq s}$ satisfies the following SDE:

$$\begin{cases}
dX_t^M = b(t, X_{t-}^M, u_t)dt + \sigma(t, X_{t-}^M, u_t)dW_t + \displaystyle\int_{0<|\eta|<1} \gamma(t, X_{t-}^M, u_t, \eta)\tilde{N}(dt, d\eta) \\
\qquad + \displaystyle\int_{1\leq|\eta|<M} \gamma(t, X_{t-}^M, u_t, \eta)N(dt, d\eta) \\
X_s^M = x, \qquad\qquad 0 \leq s \leq t \leq T.
\end{cases} \tag{2.8}$$

Again, to emphasize dependence on initial conditions and the control, we may write X_t^M as $X_t^{u,s,x,M}$.

Following a standard argument, for example similar as in Kunita (2004) (see pp340-341 in Kunita (2004)), we can obtain the estimates below.

Lemma 2.3.3. *For every $M \geq 1$, and all $p \geq 2$, there exists a $C_{T,p,M} > 0$ such that*

1. $\mathbb{E}\Big(\sup_{t\in[s,T]} \big|X_t^{u,s,x,M}|^p\Big) \leq C_{T,p,M}\Big(1+|x|^p\Big)$,
2. $\mathbb{E}\Big(\sup_{t\in[s,T]} \big|X_t^{u,s,x,M} - X_t^{u,\hat{s},\hat{x},M}|^p\Big) \leq C_{T,p,M}\Big(|x-\hat{x}|^p + \big(1+|x|^p\big)|s-\hat{s}|\Big)$.

Remark 2.3.4. *We may extend $X^{u,\hat{s},\hat{x},M}$ by setting $X_t^{u,\hat{s},\hat{x},M} = \hat{x}$ for all $t \in [s,\hat{s}]$ (see p175 in Zălinescu (2011)).*

For the sequence of state processes $(X_t^M)_{t\geq s}$, we define their corresponding revenual functionals $V^{u,M}$ as

$$V^{u,M}(s,x) \quad = \quad \mathbb{E}\Big(\int_s^T f(t, X_t^{u,s,x,M}, u_t)dt + h(X_T^{u,s,x,M})\Big).$$

The value functions V^M is given by

$$V^M(s,x) \quad = \quad \sup_{u\in\mathcal{A}_s} V^{u,M}(s,x). \tag{2.9}$$

Next, we obtain the following lemma.

Lemma 2.3.5. *For every $(s,x) \in [0,T]\times\mathbb{R}^d$, as $M\to\infty$, $V^M(s,x) \to V(s,x)$.*

Proof. Since f and h are bounded and $1_{\{\tau_M>T\}}X_t^{u,s,x} = 1_{\{\tau_M>T\}}X_t^{u,s,x,M}$ $\mathbb{P}$-a.s. for every $t\in[s,T]$ and $u\in\mathcal{A}_s$, we see that

$$\begin{aligned}
V^u(s,x) \quad &\leq \quad \mathbb{E}\Bigg(\int_s^T \Big(f(t, X_t^{u,s,x}, u_t)1_{\{\tau_M>t\}} + C1_{\{\tau_M\leq t\}}\Big)dt \\
&\qquad + 1_{\{\tau_M>T\}}h(X_T^{u,s,x}) + C1_{\{\tau_M\leq T\}}\Bigg) \\
&\leq \quad \mathbb{E}\Bigg(\int_s^T \Big(f(t, X_t^{u,s,x,M}, u_t) + h(X_T^{u,s,x,M})\Big)dt + C1_{\{\tau_M\leq T\}}\Bigg) \\
&\leq \quad V^{u,M}(s,x) + C\mathbb{E}(1_{\{\tau_M\leq T\}}) \\
&\leq \quad V^M(s,x) + C\mathbb{E}(1_{\{\tau_M\leq T\}}).
\end{aligned}$$

As $M\to\infty$, and by Lemma 2.3.2, we have

$$V^u(s,x) \quad \leq \quad \liminf_{M\to\infty} V^M(s,x).$$

Taking supremum over $\mathcal{A}_s$, we have

$$V(s,x) \quad \leq \quad \liminf_{M\to\infty} V^M(s,x).$$

To show the converse inequality, we observe that for every $M\geq 1$ and $\epsilon>0$ there exists an ϵ-optimal control $u^{\epsilon,M}\in\mathcal{A}_s$ such that

$$V^M(s,x) \quad \leq \quad V^{u,M}(s,x) + \epsilon. \tag{2.10}$$

By (2.10) and again note that $1_{\{\tau_M>T\}}X_t^{u^{\epsilon,M},s,x} = 1_{\{\tau_M>T\}}X_t^{u^{\epsilon,M},s,x,M}$ $\mathbb{P}$-a.s. we see that

$$\begin{aligned}
V^M(s,x) &\leq V^{u^{\epsilon,M},M}(s,x)+\epsilon \\
&\leq \mathbb{E}\bigg(\int_s^T \Big(f(t,X_t^{u^{\epsilon,M},s,x},u_t^{\epsilon,M})1_{\{\tau_M>t\}}+C1_{\{\tau_M\leq t\}}\Big)dt \\
&\quad +1_{\{\tau_M>T\}}h(X_T^{u^{\epsilon,M},s,x})+C1_{\{\tau_M\leq T\}}\bigg)+\epsilon \\
&\leq \mathbb{E}\bigg(\int_s^T \Big(f(t,X_t^{u^{\epsilon,M},s,x},u_t^{\epsilon,M})+h(X_T^{u^{\epsilon,M},s,x})\Big)dt+C1_{\{\tau_M\leq T\}}\bigg)+\epsilon \\
&\leq V^{u^{\epsilon,M}}(s,x)+C\mathbb{E}(1_{\{\tau_M\leq T\}})+\epsilon \\
&\leq V(s,x)+C\mathbb{E}(1_{\{\tau_M\leq T\}})+\epsilon
\end{aligned}$$

As $M\to\infty$, and by Lemma 2.3.2, we obtain

$$\limsup_{M\to\infty} V^M(s,x) \leq V(s,x)+\epsilon.$$

Since ϵ is arbitrary, the proof is completed. □

Next, we present two results which we borrowed from Zălinescu (2011) (modified version of Lemma 2.3 in Zălinescu (2011)). Since the author does not provide a proof, we prove it here in our context.

Now, under the assumption that f and h are continuous, we know that the functions f and h admit a joint modulus of continuity (see Lemma 2.3 in Zălinescu (2011)):

$$\rho(\alpha,\beta) = \sup_{\substack{t\in[0,T],u\in A,\\ x,\hat{x}\in\overline{B(0,\beta)},|x-\hat{x}|\leq\alpha}} \Big(|f(t,x,u)-f(t,\hat{x},u)|+|h(x)-h(\hat{x})|\Big),$$

such that $\lim_{\beta\to\infty}\lim_{\alpha\to 0}\rho(\alpha,\beta)=0$. Thus, we have the first result below.

Proposition 2.3.6. *There exists constants $C>0$ and $C_{T,p,M}>0$, such that for every $u\in\mathcal{A}_s$, $(s,x),(\hat{s},\hat{x})\in[0,T]\times\mathbb{R}^d$, and all $p\geq 2$, $\alpha>0$, $\beta>0$,*

$$\begin{aligned}
\big|V^{u,M}(s,x)-V^{u,M}(\hat{s},\hat{x})\big| &\leq C_T\rho(\alpha,\beta)+C_{T,p,M}\frac{|x-\hat{x}|^p+(1+|\hat{x}|^p)|s-\hat{s}|}{\alpha^p} \\
&\quad +C_{T,p,M}\frac{(1+|x|^p+|\hat{x}|^p)}{\beta^p},
\end{aligned}$$

Proof. For $u\in\mathcal{A}_s$, and $(s,x),(\hat{s},\hat{x})\in[0,T]\times\mathbb{R}^d$, we see that

$$\begin{aligned}
\big|V^{u,M}(s,x)-V^{u,M}(\hat{s},\hat{x})\big| &\leq \big|V^{u,M}(s,x)-V^{u,M}(s,\hat{x})\big| \\
&\quad +\big|V^{u,M}(s,\hat{x})-V^{u,M}(\hat{s},\hat{x})\big| \\
&= (I_1)+(I_2). \qquad (2.11)
\end{aligned}$$

The first term in (2.11) yields

$$
\begin{aligned}
(I_1) &= \left|V^{u,M}(s,x) - V^{u,M}(s,\hat{x})\right| \\
&\leq \mathbb{E}\left(\int_s^T \left|f(t, X_t^{u,s,x,M}, u_t) - f(t, X_t^{u,s,\hat{x},M}, u_t)\right|dt + \left|h(X_T^{u,s,x,M}) - h(X_T^{u,s,\hat{x},M})\right|\right) \\
&= (I_{1,1}) + (I_{1,2}).
\end{aligned}
$$

The second term in (2.11) can be estimated as

$$
\begin{aligned}
(I_2) &= \left|V^{u,M}(s,\hat{x}) - V^{u,M}(\hat{s},\hat{x})\right| \\
&\leq \mathbb{E}\left(\int_s^{\hat{s}} \left|f(t, X_t^{u,s,\hat{x},M}, u_t) - f(t, X_t^{u,\hat{s},\hat{x},M}, u_t)\right|dt + \left|h(X_T^{u,s,\hat{x},M}) - h(X_T^{u,\hat{s},\hat{x},M})\right|\right) \\
&= (I_{1,3}) + (I_{1,4}).
\end{aligned}
$$

Each of $(I_{1,1}) - (I_{1,4})$ can be estimated by using the bounds of f and h, the Markov inequality, and Lemma 4.3.1. For example, for (I) we have

$$
\begin{aligned}
(I_{1,1}) &= \mathbb{E}\left(\int_s^T \left|f(t, X_t^{u,s,x,M}, u_t) - f(t, X_t^{u,s,\hat{x},M}, u_t)\right|dt\right) \\
&\leq C_T \mathbb{P}\left(\sup_{t\in[s,T]} \left|X_t^{u,s,x,M} - X_t^{u,s,\hat{x},M}\right|^p > \alpha^p\right) + C_T\rho(\alpha,\beta) \\
&\quad + C_T \mathbb{P}\left(\sup_{t\in[s,T]} \left|X_t^{u,s,x,M}\right|^p \geq \beta^p\right) + C_T \mathbb{P}\left(\sup_{t\in[s,T]} \left|X_t^{u,s,\hat{x},M}\right|^p \geq \beta^p\right) \\
&\leq C_T\rho(\alpha,\beta) + C_{T,p,M}\frac{|x-\hat{x}|^p}{\alpha^p} + C_{T,p,M}\frac{(1+|x|^p+|\hat{x}|^p)}{\beta^p}.
\end{aligned}
$$

In a similar way, we obtain

$$
\begin{aligned}
(I_{1,2}) &\leq C_T\rho(\alpha,\beta) + C_{T,p,M}\frac{|x-\hat{x}|^p}{\alpha^p} + C_{T,p,M}\frac{(1+|x|^p+|\hat{x}|^p)}{\beta^p}, \\
(I_{1,3}) &\leq C_T\rho(\alpha,\beta) + C_{T,p,M}\frac{(1+|\hat{x}|^p)|s-\hat{s}|}{\alpha^p} + C_{T,p,M}\frac{(1+|\hat{x}|^p)}{\beta^p}, \\
(I_{1,4}) &\leq C_T\rho(\alpha,\beta) + C_{T,p,M}\frac{(1+|\hat{x}|^p)|s-\hat{s}|}{\alpha^p} + C_{T,p,M}\frac{(1+|\hat{x}|^p)}{\beta^p}.
\end{aligned}
$$

Combing $(I) - (IV)$, we complete the proof. □

Since

$$
\begin{aligned}
|V^M(s,x) - V^M(s,\hat{x})| &= \left|\sup_{u\in\mathcal{A}_s} V^{u,M}(s,x) - \sup_{u\in\mathcal{A}_s} V^{u,M}(s,\hat{x})\right| \\
&\leq \sup_{u\in\mathcal{A}_s}\left|V^{u,M}(s,x) - V^{u,M}(s,\hat{x})\right|,
\end{aligned}
$$

the following corollary is a direct consequence of Proposition 2.3.6.

Corollary 2.3.7. *For all $p \geq 2$, there exists constants $C_T > 0$ and $C_{T,p,M} > 0$ such that* *for $\alpha > 0$, $\beta > 0$, and $(s,x),(\hat{s},\hat{x}) \in [0,T]\times\mathbb{R}^d$, we have*

$$
\begin{aligned}
|V^M(s,x) - V^M(\hat{s},\hat{x})| \quad \leq \quad & C_T\rho(\alpha,\beta) + C_{T,p,M}\frac{|x-\hat{x}|^p + (1+|\hat{x}|^p)|s-\hat{s}|}{\alpha^p} \\
& + C_{T,p,M}\frac{(1+|x|^p+|\hat{x}|^p)}{\beta^p}.
\end{aligned}
$$

In order to prove the DPP, the Markov characterization of the state process (see for example, Lemma 3.2 in Zălinescu (2011)) plays an important role. The next lemma states the controlled Markovian property for jump processes.

Lemma 2.3.8. *The following two assertions hold.*

1. *For almost every $\omega \in \Omega$, all $\tau \in \mathcal{T}_{[s,T]}$, and $u \in \mathcal{A}_s$, there exists a control $\hat{u}^\omega \in \mathcal{A}_\tau$ such that*

$$\mathbb{E}\Big(\int_\tau^T f(r, X_r^{u,s,x,M}, u_r)dr + h(X_T^{u,s,x,M})\big|\mathcal{F}_\tau\Big)(\omega) = V^{\hat{u}^\omega,M}(\tau(\omega), X_\tau^{u,s,x,M}(\omega)).$$

2. *For every $t \in [s,T]$, and all $\tau \in \mathcal{T}_{[s,t]}$, there exists a control $\hat{u} \in \mathcal{A}_s$, where*

$$\hat{u}_r := u_r 1_{\{r\in[s,\tau]\}} + \tilde{u}_r 1_{\{r\in(\tau,T]\}},$$

and $\tilde{u} \in \mathcal{A}_t$, such that

$$\mathbb{E}\Big(\int_\tau^T f(r, X_r^{\hat{u},s,x,M}, u_r)dr + h(X_T^{\hat{u},s,x,M})\big|\mathcal{F}_\tau\Big)(\omega) = V^{\tilde{u},M}(\tau(\omega), X_\tau^{u,s,x,M}(\omega)) \quad \mathbb{P}\text{-a.s.}$$

Proof. The proof follows from Remark 3.10 and the proof of Proposition 5.4 in Bouchard and Touzi (2011). □

2.3.2 The Proof of DPP

We now proceed to the prove of DPP. We will follow Bouchard and Touzi (2011) and Zălinescu (2011).

Proof. We start from the easy direction. For $\tau \in \mathcal{T}_{[s,T]}$, $u \in \mathcal{A}_s$, and by the first assertion of Lemma 2.3.8, we see that for $M \geq 1$ there exists a control $\hat{u} \in \mathcal{A}_\tau$ such that

$$
\begin{aligned}
& \mathbb{E}\Big(\int_s^T f(t, X_t^{u,s,x,M}, u_t)dt + h(X_T^{u,s,x,M})\Big) \\
= \; & \mathbb{E}\Big(\int_s^\tau f(t, X_t^{u,s,x,M}, u_t)dt + V^{\hat{u},M}(\tau, X_\tau^{u,s,x,M})\Big) \\
\leq \; & \mathbb{E}\Big(\int_s^\tau f(t, X_t^{u,s,x}, u_t)1_{\{\tau_M>\tau\}}dt + C_T 1_{\{\tau_M\leq\tau\}} + V^M(\tau, X_\tau^{u,s,x})1_{\{\tau_M>\tau\}}\Big).
\end{aligned}
$$

In the last line, we have used the fact that $1_{\{\tau_M>\tau\}}X_t^{u,s,x,M} = 1_{\{\tau_M>\tau\}}X_t^{u,s,x}$ ($\mathbb{P}$-a.s.) for every $t \in [s,\tau]$. As $M \to \infty$, by boundedness of f, and h, we can apply the Dominated Convergence Theorem. Thus, together with Lemma 2.3.2 and Lemma 2.3.5, we have

$$\mathbb{E}\Big(\int_s^T f(t, X_t^{u,s,x}, u_t)dt + h(X_T^{u,s,x})\Big) \quad \leq \quad \mathbb{E}\Big(\int_s^\tau f(t, X_t^{u,s,x}, u_t)dt + V(\tau, X_\tau^{u,s,x})\Big).$$

Taking supremum over $\mathcal{A}_s$, we obtain

$$V(s,x) \quad \leq \quad \sup_{u\in\mathcal{A}_s} \mathbb{E}\Big(\int_s^\tau f(t, X_t^{u,s,x}, u_t)dt + V(\tau, X_\tau^{u,s,x})\Big). \tag{2.12}$$

To show the converse, fix $\epsilon \in (0,1)$ and choose $\alpha < \epsilon$. Next, choose $\beta > (\frac{1}{\epsilon})^{\frac{1}{p}}$ such that $\rho(\alpha,\beta) < \epsilon$. For a fixed $p \geq 2$, let us take a Borel partition $\{B_j\}_{j\geq 1}$ of $\mathbb{R}^d$ such that

$$\sup_{x_j, \hat{x}_j \in B_j} |x_j - \hat{x}_j|^p \leq \alpha^p \epsilon. \tag{2.13}$$

For $M \geq 1$, $t \in [s,T]$ and $x \in \mathbb{R}^d$, we know that there exists an ϵ-optimal control $\tilde{u}^{\epsilon,M} \in \mathcal{A}_t$ such that

$$V^M(t,x) \leq V^{M,\tilde{u}^{\epsilon,M}}(t,x) + \epsilon. \tag{2.14}$$

By Corollary 2.3.7, (2.13)-(2.14), and Proposition 2.3.6, we see that for every $x_j, \hat{x}_j \in B_j$, there exists an ϵ-optimal control $\tilde{u}^{j,\epsilon,M} \in \mathcal{A}_t$ such that

$$\begin{aligned} V^M(t,x_j) \quad &\leq \quad V^M(t,\hat{x}_j) + \epsilon C_{T,p,M}(1 + |x_j|^p) \\ &\leq \quad V^{M,\tilde{u}^{j,\epsilon,M}}(t,x_j) + \epsilon C_{T,p,M}(1 + |x_j|^p) + \epsilon \\ &\leq \quad V^{M,\tilde{u}^{j,\epsilon,M}}(t,x_j) + \epsilon C_{T,p,M}(1 + |x_j|^p). \end{aligned} \tag{2.15}$$

For $u \in \mathcal{A}_s$, we take a sequence of controls

$$\hat{u}_r^{j,\epsilon,M} = \begin{cases} u_r, & \text{if } r \in [s,t], \\ \tilde{u}_r^{j,\epsilon,M}, & \text{if } r \in (t,T] \text{ and } X_t^{u,s,x,M} \in B_j, \end{cases}$$

where $\tilde{u}^{j,\epsilon,M} \in \mathcal{A}_t$. It is easy to see that $\hat{u}^{j,\epsilon,M} \in \mathcal{A}_s$ which is a consequence of the measurability of $X_t^{u,s,x,M}$ and the fact that $\mathcal{F}_r^t \subset \mathcal{F}_r^s$ for all $s \leq t \leq r$. By uniqueness of solution, the second assertion of Lemma 2.3.8 and (2.15) we then obtain

$$\begin{aligned} V^M(s,x) \quad &\geq \quad \mathbb{E}\left(\int_s^T f(r, X_r^{\hat{u}^{j,\epsilon,M},s,x,M}, u_r)dr + h(X_T^{\hat{u}^{j,\epsilon,M},s,x,M})\right) \\ &= \quad \mathbb{E}\left(\int_s^t f(r, X_r^{u,s,x,M}, u_r)dr\right) + \sum_{j\geq 1} \mathbb{E}\Bigg(\mathbb{E}\Big(\int_t^T f(r, X_r^{\tilde{u}^{j,\epsilon,M},t,X_t^{u,s,x,M}}, \tilde{u}_r^{j,\epsilon,M})dr \\ &\quad + h(X_T^{\tilde{u}^{\epsilon,j,M},t,X_t^{u,s,x,M}})\big|\mathcal{F}_t\Big)1_{\{X_t^{u,s,x,M}\in B_j\}}\Bigg) \\ &= \quad \mathbb{E}\left(\int_s^t f(r, X_r^{u,s,x,M}, u_r)dr\right) + \sum_{j\geq 1} \mathbb{E}\Big(V^{u^{\epsilon,j,M},M}(t, X_t^{u,s,x,M})1_{\{X_t^{u,s,x,M}\in B_j\}}\Big) \end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{E}\bigg(\int_s^t f(r, X_r^{u,s,x,M}, u_r)dr\bigg) + \sum_{j\geq 1}\mathbb{E}\bigg(\Big(V^M(t, X_t^{u,s,x,M})\\
&\quad -\epsilon C_{T,p,M}(1+|X_t^{u,s,x,M}|^p)\Big)1_{\{X_t^{u,s,x,M}\in B_j\}}\bigg)\\
&= \mathbb{E}\bigg(\int_s^t f(r, X_r^{u,s,x,M}, u_r)dr + V^M(t, X_t^{u,s,x,M})\bigg) - \epsilon C_{T,p,M}\mathbb{E}\Big(1+|X_t^{u,s,x,M}|^p\Big).
\end{aligned}
$$

Since ϵ is arbitrary, we then have

$$
V^M(s,x) \geq \mathbb{E}\Big(\int_s^t f(r, X_r^{u,s,x,M}, u_r)dr + V^M(t, X_t^{u,s,x,M})\Big).
$$

Let $\mathcal{G}(t) := \int_s^t f(X_r^{u,s,x})dr + V^M(t, X_t^{u,s,x})$. For every $t_1, t_2 \in [s,T)$, and

$$
\hat{u}_r = \begin{cases} u_r, & \text{if } r\in[s,t_1],\\ \tilde{u}_r, & \text{if } r\in(t_1,T], \end{cases}
$$

where $u \in \mathcal{A}_s$ and $\tilde{u} \in \mathcal{A}_{t_1}$, we have, by uniqueness of solution and the second assertion of Lemma 2.3.8,

$$
\begin{aligned}
\mathbb{E}(\mathcal{G}(t_2)|\mathcal{F}_{t_1}) &= \int_s^{t_1} f(t, X_t^{u,s,x,M}, u_t)dt\\
&\quad +\mathbb{E}\bigg(\int_{t_1}^{t_2} f(t, X_t^{u,s,X_{t_1}^{u,s,x,M},u_t}, \tilde{u}_t)dt + V^M(t_2, X_{t_2}^{u,s,X_{t_1}^{u,s,x,M}})\big|\mathcal{F}_{t_1}\bigg)\\
&\leq \int_s^{t_1} f(t, X_t^{u,s,x,M}, u_t)dt + V^M(t_1, X_{t_1}^{u,s,x,M})\\
&= \mathcal{G}(t_1).
\end{aligned}
$$

Thus, $\mathcal{G}$ is a supermartingale, and by Doob's Optional Sampling Theorem we know that, for every stopping time $\tau \in \mathcal{T}_{[s,T]}$ and $u \in \mathcal{A}_s$, we have

$$
V^M(s,x) \geq \mathbb{E}\Big(\int_s^\tau f(t, X_t^{u,s,x,M}, u_t)dt + V^M(\tau, X_\tau^{u,s,x,M})\Big). \tag{2.16}
$$

Without loss of generality, we assume that $f, h > 0$ for all $(s,x) \in [s,T]\times\mathbb{R}^d$. Then, (2.16) implies

$$
V^M(s,x) \geq \mathbb{E}\Big(\int_s^\tau f(t, X_t^{u,s,x}, u_t)1_{\{\tau_M>\tau\}}dt + V^M(\tau, X_\tau^{u,s,x})1_{\{\tau_M>\tau\}}\Big).
$$

Here, we use the fact that $1_{\{\tau_M>\tau\}}X_t^{u,s,x,M} = 1_{\{\tau_M>\tau\}}X_t^{u,s,x}$ ($\mathbb{P}$-a.s.) for every $t\in[s,\tau]$. As $M\to\infty$, thanks to the boundedness of f and h, the Dominated Convergence Theorem can be applied. Together with Lemma 2.3.2 and Lemma 2.3.5, the above yields

$$
V(s,x) \geq \mathbb{E}\Big(\int_s^\tau f(t, X_t^{u,s,x}, u_t)dt + V(\tau, X_\tau^{u,s,x})\Big).
$$

Taking supremum over $\mathcal{A}_s$, and combining with (2.12) we obtain the desired result. $\square$

Finally, as a consequence of (2.12) and (2.16), we obtain the following result.

Corollary 2.3.9. *For every $\tau \in \mathcal{T}_{[s,T]}$, and all $x \in \mathbb{R}^d$. the following holds.*

$$V^M(s,x) = \sup_{u \in \mathcal{A}_s} \mathbb{E}\Big(\int_s^\tau f(r, X_r^{u,s,x,M}, u_r)dr + V^M(\tau, X_\tau^{u,s,x,M}) \Big). \tag{2.17}$$

2.4 A Brief Concluding Remark

In this chapter, we have extended the proof of DPP for standard stochastic control problems with Lévy noise. In doing so, we do not assume the state process possess any moments and we do not impose any restrictions on Lévy measures. It worth note that our result is under the restriction that the set of admissible controls are independent of $\mathcal{F}_s$. However, as remarked in Bouchard and Touzi (2011), this is not necessary (see Remark 5.2 in Bouchard and Touzi (2011)). For future work, one possible extension of our work is to consider a combined optimal stopping and stochastic control problem, and show that similar result can be established.

Chapter 3

The Associated Hamilton-Jacobi-Bellman Equation for Stochastic Control Problems driven by Lévy Noise

The infinitesimal version of DPP, that is the Hamilton-Jacobi-Bellman (HJB) equation, provides a necessary and sufficient condition for optimality. It is necessary, since as a direct consequence from last chapter, we can show that the value function satisfies the corresponding integro-HJB (henceforth just HJB) equation in the viscosity sense. It is sufficient, since the value function is the unique viscosity solution of the corresponding HJB equation.

The proof of the uniqueness of the solution of the HJB equation often relies on a comparison theorem while prove of the comparison theorem generally relies a maximum principle and a carefully chosen test function. There are many works devoted to prove an integro-version of maximum principle and/or a comparison theorem. To name a few, we mention Jakobsen and Karlsen (2005, 2006); Barles and Imbert (2008) who consider the theory of general integro-partial differential equations (henceforth integro-PDEs), Pham (1998); Zălinescu (2011) who consider HJB equations (or variational inequalities) arising from combined optimal stopping and stochastic control problems, and Barles et al. (2009); Dumitrescu et al. (2015) who consider integro-PDEs (or variational inequalities) related to backward stochastic differential equations. In the context of HJB equations (arising from stochastic control problems), Pham (1998) has proved a comparison theorem for uniform viscosity super and subsolutions. In his proof, the viscosity solution is defined in terms of parabolic semijets. Zălinescu (2011) has taken a further step and proved the comparison theorem for semi-continuous viscosity super and subsolutions. Since he considered process with no Brownian part, the proof is simplified a little bit and does not require the maximum principle. Both proofs have some restrictions on the Lévy measure. In contrast, a more general Lévy measure was considered by Jakobsen and Karlsen (2005, 2006).

To extend the work of Larsy and Lions, the comparison theorem for HJB equations also plays an important role. In this chapter, we apply the DPP (Corollary 2.3.9) to show that the value function (which we defined in section 2.2) is a viscosity solution of the corresponding HJB equation. Moreover, we borrow the nonlocal version of maximum principle from Jakobsen and Karlsen (2006, 2005), and formulate a comparison theorem by choosing an appropriate test function. An immediate consequence of this comparison theorem is a uniqueness result which states that there is a unique viscosity solution to the HJB equation (in the class of bounded semicontinuous functions). We emphasize that although there is not much novelty in this chapter, the work in this chapter is very important for us. The aforementioned works have covered our needs from various directions. However, we can't directly apply their result, and there is some work need to be done to fit into our framework. Here, our main references are Yong and Zhou (1999); Zălinescu (2011); Pham (1998); Jakobsen and Karlsen (2006, 2005); Barles et al. (2009); Ishii (1984); Barles and Imbert (2008).

3.1 Theory of Viscosity Solution of HJB equation

Through this chapter, we assume Assumption 2.1 holds, and we work under the framework of the previous chapter. In addition, we make a stronger assumption on functions f and h (recall that these are defined in (4.3)), and an extra assumption on functions b, σ, and γ (recall that these are defined in (2.3)). This is presented below.

Assumption 3.1. There exists constants $C > 0$, $C_M > 0$ and a modulus of continuity ρ, such that for all $s_1, s_2 \in [0,T]$, $u \in A$, $x_1, x_2 \in \mathbb{R}^d$, and $0 < |\eta| < M$, we have

$$\begin{aligned}
&|f(s_1,x_1,u) - f(s_2,x_2,u)| + |h(x_1) - h(y_2)| \leq C\rho\big((s_1,x_1) - (s_2,y_2)\big),\\
&|b(s_1,x_1,u) - b(s_2,x_1,u)| + |\sigma(s_1,x,u) - \sigma(2_2,y,u)| \leq C|s_1 - s_2|,\\
&|\gamma(s_1,x,u,\eta) - \gamma(s_2,x,u,\eta)| \leq C_M|\eta||s_1 - s_2|.
\end{aligned}$$

We first recall some basic facts from the theory of viscosity solutions. Fix $M \geq 1$. For $s \in (0,T)$, $x \in \mathbb{R}^d$, $u \in A$ and $\phi \in C_b^{1,2}((0,T] \times \mathbb{R}^d; \mathbb{R})$, define an operator

$$\begin{aligned}
I^M[\phi](s,x,u) \;=\; & \int_{0\leq|\eta|<M} \Big(\phi\big(s, x + \gamma(s,x,u,\eta)\big) - \phi(s,x)\\
& - < D_2\phi(s,x), \gamma(s,x,u,\eta) > 1_{\{0<|\eta|<1\}}\Big)\nu(d\eta).
\end{aligned}$$

The so-called (generalized) Hamiltonian is defined as

$$\begin{aligned}
& \mathcal{H}\Big(s,x,u,D_2\phi(s,x),D_2^2\phi(s,x),I^M[\phi](s,x,u)\Big)\\
= \;& < -D_2\phi(s,x), b(s,x,u) > + \frac{1}{2}tr\Big(-D_2^2\phi(s,x)\sigma(s,x,u)\sigma(s,x,u)^T\Big)\\
& -I^M[\phi](s,x,u) - f(s,x,u). \qquad (3.1)
\end{aligned}$$

Fix $M \geq 1$. For $s \in (0,T)$ and $x \in \mathbb{R}^d$, we consider the HJB equation:

$$\begin{cases} -D_1\phi(s,x) + \sup_{u\in A} \mathcal{H}\Big(s,x,u,D_2\phi(s,x),D_2^2\phi(s,x),I^M[\phi](s,x,u)\Big) = 0, \\ \phi(T,x) = h(x). \end{cases} \tag{3.2}$$

It is clear that we may not able to find a (classical) solution in the class of smooth functions (whose partial derivatives exist as required in (3.2)). Thus, the notion of viscosity solutions allows us to seek for a (weaker) solution in a larger class of functions. There are several equivalent ways to define viscosity solutions (see for example Barles and Imbert (2008); Arisawa (2008)). Here, we present two of them.

Definition 3.1.1. *A function $w \in USC([0,T]\times\mathbb{R}^d;\mathbb{R})$ (respectively $LSC([0,T]\times\mathbb{R}^d;\mathbb{R})$) is a viscosity subsolution (respectively supersolution) to equation (3.2) if*

1. *$w(T,x) \le h(x)$ (respectively $w(T,x) \ge h(x)$);*
2. *whenever $\phi \in C_b^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$, and $\phi - w$ attains a global maximum (respectively global minimum) at $(s^*,x^*) \in (0,T)\times\mathbb{R}^d$, we have*

$$-D_1\phi(s^*,x^*) + \sup_{u\in A}\mathcal{H}\Big(s^*,x^*,u,D_2\phi(s^*,x^*),D_2^2\phi(s^*,x^*),I^M[\phi](s^*,x^*,u)\Big) \le 0,$$
$$(\text{respectively } \ge 0).$$

If w is both a viscosity supersolution and subsolution, then it is a viscosity solution, and we say that w satisfies the corresponding equation in the viscosity sense. This definition allows us to show that value function is a viscosity solution of the HJB equation. However, in order to prove the comparison theorem and hence formulate the uniqueness result, we need another definition of viscosity solutions. To make some preparations, we define some new terms. Fix $M \ge 1$. For $s \in (0,T)$, $x \in \mathbb{R}^d$, $\beta \in (0,1)$, $u \in A$, $q \in \mathbb{R}$, $\phi \in C_b^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$, and $w \in SC([0,T]\times\mathbb{R}^d;\mathbb{R})$, we define two operators

$$\begin{aligned} I^\beta[\phi]\big(s,x,u\big) &= \int_{0<|\eta|<\beta}\Big(\phi\big(s,x+\gamma(s,x,u,\eta)\big) - \phi(s,x) \\ &\quad - < D_2\phi(s,x),\gamma(s,x,u,\eta) > \Big)\nu(d\eta), \end{aligned}$$

and

$$\begin{aligned} I_\beta^M[w]\big(s,x,u,q\big) &= \int_{\beta\le|\eta|<M}\Big(w\big(s,x+\gamma(s,x,u,\eta)\big) - w(s,x) \\ &\quad - < q,\gamma(s,x,u,\eta) > 1_{\{0<|\eta|<1\}}\Big)\nu(d\eta). \end{aligned}$$

Furthermore, we define the function

$$\begin{aligned} &\hat{\mathcal{H}}\Big(s,x,u,I^\beta[\phi](s,x,u),I_\beta^M[w]\big(s,x,u,D_2\phi(s,x)\big)\Big) \\ = \;& < -D_2\phi(s,x),b(s,x,u) > + \frac{1}{2}tr\Big(-D_2^2\phi(s,x)\sigma(s,x,u)\sigma(s,x,u)^T\Big) \\ & -I^\beta[\phi]\big(s,x,u\big) - I_\beta^M[w]\big(s,x,u,\phi_2(s,x)\big) - f(s,x,u). \end{aligned} \tag{3.3}$$

Next, we present the second definition.

