LOCALISATION AND HIGHER ORDER AVERAGING FOR BOUNDARY INTEGRAL EQUATIONS

by

Thanh Tran

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CERTIFICATE OF ORIGINALITY

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgement is made in the text.

(Signed)... ....

CONTENTS

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ABSTRACT

The behaviour of the solutions of an elliptic boundary value problem in a plane domain depends heavily on the settings of that problem. If the boundary of the domain is a non-smooth curve or an open arc, or if the given data are non-smooth functions, the solutions will have singularities. When the problem is reformulated, via the direct method, into a boundary integral equation, the solutions of the latter inherit those singularities. This bad behaviour then affects the accuracy when an approximation method , e.g. the Galerkin method, is used for this integral equation. The rate of the global convergence is significantly reduced compared to the case when the boundary is a smooth and closed curve and the given data are smooth. However, regardless of the properties of the curves, if the given data are smooth, then the solutions are smooth locally, i.e., away from the singularities. The comparison of the local to global accuracy therefore deserves a careful study.

The first part of this thesis gives a complete analysis of the local errors of the Galerkin approximation to solutions of strongly elliptic integral equations on smooth curves, closed or open. The analysis will lead to error estimates in a large range of Sobolev norms. In most of the cases, the local error in a smooth region of the solutions is more accurate than the global error.

Another problem occurs when the highest order of convergence achieved for an approximation method is hidden in a negative norm. If this is the case, that high order is not easily observed. We shall establish a post-processing method to force that order to appear locally in the L^2-norm. At first the study is carried out for

the Galerkin method applied to equations on smooth curves, closed or open. In this case, the mesh is required to be uniform on the interval under consideration and can be freely defined away from this interval. Then the post processing is used for a semi-discrete method, the qualocation method, for equations on a smooth, closed curve on which a uniform mesh is defined.

NOTATION

The Sobolev spaces on a curve in $\mathbb{R}^2$, closed or open, and the corresponding norms are defined in Section 2 of Chapter II while the periodic Sobolev spaces and the corresponding norms are defined in Section 2 of Chapter III. Similarly, the spline spaces on a plane curve are defined in Section 2 of Chapter II whereas those for periodic splines are defined in Section 2 of Chapter III.

Throughout this thesis, c denotes a generic constant which can take different values at different occurences.

CHAPTER I

INTRODUCTION

1. Prologue: The Matter at Issue

The history of the study of local behaviour perhaps goes back to the 19^{th} century with Riemann's well-known localisation principle. It tells us that the convergence or divergence at a particular point of a Fourier series of a function is governed entirely by the behaviour of that function in an arbitrarily small neighbourhood of that point. The story turns out to be more interesting in the study of approximation methods. It is well known that the behaviour of the approximations depends heavily on the characteristics of the settings of the problems. Non-smoothness of domains and singularities of given data incur singularities of the exact solutions, and reduce the global rate of convergence of the approximations. It is then natural to ask whether the accuracy of the approximation is better in regions of smooth behaviour of the exact solutions. Another interesting question is whether there are efficient ways to recover the loss of accuracy due to singularities. These problems are clearly pointed out in the foreword of L. Wahlbin's article on local behaviour of the finite element methods [54]

When facing various types of singularities one is forced to consider the local behavior of an approximation method .... An archetypical question is the following: The problem at hand contains isolated singularities and we know, a priori, that our approximation method cannot resolve these singularities (to solve them may be too costly, we do not know where the singularities are, or even what is *their nature). Assuming then that we have given up on resolving the singular behaviors, can we at last assert how good our approximation is in regions of smooth behavior? Can we precisely account for the spread of errors emanating at the singularities into smooth regions? And, can the analysis indicate an efficient way of resolving the singularities, e.g. by mesh refinement, inclusion of special functions mimicking a singularity, tracking of its unknown location, or by a posteriori processing? Regrettably, "... questions are abundant and answers are rare."*

We shall not give a traditional discussion of boundary element methods, i.e., we shall not mention the role of boundary element methods in solving boundary value problems. The literature is copious [11, 12, 13, 22, 25, 28, 29, 31, 33, 42, 56, 58] and more discussion may be redundant. Our main concerns in this dissertation are *local error estimates* and *a posteriori processing* for the Galerkin approximations to strongly elliptic integral equations on smooth curves in the plane, either closed or open.

To illustrate the necessity of the study of local estimates and post-processing, let us begin with the most common equation, Symm's equation,

$$V\psi(x) := -\frac{1}{2\pi}\int_\Gamma \log|x-y|\psi(y)\,ds(y) = f(x) \quad \text{for } x \in \Gamma, \tag{1.1.1}$$

where Γ is a smooth closed or open curve in $\mathbb{R}^2$.

It is well known that if Γ is smooth and closed then V is a continuous and bijective mapping from $H^\tau(\Gamma)$ to $H^{\tau+1}(\Gamma)$ for any real value of τ, provided that the transfinite diameter of Γ is different from 1 (see [26, 42, 55, 56]). (All the Sobolev spaces mentioned in this chapter will be defined in Chapter II.) It follows in turn that if piecewise-constant functions are used as test and trial functions for the Galerkin approximation to (1.1.1) then the following global error estimates hold:

$$\|e\|_{H^t(\Gamma)} \le ch^{s-t}\|\psi\|_{H^s(\Gamma)}, \tag{1.1.2}$$

for $-2 \le t \le s \le 1$, $t < 1/2$ and $s \ge -1/2$. Therefore, the orders of convergence in the L^2-norm and energy norm (i.e., $H^{-1/2}$-norm) are $O(h)$ and $O(h^{3/2})$ respectively. The highest order achievable is in the H^{-2}-norm, which is $O(h^3)$.

When the curve is open it was proved in [46] that if the transfinite diameter of the curve is different from 1 then $V\colon \widetilde{H}^\tau(\Gamma) \to H^{\tau+1}(\Gamma)$ is a continuous and bijective mapping for $-1 < \tau < 0$ and that, no matter how smooth the given data F is, the exact solution ψ of (1.1.1) has a singularity of the form $d^{-1/2}$ at each end point of Γ, where d is the distance to the end point (see also [59]). It is clear that $\psi \notin H^0(\Gamma) = L^2(\Gamma)$ and therefore, with piecewise-constant functions used as test and trial functions, only the following estimates [20] hold

$$\|e\|_{\widetilde{H}^t(\Gamma)} \le ch^{s-t}\|\psi\|_{\widetilde{H}^s(\Gamma)} \quad \text{for } -1 < t \le s < 0. \tag{1.1.3}$$

In the case of an open arc Γ, even though the global norm $\|\psi_h - \psi\|_{L^2(\Gamma)}$ is undefined, the solution ψ can be smooth in any sub-arc of Γ (e.g., if Γ is the interval $(-1,1)$ and if $F(x) = x$ then $\psi(x) = 2x(1-x^2)^{-1/2}$ for $x \in \Gamma$), which gives rise to the question of how ψ_h approximates ψ in the L^2-norm on some sub-arc of Γ. In the energy norm (i.e., $H^{-1/2}$-norm) or other norms defined globally, it is also natural to ask whether local convergence is better than the global convergence.

When the curve is smooth and closed the answers are known. Results were shown by J. Saranen in [36] for strongly elliptic pseudo-differential equations on a smooth, closed curve in $\mathbb{R}^2$ with smoothest splines used as test and trial functions for the Galerkin approximation. They can be briefly stated as follows: if the exact solution is smooth in some sub-arc of Γ, local convergence in Sobolev norms greater than the energy norm is not affected by the lack of global smoothness of the exact solution (provided that the solution lies in the energy space); whereas additional global regularity is needed to obtained optimal local convergence in lower order norms. For example, if piecewise-constant functions on a quasi-uniform mesh are used as test and trial functions and if the exact solution of (1.1.1) belongs to $H^{s_1}(\Gamma^*) \cap H^{s_2}(\Gamma)$ with $-1/2 \leq s_2 \leq s_1 \leq 1$ then local convergence in the H^t-norm in Γ_0 for $t \geq -1/2$ is of order $O(h^{s_1-t})$, even if $s_2 = -1/2$; whereas for $t < -1/2$ the order is $O(h^{\min(s_1-t, s_2+2)})$. Here Γ_0 and Γ^* are sub-arcs of Γ with the property that the closure of Γ_0 is contained in the interior of Γ^*. The results for the case of open arcs Γ were not mentioned in that paper. Any endeavour to fill the gap is therefore worthwhile.

There are efficient ways of accommodating the singularity of the exact solution ψ of (1.1.1) so as to increase the order of global convergence in (1.1.3). The augmented-Galerkin procedure, used by E. Stephan and W. Wendland [46], is designed to include, in addition to regular finite elements, appropriate singular elements mimicking the singular part in the decomposition of ψ. This approach for the approximation of equation (1.1.1) yields [20, 46] convergence of order $O(h^{3/2})$ in the energy norm (i.e., $H^{-1/2}$-norm) and order $O(h^3)$ in the H^{-2}-norm, which is the same as in the case of a smooth, closed curve. Another method, which is simpler to implement, is mesh grading. This method has been used extensively in the boundary element literature for weakly singular integral equations and for second kind boundary integral equations arising from the Dirichlet problem on a polygon through the double-layer potential formulation [16, 23, 53, 60]. It was shown in [53, 60] that by appropriately grading the mesh at the two ends of the open curve Γ in the above example, convergence of the same order $O(h^{3/2})$ is achieved in the energy norm. A simple argument (see details in Chapter III) then yields convergence of order almost $O(h^3)$ in the H^{-2}-norm. In this dissertation we consider only the latter method.

The highest order of convergence $O(h^3)$ in the case of smooth closed curves or even in the case of open curves after using mesh grading is lurking in the negative H^{-2}-norm and hence, at first sight, may not be observed. Indeed, as stressed by I. Sloan in [42], this high order of accuracy is beneficial if we finally are not interested in ψ but in the integral $\int_\Gamma g(y)\psi(y)\,dl(y)$, where g is a reasonably smooth function. For by using the duality of $H^{-2}(\Gamma)$ and $H^2(\Gamma)$ we have

$$\begin{aligned}\left|\int_\Gamma g(y)\psi(y)\,ds(y) - \int_\Gamma g(y)\psi_h(y)\,ds(y)\right| &= \left|\int_\Gamma g(y)\left(\psi(y) - \psi_h(y)\right)\,ds(y)\right| \\ &\le \|g\|_{H^2(\Gamma)}\|\psi - \psi_h\|_{H^{-2}(\Gamma)},\end{aligned}$$

so that the $O(h^3)$ order of convergence is observable if $g \in H^2(\Gamma)$. Our main focus in this thesis is on direct boundary integral equations, of which the solutions have immediate physical meanings (see [29, 42, 56]). Therefore we are concerned with the approximation of ψ itself, in which this $O(h^3)$ order of convergence is not easily observed. Hence it is useful to establish a post-processing method so that the same order of accuracy emerges in the L^2-norm.

2. The Scope of the Thesis

In the following chapters, we try to give some decent answers to the above questions. However, we shall consider strongly elliptic integral equations rather than just Symm's equation, which is then a particular case. Therefore, boundary value problems with the Dirichlet or Neumann conditions for various types of equations (Laplace's equation, the Helmholtz equation, the Stokes and Navier-Stokes equations,..., see e.g., [56]) are covered in the discussion, and we will not mention them again. We shall prove local error estimates for the Galerkin approximations to integral equations on smooth curves (closed and open), and establish a post-processing method to increase the order of local convergence in the L^2-norm. We even discuss that post-processing method for the qualocation approximation to strongly elliptic integral equations on smooth closed curves.

We are inspired by the works of J. Bramble and A. Schatz [7, 9] and V. Thomée [48] in the finite element literature. The original idea is due to Bramble and Schatz. These two authors considered the Galerkin approximation to some elliptic boundary value problems in which a high order of accuracy is concealed in some negative norm. They introduced a local, simple and systematic way of averaging the values of the Galerkin solution, using the so-called *K-operator*, so that the high order of accuracy no longer lingers in the negative norm but emerges in the L^2-norm. That operator acting on the Galerkin solution is defined as a convolution of the solution with a special kind of spline with small support. That spline function is chosen so that it reproduces certain polynomials under convolution. This method is applicable to a very general class of locally uniform meshes. Thomée gave an alternative definition (and therefore an alternative proof) for the K-operator, and considered the error estimates not only for the approximate solutions but also for the derivatives. The salient features of the K-operator are well elucidated in [54].

When the K-operator is applied to boundary integral equations, difficulties appear if the mesh is only locally uniform, due to the non-local property of integral operators. In fact, since the method relies on, besides other factors, the translational invariance of the trial space, and since non-uniformity spoils this property of the space of piecewise-polynomial functions on a mesh of the curve Γ, closed or open, one has to restrict the trial space to a subspace of splines with compact supports in a sub-arc of Γ where the mesh is uniform. This space is now invariant under translation by the mesh step of this sub-arc. It turns out that the local estimates given in [36] for approximation equations defined on the whole closed curve Γ using the whole test and trial spaces are not suitable for our use. Following the idea of J. Nitsche, Schatz and Wahlbin [34, 38] in the finite element environment we consider an ***interior approximate equation*** which can be thought of as an equation to define an approximation of the exact solution in the interior (i.e., away from the singularities). Local estimates for this approximation can be obtained by modifying the proof of Saranen in [36]. An interesting outcome is then achieved. We obtain local error estimates for integral equations on an open arc since it can be embedded in a smooth closed curve.

Chapter II gives details of the modification of the proof of Saranen [36] to obtain interior local estimates for interior approximate equations. Local error estimates for integral equations on open curves are then deduced. We shall in particular discuss the two most common examples, weakly singular and hypersingular integral equations on the interval $[-1, 1]$, to illustrate the significant difference between global convergence and local convergence.

We shall introduce in Chapter III the K-operator method. When the curve is smooth and closed, we try to be as general as possible when considering the pseudo-differential operators. The perturbation part of the operator is rather free so as to allow our method to be applicable to boundary value problems of various types of equations (e.g., Laplace's equation, the Helmholtz equation ...). When the curve is open, for simplicity we restrict our discussion to Symm's equation on the interval $(-1, 1)$, and give comments on when the method is applicable. Two kinds of meshes are considered: quasi-uniform and graded meshes. We shall recover the order of convergence achieved for smooth closed curves by an appropriate grading at the two ends of the arc.

We reserve Chapter IV to discuss a semi-discrete method, the qualocation method, to see how well the K-operator works in a different setting. A perturbation argument is used to widen the class of operators that can be considered. The application of the K-operator is considered only with a uniform mesh and a closed curve, as the qualocation error estimates have been proved only for this kind of mesh [17, 39, 41, 42, 45].

Numerical results are supplied in each chapter to convince the reader that the theoretically predicted increase in accuracy can be observed in practice. This applies both to local errors compared to the global ones (Chapter II) and to the K-operator method (Chapters III and IV).

CHAPTER II

LOCAL ERROR ESTIMATES

1. Introduction

In this chapter we shall study local convergence properties of the Galerkin method applied to strongly elliptic pseudo-differential equations given on smooth curves in the plane, either closed or open. The equations are of the form

$$Au = f, \tag{2.1.1}$$

where A is a strongly elliptic pseudo-differential operator of real order 2α on a smooth curve Γ in $\mathbb{R}^2$. Common examples of these equations are Symm's first-kind integral equation with logarithmic kernel and the hypersingular integral equation, which are defined respectively as

$$Vu(x) := -\frac{1}{2\pi}\int_\Gamma \log|x-y|u(y)\,ds(y) = f(x) \quad \text{for } x \in \Gamma, \tag{2.1.2}$$

and

$$Du(x) := -\frac{1}{\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}(\log|x-y|)u(y)\,ds(y) = f(x) \quad \text{for } x \in \Gamma, \tag{2.1.3}$$

where $ds(y)$ is the element of arc-length, and $\partial/\partial n_z$ denotes the directional derivative operator in the direction of the outside normal (at the point z) of Γ in the case Γ is closed, or of a closed curve $\tilde{\Gamma}$ containing Γ in case Γ is open. The operators V and D are of order -1 and 1 respectively. Another example is the singular integral equation of Cauchy type. These integral equations are of fundamental importance in solving boundary value problems of potential theory.

For the Galerkin method global error estimates in Sobolev norms are known (for various types of equations) in the case of smooth closed curves [25, 26, 27], smooth open curves and polygonal curves [4, 15, 18, 19, 20, 21, 43, 44, 46, 53, 57, 60]. However, local estimates have been proved only for positive definite operators [10] and for strongly elliptic operators [36] on smooth and closed curves.

In this chapter, we derive local error estimates for the Galerkin approximation to strongly elliptic equations on smooth open curves, when smoothest splines are used as trial and test functions. These results for the open curves follow by modifying the proof of Saranen [36] to obtain a kind of *interior local estimate* for an *interior approximate equation* on smooth closed curves. Another application of these interior local estimates will be discussed in the next chapter, where a post processing method using the *K-operator* is studied.

The chapter consists of 5 sections. Notations to be used in this chapter are given in Section 2. The result on interior local estimates is proved in Section 3. That result is then applied in Section 4 to achieve local error estimates for the case of open curves. In particular, we consider a weakly singular integral equation (Symm's equation) and a hypersingular integral equation. Section 5 is devoted to some numerical experiments.

The main result in the chapter is Theorem 3.7 for closed curves and then Theorem 4.2 for open curves. These give for the local error a bound comprising two parts. The first part represents the local approximation property of the splines. The second part consists of the global error term in the deepest Sobolev norm. The theorems suggest that even though the local convergence in a given norm is often better than the global convergence in that norm, that local order cannot exceed the order achieved globally in the deepest negative norm.

2. Notations and Some Preliminaries

The definition of the Sobolev spaces to be used in this chapter and in Section 4 of the next chapter is as follows. Let Γ be a smooth, closed curve in $\mathbb{R}^2$. As in [24, 32], we define

$$H^s(\Gamma) = \begin{cases} \{u|_\Gamma : u \in H^{s+1/2}(\mathbb{R}^2)\} & \text{for } s > 0, \\ L^2(\Gamma) & \text{for } s = 0, \\ (H^{-s}(\Gamma))', \text{ (dual space)}, & \text{for } s < 0. \end{cases} \tag{2.2.1}$$

Moreover, if Γ' is a sub-arc of Γ we define, for $s \geq 0$,

$$\begin{aligned} H^s(\Gamma') &= \{u|_{\Gamma'} : u \in H^s(\Gamma)\}, \\ \widetilde{H}^s(\Gamma') &= \{u \in H^s(\Gamma') : u^\star \in H^s(\Gamma)\}, \end{aligned} \tag{2.2.2}$$

where

$$u^\star = \begin{cases} u & \text{on } \Gamma' , \\ 0 & \text{on } \Gamma \setminus \Gamma' , \end{cases}$$

and for $s < 0$,

$$\begin{aligned} H^s(\Gamma') &= (\widetilde{H}^{-s}(\Gamma'))', \\ \widetilde{H}^s(\Gamma') &= (H^{-s}(\Gamma'))'. \end{aligned} \tag{2.2.3}$$

For $s > 0$, the norms in $H^s(\Gamma)$, $H^s(\Gamma')$ and $\widetilde{H}^s(\Gamma')$ are defined respectively as

$$\|u\|_{H^s(\Gamma)} = \inf\{\|U\|_{H^{s+1/2}(\mathbb{R}^n)} : U|_\Gamma = u|_\Gamma\},$$

$$\|u\|_{H^s(\Gamma')} = \inf\{\|v\|_{H^s(\Gamma)} : v|_{\Gamma'} = u|_{\Gamma'}\},$$

$$\|u\|_{\widetilde{H}^s(\Gamma')} = \|u^\star\|_{H^s(\Gamma)}.$$

For $s < 0$, the norms are defined by duality.

In the analysis of local estimates, we will repeatedly use a number of sub-arcs and cut-off functions, so we fix the notations right from here:

$$\Gamma_0 \Subset \Gamma_1 \Subset \cdots \Subset \Gamma_J \Subset \Gamma_* \subset \Gamma, \tag{2.2.4}$$

$$\omega_j \in C_0^\infty(\Gamma_{j+1}) \text{ and } \omega_j \equiv 1 \text{ on } \Gamma_j \text{ for } j = 0, \ldots , J-1, \tag{2.2.5}$$

where $X \Subset Y$ means that the closure of X is contained in the interior of Y.

Let us introduce a family of boundary elements on Γ in the sense of Babuška and Aziz [5]. Let γ: $[0,1] \to \mathbf{R}^2$ be the parametric representation of Γ by the arc-length. With a quasi-uniform mesh Δ on the interval $[0,1]$, we can define 1-periodic smoothest splines of order r (or degree $r-1$) with breakpoints Δ. Then, with the parametric representation γ, we transplant the splines onto Γ and denote the space of these splines by S_h^r, where h is the maximum value of the step-sizes. This space will be used as both test and trial space for the Galerkin approximation. The order r is chosen such that the conformity condition $S_h^r \subset H^\alpha(\Gamma)$ is satisfied, i.e., $\alpha < r - 1/2$, where 2α is the order of the operator A. We will also consider the following spaces:

$$\begin{aligned} \overset{\circ}{S_h^r}(\Gamma_j) &= \{\phi \in S_h^r : \text{ supp } \phi \subset \Gamma_j\}, \\ S_h^r(\Gamma_j) &= \{v \in H^\alpha(\Gamma) : v|_{\Gamma_j} = \phi|_{\Gamma_j} \text{ for some } \phi \in S_h^r\}, \end{aligned} \tag{2.2.6}$$

for $j = 0, \dots, J$. The following properties of the spline space, which shall be frequently used in this thesis, were proved in [5, 36].

