The Generalized Continuous Wavelet Transform on Hilbert Modules

by

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A thesis submitted for the degree of

Doctor of Philosophy

at the University of New South Wales

January 2008

Abstract

The construction of the generalized continuous wavelet transform (GCWT) on Hilbert spaces is a special case of the coherent state transform construction, where the coherent state system arises as an orbit of an admissible vector under a strongly continuous unitary representation of a locally compact group.

In this thesis we extend this construction to the setting of Hilbert C^*-modules. In particular, we define a coherent state transform and a GCWT on Hilbert modules. This construction gives a reconstruction formula and a resolution of the identity formula analogous to those found in the Hilbert space setting. Moreover, the existing theory of standard normalized tight frames in finite or countably generated Hilbert modules can be viewed as a discrete case of this construction.

We also show that the image space of the coherent state transform on Hilbert module is a reproducing kernel Hilbert module. We discuss the kernel and the intertwining property of the group coherent state transform.

Acknowledgements

I would like to take this opportunity to express my gratitude to those who have supported me during the completion of this thesis.

First of all, I would like to thank God who provides me with the opportunity to do my research while being a mom of my two beloved children Naura and Hanan, and being a wife of my dearest husband Arifin. I would like to thank my Mum and Dad, my mother-in-law and all my sisters and brothers for all their support, hope, and pray for my completion.

I would like to express my gratitude to my supervisor Prof. Tony Dooley whose guidance, suggestions, and encouragement helped me in all the time of research for and writing of this thesis. Once more, I thank Prof. Dooley for his inspirations, especially on some examples in the discussion of $\mathbb{L}^2(X, \mathcal{A})$ and also for his ongoing support.

I am also very thankful to my co-supervisor Dr. Astrid An Huef for her interest, helpful advice and for answering my questions especially on C^*-algebra theory. I thank Dr. Brian Jefferies for his suggestion reading on vector valued integral. I also thank Prof. Hartmut Führ for answering my questions on his works on GCWT on Hilbert space. I also thank my fellow postgrads, especially Venta, Raed and Dalia.

I would like to thank my other brothers and sisters for all of their support and prayer, in particular, Mira and Bu Evy. I thank Teddy for his help on picture for example 4.1.2.11, Alhaji and Pak Mico for the computer matters.

Last but not least, I thank Lorraine and Bernadette for all of their help and support.

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Contents

Preface

Given a system $(\eta_x)_{x\in X}$ in a Hilbert space $\mathcal{H}$, indexed by a measure space X, we can study the map V_η, which maps each $\varphi \in \mathcal{H}$ to a bounded continuous function $V_\eta\varphi$ on X, defined by

$$V_\eta\varphi(x) = (\varphi \mid \eta_x)\,.$$

We call $(\eta_x)_{x\in X}$ a coherent state system whenever $V_\eta\varphi$ is measurable for each φ. A coherent state system is admissible if V_η is an isometry of $\mathcal{H}$ into $L^2(X)$.

The construction of the generalized wavelet transform (GCWT) is a special case of the coherent state transform construction. In this case, the coherent state system arises as an orbit of an admissible vector under a strongly continuous unitary representation $(\pi, \mathcal{H}_\pi)$ of a locally compact group G.

A Hilbert C^*-module is a natural generalization of Hilbert space as a complete inner product space. While the inner product in Hilbert space is complex valued, in a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$, the inner product is $\mathcal{A}$-valued. As is well-known, Hilbert C^*-modules behave like Hilbert spaces in many respects. But there is a fundamental difference: not every closed submodule has an orthogonal complement. This requires extra care in developing an analogous theory for Hilbert C^*-modules.

It is clearly interesting to consider whether the GCWT construction can be generalized to the Hilbert module setting. In this thesis, we will show

how this can be done. Specifically we will show how the construction may be generalized to separable Hilbert modules over a unital C^*-algebra $\mathcal{A}$.

There are some results in the literature which generalize wavelet transforms in the sense of wavelet frames to the setting of Hilbert modules, [24, 64]. In this thesis, we generalize the continuous version of the wavelet transform to the setting of Hilbert modules and also generalize results due to Führ on the GCWT on Hilbert spaces.

The generalization of the continuous wavelet transforms to the setting of Hilbert modules is not immediately obvious, since we need a weaker integral than the Bochner integral to work with the functions in the Hilbert modules $\mathbb{L}^2(X, \mathcal{A})$. We are able to achieve this by defining what we call the $\mathcal{A}$-integral on $\mathcal{A}$-valued functions.

In fact, the discussion on Hilbert module $\mathbb{L}^2(X, \mathcal{A})$ is interesting by itself. We show that extra care is needed in generalizing the standard Hilbert space ℓ_2 to the standard Hilbert module $\mathbb{H}_{\mathcal{A}}$, by providing examples, which then motivate us to develop the $\mathcal{A}$-integral theory.

Chapter 1 is a preliminary chapter, that provides notations, conventions and basic results on integration and Hilbert space theory.

Chapter 2 contains some basic results on C^*-algebras and Hilbert C^*-module theory which will be used in our generalization of continuous wavelet transform into the Hilbert module setting.

Chapter 3 gives an introduction to wavelet transform theory. The first part of this chapter contains the historical background of the continuous wavelet transform on the real line, and reviews some results related to wavelet transforms, particularly those related to Hilbert C^*-modules. The second part of this chapter includes a discussion on GCWT on Hilbert space based on work of Führ.

Finally, our results are presented in chapter 4 and 5. In chapter 4, we use results on C^*-algebras and the theory of Bochner integrals to generalize the results which we will need, from the Hilbert space $L^2(X)$ to the Hilbert module $\mathbb{L}^2(X, \mathcal{A})$.

In chapter 5, we use the results in chapter 4 to generalize the theory of chapter 3 to the setting of Hilbert modules. We are able to recapture the results of [24, 64] in the case where the Hilbert module is a finitely or countably generated Hilbert module and X is a discrete space.

Chapter 1

Preliminaries

In this chapter, we introduce some notation and conventions that we use in the entire thesis.

We use this chapter to state some definitions, and notation from basic concepts of the theory of groups and their representations. We include the theory of Hilbert spaces and their tensor products. Finally, we also list some results related to the measurability and integrability of vector-valued integrals.

1.1 Notation and conventions

Let $\mathcal{X}$ be a locally compact space, i.e. a topological space which is locally compact, Hausdorff (for every two distinct points $x, y \in \mathcal{X}$ there are neighborhoods E of x and F of y such that $E \cap F = \emptyset$) and second countable (has a countable basis).

If $Y \subset \mathcal{X}$, the intersection of all closed sets containing Y is called the **closure** of Y and denoted by $\overline{Y}$. If $\overline{Y} = \mathcal{X}$, we say Y is **dense** in $\mathcal{X}$. We call $\mathcal{X}$ **separable** if it contains a countable dense subset.

We say a group G is a **locally compact group** if it is a topological group (a group with a topology such that the group operations are continuous) which is also a locally compact space.

Notation. From now on, we reserve the symbol G for locally compact group, unless stated otherwise.

In chapter 3 we will discuss the original wavelet transform, which is related to the following locally compact group.

Example 1.1.0.1. The ax+b group. Let $G = \mathbb{R} \rtimes \mathbb{R}^+$. then G is a locally compact group under the product topology, and under the multiplication (group law) defined by:

$$(b, a)\,(b', a') = (b + ab', aa')$$

where the inverse is given by

$$(b, a)^{-1} = \left(-\frac{b}{a}, \frac{1}{a}\right).$$

Notation. Some examples in chapter 4 use the space of continuous functions. We denote by $C\,(\mathcal{X})$ the space of continuous functions on $\mathcal{X}$.

We can endow $C\,(\mathcal{X})$ by the **supremum norm**

$$\|f\| = \sup_{x\in\mathcal{X}} |f(x)|\,. \tag{1.1}$$

Suppose that $\mathcal{V}$ and $\mathcal{W}$ are linear spaces over $\mathbb{F}$. By a **linear operator** (or **linear transformation**) from $\mathcal{V}$ into $\mathcal{W}$, we mean a mapping $T : \mathcal{V} \to \mathcal{W}$ such that

$$T(\alpha\varphi + \beta\psi) = \alpha T\varphi + \beta T\psi.$$

whenever $\varphi, \psi \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{F}$ (the notation $T : \mathcal{V} \to \mathcal{W}$ indicates that T is defined on $\mathcal{V}$ and takes values in $\mathcal{W}$; it can be read T, from $\mathcal{V}$ to $\mathcal{W}$).

In this case, the **kernel** of T is the linear subspace $\{\varphi \in \mathcal{V} : T\varphi = 0\}$ of $\mathcal{V}$ and denote it by $\ker(T)$. We call the space $T(\mathcal{V}) = \{T\varphi : \varphi \in \mathcal{V}\}$ the **image** (or **range**) and denote it by $\mathrm{range}(T)$. In general we call the space where a mapping T is defined on, the **domain** of T and denoted by $\mathcal{D}(T)$.

1.2 Measure spaces

Since we will deal with integration of scalar or vector-valued functions on a locally compact group, we will recall here some basics on measure spaces.

We will begin with the term **σ-algebra** Σ of a set X, that is a nonempty set of subsets of X (including the set X itself) which is closed under taking complements and countable unions. If X is a topological space, and the σ-algebra Σ is generated by the family of open sets in X it is also called **Borel σ-algebra** on X, and its elements are called **measurable (or Borel) sets**. If Σ contains all singletons, we say the σ-algebra **separates points**. A set X together with its σ-algebra Σ is called a **measurable space**, and denote it by (X, Σ) . Let us mention some examples of (Borel) σ-algebra. For a locally compact space, it is given by smallest σ-algebra containing all the open sets. For a countable set, it is given by the power set of the space.

We define a **measure** μ as a non-negative and completely additive set function. If the σ-algebra is a Borel σ-algebra then the measure is called a **Borel measure**. We define a **(Borel) measure space** (X, Σ, μ), as a space X together with a measure μ on its (Borel) σ-algebra Σ. We also define a set **μ-nullset** as $\{E \subset X \mid \mu(E) = 0\}$. With this terminology, we say a subset E in the σ-algebra Σ of a measure space (X, Σ, μ), is called μ-measurable (or sometimes just measurable). If $\mu(X) < \infty$ (which implies that $\mu(E) < \infty$ for all $E \in \Sigma$ since $\mu(X) = \mu(E) + \mu(E^c)$), μ is called **finite**. If $X = \bigcup_1^\infty E_j$

where $E_j \in \Sigma$ and $\mu(E_j) < \infty$ for all j, μ is called **σ-finite**.

If a certain relation holds for all points $x \in X \setminus E$ where E is a μ-nullset then we say that this relation holds **μ-almost everywhere** (or sometimes just almost everywhere) on X.

If $X = G$ is a group, then we say a measure μ on G is **left-invariant** if $\mu(xE) = \mu(E)$ for all $x \in G$ and $E \subset G$ measurable. We say that μ is **right-invariant** if $\mu(Ex) = \mu(E)$ for all $x \in G$ and $E \subset G$ measurable.

A measure μ on a topological space X is called **regular** if for each open set $E \subset X$, $\mu(E) = \sup\{\mu(K) \mid K \subset E \text{ and } K \text{ is compact }\}$ and for each Borel (measurable) set F, $\mu(F) = \inf\{\mu(E) \mid F \subset E \text{ and } E \text{ is open}\}$. We call a regular Borel measure which is finite on all compact sets, a **Radon measure**. For a further discussion on Radon measures, we refer to Chapter 7 in [18].

Notation. We reserve X to denote a measure space (X, Σ, μ), where X is also a locally compact space.

The main function space in Chapter 3 is the Hilbert space of square integrable scalar-valued functions on X with respect to a given measure. We denote it by $L^2(X)$.

1.2.1 Haar measure

In Chapter 3 we discuss Führ's generalized wavelet transform, which is a certain class of coherent state systems arising from certain representations of locally compact groups endowed with Haar measure. In chapter 5 we use the same groups to generalize the concept. Therefore, here, we include some definitions and basic results about Haar measure.

Definition 1.2.1.1. A nonzero Radon measure μ on a topological group

G is a **left Haar measure** if it is left-invariant, and it is a **right Haar measure** if it is right-invariant. In what follows, we shall reserve the words Haar measure for the left Haar measure.

The important results are the following:

Theorem 1.2.1.2. *Every locally compact group G has a unique Haar measure up to scalar multiplication.*

Remark 1.2.1.3. Since our G is always second countable, then its Haar measure is certainly σ-finite.

Theorem 1.2.1.4. *If μ and λ are left Haar measures on G, there exists $c \in (0, \infty)$ such that $\lambda = c\mu$.*

The proofs can be found in Folland's book [17].

Once we have a Haar measure μ, we can define a right Haar measure related to it by $\tilde{\mu}(E) = \mu(E^{-1})$. The relation between these two Haar measure is given by the **modular function** $\Delta : G \to \mathbb{R}_+$, which is defined by:

$$\Delta(x) = \frac{\mu(Ex)}{\mu(E)} \quad \text{for } E \text{ any measurable set}$$

Note that μ_x which is defined by $\mu_x(E) = \mu(Ex)$ is also a Haar measure, therefore the existence of a number $\Delta(x)$ is guaranteed by theorem 1.2.1.4 of the uniqueness of the Haar measure, and is independent of the choice of μ. Moreover, a group G is said to be **unimodular** if $\Delta(x) = 1$ for all $x \in G$, i.e. the left Haar measure of G is also a right Haar measure. It is easy to see that abelian groups and discrete groups are unimodular. In fact, the modular function is a continuous homomorphism from G to $\mathbb{R}$, see proposition 2.24 in [17]. Moreover, by proposition 2.31 in [17] we can view the modular function as

$$\Delta = \frac{d\mu}{d\tilde{\mu}}. \tag{1.2}$$

By equation 1.2, we obtain the following formula:

$$\int_G f(x)d\mu(x) = \int_G f(x^{-1})\Delta(x^{-1})d\mu(x). \tag{1.3}$$

Example 1.2.1.5. Let G be the $ax+b$ group. Then $dadb/|a|^2$ is a left Haar measure of G, and $dadb/|a|$ is a right Haar measure of G so that $1/|a|$ is the modular function.

1.3 Hilbert spaces

We will assume that all vector spaces in this thesis are complex vector spaces, unless stated otherwise. We call a vector space equipped with a complex inner product, an inner product space. We call an inner product space $\mathcal{H}$ that is complete in the norm induced from the inner product, a Hilbert space. We denote the space of bounded operators on Hilbert space $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. Furthermore, $\dim \mathcal{H}$ denotes the dimension of $\mathcal{H}$ which is the cardinality of an arbitrary orthonormal basis of $\mathcal{H}$. For separable Hilbert spaces $\mathcal{H}$, $\dim \mathcal{H} \in \mathbb{N} \cup \{\infty\}$,

Notation. We reserve $\mathcal{H}$ to denote a Hilbert space and $I_{\mathcal{H}}$ as the identity operator in $\mathcal{B}(\mathcal{H})$.

There are two important topologies that we use in $\mathcal{B}(\mathcal{H})$. The first is the **strong operator topology**, where the sequence (T_n) converges to T if and only if $T_n\varphi$ converges to $T\varphi$ for any $\varphi \in \mathcal{H}$. The other one is the **weak operator topology**, where sequence (T_n) converges to T if and only if $(\varphi \mid T_n\psi)$ converges to $(\varphi \mid T\psi)$ for any $\varphi, \psi \in \mathcal{H}$.

1.3.1 Unbounded operators on Hilbert spaces

In the discussion of coherent state systems in Hilbert space in chapter 3, we will deal with coefficient operators which are possibly unbounded. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and T be a linear operator with domain of definition $\mathcal{D}(T)$ a linear submanifold (not necessarily closed), of $\mathcal{H}$ into $\mathcal{K}$. If $\mathcal{D}(T)$ is dense in $\mathcal{H}$, T is said to be **densely defined**. We define the **graph** of T as the set $\mathcal{G}(T) \equiv \{h \oplus k \in \mathcal{H} \oplus \mathcal{K} : h \in \mathcal{D}(T)\}$. The operator T is said to be **closed operator** if its graph is closed in $\mathcal{H} \oplus \mathcal{K}$.

1.3.2 Unitary representations

In general, when people discuss the theory of unitary representations of locally compact groups, they are talking about representations of the group as unitary operators on Hilbert spaces. This is what we recall in this section (In chapter 2, we recall the notion of group representations in Hilbert modules). Our main reference for this section is [17].

Definition 1.3.2.1. Let $\mathcal{H}$ be a Hilbert space, and define

$$\mathcal{U}(\mathcal{H}) := \{U \in \mathcal{B}(\mathcal{H}) \mid U^*U = UU^* = I_{\mathcal{H}}\}.$$

Then $\mathcal{U}(\mathcal{H})$ is a topological group, with the strong operator topology.

Remark 1.3.2.2. The strong and weak operator topologies coincide in $\mathcal{U}(\mathcal{H})$.

Definition 1.3.2.3. A representation π of G in a Hilbert space $\mathcal{H}_\pi$ is called **a unitary representation** if it is a homomorphism from G into the group $\mathcal{U}(\mathcal{H}_\pi)$ of unitary operators on $\mathcal{H}_\pi$ that is continuous in the strong operator topology.

Equivalently, if π is a unitary representation of G in a Hilbert space $\mathcal{H}_\pi$, then the map $\pi : G \to \mathcal{U}(\mathcal{H}_\pi)$ satisfies $\pi(xy) = \pi(x)\pi(y)$ and $\pi(x^{-1}) =$

$\pi(x)^{-1} = \pi(x)^*$, and $x \mapsto \pi(x)\varphi$ is continuous from G to $\mathcal{H}_\pi$ for any $\varphi \in \mathcal{H}_\pi$. We call $\mathcal{H}_\pi$ the **representation space** of π, and call the dimension of $\mathcal{H}_\pi$ the **dimension** or **degree** of representation π, which is possibly infinite. Since the weak and strong operator topologies coincide on $\mathcal{U}(\mathcal{H}_\pi)$, the continuity requirement is equivalent to the condition that the map $x \mapsto (\varphi \mid \pi(x)\eta)$ is continuous for all $x \in G$ and $\varphi, \eta \in \mathcal{H}_\pi$.

Example 1.3.2.4. Let G be a locally compact group. Let μ be its left invariant measure. The **left regular representation** λ_G is defined by:

$$\lambda_G(x)f(y) = f(x^{-1}y).$$

Now, we will list some standard terminology and properties related to unitary representations. Let π and σ be unitary representations, a bounded operator $T : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ is called an **intertwining operator** for π and σ if $T\pi(x) = \sigma(x)T$ for every $x \in G$. We say that π and σ are **disjoint** if there is no nonzero intertwining operator in either direction. In the case there exist T which is unitary, we say that π and σ are **unitarily equivalent**. The set of all intertwining operators for π and σ which is denoted by $\mathcal{C}(\pi, \sigma)$, contains unitary operators U, such that $\sigma(x) = U\pi(x)U^{-1}$. If $\pi = \sigma$ then we write $\mathcal{C}(\pi)$ for $\mathcal{C}(\pi, \pi)$, and call it **commutant** or **commuting algebra** of π. It can be shown that this commuting algebra is closed under the weak operator topology, also closed under taking adjoint, hence it is a Von Neumann algebra.

Remark 1.3.2.5. In the rest of this section we shall use the term representation to refer to unitary representation unless stated otherwise.

Now, let $\mathcal{K}$ be a closed subspace of $\mathcal{H}_\pi$. We call $\mathcal{K}$ an **invariant subspace** for π if $\pi(x)\mathcal{K} \subset \mathcal{K}$ for all $x \in G$. In this case, its **orthogonal complement** $\mathcal{K}^\perp$ is also an invariant subspace and therefore π is the direct sum of $\pi_\mathcal{K}$

and $\pi_{\mathcal{K}^\perp}$. The restriction of π to $\mathcal{K}$ defines a representation of G on $\mathcal{K}$, denoted by $\pi|_{\mathcal{K}}$ and is called a **subrepresentation** of π. Furthermore, we can define the direct sum: $\bigoplus \pi_i$, as a representation π on $\mathcal{H} = \bigoplus_i \mathcal{H}_{\pi_i}$ defined by $\pi(x)\left(\sum \varphi_i\right) = \sum \pi_i(x)\varphi_i$. If π has invariant subspaces which are nontrivial (neither 0 nor $\mathcal{H}_\pi$), we say π is **reducible**, otherwise π is said to be **irreducible**. A vector $\varphi \in \mathcal{H}_\pi$ is **cyclic** if $\pi(G)\varphi$ spans a dense subspace of $\mathcal{H}_\pi$. A representation having such vectors is called a **cyclic representation.** If every nonzero vector is cyclic, then the representation π is irreducible. We write $\sigma < \pi$ if a representation σ is unitarily equivalent to a subrepresentation of π.

1.4 Tensor products

In this section, we discuss the notion of vector spaces tensor products and Hilbert space tensor product, which the second term is the completion of vector space tensor product in the norm induced from the inner product. Similar techniques are used to define the Hilbert module tensor product $\mathcal{H}\widehat{\otimes}\mathcal{A}$ in example 2.2.4.25.

1.4.1 Vector space tensor products

We state the definition of vector space tensor product. For detailed information on this, the reader may consult [38].

Definition 1.4.1.1. A vector space L and a bilinear mapping $\otimes$ of H, K into L is a **tensor product** of H, K and we write $\mathrm{L} = \mathrm{H} \otimes \mathrm{K}$ if the pair $(\mathrm{L}, \otimes)$ satisfy the condition:
If L' is any vector space and $\times'$ is a bilinear mapping of $\mathrm{H} \times \mathrm{K}$ into L', then there exists a unique linear mapping T of L into L' such that $T\left(\varphi \otimes \psi\right) =$

$\varphi \times' \psi$. In this case we say that the pair $(\otimes, \mathrm{L})$ is universal for bilinear mappings of H and K.

For $\varphi \in \mathrm{H}$ and $\psi \in \mathrm{K}$, we call the element $\varphi \otimes \psi$ in L an **elementary tensor**. The tensor product is unique, and a basis of the tensor product L comes from the basis of H and K. Formally, we will state this in the following lemma.

Lemma 1.4.1.2. *Let* H *and* K *be the vector spaces with the basis* $\{\varphi_\gamma\}$ *and* $\{\psi_\lambda\}$ *respectively. Then the set* $\{\varphi_\gamma \otimes \psi_\lambda\}$ *is a basis for the tensor space* $\mathrm{H} \otimes \mathrm{K}$.

1.4.2 Hilbert space tensor products

It is a natural expectation that when the vector spaces are also Hilbert spaces, the algebraic tensor product can be densely embedded in a Hilbert space. And fortunately, that is the case. We will include here the definition of the Hilbert space tensor product, and some facts related to basis of the space. Our main references for this section are [58] and [66].

Theorem 1.4.2.1. *Let* $\mathcal{H}$ *and* $\mathcal{K}$ *be Hilbert spaces. There exist an inner product on* $\mathcal{H} \otimes \mathcal{K}$ *defined as:*

$$(\varphi_1 \otimes \psi_1 \mid \varphi_2 \otimes \psi_2) = (\varphi_1 \mid \varphi_2)\,(\psi_1 \mid \psi_2) \quad \varphi_1\varphi_2 \in \mathcal{H}, \psi_1\psi_2 \in \mathcal{K}$$

Definition 1.4.2.2. For any two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, We define the **Hilbert space tensor product** $\mathcal{H}\widehat{\otimes}\mathcal{K}$ as the completion of the vector space tensor product in the norm induced by the inner product defined above.

There are several facts to note. Suppose that $\mathcal{H}, \mathcal{K}$ are Hilbert spaces. For any $\varphi \in \mathcal{H}, \psi \in \mathcal{K}$,

$$\|\varphi \otimes \psi\| = \|\varphi\|\,\|\psi\|\,.$$

Moreover, if $\{\varepsilon_\gamma\}$ and $\{\nu_\lambda\}$ are orthonormal basis for $\mathcal{H}$ and $\mathcal{K}$ respectively, then $\{\varepsilon_\gamma \otimes \nu_\lambda\}$ is an orthonormal basis for $\mathcal{H}\widehat{\otimes}\mathcal{K}$.

1.5 Vector-valued functions

Let X denote a measure space (X, Σ, μ) σ-algebra Σ and σ-finite measure μ and let $\mathcal{V}$ be a Banach space. We denote by $\mathcal{V}'$ the space of continuous linear functionals on $\mathcal{V}$. Here we will provide some theory related to functions on X having values in $\mathcal{V}$. The main references for this discussion are [36, 17, 76]

1.5.1 Measurable functions

Definition 1.5.1.1. A function f is **countably-valued** if it assumes at most a countable set of non zero distinct values in $\mathcal{V}$, each on a measurable set. If f has finite number of distinct non-zero values, each on a measurable set of finite measure, it is called **simple function**.

Definition 1.5.1.2. A vector-valued function f is **weakly measurable** in X if the scalar valued function $v'(f(x))$ is **μ-measurable**, for each $v' \in \mathcal{V}'$. We said that a vector-valued f is **strongly-measurable** if there exists a sequence of countably-valued functions converging almost everywhere in X to f.

Below are some useful results from [36, Corollary 1, Corollary 2, Theorem 3.5.4, and the following paragraph].

Lemma 1.5.1.3. *A function f is strongly measurable if and only if it is the uniform limit almost everywhere of a sequence of countably-valued functions.*

Lemma 1.5.1.4. *If $\mathcal{V}$ is separable, then strong and weak measurability are equivalent.*

Theorem 1.5.1.5. *(1) If f and g are strongly measurable and α_1 and α_2 are scalar, then $\alpha_1 f + \alpha_2 g$ is strongly measurable. (2) If f is the limit almost everywhere of a sequence of strongly measurable functions, then f is strongly measurable. (3) If $\mathcal{V}$ is a Banach algebra and f, g are strongly measurable, then the product fg is strongly measurable.*

1.5.2 Vector-valued integral

Weak Integral

The notion of weak integral was originally known as the Pettis Integral. The integral is defined using reflexivity of the Hilbert space and the independent result of Gelfand and Dunford which is given in the following theorem, [36, Theorem 3.7.1.].

Theorem 1.5.2.1. *If f is weakly measurable and if $v'(f) \in L^1(X, \mathcal{V})$ for each $v' \in \mathcal{V}'$, then there exists $w'' \in \mathcal{V}''$ such that*

$$w''(v') = \int_X v'(f(x))\, d\mu(x) \tag{1.4}$$

for all $v' \in \mathcal{V}'$.

The theorem allows us to define $w'' = \int_X f(x)\, d\mu(x)$. In general, $\mathcal{V}'' \neq \mathcal{V}$. When equality holds, the integral is called the weak integral. The following is the definition of the weak integral, c.f [36, Definiton 3.7.1] and [17, Appendix 3].

Definition 1.5.2.2. A function f on X to $\mathcal{V}$ is **weakly integrable** if and only if $v'(f(\cdot))$ is Lebesgue integrable for all $v' \in \mathcal{V}'$, and there is an element w of $\mathcal{V}$ such that

$$v'(w) = \int v'(f(x))\, d\mu(x). \tag{1.5}$$

By definition,

$$\int f(x)\, d\mu(x) = w. \tag{1.6}$$

Remark 1.5.2.3. The Riesz representation theorem holds in a reflexive space $\mathcal{V}$: we can represent each functional linear in its dual $\mathcal{V}'$ as a unique element of $\mathcal{V}$. We will write an element in $\mathcal{V}'$ represented by a $v \in \mathcal{V}$, $v' \in \mathcal{V}'$. Hence, we can rewrite equation 1.5 as:

$$\left(v \mid \int_X f(x)\, d\mu(x)\right) = \int (v \mid f(x))\ d\mu(x). \tag{1.7}$$

The Bochner integral

The Bochner integral is one of the generalization of Lebesgue integral for vector-valued functions. We follow the definitions from [36].

Definition 1.5.2.4. A countably-valued function f on X to $\mathcal{V}$ is integrable if and only if $\|f(\,.\,)\|$ is Lebesgue integrable. By definition

$$\mathbf{B}\text{ - }\int_E f(x)\, d\mu(x) = \sum_{k=1}^{\infty} v_k \mu(E_k \cap E)$$

where $f(x) = v_k$ on $E_k \in \Sigma(k = 1, 2, 3...)$.

Definition 1.5.2.5. A function f on X to $\mathcal{V}$ is Bochner integrable if and only if there exists a sequence of countably-valued integrable functions $\{f_n\}$ converging almost everywhere to f and such that

$$\lim_{n\to\infty} \int \|f(x) - f_n(x)\|\ d\mu(x) = 0. \tag{1.8}$$

By definition,

$$\mathbf{B}\text{ - }\int_E f(x)\, d\mu(x) = \lim_{n\to\infty} \mathbf{B}\text{ - }\int_E f_n(x)\, d\mu(x) \tag{1.9}$$

for each $E \in \Sigma$.

Notation. We use $L^1(X, \mathcal{V})$ to denote the class of Bochner integrable $\mathcal{V}$-valued functions on X.

It is clear from the definition that the space of countably $\mathcal{V}$-valued Bochner integrable functions on X is dense in $L^1(X, \mathcal{V})$. Now, every simple $\mathcal{V}$-valued function is countably-valued Bochner integrable. Furthermore, each countably-valued function can be approximated by simple functions, therefore the space of simple functions is dense in the space of countably-valued functions. Hence it is also dense in $L^1(X, \mathcal{V})$.

There is a useful characterization of Bochner integrable functions [36, Theorem 3.7.4]

Theorem 1.5.2.6. *A necessary and sufficient condition that f on X to $\mathcal{V}$ be Bochner integrable is that f be strongly measurable and that*

$$\int_X \|f(x)\| \; d\mu(x) < \infty.$$

Now, the following holds, [36, Theorem 3.7.5].

Theorem 1.5.2.7. *If f and g are in $L^1(X, \mathcal{V})$ and α_1 and α_2 are scalars, then $\alpha_1 f(x) + \alpha_2 g(x) \in L^1(X, \mathcal{V})$ and*

$$\mathbf{B}\text{ - }\int_X (\alpha_1 f(x) + \alpha_2 g(x)) = \alpha_1 \mathbf{B}\text{ - }\int_X f(x)\; d\mu(x) + \alpha_2 \mathbf{B}\text{ - }\int_X g(x)\; d\mu(x).$$

If the norm of a function f is defined by

$$\|f\|_1 = \int_X \|f(x)\| \quad d\mu(x) \tag{1.10}$$

then $L^1(X, \mathcal{V})$ will be a Banach space, [36, Theorem 3.7.6 and Theorem 3.7.8]

Theorem 1.5.2.8. *If $f \in L^1(X, \mathcal{V})$, then*

$$\left\| \mathbf{B}\text{ - }\int_X f(x) d\mu(x) \right\| \leq \int_X \|f(x)\| \; d\mu(x) \tag{1.11}$$

Theorem 1.5.2.9. *The set of functions $L^1(X, \mathcal{V})$ becomes a Banach space if we identify functions which differ only on sets of measure zero.*

Remark 1.5.2.10. If for $1 \leq p < \infty$ we define the norm $\|f\|_p = \left(\int \|f(x)\|^p\right)^{1/p}$, we can define Banach spaces $L^p(X, \mathcal{V})$ relative to the norm $\|\cdot\|_p$. See [35, Exercise 7.5.3 and 7.5.4]. These spaces are also known as Lebesgue spaces and the space of simple functions is also dense in these spaces relative to the norm $\|\cdot\|_p$. In the case $p = 2$, we call the Banach space $L^2(X, \mathcal{V})$ as the space of **$\mathcal{V}$-valued norm square integrable functions**.

We also have a Dominated Convergence Theorem for Bochner integrals similar to the Lebesgue integral case, [35, Theorem 7.5.9].

Theorem 1.5.2.11. ***Dominated Convergence Theorem for Bochner Integrals*** *If $f_n \in L^1(X, \mathcal{V})$ and $\|f_n(x)\| \leq g(x)$ for $n = 1, 2, \cdots$, where g is Bochner integrable over X, and if f_n converges to f a.e. then $f \in L^1(X, \mathcal{V})$ and*

$$\lim_{n\to\infty} \int_X \|f(x) - f_n(x)\| \; d\mu(x) = 0.$$

In particular,

$$\mathbf{B}\text{ -}\int_X f(x)\; d\mu(x) = \lim_{n\to\infty} \mathbf{B}\text{ -}\int_X f_n(x)\; d\mu(x).$$

Remark 1.5.2.12. A more general theory of vector-valued integrals is discussed in [16, Section II.5.] and [14, Chapter III].

Chapter 2

Hilbert C^*-modules

Since the aim of the thesis is to generalize the continuous wavelet transform from the setting of Hilbert spaces, to the setting of Hilbert C^*-modules (a natural generalization of Hilbert space for which the inner product takes its values in a C^*-algebra instead of the complex number), we will include here the basics of the theory of C^*-algebras and some results related to Hilbert C^*-modules. Our references for this chapter are [49, 12, 75, 66, 39, 43].

2.1 C^*-Algebras

We will introduce here the basic theory of C^*-algebras. We begin with a discussion of involutive algebras.

2.1.1 Basics

Definition 2.1.1.1. Let $\mathcal{A}$ be a linear space over a field $\mathbb{F}$. We say $\mathcal{A}$ is an **associative algebra** over $\mathbb{F}$ if for each $a, b, c \in \mathcal{A}$ and $\alpha \in \mathbb{F}$

1. $a(bc) = (ab)c$;

2. $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$;

3. $\alpha(ab) = (\alpha a)b = a(\alpha b)$.