Definition 3.1.2. *A function $w \in USC_b([0,T]\times\mathbb{R}^d;\mathbb{R})$ (respectively $LSC_b([0,T]\times\mathbb{R}^d;\mathbb{R})$) is a viscosity subsolution (respectively supersolution) to equation (3.2) if and only if*

1. *$w(T,x) \le h(x)$ (respectively $w(T,x) \ge h(x)$);*
2. *whenever $\phi \in C^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$, and $\phi - w$ attains a global maximum (respectively global minimum) at $(s^*,x^*) \in (0,T)\times\mathbb{R}^d$, we have*

$$-D_1\phi(s^*,x^*) + \sup_{u\in A}\hat{\mathcal{H}}\Big(s^*,x^*,u,D_2\phi(s^*,x^*),D_2^2\phi(s^*,x^*),I^{\beta}[\phi](s^*,x^*,u),$$
$$I_{\beta}^{M}[w]\big(s^*,x^*,u,D_2\phi(s^*,x^*)\big)\Big) \le 0,$$
$$(\text{respectively } \ge 0).$$

Now, to conclude this long summary of viscosity solutions, we include the following two results without proofs (for **(C1)**-**(C4)** and **(F0)** -**(F3)**, see Example 4.1 and Example 4.7 in Jakobsen and Karlsen (2006) and the proof of Theorems 4.1 - 4.3 in Jakobsen and Karlsen (2005); for **(F4)**, see p193 in Zălinescu (2011)). These results summaries some properties of viscosity solutions and ensure the validity of the maximum principle.

Lemma 3.1.3. *For $s \in (0,T)$, $x,y \in \mathbb{R}^d$, $q \in \mathbb{R}^d$, $X,Y \in \mathbb{S}^d$, and $\phi^k,\phi,\psi \in C_b^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$, the following hold.*

(C1). *If $(t,y,Y) \to (s,x,X)$, then*

$$\mathcal{H}\Big(t,y,u,D_2\phi(t,y),Y,I^M[\phi](t,y,u)\Big) \to \mathcal{H}\Big(s,x,u,D_2\phi(s,x),X,I^M[\phi](s,x,u)\Big)$$

uniformly in u.

(C2). *If $(s_k,x_k) \to (s,x)$, and*

$$\begin{aligned}\phi^k &\to \phi\\ D_2\phi^k &\to D_2\phi\\ D_2^2\phi^k &\to D_2^2\phi\end{aligned}$$

locally uniformly in $(0,T)\times\mathbb{R}^d$, then

$$\begin{aligned}&\mathcal{H}\Big(s_k,x_k,u,D_2\phi^k(s_k,x_k),D_2^2\phi(s_k,x_k),I^M[\phi^k](s_k,x_k,u)\Big)\\ \to\ &\mathcal{H}\Big(s,x,u,D_2\phi(s,x),D_2^2\phi(s,x),I^M[\phi](s,x,u)\Big)\end{aligned}$$

uniformly in u.

(C3). *If $X \le Y$ (i.e., $Y-X$ is positive semidefinite) and $(\phi-\psi)(s,\cdot)$ has global maximum at x^*, then*

$$\sup_{u\in A}\mathcal{H}\Big(s,x^*,u,q,X,I^M[\phi](s,x^*,u)\Big) \ \ge\ \sup_{u\in A}\mathcal{H}\Big(s,x^*,u,q,Y,I^M[\psi](s,x^*,u)\Big).$$

(C4). *For every constant $C \in \mathbb{R}$,*

$$\sup_{u\in A} \mathcal{H}\Big(s,x,u,q,X,I^M[\phi+C](s,x,u)\Big) \;=\; \sup_{u\in A} \mathcal{H}\Big(s,x,u,q,X,I^M[\phi](s,x,u)\Big).$$

Lemma 3.1.4. *For $\beta \in (0,1)$, $s \in (0,T)$, $x,y \in \mathbb{R}^d$, $q \in \mathbb{R}^d$, $X,Y \in \mathbb{S}^d$, $w,-v \in USC_b([0,T]\times\mathbb{R}^d;\mathbb{R})$ and $\phi^k,\psi^k,\phi,\psi \in C^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$ the following hold.*

(F0). *The function $\hat{\mathcal{H}}$ satisfies the property*

$$\begin{aligned} &\sup_{u\in A} \hat{\mathcal{H}}\Big(s,x,u,D_2\phi(s,x),D_2^2\phi(s,x),I^\beta[\phi](s,x,u),I^M_\beta[\phi]\big(s,x,u,D_2\phi(s,x)\big)\Big), \\ =\;& \sup_{u\in A} \mathcal{H}\Big(s,x,u,D_2\phi(s,x),D_2^2\phi(s,x),I^M[\phi](s,x,u)\Big). \end{aligned}$$

*where $\mathcal{H}$ satisfies **(C1)**.*

(F1). *If $X \le Y$ and $(w-v)(s,\cdot)$ and $(\phi-\psi)(s,\cdot)$ have global maximum at x^*, then*

$$\begin{aligned} &\sup_{u\in A} \hat{\mathcal{H}}\Big(s,x^*,u,q,X,I^\beta[\phi](s,x^*,u),I^M_\beta[w]\big(s,x^*,u,q\big)\Big) \\ =\;& \sup_{u\in A} \hat{\mathcal{H}}\Big(s,x^*,u,q,Y,I^\beta[\psi](s,x^*,u),I^M_\beta[v]\big(s,x^*,u,q\big)\Big). \end{aligned}$$

(F2). *For every constant $C,\hat{C} \in \mathbb{R}$,*

$$\begin{aligned} &\sup_{u\in A} \hat{\mathcal{H}}\Big(s,x,u,q,X,I^\beta[\phi+C](s,x,u),I^M_\beta[w+\hat{C}]\big(s,x,u,q\big)\Big) \\ =\;& \sup_{u\in A} \hat{\mathcal{H}}\Big(s,x,u,q,X,I^\beta[\phi](s,x,u),I^M_\beta[w]\big(s,x,u,q\big)\Big). \end{aligned}$$

(F3). *If $\psi^k \to w$ then*

$$\begin{aligned} &\hat{\mathcal{H}}\Big(s,x,u,q,X,I^\beta[\phi](s,x,u),I^M_\beta[\psi^k]\big(s,x,u,q\big)\Big) \\ \to\;& \hat{\mathcal{H}}\Big(s,x,u,q,X,I^\beta[\phi](s,x,u),I^M_\beta[w]\big(s,x,u,q\big)\Big). \end{aligned}$$

uniformly in u.

(F4). *The function*

$$\hat{\mathcal{H}}: \quad u \to \quad \hat{\mathcal{H}}\Big(s,x,u,q,X,I^\beta[\phi]\big(s,x,u\big),I^M_\beta[w]\big(s,x,u,q\big)\Big)$$

is upper semicontinuous in u.

3.2 Value Function: Viscosity Solution of HJB Equation

Now, we show that, for every $M \ge 1$, the value function V^M (which is defined in (2.9)) is a viscosity solution of (3.2).

Theorem 3.2.1. *For every $M \ge 1$, V^M defined in (2.9) is a viscosity solution of (3.2).*

The proof follows from a standard argument (see for example Yong and Zhou (1999); Pham (1998); Zălinescu (2011)) which need the following proposition.

Proposition 3.2.2. *The value function V^M is continuous.*

Proof. This proposition is a consequence of Corollary 2.3.7, and the its proof is identical to the proof of Proposition 2.4 in Zălinescu (2011). ☐

We now prove Theorem 3.2.1.

Proof. Fix $M \geq 1$. By (2.9) and Proposition 3.2.2, to show the desired result, it is enough to verify the second condition of Definition 3.1.1.

Fix $s \in (0,T)$, and $x \in \mathbb{R}^d$, for $u \in \mathcal{A}_s$ and $N > |x|$, define

$$\theta_{N,u} := \theta(N,u,s,x) = \inf \Big\{ t \in (s,T) : \; X_t^{u,s,x} \notin B(x,N) \Big\}.$$

We split the proof into two parts. Firstly, we prove that V^M is a viscosity subsolution. Suppose that $V^M - \phi$ attains a maximum at $(s^*,x^*) \in (0,T) \times \mathbb{R}^d$. For every $\epsilon > 0$, we can choose $\hat{s} \in (s^*,T)$ such that $\delta = \hat{s} - s^*$ are small enough so that by Corollary 2.3.9 we can find a control $u^{\epsilon,M} := u(\epsilon, M, \hat{s}) \in \mathcal{A}_{s^*}$ such that

$$V^M(s^*,x^*) \;\; \leq \;\; \mathbb{E}\Bigg(\int_{s^*}^{\hat{s}\wedge\theta_{N,u^\epsilon}} f\Big(t, X_t^{u^{\epsilon,M},s^*,x^*,M}, u^{\epsilon,M}\Big) dt + V^M\big(\hat{s}\wedge\theta_{N,u^\epsilon}, X_{\hat{s}\wedge\theta_{N,u^\epsilon}}^{u^{\epsilon,M},s^*,x^*,M}\big)\Bigg) + \epsilon\delta.$$

Rearranging terms and dividing both sides by δ we obtain

$$\frac{1}{\delta}\mathbb{E}\Bigg(V^M(s^*,x^*) - V^M(\hat{s}\wedge\theta_{N,u^\epsilon}, X_{\hat{s}\wedge\theta_{N,u^\epsilon}}^{u^{\epsilon,M}s^*,x^*,M}) - \int_{s^*}^{\hat{s}\wedge\theta_{N,u^\epsilon}} f(t, X_t^{u^{\epsilon,M},s^*,x^*}, u_t^{\epsilon,M}) dt \Bigg) \;\; \leq \;\; \epsilon.$$

Since $V^M - \phi$ attains a maximum at $(s^*,x^*) \in (0,T) \times \mathbb{R}^d$, we see that

$$\phi(s^*,x^*) - \phi\Big(\hat{s}\wedge\theta_{N,u^\epsilon}, X_{\hat{s}\wedge\theta_{N,u^\epsilon}}^{u^{\epsilon,M},s^*,x^*,M}\Big) \leq V^M(s^*,x^*) - V^M\Big(\hat{s}\wedge\theta_{N,u^\epsilon}, X_{\hat{s}\wedge\theta_{N,u^\epsilon}}^{u^{\epsilon,M},s^*,x^*,M}\Big).$$

This implies

$$\frac{1}{\delta}\mathbb{E}\Bigg(\phi(s^*,x^*) - \phi\Big(\hat{s}\wedge\theta_{N,u^\epsilon}, X_{\hat{s}\wedge\theta_{N,u^\epsilon}}^{u^{\epsilon,M},s^*,x^*,M}\Big) - \int_{s^*}^{\hat{s}\wedge\theta_{N,u^\epsilon}} f(t, X_t^{u^{\epsilon,M},s^*,x^*,M}, u_t^{\epsilon,M}) dt \Bigg) \;\; \leq \;\; \epsilon.$$

Since X_t and X_{t-} differs only on a countable number of points, applying Ito's formula, rearranging and taking expectation of both sides yields

$$\begin{aligned}
& \mathbb{E}\Bigg(\phi(s^*,x^*) - \phi\Big(\hat{s}, X_{\hat{s}\wedge\theta_{N,u^\epsilon}}^{u^{\epsilon,M},s^*,x^*,M}\Big)\Bigg) \\
= \;\; & \mathbb{E}\Bigg(\int_{s^*}^{\hat{s}\wedge\theta_{N,u^\epsilon}} \Big(- D_1\phi(t, X_t^{u^{\epsilon,M},s^*,x^*,M}) + < -D_2\phi(t, X_t^{u^{\epsilon,M},s^*,x^*,M}), b(t, X_t^{u^{\epsilon,M},s^*,x^*,M}, u_t^{\epsilon,M}) > \\
& + \frac{1}{2} tr\Big(- D_2^2\phi(t, X_t^{u^{\epsilon,M},s^*,x^*,M}) \sigma(t, X_t^{u^{\epsilon,M},s^*,x^*,M}, u_t^{\epsilon,M}) \sigma(t, X_t^{u^{\epsilon,M},s^*,x^*,M}, u_t^{\epsilon,M})^T \Big)\Big) dt
\end{aligned}$$

$$
\begin{aligned}
&-\int_{s^*}^{\hat{s}\wedge\theta_{N,u^\epsilon}}\int_{1\le|\eta|<M}\Big(\phi(t,X_t^{u^{\epsilon,M},s^*,x^*,M}+\gamma(t,X_t^{u^{\epsilon,M},s^*,x^*,M},u_t^{\epsilon,M},\eta))\\
&-\phi(t,X_t^{u^{\epsilon,M},s^*,x^*,M})\Big)\nu(d\eta)dt-\int_{s^*}^{\hat{s}\wedge\theta_{N,u^\epsilon}}\int_{0<|\eta|<1}\Big(\phi(t,X_t^{u^{\epsilon,M},s^*,x^*,M}\\
&+\gamma(t,X_t^{u^{\epsilon,M},s^*,x^*,M},u_t^{\epsilon,M},\eta))-\phi(t,X_t^{u^{\epsilon,M},s^*,x^*,M})\\
&-<D_2\phi(t,X_t^{u^{\epsilon,M},s^*,x^*,M}),\gamma(t,X_t^{u^{\epsilon,M},s^*,x^*,M},u_t^{\epsilon,M},\eta)>\nu(d\eta)dt\Big)\Bigg).
\end{aligned}
$$

Since $\phi\in C_b^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$, we may apply the Dominated Convergence Theorem. This then yields

$$
\begin{aligned}
&\lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\Bigg(\phi(s^*,x^*)-\phi\Big(\hat{s}\wedge\theta_{N,u^\epsilon},X^{u,s^*,x^*}_{s\wedge\hat{\theta}_{N,u^\epsilon}}\Big)\Bigg)\\
=\;&-D_1\phi(s^*,x^*)+<-D_2\phi(s^*,x^*),b(s^*,x^*,u)>\\
&+\frac{1}{2}tr\Big(-D_2^2\phi(s^*,x^*)\sigma(s^*,x^*,u)\sigma(s^*,x^*,u)^T\Big)\\
&-\int_{1\le|\eta|<M}\Big(\phi\big(s^*,x^*+\gamma(s^*,x^*,u,\eta)\big)-\phi(s^*,x^*)\Big)\nu(d\eta)\\
&-\int_{0<|\eta|<1}\Big(\phi\big(s^*,x^*+\gamma(s^*,x^*,u,\eta)\big)-\phi(s^*,x^*)\\
&-<D_x\phi(s^*,x^*),\gamma(s^*,x^*,u,\eta)>\Big)\nu(d\eta)
\end{aligned}
$$

and

$$
\lim_{\delta\to 0}\frac{1}{\delta}\mathbb{E}\Bigg(\int_{s^*}^{\hat{s}\wedge\wedge\theta_{N,u^\epsilon}}f(t,X_t^{u^\epsilon,s^*,x^*},u_t^\epsilon)dt\Bigg)=f(s^*,x^*,u)
$$

for almost every $s^*\in(0,T)$. Thus, we have

$$
-D_1\phi(s^*,x^*)+\mathcal{H}\Big(s^*,x^*,u,D_2\phi(s^*,x^*),D_2^2\phi(s^*,x^*),I^M[\phi](s^*,x^*,u)\Big)\quad\le\quad\epsilon. \tag{3.4}
$$

Finally, by taking supremum of u over A and letting $\epsilon\to 0$, we obtain

$$
-D_1\phi(s^*,x^*)+\sup_{u\in A}\mathcal{H}\Big(s^*,x^*,u,D_2\phi(s^*,x^*),D_2^2\phi(s^*,x^*),I^M[\phi](s^*,x^*,u)\Big)\quad\le\quad 0
$$

for almost every $s^*\in(0,T)$. Since ϕ is $C_b^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$ and by **(C1)** the above claim holds for every $s^*\in(0,T)$.

Next, we show that V^M is a viscosity supersolution. Suppose that $V^M-\phi$ attains a minimum at $(s^*,x^*)\in(0,T)\times\mathbb{R}^d$. Take $\hat{s}\in[s^*,T]$, and a constant control u (i.e., $u_t=u$ for all $t\in[0,T]$). By Corollary 2.3.9, we have

$$
V^M(s^*,x^*)\quad\ge\quad\mathbb{E}\Bigg(\int_{s^*}^{\hat{s}\wedge\theta_{N,u}}f(t,X_t^{u,s^*,x^*},u)dt+V^M(\hat{s}\wedge\theta_{N,u},X^{u,s^*,x^*}_{\hat{s}\wedge\wedge\theta_{N,u}})\Bigg).
$$

Rearranging terms and dividing both sides by $\hat{s} - s^*$ we obtain

$$0 \;\leq\; \frac{1}{\hat{s} - s^*}\mathbb{E}\bigg(V^M(s^*, x^*) - V^M\Big(\hat{s} \wedge \theta_{N,u}, X^{u,s^*,x^*}_{\hat{s}\wedge\theta_{N,u}}\Big) - \int_{s^*}^{\hat{s}\wedge\theta_{N,u}} f(t, X^{u,s^*,x^*}_t, u)dt\bigg).$$

Since $V^M - \phi$ attains a minimum at $(s^*, x^*) \in [0, T) \times \mathbb{R}^d$, we see that

$$V^M(s^*, x^*) - V^M\Big(\hat{s}, X^{u,s^*,x^*}_{\hat{s}\wedge\theta_{N,u}}\Big) \leq \phi(s^*, x^*) - \phi\Big(\hat{s} \wedge \theta_{N,u}, X^{u,s^*,x^*}_{\hat{s}\wedge\theta_{N,u}}\Big).$$

This implies

$$0 \;\leq\; \frac{1}{\hat{s} - s^*}\mathbb{E}\bigg(\phi(s^*, x^*) - \phi\Big(\hat{s} \wedge \theta_{N,u}, X^{u,s^*,x^*}_{\hat{s}\wedge\theta_{N,u}}\Big) - \int_{s^*}^{\hat{s}\wedge\theta_{N,u}} f(t, X^{u,s^*,x^*}_t, u)dt\bigg).$$

Now, Ito's formula implies

$$\begin{aligned}
0 \;&\leq\; \frac{1}{\hat{s} - s^*}\mathbb{E}\bigg(\phi(s^*, x^*) - \phi\Big(\hat{s} \wedge \theta_{N,u}, X^{u,s^*,x^*}_{\hat{s}\wedge\theta_{N,u}}\Big) - \int_{s^*}^{\hat{s}\wedge\theta_{N,u}} f(t, X^{u,s^*,x^*}_t, u)dt\bigg) \\
&= \frac{1}{\hat{s} - s^*}\mathbb{E}\bigg(\int_{s^*}^{\hat{s}\wedge\theta_{N,u}} -D_1\phi(t, X^{u,s^*,x^*}_t) + \mathcal{H}\Big(t, X^{u,s^*,x^*}_t, u, D_2\phi(t, X^{s^*,x^*,x}_t), \\
&\quad D_2^2\phi(t, X^{s^*,x^*,x}_t), I^M[\phi](t, X^{s^*,x^*,x}_t, u)\Big)dt\bigg) \\
&\leq \frac{1}{\hat{s} - s^*}\mathbb{E}\bigg(\int_{s^*}^{\hat{s}\wedge\tau^N} -D_1\phi(t, X^{u,s^*,x^*}_t) + \sup_{u\in A}\mathcal{H}\Big(t, X^{u,s^*,x^*}_t, u, D_2\phi(t, X^{u,s^*,x^*}_t), \\
&\quad D_2^2\phi(t, X^{u,s^*,x^*}_t), I^M[\phi](t, X^{u,s^*,x^*}_t, u)\Big)dt\bigg).
\end{aligned}$$

As $\hat{s} \to s^*$, the result follows from **(C1)**, and the fact that ϕ is $C_b^{1,2}((0, T] \times \mathbb{R}^d; \mathbb{R})$, and $(X_t)_{t\geq s}$ is cádlág. □

3.3 Comparison Theorem and Uniqueness of Viscosity Solution of HJB Equation

In this section, we establish a comparison theorem for viscosity solutions of (3.2). We will first present two preliminary results. The first one is a classical lemma which summaries some properties of a chosen test function. Different versions of this result are taken and proved for different purpose, see for example Pham (1998); Zălinescu (2011); Dumitrescu et al. (2015); Yong and Zhou (1999); Ishii (1984); Crandall et al. (1992). Here, we repeat some of the arguments from those works. The second result is a maximum principle for HJB equation (see for example Theorem 2.2 in Jakobsen and Karlsen (2005)). By using these two results we obtain a comparison theorem. We present a proof of the comparison theorem which follows from Pham (1998); Zălinescu (2011); Jakobsen and Karlsen (2006). An application of this comparison theorem then implies that there exists a unique solution to (3.2) in the viscosity sense.

Lemma 3.3.1. *Suppose $w_{sub} \in USC_b([0,T] \times \mathbb{R}^d;\mathbb{R})$ and $w^{sup} \in LSC_b([0,T] \times \mathbb{R}^d;\mathbb{R})$ are viscosity subsolution and viscosity supersolution of equation (3.2) respectively. Assume that there exists some $(s_0, x_0) \in (0,T) \times \mathbb{R}^d$ such that*

$$w_{sub}(s_0, x_0) - w^{sup}(s_0, x_0) = \theta > 0, \tag{3.5}$$

For $\delta > 0$, $\epsilon > 0$, $\lambda > 0$ and $(s,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$, define a function

$$\Phi^{\delta,\epsilon,\lambda}(s,x,y) \quad = \quad w_{sub}(s,x) - w^{sup}(s,y) - \phi^{\delta,\epsilon,\lambda}(s,x,y), \tag{3.6}$$

where

$$\phi^{\delta,\epsilon,\lambda}(s,x,y) \quad = \quad \frac{1}{2\delta}|x-y|^2 + \frac{\epsilon}{2}\exp\big(\lambda(T-s)\big)\big(|x|^2+|y|^2\big) + \theta\frac{s_0}{2s}. \tag{3.7}$$

Further, we define

$$\chi^{\delta,\epsilon,\lambda} \quad = \quad \sup_{(s,x,y)\in(0,T]\times\mathbb{R}^d\times\mathbb{R}^d} \Phi^{\delta,\epsilon,\lambda}(s,x,y), \tag{3.8}$$

$$\chi \quad = \quad \sup_{(s,x,y)\in(0,T]\times\mathbb{R}^d\times\mathbb{R}^d} \left(w_{sub}(s,x) - w^{sup}(s,y) - \theta\frac{s_0}{2s}\right), \tag{3.9}$$

then the following hold.

1. *The supremum $\chi^{\delta,\epsilon,\lambda}$ is finite, and we can find an $\epsilon_0 > 0$ such that for all $\epsilon < \epsilon_0$ there exists $(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$ such that*

$$\chi^{\delta,\epsilon,\lambda} = \Phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}). \tag{3.10}$$

2. *As $\delta \to 0$, there is a subsequence (also denoted as $s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}$) such that $s^{\delta,\epsilon,\lambda} \to s^{\epsilon,\lambda}$, $x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda} \to x^{\epsilon,\lambda}$ and*

$$\lim_{\delta\to 0}\frac{1}{\delta}|x^{\delta,\epsilon,\lambda} - y^{\delta,\epsilon,\lambda}|^2 \quad = \quad 0. \tag{3.11}$$

3. *Letting $\delta \to 0$ followed by letting $\epsilon \to 0$, we have*

$$\lim_{\epsilon\to 0}\lim_{\delta\to 0}\chi^{\delta,\epsilon,\lambda} \quad = \quad \chi. \tag{3.12}$$

Proof. Since w_{sub} and $-w^{sup}$ are bounded, and $-\phi^{\delta,\epsilon,\lambda}(s,x,y) < 0$ for all $\delta > 0$, $\epsilon > 0$, $\lambda > 0$, $(s,x,y) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$, we see that $\Phi^{\delta,\epsilon,\lambda}$ is bounded above. Hence, $\chi^{\delta,\epsilon,\lambda}$ is finite. In addition, we see that

$$\begin{aligned}
\lim_{|x|+|y|\to\infty} \Phi^{\delta,\epsilon,\lambda}(s,x,y) \quad &= \quad -\infty, \quad \text{uniformly in } s \in (0,T], \\
\lim_{s\to 0} \Phi^{\delta,\epsilon,\lambda}(s,x,y) \quad &= \quad -\infty, \quad \text{uniformly in } x, y \in \mathbb{R}^d.
\end{aligned}$$

Thus, we can find a compact subset $K_{\epsilon,\lambda} \subset (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ such that for all $(s,x,y) \in K^c_{\epsilon,\lambda}$, we have $\Phi^{\delta,\epsilon,\lambda}(s,x,y) < 0$. Moreover for all $\epsilon < \epsilon_0 := \dfrac{\theta}{2\exp\big(\lambda(T-s_0)\big)|x_0|^2}$, we have

$$\chi^{\delta,\epsilon,\lambda} \quad \geq \quad \Phi^{\delta,\epsilon,\lambda}(s_0,x_0,x_0) = w_{sub}(s_0,x_0) - w^{sup}(s_0,x_0) - \epsilon|x_0|^2 - \theta\frac{1}{2} > 0. \quad (3.13)$$

Since w_{sub} and $-w^{sup}$ are upper semicontinuous and $\phi^{\delta,\epsilon,\lambda}$ is continuous, there exists some $(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) \in K_{\epsilon,\lambda}$ such that

$$\chi^{\delta,\epsilon,\lambda} = \Phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}).$$

Now, it is clear that we have

$$\Phi^{\delta,\epsilon,\lambda}(T,0,0) \leq \Phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) = \chi^{\delta,\epsilon,\lambda}, \quad (3.14)$$

we may choose a constant $C > 0$ which is independent of δ, ϵ and λ such that for all $\epsilon < \epsilon_0$ we have

$$\epsilon|x^{\delta,\epsilon,\lambda}|^2 \leq C, \quad \text{and} \quad \epsilon|y^{\delta,\epsilon,\lambda}|^2 \leq C. \quad (3.15)$$

For every $\epsilon > 0$, we see that $\chi^{\delta,\epsilon,\lambda}$ is increasing in δ. For this reason, we observe that

$$\begin{aligned}
\chi^{2\delta,\epsilon,\lambda} \quad &= \quad \sup_{(s,x,y)\in(0,T]\times\mathbb{R}^d\times\mathbb{R}^d} \Phi^{2\delta,\epsilon,\lambda}(s,x,y) \\
&\geq \quad \Phi^{2\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) \\
&= \quad \chi^{\delta,\epsilon,\lambda} + \frac{1}{4\delta}|x^{\delta,\epsilon,\lambda} - y^{\delta,\epsilon,\lambda}|^2. \qquad (3.16)
\end{aligned}$$

Hence, as $\delta \to 0$ together with (3.13), (3.15), and $s^{\delta,\epsilon,\lambda} \in (0,T]$, the second claim is proved. At the same time, we obtain

$$\chi^{\epsilon} \quad := \quad \lim_{\delta\to 0} \chi^{\delta,\epsilon,\lambda}. \quad (3.17)$$

Moreover, from (3.13) and the fact that $w_{sub}(T,x) \leq w^{sup}(T,x)$ for all $x \in \mathbb{R}^d$, we know that $s^{\delta,\epsilon,\lambda} \in (0,T)$ (i.e., it cannot occur at T). This completes the proof of the first claim.

Next, by definition, we know that for every $\xi > 0$, there exists a $(s^{\xi}, x^{\xi}) \in (0,T] \times \mathbb{R}^d$ such that

$$\chi - \xi \leq w_{sub}(s^{\xi}, x^{\xi}) - w^{sup}(s^{\xi}, x^{\xi}) - \theta\frac{1}{s^{\xi}}.$$

This implies

$$\begin{aligned}
\chi^{\delta,\epsilon,\lambda} \quad &= \quad \Phi^{\delta,\epsilon}(s^{\delta,\epsilon}, x^{\delta,\epsilon}, y^{\delta,\epsilon}) \\
&\geq \quad \Phi^{\delta,\epsilon}(s^{\xi}, x^{\xi}, x^{\xi})
\end{aligned}$$

$$
\begin{aligned}
&= w_{sub}(s^\xi, x^\xi) - w^{sup}(s^\xi, x^\xi) - \epsilon|x^\xi|^2 - \theta\frac{1}{s^\xi} \\
&\geq \chi - \xi - \epsilon|x^\xi|^2.
\end{aligned}
$$

Taking limit as $\delta \to 0$ together with (3.17) we obtain

$$
\chi - \xi - \epsilon|x^\xi|^2 \leq \chi^\epsilon \leq \chi.
$$

As $\epsilon \to 0$, the last claim is proved. $\square$

Theorem 3.3.2. ***(Maximum Principle)**: Suppose $w_{sub} \in USC_b([0,T] \times \mathbb{R}^d; \mathbb{R})$ and $w^{sup} \in LSC_b([0,T] \times \mathbb{R}^d; \mathbb{R})$ are viscosity subsolution and viscosity supersolution of (3.2) respectively. Let $\phi \in C^{2,2,2}((0,T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$ and $\Phi(s,x,y)$ admits a maximum at $(s^*, x^*, y^*) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$, where*

$$
\Phi(s,x,y) \quad = \quad w_{sub}(s,x) - w^{sup}(s,y) - \phi(s,x,y).
$$

If there exists $g_0 \in C((0,T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R})$, $g_1 \in C((0,T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{S}^d)$, and $g_2 \in C((0,T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{S}^d)$ with $g_0(s^*, x^*, y^*) > 0$ such that

$$
\begin{aligned}
&\begin{pmatrix} D_{22}\phi(s^*, x^*, y^*) & D_{23}\phi(s^*, x^*, y^*) \\ D_{32}\phi(s^*, x^*, y^*) & D_{33}\phi(s^*, x^*, y^*) \end{pmatrix} \\
\leq \;& g_0(s,x,y) \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \begin{pmatrix} g_1(s,x,y) & 0 \\ 0 & g_2(s,x,y) \end{pmatrix}
\end{aligned}
$$

for all $(s,x,y) \in Q^r$, where

$$
Q^r = \big\{(s,x,y) \mid |s^* - s|^2 + |x^* - x|^2 + |y^* - y|^2 \leq r \;\text{ for some } r > 0\big\}.
$$

Then, for every $\alpha \in (0, \frac{1}{2})$, there exists two matrices $X \in \mathbb{S}^d$ and $Y \in \mathbb{S}^d$ such that

$$
\begin{aligned}
\frac{g_0((s^*, x^*, y^*))}{\alpha} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} - \begin{pmatrix} g_1(s^*, x^*, y^*) & 0 \\ 0 & g_2(s^*, x^*, y^*) \end{pmatrix} \\
&\leq \frac{g_0(s^*, x^*, y^*)}{1 - 2\alpha} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix}
\end{aligned}
$$

and

$$
\begin{aligned}
& a + \sup_{u \in A} \hat{\mathcal{H}}\Big(s^*, x^*, u, D_2\phi(s^*, x^*, y^*), X, I^\beta[\phi](s^*, x^*, u), \\
&\qquad\qquad I^M_\beta[w_{sub}](s^*, x^*, u, D_2\phi(s^*, x^*, y^*))\Big) \leq 0, \\
& b + \sup_{u \in A} \hat{\mathcal{H}}\Big(s^*, y^*, u, -D_3\phi(s^*, x^*, y^*), Y, I^\beta[-\phi](s^*, y^*, u), \\
&\qquad\qquad I^M_\beta[w^{sup}](s^*, y^*, u, -D_3\phi(s^*, x^*, y^*))\Big) \geq 0,
\end{aligned}
$$

where $D_1\phi(s^*, x^*, y^*) = b - a$.