Lemma 2.1. (Approximation property). *Assume that $t_0 < r - \frac{1}{2}$ and $q \in \mathbb{N}$. Let $u \in H^s(\Gamma)$ with $t_0 \le s \le r$.*

(a) There exists $\zeta \in S_h^r$ such that

$$\|u - \zeta\|_{H^t(\Gamma)} \le ch^{s-t}\|u\|_{H^s(\Gamma)} \quad \textit{for all } t \le t_0.$$

(b) For any $j = 0, \dots, J-2$, there exist $h_0 > 0$ and $\zeta \in \overset{\circ}{S_h^r}(\Gamma_{j+2})$ such that

$$\|\omega_j u - \zeta\|_{H^t(\Gamma)} \le ch^{s-t}\|u\|_{H^s(\Gamma_{j+1})},$$

for all $t \in [-q, t_0]$ and $h \in (0, h_0]$.

Lemma 2.2. (Inverse property). *For all $t \le s < r - \frac{1}{2}$, $j = 0, \dots, J-1$, and $\phi \in S_h^r$, there hold*

$$\|\phi\|_{H^s(\Gamma)} \le ch^{t-s}\|\phi\|_{H^t(\Gamma)},$$
$$\|\omega_j\phi\|_{H^s(\Gamma)} \le ch^{t-s}\|\phi\|_{H^t(\Gamma_{j+1})}.$$

Lemma 2.3. (Super-approximation property). *Assume that $t_0 < r - \frac{1}{2}$ and $q \in \mathbb{N}$. Then for any $\phi \in S_h^r$ and $j = 0, \dots, J-2$ there exist $h_0 > 0$ and $\zeta \in \overset{\circ}{S}{}_h^r(\Gamma_{j+2})$ such that*

$$\|\omega_j\phi - \zeta\|_{H^t(\Gamma)} \le ch^{s+1-t}\|\phi\|_{H^s(\Gamma_{j+1})},$$

for all $t \in [-q, t_0]$, $s \le r-1$ and $h \in (0, h_0]$.

3. Interior Local Estimates

In this section Γ denotes a smooth, closed curve. We assume that A is an isomorphism from $H^s(\Gamma)$ to $H^{s-2\alpha}(\Gamma)$ for any $s \in \mathbb{R}$. Moreover, we assume that A has the representation

$$A = A_0 + A_1,$$

where A_0 satisfies, with some $\gamma > 0$,

$$\langle A_0 u, u\rangle \geq \gamma \|u\|^2_{H^\alpha(\Gamma)} \quad \text{for all } u \in H^\alpha(\Gamma),$$

and where $A_1 \colon H^s(\Gamma) \to H^{s-2\alpha}(\Gamma)$ is compact for any $s \in \mathbb{R}$. Here $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2(\Gamma)$. We then deduce the stability condition (see e.g. [56]): for some $h_0 > 0$,

$$\inf_{\phi \in S_h^r} \sup_{\psi \in S_h^r} \frac{|\langle A\phi, \psi\rangle|}{\|\phi\|_{H^\alpha(\Gamma)} \|\psi\|_{H^\alpha(\Gamma)}} \geq c > 0 \quad \text{for } 0 < h < h_0. \tag{2.3.1}$$

In [36], Saranen studied the local error $\|u - u_h\|_{H^t(\Gamma_0)}$ for a large range of t, where u globally lies in the energy space $H^\alpha(\Gamma)$ and is smoother in some subarc Γ_* properly containing Γ_0 (see definition (2.2.4)). More precisely, he assumed that

(i) $u \in H^\alpha(\Gamma) \cap H^s(\Gamma_*) \quad$ for $\alpha \leq s \leq r$;

(ii) $u_h \in S_h^r$;

(iii) $\langle A(u - u_h), \phi\rangle = 0 \quad$ for all $\phi \in S_h^r$.

We are concerned with the error $\|u - u_h\|_{H^t(\Gamma_0)}$, where u and u_h now satisfy

(A1) $u \in H^\alpha(\Gamma) \cap H^s(\Gamma_*) \quad$ for $\alpha \leq s \leq r$;

(A2) $u_h \in S_h^r(\Gamma_*)$;

(A3) $\langle A(u - u_h), \varphi\rangle = 0 \quad$ for all $\varphi \in \overset{\circ}{S}{}_h^r(\Gamma_*)$.

For technical reasons, as will be seen in the remainder of this chapter and in the next chapter, we only assume in (A2) that u_h is a spline on the sub-arc Γ_* and can be arbitrarily extended onto Γ provided that the extension is still in $H^\alpha(\Gamma)$. Later this will allow us to choose the mesh freely away from the interval under consideration. The interior equation (A3) is analogous to the interior equation studied by Nitsche, Schatz and Wahlbin in [34, 38] for the finite element method.

The result obtained under the assumptions (A1)–(A3) will be used to deduce local error estimates for equations on open curves. It is also useful for the application of the K-operator in Chapter III.

In the remainder of this section we will follow the techniques used in the proof of Saranen [36] to obtain our result. We consider the following auxiliary problem (which defines the Galerkin solution):

For any $v \in H^\alpha(\Gamma)$, find $Gv \in S_h^r$ such that

$$\langle A(v - Gv), \phi \rangle = 0 \quad \text{for all } \phi \in S_h^r. \tag{2.3.2}$$

It follows from (2.3.1) that

$$\|Gv - v\|_{H^\alpha(\Gamma)} \leq c \inf_{\phi \in S_h^r} \|\phi - v\|_{H^\alpha(\Gamma)} \quad \text{for any } v \in H^\alpha(\Gamma). \tag{2.3.3}$$

Let $e = u - u_h$. Introducing the notation $\tilde{v} = \omega_0 v$ for any function v, we decompose the local error $\tilde{e}$ as

$$\tilde{e} = (\tilde{u} - G\tilde{u}) + (G\tilde{u} - G\tilde{u}_h) + (G\tilde{u}_h - \tilde{u}_h), \tag{2.3.4}$$

and estimate each of the terms in parentheses separately.

Lemma 3.1. *Let $u \in H^\alpha(\Gamma) \cap H^s(\Gamma_*)$ with $\alpha \leq s \leq r$. Then there exists an $h_0 > 0$ such that*

$$\|\tilde{u} - G\tilde{u}\|_{H^\alpha(\Gamma)} \leq ch^{s-\alpha}\|u\|_{H^s(\Gamma_*)} \quad \textit{for } 0 < h \leq h_0.$$

Proof. The result follows from (2.3.3) and Lemma 2.1(b). □

As for the second term of the decomposition (2.3.4), by noting the support of ζ given by Lemma 2.3, we can prove similarly to [36, Lemma 3.3]

Lemma 3.2. *Under the assumptions* (A1)–(A3) *with* $\alpha \le s \le r$, *for any fixed* $\beta \le \alpha$ *there holds*

$$\|G\tilde{u} - G\tilde{u}_h\|_{H^\alpha(\Gamma)} \le c\{h^\lambda \|e\|_{H^\alpha(\Gamma_*)} + \|e\|_{H^{\alpha-1}(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\},$$

for $0 < h \le h_0$ *with some* $h_0 > 0$, *where* $\lambda = \min(1, r-\alpha)$.

Proof. From (2.3.1) we have

$$\|G\tilde{e}\|_{H^\alpha(\Gamma)} \le c \sup_{\psi \in S_h^r} \frac{|\langle AG\tilde{e}, \psi\rangle|}{\|\psi\|_{H^\alpha(\Gamma)}}. \tag{2.3.5}$$

The equation (2.3.2) yields

$$\langle AG\tilde{e}, \psi\rangle = \langle A\tilde{e}, \psi\rangle = \langle Ae, \tilde{\psi}\rangle + \langle [A, \omega_0]e, \psi\rangle, \tag{2.3.6}$$

where $[A, \omega_0] = A\omega_0 - \omega_0 A$. By Lemma 2.3 there exists $\zeta \in \overset{\circ}{S}{}_h^r(\Gamma_2)$ such that

$$\|\tilde{\psi} - \zeta\|_{H^\alpha(\Gamma)} \le ch^\lambda \|\psi\|_{H^\alpha(\Gamma)}. \tag{2.3.7}$$

Since $\operatorname{supp}\zeta \subset \Gamma_2 \Subset \Gamma_*$, from the assumption (A3) and the fact that $\omega_2 \equiv 1$ on $\operatorname{supp}(\tilde{\psi} - \zeta)$ there follows

$$\begin{aligned}\langle Ae, \tilde{\psi}\rangle &= \langle Ae, \tilde{\psi} - \zeta\rangle = \langle \omega_2 Ae, \tilde{\psi} - \zeta\rangle \\ &= \langle \omega_2 A\omega_3 e, \tilde{\psi} - \zeta\rangle + \langle \omega_2 A(1-\omega_3)e, \tilde{\psi} - \zeta\rangle.\end{aligned}$$

Since $\omega_2(1-\omega_3) \equiv 0$, from the theory of pseudo-differential operators we know that $\omega_2 A(1-\omega_3)$ is a pseudo-differential operator of order $-\infty$ (see [47]). Hence from the Cauchy-Schwarz inequality and (2.3.7) we infer

$$\begin{aligned}|\langle Ae, \tilde{\psi}\rangle| &\le c\{\|\omega_2 A\omega_3 e\|_{H^{-\alpha}(\Gamma)} + \|\omega_2 A(1-\omega_3)e\|_{H^{-\alpha}(\Gamma)}\}\|\tilde{\psi} - \zeta\|_{H^\alpha(\Gamma)} \\ &\le ch^\lambda\{\|e\|_{H^\alpha(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\}\|\psi\|_{H^\alpha(\Gamma)}.\end{aligned} \tag{2.3.8}$$

The last term of (2.3.6) can be rewritten as

$$\begin{aligned}\langle [A,\omega_0]e,\psi\rangle &= \langle [A,\omega_0]\omega_2 e,\psi\rangle + \langle [A,\omega_0](1-\omega_2)e,\psi\rangle\\ &= \langle [A,\omega_0]\omega_2 e,\psi\rangle - \langle \omega_0 A(1-\omega_2)e,\psi\rangle.\end{aligned}$$

Since $[A,\omega_0]$ and $\omega_0 A(1-\omega_2)$ are pseudo-differential operators of order $2\alpha-1$ and $-\infty$ respectively (see [47]), we obtain, by using the Cauchy-Schwarz inequality,

$$\begin{aligned}|\langle [A,\omega_0]e,\psi\rangle| &\le \{\|[A,\omega_0]\omega_2 e\|_{H^{-\alpha}(\Gamma)} + \|\omega_0 A(1-\omega_2)e\|_{H^{-\alpha}(\Gamma)}\}\|\psi\|_{H^{\alpha}(\Gamma)}\\ &\le c\{\|e\|_{H^{\alpha-1}(\Gamma_*)} + \|e\|_{H^{\beta}(\Gamma)}\}\|\psi\|_{H^{\alpha}(\Gamma)}. \qquad (2.3.9)\end{aligned}$$

Inequalities (2.3.5), (2.3.6), (2.3.8) and (2.3.9) now give the desired result. □

To estimate the last term in (2.3.4) we slightly modify the proof of Saranen [36].

Lemma 3.3. *Under the assumptions* (A1)–(A3) *with* $\alpha \le s \le r$ *we have*

$$\|G\tilde{u}_h - \tilde{u}_h\|_{H^{\alpha}(\Gamma)} \le c\{h^{s-\alpha}\|u\|_{H^{s}(\Gamma_*)} + h^{\lambda}\|e\|_{H^{\alpha}(\Gamma_*)}\},$$

for $0 < h \le h_0$ *with some* $h_0 > 0$, *where* $\lambda = \min(1, r-\alpha)$.

Proof. We only need to note that by the definition of $S_h^r(\Gamma_*)$ there exists $u_h^* \in S_h^r$ such that $u_h^*|_{\Gamma_*} = u_h|_{\Gamma_*}$. The proof then follows in the same way as for [36, Lemma 3.4] if we replace u_h by u_h^*.

In fact, assume first that $\alpha \le r-1$ and consider the case $\alpha \le s \le r-1$. Since $\tilde{u}_h = \tilde{u}_h^*$ (where $\tilde{u}_h^* = \omega_0 u_h^*$), by using (2.3.3) and Lemma 2.3, and noting that $\tilde{u}_h^* \in H^s(\Gamma)$, we obtain

$$\begin{aligned}\|G\tilde{u}_h - \tilde{u}_h\|_{H^{\alpha}(\Gamma)} = \|G\tilde{u}_h^* - \tilde{u}_h^*\|_{H^{\alpha}(\Gamma)} &\le c \inf_{\phi\in S_h^r}\|\tilde{u}_h^* - \phi\|_{H^{\alpha}(\Gamma)}\\ &\le ch^{s-\alpha+1}\|u_h^*\|_{H^{s}(\Gamma_1)}. \qquad (2.3.10)\end{aligned}$$

Noting that $\omega_2 \equiv 1$ on Γ_2 and hence on Γ_1, we can write

$$\|u_h^*\|_{H^s(\Gamma_1)} \le \|\omega_2 u_h^*\|_{H^s(\Gamma)} \le \|\omega_2(u_h^* - \phi)\|_{H^s(\Gamma)} + \|\omega_2\phi\|_{H^s(\Gamma)}, \tag{2.3.11}$$

with

$$\|\omega_2\phi\|_{H^s(\Gamma)} \le \|\phi\|_{H^s(\Gamma)} \le \|\omega_3 u - \phi\|_{H^s(\Gamma)} + \|\omega_3 u\|_{H^s(\Gamma)} \tag{2.3.12}$$

for any $\phi \in S_h^r$. By Lemma 2.1(b) we can choose ϕ such that

$$\|\omega_3 u - \phi\|_{H^t(\Gamma)} \le ch^{s-t}\|u\|_{H^s(\Gamma_*)} \quad \text{for } \alpha \le t \le s. \tag{2.3.13}$$

Inequalities (2.3.12) and (2.3.13) give

$$\|\omega_2\phi\|_{H^s(\Gamma)} \le c\|u\|_{H^s(\Gamma_*)}. \tag{2.3.14}$$

Furthermore, using Lemma 2.2 and the triangle inequality, and noting that $\omega_3 \equiv 1$ on Γ_3 we infer

$$\begin{aligned}\|\omega_2(u_h^* - \phi)\|_{H^s(\Gamma)} &\le ch^{\alpha-s}\|u_h^* - \phi\|_{H^\alpha(\Gamma_3)} \\ &\le ch^{\alpha-s}\{\|u - u_h^*\|_{H^\alpha(\Gamma_3)} + \|u - \phi\|_{H^\alpha(\Gamma_3)}\} \\ &\le ch^{\alpha-s}\{\|u - u_h^*\|_{H^\alpha(\Gamma_3)} + \|\omega_3 u - \phi\|_{H^\alpha(\Gamma)}\}.\end{aligned} \tag{2.3.15}$$

Inequalities (2.3.13) and (2.3.15) give

$$\|\omega_2(u_h^* - \phi)\|_{H^s(\Gamma)} \le c\{h^{\alpha-s}\|u - u_h^*\|_{H^\alpha(\Gamma_3)} + \|u\|_{H^s(\Gamma_*)}\}. \tag{2.3.16}$$

Combining (2.3.10), (2.3.11), (2.3.14) and (2.3.16) we obtain, by noting that $u_h^*|_{\Gamma_*} = u_h|_{\Gamma_*}$,

$$\|G\tilde{u}_h - \tilde{u}_h\|_{H^\alpha(\Gamma)} \le c\{h^{s-\alpha+1}\|u\|_{H^s(\Gamma_*)} + h\|e\|_{H^\alpha(\Gamma_*)}\}. \tag{2.3.17}$$

Next consider the case $\alpha \le r-1 < s \le r$. Then using (2.3.17) with $s = r-1$ we have

$$\begin{aligned}\|G\tilde{u}_h - \tilde{u}_h\|_{H^\alpha(\Gamma)} &\le c\{h^{r-\alpha}\|u\|_{H^{r-1}(\Gamma_*)} + h\|e\|_{H^\alpha(\Gamma_*)}\} \\ &\le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h\|e\|_{H^\alpha(\Gamma_*)}\} \\ &\le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^\lambda\|e\|_{H^\alpha(\Gamma_*)}\}.\end{aligned}$$

Finally, if $r-1 \le \alpha \le s \le r$, we have by (2.3.3) and Lemma 2.3

$$\begin{aligned} \|G\tilde{u}_h - \tilde{u}_h\|_{H^\alpha(\Gamma)} &\le ch^{r-\alpha}\|u_h\|_{H^{r-1}(\Gamma_1)} \\ &\le ch^{r-\alpha}\{\|u\|_{H^{r-1}(\Gamma_1)} + \|e\|_{H^{r-1}(\Gamma_1)}\} \\ &\le c\{h^{s-\alpha}\|u\|_{H^{r-1}(\Gamma_*)} + h^{r-\alpha}\|e\|_{H^\alpha(\Gamma_*)}\} \\ &\le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^\lambda\|e\|_{H^\alpha(\Gamma_*)}\}. \quad \square \end{aligned}$$

Summing up the results in Lemmas 3.1–3.3 we achieve

Lemma 3.4. *Assume that* (A1)–(A3) *hold with* $\alpha \le s \le r$. *Let* $\beta \le \alpha$ *be arbitrary but fixed. Then there exists* $h_0 > 0$ *such that*

$$\|e\|_{H^\alpha(\Gamma_0)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^\lambda\|e\|_{H^\alpha(\Gamma_*)} + \|e\|_{H^{\alpha-1}(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\},$$

for $0 < h \le h_0$, *where* $\lambda = \min(1, r-\alpha)$.

We next use Nitsche's trick to obtain local estimates for lower order norms.

Lemma 3.5. *Let the assumptions* (A1)–(A3) *hold with* $\alpha \le s \le r$. *Let* $\beta \le \alpha$ *be fixed. Then there exists* $h_0 > 0$ *such that*

$$\|e\|_{H^t(\Gamma_0)} \le c\{h^\mu\|e\|_{H^\alpha(\Gamma_*)} + \|e\|_{H^{t-1}(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\},$$

for $t \le \alpha$ *and* $0 < h \le h_0$, *where* $\mu = \min(\alpha - t, r-\alpha)$.

Proof. The proof is somewhat similar to that of Lemma 3.2. However, instead of (2.3.5) we will make use of the identity

$$\|\tilde{e}\|_{H^t(\Gamma)} = \sup_{w \in H^{-t}(\Gamma)} \frac{|\langle \tilde{e}, w\rangle|}{\|w\|_{H^{-t}(\Gamma)}}. \tag{2.3.18}$$

For any $w \in H^{-t}(\Gamma)$, let y be the solution of $A^*y = w$. Then $y \in H^{2\alpha-t}(\Gamma)$ and

$$\|y\|_{H^{2\alpha-t}(\Gamma)} \leq c\|w\|_{H^{-t}(\Gamma)}. \tag{2.3.19}$$

(Recall that A^*, the adjoint of A, is an isomorphism from $H^s(\Gamma)$ to $H^{s-2\alpha}(\Gamma)$ for any $s \in \mathbf{R}$.) Moreover, we can write

$$\langle \tilde{e}, w\rangle = \langle \tilde{e}, A^*y\rangle = \langle A\tilde{e}, y\rangle = \langle Ae, \tilde{y}\rangle + \langle [A, \omega_0]e, y\rangle. \tag{2.3.20}$$

By Lemma 2.1(b), there exists $\zeta \in \overset{\circ}{S}{}_h^r(\Gamma_2)$ such that

$$\|\tilde{y} - \zeta\|_{H^\alpha(\Gamma)} \leq ch^\mu \|y\|_{H^{2\alpha-t}(\Gamma)}.$$

Hence the first term on the right hand side of (2.3.20) can be estimated as

$$\begin{aligned}
|\langle Ae, \tilde{y}\rangle| &= |\langle Ae, \tilde{y} - \zeta\rangle| = |\langle \omega_2 Ae, \tilde{y} - \zeta\rangle| \\
&\leq |\langle \omega_2 A\omega_3 e, \tilde{y} - \zeta\rangle| + |\langle \omega_2 A(1-\omega_3)e, \tilde{y} - \zeta\rangle| \\
&\leq \Big(\|\omega_2 A\omega_3 e\|_{H^{-\alpha}(\Gamma)} + \|\omega_2 A(1-\omega_3)e\|_{H^{-\alpha}(\Gamma)}\Big)\|\tilde{y} - \zeta\|_{H^\alpha(\Gamma)} \\
&\leq ch^\mu\Big(\|e\|_{H^\alpha(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\Big)\|y\|_{H^{2\alpha-t}(\Gamma)}.
\end{aligned} \tag{2.3.21}$$

In the last step we used again the fact that $\omega_2\ A\ (1 - \omega_3\ )$ is a pseudo-differential operator of order $-\infty$. The last term of (2.3.20) can be estimated as:

$$\begin{aligned}
|\langle [A, \omega_0]e, y\rangle| &= |\langle [A, \omega_0]\omega_2 e, y\rangle + \langle [A, \omega_0](1-\omega_2)e, y\rangle| \\
&= |\langle [A, \omega_0]\omega_2 e, y\rangle - \langle \omega_0 A(1-\omega_2)e, y\rangle| \\
&\leq \Big(\|[A, \omega_0]\omega_2 e\|_{H^{t-2\alpha}(\Gamma)} + \|\omega_0 A(1-\omega_2)e\|_{H^{t-2\alpha}(\Gamma)}\Big)\|y\|_{H^{2\alpha-t}(\Gamma)} \\
&\leq c\Big(\|e\|_{H^{t-1}(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\Big)\|y\|_{H^{2\alpha-t}(\Gamma)}.
\end{aligned} \tag{2.3.22}$$

Inequalities (2.3.18)–(2.3.22) now give the desired result. □

Remark. Lemma 3.5 is very slightly different from [36, Lemma 4.1] in that we allow t to be less than $2\alpha - r$, and therefore allowing β to be smaller than $2\alpha - r$ (as will be seen in the proof of the next lemma), which is necessary in the application of the K-operator in the next chapter.

