An electronic method of integration with respect to variables other than time

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AN ELECTRONIC METHOD OF INTEGRATION WITH RESPECT TO VARIABLES OTHER THAN TIME

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WITH RESPECT TO VARIABLES OTHER THAN TIME

CONTENTS

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1.0 SUMMARY

THE RANGE OF PROBLEMS WHICH CAN BE DONE ON CONVENTIONAL ANALOGUE COMPUTERS IS RESTRICTED BECAUSE THESE MACHINES INTEGRATE WITH RESPECT TO TIME ONLY.

THIS THESIS DESCRIBES THE DESIGN AND CONSTRUCTION OF AN ANALOGUE INTEGRATOR, OF REASONABLE ACCURACY AND BANDWIDTH, WHICH WILL INTEGRATE WITH RESPECT TO ANY VARIABLE FOR WHICH A VOLTAGE ANALOGUE IS AVAILABLE.

The method consists essentially of summing a series of equally spaced ordinates under a curve. This sum is an approximation to the area under the curve and hence to the integral of the ordinate variable with respect to the abscissa variable. Neither the inputs nor the outputs are restricted as to sign; that is, the integrator operates in all four quadrants.

The abscissa variable is divided into small equal increments by storing the difference between its present value and the present sum of all the elapsed increments on a condenser. When the difference between these two values reaches a definite and fixed increment a discriminator closes several switches thereby charging the condenser to the new value of the abscissa minus the sum of the increments. At the same time the present value of the ordinate variable is added to the output of the integrator which accumulates

the ordinates. The whole process is repeated until the end of the problem, the output of the integrator at any instant being an approximation to the area under the curve.

The integrator handles signals within the range of plus and minus 50 volts and is arranged for use with a typical electronic analogue computer which handles problems up to 100 seconds in duration. Its accuracy is better than 1% of maximum output for inputs within the frequency range from zero to 8 cycles per second.

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2.0 SYMBOLS

a	A constant.
A	Stage gain.
	Amplitude of digitizer reset pulse in volts.
b	A constant.
c	A constant.
C	Capacity.
d	A constant.
e	Base of natural logarithms.
	Electronic charge.
	"Virtual earth" voltage.
	Diode voltage in y switch.
E	Amplitude of a unit step of voltage.
	Error in trapezoidal rule of integration.
f	Frequency in cycles per second.
	Denoting a function of a variable.
F	Frequency in cycles per minute.
	Desired output of Bekey's integrator.
g	Denoting a function of a variable.
	Electrical conductance.
h	A small interval of x.
i	Electrical current.
	An integer.
j	An integer.
k	An integer.
	Boltzman constant.
K	A scale factor.
	Amplifier gain.
m	An integer.
M	Dynamic figure of merit of an integrator.
n	An integer.
N	Number of revolutions.
r	Radius of wheel of wheel and disc integrator.
	Electrical resistance.
R	Electrical resistance.
s	Complex variable in Laplace transforms.
t	Time.
T	Absolute temperature.
	Error time constant of an R.C. integrator.
u	A parameter.
U	Input voltage of an R.C. integrator.
V	Output voltage of an R.C. integrator.
w	Frequency in radians per second.
x	An independent variable - usually the argument of integration.
y	An independent variable - usually the integrand.
z	Integrator output.

Δ	Denoting a small increment of a variable.
θ	Shaft rotation.
μ	Amplification factor.
ζ	Damping factor.
ε	Error.
ξ	A value of x within a defined interval.
δ	A difference operator.
τ	A small fixed time delay.

NOTE: Suffixes where used are defined in the text. The various units used are also defined in the relevant sections of the text.

3.0 INTRODUCTION

The mechanical differential analyser, suggested in 1876 by W. Thomson (Lord Kelvin) [1,2] and first constructed by Dr. Vannevar Bush in 1930 [3] was largely replaced by electronic analogue computers from about 1940 onwards. The main reasons for its gradual displacement are its large size, its limited bandwidth or speed of response, the physical difficulties of setting up a problem on it and its greater cost in comparison with the electronic analogue machine.

However electronic analogue computers have not completely superseded the mechanical differential analyser mainly because, until recently [4], no satisfactory electronic analogue technique has been available for integrating with respect to variables other than time. The ability to integrate with respect to any dependent variable which may arise from the solution of a differential equation or set of differential equations greatly enhances the power of an analogue computer. There are many examples of problems where such an ability is either necessary to a solution or simplifies the solution with a resultant decrease in the complexity of equipment required to achieve the solution.

For example:

(1) The solution of problems in polar coordinates where simultaneous integration with respect to radius and angle is required.

(2) The solution of simultaneous differential equations involving two or more different arguments.

(3) Problems involving the generation of the following functions - sin x, cos x, sinh x, cosh x, x^2, e^x, log x, $\sqrt{x}$, where x is not a linear function of time.

The abovementioned deficiency of the electronic analogue computer was first drawn to the author's attention by C.P. Gilbert in 1956 when he gave a series of lectures on Analogue computers to Post Graduate students at the School of Electrical Engineering of the University of New South Wales [11,13,16]. Later when the author was working on diode pump counters [5] in December 1958 it occurred to him that if these counters could be made to operate in an aperiodic manner then it should be possible to construct an electronic integrator operating on the principle described in the summary.

An investigation of the existing methods of integration was then carried out and work commenced on the design and construction of the integrator. When the first model of the integrator was completed in June 1959 Bekey published his paper [4] on another electronic method of integration which is broadly similar to the one under discussion. However Bekey's integrator depends for its accuracy on the accurate measurement of a time interval and its bandwidth is restricted by the method used to measure increments of the abscissa variable.

A description of the earlier analogue methods of integration will be given followed by a discussion of the design and performance of the new integrator. Particular attention will be given to the mechanical integrator because the methods of solving problems with it are identical in principle to those used with the new integrator.

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4.0 EARLIER FORMS OF ANALOGUE INTEGRATORS

The earliest form of Analogue Integrator appears to be the wheel and disk integrator introduced in 1814 [6]. The method of using this integrator for solving differential equations was proposed by Thomson [1,2] in 1876 but the first successful computer using these principles was not built until 1930 at M.I.T. [3] The ball and disk integrator is now used in mechanical computers where the greater accuracy of the wheel and disk integrator is not required. These integrators do not provide a large output torque so the earlier differential analysers used torque amplifiers between integrators. The most common form of torque amplifier being that based on the principle of the capstan [7], giving a torque amplification of 10,000. In later years many forms of electro-mechanical torque amplifiers have appeared [6,12,13] and indeed it is often quite difficult to classify an integrator as mechanical or electro-mechanical when these amplifiers are used. For the purposes of this thesis any integrator which uses the wheel (or ball) and disk principle will be classified as 'mechanical' regardless of the type of torque amplifier used.

Electromechanical integrators which will integrate with respect to time are easily constructed using velocity servomechanisms and these appeared in their present form during the 1940-1950 decade [8].

The common induction disk watthour meter is also an example of an accurate and commercially available integrator. Electromechanical integration with respect to variables other than time is achieved by making use of the relation [6]

$$\int y.dx = \int y.dx/dt.dt \tag{4.1}$$

Here the result is obtained by making use of a differential with respect to time, a multiplication and an integration with respect to time - all of which are readily instrumented using electromechanical components. However the results obtained are of poor accuracy because of the restricted bandwidth and high noise of electromechanical differentiators. Generally it is preferable to use a ball and disk integrator with electromechanical torque convertors where integration with respect to any argument other than time is required in an electromechanical system.

Electronic integration with respect to time is performed using the well known result that the voltage across a condenser is proportional to the time integral of the current flowing in it. Integrals of short duration may be performed using a simple combination of resistors and condensers. Accuracy may be maintained for larger periods if the resistors and condensers are used to provide feedback around a high gain direct coupled amplifier [6,14,15,24]. Integration with respect to any dependent variable may be performed electronically using equation (4.1) above. However differentiation and multiplication usually result in rather large errors in the final result. Bekey [4] has described an integrator based on a simple numerical integration formula and this integrator is the first successful attempt to make an accurate electronic equivalent of the ball and disk or wheel and disk integrator. It will be described in more detail in section 4.3.3.