Proof. The proof is outlined in Jakobsen and Karlsen (2005) (see the proof of Theorem 2.2 in Jakobsen and Karlsen (2005)), which follows from Ishii and Lions (1990), Lemma 7.8 and the proof of Theorem 4.9 in Jakobsen and Karlsen (2006). ☐

Theorem 3.3.3. ***(Comparison Theorem)****: If $w_{sub} \in USC_b([0,T] \times \mathbb{R}^d; \mathbb{R})$ and $w^{sup} \in LSC_b([0,T] \times \mathbb{R}^d; \mathbb{R})$ are viscosity subsolution and supersolution of (3.2) respectively, then $w_{sub}(s,x) \leq w^{sup}(s,x)$ for all $(s,x) \in [0,T] \times \mathbb{R}^d$.*

Proof. By Definition 3.1.2, we have $w_{sub}(T,x) \leq h(x) \leq w^{sup}(T,x)$ for all $x \in \mathbb{R}^d$. By the semicontinuity of viscosity subsolution and viscosity supersolution, we can obtain the desired result at $s = 0$ provided $w_{sub}(s,x) \leq h(x) \leq w^{sup}(s,x)$ for all $(s,x) \in (0,T) \times \mathbb{R}^d$. Indeed, if we have $w_{sub}(s,x) \leq w^{sup}(s,x)$ for all $s \in (0,T)$. then we see that

$$w_{sub}(0,x) \leq \liminf_{s \to 0} w_{sub}(s,x) \leq \liminf_{s \to 0} w^{sup}(s,x) \leq \limsup_{s \to 0} w^{sup}(s,x) \leq w^{sup}(0,x).$$

Thus, we only need to prove the desired results for all $(s,x) \in (0,T) \times \mathbb{R}^d$ for which we prove by contradiction. To this end, let us assume there exists a point $(s_0, x_0) \in (0,T) \times \mathbb{R}^d$ such that

$$\theta := w_{sub}(s_0,x_0) - w^{sup}(s_0,x_0) > 0. \tag{3.18}$$

We choose a test function $\phi^{\delta,\epsilon,\lambda}(s,x,y)$ such that

$$\phi^{\delta,\epsilon,\lambda}(s,x,y) = \frac{1}{2\delta}|x-y|^2 + \exp\big(\lambda(T-s)\big)\frac{\epsilon}{2}\big(|x|^2 + |y|^2\big) + \theta\frac{s_0}{2s}.$$

It is easy to see that $\phi^{\delta,\epsilon,\lambda} \in C^{2,2,2}\big((0,T] \times \mathbb{R}^d \times \mathbb{R}^d; \mathbb{R}\big)$. Next, we define

$$\Phi^{\delta,\epsilon,\lambda}(s,x,y) = w_{sub}(s,x) - w^{sup}(s,y) - \phi^{\delta,\epsilon,\lambda}(s,x,y).$$

By the first claim of Lemma 3.3.1, we know that the supremum of $\Phi^{\delta,\epsilon,\lambda}$ is achieved at some $(s^{\delta,\epsilon}, x^{\delta,\epsilon}, y^{\delta,\epsilon}) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$. Also, we have

$$\begin{aligned} &\begin{pmatrix} D_2^2\phi(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) & D_{23}\phi(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) \\ D_{32}\phi(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) & D_3^2\phi(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) \end{pmatrix} \\ &= \frac{1}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \exp\big(\lambda(T-s)\big)\begin{pmatrix} \epsilon I & 0 \\ 0 & \epsilon I \end{pmatrix}. \end{aligned}$$

By Theorem 3.3.2, for all $0 < \delta < 1$ there exists two matrices $X \in \mathbb{S}^d$ and $Y \in \mathbb{S}^d$ such that

$$\begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \frac{1}{\delta}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \exp\big(\lambda(T-s)\big)\begin{pmatrix} \epsilon I & 0 \\ 0 & \epsilon I \end{pmatrix},$$

and

$$a+\sup_{u\in A}\hat{\mathcal{H}}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u,D_2\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}),X,I^{\beta}[\phi^{\delta,\epsilon,\lambda}]\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u\Big),$$
$$I^{M}_{\beta}[w_{sub}]\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u,D_2\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\Big)\Big)\le 0, \tag{3.19}$$

$$b+\sup_{u\in A}\hat{\mathcal{H}}\Big(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u,-D_3\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}),Y,I^{\beta}[-\phi^{\delta,\epsilon,\lambda}]\Big(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u\Big),$$
$$I^{M}_{\beta}[w^{sup}]\Big(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u,-D_3\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\Big)\Big)\ge 0, \tag{3.20}$$

where $b-a=D_1\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})$.

By **(F4)** and since A is compact, there exists a control $u^*:=u(s,x,M)$ such that the supremum is attained. Using (3.20) - (3.19), we obtain

$$\frac{\theta s_0}{2(s^{\delta,\epsilon,\lambda})^2}\quad\le\quad\Big(J_1^{u^*}+J_2^{u^*}+J_3^{u^*}+J_4^{u^*,M}+J_5^{u^*}\Big)-\lambda\exp\big(\lambda(T-s)\big)\frac{\epsilon}{2}\big(|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big).$$

where

$$\begin{aligned}
J_1^{u^*} &= <D_3\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}),b(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*)>\\
&\quad +<D_2\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}),b(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*)>,\\
J_2^{u^*} &= \frac{1}{2}tr\Big(X\sigma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*)\sigma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*)^T\Big)\\
&\quad -\frac{1}{2}tr\Big(Y\sigma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*)\sigma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*)^T\Big),\\
J_3^{u^*} &= \int_{0<\eta<\beta}\phi^{\delta,\epsilon,\lambda}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta),y^{\delta,\epsilon,\lambda}\Big)-\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\\
&\quad -<D_2\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}),\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)>\nu(d\eta)\\
&\quad -\int_{0<\eta<\beta}\phi^{\delta,\epsilon}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big)-\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\\
&\quad -<-D_3\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}),\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)>\nu(d\eta),\\
J_4^{u^*,M} &= \int_{\beta\le|\eta|<M}w_{sub}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)\Big)-w_{sub}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda})\\
&\quad -<D_2\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}),\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)>\nu(d\eta)\\
&\quad -\int_{\beta\le|\eta|<M}w^{sup}\Big(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big)-w^{sup}(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\\
&\quad -<-D_3\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}),\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)>\nu(d\eta),\\
J_5^{u^*} &= f(s,x,u^*)-f(s,y,u^*).
\end{aligned}$$

Now, let us look at each of these terms. The first term implies

$$
\begin{aligned}
J_1^{u^*} &= < D_3\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}), b(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*) > \\
&\quad + < D_2\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}), b(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*) > \\
&= < -\frac{1}{\delta}(x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}) + \epsilon y^{\delta,\epsilon,\lambda}\exp\big(\lambda(T-s)\big), b(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*) > \\
&\quad + < \frac{1}{\delta}(x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}) + \epsilon x^{\delta,\epsilon,\lambda}\exp\big(\lambda(T-s)\big), b(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*) > \\
&= \sup_{u\in A}\Big(< \frac{1}{\delta}(x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}), b(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u) - b(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u) > \\
&\quad + < \epsilon x^{\delta,\epsilon,\lambda}\exp\big(\lambda(T-s)\big), b(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*) > \\
&\quad + < \epsilon y^{\delta,\epsilon,\lambda}\exp\big(\lambda(T-s)\big), b(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*) > \Big) \\
&\leq C\frac{1}{\delta}|x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}|^2 + C\epsilon\exp\big(\lambda(T-s)\big)\big(1+|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big).
\end{aligned}
$$

By using (3.19), and follow a similar argument as in Fleming and Soner (1993) (see Lemma 6.2 on p240 in Fleming and Soner (1993)), the second term yields

$$
\begin{aligned}
J_2^{u^*} &= \frac{1}{2}tr\Big(X\sigma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*)\sigma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*)^T\Big) \\
&\quad -\frac{1}{2}tr\Big(Y\sigma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*)\sigma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*)^T\Big) \\
&\leq \frac{1}{\delta}\big|\sigma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*) - \sigma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*)\big|^2 \\
&\quad +\frac{\epsilon}{2}\exp\big(\lambda(T-s)\big)\Big(\big|\sigma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*)\big|^2 + \big|\sigma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*)\big|^2\Big) \\
&\leq C\frac{1}{\delta}|x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}|^2 + C\epsilon\exp\big(\lambda(T-s)\big)\big(1+|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big).
\end{aligned}
$$

Next, we observe that the third terms gives

$$
\begin{aligned}
J_3^{u^*} &= \int_{0<\eta<\beta}\phi^{\delta,\epsilon,\lambda}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta),y^{\delta,\epsilon,\lambda}\Big) \\
&\quad -\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}) \\
&\quad - < D_2\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}), \gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta) > \nu(d\eta) \\
&\quad - \int_{0<\eta<\beta} -\phi^{\delta,\epsilon,\lambda}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big) \\
&\quad -\Big(-\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\Big) \\
&\quad - < -D_3\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}), \gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta) > \nu(d\eta) \\
&\leq \int_{0<\eta<\beta}\Big|\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)^T\int_0^1(1-\xi)D_2^2\phi^{\delta,\epsilon,\lambda}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda} \\
&\quad +\xi\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta),y^{\delta,\epsilon,\lambda}\Big)d\xi\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)\Big|\nu(d\eta) \\
&\quad + \int_{0<\eta<\beta}\Big|\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)^T\int_0^1(1-\xi)D_3^2\phi^{\delta,\epsilon,\lambda}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda} \\
&\quad +\xi\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big)d\xi\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big|\nu(d\eta) \\
&\leq \int_{0<\eta<\beta}|\eta|^2\nu(d\eta)C\Big(\frac{1}{\delta}+\epsilon\exp\big(\lambda(T-s)\big)\Big)\big(1+|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big).
\end{aligned}
$$

For the forth term, we first calculate the expression below by using the definition of $\Phi^{\delta,\epsilon,\lambda}$, the fact that $\Phi^{\delta,\epsilon,\lambda}$ attains its supremum at $(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda})$ and polarization identity of inner product.

$$\begin{aligned}
& \Big(w_{sub}\Big(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u^*, \eta)\Big) - w_{sub}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}) \\
& - < D_2\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}), \gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u^*, \eta) > \Big) \\
& -\Big(w^{sup}\Big(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u^*, \eta)\Big) - w^{sup}(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) \\
& - < -D_3\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}), \gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u^*, \eta) > \Big) \\
= & \Bigg(w_{sub}\Big(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda} + \gamma\big(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u^*, \eta\big)\Big) - w^{sup}\Big(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda} + \gamma\big(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u^*, \eta\big)\Big)\Bigg) \\
& -\Big(w_{sub}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}) - w^{sup}(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda})\Big) \\
& -\Big(< \frac{1}{\delta}(x^{\delta,\epsilon,\lambda} - y^{\delta,\epsilon,\lambda}) + \epsilon \exp\big(\lambda(T-s)\big)x^{\delta,\epsilon,\lambda}, \gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u^*, \eta) > \\
& - < \frac{1}{\delta}(x^{\delta,\epsilon,\lambda} - y^{\delta,\epsilon,\lambda})(-1) + \epsilon \exp\big(\lambda(T-s)\big)y^{\delta,\epsilon,\lambda}, \gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u^*, \eta) > \Big) \\
\leq & \Bigg(\Phi^{\delta,\epsilon,\lambda}\Big(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u^*, \eta), y^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u^*, \eta)\Big) \\
& -\Phi^{\delta,\epsilon,\lambda}\big(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}\big)\Bigg) + \frac{1}{2\delta}\big|\gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u^*, \eta) - \gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u^*, \eta)\big|^2 \\
& +\frac{\epsilon}{2}\exp\big(\lambda(T-s)\big)\Big(\big|\gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u^*, \eta)\big|^2 + \big|\gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u^*, \eta)\big|^2\Big) \\
\leq & C\frac{1}{\delta}|\eta|^2\big|x^{\delta,\epsilon,\lambda} - y^{\delta,\epsilon,\lambda}\big|^2 + C\epsilon|\eta|^2\exp\big(\lambda(T-s)\big)\Big(1 + |x^{\delta,\epsilon,\lambda}|^2 + |y^{\delta,\epsilon,\lambda}|^2\Big).
\end{aligned} \tag{3.21}$$

Then, we calculate the following expression by using the definition of Φ.

$$\begin{aligned}
& \Big(w_{sub}\big(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u, \eta)\big) - w_{sub}\big(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}\big)\Big) \\
& -\Big(w^{sup}\big(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u, \eta)\big) - w^{sup}\big(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}\big)\Big) \\
\leq & \Big(w_{sub}\big(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u, \eta)\big) - w^{sup}\big(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u, \eta)\big) \\
& -\theta\frac{s_0}{2s^{\delta,\epsilon,\lambda}}\Big) - \Phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}).
\end{aligned} \tag{3.22}$$

By (3.21) and (3.22), the forth term becomes

$$\begin{aligned}
J_4^{u^*} = & \int_{\beta\leq|\eta|<1}\Bigg(w_{sub}\Big(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u^*, \eta)\Big) - w_{sub}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}) \\
& - < D_2\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}), \gamma(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, u^*, \eta) > \Bigg) \\
& -\Bigg(w^{sup}\Big(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda} + \gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u^*, \eta)\Big) - w^{sup}(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}) \\
& - < D_3\phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda}, x^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}), \gamma(s^{\delta,\epsilon,\lambda}, y^{\delta,\epsilon,\lambda}, u^*, \eta) > \Bigg)\nu(d\eta)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{1\leq|\eta|<M}\Big(w_{sub}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)\Big)-w_{sub}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda})\Big)\\
&-\Big(w^{sup}\Big(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big)-w^{sup}(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\Big)\nu(d\eta)\\
\leq\ & C\int_{0<|\eta|<1}|\eta|^2\nu(d\eta)\frac{1}{\delta}|x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}|^2\\
&+C\int_{0<|\eta|<1}|\eta|^2\nu(d\eta)\epsilon\exp\big(\lambda(T-s)\big)\big(1+|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big).\\
&+\int_{1\leq|\eta|<M}\Big(w_{sub}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)\Big)\\
&-w^{sup}\Big(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big)-\theta\frac{s_0}{2s^{\delta,\epsilon,\lambda}}\\
&-\Phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\Big)\nu(d\eta).
\end{aligned}
\tag{3.23}
$$

The last term yields

$$
J_5^u\ =\ \big|f(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*)-f(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*)\big|\leq C\rho\big(|x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}|\big).
$$

Combining all of these terms and note that $s^{\delta,\epsilon,\lambda}\in(0,T)$ we obtain

$$
\begin{aligned}
\frac{\theta s_0}{2T^2}\ \leq\ & \int_{0<\eta<\beta}|\eta|^2\nu(d\eta)C\Big(\frac{1}{\delta}+\epsilon\exp\big(\lambda(T-s)\big)\Big)\big(1+|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big)\\
&+C\frac{1}{\delta}|x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}|^2+C\int_{0<|\eta|<1}|\eta|^2\nu(d\eta)\frac{1}{\delta}|x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}|^2\\
&+C\omega\big(|x^{\delta,\epsilon,\lambda}-y^{\delta,\epsilon,\lambda}|\big)+\int_{1\leq|\eta|<M}\Big(w_{sub}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)\Big)\\
&-w^{sup}\Big(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big)-\theta\frac{s_0}{2s^{\delta,\epsilon,\lambda}}-\Phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\Big)\nu(d\eta)\\
&+\epsilon\exp\big(\lambda(T-s)\big)\Bigg(C\big(1+|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big)-\lambda\frac{1}{2}\big(|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big)\Bigg).
\end{aligned}
\tag{3.24}
$$

Taking $\beta\to 0$, the first term on the right hand side of (3.24) vanishes. Next, we take limsup as $\delta\to 0$, by the second claim of Lemma 3.3.1, the second, third and fourth terms of (3.24) vanish, and the last two terms become

$$
\begin{aligned}
&\int_{1\leq|\eta|<M}\Bigg(w_{sub}\Bigg(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)\Big)\\
&-w^{sup}\Big(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big)-\theta\frac{s_0}{2s^{\delta,\epsilon,\lambda}}-\Phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\Bigg)\nu(d\eta)\\
\to\ &\int_{1\leq|\eta|<M}\limsup_{\delta\to 0}\Bigg(w_{sub}\Big(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},u^*,\eta)\Big)\\
&-w^{sup}\Big(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda}+\gamma(s^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda},u^*,\eta)\Big)-\theta\frac{s_0}{2s^{\delta,\epsilon,\lambda}}-\Phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\Bigg)\nu(d\eta)
\end{aligned}
$$

$$\leq \int_{1\leq|\eta|<M}\Bigg(w_{sub}\Big(s^{\epsilon,\lambda},x^{\epsilon,\lambda}+\gamma(s^{\epsilon,\lambda},x^{\epsilon,\lambda},u^*,\eta)\Big)-w^{sup}\Big(s^{\epsilon,\lambda},y^{\epsilon,\lambda}+\gamma(s^{\epsilon,\lambda},y^{\epsilon,\lambda},u^*,\eta)\Big)$$
$$-\theta\frac{s_0}{2s^{\epsilon,\lambda}}-\limsup_{\delta\to 0}\Phi^{\delta,\epsilon,\lambda}(s^{\delta,\epsilon,\lambda},x^{\delta,\epsilon,\lambda},y^{\delta,\epsilon,\lambda})\Bigg)\nu(d\eta) \tag{3.25}$$

and

$$\epsilon\exp\big(\lambda(T-s)\big)\Bigg(C\big(1+|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big)-\lambda\frac{1}{2}\big(|x^{\delta,\epsilon,\lambda}|^2+|y^{\delta,\epsilon,\lambda}|^2\big)\Bigg)$$
$$\to \epsilon\exp\big(\lambda(T-s)\big)\Big(C\big(1+2|x^{\epsilon,\lambda}|^2\big)-\lambda|x^{\epsilon,\lambda}|^2\Big). \tag{3.26}$$

Taking λ sufficient large, and taking limsup as $\epsilon\to 0$, by the last claim of Lemma 3.3.1, and (3.24) - (3.26), we obtain

$$\frac{\theta s_0}{2T^2}\leq 0.$$

This contradicts (3.18). The proof is then completed. □

Remark 3.3.4. *If f is independent of s and u, then Assumption 3.1 can be dropped.*

The comparison theorem then implies the following uniqueness result.

Corollary 3.3.5. *There exists a unique viscosity solution of equation (3.2) in the class of bounded semicontinuous functions.*

Remark 3.3.6. *In the above analysis, we take $1\leq M<\infty$, we may repeat the above argument to establish the corresponding result in the case $M=\infty$.*

Chapter 4

On a Class of Singular Stochastic Control Problems driven by Lévy Noise

We have made some preparations in Chapter 2 and Chapter 3. In this chapter, we return to the main task of the first part of the thesis. We again work on the Wiener-Poisson space. Let us recall, from the Introduction, that the main difficulty to extend the work of Lasry and Lions (2000) is the fact that their approach requires that the state process possess some finite moments. This is certainly true for Brownian noise with appropriate assumptions. However, for general Lévy type noise, this fails. Thus, to overcome this difficulty, we again use the approximation of the state process which we presented in Chapter 2. A version of this work has submitted to Stochastic Processes and Their Applications.

4.1 The Result of Lasry and Lions

To start this chapter, let us briefly recall the original result proved by Lasry and Lions (2000) (in the case of Brownian noise). All detailed assumptions required in this section will be presented under a more general framework in later sections. On the Wiener space $(\Omega_W, \mathbb{F}^W, \mathbb{P})$, consider the stochastic control problem with state process satisfies

$$\begin{cases} dX_t = a(X_{t-})dt + b(X_{t-})u_t dt + \sigma(X_{t-})dW_t, \\ X_0 = x \in \mathbb{R}^d. \qquad\qquad\qquad\qquad 0 \le t \le T, \end{cases}$$

and admissible control set

$$\mathcal{A} = \Big\{ u : [0,T] \times \Omega \to A \subseteq \mathbb{R} \mid u \text{ is predictable with respect to } (\mathcal{F}_t)_{t\ge 0}, \text{ and } \int_0^T |u_t| dt \le C \ \mathbb{P}-\text{a.s.} \Big\}.$$

The revenue functional for a given $u \in \mathcal{A}$ is defined as

$$V^u(s,x) = \mathbb{E}\Big(\int_0^T f(X_t)dt + h(X_T)\Big),$$

where $f : \mathbb{R}^d \to \mathbb{R}$, $h : \mathbb{R}^d \to \mathbb{R}$. We will say that

$$V(s,x) = \sup_{u \in \mathcal{A}} V^u(s,x).$$

is the value function. If there exists a maximizer $u^* \in \mathcal{A}$, then

$$V(s,x) \quad = \quad V^{u^*}(s,x).$$

Let φ be the flow associated to b, consider a new control problem with state process satisfies

$$\begin{cases} dZ_t = \tilde{\sigma}(\mu_t, Z_{t-})dW_t + \tilde{b}(\mu_t, Z_{t-})dt, \\ \\ Z_0 = x \in \mathbb{R}^d, \qquad\qquad\qquad 0 \le t \le T, \end{cases}$$

where

$$\begin{aligned} \tilde{\sigma}(\kappa, z) &= D_2\varphi(\kappa, \varphi(-\kappa, z))\sigma(\varphi(-\kappa, z)), \\ \tilde{b}(\kappa, z) &= D_2\varphi(\kappa, \varphi(-\kappa, z))a(\varphi(-\kappa, z)) + \frac{1}{2}\hat{b}(\kappa, z), \\ \hat{b}(\kappa, z) &= \begin{pmatrix} tr\Big(D_2^2\varphi^1(\kappa, \varphi(-\kappa, z))\sigma(\varphi(-\kappa, z))\sigma(\varphi(-\kappa, z))^T\Big) \\ \vdots \\ tr\Big(D_2^2\varphi^n(\kappa, \varphi(-\kappa, z))\sigma(\varphi(-\kappa, z))\sigma(\varphi(-\kappa, z))^T\Big) \end{pmatrix}. \end{aligned}$$

Fix a compact set $\Upsilon_C \subset \mathbb{R}$, where C is given in the definition of $\mathcal{A}$. We take the set of admissible controls of the new control problem to be

$$\mathcal{M} \quad = \quad \Big\{\mu : [0,T] \times \Omega \to \Upsilon_C \mid \mu \text{ is predictable with respect to } (\mathcal{F}_t)_{t \ge 0}\Big\}.$$

For $\mu \in \mathcal{M}$, the revenue functional of the new control problem is defined as

$$\mathcal{V}^\mu(s,x) = \mathbb{E}\Big(\int_s^T \tilde{f}(\mu_t, Z_t)dt + \tilde{h}(Z_T)\Big),$$

where

$$\tilde{f}(\kappa, z) = f(\varphi(-\kappa, z)), \quad \tilde{h}(z) = \sup_{\kappa \in \mathbb{R}} h(\varphi(-\kappa, z)).$$

The value function of the new control problem is given by

$$\mathcal{V}(s,x) = \sup_{\mu \in \mathcal{M}} \mathcal{V}^\mu(s,x).$$

Then, the result of Larsy and Lions is presented below.

Theorem 4.1.1. *For $s \in [0,T)$, and $x \in \mathbb{R}^d$ the following hold for all $\kappa \in \mathbb{R}$.*

$$V(s,x) = V(s, \varphi(\kappa, x)) = \mathcal{V}(s,x).$$

Starting from the next section, we will extend this result to the case of Lévy noise.

4.2 Problem Formulation

Consider the following stochastic control problem. We will call this control problem the original control problem. Fix $s \in [0,T)$, the state process $(X_t)_{t\geq s}$ is assumed to follow the stochastic differential equation (SDE):

$$\begin{cases} dX_t = a(X_{t-})dt + b(X_{t-})u_t dt + \sigma(X_{t-})dW_t + \displaystyle\int_{0<|\eta|<1} \gamma(X_{t-},\eta)\tilde{N}(dt,d\eta) \\ \qquad + \displaystyle\int_{|\eta|\geq 1} \gamma(X_{t-},\eta)N(dt,d\eta) \\ \\ X_s = x \in \mathbb{R}^d, \qquad\qquad 0 \leq s \leq t \leq T, \end{cases} \tag{4.1}$$

where X_{t-} is the left limit of X_t. Here, $u : [0,T]\times\Omega \to \mathbb{R}^\ell$ is a predictable process which acts as a control. Moreover, $a : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function, $b : \mathbb{R}^d \to \mathbb{R}^d$ is a continuous function, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ is a continuous function, $\gamma : \mathbb{R}^d \times \mathbb{R}^q_0 \to \mathbb{R}^d$ is a Borel measurable function, and γ is continuous in x for every $\eta \in \mathbb{R}^q_0$. As usual, $\tilde{N}$ denotes the compensated Poisson random measure associated to N, i.e., $\tilde{N}(dt,d\eta) = N(dt,d\eta) - \nu(d\eta)dt$.

The set of admissible controls $(u_t)_{t\in[0,T]}$ is denoted by $\mathcal{A}_s$, where

$$\begin{aligned} \mathcal{A}_s \;=\; & \Big\{ u : [0,T]\times\Omega \to A \subseteq \mathbb{R} \mid u \text{ is predictable with respect to } (\mathcal{F}^s_t)_{t\geq s}, \\ & u_r = 0 \text{ for all } r \in [0,s], \text{and } \int_s^T |u_t|dt \leq C \ \mathbb{P}-\text{a.s.} \Big\} \end{aligned} \tag{4.2}$$

It is worth to note that $\mathcal{A}_s \subset \mathcal{A}_t$ for all $0 \leq t \leq s \leq T$.

In the rest of this chapter, we impose the following assumptions.

Assumption 4.1. There exist constants $C > 0$ and $C_M > 0$ such that for all $x_1, x_2 \in \mathbb{R}^d$, and $0 < |\eta| < M$, we have

$$\begin{aligned} \big|\sigma(x_1) - \sigma(x_2)\big| + \big|a(x_1) - a(x_2)\big| &\leq C|x_1 - x_2|, \\ \big|\gamma(x_1,\eta) - \gamma(x_2,\eta)\big| &\leq C_M|\eta||x_1 - x_2|, \\ \big|\sigma(x)\big| &\leq C\big(1 + g(|x|^{\frac{1}{2}})\big), \\ \big|\gamma(x,\eta)\big| &\leq C_M|\eta|\big(1 + g(|x|^{\frac{1}{2}})\big), \end{aligned}$$

where $g(x) = x \wedge x^2$.

Assumption 4.2. $Db : \mathbb{R}^d \to \mathbb{R}^d$ is bounded, twice continuously differentiable functions with bounded derivatives up to second order.

By Theorem 1.3.2, we know that under Assumption 4.1 there exists a unique cádlág and adapted solution $(X_t)_{t \geq s}$ to (4.12). To emphasize dependence on initial conditions and the control, we may write X_t as $X_t^{u,s,x}$.

The revenue functional for a given $u \in \mathcal{A}_s$ is defined as

$$V^u(s,x) = \mathbb{E}\Big(\int_s^T f(X_t^{u,s,x})dt + h(X_T^{u,s,x}) \Big), \tag{4.3}$$

where $f : \mathbb{R}^d \to \mathbb{R}$, $h : \mathbb{R}^d \to \mathbb{R}$ are continuous bounded functions. We will say that

$$V(s,x) = \sup_{u \in \mathcal{A}_s} V^u(s,x). \tag{4.4}$$

is the value function. If there exists a maximizer $u^* \in \mathcal{A}_s$, then

$$V(s,x) \quad = \quad V^{u^*}(s,x).$$

If A is compact, as in Chapter 3, we can show that the value function V satisfies the following HJB equation in the viscosity sense :

$$\begin{aligned}
&-D_1 V(s,x) + \sup_{u \in A} \Big(< -D_2 V(s,x), a(x) + b(x)u > \Big) \\
&+\frac{1}{2} tr\big(-D_2^2 V(s,x)\sigma(x)\sigma(x)^T \big) - \int_{0<|\eta|<\infty} \Big(V(s, x + \gamma(x,\eta)) \\
&-V(s,x) - < D_2 V(s,x), \gamma(x,\eta) > 1_{\{0<|\eta|<1\}} \Big)\nu(d\eta) - f(x) = 0,
\end{aligned} \tag{4.5}$$

When A fails to be bounded, the term under the supremum may be infinite. This can be seen through the following example. Let $A = \mathbb{R}, a(x) = 0, b(x) = 1, \sigma(x) = 0, \gamma(x,\eta) = 0, f(x) = 0$, and $h(x) = x$. In this case, the value function becomes

$$V(s,x) = x + \sup_{u \in \mathcal{A}_s} \Big(\int_s^T u_t dt \Big). \tag{4.6}$$

The supremum part of (4.5) yields

$$\sup_{u \in A} \Big(< -D_2 V(s,x), a(x) + b(x)u > \Big) = \sup_{u \in \mathbb{R}} \big(-u \big). \tag{4.7}$$

Now, it is clear that this supremum may be infinite. Thus, (4.5) only makes sense provided we have

$$< D_2 V(s,x), b(x) >= 0. \tag{4.8}$$

Let $\varphi : (\kappa, x) \to \varphi(\kappa, x)$ be the integral flow associate to the drift coefficient b, i.e., φ is the unique solution of the following ordinary differential equation:

$$\begin{cases} \dfrac{d\varphi(\kappa)}{d\kappa} = b(\varphi(\kappa)), \quad \kappa \in \mathbb{R}, \\ \\ \varphi(0, x) = x. \end{cases} \tag{4.9}$$

Now, if we can prove that

$$V(s, \varphi(\kappa, x)) = V(s, x), \tag{4.10}$$

by differentiating (4.10) with respect to κ and invoking (4.9) we obtain (4.8). This leads to the observation made by Lasry and Lions (2000) that the original control problem should be invariant under the flow φ. This invariance property motivates the construction of a new control problem.

The new control problem is a standard stochastic control problem whose controls take values in a compact set and whose value function is equal to the value function of the original singular stochastic control problem. Now, fix a compact set $\Upsilon_C \subset \mathbb{R}$, where C is given in (4.2). We take the set of admissible controls of the new control problem to be

$$\begin{aligned} \mathcal{M}_s \quad = \quad & \Big\{ \mu : [0, T] \times \Omega \to \Upsilon_C \mid \mu \text{ is predictable with respect to } (\mathcal{F}_t^s)_{t \geq s}, \\ & \text{and } \mu_r = 0 \text{ for all } r \in [0, s] \Big\}. \end{aligned} \tag{4.11}$$

Again, we note that $\mathcal{M}_s \subset \mathcal{M}_t$ for all $0 \leq t \leq s \leq T$.

The state process $(Z_t)_{t \geq s}$ of the new control problem satisfies the SDE:

$$\begin{cases} dZ_t = \tilde{\sigma}(\mu_t, Z_{t-})dW_t + \tilde{b}(\mu_t, Z_{t-})dt + \displaystyle\int_{0<|\eta|<1} \tilde{\gamma}(\mu_t, Z_{t-}, \eta)\tilde{N}(dt, d\eta) \\ \qquad + \displaystyle\int_{|\eta| \geq 1} \tilde{\gamma}(\mu_t, Z_{t-}, \eta) N(dt, d\eta), \\ \\ Z_s = x \in \mathbb{R}^d, \qquad\qquad 0 \leq s \leq t \leq T, \end{cases} \tag{4.12}$$

where

$$\begin{aligned} \tilde{\sigma}(\kappa, z) \quad &= \quad D_2\varphi(\kappa, \varphi(-\kappa, z))\sigma(\varphi(-\kappa, z)), \\ \tilde{b}(\kappa, z) \quad &= \quad D_2\varphi(\kappa, \varphi(-\kappa, z))a(\varphi(-\kappa, z)) + \frac{1}{2}\hat{b}(\kappa, z) \\ & \quad + \int_{0<|\eta|<1} \Big(\varphi(\kappa, \varphi(-\kappa, z) + \gamma(\varphi(-\kappa, z), \eta)) \\ & \quad - z - D_2\varphi(\kappa, \varphi(-\kappa, z))\gamma(\varphi(-\kappa, z), \eta) \Big) \nu(d\eta), \end{aligned}$$

$$\begin{aligned}
\hat{b}(\kappa,z) &= \begin{pmatrix} tr\Big(D_2^2\varphi^1(\kappa,\varphi(-\kappa,z))\sigma(\varphi(-\kappa,z))\sigma(\varphi(-\kappa,z))^T\Big) \\ \vdots \\ tr\Big(D_2^2\varphi^n(\kappa,\varphi(-\kappa,z))\sigma(\varphi(-\kappa,z))\sigma(\varphi(-\kappa,z))^T\Big) \end{pmatrix}, \\
\tilde{\gamma}(\kappa,z,\eta) &= \varphi(\kappa,\varphi(-\kappa,z)+\gamma(\varphi(-\kappa,z),\eta))-z.
\end{aligned}$$

Indeed, the process $(Z_t)_{t\geq s}$ is induced by the process $(Y_t)_{t\geq s}$ through the flow φ, where the process $(Y_t)_{t\geq s}$ satisfies the SDE:

$$\begin{cases} dY_t = a(Y_{t-})dt + \sigma(Y_{t-})dW_t + \displaystyle\int_{0<|\eta|<1}\gamma(Y_{t-},\eta)\tilde{N}(dt,d\eta) \\ \qquad\quad + \displaystyle\int_{|\eta|\geq 1}\gamma(Y_{t-},\eta)N(dt,d\eta), \\ \\ Y_s = \varphi(-\kappa,x), \qquad\qquad\qquad\qquad 0\leq s\leq t\leq T. \end{cases} \tag{4.13}$$

and φ is the flow associated to the function b (as defined in equation (4.9)). We note that $(Y_t)_{t\geq s}$ is the solution of (4.1) with the constant control process u such that $u_t=0$, $\forall t\in[s,T]$. The intuition behind the construction of the process $(Z_t)_{t\geq s}$ is the following. For each $i=1,...,n$, we apply Ito's formula to $\varphi^i(\kappa,Y_t)$ which yields

$$\begin{aligned}
d\varphi^i(\kappa,Y_t) &= <D_2\varphi^i(\kappa,Y_t),a(Y_{t-})dt> + <D_2\varphi^i(\kappa,Y_t),\sigma(Y_{t-})dW_t> \\
&\quad +\frac{1}{2}tr\Big(D_2^2\varphi^i(\kappa,Y_t)\sigma(Y_{t-})\sigma(Y_{t-})^T\Big)dt + \int_{0<|\eta|<1}\Big(\varphi^i(\kappa,Y_{t-}+\gamma(Y_{t-},\eta)) \\
&\quad -\varphi^i(\kappa,Y_{t-}) - <D_2\varphi^i(\kappa,Y_{t-}),\gamma(Y_{t-},\eta)>\Big)\nu(d\eta)dt \\
&\quad + \int_{0<|\eta|<1}\Big(\varphi^i(\kappa,Y_{t-}+\gamma(Y_{t-},\eta)) - \varphi^i(\kappa,Y_{t-})\Big)\tilde{N}(d\eta,dt) \\
&\quad + \int_{|\eta|\geq 1}\Big(\varphi^i(\kappa,Y_{t-}+\gamma(Y_{t-},\eta)) - \varphi^i(\kappa,Y_{t-})\Big)N(d\eta,dt).
\end{aligned}$$

Let $\tilde{Y}_t=\varphi^i(\kappa,Y_t)$, by the flow property, we have $Y_t=\varphi^i(-\kappa,\tilde{Y}_t)$. By replacing Y_t with $\varphi^i(-\kappa,\tilde{Y}_t)$ and κ with a predictable process $(\mu_t)_{t\geq s}$, the state process $(Z_t)_{t\geq s}$ of the new control problem is then taken be the solution of the resulting SDE. We will call $(Z_t)_{t\geq s}$ the state process induced by the process $(Y_t)_{t\geq s}$ through the flow φ.