Combining Lemmas 3.4 and 3.5 we obtain an explicit estimate in the energy norm:

Lemma 3.6. *Assume that* (A1)–(A3) *hold with* $\alpha \le s \le r$. *Let* $\beta \le \alpha$ *be arbitrary but fixed. Then there exists* $h_0 > 0$ *such that*

$$\|e\|_{H^\alpha(\Gamma_0)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\} \quad \textit{for } 0 < h \le h_0.$$

Proof. From Lemma 3.4 we have

$$\|e\|_{H^\alpha(\Gamma_0)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^\lambda\|e\|_{H^\alpha(\Gamma_1)} + \|e\|_{H^{\alpha-1}(\Gamma_1)} + \|e\|_{H^\beta(\Gamma)}\},$$

where $\lambda = \min(1, r-\alpha)$, and we have now redefined $\Gamma_1, \Gamma_2, etc.$ Using Lemma 3.5 with $t = \alpha - 1$ we then deduce

$$\|e\|_{H^{\alpha-1}(\Gamma_1)} \le c\{h^{\mu_1}\|e\|_{H^\alpha(\Gamma_2)} + \|e\|_{H^{\alpha-2}(\Gamma_2)} + \|e\|_{H^\beta(\Gamma)}\},$$

where $\mu_1 = \min(\alpha - (\alpha - 1), r - \alpha) = \min(1, r - \alpha) = \lambda$. Hence

$$\|e\|_{H^\alpha(\Gamma_0)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^\lambda\|e\|_{H^\alpha(\Gamma_2)} + \|e\|_{H^{\alpha-2}(\Gamma_2)} + \|e\|_{H^\beta(\Gamma)}\}.$$

Again we use Lemma 3.5 with $t = \alpha - 2$ to obtain

$$\|e\|_{H^{\alpha-2}(\Gamma_2)} \le c\{h^{\mu_2}\|e\|_{H^\alpha(\Gamma_3)} + \|e\|_{H^{\alpha-3}(\Gamma_3)} + \|e\|_{H^\beta(\Gamma)}\},$$

where $\mu_2 = \min(\alpha - (\alpha - 2), r - \alpha) = \min(2, r - \alpha) \ge \lambda$. Hence

$$\|e\|_{H^\alpha(\Gamma_0)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^\lambda\|e\|_{H^\alpha(\Gamma_3)} + \|e\|_{H^{\alpha-3}(\Gamma_3)} + \|e\|_{H^\beta(\Gamma)}\}.$$

Repeating the argument, we achieve

$$\|e\|_{H^\alpha(\Gamma_0)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^\lambda\|e\|_{H^\alpha(\Gamma_J)} + \|e\|_{H^{\alpha-J}(\Gamma_J)} + \|e\|_{H^\beta(\Gamma)}\}.$$

Taking J sufficiently large so that $\alpha - J \le \beta$ we then obtain

$$\|e\|_{H^{\alpha-J}(\Gamma_J)} \le \|e\|_{H^{\alpha-J}(\Gamma)} \le \|e\|_{H^\beta(\Gamma)},$$

and therefore

$$\|e\|_{H^\alpha(\Gamma_0)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^\lambda\|e\|_{H^\alpha(\Gamma_J)} + \|e\|_{H^\beta(\Gamma)}\}. \tag{2.3.23}$$

Using (2.3.23) for the term $\|e\|_{H^\alpha(\Gamma_J)}$ (by considering more sub-arcs properly contained in Γ_*) we obtain

$$\|e\|_{H^\alpha(\Gamma_J)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^\lambda\|e\|_{H^\alpha(\Gamma_{J+1})} + \|e\|_{H^\beta(\Gamma)}\}. \qquad (2.3.24)$$

Inequalities (2.3.23) and (2.3.24) give

$$\|e\|_{H^\alpha(\Gamma_0)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^{2\lambda}\|e\|_{H^\alpha(\Gamma_{J+1})} + \|e\|_{H^\beta(\Gamma)}\}.$$

Continuing the process we finally achieve

$$\|e\|_{H^\alpha(\Gamma_0)} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + h^{m\lambda}\|e\|_{H^\alpha(\Gamma_{J+m-1})} + \|e\|_{H^\beta(\Gamma)}\}.$$

With m chosen sufficiently large so that

$$h^{m\lambda}\|e\|_{H^\alpha(\Gamma_{J+m-1})} \le c\{h^{s-\alpha}\|u\|_{H^s(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\},$$

the desired result will be proved. This can be done by using Lemma 2.2 for the spline u_h^* defined in the proof of Lemma 3.3. In fact we have

$$\begin{aligned} h^{m\lambda}\|e\|_{H^\alpha(\Gamma_{J+m-1})} &\le h^{m\lambda}\big\{\|u\|_{H^\alpha(\Gamma_{J+m-1})} + \|u_h\|_{H^\alpha(\Gamma_{J+m-1})}\big\} \\ &\le h^{m\lambda}\big\{\|u\|_{H^s(\Gamma_*)} + \|\omega_{J+m-1}u_h^*\|_{H^\alpha(\Gamma)}\big\} \\ &\le h^{m\lambda}\big\{\|u\|_{H^s(\Gamma_*)} + h^{\beta-\alpha}\|u_h\|_{H^\beta(\Gamma_{J+m})}\big\} \\ &\le h^{m\lambda+\beta-\alpha}\big\{\|u\|_{H^s(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\big\}. \end{aligned}$$

By choosing m so that $m\lambda + \beta \ge s$, we complete the proof. □

Theorem 3.7. *Assume that* (A1)–(A3) *hold with* $\alpha \le s \le r$. *Let* $\beta \le \alpha$ *be arbitrary but fixed. Then for* $-r + 2\alpha \le t \le s \le r$ *and* $t < r - \frac{1}{2}$ *we have*

$$\|e\|_{H^t(\Gamma_0)} \le c\{h^{s-t}\|u\|_{H^s(\Gamma_*)} + h^\sigma\|e\|_{H^\beta(\Gamma)}\},$$

where $0 < h \le h_0$ *for some* $h_0 > 0$ *and where*

$$\sigma = \begin{cases} 0 & \text{if } t \le \alpha, \\ \alpha - t & \text{if } \alpha < t. \end{cases}$$

Proof. Consider first the case $t \le \alpha$. Using Lemmas 3.5 and 3.6 and noting that

$$\mu = \min(\alpha - t, r - \alpha) = \begin{cases} \alpha - t & \text{if } t \ge 2\alpha - r, \\ r - \alpha & \text{if } t < 2\alpha - r, \end{cases}$$

we obtain

$$\begin{aligned} \|e\|_{H^t(\Gamma_0)} &\le c\{h^\mu \|e\|_{H^\alpha(\Gamma_1)} + \|e\|_{H^{t-1}(\Gamma_1)} + \|e\|_{H^\beta(\Gamma)}\} \\ &\le c\{h^{\mu+s-\alpha} \|u\|_{H^s(\Gamma_*)} + \|e\|_{H^{t-1}(\Gamma_1)} + \|e\|_{H^\beta(\Gamma)}\} \\ &\le c\{h^\nu \|u\|_{H^s(\Gamma_*)} + \|e\|_{H^{t-1}(\Gamma_1)} + \|e\|_{H^\beta(\Gamma)}\}, \end{aligned} \tag{2.3.25}$$

where

$$\nu = \begin{cases} s - t & \text{if } t \ge 2\alpha - r, \\ s + r - 2\alpha & \text{if } t < 2\alpha - r. \end{cases}$$

Using the same argument for $\|e\|_{H^{t-1}(\Gamma_1)}$ and then inserting into (2.3.25) we have

$$\|e\|_{H^t(\Gamma_0)} \le c\{h^\nu \|u\|_{H^s(\Gamma_*)} + \|e\|_{H^{t-2}(\Gamma_2)} + \|e\|_{H^\beta(\Gamma)}\}.$$

Repeating the argument, we infer

$$\|e\|_{H^t(\Gamma_0)} \le c\{h^\nu \|u\|_{H^s(\Gamma_*)} + \|e\|_{H^{t-J}(\Gamma_J)} + \|e\|_{H^\beta(\Gamma)}\}.$$

Taking J sufficiently large so that $t - J \le \beta$ we arrive at

$$\|e\|_{H^t(\Gamma_0)} \le c\{h^\nu \|u\|_{H^s(\Gamma_*)} + \|e\|_{H^\beta(\Gamma)}\}. \tag{2.3.26}$$

Consider now the case $\alpha < t < r - 1/2$ and $t \le s$. For any $\zeta \in S_h^r$ we have, by noting that $\omega_j \equiv 1$ on Γ_j and using Lemma 2.2,

$$\begin{aligned} \|e\|_{H^t(\Gamma_0)} &\le \|\omega_1 u - \zeta\|_{H^t(\Gamma_0)} + \|u_h - \zeta\|_{H^t(\Gamma_0)} \\ &\le \|\omega_1 u - \zeta\|_{H^t(\Gamma)} + \|\omega_0(u_h - \zeta)\|_{H^t(\Gamma)} \\ &\le \|\omega_1 u - \zeta\|_{H^t(\Gamma)} + ch^{\alpha - t}\|u_h - \zeta\|_{H^\alpha(\Gamma_1)} \\ &\le \|\omega_1 u - \zeta\|_{H^t(\Gamma)} + ch^{\alpha - t}\{\|e\|_{H^\alpha(\Gamma_1)} + \|u - \zeta\|_{H^\alpha(\Gamma_1)}\} \\ &\le \|\omega_1 u - \zeta\|_{H^t(\Gamma)} + ch^{\alpha - t}\{\|e\|_{H^\alpha(\Gamma_1)} + \|\omega_1 u - \zeta\|_{H^\alpha(\Gamma)}\}. \end{aligned}$$

Using Lemma 2.1(b) and using (2.3.26) for the term $\|e\|_{H^{\alpha}(\Gamma_1)}$ we infer

$$\|e\|_{H^t(\Gamma_0)} \le c\{h^{s-t}\|u\|_{H^s(\Gamma_*)} + h^{\alpha-t}\|e\|_{H^{\beta}(\Gamma)}\},$$

and the theorem is proved. □

Remark. In the proof of this theorem, the inverse property is used only on the sub-arc Γ^* of Γ. Therefore, the quasi-uniformity of the mesh is required only on Γ^*. This remark is important for the next chapter when we use mesh grading on $\Gamma \setminus \Gamma^*$ to improve the convergence in the case that Γ is an open curve.

4. Local Error Estimates for Equations on Open Curves

In this section Γ is a smooth, simple, open curve and $\widetilde{\Gamma}$ is a smooth, simple, closed curve containing Γ. Let $\tilde{A}\colon H^{\alpha}(\widetilde{\Gamma}) \to H^{-\alpha}(\widetilde{\Gamma})$ be a strongly elliptic integral operator such that $A\colon \widetilde{H}^{\alpha}(\Gamma) \to H^{-\alpha}(\Gamma)$ is the restriction of $\tilde{A}$. We will assume that both A and $\tilde{A}$ are invertible. For example, if $A = V$ or $A = D$, then $\tilde{A}$ is defined by (2.1.2) or (2.1.3) with Γ replaced by $\widetilde{\Gamma}$. In case $A = V$, the above assumption is that the transfinite diameters (or logarithmic capacities) of Γ and $\widetilde{\Gamma}$ are both different from 1. The defining Galerkin equation is

$$\langle A(u_h - u), \phi\rangle_{L^2(\Gamma)} = 0 \quad \text{for } \phi \in S_h^r \cap \widetilde{H}^{\alpha}(\Gamma), \tag{2.4.1}$$

where $u_h \in S_h^r \cap \widetilde{H}^{\alpha}(\Gamma)$. We will use the result of Theorem 3.7 to deduce local error estimates for this open curve case. Discussion is then concentrated on the case $A = V$ and $A = D$. Since local error estimates are only valuable when the exact solution of the equation is smoother in some sub-arc than on the whole curve Γ, it is worth considering the local regularity of the solution. In the analysis, we shall use sub-arcs and cut-off functions defined by (2.2.4) and (2.2.5) with the addition that now we assume $\Gamma_* \Subset \Gamma$ since Γ is open.

Lemma 4.1. *If $f \in H^{\tau}(\Gamma)$ for some $\tau \geq -\alpha$ then the solution u of* (2.1.1) *is in $H^{\tau+2\alpha}(\Gamma_0)$ and there holds the following a priori estimate*

$$\|u\|_{H^{\tau+2\alpha}(\Gamma_0)} \leq c\{\|u\|_{\widetilde{H}^{\alpha}(\Gamma)} + \|f\|_{H^{\tau}(\Gamma)}\}. \tag{2.4.2}$$

Proof. The proof follows that of [21, Lemma 4.1]. From the equation (2.1.1) we deduce, for $j = 0, \dots, J-1$,

$$\omega_j A \omega_{j+1} u = -\omega_j A(1 - \omega_{j+1})u + \omega_j f. \tag{2.4.3}$$

Since $\omega_j A(1-\omega_{j+1})$ is a pseudo-differential operator of order $-\infty$, if we let $g_j := -\omega_j A(1-\omega_{j+1})u + \omega_j f$, then

$$\|g_j\|_{H^\tau(\Gamma)} \le c\{\|u\|_{\widetilde{H}^\alpha(\Gamma)} + \|f\|_{H^\tau(\Gamma)}\} \quad \text{for } j = 0, \dots, J-1. \tag{2.4.4}$$

The equation(2.4.3) can be understood as an equation on the smooth closed curve $\widetilde{\Gamma}$. To clarify this point, we rewrite that equation as

$$\omega_j \tilde{A} \omega_{j+1} u = g_j. \tag{2.4.5}$$

Since $\tilde{A}^{-1}$ exists as a pseudo-differential operator of order -2α, and since

$$\tilde{A}^{-1}\omega_j \tilde{A}\omega_{j+1}u = \omega_j\omega_{j+1}u + (\tilde{A}^{-1}\omega_j - \omega_j\tilde{A}^{-1})\tilde{A}\omega_{j+1}u$$

we deduce from the equation (2.4.5) that

$$\omega_j\omega_{j+1}u = -(\tilde{A}^{-1}\omega_j - \omega_j\tilde{A}^{-1})\tilde{A}\omega_{j+1}u + \tilde{A}^{-1}g_j.$$

By noting that the commutator $(\tilde{A}^{-1}\omega_j - \omega_j\tilde{A}^{-1})$ is a pseudo-differential operator of order $-2\alpha-1$ and by using (2.4.4), we obtain, for any $t \le \tau$,

$$\begin{aligned}\|\omega_j u\|_{H^{t+2\alpha}(\Gamma)} &\le c\{\|\omega_{j+1}u\|_{H^{t+2\alpha-1}(\Gamma)} + \|g_j\|_{H^t(\Gamma)}\}\\ &\le c\{\|\omega_{j+1}u\|_{H^{t+2\alpha-1}(\Gamma)} + \|u\|_{\widetilde{H}^\alpha(\Gamma)} + \|f\|_{H^\tau(\Gamma)}\}.\end{aligned}$$

By using the above estimate repeatedly, starting with $j = 0$ and $t = \tau$, we find the estimate

$$\|\omega_0 u\|_{H^{\tau+2\alpha}(\Gamma)} \le c\{\|\omega_J u\|_{H^{\tau+2\alpha-J}(\Gamma)} + \|u\|_{\widetilde{H}^\alpha(\Gamma)} + \|f\|_{H^\tau(\Gamma)}\}.$$

By choosing J sufficiently large so that $\tau + 2\alpha - J \le \alpha$ and by noting that $\omega_0 \equiv 1$ on Γ_0 we obtain the desired result. $\square$

We are now able to consider the local convergence properties.

Theorem 4.2. *Assume that the solution of the equation* (2.1.1) *satisfies* $u \in \widetilde{H}^{\alpha}(\Gamma) \cap H^{s}(\Gamma_{*})$ *for* $\alpha \le s \le r$. *Let* $u_h \in S_h^r \cap \widetilde{H}^{\alpha}(\Gamma)$ *satisfy* (2.4.1). *Let* $\beta \le \alpha$ *be fixed. Then for* $-r + 2\alpha \le t \le s \le r$ *and* $t < r - \frac{1}{2}$ *we have*

$$\|e\|_{H^t(\Gamma_0)} \le c\{h^{s-t}\|u\|_{H^s(\Gamma_*)} + h^{\sigma}\|e\|_{\widetilde{H}^{\beta}(\Gamma)}\}, \tag{2.4.6}$$

where $e = u_h - u$ *and*

$$\sigma = \begin{cases} 0 & \text{if } t \le \alpha, \\ \alpha - t & \text{if } \alpha < t. \end{cases}$$

Proof. From the Galerkin equation (2.4.1) we deduce

$$\langle A(u_h - u), \varphi\rangle_{L^2(\Gamma)} = 0 \quad \text{for any } \varphi \in \overset{\circ}{S}{}_h^r(\Gamma). \tag{2.4.7}$$

For any function (or distribution) v defined on Γ we denote by $v^\star$ the extension of v onto $\widetilde{\Gamma}$ by 0, i.e.,

$$v^\star = \begin{cases} v & \text{on } \Gamma, \\ 0 & \text{on } \widetilde{\Gamma} \setminus \Gamma. \end{cases}$$

We then have $u^\star \in H^{\alpha}(\widetilde{\Gamma}) \cap H^{s}(\Gamma_*)$ and $u_h^\star \in S_h^r(\Gamma) \cap H^{\alpha}(\tilde{\Gamma})$. Equation (2.4.7) implies

$$\langle \tilde{A}(u_h^\star - u^\star), \varphi^\star\rangle_{L^2(\tilde{\Gamma})} = 0 \quad \text{for any } \varphi^\star \in \overset{\circ}{S}{}_h^r(\Gamma).$$

Theorem 3.7 then gives

$$\begin{aligned} \|u_h - u\|_{H^t(\Gamma_0)} = \|u^\star - u_h^\star\|_{H^t(\Gamma_0)} &\le c\{h^{s-t}\|u^\star\|_{H^s(\Gamma_*)} + h^{\sigma}\|u_h^\star - u^\star\|_{H^{\beta}(\tilde{\Gamma})}\} \\ &\le c\{h^{s-t}\|u\|_{H^s(\Gamma_*)} + h^{\sigma}\|e\|_{\widetilde{H}^{\beta}(\Gamma)}\}. \end{aligned}$$

The theorem is proved. □

Remark. The global error term in (2.4.6) controls the highest possible order of convergence, even though for convenience in the theorem we allow t to go down to $-r + 2\alpha$.

As examples, we will now consider the weakly singular and hypersingular integral equations on the interval $\Gamma = [-1, 1]$.

Weakly Singular Integral Equation. The equation, as given by (2.1.2), is

$$Vu(x) = -\frac{1}{2\pi}\int_\Gamma \log|x-y|u(y)\,ds(y) = f(x) \quad \text{for } x \in \Gamma.$$

A physical interpretation of u is that it is the jump in the normal derivative of the solution of a Dirichlet problem for the Laplacian in $\mathbb{R}^2 \setminus \Gamma$ with boundary values f on Γ and vanishing at infinity [46]. It is known [46, Theorem 1.5] that $V\colon \widetilde{H}^\tau(\Gamma) \to H^{\tau+1}(\Gamma)$ is a continuous and bijective mapping for $-1 < \tau < 0$. If we use piecewise-constant functions as trial and test functions for the Galerkin equation (2.4.1), the following estimates hold (see [20, 46])

$$\|e\|_{\widetilde{H}^t(\Gamma)} \le ch^{\tau-t}\|u\|_{\widetilde{H}^\tau(\Gamma)} \quad \text{for } -1 < t \le \tau < 0.$$

Therefore, for any ϵ satisfying $0 < \epsilon < 1/2$, provided that the boundary data are sufficiently smooth we have

$$\|e\|_{\widetilde{H}^t(\Gamma)} \le ch^{-\epsilon-t}\|u\|_{\widetilde{H}^{-\epsilon}(\Gamma)} \quad \text{for } -1+\epsilon \le t \le -\epsilon,$$

whereas, by applying Theorem 4.2 (with $s = 1/2$ and $\beta = -1+\epsilon$) we obtain

$$\|e\|_{H^t(\Gamma_0)} \le \begin{cases} ch^{1-2\epsilon}(\|u\|_{H^{1/2}(\Gamma_*)} + \|u\|_{\widetilde{H}^{-\epsilon}(\Gamma)}) & \text{for } -1+\epsilon \le t \le -\frac{1}{2}, \\ ch^{1/2-2\epsilon-t}(\|u\|_{H^{1/2}(\Gamma_*)} + \|u\|_{\widetilde{H}^{-\epsilon}(\Gamma)}) & \text{for } -\frac{1}{2} < t < \frac{1}{2}. \end{cases} \tag{2.4.8}$$

In particular, in the L^2-norm we have local convergence of order $O(h^{1/2-2\epsilon})$ even though the global L^2-norm of e is not defined. In the energy norm ($H^{-1/2}$-norm) we have convergence of order $O(h^{1-2\epsilon})$ locally, compared to $O(h^{1/2-\epsilon})$ globally.