4.1 The Mechanical Integrator

Fig. (4.1a) shows the principle of the wheel and disk integrator. Since the surface velocity of the wheel is equal to that of a point on the disk at radius y we see that

$$N_z = (1/r)y\ N_x \qquad (4.2)$$

where N_z and N_x are the number of turns made by the shafts z and x when no slip occurs between the wheel and the disk. If we consider small movements of x and z we may write $\Delta z = (1/r)y \Delta x$ and in the limit as $\Delta x \longrightarrow 0$ we have

$$z = 1/r.\int_{x_0}^{x} y.dx \qquad (4.3)$$

In normal use the disk is rotated by the input x and a lead screw is provided to enable the other input to control y, the distance of the wheel from the centre of the disk. Thus the device will perform the integration of y with respect to x. The constant l/r may be regarded as a scale factor K which would, in practice, also include a factor accounting for the pitch of the leadscrew. Thus:

$$z = K\int_{x_0}^{x} y.dx \qquad (4.4)$$

x y and z each being represented in the machine by the numbers of turns made by their respective shafts.

The accuracy of a good wheel and disk integrator, made from a highly polished hardened steel or plate glass disk and a hardened steel wheel with a sharp edge of 0.002" radius, is about 0.02% but the output torque is only of the order of 0.1 ounce inches [6].

Where a more robust but less accurate device is required such as in simple fire control computers for gunnery the ball and disk integrator of figure (4.1.b) may be used. Here the sharp edged wheel is replaced by a roller and two ball bearings held in a cage which may be moved back and forth along a diameter of the disk by means of the y lead screw. Equations (4.3) and (4.4) given for the wheel and disk integrator also apply to the ball and disk integrator. A high quality ball and disk integrator will give an accuracy of 0.1% and an output torque several times that of the wheel and disk integrator [6,9,13].

Both the above forms of integrator are slow in operation. The upper limit of frequencies occuring in the input or output variables is of the order of several cycles per minute. This low speed of operation results from the fact that the surface velocity of the wheel must be limited to prevent slip. Consider for example a problem scaled so that the maximum value of z is represented by 10 turns of the output shaft the maximum speed of the wheel will be 10 w turns per second, where w is the frequency, in radians/second, of a sinusoidal output. Now $10 w = 20 \pi f$ where f is the output frequency in cycles per second. Thus the wheel speed in turns/minute $= 20 \times 60 \pi f = 20 \pi F$ where F is the output frequency in cycles per minute. Thus the wheel would have to revolve at a speed of 600 revolutions per minute to follow a 10 cycle per minute output of maximum amplitude. The danger of errors from slip is readily apparent at wheel speeds much in excess of this. The wheel speed could, of course, be reduced by scaling the problem so that maximum output was represented by only 1 turn. However it is usually only possible to resolve one hundredth of a turn of the output shaft by means of mechanical counters and so the accuracy of the calculation would be limited to 1%. (If it is desired to drive a plotting table from the output a maximum value of z of 100 turns is usually desirable). Thus accuracy can be traded for speed and if an upper limit is set for the wheel speed a figure of merit M may be assigned to an integrator; M being the quotient of maximum output frequency in cycles per second divided by the resolution accuracy of the output in per cent of maximum output. Such a figure of merit is important because it facilitates comparison of the dynamic performance of integrators such as these with Bekey's integrator and the new integrator described in this thesis. Typically a mechanical integrator may handle a frequency of ten cycles per minute with an accuracy of 0.1%. Therefore:

$$M = (1^{o}/60)/0.1 = 1^{o}/6 = 1.6$$

cycles per second per per cent error.

Kuehni and Peterson [12] describe an integrator in which beams of polarized light are used to make a powerful servomotor follow the disk rotation. Thus a negligible torque is required to rotate the disk against only bearing friction and a small inertia. This machine has a glass disk of $8\frac{1}{2}''$ diameter, and a stainless steel wheel of $2''$ diameter, with a rim of $\frac{1}{32}$ inch radius of curvature. The lead screw pitch is 16 threads per inch and the maximum rotation of the lead screw is $\pm$ 65 turns, resolved to $\frac{1}{100}$ of a turn by means of a scale on the shaft. This integrator will operate at disk speeds of up to 1,000 revolutions per minute with less than 0.03% slip. Under these conditions the maximum wheel speed is 4,000 revolutions per minute. For 0.1% accuracy and a resolution of $\frac{1}{100}$ of a turn the maximum value of the output should be represented by 10 turns. Thus maximum output speed is 10 turns/radian of output variable or 62.8 turns/cycle of output variable. Hence maximum frequency of output variable is $\frac{4,000}{62.8} \doteqdot 62$ cycles/minute or approximately 1 cycle per second. Therefore $M = \frac{1}{0.1} = 10$, for this machine.

Appendix I shows the performances of various differential analysers. It is not normal practice to scale the output so that limiting machine accuracy is just resolved. Normally maximum values of the output variable are represented by 100 to 1,000 turns of the output shaft.

It should be kept in mind that speed of response and hence of computation is not of great importance unless many solutions of the one set of equations are required or the machine is to be used for a "real time" simulation of a system or system component. Usually the problem preparation time is very much in excess of that required for solution on the machine.

4.1.1 An outline of the methods of solution of some typical Problems on the mechanical differential analyser.

Figure 4.2 shows the most commonly used symbolic representation of the mechanical integrator [6,9,10,14]. Addition is performed by means of a mechanical differential gear and constant multiplication by means of a gear box. Both these processes are represented by a rectangle containing the appropriate symbols. Torque amplifiers are not shown in what follows but are assumed to be inserted where required. The following treatment is essentially that of Soroka [6], Hartree [10] and Crank [9].

A variety of important and commonly used functions may be generated with one or two integrators as follows:

An exponential may be generated by integrating the equation

$$dy/dx = y \tag{4.5}$$

to obtain

$$y = \int y.dx \tag{4.6}$$

which is readily solved using one integrator as in fig. 4.3. The solution of 4.5 and 4.6 is

$$y = e^x \tag{4.7}$$

and taking logs $\log_e y = x$

or

$$y = \text{antilog}_e\, x \tag{4.8}$$

is another useful result of the same connection.

A variable may be squared using the connection of fig. 4.4.

Two variables may be multiplied together using the formula for integration by parts

$$\int y.dx = yx - \int x.dy \tag{4.9}$$

i.e. $$yx = \int y.dx + \int x.dy \tag{4.10}$$

Fig. 4.5 shows the arrangement used. This is perhaps the best analogue multiplier available - when judged on accuracy and simplicity.

The integral of a product may be produced with the arrangement of figure 4.6 using the relation

$$\int xy.dt = \int x.d\left[\int y.dt\right] \tag{4.11}$$

Note that t is any dependent variable, not necessarily time, and that only two integrators are needed.

Both log x and 1/x can be made available from the same pair of integrators (See fig. 4.7) if the following relations are used

$$\int (1/x^2)dx = -1/x \tag{4.12}$$

& $$\int (1/x)dx = \log x \tag{4.13}$$

hence $$\log x = \int (1/x)dx = -\int\left[\int (1/x^2)dx\right]dx$$

but $$\int (1/x^2)dx = \int (1/x)(dx/x) = \int (1/x)d(\log x)$$

therefore $$\log x = -\int\left[\int (1/x)d(\log x)\right]dx \tag{4.14}$$

Division may be carried out using the above arrangement and two more integrators to multiply z by 1/x to get z/x.

Sine x and Cosine x may be generated by solving

$$d^2y/dx^2 = y \tag{4.15}$$

and sinh x and cosh x by solving

$$d^2y/dx^2 = -y \tag{4.16}$$

See figure 4.8.

A further variety of useful functions may be generated by connecting integrators in a regenerative mode [25]. That is in such a way that an integrator output contributes directly to the rotation of its own disk. However only one example will be given here to illustrate the method. The example chosen is that of a divider shown in fugure 4.9. From the figure we may write:

$$dw = -z.dx \tag{4.17}$$

$$z = w+y+u \tag{4.18}$$

$$du = (1-x).dz \tag{4.19}$$

Equation 4.18 gives

$$du = dz - dw - dy$$

Substituting from 4.19 we have:

$$dy = x\ dz - dw$$

which with 4.17 gives:

$$dy = x\ dz + z\ dx \tag{4.20}$$

the solution of which is

$$y = zx \qquad \text{or} \qquad z = y/x$$

Thus if y and x are inputs the output z is the quotient of y over x, and this has been achieved with only 2 integrators.

This concludes the discussion on function generation and we pass now to the more general use of the mechanical integrator, namely, the solution of differential equations. It was this use of a number of integrators which gave rise to the name 'differential analyser' and the method used was proposed by Lord Kelvin [1,2] in 1876. In its most general form an ordinary linear differential equation may be written

$$f[(d^ny/dx^n),(d^{n-1}y/dx^{n-1}), \ldots \tfrac{dy}{dx}, y, x] = 0 \quad (4.21)$$

where f represents a linear function of the variables within the square brackets.