Remark 4.2.1. *It can be checked that there exist constants $C>0$ and $C_M>0$ such that for all $|z_1|,|z_2|\leq N$, $\kappa_1,\kappa_2\in\Upsilon_C$, and $0<|\eta|<M$,*

$$\begin{aligned}
&\big|\tilde{\sigma}(\kappa_1,z_1)-\tilde{\sigma}(\kappa_2,z_2)\big| + \big|\tilde{b}(\kappa_1,z_1)-\tilde{b}(\kappa_2,z_2)\big| \\
\leq\ & C|z_1-z_2| + C\big(1+|z_1|\wedge|z_2|\big)|\kappa_1-\kappa_2|,
\end{aligned}$$

and

$$\begin{aligned}
&\big|\tilde{\gamma}(\kappa_1,z_1,\eta)-\tilde{\gamma}(\kappa_2,z_2,\eta)\big| \\
\leq\ & C_M|\eta||z_1-z_2| + C_M|\eta|\big(1+|z_1|\wedge|z_2|\big)|\kappa_1-\kappa_2|,
\end{aligned}$$

and there exists constants $C > 0$ and $C_M > 0$ such that for all $z \in \mathbb{R}^d$, $\kappa \in \Upsilon_C$, and $0 < |\eta| < M$,

$$\begin{aligned} \big|\tilde{\sigma}(\kappa, z)\big| + \big|\tilde{b}(\kappa, z)\big| &\leq C\big(1 + |z|\big), \\ \big|\tilde{\gamma}(\kappa, z, \eta)\big| &\leq C_M|\eta|\big(1 + |z|\big). \end{aligned}$$

Proposition 4.2.2. *There exists a unique càdlàg and adapted solution to (4.12).*

Proof. By the construction of solution of SDE with Lévy noise (see for example Theorem 6.2.9 on p374 in Applebaum (2009) or pp354-355 in Kunita (2004)), to show this proposition, it reduces to show the existence and uniqueness of solution of the SDE:

$$\begin{cases} dZ_t = \tilde{\sigma}(\mu_t, Z_{t-})dW_t + \tilde{b}(\mu_t, Z_{t-})dt + \displaystyle\int_{0<|\eta|<1} \tilde{\gamma}(\mu_t, Z_{t-}, \eta)\tilde{N}(dt, d\eta), \\ \\ Z_s = x \in \mathbb{R}^d, \qquad\qquad\qquad\qquad 0 \leq s \leq t \leq T. \end{cases} \tag{4.14}$$

Next, to show the existence and uniqueness of solution of the above SDE, it is sufficient to show that its coefficients satisfy the conditions (5.6) and (5.7) on p495 in Menaldi (2014). This then follows from Remark 4.2.1. □

For $\mu \in \mathcal{M}_s$, the revenue functional of the new control problem is defined as

$$\mathcal{V}^\mu(s, x) = \mathbb{E}\Big(\int_s^T \tilde{f}(\mu_t, Z_t^{\mu,s,x})dt + \tilde{h}(Z_T^{\mu,s,x})\Big), \tag{4.15}$$

where

$$\tilde{f}(\kappa, z) = f(\varphi(-\kappa, z)), \quad \tilde{h}(z) = \sup_{\kappa \in \mathbb{R}} h(\varphi(-\kappa, z)),$$

and f, h are defined in (4.3). The value function of the new control problem is then given by

$$\mathcal{V}(s, x) = \sup_{\mu \in \mathcal{M}_s} \mathcal{V}^\mu(s, x). \tag{4.16}$$

To conclude this section, we present our main result.

Theorem 4.2.3. *For $s \in [0, T)$, and $x \in \mathbb{R}^d$ the following hold for all $\kappa \in \mathbb{R}$.*

- ***Invariance**: The value function of the original singular control problem has the invariance property:*

$$V(s, x) = V(s, \varphi(\kappa, x)). \tag{4.17}$$

- ***Equivalence**: The original singular control problem and the new standard control problem are equivalent in the sense that*

$$V(s, x) = \mathcal{V}(s, x). \tag{4.18}$$

4.3 Auxiliary Results

In order to prove Theorem 4.2.3, in this section, we include some auxiliary results.

4.3.1 More on the Approximation of the State Process

We first recall some facts from Section 2.3.1. Let $(X_t^M)_{t\geq s}$ be the approximation of $(X_t)_{t\geq s}$ in the sense of (2.3.2), where $(X_t)_{t\geq s}$ is a solution of (4.1). Again, to emphasize dependence on initial conditions and the control, we may write X_t^M as $X_t^{u,s,x,M}$.

Adapting the proof of Lemma 2.2 in Dufour and Miller (2002), we can obtain the estimates below.

Lemma 4.3.1. *Let*

$$\tau_N := \tau(N, M, u, s, x) = \inf\{t \geq s : |X_t^{u,s,x,M}| \notin B(x, N)\}. \tag{4.19}$$

For every $M \geq 1$, $t \in [s, T]$, and all $p \geq 2$, there exist a constant $C_{T,p,M} > 0$ such that

1. $\mathbb{E}\Big(\big|X_t^{u,s,x,M}\big|^p\Big) \leq C_{T,p,M}\Big(1 + |x|^p\Big)$;
2. $\mathbb{E}\Big(\sup\limits_{r\in[s,t]} 1_{\{t<\tau_N\}}\big|X_r^{u,s,x,M}\big|^p\Big) \leq C_{T,p,M}\Big(1 + |x|^p\Big)$;

Proof. We will only prove the first result, since the proof for the second result follows in the same fashion.

Firstly, by a straightforward calculation we obtain

$$\begin{aligned}
1_{\{t<\tau_N\}}\big|X_t^{u,s,x,M}\big| \quad &\leq \quad 1_{\{t<\tau_N\}}|x| + \Big|\int_s^t 1_{\{\xi<\tau_N\}} a(X_{\xi-}^{u,s,x,M})d\xi\Big| \\
&\quad + \Big|\int_s^t 1_{\{\xi<\tau_N\}}\sigma(X_{\xi-}^{u,s,x,M})dW_\xi\Big| \\
&\quad + \Big|\int_s^t\int_{0<|\eta|<M} 1_{\{\xi<\tau_N\}}\gamma(X_{\xi-}^{u,s,x,M},\eta)\tilde{N}(d\xi,d\eta)\Big| \\
&\quad + \Big|\int_s^t\int_{1\leq|\eta|<M} 1_{\{\xi<\tau_N\}}\gamma(X_{\xi-}^{u,s,x,M},\eta)\nu(d\eta)d\xi\Big| \\
&\quad + C\int_s^t |u_\xi|d\xi + \int_s^t C|u_\xi|I_{\{\xi<\tau_N\}}|X_\xi^{u,s,x,M}|d\xi \\
&:= \quad g_t + \int_s^t C|u_\xi|I_{\{\xi<\tau_N\}}|X_\xi^{u,s,x,M}|d\xi.
\end{aligned}$$

By Gronwall's inequality, and invoking

$$\int_0^T |u_t|dt \leq C \quad \mathbb{P}-\text{a.s.} \tag{4.20}$$

we obtain

$$
\begin{aligned}
1_{\{t<\tau_N\}}|X_t^{u,s,x,M}| \quad \leq \quad & C(1+|x|) + C\Big|\int_s^t 1_{\{\xi<\tau_N\}} a(X_{\xi-}^{u,s,x,M})d\xi\Big| \\
& + C\Big|\int_s^t 1_{\{\xi<\tau_N\}} \sigma(X_{\xi-}^{u,s,x,M})dW_\xi\Big| \\
& + C\Big|\int_s^t \int_{0<|\eta|<M} 1_{\{\xi<\tau_N\}} \gamma(X_{\xi-}^{u,s,x,M},\eta)\tilde{N}(d\xi,d\eta)\Big| \\
& + C\Big|\int_s^t \int_{1\leq|\eta|<M} 1_{\{\xi<\tau_N\}} \gamma(X_{\xi-}^{u,s,x,M},\eta)\nu(d\eta)d\xi\Big| \quad \mathbb{P}-\text{a.s.}
\end{aligned}
$$

Next, we apply Kunita's inequality (see for example, Theorem 2.11 on p332 of Kunita (2004)). The above then implies

$$
\mathbb{E}\Big(1_{\{t<\tau_N\}}|X_t^{u,s,x,M}|^p\Big) \quad \leq \quad C_{T,p,M}(1+|x|^p) + C_{p,M}\int_s^t \mathbb{E}\Bigg(1_{\{r<\tau_N\}}\big|X_r^{u,s,x,M}\big|^p\Bigg)dr.
$$

Applying Gronwall's inequality again, we obtain

$$
\mathbb{E}\Big(1_{\{t<\tau_N\}}|X_t^{u,s,x,M}|^p\Big) \quad \leq \quad C_{T,p,M}(1+|x|^p).
$$

Letting $N \to \infty$, the results then follows by noting that $1_{\{t<\tau_N\}} \to 1$ $\mathbb{P}$-a.s. (for every $M \geq 1$).

□

For the sequence of state processes $(X_t^{u,s,x,M})_{t\geq s}$, we define their corresponding revenue functionals $V^{u,M}$ by

$$
V^{u,M}(s,x) \quad = \quad \mathbb{E}\Big(\int_s^T f(X_t^{u,s,x,M})dt + h(X_T^{u,s,x,M})\Big),
$$

and the value functions V^M are given by

$$
V^M(s,x) \quad = \quad \sup_{u\in\mathcal{A}_s} V^{u,M}(s,x).
$$

Then, the following result holds.

Lemma 4.3.2. *For every $(s,x) \in [0,T]\times\mathbb{R}^d$, as $M\to\infty$, $V^M(s,x)\to V(s,x)$.*

Proof. Proof is identical to Lemma 2.3.5.

□

Next, we define a process $(Z_t^M)_{t\geq s}$ as an approximation of $(Z_t)_{t\geq s}$ in the sense of (2.3.2).

Remark 4.3.3. *By a construction of solution of SDE with Lévy noise, we see that $(Z_t^M)_{t\geq s}$ satisfies the following SDE:*

$$\begin{cases} dZ_t^M = \tilde{\sigma}(\mu_t, Z_{t-}^M)dW_t + \tilde{b}(\mu_t, Z_{t-}^M)dt + \displaystyle\int_{0<|\eta|<1} \tilde{\gamma}(\mu_t, Z_{t-}^M, \eta)\tilde{N}(dt, d\eta) \\ \qquad\qquad + \displaystyle\int_{1\le|\eta|<M} \tilde{\gamma}(\mu_t, Z_{t-}^M, \eta)N(dt, d\eta), \\ \\ Z_s^M = x, \qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \le s \le t \le T, \end{cases} \tag{4.21}$$

Since

$$\int_{1\le|\eta|<M} |\eta|^p \nu(d\eta) < \infty, \tag{4.22}$$

for every $M > 1$, and all $p \ge 2$, a consequence of the existence of solutions is

$$\mathbb{E}\Big(\sup_{t\in[s,T]} |Z_t^{\mu,s,x,M}|^p\Big) < \infty. \tag{4.23}$$

We define the corresponding revenue functional $\mathcal{V}^{\mu,M}$ and value function and $\mathcal{V}^M$ as

$$\mathcal{V}^{\mu,M}(s,x) = \mathbb{E}\Big(\int_s^T \tilde{f}(\mu_t, Z_t^{\mu,s,x,M})dt + \tilde{h}(Z_T^{\mu,s,x,M})\Big),$$

and

$$\mathcal{V}^M(s,x) = \sup_{\mu\in\mathcal{M}_s} \mathcal{V}^{\mu,M}(s,x).$$

Here, the functions $\tilde{f}$ and $\tilde{h}$ are defined in the new control problem (see (4.15)).

Remark 4.3.4. *One may check that Lemma 2.3.5 remains valid with V^M and V replaced by $\mathcal{V}$ and $\mathcal{V}^M$.*

In a similar way, we define $(Y_t^M)_{t\ge s}$ as an approximation of $(Y_t)_{t\ge s}$ in the sense of (2.3.2). Let $(\tilde{Z}_t^M)_{t\ge s}$ be the state process induced by the process $(Y_t^M)_{t\ge s}$ through the flow φ. We define the corresponding revenual functional $\tilde{\mathcal{V}}^{\mu,M}$ and value function $\tilde{\mathcal{V}}^M$ as

$$\tilde{\mathcal{V}}^{\mu,M}(s,x) = \mathbb{E}\Big(\int_s^T \tilde{f}(\mu_t, \tilde{Z}_t^{\mu,s,x,M})dt + \tilde{h}(\tilde{Z}_T^{\mu,s,x,M})\Big),$$

and

$$\tilde{\mathcal{V}}^M(s,x) = \sup_{\mu\in\mathcal{M}_s} \tilde{\mathcal{V}}^{\mu,M}(s,x).$$

Then, the following lemma holds.

Lemma 4.3.5. *For every $t \in [s,T]$, we have $Z_t^{\mu,s,x,M} = \tilde{Z}_t^{\mu,s,x,M}$ $\mathbb{P}$-a.s.*

Proof. Firstly, we observe that $(Y_t^M)_{t\ge s}$ satisfies

$$\begin{cases} dY_t^M = a(Y_{t-}^M)dt + \sigma(Y_{t-}^M)dW_t + \int_{0<|\eta|<1} \gamma(Y_{t-}^M, \eta)\tilde{N}(dt, d\eta) \\ \qquad + \int_{1\le|\eta|<M} \gamma(Y_{t-}^M, \eta)N(dt, d\eta) \\ \\ Y_s^M = \varphi(-\kappa, x), \qquad\qquad 0 \le s \le t \le T, \end{cases}$$

By the definition of $(\tilde{Z}^M)_{t\ge s}$, it is easy to see that it also satisfies (4.21). The argument then follows from the uniqueness of solution of SDE. $\square$

As a consequence we have the following result.

Corollary 4.3.6. *For every $\mu \in \mathcal{U}$, and $(s, x) \in [0, T] \times \mathbb{R}^d$, we have*

$$\begin{aligned} \mathcal{V}^{\mu,M}(s, x) &= \tilde{\mathcal{V}}^{\mu,M}(s, x), \\ \mathcal{V}^M(s, x) &= \tilde{\mathcal{V}}^M(s, x). \end{aligned}$$

4.3.2 Piecewise Constant Controls

In this subsection, we recall some results from the theory of piecewise constant controls which we mainly collect from Krylov (2009); Ishikawa (2004) (possibly with minor modifications).

With the Euclidean metric $|\cdot|$, we know that $(\Upsilon_C, |\cdot|)$ is a compact metric space, where Υ_C is defined in (4.11). Let $S = \{\alpha_1, \alpha_2, \dots\}$ be a countable dense subset in Υ_C, and take $S_N = \{\alpha_1, \alpha_2, \dots, \alpha_N\} \subset S$. Next, take a partition $P_n := P_n(s, T) = \big\{s = t_0 < t_1 < \cdots < t_n = T\big\}$. The piecewise constant control is recalled next (see pp142-143, Definition 5 in Krylov (2009)).

Definition 4.3.7. *The control $\mu \in \mathcal{M}_s$ is called a piecewise constant control, if for all $\omega \in \Omega$, it takes values in S_N and satisfies $\mu_t(\omega) = \mu_{t_i}(\omega)$ for $t \in (t_i, t_{i+1}], i = 0, 1, \dots, n-1$. The set of all piecewise constant controls corresponding to partition P_n taking values in S_N is denoted by $\mathcal{M}_s^{pc}(P_n, N)$, and we will sometimes denote the piecewise constant control as μ^{pc} to emphasize the fact that it is a piecewise constant control.*

Further we define

$$\mathcal{M}_s^{pc}(P_n) = \bigcup_N \mathcal{M}_s^{pc}(P_n, N), \quad \text{and} \quad \mathcal{M}_s^{pc} = \bigcup_{P_n} \mathcal{M}_s^{pc}(P_n).$$

In addition, we define the convergence of the control process in the following sense (See Definition 3 on p142 in Krylov (2009)).

Definition 4.3.8. *Define a metric d on $\mathcal{M}_0$ such that*

$$d(\mu^1, \mu^2) = \mathbb{E}\Big(\int_0^T |\mu_t^1 - \mu_t^2| dt\Big), \tag{4.24}$$

for $\mu^1, \mu^2 \in \mathcal{M}_0$. We say that a sequence of controls $(\mu^n)_{n\in\mathbb{N}}$, where $\mu^n \in \mathcal{M}_0$, converges to $u \in \mathcal{M}_0$ if $d(\mu^n, \mu) \to 0$ as $n \to \infty$.

Remark 4.3.9. *With minor adjustments, we can define convergence of sequence of controls of the original control problem (those controls in $\mathcal{A}_0$) in the same sense as in Definition 4.3.8. Thus, from now on, when we refer to as the convergence of control process, we will always mean the convergence in the sense of Definition 4.3.8.*

Next, we have the following convergence lemma. (see Lemma 6 on p143 in Krylov (2009), and see also Proposition 4.2 in Ishikawa (2004)).

Lemma 4.3.10. *Take a sequence of partition $P_n^j := P_n^j(s,T) = \{s = t_0^j < t_1^j < \cdots < t_{n(j)}^j = T\}$, $j = 1, 2, \ldots$, each of which is a refinement of the previous one. Suppose that as $j \to \infty$ (or equivalently as $n \to \infty$), $\max_i(t_{i+1}^j - t_i^j) \to 0$. Then, for every $\mu \in \mathcal{M}_s$ there exists a sequence of controls $\mu^n \in \mathcal{M}_s^{pc}(P_n^j)$ converging to the control $\mu \in \mathcal{M}_s$.*

Proof. This lemma follows from the proof of Lemma 6, on pp143-144, in Krylov (2009). ☐

For $r \le s$, we set $u_r = 0$ for $\mu_s \in \mathcal{M}_s$. The convergence of the control process leads to the following convergence result.

Proposition 4.3.11. *Suppose there exists a sequence of controls $\mu^n \in \mathcal{M}_s$ such that $\mu^n \to \mu \in \mathcal{M}_s$ as $n \to \infty$, then*

$$\lim_{n\to\infty} \mathbb{E}\Big(\sup_{t\in[s,T]} |Z_t^{\mu^n,s,x,M} - Z_t^{\mu,s,x,M}|^2 \Big) = 0.$$

In particular, for every $t \in [s,T]$, $M \ge 1$, and all $x \in \mathbb{R}^d$, $Z_t^{\mu^n,s,x,M} \to Z_t^{\mu,s,x,M}$ $\mathbb{P}$-a.s. (along a subsequence if necessary).

Proof. The proof follows from a standard localization argument (see for example pp499 - 500 in Menaldi (2014)). Let

$$\Pi_N(z) = \begin{cases} z, & \text{if } |z| \le N; \\ N\frac{z}{|z|}, & \text{if } |z| > N. \end{cases} \tag{4.25}$$

Define the following coefficients:

$$\begin{aligned} \tilde{b}^N(\mu, z) &= \tilde{b}(\mu, \Pi_N(z)), \\ \tilde{\sigma}^N(\mu, z) &= \tilde{\sigma}(\mu, \Pi_N(z)), \\ \tilde{\gamma}^N(\mu, z, \eta) &= \tilde{\gamma}(\mu, \Pi_N(z), \eta). \end{aligned}$$

Let $(Z_t^{\mu,s,x,M,N})_{t\in[s,T]}$ be the solution of (4.21) with $\tilde{b}, \tilde{\sigma}, \tilde{\gamma}$ replaced by $\tilde{b}^N, \tilde{\sigma}^N, \tilde{\gamma}^N$. We split the proof in two steps.

Step 1. We will show that for every $\mu \in \mathcal{M}_s$ and $M \ge 1$, the sequence $(f_N)_N$ is uniformly integrable, where

$$f_N := \sup_{t\in[s,T]} |Z_t^{\mu,s,x,,M} - Z_t^{\mu,s,x,,M,N}|^2.$$

By a straightforward calculation together with Remark 4.2.1, Kunita's inequality, and Gronwall's inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[s,T]}|Z_t^{\mu,s,x,M,N}|^4\Big)+\mathbb{E}\Big(\sup_{t\in[s,T]}|Z_t^{\mu,s,x,M}|^4\Big)\\
\leq\;& C_{T,M}(1+|x|^4). \qquad (4.26)
\end{aligned}$$

Then, we see that

$$\begin{aligned}
\mathbb{E}(f_N^2) &= \mathbb{E}\Big(\sup_{t\in[s,T]}|Z_t^{\mu,s,x,,M}-Z_t^{\mu,s,x,,M,N}|^4\Big)\\
&\leq C_{T,M}(1+|x|^4).
\end{aligned}$$

By Theorem 4.2 (on p214) in Gut (2013), $(f_N)_N$ is uniformly integrable.

Step 2. Next, we will show the desired result holds.

By Remark 4.2.1, Kunita's inequality, and Gronwall's inequality we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[s,T]}\big|Z_t^{\mu,s,x,M,N}-Z_t^{\mu^n,s,x,M,N}\big|^2\Big)\\
\leq\;& C_{T,M}\mathbb{E}\Big(\int_s^T\big|\tilde{b}^N(\mu_t,Z_{t-}^{\mu,s,x,M,N})-\tilde{b}^N(\mu_t^n,Z_{t-}^{\mu,s,x,M,N})\big|^2dt\Big)\\
&+C_M\mathbb{E}\Big(\int_s^T\big|\tilde{\sigma}^N(\mu_t,Z_{t-}^{\mu,s,x,M,N})-\tilde{\sigma}^N(\mu_t^n,Z_{t-}^{\mu,s,x,M,N})\big|^2dt\Big)\\
&+C_M\mathbb{E}\Big(\int_s^T\int_{0<|\eta|<M}\big|\tilde{\gamma}^N(\mu_t,Z_{t-}^{\mu,s,x,M,N},\eta)-\tilde{\gamma}^N(\mu_t^n,Z_{t-}^{\mu,s,x,M,N},\eta)\big|^2\nu(d\eta)dt\Big).
\end{aligned}$$

Since control takes values in a compact set, by Remark 4.2.1, Hölder's inequality, and by invoking (4.26), we obtain

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[s,T]}\big|Z_t^{\mu,s,x,M,N}-Z_t^{\mu^n,s,x,M,N}\big|^2\Big)\\
\leq\;& C_{T,M}\bigg(1+\mathbb{E}\Big(\sup_{t\in[s,T]}|Z_t^{\mu,s,x,M,N}|^4\Big)\bigg)^{\frac{1}{2}}\bigg(\mathbb{E}\Big(\int_s^T|\mu_t^n-\mu_t|dr\Big)\bigg)^{\frac{1}{2}}\\
\leq\;& C_{T,M}\Big((1+|x|^4)\Big)^{\frac{1}{2}}\bigg(\mathbb{E}\Big(\int_s^T|\mu_t^n-\mu_t|dt\Big)\bigg)^{\frac{1}{2}} \qquad (4.27)
\end{aligned}$$

Then, by invoking (4.27), we see that for every $\epsilon>0$ there exists n_ϵ such that for all $n\geq n_\epsilon$,

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{t\in[s,T]}\big|Z_t^{\mu^n,s,x,M}-Z_t^{\mu,s,x,M}\big|^2\Big)\\
\leq\;& 3\mathbb{E}\Big(\sup_{t\in[s,T]}\big|Z_t^{\mu^n,s,x,M}-Z_t^{\mu^n,s,x,M,N}\big|^2\Big)+3\mathbb{E}\Big(\sup_{t\in[s,T]}\big|Z_t^{\mu^n,s,x,M,N}-Z_t^{\mu,s,x,M,N}\big|^2\Big)
\end{aligned}$$

$$
\begin{aligned}
&+3\mathbb{E}\Big(\sup_{t\in[s,T]}\big|Z_t^{\mu,s,x,M,N}-Z_t^{\mu,s,x,M}\big|^2\Big)\\
\leq\ & 3\mathbb{E}\Big(\sup_{t\in[s,T]}\big|Z_t^{\mu^n,s,x,M}-Z_t^{\mu^n,s,x,M,N}\big|^2\Big)+\epsilon C_{T,M}\Big((1+|x|^4)\Big)^{\frac{1}{2}}\\
&+3\mathbb{E}\Big(\sup_{t\in[s,T]}\big|Z_t^{\mu,s,x,M,N}-Z_t^{\mu,s,x,M}\big|^2\Big).
\end{aligned}
$$

It is well known (see for example p500 in Menaldi (2014)) that for every $\mu \in \mathcal{M}_s$, and $0 < N \leq N'$,

$$
\mathbb{P}\Big(\sup_{t\in[s,T]}|Z_t^{\mu,s,x,M,N}-Z_t^{\mu,s,x,M,N'}|>0\Big)\to 0,\quad \text{as } N\to\infty.
$$

From Step 1 and since probability measure is finite, we can apply Vitali convergence theorem (see for example, Theorem 2.18 on p38 in Da Prato et al. (2011)). Thus, as $N \to \infty$, we found

$$
\mathbb{E}\Big(\sup_{t\in[s,T]}|Z_t^{\mu^n,s,x,M}-Z_t^{\mu,s,x,M}|^2\Big)\ \leq\ \epsilon C_{T,M}\Big((1+|x|^4)\Big)^{\frac{1}{2}}.
$$

(along a subsequence if necessary). Since ϵ is arbitrary, we complete the proof.

$\square$

Next, as a consequence of Proposition 4.3.11, we obtain the lemma below.

Proposition 4.3.12. *For every $(s,x) \in [0,T]\times\mathbb{R}^d$, $\mathcal{V}^{\mu^n,M}(s,x)\to\mathcal{V}^{\mu,M}(s,x)$ as $n\to\infty$.*

Proof. Fix $s \in [0,T]$, and $x \in \mathbb{R}^d$. Take a control $\mu \in \mathcal{M}_s$, then by Lemma 4.3.10, there exists a sequence of control $\mu^n \in \mathcal{M}_s^{pc}(P_n)$ such that $\mu^n \to \mu$, as $n \to \infty$. Next, we observe, by definition of the revenue functional $\mathcal{V}^{\mu,M}(s,x)$, that

$$
\begin{aligned}
&\Big|\mathcal{V}^{\mu^n,M}(s,x)(s,x)-\mathcal{V}^{\mu,M}(s,x)(s,x)\Big|\\
\leq\ & \mathbb{E}\Bigg(\int_s^T\Big|\tilde{f}(Z_t^{\mu^n,s,x,M})-\tilde{f}(Z_t^{\mu,s,x,M})\Big|dt+\Big|\tilde{h}(Z_T^{\mu^n,s,x,M})-\tilde{h}(Z_T^{\mu,s,x,M})\Big|\Bigg).
\end{aligned}
$$

By boundedness and continuity of f and h, and continuity of flow, we see that $\tilde{f}$ and $\tilde{h}$ are bounded and continuous. Thus, as $n \to \infty$, by Proposition 4.3.11, we complete the proof. $\square$

Moreover, by Proposition 4.3.12, we see that the following result hold.

Lemma 4.3.13. *Let $\mu^n \in \mathcal{M}_s^{pc}(P_n)$, we then see that:*

$$
\mathcal{V}^M(s,x)=\lim_{n\to\infty}\sup_{\mu^n\in\mathcal{M}_s^{pc}(P_n)}\mathcal{V}^{\mu^n,M}(s,x); \tag{4.28}
$$

Proof. see p144, Corollary 9 in Krylov (2009) for the diffusion case, and the proof of Proposition 4.3 in Ishikawa (2004) for the jump case. $\square$

4.3.3 Approximation of Admissible Control Set

In this subsection, two auxiliary results which are due to Lasry and Lions (see Section 3.1 and Lemma in Section 3.2 in Lasry and Lions (2000)) are presented. Since they only sketched the proof (in the case of Brownian noise), we fill in the gaps in the case of Lévy noise here.

Let

$$V^{n,M}(s,x) = \sup_{u\in\mathcal{A}_s^n} V^{u,M}(s,x), \tag{4.29}$$

where

$$\mathcal{A}_s^n \quad = \quad \Big\{u \in \mathcal{A}_s : u_t \in \overline{B(0,n)} \text{ for all } t \in [s,T]\Big\}.$$

Remark 4.3.14. *For any sequence of $u^n \in \mathcal{A}_s^n$ such that $u^n \to u \in \mathcal{A}_s$ (see Remark 4.3.9 for the meaning of this convergence), Lemma 4.3.13 remains valid if we replace $\mathcal{V}^{u^n,M}$, and $\mathcal{V}^M$ by $V^{u^n,M}$ and V^M respectively. Indeed, we may compute, in the same way as in Lemma 4.3.1 by invoking (4.20),*

$$\begin{aligned}
&1_{\{t<\tau_{N,n}\}}\big|X_t^{u,s,x,M} - X_t^{u^n,s,x,M}\big| \\
\leq\ & \Big|\int_s^t 1_{\{\xi<\tau_{N,n}\}}\Big(a(X_{\xi-}^{u,s,x,M}) - a(X_{\xi-}^{u^n,s,x,M})\Big)d\xi\Big| \\
&+\Big|\int_s^t 1_{\{\xi<\tau_{N,n}\}}\Big(\sigma(X_{\xi-}^{u,s,x,M}) - \sigma(X_{\xi-}^{u^n,s,x,M})\Big)dW_\xi\Big| \\
&+\Big|\int_s^t\int_{0<|\eta|<M} 1_{\{\xi<\tau_{N,n}\}}\Big(\gamma(X_{\xi-}^{u,s,x,M},\eta) - \gamma(X_{\xi-}^{u^n,s,x,M},\eta)\Big)\tilde{N}(d\xi,d\eta)\Big| \\
&+\Big|\int_s^t\int_{1\leq|\eta|<M} 1_{\{\xi<\tau_{N,n}\}}\Big(\gamma(X_{\xi-}^{u,s,x,M},\eta) - \gamma(X_{\xi-}^{u^n,s,x,M},\eta)\Big)\nu(d\eta)d\xi\Big| \\
&+C\int_s^t |u_\xi - u_\xi^n|d\xi + C\int_s^t 1_{\{\xi<\tau_{N,n}\}}|X_{\xi-}^{u,s,x,M}||u_\xi - u_\xi^n|d\xi \\
&+C\int_s^t |u_\xi^n|1_{\{\xi<\tau_{N,n}\}}|X_{\xi-}^{u,s,x,M} - X_{\xi-}^{u^n,s,x,M}|d\xi \\
:=\ & g_t + C\int_s^t |u_\xi^n|1_{\{\xi<\tau_{N,n}\}}|X_{\xi-}^{u,s,x,M} - X_{\xi-}^{u^n,s,x,M}|d\xi
\end{aligned}$$

where

$$\begin{aligned}
\tau_{N,n} &:= \tau(M,N,u,u^n,s,x) \\
&= \inf\{t \geq s : |X_t^{u,s,x,M}| + |X_t^{u^n,s,x,M}| \notin B(x,N)\}.
\end{aligned}$$

By Gronwall's inequality, and note that

$$\int_0^T |u_\xi - u_\xi^n|d\xi \leq C \quad \mathbb{P}-\text{a.s.} \tag{4.30}$$

we obtain

$$
\begin{aligned}
& 1_{\{t<\tau_{N,n}\}}\big|X_t^{u,s,x,M}-X_t^{u^n,s,x,M}\big| \\
\leq\ & C\Big|\int_s^t 1_{\{\xi<\tau_{N,n}\}}\Big(a(X_{\xi-}^{u,s,x,M})-a(X_{\xi-}^{u^n,s,x,M})\Big)d\xi\Big| \\
& +C\Big|\int_s^t 1_{\{\xi<\tau_{N,n}\}}\Big(\sigma(X_{\xi-}^{u,s,x,M})-\sigma(X_{\xi-}^{u^n,s,x,M})\Big)dW_\xi\Big| \\
& +C\Big|\int_s^t\int_{0<|\eta|<M} 1_{\{\xi<\tau_{N,n}\}}\Big(\gamma(X_{\xi-}^{u,s,x,M},\eta)-\gamma(X_{\xi-}^{u^n,s,x,M},\eta)\Big)\tilde{N}(d\xi,d\eta)\Big| \\
& +C\Big|\int_s^t\int_{1\leq|\eta|<M} 1_{\{\xi<\tau_{N,n}\}}\Big(\gamma(X_{\xi-}^{u,s,x,M},\eta)-\gamma(X_{\xi-}^{u^n,s,x,M},\eta)\Big)\nu(d\eta)d\xi\Big| \\
& +C+C\sup_{r\in[s,t]}1_{\{r<\tau_{N,n}\}}|X_r^{u,s,x,M}|\int_s^t|u_\xi-u_\xi^n|d\xi \qquad \mathbb{P}-\text{a.s.}
\end{aligned}
$$

Next, by Kunita's inequality, Hölder's inequality and (4.30), the above implies, for all $p\geq 2$,

$$
\begin{aligned}
& \mathbb{E}\Big(1_{\{t<\tau_{N,n}\}}\big|X_t^{u,s,x,M}-X_t^{u^n,s,x,M}\big|^p\Big) \\
\leq\ & C\bigg(\mathbb{E}\Big(\sup_{t\in[s,T]}1_{\{t<\tau_{N,n}\}}|X_t^{u,s,x,M}|^{2p}\Big)\bigg)^{\frac12}\bigg(\mathbb{E}\Big(\int_s^T|u_r-u_r^n|dr\Big)\bigg)^{\frac12} \\
& +C_{p,M}\int_s^t\mathbb{E}\bigg(1_{\{r<\tau_{N,n}\}}\big|X_t^{u,s,x,M}-X_t^{u^n,s,x,M}\big|^p\bigg)dr.
\end{aligned}
$$

Applying Gronwall's inequality again and Lemma 4.3.1, we obtain

$$
\mathbb{E}\Big(1_{\{t<\tau_{N,n}\}}\big|X_t^{u,s,x,M}-X_t^{u^n,s,x,M}\big|^p\Big)\ \leq\ C_{T,p,M}\big(1+|x|^{2p}\big)^{\frac12}\bigg(\mathbb{E}\Big(\int_s^T|u_r-u_r^n|dr\Big)\bigg)^{\frac12}.
$$

Letting $N\to\infty$, we see that $1_{\{t<\tau_{N,n}\}}\to 1$ $\mathbb{P}$-a.s. for every n. Then, by letting $n\to\infty$, we obtain

$$
\lim_{n\to 0}\mathbb{E}\Big(\big|X_t^{u,s,x,M}-X_t^{u^n,s,x,M}\big|^p\Big)\ =\ 0.
$$

Thus, for every $t\in[s,T]$, we see that

$$
X_t^{u^n,s,x,M}\to X_t^{u^n,s,x,M}\quad \mathbb{P}\text{-a.s. (along a subsequence if necessary), as } n\to\infty.
$$

As a immediate consequence, we see that Proposition 4.3.12 remains valid if we replace $\mathcal{V}^{u^n,M}$ and $\mathcal{V}^{u,M}$ by $V^{u^n,M}$ and $V^{u,M}$ respectively. The claim then follows.