Hypersingular Integral Equation. The equation, given by (2.1.3), is

$$Du(x) = -\frac{1}{\pi}\frac{\partial}{\partial n_x}\int_\Gamma \frac{\partial}{\partial n_y}(\log|x-y|)u(y)\,ds(y) = f(x) \quad \text{for } x \in \Gamma.$$

This equation arises as a reformulation of the Neumann problem for the Laplacian in $\mathbb{R}^2 \setminus \Gamma$ with boundary values f on Γ. The function u is the jump of the solution of that problem (see [19, 57]). It is known that $D\colon \widetilde{H}^\tau(\Gamma) \to H^{\tau-1}(\Gamma)$ is continuous and bijective for $0 < \tau < 1$ (see [19, Theorem 1.7], [57, Corollary 1.7 and the remark after that]). If $S_2'(\Gamma)$ —the space of continuous piecewise linear functions vanishing at ± 1— is used as both test and trial space, then

$$\|e\|_{\widetilde{H}^t(\Gamma)} \le ch^{\tau-t}\|u\|_{\widetilde{H}^\tau(\Gamma)} \quad \text{for } 0 \le t \le \tau < 1.$$

Hence, for any $\epsilon > 0$ we have

$$\|e\|_{\widetilde{H}^t(\Gamma)} \le ch^{1-\epsilon-t}\|u\|_{\widetilde{H}^{1-\epsilon}(\Gamma)} \quad \text{for } 0 \le t \le 1-\epsilon.$$

Using Theorem 4.2 (with $s = 3/2$ and $\beta = 0$) we obtain

$$\|e\|_{H^t(\Gamma_0)} \le \begin{cases} ch^{1-\epsilon}(\|u\|_{H^{3/2}(\Gamma_*)} + \|u\|_{\widetilde{H}^{1-\epsilon}(\Gamma)}) & \text{for } 0 \le t \le \frac{1}{2}, \\ ch^{3/2-t-\epsilon}(\|u\|_{H^{3/2}(\Gamma_*)} + \|u\|_{\widetilde{H}^{1-\epsilon}(\Gamma)}) & \text{for } \frac{1}{2} < t < \frac{3}{2}. \end{cases} \tag{2.4.9}$$

In particular, we have in the energy norm ($H^{1/2}$-norm) convergence of order $O(h^{1-\epsilon})$ locally, compared to order $O(h^{1/2-\epsilon})$ globally. In the H^1- norm, there is local convergence of order $O(h^{1/2-\epsilon})$, whereas there is no result for the global error in that norm because the exact solution u may not be in $H^1(\Gamma)$.

5. Numerical Experiments

We tested the convergence of the two cases discussed above. In the experiments we made use of the program of Manfred Hahne (University of Hannover) to find the Galerkin solutions.

Experiment 1. We considered the weakly singular equation (2.1.2) with $f(x) = x$. The exact solution is then $u(x) = 2x(1-x^2)^{-1/2}$. We calculated analytically the L^2-norm of the error on $\Gamma_m = (-1+\frac{1}{m}, 1-\frac{1}{m})$, and investigated various values of m, even with m sufficiently large to see a deterioration of the convergence process. The empirical convergence rate (for small m) is higher than what we expect from our analysis: we achieved convergence of order $O(h)$ instead of $O(h^{1/2-2\epsilon})$ (see Table 1).

N	$\|e\|_{L^2(-0.5,0.5)}$		$\|e\|_{L^2(-0.9,0.9)}$		$\|e\|_{L^2(-0.99,0.99)}$	
4	4.36e-01		1.51e-00		2.20e-00	
8	1.70e-01	1.36	1.40e-00	0.11	1.77e-00	0.31
16	8.41e-02	1.02	9.70e-01	0.53	1.49e-00	0.25
32	4.22e-02	0.99	1.38e-01	2.81	1.37e-00	0.12
64	2.11e-02	1.00	7.37e-02	0.91	1.35e-00	0.02
128	1.06e-02	1.00	3.93e-02	0.91	1.17e-00	0.21
256	5.30e-03	1.00	1.99e-02	0.98	1.88e-01	2.63
512	2.65e-03	1.00	9.87e-03	1.01	1.04e-01	0.85
1024	1.32e-03	1.00	4.89e-03	1.01	4.19e-02	1.31

TABLE 1. L^2-errors on indicated intervals and empirical orders of convergence for Experiment 1

Experiment 2. The hypersingular equation (2.1.3) was tested with the right hand side $f(x) = 2$. The exact solution is $u(x) = -2(1-x^2)^{1/2}$. In the L^2-norm, the errors were calculated on various subintervals and even on the whole

N	$\|e\|_{L^2(-0.5,0.5)}$		$\|e\|_{L^2(-0.9,0.9)}$		$\|e\|_{L^2(-0.99,0.99)}$		$\|e\|_{L^2(-1,1)}$	
4	8.25e-02		3.37e-01		3.85e-01		3.86e-01	
8	5.96e-02	0.47	1.29e-01	1.38	1.92e-01	1.00	1.94e-01	1.00
16	2.96e-02	1.01	4.28e-02	1.59	9.59e-02	1.00	9.84e-02	0.98
32	1.49e-02	0.99	2.39e-02	0.81	4.65e-02	1.04	5.03e-02	0.97
64	7.51e-03	0.99	1.22e-02	0.98	2.06e-02	1.17	2.58e-02	0.96
128	3.76e-03	1.00	6.13e-03	0.99	8.01e-03	1.36	1.32e-02	0.96
256	1.88e-03	1.00	3.07e-03	0.99	3.98e-03	1.01	6.77e-03	0.96
512	9.42e-04	1.00	1.54e-03	1.00	2.04e-03	0.96	3.47e-03	0.97
1024	4.71e-04	1.00	7.70e-04	1.00	1.03e-03	0.99	1.77e-03	0.97

TABLE 2. L^2-errors on indicated intervals and empirical orders of convergence for Experiment 2

N	$\|e\|_{H^{1/2}(-0.5,0.5)}$		$\|e\|_{H^{1/2}(-0.9,0.9)}$		$\|e\|_{H^{1/2}(-0.99,0.99)}$		$\|e\|_{H^{1/2}(-1,1)}$	
4	1.90e-01		7.12e-01		9.20e-01		9.01e-01	
8	1.01e-01	0.91	4.26e-01	0.74	5.84e-01	0.66	6.18e-01	0.54
16	4.99e-02	1.01	2.04e-01	1.06	3.78e-01	0.63	4.31e-01	0.52
32	2.51e-02	0.99	5.76e-02	1.82	2.53e-01	0.58	3.03e-01	0.51
64	1.26e-02	1.00	2.99e-02	0.94	1.67e-01	0.60	2.13e-01	0.50
128	6.31e-03	1.00	1.55e-02	0.95	9.67e-02	0.79	1.51e-01	0.50
256	3.16e-03	1.00	7.82e-03	0.99	2.74e-02	1.82	1.06e-01	0.50
512	1.59e-03	1.00	3.90e-03	1.00	1.46e-02	0.91	7.52e-02	0.50
1024	7.90e-04	1.00	1.94e-03	1.01	6.57e-03	1.15	5.31e-02	0.50

TABLE 3. Energy norm errors on indicated intervals and empirical orders of convergence for Experiment 2

interval $[-1,1]$. In every case we observe the expected convergence rate $O(h^{1-\epsilon})$, as one can see in Table 2. We also considered the energy norm ($H^{1/2}$-norm). The global errors were evaluated using the formula $\|e\|_{\tilde{H}^{1/2}(\Gamma)} = \langle De, e\rangle^{1/2}_{L^2(\Gamma)} = \langle g, e\rangle^{1/2}_{L^2(\Gamma)}$, whereas the local norm was approximated by a bound given by the interpolation theory: $\|e\|^{1/2}_{L^2(\Gamma_m)}\|e'\|^{1/2}_{L^2(\Gamma_m)}$, where $'$ denotes the derivative with

N	$\|e\|_{H^1(-0.5,0.5)}$		$\|e\|_{H^1(-0.9,0.9)}$		$\|e\|_{H^1(-0.99,0.99)}$	
4	4.44e-01		1.54e-00		2.23e-00	
8	1.80e-01	1.30	1.41e-00	0.13	1.78e-00	0.32
16	8.91e-02	1.02	9.71e-01	0.53	1.50e-00	0.25
32	4.48e-02	0.99	1.41e-01	2.79	1.37e-00	0.12
64	2.24e-02	1.00	7.47e-02	0.91	1.35e-00	0.02
128	1.12e-02	1.00	3.97e-02	0.91	1.17e-00	0.21
256	5.62e-03	1.00	2.01e-02	0.98	1.88e-01	2.63
512	2.81e-03	1.00	9.99e-03	1.01	1.04e-01	0.85
1024	1.41e-03	1.00	4.95e-03	1.01	4.20e-02	1.31

TABLE 4. H^1-errors on indicated intervals and empirical orders of convergence for Experiment 2

respect to x. Convergence rates $O(h^{1/2-\epsilon})$ globally and $O(h^{1-\epsilon})$ locally match the analysis (see Table 3). The H^1-norm of the local errors was computed on Γ_m with various values of m. We obtained an apparent order of $O(h)$ when m is not too large, even though the predicted order is only $O(h^{1/2-\epsilon})$, and again when we increased m the convergence declined. The numerical results are given in Table 4.

CHAPTER III

A POST-PROCESSING METHOD

1. Introduction

In this chapter we shall study a way of increasing the order of local convergence in the L^2-norm of the Galerkin approximation to the solution of a strongly elliptic pseudo-differential equation on a smooth curve in $\mathbb{R}^2$, closed or open. This better approximation is a legacy of the highest order of global convergence achieved in a negative norm.

Consider for example Symm's equation. With piecewise-constant functions used as trial and test functions, it was proved that the local L^2-error converges with order $O(h)$ in the case of smooth closed curves [36] and with order almost $O(h^{1/2})$ in the case of smooth open curves (see (2.4.8)). However, it is well known that the highest orders of global convergence achieved (in negative norms) are $O(h^3)$ for the closed smooth case [27] and nearly $O(h)$ for the open smooth case [20, 46].

We shall construct, from the Galerkin solution, a better approximate solution which inherits the highest possible orders of global convergence to give best local convergence in the L^2-norm. For example, for Symm's equation mentioned above, order $O(h^3)$ for the closed case and almost $O(h)$ for the open case can be achieved locally in the L^2-norm. That better approximation is constructed by averaging the values of the Galerkin solution, using the K-*operator*.

The K-operator was proved to be an effective post-processing method in the finite element environment [7, 9, 48]. Its main features are clarified in [54]. It produces an easily computed new approximant in the form of a convolution of the Galerkin solution with a special kind of spline with small support. This method is applicable when the problem and the trial space are translationally invariant and when an estimate in a negative norm is available.

If the trial space is chosen to be the space of piecewise-polynomial functions (in this thesis we always consider this kind of trial space), in order that it is invariant under translation by a mesh step, it is essential that the mesh be uniform or at least locally uniform. In the case that the mesh is locally uniform, all the error estimates considered are local estimates; therefore the estimates proved in Chapter II are important for the application of the K-operator method in this chapter.

However, unlike partial differential operators, pseudo-differential operators have only a pseudo-local property. Hence, the effectiveness of the K-operator is not obvious in boundary element methods. A careful study is therefore necessary. This study will give light on the reason for the peculiar assumptions (A2)–(A3) in the previous chapter.

It is worth noting that for Fredhom integral equations of the second kind, Chandler [14] has used a method analogous to the K-operator (which he referred to as 'superinterpolation') to obtain superconvergence. The mesh used there is uniform and only global errors were investigated.

This chapter has 5 sections. Section 2 gives some notations to be used and a review of the global property of the Galerkin approximation. The definition and some properties of the K-operator are given in Section 3. Its application to the case of smooth and closed curves can then be found in Section 4. Section 5 is devoted to a consideration of a special kind of equation on open curves: Symm's equation on an interval. Both quasi-uniform and graded meshes (graded at the ends of the interval) are discussed in this section. Using mesh grading, we will obtain local convergence of order almost $O(h^3)$ in the L^2-norm, which is the same as in the case of a smooth and closed curve. Numerical examples are given in Section 6.

2. Notations and Some Preliminaries

Notations introduced in this section are to be used in Section 3 for the study of the smooth, closed curve case. Let Γ be a plane smooth and closed curve given by a parametric representation $\gamma\colon [0,1] \to \mathbb{R}^2$ such that $|\gamma'| > c$ for some $c > 0$. In boundary element methods, Γ is the boundary of a given domain associated with some boundary value problem. Via the parametrization we have a one-to-one correspondence between functions on Γ and 1-periodic functions. We thus restrict ourselves without loss of generality to equations of the form

$$Lu = f, \tag{3.2.1}$$

where u and f are 1-periodic functions. Each periodic function u has a Fourier expansion

$$u(x) \sim \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{2\pi inx},$$

where the Fourier coefficients are given by the formula

$$\hat{u}(n) = \int_0^1 u(x)e^{-2\pi inx}\,dx,$$

provided u is in $L^1(0,1)$. For $s \in \mathbb{R}$ we define the norm

$$\|u\|_s^2 = |\hat{u}(0)|^2 + \sum_{n\neq 0} |n|^{2s}|\hat{u}(n)|^2.$$

The Sobolev space H_p^s consists of all periodic distributions u for which the norm $\|u\|_s$ is finite. If I' is an open subset of $I = [0,1]$, we also consider the space $H^s(I')$ with norm denoted by $\|\cdot\|_{s,I'}$ (see the definition in e.g. (2.2.2)).

The operator L is assumed to be of the form

$$L = L_0 + L_1,$$

where the principal part L_0 is defined by

$$L_0 u(x) := \sum_{n\in\mathbb{Z}} [n]_\alpha \hat{u}(n) e^{2\pi i n x}, \tag{3.2.2}$$

with $\alpha \in \mathbb{R}$ and $[n]_\alpha$ defined either by

$$[n]_\alpha := \begin{cases} 1 & \text{for } n = 0, \\ |n|^{2\alpha} & \text{for } n \neq 0, \end{cases} \tag{3.2.3}$$

or by

$$[n]_\alpha := \begin{cases} 1 & \text{for } n = 0, \\ (\text{sign } n)|n|^{2\alpha} & \text{for } n \neq 0. \end{cases} \tag{3.2.4}$$

In either case L_0 is a pseudo-differential operator of order 2α, and is an isometry from H_p^s to $H_p^{s-2\alpha}$ for all $s \in \mathbb{R}$. The operator L_1 is assumed to be bounded from H_p^s to $H_p^{s-2\alpha+\eta}$ for all $s \in \mathbb{R}$ and some positive number η to be specified later. We then have $L_0^{-1}L_1$ bounded from H_p^s to $H_p^{s+\eta}$ and compact on H_p^s for all $s \in \mathbb{R}$. We also assume that L is 1-1, and thus by the Fredholm alternative

$$(I + L_0^{-1}L_1)^{-1} : H_p^s \longrightarrow H_p^s$$

is bounded for all $s \in \mathbb{R}$.

Since the boundary integral operators associated with regular elliptic boundary value problems on smooth closed curves are pseudo-differential operators of integer order (see [56, Theorem 2.1]), we assume for simplicity in the sequel that the operator L has integer order 2α, even though our results are still correct for any real α.

Let $I_0, \ldots, I_4$, I_* and I^* be intervals such that $I_i \Subset I_{i+1} \Subset I_* \Subset I^* \Subset I = [0,1]$, for $i = 0, \ldots, 3$. Let $\Delta = \{x_k\}$, $x_k < x_{k+1}$ for $k \in \mathbb{Z}$, be a set of points on the real axis such that $x_{k+N} = x_k + 1$ for some $N \in \mathbb{N}$ and all $k \in \mathbb{Z}$. We consider for $r \geq 2$ the space $S^r_{h,p}$ of 1-periodic smoothest splines, i.e., $\varphi \in S^r_{h,p}$ if φ is a polynomial of degree at most $r-1$ in every subinterval (x_k, x_{k+1}) and has continuous derivatives up to order $r-2$. Here h is the maximum value of the step-sizes. The space $S^1_{h,p}$ means the space of 1-periodic piecewise-constant functions. The order r is assumed to be chosen so that the conformity condition $S^r_{h,p} \subset H^\alpha_p$ is satisfied, i.e., $\alpha < r - 1/2$, and so that u, the exact solution to (3.2.1), belongs to H^r_p. We shall also consider the following spaces:

$$\overset{\circ}{S}{}^r_{h,p}(I_i) = \{\varphi \in S^r_{h,p} : \text{ supp } (\varphi|_I) \subset I_i\},$$

$$S^r_{h,p}(I_i) = \{v \in H^\alpha_p : v|_{I_i} = \phi|_{I_i} \text{ for some } \phi \in S^r_{h,p}\}, \quad i = 0, \dots, 4.$$

We shall assume that the mesh is uniform in the interval I^*. Then there exists an $h_0 > 0$ such that, for any $h \in (0, h_0]$, for $i = 0, \dots, 3$, and for $j = 1, \dots, r - 2\alpha$,

$$T^j_{\pm h}\varphi \in \overset{\circ}{S}{}^r_{h,p}(I_{i+1}) \quad \forall \varphi \in \overset{\circ}{S}{}^r_{h,p}(I_i), \tag{3.2.5}$$

where T_h denotes the translation operator $T_h v(x) = v(x+h)$.

Let $u_h \in S^r_{h,p}$ satisfy

$$\langle Lu_h, \phi \rangle = \langle Lu, \phi \rangle \quad \text{for any } \phi \in S^r_{h,p}, \tag{3.2.6}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $H^0_p = L^2_p(I)$. It is known [27] that for $2\alpha - r \leq t \leq s \leq r$ and $t < r - \frac{1}{2}$,

$$\|u_h - u\|_t \leq ch^{s-t}\|u\|_s. \tag{3.2.7}$$

In particular, in the L^2-norm we have

$$\|u_h - u\|_0 \leq ch^r\|u\|_r,$$

whereas in the most extreme negative norm we can obtain

$$\|u_h - u\|_{2\alpha - r} \leq ch^{2(r-\alpha)}\|u\|_r. \tag{3.2.8}$$

In addition to u_h itself, we will consider $K_h * u_h$ as an approximation to u (where $*$ denotes the convolution, and the function K_h is to be defined later) in such a way that if $2\alpha - r < 0$ and if u is smoother than previously assumed in some sub-interval of I, i.e., $u \in H^{r_1}(I_*) \cap H_p^r$, for some $r_1 > r$ to be specified later, then

$$\|K_h * u_h - u\|_{0,I_0} \leq ch^{2(r-\alpha)}(\|u\|_{r_1,I_*} + \|u\|_r).$$

Since we are in the periodic context, we need a periodic version of Theorem 3.7 in Chapter II. In this context, that theorem can be interpreted as:

Theorem 2.1. *Let $v \in H_p^\alpha \cap H^s(I_*)$, $\alpha \leq s \leq r$, and $v_h \in S_{h,p}^r(I_*)$ satisfy*

$$\langle L(v_h - v), \varphi\rangle = 0 \quad \text{for all } \varphi \in \overset{\circ}{S}{}_{h,p}^r(I_*).$$

Let $\beta \leq \alpha$ with β arbitrary but fixed. Then there exists $h_0 > 0$ such that for any $h \in (0, h_0]$

$$\|v_h - v\|_{t,I_0} \leq c(h^{s-t}\|v\|_{s,I_*} + h^\sigma\|v_h - v\|_\beta),$$

with $2\alpha - r \leq t \leq s \leq r$, $t < r - \frac{1}{2}$ and

$$\sigma = \begin{cases} 0 & \text{if } t \leq \alpha, \\ \alpha - t & \text{if } \alpha < t. \end{cases}$$

We shall in the next section give a full description of the K-operator.

3. The K-Operator and Its Properties

The K-operator acting on u_h is defined by the convolution of u_h with a function K_h defined as a linear combination of B-splines such that it reproduces polynomials (up to some degree) under convolution. For the application to our problem we will give here its definition in the 1-dimensional case only.

Let

$$\chi(x) = \begin{cases} 1 & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \text{ or } x = -\frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$\psi^{(l)} = \chi * \chi * \cdots * \chi, \quad \chi \text{ occuring } l \text{ times }, l \geq 1.$$

It is well known that $\psi^{(l)}$ is the B-spline of order l symmetric about 0, with integer or half-integer knots, and with support $[-\frac{l}{2}, \frac{l}{2}]$. Let q, l be arbitrary but fixed positive integers. We define

$$K_q^l(x) = \sum_{j=-(q-1)}^{q-1} k_j \psi^{(l)}(x-j), \tag{3.3.1}$$

and try to choose k_j, $j = -(q-1), \ldots, q-1$ in such a way that

$$\int_{-\infty}^{\infty} K_q^l(x) x^i \, dx = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i = 1, \ldots, 2q-1. \end{cases} \tag{3.3.2}$$

Since $\psi^{(l)}$ is an even function and since we want K_q^l to have the same property, we impose the symmetry condition

$$k_{-j} = k_j, \quad j = 1, \ldots, q-1. \tag{3.3.3}$$

Then the condition (3.3.2) is equivalent to

$$\int_{-\infty}^{\infty} K_q^l(x) x^{2m} \, dx = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \ldots, q-1. \end{cases} \tag{3.3.4}$$

In fact (3.3.4) can be written as

$$\sum_{j=0}^{q-1} k'_j \int_{-\infty}^{\infty} \psi^{(l)}(x)(x+j)^{2m}\, dx = \begin{cases} 1 & \text{if } m = 0, \\ 0 & \text{if } m = 1, \dots, q-1, \end{cases} \tag{3.3.5}$$

where $k'_0 = k_0$, $k'_j = 2k_j$, $j = 1, \dots, q-1$. The system (3.3.5) is a system of q equations with q unknowns $k'_0, \dots, k'_{q-1}$. It was proved in [8, Lemma 8.1] that the solutions exist uniquely.