The principle used for the solution is, basically, to assume that the highest derivative is known. Successive integration will then produce all lower derivatives and finally y. If the highest derivative is then transposed to the R.H.S. of equation 4.2.1 we can sum all the functions of the L.H.S. and make them equal to the highest derivative by carrying out the correct mechanical connections. The process is best illustrated by an example such as the following

$$a(d^3y/dx^3) + b(d^2y/dx^2) + c(dy/dx) + dy + x = 0 \quad (4.22)$$

Isolating the highest derivative we have,

$$d^3y/dx^3 = -\tfrac{b}{a}(d^2y/dx^2) - \tfrac{c}{a}(dy/dx) - \tfrac{d}{a}y - \tfrac{x}{a} \quad (4.23)$$

also

$$d^2y/dx^2 = \int (d^3y/dx^3)dx \quad (4.24)$$

$$dy/dx = \int (d^2y/dx^2)dx \quad (4.25)$$

$$y = \int (dy/dx)dx \quad (4.26)$$

Fig. 4.10 shows how the successive integration may be done and finally the R.H.S. of equation 4.23 made equal to d^3y/dx^3 by means of a gear box. The required solution of $y = g(x)$ is thus produced at the y shaft. Initial conditions are set in before the solution is commenced.

The differential analyser is not limited to the solution of linear differential equations; it can also solve equations such as

$$d^2y/dx^2 + \sin y\,(dy/dx) = x \sin y \qquad (4.27)$$

using principles already described herein. Many forms of non-linear differential equations may be solved when plotting tables are available for use with the integrators.

One final example of the versatility of the differential analyser is shown in figure 4.11 which illustrates the solution of the simultaneous differential equations:

$$d^2y/dx^2 - z(dy/dx) = y \qquad (4.28)$$

$$dz/dy = y^2 + x \qquad (4.29)$$

These are solved in the form

$$d^2y/dx^2 = z(dy/dx) + y$$

$$dz/dy = y^2 + x$$

The solution involves squaring, multiplying and integration with respect to two different arguments.

4.2 Electromechanical Integrators

Figure 4.12 shows an electromechanical integrator which uses a velocity servo known as a Velodyne [8]. G is a tachometer generator whose output voltage is proportional to the output shaft speed. The difference between this output voltage and the voltage picked off the input potentiometer, by the input shaft position, is kept to a very low value by the high gain difference amplifier and the motor. Thus the output shaft speed is held proportional to the input shaft position and the output shaft position is thus the time integral of the input shaft position or

$$\theta_o = k \int \theta_i \,.\, dt \qquad (4.30)$$

k being a constant.

θ_o = output shaft position.

θ_i = input shaft position.

t = real time.

This integrator may also be arranged as in figure 4.13 so that the input and output are voltages.

Electromechanical integration with arguments other than time is almost always done using a variant of the ball and disk integrator with electromechanical torque amplifiers. Methods based on $z = \int y\frac{dx}{dt}.dt$ are unsatisfactory because of jitter and noise arising in electromechanical differentiators. Electromechanical integrators using time as the argument may be used to solve linear differential equations such as equation (4.22) by letting x be represented by time. The process is similar to that for the differential analyser in that the highest order derivative is isolated on the L.H.S. of the equation and the other derivatives are found by successive integration. However these integrators cannot be used to generate most of the functions described under mechanical integrators in section 4.11 nor can they be used to solve equations such as (4.27), (4.28) and (4.29). Their field of application is similar to that of the conventional electronic integrator using capacitive feedback over a high gain amplifier, but they have the advantage of being able to use shaft rotation or voltage as the analogue. Electromechanical integrators generally have an accuracy of 1% of maximum output or better. Precision helical potentiometers used in conjunction with carefully designed servomechanisms will result in accuracies of the order of 0.1% of maximum output but the bandwidth extends only to several cycles per second.

4.3 Electronic Integrators

This section will describe electronic integration with respect to time followed by several electronic methods of integration with respect to variables other than time.

Electronic integrators may be used to solve linear differential equations in the same manner as electromechanical integrators i.e. by letting x be represented by time. In this respect they suffer the same restrictions as electromechanical integrators.

4.3.1 Electronic integration with respect to time:

The simplest method of integration with respect to time is that shown in figure 4.14 where the input voltage is applied to a resistor and condenser in series and the output is taken across the condenser. Assuming that no current is taken by the output device we have

$$i = \sum i_n = \sum \frac{U_n - V}{R_n} \tag{4.31}$$

and

$$V = \int_0^t (1/c).i.dt \tag{4.32}$$

hence

$$V = 1/c.\int_0^t [(U_1 - V)/R_1 + (U_2 - V)/R_2 + \dots (U_n - V)/R_n]dt \tag{4.33}$$

or for one input only

$$V = 1/c.\int_0^t [(U_1 - V)/R_1]dt$$

We see immediately that for accurate results V must be small compared with U_1, otherwise the output departs too much from the desired value

$$V = (1/CR_1)\int_0^t U_1.dt \qquad (4.35)$$

The response to a unit step of voltage at one input is shown in figure 4.14 and solution of (4.34) gives

$$V = E[1 - \exp(-t/CR_1)] \qquad (4.36)$$

instead of the desired output

$$V = (E/RC)t \qquad (4.37)$$

expansion of (4.36) gives

$$V = (Et/RC)(1 - t/2RC + t^2/6R^2C^2 \text{ ------}) \qquad (4.38)$$

The correct output is V = Et/RC and thus we see from equation 4.38 that t must be very much less than RC and the fractional error is t/2RC which is equivalent to an error of 50t/RC per cent. Thus for errors less than 0.1% t must be less than 1/500 of RC.

Integration may be carried on for a far greater time, for a given accuracy, if the arrangement of fig. 4.15 is used. Here a high gain direct coupled amplifier is used to hold the junction of the input resistors and the integrating capacity at a voltage very close to zero. Dr. F.C. Williams [17] has given this junction the name "Virtual Earth" and has demonstrated how this concept can simplify calculation of the properties of "see-saw" feedback amplifiers such as that of fig. 4.15. However for the purposes of error analysis the treatment given by Gilbert [16] has several advantages arising from the way in which it separates the errors due to various causes.

Referring to figure 4.15 we may write

$$i = (U_1 - e)/R_1 + (U_2 - e)/R_2 + \text{---} + (U_n - e)/R_n \qquad (4.39)$$

and

$$(e - V) = (1/C)\int_0^t i.dt \qquad (4.40)$$

Therefore

$$(e - V) = (1/C)\int_0^t [(U_1 - e)/R_1 + \text{---} + (U_n - e)/R_n]dt \qquad (4.41)$$

but

$$e = -(V/K) \qquad (4.42)$$

where K is the amplifier gain.

From equations (4.41) and (4.42) we may show that

$$-V = K/(1 + K)[(1/CR_1)\int_0^t U_1dt + (1/CR_2)\int_0^t U_2dt + \text{--}\tfrac{1}{CR_n}\int_0^t U_ndt]$$

$$+ 1/(1 + K).\ 1/C.\ (1/R_1 + 1/R_2 + \text{---}\ 1/R_n).\int_0^t V\,dt \qquad (4.43)$$

For simplicity we shall omit the limits of integration and consider only one input:

$$-V = K/(1 + K).\ 1/CR_1.\int U_1dt + 1/(1 + K).\ 1/CR_1.\int Vdt \qquad (4.44)$$

If K is large and remains constant the first term becomes the required output $1/CR_1.\int U_1dt$ but the second term represents an error which for a given integrator is a function of the output voltage only. It is equivalent to the integral of the output voltage divided by the "Error time constant" $(1 + K)CR_1 = T$.

Consider the case where the integrator is expected to hold a fixed output for some time. The output will fall exponentially towards zero with a time constant $(1 + K)CR_1$ thus

$$V_o = V \exp\left(-\frac{t}{T}\right) \tag{4.45}$$

For a 0.1% error $\exp(-t/T) = 0.999$

giving $t/T = 0.00095$

i.e. $$t = 0.0095\,(1 + K)CR_1 \tag{4.46}$$

Following the above procedure we can prepare a table showing t/CR for various values of K and the error:

K	10	100	1000	10,000	100,000
t/CR (0.1% error)	0.0105	0.096	0.95	9.5	95
t/CR (1% error)	0.109	1	9.96	99.5	995

and from it we see that for a typical integration time of 100 seconds, time constant 1 second and error 0.1% we require a gain of at least 100,000.