Lemma 4.3.15. *For every $(s,x)\in[0,T]\times\mathbb{R}^d$, we have*

$$
\lim_{n\to\infty}V^{n,M}(s,x)=V^M(s,x). \tag{4.31}
$$

As a consequence, $V^M:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ is lower semi-continuous.

Proof. First, for every $s\in[0,T]$, $x\in\mathbb{R}^d$ and $M\geq 1$, we see that $V^{n,M}$ is non-decreasing and bounded by V^M, hence the limit exists.

Take a sequence of control $u^{n,\epsilon,M}$ such that $u^{n,\epsilon,M} = u^{\epsilon,M} \wedge n$, where $u^{\epsilon,M} \in \mathcal{A}_s$ is an ϵ-optimal control for V^M. It is easy to see that $u^{n,\epsilon,M} \in \mathcal{A}_s^n$. Thus, by Proposition 4.3.12 (and Remark 4.3.14), we see that for all $\epsilon > 0$, we have

$$V^M - \epsilon \leq V^{u^{\epsilon,M},M} = \lim_{n\to\infty} V^{u^{n,\epsilon,M},M} \leq V^{n,M}.$$

Letting $\epsilon \to 0$, we obtain (4.31).

Furthermore, as a consequence of Corollary 2.3.7 we know that $V^{n,M}$ are continuous (see for example the proof of Proposition 2.4 in Zălinescu (2011)). This then concludes that V^M is lower semi-continuous as a pointwise limit. □

Next, for every $s \in [0,T]$, $\kappa \in \mathbb{R}$ and $\epsilon \in (s,T]$, define a control u^ϵ by

$$u_t^\epsilon := u(\kappa, s, t, \epsilon) = \begin{cases} \dfrac{\kappa}{\epsilon - s}, & \text{for all } t \in [s,\epsilon]; \\ 0, & \text{for all } t \in (\epsilon, T]. \end{cases} \tag{4.32}$$

Set $N_\epsilon = \left|\dfrac{\kappa}{\epsilon - s}\right|$, then for all $n \geq N_\epsilon$, we see that $u^\epsilon \in \mathcal{A}_n$. Moreover, we have the following lemma.

Lemma 4.3.16. *The sequence $(X_\epsilon^{u^\epsilon,s,x})_\epsilon$ converges to $\varphi(\kappa,x)$ $\mathbb{P}$-a.s. as $\epsilon \downarrow s$, where u^ϵ is defined in (4.32).*

Proof. For $i = 1, ..., n$ and $t \in [s,\epsilon]$, apply Ito's formula to $\varphi^i\Big(\frac{-\kappa}{\epsilon - s}(t-s), X_t^{u^\epsilon,s,x,M}\Big)$, and substitute $t = \epsilon$. This then yields

$$\begin{aligned}
&\varphi^i\Big(-\kappa, X_\epsilon^{u^\epsilon,s,x,M}\Big) - x^i \\
= \;& \int_s^\epsilon \Big(\frac{-\kappa}{\epsilon - s}\Big)\varphi_t^i\Big(\frac{-\kappa}{\epsilon - s}(r-s), X_r^{u^\epsilon,s,x,M}\Big)dr \\
&+ \int_s^\epsilon < D_2\varphi^i\Big(\frac{-\kappa}{\epsilon - s}(r-s), X_r^{u^\epsilon,s,x,M}\Big), a(X_r^{u^\epsilon,s,x,M}) > dr \\
&+ \int_s^\epsilon u_r^\epsilon < D_2\varphi^i\Big(\frac{-\kappa}{\epsilon - s}(r-s), X_r^{u^\epsilon,s,x,M}\Big), b(X_r^{u^\epsilon,s,x,M}) > dr \\
&+ \frac{1}{2}\int_s^\epsilon tr\Big(D_2^2\varphi^i\Big(\frac{-\kappa}{\epsilon - s}(r-s), X_r^{u^\epsilon,s,x,M}\Big)\sigma(X_r^{u^\epsilon,s,x,M})\sigma^T(X_r^{u^\epsilon,s,x,M})\Big)dr \\
&+ \int_s^\epsilon < D_2\varphi^i\Big(\frac{-\kappa}{\epsilon - s}(r-s), X_r^{u^\epsilon,s,x,M}\Big), \sigma(X_r^{u^\epsilon,s,x,M})dW_r > \\
&+ \int_s^\epsilon \int_{0<|\eta|<M} \Big(\varphi^i\Big(\frac{-\kappa}{\epsilon - s}(r-s), X_{r-}^{u^\epsilon,s,x,M} + \gamma(X_{r-}^{u^\epsilon,s,x^\epsilon,M}, \eta)\Big) \\
&- \varphi^i\Big(\frac{-\kappa}{\epsilon - s}(r-s), X_{r-}^{u^\epsilon,s,x,M}\Big)\Big)\tilde{N}(d\eta, dt) \\
&+ \int_s^\epsilon \int_{1\leq|\eta|<M} \Big(\varphi^i\Big(\frac{-\kappa}{\epsilon - s}(r-s), X_{r-}^{u^\epsilon,s,x,M} + \gamma(X_{r-}^{u^\epsilon,s,x,M}, \eta)\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\varphi^i\Big(\frac{-\kappa}{\epsilon-s}(r-s), X_{r-}^{u^\epsilon,s,x,M}\Big)\Big)\nu(d\eta)dr \\
&+\int_s^\epsilon\int_{0<|\eta|<1}\Big(\varphi^i\Big(\frac{-\kappa}{\epsilon-s}(r-s), X_{r-}^{u^\epsilon,s,x,M}+\gamma(X_{r-}^{u^\epsilon,s,x,M},\eta)\Big) \\
&-\varphi^i\Big(\frac{-\kappa}{\epsilon-s}(r-s), X_{r-}^{u^\epsilon,s,x,M}\Big) \\
&- < D_2\varphi^i\Big(\frac{-\kappa}{\epsilon-s}(r-s), X_{r-}^{u^\epsilon,s,x,M}\Big), \gamma(X_{r-}^{u^\epsilon,s,x,M},\eta) > \Big)\nu(d\eta)dr.
\end{aligned}
$$

Letting $\epsilon \to s$, by the right continuity of X and the continuity of φ_t^i, b and $D_2\varphi^i$, we see that, the first term,

$$
\int_s^\epsilon \Big(\frac{-\kappa}{\epsilon-s}\Big)\varphi_t^i\Big(\frac{-\kappa}{\epsilon-s}(r-s), X_r^{u^\epsilon,s,x,M}\Big)dr \quad\to\quad -\kappa b(x) \ \ \mathbb{P}\text{-a.s.}
$$

and the third term,

$$
\int_s^\epsilon u_r^\epsilon < D_2\varphi^i\Big(\frac{-\kappa}{\epsilon-s}(r-s), X_r^{u^\epsilon,s,x,M}\Big), b(X_r^{u^\epsilon,s,x,M}) > dr \quad\to\quad \kappa b(x) \ \ \mathbb{P}\text{-a.s.}
$$

By Kunita's inequality, the right continuity of X and the continuity of σ and $D_2\varphi^i$, the Brownian term yields

$$
\begin{aligned}
&\lim_{\epsilon\to s}\mathbb{E}\Big(\Big|\int_s^\epsilon < D_2\varphi^i\Big(\frac{-\kappa}{\epsilon-s}(r-s), X_r^{u^\epsilon,s,x,M}\Big), \sigma(X_r^{u^\epsilon,s,x,M})dW_r > \Big|^2\Big) \\
\leq\ & C_{T,M}\lim_{\epsilon\to s}\mathbb{E}\Big(\int_s^\epsilon \Big|D_2\varphi^i\Big(\frac{-\kappa}{\epsilon-s}(r-s), X_r^{u^\epsilon,s,x,M}\Big)\sigma(X_r^{u^\epsilon,s,x,M})\Big|^2dr\Big) = 0.
\end{aligned}
$$

Thus,

$$
\int_s^\epsilon < D_2\varphi^i\Big(\frac{-\kappa}{\epsilon-s}(r-s), X_r^{u^\epsilon,s,x,M}\Big), \sigma(X_r^{u^\epsilon,s,x,M})dW_r \to 0, \ \mathbb{P}\text{-a.s.}
$$

along a subsequence (if necessary). Similarly, the jump term goes to zero. Finally, it is easy to see that all other terms also go to zero by the right continuity of the integrands. Thus, we have

$$
\varphi^i(-\kappa, X_\epsilon^{u^\epsilon,s,x,M}) \to x^i, \quad \mathbb{P}\text{-a.s.}
$$

(along a subsequence if necessary). By definition and the continuity of flow, we obtain

$$
X_\epsilon^{u^\epsilon,s,x,M} \to \varphi(\kappa, x),
$$

which completes the proof. □

Next, let us consider a different control $\hat{u}^\epsilon$ such that

$$
\hat{u}_t^\epsilon := \hat{u}(\kappa, s, t, \epsilon) = \begin{cases} \dfrac{-\kappa}{\epsilon-s}, & \text{for all } t\in[s,\epsilon]; \\ 0, & \text{for all } t\in(\epsilon,T], \end{cases}
$$

where $\epsilon\in(s,T]$, and $\kappa\in\mathbb{R}$. We can find $\hat{N}_\epsilon$ such that for all $n\geq\hat{N}_\epsilon$, $u^\epsilon\in\mathcal{A}_n$. Arguing as in the proof of Lemma 4.3.16 we obtain the following result.

Corollary 4.3.17. *The sequence* $(X_\epsilon^{u^\epsilon,s,\varphi(\kappa,x)})_\epsilon$ *converges to* x $\mathbb{P}$*-a.s. as* $\epsilon \downarrow s$.

4.4 Invariance and Equivalence

4.4.1 Proof of Invariance

We now proceed to the proof of Theorem 4.2.3. We adapt the approach in Lasry and Lions (2000) and extend their argument to the Lévy case.

We first present the proof of the Invariance property.

Proof. By Lemma 2.3.5, it is easy to see that to show (4.17) it is enough to show

$$V^M(s,x) = V^M(s,\varphi(\kappa,x)),$$

where $\kappa \in \mathbb{R}$. Let u^ϵ be the control defined in Lemma 4.3.16. For $n \geq N_\epsilon$, we have

$$V^M(s,x) \geq V^{n,M}(s,x),$$

where $V^{n,M}$ is defined in (4.29). By Corollary 2.3.9, we see that

$$V^M(s,x) \;\; \geq \;\; \mathbb{E}\Big(\int_s^\epsilon f(X_t^{u^\epsilon,s,x,M})dt + V^{n,M}(\epsilon, X_\epsilon^{u^\epsilon,s,x,M})\Big).$$

As $n \to \infty$, by boundedness of f and h, we see that the Dominated Convergence Theorem can be applied here. By Lemma 4.3.15, we then obtain

$$V^M(s,x) \;\; \geq \;\; \mathbb{E}\Big(\int_s^\epsilon f(X_t^{u^\epsilon,s,x,M})dt + V^{M}(\epsilon, X_\epsilon^{u^\epsilon,s,x,M})\Big).$$

Letting $\epsilon \to s$, by Fatou's Lemma, lower semi-continuity of value function (see Lemma 4.3.15), and Lemma 4.3.16, we conclude

$$\begin{aligned} V^M(s,x) \;\; &\geq \;\; \mathbb{E}\Big(\liminf_{\epsilon \to s}\big(\int_s^\epsilon f(X_t^{u^\epsilon,s,x,M})dt + V^{M}(\epsilon, X_\epsilon^{u^\epsilon,s,x,M})\big)\Big) \\ &= \;\; V^M(s,\varphi(\kappa,x)). \end{aligned}$$

To show the converse, we repeat the above argument. Indeed, by considering the control defined in Corollary 4.3.17, we see that for all $n \geq \hat{N}^\epsilon$, we have

$$V^M(s,\varphi(\kappa,x)) \;\; \geq \;\; V^{n,M}(s,\varphi(\kappa,x)).$$

Corollary 2.3.9 then implies

$$V^M(s,\varphi(\kappa,x)) \;\; \geq \;\; \mathbb{E}\Big(\int_s^\epsilon f(X_t^{u^\epsilon,s,\varphi(\kappa,x),M})dt + V^{n,M}(\epsilon, X_\epsilon^{u^\epsilon,s,\varphi(\kappa,x),M})\Big).$$

Thus, by first letting $n \to \infty$, then letting $\epsilon \to s$, the result follows from the Dominated Convergence Theorem, Fatou's Lemma, lower semi-continuity of value function, Lemma

4.3.15, and Corollary 4.3.17. This completes the proof. $\square$

4.4.2 Proof of Equivalence

To show the equivalence, we need another invariance result. This is summarized in the following result.

Lemma 4.4.1. *For every $s \in [0,T]$, $x \in \mathbb{R}^d$, and all $\kappa \in \mathbb{R}$ such that $2\kappa \in \Upsilon_C$, we have*

$$\tilde{\mathcal{V}}^M(s,x) = \tilde{\mathcal{V}}^M(s,\varphi(\kappa,x)). \tag{4.33}$$

Proof. To see the validity of this lemma, we will consider two cases. Firstly, we see that when $s=T$ the claim holds trivially. Indeed, we have

$$\begin{aligned}
\tilde{\mathcal{V}}^M(T,x) &= \max_{\mu\in\mathbb{R}} h\Big(\varphi(-\mu, \tilde{Z}_T^{\mu,T,x,M})\Big) \\
&= \max_{\mu\in\mathbb{R}} h\Big(\varphi(-\mu, x)\Big) \\
&= \max_{\mu\in\mathbb{R}} h\Big(\varphi(-\mu+\kappa, x)\Big) \\
&= \max_{\mu\in\mathbb{R}} h\Big(\varphi\big(-\mu, \varphi(\kappa,x)\big)\Big) \\
&= \tilde{\mathcal{V}}\big(T,\varphi(\kappa,x)\big).
\end{aligned}$$

Next, fix $s \in [0,T)$, and take a partition $P_n := P_n(s,T) = \{s = t_0 < t_1 < \dots < t_{n-1} < t_n = T\}$. Let $\tilde{\mu}$ be a constant control such that $\tilde{\mu}_t = 0$ for all $t \in [s,T]$. By definition of $\tilde{Z}^M$, we observe that

$$\begin{aligned}
\tilde{\mathcal{V}}(s,x) &\geq \mathbb{E}\Bigg(\int_s^T f\big(\varphi(-\tilde{\mu}, \tilde{Z}_t^{\tilde{\mu},s,x,M})\big)dt + h\big(\varphi(-\tilde{\mu}, \tilde{Z}_T^{\tilde{\mu},s,x,M})\big)\Bigg) \\
&= \mathbb{E}\Bigg(\int_s^{t_1} f(Y_t^{0,s,x,M})dt + \int_{t_1}^{t_2} f(Y_t^{0,s,x,M})dt + \cdots + \int_{t_{n-1}}^{T} f(Y_t^{0,s,x,M})dt + h(Y_T^{0,s,x,M})\Bigg).
\end{aligned}$$

Take an arbitrary piecewise constant control $\mu^n \in \mathcal{M}_s^{pc}(P_n)$. By definition, we see that $Y_t^{0,s,x,M} = \varphi(\mu_t^n, \tilde{Z}_t^{\mu_t^n,s,\varphi(\kappa,x),M})$ over $t \in [t_i, t_{i+1})$, $i = 0, \dots, n-1$. This implies

$$\begin{aligned}
\tilde{\mathcal{V}}(s,x) &\geq \mathbb{E}\Bigg(\int_s^{t_1} \tilde{f}\big(\mu_t^n, \tilde{Z}_t^{\mu^n,s,\varphi(\kappa,x),M}\big)dt + \int_{t_1}^{t_2} \tilde{f}\big(\mu_t^n, \tilde{Z}_t^{\mu^n,s,\varphi(\kappa,x),M}\big)dt \\
&\qquad + \cdots + \int_{t_{n-1}}^{T} \tilde{f}\big(\mu_t^n, \tilde{Z}_t^{\mu^n,s,\varphi(\kappa,x),M}\big)dt + \tilde{h}\big(\tilde{Z}_T^{\mu^n,s,\varphi(\kappa,x),M}\big)\Bigg) \\
&= \tilde{\mathcal{V}}^{\mu^n}\big(s,\varphi(\kappa,x)\big).
\end{aligned}$$

Taking supremum over $\mathcal{M}_s^{pc}(P_n)$, and letting $n \to \infty$ with Lemma 4.3.13 yields

$$\tilde{\mathcal{V}}(s,x) \geq \lim_{n\to\infty} \tilde{\mathcal{V}}^{\mu^n}(s,\varphi(\kappa,x)) = \tilde{\mathcal{V}}(s,\varphi(\kappa,x)).$$

The reverse inequality can be proved in a similar way. Take a constant control $\hat{\mu}$ such that

$\hat{\mu}_t = 2\kappa$ for all $t \in [s,T]$. For an arbitrary piecewise constant control $\mu^n \in \mathcal{M}_s^{pc}(P_n)$, we see that

$$\begin{aligned}
\tilde{\mathcal{V}}(s,\varphi(\kappa,x)) &\geq \mathbb{E}\bigg(\int_s^T f\big(\varphi(-\hat{\mu},\tilde{Z}_t^{\hat{\mu},s,\varphi(\kappa,x),M})\big)dt + h\big(\varphi(-\hat{\mu},\tilde{Z}_T^{\hat{\mu},s,\varphi(\kappa,x),M})\big)\bigg) \\
&= \mathbb{E}\bigg(\int_s^{t_1} f(Y_t^{0,s,\varphi(-\kappa,x)})dt + \int_{t_1}^{t_2} f(Y_t^{0,s,\varphi(-\kappa,x)})dt + \cdots \\
&\quad + \int_{t_{n-1}}^T f(Y_t^{0,s,\varphi(-\kappa,x)})dt + h(Y_T^{0,s,\varphi(-\kappa,x)})\bigg) \\
&= \mathbb{E}\bigg(\int_s^{t_1} \tilde{f}(\mu_t^n,\tilde{Z}_t^{\mu^n,s,x,M})dt + \int_{t_1}^{t_2} \tilde{f}(\mu_t^n,\tilde{Z}_t^{\mu^n,s,x,M})dt + \cdots \\
&\quad + \int_{t_{n-1}}^T \tilde{f}(\mu_t^n,\tilde{Z}_t^{\mu^n,s,x,M})dt + \tilde{h}(\tilde{Z}_T^{\mu^n,s,x,M})\bigg) \\
&= \tilde{\mathcal{V}}^{\mu^n}(s,x).
\end{aligned}$$

Taking supremum over $\mathcal{M}_s^{pc}(P_n)$, and letting $n \to \infty$ together with Lemma 4.3.13, we complete the proof. ☐

Remark 4.4.2. *It is worth to note that the invariance of the value function of the new control problem stated in Lemma 4.4.1 does not hold for all $\kappa \in \mathbb{R}$. However, the invariance of the value function of the original control problem (see Theorem 4.2.3) holds for all $\kappa \in \mathbb{R}$.*

We now prove the equivalence.

Proof. Replacing x by $\varphi(-\kappa,x)$, we obtain, from (4.33),

$$V^M(s,x) = V^M\big(s,\varphi(-\kappa,x)\big).$$

Take a partition $P_n := P_n(s,T) = \{s = t_0 < t_1 < ... < t_{n-1} < t_n = T\}$. By definition of V^M and $(Y_t^M)_{t\in[s,T]}$, we see that

$$\begin{aligned}
V^M(s,x) &\geq \mathbb{E}\bigg(\int_s^T f(Y_t^{0,s,\varphi(-\kappa,x),M})dt + h(Y_T^{0,s,\varphi(-\kappa,x),M})\bigg) \\
&= \mathbb{E}\bigg(\int_s^{t_1} f(Y_t^{0,s,\varphi(-\kappa,x),M})dt + \int_{t_1}^{t_2} f(Y_t^{0,s,\varphi(-\kappa,x),M})dt \\
&\quad + \cdots + \int_{t_{n-1}}^T f(Y_t^{0,s,\varphi(-\kappa,x),M})dt + h(Y_T^{0,s,\varphi(-\kappa,x),M})\bigg).
\end{aligned}$$

By Lemma 4.3.10, for every $\mu \in \mathcal{M}_s$, there exists a sequence of controls $(\mu^n)_n$, where $\mu^n \in \mathcal{M}_s^{pc}(P_n)$ such that $\mu^n \to \mu$ as $n \to \infty$ (convergence is in the sense of Lemma 4.3.10). We again use $Y_t^{0,s,\varphi(-\kappa,x),M} = \varphi(\mu_t^n,\tilde{Z}_t^{\mu_t^n,s,x,M})$ which yields

$$
\begin{aligned}
V^M(s,x) &\geq \mathbb{E}\bigg(\int_s^{t_1}\tilde{f}(\mu_t^n,\tilde{Z}_t^{\mu^n,s,x,M})dt+\int_{t_1}^{t_2}\tilde{f}(\mu_t^n,\tilde{Z}_t^{\mu^n,s,x,M})dt\\
&\qquad +\cdots+\int_{t_{n-1}}^{T}\tilde{f}(\mu_t^n,\tilde{Z}_t^{\mu^n,s,x,M})dt+\tilde{h}(\tilde{Z}_T^{\mu^n,s,x,M})\bigg)=\tilde{\mathcal{V}}^{\mu^n,M}(s,x).
\end{aligned}
$$

Taking supremum over $\mathcal{M}_s^{pc}(P_n)$, letting $n\to\infty$, and by invoking Corollary 4.3.6 and Lemma 4.3.13, we obtain

$$
\begin{aligned}
V^M(s,x) &\geq \lim_{n\to\infty}\tilde{\mathcal{V}}^{\mu^n,M}(s,x)\\
&= \mathcal{V}^M(s,x).
\end{aligned}
$$

As $M\to\infty$, by Lemma 2.3.5 we have

$$
V(s,x)\geq\mathcal{V}(s,x).
$$

To show the converse, similar as in Chapter 3, we can show that whenever $\mathcal{V}^M-\phi$ attains a global maximum at $(s^*,\varphi(\kappa^*,x^*))\in(0,T)\times\mathbb{R}^d$, where $\phi\in C_b^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$, the function ϕ satisfies the following inequality:

$$
\begin{aligned}
& I_1+I_2+I_3+I_4+I_5+I_6\\
= & -D_1\phi(s^*,\varphi(\kappa^*,x^*))+<-D_2\phi(s^*,\varphi(\kappa^*,x^*)),\tilde{b}(\kappa^*,\varphi(\kappa^*,x^*))>\\
& +\frac{1}{2}tr\Big(-D_2^2\phi(s^*,\varphi(\kappa^*,x^*))\tilde{\sigma}(\kappa^*,\varphi(\kappa,x^*))\tilde{\sigma}(\kappa^*,\varphi(\kappa^*,x^*))^T\Big)\\
& -\int_{1\leq|\eta|<M}\Big(\phi(s^*,\varphi(\kappa^*,x^*)+\tilde{\gamma}(\kappa^*,\varphi(\kappa^*,x^*),\eta))-\phi(s^*,\varphi(\kappa^*,x^*))\Big)\nu(d\eta)\\
& -\int_{0<|\eta|<1}\Big(\phi(s^*,\varphi(\kappa^*,x^*)+\tilde{\gamma}(\kappa^*,\varphi(\kappa^*,x^*),\eta))-\phi(s^*,\varphi(\kappa^*,x^*))\\
& -<D_2\phi(s^*,\varphi(\kappa^*,x^*)),\tilde{\gamma}(\kappa^*,\varphi(\kappa^*,x^*),\eta)>\Big)\nu(d\eta)-\tilde{f}(\varphi(\kappa^*,x^*))\geq 0.
\end{aligned}
$$

Substituting the value of $\tilde{b},\tilde{\sigma}$, $\tilde{\gamma}$, $\tilde{f}$, and by definition of flow, we see that

$$
\begin{aligned}
I_2 &= <-D_2\phi\big(s^*,\varphi(\kappa^*,x^*)\big),D_2\varphi(\kappa^*,x^*)a(x^*)>\\
&\quad +\frac{1}{2}tr\Big(\sum_{i=1}^{n}-D_2(\phi)^i(s^*,\varphi(s^*,x^*))D_2^2\varphi^i(\kappa^*,x^*)\sigma(x^*)\sigma(x^*)^T\Big)\\
&\quad +<-D_2\phi(s^*,\varphi(\kappa^*,x^*)),\int_{0<|\eta|<1}\Big(\varphi(\kappa^*,x^*+\gamma(x^*,\eta))-\varphi(\kappa^*,x^*)\\
&\quad -D_2\varphi(\kappa^*,x^*)\gamma(x^*,\eta)\Big)\nu(d\eta)>,\\
I_3 &= \frac{1}{2}tr\Big(-D_2\varphi(\kappa^*,x^*)^TD_2^2\phi(s^*,\varphi(\kappa^*,x^*))D_2\varphi(\kappa^*,x^*)\sigma(x^*)\sigma(x^*)^T\Big),\\
I_4 &= -\int_{1\leq|\eta|<M}\Big(\phi\big(s^*,\varphi(\kappa^*,x^*+\gamma(x^*,\eta))\big)-\phi\big(s^*,\varphi(\kappa^*,x^*)\big)\Big)\nu(d\eta),
\end{aligned}
$$

$$
\begin{aligned}
I_5 &= -\int_{0<|\eta|<1} \Big(\phi\big(s^*, \varphi(\kappa^*, x^* + \gamma(x^*, \eta))\big) - \phi(s^*, \varphi(\kappa^*, x^*)) \\
&\quad - < D_2\phi(s^*, \varphi(\kappa^*, x^*)), \varphi(\kappa^*, x^* + \gamma(x^*, \eta)) - \varphi(\kappa^*, x^*) > \Big)\nu(d\eta), \\
I_6 &= -f(x^*).
\end{aligned}
$$

Combining all these terms, we obtain

$$
\begin{aligned}
&-D_1\phi(s^*, \varphi(\kappa^*, x^*)) + < -D_2\varphi(\kappa^*, x^*)D_2\phi(s^*, x^*), a(x^*) > \\
&-\frac{1}{2}tr\Big(\sum_{i=1}^{n} D_2(\phi)^i(s^*, \varphi(s^*, x^*))D_2^2\varphi^i(\kappa^*, x^*)\sigma(x^*)\sigma(x^*)^T \\
&+(D_2\varphi(\kappa^*, x^*))^T D_2^2\phi(s^*, \varphi(\kappa^*, x^*))D_2\varphi(\kappa^*, x^*)\sigma(x^*)\sigma(x^*)^T\Big) \\
&-\int_{1\le|\eta|<M} \Big(\phi\big(s^*, \varphi(\kappa^*, x^* + \gamma(x^*, \eta))\big) - \phi(s^*, \varphi(\kappa^*, x^*))\Big)\nu(d\eta) \\
&-\int_{0<|\eta|<1} \Big(\phi\big(s^*, \varphi(\kappa^*, x^* + \gamma(x^*, \eta))\big) - \phi(s^*, \varphi(\kappa^*, x^*)) \\
&- < D_2\phi(s^*, \varphi(\kappa^*, x^*))^T D_2\varphi(\kappa^*, x^*), \gamma(x^*, \eta) > \Big)\nu(d\eta) - f(x^*) \ge 0.
\end{aligned}
$$

Moreover, for every $\phi \in C_b^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$, we may define a new function $\hat{\phi} \in C_b^{1,2}((0,T]\times\mathbb{R}^d;\mathbb{R})$ such that

$$
\hat{\phi}(s,x) := \phi(s, \varphi(\kappa^*, x)).
$$

This then yields

$$
\begin{aligned}
D_1\phi(s^*, \varphi(\kappa^*, x^*)) &= D_1\hat{\phi}(s^*, x^*), \\
D_2\phi(s^*, \varphi(\kappa^*, x^*))^T D_2\varphi(\kappa^*, x^*) &= D_2\hat{\phi}(s^*, x^*), \\
\sum_{i=1}^{n} D_2(\phi)^i\big(s^*, \varphi(s^*, x^*)\big)D_2^2\varphi^i(\kappa^*, x^*) &+ (D_2\varphi(\kappa^*, x^*))^T D_2^2\phi\big(s^*, \varphi(\kappa^*, x^*)\big)D_2\varphi(\kappa^*, x^*) \\
&= D_2^2\hat{\phi}(s^*, x^*), \\
\phi\Big(s^*, \varphi\big(\kappa^*, x^* + \gamma(x^*, \eta)\big)\Big) &= \hat{\phi}\big(s^*, x^* + \gamma(x^*, \eta)\big).
\end{aligned}
$$

Hence, whenever $\mathcal{V}^M - \phi$ attains a global maximum at $(s^*, \varphi(\kappa^*, x^*)) \in (0,T) \times \mathbb{R}^d$, $\mathcal{V}^M(\cdot, \varphi(\kappa^*, \cdot)) - \hat{\phi}$ attains a global maximum at $(s^*, x^*) \in (0,T) \times \mathbb{R}^d$. Moreover, the function $\hat{\phi}$ satisfies the inequality

$$
\begin{aligned}
&-D_1\hat{\phi}(s^*, x^*) + < -D_2\hat{\phi}(s^*, x^*), a(x) > + \frac{1}{2}tr\Big(-D_2^2\hat{\phi}(s^*, x^*)\sigma(x^*)\sigma(x^*)^T\Big) \\
&-\int_{1\le|\eta|<M} \Big(\hat{\phi}\big(s^*, x^* + \gamma(x^*, \eta)\big) - \hat{\phi}(s^*, x^*)\Big)\nu(d\eta) - \int_{0<|\eta|<1} \Big(\hat{\phi}\big(s^*, x^* + \gamma(x^*, \eta)\big) \\
&-\hat{\phi}(s^*, x^*) - < D_2\hat{\phi}(s^*, x^*), \gamma(x^*, \eta) > \Big)\nu(d\eta) - f(x^*) \ge 0. \qquad (4.34)
\end{aligned}
$$

Also, we note that $\mathcal{V}^M$ is lower semi-continuous. Indeed, to verify this, it is enough to verify that for every $u \in \mathcal{M}_s$, the function $\mathcal{V}^{u,M} : (s,x) \to \mathcal{V}^{u,M}(s,x)$ is continuous.