Now for $0 < h < 1$, we define

$$K_h(x) = K^l_{h,q}(x) = \frac{1}{h} K^l_q\left(\frac{x}{h}\right). \tag{3.3.6}$$

Then we have supp $K^l_{h,q} = [-(q-1+\frac{l}{2})h, (q-1+\frac{l}{2})h]$ and

$$\int_{-\infty}^{\infty} K_h(x) x^i\, dx = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i = 1, \dots, 2q-1. \end{cases} \tag{3.3.7}$$

As an example, we give here the graph of K^4_3 (Figure 1), a cubic spline. The coefficients k_j in that case are $k_0 = \frac{181}{120}$, $k_1 = k_{-1} = -\frac{17}{60}$, $k_2 = k_{-2} = \frac{7}{240}$.

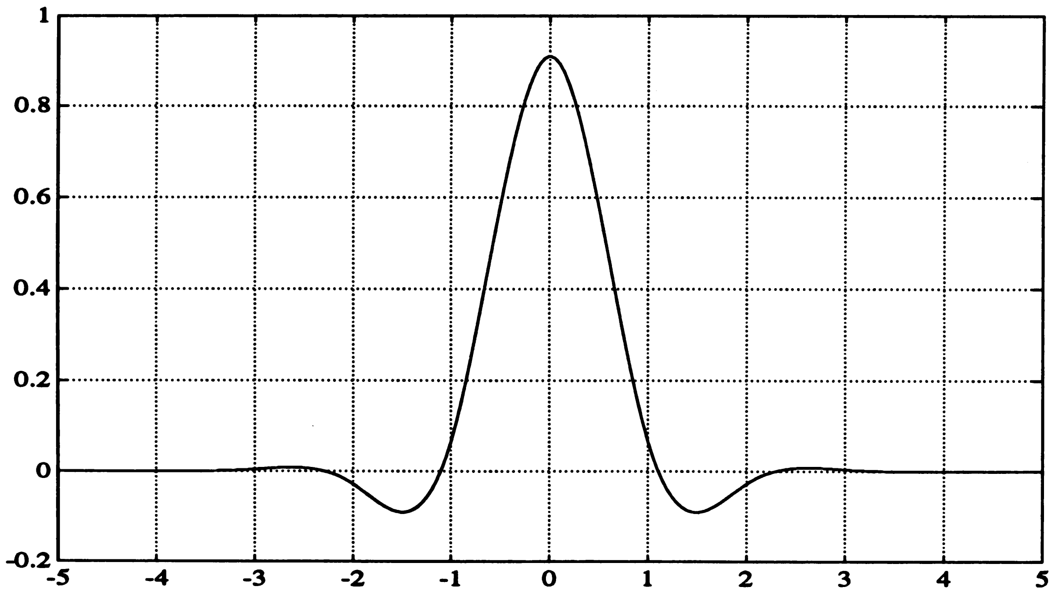

FIGURE 1. Graph of K^4_3

Representation of $K_h * u_h$. We give here the representation of $K_h * u_h$ in case the mesh is uniform.

Let $\psi_{h,p}^{(l')}$ be 1-periodic functions defined by

$$\psi_{h,p}^{(l')}(x) = \psi^{(l')}(\frac{x}{h} - \frac{l'}{2}) \quad \text{for } x \in [0,1), \quad l' = 1, \dots, N, \quad h = 1/N.$$

If u_h is a solution to the equation (3.2.6), then since $u_h \in S_{h,p}^r$ we can write u_h in the form

$$u_h(x) = \sum_{i=0}^{N-1} c_i \psi_{h,p}^{(r)}(x - ih).$$

Hence $K_h * u_h$ can be represented as

$$K_h * u_h(x) = \sum_{i=0}^{N-1} c_i \phi_{h,p}^{(r+l)}(x - (i + \frac{r}{2})h),$$

where $\phi_{h,p}^{(l')}$ is a 1-periodic, even function defined by

$$\phi_{h,p}^{(l')}(x) = \sum_{j=-(q-1)}^{q-1} k_j \psi_{h,p}^{(l')}(x - jh + \frac{l'h}{2}).$$

To ensure that $\phi_{h,p}^{(r+l)}$, and hence $K_h * u_h$, is a periodic spline of order $r + l$, we require $q - 1 + \frac{r+l}{2} \leq \frac{N}{2}$. If the inequality is strict, then the support of $\phi_{h,p}^{(r+l)}$ in $[-\frac{1}{2}, \frac{1}{2}]$ is $[-(q - 1 + \frac{r+l}{2})h, (q - 1 + \frac{r+l}{2})h]$. As an example, the graph of $\phi(t) = \phi_{h,p}^{(5)}(th)$ for the case $l = 4$, $q = 3$, and $r = 1$ is given in Figure 2. Its support in $[-\frac{N}{2}, \frac{N}{2}]$ is $[-\frac{9}{2}, \frac{9}{2}]$.

Stability Discussion. Assume that

$$\bar{u}_h(x) = \sum_{i=0}^{N-1} \bar{c}_i \psi_{h,p}^{(r)}(x - ih)$$

and that

$$|c_i - \bar{c}_i| \leq \epsilon \quad \text{for } i = 0, \dots, N-1.$$

Then

$$|K_h * u_h(x) - K_h * \bar{u}_h(x)| \leq \epsilon \sum_{i=0}^{N-1} |\phi_{h,p}^{(r+l)}(x - (i + \frac{r}{2})h)|.$$

In case $l = 4$, $q = 3$ and $r = 1$, elementary but lengthy calculation gives us

$$\max_{0 \le x \le 1} \sum_{i=0}^{N-1} |\phi_{h,p}^{(5)}(x - (i + \frac{1}{2})h)| = \sum_{i=0}^{N-1} |\phi_{h,p}^{(5)}((i + \frac{1}{2})h)|$$
$$= 1.2146.$$

Hence

$$|K_h * u_h(x) - K_h * \bar{u}_h(x)| \le 1.2146\epsilon,$$

i.e. the K-operator method is quite stable in this case.

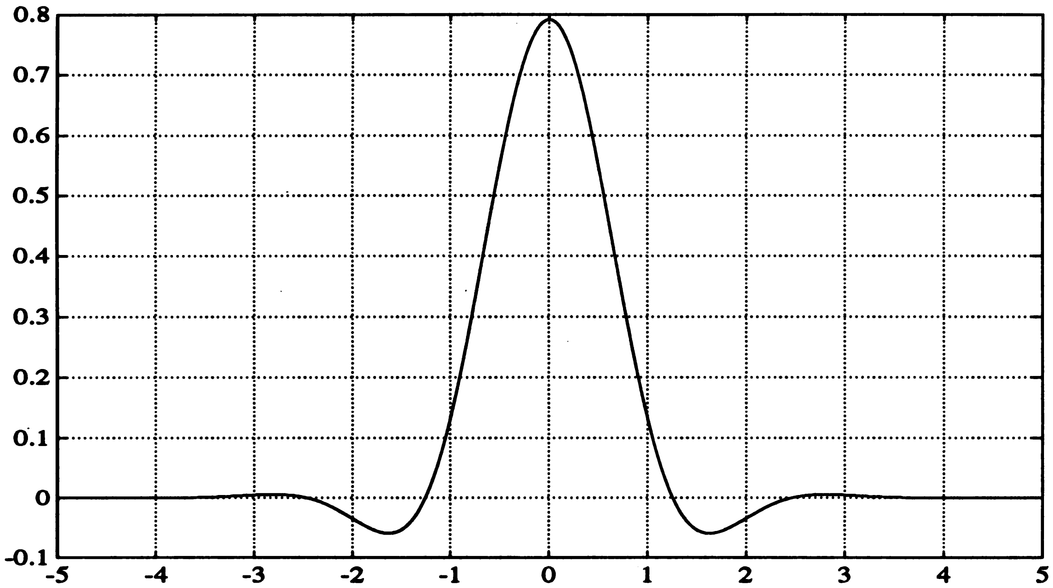

FIGURE 2. Graph of ϕ

We will give here some properties of the K-operator.

Lemma 3.1. *K_h reproduces polynomials of order no greater than $2q$ (i.e. of degree no greater than $2q - 1$) under convolution, i.e.,*

$$K_h * v = v \quad \textit{if } v \in \mathbb{P}_{2q}.$$

Proof. Let $v(x) = \sum_{j=0}^{2q-1} a_j x^j$. Then by (3.3.7)

$$\begin{aligned}
K_h * v(x) &= \int_{\mathbb{R}} K_h(y) \sum_{j=0}^{2q-1} a_j (x-y)^j \, dy \\
&= \sum_{j=0}^{2q-1} a_j \int_{\mathbb{R}} K_h(y) \sum_{i=0}^{j} \binom{j}{i} x^i y^{j-i} \, dy \\
&= \sum_{j=0}^{2q-1} a_j \sum_{i=0}^{j} \binom{j}{i} x^i \int_{\mathbb{R}} K_h(y) y^{j-i} \, dy \\
&= \sum_{j=0}^{2q-1} a_j x^j \\
&= v(x). \quad \square
\end{aligned}$$

Using the above property and the Bramble-Hilbert lemma [6, Theorem 2], we prove the following lemma, which was stated in [9].

Lemma 3.2. ([9, Lemma 5.2]) *For any $i = 0, \ldots, 3$ and s with $0 \leq s \leq 2q$, there exist $c > 0$ and $h_0 > 0$ such that for any $v \in H^s(I_{i+1})$*

$$\|K_h * v - v\|_{0,I_i} \leq c h^s \|v\|_{s,I_{i+1}} \quad \text{for } 0 < h \leq h_0.$$

Proof. We adapt the proof of [6, Theorem 3] to prove the lemma, only in the case $i = 0$ and $s = 2q$, since the other cases can be proved similarly. Let I' be an interval such that $I_0 \Subset I' \Subset I_1$. Recall that supp $K_h = [-(q-1+l/2)h, (q-1+l/2)h]$. We can choose $h_0 > 0$ such that for any $h \in (0, h_0]$

$$I'' := I' + \text{supp } K_h = \{x + y : x \in I' \text{ and } y \in \text{ supp } K_h\} \subset I_1.$$

For any $h \in (0, h_0]$, there are a finite number of intervals $J_k = (x_k, x_{k+1})$, $k \in N_h \subset \mathbb{N}$, such that

$$I_0 \subset \cup_{k \in N_h} J_k \subset I'.$$

For any fixed $x \in J_k$ we define

$$F(x;v) = K_h * v(x) - v(x).$$

Then $F(x;\cdot)$ is a linear functional on $C^{2q-1}(I_1)$, the space of continuous functions on I_1 with continuous derivatives up to the order $2q-1$. Lemma 3.1 implies that $F(x;v) = 0$ for any $v \in \mathbf{P}_{2q}$. Since $\int_{-\infty}^{\infty} K_h(y)\,dy = 1$ we can write

$$F(x;v) = \int_{-\infty}^{\infty} K_h(y)(v(x-y) - v(x))\,dy.$$

By Taylor's theorem we have, for $1 \le m \le 2q-2$,

$$\begin{aligned}|F(x;v)| \le &\Big|\sum_{j=1}^{m}(-1)^j \frac{D^j v(x)}{j!}\int_{-\infty}^{\infty} K_h(y) y^j\,dy\Big| \\ &+ \frac{1}{m!}\Big|\int_{-\infty}^{\infty}\Big(K_h(y) y^{m+1}\int_0^1 (1-t)^m D^{m+1}v(x-ty)\,dt\Big)\,dy\Big|.\end{aligned}$$

The first term on the right side vanishes due to (3.3.7). For the second term, we note that $x - y \in J'_k \subset I'' \subset I_1$, where $J'_k = J_k + \operatorname{supp} K_h$. Hence

$$|F(x;v)| \le c h^{m+1} |D^{m+1} v|_{0,J'_k} \quad \text{for } 0 < h \le h_0,$$

where c is independent of x, v, k and h and where, for any $j \in \mathbb{N}$,

$$|D^j v|_{0,J'_k} := \sup_{z \in J'_k} |D^j v(z)|.$$

It follows that

$$|F(x;v)| \le c \sum_{j=0}^{2q-1} h^j |D^j v|_{0,J'_k} \quad \text{for } 0 < h \le h_0.$$

The Corollary following [6, Theorem 2] (the Bramble-Hilbert lemma) assures us that there exists a constant c independent of x, v, k and h such that

$$|F(x;v)| \le c h^{2q-1/2} \|D^{2q} v\|_{0,J'_k} \quad \text{for } 0 < h \le h_0.$$

Therefore,

$$\begin{aligned}\|K_h * v - v\|_{0,J_k} &= \Big(\int_{J_k} |F(x;v)|^2\,dx\Big)^{1/2} \\ &\le c h^{2q} \|D^{2q} v\|_{0,J'_k} \le c h^{2q} \|v\|_{2q,J'_k}.\end{aligned}$$

Here again c is independent of v and k. Summing up over $k \in N_h$ we then achieve

$$\|K_h * v - v\|_{0,I_0} \leq ch^{2q}\|v\|_{2q,I_1}. \quad \square$$

Another interesting property of K_h is that its derivative equals a central difference of a similar function. More precisely, letting

$$\begin{aligned} \tilde{\partial}_h v(x) &= \frac{1}{h}\{v(x+\frac{h}{2}) - v(x-\frac{h}{2})\}, \\ \partial_h v(x) &= \frac{1}{h}\{v(x+h) - v(x)\}, \end{aligned} \tag{3.3.8}$$

we have the following:

Lemma 3.3. ([9, Lemma 5.3]) *For any $j = 0, 1, \ldots, l$ and $i = 0, \ldots, 3$ we have*

$$D^j K_h = \tilde{\partial}_h^j V_{h,q}^{l-j},$$

and

$$\|D^j(K_h * v)\|_{s,I_i} \leq c\|\partial_h^j v\|_{s,I_{i+1}},$$

where

$$V_{h,q}^{l'}(x) = \frac{1}{h} \sum_{j=-(q-1)}^{q-1} k_j \psi^{(l')}(\frac{x}{h} - j). \tag{3.3.9}$$

Note that in this notation $K_h = V_{h,q}^l$. Before going to the main results of this chapter we state the following lemma, which was proved in [9]:

Lemma 3.4. ([9, Lemma 2.2]) *Let τ be a non-negative integer. Then for any $i = 0, \ldots, 3$ there exists a constant c such that*

$$\|v\|_{0,I_i} \leq c \sum_{j=0}^{\tau} \|D^j v\|_{-\tau, I_{i+1}}.$$

4. The Case of Smooth Closed Curves

We shall in this section exploit the highest order of convergence in a negative norm, given by (3.2.8), to increase the order of convergence in the L^2-norm using post-processing. It is therefore sensible to consider only the case $r - 2\alpha > 0$.

Theorem 4.1. *Let the mesh Δ be uniform in the interval I^*. Assume that $u \in H^{2(r-\alpha)}(I_*) \cap H_p^r$. Assume further that L_1 is bounded from H_p^s to $H_p^{s-2\alpha+\eta}$ for any $s \in \mathbb{R}$ and for some $\eta \geq r - 2\alpha$. If K_h is defined by (3.3.6) with $l = r - 2\alpha$ and $q \geq r - \alpha$ then there exists an $h_0 > 0$ such that for $h \in (0, h_0]$,*

$$\|K_h * u_h - u\|_{0,I_0} \leq ch^{2(r-\alpha)}(\|u\|_{2(r-\alpha),I_*} + \|u\|_r). \tag{3.4.1}$$

Proof. By the triangle inequality we have

$$\|K_h * u_h - u\|_{0,I_0} \leq \|K_h * u - u\|_{0,I_0} + \|K_h * (u_h - u)\|_{0,I_0} = I + II. \tag{3.4.2}$$

We will prove separately that I and II satisfy (3.4.1). The result for the first term comes easily from Lemma 3.2 and the conditions $q \geq r - \alpha$:

$$I \leq ch^{2(r-\alpha)}\|u\|_{2(r-\alpha),I_*}. \tag{3.4.3}$$

For the second term since $l = r - 2\alpha$ it is possible (see Lemma 3.3) to differentiate K_h up to the order $r - 2\alpha$. Therefore, by using Lemmas 3.3 and 3.4 we are able to go from the L^2-norm down to the $H^{2\alpha-r}$-norm and then obtain

$$\begin{aligned} II &\leq c\sum_{j=0}^{r-2\alpha} \|D^j K_h * (u_h - u)\|_{2\alpha-r,I_1} \leq c\sum_{j=0}^{r-2\alpha} \|\partial_h^j(u_h - u)\|_{2\alpha-r,I_2} \\ &\leq c\sum_{j=0}^{r-2\alpha} \|\partial_h^j(u_h - u + L_0^{-1}L_1(u_h - u))\|_{2\alpha-r,I_2} \\ &\quad + c\sum_{j=0}^{r-2\alpha} \|\partial_h^j L_0^{-1}L_1(u_h - u)\|_{2\alpha-r,I_2} \\ &= III + IV. \end{aligned} \tag{3.4.4}$$

The term IV is easily estimated by noting that $\|\partial_h^j v\|_s \le \|v\|_{s+j}$ for any v and that $L_0^{-1}L_1$ is bounded from H_p^s to $H_p^{s+\eta}$ for any $s \in \mathbf{R}$:

$$IV \le c\|L_0^{-1}L_1(u_h - u)\|_0 \le c\|u_h - u\|_{-\eta}.$$

Since $\eta \ge r - 2\alpha$ we then deduce from (3.2.8)

$$IV \le c\|u_h - u\|_{2\alpha - r} \le ch^{2(r-\alpha)}\|u\|_r. \tag{3.4.5}$$

For the term III let us note that from (3.2.6) we have

$$\langle L_0(u_h - u + L_0^{-1}L_1(u_h - u)), \varphi\rangle = 0 \text{ for any } \varphi \in \overset{\circ}{S}{}^r_{h,p}(I_3). \tag{3.4.6}$$

We shall prove that for any $j = 0, \dots, r - 2\alpha$

$$\langle L_0\partial_h^j(u_h - u + L_0^{-1}L_1(u_h - u)), \varphi\rangle = 0 \text{ for any } \varphi \in \overset{\circ}{S}{}^r_{h,p}(I_2). \tag{3.4.7}$$

From the definitions of ∂_h^j, L_0 and T_h there follows

$$\begin{aligned}
&\langle L_0\partial_h^j(u_h - u + L_0^{-1}L_1(u_h - u)), \varphi\rangle \\
&\qquad = \frac{1}{h^j}\sum_{i=0}^{j}\binom{j}{i}\langle L_0T_h^i(u_h - u + L_0^{-1}L_1(u_h - u)), \varphi\rangle \\
&\qquad = \frac{1}{h^j}\sum_{i=0}^{j}\binom{j}{i}\langle T_h^iL_0(u_h - u + L_0^{-1}L_1(u_h - u)), \varphi\rangle \\
&\qquad = \frac{1}{h^j}\sum_{i=0}^{j}\binom{j}{i}\langle L_0(u_h - u + L_0^{-1}L_1(u_h - u)), T_{-h}^i\varphi\rangle.
\end{aligned}$$

Equation (3.4.7) now follows from (3.2.5) and (3.4.6). It follows in turn that for any $\zeta \in S^r_{h,p}$ we have

$$\langle L_0\{[\partial_h^j u_h - \zeta] - [\partial_h^j(u - L_0^{-1}L_1(u_h - u)) - \zeta]\}, \varphi\rangle = 0 \quad \forall\varphi \in \overset{\circ}{S}{}^r_{h,p}(I_2).$$

This equation, together with the boundedness of $L_0^{-1}L_1$ from H_p^s to $H_p^{s+\eta}$ with $\eta > 0$ and the condition (3.2.5), assure that we can use Theorem 2.1 with