If the input U_1 is a constant we expect the output to be a ramp function and it can be shown that for an 0.1% error $t/CR_1 = (1 + K)/500$

$$\tag{4.47}$$

Thus with a gain of 100,000 and a time constant of 1 second we can integrate for 200 seconds with less than 0.1% error.

In fact holding a constant output is the most stringent test for such an integrator and if the necessary gain is calculated from equation (4.46) it will be sufficient for all other functions for which the integrator will be used.

Considering the first term of equation (4.44) we see that errors may arise if K is not accurately known or if it changes during the problem. Assume that K decreases to $\frac{1}{2}K$ as the worst practical case;

the fractional change in $(1 + K)/K$ is

$$[-(1 + K)/K + (1 + K/2)/(K/2)]/[(1 + K)/K] = 1/(1 + K)$$

Thus for an error of 0.1% or less K must be 1,000 or greater but will normally be much higher than this, for reasons given above.

So far we have considered perfect condensers and a perfect amplifier we now consider errors arising from amplifier and condenser defects:

(a) Condenser leakage:

In practice wound polystyrene-foil condensers are used which have a leakage resistance-capacity product of 10^6 megohm microfarad. For example, a 1 microfarad condenser would have a leakage resistance R_L of 10^6 megohms. R_L may be taken account of by using equation (4.43) and a value of K determined by the ratio of R_L to the parallel resistance Ri of all the input resistors. Thus if $R_i = 1$ megohm and $R_L = 10^6$ megohm $K^1 = 10^6$ which is of no consequence unless the amplifier gain was greater than 10^6 in the first place.

(b) Amplifier drift and zero error:

Figure 4.16 shows how amplifier drift and zero error may be taken into account when determining the output voltage viz:

$$V = -eK + gfK \qquad (4.48)$$

$$i = (U_1 - e)/R_1 \qquad (4.49)$$

$$e - V = 1/C.\int i \, dt \qquad (4.50)$$

$$= 1/CR_1.\int (U_1 - e) \, dt \qquad (4.51)$$

From equation (4.48) we have

$$e = gf - V/K$$

Substitution in 4.50 gives

$$-V = -gfK/(1 + K) + K/CR_1(1 + K).\int U_1 dt$$
$$+ 1/CR_1(1 + K).\int (V - gfk)dt - \qquad (4.52)$$

Since zero error and drift are of the order of 1 mV or less the first term may normally be neglected, the second term is the desired answer and the third term is the "true error". The "true error" can be minimized by setting gfK = 0 during the balancing of the amplifier. The "true error" then becomes the value given by equation (4.46) from which we can set an upper limit to the integrating time for a given accuracy.

(c) Amplifier grid current:

If we refer again to figure (4.16) and in addition to the zero error and drift we have an amplifier which takes a grid current i_g, an analysis similar to that for (b) above gives

$$-V = -gfK/(1 + K) + K/CR_1(1 + K).\int U_1 dt + 1/CR_1(1 + K).\int V dt$$
$$+ K/CR_1(1 + K).\int (R_1 i_g - gf)dt \qquad (4.53)$$

We have previously discussed the negligible first term, the required second term and the error due to the third term. The last term is interesting in that it shows us how to offset the grid current error against the zero error gf. The procedure for a manually balanced amplifier is simply to earth all the input resistances and balance for zero rate of change of output near zero output. However if a chopper stabilized amplifier (28) is used this procedure is not possible and every effort is made to keep the grid current low for such amplifiers.

Electronic integration with respect to time, and the errors arising therein, has been discussed in some detail so that frequent reference may be made to this section during the discussion on integration with respect to arguments other than time.

4.3.2 Methods of electronic integration with respect to arguments other than time, based on differentiation and multiplication:

These methods all use the relation

$$z = \int_{x_1}^{x_2} y \, dx = \int_{t_1}^{t_2} y(dx/dt)dt \qquad (4.54)$$

Bekey [4] states that x(t) must be monotonic so that dx/dt is never multivalued. This appears to be an unnecessary restriction, particularly considering Graves [30] definition of a monotonic function: "A monotonic function is either non-decreasing or non-increasing". This definition excludes x = A sin wt, which Bekey uses as a practical example of the method. Further - Apostal [31] states that: "The only kind of discontinuities which may exist in monotonic functions are jump discontinuities". Jump discontinuities in x(t) would give rise to multiple values of dx/dt.

The integrator would not, in practical use, be presented with x(t)'s of a form which would cause $\frac{dx}{dt}$ to be multivalued. Jump functions and discontinuities in x(t) would simply cause an error similar to that of any device with the same restricted bandwidth as the integrator. A class of functions which it might be thought necessary to exclude are those such as the one shown in Fig. 4.23.

Here $x(t)$ takes on several values at the time t_2. However, these functions cannot occur in practice as it is not possible to return to an instant of time once it has passed. Therefore it is unnecessary to exclude them. Fig. 4.17 shows how equation 4.54 may be instrumented using a differentiator, a multiplier and an integrator. This represents quite a large amount of equipment for one basic operation of a computer and the cumulative errors of the three units are appreciable. Techniques of electronic multiplication (14, 15, 18, 24, 26) are well established and will not be further discussed here. The differentiator is by far the most difficult unit of fig. 4.17 to construct. Differentiators are avoided, where feasible, in analogue machines because of their inherent tendency to instability, their sensitivity to noise and because of these the difficulty of maintaining the required gain at high frequencies without overloading the device. Fig. 4.18 shows two approximate methods of differentiation which have been tried and found unsatisfactory for signals of relatively wide bandwidth but are useful for low frequency signals.

Bekey and Whittier (4) have thoroughly investigated another method of differentiation using the relation

$$dx/dt = \lim_{\tau \to 0} [x(t) - x(t - \tau)]/\tau \qquad (4.55)$$

Figure 4.19 shows how this is carried out and the result used to perform "generalized integration". Appendix II shows how the error in the differentiator may be calculated for a sinusoidal input. Bekey used a 4^{th} order Pade approximation to the function

$$\text{Laplace } [f(t - \tau)] = F(s) \exp. (-\tau S) \qquad (4.56)$$

to construct his time delay [4, 19, 20, 21, 22].

The approximation is:

$$\exp(-\tau s) \doteqdot \frac{(\tau s)^4 - 20\ (\tau s)^3 + 180(\tau s)^2 - 840\tau s + 1680}{(\tau s)^4 + 20\ (\tau s)^3 + 180(\tau s)^2 + 840\tau s + 1680} \qquad (4.57)$$

within a bandwidth of approximately 0 to $1.2/\tau$ cycles per second. Appendix III shows Thomson's method [20] of realizing such a function with analogue components.

The more usual forms of analogue delay units depend for their operation on some incremental memory device whose output changes in descreet steps. These are all unsuitable for this application because of the noise, or jitter, which they would introduce in the differentiated output. One example of such a delay unit is the synchronously switched bank of condensers described in reference 23. Bekey found that with the analogue amplifiers available to him the shortest delay he could achieve was 0.01 second. From Appendix II we see that we need w = 0.02 for a 1% error and hence the maximum frequency useable is only 2 radians per second or 1/3 of a cycle per second. Any generalized integrator based the above method of differentiation will therefore be limited to slow inputs and will exhibit a low signal to noise ratio as the output of the differentiator will contain $\frac{1}{\tau}$ times the noise of the first amplifier.