Then, the claim follows from the fact that the supremum of an arbitrary set of continuous functions is lower semi-continuous. To show that for every $u \in \mathcal{M}_s$, the function $\mathcal{V}^{u,M} : (s,x) \to \mathcal{V}^{u,M}(s,x)$ is continuous, by Remark 5.2 in Bouchard and Touzi (2011), it is enough to show that for every $u \in \mathcal{M}_0$, the function $\mathcal{V}^{u,M} : (s,x) \to \mathcal{V}^{u,M}(s,x)$ is continuous. The proof of this is similar to Proposition 2.4 in Zălinescu (2011).

The above facts make $\mathcal{V}^M : (s,x) \to \mathcal{V}^M(s,\varphi(\kappa,x))$ a viscosity super-solution of

$$\begin{aligned} 0 \;=\; & -D_1\hat{\phi}(s^*,x^*) + n\big| < D_2\hat{\phi}(s^*,x^*), b(x^*) > \big| + < -D_2\hat{\phi}(s^*,x^*), a(x^*) > \\ & + \frac{1}{2} tr\Big(- D_2^2\hat{\phi}(s^*,x^*)\sigma(x^*)\sigma(x^*)^T\Big) - \int_{1\leq|\eta|<M} \Big(\hat{\phi}(s^*,x^*+\gamma(x^*,\eta)) \\ & - \hat{\phi}(s^*,x^*)\Big)\nu(d\eta) - \int_{0<|\eta|<1} \Big(\hat{\phi}(s^*,x^*+\gamma(x^*,\eta)) - \hat{\phi}(s^*,x^*) \\ & - < D_2\hat{\phi}(s^*,x^*), \gamma(x^*,\eta) > \Big)\nu(d\eta) - f(x^*). \end{aligned}$$

with terminal condition

$$\mathcal{V}^M(T,\varphi(\kappa,x)) \;=\; \max_{\mu\in\mathbb{R}} h(\varphi(-\mu,\varphi(\kappa,x))) \geq h(x).$$

for which $V^{n,M}(s,x)$ is a viscosity solution.

By Theorem 3.3.3, we conclude that $V^{n,M}(s,x) \leq \mathcal{V}^M(s,\varphi(\kappa,x))$. Thus, from Lemma 4.4.1, we have $V^{n,M}(s,x) \leq \mathcal{V}^M(s,x)$. Taking limit as $n \to \infty$ followed by letting $M \to \infty$, the claim follows from Lemma 4.3.15, Lemma 2.3.5, and Remark 4.3.4. This completes the proof. $\square$

Remark 4.4.3. *If the flow φ is linear we may repeat the above argument with a slightly weaker assumption. Indeed, in this case the coefficients $\tilde{b}$, $\tilde{\sigma}$, and $\tilde{\gamma}$ reduce to*

$$\tilde{b}(\kappa,z) = Ga(\varphi(-\kappa,z)),\;\; \tilde{\sigma}(\kappa,z) = G\sigma(\varphi(-\kappa,z)),\;\; \text{and}\;\; \tilde{\gamma}(\kappa,z,\eta) = G\gamma(\varphi(-\kappa,z),\eta).$$

where G is a constant matrix. Thus, we can replace Assumption 4.1 with

Assumption 4.1**. *There exist constants $C>0$ and $C_M>0$ such that for all $x_1, x_2 \in \mathbb{R}^d$, and $0<|\eta|<M$, we have*

$$\begin{aligned} \big|\sigma(x_1)-\sigma(x_2)\big| + \big|a(x_1)-a(x_2)\big| &\leq C|x_1-x_2|, \\ \big|\gamma(x_1,\eta)-\gamma(x_2,\eta)\big| &\leq C_M|\eta||x_1-x_2|, \\ \big|\gamma(\eta)\big| &\leq C_M|\eta|\big(1+|x|\big). \end{aligned}$$

4.5 Application: An Optimal Trade Execution Problem

In this section, we apply our main result to an optimal trade execution problem. We will show that the optimal payoff of the investor can be computed in close form which generalizes Theorem 2.3 in Lasry and Lions (2007b) (see also Theorem 4.2 (ii) in Kato (2014)) to the case of Lévy noise. Also, we note, in the case of Lévy noise, a closely related problem was studied by Ishitani and Kato (2012) (see Theorem 7 in Ishitani and Kato (2012)). In some sense, we also generalize their results.

We consider a "large" investor who trades an asset during a finite investment horizon $[0,T]$. For a large investor, here, we mean an investor who is large enough so that his trading will influence the price of an asset. The price of the asset is assumed to follow the SDE:

$$\begin{cases} dS_t &= S_{t-}\Big(-\theta u_t dt + \sigma dW_t + \int_{0<|\eta|<1} \big(\exp(\eta)-1\big)\tilde{N}(dt,d\eta) \\ & \qquad + \int_{|\eta|\geq 1} \big(\exp(\eta)-1\big) N(dt,d\eta)\Big), \\ S_0 &= S > 0, \end{cases}$$

where $\sigma > 0$ is a constant which denotes the volatility of the asset price, $\theta > 0$ is a parameter which quantifies investor's influence on the price of the asset. Here and after, all random quantifies are defined on the Wiener-Poisson space $(\Omega, \mathcal{F}, \mathbb{P})$.

At any time $t \in [0,T]$, the investor holds the amount of asset μ_t. We assume its dynamics is given by

$$\begin{cases} d\mu_t &= -u_t dt, \\ \mu_0 &= \mu, \end{cases} \tag{4.35}$$

where the process $(u_t)_{t\geq 0}$ acts as the control. We may interpret u_t as the instantaneous trading rate of the investor, and trading executed at the price S_{t-}. If $u_t > 0$, it represents a selling, and if $u_t < 0$, it represents a buying. We denote the set of admissible controls as $\mathcal{A}$ which contains a set of càglàd processes $u : [0,T] \times \Omega \to \mathbb{R}$ adapted to the filtration $(\mathcal{F}_t)_{t\geq s}$ such that the following hold $\mathbb{P}$-a.s.

$$\int_0^T |u_t| dt \leq C \text{ for some constant } C > 0; \tag{4.36}$$

$$\int_0^T u_t dt = 0; \tag{4.37}$$

Because the investor's trading will influence the price of the asset, it is natural to restrict investor's trading intensity. This is given in condition (4.36). Thus, the amount of the assets held by the investor is restricted, i.e.

$$|\mu_t| \leq \mu + C.$$

Also, we require the investor to close out his position at the end of his investment horizon which is condition (4.37).

The investor holds a cash account K which is used to trade the asset. We assume that this account does not generate any interest. The cash account is assumed to follow

$$\begin{cases} dK_t & = & S_{t-}u_t dt, \\ K_0 & = & K. \end{cases} \tag{4.38}$$

The investor wishes to maximize

$$\begin{aligned} V^u(0,K,S,\mu) &= \mathbb{E}\Big(U\Big((1,0,0)(K_T,S_T,\mu_T)^T\Big)|S_0=S,K_0=K,\mu_0=\mu\Big) \\ &= \mathbb{E}\Big(U(K_T)|S_0=S,K_0=K,\mu_0=\mu\Big), \\ &= \mathbb{E}\Big(U(K_T^{u,0,S,K,\mu})\Big), \end{aligned}$$

where $U:\mathbb{R}\to\mathbb{R}_+$ is a continuous, non-decreasing and bounded utility function such that:

- U is concave if $x>C$, and $U(x)=0$ if $x\le C$, where $C>0$ is a constant.

The optimal payoff of the investor (or the value function) is thus given by

$$V(0,K,S,\mu) = \sup_{u\in\mathcal{A}} V^u(0,K,S,\mu).$$

Next, we propose the following two conditions.

Condition 4.1. $\mu>0$, $u_t\ge 0$ for all $t\in[0,T]$, and $\nu\big((-\infty,0)\big)=0$.

Condition 4.2. $\mu<0$, $u_t\le 0$ for all $t\in[0,T]$, and $\nu\big((0,\infty)\big)=0$.

Remark 4.5.1. *The first (respectively the second) condition is an example of investor who has a positive (respectively negative) position at the beginning of his investment horizon and tries to liquidate (respectively recover) his position at the end of his investment horizon. In doing so, he is not allowed to short sell (respectively purchase) more of the assets during this period of time.*

Then, we obtain the following theorem.

Theorem 4.5.2. *Assume Condition 4.1 or Condition 4.2 hold, the optimal payoff*

$$V(s,K,S,\mu) = U\Big(K+\frac{1-\exp(-\theta\mu)}{\theta}S\Big). \tag{4.39}$$

Proof. We note that we have a singular control problem with state process $(K_t,S_t,\mu_t)_{t\in[0,T]}$. The drift term is given by

$$b(K,S,\mu) = \begin{pmatrix} S \\ -\theta S \\ -1 \end{pmatrix},$$

with the associated flow

$$\varphi\big(\kappa,(K,S,\mu)\big) = \begin{pmatrix} K+\frac{1-\exp(-\theta\kappa)}{\theta}S \\ \exp(-\theta\kappa)S \\ \mu-\kappa \end{pmatrix}.$$

Since the flow is linear, one can check that Assumption 1* and Assumption 2 are satisfied. By the first claim in Theorem 4.2.3, we know that V is invariant under the flow φ. We note that the invariance claimed in Theorem 4.2.3 does not require (4.37) holds.

To prove this theorem, we again construct a new control problem as in Section 4.4. Let us recall briefly what we have done in Section 4.4. We construct the new control problem by first setting the old control process equal to zero. Then, we define a new state process through the flow with a replacement of the time parameter in the flow by a predictable process (which acts as the new control process). The purpose of setting the old control equal to zero is to remove the old control, and the purpose of replacing the time parameter in the flow by a predictable process is to introduce a new control. To keep this in mind, we define a new state process $(P_t, L_t)_{t\in[0,T]}$, by using the flow φ and by replacing the time parameter κ by the process $(\mu_t)_{t\in[0,T]}$, such that

$$P_t = K_t + \frac{1-\exp\left(-\theta\mu_t\right)}{\theta}S_t, \quad \text{and} \quad L_t = \exp\left(-\theta\mu_t\right)S_t.$$

Thus, the component μ of the old state process now becomes the new control process. We use $\mathcal{M}$ to denote the set of all new controls μ. By Ito's formula we compute the dynamics of the new state process

$$\begin{aligned}
dP_t &= S_{t-}\Big(\frac{1-\exp(-\theta\mu_t)}{\theta}\Big)\Big(\sigma dW_t + \int_{0<|\eta|<1}\big(\exp(\eta)-1\big)\tilde{N}(dt,d\eta) \\
&\quad + \int_{|\eta|\geq 1}\big(\exp(\eta)-1\big)N(dt,d\eta)\Big), \\
dL_t &= S_{t-}\exp(-\theta\mu_t)\Big(\sigma dW_t + \int_{0<|\eta|<1}\big(\exp(\eta)-1\big)\tilde{N}(dt,d\eta) \\
&\quad + \int_{|\eta|\geq 1}\big(\exp(\eta)-1\big)N(dt,d\eta)\Big).
\end{aligned}$$

We may add superscripts to emphasize the dependence of state process on the control and its initial conditions, for example $P_t^{\mu,0,P}$.

Define $(S_t^M)_{t\in[0,T]}$ as the approximation of $(S_t)_{t\in[0,T]}$ in the sense of (2.3.2), and define

$$P_t^M = K_t^M + \frac{1-\exp\left(-\theta\mu_t\right)}{\theta}S_t^M,$$

where

$$K_t^M = K + \int_s^t S_{r-}^M u_r dr.$$

By the second claim in Theorem 4.2.3, we need to compute

$$\max_{\kappa\in\mathbb{R}} U\Big((1,0,0)\varphi\Big(-\kappa, (P_T^{\mu,0,P}, L_T, 0)\Big)\Big) = U(P_T^{\mu,0,P}),$$

where we use the fact

$$\mu_T = \int_0^T -u_t dt = 0. \tag{4.40}$$

Thus we obtain the equivalent control problem as

$$\mathcal{V}(0, P, L) \quad = \quad \sup_{\mu \in \mathcal{M}} \mathbb{E}\Big(U\big(P_T^{\mu,0,P}\big)\Big).$$

We further define

$$\hat{\tau}_{N,M} = \inf\{t > 0 : |S_t^M| \geq N \quad \mathbb{P} - a.s.\}$$

Since, we require that $\mu_T = 0$ (i.e., the investor has to close his positions at the end of his investment horizon), it is easy to see that for all $t < \hat{\tau}_{N,M}$, we have

$$P_t^M = K + \int_0^t S_{r-}^M u_r dr > K - NC \quad \mathbb{P} - a.s.$$

Since $U\big(P_T^{\mu,0,P}\big)1_{\{\tau_M > T\}} = U\big(P_T^{\mu,0,P,M}\big)1_{\{\tau_M > T\}}$ ($\mathbb{P}$-a.s.), we see that

$$\begin{aligned}
\mathbb{E}\Big(U\big(P_T^{\mu,0,P}\big)\Big) \quad &= \quad \mathbb{E}\Big(1_{\{\tau_M > T\}} U\big(P_T^{\mu,0,P,M}\big)\Big) + \mathbb{E}\Big(1_{\{\tau_M \leq T\}} U\big(P_T^{\mu,0,P}\big)\Big) \\
&\leq \quad \mathbb{E}\Big(U\big(P_T^{\mu,0,P,M}\big)\Big) + \mathbb{E}\Big(1_{\{\tau_M \leq T\}} U\big(P_T^{\mu,0,P}\big)\Big),
\end{aligned}$$

where τ_M is defined in Lemma 2.3.2. A further split yields

$$\begin{aligned}
\mathbb{E}\Big(U\big(P_T^{\mu,0,P}\big)\Big) \quad &\leq \quad \mathbb{E}\Big(1_{\{\hat{\tau}_{N,M} > T\}} U\big(P_T^{\mu,0,P,M} 1_{\{\hat{\tau}_{N,M} > T\}}\big)\Big) + \mathbb{E}\Big(1_{\{\hat{\tau}_{N,M} \leq T\}} U\big(P_T^{\mu,0,P,M} 1_{\{\hat{\tau}_{N,M} \leq T\}}\big)\Big) \\
&\quad + \mathbb{E}\Big(1_{\{\tau_M \leq T\}} U\big(P_T^{\mu,0,P}\big)\Big) \\
&\leq \quad \mathbb{E}\Big(U\big(P_T^{\mu,0,P,M} 1_{\{\hat{\tau}_{N,M} > T\}}\big)\Big) + \mathbb{E}\Big(U\big(P_T^{\mu,0,P,M} 1_{\{\hat{\tau}_{N,M} \leq T\}}\big)\Big) \\
&\quad + \mathbb{E}\Big(1_{\{\tau_M \leq T\}} U\big(P_T^{\mu,0,P}\big)\Big).
\end{aligned}$$

By concavity of U, we have

$$\begin{aligned}
\mathbb{E}\Big(U\big(P_T^{\mu,0,P}\big)\Big) \quad &\leq \quad U\Big(\mathbb{E}\Big(P_T^{\mu,0,P,M} 1_{\{\hat{\tau}_{N,M} > T\}}\Big)\Big) + \mathbb{E}\Big(U\big(P_T^{\mu,0,P,M} 1_{\{\hat{\tau}_{N,M} \leq T\}}\big)\Big) \\
&\quad + \mathbb{E}\Big(1_{\{T \leq \tau_M\}} U\big(P_T^{\mu,0,P}\big)\Big).
\end{aligned}$$

Taking $N \to \infty$, by a similar argument as in Lemma 2.3.2, we see that $1_{\{\hat{\tau}_{N,M} > T\}} \to 1$ $\mathbb{P}$-a.s. By the Dominated Convergence Theorem and continuity of U, we obtain

$$\begin{aligned}
\mathbb{E}\Big(U\big(P_T^{\mu,0,P}\big)\Big) \quad &\leq \quad U\Big(\mathbb{E}\Big(P_T^{\mu,0,P,M}\Big)\Big) + \mathbb{E}\Big(1_{\{T \leq \tau_M\}} U\big(P_T^{\mu,0,P}\big)\Big) \\
&\leq \quad U\bigg(P + \mathbb{E}\Big(\int_0^T S_{r-}^M \frac{(1 - \exp(-\theta \mu_t))}{\theta} dt \int_{1 < |\eta| \leq M} \big(\exp(\eta) - 1\big)\nu(d\eta)\Big)\bigg) \\
&\quad + \mathbb{E}\Big(U\big(P_T^{\mu,0,P}\big) 1_{\{T \leq \tau_M\}}\Big).
\end{aligned}$$

By Condition 4.1 (or Condition 4.2), the fact that $S_t^M > 0$, and since U is non-decreasing, we obtain

$$\mathbb{E}\Big(U\big(P_T^{\mu,0,P}\big)\Big) \quad \leq \quad U(P) + \mathbb{E}\Big(U\big(P_T^{\mu,0,P}\big)1_{\{T\leq\tau_M\}}\Big).$$

Letting $M \to \infty$, we first apply the Dominated Convergence Theorem together with the continuity of U, and then take supremum over $\mathcal{M}_0$. This yields

$$V(0,K,S,\mu) = \mathcal{V}(0,P,L) \leq U(P) = U\Big(K + \frac{1-\exp\left(-\theta\mu\right)}{\theta}S\Big). \tag{4.41}$$

Now, to obtain the utility on the right hand side of (4.41), we apply strategy (4.32) for $\epsilon > 0$ small enough. For more details of this strategy, we refer to Lasry and Lions (2007b); Kato (2014). By applying this ϵ-optimal strategy, we obtain the right hand side of (4.41). Hence, the proof is completed.

□

4.6 Future Extensions

There are a few possible extensions of our current work. Firstly, we can include dependence of controls to the Lévy term and try to establish a similar equivalent result. This would allow us to study the optimal liquidation problem in Ishitani and Kato (2012) without any moments assumptions. Another possibility is to apply our main result to study the option hedging problem initiated by Lasry and Lions (2007b) in the Brownian case. In addition, Larsy and Lions have mentioned a few extensions in Lasry and Lions (2000, 2007b) in the Brownian case. For example, one possible extension is to allow controls take values in $\mathbb{R}^\ell$. These extensions are also interesting topics in case of Lévy noise.

Part II

Limitation of Dynamic Programming Approach: Time Inconsistency

Chapter 5

Mean-Standard-Deviation Time Consistent Portfolio Selection: a Discrete Time Model

As mentioned in the Introduction, the second part of this thesis studies a time consistent portfolio selection problem. This work is based upon a project that completed by the candidate during the Australian Mathematical Science Institute (AMSI) Internship with the industry sponsor, Optimo Financial. A version of this work has been submitted to Automatica.

5.1 Problem Formulation

5.1.1 The Market and the Investor

Consider a market which has a finite number of different states such as "Normal", "Bull" and "Bear". From time to time the market may shift from one state to another. The transitions of the market are captured by a discrete time homogeneous Markov Chain $\{\theta_n, n \geq 0\}$, with a state space $S = \{1, ..., k\}$, and a transition matrix $\boldsymbol{Q} = (q_{ij})_{k\times k}$. There are $d > 1$ risky assets in the market with random return rates $\boldsymbol{r}_n^1, \ldots, \boldsymbol{r}_n^d$ evolving over time interval $[0, N]$. The vector process of return rates $(\boldsymbol{r}_n^1, \ldots, \boldsymbol{r}_n^d)^T$ will be denoted by $\boldsymbol{r}_n$ whose dynamics is given by an equation

$$\boldsymbol{r}_{n+1}(\theta_n) = \boldsymbol{m}_n(\theta_n) + \boldsymbol{s}_n(\theta_n)\boldsymbol{\epsilon}_{n+1} \in \mathbb{R}^d, \tag{5.1}$$

(see for example, Costa and Araujo (2008), for this commonly used model). The process $(\boldsymbol{\epsilon}_n)_{n>0}$ is a sequence of independent identically distributed d-dimensional zero mean random vectors, with covariance matrix I. The functions $\boldsymbol{m}_n : S \to \mathbb{R}^d$ and $\boldsymbol{s}_n : S \to \mathbb{R}^{d\times d}$ are deterministic for each $n = 0, ..., N-1$. In what follows it will be sometimes convenient to use the notation $\boldsymbol{r}_n(\theta_n)$ for $\boldsymbol{r}_n$. Then, for a given market state $\theta_n = j$, the ith component $\boldsymbol{r}_{n+1}^i(j)$ of $\boldsymbol{r}_{n+1}(j)$ represents the return of the ith risky asset over time period $[n, n+1]$. Thus, for every one dollar, we obtain

$$\boldsymbol{R}_{n+1}(\theta_n) = \boldsymbol{1} + \boldsymbol{r}_{n+1}(\theta_n), \tag{5.2}$$

where $\mathbf{1} \in \mathbb{R}^d$ is a vector of ones. An investor, who has a finite investment horizon $[0, N]$, chooses a strategy at time 0, and adjusts his strategy at times $n = 1, ..., N-1$. We denote the strategy of the investor as

$$\boldsymbol{u} = (\boldsymbol{u}_0(\theta_0), ..., \boldsymbol{u}_{N-1}(\theta_{N-1}))^T, \tag{5.3}$$

where each $\boldsymbol{u}_n : S \to U$ is a deterministic function and

$$U = \{\boldsymbol{u} \in \mathbb{R}^d : \mathbf{1}^T \boldsymbol{u} = 1\}. \tag{5.4}$$

For any given market state $\theta_n = j$, the ith component $\boldsymbol{u}_n^i(j) \in \mathbb{R}$ represents the proportions of wealth allocated by the investor to the ith asset. The set $\mathcal{U}^0$ of all such strategies will be interpreted as a set of strategies admissible at time 0. For all $m > 0$, we call $\boldsymbol{u}^m = (\boldsymbol{u}_n)_{n \geq m}$ a sub-strategy of $\boldsymbol{u}$, and use $\mathcal{U}^m$ to denote the set of such admissible sub-strategies.

In this work we assume that at every stage the investor can make cash injections and offtakes. More precisely, at time $n \leq N-1$, the investor can manually change his account by amount of $C_n := C_n(\theta_n)$, where $C_n : S \to \mathbb{R}$ is a deterministic function. If $C_n \geq 0$, this represents a net cash injection, and if $C_n < 0$, this represents a net offtake. Thus, if the market is in a good state, the investor may choose to add money to his portfolio and if the market is in a bad state he may wish to take some cash out. When injections and offtakes are deterministic, this can be interpreted as those investments and/or consumptions which the investor has already planned to add and/or withdraw at the beginning of his investment horizon.

The wealth process $(W_n)_{n \geq 0}$ of the investor is modeled by an $\mathbb{R}$-valued discrete time stochastic process with the dynamics

$$W_{n+1} = W_n \boldsymbol{R}_{n+1}^T(\theta_n)\boldsymbol{u}_n(\theta_n) + C_n(\theta_n). \tag{5.5}$$

Moreover, for $n = 0, ..., N-1$, and all $i \in S$, we write

$$\boldsymbol{M}_n(i) := \mathbb{E}(\boldsymbol{R}_{n+1}(\theta_n)|\theta_n = i) = \mathbf{1} + \boldsymbol{m}_n(i),$$
$$\boldsymbol{\Sigma}_n(i) := Var(\boldsymbol{R}_{n+1}(\theta_n)|\theta_n = i) = \boldsymbol{s}_n(i)\boldsymbol{s}_n^T(i).$$

5.1.2 Assumptions

Now, in the rest of this work, we make the following assumptions.

Assumption 5.1. Suppose θ_0 is deterministic. Assume that the Markov chain $\{\theta_n, n \geq 0\}$ is generated by

$$\theta_{n+1} = F(\theta_n, \xi_{n+1}),$$

where $F : S \times \mathbb{R}^d \to S$ is a mapping , and $(\xi_n)_{n>0}$ is a sequence of independent $\mathbb{R}$-

valued random variables (see Theorem 58.1 in Levine (2010) for this way of generating Markov chain).

Assumption 5.2. The sequences $(\xi_n)_{n>0}$ and $(\boldsymbol{\epsilon}_n)_{n>0}$ are independent.

Assumption 5.3. All random quantities are defined on a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a discrete time filtration $\{\mathcal{F}_n\}_{n\geq 0}$. We assume $\mathcal{F}_n = \sigma\Big(\mathcal{G}_n \vee \mathcal{Y}_n\Big)$ for $n \geq 1$, where $\mathcal{G}_n = \sigma(\xi_m, 1 \leq m \leq n)$, and $\mathcal{Y}_n = \sigma(\boldsymbol{\epsilon}_m, 1 \leq m \leq n)$, and $\mathcal{F}_0$ is trivial.

Assumption 5.4. Without loss of generality, we assume W_0 is deterministic.

Assumption 5.5. The matrices $\boldsymbol{\Sigma}_n(i)$ are positive definite for $n = 0, ..., N-1$, and all $i \in S$.

Assumption 5.6. Short selling is allowed, and there are no tax and transaction costs.

We note that, under Assumption 5.1 - Assumption 5.3, W_n and θ_{n+1} are conditionally independent (given θ_n). Under Assumption 5.5, we know that $\boldsymbol{\Sigma}_n(i)$ is invertible and its inverse is positive definite for $n = 0, ..., N-1$, and for all $i \in S$. To simplify our presentation, for $n = 0, ..., N-1$, and all $i \in S$, we further define

$$
\begin{aligned}
a_n(i) &= \mathbf{1}^T \boldsymbol{\Sigma}_n^{-1}(i)\mathbf{1}, \qquad b_n(i) = \mathbf{1}^T \boldsymbol{\Sigma}_n^{-1}(i)\boldsymbol{M}_n(i),\\
h_n(i) &= \boldsymbol{M}_n(i)^T \boldsymbol{\Sigma}_n^{-1}(i)\boldsymbol{M}_n(i), \quad g_n(i) = h_n(i) - \frac{b_n^2(i)}{a_n(i)}.
\end{aligned}
$$

Since $\boldsymbol{\Sigma}_n^{-1}(i)$ is positive definite, it is clear that $a_n(i) > 0$.

5.1.3 The Control Problem

Firstly, we introduce a definition of a single period selection criterion. We adapt the definition of probability functional and separable expected conditional mapping proposed by Kovacevic and Pflug (2009), and extended by Chen et al. (2013).

Definition 5.1.1. *A single period selection criterion over a given time period* $[n, n+1]$*, where* $n \geq 0$*, is an* $\mathcal{F}_n$*-measurable functional* $\mathcal{J}_n(\cdot) : L^2(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$.

For a fixed $x \in \mathbb{R}$ and $i \in S$, a single period mean-standard-deviation (MSD) criterion over $[n, n+1]$ takes the form:

$$
\mathcal{J}_{n,x,i}(W_{n+1}) \quad = \quad \mathbb{E}_{n,x,i}(W_{n+1}) - \kappa_n(i)\sqrt{Var_{n,x,i}(W_{n+1})}, \tag{5.6}
$$

where

$$
\begin{aligned}
\mathbb{E}_{n,x,i}(W_{n+1}) &= \mathbb{E}(W_{n+1}|W_n = x, \theta_n = i),\\
Var_{n,x,i}(W_{n+1}) &= Var(W_{n+1}|W_n = x, \theta_n = i).
\end{aligned}
$$

We may add a superscript $\boldsymbol{u}$ to W_n when we wish to emphasize the dependence of the wealth process on a (corresponding) strategy.

It is easy to see that in the presence of the variance term we can not apply the law of iterated expectations to single period MSD criterion. Thus, the standard dynamic programming principle (DPP) fails. This is referred to as non-separability (see Li and Ng (2000)). To this end, we introduce a separable multiperiod selection criterion of MSD type. Then, this allows to formulate the problem of interest as a discrete time stochastic optimal control problem. By applying the pseudo DPP, which we shall present later, we can solve this optimal control problem.

Now, for any starting time $n = 0, ..., N-1$, we define the separable multiperiod selection criterion of MSD type as

$$\begin{aligned} & J_{n,x,i}(\boldsymbol{u}^n) \\ = \ & \mathbb{E}\Big(\sum_{m=n}^{N-2} \mathcal{J}_{m,W_m,\theta_m}(W_{m+1}) + \mathcal{J}_{N-1,W_{N-1},\theta_{N-1}}(W_N)|W_n = x, \theta_n = i\Big). \end{aligned} \tag{5.7}$$

Here and after, we use superscripts to emphasize the dependence of J on n, x, and i.

Next, we borrow the definition of time consistency from Kang and Filar (2006).

Definition 5.1.2. *Given any starting time $n = 0, ..., N-1$, a strategy*

$$\boldsymbol{u}^{n,*} = (\boldsymbol{u}_n^*(\theta_n), ..., \boldsymbol{u}_{N-1}^*(\theta_{N-1}))$$

is said to be a strongly time consistent optimal strategy with respect to $J_{n,x,i}(\boldsymbol{u}^n)$ if it satisfies the following two conditions.

Condition 5.1. *Let $\mathcal{A}^n \subset \mathcal{U}^n$ be a set of strategies of the form*

$$\boldsymbol{u}^n = (\boldsymbol{v}(i), \boldsymbol{u}_{n+1}^*(\theta_{n+1}), ..., \boldsymbol{u}_{N-1}^*(\theta_{N-1})) \tag{5.8}$$

where $\boldsymbol{v}(i) \in \mathbb{R}^d$ is arbitrary. Then, we have

$$\sup_{\boldsymbol{u}^n \in \mathcal{A}^n} J_{n,x,i}(\boldsymbol{u}^n) = J_{n,x,i}(\boldsymbol{u}^{n,*}). \tag{5.9}$$

Condition 5.2. *For $m = n+1, .., N-1$,*

$$\sup_{\boldsymbol{u}^m \in \mathcal{U}^m} J_{m,x,i}(\boldsymbol{u}^m) = J_{m,x,i}(\boldsymbol{u}^{m,*}), \tag{5.10}$$

where $\boldsymbol{u}^{m,*} = (\boldsymbol{u}_m^*(i), ..., \boldsymbol{u}_{N-1}^*(\theta_{N-1}))$.

If Condition 5.1 is satisfied, then we say that the strategy is a weakly time consistent optimal strategy with respect to $J_{n,x,i}(\cdot)$.

Condition 5.1 states that a weakly time consistent optimal strategy is obtained through a period-wise optimization and backward recursion. This formulates a pseudo DPP and provides a way of selecting an optimal strategy. In contrast, a strongly time consistent optimal strategy has an extra property which guarantees that any sub-strategy of a weakly time consistent optimal strategy is also optimal for the corresponding subsequent period. This extra property makes a strongly time consistent optimal strategy similar to an optimal strategy obtained by standard DPP.

Let

$$V(n, x, i) = \sup_{\boldsymbol{u}^n \in \mathcal{A}^n} J_{n,x,i}(\boldsymbol{u}^n).$$

Based on the arguments in Chen et al. (2013), if there exists a weakly time consistent optimal strategy for the above problem, then such strategy satisfies the strong time consistency conditions presented in Definition 5.1.2. Our mission is to solve this optimal control problem and find such optimal strategy.

5.2 Some Discussions of the Single Period Problem

The name single period MSD criterion is actually a special term taken from Actuarial science, and its negative opposite in some sense is called the standard deviation premium (see Landsman (2008) and the reference therein). In this section, we briefly outline some properties of the single period MSD criterion, and discuss the issue of the presence of risk free asset. Without loss of generality, we consider the first period. Since the cash injections (and offtakes) are no longer relevant, we have

$$\mathcal{J}_{0,x,i}(W_1^{\boldsymbol{u}}) = \mathbb{E}_{0,x,i}(W_1^{\boldsymbol{u}}) - \kappa_0 \sqrt{Var_{0,x,i}(W_1^{\boldsymbol{u}})}, \tag{5.11}$$

where

$$\begin{aligned} \mathbb{E}_{0,x,i}(W_1^{\boldsymbol{u}}) &= x\boldsymbol{u}_0^T(i)\boldsymbol{M}_0(i), \\ Var_{0,x,i}(W_1^{\boldsymbol{u}}) &= x^2\boldsymbol{u}_0^T(i)\boldsymbol{\Sigma}_0(i)\boldsymbol{u}_0(i). \end{aligned}$$

For a single period problem, the market transitions and time dependence are also irrelevant. To simplify notations, we drop the subscripts in κ, $\boldsymbol{M}$, $\boldsymbol{\Sigma}$, $\boldsymbol{u}$, W, and $\mathcal{J}$, and also drop the market state argument i in $\boldsymbol{M}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{u}$. Then, we have the following lemma.

5.2.1 Properties of Single Period Mean-Standard-Deviation Selection Criterion

We summarize the properties of the single period MSD criterion in the following lemma.