$v = \partial_h^j(u + L_0^{-1}L_1(u - u_h)) - \zeta$, $v_h = \partial_h^j u_h - \zeta$, $t = 2\alpha - r$ and $s = r - 1/2 - \epsilon$ with $\epsilon > 0$, to obtain for the j^{th} term III_j of III

$$\begin{aligned}
III_j &= \|[\partial_h^j u_h - \zeta] - [\partial_h^j(u - L_0^{-1}L_1(u_h - u)) - \zeta]\|_{2\alpha - r, I_1} \\
&\le c\{h^{2(r-\alpha)-1/2-\epsilon}\|\partial_h^j(u - L_0^{-1}L_1(u_h - u)) - \zeta\|_{r-1/2-\epsilon, I_2} \\
&\qquad + \|\partial_h^j(u_h - u + L_0^{-1}L_1(u_h - u))\|_\beta\} \\
&\le c\{h^{2(r-\alpha)-1/2-\epsilon}(\|\partial_h^j u - \zeta\|_{r-1/2-\epsilon, I_2} + \|\partial_h^j L_0^{-1}L_1(u_h - u)\|_{r-1/2-\epsilon, I_2}) \\
&\qquad + \|\partial_h^j(u_h - u + L_0^{-1}L_1(u_h - u))\|_\beta\} \\
&\le c\{h^{2(r-\alpha)-1/2-\epsilon}(\|\omega_2\partial_h^j u - \zeta\|_{r-1/2-\epsilon} + \|L_0^{-1}L_1(u_h - u)\|_{r-1/2-\epsilon+j}) \\
&\qquad + \|u_h - u + L_0^{-1}L_1(u_h - u)\|_{\beta+j}\} \quad \text{for arbitrary } \zeta \in S_{h,p}^r,
\end{aligned}$$

where ω_2 is a cut-off function satisfying $\omega_2 \in C_0^\infty(I_3)$ and $\omega_2 \equiv 1$ on I_2. Therefore,

$$\begin{aligned}
III_j &\le c\{h^{2(r-\alpha)-1/2-\epsilon}(\|\omega_2\partial_h^j u - \zeta\|_{r-1/2-\epsilon} + \|u_h - u\|_{r-1/2-\epsilon}) \\
&\qquad + \|u_h - u\|_{\beta+r-2\alpha}\} \quad \text{for any } j = 0, \dots, r - 2\alpha.
\end{aligned}$$

Here again we have used the boundedness of $L_0^{-1}L_1$ from H_p^s to $H_p^{s+\eta}$ for any $s \in \mathbf{R}$ with $\eta \ge r - 2\alpha$. Lemma 2.1(b) of Chapter II assures us that we can choose ζ so that

$$\begin{aligned}
\|\omega_2\partial_h^j u - \zeta\|_{r-1/2-\epsilon} &\le ch^{1/2+\epsilon}\|\partial_h^j u\|_{r, I_3} \le ch^{1/2+\epsilon}\|u\|_{r+j, I_3} \\
&\le ch^{1/2+\epsilon}\|u\|_{2(r-\alpha), I_3} \quad \text{for } j = 0, \dots, r - 2\alpha.
\end{aligned}$$

The estimate (3.2.7) then implies, for any $j = 0, \dots, r - 2\alpha$,

$$III_j \le c\{h^{2(r-\alpha)}(\|u\|_{2(r-\alpha), I_*} + \|u\|_r) + \|u_h - u\|_{\beta+r-2\alpha}\}. \tag{3.4.8}$$

Summing up the result in (3.4.8), combining with inequalities (3.4.2)–(3.4.5), we deduce

$$\|K_h * u_h - u\|_{0, I_0} \le c\{h^{2(r-\alpha)}(\|u\|_{2(r-\alpha), I_*} + \|u\|_r) + \|u_h - u\|_{\beta+r-2\alpha}\}.$$

Let $\beta = 4\alpha - 2r$. The desired result then follows from (3.2.8). $\square$

Remark 1. If L is the operator associated with Symm's equation on a smooth closed curve then L_0 is given by (3.2.2) and (3.2.3) with $\alpha = -1/2$ and L_1 is bounded from H_p^s to H_p^t for any $s, t \in \mathbb{R}$ (see e.g. [42]). The condition of Theorem 4.1 on L_1 is obviously satisfied. If L is the operator associated with the Dirichlet boundary value problem for the Hemholtz equation then L_1 is only bounded from H_p^s to H_p^{s+3} for any $s \in \mathbb{R}$ (see [30]). Nevertheless, if we use piecewise-constant functions to approximate the solution u, the condition on L_1 is satisfied with $\eta = 2$ (since $2\alpha = -1$), and hence the K-operator method is applicable to this problem.

Remark 2. As can be seen from the proof of Theorem 4.1, the parameter l in the definition of the function K_h is of the same magnitude (with opposite sign) as the order of the Sobolev norm which gives best convergence order (i.e., $2\alpha - r$ for the smooth case discussed above or -1 for Symm's equation on a slit); whereas the parameter q is determined (via Lemma 3.2) by the rate of convergence to be achieved for the K-operator.

Remark 3. The proof of Theorem 4.1 explains the necessity of the modification of Saranen's result [36] with assumptions (A2)–(A3) as presented in Chapter II. In fact, one can see that $\partial_h^j u_h$, the forward difference of u_h, is a spline only on I_*, if the mesh is uniform on I^*, and approximates $\partial_h^j(u - L_0^{-1}L_1(u_h - u))$ only in the sense of equation (3.4.7).

5. The Case of Smooth Open Curves

In this section we will study a special equation on an open curve: Symm's equation on the interval $\Gamma = [-1, 1]$. The method is, however, applicable to any pseudo-differential equations on any smooth open curve, provided that negative norm error estimates are available. The equation, as defined in the previous chapter, is

$$V\psi(x) := -\frac{1}{2\pi}\int_\Gamma \log|x-y|\psi(y)\,ds(y) = f(x) \quad \text{for } x \in \Gamma. \tag{3.5.1}$$

It was proved in [46] that V is a continuous and bijective mapping from $\tilde{H}^\tau(\Gamma)$ onto $H^{\tau+1}(\Gamma)$ for $\tau \in (-1, 0)$. Moreover, the following results were proved:

Lemma 5.1. ([46, Lemma 2.1 and Theorem 2.3]) *Let $\delta \in (-1/2, 1/2)$ be fixed. For $i = 1, 2$, let $d_i(x) = |x - (-1)^i|$ and let χ_i be cut-off functions satisfying $0 \le \chi_i \le 1$, $\chi_i \equiv 1$ near $(-1)^i$ and $\chi_i \equiv 0$ elsewhere.*

(i) If $f \in H^{3/2+\delta}(\Gamma)$ then the solution ψ has the form

$$\psi = \sum_{i=1}^{2} \alpha_i d_i^{-1/2}\chi_i + \psi_0 \quad \textit{with } \psi_0 \in \tilde{H}^{1/2+\delta}(\Gamma) \textit{ and } \alpha_i \in \mathbb{R},$$

and there holds the a-priori estimate

$$\sum_{i=1}^{2} |\alpha_i| + \|\psi_0\|_{\tilde{H}^{1/2+\delta}(\Gamma)} \le c\|f\|_{H^{3/2+\delta}(\Gamma)}.$$

(ii) If $f \in H^{5/2+\delta}(\Gamma)$ then the solution ψ has the form

$$\psi = \sum_{i=1}^{2} (\alpha_i d_i^{-1/2} + \beta_i d_i^{1/2})\chi_i + \psi_1 \quad \textit{with } \psi_1 \in \tilde{H}^{3/2+\delta}(\Gamma) \textit{ and } \alpha_i, \beta_i \in \mathbb{R},$$

and there holds the a-priori estimate

$$\sum_{i=1}^{2} (|\alpha_i| + |\beta_i|) + \|\psi_1\|_{\tilde{H}^{3/2+\delta}(\Gamma)} \le c\|f\|_{H^{5/2+\delta}(\Gamma)}.$$

Let $\Delta = \{x_i\}$, with $x_i < x_{i+1}$, $i = 1, \ldots, N$, $N \in \mathbb{N}$, be a mesh on Γ. Let S_h be the space of piecewise-constant functions on Γ with breakpoints Δ, where $h = 1/N$. The Galerkin approximation for the solution of equation (3.5.1) is defined as: $\psi_h \in S_h$ such that

$$\langle V\psi_h, \varphi\rangle_{L^2(\Gamma)} = \langle f, \varphi\rangle_{L^2(\Gamma)} \quad \text{for any } \varphi \in S_h. \tag{3.5.2}$$

Quasi-Uniform Mesh. If the mesh Δ is quasi-uniform, the following global error estimates hold (see [20, 46])

$$\|\psi - \psi_h\|_{\tilde{H}^t(\Gamma)} \leq ch^{\tau - t}\|\psi\|_{\tilde{H}^\tau(\Gamma)} \quad \text{for } -1 < t \leq \tau < 0.$$

The condition $\tau < 0$ is necessary because in general $\psi \notin H^0(\Gamma) = L^2(\Gamma)$. Therefore, for any $\epsilon > 0$, provided that the boundary data are sufficiently smooth we have

$$\|\psi - \psi_h\|_{\tilde{H}^{-1}(\Gamma)} \leq ch^{1-2\epsilon}\|\psi\|_{\tilde{H}^{-\epsilon}(\Gamma)}. \tag{3.5.3}$$

As proved in Chapter II (see (2.4.8)), the local L^2-error converges as

$$\|\psi - \psi_h\|_{L^2(\Gamma_0)} \leq ch^{1/2-\epsilon}(\|\psi\|_{H^{1/2}(\Gamma_*)} + \|\psi\|_{\tilde{H}^{-\epsilon}(\Gamma)}) \tag{3.5.4}$$

for some $\epsilon > 0$ even though the global L^2-norm of $\psi - \psi_h$ is not defined. Here and in the sequel we use nested sub-intervals

$$\Gamma_i \Subset \Gamma_{i+1} \Subset \Gamma_* \Subset \Gamma^* \Subset \Gamma \quad \text{for } i = 0, 1, 2.$$

We are led by the Remark 2 following Theorem 4.1 to use a K_h spline of order 1, i.e., $l = 1$, in the hope that $K_h * u_h$ is an approximation to ψ which gives local convergence of order $O(h)$ in the L^2-norm. That function K_h, defined by (3.3.6) is

$$K_h(x) = K^1_{h,1}(x) = \frac{1}{h}\chi(\frac{x}{h}).$$

To define the convolution, we extend each function v on Γ by 0 onto $\mathbb{R} \setminus \Gamma$ and denote it by $\tilde{v}$. The K-operator acting on ψ_h is now given by

$$K_h(\psi_h) = K_h * \tilde{\psi}_h. \tag{3.5.5}$$

Theorem 5.2. *Assume that the mesh is uniform on Γ^* and that the exact solution ψ satisfies $\psi \in H^{1-\epsilon}(\Gamma_*) \cap \widetilde{H}^{-\epsilon}(\Gamma)$ for some $\epsilon > 0$. Let $h_0 > 0$ be such that $T_{\pm h_0}(\Gamma_1) = \{x \pm h_0 : x \in \Gamma_1\} \subset \Gamma_*$. Then for $h \in (0, h_0]$*

$$\|K_h(\psi_h) - \psi\|_{L^2(\Gamma_0)} \leq ch^{1-2\epsilon}(\|\psi\|_{H^{1-\epsilon}(\Gamma_*)} + \|\psi\|_{\widetilde{H}^{-\epsilon}(\Gamma)}). \tag{3.5.6}$$

Proof. Following the line of the proof of Theorem 4.1 we shall prove (3.5.6) by using Lemma 3.2 and the fact that the forward difference of ψ_h approximates that of ψ (in some sense). However, now that $\tilde{\psi} \notin L^2(\mathbb{R})$ it is not useful to define $K_h(\psi)$ as in (3.5.5). We will make use of the function $\psi_* = \omega_* \tilde{\psi}$ where ω_* is a cut-off function satisfying

$$\omega_* \equiv 1 \text{ on } \Gamma_* \quad \text{and} \quad \omega_* \in C_0^\infty(\Gamma^*).$$

By noting that $\psi_* = \psi$ on Γ_0 and using the triangle inequality we obtain

$$\begin{aligned} \|K_h(\psi_h) - \psi\|_{L^2(\Gamma_0)} &\leq \|K_h * \psi_* - \psi_*\|_{L^2(\Gamma_0)} + \|K_h * (\tilde{\psi}_h - \psi_*)\|_{L^2(\Gamma_*)} \\ &= I + II. \end{aligned} \tag{3.5.7}$$

That I is bounded by the right hand side of (3.5.6) comes from the local smoothness of ψ and Lemma 3.2. To obtain the same estimate for II, again we use Lemmas 3.3 and 3.4, so obtaining, with (3.5.3),

$$\begin{aligned} II &\leq c(\|\tilde{\psi}_h - \psi_*\|_{H^{-1}(\Gamma_1)} + \|\partial_h(\tilde{\psi}_h - \psi_*)\|_{H^{-1}(\Gamma_1)}) \\ &\leq c(\|\psi_h - \psi\|_{H^{-1}(\Gamma_1)} + \|\partial_h(\tilde{\psi}_h - \tilde{\psi})\|_{H^{-1}(\Gamma_1)}) \\ &\leq c(h^{1-2\epsilon}\|\psi\|_{\widetilde{H}^{-\epsilon}(\Gamma)} + \|\partial_h(\tilde{\psi}_h - \tilde{\psi})\|_{H^{-1}(\Gamma_1)}) \quad \text{for } 0 < h \leq h_0. \end{aligned} \tag{3.5.8}$$

In the second last step we have used the assumption $T_{\pm h}(\Gamma_1) \subset \Gamma_*$ to obtain $\partial_h \psi_* = \partial_h \tilde{\psi}$. To estimate the last term of (3.5.8), let $\tilde{\Gamma}$ be a smooth closed curve containing the interval $[-2,2]$ and define $V_{\tilde{\Gamma}}$ by (3.5.1) with Γ replaced by $\tilde{\Gamma}$. We extend $\tilde{\psi}_h - \tilde{\psi}$ and $\partial_h(\tilde{\psi}_h - \tilde{\psi})$ by 0 onto $\tilde{\Gamma} \setminus [-2,2]$. Then by using the equation (3.5.2), and by noting that $\tilde{\psi}$ and $\tilde{\psi}_h$ vanish outside $\Gamma = [-1,1]$, we obtain, for any $\varphi \in \overset{\circ}{S}_h(\Gamma_2)$ (see definition (2.2.6)) and $h \in (0, h_0]$ with $h_0 < 1$,

$$
\begin{aligned}
\langle V_{\tilde{\Gamma}} \partial_h(\tilde{\psi}_h - \tilde{\psi}), \varphi \rangle_{L^2(\tilde{\Gamma})} &= \frac{1}{h}\{\langle V_{\tilde{\Gamma}} T_h(\tilde{\psi}_h - \tilde{\psi}), \varphi \rangle_{L^2(\tilde{\Gamma})} - \langle V_{\tilde{\Gamma}}(\tilde{\psi}_h - \tilde{\psi}), \varphi \rangle_{L^2(\tilde{\Gamma})}\} \\
&= -\frac{1}{\pi h} \int_{-2}^{2} \int_{-2}^{2} \log|x-y| (\tilde{\psi}_h - \tilde{\psi})(y+h) \varphi(x)\, dy\, dx \\
&= -\frac{1}{\pi h} \int_{-2}^{2} \int_{-2}^{2} \log|x-y| (\tilde{\psi}_h - \tilde{\psi})(y) \varphi(x-h)\, dy\, dx \\
&= \frac{1}{h} \langle V_{\tilde{\Gamma}}(\tilde{\psi}_h - \tilde{\psi}), T_{-h}\varphi \rangle_{L^2(\tilde{\Gamma})} \\
&= \frac{1}{h} \langle V(\psi - \psi_h), T_{-h}\varphi \rangle_{L^2(\Gamma)}.
\end{aligned}
\tag{3.5.9}
$$

Since the mesh is uniform on Γ^*, we have $T_{-h}\varphi \in \overset{\circ}{S}_h(\Gamma_3) \subset S_h$ for any $\varphi \in \overset{\circ}{S}_h(\Gamma_2)$. Equations (3.5.2) and (3.5.9) then imply

$$
\langle V_{\tilde{\Gamma}} \partial_h(\tilde{\psi}_h - \tilde{\psi}), \varphi \rangle_{L^2(\tilde{\Gamma})} = 0 \quad \text{for any } \varphi \in \overset{\circ}{S}_h(\Gamma_2).
$$

We can now use Theorem 4.2 of Chapter II to obtain

$$
\begin{aligned}
\|\partial_h(\tilde{\psi}_h - \tilde{\psi})\|_{H^{-1}(\Gamma_1)} &\leq c(h^{1-2\epsilon}\|\partial_h \tilde{\psi}\|_{H^{-\epsilon}(\Gamma_*)} + \|\partial_h(\tilde{\psi}_h - \tilde{\psi})\|_{H^\beta(\tilde{\Gamma})}) \\
&\leq c(h^{1-2\epsilon}\|\tilde{\psi}\|_{H^{1-\epsilon}(\Gamma_*)} + \|\tilde{\psi}_h - \tilde{\psi}\|_{H^{\beta+1}(\tilde{\Gamma})}) \\
&\leq c(h^{1-2\epsilon}\|\psi\|_{H^{1-\epsilon}(\Gamma_*)} + \|\psi_h - \psi\|_{\tilde{H}^{\beta+1}(\Gamma)}).
\end{aligned}
$$

Choosing $\beta = -2$ and using (3.5.3) again, we get the desired estimate and hence the theorem is proved. □

Graded Mesh. To recover the order $O(h^3)$ of the smooth case, mesh grading is necessary. We note that in the proof of Theorem 4.2 of Chapter II, the mesh is required to be quasi-uniform only in some sub-interval of I (e.g., on I^*). Hence a consideration of mesh grading on $\Gamma \setminus \Gamma^*$ is permissible. For example, in the case $\Gamma = [-1, 1]$, we can define a mesh which is uniform on $[-3/4, 3/4]$ and graded on the other sub-intervals. More precisely, we can define $\Delta = \{x_k : k = 0, \ldots, N\}$ as

$$x_k = \begin{cases} -1 + 4^{\varrho-1}(kh)^{\varrho} & \text{if } 0 \le k \le N/8 - 1, \\ -1 + kh & \text{if } N/8 \le k \le 7N/8 - 1, \\ 1 - 4^{\varrho-1}(2 - kh)^{\varrho} & \text{if } 7N/8 \le k \le N, \end{cases}$$

where $N = 8n$, $n \in \mathbb{N}$, $h = 2/N$ and $\varrho \ge 1$. Note that $x_{N/8} = -3/4$, $x_{7N/8} = 3/4$ and that the mesh is uniform when $\varrho = 1$. The mesh being dependent on ϱ, we shall denote the spline space by S_h^{ϱ}. We will need the following technical inequalities, which were also used in [53]: For any $\mu \in [1 - 1/\varrho, 1]$ and $k = 1, \ldots, N/8$

$$\begin{aligned} h_k := x_k - x_{k-1} &= 4^{\varrho-1}h^{\varrho}[k^{\varrho} - (k-1)^{\varrho}] \le \varrho 4^{\varrho-1}h^{\varrho}k^{\varrho-1} \\ &= \varrho 4^{\varrho-1}h^{\varrho}k^{\varrho-1}(1 + x_k)^{\mu}4^{(1-\varrho)\mu}h^{-\varrho\mu}k^{-\varrho\mu} \\ &= \varrho 4^{(\varrho-1)(1-\mu)}(1 + x_k)^{\mu}h^{\varrho(1-\mu)}k^{\varrho(1-\mu)-1} \\ &\le \varrho 4^{(\varrho-1)(1-\mu)}(1 + x_k)^{\mu}h^{\varrho(1-\mu)}, \end{aligned} \tag{3.5.10}$$

and

$$1 + x_k = \left(\frac{k}{k-1}\right)^{\varrho}(1 + x_{k-1}) \le 2^{\varrho}(1 + x) \quad \text{for } x \in J_k := [x_{k-1}, x_k). \tag{3.5.11}$$

For the application of the K-operator in this case, the availability of the error estimate in the deepest negative norm is necessary. By slightly modifying a result of von Petersdorff [53, Satz 3.7] and using the *a priori* estimates given in Lemma 5.1 we can prove

Lemma 5.3. *Let $\epsilon > 0$ be given. Then*

$$\|\psi - \psi_h\|_{\tilde{H}^{-1/2}(\Gamma)} \leq \begin{cases} ch^{\varrho/2}\|f\|_{H^{(\varrho+1)/2}(\Gamma)} & \text{if } 1 < \varrho < 3, \\ ch^{3/2}(\log \frac{1}{h})^{1/2}\|f\|_{H^{2+\epsilon}(\Gamma)} & \text{if } \varrho = 3, \\ ch^{3/2}\|f\|_{H^{2+\epsilon}(\Gamma)} & \text{if } \varrho > 3. \end{cases} \tag{3.5.12}$$

Proof. By Lemma 5.1, if f is sufficiently smooth we can express ψ as

$$\psi = \sum_{i=1}^{2} \alpha_i d_i^{-1/2}\chi_i + \psi_0 \quad \text{with } \psi_0 \in \tilde{H}^{1/2+\delta}(\Gamma) \text{ and } \alpha_i \in \mathbb{R},$$

where $\delta \in (-1/2, 1/2)$. It is essential to estimate the terms involving $d_i^{-1/2}$.