4.3.3 Bekey's Generalized Integrator [4]

Bekey has overcome most of the defects of the method described in section 4.3.2 by using a simple numerical integration formula

$$F = \int_{x_1}^{x_n} y\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{n} y_i\,\Delta x \qquad (4.58)$$

Reference to fig. 4.21 shows that this corresponds to summing strips of constant width, x, as an approximation to the area under the curve. If the strips are sufficiently narrow then

$$F \doteqdot \Delta x \sum_{i=1}^{n} y_i \tag{4.59}$$

Bekey used standard analog components to mechanize this formula which is a summation of a series of values of y. Fig. 4.22 shows his arrangement. The operation is as follows: The value of x when the previous value of y was summed is held at point A by the sampling circuit store. The difference between this and the current value of x is used to control relay 1 so that the accumulator input is connected to +y or -y according to the sign of Δ x, i.e., whether $(x-x_i)$ is positive or negative. Relay 2 earths the input to the timing circuit when the absolute value of $(x-x_i)$ exceeds the chosen fixed value of Δ x. The timing circuit then controls relay 3 so that the input to the accumulator integrator is switched from earth to either plus or minus y for a definite and constant period of time Δ t. Thus each time relay three operates the output of the accumulator changes by $y_i \Delta t$ and hence

$$e_{o_n} = (\Delta t) \sum_{i=1}^{n} y_i \tag{4.60}$$

but $$F \doteqdot (\Delta x) \sum_{i=1}^{n} y_i \quad \text{(from equation 4.59)}$$

therefore $$F \doteqdot \frac{\Delta x}{\Delta t} . \, e_{o_n} \tag{4.61}$$

Since both Δ x and Δ t are fixed and chosen by design the output e_{o_n} of the integrator is proportional to the desired integral F. It should be noted that the integrator inputs and output can each be of either sign. The relays used for switching had changeover times of the order of 1 millisecond and integration periods of 30 milliseconds were used to ensure accuracy of timing. The time constant of the sampling circuit was chosen to be $1/20^{th}$ of the 30 millisecond sampling period to ensure that the output would exponentially change to the input voltage with a very small error. The sampling interval Δx (in volts) must be chosen so that 30 milliseconds is small compared with the shortest time required for x to change by Δx. Thus suppose we wish to integrate to an accuracy of 1% of peak amplitude with respect to an x which is a sine wave of 100 volt amplitude. To produce 1% accuracy Δx must be no larger than 1 volt, and to ensure that x changes by less than 1% of Δx during the charging time of the sampling circuit charging must be complete in the time required for x to change by $\frac{\Delta x}{100}$, otherwise x information is lost during the charging period. Hence the rate of change of x is limited to 0.01 volts in 30 milliseconds, i.e., $\frac{1}{3}$ volt per second or $\frac{1}{300}$ radian per second of 100 volt peak sine wave. Bekey has published (4) a series of results from a chart recorder and an X - Y plotter. Unfortunately he has not labelled the amplitudes on the chart recordings and the time scale on the plotter records. We can however say, from the appearance of his results for $F = \int C\,dt$, that a change in x of 1 volt requires about 20 Δt, i.e. 20 x 0.03 sec. = 0.6 seconds, corresponding to a rate of change of x of 1.7 volts per second. This is better than the $\frac{1}{3}$ volt per second which was arrived at on the assumption that changing should take place during the time required for x to change by $\frac{\Delta x}{100}$. If the input were of constant slope (i.e. $\frac{dx}{dt}$ = Const.) then errors due to loss of dx information during changing could be scaled out. The x input is rarely of constant slope in any practical problem. It is not possible to deduce the accuracy of integration from the results which Bekey has published, he has however stated that the integrator which he constructed was unduly slow and he suggests the use of fast electronic switches instead of relays. He used an Electronic Associates amplifier whose response was such that the rise time of the amplifier affected the results if a sampling time constant of less than 0.05 milliseconds was used. The sampling time should therefore be not less than 0.5 milliseconds if the output of the sampler is to approach the input with an error of 0.01 percent. He then arbitrarily assumes that the samples in x should be spaced so that the minimum time between samples is twice the sampling time. This will surely lead to large errors as information about the rate of change of x will be lost during sampling time and the time between samples should be at least 5 milliseconds if 1% accuracy is required. Proceeding with Bekey's assumption of 1 millisecond between samples and $\Delta x = 0.1$ volt then $(\frac{dx}{dt})_{max} = \frac{0.1 \text{ volt}}{0.001 \text{ sec.}} = 100$ volts per second which is equivalent to a 1 radian per second 100 volt peak sine wave. Bekey has pointed out that the main restriction on the speed of operation of his integrator is the sampling circuit time constant. He proposes to improve this and suggests that with the four diode gates proposed by Diamantides [27] accuracies of the order of 0.01 percent should be obtained at frequencies below 1 cycler per second. This accuracy presumably applies to the switches only and not to the whole generalized integrator.

Thus Bekey achieved a dynamic figure of merit of $M = 1.5$ and suggested means of achieving a figure of merit $M = 10$.

5.0 THE PRINCIPLE OF THE NEW METHOD OF INTEGRATION

The integrator uses the same numerical integration formula as Bekey's integrator does (see Fig. 4.21)

$$F = \int_{x_1}^{x_2} y dx \doteq \sum_{x = x_1}^{x_2} y_i \cdot \Delta x = \Delta x . \sum_{x = x_1}^{x_2} y_i \qquad (5.1)$$

Figure 5.1 shows how this is mechanized for positive values only of x and y. The integrator consists of two main components, the Digitizer and the Accumulator.

The Digitizer supplies the switching pulses to operate the Accumulator whenever x has changed by a fixed amount Δ x. These pulses cause the output of the accumulator to increase by an amount equal to y each time x increases by Δ x. Fig. 5.2 shows the output z for inputs x = kt and y = const.

The Digitizer consists of a high gain direct coupled amplifier with condensers as feedback components. In the absence of C_3 the output V of this amplifier would be -ax (where a is the ratio C_1/C_2). However as soon as -V reaches a value equal to $-\Delta xC_1/C_2$ the voltage discriminator operates and applies a negative pulse of fixed amplitude A to C_3. During the fall time of this pulse diode D_2 conducts and the voltage V rises by AC_3/C_2 which by choice of C_3 is made equal to $- \Delta xC_1/C_2$. V has thus been returned to zero except for a small change due to changes in x occurring during the fall time of the pulse. The pulse makes no contribution to V during its flat top or during the rise time when D_1 conducts and discharges C_3. This whole process is repeated producing a pulse each time x increases by an amount Δx. Further pulses are produced to operate the Accumulator switches.

The accumulator consists of a high gain direct coupled amplifier with feedback components C_4 & C_5. Each time the digitizer supplies operating pulses to the accumulator switch S_2 connects C_4 to the input of amplifier 2 and shortly afterwards S_1 connects the other end of C_4 to y. This causes the output voltage z to decrease by yC_4/C_5. The switches remain in this position for a sufficient time for C_4 to be completely changed to y. At the end of the discriminator pulse S_2 switches to earth again and is followed by S_1 switching to earth, C_4 then looses its charge rapidly and is ready for the next cycle after x has changed by a further Δx. The amplifier 2 together with C_5 acts as a store and its output voltage will not change unless further current is fed in via S_2. The output is thus a series of steps $(C_4/C_5) \sum_{n=1}^{m} y_n$ where n is independent of t but corresponds to equal intervals Δx of x. The required summation is

$$z_r = \Delta x \sum_{h=0}^{m} y_n \tag{5.2}$$

where $$m = x/\Delta x$$

The actual output is $$z_a = (C_4/C_5) \sum_{n=0}^{m} y \tag{5.3}$$

Now let $$z_r = kz_a \tag{5.4}$$

where k is a scale factor.

Hence $$k = \Delta x.\ C_5/C_4 \tag{5.5}$$

To find the true output from the actual output we therefore must multiply by $\Delta x . C_5/C_4$. Typically $\Delta x = 1$ unit for a maximum value of x and of y 100 units $. C_5/C_4$ is typically chosen as 100 so that Z_A will not exceed the 100 units which is the maximum output obtainable from the integrator.

The above evaluation of the scale factor k takes no account of the error inherent in writing $\int_{x_1}^{x_2} y\, dx$ as $\Delta x \sum_{n=0}^{m} y_n$. This error will be examined later in the thesis.

Figure 5.3 shows how the principle may be extended to allow x, y and z to assume positive and/or negative values.

The Digitizer now has two voltage discriminators together with their reset circuits via C_3 and C_4. When x increases by Δx the positive discriminator operates to generate a reset pulse and pulses to change S_3 over from earth to accumulator grid and then S_2 from y to earth. This results in z changing by $(C_7/C_8)y$. After a short period the trailing edges of the discriminator pulses return S_3 to earth and then S_2 to y. This recharges C_7 without altering z - there being no current fed to the accumulator as S_3 is disconnected from the input grid. When x decreases by Δx the negative voltage discriminator operates S_3 and S_1 in a similar manner to above causing z to change by $-(C_6/C_8)y$. The above operations are independent of the polarity of y and the polarity of the change in z depends on those of x and of y as desired.

6.0 BROAD OBJECTIVES FOR THE DESIGN OF THE INTEGRATOR

The main aim was to produce a generalized integrator, of wide bandwidth, for use with a conventional electronic analogue computer handling problems lasting for up to 100 seconds.

It was intended to keep the size of the unit down to the space ordinarily occupied by three operational amplifiers and to keep the control circuitry essentially simple. Thus switches and discriminators using operational amplifiers [27] were precluded.

Electronic switches were proposed instead of relays in order to achieve greater bandwidth than that of previous generalized integrators [4].