We say that $\mathcal{A}$ is **commutative** if $ab = ba$ for all $a, b \in \mathcal{A}$. We say that $\mathcal{A}$ is **unital** if there exists an element $1_{\mathcal{A}} \in \mathcal{A}$ such that $1_{\mathcal{A}} a = a = a 1_{\mathcal{A}}$ for all $a \in \mathcal{A}$.

In this thesis we will always assume that our associative algebra $\mathcal{A}$ is over the complex field $\mathbb{C}$, unless stated otherwise and shall write algebra for the associative algebra. Sometimes, an algebra possesses a norm, and it may also be a Banach space.

Definition 2.1.1.2. An algebra $\mathcal{A}$ is said to be a **normed algebra** if it is a normed linear space such that

$$\|ab\| \leq \|a\| \, \|b\| \, .$$

Certainly, if $\mathcal{A}$ is unital, then $\|1_{\mathcal{A}}\| = 1$. If $\mathcal{A}$ is a Banach space relative to this norm, $\mathcal{A}$ is said to be **Banach algebra**.

Example 2.1.1.3. The space $L^1(X)$ of absolutely integrable functions on a locally compact group is a Banach algebra.

Some algebras have involution mappings.

Definition 2.1.1.4. We call an algebra $\mathcal{A}$ a *-algebra or an involutive algebra, whenever $\mathcal{A}$ has a bijective mapping $a \mapsto a^*$ from $\mathcal{A}$ to $\mathcal{A}$ such that for any $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$

1. $(a+b)^* = a^* + b^*$;

2. $(ab)^* = b^* a^*$;

3. $(\alpha a)^* = \bar{\alpha} a^*$;

4. $(a^*)^* = a$.

We call such map an **involution**.

For some normed *-algebras, there exists an additional condition, called the C^*-condition.

Definition 2.1.1.5. Let $\mathcal{A}$ be a normed *-algebra. The C^*-condition holds if for any $a \in \mathcal{A}$,

$$\|a^*a\| = \|a\|^2 . \tag{2.1}$$

Remark 2.1.1.6. The C^*-condition ensures that the involution in a C^*-algebra preserves norm: $\|a\|^2 = \|a^*a\| \leq \|a^*\| \|a\|$, hence $\|a\| \leq \|a^*\|$. Replacing a by a^* implies the reverse inequality. Therefore, $\|a\| = \|a^*\|$.

A Banach *-algebra which satisfies the C^*-condition is called a C^***-algebra**, while a normed *-algebra whose norm satisfies the C^*-condition is called a pre-C^*-algebra. Since the C^*-condition ensures that the involution in both C^*-algebras and pre-C^*-algebras preserves the norm, the involution is continuous. There are two important examples of C^*-algebras. First, if $\mathcal{H}$ is a Hilbert space, then $\mathcal{B}(\mathcal{H})$ is a C^*-algebra, with the adjoint operation as its involution. Second, if X is a locally compact Hausdorff space, the algebra $C_0(X)$, the space of continuous functions on X vanishing at infinity, is a C^*-algebra with the involution given by complex conjugation and the supremum norm. Theorem 2.1.1.12 below shows us how important those two C^*-algebras are. Now, we introduce some terminology related to elements of Banach algebra $\mathcal{A}$, with or without involution. First, let us introduce a definition of spectrum.

Definition 2.1.1.7. If $a \in \mathcal{A}$, we say that a complex number α is a **spectral value** of a relative to $\mathcal{A}$ when $a - \alpha 1$ does not have a two-sided inverse in $\mathcal{A}$. The set of spectral values of a is called the **spectrum** of a and is denoted by $\sigma_{\mathcal{A}}(a)$. We define the **spectral radius** $r_{\mathcal{A}}(a)$ by

$$\sup\{|\alpha|\,|\alpha \in \sigma(a)\}.$$

If $a \in \mathcal{A}$, we call a^* the **adjoint** of a, and we say a is **self-adjoint** if $a = a^*$, **normal** if $aa^* = a^*a$, **unitary** if $aa^* = a^*a = 1_{\mathcal{A}}$ and an idempotent element, if $a = a^2$. A self-adjoint idempotent element is called a **projection**. We say that two projections p, q are orthogonal when $pq = 0$, and we denote the orthogonal sum of two projection by $p \oplus q$. If a^*a is a projection, then we say a is a **partial isometry**. It is clear that the unit element $1_{\mathcal{A}}$ is both self adjoint and unitary. The set of all self-adjoint elements of $\mathcal{A}$ is a real vector space, while the unitary elements form a multiplicative group, called the unitary group of $\mathcal{A}$. Each $a \in \mathcal{A}$ can be expressed (uniquely) in the form $a_r + a_{im}i$, where $a_r(=\frac{1}{2}(a+a^*))$ and $a_{im}(=\frac{1}{2}(a-a^*))$ are self-adjoint elements of $\mathcal{A}$. We call a_r and a_{im} the real and imaginary parts of $\mathcal{A}$ respectively.

An element $a \in \mathcal{A}$ is invertible if and only if a^* is, and $(a^{-1})^* = (a^*)^{-1}$. Hence, it is easy to see that

$$\sigma(a^*) = \{\bar{\alpha}|a \in \sigma(a)\},$$

and hence, $r(a^*) = r(a)$. In what follows is an important property of elements of Banach algebras, [39, Theorem 3.2.3].

Theorem 2.1.1.8. *If a is an element of the Banach algebra $\mathcal{A}$ then the spectrum $\sigma(a)$ is a non-empty closed subset of the closed disk in $\mathbb{C}$ with center 0 and radius $\|a\|$.*

Remark 2.1.1.9. It is possible to replace the radius $\|a\|$ by the spectral radius $r(a)$, recalling that the radius spectral is entirely definable in terms of the norm as $lim_{n\to\infty} \|a^n\|^{1/n}$.

Below some properties of the spectrum of elements of a C^*-algebra $\mathcal{A}$.

Proposition 2.1.1.10. *[39, Proposition 4.1.1] Suppose that $a \in \mathcal{A}$.*

1. *If a is normal, $r(a) = \|a\|$.*
2. *If a is a self-adjoint, $\sigma(a)$ is a compact subset of the real line $\mathbb{R}$, and contains at least one of the two real numbers $\pm \|a\|$.*
3. *If a is unitary, $\|a\| = 1$ and $\sigma(a)$ is a compact subset of the unit circle $\{a \in \mathbb{C} |\, |a| = 1\}$.*

If $\mathcal{A}$ and $\mathcal{B}$ are involutive Banach algebras, we say a mapping π from $\mathcal{A}$ to $\mathcal{B}$ is a ***-homomorphism** if it is a homomorphism such that $\pi(a^*) = \pi(a)^*$ for each $a \in \mathcal{A}$. In addition, if π is one to one, it is called a ***-isomorphism**. Furthermore, if both $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, *-homomorphisms do not increase the norm and *-isomorphisms are norm preserving c.f [39, Theorem 4.1.8].

Definition 2.1.1.11. A **representation** of a C^*-algebra $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ is a *-homomorphism of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$.

Now we are ready to include the theorem shows the importance of our examples mentioned above.

Theorem 2.1.1.12. ***(Gelfand-Naĭmark Theorems)***

1. *Let $\mathcal{A}$ be a commutative C^*-algebra. There is a locally compact Hausdorff space X such that $\mathcal{A}$ is isometrically *-isomorphic to $C_0(X)$.*

2. *Let $\mathcal{A}$ be a C^*-algebra. Then $\mathcal{A}$ is isometrically *-isometric to a norm-closed *-subalgebra of the bounded linear operators on some Hilbert space.*

Notation. For the rest of this thesis, $\mathcal{A}$ will be a C^*-algebra over a complex field, unless stated otherwise.

Any subalgebra of a C^*-algebra which is closed under involutions and which is norm closed is also a C^*-algebra. An important subalgebra of $\mathcal{B}(\mathcal{H})$ is $\mathcal{K}(\mathcal{H})$, the algebra of compact operators on a Hilbert space $\mathcal{H}$. In fact, $\mathcal{K}(\mathcal{H})$ is not only a subalgebra but also a two sided ideal of $\mathcal{B}(\mathcal{H})$. There exist a standard characterization of $\mathcal{K}(\mathcal{H})$ using the finite-rank operators, see Proposition 2.1.1.13 below. Recall that the **rank** of an operator T on $\mathcal{H}$ is the dimension of $T(\mathcal{H})$, denoted by dim $T(\mathcal{H})$ and thus a **finite rank operator** T is one with dim $T(\mathcal{H})$ finite.

Proposition 2.1.1.13. *([66], proposition 1.1) Suppose $\mathcal{H}$ is a Hilbert space. Then every finite rank operator is compact; a bounded operator is compact if and only if it is the norm-limit of a sequence of finite rank operators. Indeed, if $h \otimes \bar{k}$ denotes the rank one operator $g \mapsto (g \mid k)\, h$, then*

$$\mathcal{K}(\mathcal{H}) = \overline{\operatorname{span}\left\{h \otimes \bar{k} | h, k \in \mathcal{H}\right\}}$$

2.1.2 Functional calculus

Now, we will introduce the notion of functional calculus. Let $C(\sigma(a))$ be the C^*-algebra of all complex-valued functions on the spectrum $\sigma(a)$. If $a \in \mathcal{A}$ is self-adjoint, then we can define the **functional calculus** for a by a mapping that associates with each f in $C(\sigma(a))$ an element $f(a)$ in $\mathcal{A}$. Below is the existence theorem, see [39, Theorem 4.1.3] for the proof.

Theorem 2.1.2.1. *If a is a self-adjoint element of $\mathcal{A}$, there is a unique continuous mapping $f \mapsto f(a) : C(\sigma(a)) \to \mathcal{A}$ such that for any $f, g \in C(\sigma(a))$ and $\alpha, \beta \in \mathbb{C}$*

1. *$f(a)$ has its elementary meaning when f is a polynomial,*
2. *$\|f(a)\| = \|f\|$,*
3. *$(\alpha f + \beta g)(a) = \alpha f(a) + \beta g(a)$,*
4. *$(fg)(a) = f(a)g(a)$,*
5. *$\bar{f}(a) = (f(a))^*$, and $f(a)$ is self adjoint if and only if f takes real values throughout $\sigma(a)$.*
6. *$f(a)$ is normal,*
7. *$f(a)b = bf(a)$ whenever $b \in \mathcal{A}$ and $ab = ba$.*

2.1.3 Positive elements and ordering structure

We will include here the notion of positivity, since we will use it to study Hilbert C^*-modules. In the Hilbert space case, one property of its inner product use the positivity of its scalars. In the Hilbert C^*- module we will use the positive elements in the C^*-algebra. An element $a \in \mathcal{A}$ is **positive** if a is self-adjoint and $\sigma(a) \subseteq \mathbb{R}^+$. We denote by $\mathcal{A}^+$ the set of all positive elements of $\mathcal{A}$. Below are several characterizations of a positive element $a \in \mathcal{A}^+$. See [39, Theorem 4.2.6] for the proof.

Theorem 2.1.3.1. *If $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$, the following conditions are equivalent:*

1. *$a \in \mathcal{A}^+$.*

2. $a = h^2$ *for a unique* $h \in \mathcal{A}^+$.

3. $a = b^*b$, *for some* $b \in \mathcal{A}$.

Remark 2.1.3.2. The result in 2.1.3.1 (3). which is similar to [58, Section 2.2.] will be used in proving that the form $\langle f, g\rangle$ where f, g are C^*-algebra valued square integrable functions, fulfills the fourth condition for right inner product over a C^*-algebra. That is $\langle f, f\rangle \geq 0$ for every square integrable function f.

When $a \in \mathcal{A}^+$, the element h in condition 2 of Theorem 2.1.3.1 is called the **positive square root** of a and denoted by $a^{1/2}$. In fact, we can used a similar procedure to introduce an element $a^\alpha \in \mathcal{A}^+$, for all real values of α. Given the function f_α in $C(\sigma(a))$ defined by $f_\alpha(x) = x^\alpha$, we can define $a^\alpha \in \mathcal{A}^+$ by $f_\alpha(a)$, when $\alpha > 0$ or for all real α if a is invertible. Note that $a^\alpha a^\beta = a^{\alpha+\beta}$ and $a^1 = a$. If a is invertible, we can define $a^0 = 1$ and $a^{-1} = f_{-1}(a)$ of $\mathcal{A}^+$. We shall list here the properties of $\mathcal{A}^+$. See [39, Theorem 4.2.2.] and also the proof.

Theorem 2.1.3.3. *For all* $\mathcal{A}$,

1. $\mathcal{A}^+ = \{a \in \mathcal{A} | a = a^* \text{ and } \|a - \|a\| 1\| \leq \|a\|\}$;

2. $\mathcal{A}^+$ *is closed in* $\mathcal{A}$;

3. $\alpha a \in \mathcal{A}^+$ *if* $a \in \mathcal{A}^+$ *and* $\alpha \in \mathbb{R}^+$;

4. $a + b \in \mathcal{A}^+$ *if* $a, b \in \mathcal{A}^+$;

5. $ab \in \mathcal{A}^+$ *if* $a, b \in \mathcal{A}^+$ *and* $ab = ba$;

6. *if* $a \in \mathcal{A}^+$ *and* $-a \in \mathcal{A}^+$ *then* $a = 0$.

Notation. Sometimes, we write $a \geq 0$ when $a \in \mathcal{A}^+$.

It is easy to see from Theorem 2.1.3.3 (1) that $1_{\mathcal{A}}$ is in $\mathcal{A}^+$. Now, let $\mathcal{A}_{sa}$ be the set of all self adjoint elements of $\mathcal{A}$. It is a real Banach space which is a partially ordered vector space with a closed positive cone $\mathcal{A}^+$, c.f. [39, page 249]. That is for any $a, b \in \mathcal{A}_{sa}$, $a \leq b$ if and only if $b - a \in \mathcal{A}^+$; and $\mathcal{A}^+ = \{a \in \mathcal{A}_{sa} |\ a \geq 0\}$. Let us give the definition of order unit of the partially ordered vector space. We will prove in Section 4.1.1 the existence of order unit of the space of self adjoint elements of a unital C^*-algebra $\mathcal{A}$. The existence of this order unit will be very useful in constructing an increasing subsequence of positive element in $\mathcal{A}$, which will be used in several proofs of results in the integration of C^*-algebra valued function in Section 4.1.1.

Definition 2.1.3.4. An element i in a partially ordered vector space $\mathcal{V}$ is said to be an **order unit** when given any $v \in \mathcal{V}$ we have $-\alpha i \leq v \leq \alpha i$ for a suitable scalar α. We may choose α to be $\|v\|$.

To prove the existence of order units we need the following proposition from [65, Corollary 20.].

Proposition 2.1.3.5. *Let $\mathcal{A}$ be a unital C^*-algebra with unit $1_{\mathcal{A}}$, and $a \in \mathcal{A}$ is self adjoint. Then $0 \leq a \leq \lambda 1_{\mathcal{A}}$ if and only if $\sigma(a) \subseteq [0, \lambda]$. Thus $0 \leq a \leq \lambda 1_{\mathcal{A}}$ implies $\|a\| \leq \lambda$, and we have $\|b\|^2 1_{\mathcal{A}} - b^* b \geq 0$ for all $b \in \mathcal{A}$.*

In what follows, we will give some more properties of the order structure of $\mathcal{A}_{sa}$, [39, Proposition 4.2.8].

Proposition 2.1.3.6. *Suppose that a and b are self adjoint elements of $\mathcal{A}$.*

a. *If $ab \leq a \leq b$, then $\|a\| \leq \|b\|$.*

b. *If $0 \leq a \leq b$, then $a^{1/2} \leq b^{1/2}$.*

c. *If $0 \leq a \leq b$ and a is invertible, then b is invertible and $b^{-1} \leq a^{-1}$.*

The following is also from [39].

Proposition 2.1.3.7. *Suppose that a is a self-adjoint element of $\mathcal{A}$, a can be expressed in the form $a^+ - a^-$, where $a^+, a^- \in \mathcal{A}^+$ and $a^+a^- = a^-a^+ = 0$. These conditions determine a^+ and a^- uniquely, and $\|a\| = \max(\|a^+\|, \|a^-\|)$.*

Corollary 2.1.3.8. *Each element a of $\mathcal{A}$ is a linear combination of at most four members of $\mathcal{A}^+$.*

2.2 Hilbert modules

A Hilbert C^*-module is a natural generalization of Hilbert space, where the field of scalars is replaced by a C^*-algebra. A discussion of this generalization can be found in [40, 71, 63, 67]. Over the years, there have been many applications of the theory of Hilbert C^*-modules. A brief review can be found inthe preface of [49]. The reader interested in a more detailed bibliography is referred to [21]. While many familiar properties of Hilbert spaces continue to hold in this setting, other properties such as self-duality and decomposition into orthogonal complements no longer hold. A theory of operators on Hilbert modules generalizing the theory of bounded operator on Hilbert space is also available. However, here, the existence of adjoint operators is not automatic.

Throughout this section, $\mathcal{A}$ is a C^*-algebra, with or without a unit, and all our modules will be right $\mathcal{A}$ modules.

2.2.1 Pre-Hilbert modules

An action of an element $a \in \mathcal{A}$ on $\mathbb{H}$ is denoted by $x \cdot a$, where $x \in \mathbb{H}$.

Definition 2.2.1.1. A **pre-Hilbert $\mathcal{A}$-module** is a (right) $\mathcal{A}$-module $\mathbb{H}$ (which is at the same time a complex vector space) equipped with a sesquilinear form $\langle \cdot, \cdot \rangle : \mathbb{H} \times \mathbb{H} \to \mathcal{A}$ which respects the module action and is positive definite:

1. $\langle \varphi, \psi_1 + \psi_2 \rangle = \langle \varphi, \psi_1 \rangle + \langle \varphi, \psi_2 \rangle$, for $\varphi, \psi_1, \psi_2 \in \mathbb{H}$;

2. $\langle \varphi, \psi \cdot a \rangle = \langle \varphi, \psi \rangle a$, for $\varphi, \psi \in \mathbb{H}$, and $a \in \mathcal{A}$

3. $\langle \varphi, \alpha\psi \rangle = \alpha \langle \varphi, \psi \rangle$, for $\varphi, \psi \in \mathbb{H}$, and $\alpha \in \mathbb{C}$;

4. $\langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle^*$, for $\varphi, \psi \in \mathbb{H}$;

5. $\langle \varphi, \varphi \rangle \geq 0$, for $\varphi \in \mathbb{H}$, and $\langle \varphi, \varphi \rangle = 0$ if and only if $\varphi = 0$.

The map $\langle \cdot, \cdot \rangle$ is called an **$\mathcal{A}$-inner product**.

Sometimes, we will write pre-Hilbert module for pre-Hilbert $\mathcal{A}$-module when $\mathcal{A}$ is understood.

Remark 2.2.1.2. The positivity condition (5) above is in the sense of positive elements in $\mathcal{A}$. Using condition (1) and (4), we also know that $\langle \varphi \cdot a, \psi \rangle = a^* \langle \varphi, \psi \rangle$ and $\langle \alpha\varphi, \psi \rangle = \bar{\alpha} \langle \varphi, \psi \rangle = \langle \varphi, \bar{\alpha}\psi \rangle$.

More general examples can be found in [49, example 1.2.2].

Example 2.2.1.3. Let $\mathbb{H} = \ell_2(I, \mathcal{A})$ be the linear space of all sequences $(a_i)_{i \in I}, a_i \in \mathcal{A}$ satisfying the condition $\sum_{i \in I} \|a_i\|^2 < \infty$. Then $\ell_2(I, \mathcal{A})$ becomes a right $\mathcal{A}$-module if the action of $\mathcal{A}$ is defined by $(a_i) \cdot a = (a_i a)$ for $(a_i)_{i \in I} \in \ell_2(I, \mathcal{A}), a_i \in \mathcal{A}$. It becomes a pre-Hilbert module if the inner product of elements $(a_i), (b_i) \in \ell_2(I, \mathcal{A})$ is defined by $\langle (a_i), (b_i) \rangle = \sum_i a_i^* b_i$. If $I = \mathbb{N}$, we denote $\ell_2(\mathbb{N}, \mathcal{A})$ by $\ell_2^{\mathcal{A}}$.

Fortunately, we have an inequality analogous to the Cauchy-Schwartz inequality and an analogous result to the triangle inequality.

Lemma 2.2.1.4. *(**Cauchy-Schwartz inequality**) If $\mathbb{H}$ is a pre-Hilbert module and $\varphi, \psi \in \mathbb{H}$, then*

$$\|\langle \varphi, \psi \rangle\| \leq \|\langle \varphi, \varphi \rangle\| \, \|\langle \psi, \psi \rangle\| \, .$$

Lemma 2.2.1.5. *(**Triangle inequality**) If $\mathbb{H}$ is a pre-Hilbert module and $\varphi, \psi \in \mathbb{H}$, then*

$$\|\langle \varphi + \psi, \varphi + \psi \rangle\|^{1/2} \leq \|\langle \varphi, \varphi \rangle\|^{1/2} + \|\langle \psi, \psi \rangle\|^{1/2} \, .$$

Therefore, we also have a norm in a pre-Hilbert $\mathcal{A}$-module $\mathbb{H}$ induced by the inner product.

Definition 2.2.1.6. The **norm** of an element of pre-Hilbert $\mathcal{A}$-module $\varphi \in \mathbb{H}$ is defined as

$$\|\varphi\|_{\mathcal{A}} = \|\langle \varphi, \varphi \rangle\|^{1/2} \, .$$

The inner product is separately continuous in each variable:

$$\|\langle \varphi_i, \psi \rangle - \langle \varphi, \psi \rangle\| = \|\langle \varphi - \varphi_i, \psi \rangle\| \leq \|\varphi - \varphi_i\|_{\mathcal{A}} \, \|\psi\|_{\mathcal{A}} \, .$$

Because of this, the inner product of a pre-Hilbert module extends to an inner product of its completion.

2.2.2 Hilbert modules

Definition 2.2.2.1. A pre-Hilbert $\mathcal{A}$-module $\mathbb{H}$ is called a **Hilbert C^*-module** over $\mathcal{A}$ (or sometimes just **Hilbert $\mathcal{A}$-module**, or **Hilbert module** if $\mathcal{A}$ is understood) if it is complete with respect to the norm $\|\cdot\|_{\mathcal{A}}$. A **Hilbert submodule** of a Hilbert module $\mathbb{H}$ is a closed submodule of $\mathbb{H}$. It is a **full Hilbert module** if the ideal

$$I = \operatorname{span} \{ \langle \varphi, \psi \rangle \, | \varphi, \psi \in \mathbb{H} \}$$

is dense in $\mathcal{A}$.

Notation. From now on, let $\mathbb{H}$ denote a Hilbert C^*-module over $\mathcal{A}$. We will refer to $\mathbb{H}$ as Hilbert $\mathcal{A}$-module or simply as a Hilbert module when $\mathcal{A}$ is understood.

2.2.3 Examples

Example 2.2.3.1. (**Hilbert $\mathcal{A}$-module $\mathcal{A}$**) A C^*-algebra $\mathcal{A}$ is itself a Hilbert $\mathcal{A}$-module with action $a \cdot b = ab$ and $\langle a, b\rangle = a^*b$. It is a full Hilbert module, cf. [66, Example 2.10].

Example 2.2.3.2. (**Hilbert $\mathbb{C}$-module**) A Hilbert module over $\mathbb{C}$ is a Hilbert space with the usual scalar multiplication and inner product $\langle \varphi, \psi\rangle = (\psi \mid \varphi)$, where $(\cdot \mid \cdot)$ is the inner product on the Hilbert space.

Example 2.2.3.3. (**Hilbert modules $\mathbb{H}^n$**) If $\{\mathbb{H}_i\}$ is a finite set of Hilbert $\mathcal{A}$-modules, we can define the direct sum $\oplus\mathbb{H}_i$. It is a Hilbert $\mathcal{A}$-module with action $(\varphi_i) \cdot a = (\varphi_i \cdot a)$ and inner product is given by $\langle \varphi, \psi\rangle = \sum_i \langle \varphi_i, \psi_i\rangle$, where $\varphi = (\varphi_i), \psi = (\psi_i) \in \oplus\mathbb{H}_i$. We denote the direct sum of n copies of Hilbert module $\mathbb{H}$ by $\mathbb{H}^n$. If $\mathbb{H} = \mathcal{A}$, we write $\mathcal{A}^n$. In this case, when $\mathcal{A}$ is unital, the vectors $\varphi_i = 0 \oplus \cdots \oplus 0 \oplus 1_{\mathcal{A}} \oplus 0 \oplus \cdots \oplus 0$, where $1_{\mathcal{A}}$ is at i position and 0 elsewhere, form an orthonormal basis for $\mathcal{A}^n$.

Remark 2.2.3.4. The set of vectors $\{\xi_i | i \in I\}$ in $\mathbb{H}$ is **orthonormal** if

$$\langle \xi_i, \xi_j\rangle = \begin{cases} 1_{\mathcal{A}} & i = j \\ 0 & i \neq j. \end{cases} \tag{2.2}$$

If 0 is the only vector that orthogonal to all the $\{\xi_i\}$ then we say that the set is an **orthonormal basis** for $\mathbb{H}$.

Example 2.2.3.5. (**Hilbert module $\bigoplus_{i\in I} \mathbb{H}_i$**) Let $\{\mathbb{H}_i\}$ be a collection of Hilbert $\mathcal{A}$-modules indexed by an infinite set I. We may generalize the Hilbert space $\ell_2(I)$ as follows. The set

$$\bigoplus_{i\in I}\mathbb{H}_i=\left\{(\varphi_i)\in\prod_1^\infty\mathcal{A}\mid\sum\varphi_i^*\varphi_i\ \text{ converges in norm in }\mathcal{A}\right\}$$

is a Hilbert $\mathcal{A}$-module with action $(\varphi_i)\cdot a$ and inner product

$$\langle\varphi,\psi\rangle=\sum_i\langle\varphi_i,\psi_i\rangle\,.$$

In particular, if I is countable and $\mathbb{H}=\mathcal{A}$, we denote it by $\mathbb{H}_{\mathcal{A}}$, and call it **a standard Hilbert module**. See [43, page 6]. If $\mathcal{A}$ is unital then the Hilbert module $\mathbb{H}_{\mathcal{A}}$ possesses a standard basis $\{e_i\}_{i\in\mathbb{N}}$, where $e_i=(0,\cdots,0,1_{\mathcal{A}},0,\cdots)$.

Remark 2.2.3.6. By [75, page 239], the common mistakes defining the standard Hilbert module are as follows.

$$\begin{aligned}\mathbb{H}_{\mathcal{A}}^{wrong1}&=\left\{(a_i)\in\prod_1^\infty\mathcal{A}|\sum_{\mathbb{N}}\|a_i\|^2<\infty\right\},\\ \mathbb{H}_{\mathcal{A}}^{wrong2}&=\left\{(a_i)\in\prod_1^\infty\mathcal{A}|\left\|\sum_{\mathbb{N}}a_i^*a_i\right\|<\infty\right\}.\end{aligned}$$

Note that $\mathbb{H}_{\mathcal{A}}^{wrong1}$ is exactly the pre-Hilbert $\mathcal{A}$-module ℓ_2, see example 2.2.1.3. It is proved that $\sum_{\mathbb{N}}a_i^*a_i$ is norm convergent in $\mathcal{A}$ when $(a_i)\in\mathbb{H}_{\mathcal{A}}^{wrong1}$ and strongly convergent in $\mathcal{A}^{**}$ when $(a_i)\in\mathbb{H}_{\mathcal{A}}^{wrong2}$. Hence, $\mathbb{H}_{\mathcal{A}}^{wrong1}\subseteqq\mathbb{H}_{\mathcal{A}}\subseteqq\mathbb{H}_{\mathcal{A}}^{wrong2}$. It is also cited from [20, 4.3], that $\mathbb{H}_{\mathcal{A}}^{wrong1}=\mathbb{H}_{\mathcal{A}}$ precisely when $\mathcal{A}$ is finite dimensional. Meanwhile, Wegge-Olsen in [75] gives an example where $\mathbb{H}_{\mathcal{A}}\neq\mathbb{H}_{\mathcal{A}}^{wrong2}$. Specifically, let $\mathcal{A}=\mathcal{K}(\mathcal{H})$ and consider a sequence of mutually orthogonal rank 1 projections $\{p_i\}$. The series $\sum_{\mathbb{N}}p_i^*p_i=\sum_{\mathbb{N}}p_i=I_{\mathcal{H}}\notin\mathcal{K}(\mathcal{H})$, where the convergence is in the sense strong operator topology, and the norm of the sum is equal to 1. In Chapter 4 we will generalize $L^2(X)$ to Hilbert module developed from $\mathcal{A}$-valued functions. As explained above, care is needed in defining those generalizations.

Example 2.2.3.7. The Hilbert $\mathcal{A}$-module $\mathcal{H}\widehat{\otimes}\mathcal{A}$. If $\mathcal{H}$ is a Hilbert space and $\mathcal{A}$ is a C^*-algebra, the algebraic tensor product $\mathcal{H}\otimes\mathcal{A}$ has an $\mathcal{A}$-valued inner product given on simple tensors by:

$$\langle \varphi\otimes a, \psi\otimes b\rangle = \langle \varphi,\psi\rangle\, a^*b\ (\varphi,\psi\in\mathcal{H}, a,b\in\mathcal{A}),$$

and the action of $\mathcal{A}$ given on simple tensors by:

$$(\varphi\otimes a)\cdot b = \varphi\otimes ab\ (\varphi\in\mathcal{H}, a,b\in\mathcal{A}).$$

Thus $\mathcal{H}\otimes\mathcal{A}$ is a pre-Hilbert $\mathcal{A}$-module and we denote its completion by $\mathcal{H}\widehat{\otimes}\mathcal{A}$. Let $\{\varepsilon_i\}_{i\in I}$ be an orthonormal basis for $\mathcal{H}$. If $\mathcal{H}$ is finite dimensional, e.g $\dim(\mathcal{H}) = n$, then $\mathcal{H}\widehat{\otimes}\mathcal{A}$ can be naturally identified with the Hilbert module $\mathcal{A}^n$. If $\mathcal{H}$ is infinite-dimensional Hilbert space, $\mathcal{H}\widehat{\otimes}\mathcal{A}$ is often denoted by $\bigoplus_i \mathcal{A}_i$. In the case $\mathcal{H}$ is a separable, infinite-dimensional Hilbert space, $\mathcal{H}\widehat{\otimes}\mathcal{A}$ is often denoted by $\mathbb{H}_{\mathcal{A}}$. See [43, page 6].

2.2.4 Operators on Hilbert modules

Definition 2.2.4.1. If $\mathbb{H}$ and $\mathbb{K}$ are both Hilbert $\mathcal{A}$-modules, a **Hilbert module map** from $\mathbb{H}$ to $\mathbb{K}$ is a linear map $T:\mathbb{H}\to\mathbb{K}$ that respects the module action: $T(\phi\cdot a) = T(\phi)\cdot a$. In this case we say that T is an $\mathcal{A}$-linear map.

Bounded and adjointable operators

Definition 2.2.4.2. Suppose that $\mathbb{H},\mathbb{K}$ are Hilbert modules. We define $\mathcal{L}(\mathbb{H},\mathbb{K})$ to be the set of all linear maps $T:\mathbb{H}\to\mathbb{K}$ such that for each T there exist an $\mathcal{A}$-linear map T^* such that $\langle T\varphi,\psi\rangle = \langle\varphi, T^*\psi\rangle$ for all $\varphi\in\mathbb{H},\psi\in\mathbb{K}$. We call T^* the adjoint of T, and furthermore, we call $\mathcal{L}(\mathbb{H},\mathbb{K})$ the set of **adjointable** maps from $\mathbb{H}$ to $\mathbb{K}$. When $\mathbb{K}=\mathbb{H}$, we denote $\mathcal{L}(\mathbb{H},\mathbb{H}) = \mathcal{L}(\mathbb{H})$.

Notation. We denote the identity operator in $\mathcal{L}(\mathbb{H})$ by $I_{\mathbb{H}}$.

Lemma 2.2.4.3. *Every element T of $\mathcal{L}(\mathbb{H},\mathbb{K})$ is a bounded $\mathcal{A}$-linear map (and T^* is as well).*

The following lemma is from [75, lemma 15.2.3].

Lemma 2.2.4.4. *If T is adjointable, then its adjoint is unique and adjointable with $T^{**} = T$. If both T and S are adjointable, then, so is ST with $(ST)^* = T^*S^*$.*

Remark 2.2.4.5. A bounded $\mathcal{A}$-linear map need not be adjointable. See [43, Page 8]

Notation. We denote the set of bounded $\mathcal{A}$-linear map from $\mathbb{H}$ to $\mathbb{K}$ by $\mathcal{B}(\mathbb{H},\mathbb{K})$ and write $\mathcal{B}(\mathbb{H})$ for $\mathcal{B}(\mathbb{H},\mathbb{H})$.

Lemma 2.2.4.6. *([75, Proposition 15.2.4.]) When equipped with the operator norm*

$$\|T\| = \sup\{\|T\varphi\| \mid \|\varphi\| \leq 1\},$$

$\mathcal{B}(\mathbb{H})$ is a Banach algebra and $\mathcal{L}(\mathbb{H})$ is a C^*-algebra.