Lemma 5.2.1. *The single period MSD criterion satisfies the following properties.*

- ***(Translation Invariance)**: For all $\boldsymbol{u} \in \mathbb{R}^d$, and $y \in \mathbb{R}$ we have*

$$\mathcal{J}(W^{\boldsymbol{u}} + y) = \mathcal{J}(W^{\boldsymbol{u}}) + y.$$

- ***(Positive Homogeneity)**: For all $\boldsymbol{u} \in \mathbb{R}^d$, and $\alpha \geq 0$ we have*

$$\mathcal{J}(\alpha W^{\boldsymbol{u}}) = \alpha \mathcal{J}(W^{\boldsymbol{u}}).$$

- ***(Scaling Property)**: There exists some functional $\hat{J}$ such that for all $x \in \mathbb{R}$,*

$$\mathcal{J}(W^{\boldsymbol{u}}) = x\hat{J}(\boldsymbol{u}).$$

- ***(Concavity)**: Assume that $x \in (0, \infty)$. Then, $\mathcal{J} : \boldsymbol{u} \to x\big(\boldsymbol{u}^T\boldsymbol{M} - \sqrt{\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}}\big)$ is a strictly concave function of $\boldsymbol{u}$, i.e., for all $\boldsymbol{u}, \hat{\boldsymbol{u}} \in \mathbb{R}^d$, $\boldsymbol{u} \neq \hat{\boldsymbol{u}}$ and $\xi \in (0,1)$, we have*

$$\xi\mathcal{J}\big(\boldsymbol{u}\big) + (1-\xi)\mathcal{J}\big(\hat{\boldsymbol{u}}\big) < \mathcal{J}\big(\xi\boldsymbol{u} + (1-\xi)\hat{\boldsymbol{u}}\big). \tag{5.12}$$

Proof. The first three properties can be trivially verified. For concavity, we can either follow the argument in Landsman (2008) (see pp319 - 320) or in Owadally (2012) (see p4435). Alternatively, we could also argue from the Hessian matrix.

Indeed, since x and κ are positive deterministic scalars, and $\boldsymbol{u}^T\boldsymbol{M}_0$ is linear, it is enough to show that

$$H(\boldsymbol{u}) \;:=\; \sqrt{G(\boldsymbol{u})} := \sqrt{\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}}$$

is strictly convex. The Hessian of H is given by

$$D^2_{\boldsymbol{u}}H \;=\; \frac{2GD^2_{\boldsymbol{u}}G - D_{\boldsymbol{u}}G(D_{\boldsymbol{u}}G)^T}{4G\sqrt{G}} = \frac{4\Big(\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{\Sigma}\Big)}{4\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}\sqrt{\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}}}.$$

Note that U is convex, and $\boldsymbol{\Sigma}$ is positive definite (i.e., $\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u} > 0$), thus it is easy to see that H is strictly convex if the matrix $\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{\Sigma}$ is positive definite. Since $< \boldsymbol{x}, \boldsymbol{y} >= \boldsymbol{x}^T\boldsymbol{\Sigma}\boldsymbol{y}$ defines an inner product for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$, by Cauchy-Schwarz inequality we have

$$\boldsymbol{x}^T\Big(\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}\boldsymbol{u}\boldsymbol{u}^T\boldsymbol{\Sigma}\Big)\boldsymbol{x} \;=\; \big(\boldsymbol{u}^T\boldsymbol{\Sigma}\boldsymbol{u}\big)\big(\boldsymbol{x}^T\boldsymbol{\Sigma}\boldsymbol{x}\big) - \big(\boldsymbol{x}^T\boldsymbol{\Sigma}\boldsymbol{u}\big)^2 > 0,$$

We note here that the use of the strict Cauchy-Schwarz inequality relies on the assumption that vectors used in the inner product are not collinear. This hold true in the present case since $\boldsymbol{u}$ takes values in U. □

The first two properties are not surprising since we have mentioned in the Introduction , that under the assumption that returns follow a (joint) elliptical distribution, optimizing a risk measure from the translation-invariant and positive-homogeneous (TIPH) risk measure class is equivalent to optimizing the single period MSD selection criterion. One may note that by definition the negative of a single period selection criterion is a risk measure. For different members of the TIPH risk measure class, the form of single MSD selection criterion only varies through the parameter κ. As shown in Landsman and Makov (2011), for example, for Value at Risk (VaR) we have $\kappa = F^{-1}(q)$ for some $q \in [0,1]$, where F^{-1} is the inverse of the distribution function of a standard univariate elliptical random variable. The parameter κ characterizes the investor's risk tolerance. The larger the κ, the more risk averse the investor is. For any $\kappa > 0$, $\mathcal{J}(W^{\boldsymbol{u}})$ represents a quantile value, that is κ standard deviations of the wealth away from the investor's expected terminal wealth $\mathbb{E}(W^{\boldsymbol{u}})$ to the left (see Figure 5.1). This quantile value in turn corresponds to a probability p, where

$$p = \mathbb{P}\Big(W^{\boldsymbol{u}} \geq \mathcal{J}(W^{\boldsymbol{u}})\Big).$$

Thus, the investor will have with probability p at least $\mathcal{J}(W^{\boldsymbol{u}})$ at the end of the investment period (the green area in Figure 5.1). This provides a confidence level to the investor which is somewhat inherited from VaR. Hence, it is possible to choose an appropriate risk aversion parameter, provide the returns are reasonable, so that his wealth is above zero with a high probability.

The scaling property and the concavity form an important aspect of the single period MSD selection criterion. Unlike the single period mean-variance selection criterion, the MSD selection criterion does not possess a nice quadratic structure, however, these two nice properties make it a suitable candidate in both single and in multiperiod portfolio selection.

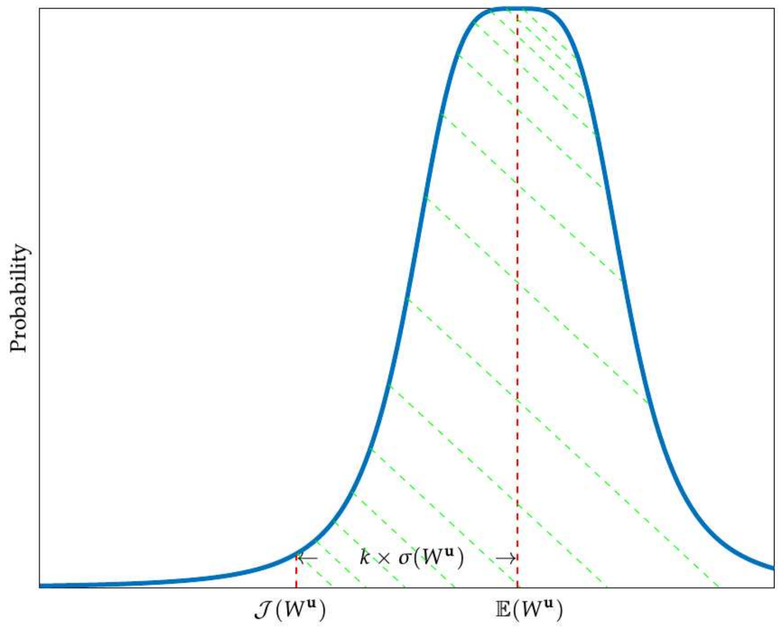

Figure 5.1: Investor's Risk Characterization

5.2.2 The Presence of Risk Free Assets

Without loss of generality, we set $x = 1$. Then, from (5.11), we see that the single period problem becomes

$$\begin{aligned} \max_{\boldsymbol{u}} \quad & \boldsymbol{u}^T \boldsymbol{M} - \kappa\sqrt{\boldsymbol{u}^T \boldsymbol{\Sigma} \boldsymbol{u}}, \\ \text{s.t.} \quad & \mathbf{1}^T \boldsymbol{u} = 1. \end{aligned} \tag{5.13}$$

As mentioned in Introduction, there are more than one way to solve the single period problem. In fact, the solution can be obtained from the result in the following section. For alternative ways of solving this problem, we refer to Landsman (2008); Owadally (2012).

It worth to note that we only consider the market of risky assets. For the single period problem, this has been work devoted to study the market in the presence of risk free asset. In the rest of this section, we will discuss the issue of the risk free asset. To alleviate the presentation, we consider the case of two assets: one risky and one risk free (see Kronborg and Steffensen (2015)). Let r denote the return of the risky asset, and r_0 denote the return of the risk free asset. Suppose that we allocate a proportion of u to the risky asset. By (5.14), we know $1 - u$ proportion will be invested into risk free asset. Thus, our problem reduces to the following optimization problem:

$$\sup_{u \in \mathbb{R}} \Big((\mathbb{E}(r) - r_0)u - \kappa\sigma(r)|u| \Big),$$

where $\sigma(r)$ denotes the standard deviation of the risky asset, and κ is the risk aversion. We make a trivial assumption that

$$\mathbb{E}(r) > r_0.$$

After some simple algebra we obtain the optimal solution

$$u^* = \begin{cases} \infty, & \text{if } \Delta > \kappa, \\ 0, & \text{if } \Delta < \kappa, \\ \text{any admissible } u \geq 0, & \text{if } \Delta = \kappa, \end{cases}$$

where

$$\Delta = \frac{\big(\mathbb{E}(r) - r_0\big)}{\sigma(r)}.$$

We may interpret Δ as the Sharpe ratio of the risky asset (with reference to the risk free asset). If the reward is large enough, it will be optimal to invest as much as possible into the risky asset. Similarly, if the reward is so little in comparison to investor's risk tolerance, investor will prefer to invest into the risk free asset only. Thus, in the presence of risk free asset, the solution does not tell us much. This trivial result coincides with the more general one period model in Landsman and Makov (2012). In fact, as in Landsman and Makov

(2012), if there are other requirement for the investment which leads to extra restrictions, this may leads to more interesting result. Thus in the remaining of this chapter, we shall only consider the market with risky assets only.

Remark 5.2.2. *One may note that the time length we consider here is one unit, while Kronborg and Steffensen (2015) argue that we can always find a time interval small enough, so that the optimal solution is always given by* 0*, and thus in the continuous time the optimal solution is* 0*. Also, the similar argument can be extended to multiperiod problem. However, this does not seem to make sense as the investment in risky asset suddenly changes from* ∞ *to* 0 *by just taking a slightly shorter rebalanced period.*

5.3 Optimal Portfolio Selection under Market Transitions and Intermediate Cash Injections

Now, we turn to the multiperiod problem. In order to build our portfolio selection scheme, we first calculate our optimal strategy.

5.3.1 Optimal Strategy and Value Function

Theorem 5.3.1. *For any given time* $n = 0, ..., N-1$*, a market state* $\theta_n = i \in S$*, and* $x \in (0, \infty)$*, assume that* $\kappa_n(i) > \hat{\kappa}_n(i)$*, where*

$$\hat{\kappa}_n(i) \quad = \quad \sqrt{g_n(i)(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i))^2}, \tag{5.14}$$

the optimal strategy is given by

$$\begin{aligned} \boldsymbol{u}_n^*(i) \quad &= \quad \frac{\big(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i)\big)f_n(i)}{\kappa_n(i)}\left(\boldsymbol{\Sigma}_n^{-1}(i)\boldsymbol{M}_n(i) - \frac{b_n(i)\boldsymbol{\Sigma}_n^{-1}(i)\mathbf{1}}{a_n(i)}\right) \\ &\quad + \frac{\boldsymbol{\Sigma}_n^{-1}(i)\mathbf{1}}{a_n(i)}, \end{aligned} \tag{5.15}$$

and the corresponding value function is given by

$$\begin{aligned} V(n,x,i) \quad &= \quad xA_n(i) + C_n(i)\big(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\big) + \sum_{m=n+1}^{N-1} \overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m}}(i) \\ &\quad + \sum_{m=n+1}^{N-2} \overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m} Q_{\boldsymbol{A}_{m+1}}}(i), \end{aligned}$$

where

$$f_n(i) = \sqrt{\frac{\frac{1}{a_n(i)}}{1-\frac{g_n(i)}{\kappa_n^2(i)}\big(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i)\big)^2}} = \sqrt{\frac{\frac{1}{a_n(i)}}{1-\frac{\hat{\kappa}_n^2(i)}{\kappa_n^2(i)}}},$$

$$A_n(i) = \kappa_n(i)f_n(i)\left(\frac{\hat{\kappa}_n^2(i)}{\kappa_n^2(i)} - 1\right) + \big(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i)\big)\frac{b_n(i)}{a_n(i)}, \quad A_N = 0,$$

$$\boldsymbol{A}_n = \Big(A_n(1), ..., A_n(k)\Big)^T, \boldsymbol{C}_n = \Big(C_n(1), ..., C_n(k)\Big)^T.$$

Proof. **Step 1**: For $n = N-1$, we have the following static optimization problem

$$\begin{array}{ll} \max_{\boldsymbol{u}_{N-1}(i)} & \left(\mathbb{E}_{N-1,x,i}(W_N) - \kappa_{N-1}(i)\sqrt{Var_{N-1,x,i}(W_N)}\right), \\ \text{s.t.} & W_N = W_{N-1}\boldsymbol{R}_N^T(i)\boldsymbol{u}_{N-1}(i) + C_{N-1}(i), \\ & \mathbf{1}^T\boldsymbol{u}_{N-1}(i) = 1. \end{array}$$

Since x and $C_{N-1}(i)$ are known at $n = N-1$, simple calculations yield the conditional expectation

$$\mathbb{E}_{N-1,x,i}(W_N) = x\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}(i) + C_{N-1}(i), \tag{5.16}$$

and the conditional variance

$$Var_{N-1,x,i}(W_N) = x^2\boldsymbol{u}_{N-1}^T(i)\boldsymbol{\Sigma}_{N-1}(i)\boldsymbol{u}_{N-1}(i). \tag{5.17}$$

Let

$$\begin{aligned} S_{N-1}(i) &= S_{N-1}(i, \boldsymbol{u}_{N-1}(i)) \\ &:= \sqrt{\boldsymbol{u}_{N-1}^T(i)\boldsymbol{\Sigma}_{N-1}(i)\boldsymbol{u}_{N-1}(i)}. \end{aligned} \tag{5.18}$$

Thus, we have

$$\begin{aligned} & \mathbb{E}_{N-1,x,i}(W_N) - \kappa_{N-1}(i)\sqrt{Var_{N-1,x,i}(W_N)} \\ = \ & x\left(\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}(i) - \kappa_{N-1}(i)S_{N-1}(i)\right) + C_{N-1}(i). \end{aligned} \tag{5.19}$$

Then, it is easy to see that optimizing the above function is equivalent to optimizing

$$\Big(\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}(i) - \kappa_{N-1}(i)S_{N-1}(i)\Big),$$

(which is really the **Scaling Property** in Lemma 5.2.1).

Thus, we obtain an equivalent optimization problem:

$$\begin{array}{ll} \max\limits_{\boldsymbol{u}_{N-1}(i)} & \Big(\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}(i) - \kappa_{N-1}(i)S_{N-1}(i)\Big), \\ \text{s.t.} & \mathbf{1}^T\boldsymbol{u}_{N-1}(i) = 1. \end{array}$$

Let $\lambda_{N-1}(i)$ be the Lagrange multiplier. Since $x > 0$, the **Concavity** in Lemma 5.2.1 holds. Thus, the following first order conditions yield a unique global optimum, if there exists a solution:

$$\boldsymbol{M}_{N-1}(i) - \kappa_{N-1}(i)\frac{\boldsymbol{\Sigma}_{N-1}(i)\boldsymbol{u}_{N-1}(i)}{S_{N-1}(i)} - \lambda_{N-1}(i)\mathbf{1} = 0, \tag{5.20}$$

$$\mathbf{1}^T\boldsymbol{u}_{N-1}(i) = 1. \tag{5.21}$$

From (5.20), we obtain

$$\boldsymbol{u}_{N-1}(i) \quad = \quad \frac{S_{N-1}(i)}{\kappa_{N-1}(i)}\boldsymbol{\Sigma}^{-1}_{N-1}(i)\Big(\boldsymbol{M}_{N-1}(i) - \lambda_{N-1}(i)\mathbf{1}\Big). \tag{5.22}$$

Substituting (5.22) into (5.21) we find $\lambda_{N-1}(i)$ as

$$\lambda_{N-1}(i) = \frac{\frac{S_{N-1}(i)}{\kappa_{N-1}(i)}\mathbf{1}^T\boldsymbol{\Sigma}^{-1}_{N-1}(i)\boldsymbol{M}_{N-1}(i) - 1}{\frac{S_{N-1}(i)}{\kappa_{N-1}(i)}\mathbf{1}^T\boldsymbol{\Sigma}^{-1}_{N-1}(i)\mathbf{1}}. \tag{5.23}$$

Substituting (5.23) into (5.22) gives the optimal strategy

$$\begin{aligned}\boldsymbol{u}^*_{N-1}(i) \quad &= \quad \frac{S^*_{N-1}(i)}{\kappa_{N-1}(i)}\left(\boldsymbol{\Sigma}^{-1}_{N-1}(i)\boldsymbol{M}_{N-1}(i) - \frac{b_{N-1}(i)\boldsymbol{\Sigma}^{-1}_{N-1}(i)\mathbf{1}}{a_{N-1}(i)}\right)\\ &\quad + \frac{\boldsymbol{\Sigma}^{-1}_{N-1}(i)\mathbf{1}}{a_{N-1}(i)},\end{aligned} \tag{5.24}$$

where $S^*_{N-1}(i) = S_{N-1}(i, \boldsymbol{u}^*_{N-1}(i))$.

One can obtain $S^*_{N-1}(i)$ by substituting (5.24) into (5.18). Thus we have

$$\begin{aligned}S^*_{N-1}(i) \quad &= \quad \sqrt{\frac{\frac{1}{a_{N-1}(i)}}{1 - \frac{h_{N-1}(i)}{\kappa^2_{N-1}(i)} + \frac{b^2_{N-1}(i)}{\kappa^2_{N-1}(i)a_{N-1}(i)}}}\\ &:= \quad f_{N-1}(i),\end{aligned} \tag{5.25}$$

provided that $\kappa_{N-1}(i) > \sqrt{g_{N-1}(i)} := \hat{\kappa}_{N-1}(i)$. Substituting $S^*_{N-1}(i)$ into (5.24), we obtain the desired form of $\boldsymbol{u}^*_{N-1}(i)$.

One may note that we always have $g_{N-1}(i) \geq 0$, by Cauchy-Schwartz inequality, and the fact that $\boldsymbol{\Sigma}^{-1}_{N-1}(i)$ is positive definite (which is a consequence of the assumption that $\boldsymbol{\Sigma}_{N-1}(i)$ is positive definite). Indeed, we see that

$$\begin{aligned}g_{N-1}(i) \quad &= \quad h_{N-1}(i) - \frac{b^2_{N-1}(i)}{a_{N-1}(i)}\\ &= \quad \frac{1}{\mathbf{1}^T\boldsymbol{\Sigma}^{-1}_{N-1}(i)\mathbf{1}}\Big(\boldsymbol{M}^T_{N-1}(i)\boldsymbol{\Sigma}^{-1}_{N-1}(i)\boldsymbol{M}_{N-1}(i)\mathbf{1}^T\boldsymbol{\Sigma}^{-1}_{N-1}(i)\mathbf{1}\\ &\qquad -\big(\mathbf{1}^T\boldsymbol{\Sigma}^{-1}_{N-1}(i)\boldsymbol{M}_{N-1}(i)\big)^2\Big) \geq 0.\end{aligned}$$

Next, let us calculate the value function $V(N-1, x, i)$ at time $N-1$. By (5.24) and (5.25), we see that

$$\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}^*(i) = \frac{f_{N-1}(i)g_{N-1}(i)}{\kappa_{N-1}(i)} + \frac{b_{N-1}(i)}{a_{N-1}(i)},$$

and

$$\sqrt{(\boldsymbol{u}_{N-1}^*)^T(i)\boldsymbol{\Sigma}_{N-1}(i)\boldsymbol{u}_{N-1}^*(i)} = S_{N-1}^*(i) = f_{N-1}(i).$$

Thus, we have

$$\begin{aligned} V(N-1,x,i) &= x\Big(\boldsymbol{M}_{N-1}^T(i)\boldsymbol{u}_{N-1}^*(i) - \kappa_{N-1}(i)S_{N-1}^*(i)\Big) + C_{N-1}(i) \\ &= xA_{N-1}(i) + C_{N-1}(i). \end{aligned} \tag{5.26}$$

Step 2: For $n = N-2$, we have the following static optimization problem:

$$\begin{aligned} \max_{\boldsymbol{u}_{N-2}(i)} \quad & \Bigg(\mathbb{E}_{N-2,x,i}(W_{N-1}) - \kappa_{N-2}(i)\sqrt{Var_{N-2,x,i}(W_{N-1})} \\ & +\mathbb{E}_{N-2,x,i}\Big(V(N-1,W_{N-1},\theta_{N-1})\Big)\Bigg), \\ \text{s.t.} \quad & W_{N-1} = W_{N-2}\boldsymbol{R}_{N-1}^T(i)\boldsymbol{u}_{N-2}(i) + C_{N-2}(i), \\ & \boldsymbol{1}^T\boldsymbol{u}_{N-2}(i) = 1. \end{aligned} \tag{5.27}$$

Similarly, as in (5.16) and (5.17), we calculate expressions for

$$\mathbb{E}_{N-2,x,i}(W_{N-1}) \text{ and } \sqrt{Var_{N-2,x,i}(W_{N-1})},$$

which yield

$$\mathbb{E}_{N-2,x,i}(W_{N-1}) = x\boldsymbol{M}_{N-2}^T(i)\boldsymbol{u}_{N-2}(i) + C_{N-2}(i), \tag{5.28}$$

$$\sqrt{Var_{N-2,x,i}(W_{N-1})} = x\sqrt{\boldsymbol{u}_{N-2}^T(i)\boldsymbol{\Sigma}_{N-2}(i)\boldsymbol{u}_{N-2}(i)}. \tag{5.29}$$

Moreover, by using (5.26) and (5.27), we find

$$\begin{aligned} \mathbb{E}_{N-2,x,i}\Big(V(N-1,W_{N-1},\theta_{N-1})\Big) &= x\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\boldsymbol{M}_{N-2}^T(i)\boldsymbol{u}_{N-2}(i) \\ &\quad +C_{N-2}(i)\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i) + \overline{\boldsymbol{Q}}_{\boldsymbol{C}_{N-1}}(i). \end{aligned} \tag{5.30}$$

Combining these three expressions, and by recognizing that x and

$$C_{N-2}(i)\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i) + \overline{\boldsymbol{Q}}_{\boldsymbol{C}_{N-1}}(i)$$

are known at $n = N-2$, thus we obtain an equivalent optimization problem:

$$\begin{aligned} \max_{\boldsymbol{u}_{N-2}(i)} \quad & \Big(\boldsymbol{M}_{N-2}^T(i)\boldsymbol{u}_{N-2}(i) - \kappa_{N-2}(i)S_{N-2}(i) + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\boldsymbol{M}_{N-2}^T(i)\boldsymbol{u}_{N-2}(i)\Big), \\ \text{s.t.} \quad & \boldsymbol{1}^T\boldsymbol{u}_{N-2}(i) = 1, \end{aligned}$$

where

$$S_{N-2}(i) = S_{N-2}(i, \boldsymbol{u}_{N-2}(i)) \quad := \quad \sqrt{\boldsymbol{u}_{N-2}^T(i)\boldsymbol{\Sigma}_{N-2}(i)\boldsymbol{u}_{N-2}(i)}. \tag{5.31}$$

We repeat the procedure from **Step 1**. Let $\lambda_{N-2}(i)$ be the corresponding Lagrange multiplier. The solution of the following system of equations, if exists, yields an unique global optimum:

$$\begin{aligned}
&\big(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big)\boldsymbol{M}_{N-2}(i) - \kappa_{N-2}(i)\frac{\boldsymbol{\Sigma}_{N-2}(i)\boldsymbol{u}_{N-2}(i)}{S_{N-2}(i)} - \lambda_{N-2}(i)\mathbf{1} = 0\\
&\mathbf{1}^T\boldsymbol{u}_{N-2}(i) = 1.
\end{aligned}$$

Solving this system yields

$$\begin{aligned}
\boldsymbol{u}_{N-2}^*(i) \quad &= \quad \frac{\big(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big)}{\kappa_{N-2}(i)} S_{N-2}^*(i)\left(\boldsymbol{\Sigma}_{N-2}^{-1}(i)\boldsymbol{M}_{N-2}(i) - \frac{b_{N-2}(i)\boldsymbol{\Sigma}_{N-2}^{-1}(i)\mathbf{1}}{a_{N-2}(i)}\right)\\
&\quad + \frac{\boldsymbol{\Sigma}_{N-2}^{-1}(i)\mathbf{1}}{a_{N-2}(i)},
\end{aligned}$$

where $S_{N-2}^*(i) = S_{N-2}(i, \boldsymbol{u}_{N-2}^*(i))$. Similarly to **Step 1**, we obtain an expression of $S_{N-2}^*(i)$ by using (5.31) and (5.32). This yields

$$S_{N-2}^*(i) \quad = \quad \sqrt{\frac{\frac{1}{a_{N-2}(i)}}{1-\frac{g_{N-2}(i)}{\kappa_{N-2}^2(i)}\big(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big)^2}} \quad := f_{N-2}(i),$$

provided that $\kappa_{N-2}(i) > \sqrt{g_{N-2}(i)\big(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big)^2}$. This gives the desired form of optimal strategy at $n = N-2$. Arguing in the same way as in **Step 1**, we see that $g_{N-2}(i) \geq 0$.

By knowing the optimal strategy, we can easily obtain the value function $V(N-2, x, i)$ as in **Step 1**, in which we need (5.28), (5.29), and (5.30). Thus, simple algebra yields

$$V(N-2, x, i) \quad = \quad xA_{N-2}(i) + C_{N-2}(i)\big(1+\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{N-1}}(i)\big) + \overline{\boldsymbol{Q}}_{\boldsymbol{C}_{N-1}}(i).$$

Step 3: For $n = N-2, ..., 0$, we use backward induction. Since we have proved the claim holds at $N-2$, let us assume it holds up to $n+1$. Over any time period $[n, n+1]$, we solve the following static optimization problem:

$$\begin{aligned}
\max_{\boldsymbol{u}_n(i)} \quad & \left(\mathbb{E}_{n,x,i}(W_{n+1}) - \kappa_n(i)\sqrt{Var_{n,x,i}(W_{n+1})} + \mathbb{E}_{n,x,i}\Big(V(n+1, W_{n+1}, \theta_{n+1})\Big)\right),\\
\text{s.t.} \quad & W_{n+1} = W_n\boldsymbol{R}_{n+1}^T(i)\boldsymbol{u}_n(i) + C_n(i),\\
& \mathbf{1}^T\boldsymbol{u}_n(i) = 1.
\end{aligned}$$

We repeat the procedure in **Step 1** (or **Step 2**). Firstly, we calculate

$$\mathbb{E}_{n,x,i}(W_{n+1}) = x\boldsymbol{M}_n^T(i)\boldsymbol{u}_n(i) + C_n(i), \tag{5.32}$$

$$\sqrt{Var_{n,x,i}(W_{n+1})} = x\sqrt{\boldsymbol{u}_n^T(i)\boldsymbol{\Sigma}_n(i)\boldsymbol{u}_n(i)}. \tag{5.33}$$

By induction hypothesis and the proof of Lemma 3 in Wu and Li (2012), we obtain

$$\begin{aligned}\mathbb{E}_{n,x,i}\Big(V(n+1, W_{n+1}, \theta_{n+1})\Big) &= x\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\boldsymbol{M}_n^T(i)\boldsymbol{u}_n(i) + C_n(i)\overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i) \\ &\quad + \sum_{m=n+1}^{N-1} \overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m}}(i) \\ &\quad + \sum_{m=n+1}^{N-2} \overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m}\boldsymbol{Q}_{\boldsymbol{A}_{m+1}}}(i). \end{aligned} \tag{5.34}$$

Then, we obtain the first order (necessary and sufficient) condition for the equivalent optimization problem with the Lagrange multiplier $\lambda_n(i)$:

$$\begin{aligned}&\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\big)\boldsymbol{M}_n(i) - \kappa_n(i)\frac{\boldsymbol{\Sigma}_n(i)\boldsymbol{u}_n(i)}{S_n(i)} - \lambda_n(i)\mathbf{1} = 0, \\ &\mathbf{1}^T\boldsymbol{u}_n(i) = 1,\end{aligned}$$

where $S_n(i) = S_n(i, \boldsymbol{u}_n(i)) := \sqrt{\boldsymbol{u}_n^T(i)\boldsymbol{\Sigma}_n(i)\boldsymbol{u}_n(i)}$.

The unique solution of this system is then given by

$$\boldsymbol{u}_n^*(i) = \frac{\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\big)}{\kappa_n(i)} S_n^*(i)\left(\boldsymbol{\Sigma}_n^{-1}(i)\boldsymbol{M}_n(i) - \frac{b_n(i)\boldsymbol{\Sigma}_n^{-1}(i)\mathbf{1}}{a_n(i)}\right) + \frac{\boldsymbol{\Sigma}_n^{-1}(i)\mathbf{1}}{a_n(i)},$$

where $S_n^*(i) = S_n(i, \boldsymbol{u}_n^*(i))$.

We can obtain $S_n^*(i)$ in the same way as in **Step 1** (or **Step 2**). Thus, we have

$$S_n^*(i) = \sqrt{\frac{\frac{1}{a_n(i)}}{1 - \frac{g_n(i)}{\kappa_n^2(i)}\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\big)^2}} := f_n(i),$$

provided that $\kappa_n(i) > \sqrt{g_n(i)\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\big)^2}$.

Furthermore, by using the optimal strategy, (5.32), (5.33) and (5.34) we obtain the value function

$$\begin{aligned}V(n, x, i) &= xA_n(i) + C_n(i)\big(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A}_{n+1}}(i)\big) + \sum_{m=n+1}^{N-1} \overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m}}(i) \\ &\quad + \sum_{m=n+1}^{N-2} \overline{\boldsymbol{Q}^{m-(n+1)}\boldsymbol{Q}_{\boldsymbol{C}_m}\boldsymbol{Q}_{\boldsymbol{A}_{m+1}}}(i).\end{aligned}$$

This completes the proof. $\square$

5.3.2 The Multiperiod Portfolio Selection Scheme

From the proof of Theorem 5.3.1, we see that when the separable multiperiod selection criterion of MSD type is used for portfolio selection, there exists an optimal strategy if:

1. The investor's risk aversion parameter is above a given lower bound.
2. For all market states $i \in S$ and all periods $n = 0, ..., N-1$ the wealth of the investor is positive.

The first condition is crucial. The investor has to be risk averse enough in order to obtain an optimal strategy (recall that the larger the risk aversion parameter, the more risk averse the investor is). We note that for any $n = 0, ..., N-1$, one has to specify their future risk aversion $\kappa_{n+1}(\theta_{n+1}), ..., \kappa_{N-1}(\theta_{N-1})$ for all market states $\theta_{n+1}, ..., \theta_{N-1}$ in order to obtain the lower bound for the current risk aversion parameter κ_n. This reflects the time consistency idea, according to which the investor maintains a risk aversion consistently through time.

```
set abandon = false;
for n = N-1, ..., 0 do
    for θ_n = 1, ..., k do
        set W_n = 1;
        calculate κ̂_n(θ_n) by using (5.14);
        choose an δ_n(θ_n) > 0;
        set κ_n(θ_n) = κ̂_n(θ_n) + δ_n(θ_n);
        calculate u_n(θ_n) by using (5.15);
        calculate p_n(u_n, θ_n) = P(W^u_{n+1} > 0);
        if p_n(u_n, θ_n) > 1 - exp(-δ_n) then
            keep the strategy u_n(θ_n);
        else
            abandon = true;
        end
    end
end
if abandon == false then
    take the investment;
else
    abandon the investment;
end
```

Algorithm 5.3.1: Multiperiod MSD Portfolio Selection Scheme

Mathematically, we could remove the second condition, however, to reflect the reality, this is a must. Of course, there is no guarantee that the wealth of the investor always stays positive, although this may not be a big issue in practice (see Remark 8 in Wu (2013)). However, the whole point of portfolio selection is to make a decision. Depending on the risk aversion, the investor may still wish to take the risk and go for the investment, provided that his wealth stays above zero with a high probability (under our optimal strategy). This leads to Algorithm 5.3.1.

By Algorithm 5.3.1, our portfolio selection process proceeds in the following way. Over the period $[n, n+1]$, and given a market state θ_n, we pick a number $\delta_n(\theta_n) > 0$ in such a way that $\kappa_n(\theta_n) > \hat{\kappa}_n(\theta_n)$ holds. Next, with our strategy in (5.15), we calculate the probability $p_n(\boldsymbol{u}_n, \theta_n)$ that the wealth at the end of this period is positive for every 1 dollar which we invest at the beginning of this period. Thanks to the **Scaling Property**, this will be enough to determine whether to abandon the investment. Finally, we choose a threshold equal to $\big(1 - \exp(-\delta_n)\big)$. If the probability is larger than this threshold, we keep our strategy, otherwise we give up the investment. The choice of threshold is arbitrary. We only need an increasing function (since the larger the risk aversion parameter, the more risk averse the investor is) whose range is between 0 and 1.