It was decided to design two new operational amplifiers for the integrator as the work was to be carried out remote from the computer laboratory. These amplifiers and in fact the complete integrator were to operate from a high tension supply of $\pm$ 300 volts d.c. this being available both in the computer laboratory and in the place where the work was undertaken.

When the project was started it appeared unlikely that transistor operational amplifiers of sufficiently low drift and input current could be designed without a great deal of development work. It was therefore decided to use valve amplifiers and, because of the difficulty of obtaining high voltage switching transistors, to use valve discriminators also. However a recent paper [32] shows that it is now possible to produce a transistor operational amplifier of only 10 microvolt input circuit drift and an input current of $\pm 4 \times 10^{-9}$

amps, which is adequate for the purpose. Switching transistors withstanding 80 volts between collector and base are now available also. Therefore it seems that a fully transistorized model of the integrator could be made in the near future.

The computer with which the integrator is intended to work has maximum amplitude of problem variable of 60 volts and the integrator was designed to handle amplitudes of x, y and z of $\pm$ 50 volts. An initial consideration of switching speeds showed that 1% of maximum z was a reasonable accuracy to expect from the integrator. This can be closely approached using a Δx of 1/100 of x max; i.e., $\Delta x = \frac{1}{2}$ volt. This value was therefore adopted for Δx.

In general an attempt was made to keep errors in each component of the integrator to less than 0.1% so that the overall 1% accuracy would not be significantly degraded.

7.0 DESIGN AND PERFORMANCE OF THE ACCUMULATOR AMPLIFIER

The design follows current computer amplifier practice. Chopper stabilization of zero and drift was not used because it was known that long term drift could be reduced to less than 1 millivolt by simple compensation techniques. In an accumulator such as this, grid current causes serious errors also. A chopper [28] makes no contribution to reducing grid current unless Buckerfield's [33] parallel tee principle is used; in which case the main amplifier is capacitively coupled to the "virtual earth" of the computing components.

In order that advantage could be taken of the technique outlined in section 4.3.1, for compensation of grid current errors against zero errors, a one megohm resistor is connected from input grid to earth. Assuming that a 1 microfarad condenser will be used for integration periods of 100 seconds, equation 4.46 shows that a gain of 100,000 is necessary to ensure storage errors of less than 0.1%. Use of an 0.1 microfarad condenser would allow integration for a period of 10 seconds for 0.1% error.

7.1 Circuit Design

Figure 7.1 shows the complete circuit. Cathode coupled balanced amplifier stages were used because it was feared that poor regulation of the $\pm$ 300 volt power supplies might cause too much drift if unsymmetrical amplifier stages were used. A further advantage is that heater drifts are also reduced by this arrangement. EF86 low noise pentodes were used because of their high gain and relative freedom from microphony together with reasonably low grid currents.

In order that drifts of subsequent stages will have little effect on the total amplifier drift the first stage should have low drift and a very high gain. For this reason a balanced "starvation amplifier" V_1 and V_2 was chosen for the first stage. Using the curves given by Ferguson (34) the stage was designed to have a gain of 400 from input grid to one anode. Using a 5 megohm load, an anode voltage of 50 volts and an anode current of 50 microamps (to give the necessary drop in the anode load) we find (from Ferguson) that 22 volts is the optimum screen voltage and under these conditions:

$$\mu = 2000$$

$$r_a = 8 \text{ megohms}$$

The stage gain from input grid to one anode is

$$A_1 = \mu R_L/(2r_a + 2R_L) \qquad (7.1)$$

and hence $$A_1 = 2000 \times 5 /(16 + 10) \doteqdot 400$$

The actual measured first stage gain was 440.

This stage is coupled to the second stage by direct connection of its anodes to the second stage grids. This arrangement has the advantages of no gain loss between stages, the avoidance of yet another frequency dependent coupling network, and a high impedance load for the starvation amplifier.

The second stage cannot be run under starvation conditions because of the high voltage swings occuring at its anodes. A conventional cathode coupled pair was therefore used. Choice of an anode current of 0.5 milliamp, screen voltage of 200 and an anode voltage 150 volts leads to a load resistance of 220K, $g_m = 0.7$ milliamps per volt $r_a = 2$ megohms. Therefore the gain from a grid to an anode is $A_2 = g_m R_L/2 = 0.7 \times 220 \times 0.5 = 77$. Actual measured gain was 70.

A 6BM8 valve was chosen for the output cathode follower because the low power triode (enclosed in the same envelope as the high dissipation pentode) could be used as a constant current source for the coupling network between the second stage and the cathode follower. This again avoids a gain loss between stages as the current in the coupling resistor to the cathode follower grid is held constant by the triode. The coupling resistor is bypassed by a condenser, which is large compared with the wiring and valve input capacity at the cathode follower grid, thus ensuring that there is negligible gain loss at high frequencies. The impedance looking into the triode anode is $Z_a = r_a + (1 + \mu) R_k \doteqdot \mu R_k$ when R_k and μ are large (35). R_k is the resistance from cathode to earth. $\mu = 70$ and $R_k = 1.5$ megohm. Therefore $Z_a =$ 105 megohm and the low frequency gain of the coupling network is thus $= 105/(105 + 2.2) \doteqdot 1$ as required.

The cathode follower is arranged to take 10 milliamps from the supply and hence to be capable of supplying ± 8 milliamps to the load. This results in an output impedance of $1/g_m$ = 100 ohms (when no feedback is applied to the amplifier).

The resultant calculated overall gain of the amplifier is 2 x 400 x 77 ≑ 62,000 which compares well with the measured gain of 2 x 440 x 70 = 66,000. Neither are as great as the 100,000 which was the design aim; but a gain of 60,000 will be adequate for the purposes of demonstrating the integrator accuracy and operation. The desired increase in gain could be obtained by altering the configuration of the amplifier but this did not appear to be warranted.

The linearity of the second stage and cathode follower was observed by feeding a sine wave into the input grid of the second stage and noting that crushing of the peaks only became apparent beyond ± 60 volts peak output.

7.2 Stabilization of the amplifier against oscillation

When the amplifier is operating as an accumulator the storage C will act as a short circuit from output to input, at high frequencies. Therefore frequency correction networks must be included in the amplifier to ensure that Nyquist's criterion [39,40] for stability is satisfied.

With S_3 of figure 5.3 open the amplifier is connected as an integrator with a time constant equal to the product of C_8 and the input resistance. This time constant is the only low frequency one occuring within the feedback loop and therefore the low frequency phase shift cannot exceed 90°. The amplifier will thus be stable at low frequencies, but at high frequencies C_8 is of negligible impedance and the stability of the amplifier must be examined carefully.

When S_3 is closed the input feedback component is C_6 or C_7. Since C_6 and C_7 are commonly one hundredth of C_8 the effect of the feedback circuit is to reduce the loop gain by approximately 1% at all frequencies. If the amplifier is stable as an integrator it will therefore be stable under this condition also. In practice S_1, S_2 and S_3 each have some internal resistance, leading to a condition intermediate between an integrator and the pure capacity feedback network described above.

To examine the high frequency stability of the network we may therefore consider C_8 as a short circuit. If the amplifier is stable with a direct connection from input to output it will be stable under all the above conditions.

Several graphical methods of analysis are available for analysing and correcting the stability of linear feedback amplifiers. These are:

(1) Nyquist polar plots of gain and phase [39].

(2) Bode straight line approximations to the variation of gain with frequency [41]. These are used in conjunction with phase change curves.

(3) Evans root locus method.[42]

Nyquist plots are not convenient for high loop gain systems as the gain scale is linear and it is difficult to display the system gain from a large value down to the point where it is unity. Bode plots can be drawn very simply and have no practical limit to their dynamic range. Root locus diagrams are somewhat more difficult to draw but are of great use if the transient response of a system is to be studied.

In the amplifier under consideration the transient response has such a short rise time, compared with that of any pulses fed to it, that a detailed study of transient response is unnecessary. The stability has therefore been studied using Bode's techniques.