The following lemma is from [49]. It shows the self duality of unital Hilbert Module $\mathcal{A}$ over $\mathcal{A}$.

Lemma 2.2.4.7. *Let $\mathcal{A}$ be a unital C^*-algebra and let $\phi : \mathcal{A} \to \mathcal{A}$ be a bounded $\mathcal{A}$-linear map (i.e., for some constant $K \geq 0$, the inequality $\phi(a)^*\phi(a) \leq Ka^*a$ holds for all $a \in \mathcal{A}$). Then $\phi(a) = \phi(1)a$ for all $a \in \mathcal{A}$.*

Lemma 2.2.4.8. *Let $\mathbb{H} \cdot \mathcal{A} = \{x \cdot a | x \in \mathbb{H}, a \in \mathcal{A}\}$. The closed linear span of $\mathbb{H} \cdot \mathcal{A}$ over $\mathbb{C}$ is equal to $\mathbb{H}$.*

Notation. If $\mathbb{J} \subset \mathbb{H}$, we denote the linear span over $\mathbb{C}$ and $\mathcal{A}$ of this set by $\operatorname{span}\{\mathbb{J}\}$.

Definition 2.2.4.9. A Hilbert $\mathcal{A}$-module $\mathbb{H}$ is called **finitely generated** if there exists a finite set $\{x_i\} \subset \mathbb{H}$ such that $\mathbb{H} = \operatorname{span}\{x_i\}$. If $\{x_i\}$ is a countable subset of $\mathbb{H}$, and $\mathbb{H} = \operatorname{span}\{x_i\}$ then we call $\mathbb{H}$ a **countably generated** Hilbert $\mathcal{A}$-module.

The following theorem is from [75, Theorem 15.4.6.]

Theorem 2.2.4.10. ***Kasparov stabilization theorem*** *If $\mathbb{H}$ is a countably generated Hilbert $\mathcal{A}$-module then $\mathbb{H} \oplus \mathbb{H}_{\mathcal{A}} \cong \mathbb{H}_{\mathcal{A}}$.*

Theorem 2.2.4.11. *Let $\mathbb{K}$ be a finitely generated Hilbert submodule in a Hilbert $\mathcal{A}$-module $\mathbb{H}$. If $\mathcal{A}$ is unital, then $\mathbb{K}$ is an orthogonal direct sumand in $\mathbb{H}$.*

Projections and unitaries

Our main source for this section is [43, Chapter 3].

Definition 2.2.4.12. Given a closed submodule $\mathbb{K}$ of a Hilbert $\mathcal{A}$-module $\mathbb{H}$, define

$$\mathbb{K}^{\perp} = \{\varphi \in \mathbb{H} | \langle \varphi, \psi \rangle = 0, \psi \in \mathbb{K}\}.$$

Then $\mathbb{K}^{\perp}$ is also a closed submodule of $\mathbb{H}$.

Unlike the case when $\mathcal{A} = \mathbb{C}$, it is not always the case that for any Hilbert submodule $\mathbb{K}$ of $\mathbb{H}$, $\mathbb{K} \oplus \mathbb{K}^{\perp}$ equals $\mathbb{H}$.

Definition 2.2.4.13. A closed submodule $\mathbb{K}$ in a Hilbert $\mathcal{A}$-module $\mathbb{H}$ is called **complementable** if $\mathbb{H} = \mathbb{K} \oplus \mathbb{K}^{\perp}$.

Definition 2.2.4.14. A closed submodule $\mathbb{K}$ in a Hilbert $\mathcal{A}$-module $\mathbb{H}$ is called **(topologically) complementable** if there exists a closed submodule $\mathbb{J}$ in $\mathbb{H}$ such that $\mathbb{K} + \mathbb{J} = \mathbb{H}$, $\mathbb{K} \cap \mathbb{J} = \{0\}$. We denote the nonorthogonal direct sum of $\mathbb{H}$ by $\mathbb{K} \tilde{\oplus} \mathbb{J}$.

Lemma 2.2.4.15. *Let $\mathbb{H}$ be a Hilbert module. Then*

$$\mathbb{H}^{\perp} = \{0\} \qquad \{0\}^{\perp} = \mathbb{H}.$$

If $\mathbb{K}_1$ and $\mathbb{K}_2$ are orthogonal submodules with $\mathbb{K}_1 \oplus \mathbb{K}_2 = \mathbb{H}$, then they are closed and $\mathbb{K}_2^{\perp} = \mathbb{K}_1$, $\mathbb{K}_1^{\perp} = \mathbb{K}_2$, and $\mathbb{K}_k^{\perp\perp} = \mathbb{K}_k$, $k = 1, 2$.

Theorem 2.2.4.16. *Let $\mathbb{H}, \mathbb{K}$ be Hilbert $\mathcal{A}$-modules and suppose that T belongs to $\mathcal{L}(\mathbb{H}, \mathbb{K})$ and has closed range. Then*

1. $\ker(T)$ *is a complementable submodule of* $\mathbb{H}$,
2. $\text{range}(T)$ *is a complementable submodule of* $\mathbb{K}$,
3. *the mapping $T^* \in \mathcal{L}(\mathbb{K}, \mathbb{H})$ also has closed range.*

Corollary 2.2.4.17. *If $P \in \mathcal{L}(\mathbb{H})$ is an idempotent, then* $\text{range}(P)$ *is a complementable submodule in* $\mathbb{H}$.

For a general $T \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ it is easy to verify that $\text{range}(T)^{\perp} = \ker(T^*)$. However, it need not be the case that $\ker(T^*)^{\perp} = \overline{\text{range}(T)}$.

Definition 2.2.4.18. We call $U \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ **unitary** if it is an isomorphism which preserves inner products: $\langle U(\varphi), U(\psi)\rangle = \langle \varphi, \psi\rangle$. Equivalently, if

$$U^*U = I_{\mathbb{H}}, \;\; UU^* = I_{\mathbb{K}}.$$

If there exists a unitary element of $\mathcal{L}(\mathbb{H}, \mathbb{K})$ then we say $\mathbb{H}$ and $\mathbb{K}$ are **unitarily equivalent** Hilbert $\mathcal{A}$-modules, and we write $\mathbb{H} \cong \mathbb{K}$.

Example 2.2.4.19. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{\varepsilon_i\}$ then $\mathcal{H}\widehat{\otimes}\mathcal{A} \cong \bigoplus_i \mathcal{A}_i$, where each $\mathcal{A}_i$ is a copy of $\mathcal{A}$. The unitary that gives this equivalence is the map U that takes $\varepsilon_i \otimes a$ to the element of $\bigoplus_i \mathcal{A}_i$ that has a in the ith coordinate and zeros elsewhere.

It is clear that if $U \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ is unitary then U is a surjective $\mathcal{A}$-linear map, that is isometric. For the converse, we include the following result.

Theorem 2.2.4.20. *Let $\mathbb{H}, \mathbb{K}$ be Hilbert $\mathcal{A}$-modules and let U be a linear map from $\mathbb{H}$ to $\mathbb{K}$. Then the following conditions are equivalent:*

1. *U is an isometric, surjective $\mathcal{A}$-linear map;*

2. *U is a unitary element of $\mathcal{L}(\mathbb{H}, \mathbb{K})$.*

Proposition 2.2.4.21. *Let V be a linear map from $\mathbb{H}$ to $\mathbb{K}$. The following conditions are equivalent:*

1. *V is an isometric $\mathcal{A}$-linear map with complemented range;*

2. *$V \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ and $V^*V = I_{\mathbb{H}}$.*

Corollary 2.2.4.22. *Let V be a linear map from $\mathbb{H}$ to $\mathbb{K}$. If $V \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ and $V^*V = I_{\mathbb{H}}$, then VV^* is a projection onto the range of V, whose kernel is an orthogonal complement for* range(V).

Compact operators in Hilbert modules

Recall from Proposition 2.1.1.13, that $\mathcal{K}(\mathcal{H})$ is the closed span of the rank-one operators $\psi \otimes \bar{\varphi} : \eta \to \psi \cdot (\eta \mid \varphi)_{\mathbb{C}} = \psi \cdot \langle \varphi, \eta \rangle_{\mathbb{C}}$. By analogy, given Hilbert $\mathcal{A}$-modules $\mathbb{H}$ and $\mathbb{K}$, $\varphi \in \mathbb{H}$ and $\psi \in \mathbb{K}$, we define $\psi \otimes \bar{\varphi} : \mathbb{H} \to \mathbb{K}$ by

$$\psi \otimes \bar{\varphi}(\eta) = \psi \cdot \langle \varphi, \eta \rangle \, .$$

It is easy to check that $\varphi \otimes \bar{\psi} \in \mathcal{L}(\mathbb{H}, \mathbb{K})$ with $(\varphi \otimes \bar{\psi})^* = \psi \otimes \bar{\varphi}$.

Notation. We denote by $\mathcal{K}(\mathbb{H}, \mathbb{K})$ the closed linear subspace of $\mathcal{L}(\mathbb{H}, \mathbb{K})$ spanned by $\left\{ \varphi \otimes \bar{\psi} | \varphi \in \mathbb{H}, \psi \in \mathbb{K} \right\}$, and we write $\mathcal{K}(\mathbb{H})$ for $\mathcal{K}(\mathbb{H}, \mathbb{H})$.

Remark 2.2.4.23. Elements of $\mathcal{K}(\mathbb{H},\mathbb{K})$ need not be compact. For example, If $\mathcal{A}$ is unital, then $1_{\mathcal{A}} \otimes \overline{1_{\mathcal{A}}} = I_{\mathcal{A}}$, the identity operator on $\mathcal{A}$ belongs to $\mathcal{K}(\mathcal{A})$, but it is not a compact operator (unless $\mathcal{A}$ is finite-dimensional).

Example 2.2.4.24. If $\mathbb{H} = \mathcal{A}$, then $\mathcal{K}(\mathcal{A}) \cong \mathcal{A}$. The isomorphism is given by identifying $a \otimes \bar{b}$ with L_{ab^*}, the left multiplication by ab^*. Moreover, if $\mathcal{A}$ unital, $\mathcal{K}(\mathcal{A}) = \mathcal{L}(\mathcal{A})$.

Example 2.2.4.25. For the Hilbert module $\mathcal{H}\widehat{\otimes}\mathcal{A}$, we have $\mathcal{K}(\mathcal{H}\widehat{\otimes}\mathcal{A}) \cong \mathcal{K}(\mathcal{H})\widehat{\otimes}\mathcal{A}$, where $\mathcal{K}(\mathcal{H})\widehat{\otimes}\mathcal{A}$ denotes the C^*-algebraic tensor product of $\mathcal{K}(\mathcal{H})$ and $\mathcal{A}$ (the completion of algebraic tensor product, to the spatial, or minimal, C^*-norm). The identification given by the map developed from $(\varphi \otimes a) \otimes \overline{(\psi \otimes b)} \mapsto (\varphi \otimes \bar{\psi}) \otimes (a \otimes \bar{b}) = \big(\varphi \otimes \bar{\psi}\big) \otimes ab^*$.

An example of this kind of Hilbert module tensor product is the Hilbert module $\mathcal{L}(X)\widehat{\otimes}\mathcal{A}$. In Theorem 4.2.1.1 we will show that the Hilbert module $\mathbb{L}^2(X,\mathcal{A})$ is isomorphic to Hilbert module $\mathcal{L}(X)\widehat{\otimes}\mathcal{A}$.

Dual modules, self-duality and Riesz-Fréchet theorem

Definition 2.2.4.26. Let $\mathbb{H}$ be a Hilbert $\mathcal{A}$-module and denote by $\mathbb{H}'$ the **dual** of $\mathbb{H}$, i.e the set of bounded module maps from $\mathbb{H}$ to $\mathcal{A}$, $\mathcal{B}(\mathbb{H},\mathcal{A})$.

The dual $\mathbb{H}'$ is a linear vector space and a right $\mathcal{A}$-module. For every $\varphi \in \mathbb{H}$, the map $\psi \mapsto \langle \varphi, \psi \rangle$ belongs to $\mathbb{H}'$, and the map defined by $\varphi \mapsto \langle \varphi, \cdot \rangle$ is an injective $\mathcal{A}$-module map. Call a Hilbert $\mathcal{A}$-module $\mathbb{H}$ **self-dual** or **reflexive** when every module map in $\mathbb{H}'$ arises by taking the inner product with some fixed element of $\mathbb{H}$, equivalently $\mathbb{H}' \cong \mathbb{H}$.

Remark 2.2.4.27. Certainly that $\mathbb{H}$ itself is a vector space in its own right, and therefore, there exists a dual space in the sense of the space of all functional linear from $\mathbb{H}$ to $\mathbb{C}$. However, since in this thesis we do not use the notion dual in this sense, we will always refer to definition 2.2.4.26 for the term dual of a Hilbert module.

Example 2.2.4.28. The standard Hilbert module $\mathbb{H}_{\mathcal{A}}$ is not reflexive unless $\mathcal{A}$ is finite dimmensional. See [75, 15.I].

Lemma 2.2.4.29. *If* $\mathbb{H}$ *is a self-dual module then every bounded module map* $T : \mathbb{H} \to \mathbb{K}$ *has an adjoint* $T^* : \mathbb{K} \to \mathbb{H}$*. In particular, this implies* $\mathcal{L}(\mathbb{H}) = \mathcal{B}(\mathbb{H})$.

If $\mathcal{A}$ is unital, then $\mathcal{A}$ is self dual, c.f. lemma 2.2.4.7. Furthermore, $\mathcal{A}^n$ is self dual if and only if $\mathcal{A}$ is self dual if and only if $\mathcal{A}$ is unital. See [75, exercise 15.I].

Let $\mathbb{H}$ be a Hilbert $\mathcal{A}$-module, fix $\varphi \in \mathbb{H}$ and let

$$T_\varphi \psi = \langle \varphi, \psi \rangle \tag{2.3}$$

for all $\psi \in \mathbb{H}$. It is easy to see that T_φ belongs to $\mathcal{L}(\mathbb{H}, \mathcal{A})$. In fact it belongs to $\mathcal{K}(\mathbb{H}, \mathcal{A})$. The following theorem gives a generalization of **Riesz-Fréchet theorem** for Hilbert C^*-module. See [43, page 13].

Theorem 2.2.4.30. ***Riesz-Fréchet theorem for Hilbert*** C^****-modules*** *Every element* T *of* $\mathcal{K}(\mathbb{H}, \mathcal{A})$ *is given by an inner product as in (2.3), or equivalently,* $T = T_\varphi$ *for some* φ *in* $\mathbb{H}$.

Proposition 2.2.4.31. *If* $\mathcal{A}$ *is unital, then every element of* $\mathcal{L}(\mathbb{H}, \mathcal{A})$ *is given by the inner product, and hence,* $\mathcal{K}(\mathbb{H}, \mathcal{A}) = \mathcal{L}(\mathbb{H}, \mathcal{A})$*. Furthermore, if* $T \in \mathcal{L}(\mathbb{H}, \mathcal{A})$ *then* $T = T_\varphi$ *where* $\varphi = T^*(1_{\mathcal{A}})$.

Group representation in Hilbert Modules

Notation. We denote the group of unitary elements of $\mathcal{L}(\mathbb{H})$ by $\mathcal{U}(\mathbb{H})$.

Definition 2.2.4.32. Let G be a locally compact group. A homomorphism $U : G \to \mathcal{U}(\mathbb{H})$ is called a **unitary representation** of G on $\mathcal{L}(\mathbb{H})$ if it is continuous in the strong operator topology, i.e. $x \mapsto U(x)\varphi$ is continuous for all $\varphi \in \mathbb{H}$.

Chapter 3

The wavelet transform on Hilbert spaces

This chapter discuss the theory of continuous wavelet transforms and their generalizations, in the Hilbert space setting. We think it is important to include a discussion of the discrete wavelet transform and its existing generalizations, both in the Hilbert space and the Hilbert module settings before passing to our generalizations in Chapter 4 and 5. We also mention here some results from the literature in wavelet theory related to the theory of Hilbert C^*-modules. Beside trying to give a global idea of the terms wavelet and wavelet transform, we also want to show that our generalization of the continuous wavelet transform to Hilbert C^*-modules is original, and different from the existing results in wavelet theory in Hilbert C^*-modules. This chapter also discusses Führ's generalized continuous wavelet transform in the Hilbert space setting. We will generalize this to the Hilbert module setting in the next chapter.

The first section of this chapter presents an introduction to wavelet transforms. In the second section we discuss Führ's generalized continuous wavelet transform, arising from continuous unitary representations of locally compact groups.

3.1 Introduction to wavelet transforms

In this section, we present a historical review of the notions of wavelets and wavelet transforms. In particular, in Section 3.1.1, we discuss the continuous wavelet transform on the real line. In Section 3.1.2 we list some types of wavelets and their generalizations. We hope that this will clarify the relationship of the results of this thesis to other results in wavelet theory.

3.1.1 Historical background

In the last twenty or twenty five years, wavelets have been considered as an interesting topic for researchers from many different disciplines. There are some examples of scientific discoveries or technological improvements that implemented wavelet-like techniques and were discovered sometime before the general theory of wavelets became well-known.

[52] mentions at least sixteen related concepts or approaches to wavelets, which had previously been known by other names. A similar comment is available in [10]. For example, in pure harmonic analysis there exist Calderón's formula, in physics we have affine coherent states; in electrical engineering there are subband coding and constant Q-filters; and in image processing we have multiscale representation. Anyone interested in those mentioned and other possible application of wavelets may consult [5, 8, 51, 70, 11].

In this thesis, we will start from wavelets as a special case of coherent state systems in physics, related to the group representations in Hilbert spaces, as described by Führ. We will generalize the concept using group representations in Hilbert modules to obtain a generalization of the continuous wavelet transform to the setting of Hilbert modules, in particular the GCWT on the Hilbert module $\mathbb{L}^2(X, \mathcal{A})$.

Now, we will discuss the continuous wavelet transform in $L^2(\mathbb{R})$, which is known as **the original continuous wavelet transform.** To give us a better sense of the theory, we shall include here a comparison between Fourier analysis and wavelet analysis on a signal or function. In fact the term wavelets was first introduced in this context. An introduction of this theory can be found in [11, Chapter 2 and 3], [34] and [41].

In this section, our signals are real-valued square integrable functions f on $\mathbb{R}$, and the Fourier transform $\mathcal{F}f$ of f will be given by the (normalized) standard Fourier transform on the real line:

$$\mathcal{F}f(\omega) = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t)e^{-i\omega t}dt. \tag{3.1}$$

While we call the domain of f the **time domain**, we call the domain of its transform the **frequency domain**. Recall that we can always extend this transform as an isometry from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ by using the Plancherel theorem. See [18, Theorem 8.29].

Using Fourier analysis, one can analyze a signal or function by cutting it up into different frequency components (sines and cosines), and reconstructing the signal or function using a sum (or integral) of its components, known as Fourier series (or inverse Fourier transform). See Section 2.1 and 2.2 of [11] for some examples.

A major problem in using Fourier analysis is that although we may be able to determine all the frequencies which occur in a signal, we cannot do this while retaining information concerning time-localization. In other words, we only have frequency resolution of the signal, not time resolution. Thus, if one is interested in the frequency content locally in time, one solution of this problem is to cut the signal f of interest into several parts and use the Fourier analysis on each part. The formulation of this transform $\mathcal{F}^{win}$ is:

$$\mathcal{F}^{win} f(\omega, t) = \int f(s) g(s-t) e^{-i\omega s}\, dt.$$

This is called the windowed Fourier transform with the window g. Of course the transform depends on how we cut the signal. Unfortunately, it leads to a fundamental problem as well; that is, when we want to analyze a signal at a certain moment in time, or, equivalently, when we use a Dirac pulse function as a window, we always find that the frequency components of the signal are spread over all frequencies. Thus, we cannot represent a signal as a point in the time-frequency plane, where both time and frequency are limited in the plane. In fact, the Heisenberg uncertainty principle limits the extent to which time and frequency can be localized. More discussion about this can be found in [48, chapter 2 and 4].

It was J. Morlet who had proposed wavelets as an alternative tool for the analysis of seismic data, since the standard technique of windowed Fourier transform, could not meet the needs of the application [57]. In fact, the word wavelet itself was used for the first time by A. Grossmann and J. Morlet in the early 80's, [37, 57, 32].

As an alternative for the solution to the time-frequency localization, the wavelet transform or wavelet analysis, instead of using the same shifted window as in the window Fourier transform, the wavelet transform uses a fully scalable modulated window $\psi \in L^2(\mathbb{R})$ shifted along the signal. In other words we assume $\int \psi(t)\ dt = 0$, dilate ψ to get different scales, and then translate ψ to get a collection of functions

$$\psi_{b,a}(t) = |a|^{-1/2}\, \psi\left(\frac{t-b}{a}\right) \tag{3.2}$$

in $L^2(\mathbb{R})$.

We calculate the spectrum of a signal $f \in L^2(\mathbb{R})$ for each position and scale of the window determined by b and a respectively. As a result, we have a collection of time-scale representations of f given by a **coefficient function** $V_\psi f$ in $L^2(\mathbb{R} \times \mathbb{R}', \frac{da}{|a|^2} db)$, where $\mathbb{R}' = \mathbb{R} \setminus \{0\}$ by:

$$V_\psi f(b, a) = |a|^{-1/2} \int f(t) \psi \left(\frac{t-b}{a} \right) dt. \tag{3.3}$$

Note that we can rewrite the coefficient function of f as an inner product with the translated and dilated ψ as follows:

$$V_\psi f(b, a) = (f \mid \psi_{b,a}).$$

This is called the **matrix coefficient** of f.

Any function ψ satisfying the admissibility condition

$$\int_{\mathbb{R}'} \frac{|\widehat{\psi}(\omega)|}{|\omega|} d\omega = 1, \tag{3.4}$$

is called an **admissible function**. In this case there exists a **reconstruction formula** or an **inversion formula** given by:

$$f = \int_{\mathbb{R}} \int_{\mathbb{R}'} (f \mid \psi_{b,a}) \, \psi_{b,a} \, \frac{dadb}{|a|^2}, \tag{3.5}$$

to be understood in the weak sense as in definition 1.5.2.2. See also remark 1.5.2.3.

The map $f \mapsto V_\psi f$ from $L^2(\mathbb{R})$ to $L^2(X, \mathbb{R} \times \mathbb{R}') \frac{da}{|a|^2} db$ is an isometry and is called the **continuous wavelet transform**. In this case, the function ψ is called the **mother wavelet**, and the system $\{\psi_{b,a}\}$ the wavelets or the wavelet system.

Not long after its first appearance, A. Grossmann recognized a family of coherent states associated with the $ax + b$-group in the construction introduced by J. Morlet. In particular, it is an affine coherent state system such in the sense first introduced by Aslaksen and Klauder in [1, 2]. In this setting, the admissible vectors known as fiducial vectors, and the reconstruction formula is equivalent to what is called the **resolution of the identity**. Since then, there have been many advanced studies of this continuous wavelet transform and its application using the theory of coherent state due to Grossmann, Morlet and other collaborators. See [32, 55, 56, 42].

In fact after it was realized that there is a connection between wavelets and representations of groups, researchers in harmonic analysis was motivated to study wavelets and their applications, and this research continues to bear fruit.

3.1.2 Types of wavelets and their generalization

In practice, some people use discrete subsets of the dilation parameter b and the translation parameter a of the continuous wavelet transform, to obtain what is called the discrete wavelet transform. We call such a process a **discretization**. Therefore, as well as the continuous wavelet transform, the **discrete wavelet transform** is a common object to study.

There are two main types of discrete transform obtained from the discretization process: wavelet frames and the orthonormal wavelet bases. In brief, a family $\{\psi_{b_m,a_n}\}_{m,n\in\mathbb{N}}$ in a Hilbert space $\mathcal{H}$ is called a **frame** if there exist constants $0 < A \leq B$ such that

$$A \sum_{m,n\in\mathbb{N}} |(f \mid \psi_{b_m,a_n})|^2 \leq \|f\|^2 \leq B \sum_{m,n\in\mathbb{N}} |(f \mid \psi_{b_m,a_n})|^2 .$$

If $\{\psi_{b_m,a_n}\}_{m,n\in\mathbb{N}}$ constitute an orthonormal basis for $L^2(\mathbb{R})$ then it is called an **orthonormal wavelet basis**. An important family of orthonormal bases [47] are certain compactly supported functions, some of which arise from multiresolution analysis, although in general wavelets that come from multiresolution analysis need not have compact support. For more detail information and intuitive description of orthogonal wavelets and multiresolution analysis, the reader may consult [73, 72, 11].

Different backgrounds, interest, purposes or needs of the researchers affect which part of the wavelet theory are studied or generalized. Hence, we often find the word wavelet or wavelets used with different meanings. For example, some people define wavelet frame structures for any system in Hilbert space or even in Hilbert C^*-modules as a system satisfying a boundedness condition similar to the condition for a frame derived from the discretization process above. Therefore, in general, not all wavelet frames known arise as discretization of continuous transform, and hence they do not arise from an admissible vector in the sense we have discussed above, nor do they have a resolution of the identity. In this general sense, a frame $\{\psi_{b_m,a_n}\}_{m,n\in\mathbb{N}}$ in Hilbert space is said to be a **tight frame** if $A = B$, and said to be **normalized** if $A = B = 1$. In fact, in the terminology established in Section 3.2, a **normalized tight frame** is an admissible coherent state system based on discrete space $X = \mathbb{N}$ with counting measure. Using the results in Chapter 5 and [24], we will show an analogous result holds in the setting of Hilbert modules, for a unital C^*-algebra $\mathcal{A}$ and a finitely and countably generated Hilbert module.

We now discuss further generalizations of discrete and continuous transforms and results on wavelets related to other fields.

One of the earliest studies of discrete wavelet transforms was the dyadic orthonormal wavelet. In this case, the family $\left\{2^{-j/2}\psi(2^j x - k)\right\}_{j,k\in\mathbb{N}}$ is an orthonormal basis for $L^2(\mathbb{R})$, which leads to a multiresolution structure on this Hilbert space. We can relate the translation and dilation with unitary operators T and D in $\mathcal{B}(L^2(\mathbb{R}))$ given by $(Tf)(t) = f(t-1)$ and

$(Df)(t) = \sqrt{2}f(2t)$. Using this setting, there has been some work generalizing this kind of wavelet to other Hilbert space [46, 54, 50, 62]. Furthermore, the applications of wavelets in other fields has been studied: harmonic analysis [3], operator algebra and operator theory [44, 45, 33, 9].

In particular, the relation between wavelets and Hilbert modules has also been investigated. One of the earlier ideas which includes a construction of Hilbert C^*-module related to wavelets, was given by M. A. Rieffel in 1997 [68], and was developed by J. Packer and M. A. Rieffel in [60, 61]. A similar construction was used by Wood in [77]. In these papers, Hilbert C^*-modules associated to wavelets are constructed, which are then used to study the properties of the wavelet. There are also some generalizations of wavelet frame theory in Hilbert modules given by M. Frank and D. Larson [22, 23, 24] and by I. Raeburn and S. Thompson [64]. A brief review of frames for Hilbert C^*-modules was given by Frank in [19].

On the other hand, it was realized by Grossmann, Morlet and Paul in [55, 56] that the original continuous wavelet transform on $L^2(\mathbb{R})$ and its inversion formula are related to certain representations on the $ax+b$-group. Furthermore, they consider the coherent state systems which arise as orbits of continuous unitary irreducible representations π of locally compact group G on Hilbert spaces $\mathcal{H}_\pi$, and define the wavelet transforms as the transforms related to (irreducible) square integrable representations in the sense [13] or [6]. They showed that the transforms are isometries between $\mathcal{H}_\pi$ and $L^2(G)$.

Inspired by those facts and viewing the $ax+b$-group as the semidirect product $\mathbb{R} \ltimes \mathbb{R}^+$ acting on $\mathbb{R}$, gives a natural idea for generalization of the notion of wavelets using similar square integrable representations of the more general semidirect product groups [4, 25, 74] or the more specific semidirect products: one coming from abelian dilation groups [26] or one coming from a closed subgroups of $GL(n, \mathbb{R})$ [15]. In these cases, we obtain continuous wavelet transforms in higher dimensions: $L^2(\mathbb{R}^n)$. Furthermore there has been some work where one drops the irreducibility condition of the group representation. For example, Führ use Plancherel theory to characterize the admissibility condition for the regular and cyclic representations of a type I group [29, 27, 30], and others define a continuous wavelet transform on a special homogeneous space [59].

Since the Hilbert C^*-module is a natural generalization of Hilbert space, with the inner product taking values in a C^*-algebra, in this thesis, we aim to generalize the continuous wavelet transform on Hilbert spaces due to Führ to the continuous wavelet transform on Hilbert C^*-modules. We will use the group theoretic approach to define wavelets as a special case of coherent state systems in Hilbert C^*-modules.

3.2 Führ's GCWT on Hilbert spaces

We now discuss generalized wavelet transforms on Hilbert spaces based on [28]. In the last two chapters, we shall further generalize this approach to Hilbert C^*-modules. In this section, we define wavelets as a special case of coherent state systems which come from strongly continuous unitary representations of locally compact groups. Most of the results here are stated explicitly in [28]. Therefore we will refer those restated here to the original source. Our contributions are to give greater detail in the proofs of some of the results, in particular those we will later generalize to the setting of Hilbert modules. We will also use a slightly different approach to the proofs for the synthesis operator and the resolution of the identity formula, and will restate explicitly some of Führ's statements or lemmas and provide the proofs, in particular those that will be generalized to the Hilbert module setting.

There are several reasons why we are interested in Führ's approach. First, this construction gives a systematic and powerful approach which can be applied to the original continuous wavelet transform and to other related transforms (windowed Fourier transform, two dimension continuous wavelet transform, Gabor system). Secondly, this approach uses more general group representations (not only the irreducible ones) to define wavelets and leads to a complete explanation of which possible representations and admissible vectors can be chosen in Hilbert space. The last and the most important reason is that strongly continuous unitary representations of groups in Hilbert spaces and isometries arise naturally in the setting of Hilbert C^*-modules. In fact, those notions will give a possible generalization of the continuous wavelet transform in Hilbert C^*-module. We will develop this approach in the next chapter.

In what follows, we will reserve the notation π for the designation of a strongly continuous unitary representation of G on a Hilbert space $\mathcal{H}_\pi$. We will assume that the groups are second countable and all Hilbert spaces are separable.

3.2.1 Coherent state systems and the GCWT

In this section we present a general notion of coherent state systems due to Führ [28]. Included here is the definition of coherent state systems, admissible vectors, coefficient operators and their adjoints and the resolution of the identity. We also include the realization of the images of the coefficient operators related to admissible vectors as reproducing kernel Hilbert spaces. Finally we define the generalized continuous wavelet transform.

Coherent state systems

Briefly, a coherent state system is defined as an expansion of Hilbert space elements with respect to a system of its vectors. As stated in [28, Section 2.2], the blueprint of such an expansion is the expansion of elements of a Hilbert space with respect to an orthonormal basis (ONB). More precisely, if $\eta = (\eta_i)_{i\in\mathbb{N}}$ is an ONB of a Hilbert space $\mathcal{H}$, for any vector $\varphi \in \mathcal{H}$ we can write

$$\varphi = \sum_{i\in\mathbb{N}} (\varphi \mid \eta_i)\, \eta_i$$

and define a mapping $V_\eta : \mathcal{H} \to \ell_2(\mathbb{N})$ given by $\varphi \mapsto ((\varphi \mid \eta_i))_{i\in N}$. The notion of coherent state system will be a generalization of such expansions where the index set $\mathbb{N}$ will be replaced by a measure space X. Thus most of the time we will need to replace the summation with integration over X.

In this section $\mathcal{H}$ will denote a separable Hilbert space and X will denote a measure space $(X, \mathcal{B}, \mu)$.

We start with the definitions of coherent state system, coefficient operator and admissibility criteria in the sense of [28, Definition 2.7], and then we list some properties of the coefficient operators.

Definition 3.2.1.1. Let $\eta = (\eta_x)_{x\in X}$ denote a family of vectors in $\mathcal{H}$, indexed by the elements of a measure space X.

a. For any $\varphi \in \mathcal{H}$ define a complex valued function $V_\eta\varphi$ on X by

$$V_\eta\varphi(x) = (\varphi \mid \eta_x)\,.$$

We call this function the **coefficient function**.

b. If for all $\varphi \in \mathcal{H}$, the coefficient function $V_\eta\varphi$ is μ-measurable, we call η a **coherent state system**.

Note that since inner product in Hilbert space is continuous, the coefficient function is continuous. However, we are interested in defining an operator on Hilbert spaces, hence we will require that the coefficient functions be square integrable.

Definition 3.2.1.2. Let $\eta = (\eta_x)_{x\in X}$ be a coherent state system in $\mathcal{H}$, indexed by the elements of a measure space X. Define

$$\mathcal{D}_\eta = \left\{\varphi \in \mathcal{H} | V_\eta\varphi \in L^2(X)\right\}. \tag{3.6}$$

We denote by $V_\eta : \mathcal{H} \to L^2(X)$ the (possibly unbounded) operator defined by the mapping $\varphi \mapsto V_\eta\varphi$ from $\mathcal{D}_\eta$ to $L^2(X)$. It is a linear operator with domain $\mathcal{D}_\eta$ and we call it **coefficient operator**.