5.3.3 Optimal Conditional Expectation and Conditional Variance of Terminal Wealth

Apart from the optimal strategy, we derive the optimal conditional expectation and conditional variance of the terminal wealth. For the purpose of this section, we introduce some additional definitions. For $n = 0, ..., N-1$, $i \in S$, define

$$D_n(i) = \frac{(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i)) f_n(i) g_n(i)}{\kappa_n(i)} + \frac{b_n(i)}{a_n(i)}, \quad \hat{D}_n(i) = 2C_n(i)D_n(i), \quad \tilde{C}_n(i) = C_n^2(i),$$

$$\boldsymbol{D_n} = (D_n(1), ..., D_n(k)), \quad \boldsymbol{\hat{D}_n} = (\hat{D}_n(1), ..., \hat{D}_n(k)), \quad \boldsymbol{\tilde{C}_n} = (\tilde{C}_n(1), ..., \tilde{C}_n(k)),$$

$$G_n(i) = \frac{(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i)) f_n(i) g_n(i)}{\kappa_n(i)} \left(\frac{(1 + \overline{\boldsymbol{Q}}_{\boldsymbol{A_{n+1}}}(i)) f_n(i)}{\kappa_n(i)} + D_n(i) + \frac{b_n(i)}{a_n(i)} \right) + \frac{a_n(i) + b_n^2(i)}{a_n^2(i)}, \qquad \boldsymbol{G_n} = (G_n(1), ..., G_n(k)).$$

Proposition 5.3.2. *Given $W_0 = x > 0$, and $\theta_0 = i \in S$, the optimal conditional expectation and conditional second moments of the terminal wealth are given by*

$$\mathbb{E}_{0,x,i}(W_N^*) = \alpha(i)x + \beta(i), \tag{5.35}$$
$$\mathbb{E}_{0,x,i}\big((W_N^*)^2\big) = \gamma(i)x^2 + \delta(i)x + \eta(i), \tag{5.36}$$

where

$$\alpha(i) = D_0(i)\overline{\prod_{n=1}^{N-1} \boldsymbol{Q}_{\boldsymbol{D}_n}}(i), \quad \beta(i) = C_0(i)\overline{\prod_{n=1}^{N-1} \boldsymbol{Q}_{\boldsymbol{D}_n}}(i) + \sum_{m=1}^{N-1} \overline{\boldsymbol{Q}^{m-1}\boldsymbol{Q}_{\boldsymbol{C}_m} \prod_{n=m+1}^{N-1} \boldsymbol{Q}_{\boldsymbol{D}_n}}(i),$$

$$\gamma(i) = G_0(i)\tau(i), \quad \delta(i) = \hat{D}_0(i)\tau(i) + D_0(i)\psi(i), \quad \eta(i) = \tilde{C}_0(i)\tau(i) + C_0(i)\psi(i) + \lambda(i),$$

$$\tau(i) = \overline{\prod_{n=1}^{N-1} \boldsymbol{Q}_{\boldsymbol{G}_n}}(i), \qquad \psi(i) = \sum_{\ell=1}^{N-1} \overline{\prod_{m=1}^{\ell-1} \boldsymbol{Q}_{\boldsymbol{D}_m} \boldsymbol{Q}_{\hat{\boldsymbol{D}}_\ell} \prod_{n=\ell+1}^{N-1} \boldsymbol{Q}_{\boldsymbol{G}_n}}(i),$$

$$\lambda(i) = \sum_{m=1}^{N-1} \overline{\boldsymbol{Q}^{m-1} \boldsymbol{Q}_{\tilde{\boldsymbol{C}}_m} \prod_{n=m+1}^{N-1} \boldsymbol{Q}_{\boldsymbol{G}_n}}(i) + \sum_{\ell=1}^{N-2} \sum_{m=\ell+1}^{N-1} \overline{\boldsymbol{Q}^{\ell-1} \boldsymbol{Q}_{\boldsymbol{C}_\ell} \prod_{t=\ell+1}^{m-1} \boldsymbol{Q}_{\boldsymbol{D}_t} \boldsymbol{Q}_{\hat{\boldsymbol{D}}_m} \prod_{n=m+1}^{N-1} \boldsymbol{Q}_{\boldsymbol{G}_n}}(i).$$

Proof. For every $n = 0, ..., N-1$, we have

$$W_{n+1}^* \quad = \quad W_n^* \boldsymbol{R}_{n+1}^T(\theta_n) \boldsymbol{u}_n^*(\theta_n) + C_n(\theta_n).$$

Taking conditional expectation with respect to the condition $(W_0 = x, \theta_0 = i, \theta_1, ..., \theta_n)$, and by taking into account (5.15) together with the conditional independence between W_n^* and $\boldsymbol{R}_{n+1}$, we obtain

$$\begin{aligned} & \mathbb{E}\Big(W_{n+1}^* | W_0 = x, \theta_0 = i, \theta_1, ..., \theta_n\Big) \\ = \quad & D_n(\theta_n) \mathbb{E}\Big(W_n^* | W_0 = x, \theta_0 = i, \theta_1, ..., \theta_{n-1}\Big) + C_n(\theta_n). \end{aligned} \tag{5.37}$$

Using (5.37) and backward recursion, we find that

$$\begin{aligned} & \mathbb{E}\Big(W_N^* | W_0 = x, \theta_0 = i, \theta_1, ..., \theta_{N-1}\Big) \\ = \quad & x \prod_{n=0}^{N-1} D_n(\theta_n) + \sum_{m=0}^{N-1} \left(C_m(\theta_m) \prod_{n=m+1}^{N-1} D_n(\theta_n) \right). \end{aligned}$$

Iterated conditional expectation then implies

$$\begin{aligned} \mathbb{E}_{0,x,i}(W_N^*) \quad = \quad & x D_0(i) \mathbb{E}_{0,x,i}\left(\prod_{n=1}^{N-1} D_n(\theta_n) \right) + \sum_{m=1}^{N-1} \mathbb{E}_{0,x,i}\left(C_m(\theta_m) \prod_{n=m+1}^{N-1} D_n(\theta_n) \right) \\ & + C_0(i) \mathbb{E}_{0,x,i}\left(\prod_{n=1}^{N-1} D_n(\theta_n) \right). \end{aligned}$$

By Lemma 3 in Wu and Li (2012), we have

$$\mathbb{E}_{0,x,i}\left(\prod_{n=1}^{N-1} D_n(\theta_n) \right) \quad = \quad \overline{\prod_{n=1}^{N-1} \boldsymbol{Q}_{\boldsymbol{D}_n}}(i),$$

and for all $m = 1, ..., N-1$ we have

$$\mathbb{E}_{0,x,i}\left(C_m(\theta_m) \prod_{n=m+1}^{N-1} D(\theta_n) \right) \quad = \quad \overline{\boldsymbol{Q}^{m-1} \boldsymbol{Q}_{\boldsymbol{C}_m} \prod_{n=m+1}^{N-1} \boldsymbol{Q}_{\boldsymbol{D}_n}}(i).$$

This then yields (5.35).

To compute the conditional second moments, we compute the square of the optimal wealth process:

$$(W_{n+1}^*)^2 \quad = \quad \Big(W_n^* \boldsymbol{R}_{n+1}^T(\theta_n)\boldsymbol{u}_n^*(\theta_n) + C_n(\theta_n)\Big)^2.$$

Again, we take conditional expectation with respect to the condition $(W_0 = x, \theta_0 = i, \theta_1, ..., \theta_n)$. By (5.15) and a similar independence argument as in calculating the conditional expectation we obtain

$$\begin{aligned}
& \mathbb{E}\Big((W_{n+1}^*)^2 | W_0 = x, \theta_0 = i, \theta_1, ..., \theta_n\Big) \\
= \; & G_n(\theta_n)\mathbb{E}\Big((W_n^*)^2 | W_0 = x, \theta_0 = i, \theta_1, ..., \theta_{n-1}\Big) \\
& + \hat{D}_n(\theta_n)\mathbb{E}\Big(W_n^* | W_0 = x, \theta_0 = i, \theta_1, ..., \theta_{n-1}\Big) + \tilde{C}_n(\theta_n).
\end{aligned} \tag{5.38}$$

Using (5.38) and backward recursion, we find that

$$\begin{aligned}
& \mathbb{E}\Big((W_N^*)^2 | W_0 = x, \theta_0 = i, \theta_1, ..., \theta_{N-1}\Big) \\
= \; & x^2 \prod_{n=0}^{N-1} G_n(\theta_n) + x\Bigg(\hat{D}_0(\theta_0) \prod_{n=1}^{N-1} G_n(\theta_n) \\
& + D_0(\theta_0) \sum_{\ell=1}^{N-1} \Bigg(\prod_{m=1}^{\ell-1} D_m(\theta_m)\hat{D}_\ell(\theta_\ell) \prod_{n=\ell+1}^{N-1} G_n(\theta_n)\Bigg)\Bigg) \\
& + \tilde{C}_0(\theta_0) \prod_{n=1}^{N-1} G_n(\theta_n) + \sum_{m=1}^{N-1} \Bigg(\tilde{C}_m(\theta_m) \prod_{n=m+1}^{N-1} G_n(\theta_n)\Bigg) \\
& + C_0(\theta_0) \sum_{\ell=1}^{N-1} \Bigg(\prod_{m=1}^{\ell-1} D_m(\theta_m)\hat{D}_\ell(\theta_\ell) \prod_{n=\ell+1}^{N-1} G_n(\theta_n)\Bigg) \\
& + \sum_{\ell=1}^{N-2} \Bigg(C_\ell(\theta_\ell) \sum_{m=\ell+1}^{N-1} \hat{D}_m(\theta_m) \prod_{t=\ell+1}^{m-1} D_t(\theta_t) \prod_{n=m+1}^{N-1} G_n(\theta_n)\Bigg).
\end{aligned}$$

In the above expression, we may arrange and combine terms in different ways. However, we have chosen this form deliberately, so that when we calculate this conditional expectation later, we can directly apply Lemma 3 in Wu and Li (2012). Next, by iterated conditional expectation, we have

$$\begin{aligned}
\mathbb{E}_{0,x,i}\Big(\big(W_N^*\big)^2\Big) \quad = \quad & x^2 G_0(i)\mathbb{E}_{0,x,i}\Bigg(\prod_{n=1}^{N-1} G_n(\theta_n)\Bigg) + x\hat{D}_0(i)\mathbb{E}_{0,x,i}\Bigg(\prod_{n=1}^{N-1} G_n(\theta_n)\Bigg) \\
& + xD_0(i) \sum_{\ell=1}^{N-1} \mathbb{E}_{0,x,i}\Bigg(\prod_{m=1}^{\ell-1} D_m(\theta_m)\hat{D}_\ell(\theta_\ell) \prod_{n=\ell+1}^{N-1} G_n(\theta_n)\Bigg) \\
& + \tilde{C}_0(i)\mathbb{E}_{0,x,i}\Bigg(\prod_{n=1}^{N-1} G_n(\theta_n)\Bigg) + \sum_{m=1}^{N-1} \mathbb{E}_{0,x,i}\Bigg(\tilde{C}_m(\theta_m) \prod_{n=m+1}^{N-1} G_n(\theta_n)\Bigg)
\end{aligned}$$

$$+C_0(i)\sum_{\ell=1}^{N-1}\mathbb{E}_{0,x,i}\left(\prod_{m=1}^{\ell-1}D_m(\theta_m)\hat{D}_\ell(\theta_\ell)\prod_{n=\ell+1}^{N-1}G_n(\theta_n)\right)$$
$$+\sum_{\ell=1}^{N-2}\sum_{m=\ell+1}^{N-1}\mathbb{E}_{0,x,i}\left(C_\ell(\theta_\ell)\prod_{t=\ell+1}^{m-1}D_t(\theta_t)\hat{D}_m(\theta_m)\prod_{n=m+1}^{N-1}G_n(\theta_n)\right). \tag{5.39}$$

Similarly, as in calculating $\mathbb{E}_{0,x,n}(W_N^*)$, we apply Lemma 3 in Wu and Li (2012), which then yields

$$\mathbb{E}_{0,x,i}\left(\prod_{n=1}^{N-1}G_n(\theta_n)\right)=\overline{\prod_{n=1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}}(i),$$
$$\mathbb{E}_{0,x,i}\left(\prod_{m=1}^{\ell-1}D_m(\theta_m)\hat{D}_\ell(\theta_\ell)\prod_{n=\ell+1}^{N-1}G_n(\theta_n)\right)=\overline{\prod_{m=1}^{\ell-1}\boldsymbol{Q}_{\boldsymbol{D}_m}\boldsymbol{Q}_{\hat{\boldsymbol{D}}_\ell}\prod_{n=\ell+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_m}}(i),$$
$$\mathbb{E}_{0,x,i}\left(\tilde{C}_m(\theta_m)\prod_{n=m+1}^{N-1}G_n(\theta_n)\right)=\overline{\boldsymbol{Q}^{m-1}\boldsymbol{Q}_{\tilde{\boldsymbol{C}}_m}\prod_{n=m+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}}(i),\ \text{for } m=1,...,N-1,$$

and

$$\mathbb{E}_{0,x,i}\left(C_\ell(\theta_\ell)\prod_{t=\ell+1}^{m-1}D_t\hat{D}_m(\theta_m)\prod_{n=m+1}^{N-1}G_n(\theta_n)\right)$$
$$=\overline{\boldsymbol{Q}^{\ell-1}\boldsymbol{Q}_{\boldsymbol{C}_\ell}\prod_{t=\ell+1}^{m-1}\boldsymbol{Q}_{\boldsymbol{D}_t}\boldsymbol{Q}_{\hat{\boldsymbol{D}}_m}\prod_{n=m+1}^{N-1}\boldsymbol{Q}_{\boldsymbol{G}_n}}(i)$$
$$\text{for } m=\ell+1,...,N-1,\ell=1,...,N-2.$$

Finally, by substituting the above results into (5.39), we obtain (5.36). This completes the proof. □

A direct consequence of this proposition leads to

$$\begin{aligned}Var_{0,x,i}(W_N^*) &= \mathbb{E}_{0,x,i}\big((W_N^*)^2\big)-\Big(\mathbb{E}_{0,x,i}(W_N^*)\Big)^2\\ &= \big(\gamma(i)-\alpha^2(i)\big)x^2+\big(\delta(i)-2\alpha(i)\beta(i)\big)x+\big(\eta(i)-\beta(i)^2\big).\end{aligned}$$

5.4 Numerical Illustrations

We collect (weekly) stock prices of ANZ, BHP, and Telstra, which traded on Australian Securities Exchange [1], during two periods of time [2]. We calculate their expected returns and the corresponding covariance matrices. Thus, in all examples that follow, we assume that there are two market states ($S=1,2$) and three risky assets ($d=3$). We further assume that the investment horizon consists of five periods ($N=5$), and for $n=0,...,N-1$, the expected returns $\mathbf{m}_n$ and covariance matrices $\mathbf{\Sigma}_n$ are equal. Thus, we drop the dependence on time, and for each market state we simply write

[1]Data obtained from Yahoo Finance https://au.finance.yahoo.com/
[2]Two periods are 01 01 2008 - 23 05 2011 and 02 01 2012 - 25 05 2015.

State 1:

$$\mathbf{m}(1) = \begin{pmatrix} -0.000566 \\ 0.000180 \\ -0.002364 \end{pmatrix}, \quad \boldsymbol{\Sigma}(1) = \begin{pmatrix} 0.002203 & 0.000848 & 0.000330 \\ 0.000848 & 0.002971 & 0.000248 \\ 0.000330 & 0.000248 & 0.000884 \end{pmatrix},$$

and

State 2:

$$\mathbf{m}(2) = \begin{pmatrix} 0.002425 \\ -0.000633 \\ 0.003943 \end{pmatrix}, \quad \boldsymbol{\Sigma}(2) = \begin{pmatrix} 0.000537 & 0.000261 & 0.000195 \\ 0.000261 & 0.000730 & 0.000105 \\ 0.000195 & 0.000105 & 0.000311 \end{pmatrix}.$$

Moreover, without loss of generality, we will assume that the initial wealth is 1 dollar, i.e., $W_0 = 1$.

We use three examples to illustrate our model. In the first example, we investigate the effect of different risk aversion parameters. In the second example, we investigate the effect of cash injections. In the last example, we investigate the difference between investment strategy with and without cash injections and offtakes under multiple market states.

Example 5.4.1. *We assume that there is only one market state, which we take to be State 2. There are no cash injections or offtakes.*

	1st period	2nd period	3rd period	4th period	last period
constant $\kappa_n(2)$ through time					
$\kappa_n(2)$	1	1	1	1	1
$\hat{\kappa}_n(2)$	0.789058	0.630508	0.472666	0.315436	0.158743
increasing $\kappa_n(2)$ through time					
$\kappa_n(2)$	1	1.1	1.6	2	2.2
$\hat{\kappa}_n(2)$	0.781398	0.623197	0.467012	0.312385	0.158743
decreasing $\kappa_n(2)$ through time					
$\kappa_n(2)$	2.2	2	1.6	1.1	1
$\hat{\kappa}_n(2)$	0.784313	0.628606	0.472401	0.315436	0.158743
random $\kappa_n(2)$ through time					
$\kappa_n(2)$	1.6	1	2	2.2	1.1
$\hat{\kappa}_n(2)$	0.782971	0.624450	0.469305	0.315180	0.158743

Table 5.1: Risk Aversions and its Lower Bounds

Now, we apply Algorithm 5.3.1. We choose four sequences of $\delta_n(2)$ such that we have: a constant $\kappa_n(2)$ through time, an increasing sequence of $\kappa_n(2)$ through time, a decreasing sequence of $\kappa_n(2)$ through time, and a random sequence of $\kappa_n(2)$ through time. This is summarized in Table 5.1. It seems that the influence on the actual value of the lower bound $\hat{\kappa}_n(2)$ is quite small, and the general trend remains the same for all four cases. Thus, from now on we will consider only a constant $\kappa_n(2)$ through time, and will simply write $\kappa(2)$ instead. One may wonder whether we should take this investment. In order to make this decision, we need to specify the distribution of the return $\boldsymbol{r}(2)$. For example, if we assume

$\boldsymbol{r}(2) \sim N(\mathbf{m}(2), \mathbf{\Sigma}(2))$ i.e., normally distributed with mean $\mathbf{m}(2)$ and covariance matrix $\mathbf{\Sigma}(2)$, Algorithm 5.3.1 then implies that we will take this investment under each of these four cases. From now on, we will always assume that we are given a distribution of return such that we will take the given investment, and focus on the analysis of our model.

$\kappa(2) = 1$					
	1st period	2nd period	3rd period	4th period	last period
ANZ	0.124591	0.127184	0.128699	0.129819	0.130761
BHP	-0.502347	-0.242418	-0.090490	0.021798	0.116286
Telstra	1.377756	1.115234	0.961791	0.848383	0.752952
$\kappa(2) = 3$					
	1st period	2nd period	3rd period	4th period	last period
ANZ	0.130190	0.130494	0.130788	0.131073	0.131353
BHP	0.059000	0.089497	0.118914	0.147541	0.175632
Telstra	0.810810	0.780009	0.750298	0.721386	0.693015

Table 5.2: Optimal Strategies ($\kappa(2) = 1$ and $\kappa(2) = 3$)

Now, fix two risk aversion parameters, $\kappa(2) = 1$ and $\kappa(2) = 3$. The former represents a more risky choice than the latter. For each case, we compute the optimal strategies (see Table 5.2), and the optimal conditional expectations and conditional variances of the investor's terminal wealth for different time length of investment (see Table 5.3).

$\kappa(2) = 1$					
	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
$\mathbb{E}_{0,1,2}(W_N^*)$	1.006053	1.010941	1.015149	1.018850	1.022123
$Var_{0,1,2}(W_N^*)$	0.000672	0.001105	0.001448	0.001749	0.002031
$\kappa(2) = 3$					
	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
$\mathbb{E}_{0,1,2}(W_N^*)$	1.003475	1.006822	1.010044	1.013144	1.016123
$Var_{0,1,2}(W_N^*)$	0.000271	0.000540	0.000807	0.001073	0.001341

Table 5.3: Optimal Conditional Expectations and Conditional Variances of Investor's Terminal Wealth

In addition, we plot the optimal strategies for each case (see Figure 5.2 - Figure 5.3). As expected, in the less risky case (i.e., $\kappa(2) = 3$), investor's optimal strategy is more conservative along the way. This is reflected in the fact that he keeps a large and steady proportion in the less risky asset (i.e., Telstra). We can also see this by plotting the single period optimal expectation of the portfolio value and its variance for each time (see Figure 5.4 - Figure 5.5). The more risky case (i.e., $\kappa(2) = 1$) dominates the less risky case (i.e., $\kappa(2) = 3$) through time in the sense that it always has a higher expected return and a higher standard deviation.

Example 5.4.2. *We assume that there is only one market state which we take to be State 2. Assume that there is a constant cash injection of \$0.1 at each periods. The risk aversion parameter is assumed to be* $\kappa(2) = 3$.

As we have seen in Theorem 5.3.1, the presence of cash (in the form of our model) does not affect the optimal strategies, however, it affects the optimal conditional expectation and the conditional variance of investor's terminal wealth. We calculate investor's optimal

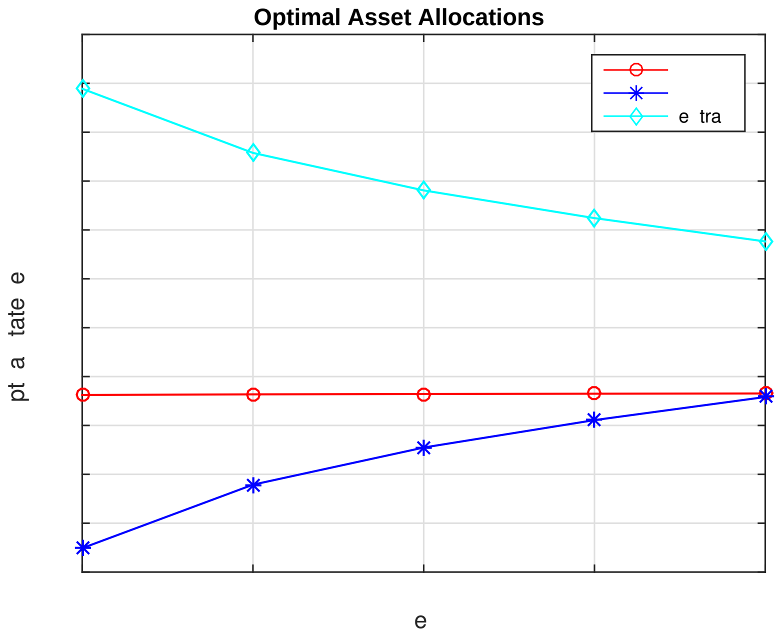

Figure 5.2: Optimal Strategies ($\kappa(2) = 1$)

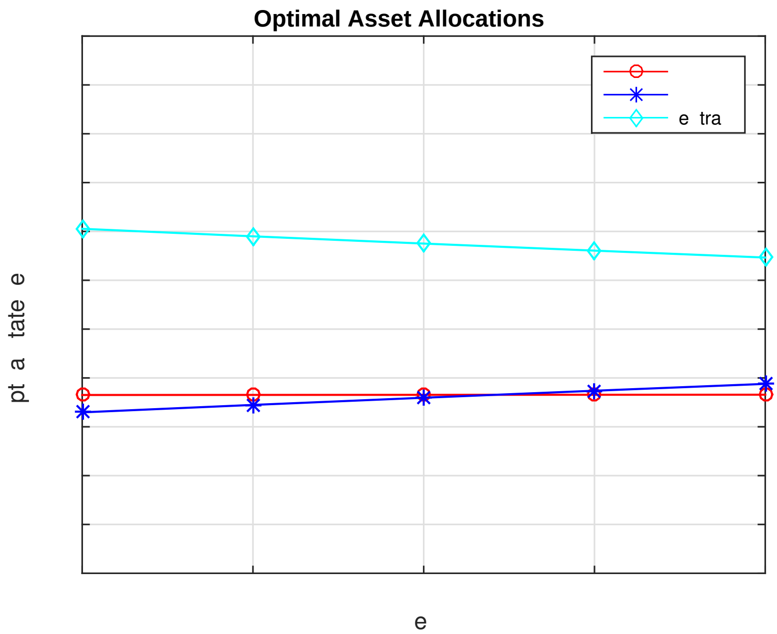

Figure 5.3: Optimal Strategies ($\kappa(2) = 3$)

conditional expectation and conditional variance of the terminal wealth, which yields

$$E_{0,1,2}(W_5^*) = 1.519203, \quad Var_{0,1,2}(W_5^*) = 0.001943.$$

Let us compare this result with the case of no cash injections (the last column of Table 5.3 when $\kappa(2) = 3$). After subtracting the extra cash injections and by ignoring the time value of money, we see that the optimal conditional expectation and variance of investor's wealth are higher if there are cash injections. This makes sense in this example. As he injects extra amount into his portfolio he will invest more in these risky assets. On one hand, this increases the expectation of his wealth and on the other hand, he also exposes himself to uncertain environment.

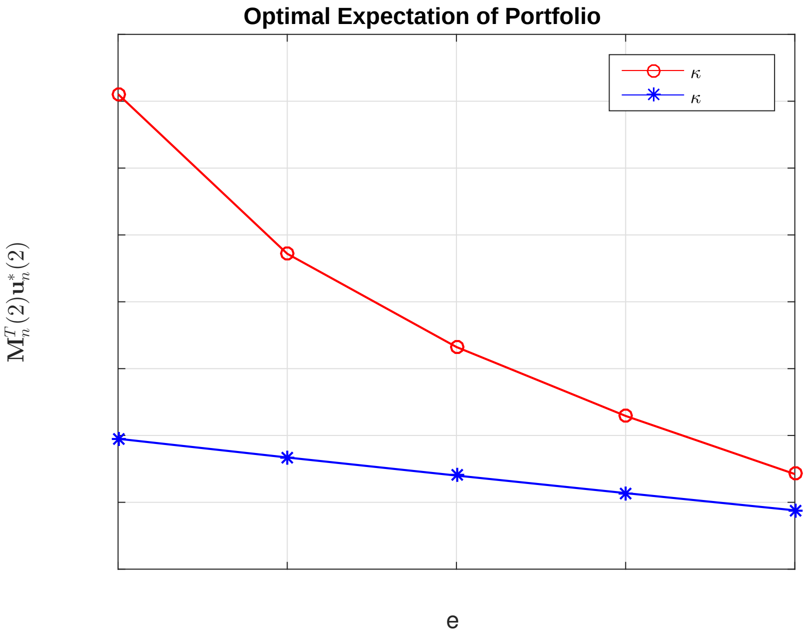

Figure 5.4: Single Period Optimal Expectations of Portfolio ($\kappa(2) = 1$, and $\kappa(2) = 3$)

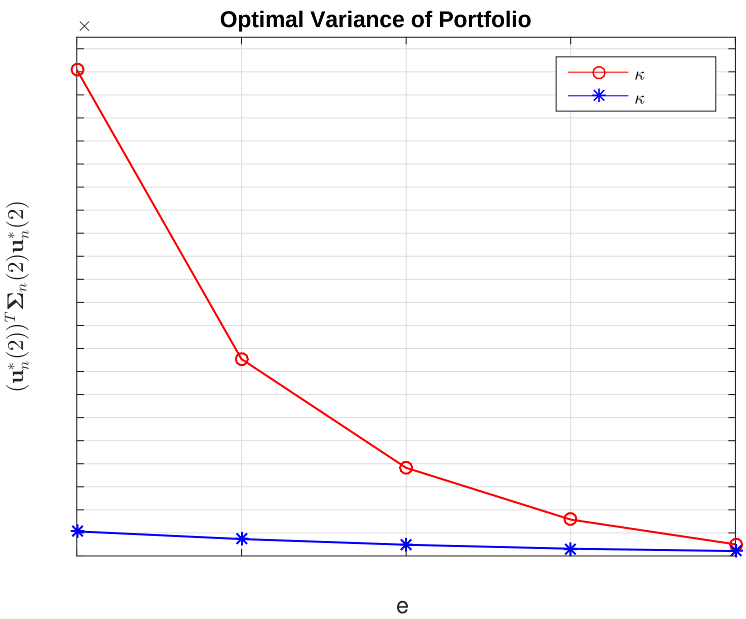

Figure 5.5: Single Period Optimal Variances of Portfolio ($\kappa(2) = 1$, and $\kappa(2) = 3$)

It is worth noting that in this example we only consider cash injections but no takeoffs. It becomes an interesting question if the investor has a plan at the beginning of his investment horizon to withdraw certain (deterministic) amount of money at some future time. The question is how much he is able to withdraw. It may happen that the investor wishes to withdraw large amount, but it turns out that he does not have enough money in his portfolio (at the time he wishes to withdraw). In Table 5.3, we have calculated the optimal conditional expectation and conditional variance of investor's terminal wealth for different time length of investment. Thus, we can then calculate the corresponding MSD values of the wealth position. This is summarized in Table 5.4. Similarly to the single period MSD selection criterion (see Section 5.2.1), for every risk aversion parameter $\kappa(2)$, we can attach a probability p such that

$$p = \mathbb{P}\Big(W_N^* \geq E_{0,1,2}(W_N^*) - \kappa(2) * \sqrt{Var_{0,1,2}(W_N^*)}\Big).$$

Thus, for any given distribution of asset returns, one can calculate such p. This provides some confidence level to the investor about the amount he would be able to withdraw without going bankrupt.

	$\kappa(2) = 3$				
	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
$\mathbb{E}_{0,1,2}(W_N^*) - \kappa(2) * \sqrt{Var_{0,1,2}(W_N^*)}$	0.954089	0.937108	0.924821	0.914874	0.906264

Table 5.4: Optimal Conditional MSD Values of Investor's Terminal Wealth

Example 5.4.3. *We assume that there are two market states. If the market is in State 1 ("bad state")*[3]*, we take out* 0.1 *dollar, and if the market is in State 2 ("good state"), we add* 0.1 *dollar. The risk aversion is assumed to be* 3 *for both states and all time. We also assume the transition matrix to be given by*

$$\boldsymbol{Q} = \begin{pmatrix} 0.1 & 0.9 \\ 0.15 & 0.85 \end{pmatrix}.$$

For each market state, we calculated the optimal strategies which we summarized in Table 5.5.

	State 1				
	1st period	2nd period	3rd period	4th period	last period
ANZ	0.184722	0.180831	0.176960	0.173108	0.169264
BHP	0.158328	0.153187	0.148075	0.142987	0.137910
Telstra	0.656950	0.665982	0.674965	0.683906	0.692827
	State 2				
	1st period	2nd period	3rd period	4th period	last period
ANZ	0.130198	0.130500	0.130791	0.131075	0.131353
BHP	0.059815	0.090056	0.119259	0.147704	0.175632
Telstra	0.809986	0.779444	0.749950	0.721221	0.693015

Table 5.5: Optimal Strategies (State 1 and State 2)

Next, let us have a look how market transitions affect the choice of the optimal strategy. Given we are in State 1 at the beginning (i.e., $n = 0$), we follow the corresponding optimal strategy for State 1. When we move to the second period, if the market state switches to the State 2, we use the corresponding optimal strategy for State 2 (by treating the initial state as State 2 and dealing with a four period problem). We continue this process until

[3]This has been classified as a "bad state" since majority of the assets in this state have less expected return and all assets have larger standard deviation than in State 2".

we select the optimal strategy for each time period.

To give a concrete example, let us assume that the market has the following transitions.

$$\text{State } 1 \to \text{State } 2 \to \text{State } 2 \to \text{State } 1 \to \text{State } 2.$$

The corresponding optimal strategy will be

$$
\begin{aligned}
& \boldsymbol{u}_0^*(1) = \begin{pmatrix} 0.184722 \\ 0.158328 \\ 0.656950 \end{pmatrix} \quad \text{(1st period of State 1)} \\
\to \quad & \boldsymbol{u}_1^*(2) = \begin{pmatrix} 0.130500 \\ 0.090056 \\ 0.779444 \end{pmatrix} \quad \text{(2nd period of State 2)} \\
\to \quad & \boldsymbol{u}_2^*(2) = \begin{pmatrix} 0.130791 \\ 0.119259 \\ 0.749950 \end{pmatrix} \quad \text{(3rd period of State 2)} \\
\to \quad & \boldsymbol{u}_3^*(1) = \begin{pmatrix} 0.173108 \\ 0.142987 \\ 0.683906 \end{pmatrix} \quad \text{(4th period of State 1)} \\
\to \quad & \boldsymbol{u}_4^*(2) = \begin{pmatrix} 0.131353 \\ 0.175632 \\ 0.693015 \end{pmatrix} \quad \text{(last period of State 2).}
\end{aligned}
$$

Next, we calculate the optimal conditional expectations and conditional variances of the investor's terminal wealth with cash injections and offtakes (as described in Example 5.4.3):

$$
\begin{aligned}
E_{0,1,1}(W_5^*) &= 1.202311, \quad Var_{0,1,1}(W_5^*) = 0.020094, \\
E_{0,1,2}(W_5^*) &= 1.399611, \quad Var_{0,1,2}(W_5^*) = 0.021796.
\end{aligned}
$$

and without cash injections and offtakes:

$$
\begin{aligned}
E_{0,1,1}(W_5^*) &= 1.008384, \quad Var_{0,1,1}(W_5^*) = 0.002028, \\
E_{0,1,2}(W_5^*) &= 1.013296, \quad Var_{0,1,2}(W_5^*) = 0.001612.
\end{aligned}
$$

By taking extra positions during the "good" market state and reducing positions during the "bad" state, we see that he obtains a higher expected wealth (as it can be checked, by ignoring the time value of money, this holds even after subtracting the expected cash injections). However, like in Example 5.4.2, this has created more variations (variance has increased significantly).

5.5 Conclusion

In this second part of this thesis, we develop a portfolio selection scheme where a multiperiod selection criterion of MSD type is considered. We perform the analysis in a market of risky assets and obtain a closed form optimal strategy in which market transitions and intermediate cash injections and offtakes are allowed. This model forms a good base to further study multiperiod portfolio selection problem in which a multiperiod selection criterion is of a type from the TIPH risk measure class. It is also interesting to see the effect of short selling and transaction costs to our model. These questions are left as future areas of research.

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