Reference to figure 7.2 shows the amplitude response OBC from the input grid to the cathodes of V_3 and V_4. The point B was found by measuring the frequency response and finding the point B where it was 3 db below the low frequency gain. This corresponds to the point where the reactances of the stray capacities on the anodes of V_1 and V_2 (Fig. 7.1) are equal to the 5 megohm anode loads. Similarly the curve A D E represents the straight line approximation to the frequency response of V_4 with its stray capacity. A U V is the frequency response of the cathode follower with 300 micromicro-farads stray capacity on the cathode. The above represent all the significant frequency sensitive networks in the open loop transfer function. The net open loop gain curve is then A B T G H. This crosses the unity gain line at a slope of 12 db per octave and will therefore have a phase margin of almost zero. In order to increase the phase margin at the unity gain point the anode load of V_4 was split into two parts, 3.3k ohms and 220 k ohms. The junction of these two resistors was then connected to earth via a 1500 micromicrofarad condenser. The anode circuit of V_4 then has the frequency response A J K L E resulting from a transfer function of the form

$$A(w) = \frac{(R + R_1)(1 + SC \frac{RR_1}{R + R_1})}{1 + s(CR + C_sR_1 + C_sR) + S^2(CRC_sR_1)} \qquad (7.2)$$

where the values of R R_1 and C are shown on figure 7.1. C_s is the stray capacity at the anode of V_4. The denominator factorizes to give two real poles when the practical values are substituted.

The open loop frequency response then becomes the sum of A U V, A B C and J K L E. This results in sufficient phase margin at the unity gain point but the phase margin at approximately 6 kilocycles per second is only thirty degrees. This latter margin is increased by networks in the anodes of V_1 and V_2 similar to that in V_4. The resulting frequency response of the V_1, V_2 anode circuits is A M N F C. Addition of A U V, A M N F C and A J K L E results in a straight line approximation to the open loop frequency response of the complete amplifier. In Fig. 7.2 this has had the necessary small corrections added to give the true open loop frequency response and the open loop phase lag has also been calculated according to Bode's method. The phase margin at the unity gain point is 38°. This gives adequate stability with some allowance for variations in components and amplifier gain. There is only one other point at which a small phase margin results while the amplifier gain is still greater than unity and this is at 4500 cycles per second, where the phase margin is an adequate 42°.

If the output of the accumulator was driving the x input of another integrator it would have condenser (possibly as large as 0.1 microfarad) connected from the cathode of V_5 to earth. Let

C = capacity of the condenser

r = output impedance of the cathode follower (without amplifier feedback).

The transfer function of the cathode follower will then be

$$A(w) = \frac{1}{1 + SCR} \qquad (7.3)$$

With r = 100 ohms and C = 0.1 microfarad this will cause the point U (Fig. 7.2) to move to 16 kilocycles per second and hence make the amplifier unstable.

This difficulty may be overcome by placing a 100 ohm resistor R in series with C external to the accumulator. The transfer function of the cathode follower will then be

$$A(w) = \frac{(1 + SCR)}{1 + SC(R + r)} \cdot \frac{1}{1 + SC_s rR/(r + R)} \qquad (7.4)$$

This will result in an amplitude and phase response as shown in figure 7.3. This response when added to the overall response of figure 7.2 will increase the phase margin at the unity gain point and cause the phase margin at 16 kilocycles per second to reduce to 38°. The amplifier performance will then be satisfactory. The error introduced in the following integrator by the 100 ohm resistor will be 0.05% at maximum dx/dt.

The bandwidth of the amplifier under closed loop conditions is seen from Fig. 7.2 to be approximately 1 megacycle per second. The response to a unit step of voltage input will therefore have a rise time of approximately $t_r = \frac{0.35}{f_n}$ where f_n is the upper half power frequency [14]. Thus the amplifier rise time will be approximately 0.35 microseconds. In operation the fastest rise time of any input pulse will probably be several microseconds and the amplifier will have from 20 to 100 microseconds to settle to the final value. With a 38° phase margin there will be some overshoot in the closed loop response to a unit step input. However, rise time of several microseconds will probably not give rise to an oscillatory response and if it does there is ample time for it to settle to the final steady value.

7.3 Reduction of zero error and drift

Many compensation techniques have been proposed for the reduction of amplifier drift due to heater temperature changes. Most workers [14,15,36] take the first step of using cathode coupled pairs of valves so that the heater variations of two ostensibly identical valves will tend to cancel one another.[37] The well known

Miller compensation circuit [36] is usually resorted to for further reduction of drift. Martin [38] claims balancing factors greater than 100 for a method which takes advantage of the fact that a valve is about twice as sensitive to heater current changes as to heater voltage changes. This method was used in the amplifier (see Fig. 7.1) because of its high balancing factor and the fact that it does not require any valves to be used solely for the purpose of heater compensation, as does the Miller circuit.

Reference to figure 7.1 shows Martin's circuit applied to the heaters of V_1 and V_2. The heaters are fed from a source voltage of $3V_h$ through individual resistances each of $2R_h$. The live ends of the heaters are connected together through a 100 Ω potentiometer whose slider is connected to a source of the normal heater voltage V_h. The potentiometer adjustment is facilitated by an 8% tap changing switch on the primary of the heater transformer. The potentiometer is adjusted for minimum amplifier drift when the tap changer is switched from one position to another. Several successive adjustments are needed to obtain the optimum setting. It is necessary to allow a period of approximately 2 minutes between each adjustment because of thermal lags in the heaters of the valves. This adjustment was found to be necessary only once a week unless a valve was changed. As the valves aged the adjustment was required even less frequently. The balance control on the grid of V_2 was used to take care of all short term drifts.

Amplifier drift was measured by connecting a one megohm resistor from input to output and a one thousand ohm resistor from input to earth. It can be shown (15) that the output voltage is then 1000 times the equivalent input drift or zero error. A ten volt F.S.D. voltmeter was connected from output to earth to measure drift.

An 8% change in heater volts may be expected to produce an equivalent change in grid to cathode volts of 80 millivolts for a single valve. In the starvation circuit used 8% change in heater volts caused an equivalent input drift of 30 millivolts. The heater compensation circuit enabled this to be reduced 0.5 of a millivolt for 8% change in heater volts. Further measurements showed that it was possible to operate the heaters from an unregulated heater supply. This resulted in drifts of 0.25 millivolt for periods of up to two minutes and 1 millivolt during any 10 minute period. This was considered quite satisfactory for the accumulator amplifier as the one megohm input resistor would undergo a current change of only 10^{-9} amps for a 1 millivolt drift. In a 1 microfarad accumulating condenser this would result in an error of 1 millivolt per second.

A change of 10% in the +300V supply was found to cause an equivalent input drift of 5 millivolts. A change of 10% in the negative supply caused a drift of 4 millivolts. Both these supplies are regulated to better than 1% so drifts resulting from their variations will not seriously affect the accumulator.

7.4 Grid Current

The input grid current of the amplifier was measured by removing the 1 megohm input resistor and connecting an 0.01 microfarad polystyrene condenser from input grid to amplifier output. It was noted that the rate of change of the output voltage was unaffected by rotation of the balance control and was a linear function of time. Thus the amplifier input resistance and the condenser leakage resistance were shown to be sufficiently high to be neglected.

The rate of change of output was 0.5 volt per second from which the grid current, being the only current flowing in the condenser, was calculated as 0.5×10^{-8} amps. Even if this were not compensated against zero error, as outlined in section 7.0, it would cause errors of only 5 millivolts per second in a 1 microfarad accumulating condenser. At this stage the resistors for the starvation amplifier had already been chosen to give maximum gain and so electrode voltages could not easily be varied to reduce grid current. Similarly reduction of heater volts would reduce gain and upset the heater compensation circuit. Accordingly, since the resultant error was acceptable no further attempt was made to reduce grid current.

The technique of minimizing grid current by variation of anode voltage is described later, in connection with the amplifier for the Digitizer.

8.0 THE y SWITCHES S_1 AND S_2.

These switches are changeover switches which should have a low "on" resistance; preferably less than 1000 ohms. The resistance to the "off" terminal should be high compared with the "on" resistance, but need not be unduly high because it is driven by a low impedance source through the "on" resistance. This allows the use of semiconductor switches because the higher back resistance of thermionic switches is not needed.

Bidirectional operation is necessary and hence the simple single valve (or semiconductor) gates are excluded. An example of such a gate is the single suppressor gated pentode valve.

Millman and Taub [35] have described several double triode gates (which could equally well be double transistor) which all superimpose the output on a pedestal. Means are available for pedestal compensation but they are dependent on the similarity of valve or transistor characteristics. These gates have a transmission gain which is highly dependent on valve or transistor characteristics.

Many authors [35,43,44,45] have described 2, 4 and 6 diode gates taking the form of a bridge circuit. Accuracies [35] of better than 0.1% have been achieved with six diode gates using thermionic diodes. Another class of accurate and stable 4 diode gates has been described by Diamantides [27]. These gates make use of a high gain operational amplifier to produce large accurate gating pulses of equal amplitudes, balanced about earth. The diodes are connected in a bridge circuit which is on when all diodes are conducting. Accuracies of 0.01% are possible using these gates.