For each function v on Γ we define by v_k, $k = 1, \dots, N$, the mean of v on $J_k = [x_{k-1}, x_k)$, i.e.,

$$v_k = \frac{1}{h_k}\int_{J_k} v(x)\,dx,$$

provided that the integral exists. Moreover, let $P_h v$ be defined as

$$P_h v = v_k \quad \text{on } J_k \text{ for } k = 1, \dots, N.$$

Then $P_h v$ is indeed the L^2-projection of v on S_h^{ϱ}. Since P_h is linear, it is possible to decompose $\psi - P_h\psi$ as

$$\psi - P_h\psi = \sum_{i=1}^{2} \alpha_i\Big(d_i^{-1/2}\chi_i - P_h(d_i^{-1/2}\chi_i)\Big) + \Big(\psi_0 - P_h(\psi_0)\Big). \tag{3.5.13}$$

We will prove that each of the term on the right hand side of (3.5.13) satisfies (3.5.12). Consider first $g - P_h g := d_1^{-1/2}\chi_1 - P_h(d_1^{-1/2}\chi_1)$. It was proved in [53, Lemma 3.2] that

$$\begin{aligned} \|g - P_h g\|^2_{\tilde{H}^{-1/2}(\Gamma)} &\leq \|g - P_h g\|^2_{\tilde{H}^{-1/2}(J_1)} + \sum_{k=2}^{N} \|g - P_h g\|^2_{\tilde{H}^{-1/2}(J_k)} \\ &= I + II. \end{aligned} \tag{3.5.14}$$

It was also proved in [53, Lemma 3.6] that

$$I \leq ch_1 = c4^{\varrho-1}h^{\varrho}. \tag{3.5.15}$$

For the term II, let us note that $g \in H^1(J_k)$ for $k = 2, \dots, N$ and hence (see [5])

$$\|g - g_k\|_{\tilde{H}^{-1/2}(J_K)} \le c h_k^{3/2} \|g'\|_{L^2(J_K)}.$$

Therefore, by choosing χ_1 with supp $\chi_1 \subset [-1, -3/4]$ so that $7N/8$ last terms in the sum II vanish and by using (3.5.10), (3.5.11) we obtain

$$\begin{aligned}
II &\le \sum_{k=2}^{N/8} \|g - g_k\|^2_{\tilde{H}^{-1/2}(J_k)} \le c \sum_{k=2}^{N/8} h_k^3 \|g'\|^2_{L^2(J_k)} \\
&\le c \sum_{k=2}^{N/8} h^{3\varrho(1-\mu)} (1 + x_k)^{3\mu} \|g'\|^2_{L^2(J_k)} \\
&\le c h^{3\varrho(1-\mu)} \sum_{k=2}^{N/8} \int_{J_k} (1+x)^{3\mu} |g'(x)|^2 \, dx \\
&\le c h^{3\varrho(1-\mu)} \int_{-1+h_1}^{0} (1+x)^{-3(1-\mu)} \, dx,
\end{aligned} \tag{3.5.16}$$

where in the last step we have used $|g'(x)| \le c(1+x)^{-3/2}$. Since

$$\int_{-1+h_1}^{0} (1+x)^{-3/\varrho} \, dx \le \begin{cases} c h^{\varrho - 3} & \text{if } 1 \le \varrho < 3, \\ c \log(1/h) & \text{if } \varrho = 3, \\ c & \text{if } \varrho > 3, \end{cases}$$

where c is independent of h but depends on ϱ, by choosing μ so that $1 - \mu = 1/\varrho$ we infer from (3.5.16) that

$$II \le \begin{cases} c h^{\varrho} & \text{if } 1 \le \varrho < 3, \\ c h^3 \log(1/h) & \text{if } \varrho = 3, \\ c h^3 & \text{if } \varrho > 3. \end{cases} \tag{3.5.17}$$

Inequalities (3.5.14), (3.5.15) and (3.5.17) imply

$$\|g - P_h g\|^2_{\tilde{H}^{-1/2}(\Gamma)} \le \begin{cases} c h^{\varrho} & \text{if } 1 \le \varrho < 3, \\ c h^3 \log(1/h) & \text{if } \varrho = 3, \\ c h^3 & \text{if } \varrho > 3. \end{cases}$$

Hence

$$\Big\|\sum_{i=1}^{2} \alpha_i\Big(d_i^{-1/2}\chi_i - P_h(d_i^{-1/2}\chi_i)\Big)\Big\|_{\widetilde{H}^{-1/2}(\Gamma)}$$

$$\leq \begin{cases} ch^{\varrho/2}\sum_{i=1}^{2}|\alpha_i| & \text{if } 1 \leq \varrho < 3, \\ ch^{3/2}(\log(1/h))^{1/2}\sum_{i=1}^{2}|\alpha_i| & \text{if } \varrho = 3, \\ ch^{3/2}\sum_{i=1}^{2}|\alpha_i| & \text{if } \varrho > 3. \end{cases} \qquad (3.5.18)$$

Consider now the smooth part in (3.5.13). For $1 < \varrho < 3$, we can choose $\delta \in (-1/2, 1/2)$ so that $1/2 + \delta = (\varrho - 1)/2$. It was proved in [5] that

$$\|\psi_0 - P_h\psi_0\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq ch^{\varrho/2}\|\psi_0\|_{H^{(\varrho-1)/2}(\Gamma)}. \qquad (3.5.19)$$

For $\varrho \geq 3$ we will use the expansion (ii) in Lemma 5.1 to decompose $\psi - P_h\psi$ as

$$\begin{aligned}\psi - P_h\psi = & \sum_{i=1}^{2} \alpha_i\Big(d_i^{-1/2}\chi_i - P_h(d_i^{-1/2}\chi_i)\Big) \\ & + \sum_{i=1}^{2} \beta_i\Big(d_i^{1/2}\chi_i - P_h(d_i^{1/2}\chi_i)\Big) + \Big(\psi_1 - P_h(\psi_1)\Big).\end{aligned}$$

The middle term on the right hand side is smoother than the first term, so can be estimated in the same way as for the first term. For the third term, now we have [5] for any $\epsilon > 0$

$$\|\psi_1 - P_h\psi_1\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq ch^{3/2+\epsilon}\|\psi_1\|_{H^{1+\epsilon}(\Gamma)}. \qquad (3.5.20)$$

Combining (3.5.18), (3.5.19), (3.5.20) and the *a-priori* estimates given in Lemma 5.1 we obtain the desired result. □

Following the line of reasoning in [20] using Nitsche's trick, we can now prove the following:

Lemma 5.4. *Let $\epsilon > 0$ be given. Let*

$$\tau = \begin{cases} \frac{\varrho+1}{2} & \text{for } 1 < \varrho < 3, \\ 2+\epsilon & \text{for } \varrho \geq 3. \end{cases}$$

Then for $1/2 \leq s \leq \tau$ there holds

$$\|\psi - \psi_h\|_{\widetilde{H}^{-s}(\Gamma)} \leq \begin{cases} ch^{s+(\varrho-1)/2} & \text{if } 1 < \varrho < 3, \\ ch^{3\theta}(\log \frac{1}{h})^{\theta} & \text{if } \varrho = 3, \\ ch^{3\theta} & \text{if } \varrho > 3, \end{cases}$$

where $\theta = (1+\epsilon+s)/(3+2\epsilon)$ and where c may depend on ψ. In particular, when $\varrho \geq 3$ we have

$$\|\psi - \psi_h\|_{\widetilde{H}^{-2}(\Gamma)} \leq ch^{3-\epsilon}. \tag{3.5.21}$$

Proof. We will only give the proof for the case $1 < \varrho < 3$ and $s = (\varrho+1)/2$. The other cases can be proved similarly. First note that by definition we have

$$\|\psi - \psi_h\|_{\widetilde{H}^{-(\varrho+1)/2}(\Gamma)} = \sup_{\zeta \in H^{(\varrho+1)/2}(\Gamma)} \frac{\langle \psi - \psi_h, \zeta \rangle}{\|\zeta\|_{H^{(\varrho+1)/2}(\Gamma)}}. \tag{3.5.22}$$

For any $\zeta \in H^{(\varrho+1)/2}(\Gamma)$, we have $\zeta \in H^{1/2}(\Gamma)$. Hence there exists $\xi \in \widetilde{H}^{-1/2}(\Gamma)$ so that $V\xi = \zeta$. Let $\xi_h \in S_h^{\varrho}$ satisfy

$$\langle V(\xi - \xi_h), \phi \rangle = 0 \quad \text{for any } \phi \in S_h^{\varrho}.$$

It was proved in Lemma 5.3 that

$$\|\xi - \xi_h\|_{\widetilde{H}^{-1/2}(\Gamma)} \leq ch^{\varrho/2}\|\zeta\|_{H^{(\varrho+1)/2}(\Gamma)}. \tag{3.5.23}$$

Lemma 5.3, (3.5.22) and (3.5.23) then imply

$$\begin{aligned} \|\psi - \psi_h\|_{\widetilde{H}^{-(\varrho+1)/2}(\Gamma)} &= \sup_{\zeta \in H^{(\varrho+1)/2}(\Gamma)} \frac{\langle \psi - \psi_h, V\xi \rangle}{\|\zeta\|_{H^{(\varrho+1)/2}(\Gamma)}} \\ &= \sup_{\zeta \in H^{(\varrho+1)/2}(\Gamma)} \frac{\langle V(\psi - \psi_h), \xi \rangle}{\|\zeta\|_{H^{(\varrho+1)/2}(\Gamma)}} \\ &= \sup_{\zeta \in H^{(\varrho+1)/2}(\Gamma)} \frac{\langle V(\psi - \psi_h), \xi - \xi_h \rangle}{\|\zeta\|_{H^{(\varrho+1)/2}(\Gamma)}} \\ &\leq \sup_{\zeta \in H^{(\varrho+1)/2}(\Gamma)} \frac{\|\psi - \psi_h\|_{\widetilde{H}^{-1/2}(\Gamma)}\|\xi - \xi_h\|_{\widetilde{H}^{-1/2}(\Gamma)}}{\|\zeta\|_{H^{(\varrho+1)/2}(\Gamma)}} \\ &\leq ch^{\varrho}. \quad \square \end{aligned}$$

Following the Remark 2 coming after Theorem 4.1, we will now use the spline $K_h = K^l_{h,q}$ with $l = 2$ and $q = 2$ to establish the new approximant. That function K_h has the form (see [9, 49])

$$K_h(x) = \frac{1}{12h}\Big\{-\psi^{(2)}(x/h-1) + 14\psi^{(2)}(x/h) - \psi^{(2)}(x/h+1)\Big\}. \qquad (3.5.24)$$

Replacing (3.5.3) by (3.5.21) and using the same argument as in the proof of Theorem 5.2 we can prove

Theorem 5.5. *Let $\varrho \geq 3$. Assume that $\psi \in H^{3-\epsilon}(\Gamma_*) \cap \widetilde{H}^{-\epsilon}(\Gamma)$ for some $\epsilon > 0$. Let $h_0 > 0$ be such that $T_{\pm 2h_0}(\Gamma_1) \subset \Gamma_*$. Then for $h \in (0, h_0]$*

$$\|K_h(\psi_h) - \psi\|_{L^2(\Gamma_0)} = O(h^{3-\epsilon}).$$

6. Numerical Experiments

Experiment 1. We tested the K-operator method when L is the logarithmic-kernel integral operator arising from the boundary value problem

$$\begin{aligned} \Delta U &= 0 \quad \text{in } \Omega, \\ U &= F \quad \text{on } \Gamma, \end{aligned} \tag{3.6.1}$$

where $\Gamma = \partial\Omega$ is the ellipse $16t_1^2 + 64t_2^2 = 1$ and $F(t) = t_1 + t_2$ with $t = (t_1, t_2)$. It is known [see e.g., 29, 42, 56] that by using the direct method the problem (3.6.1) can be reformulated as

$$-\frac{1}{\pi}\int_\Gamma \log|t-s| z(s)\, dl_s = F(t) - \frac{1}{\pi}\int_\Gamma (\frac{\partial}{\partial n_s}\log|t-s|)F(s)\, dl_s, \quad t \in \Gamma, \tag{3.6.2}$$

where $z = \partial U/\partial n$ is the directional derivative of U with respect to the outward normal vector n. Using a parametrisation $\gamma\colon [0,1] \to \mathbb{R}^2$ for the curve Γ we can rewrite (3.6.2) in the form

$$Lu(x) = f(x) \quad \text{for } x \in [0,1], \tag{3.6.3}$$

where

$$\begin{aligned} u(x) &= (2\pi)^{-1} z[\gamma(x)]|\gamma'(x)|, \\ Lu(x) &= -2\int_0^1 \log(|\gamma(x) - \gamma(y)|)u(y)\, dy, \end{aligned} \tag{3.6.4}$$

and where f is obtained from the right side of (3.6.2) by using the parametrisation. It is known (see e.g. [42]) that $L = L_0 + L_1$ with L_0 expressible as

$$L_0u(x) = \hat{u}(0) + \sum_{n\neq 0} \frac{1}{|n|}\hat{u}(n)e^{2\pi i n x},$$

and with L_1 bounded from H_p^s to H_p^t for any $s, t \in \mathbb{R}$.

There being no need to consider a non-uniform mesh, we used a uniform mesh and investigated the global errors in this example. We chose piecewise-constant functions as test and trial functions in the Galerkin approximation for (3.6.3),

N	$\|u_h - u\|_0$		$\|K_h * u_h - u\|_0$	
8	9.04e-02		4.49e-03	
16	4.49e-02	1.00	4.51e-04	3.31
32	2.24e-02	1.00	4.98e-05	3.18
64	1.12e-02	1.00	5.83e-06	3.09
128	5.60e-03	1.00	7.05e-07	3.05

TABLE 1. Errors and empirical orders of convergence for Experiment 1

and used $K_h = K_{h,2}^2$ given by (3.5.24) to average the values of u_h (see Theorem 4.1 and Remark 2 after that). The empirical orders of convergence obtained for $\|u - u_h\|_0$ and for $\|K_h * u_h - u\|_0$ were $O(h)$ and $O(h^3)$ respectively (see Table 1), which match the analysis.

Experiment 2. We considered in this experiment the weakly singular integral equation (3.5.1) with $f(x) = x$ and tested the local convergence on $\Gamma_0 = (-1/2, 1/2)$ for the errors $e = \psi - \psi_h$ and $E = \psi - K_h(\psi_h)$ with various values of ϱ. When $\varrho = 1$ (uniform mesh) we achieved convergence of apparent order $O(h)$ for both errors (see Table 2), instead of the predicted orders of $O(h^{1/2})$ for $\|e\|_{L^2(\Gamma_0)}$ and $O(h)$ for $\|E\|_{L^2(\Gamma_0)}$. However, one can see that $\|E\|_{L^2(\Gamma_0)}$ is smaller than $\|e\|_{L^2(\Gamma_0)}$ by an order of magnitude. When $\varrho = 3$ or $\varrho = 3.2$ almost nothing changed for $\|e\|_{L^2(\Gamma_0)}$ whereas the empirical rate of convergence for $\|E\|_{L^2(\Gamma_0)}$ is slowly asymptotic to $O(h^3)$. When ϱ is increased to 3.5 the asymptotic $O(h^3)$ order is obtained much more quickly (see Table 2).

	$\|e\|_{L^2(\Gamma_0)}$		$\|E\|_{L^2(\Gamma_0)}$							
N	$\varrho = 1$		$\varrho = 1$		$\varrho = 3$		$\varrho = 3.2$		$\varrho = 3.5$	
8	1.1e-1		1.7e-2		4.1e-2		4.1e-2		4.1e-2	
16	5.4e-2	1.00	5.8e-3	1.58	3.0e-3	3.77	3.1e-3	3.70	3.4e-3	3.60
32	2.7e-2	1.00	2.8e-3	1.06	8.2e-5	5.18	4.9e-5	5.98	7.9e-5	5.40
64	1.4e-2	1.00	1.3e-3	1.06	1.5e-5	2.42	9.0e-6	2.46	1.4e-5	2.51
128	6.8e-3	1.00	6.5e-4	1.03	2.8e-6	2.44	1.5e-6	2.61	1.6e-6	3.15
256	3.4e-3	1.00	3.2e-4	1.02	4.8e-7	2.57	2.4e-7	2.61	1.7e-7	3.18
512	1.7e-3	1.00	1.6e-4	1.01	7.5e-8	2.67	3.7e-8	2.70	2.1e-8	3.02

TABLE 2. Errors and empirical orders of convergence for Experiment 2

CHAPTER IV

A SEMI-DISCRETE METHOD

1. Introduction

In this chapter we shall see how the K-operator method works in a different setting, where a semi-discrete method (the qualocation method) is used instead of the Galerkin method.

The qualocation method (see [17, 39, 40, 41, 42, 45]), which can be explained in short terms as a quadrature-based modification of the collocation method with unusual quadrature rules, aims to increase the order of convergence given by the collocation method while reducing the difficulty in implementation of the Galerkin method. Formally, the qualocation method is obtained from the Galerkin method by replacing the 'outer' integral with a well-chosen quadrature rule. In some particular cases, it even gives higher order convergence than the Galerkin method itself.

To illustrate, consider for example the logarithmic-kernel integral equation on a smooth and closed curve Γ in the plane. With the trial and test spaces being the space of piecewise-constant functions on a uniform mesh, the Galerkin and the collocation methods yield an $O(h^3)$ order of convergence in suitable negative norms (see e.g. [2, 3, 27, 37, 56]). Yet, it is shown in [17] that the quadrature rule for the qualocation method can be chosen so that the qualocation method yields an order $O(h^5)$ (in a suitable negative norm). More precisely, a Simpson-type rule that achieves order $O(h^5)$ has just two points per interval, one at the break-point where the weight is 3/7, and the other at the midpoint where the weight is 4/7. For a systematic review of the qualocation method, see [41, 42].

In this chapter, we will exploit the highest order convergence (in a negative norm) of the qualocation method to obtain, by using the K-operator, a higher order of convergence in the L^2-norm and the max-norm. We even consider the convergence of approximations to the derivatives of the solution. However, since the error analysis for the qualocation approximation is so far proved with a uniform mesh and in the global sense, we shall consider only the global errors.

This chapter contains 4 sections. Section 2 gives a brief review of the qualocation method. In an attempt to make the K-operator be widely applicable for this approximation method, we shall in this section use the perturbation argument used in [30] to widen the class of operators that can be considered. The main result of the chapter is in Section 3, where higher orders of convergence of the approximate solutions and derivatives, both in the L^2 and in the maximum norms, are proved. Section 4 is devoted to a numerical experiment.

2. The Qualocation Method

We recall in this section some basic notion of the qualocation method for the equation

$$Lu = f, \tag{4.2.1}$$

with L, u and f are defined as in Section 2 of Chapter III. Other notations to be used are also the same as introduced in that Section, except when otherwise identified. However, the mesh considered now is assumed to be uniform, the trial space is also $S^r_{h,p}$, but the test space is taken to be $S^{r'}_{h,p}$, i.e., the space of smoothest splines of order r'.

The qualocation method is a discrete version of (3.2.6) in which the outer integral is approximated by a composite quadrature rule determined by points ξ_j and weights ϖ_j with $j = 1, \dots, J$, where

$$0 \le \xi_1 < \xi_2 < \ldots < \xi_J < 1, \tag{4.2.2}$$

and

$$\varpi_j > 0, \quad \sum_{j=1}^{J} \varpi_j = 1.$$

The qualocation rule is defined by

$$Q_N(g) := h \sum_{i=1}^{N} \sum_{j=1}^{J} \varpi_j g(x_i + h\xi_j). \tag{4.2.3}$$

This in turn allows us to define the discrete inner product

$$\langle u, v \rangle_N := Q_N(u\bar{v}), \tag{4.2.4}$$

where $\bar{v}$ denotes the complex conjugate of v. The qualocation solution to the equation (4.2.1) is then defined by

$$u_h \in S^r_{h,p} \quad \text{and} \quad \langle Lu_h, \phi' \rangle_N = \langle f, \phi' \rangle_N \quad \forall \phi' \in S^{r'}_{h,p}. \tag{4.2.5}$$

After choosing bases for $S^r_{h,p}$ and $S^{r'}_{h,p}$, we obtain from (4.2.5) a system of N linear equations in N unknowns, which is referred to as the qualocation equation.

Definition 2.1. *The qualocation method is well defined if either*

$$r > 2\alpha + 1$$

or

$$r > 2\alpha + 1/2 \quad \text{and} \quad \xi_1 > 0.$$

The condition $\xi_1 > 0$ in the latter alternative is necessary because of the fact that if

$$2\alpha + 1/2 < r \leq 2\alpha + 1,$$

then $L\phi$ for $\phi \in S^r_{h,p}$ is in general not continuous at the knot points, so that in this case the knot points are not allowed as quadrature points. The condition $r > 2\alpha + 1/2$ ensures the continuity of $L\phi$ at points other than knot points for $\phi \in S^r_{h,p}$ (see [3, 17] for more details).

For $y \in [-1/2, 1/2]$, let

$$D(y) := \sum_{j=1}^{J} \varpi_j (1 + \Omega(\xi_j, y))(1 + \overline{\Delta'(\xi_j, y)}),$$

and let

$$E(y) := \sum_{j=1}^{J} \varpi_j \Omega(\xi_j, y)(1 + \overline{\Delta'(\xi_j, y)}),$$

where

$$\Delta'(\xi, y) = y^{r'} \sum_{l \neq 0} \frac{1}{(l+y)^{r'}} e^{2\pi i l \xi},$$

and where

$$\Omega(\xi, y) = |y|^{r-2\alpha} \sum_{l \neq 0} \frac{1}{|l+y|^{r-2\alpha}} e^{2\pi i l \xi}$$

if r and L_0 are both even or both odd, or

$$\Omega(\xi, y) = (\text{sign } y)|y|^{r-2\alpha} \sum_{l \neq 0} \frac{\text{sign } l}{|l+y|^{r-2\alpha}} e^{2\pi i l \xi}$$

if r and L_0 are of opposite parity.