In what follows, we will see that coefficient operators are closed operators, [28, Proposition 2.8].

Proposition 3.2.1.3. *Any coefficient operator is a closed operator.*

Proof. Let $\eta = (\eta_x)_{x\in X}$ be a coherent state system and (φ_n) be a sequence in $\mathcal{D}_\eta$. Assume that (φ_n) converges to a vector $\varphi \in \mathcal{H}$, and $V_\eta\varphi_n$ converge to an element $f \in L^2(X)$. Then there exist a subsequence $V_\eta\varphi_{n_i}$ which converges to f almost every where, see [16, Corollary 3.5.]. By the Cauchy-Schwarz-inequality we also have for any $x \in X$,

$$\begin{aligned}
|V_\eta\varphi_{n_i}(x) - V_\eta\varphi(x)| &= |(\varphi_{n_i} \mid \eta_x) - (\varphi \mid \eta_x)| \\
&= |(\varphi_{n_i} - \varphi \mid \eta_x)| \\
&\leq \|\varphi_{n_i} - \varphi\| \, \|\eta_x\|
\end{aligned}$$

Since $\varphi_{n_i} \to \varphi$ then the last line goes to zero, so $V_\eta\varphi_{n_i} \to V_\eta\varphi$ a.e. By the uniqueness of the limit, $V_\eta\varphi = f$ a.e. and hence V_η is also in $L^2(X)$ and therefore $\varphi \in \mathcal{D}_\eta$. $\square$

Adjoints of coefficient operators

As coefficient operators are closed, if the domain of a coefficient operator is the whole space, the closed graph theorem implies that it is also bounded. In this case, we can discuss its adjoint operator [28, Proposition 2.10]. We prefer to split Führ's result for this case into two following corollaries.

Corollary 3.2.1.4. *A coefficient operator is a bounded operator on the underlying Hilbert space $\mathcal{H}$ if and only if $\mathcal{D}_\eta = \mathcal{H}$.*

Proof. The statement follows from proposition 3.2.1.3 and the closed graph theorem. □

We will give a more detailed proof for the following corollary than the one given in [28, Proposition 2.10]. In fact we will use a slightly different approach here, that is using the definition of weakly vector valued integral and the reflexiveness of the Hilbert space $\mathcal{H}$.

Corollary 3.2.1.5. *If the domain of a coefficient operator is the whole Hilbert space $\mathcal{H}$ then it has an adjoint which is given pointwise by the weak operator integral*

$$V_\eta^*(f) = \int_X f(x)\eta_x \, d\mu(x). \tag{3.7}$$

We call this adjoint operator, the ***synthesis operator****.*

Proof. By corollary 3.2.1.4, the coefficient operator V_η is bounded, hence it has an adjoint. Now, for an element $\varphi \in \mathcal{H}$, let ϕ_φ be an element of $\mathcal{H}^*$, defined by $\phi_\varphi(\psi) = (\varphi \mid \psi)$. Since $\mathcal{H}$ is reflexive any element of $\mathcal{H}^*$ always has the form ϕ_φ for some element $\varphi \in \mathcal{H}$. Let $f \in L^2(X)$. Note that $x \mapsto f(x)\eta_x$

defines an $\mathcal{H}$-valued function. Now, for any $\varphi \in \mathcal{H}$ we calculate

$$\begin{aligned}
\left(\varphi \mid V_\eta^* f\right) &= (V_\eta \varphi \mid f) && (3.8)\\
&= \int_X V_\eta \varphi(x) \overline{f(x)}\, d\mu(x) && (3.9)\\
&= \int_X (\varphi \mid \eta_x)\, \overline{f(x)}\, d\mu(x) && (3.10)\\
&= \int_X (\varphi \mid f(x)\eta_x)\ d\mu(x). && (3.11)\\
&= \int_X \phi_\varphi\, (f(x)\eta_x)\ d\mu(x) && (3.12)
\end{aligned}$$

This calculation shows that the last integral converges for each element $\phi_\varphi \in \mathcal{H}^*$, and by definition $x \mapsto f(x)\eta_x$ is weakly integrable. Thus, there exists an element of $\mathcal{H}$ which we denote by $\int f(x)\eta_x\, d\mu(x)$ such that

$$\phi_\varphi \left(\int_X f(x)\eta_x\, d\mu(x) \right) = \int_X \phi_\varphi\, (f(x)\eta_x)\ d\mu(x). \tag{3.13}$$

Hence, from equality (3.12) and equation (3.13):

$$\begin{aligned}
\left(\varphi \mid V_\eta^* f\right) &= \int_X \phi_\varphi\, (f(x)\eta_x)\ d\mu(x)\\
&= \phi_\varphi \left(\int_X f(x)\eta_x\, d\mu(x) \right)\\
&= \left(\varphi \mid \int_X f(x)\eta_x\, d\mu(x) \right).
\end{aligned}$$

□

Admissible coherent state system

Here we define a special coherent state system, called an admissible coherent state system.

Definition 3.2.1.6. The coherent state system $\eta = (\eta_x)_{x \in X}$ in $\in \mathcal{H}$ is called **admissible** if the associated coefficient operator V_η is an isometry with domain $\mathcal{H}$, i.e. $\mathcal{D}_\eta = \mathcal{H}$.

The reconstruction formula

It is easy to see that the isometry property of the coefficient operator V_η for an admissible coherent state η, gives that $V_\eta^* V_\eta$ is the identity operator on $\mathcal{H}$ and that $V_\eta V_\eta^*$ is a projection onto the range of V_η. Let us state these facts formally in the following lemma.

Lemma 3.2.1.7. *Let $(\eta_x)_{x\in X}$ be an admissible coherent state system. Then the coefficient operator V_η satisfies the following conditions:*

a. *The operator $V_\eta^* V_\eta$ is the identity operator on $\mathcal{H}$.*

b. *The operator $V_\eta V_\eta^*$ is a projection onto the range of V_η.*

Proof. For any $\varphi, \psi \in \mathcal{H}$ we have $\left(V_\eta^* V_\eta \varphi \mid \psi\right) = (V_\eta\varphi \mid V_\eta\psi) = (\varphi \mid \psi)$. A straightforward implication is that $\left(V_\eta V_\eta^*\right)^2 = V_\eta V_\eta^* V_\eta V_\eta^* = V_\eta \left(V_\eta^* V_\eta\right) V_\eta^* = V_\eta V_\eta^*$ □

In fact, Lemma 3.2.1.7 (a) leads to an inversion or reconstruction formula which can be read as an expansion of a given vector in terms of the coherent state system. This is written precisely in the following theorem, [28, Proposition 2.11].

Theorem 3.2.1.8. *Let $(\eta_x)_{x\in X}$ be an admissible coherent state system. Then for $\varphi \in \mathcal{H}$ we have the following reconstruction formula*

$$\varphi = \int_X (\varphi \mid \eta_x)\, \eta_x \, d\mu(x) \tag{3.14}$$

to be read in the weak sense.

Proof. Note that the admissibility of the coherent state system means that the domain of the operator is the whole space $\mathcal{H}$, and hence implies boundedness of the operator and the existence of its adjoint operator. From 3.2.1.7

(b) we know that $V_\eta^* V_\eta = I_{\mathcal{H}}$. Therefore, for any $\varphi \in \mathcal{H}$, using the definition of coefficient function and the synthesis operator formula (5.13), we obtain the following weak integral:

$$\varphi = \left(V_\eta^* V_\eta\right)(\varphi) = V_\eta^*\left(V_\eta(\varphi)\right) = \int_X V_\eta\varphi(x)\eta_x \, d\mu(x) = \int_X \left(\varphi \mid \eta_x\right)\eta_x \, d\mu(x). \tag{3.15}$$

□

Remark 3.2.1.9. Here, we give a different approach to the proof of 3.2.1.8 from that given in [28, Proposition 2.11]. Führ's approach is to show that the equality holds in the weak sense. Here, we are interested in how we can obtain the equality by the definition and the properties of the coefficient operator, in particular the isometry and the adjoint properties. The latter approach will be very helpful for the generalization in the next chapter.

Note that by using the rank-one operator notation as in 2.1.1.13, we can re-express the inversion formula in the following form:

$$\varphi = \int_X \left(\eta_x \otimes \bar{\eta}_x\right)\varphi \, d\mu(x). \tag{3.16}$$

The resolution of the identity formula

Now we describe a formula involving a resolution of the identity, as an alternative way to describe the expansion property of the coherent state system, c.f [28, Proposition 2.11]. In fact this is how we can express the identity operator using the rank-one operators.

Before we continue the discussion of the resolution of the identity, let us include the definition of the weak operator integral from [28, page 20].

Definition 3.2.1.10. For a family of operators $(T_x)_{x\in X} \subset \mathcal{B}(\mathcal{H})$, if the integral $\int_X T_x(\varphi)\ d\mu(x)$ converges weakly for every $\varphi \in \mathcal{H}$, we define the

weak operator integral $\int_X T_x \, d\mu(x)$ pointwise as

$$\left(\int_X T_x \, d\mu(x)\right)(\varphi) = \int_X T_x(\varphi) \, d\mu(x). \tag{3.17}$$

Theorem 3.2.1.11. *Let $(\eta_x)_{x\in X}$ be an admissible coherent state system. Then we can rewrite the identity operator $I_{\mathcal{H}}$ as a weak operator integral which is known as a* ***resolution of the identity*** *formula:*

$$\int_X \eta_x \otimes \bar{\eta}_x \, d\mu(x) = I_{\mathcal{H}}. \tag{3.18}$$

Proof. Consider the family of the rank-one operators in equation (3.16), which we know converges for any $\varphi \in \mathcal{H}$. By definition 3.2.1.10, we can rewrite the identity operator as the integral of the rank-one operators. It is a weak operator integral well known as a resolution of the identity

$$\int_X \eta_x \otimes \bar{\eta}_x \, d\mu(x) = I_{\mathcal{H}}.$$

□

Image spaces of coefficient operators

Finally, we will discuss the image of the coefficient operator V_η for which η is an admissible coherent state system. Before that, let us review the definition of reproducing kernel Hilbert space. A **reproducing kernel Hilbert space** is a function space that can be defined by a reproducing kernel.

Definition 3.2.1.12. Let X be a measure space, and $\mathcal{H}$ be a Hilbert space of functions $f : X \to \mathbb{C}$ with some inner product $(\cdot \mid \cdot)$. The space $\mathcal{H}$ is called a reproducing kernel Hilbert space if there is a function $K : X \times X \to \mathbb{C}$ such that:

1. The function K_x defined by

$$K_x(y) = K(x, y)$$

lies in $\mathcal{H}$ for all x in X.

2. For all f in $\mathcal{H}$, $f(x) = (K_x \mid f)$.

Definition 3.2.1.13. The map $K : X \times K \to \mathbb{C}$ is called the **reproducing kernel** of $\mathcal{H}$.

Hence, the projection onto the space is given by an integral operator where the kernel is a reproducing kernel. For more information about the definition and properties of reproducing kernel Hilbert space, the reader may consult [7, 15].

As we have noticed before, the isometry property of a coefficient operator of an admissible coherent state system gives a projection on its image, we will see that the projection is an integral operator which is defined by a reproducing kernel, c.f. [28, Proposition 2.12].

Proposition 3.2.1.14. *Let $(\eta_x)_{x\in X}$ be an admissible coherent state system. Then the image space of V_η is a reproducing kernel Hilbert space, that is, the projection on its image is given by an integral operator with a reproducing kernel.*

Proof. Let $f \in L^2(X)$ be arbitrary. The projection of f on the image space of V_η is given by:

$$V_\eta V_\eta^* f(x) = \left(V_\eta^* f \mid \eta_x\right) = \int_X f(y)\,(\eta_y \mid \eta_x)\; d\mu(y). \tag{3.19}$$

For $f \in V_\eta(\mathcal{H})$,

$$f(x) = \left(V_\eta^* f \mid \eta_x\right) = \int_X f(y)\,(\eta_y \mid \eta_x)\; d\mu(y). \tag{3.20}$$

Thus the projection $V_\eta V_\eta^*$ is an integral operator with reproducing kernel $K(x, y) = (\eta_y \mid \eta_x)$. □

Remark 3.2.1.15. In practice, such reproducing kernel Hilbert spaces are useful in a various contexts. For example, they describe the space of band limited functions whose Fourier transforms has compact support [11, Section 2.1 and 2.2], and the image spaces of continuous wavelet transforms [11, Section 2.5].

Generalized continuous wavelet transforms (GCWT)

In what follows, we will include the definition of Führ's generalized continuous wavelet transform, c.f. [28, definition 2.13]. First, let us state the following lemma.

Lemma 3.2.1.16. *Let $(\pi, \mathcal{H}_\pi)$ denote a strongly continuous unitary representation of the locally compact group G with left Haar measure μ. For an element $\eta \in \mathcal{H}_\pi$, we define a coherent state system $(\eta_x)_{x\in G}$ as the orbit $(\pi(x)\eta)_{x\in G}$. We call this system the* ***group coherent state system****, and name the coefficient operator the* ***group coherent state transform****.*

Proof. Since the weak and strong operator topologies coincide on $\mathcal{U}(\mathcal{H}_\pi)$, the strong continuity of the representation is equivalent to the continuity of all coefficient functions $V_\eta\varphi$ for any $\varphi \in \mathcal{H}_\pi$. Since continuous functions are μ-measurable, by definition 3.2.1.1 $(\pi(x)\eta)_{x\in X}$ is a coherent state system. □

In what follows, for an element $\eta \in \mathcal{H}_\pi$, we will write $(\eta_x)_{x\in G}$ for the coherent state system $(\pi(x)\eta)_{x\in G}$ related to the representation $(\pi, \mathcal{H}_\pi)$ of the locally compact group G on $\mathcal{H}_\pi$.

Definition 3.2.1.17. Let $(\pi, \mathcal{H}_\pi)$ denote a strongly continuous unitary representation of the locally compact group G with left Haar measure μ. Let $\eta \in \mathcal{H}_\pi$ be arbitrary.

1. The vector η is called an **admissible vector** if and only if the coherent state $(\eta_x)_{x \in G}$ is admissible.

2. The coefficient operator V_η related to an admissible vector η, is called the **generalized continuous wavelet transform**

3. If the coefficient operator V_η is bounded on $\mathcal{H}_\pi$ then η is called a **bounded vector**.

3.2.2 The GCWT and the left regular representation

In this section we continue the discussion of Führ's results on the connection between the group coherent state transform and the left regular representation of the group. In particular, we remark that all representations which give admissible vectors are subrepresentations of the left regular representation. In order to explain the connection, we will discuss the kernel of the group coherent state transform, the connection between the injectivity of the transform and the cyclic property of the vector, the projection of the cyclic, bounded, and admissible vector, the intertwining property of the transform between the representation and the left regular representation. These results, will be generalized in Chapter 5.

To complete the discussion in this section, we include the discussion of relationship of bounded cyclic vectors and the left regular representation, and also representations with admissible vectors. We will generalize these to the setting of Hilbert modules in future work.

The kernel and intertwining property of the group coherent state transform

In what follows we will characterize the kernel of the group coherent state transform. As discussed in [28, page 22], the kernel of the group coherent state transform is the orthogonal complement of the span of $\{\pi(x)\eta\}_{x\in G}$ and the injectivity of the transform is equivalent to the cyclicity of the related vector. We will restate these facts here and provide proofs.

Lemma 3.2.2.1. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of G on $\mathcal{H}_\pi$ and η be an element of $\mathcal{H}_\pi$. Let $\{\pi(G)\eta\}$ be the related coherent state system and $K = \operatorname{span}\{\pi(G)\eta\}$. Then the kernel of the group coherent state transform V_η is the orthogonal complement of the closure of $K, \ker(V_\eta) = \overline{K}^{\perp}$.*

Proof. We have noticed before that for any $\varphi \in \mathcal{H}_\pi$ the coefficient function $V_\eta\varphi$ is continuous. It is also straightforward to see that it is a bounded function. Hence, without loss of generality, we may consider V_η as an operator from $\mathcal{H}_\pi$ to $C_b(G)$, the set of bounded continuous functions on G. In this setting, for any $\varphi \in \ker(V_\eta)$, $V_\eta\varphi = 0$ means $V_\eta\varphi(x) = 0$, for all $x \in G$. By definition of the coefficient function,

$$0 = V_\eta\varphi(x) = (\varphi \mid \pi(x)\eta).$$

Therefore for any $k = \sum_{i=1}^{n} \pi(x_i)\eta$ in K,

$$\begin{aligned}
(\varphi \mid k) &= \left(\varphi \mid \sum_{i=1}^{n} \pi(x_i)\eta\right) \\
&= \sum_{i=1}^{n} (\varphi \mid \pi(x_i)\eta) \\
&= \sum_{i=1}^{n} 0 \\
&= 0.
\end{aligned}$$

Since $(\cdot \mid \cdot)$ is continuous then for every $k \in \overline{K}$, $(\varphi \mid k) = 0$, i.e. If $k_n \underset{n}{\rightarrow} k$ where $k_n \in K$ then

$$(\varphi \mid k) = \left(\varphi \mid \lim_{n\to\infty} k_n\right) = \lim_{n\to\infty} (\varphi \mid k_n) = \lim_{n\to\infty} 0 = 0.$$

To prove the other direction, let $\varphi \in \overline{K}^{\perp}$. It is obvious that for every $x \in G$ $V_\eta\varphi(x) = (\pi(x)\eta \mid \varphi) = 0$. Hence we can conclude that $V_\eta\varphi = 0$ i.e. $\varphi \in ker(V_\eta)$. □

Lemma 3.2.2.2. *A vector η is a cyclic vector of the representation $(\pi, \mathcal{H}_\pi)$ of a group G on a Hilbert space $\mathcal{H}_\pi$ if and only if the coefficient operator V_η is injective.*

Proof. Let $K = \operatorname{span}\{\pi(G)\eta\}$. If we assume that η is cyclic, then by Lemma 3.2.2.1 the kernel of the transform $\ker V_\eta = \overline{K}^{\perp} = {\mathcal{H}_\pi}^{\perp} = \{0\}$. That is, the coefficient operator V_η is injective.

For the other direction, suppose that V_η is injective, i.e. $\ker(V_\eta) = \{0\}$. Suppose η is not cyclic, then $\overline{K}$ is a closed proper subset of $\mathcal{H}_\pi$ and hence $\overline{K}^{\perp} \neq \{0\}$. Together with Lemma 3.2.2.1 we have the following calculation $\{0\} = \ker(V_\eta) = \overline{K}^{\perp} \neq \{0\}$ which is a contradiction. □

Remark 3.2.2.3. We have seen that the proof of the previous lemma is based on the use of orthogonal complements, a basic property of Hilbert space. We will see that the situation is different in the setting of Hilbert C^*-modules.

It is also shown in [28, page 22] that domain of the group coherent state transform is invariant and that the transform has an intertwining property. We restate these in the following lemmas. We will also give the proof as in [28] which is a direct consequence of the definition of coefficient function.

Lemma 3.2.2.4. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of G on $\mathcal{H}_\pi$, and $\eta \in \mathcal{H}_\pi$. Then the domain $\mathcal{D}_\eta$ of the coefficient operator V_η is closed under the action of G via π.*

Proof. Let x, y be elements of G and φ an element of $\mathcal{D}_\eta \subset \mathcal{H}_\pi$. By definition

$$\begin{aligned} V_\eta(\pi(x)\varphi)(y) &= (\pi(x)\varphi \mid \pi(y)\eta) \\ &= \big(\varphi \mid \pi(x^{-1}y)\eta\big) \\ &= V_\eta\varphi(x^{-1}y) \\ &= \lambda_G(x)V_\eta\varphi(y). \end{aligned}$$

Hence $V_\eta\,(\pi(x)\varphi)$ is square integrable, and so $\pi(x)\varphi$ belongs to $\mathcal{D}_\eta$. □

With the same proof we can prove the following corollary.

Corollary 3.2.2.5. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of G on $\mathcal{H}_\pi$, and $\eta \in \mathcal{H}_\pi$. Suppose that V_η is the related coefficient operator. Then V_η intertwines π with the left regular representation.*

In fact, the intertwining property shows that we must concentrate on representations which are equivalent to subrepresentations of the left regular representation if we wish to obtain a GCWT via its admissible vectors. This is because every admissible vector is a bounded cyclic vector. The existence of a bounded cyclic vector for a representation means that the representation is equivalent to the subrepresentation of the left regular representation.

Lemma 3.2.2.6. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of G on $\mathcal{H}_\pi$, and $\eta \in \mathcal{H}_\pi$ be admissible. Then η is a bounded cyclic vector.*

Proof. By definition of an admissible coherent state system, Definition 3.2.1.1, the domain of the related group coherent state transform is the whole space, hence η is a bounded vector. Now, suppose that φ is in the kernel of the transform. By the isometry property of the transform, we have

$$0 = \|V_\eta \varphi\| = \|\varphi\| .$$

That is V_η is injective. Hence by Lemma 3.2.2.2, η is a cyclic vector. $\square$

The commuting algebra and bounded, cyclic or admissible vectors

In order to focus on subrepresentations of the left regular representation, we will need Führ's result on the action of the commuting algebra of admissible, cyclic or bounded vectors.

Proposition 3.2.2.7. *Let $(\pi, \mathcal{H}_\pi)$ be a representation of G on $\mathcal{H}_\pi$ and $\eta \in \mathcal{H}_\pi$. If T is in the commuting algebra $\pi(G)'$ then*

$$V_{T\eta} = V_\eta \circ T^*.$$

Proof. Suppose that $\varphi \in \mathcal{H}_\pi$ and $x \in G$ are arbitrary. By definition of the coefficient function,

$$\begin{aligned}
(V_{T\eta}\, \varphi)(x) &= (\varphi \mid \pi(x) T\eta) \\
&= (\varphi \mid T\pi(x)\eta) \\
&= (T^*\varphi \mid \pi(x)\eta) \\
&= (V_\eta T^* \varphi)(x) \\
&= (V_\eta \circ T^*\, \varphi)(x) .
\end{aligned}$$

From this we can see that $V_{T\eta} = V_\eta \circ T^*$. $\square$

Corollary 3.2.2.8. *Suppose that K is an invariant closed subspace of $\mathcal{H}_\pi$, with projection operator P_K. If $\eta \in \mathcal{H}_\pi$ is admissible (respectively bounded or cyclic) for $(\pi, \mathcal{H}_\pi)$ then $P_K\eta$ has the same property for the subrepresentation $(\pi\mid_K, K)$.*

Proof. Since P_K is a projection, $P_K = P_K^*$, and hence by proposition 5.2.2.11

$$V_{P_K \eta} = V_\eta \circ P_K^* = V_\eta \circ P_K.$$

Now let us calculate the domains of these operators. By definition, the domain $\mathcal{D}_{P_K \eta} = \mathcal{D}(V_\eta \circ P_K) = \{\varphi \in \mathcal{H}_\pi \mid P_K \varphi \in \mathcal{D}_\eta\}$. Hence it is straightforward to see that as an operator on $K, V_{P_K \eta} = V_\eta \mid_K$.

If η is an admissible vector, then, by definition the domain $\mathcal{D}_\eta$ of V_η is equal to $\mathcal{H}_\pi$. Since $K \subset \mathcal{H}_\pi = \mathcal{D}_\eta$ then as an operator on K, $\mathcal{D}_{P_K\eta} = \mathcal{D}_\eta \cap K = \mathcal{H}_\pi \cap K = K$. Since the restriction of an isometry is also an isometry, $V_{P_K\eta}$ is an isometry on K with domain the whole space. Hence by definition $P_K\eta$ is an admissible vector. Furthermore by the same argument, $P_K\eta$ is bounded if η is a bounded vector (η is bounded if V_η is bounded i.e. $\mathcal{D}_\eta = \mathcal{H}_\pi$). Finally, by Lemma 3.2.2.1, and by the fact that the restriction of an injection is an injection, if η is cyclic for $(\pi, \mathcal{H}_\pi)$ then $P_K\eta$ also cyclic for $(\pi \mid_K, K)$. □

A similar result also holds for unitary intertwining operators.

Corollary 3.2.2.9. *Let T be a unitary operator intertwining the representations π and σ. Then η is admissible (respectively bounded or cyclic) if and only if $T\eta$ has the same property.*

Proof. By definition, T is a map $T : \mathcal{H}_\pi \to \mathcal{H}_\sigma$ which is unitary, and for any $x \in G$, $T\pi(x) = \sigma(x)T$. By a similar argument as before, $\psi \in \mathcal{D}_{T\eta}$ if and only if $T^*\psi \in \mathcal{D}_\eta$. We know that η is admissible if $\mathcal{D}_\eta = \mathcal{H}_\pi$, and by definition of adjoint operator $T^*(\psi) \in \mathcal{H}_\pi$, for all $\psi \in \mathcal{H}_\sigma$. Thus, $T^*(\psi) \in \mathcal{D}_\eta$ and hence $\psi \in \mathcal{D}_{T\eta}$ for all $\psi \in \mathcal{H}_\sigma$. It follows that $\mathcal{D}_{T\eta} = \mathcal{H}_\sigma$. From this we can see that if η is a bounded vector for π then $T\eta$ is a bounded vector for σ.

Now, if V_η is an isometry, then

$$\begin{aligned}\|V_{T\eta}\psi\| &= \|(V_\eta \circ T^*)\,\psi\| \\ &= \|V_\eta(T^*\psi)\| \\ &= \|T^*\psi\|\,.\end{aligned}$$

Now, T^* is unitary, hence is an isometry and so

$$\|T^*\psi\| = \|\psi\|\,.$$

This implies that $V_{T\eta}$ is also an isometry. Together with the result of the previous paragraph, we see that if η is an admissible vector then $T\eta$ is also admissible.

Now, suppose $\psi \in \ker(V_{T\eta})$. We then have $0 = V_{T\eta}\,(\psi) = (V_\eta \circ T^*)\,(\psi) = V_\eta(T^*\psi)$. If V_η is an injection, then $T^*\psi = 0$. since T is unitary,

$$\psi = TT^*\psi = T(0) = 0.$$

In other words $\ker(V_{T\eta}) = \{0\}$, or $V_{T\eta}$ is an injection.

For the other direction, we use the same argument for T^* as a unitary intertwining operator, i.e. we prove that if $T\eta$ is admissible (respectively bounded or cyclic) then $\eta = T^*T\eta$ has the same property.

□

Bounded cyclic vectors and the left regular representation

Here we review the fact that representations having bounded cyclic vectors, are equivalent to subrepresentations of the left regular representation, [28, Proposition 2.16 (b)]

Proposition 3.2.2.10. *If a representation π of G on $\mathcal{H}_\pi$ has a cyclic vector η for which V_η is densely defined, then the group coherent state transform V_η is an isometric intertwining operator between $\mathcal{H}_\pi$ to $L^2(G,\mu)$, and hence $\pi < \lambda_G$.*

For the left regular representation λ_G and its subrepresentations we have the following existence theorem for bounded cyclic vectors.

Corollary 3.2.2.11. *If a representation π of G on $\mathcal{H}_\pi$ has a bounded cyclic vector η, then $\pi < \lambda_G$.*

Theorem 3.2.2.12. *There exists a bounded cyclic vector for λ_G. Hence, an arbitrary representation π has a bounded cyclic vector if and only if $\pi < \lambda_G$.*

Remark 3.2.2.13. The existence of a bounded cyclic vector for λ_G has been proved by Losert and Rindler, for the first countable group G. By recalling that every second countable group is first countable, we have the first result in the above theorem. See proof of [28, Theorem 2.21].

Representations with admissible vectors

We have seen in the previous section, that an admissible vector is bounded and also cyclic. On the other hand, the existence of bounded cyclic vector of a representation guarantees that the representation is unitarily equivalent to a subrepresentation of the left regular representation. This fact allows us to concentrate on the left regular representation and its subrepresentations to answer the question: Which representations π have admissible vectors?

Subrepresentations of λ_G which are irreducible are called **discrete series representations**. The existence of admissible vectors for this kind of representations is guaranteed. More generally an irreducible representation π has admissible vectors if and only if $\pi < \lambda_G$. A complete characterization of admissible vectors for such representations is given in [28, Theorem 2.25]. This theory was originally proved by Grossman, Morlet and Paul [55], using tools established by Duflo and Moore in [13].

Furthermore, one of the conclusions in the discussion of relations between continuous wavelet transforms and λ_G in [28, Section 2.5] says that a necessary condition for a representation π to have admissible vectors is that $\pi < \lambda_G$. For nondiscrete unimodular groups, this property is not sufficient. A detailed characterization of which subrepresentations of λ_G have admissible vectors is given in [28, Theorem 4.22].

3.2.3 An example

As an example, we will study the original continuous wavelet transform and its admissibility criteria defined using the approach introduce in this section.

Example 3.2.3.1. 1D-CWT Let $G = \mathbb{R} \rtimes \mathbb{R}^+$ the $ax+b$-group. Recall that the group multiplication is given by $(b,a)(b',a') = (b+ab', aa')$ and the left Haar measure is given by $|a|^{-2}dadb$.

Let π be the **quasi-regular representation** of G acting on $L^2(\mathbb{R})$ via

$$(\pi(b,a)f)(x) = |a|^{-1/2} f\left(\frac{x-b}{a}\right) \text{ and } (\pi(b,a)f)\hat{}(\omega) = |a|^{1/2} e^{-i\omega b}\hat{f}(a\omega).$$

Note that $L^2(\mathbb{R})$ here means $L^2(\mathbb{R},\mathbb{R})$, the set of square integrable real valued functions on $\mathbb{R}$.

Using the Plancherel theorem, the Parseval identity and Fubini's theorem we see that for any functions $f, g \in L^2(\mathbb{R})$, $\|V_f\, g\|_2^2 = \|g\|^2 {c_f}^2$ where

$${c_f}^2 = \int \frac{|\hat{f}(\omega)|^2}{|\omega|} d\omega.$$

In this case, the admissibility condition arises from the isometry property of V_f i.e.

$$f \in L^2(\mathbb{R}) \text{ is admissible } \Leftrightarrow c_f = 1.$$

In the calculation, we will assume that $\phi_a(\omega) = \hat{g}(\omega)\hat{f}(a\omega)$. Then

$$\|V_f\, g\|_2^2 = \int_G |(g \mid \pi(b,a)f)|^2 \, d\, \mu_G\left((b,a)\right).$$

Using Parseval's identity we have

$$\begin{aligned}
\|V_f\, g\|_2^2 &= \int_G |(\hat{g} \mid \pi(b,a)\hat{f})|^2 \, d\, \mu_G\left((b,a)\right) \\
&= \int_{\mathbb{R}'}\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \hat{g}(\omega)|a|^{1/2} e^{-i\omega b}\hat{f}(a\omega) \right|^2 |a|^{-2} da db. \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \hat{g}(\omega) e^{i\omega b}\bar{\hat{f}}(a\omega) \right|^2 |a|^{-1} da db.
\end{aligned}$$

Note that f is real-valued, therefore,

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \hat{g}(\omega) e^{i\omega b}\bar{\hat{f}}(a\omega) \right|^2 |a|^{-1} da db &= \int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \hat{g}(\omega) e^{i\omega b}\hat{f}(a\omega) \right|^2 |a|^{-1} da db \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \phi_a(\omega) e^{i\omega b} \right|^2 |a|^{-1} da db \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \hat{\phi}_a(-b) \right|^2 |a|^{-1} da db.
\end{aligned}$$

The measure db is unimodular, hence

$$\int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \hat{\phi}_a(-b) \right|^2 |a|^{-1} da db = \int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \hat{\phi}_a(b) \right|^2 |a|^{-1} da db.$$

Applying the Plancherel formula to the last equation, we get:

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \hat{\phi}_a(b) \right|^2 |a|^{-1} da db &= \int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \phi_a(b) \right|^2 |a|^{-1} da db \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}'} \left| \int_{\mathbb{R}} \hat{g}(b)\hat{f}(ab) \right|^2 |a|^{-1} da db.
\end{aligned}$$

Finally, apply Fubini's theorem and use the fact that $a^{-1}da$ is Haar measure of $\mathbb{R}'$. We can thus re-write the last equation as:

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}'}\left|\int_{\mathbb{R}}\hat{g}(b)\hat{f}(ab)\right|^2|a|^{-1}dadb &= \int_{\mathbb{R}'}\left|\hat{f}(ab)\right|^2|a|^{-1}\,da\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\hat{g}(b)\right|^2\,db\\ &= {c_f}^2\left\|\hat{g}\right\|^2\\ &= {c_f}^2\left\|g\right\|^2.\end{aligned}$$

Chapter 4

The Hilbert $\mathcal{A}$-Module $\mathbb{L}^2(X, \mathcal{A})$

From now on we will assume that our C^*-algebra $\mathcal{A}$ is unital.

Basically, this chapter introduces the Hilbert module $\mathbb{L}^2(X, \mathcal{A})$ over a C^*-algebra $\mathcal{A}$, which contains the space of norm-square integrable $\mathcal{A}$-valued functions $L^2(X, \mathcal{A})$. This is the main Hilbert module that is used in Chapter 5. If in Chapter 3, the Hilbert space $L^2(X)$ of scalar-valued square integrable function plays the main rule in the definition of continuous wavelet transform, in Chapter 5, the Hilbert module $\mathbb{L}^2(X, \mathcal{A})$ plays a similar rule.

Instead of dealing with scalar complex valued coefficient functions, the generalization process of GCWT to Hilbert $\mathcal{A}$-modules defined in Chapter 5 naturally leads us to work with $\mathcal{A}$-valued coefficient functions, c.f. Definition 3.2.1.1.a and Definition 5.2.1.5.a. This is because the inner product of our Hilbert modules take values in $\mathcal{A}$. Therefore, we need to use vector-valued integration theory in the generalization, specifically, integration theory for $\mathcal{A}$-valued functions.

We will use the Bochner integral as a generalization of Lebesgue integral for vector-valued integrals, to work with our $\mathcal{A}$-valued functions. As a result, we show that the existence of involution and the notion of positive elements in $\mathcal{A}$ lead to further properties of the Bochner integral for $\mathcal{A}$-valued functions. We discuss these results in section 4.1.1.

To generalize the Hilbert space $L^2(X)$, to a Hilbert $\mathcal{A}$-module developed from the space of $\mathcal{A}$-valued functions, $\mathbb{L}^2(X,\mathcal{A})$, we follow the process used to generalize the standard Hilbert space ℓ_2 to the standard Hilbert module $\mathbb{H}_{\mathcal{A}}$. We start by defining a space of $\mathcal{A}$-valued functions f for which the Bochner integral $\mathbf{B}$ - $\int_X f(x)^* f(x)\ d\mu(x)$ converges. As a result, we show that the C^*-condition implies the space defined is precisely the space of norm-square integrable $\mathcal{A}$-valued functions $L^2(X,\mathcal{A})$, see Remark 4.1.2.3. We then show that the space of $\mathcal{A}$-valued simple functions which we denote by $F(X,\mathcal{A})$, is a pre-Hilbert module such that its Hilbert module norm $\|\cdot\|_{\mathcal{A}} \leq \|\cdot\|_2$. This shows that its completion with the norm $\|\cdot\|_{\mathcal{A}}$ is possibly bigger than the completion with the norm $\|\cdot\|_2$, i.e. $L^2(X,\mathcal{A}) \subseteq \mathbb{L}^2(X,\mathcal{A})$. We show that the inner product in $L^2(X,\mathcal{A})$ has the form $\langle f,g\rangle = \mathbf{B}$ - $\int_X f(x)^* g(x)\ d\mu(x)$ for any $f,g \in L^2(X,\mathcal{A})$. We know that if $\mathcal{A} = \mathbb{C}$, the $\|\cdot\|_{\mathcal{A}} = \|\cdot\|_2$ and thus space $L^2(X,\mathcal{A})$ is complete with the norm induced by the $\mathcal{A}$-inner product. However we give an example to show that for some $\mathcal{A}$, $L^2(X,\mathcal{A})$ is not necessarily complete. Discussion concerning this matter can be found in section 4.1.2.

Finally, in section 4.2, we explore more properties of the Hilbert module $\mathbb{L}^2(X,\mathcal{A})$. We give a calculation to show that $\mathbb{L}^2(X,\mathcal{A})$ is equivalent to the Hilbert module tensor product $L^2(X)\widehat{\otimes}\mathcal{A}$. It will follow that $\mathbb{L}^2(X,\mathcal{A})$ is the completion of the pre-Hilbert module $C_c(X,\mathcal{A})$, (the continuous compactly supported functions from X to $\mathcal{A}$,) in the sense of [53, Definition 2.2].

Actually, we show that it is possible to generalize the concept to develop the theory of Hilbert $\mathcal{A}$-modules starting from the space of norm-square integrable Hilbert $\mathcal{A}$-module-valued functions. Using the definition of the

Hilbert $\mathcal{A}$-module norm as a generalization of the C^*-condition, we can show the result that the pre-Hilbert modules defined using Bochner integral theory are precisely the norm-square integrable Hilbert $\mathcal{A}$-module-valued functions. However, here, we restrict ourselves to the Hilbert module $\mathbb{L}^2(X, \mathcal{A})$ which is the completion of the pre-Hilbert module $L^2(X, \mathcal{A})$ of norm-square integrable $\mathcal{A}$-valued functions.

We further show that $\mathbb{L}^2(X, \mathcal{A})$ is a full Hilbert module, and give some results that characterize $\mathbb{L}^2(X, \mathcal{A})$ which are related to the Hilbert space $L^2(X)$ which is separable.

4.1 The $L^2(X, \mathcal{A})$

Here we will discuss some results of $\mathcal{A}$-valued Bochner integral functions and use them to show that the norm-square integrable functions, $L^2(X, \mathcal{A})$ is a pre-Hilbert $\mathcal{A}$-module. We will denote its completion $\mathbb{L}^2(X, \mathcal{A})$.

4.1.1 The Bochner integral of $\mathcal{A}$-valued functions

Since in this chapter our functions will be $\mathcal{A}$-valued, we will need some results related to $\mathcal{A}$-valued Bochner integrable functions, $L^1(X, \mathcal{A})$, in addition to the distribution property for general vector valued integrable functions in Theorem 1.5.2.7.

Notation. For any $\mathcal{A}$-valued function f on X and $a \in \mathcal{A}$, let us define

$$(f \cdot a)(x) = f(x)a \quad \text{and} \quad \tilde{f}(x) = f(x)^*, \quad x \in X.$$

Lemma 4.1.1.1. *Let f be an $\mathcal{A}$-valued strongly measurable function on X. Then $\tilde{f}$ is strongly measurable.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of countably-valued functions which converges to f a.e. It is clear that $\tilde{f}_n$ is countably-valued. Since the involution is continuous, for any $x \in X$, $f_n(x) \to f(x)$ implies $f_n(x)^* \to f(x)^*$. Hence $\tilde{f}_n \to \tilde{f}$ a.e. By definition, $\tilde{f}$ is strongly measurable. □

Lemma 4.1.1.2. *Let f be an $\mathcal{A}$-valued strongly measurable function on X, and $a \in \mathcal{A}$, then $f \cdot a$ is strongly measurable.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of countably-valued functions that converges a.e to f. Since multiplication in $\mathcal{A}$ is continuous, then for any $x \in X$, $f_n(x)a \to f(x)a$. Hence $f_n \cdot a \to f \cdot a$ a.e. □

Theorem 4.1.1.3. *Let f be an element in $L^1(X, \mathcal{A})$ and $(f_n)_{n\in\mathbb{N}}$ be a sequence that gives the Bochner integral of f. Let $a \in \mathcal{A}$. The following are satisfied:*

a. *The function $f \cdot a$ belongs to $L^1(X, A)$ and*

$$\mathbf{B}\text{ - }\int_X f(x)a \; d\mu(x) = \left(\mathbf{B}\text{ - }\int_X f(x) \; d\mu(x)\right) a.$$

b. *$\tilde{f} \in L^1(X, \mathcal{A})$ and $\mathbf{B}$ - $\int_X \tilde{f}(x)\; d\mu(x) = \lim_{n\to\infty} \mathbf{B}$ - $\int_X \tilde{f}_n(x)\; d\mu(x)$.*

Proof. **a.** The theorem is true for countably-valued integrable functions f; $f \cdot a$ is countably-valued and

$$\begin{aligned}
\int_X \|(f \cdot a)(x)\| \; d\mu(x) &= \int_X \|f(x)a\| \; d\mu(x) \\
&\leq \int_X \|f(x)\| \, \|a\| \; d\mu(x) \\
&= \int_X \|f(x)\| \; d\mu(x) \, \|a\| \\
&= \|f\|_1 \, \|a\| < \infty.
\end{aligned}$$

This implies that $f \cdot a$ is integrable. Now, let us write

$$f = \sum_{k=1}^{\infty} v_k \chi_k$$

where $f(x) = v_k$ on disjoint sets $E_k \in \Sigma$ $(k = 1, 2, 3...)$. Then, by definition, for any $E \in \Sigma$:

$$\begin{aligned}
\mathbf{B}\text{ - }\int_E f(x)a \; d\mu(x) &= \lim_{n\to\infty} \left(\sum_{k=1}^{n} v_k a \mu(E_k \cap E) \right) \\
&= \lim_{n\to\infty} \left(\left(\sum_{k=1}^{n} v_k \mu(E_k \cap E) \right) a \right) \\
&= \left(\lim_{n\to\infty} \sum_{k=1}^{n} v_k \mu(E_k \cap E) \right) a \\
&= \mathbf{B}\text{ - }\int_E f(x) \; d\mu(x) a.
\end{aligned}$$

For the general case, let $(f_n)_{n\in\mathbb{N}}$ be the sequence of countably-valued integrable functions that defines the Bochner integral of f. Hence, it is convergent a.e. to f and $\lim_{n\to\infty} \int_X \|f(x) - f_n(x)\| \; d\mu(x) = 0$. By Lemma 4.1.1.2 and its proof, the function $f \cdot a$ is strongly measurable and $\{f_n \cdot a\}$ are countably-valued integrable functions such that $f_n \cdot a \to f \cdot a$ a.e. Therefore

$$\begin{aligned}
\int_X \|f_n(x)a - f(x)a\| \, d\mu(x) &= \int_X \|(f_n(x) - f(x))\, a\| \; d\mu(x) \\
&\leq \int_X \|(f_n(x) - f(x))\| \, \|a\| \; d\mu(x) \\
&= \int_X \|(f_n(x) - f(x))\| \; d\mu(x) \, \|a\| \to 0.
\end{aligned}$$

Using the result for the countably-valued case, and the continuity of multiplication in $\mathcal{A}$, we obtain also

$$\begin{aligned}
\mathbf{B}\text{ - }\int_X f(x)a\, d\mu(x) &= \lim_{n\to\infty} \mathbf{B}\text{ - }\int_X f_n(x)a\, d\mu(x) \\
&= \lim_{n\to\infty}\left(\mathbf{B}\text{ - }\int_X f_n(x)\, d\mu(x)a\right) \\
&= \left(\lim_{n\to\infty}\mathbf{B}\text{ - }\int_X f_n(x)\, d\mu(x)\right)a \\
&= \left(\mathbf{B}\text{ - }\int_X f(x)\, d\mu(x)\right)a.
\end{aligned}$$

b. By Lemma 4.1.1.1, $\tilde{f}$ is strongly measurable. Since the involution in $\mathcal{A}$ preserves the norm, $\|f(x)^*\| = \|f(x)\|$, for each $x \in X$. We know that $\|f(\cdot)\|$ is Lebesgue integrable, and so $\|f(\cdot)^*\| = \left\|\tilde{f}(\cdot)\right\|$ is also Lebesgue integrable. Hence by the characterization of Bochner integrable function given in Theorem 1.5.2.6, $\tilde{f}$ is Bochner integrable. If $\{f_n\}_{n\in\mathbb{N}}$ defines the Bochner integral of f, $f_n \to f$ a.e. and

$$\begin{aligned}
&\left\|\mathbf{B}\text{ - }\int_X f(x)^*\, d\mu(x) - \mathbf{B}\text{ - }\int_X f_n(x)^*\, d\mu(x)\right\| \\
&\leq \int_X \|f(x)^* - f_n(x)^*\|\ d\mu(x) \\
&= \int_X \|(f(x) - f_n(x))^*\|\ d\mu(x) \\
&= \int_X \|f(x) - f_n(x)\|\ d\mu(x) \to 0.
\end{aligned}$$

By definition, $\mathbf{B}\text{ - }\int_X f(x)^*\, d\mu(x) = \lim_{n\to\infty} \mathbf{B}\text{ - }\int f_n(x)^*\, d\mu(x)$.

□

Theorem 4.1.1.4. *Let f and g be strongly measurable functions. If $fg \in L^1(X, A)$, then* $\left(\mathbf{B}\text{ - }\int_X f(x)g(x)\, d\mu(x)\right)^* = \mathbf{B}\text{ - }\int_X g(x)^* f(x)^*\, d\mu(x)$

Proof. It is easy to see that by the continuity of the involution, the theorem holds for countably-valued functions f, g. Now, we will prove the more general case. Since f, g are strongly measurable, then by Theorem 1.5.1.5 fg is strongly measurable. Let $\{f_n\}$ and $\{g_n\}$ be sequences of countably-valued functions that converge a.e. to f and g respectively, such that $\{f_n g_n\}$ be a sequence of countably-valued integrable functions that define the Bochner integral of fg. Note that by Theorem 4.1.1.3 (b), $\tilde{g}\tilde{f} = \widetilde{fg}$ is integrable and

$$\mathbf{B}\text{ - }\int g(x)^* f(x)^*\, d\mu(x) = \mathbf{B}\text{ - }\int_X \widetilde{fg}(x)\, d\mu(x) \tag{4.1}$$
$$= \lim_{n\to\infty} \mathbf{B}\text{ - }\int_X \widetilde{f_n g_n}(x)\, d\mu(x) \tag{4.2}$$
$$= \lim_{n\to\infty} \mathbf{B}\text{ - }\int_X g_n(x)^* f_n(x)^*\, d\mu(x). \tag{4.3}$$

Since involution in $\mathcal{A}$ is continuous, together with the definition of Bochner integral, equation (4.3) gives

$$\left(\mathbf{B}\text{ - }\int_X f(x)g(x)\, d\mu(x)\right)^* = \left(\lim_{n\to\infty} \mathbf{B}\text{ - }\int_X f_n(x)g_n(x)\, d\mu(x)\right)^* \tag{4.4}$$
$$= \lim_{n\to\infty} \left(\mathbf{B}\text{ - }\int_X f_n(x)g_n(x)\, d\mu(x)\right)^* \tag{4.5}$$
$$= \lim_{n\to\infty} \left(\mathbf{B}\text{ - }\int_X g_n(x)^* f_n(x)^*\, d\mu(x)\right) \tag{4.6}$$
$$= \mathbf{B}\text{ - }\int_X g(x)^* f(x)^*\, d\mu(x). \tag{4.7}$$

□

4.1.2 The Pre-Hilbert $\mathcal{A}$-module $L^2(X,\mathcal{A})$

Here, we will introduce $L^2(X,\mathcal{A})$, the norm-square integrable $\mathcal{A}$-valued functions, as a pre-Hilbert $\mathcal{A}$-module.

We start by considering a space L of $\mathcal{A}$-valued functions f on X such that the Bochner integral $\mathbf{B}\text{ - }\int_X f(x)^* f(x)\, d\mu(x)$ converges.

Definition 4.1.2.1. An $\mathcal{A}$-valued function on X is an element of L if and only if f is strongly measurable and the Bochner integral $\mathbf{B}\text{ - }\int_X f(x)^* f(x)\, d\mu(x)$ converges.

In what follows we will show that L coincides with the space of norm-square integrable $\mathcal{A}$-valued functions $L^2(X,\mathcal{A})$. It is worth noting that if $\mathcal{A}$ is a general Banach *-algebra, this is possibly not always true. On the other words, an extra work will be needed to see wether the result is applied for the general case. Therefore, we will only concentrate on C^*-algebra $\mathcal{A}$ in this thesis. The C^*-condition of $\mathcal{A}$ leads to the following results.

Theorem 4.1.2.2. *Let f be an $\mathcal{A}$-valued strongly measurable function on X. Then, the Bochner integral $\mathbf{B}$ - $\int_X f(x)^*f(x)\ d\mu(x)$ converges if and only if*

$$\int_X \|f(x)\|^2\ d\mu(x) < \infty.$$

Furthermore, if the Bochner integral $\mathbf{B}$ - $\int_X f(x)^*f(x)\ d\mu(x)$ converges then

$$\left\|\mathbf{B}\text{ - }\int_X f(x)^*f(x)\ d\mu(x)\right\| \leq \int_X \|f(x)\|^2\ d\mu(x) < \infty.$$

Proof. Let f be an $\mathcal{A}$-valued strongly measurable function on X and suppose that the Bochner integral $\mathbf{B}$ - $\int_X f(x)^*f(x)\ d\mu(x)$ converges. By Theorem 1.5.2.6 we know that

$$\int_X \|f(x)^*f(x)\|\ d\mu(x) < \infty.$$

Since the values of f belong to $\mathcal{A}$ then the C^*-condition, gives

$$\|f(x)^*f(x)\| = \|f(x)\|^2 \ \text{ for all } x \in X.$$

Hence

$$\int_X \|f(x)\|^2\ d\mu(x) = \int_X \|f(x)^*f(x)\|\ d\mu(x) < \infty.$$

For the other direction, let f be an $\mathcal{A}$-valued strongly measurable function on X and $\int_X \|f(x)\|^2\ d\mu(x) < \infty$. By Lemma 4.1.1.1 $\tilde{f}$ is strongly measurable. The product of strongly measurable functions is strongly measurable, Theorem 1.5.1.5 (3). Hence $\tilde{f}f$ is strongly measurable. The C^*-condition gives:

$$\int_X \|f(x)^* f(x)\| \ d\mu(x) = \int_X \|f(x)\|^2 \ d\mu(x) < \infty.$$

Again, we use Theorem 1.5.2.6 to conclude that the Bochner integral

$$\mathbf{B}\text{ - }\int_X f(x)^* f(x) \, d\mu(x)$$

converges. □

Remark 4.1.2.3. Theorem 4.1.2.2 says that L is precisely the space $L^2(X, \mathcal{A})$. Therefore, in what follows, we will use the notation $L^2(X, \mathcal{A})$ instead of L to denote the space defined in definition 4.1.2.1. Recall that inner product in $\mathcal{A}$ as a Hilbert $\mathcal{A}$-module is defined as $\langle a, b \rangle = a^* b$. This allows us to define a space of strongly measurable $\mathbb{H}$-valued functions f such that $\mathbf{B}$ - $\int_X \langle f(x), f(x) \rangle \ d\mu(x)$ converges. We know from the definition of the Hilbert module norm, that $\|\langle f, f \rangle\| = \|f\|^2$. In a similar way, we also have a result that the space coincides with the square integrable $\mathbb{H}$-valued functions. We will consider first the case of $\mathcal{A}$-valued functions.

Now we will discuss $L^2(X, \mathcal{A})$ as a pre-Hilbert $\mathcal{A}$-module. First we show that there is an action of $\mathcal{A}$ on $L^2(X, \mathcal{A})$, which allows us to define the structure of a right $\mathcal{A}$-module on $L^2(X, \mathcal{A})$.

Proposition 4.1.2.4. *Let $f \in L^2(X, \mathcal{A})$ and $a \in \mathcal{A}$. Then $f \cdot a \in L^2(X, \mathcal{A})$*

Proof. From Lemma 4.1.1.2 we know that $f \cdot a$ is strongly measurable. Now,

$$\begin{aligned} \int_X \|(f \cdot a)(x)\|^2 \ d\mu(x) &= \int_X \|f(x)a\|^2 \ d\mu(x) \\ &\leq \int_X \|f(x)\|^2 \|a\|^2 \ d\mu(x) \\ &= \left(\int_X \|f(x)\|^2 \ d\mu(x) \right) \|a\|^2 < \infty. \end{aligned}$$

By definition, $f \cdot a$ is norm square integrable. □

Before we discuss about the existence of an $\mathcal{A}$-inner product in $L^2(X,\mathcal{A})$, we will discuss the existence of an $\mathcal{A}$-inner product in the subspace $F(X,\mathcal{A})$ of $\mathcal{A}$-valued simple functions on X. Recall that $F(X,\mathcal{A})$ is a dense subspace of $L^2(X,\mathcal{A})$ relative to $\|\cdot\|_2$. It is also easy to show that $F(X,\mathcal{A})$ is a pre-Hilbert module with the action defined in Proposition 4.1.2.4, and an $\mathcal{A}$-inner product which is given by: for any simple functions $f = \sum_{i=1}^N \chi_{E_i} a_i$ and $g = \sum_{j=1}^M \chi_{F_j} b_j$, where $\{E_i\}$ are disjoint sets and $\{F_j\}$ are disjoint sets,

$$\begin{aligned}
\langle f, g\rangle &= \left\langle \sum_{i=1}^N \chi_{E_i} a_i, \sum_{j=1}^M \chi_{F_j} b_j \right\rangle \\
&= \mathbf{B}\text{ - }\int_X (\sum_{i=1}^N \chi_{E_i} a_i)^* \sum_{j=1}^M \chi_{F_j} b_j \; d\mu(x) \\
&= \mathbf{B}\text{ - }\int_X \sum_{i=1}^N \sum_{j=1}^M \chi_{E_i \cap F_j} a_i^* b_j \; d\mu(x) \\
&= \sum_{i=1}^N \sum_{j=1}^M a_i^* b_j \mu(E_i \cap F_j).
\end{aligned}$$

Notation. Let us denote the Hilbert module completion of $F(X,A)$ with the norm $\|\cdot\|_{\mathcal{A}}$, $\overline{F(X,\mathcal{A})}_{\|\cdot\|_{\mathcal{A}}}$, by $\mathbb{L}^2(X,\mathcal{A})$.

Lemma 4.1.2.5. *For any simple function $f \in F(X,\mathcal{A})$, $\|f\|_2 \geq \|f\|_{\mathcal{A}}$.*

Proof. Let $f = \sum_{i=1}^N \chi_{E_i} a_i$ where $\{E_i\}$ are disjoint sets,

$$\begin{aligned}
\|f\|_2^2 &= \int_X \|f(x)\|^2 \; d\mu(x) = \int_X \left\| \sum_{i=1}^N \chi_{E_i}(x) a_i \right\|^2 \; d\mu(x) \\
&= \sum_{i=1}^N \int_{E_i} \|\chi_{E_i}(x) a_i\|^2 \; d\mu(x) = \sum_{i=1}^N \|a_i\|^2 \mu(E_i) \\
&= \sum_{i=1}^N \|a_i^* a_i\| \mu(E_i) \geq \left\| \sum_{i=1}^N a_i^* a_i \mu(E_i) \right\| = \|\langle f, f\rangle\|.
\end{aligned}$$

$\square$

This implies that in $F(X,A)$ a Cauchy sequence in the $\|\cdot\|_2$ norm is Cauchy in the $\|\cdot\|_{\mathcal{A}}$ norm. This gives that the completion of $F(X,A)$ with the norm $\|\cdot\|_{\mathcal{A}}$ is the same or larger than $L^2(X,\mathcal{A})$.

Corollary 4.1.2.6. $F(X,\mathcal{A}) \subset L^2(X,\mathcal{A}) = \overline{F(X,\mathcal{A})}_{\|\cdot\|_2} \subseteq \overline{F(X,\mathcal{A})}_{\|\cdot\|_{\mathcal{A}}} = \mathbb{L}^2(X,\mathcal{A})$.

Corollary 4.1.2.7. *$L^2(X,\mathcal{A})$ is a pre-Hilbert module whose $\mathcal{A}$-inner product extends from the $\mathcal{A}$-inner product of $F(X,A)$.*

Proof. The fact that $F(X,\mathcal{A})$ is a dense submodule of the Hilbert module $\mathbb{L}^2(X,\mathcal{A})$ such that $F(X,\mathcal{A}) \subset L^2(X,\mathcal{A}) \subset \mathbb{L}^2(X,\mathcal{A})$ gives the result. □

Lemma 4.1.2.8. *For any function $f \in L^2(X,\mathcal{A})$, $\|f\|_2 \geq \|f\|_{\mathcal{A}}$.*

Proof. Let (f_n) be a Cauchy sequence in $\|\cdot\|_a$ of simple functions such that $\|f - f_n\|_2 \to 0$. By Lemma 4.1.2.5, (f_n) is also Cauchy in $\|\cdot\|_{\mathcal{A}}$ and

$$\|f - f_n\|_{\mathcal{A}} = \lim_{m\to\infty} \|f_m - f_n\|_{\mathcal{A}} \leq \lim_{m\to\infty} \|f_m - f_n\|_2 = \|f - f_n\|_{\mathcal{A}} \to 0.$$

Therefore, $\|f\|_2 = \lim_{n\to\infty} \|f_n\|_2$ and $\|f\|_{\mathcal{A}} = \lim_{n\to\infty} \|f_n\|_{\mathcal{A}}$. For any $n \in \mathbb{N}$ by Lemma 4.1.2.5

$$\|f_n\|_{\mathcal{A}} \leq \|f_n\|_2$$

therefore

$$\|f\|_{\mathcal{A}} = \lim_{n\to\infty} \|f_n\|_{\mathcal{A}} \leq \lim_{n\to\infty} \|f_n\|_2 = \|f\|_2 .$$

□

Theorem 4.1.2.9. *If we view $L^2(X,\mathcal{A})$ as a space of equivalence classes of functions which differ only on a null set, then*

$$\langle f, g\rangle = \mathbf{B}\text{ -}\int_X \langle f(x), g(x)\rangle \; d\mu(x)$$

where $f, g \in L^2(X,\mathcal{A})$, defines an $\mathcal{A}$-valued inner product in $L^2(X,\mathcal{A})$.

Proof. First of all we need to show that $\langle f,g\rangle$ converges for all $f,g\in L^2(X,\mathcal{A})$. Note that the inner product in $\mathcal{A}$ is defined by $\langle a,b\rangle = a^*b$ for all $a,b\in\mathcal{A}$. Hence,

$$\langle f,g\rangle = \mathbf{B}\text{ - }\int_X \langle f(x),g(x)\rangle\ d\mu(x) = \mathbf{B}\text{ - }\int_X f(x)^*g(x)\ d\mu(x). \tag{4.8}$$

Since $f,g\in L^2(X,\mathcal{A})$, they are both strongly measurable and are norm-square integrable: $\|f(\cdot)\|$ and $\|g(\cdot)\|$ are both in $L^2(X)$. By [69, theorem 3.8] the function $\|f(\cdot)\|\,\|g(\cdot)\|$ belongs to $L^1(X)$. Hence,

$$\int_X \|f(x)^*g(x)\|\ d\mu(x) \le \int_X \|f(x)^*\|\,\|g(x)\|\ d\mu(x) \tag{4.9}$$

$$= \int_X \|f(x)\|\,\|g(x)\|\ d\mu(x) < \infty. \tag{4.10}$$

By Lemma 4.1.1.1 and Theorem 1.5.1.5 (3), $\tilde{f}g$ is strongly measurable. Hence from Theorem 1.5.2.6,

$$\langle f,g\rangle = \mathbf{B}\text{ - }\int_X \langle f(x),g(x)\rangle\ d\mu(x)$$

is well-defined for all $f,g\in L^2(X,\mathcal{A})$.

For each $f,g\in L^2(X,\mathcal{A})$ there exist sequences $(f_n),(g_n)$ sets of simple functions such that

$$\|f-f_n\|_2\to 0 \text{ and } f_n\to f \text{ a.e.}$$

and

$$\|g-g_n\|_2\to 0 \text{ and } g_n\to g \text{ a.e.}$$

By Corollaries 4.1.2.6 and 4.1.2.7, we know that there exists an $\mathcal{A}$-inner product in $L^2(X,\mathcal{A})$ which is an extension of that in $F(X,A)$ and a restriction of that in the Hilbert module $\mathbb{L}^2(X,\mathcal{A})$.

Since $\|\cdot\|_{\mathcal{A}}\le\|\cdot\|_2$, then

$$\|f-f_n\|_{\mathcal{A}}\to 0$$

and

$$\|g - g_n\|_{\mathcal{A}} \to 0.$$

This implies

$$\begin{aligned}
\langle f, g\rangle &= \lim_{n\to\infty} \langle f_n, g_n\rangle && (4.11)\\
&= \lim_{n\to\infty} \mathbf{B}\text{-}\int_X \langle f_n(x), g_n(x)\rangle\ d\mu(x) && (4.12)\\
&= \lim_{n\to\infty} \mathbf{B}\text{-}\int_X f_n(x)^* g_n(x)\ d\mu(x). && (4.13)
\end{aligned}$$

To complete the proof, we need to show that

$$\mathbf{B}\text{-}\int_X f(x)^* g(x)\ d\mu(x) = \lim_{n\to\infty} \mathbf{B}\text{-}\int_X f_n(x)^* g_n(x)\ d\mu(x). \tag{4.14}$$

We know that for each n, $\tilde{f}_n g_n$ is a simple function, and we also know that $\tilde{f}_n g_n \to \tilde{f} g$ a.e. We will show that

$$\left\|\tilde{f} g - \tilde{f}_n g_n\right\|_1 \to 0.$$

By noticing that for any $f \in L^2(X, \mathcal{A})$, $\|f(\cdot)\|_2 \in L^2(X, \mathbb{R})$, using the ordinary Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left\|\tilde{f} g - \tilde{f}_n g_n\right\|_1\\
&= \int_X \|f(x)^* g(x) - f_n(x)^* g_n(x)\|\ dx\\
&= \int_X \|f(x)^* g(x) - f(x)^* g_n(x) + f(x)^* g_n(x) - f_n(x)^* g_n(x)\|\ dx\\
&= \int_X \|f(x)^*(g(x) - g_n(x)) + (f(x) - f_n(x))^* g_n(x)\|\ dx\\
&\le \int_X \|f(x)^*(g(x) - g_n(x))\|\ dx + \int_X \|(f(x) - f_n(x))^* g_n(x)\|\ dx\\
&\le \int_X \|f(x)\|\,\|(g(x) - g_n(x))\|\ dx + \int_X \|(f(x) - f_n(x))\|\,\|g_n(x)\|\ dx\\
&\le \|f\|_2\,\|g - g_n\|_2 + \|f - f_n\|_2\,\|g_n\|_2 \to 0
\end{aligned}$$

The definition of the Bochner integral gives equation 4.14. Therefore, together with equation 4.13, we can conclude that

$$\langle f, g\rangle = \mathbf{B}\text{ - }\int_X \langle f(x), g(x)\rangle \; d\mu(x)$$

defines an $\mathcal{A}$-inner product. $\square$

Using Proposition 4.1.2.4 and Theorem 4.1.2.9, we can summarize the results in the following theorem.

Theorem 4.1.2.10. *The linear vector space $L^2(X,\mathcal{A})$ is a pre-Hilbert $\mathcal{A}$-module, with the action of $\mathcal{A}$ defined by:*

$$(f \cdot a)(x) = f(x)a \text{ for any } x \in X \text{ and } a \in \mathcal{A}$$

and inner product defined by

$$\langle f, g\rangle = \mathbf{B}\text{ - }\int_X f(x)^* g(x)\; d\mu(x) = \mathbf{B}\text{ - }\int_X \langle f(x), g(x)\rangle \; d\mu(x)$$

for any $f, g \in L^2(X,\mathcal{A})$.

It is easy to see that if $\mathcal{A} = \mathbb{C}$ then the $\|\cdot\|_2 = \|\cdot\|_{\mathbb{C}}$. This implies that the completions in each norm coincide. Meanwhile, if X is also a countable space, such that $L^2(X,\mathcal{A}) = \ell_2^{\mathcal{A}}$, we know that the pre-Hilbert $\mathcal{A}$-module $\ell_2^{\mathcal{A}} = \mathbb{H}_{\mathcal{A}}$ if and only if $\mathcal{A}$ is finite dimensional, see [75, page 239].

By [43, page 6] a series of positive elements in $\mathcal{A}$ may converge even though it is not absolutely convergent. Therefore, it is possible that there exists a C^*-algebra $\mathcal{A}$ and an element $(a_i)_{i\in\mathbb{N}}$ of a standard Hilbert $\mathcal{A}$-module $\mathbb{H}_{\mathcal{A}}$ such that the sum of the series $\sum_{i=1}^{\infty} \|a_i^* a_i\|$ does not converge. To give a better sense of this kind of Hilbert module, let us include the following example.

Example 4.1.2.11. Let $Y = [0, 1]$ and $\mathcal{A} = C(Y)$. We will show that $\ell_2^{\mathcal{A}} \subsetneqq \mathbb{H}_{\mathcal{A}}$. Recall that the norm in $\mathcal{A}$ is the supremum norm, $\|\cdot\|_\infty$. We will construct a sequence of functions $(f_i)_{i\in\mathbb{N}}$ in the standard Hilbert $\mathcal{A}$-module $\mathbb{H}_{\mathcal{A}}$ such that the series $\sum_{i=1}^{\infty} \|f_i^* f_i\|_\infty$ does not converge. We start with defining a sequence $(p_i)_{i\in\mathbb{N}}$ of positive elements in $\mathcal{A}$, which are actually functions in $\mathcal{A}$, such that for each i, the maximum value of p_i is $1/i$ and its support is centered at $1/i$, and $\mathrm{supp}(p_i) \cap \mathrm{supp}(p_j) = \emptyset$ if $i \neq j$. We define p_1 to be linear on interval $(\frac{3}{4}, 1]$ and vanishes elsewhere, meanwhile for $i = 2, 3, 4, \cdots$, p_i is linear on both intervals $\left(\frac{1}{2}(\frac{1}{i} + \frac{1}{i+1}), \frac{1}{i}\right]$ and $\left(\frac{1}{i}, \frac{1}{2}(\frac{1}{i} + \frac{1}{i-1})\right]$ with supremum value $\frac{1}{i}$ and infimum value 0 :

$$p_i(y) = \begin{cases} 0 & y \leq \frac{1}{2}(\frac{1}{i} + \frac{1}{i+1}) \\ 2(i+1)y & \frac{1}{2}(\frac{1}{i} + \frac{1}{i+1}) < y \leq \frac{1}{i} \\ 2(1-i)y & \frac{1}{i} < y \leq \frac{1}{2}(\frac{1}{i} + \frac{1}{i-1}) \\ 0 & y > \frac{1}{2}(\frac{1}{i} + \frac{1}{i-1}). \end{cases} \tag{4.15}$$

Furthermore, let us define

$$S_k = \sum_{i=1}^{k} p_i, \quad \text{where } k \in \mathbb{N}.$$

For an illustration, see Figure 4.1.