Definition 2.2. *The qualocation method is* <u>*stable*</u> *if*

$$\inf\{|D(y)| : y \in [-1/2, 1/2]\} > 0.$$

It is said to be of <u>*order*</u> $r - 2\alpha + b$ *if*

$$E(y) = O(|y|^{r-2\alpha+b}) \quad \text{for } y \in [-1/2, 1/2].$$

Theorem 2.3. *Let* (4.2.1) *be solved by a well defined qualocation method which is stable and of order* $r - 2\alpha + b$ *with* $b \geq 0$*. Assume that* L_1 *is bounded from* H_p^s *to* $H_p^{s-2\alpha+\eta}$ *for any* $s \in \mathbb{R}$ *and for some* $\eta > b + 1/2$*. Then* u_h *is uniquely defined for all* N *sufficiently large. Moreover, for all* s, t *satisfying*

$$t < r - 1/2, \quad 2\alpha + 1/2 < s, \quad 2\alpha - b \leq t \leq s \leq r, \tag{4.2.6}$$

we have

$$\|u_h - u\|_t \leq ch^{s-t}\|u\|_{s+\max(2\alpha-t,0)}. \tag{4.2.7}$$

The case $L = L_0$ was proved in [17].

Proof for the case $L = L_0 + L_1$. We give here a slightly different argument from that in [17] by using the reasoning used in [30]. Assume for the moment that (4.2.5) has a solution $u_h \in S_{h,p}^r$. Since we can write the defining equation as

$$\langle (L_0 + L_1)u_h, \phi' \rangle_N = \langle (L_0 + L_1)u, \phi' \rangle_N \quad \text{for } \phi' \in S_{h,p}^{r'},$$

or

$$\langle L_0 u_h, \phi' \rangle_N = \langle L_0(u - L_0^{-1}L_1(u_h - u)), \phi' \rangle_N \quad \text{for } \phi' \in S_{h,p}^{r'}, \tag{4.2.8}$$

we have from Theorem 2 in [17] for the special case $L = L_0$

$$\begin{aligned}\|u_h - u + L_0^{-1}L_1(u_h - u)\|_t &\leq ch^{s-t}\|u - L_0^{-1}L_1(u_h - u)\|_{s_t} \\ &\leq ch^{s-t}(\|u\|_{s_t} + \|L_0^{-1}L_1(u_h - u)\|_{s_t}),\end{aligned}$$

where $s_t = s + \max(2\alpha - t, 0)$. The boundedness of $L_0^{-1}L_1$ from $H_p^{s_t}$ to $H_p^{s_t+\eta}$ implies

$$\|u_h - u + L_0^{-1}L_1(u_h - u)\|_t \leq ch^{s-t}(\|u\|_{s_t} + \|u_h - u\|_{s_t-\eta}). \tag{4.2.9}$$

On the other hand, since $(I + L_0^{-1}L_1)$ is an isomorphism on H_p^t we have

$$\|u_h - u\|_t \leq c\|(I + L_0^{-1}L_1)(u_h - u)\|_t. \tag{4.2.10}$$

Inequalities (4.2.9) and (4.2.10) now give

$$\|u_h - u\|_t \leq ch^{s-t}(\|u\|_{s_t} + \|u_h - u\|_{s_t-\eta}). \tag{4.2.11}$$

Note that (4.2.11) holds for all s and t satisfying (4.2.6). Also note that $2\alpha + 1/2 < s_t \leq r + b$. Since $\eta > b + 1/2$ and $r > 2\alpha + 1/2$, we can choose η' such that

$$1/2 \leq \eta' \leq \eta \quad \text{and} \quad 2\alpha \leq s_t - \eta' < r - 1/2.$$

Therefore we can write (4.2.11) with t replaced by $s_t - \eta'$ and s by $\bar{s} = \min\{r, s_t\}$ to obtain

$$\|u_h - u\|_{s_t-\eta'} \leq ch^{\bar{s}-s_t+\eta'}(\|u\|_{s^*} + \|u_h - u\|_{s^*-\eta}),$$

where $s^* = \bar{s} + \max(2\alpha - s_t + \eta', 0) = \bar{s} \leq s_t$. Since $\|u_h - u\|_{s^*-\eta} \leq \|u_h - u\|_{s_t-\eta'}$ and $\bar{s} - s_t + \eta' \geq 1/2$, we have, for sufficiently large N,

$$\|u_h - u\|_{s_t-\eta} \leq ch^{1/2}\|u\|_{s_t}. \tag{4.2.12}$$

Inequalities (4.2.11) and (4.2.12) now give the desired estimate (4.2.7). It remains to establish the existence and uniqueness of the solution u_h of (4.2.5). Assume that there are two solutions $u_h^{(1)}$ and $u_h^{(2)}$ of (4.2.5). Then $u_h = u_h^{(1)} - u_h^{(2)}$ is the solution to (4.2.5) with $f = 0$ on the right hand side. Since L is $1 - 1$, we have the exact solution $u = 0$ in that case; therefore we obtain from (4.2.7) $u_h = 0$ for large N. Uniqueness (for large N) for equation (4.2.5) is proved. The existence of u_h for large N then follows because (4.2.5) is a system of N equations in N unknowns. $\square$.

As a consequence, in the energy norm, we obtain by setting $t = \alpha$ and $s = r$ in (4.2.7)

$$\|u_h - u\|_\alpha \leq ch^{r-\alpha}\|u\|_{r+\max(\alpha,0)},$$

which is the same order as in the collocation or Galerkin method (except that an increased regularity of the exact solution is required). The special feature of this Theorem is that if the additional order of convergence b is greater than r then we may also obtain, in a suitable norm, a still higher order of convergence than the Galerkin or collocation method. In fact, in that case by letting $t = 2\alpha - b$ and $s = r$ we obtain

$$\|u_h - u\|_{2\alpha-b} \leq ch^{r+b-2\alpha}\|u\|_{r+b}. \tag{4.2.13}$$

Results on max-norm estimates have been proved for the case in which the trial space is a space of smoothest splines of odd degree, the test space is a space of trigonometric polynomials and $L = L_0$ is an even operator (see [40]). Actually the same argument can be used to prove the following theorem:

Theorem 2.4. *Let the conditions of Theorem 2.3 hold and let $\delta > 0$. If $u \in H_p^{r+\beta}$ with*

$$\beta \geq \max(2\alpha, \delta) + 1/2,$$

then

$$|u_h - u|_0 \leq ch^{\min(r,r+b-2\alpha)}\|u\|_{r+\beta}. \tag{4.2.14}$$

Here we have introduced the notation

$$|v|_0 = \max_{0\leq x\leq 1} |v(x)|. \tag{4.2.15}$$

We also define

$$|v|_s = \sum_{j=0}^{s} |D^j v|_0 \quad \text{for } s = 1, 2, \ldots . \tag{4.2.16}$$

With these notations introduced, we have similarly to Lemmas 3.2 and 3.3 of Chapter III, in the global norms:

Lemma 2.5. [9, Lemmas 5.2 and 5.3] *For any $s \in [0, 2q]$ and $j = 0, \ldots, l$,*

$$|K_h * v - v|_0 \le c h^s |v|_s,$$

$$|D^j (K_h * v)|_s \le c |\partial_h^j v|_s.$$

We will in the next section exploit the highest order of accuracy given by (4.2.13) to further develop the order of the L^2- and max-norm estimates. If $2\alpha - b < 0$, the order will be $O(h^{r+b-2\alpha})$ compared to $O(h^r)$ given by the qualocation method. The case $2\alpha - b \ge 0$ is not interesting in our analysis since for both the L^2- and max-norms the qualocation method itself gives optimal estimates of order $O(h^{r+b-2\alpha})$ (see Theorems 2.3 and 2.4). In this case the averaging method gives the same results.

3. The K-Operator and the Qualocation Method

For the reason given in the comment following Lemma 2.5, we consider only the case $2\alpha - b < 0$.

Theorem 3.1. *Assume that the conditions of Theorem 2.3 hold. Assume also that $b - 2\alpha > 0$. Let $\tau = \lceil b - 2\alpha \rceil$, the least integer greater than or equal to $b - 2\alpha$. Let m, l and q be non-negative integers satisfying*

$$l \geq \tau + m \quad \text{and} \quad 2q \geq r + \tau. \tag{4.3.1}$$

Assume further that L_1 is bounded from H_p^s to $H_p^{s-2\alpha+\eta}$ for all $s \in \mathbb{R}$ and some $\eta > b + 1/2 + \tau + m$. Then

$$\|D^m u - D^m K_h * u_h\|_0 \leq c h^{r+b-2\alpha} \|u\|_R, \tag{4.3.2}$$

where $R = r + b + \tau + m$.

Proof. By the triangle inequality we have

$$\begin{aligned} \|D^m u - D^m K_h * u_h\|_0 &\leq \|D^m u - D^m K_h * u\|_0 + \|D^m K_h * (u - u_h)\|_0 \\ &= I + II. \end{aligned}$$

We will prove separately that I and II are bounded by the right hand side of (4.3.2). Any convolution operator commutes with D, so by Lemma 3.2 of Chapter III we have

$$\begin{aligned} I &= \|D^m u - K_h * D^m u\|_0 \\ &\leq c h^s \|D^m u\|_s \leq c h^s \|u\|_{s+m} \quad \text{for } 0 \leq s \leq 2q. \end{aligned} \tag{4.3.3}$$

To estimate II, we assume first that $L = L_0$, i.e. $L_1 = 0$. Then by Lemmas 3.4 and 3.3 of Chapter III

$$\begin{aligned} II &\leq c \sum_{j=0}^{\tau} \|D^{m+j}(K_h * (u_h - u))\|_{2\alpha - b} \\ &\leq c \sum_{j=0}^{\tau} \|\partial_h^{m+j}(u_h - u)\|_{2\alpha - b}. \end{aligned} \tag{4.3.4}$$

We claim that $\partial_h^{m+j} u_h$ is the qualocation approximant to $\partial_h^{m+j} u$ for $j = 0, \ldots, \tau$, i.e., $\partial_h^{m+j} u_h \in S_{h,p}^r$ and

$$\langle L_0 \partial_h^{m+j}(u_h - u), \phi' \rangle_N = 0 \quad \text{for } \phi' \in S_{h,p}^{r'}. \tag{4.3.5}$$

First we note that $\partial_h^{m+j} u_h \in S_{h,p}^r$ since the mesh is now uniform. The proof then can be carried out in the same way as for (3.4.7) even though in this case we replace the L^2-inner product by the discrete product $\langle \cdot, \cdot \rangle_N$ defined by (4.2.4). We can now use (4.2.13) to estimate $\partial_h^{m+j}(u_h - u)$, and obtain

$$\|\partial_h^{m+j}(u_h - u)\|_{2\alpha-b} \le ch^{r+b-2\alpha} \|\partial_h^{m+j} u\|_{r+b} \le ch^{r+b-2\alpha} \|u\|_{r+b+m+j}. \tag{4.3.6}$$

Inequalities (4.3.4) and (4.3.6) give the required estimate for II and hence the theorem is proved in case $L = L_0$. For the general case, a familiar argument is used. From the equation (4.2.8), we see that u_h is the qualocation approximant to $u - L_0^{-1} L_1(u_h - u)$ in the case $L = L_0$ and hence by the first part of the proof we have

$$\|D^m\big(u - L_0^{-1} L_1(u_h - u)\big) - D^m K_h * u_h\|_0 \le ch^{r+b-2\alpha} \|u - L_0^{-1} L_1(u_h - u)\|_R.$$

By the triangle inequality and the boundedness of $L_0^{-1} L_1 : H_p^{s-\eta} \to H_p^s$ for any real value of s we have

$$\begin{aligned} \|D^m u - D^m K_h * u_h\|_0 &\le \|D^m\big(u - L_0^{-1} L_1(u_h - u)\big) - D^m K_h * u_h\|_0 \\ &\quad + \|D^m L_0^{-1} L_1(u_h - u)\|_0 \\ &\le ch^{r+b-2\alpha} \|u\|_R + ch^{r+b-2\alpha} \|u_h - u\|_{R-\eta} \\ &\quad + \|u_h - u\|_{m-\eta}. \end{aligned} \tag{4.3.7}$$

Since $\eta > b + 1/2 + \tau + m$, it follows that $R - \eta < r - 1/2$; hence Theorem 2.3 gives, for $\epsilon > 0$ sufficiently small,

$$\|u_h - u\|_{R-\eta} \le \|u_h - u\|_{r-1/2-\epsilon} \le ch^{1/2+\epsilon} \|u\|_r \le ch^{1/2} \|u\|_R, \tag{4.3.8}$$

and

$$\|u_h - u\|_{m-\eta} \le \|u_h - u\|_{2\alpha-b} \le ch^{r+b-2\alpha} \|u\|_{r+b} \le ch^{r+b-2\alpha} \|u\|_R. \tag{4.3.9}$$

Inequalities (4.3.7)–(4.3.9) now give the desired result. $\square$

Theorem 3.2. *Let the conditions of Theorem 3.1 hold. For $\delta > 0$,*

$$|D^m u - D^m K_h * u_h|_0 \le ch^{r+b-2\alpha}\|u\|_{R'}, \tag{4.3.10}$$

where $R' = r + \tau + m + \max(b+1, \max(2\alpha, \delta) + 1/2)$.

Proof. By the triangle inequality we have

$$\begin{aligned} |D^m u - D^m K_h * u_h|_0 &\le |D^m u - D^m K_h * u|_0 + |D^m K_h * (u_h - u)|_0 \\ &= I + II. \end{aligned}$$

By Lemma 2.5 we have, as in the proof of Theorem 3.1,

$$I \le ch^s|u|_{s+m} \quad \text{for } 0 \le s \le 2q. \tag{4.3.11}$$

To estimate II we use Bramble & Schatz's trick [9]. Let $k_h(x) = K^1_{h,q}(x)$. Then we have

$$\begin{aligned} II &\le |k_h * D^m K_h * (u_h - u)|_0 \\ &\quad + |k_h * D^m K_h * (u_h - u) - D^m K_h * (u_h - u)|_0 \\ &= III + IV. \end{aligned} \tag{4.3.12}$$

We will prove separately that III and IV are bounded by the right hand side of (4.3.10). Since

$$III \le c\|k_h * D^m K_h * (u_h - u)\|_1 = c\sum_{j=0}^{1} \|D^j k_h * D^m K_h * (u_h - u)\|_0,$$

from Lemmas 3.3 and 3.4 (Chapter III) we infer

$$III \le c\sum_{j=0}^{\tau+1} \|\partial_h^{m+j}(u_h - u)\|_{2\alpha-b}. \tag{4.3.13}$$

Again consider first the case $L = L_0$. By (4.3.13), (4.3.5) and (4.2.13) we have

$$III \le ch^{r+b-2\alpha}\|u\|_{r+b+\tau+1+m} \le ch^{r+b-2\alpha}\|u\|_{R'}. \tag{4.3.14}$$

To estimate IV, again we use Lemma 2.5 to obtain

$$\begin{aligned}
IV &\le ch^{\tau}|D^m K_h * (u_h - u)|_{\tau} \\
&= ch^{\tau}\sum_{j=0}^{\tau}|D^{m+j}K_h * (u_h - u)|_0 \\
&\le ch^{\tau}\sum_{j=0}^{\tau}|\partial_h^{m+j}(u_h - u)|_0. \qquad (4.3.15)
\end{aligned}$$

Using (4.3.5) and (4.2.14) we have

$$\begin{aligned}
|\partial_h^{m+j}(u_h - u)|_0 &\le ch^{r}\|\partial_h^{m+j}u\|_{r+\max(2\alpha,\delta)+1/2} \\
&\le ch^{r}\|u\|_{r+\max(2\alpha,\delta)+m+j+1/2}. \qquad (4.3.16)
\end{aligned}$$

From (4.3.15) and (4.3.16) we infer

$$IV \le ch^{r+\tau}\|u\|_{r+\max(2\alpha,\delta)+m+\tau+1/2} \le ch^{r+\tau}\|u\|_{R'}.$$

Hence the result is proved in case $L_1 = 0$. The case $L_1 \neq 0$ is treated by the familiar argument used in the proof of Theorem 3.1. □

4. Numerical Experiments

In this section we test the averaging method for the qualocation approximation to the equation (3.6.3). Recall that equation (3.6.3) arises as a boundary integral equation to solve the problem (3.6.1). In this experiment, we consider Γ as the ellipse $t_1^2/4 + t_2^2/9 = 1$ and the boundary data $F(t_1, t_2) = \sin(t_1 - 0.1)\cosh(t_2 - 0.2)$. The exact solution of equation (3.6.3) is then

$$\begin{aligned} u(x) =& 3\cos 2\pi x \cos(2\cos 2\pi x - 0.1)\cosh(3\sin 2\pi x - 0.2) \\ &+ 2\sin 2\pi x \sin(2\cos 2\pi x - 0.1)\sinh(3\sin 2\pi x - 0.2). \end{aligned}$$

By Green's theorem we can express the exact potential U, solution of (3.6.1), in the form

$$U(t) = \frac{1}{2\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\log|t-s|\right)F(s)\,dl_s - \int_0^1 \log|t-\gamma(x)|u(x)\,dx,\ t \in \Omega,$$

where dl_s is the element of arc length and $\frac{\partial}{\partial n_s}$ denotes the directional derivative operator in the direction of the outward normal at s.

We solved (3.6.3) using piecewise constant splines as trial and test functions and using the qualocation package written by B. Burn and D. Dowsett (The University of New South Wales). Let U_h be the approximate potential given by

$$U_h(t) = \frac{1}{2\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\log|t-s|\right)F(s)\,dl_s - \int_0^1 \log|t-\gamma(x)|u_h(x)\,dx,\ t \in \Omega. \quad (4.4.1)$$

As proved in [17], if we use the Simpson-type quadrature rule with just two points per interval, one at the break-point where the weight is 3/7 and the other at the mid-point where the weight is 4/7, then the additional order of convergence is $b = 3$, i.e. the highest order achieved is

$$\|u - u_h\|_{-4} \le ch^5\|u\|_4.$$

Therefore we can investigate U inside the boundary Γ by writing

$$\begin{aligned} U(t) - U_h(t) &= -\int_0^1 \log|t-\gamma(x)|(u(x) - u_h(x))\,dx \\ &= \langle u - u_h, G(t-\gamma(\cdot))\rangle \qquad \text{for } t \in \Omega, \end{aligned}$$

(where $G(t) = -\log|t|$) and then using the Cauchy-Schwarz inequality to obtain

$$|U_h(t) - U(t)| \leq \|u_h - u\|_{-4}\|G(t-\gamma(\cdot))\|_4 \leq ch^5\|u\|_4\|G(t-\gamma(\cdot))\|_4 \quad \text{for } t \in \Omega.$$

However, for $t \in \Gamma$ the use of Cauchy-Schwarz inequality is not possible because of the singularity of the logarithmic kernel on the boundary. If we approximate U by U_h^* defined by (4.4.1) with u_h replaced by $K_h * u_h$, where $K_h = K_{h,3}^4$ as given by Theorem 3.1, we can now make use of (4.3.2) (with $m = 0$) to obtain

$$\begin{aligned} |U_h^*(t) - U(t)| &= |\langle K_h * u_h - u, G(t-\gamma(\cdot))\rangle| \leq \|K_h * u_h - u\|_0\|G(t-\gamma(\cdot))\|_0 \\ &\leq ch^5\|u\|_8\|G(t-\gamma(\cdot))\|_0 \quad \text{for } t \in \Omega\cup\Gamma. \end{aligned}$$

Hence the averaging method gives an order of convergence in max-norm in $\bar{\Omega}$ for the approximation of the potential U. However, high smoothness is required for the exact solution u.

The numerical results shown in Table 1 are :

(1) The max-errors and the estimated orders of convergence for the qualocation solution,

(2) The errors and estimated orders of convergence at midpoints for the qualocation solution,

(3) The max-errors and the estimated orders of convergence given by the K-operator.

The results are as expected. Superconvergence at midpoints given by the qualocation method was proved in [40]. Slow asymptotic achievement for the K-operator is due to the requirement that $N \geq 16$ (see page 41).

N	$\|u_h - u\|_0$		$\max \|u_h(x_{i+1/2}) - u(x_{i+1/2})\|$		$\|K_h * u_h - u\|_0$	
16	8.17		0.59E-00		0.92E-00	
32	4.22	0.95	0.24E-00	1.28	4.26E-02	4.43
64	2.08	1.02	6.10E-02	2.00	9.57E-04	5.48
128	1.05	0.99	1.54E-02	1.99	1.97E-05	5.60
256	0.52	1.00	3.85E-03	2.00	4.35E-07	5.50
512	0.26	1.00	9.62E-04	2.00	1.07E-08	5.34

TABLE 1. Errors in the Approximations of the Solution

Approximation of the first derivative. To approximate $u'(x)$, by Theorem 3.2 we take $l = 5$ and $q = 3$. Hence

$$K_h(x) = \frac{1}{h} \sum_{j=-2}^{2} k_j \psi^{(5)}\left(\frac{x}{h} - j\right),$$

where

$$k_0 = \frac{319}{192}, \quad k_1 = k_{-1} = -\frac{107}{288}, \quad k_2 = k_{-2} = \frac{47}{1152}.$$

The numerical results yield the expected $O(h^5)$ convergence (see Table 2).

N	Maximum Errors	Orders of Convergence
16	39.9E-00	
32	2.29E-00	4.12
64	5.70E-02	5.33
128	1.16E-03	5.62
256	2.37E-05	5.61
512	5.38E-07	5.46

TABLE 2. Errors in the Approximation of the Derivative

The above numerical results convince us that the K-operator method works well for this approximation method.

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