We can see that for each $i \in \mathbb{N}$, $\|p_i\|_\infty = \frac{1}{i}$, and so,

$$\sum_{i=1}^{\infty} \|p_i\|_\infty = \sum_{i=1}^{\infty} \frac{1}{i} = \infty. \tag{4.16}$$

Now, let us consider the sequence of partial sums:

$$\left(S_k = \sum_{i=1}^{k} p_i \right)_{k\in\mathbb{N}}.$$

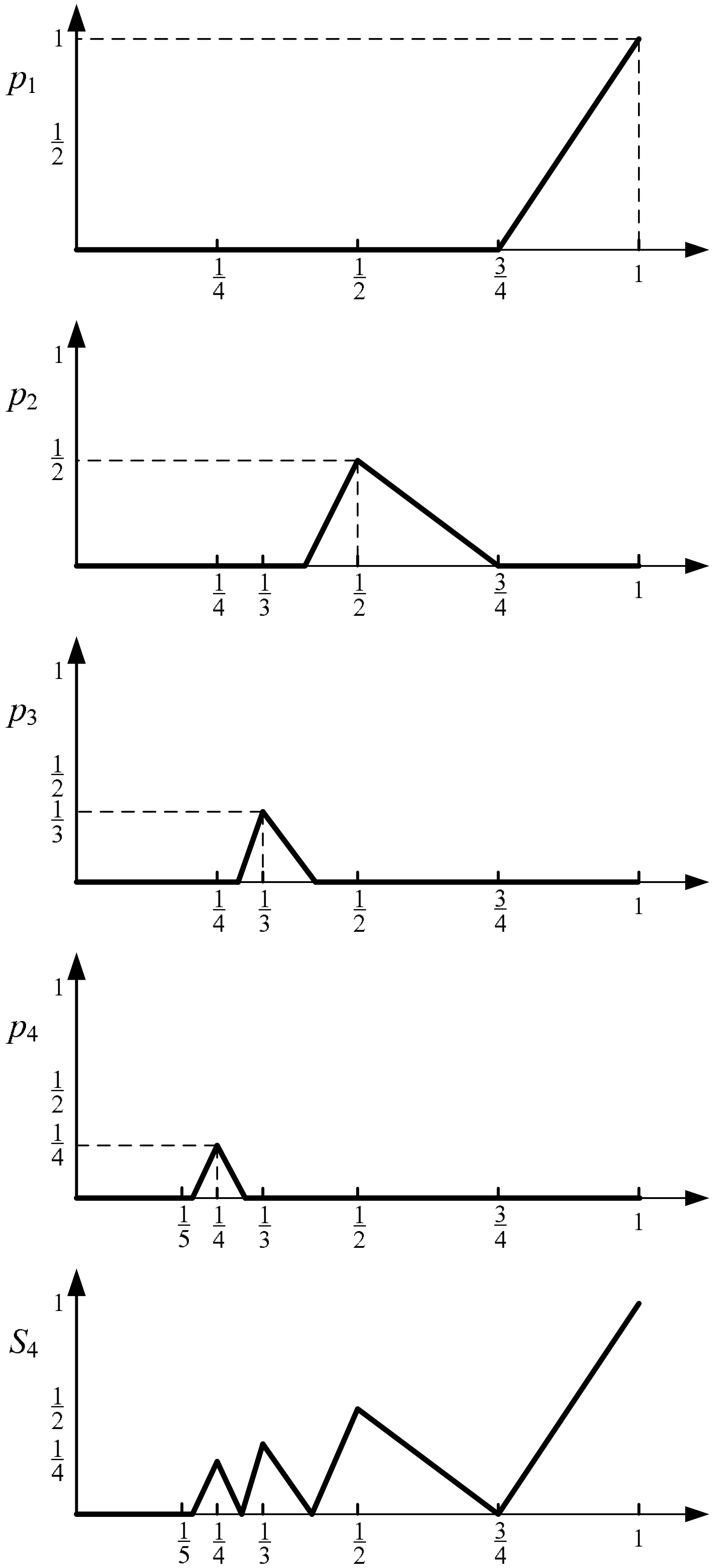

Figure 4.1: p_1, p_2, p_3, p_4 and S_4

We will show the sequence is Cauchy: Let $k, l \in \mathbb{N}$ and without loss of generality, let $k < l$. Hence,

$$\|S_l - S_k\|_\infty = \left\| \sum_{i=k+1}^{l} p_i \right\|_\infty = \sup_{y\in[0,1]} \left| \sum_{i=k+1}^{l} p_i(y) \right| = \frac{1}{k+1}$$

goes to 0 as $k, l \to \infty$ independently. Therefore the infinite sum

$$\sum_{i=1}^{\infty} p_i$$

converges in norm in $\mathcal{A}$. If we denote the sum by p_0, then, for each $y \in Y, p_0(y)$ is defined by:

$$p_0(y) = \sum_{i=1}^{\infty} p_i(y) = p_{i_0}(y)$$

If $y \in \left(\frac{1}{2}\left(\frac{1}{i_0} + \frac{1}{i_0+1}\right), \frac{1}{2}\left(\frac{1}{i_0} + \frac{1}{i_0+1}\right)\right]$ for $i_0 \in \mathbb{N}$. Moreover, since $\text{supp}(p_i)$ are disjoint, we can calculate that

$$\|p_0\|_\infty = \sup_{y\in[0,1]} \left| \sum_{i=1}^{\infty} p_i(y) \right| = \sup_{i\in\mathbb{N}} \|p_i\|_\infty = \sup_{i\in\mathbb{N}} \left\{ \frac{1}{i} \right\} = 1.$$

Since each p_i is a positive element of $\mathcal{A}$, we can take its square root $f_i = \sqrt{p_i}$ which is still an element of $\mathcal{A}$. It is easy to see that $(f_i)_{i\in\mathbb{N}}$ is an element of $\mathbb{H}_{\mathcal{A}}$:

$$\sum_{i=1}^{\infty} f_i^* f_i = \sum_{i=1}^{\infty} |f_i|^2 = \sum_{i=1}^{\infty} f_i^2 = \sum_{i=1}^{\infty} p_i = p_0 \in \mathcal{A}$$

such that

$$\sum_{i=1}^{\infty} \|f_i^* f_i\|_\infty = \sum_{i=1}^{\infty} \|p_i\|_\infty = \infty.$$

As we have discussed in Remark 2.2.3.6, by [75, Page 29] there is some confusion in defining the standard Hilbert module. Extra care is needed in discussing this module. The discussion in this chapter is a generalization of what we have in Example 2.2.3.5 for the standard Hilbert module ($L^2(X,\mathcal{A})$ is a generalization of $\mathbb{H}_{\mathcal{A}}^{wrong1}$ and its completion $\mathbb{L}^2(X,\mathcal{A})$, is a generalization of $\mathbb{H}_{\mathcal{A}}$.) Therefore, similar extra care is also needed. The following example shows that the space

$$W = \left\{ f : X \to \mathcal{A} \,|\, f \ \mu\text{-measurable and } \left\| \int_X f(x)^* f(x)\, d\mu(x) \right\| < \infty \right\},$$

sometimes is bigger than $L^2(X,\mathcal{A})$. In particular, if $X = \mathbb{R}^+$ and $\mathcal{A} = C(Y)$, then there exists a function f such that $\|f\|_2 = \infty$ and

$$\left\| \int f(x)^* f(x)\, dx \right\| < \infty,$$

but $\int f(x)^* f(x)\, dx$ is not convergent in $\mathcal{A}$.

Example 4.1.2.12. Let $X = \mathbb{R}^+$ and $Y = [0,5]$. If $\mathcal{A} = C(Y)$, a unital C^*-algebra, then for every $f \in L^2(X,\mathcal{A})$, $f(x) \in C(Y)$ for each $x \in X$. Recall that in $C(Y)$ we use the supremum norm as defined in equation (1.1). Hence, for each $\mathcal{A}$-valued function f on X, the norm $\|\cdot\|_2$ can be rewritten as

$$\|f\|_2^2 = \int_X \|f(x)\|^2 \, dx = \int_X \left(\sup_y |f(x)(y)| \right)^2 dx, \tag{4.17}$$

which is finite if f is in $L^2(X,\mathcal{A})$. On the other hand, for each $f \in L^2(X,\mathcal{A})$, we can write

$$\|f\|_{\mathcal{A}}^2 = \left\| \int_X |f(x)|^2 \, dx \right\| = \sup_y \int_X \left(|f(x)|^2 \, dx \right)(y). \tag{4.18}$$

Note that the weakly and strongly integrable coincide for $\mathbb{C}$-valued function f on $X \times Y$. Hence, if the integral $\int_X |f(x)(y)|^2 \, dx$ converges for every $y \in Y$ then we can define

$$\left(\int_X |f(x)|^2 \, dx \right)(y) = \int_X |f(x)(y)|^2 \, dx. \tag{4.19}$$

In this case, we can re-write the norm $\|f\|_{\mathcal{A}}$ as

$$\|f\|_{\mathcal{A}}^2 = \left\| \int_X |f(x)|^2 \, dx \right\| = \sup_y \int_X \left(|f(x)(y)|^2 \, dx \right). \tag{4.20}$$

Now, for each $x \in X$ let us define a function $g(x) \in \mathcal{A}$ by

$$g(x)(y) = \begin{cases} y, & 0 \le y \le \frac{1}{\sqrt{x}} \\ \frac{2}{\sqrt{x}} - y & \frac{1}{\sqrt{x}} < y \le \frac{2}{\sqrt{x}} \\ 0, & \text{otherwise.} \end{cases}$$

Using this formula, $g : x \mapsto g(x)$ define an $\mathcal{A}$-valued function. Unfortunately g is not in $L^2(X,\mathcal{A})$:

$$\begin{aligned} \|g\|_2^2 &= \int_X \|g(x)\|^2 \; dx \\ &= \int_X \sup_y |g(x)(y)|^2 \; dx \\ &= \int_X \left(\frac{1}{\sqrt{x}}\right)^2 \; dx \\ &= \int_X \frac{dx}{x} = \infty. \end{aligned}$$

Let us define a function $G : X \times Y \to \mathbb{C}$ by $G(x,y) = g(x)(y)$. It is clear that G is continuous in its second variable and the integral $\int_X |G(x,y)|^2 \; dx$ converges. Furthermore, it is equal to 0 if $y = 0$ and otherwise,

$$\begin{aligned} \int_X |G(x,y)|^2 \; dx &= \int_0^{\frac{1}{y^2}} y^2 \; dx + \int_{\frac{1}{y^2}}^{\frac{4}{y^2}} \left(\frac{2}{\sqrt{x}} - y\right)^2 \; dx \\ &= \int_0^{\frac{1}{y^2}} y^2 \; dx + \int_{\frac{1}{y^2}}^{\frac{4}{y^2}} \left(\frac{4}{x} - \frac{4}{\sqrt{x}}y + y^2\right) \; dx \\ &= \left[y^2 x\right]_0^{\frac{1}{y^2}} \; dx + \left[4\ln x - 8yx^{\frac{1}{2}} + y^2 x\right]_{\frac{1}{y^2}}^{\frac{4}{y^2}} \\ &= 1 + 4\ln\frac{4}{y^2} - 8y\frac{2}{y} + y^2\frac{4}{y^2} - 4\ln\frac{1}{y^2} + 8y\frac{1}{y} - y^2\frac{1}{y^2} \\ &= 1 + 4\ln\frac{4}{y^2} - 4\ln\frac{1}{y^2} - 8y\frac{2}{y} + 8y\frac{1}{y} + y^2\frac{4}{y^2} - y^2\frac{1}{y^2} \\ &= 1 + 4\ln 4 - 8 + 3 \\ &= 4\ln 4 - 4 \\ &= 8\ln 2 - 4. \end{aligned}$$

This shows that for each $y \in Y$, the integral $\int_X |g(x)(y)|^2 \ dx$ converges. Hence we can define a function integral $\int_X |g(x)|^2 \ dx$ from Y to $\mathbb{C}$ pointwise by:

$$\left(\int_X |g(x)|^2 \ dx\right)(y) = \int_X |g(x)(y)|^2 \ dx \tag{4.21}$$
$$= \int_X |G(x,y)|^2 \ dx \tag{4.22}$$
$$= \begin{cases} 0, & y = 0 \\ 8\ln 2 - 4, & \textit{otherwise}. \end{cases} \tag{4.23}$$

Unfortunately, $\int_X |g(x)|^2 dx$ is not continuous at $y = 0$. This implies $\int_X |g(x)|^2 dx$ is not an element of $\mathcal{A}$. Now, using equality (4.23) we calculate the norm $\|g\|_{\mathcal{A}}$ by:

$$\|g\|_{\mathcal{A}}^2 = \sup_y \left(\int_X |g(x)|^2 \ dx\right)(y) \tag{4.24}$$
$$= \sup_y \int_X |g(x)(y)|^2 \ dx \tag{4.25}$$
$$= \sup_y \{8\ln 2 - 4, 0\} \tag{4.26}$$
$$= 8\ln 2 - 4 < \infty. \tag{4.27}$$

4.2 More properties of $\mathbb{L}^2(X,\mathcal{A})$

In what follows we will show that the Hilbert module $\mathbb{L}^2(X,\mathcal{A})$ is isomorphic to the Hilbert module $L^2(X)\widehat{\otimes}\mathcal{A}$. That is, we show that there exists a unitary operator from $L^2(X)\widehat{\otimes}\mathcal{A}$ to $\mathbb{L}^2(X,\mathcal{A})$ that preserves the inner product and the action of $\mathcal{A}$ on the Hilbert modules. We also show that $\mathbb{L}^2(X,\mathcal{A})$ is a full Hilbert $\mathcal{A}$-module.

4.2.1 The $L^2(X, \mathcal{A})$ and the tensor product Hilbert module $L^2(X)\widehat{\otimes}\mathcal{A}$

Theorem 4.2.1.1. *For any measure space X, the Hilbert modules $L^2(X)\widehat{\otimes}\mathcal{A}$ and $\mathbb{L}^2(X, \mathcal{A})$ are isomorphic.*

Proof. To avoid confusion, we will use capital letters to denote $\mathcal{A}$-valued functions.

It is easy to see that for any function $f \in L^2(X)$ and $a \in \mathcal{A}$, the mapping $x \mapsto f(x)a$ defines a function F_a in $L^2(X, \mathcal{A})$. Since the set of simple function is dense in $L^2(X)$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions that converges to f almost everywhere. It is clear that for each $n \in \mathbb{N}$

$$F_n : x \mapsto f_n(x)a$$

is a sequence of countably-valued functions which converges to F_a almost everywhere by the continuity of multiplication in $\mathcal{A}$. By definition, F_a is strongly measurable. Now, let us calculate:

$$\begin{aligned}
\int_X \|F_a(x)\|^2 \, d\mu(x) &= \int_X \|f(x)a\|^2 \, d\mu(x) \\
&= \int_X |f(x)|^2 \|a\|^2 \, d\mu(x) \\
&= \int_X |f(x)|^2 \, d\mu(x) \|a\|^2 \\
&= \|f\|_2^2 \|a\|^2 .
\end{aligned}$$

It is also routine to show that the mapping $(f, a) \mapsto F_a$ is bilinear. Hence, there exists a well-defined linear map U from $L^2(X) \otimes \mathcal{A}$ to $L^2(X, \mathcal{A})$, such that

$$U(f \otimes a) = F_a. \tag{4.28}$$

Recall that the action of $\mathcal{A}$ on $L^2(X,\mathcal{A})$ is given by

$$(F \cdot a)(x) = F(x)a \text{ for all } x \in X,$$

and the action on $L^2(X) \otimes \mathcal{A}$ is given by $(f \otimes a) \cdot b = f \otimes ab$. We need to show that U preserves the action of $\mathcal{A}$ on both of the pre-Hilbert modules. That is for any $f \otimes a \in L^2(X) \otimes \mathcal{A}$ and $b \in \mathcal{A}$ then $U(f \otimes a \cdot b) = U(f \otimes a) \cdot b$. Now, let $x \in X$ be arbitrary. The following calculation gives our result: $U((f \otimes a) \cdot b)(x) = U(f \otimes ab)(x) = f(x)ab$ and $(U(f \otimes a) \cdot b)(x) = U(f \otimes a)(x) \cdot b = f(x)ab$. Next we will show that U preserves the inner product. Because of the linearity of U, it is enough to show that for any $f \otimes a$ and $g \otimes b \in L^2(X) \otimes \mathcal{A}$, then

$$\begin{aligned}
\langle U(f \otimes a), U(g \otimes b)\rangle &= \mathbf{B}\text{ - }\int_X \langle U(f \otimes a)(x), U(g \otimes b)(x)\rangle \ d\mu(x) \\
&= \mathbf{B}\text{ - }\int_X \langle f(x)a, g(x)b\rangle \ d\mu(x) \\
&= \mathbf{B}\text{ - }\int_X (f(x)a)^* g(x)b \, d\mu(x) \\
&= \mathbf{B}\text{ - }\int_X (f(x)a)^* g(x) \ d\mu(x)b \\
&= \left(\mathbf{B}\text{ - }\int_X g(x)^* f(x)a \, d\mu(x)\right)^* b \\
&= \left(\int_X g(x)^* f(x) \ d\mu(x) \ a\right)^* b \\
&= a^* \left(\int_X g(x)^* f(x) \ d\mu(x)\right)^* b \\
&= a^* \int_X f(x)^* g(x) \ d\mu(x) b \\
&= a^* \int_X \overline{f(x)} g(x) \ d\mu(x) b \\
&= \langle f, g\rangle \ a^* b \\
&= \langle f \otimes a, g \otimes b\rangle .
\end{aligned}$$

Since U preserves the inner product, it is an isometry and hence it is injective and continuous. Now, we know that U is an $\mathcal{A}$-linear operator from $L^2(X)\otimes\mathcal{A}$ to $L^2(X,\mathcal{A})$. To show that it can be extended to a unitary operator from $L^2(X)\widehat{\otimes}\mathcal{A}$ to $\mathbb{L}^2(X,\mathcal{A})$, we need to show that the range of U is dense in $\mathbb{L}^2(X,\mathcal{A})$.

If $F \in \mathbb{L}^2(X,\mathcal{A})$ is a simple function, then F is in the range of U. Recall that we can represent F by

$$F(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x) = \sum_{i=1}^{n} \chi_{E_i} a_i(x) \text{ for all } x \in X,$$

where E_i are disjoint sets with finite measure. Hence for each i, χ_{E_i} is in $L^2(X)$, and we can write $\chi_{E_i}(x)a_i = U(\chi_{E_i} \otimes a_i)(x)$. Therefore,

$$\begin{aligned} F(x) &= \sum_{i}^{n} U(\chi_{E_i} \otimes a_i)(x) \\ &= U(\sum_{i}^{n} \chi_{E_i} \otimes a_i)(x). \end{aligned}$$

We know that $F(X,A)$ is dense in $\mathbb{L}^2(X,\mathcal{A})$. Together with the fact that $F(X,A) \subset \text{range}(U) \subset L^2(X,\mathcal{A}) \subset \mathbb{L}^2(X,\mathcal{A})$, we see that the range of U is dense in $\mathbb{L}^2(X,\mathcal{A})$.

□

4.2.2 The full Hilbert module $\mathbb{L}^2(X,\mathcal{A})$

Theorem 4.2.2.1. *The Hilbert module* $\mathbb{L}^2(X,\mathcal{A})$ *is a full Hilbert module.*

Proof. We need to show that the ideal

$$I = \text{span}\left\{\langle f, g\rangle \,|\, f, g \in \mathbb{L}^2(X,\mathcal{A})\right\}$$

is dense in $\mathcal{A}$.

Here, we will view $\mathbb{L}^2(X,\mathcal{A})$ as $L^2(X)\widehat{\otimes}\mathcal{A}$.

We know that $\mathcal{A}$ is a full Hilbert module with $a\cdot b = ab$ and $\langle a,b\rangle = a^*b$. Therefore, for each $a\in\mathcal{A}$, given $\epsilon>0$, there exists

$$c=\sum_{i=1}^{n}\alpha_i\langle a_i,b_i\rangle=\sum_{i=1}^{n}\alpha_i a_i^*b_i\in I$$

such that $\|a-c\|<\epsilon$. Now, let f_0 be any function in $L^2(X)$ such that $\|f_0\|_2=1$. For each $i=1,2,\cdots,n$ $f_0\otimes a_i$ and $f_0\otimes b_i$ are elements of $L^2(X)\widehat{\otimes}\mathcal{A}$. We now can define an element $a_0\in I$ by:

$$a_0=\sum_{i=1}^{n}\alpha_i\langle a_i,b_i\rangle=\sum_{i=1}^{n}\alpha_i\langle f_0,f_0\rangle\langle a_i,b_i\rangle=\sum_{i=1}^{n}\alpha_i\langle f_0\otimes a_i,f_0\otimes b_i\rangle.$$

It is straightforward to see that $\|a-a_0\|<\epsilon$. $\square$

4.2.3 The Hilbert module $L^2(X)\widehat{\otimes}\mathcal{A}$ with separable $L^2(X)$

Suppose that $\{\varepsilon_i\}_{i\in I}$ is an orthonormal basis of $L^2(X)$, then

$$\mathbb{L}^2(X,\mathcal{A})\cong L^2(X)\widehat{\otimes}\mathcal{A}\cong\bigoplus_i\mathcal{A}_i$$

where $\mathcal{A}_i=\mathcal{A}$ for all i. Moreover, if $L^2(X)$ is separable (so that there exists a countable orthonormal basis for $L^2(X)$),

$$\mathbb{L}^2(X,\mathcal{A})\cong L^2(X)\widehat{\otimes}\mathcal{A}\cong\mathbb{H}_{\mathcal{A}}.$$

Since $\mathcal{A}$ is unital, $\{\varepsilon_i\otimes 1_{\mathcal{A}}\}_{i\in I}$ is an orthonormal basis for $L^2(X)\widehat{\otimes}\mathcal{A}$ and therefore $\{\varepsilon 1_{\mathcal{A}}\}$ is an orthonormal basis for $\mathbb{L}^2(X,\mathcal{A})$.

The other fact that we have is that $\mathcal{K}(\mathbb{L}^2(X,\mathcal{A}))\cong\mathcal{K}(L^2(X)\widehat{\otimes}\mathcal{A})\cong\mathcal{K}(L^2(X))\widehat{\otimes}\mathcal{A}$. Furthermore, since $\mathcal{A}$ is unital, $\mathcal{K}(\mathbb{L}^2(X,\mathcal{A}))=\mathcal{L}(\mathbb{L}^2(X,\mathcal{A}))$.

Now, we will show that if $L^2(X)$ is separable, each $f\in\mathbb{L}^2(X,\mathcal{A})$ is strongly measurable.

Proposition 4.2.3.1. *If* $L^2(X)$ *is separable, each function* $f \in \mathbb{L}^2(X, \mathcal{A})$ *is strongly measurable.*

Proof. Let $f \in \mathbb{L}^2(X, \mathcal{A})$ be arbitrary. Then $f = \sum_{i \in \mathbb{N}} \varepsilon_i \cdot a_i$. It is clear that for each i, $\varepsilon_i \cdot a_i$ is strongly measurable. Therefore, by Lemma 1.5.1.3 for each i there exists a sequence of countably valued functions which converges to $\varepsilon_i \cdot a_i$ uniformly almost everywhere. Hence, for each n, there exist countably valued functions g_{n_i} such that

$$\|\varepsilon_i(x)a_i - g_{n_i}(x)\| < 2^{-(n+i)}$$

for almost $x \in X$ unless for all x in a μ-nullset E_i. Now, for each $n \in \mathbb{N}$ define a function

$$g_n = \sum_i g_{n_i}.$$

Since a countable union of countable sets is countable, g_n is a countably-valued function. Moreover,

$$\|f(x) - g_n(x)\| = \left\| \varepsilon_i(x)a_i - \sum_i g_{n_i}(x) \right\| \tag{4.29}$$

$$\leq \sum_i \left\| \varepsilon_i(x)a_i - \sum_i g_{n_i}(x) \right\| \tag{4.30}$$

$$< \sum_i 2^{-(n+i)} \tag{4.31}$$

$$= 2^{-n} \sum_i 2^{-i} \tag{4.32}$$

$$< 2^{-n}. \tag{4.33}$$

for almost all $x \in X$ unless for all x in a μ-null set $E = \bigcup_i E_i$. This gives us a sequence $(g_n)_{n \in \mathbb{N}}$ of countably valued functions such that for each $m \in \mathbb{N}$,

$$\|f(x) - g_n(x)\| < 2^{-m}$$

for all $x \in X \setminus E$ if $n \geq m$, i.e. (g_n) converges to f uniformly almost everywhere. By Lemma 1.5.1.3, f is strongly measurable. $\square$

Finally, we have the following result.

Theorem 4.2.3.2. *If $L^2(X)$ is separable, then $L^2(X,\mathcal{A}) = \mathbb{L}^2(X,\mathcal{A})$ if and only if $\mathcal{A}$ is finite dimensional.*

Proof. From Theorem 4.2.1.1 and its proof, we know

$$\ell_2^{\mathcal{A}} \cong L^2(X,\mathcal{A}) \cong L^2(X) \otimes \mathcal{A},$$

and

$$\mathbb{L}^2(X,\mathcal{A}) \cong L^2(X)\widehat{\otimes}\mathcal{A} \cong \mathbb{H}_{\mathcal{A}}.$$

The result follows from the fact that $\ell_2^{\mathcal{A}} = \mathbb{H}_{\mathcal{A}}$ if and only if $\mathcal{A}$ is finite dimensional. $\square$

Chapter 5

The wavelet transform on Hilbert modules

From now, we will assume that our Hilbert modules are separable.

Here, we will generalize the ideas of coherent state systems, wavelets and the related transforms, to the setting of Hilbert modules. The definitions generalize Führ's definitions from Chapter 3. This general context brings some new problems which we must consider. For example, since the inner product in the Hilbert module is $\mathcal{A}$-valued, our coefficient functions will be $\mathcal{A}$-valued functions. Hence, our main tool here will be the $\mathcal{A}$-valued integral: we use the Bochner integral theory defined in Section 1.5.2. Moreover, by analogy, we introduce the notions of weak, semi-weak and ultra-weak integral for Hilbert module, Section 5.1. The motivation of these definitions is that we want to show that the adjoint operator of the coefficient operator, at least for those related to an admissible vector, can be read as an operator integral, (Theorem 5.2.1.6).

Following the definitions, some results that generalize the reconstruction formula, the resolution of the identity formula and the image space of the transform are obtained: Theorem 5.2.1.7, Theorem 5.2.1.10 and Theorem 5.2.1.14 respectively. We also give a result which characterizes the kernel of the coefficient operator, Lemma 5.2.2.1. Though the result is similar to its version in the setting of Hilbert space, Lemma 3.2.2.1, the following results: Lemma 5.2.2.3 and Lemma 5.2.2.4 are different. Fortunately, using these results, we still get a useful notion of admissible, bounded, and cyclic vectors related to adjointable projection, intertwining operator, and unitary operator.

Finally, we give some examples of group coherent state system and generalized continuous wavelet transform on Hilbert modules.

The notion of wavelets in Hilbert space will be a special case of our definition and, as far as possible, we use the same terminology as in the previous chapters.

5.1 Hilbert module-valued integral

Here we will define the weak integral for Hilbert module valued functions. But, first, we will introduce the notion of weak measurability which is related to the weak integrals defined in the following section.

5.1.1 Weak measurability for Hilbert module-valued functions

Definition 5.1.1.1. A $\mathbb{H}$-valued function f on a measure space X to Hilbert module $\mathbb{H}$ is **weakly measurable** relative to a dense subset $\mathfrak{D}$ of $\mathcal{B}(\mathbb{H}, \mathcal{A})$ if and only if the $\mathcal{A}$-valued function $T(f(\cdot))$ is strongly measurable for all $T \in \mathfrak{D}$.

Remark 5.1.1.2. If $\mathbb{H}$ is reflexive, then f is weakly measurable relative to a dense subset $\mathfrak{D}$ of $\mathbb{H}$ if and only if $\langle \varphi, f(\cdot)\rangle$ is strongly measurable for all $\varphi \in \mathfrak{D}$.

We know that not every Hilbert module is reflexive. See [75, 15.I]. Therefore, we introduce weaker definitions of the measurability which we call semi-weak measurability and ultra-weak measurability for vector-valued functions on a Hilbert module $\mathbb{H}$.

Definition 5.1.1.3. An $\mathbb{H}$-valued function f from a measure space X to a Hilbert module $\mathbb{H}$ is **semi-weakly measurable** relative to a dense set $\mathfrak{D}$ of $\mathcal{L}(\mathbb{H}, \mathcal{A})$ if and only if the $\mathcal{A}$-valued function $T(f(\cdot))$ strongly measurable for all $T \in \mathfrak{D}$.

Definition 5.1.1.4. An $\mathbb{H}$-valued function f from a measure space X to a Hilbert module $\mathbb{H}$ is **ultra-weakly measurable** relative to a dense subset $\mathfrak{D}$ of $\mathcal{K}(\mathbb{H}, \mathcal{A})$ if and only if the $\mathcal{A}$-valued function $T(f(\cdot))$ is stongly measurable for all $T \in \mathfrak{D}$.

Using the Riesz-Fréchet theorem for Hilbert C^*-modules, Theorem 2.2.4.30, we can rewrite the definition of ultra-weak integrability as follows:

Definition 5.1.1.5. An $\mathbb{H}$-valued function f from a measure space X to a Hilbert module $\mathbb{H}$ is **ultra-weakly measurable** relative to a dense subset $\mathfrak{D}$ of $\mathbb{H}$ if and only if $\langle \varphi, f(\cdot)\rangle$ is strongly measurable for all $\varphi \in \mathfrak{D}$.

5.1.2 Weak integrals for Hilbert module-valued functions

Definition 5.1.2.1. An $\mathbb{H}$-valued function f from a measure space X to a Hilbert module $\mathbb{H}$ which is weakly measurable relative to a dense subset $\mathfrak{D}$ of

$\mathcal{B}(\mathbb{H}, \mathcal{A})$ is **weakly integrable** relative to $\mathfrak{D}$ if and only if $T(f(\cdot))$ is Bochner integrable for all $T \in \mathfrak{D}$, and there is an element ψ of $\mathbb{H}$ such that

$$T(\psi) = \mathbf{B}\text{ -}\int T(f(x))\, d\mu(x) \;\text{ for all } T \in \mathfrak{D}. \tag{5.1}$$

We write,

$$\int f(x)\, d\mu(x) = \psi. \tag{5.2}$$

Remark 5.1.2.2. If $\mathbb{H}$ is reflexive, then (5.1) and (5.2) are equivalent to saying that f is weakly integrable relative to $\mathfrak{D}$ if and only if for all $\varphi \in \mathfrak{D}$, there is an element ψ of $\mathbb{H}$ such that

$$\langle \varphi, \psi \rangle = \mathbf{B}\text{ -}\int \langle \varphi, f(x) \rangle \; d\mu(x) \;\text{ for all } \varphi \in \mathfrak{D}. \tag{5.3}$$

We write,

$$\int f(x)\, d\mu(x) = \psi. \tag{5.4}$$

We know that not every Hilbert module is reflexive. See 2.2.4.28. Therefore, we introduce weaker definitions of the integral which we call the semi-weak integral and the ultra-weak integral for vector-valued functions on a Hilbert module $\mathbb{H}$.

Definition 5.1.2.3. An $\mathbb{H}$-valued function f from a measure space X to a Hilbert module $\mathbb{H}$ which is weakly measurable relative to a dense subset $\mathfrak{D}$ of $\mathcal{L}(\mathbb{H}, \mathcal{A})$ is **semi-weakly integrable** relative to $\mathfrak{D} \subseteq \mathcal{L}(\mathbb{H}, \mathcal{A})$ if and only if the $\mathcal{A}$-valued function $T(f(\cdot))$ is Bochner integrable for all $T \in \mathfrak{D}$, and there is an element ψ of $\mathbb{H}$ such that

$$T(\psi) = \mathbf{B}\text{ -}\int T(f(x))\, d\mu(x) \;\text{ for all } T \in \mathfrak{D}. \tag{5.5}$$

We write

$$\int f(x)\, d\mu(x) = \psi. \tag{5.6}$$

Definition 5.1.2.4. An $\mathbb{H}$-valued function f from a measure space X to a Hilbert module $\mathbb{H}$ which is weakly measurable relative to a dense subset $\mathfrak{D}$ of $\mathcal{K}(\mathbb{H}, \mathcal{A})$ is **ultra-weakly integrable** relative to $\mathfrak{D}$ if and only if the $\mathcal{A}$-valued function $T(f(\cdot))$ is Bochner integrable for all $T \in \mathfrak{D}$, and there is an element ψ of $\mathbb{H}$ such that

$$T(\psi) = \mathbf{B}\text{-}\int T(f(x))\, d\mu(x) \text{ for all } T \in \mathfrak{D}. \tag{5.7}$$

We write

$$\int f(x)\, d\mu(x) = \psi. \tag{5.8}$$

Using the Riesz-Fréchet theorem for Hilbert C^*-modules, Theorem 2.2.4.30, we can rewrite the definition of ultra-weak integrability as follows:

Definition 5.1.2.5. An $\mathbb{H}$-valued function f from a measure space X to a Hilbert module $\mathbb{H}$ which is weakly measurable relative to a dense subset $\mathfrak{D}$ of $\mathbb{H}$ is **ultra-weakly integrable** relative to a dense subset $\mathfrak{D}$ if and only if $\langle \varphi, f(\cdot)\rangle$ is Bochner integrable for all $\varphi \in \mathfrak{D}$, and there is an element ψ of $\mathbb{H}$ such that

$$\langle \varphi, \psi\rangle = \mathbf{B}\text{-}\int \langle \varphi, f(x)\rangle\ d\mu(x) \text{ for all } \varphi \in \mathfrak{D}. \tag{5.9}$$

We write

$$\int f(x)\, d\mu(x) = \psi. \tag{5.10}$$

It is clear that if an $\mathbb{H}$-valued function f is weakly integrable it is semi-weakly integrable and ultra-weakly integrable. These three definitions coincide when $\mathbb{H}$ is reflexive. Hence, in Hilbert space, these three definitions coincide. Furthermore, we know that if $\mathcal{A}$ is unital, by Proposition 2.2.4.31, $\mathcal{L}(\mathbb{H}, \mathcal{A}) = \mathcal{K}(\mathbb{H}, \mathcal{A})$. In this case, the semi-weak integral and the ultra-weak integrals also coincide.

Remark 5.1.2.6. If the Hilbert module $\mathbb{H}$ is a Hilbert space, then, the weak integral defined here is coincides to that defined in Definition 1.5.2.2.

5.2 The GCWT on Hilbert modules

In Chapter 3, our coefficient operators are operators from Hilbert spaces $\mathcal{H}$ to $L^2(X)$. We generalize this by replacing $\mathcal{H}$ by a Hilbert module $\mathbb{H}$ and $L^2(X)$ by its generalization, $\mathbb{L}^2(X, \mathcal{A})$.

5.2.1 Coherent state systems on Hilbert modules and GCWT

Here, we give definitions of coherent state system, coefficient function and coefficient operators, generalizing those in the setting of Hilbert space.

Following the structure in Chapter 3, we start with the definition of coherent state.

Definition 5.2.1.1. Let $\eta = (\eta_x)_{x \in X}$ denote a family of vectors in $\mathbb{H}$, indexed by the elements of a measure space X.

a. For any $\varphi \in \mathbb{H}$ define an $\mathcal{A}$-valued function $V_\eta \varphi$ on X by

$$V_\eta \varphi(x) = \langle \eta_x, \varphi \rangle .$$

We call this function the **coefficient function**.

b. If $V_\eta \varphi$ is strongly measurable for all $\varphi \in \mathbb{H}$, we call η a **coherent state system**.

We are interested in defining an operator in Hilbert modules, hence we will require that the coefficient functions are in the Hilbert module $\mathbb{L}^2(X, \mathcal{A})$.

Definition 5.2.1.2. Let $\eta = (\eta_x)_{x \in X}$ be a coherent state system in $\mathbb{H}$, indexed by the elements of a measure space X. Define

$$\mathcal{D}_\eta = \left\{ \varphi \in \mathbb{H} | V_\eta \varphi \in \mathbb{L}^2(X, \mathcal{A}) \right\} . \tag{5.11}$$

We denote by $V_\eta : \mathbb{H} \to \mathbb{L}^2(X, \mathcal{A})$ the (possibly unbounded) operator defined by the mapping $\varphi \mapsto V_\eta \varphi$ from $\mathcal{D}_\eta$ to $\mathbb{L}^2(X, \mathcal{A})$, call it the **coefficient operator**.

Lemma 5.2.1.3. *If η is a coherent state system in a Hilbert module $\mathbb{H}$, its coefficient operator is $\mathcal{A}$-linear.*

Proof. The following calculation shows $\mathcal{A}$-linearity of the coefficient operator V_η defined above. Let $x \in X$ be arbitrary, $a \in \mathcal{A}$ and $\varphi \in \mathbb{H}$,

$$\begin{aligned} V_\eta(\varphi \cdot a)(x) &= \langle \eta_x, \varphi \cdot a \rangle \\ &= \langle \eta_x, \varphi \rangle\, a \\ &= (V_\eta \varphi)\,(x) a \\ &= (V_\eta \varphi \cdot a)\,(x). \end{aligned}$$

□

We need to define the following space in order to define a notion of admissibility analogous to the one in the Hilbert space setting.

Definition 5.2.1.4. Let $\eta = (\eta_x)_{x \in X}$ be a coherent state system in a Hilbert module $\mathbb{H}$, indexed by the elements of a measure space X. Define

$$\mathcal{D}_\eta^\diamond = \left\{ \varphi \in \mathbb{H} | V_\eta \varphi \in L^2(X, \mathcal{A}) \right\}. \tag{5.12}$$

Note that since $L^2(X, \mathcal{A}) \subset \mathbb{L}^2(X, \mathcal{A})$ for any coherent state η, $\mathcal{D}_\eta^\diamond \subseteq \mathcal{D}_\eta$.

Admissible coherent state system

We know that not every bounded operator on Hilbert modules is adjointable. Since we are interested in finding an adjointable coefficient operator arising from coherent state systems, we introduce the following definition for admissible coherent state systems.

Definition 5.2.1.5. The coherent state system $\eta = (\eta_x)_{x\in X}$ in a Hilbert module $\mathbb{H}$ is called **admissible** if for the associated coefficient operator V_η:

1. $\mathcal{D}_\eta^\diamond$ is dense in $\mathcal{D}_\eta$, and $\mathcal{D}_\eta = \mathbb{H}$.

2. V_η is an isometry with complemented range.

Furthermore, in this case we call V_η a **coherent state transform**.

In the setting of Hilbert spaces, we have seen that if the coherent state transform is bounded then it is adjointable. In this case, Corollary 3.2.1.5 defined the adjoint operator as a weak operator integral. In the setting of Hilbert modules, we prove a similar result for the coefficient operator related to admissible coherent state system.

Theorem 5.2.1.6. *If $\{\eta_x\}_{x\in X}$ is an admissible coherent state system in a Hilbert module $\mathbb{H}$, the coherent state transform (coefficient operator) V_η has an adjoint such that for each $f \in L^2(X, \mathcal{A})$ the values $V_\eta^*(f)$ are given by the ultra-weak integrals*

$$V_\eta^*(f) = \int_X \eta_x \cdot f(x)\, d\mu(x) \tag{5.13}$$

relative to the dense set $\mathcal{D}_\eta^\diamond$ of $\mathbb{H}$. Furthermore, for any $f \in \mathbb{L}^2(X, \mathcal{A})$ and $(f_n) \subset L^2(X, \mathcal{A})$ converging to f in $\|\cdot\|_{\mathcal{A}}$,

$$V_\eta^*(f) = \lim_{n\to\infty} V_\eta^*(f_n) = \lim_{n\to\infty} \int_X \eta_x \cdot f_n(x)\, d\mu(x). \tag{5.14}$$

Proof. Since $\{\eta_x\}_{x\in X}$ is admissible, V_η is an isometry with complemented range. Hence, by Proposition 2.2.4.21, it is adjointable, i.e. there exist an $\mathcal{A}$-linear operator V_η^* from $\mathbb{L}^2(X, \mathcal{A})$ to $\mathbb{H}$ such that for every $g \in \mathbb{L}^2(X, \mathcal{A})$ and $\xi \in \mathbb{H}$,

$$\langle V_\eta \xi, g\rangle = \left\langle \xi, V_\eta^* g\right\rangle .$$

Now, suppose $f \in L^2(X, \mathcal{A})$, then $x \mapsto \eta_x \cdot f(x)$ is an $\mathbb{H}$-valued function. Let $\varphi \in \mathcal{D}_\eta^\diamond$ be arbitrary, and hence $V_\eta \varphi \in L^2(X, \mathcal{A})$. We can calculate:

$$\begin{aligned} \left\langle \varphi, V_\eta^* f \right\rangle &= \langle V_\eta \varphi, f \rangle && (5.15) \\ &= \mathbf{B}\text{ - }\int_X V_\eta \varphi(x)^* f(x)\, d\mu(x) && (5.16) \\ &= \mathbf{B}\text{ - }\int_X \langle \eta_x, \varphi \rangle^* f(x)\, d\mu(x) && (5.17) \\ &= \mathbf{B}\text{ - }\int_X \langle \varphi, \eta_x \rangle f(x)\, d\mu(x) && (5.18) \\ &= \mathbf{B}\text{ - }\int_X \langle \varphi, \eta_x \cdot f(x) \rangle \; d\mu(x). && (5.19) \end{aligned}$$

Recall that because of the admissibility of $\{\eta_x\}_{x \in X}$, $\mathcal{D}_\eta^\diamond$ is dense in $\mathbb{H}$. By definition 5.1.2.5, the function $x \mapsto \eta_x \cdot f(x)$ is ultra-weakly integrable relative to the dense subset $\mathcal{D}_\eta^\diamond$ of $\mathbb{H}$. Hence, for any $f \in L^2(X, \mathcal{A})$, $V_\eta^*(f)$ is given by the ultra-weak integral,

$$V_\eta^*(f) = \int \eta_x \cdot f(x)\, d\mu(x).$$

relative to the dense set $\mathcal{D}_\eta^\diamond$ of $\mathbb{H}$. Equation (5.14) follows from the continuity of V_η^*. □

The reconstruction formula

As in the Hilbert space setting, our notion of admissible coherent state system also gives a reconstruction formula. By Proposition 2.2.4.21, if η is an admissible coherent state system, then the coefficient operator V_η gives that $V_\eta^* V_\eta$ is the identity operator on $\mathbb{H}$ and the operator $V_\eta V_\eta^*$ is a projection onto the range of V_η.

We next show that the first fact leads us to an inversion or a reconstruction formula which is similar to the reconstruction formula in the setting of Hilbert space. In the Hilbert module setting, this formula can be read as an expansion of any vector in the dense subset $\mathcal{D}_\eta^\diamond$ of $\mathbb{H}$ in terms of the coherent state system.

Theorem 5.2.1.7. *Let $(\eta_x)_{x\in X}$ be an admissible coherent state system. Then, for $\varphi \in \mathcal{D}_\eta^\diamond \subseteqq \mathbb{H}$ we have the following reconstruction formula*

$$\varphi = \int_X \eta_x \cdot \langle \eta_x, \varphi \rangle \; d\mu(x) \tag{5.20}$$

to be read in the ultra-weak sense relative to the dense subset $\mathcal{D}_\eta^\diamond$. Furthermore, if $\varphi \in \mathbb{H}$ and $(\varphi_n) \subset \mathcal{D}_\eta^\diamond$ converges to φ, then

$$\varphi = \lim_{n\to\infty} \varphi_n = \lim_{n\to\infty} \int_X \eta_x \cdot \langle \eta_x, \varphi_n \rangle \; d\mu(x). \tag{5.21}$$

Proof. Let $\varphi \in \mathcal{D}_\eta^\diamond$ and $\psi \in \mathbb{H}$. Since η is admissible, so that $\mathcal{D}_\eta^\diamond$ is dense in $\mathbb{H}$, there exists a sequence (ψ_n) in $\mathcal{D}_\eta^\diamond$ that converges to ψ and

$$\left\langle \psi, \int_X \eta_x \cdot \langle \eta_x, \varphi \rangle \; d\mu(x) \right\rangle = \lim_{n\to\infty} \left\langle \psi_n, \int_X \eta_x \cdot \langle \eta_x, \varphi \rangle \; d\mu(x) \right\rangle. \tag{5.22}$$

Now, $\varphi \in \mathcal{D}_\eta^\diamond$, then $V_\eta \varphi \in L^2(X, \mathcal{A})$. Hence, by Theorem 5.2.1.6,

$$V_\eta^* \left(V_\eta \varphi \right) = \int_X \eta_x \cdot \langle \eta_x, \varphi \rangle \; d\mu(x)$$

is an ultra-weak integral relatives to the dense subset $\mathcal{D}_\eta^\diamond$ of $\mathbb{H}$. Therefore

$$\lim_{n\to\infty} \left\langle \psi_n, \int_X \eta_x \cdot \langle \eta_x, \varphi \rangle \; d\mu(x) \right\rangle = \lim_{n\to\infty} \mathbf{B}\text{ - }\int_X \langle \psi_n, \eta_x \cdot \langle \eta_x, \varphi \rangle \rangle \; d\mu(x) \tag{5.23}$$

$$= \lim_{n\to\infty} \mathbf{B}\text{ - }\int_X \langle \psi_n, \eta_x \rangle \langle \eta_x, \varphi \rangle \; d\mu(x) \tag{5.24}$$

$$= \lim_{n\to\infty} \mathbf{B}\text{ - }\int_X \langle \eta_x, \psi_n \rangle^* \langle \eta_x, \varphi \rangle \; d\mu(x) \tag{5.25}$$

$$= \lim_{n\to\infty} \mathbf{B}\text{ - }\int_X \left(V_\eta \psi_n(x) \right)^* V_\eta \varphi(x) \; d\mu(x). \tag{5.26}$$

For each n, $V_\eta\psi_n$ is in $L^2(X,\mathcal{A})$. Hence,

$$\mathbf{B}\text{ - }\int_X (V_\eta\psi_n(x))^* \, V_\eta\varphi(x)\; d\mu(x) = \langle V_\eta\psi_n, V_\eta\varphi\rangle\,. \tag{5.27}$$

Since V_η is an isometry,

$$\langle V_\eta\psi_n, V_\eta\varphi\rangle = \langle \psi_n, \varphi\rangle\,. \tag{5.28}$$

Hence, equations (5.28) and (5.27) give:

$$\lim_{n\to\infty}\left\langle \psi_n, \int_X \eta_x\cdot\langle\eta_x,\varphi\rangle\;\; d\mu(x)\right\rangle = \lim_{n\to\infty}\mathbf{B}\text{ - }\int_X V_\eta\psi_n(x)^* V_\eta\varphi(x)\; d\mu(x) \tag{5.29}$$

$$= \lim_{n\to\infty}\langle V_\eta\psi_n, V_\eta\varphi\rangle \tag{5.30}$$

$$= \lim_{n\to\infty}\langle \psi_n, \varphi\rangle\,. \tag{5.31}$$

By the continuity of the inner product in Hilbert modules, equations (5.22) and (5.31) give:

$$\left\langle \psi, \int_X \eta_x\cdot\langle\eta_x,\varphi\rangle\;\; d\mu(x)\right\rangle = \lim_{n\to\infty}\left\langle \psi_n, \int_X \eta_x\cdot\langle\eta_x,\varphi\rangle\;\; d\mu(x)\right\rangle \tag{5.32}$$

$$= \lim_{n\to\infty}\langle \psi_n, \varphi\rangle \tag{5.33}$$

$$= \langle \psi, \varphi\rangle\,. \tag{5.34}$$

This proves that if $\varphi\in\mathcal{D}_\eta^\diamond$, for any $\psi\in\mathbb{H}$,

$$\left\langle \psi, \int_X \eta_x\cdot\langle\eta_x,\varphi\rangle\;\; d\mu(x)\right\rangle = \langle \psi, \varphi\rangle\,.$$

Equivalently,

$$\varphi = \int_X \eta_x\cdot\langle\eta_x,\varphi\rangle\;\; d\mu(x).$$

$\square$

As in the Hilbert space setting, c.f. equation (3.16), if η is an admissible coherent state system, for each $\varphi \in \mathcal{D}_\eta^\diamond$ we can rewrite the inversion formula by using rank-one operator notation in the following form:

$$\varphi = \int_X (\eta_x \otimes \bar{\eta}_x)\, \varphi \, d\mu(x). \tag{5.35}$$

The resolution of the identity formula

Here we will introduce the resolution of the identity formula in the setting of Hilbert modules $\mathbb{H}$, as an alternative way to describe the expansion property of the admissible coherent state system. First, let us state the following definition of the ultra-weak operator integral for Hilbert modules.

Definition 5.2.1.8. For a family of operators $(T_x)_{x\in X} \subset \mathcal{L}(\mathbb{H})$, if the integral $\int_X T_x(\varphi)\, d\mu(x)$ converges ultra-weakly relative to a dense subset $\mathfrak{D}$ of $\mathbb{H}$, for every φ in a dense subset D of $\mathbb{H}$, we define the **weak operator integral** $\int_X T_x \, d\mu(x)$ pointwise as

$$\left(\int_X T_x \, d\mu(x) \right)(\varphi) = \int_X T_x(\varphi)\, d\mu(x) \tag{5.36}$$

for each $\varphi \in D$. We will use the same notation $\int_X T_x \; d\mu(x)$ to denote its extension to the whole space.

Remark 5.2.1.9. Implicit in this definition is that for each $\varphi \in D$, the function $x \mapsto T_x(\varphi)$ is ultra-weakly measurable to the set $\mathfrak{D}$.

Theorem 5.2.1.10. *Let $(\eta_x)_{x\in X}$ be an admissible coherent state system. Then we can rewrite the identity operator $I_{\mathbb{H}}$ as a weak operator integral which is called* ***the resolution of the identity****:*

$$\int_X \eta_x \otimes \bar{\eta}_x \, d\mu(x) = I_{\mathbb{H}}. \tag{5.37}$$

Proof. Consider the family of the rank-one operators in equation (5.35), which we know converges for all $\varphi \in \mathcal{D}_\eta^\diamond$. By Definition 5.2.1.8, we can rewrite the identity operator as the integral of the rank-one operators. It is an ultra-weak operator integral which we call the resolution of the identity

$$\int_X \eta_x \otimes \bar{\eta}_x \, d\mu(x) = I_{\mathcal{H}}.$$

□

Image spaces of coefficient operators

It is a well known result that the image of the continuous wavelet transform on Hilbert space, or more generally, the image of the coherent state transform is a reproducing kernel Hilbert space. See Section 3.2.1. We will show in this section that a similar result holds for the GCWT in Hilbert modules. Before that, we will introduce a definition of reproducing kernel Hilbert module in the sense of [31].

Definition 5.2.1.11. Let X be a topological space. A Hilbert $\mathcal{A}$-module $\mathbb{H}$ of functions $f : X \to \mathcal{A}$ has a **reproducing kernel** $K : X \times X \to \mathcal{A}$ if

1. for each $x \in X$ the function K_x, given by $K_x(y) = K(x, y)$ is in $\mathbb{H}$,

2. for each $f \in \mathbb{H}$, $f(x) = \langle K_x, f \rangle$.

Definition 5.2.1.12. Let X be a topological space. A Hilbert $\mathcal{A}$-module $\mathbb{H}$ of functions $f : X \to \mathcal{A}$ with a reproducing kernel is called **a reproducing kernel Hilbert module**.

We include a result from [31].

Lemma 5.2.1.13. *If $\mathbb{H}$ is a reproducing kernel Hilbert module, then a sequence that converges in the norm of $\mathbb{H}$ converges pointwise.*

Now we are ready to discuss the image of the coefficient operator V_η for which η is an admissible coherent state system as a reproducing kernel Hilbert module.

Theorem 5.2.1.14. *Let $(\eta_x)_{x\in X}$ be an admissible coherent state system. Then, the image space of V_η is a reproducing kernel Hilbert module, that is, the projection on its image is given by an inner product with a function defined by a reproducing kernel.*

Proof. Let $f \in \mathbb{L}^2(X,\mathcal{A})$ be arbitrary. Since $L^2(X,\mathcal{A})$ is dense in $\mathbb{L}^2(X,\mathcal{A})$, there exist a sequence $(f_n) \subset L^2(X,\mathcal{A})$ that converges in norm in $\mathbb{L}^2(X,\mathcal{A})$. The projection of f on the image space of V_η is given by:

$$V_\eta V_\eta^* f(x) = \left\langle \eta_x, V_\eta^* f \right\rangle \tag{5.38}$$

$$= \langle V_\eta \eta_x(y), f \rangle \,. \tag{5.39}$$

Let $K(x,y) = \langle \eta_y, \eta_x \rangle$ and $K_x(y) = V_\eta \eta_x(y)$, then $K_x(y) = K(x,y)$. Note that $f \in V_\eta(\mathbb{H})$, has the form $f = V_\eta h$ for some $h \in \mathbb{H}$. Then,

$$\begin{aligned} f(x) &= V_\eta h(x) \\ &= V_\eta (V_\eta^* V_\eta) h(x) \\ &= V_\eta V_\eta^* (V_\eta h(x)) \\ &= V_\eta V_\eta^* f(x) \\ &= \langle V_\eta \eta_x, f \rangle \\ &= \langle K_x, f \rangle \,. \end{aligned}$$

By definition, $K(x,y)$ is a reproducing kernel, hence range(V_η) is a reproducing kernel Hilbert module. □

Corollary 5.2.1.15. *Let $(\eta_x)_{x\in X}$ be an admissible coherent state system. Every sequence $(f_n) \subset$ range(V_η), that converges to a function f in norm in* range(V_η), *converges pointwise to f.*

Proof. By Theorem 5.2.1.14, range(V_η) is a reproducing kernel Hilbert module. Lemma 5.2.1.13, gives the result. $\square$

Generalized continuous wavelet transforms on Hilbert modules (GCWTHM)

In this section, we will introduce the definition of the continuous wavelet transform on Hilbert modules (GCWTHM).

Before that we will list some terminology and properties related to unitary representations. Let π and σ be unitary representations of a locally compact group G in Hilbert modules $\mathcal{H}_\pi$ and $\mathcal{H}_\sigma$ respectively. An adjointable operator $T : \mathbb{H}_\pi \to \mathbb{H}_\sigma$ is called an **intertwining operator** for π and σ if $T\pi(x) = \sigma(x)T$ for every $x \in G$. We say that π and σ are **disjoint** if there is no nonzero intertwining operator in either direction. In the case there exists T which is unitary, we say that π and σ are **unitarily equivalent**. The set of all intertwining operators for π with itself is called the **commutant** of π.

Definition 5.2.1.16. Let $(\pi, \mathbb{H}_\pi)$ denote a unitary representation of the locally compact group G with left Haar measure μ on the Hilbert module. For an element $\eta \in \mathcal{H}_\pi$, we define a coherent state system $(\eta_x)_{x\in G}$ as the orbit $(\pi(x)\eta)_{x\in G}$. We call this system the **group coherent state system**.

Remark 5.2.1.17. Since the weak and strong operator topologies coincide on $\mathcal{U}(\mathbb{H}_\pi)$, the strong continuity of the representation is equivalent to the continuity of all coefficient functions $V_\eta\varphi$ for any $\varphi \in \mathbb{H}_\pi$. Since continuous functions are strongly measurable, by Definition 5.2.1.5, $(\pi(x)\eta)_{x\in G}$ is a coherent state system.

In what follows, representation will always mean unitary representation, and for an element $\eta \in \mathbb{H}_\pi$, we will write $(\eta_x)_{x\in G}$ for the group coherent state system $(\pi(x)\eta)_{x\in G}$ related to the representation $(\pi, \mathbb{H}_\pi)$ of a locally compact group G on the Hilbert module $\mathbb{H}_\pi$.

Definition 5.2.1.18. Let $(\pi, \mathbb{H}_\pi)$ denotes a representation of the locally compact group G with the left Haar measure μ. Let $\eta \in \mathbb{H}_\pi$ be arbitrary.

1. The vector η is called an **admissible vector** if and only if the coherent state $(\eta_x)_{x\in G}$ is admissible.

2. The coefficient operator V_η related to an admissible vector η, is called the **generalized continuous wavelet transform**

3. If the coefficient operator V_η is bounded on $\mathbb{H}_\pi$ then η is called a **bounded vector**.

5.2.2 The GCWTHM and the left regular representation

As in the Hilbert space setting, we will explore the relation between a coefficient operator comes from a locally compact group G and the left regular representation of G. We shall discuss the kernel of the coefficient operator, its intertwining property, and the projection of cyclic, bounded and admissible vectors. For some terminologies that are used here, such as span or orthogonal complement, please refer to Section 2.2.4.

Kernel and intertwining property of the coefficient operator

Below is a generalization of Lemma 3.2.2.1 to the setting of Hilbert modules.

Lemma 5.2.2.1. *Let $(\pi, \mathbb{H}_\pi)$ be a representation of a locally compact group G on the Hilbert module $\mathbb{H}_\pi$ and η be an element of $\mathcal{H}_\pi$. Let $\{\pi(G)\eta\}$ be the related coherent state system and $K = \operatorname{span}\{\pi(G)\eta\}$. Then the kernel* *of the coefficient operator V_η is the orthogonal complement of the closure of K, $\ker V_\eta = \overline{K}^\perp$.*

Proof. For any $\varphi \in \ker V_\eta$, $V_\eta\varphi$ is continuous. Therefore, $V_\eta\varphi = 0$ means $V_\eta\varphi(x) = 0$, for all $x \in G$. By definition of the coefficient function,

$$\langle \pi(x)\eta, \varphi\rangle = V_\eta\varphi(x) = 0.$$

Therefore for any $k = \sum_{i=1}^n \pi(x_i)\eta \cdot a_i$ in K,

$$\begin{aligned}\langle k, \varphi\rangle &= \left\langle \sum_{i=1}^n \pi(x_i)\eta \cdot a_i, \varphi \right\rangle = \sum_{i=1}^n \langle \pi(x_i)\eta \cdot a_i, \varphi\rangle \\ &= \sum_{i=1}^n a_i^* \langle \pi(x_i)\eta, \varphi\rangle = \sum_{i=1}^n a_i^*.0 = 0\end{aligned}$$

Since $\langle \cdot, \cdot\rangle$ is continuous, for every $k \in \overline{K}$, $\langle k, \varphi\rangle = 0$, i.e. If $k_n \underset{n}{\to} k$ where $k_n \in K$ then

$$\langle k, \varphi\rangle = \left\langle \lim_{n\to\infty} k_n, \varphi \right\rangle = \lim_{n\to\infty} \langle k_n, \varphi\rangle = \lim_{n\to\infty} 0 = 0.$$

To prove the other direction, let $\varphi \in \overline{K}^\perp$. Since $\mathcal{A}$ is assumed to be unital, then for each x, $\langle \pi(x)\eta, \varphi\rangle = 1_{\mathcal{A}} \langle \pi(x)\eta, \varphi\rangle = \langle \pi(x)\eta \cdot 1_{\mathcal{A}}, \varphi\rangle = 0$. Hence, by definition, $V_\eta\varphi(x) = \langle \pi(x)\eta, \varphi\rangle = 0$, for all x. This means, $V_\eta\varphi = 0$ i.e. $\varphi \in \ker(V_\eta)$. □

Definition 5.2.2.2. An element $\varphi \in \mathbb{H}_\pi$ is a **cyclic** vector of a representation $(\pi, \mathbb{H}_\pi)$ of a locally compact group G on a Hilbert module $\mathbb{H}_\pi$ if span $\{\pi(G)\varphi\}$ is a dense submodule of $\mathbb{H}_\pi$.

Replacing Lemma 3.2.2.2, we have the following two results.

Lemma 5.2.2.3. *If a vector η is a cyclic vector of a representation $(\pi, \mathbb{H}_\pi)$ of a group G on a Hilbert module $\mathbb{H}_\pi$ then the coefficient operator V_η is injective.*

Proof. Let $K = \operatorname{span}\{\pi(G)\eta\}$. Since η is cyclic, then $\overline{K} = \mathcal{H}_\pi$. Let $\varphi \in \ker V_\eta$, then by Lemma 5.2.2.1, $\varphi \in \overline{K}^\perp$. The closure of K is the whole module, therefore by Lemma 2.2.4.15, $\overline{K}^\perp = \{0\}$. Hence $\varphi = 0$, and $ker(V_\eta) = \{0\}$ i.e. the coefficient operator V_η is injective. □

In the Hilbert space setting, the converse is also valid. However, the case is different in the setting of Hilbert modules. This is because not every closed submodule is (orthogonally) complementable. However, we do have the following result.

Lemma 5.2.2.4. *Let $(\pi, \mathbb{H}_\pi)$ be a representation of a group G on a Hilbert module $\mathbb{H}_\pi$. If η is a vector in $\mathbb{H}_\pi$ such that the coefficient operator V_η is an isometry with complementable range from $\mathbb{H}_\pi$ to $\mathbb{L}^2(G, \mathcal{A})$, then η is a cyclic vector for $(\pi, \mathbb{H}_\pi)$.*

Proof. Suppose that V_η is an isometry with complementable range. Then $\ker(V_\eta) = \{0\}$. By Proposition 2.2.4.21, $V_\eta \in \mathcal{L}(\mathbb{H}, \mathbb{L}^2(G, \mathcal{A}))$ and has a closed range. Furthermore, by Theorem 2.2.4.16, $\ker(V_\eta)$ is a complementable submodule of $\mathbb{H}$. If η is not cyclic, then $\overline{K}$ is a closed proper subset of $\mathcal{H}_\pi$. Therefore $\overline{K}^\perp = \ker V_\eta$ is a complementable submodule of $\mathbb{H}$. This implies $\overline{K}^\perp \neq \{0\}$. Let $h \in \overline{K}^\perp$ and $h \neq 0$. By Lemma 5.2.2.1, $h \in \ker(V_\eta)$. This contradicts the fact that $\ker(V_\eta) = \{0\}$. □

Definition 5.2.2.5. Let G be a locally compact group. Suppose that Λ_G acts on $\mathbb{L}^2(G, \mathcal{A})$ by

$$(\Lambda_G(x)f)(y) = f(x^{-1}y), \ x, y \in G.$$

Then, Λ_G is a strongly continuous unitary representation of G on $\mathbb{L}^2(G, \mathcal{A})$ which is called called **the left regular representation**.

Remark 5.2.2.6. It is easy to show that Λ_G is an $\mathcal{A}$-linear homomorphism. Moreover, suppose that $\{\varepsilon_i\}$ is an orthonormal basis for $L^2(G)$. Then since $\mathcal{A}$ is unital, for any $b \in \mathcal{A}$

$$\Lambda_G(x)(\varepsilon_i \cdot b) = \lambda_G(x)(\varepsilon_i)b.$$

Furthermore, $\Lambda_G(x)$ is a unitary element of $\mathcal{L}(\mathbb{L}^2(G, \mathcal{A}))$ for each $x \in G$, follows from the fact that $\Lambda_G(x)$ is $\mathcal{A}$-linear, isometric onto from $L^2(G, \mathcal{A})$ onto $L^2(G, \mathcal{A})$, which is a dense subset of $\mathbb{L}^2(G, \mathcal{A})$, hence it is an isometric, surjective $\mathcal{A}$-linear map on $\mathbb{L}^2(G, \mathcal{A})$.

Next, we will show how the coefficient operator intertwines the representation of the group G that gives the transform, with the left regular representation, c.f. Corollary 3.2.2.5.

Lemma 5.2.2.7. *Let $(\pi, \mathbb{H}_\pi)$ be a representation of a locally compact group G on the Hilbert module $\mathbb{H}_\pi$, and $\eta \in \mathbb{H}_\pi$. Suppose that V_η is the related coefficient operator. Then V_η intertwines π with the left regular representation.*

Proof. Let $x, y \in G$ and $\varphi \in \mathcal{H}_\pi$. By definition

$$\begin{aligned} V_\eta(\pi(x)\varphi)(y) &= \langle \pi(y)\eta, \pi(x)\varphi \rangle \\ &= \left\langle \pi(x^{-1}y)\eta, \varphi \right\rangle \\ &= V_\eta\varphi(x^{-1}y) \\ &= \Lambda_G(x)V_\eta\varphi(y) \end{aligned}$$

Hence, for any $x \in G$, $V_\eta\pi(x) = \Lambda_G(x)V_\eta$. $\square$

We prove that in the setting of Hilbert modules, the domain of the coefficient operator is closed under the action of G.

Corollary 5.2.2.8. *Let $(\pi, \mathbb{H}_\pi)$ be a representation of G on $\mathbb{H}_\pi$, and $\eta \in \mathbb{H}_\pi$. Then, the domain $\mathcal{D}_\eta$ is closed under the action of G via π.*

Proof. Let $\varphi \in \mathcal{D}_\eta$, since V_η intertwines π and Λ_G, then, for each $x \in G$,

$$V_\eta(\pi(x)\varphi) = \Lambda_G(x)(V_\eta\varphi)$$

is in $\mathbb{L}^2(G, \mathcal{A})$. □

As in the setting of Hilbert space, for Hilbert modules, we have a result which characterize admissible vectors.

Lemma 5.2.2.9. *Let $(\pi, \mathbb{H}_\pi)$ be a representation of G on $\mathbb{H}_\pi$, and $\eta \in \mathbb{H}_\pi$ is admissible. Then η is a bounded cyclic vector.*

Proof. By definition of an admissible coherent state system (Definition 5.2.1.5), the domain of the related coefficient operator is the whole space, hence η is a bounded vector. Moreover, the fact that η is admissible implies that V_η is an isometry with complemented range. Hence, by Lemma 5.2.2.4, η is a cyclic vector. □

Remark 5.2.2.10. Since the boundedness of an operator on Hilbert modules does not automatically imply its adjointability, this lemma is too weak to be used to find which representations give an admissible vector.

The Commuting algebra and bounded, cyclic or admissible vectors

The following is an analogous result to Proposition 5.2.2.11, in the setting of Hilbert modules.

Proposition 5.2.2.11. *Let $(\pi, \mathbb{H}_\pi)$ be a representation of a locally compact group G on a Hilbert module $\mathbb{H}_\pi$, and $\eta \in \mathbb{H}_\pi$. If T belongs to the commuting algebra $\pi(G)'$ then*

$$V_{T\eta} = V_\eta \circ T^*.$$

Proof. Suppose that $\varphi \in \mathbb{H}_\pi$ and $x \in G$ are arbitrary. By definition of the coefficient function,

$$\begin{aligned}(V_{T\eta}\,\varphi)\,(x) &= \langle \pi(x)T\eta, \varphi\rangle \\ &= \langle T\pi(x)\eta, \varphi\rangle \\ &= \langle \pi(x)\eta, T^*\varphi\rangle \\ &= (V_\eta T^*\varphi)\,(x) \\ &= (V_\eta \circ T^*\ \varphi)\,(x)\,.\end{aligned}$$

From this we can see that $V_{T\eta} = V_\eta \circ T^*$. □

The following is a result which generalizes Corollary 3.2.2.8, to the setting of Hilbert modules.

Corollary 5.2.2.12. *Suppose that $\mathbb{K}$ is an invariant closed submodule of $\mathbb{H}_\pi$, with adjointable projection operator $P_\mathbb{K}$. If $\eta \in \mathbb{H}_\pi$ is admissible (respectively bounded or cyclic) for $(\pi, \mathbb{H}_\pi)$ then $P_\mathbb{K}\eta$ has the same property for the subrepresentation $(\pi\,|_\mathbb{K}, \mathbb{K})$.*

Proof. Since $P_\mathbb{K}$ is a projection, $P_\mathbb{K} = P_\mathbb{K}^*$, and hence by Proposition 5.2.2.11

$$V_{P_\mathbb{K}\,\eta} = V_\eta \circ P_\mathbb{K}^* = V_\eta \circ P_\mathbb{K}.$$

Now let us calculate the domains of these operators. By definition, the domain $\mathcal{D}_{P_\mathbb{K}\,\eta} = \mathcal{D}(V_\eta \circ P_\mathbb{K}) = \{\varphi \in \mathbb{H}_\pi \mid P_\mathbb{K}\,\varphi \in \mathcal{D}_\eta\}$. Now, if η is admissible then $\mathcal{D}_\eta = \mathbb{H}_\pi$. Since $\mathbb{K} \subset \mathbb{H}_\pi = \mathcal{D}_\eta$ then as an operator on $\mathbb{K}$, $\mathcal{D}_{P_\mathbb{K}\eta} = \mathcal{D}_\eta \cap \mathbb{K} = \mathbb{H}_\pi \cap \mathbb{K} = \mathbb{K}$. Since the restriction of an isometry is also an isometry, it follows that $V_{P_\mathbb{K}\eta}$ is an isometry on $\mathbb{K}$ with domain the whole space. Since V_η and $P_\mathbb{K}$ are adjointable, $V_\eta \circ P_\mathbb{K}$ is adjointable with $(V_\eta \circ P_\mathbb{K})^* = P_\mathbb{K}^* \circ V_\eta^*$. Furthermore, $(V_\eta \circ P_\mathbb{K})^*(V_\eta \circ P_\mathbb{K})|_\mathbb{K} = P_\mathbb{K}|_\mathbb{K} = I_\mathbb{K}$. Equivalently, $V_{P_\mathbb{K}\eta} = V_\eta \circ P_\mathbb{K}$ is an isometry with complemented range. By viewing $V_{P_\mathbb{K}\eta}$ as a restriction of V_η on $\mathbb{K}$, and recalling that $\mathbb{K}$ is a submodule of $\mathbb{H}$ and $\mathcal{D}_\eta^\diamond$ is dense in $\mathcal{D}_\eta = \mathbb{H}_\pi$, then $\mathcal{D}_\eta^\diamond \cap \mathbb{K}$ is dense in $\mathcal{D}_\eta \cap \mathbb{K} = \mathbb{K}$. Hence by definition $P_\mathbb{K}\eta$ is an admissible vector.

Furthermore by the same argument, $P_\mathbb{K}\eta$ is bounded if η is a bounded vector (η is bounded if V_η is bounded i.e. $\mathcal{D}_\eta = \mathbb{H}_\pi$) and if η is cyclic for $(\pi, \mathbb{H}_\pi)$ then $P_\mathbb{K}\eta$ also cyclic for $(\pi \mid_\mathbb{K}, \mathbb{K})$. □

A similar result also holds for unitary intertwining operators.

Corollary 5.2.2.13. *Let T be a unitary operator intertwining the representations π and σ. Then η is admissible (respectively bounded or cyclic) if and only if $T\eta$ has the same property.*

Proof. By definition, T is a map $T : \mathbb{H}_\pi \to \mathbb{H}_\sigma$ which is unitary, and for any $x \in G, T\pi(x) = \sigma(x)T$. By a similar argument as in the proof of Corollary 5.2.2.12, the results follow.

For the other direction, we use the same argument for T^*, by viewing it as a unitary intertwining operator, i.e. we prove if $T\eta$ is admissible (respectively bounded or cyclic) then $\eta = T^*T\eta$ has the same property. □

5.2.3 Some examples

An example of an admissible coherent state system

In the setting of Hilbert space, the discretization problem can be embedded into the continuous setting, and we will see that this also works in the setting of Hilbert modules.

In [24], Frank and Larson defined the notion of a countable frame in Hilbert modules. We will show below that for a finite or countably generated Hilbert module, this is the discrete case of our construction.

Definition 5.2.3.1. ([24, Definition 2.1.]) Let $\mathcal{A}$ be a unital C^*-algebra and I be a finite or countable index subset of $\mathbb{N}$. A sequence $(\varphi_i)_{i\in I}$ of elements in a $\mathbb{H}$ is said to be a **frame** if there are real constants $\alpha, \beta > 0$ such that

$$\alpha \cdot \langle \varphi, \varphi \rangle \leq \sum_{i=1}^{\infty} \langle \varphi, \varphi_i \rangle \langle \varphi_i, \varphi \rangle \leq \beta \cdot \langle \varphi, \varphi \rangle . \tag{5.40}$$

for every $\varphi \in \mathbb{H}$. The frame $(\varphi_i)_{i\in I}$ is said to be **tight frame** if $\alpha = \beta$, and said to be **normalized** if $\alpha = \beta = 1$. We consider **standard** (normalized) frames in the main for which the sum in the middle of inequality (5.40) always converges in norm in $\mathcal{A}$.

Remark 5.2.3.2. The above definition has a simple consequences. A sequence $(\varphi_i)_{i\in I}$ is a standard normalized (tight) frame if and only if the equality

$$\langle \varphi, \varphi \rangle = \sum_{i=1}^{\infty} \langle \varphi, \varphi_i \rangle \langle \varphi_i, \varphi \rangle \tag{5.41}$$

holds for every $\varphi \in \mathbb{H}$ where the sum converges in norm in $\mathcal{A}$.

Frank and Larson found that for unital C^*-algebras $\mathcal{A}$ the frame transform operator related to a standard (normalized tight) frame in a finitely or countably generated Hilbert module is adjointable in every situation, and that the reconstruction formula holds. Moreover, they proved that the image of the frame transform is an orthogonal summand of $H_{\mathcal{A}}$. See [24, Theorem 4.1].

Recalling that our C^*-algebras $\mathcal{A}$ is unital, we can see that a standard normalized (tight) frame for a finite or countably generated Hilbert module is an admissible coherent state system based on a discrete space $\mathbb{N}$ with counting measure.

Remark 5.2.3.3. In [64] Raeburn and Thompson defined a standard normalized (tight) frame for $\mathbb{H}$ in its multiplier Hilbert module $M(\mathbb{H})$. They proved that the existence of a unit element $1_{\mathcal{A}}$ in $\mathcal{A}$ implies $M(\mathbb{H}) = \mathbb{H}$, and hence in this case the definition coincides with that of Frank and Larson, [24].

Examples of generalized continuous wavelet transform

We now give some examples of the generalized continuous wavelet transform in Hilbert modules.

Example 5.2.3.4. Let $\mathcal{A} = \mathbb{C}$; then our Hilbert modules over $\mathcal{A}$ are Hilbert spaces. Therefore, generalized continuous wavelet transforms on Hilbert space are also included in this theory.

Example 5.2.3.5. Let G be a locally compact group such that there exists an admissible vector $f \in L^2(G)$ for λ_G. Let $a \in \mathcal{A}$ be a unitary element. Suppose that $\eta = f \cdot a$. It is clear that η is in $L^2(G, \mathcal{A})$. We will show that V_η is a generalized continuous wavelet transform, and η is an admissible vector for Λ_G, where Λ_G is as in Definition 5.2.2.5.

First of all, recall that by the definition of the coefficient function, V_η is an $\mathcal{A}$-linear operator. Suppose that $\{\varepsilon_i\}$ is an orthonormal basis of $L^2(G)$ and let $b \in \mathcal{A}$ be arbitrary. Then for any $x \in G$,

$$\begin{aligned} V_\eta(\varepsilon_i \cdot b)(x) &= \langle \Lambda_G(x)(f \cdot a), \varepsilon_i \cdot b \rangle \\ &= \langle \lambda_G(x) f, \varepsilon_i \rangle \langle a, b \rangle \\ &= (V_f \varepsilon_i)(x) a^* b \\ &= (V_f \varepsilon_i) \cdot a^* b(x). \end{aligned}$$

Now, let $g \in L^2(G, \mathcal{A})$ be arbitrary. Then there exists $b = (b_i) \in \mathbb{H}_{\mathcal{A}}$ such that $\sum_i \|b_i\|^2 < \infty$ and $g = \sum_i \varepsilon_i \cdot b_i$. Hence,

$$\begin{aligned} V_\eta g(x) &= \langle \Lambda_G(x)\eta, g\rangle \\ &= \left\langle \Lambda_G(x)(f\cdot a), \sum_i \varepsilon_i \cdot b_i \right\rangle \\ &= \sum_i \langle \Lambda_G(x)(f\cdot a), \varepsilon_i \cdot b_i\rangle \\ &= \sum_i (V_f\varepsilon_i)\cdot a^* b_i(x). \end{aligned}$$

Now,

$$\begin{aligned} \|V_\eta g\|_2 &= \int_X \|V_\eta g(x)\|^2 \; d\mu(x) \\ &= \int_X \left\| \sum_i (V_f\varepsilon_i)\cdot a^* b_i(x) \right\|^2 \; d\mu(x) \\ &\leq \int_X \sum_i \|(V_f\varepsilon_i)(x)\|^2 \, \|a^* b_i\|^2 \; d\mu(x) \\ &= \int_X \sum_i \|(V_f\varepsilon_i)(x)\|^2 \, \|a\|^2 \, \|b_i\|^2 \; d\mu(x) \\ &= \sum_i \|a\|^2 \, \|b_i\|^2 \int_X \|(V_f\varepsilon_i)(x)\|^2 \; d\mu(x) \\ &= \|a\|^2 \int_X \|(V_f\varepsilon_i)(x)\|^2 \; d\mu(x) \sum_i \|b_i\|^2 \\ &= \|a\|^2 \, \|(V_f\varepsilon_i)\|_2 \sum_i \|b_i\|^2 . \end{aligned}$$

Since $\sum_i \|b_i\|^2 < \infty$ and V_f is a generalized continuous wavelet transform in $L^2(G)$, and hence is an isometry, the last expression is finite.

This proves that for each $g \in L^2(G, \mathcal{A})$, $V_\eta g \in L^2(G, \mathcal{A})$, i.e. $L^2(G, \mathcal{A}) \subset \mathcal{D}_\eta^\diamond \subset \mathcal{D}_\eta \subset \mathbb{L}^2(G, \mathcal{A})$. Since $L^2(G, \mathcal{A})$ is dense in $\mathbb{L}^2(G, \mathcal{A})$, then $\mathcal{D}_\eta^\diamond$ is dense in $\mathbb{L}^2(G, \mathcal{A})$.

Now, we show that V_η is an isometry from $L^2(G, \mathcal{A})$ to $\mathbb{L}^2(G, \mathcal{A})$, and hence it is continuous.

Let $g \in L^2(G, \mathcal{A})$ be arbitrary, and $g = \sum_i \varepsilon_i \cdot b_i$. Using the facts that V_f is an isometry and a is a unitary element, we calculate

$$\begin{aligned}
\|V_\eta g\|_{\mathcal{A}} &= \|\langle V_\eta g, V_\eta g\rangle\| \\
&= \left\| \left\langle \sum_i (V_f \varepsilon_i) \cdot a^* b_i, \sum_j (V_f \varepsilon_j) \cdot a^* b_j \right\rangle \right\| \\
&= \left\| \sum_{i,j} \langle (V_f \varepsilon_i) \cdot a^* b_i, (V_f \varepsilon_j) \cdot a^* b_j \rangle \right\| \\
&= \left\| \sum_{i,j} \langle V_f \varepsilon_i, V_f \varepsilon_j \rangle \langle a^* b_i, a^* b_j \rangle \right\| \\
&= \left\| \sum_{i,j} \langle \varepsilon_i, \varepsilon_j \rangle \langle b_i, b_j \rangle \right\| \\
&= \left\| \sum_{i,j} \langle \varepsilon_i \cdot b_i, \varepsilon_j \cdot b_j \rangle \right\| \\
&= \left\| \left\langle \sum_i \varepsilon_i \cdot b_i, \sum_j \varepsilon_j \cdot b_j \right\rangle \right\| \\
&= \|\langle g, g\rangle\| \\
&= \|g\|_{\mathcal{A}} .
\end{aligned}$$

We have seen that V_η is an isometry, hence it is continuous, from a dense subset $L^2(G, \mathcal{A})$ of $\mathbb{L}^2(G, \mathcal{A})$ to a complete space $\mathbb{L}^2(G, \mathcal{A})$. Therefore, V_η can be extended to an isometry from $\mathbb{L}^2(G, \mathcal{A}) \to \mathbb{L}^2(G, \mathcal{A})$.

We will now show that V_η is adjointable. Let $g \in \mathbb{L}^2(G, \mathcal{A})$ and $h \in L^2(G, \mathcal{A})$ be arbitrary. and $(g_n) \subset L^2(G, \mathcal{A})$ such that $\|g - g_n\|_{\mathcal{A}} \to 0$. Since V_η is continuous, this implies, $\|V_\eta g - V_\eta g_n\|_{\mathcal{A}} \to 0$.

We can suppose that $g = \sum_i \varepsilon_i \cdot b_i$. and $h = \sum_i \varepsilon_i \cdot c_i$.

$$\begin{aligned}
\langle V_\eta g, h \rangle &= \left\langle V_\eta \left(\sum_i \varepsilon_i \cdot b_i \right), \sum_j \varepsilon_j \cdot c_j \right\rangle \\
&= \left\langle \sum_i V_f \varepsilon_i \cdot a^* b_i, \sum_j \varepsilon_j \cdot c_j \right\rangle \\
&= \sum_{i,j} \langle V_f \varepsilon_i \cdot a^* b_i, \varepsilon_j \cdot c_j \rangle \\
&= \sum_{i,j} \langle V_f \varepsilon_i, \varepsilon_j \rangle \langle a^* b_i, c_j \rangle \\
&= \sum_{i,j} \langle V_f \varepsilon_i, \varepsilon_j \rangle \langle b_i, a c_j \rangle \\
&= \sum_{i,j} \langle V_f \varepsilon_i \cdot b_i, \varepsilon_j \cdot a c_j \rangle \\
&= \left\langle \sum_i V_f \varepsilon_i \cdot b_i, \sum_j \varepsilon_j \cdot a c_j \right\rangle \\
&= \left\langle g, \sum_j \varepsilon_j \cdot a c_j \right\rangle .
\end{aligned}$$

Let $k_h = \sum_j \varepsilon_j \cdot a c_j$, then $k_h \in \mathbb{L}^2(G, \mathcal{A})$. Therefore, for each $h \in L^2(G, \mathcal{A})$ there exists $k_h \in \mathbb{L}^2(G, \mathcal{A})$ such that

$$\langle V_\eta g, h \rangle = \langle g, k_h \rangle . \tag{5.42}$$

Define a mapping W from $L^2(G, \mathcal{A})$ to $\mathbb{L}^2(G, \mathcal{A})$ by $Wh = k_h$. This mapping is $\mathcal{A}$-linear.

We show that W is continuous. Let $(h_n) \subset L^2(G, \mathcal{A})$ such that

$$\|h - bh_n\|_{\mathcal{A}} \to 0.$$

Then, for any $g \in \mathbb{L}^2(G, \mathcal{A})$,

$$\|\langle g, W(h - h_n) \rangle\| = \|\langle V_\eta g, h - h_n \rangle\| \leq \|V_\eta g\|_{\mathcal{A}} \|h - h_n\|_{\mathcal{A}} = \|g\|_{\mathcal{A}} \|h - h_n\|_{\mathcal{A}} \to 0.$$

Therefore, $\|Wh - Wh_n\|_{\mathcal{A}} = \|W(h - h_n)\|_{\mathcal{A}} \to 0$.

By this, and the fact that $L^2(G, \mathcal{A})$ is dense in $\mathbb{L}^2(G, \mathcal{A})$, we can extend W to $\mathbb{L}^2(G, \mathcal{A})$. We denote the extension by V_η^*.

We will show that for each $h \in L^2(G, \mathcal{A})$ we can define $V_\eta^* h$ as an ultra-weak integral relative to the dense subset $\mathcal{D}_\eta^\diamond = L^2(G, \mathcal{A})$ of $\mathbb{L}^2(G, \mathcal{A})$.

Let $g \in L^2(G, \mathcal{A})$, then $V_\eta g \in L^2(G, \mathcal{A})$.

We calculate that

$$\begin{aligned}
\left\langle g, V_\eta^* h \right\rangle &= \langle V_\eta g, h \rangle \\
&= \mathbf{B}\text{ - }\int_X V_\eta g(x)^* h(x)\, d\mu(x) \\
&= \mathbf{B}\text{ - }\int_X \langle \eta_x, g \rangle^*\, h(x)\, d\mu(x) \\
&= \mathbf{B}\text{ - }\int_X \langle g, \eta_x \rangle\, h(x)\, d\mu(x) \\
&= \mathbf{B}\text{ - }\int_X \langle g, \eta_x \cdot h(x) \rangle\ d\mu(x).
\end{aligned}$$

By definition, $V_\eta^* h = \int_X \eta_x \cdot h(x)$ is an ultra-weak integral relative to $\mathcal{D}_\eta^\diamond$.

Finally, we show that $V_\eta^* V_\eta = I_{\mathbb{L}^2(G,\mathcal{A})}$. Let $h, g \in \mathbb{L}^2(G, \mathcal{A})$. Let $(g_n), (h_m)$ be sequences in $L^2(G, \mathcal{A})$ such that $\|h - h_m\|_{\mathcal{A}} \to 0$ and $\|g - g_n\|_{\mathcal{A}} \to 0$ as

$m, n \to \infty$. Then,

$$\begin{aligned}
\left\langle h, \left(V_\eta^* V_\eta\right)(g)\right\rangle &= \lim_{n\to\infty}\lim_{m\to\infty} \left\langle h_m, \left(V_\eta^* V_\eta\right)(g_n)\right\rangle \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \left\langle h_m, V_\eta^*\left(V_\eta g_n\right)\right\rangle \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \left\langle h_m, \int_X \eta_x \cdot \langle \eta_x, g_n\rangle \; d\mu(x)\right\rangle \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \mathbf{B}\text{ - }\int_X \langle h_m, \eta_x \cdot \langle \eta_x, g_n\rangle\rangle \; d\mu(x) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \mathbf{B}\text{ - }\int_X \langle h_m, \eta_x \cdot \langle \eta_x, g_n\rangle\rangle \; d\mu(x) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \mathbf{B}\text{ - }\int_X \langle h_m, \eta_x\rangle \langle \eta_x, g_n\rangle \; d\mu(x) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \mathbf{B}\text{ - }\int_X \langle \eta_x, h_m\rangle^* \langle \eta_x, g_n\rangle \; d\mu(x) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \mathbf{B}\text{ - }\int_X V_\eta h_m(x)^* V_\eta g_n(x) \; d\mu(x) \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \langle V_\eta h_m, V_\eta g_n\rangle \\
&= \lim_{n\to\infty}\lim_{m\to\infty} \langle h_m, g_n\rangle \\
&= \langle h, g\rangle .
\end{aligned}$$

Since, V_η is adjointable and $V_\eta^* V_\eta = I_{\mathbb{L}^2(G,\mathcal{A})}$ then it is an isometry with complementable range. Together with the fact that $\mathcal{D}_\eta^\diamond$ is dense in $\mathbb{L}^2(G, \mathcal{A})$, this shows that η is admissible for Λ_G and V_η is a generalized continuous wavelet transform on $\mathbb{L}^2(G, \mathcal{A})$.

Bibliography

[1] E. W. Aslaksen and J. R. Klauder. Unitary representations of the affine group. *J. Math. Phys.*, 9:206–211, 1968.

[2] E. W. Aslaksen and J. R. Klauder. Continuous representation theory using the affine group. *J. Math. Phys.*, 10:2267–2275, 1969.

[3] L. Baggett and K. Merrill. Abstract harmonic analysis and wavelets in $\mathbf{R}^n$. *Contemp. Math.*, 247:17–27, 1999.

[4] David Bernier and Keith F. Taylor. Wavelets from square-integrable representaions. *SIAM J. Math. Anal.*, 27(2):594–608, March 1996.

[5] P. Burt and E. Adelson. The Laplacian pyramid as a compact image code. *IEEE Trans. Comm.*, 31:532–540, 1983.

[6] A. L. Carey. Square-integrable representations of non-unimodular groups. *Bulletin Austral. Math. Soc*, 15:1–12, 1976.

[7] A. L. Carey. Group representations in reproducing kernel Hilbert spaces. *Rep. Math. Phys.*, 14(2):247–259, 1978.

[8] C. K. Chui. *Wavelets: A Tutorial in Theory and Applications*. Academic Press, New York, 1992.

[9] X. Dai and D. R. Larson. Wandering vectors for unitary systems and orthogonal wavelets. *Mem. Amer. Math. Soc.*, 640, 1998.

[10] I. Daubechies. *On Klauder's Path: A Field Trip*, pages 35–42. World Sci. Publ., River Edge, NJ, 1994.

[11] Ingrid Daubechies. *Ten Lectures on Wavelets*. CBMS-NFS Regional Conference Series in Applied Mathematics. SIAM, Philadelphia, Pennsylvania, 1992.

[12] Jacques Dixmier. *C^*-Algebras*. North-Holland Publishing Company, Amsterdam, 1977.

[13] M. Duflo and Calvin Moore. On the regular representation of a nonunimodular locally compact group. *J. Functional Anal.*, 21:209–243, 1976.

[14] Nelson Dunford and Jacob T. Schwartz. *Linear Operators Part I: General Theory*. Interscience Publishers, Inc., New York, 1964.

[15] R. Fabec and G. Ólafsson. The continuous wavelet transform and symmetric spaces. *Acta applicandae mathematicae*, 77:41–69, 2003.

[16] J. M. G. Fell and R. S. Doran. *Representations of *-Algebras, Locally Compact Groups and Banach *-Algebraic Bundles*, volume I: Basic Representation Theory of Groups and Algebras of *Pure and Applied Mathematics: 125*. Academic Press Inc., San Diego, CA, 1988.

[17] Gerald B. Folland. *A Course in Abstract Harmonic Analysis*. Studies in Advanced Mathematics. CRC Press, Inc, 2000 Corporate Blvd., N.W., Boca Raton, Florida, 1995.

[18] Gerald B. Folland. *Real Analysis, Modern Techniques and Their Applications*. Pure and Applied Mathematics. A Wiley-Interscience Publication, 2nd edition, 1999.

[19] M. Frank. Frames for Hilbert C^*-modules. Talk in a mini-workshop.

[20] M. Frank. Self-duality and C^*-reflexivity of Hilbert C^*-moduli. *Z. Anal. Anwendungen*, 9:165–176, 1990.

[21] M. Frank. Hilbert C^*-module and related subjects: A guided reference overview. (funct-an, 13), 1996. funct-an, preprint $\sharp$ 9605003.

[22] M. Frank and D. R. Larson. A module frame concept for Hilbert C^*-modules. *Contemp. Math.*, 247:207–233, 1999.

[23] M. Frank and D. R. Larson. Modular frames for Hilbert C^*-modules and symmetric approximation of frames. *Proceedings of SPIE*, 4119, 2000.

[24] M. Frank and D. R. Larson. Frames in Hilbert C^*-modules and C^*-algebras. *J. Operator Theory*, 48:273–314, 2002.

[25] Hartmut Führ. Wavelet frames and admissibility in higher dimensions. *J. Math. Phys.*, 37(12):6353–6366, December 1996.

[26] Hartmut Führ. Continuous wavelet transforms with abelian dilation groups. *J. Math. Phys.*, 39(8):3974–3986, August 1998.

[27] Hartmut Führ. Admissible vectors for the regular representation. *Proc. Amer. Math. Soc.*, 130(10):2959–2970, March 2003.

[28] Hartmut Führ. *Abstract Harmonic Analysis of Continuous Wavelet Transform*, volume 1863 of *Lecture Notes in Mathematics*. Springer Verlag, Berlin, 1st edition, 2005.

[29] Hartmut Führ and Matthias Mayer. Continuous wavelet transforms from semidirect products: Cyclic representations and Plancherel measure. *J. Fourier Anal. Appl.*, 8:375–397, 2002.

[30] S. Twareque Ali, Hartmut Führ and Anna E. Krasowska. Plancherel inversion as unified approach to wavelet transforms and Wigner functions. *Ann. Inst. Henri Poincaré*, 4:1015–1050, 2003.

[31] R. P. Gilbert and G.N. Hile. Hilbert function modules with reproducing kernels. *Nonlinear Anal.*, 1(2):135–150, 1977.

[32] A. Grossmann and J. Morlet. Decomposition of Hardy functions into square-integrable wavelets of constant shape. *SIAM J. Math. Anal.*, 15(4):723–736, July 1984.

[33] Deguang Han and David R. Larson. Frames, bases and group representations. *Mem. Amer. Math. Soc.*, 147(697), 2000.

[34] Christoper E. Heil and David F. Walnut. Continuous and discrete wavelet transforms. *SIAM review*, 31(4):628–666, Dec 1989.

[35] Einar Hille. *Methods in Classical and Functional Analysis*, volume 2844 of *Addison-Wesley Series in Mathematics*. Addison-Wesley Publ. Company, 1972.

[36] Einar Hille and Ralph S. Phillips. *Functional Analysis and Semi Groups*, volume XXXI of *American Mathematical Society Colloquium Publications*. American Mathematical Society, 1957.

[37] I. Fourgeau J. Morlet, G. Arens and D. Giard. Wave propagation and sampling theory. *Geophysics*, 47:203–236, 1982.

[38] Nathan Jacobson. *Lectures in Abstract Algebra: Theory of Fields and Galois Theory*, volume III of *The University Series in Higher Mathematics*. Van Nostrand, Princeton, New Jersey, 1964.

[39] Richard V. Kadison and John R. Ringrose. *Fundamentals of the Theory of Operators Algebras: Elementary Theory*, volume I of *Pure and Applied Mathematics:100, A Series of Monographs and Textbooks*. Academic Press, Inc, 111 Fifth Avenue, New York, New York 10003, 1983.

[40] I. Kaplansky. Modules over operator algebras. *Amer. J. Math.*, 75:839–858, 1953.

[41] Tom Hendrik Koornwinder. The continuous wavelet transform. In *Wavelets: An Elementary Treatment of Theory and Applications*, pages 27–48. World Scientific.

[42] A. Grossmann, R. Kronland-Martinet and J. Morlet. Reading and understanding continuous wavelet transform. In *Wavelets (Marseille, 1987)*, Inverse. Problem. Theoret. Imaging, pages 2–20, Berlin, 1989. Springer.

[43] E. C. Lance. *Hilbert C^*-Modules: A Toolkit for Operator Algebraists*, volume 210 of *London Mathematical Society Lecture note*. Cambridge University Press, Cambridge, 1st edition, 1995.

[44] D. R. Larson. Von Neumann algebras and wavelets. *Operator Algebra and Applications*, pages 267–312, 1997.

[45] David R. Larson. Frames and wavelets from an operator theoretic point of view. *Contemp. Math.*, 228:201–218, 1998.

[46] T. N. T. Goodman, S. L. Lee and W. S. Tang. Wavelets in wandering subspaces. *Trans. Amer. Math. Soc.*, 338(2):639–654, 1993.

[47] P. G. Lemarié. Existence de fonction père pour les ondelettes à support compact. *C. R. Acad. Sci. Paris Sér I Math.*, 314:17–19, 1992.

[48] Stéphane Mallat. *A Wavelet Tour of Signal Processing*. Academic Press, 2nd edition, 1999.

[49] V. M. Manuilov and E.V. Troitsky. *Hilbert C^*-modules*, volume 226 of *Translation of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2005.

[50] L. Baggett, H. Medina and K. Merrill. Generalised multi-resolution analyses, and a construction procedure for all wavelet sets in $\mathbf{R}^n$. *J. Fourier Anal. Appl.*, 5(6):563–573, 1999.

[51] Y. Meyer. Wavelets and applications. *Res. Notes Appl. Math.*, 20, 1991.

[52] Yves Meyer. Book review of: An Introduction to Wavelets, by Charles K. Chui. Academic Press, and Ten Lectures on Wavelets, by Ingrid Daubechies. *Bull. Amer. Math. Soc.*, 28(2):350–360, April 1993.

[53] J. A. Mingo and W.J. Phillips. Equivariant triviality theorems for Hilbert C^*-modules. *Proc. Amer. Math. Soc.*, 91(2):225–230, 1984.

[54] L. Baggett, A. Carey, W. Moran and P. Ohring. General existence theorems for orthonormal wavelets: An abstract approach. *Publ. RIMS*, 31:95–111, 1995.

[55] A. Grossmann, J. Morlet and T. Paul. Transforms associated to square integrable group representations I: General results. *J. Math. Phys.*, 26(10):2473–2479, October 1985.

[56] A. Grossmann, J. Morlet and T. Paul. Transform associated to square integrable group representation II: Examples. *Ann. Inst. H. Poincaré. Phys. Théor*, 45(3):293–309, 1986.

[57] J. Morlet. *Sampling theory and wave propagation*, volume 1, pages 233–261. Springer-Verlag, 1983. Issues in Acoustic Signal/Image Processing and Recognition.

[58] Gerald J. Murphy. *C^*-Algebras and Operator Theory.* Academic Press, Inc, 1250 Sixth Av, San Diego,CA 9210, 1990.

[59] M. Fashandi, R. A. Kamyabi Gol, A. Niknam and M. A. Pourabdollah. Continuous wavelet transform on a special homogeneous space. *J. Math. Phys.*, 44(9):4260–4266, September 2003.

[60] J. Packer and M. A. Rieffel. Wavelet filter functions, the matrix completion problem, and projective modules over $C(\mathbb{T}^n)$. *J. Fourier Anal. Appl.*, 9(2):101–116, 2003.

[61] J. A. Packer and M. A. Rieffel. Projective multi-resolution analyses for $L^2(\mathbb{R}^2)$. *J. Fourier Anal. Appl.*, 10:439–464, 2004.

[62] D. Han, D. R. Larson, M. Papadakis and Th. Stavropoulos. Multiresolution analysis of abstract Hilbert spaces and wandering subspaces. *Contemp. Math*, 247:259–283, 1999.

[63] W. L. Paschke. Inner product modules over B^*-algebra. *Trans. Amer. Math. Soc*, 182:443–468, 1973.

[64] I. Raeburn and S. J. Thompson. Countably generated Hilbert modules, the Kasparov stabilization theorem, and frames in Hilbert modules. *Proc. Amer. Math. Soc.*, 131(5):1557–1564, 2002.

[65] Iain Raeburn. C^*-algebras: Lecture notes from courses given at the University of Newcastle. December 1997.

[66] Iain Raeburn and Dana P. William. *Morita Equivalence and Continuous-Trace C^*-algebras*, volume 60 of *Mathematical Survey and Monographs.* American Mathematical Society, 1998.

[67] M. A. Rieffel. Induced representations of C^*-algebras. *Adv. in Math.*, 13:176, 1974.

[68] M. A. Rieffel. Multiwavelets and operator algebras. Talk given at AMS special session, January 1997.

[69] Walter Rudin. *Real and Complex Analysis.* McGraw Hill Series in Higher Mathematics. McGraw-Hill Inc. Book Company, New York St. Louis San Francisco Toronto London Sydney, 1966.

[70] M. B. Ruskai. *Wavelets and Their Applications.* Jones and Bartlett, Boston, MA, 1992.

[71] P. P. Saworotnov. A generalized Hilbert space. *Duke Math. J.*, 35:191–197, 1968.

[72] Gilbert Strang. Wavelet transforms versus Fourier transforms. *Bull. Amer. Math. Soc.*, 28(2):288–305, April 1993.

[73] Gilbert Strang. Wavelets. *American Scientist*, 82:250–255, April 1994.

[74] Paolo Aniello, Gianni Cassinelli, Ernesto De Vito and Alberto Levrero. Wavelet transforms and discrete frames associated to semidirect products. *J. Math. Phys.*, 39(8):3965–3973, August 1998.

[75] N. E. Wegge-Olsen. *K-theory and C^*-algebra, a Friendly Approach.* Oxford University Press, New York, 1993.

[76] Dana P. William. *Crossed Products of C^*-Algebras*, volume 134 of *Mathematical Survey and Monographs.* American Mathematical Society, 2007.

[77] P. J. Wood. *Wavelets and C^*-algebras.* PhD thesis, Flinders University of South Australia, 2003.