All the gates described in the previous paragraph are single pole, single throw gates and two of them would be needed for S_1 or S_2 (Fig. 5.3). To give 0.1% accuracy 6 diode gates would be required, i.e. 12 diodes per switch.

Fortunately a simpler 6 diode switch has been described by McKay [46]. This gate is a single pole double throw bidirectional switch and is shown in Figure 8.1. McKay used switching pulses less than 15 microseconds in width so this switch was thought to be amply fast for the 100 microsecond switching period decided on for the integrator. As used by McKay D_5 and D_6 were absent and the antiphase switching pulses were applied at the points marked V. Accuracies of an order better than 1% are indicated in McKay's paper. In view of their simplicity, accuracy and speed these switches were chosen for S_1 and S_2.

D_5 and D_6 were added to make certain that the switch accuracy would not be lessened by variations in the switching pulse amplitudes. V and V_1 are fixed positive and negative voltages respectively. They are large compared with the signal voltage y. The switching pulses have an amplitude greater than twice the maximum value of y. The switch operates as follows:

When the switching pulse at D_5 is at +60 volts D_5 is cut off and the antiphase pulse at D_6 causes D_6 to conduct and take its junction with D_3 and D_4 to -60 volts thus cutting them off. R_1 is made equal to 2R and so D_1 and D_2 conduct with approximately equal currents flowing in them. Since the load at e_o is a condenser no current is taken by the load after the initial charging current. Hence e_o is closely = y.

When the switching pulse at D_5 is -60 volts and that at D_6 is +60 volts D_1 and D_2 are cut off and D_3 and D_4 are conducting thus connecting e_o to earth. Thus the switch operates as a single pole single throw device controlled by ± 60 volt pulses.

To avoid using too much power in the control pulses and also to avoid dissipating too much power in the switch the diodes D_1, D_2, D_3, D_4, were each arranged to draw 1 milliamp in the "on" position. In order that the currents in the "on" diodes should remain as nearly equal as possible despite variations in the input y, R and R_1 and hence V and V_1 should be as large as possible. In this case V and V_1 were 300 volts and R = 150k ohms and R_1 = 330k ohms. Earlier designs of the switch used ± 60 volt pulses supplied in antiphase at points 1 and 2. This resulted in excessive variations of transmission gain with y and with switch pulse amplitude. The improved accuracy obtained was well worth the expense of two more diodes and slightly more power dissipation in the switch.

OA85 germanium diodes were chosen because of their stability, high inverse voltage (115 volts max. at 25°C.) and their low forward resistance. Silicon diodes similar to the OA202 could have been used with less variation with temperature but they have a much higher forward resistance and are more costly. High back resistance is not essential and therefore their only advantage over germanium diodes would be stability with temperature changes.

8.1 Transmission Gain and Errors in the Diode Switches S_1 and S_2.

An endeavour was made to represent the diode current-voltage curves mathematically. Shea [47] has shown that

$$I = I_o(\exp eV/kT - 1) \qquad (8.1)$$

where I = diode current.

I_o = Saturation current (inverse)

e = the electronic charge

V = voltage across the junction

k = Boltzmann constant

T = absolute temperature of the junction

For germanium at room temperature $e/kT = 39$ (8.2)

Equation 8.1 has been deduced theoretically and should represent the diode current in both the forward and reverse directions. The series resistance of the germanium (base resistance) has however been neglected and at forward voltages greater than about 0.1 volts the resistance of the barrier layer decreases so rapidly that the base resistance eventually becomes the controlling factor for the forward current. A correction must be made to equation 8.1 by adding the IR drop in the base resistance to the theoretical voltage curve [48].

For the OA85 diodes under consideration a current of 1 milliamp results in a voltage of 0.3 volts at 25°C. The resistance to small variations in current is then almost entirely base resistance and may be evaluated from the slope of the I-V curve as 200 ohms. The variation of forward current with ± 50 volt variations in y is only of the order of ± 1/6 of a milliamp so the above incremental resistance can be used to represent the diode during changes in y. The foregoing takes no account of the static voltage drops across the diodes D_1 and D_2 when $y = o$. It was intended to make these equal by matching the diodes by selection. However the narrow spread in production tolerances for the OA85 made this quite unnecessary in practice.

Figure 8.2 shows the first half of the switch in the "on" position. The reverse currents of D_5 and D_3 have been neglected.

Let r_1 = forward resistance of D_1

r_2 = " " " D_2

R and $R_1 \gg r_1$ or r_2

Therefore

$$i_1 = (V - y)/R \tag{8.3}$$

$$i_3 = (V_1 + y)/R_1 \tag{8.4}$$

$$i_2 = i_1 - i_3 \tag{8.5}$$

$$e = y + i_2 r_1 - i_3 r_2 \tag{8.6}$$

$$\therefore e = y + r_1 \; (V - y)/R - (V_1 + y)/R_1 \; - r_2(V_1 + y)/R_1 \tag{8.7}$$

If $r_1 = r_2$, $V = V_1$ and $R_1 = 2R$

then
$$e = y - 4r_1 y/R_1 \tag{8.8}$$

The error being $- 4r_1y/R_1$

In this case $r_1 = 200$, $y_{max} = 50$, $R_1 = 300K$

Max Error = 133 millivolts.

As a check on the validity of the above method the error was calculated from the published curves for the OA85. The table below shows the results of the calculation for R = 150k, V = V_1 = 300 volts and for two values of R_1.

R_1 (ohms)	y (volts)	i_1 (mA)	i_3 (mA)	i_2 (mA)	e_1 (volts)	e_2 (Volts)	Error (milli-volts)
330K	+50	1.67	1.06	0.61	0.25	0.32	70
"	-50	2.00	0.91	1.09	0.33	0.30	30
"	+50	2.33	0.76	1.57	0.40	0.29	110
300K	+50	1.67	1.17	0.50	0.25	0.35	100
"	0	2.00	1.00	1.00	0.31	0.31	0
"	-50	2.33	0.83	1.50	0.40	0.30	100

The errors for the R_1 = 300K case are reasonably well related to the theoretical 133 millivolts.

The errors for S_1 and S_2 were measured by connecting a high impedance millivoltmeter from y to e and vary in y in the range ± 50 volts. D_5 and D_3 were biased to +60 volts and -60 volts respectively so that their small reverse currents were also taken into account in the measurement.

Fig. 8.3 shows the results of these measurements for S_1 and S_2. R_1 is 330K ohms in each case. This being the nearest preferred value resistance to 300k ohms. The ambient temperature of the switch was 100°F. during the measurement. The diodes were chosen at random and since the measured maximum errors were only 80 millivolts selection of diodes was not considered necessary.

The maximum errors in the y switches are approximately 0.2% which, while not meeting the design aim of 0.1%, will be adequate for demonstration of the 1% generalized integrator. Fig. 8.3 also shows that D_3 and D_4 when conducting may introduce an error of up to 10 millivolts between e and earth. This 0.02% error is negligible.

The effects of variations in V, V_1, R and R_1 on the accuracy of the switch may be evaluated as follows:

Partial differentiation of equation 8.7 with respect to R leads to

$$\triangle e = - r_1(V - y)/R \;\triangle R/R \qquad (8.9)$$

If R changes by 10% $\qquad \triangle R/R = 0.1$

$$R = 150\text{k ohms}$$

$$r_1 = 200 \text{ ohms}$$

$$V = 300 \text{ volts}$$

$$\therefore \quad \triangle e = 40 - 0.13y \text{ millivolts}$$

which is 46.5 millivolts when y = -50 volts.

Therefore 10% change in R causes only 0.1% error in the switch.

Similarly for variations of 10% in R_1 we have.

$$\Delta e = \left[r_1(V_1 + y)/R_1 + r_2(V_1 + y)/R_1\right]\Delta R_1/R_1 \qquad (8.10)$$

$V_1 = 300$ volts, $R_1 = 300$k ohms (330k ohms in practice),

$r_2 = 200$ ohms.

$$\Delta R_1/R_1 = 0.1$$

$\therefore \quad \Delta e = 40 + 0.13y$ millivolts

$\therefore \quad$ 10% change in R_1 also causes only 0.1% error.

Partial differentiation with respect to V gives.

$$\Delta e = r_1/R.\ \Delta V = (Vr_1/R)(\Delta V/V) \qquad (8.11)$$

$\Delta V/V = 0.1$

$\therefore \quad \Delta e = 40$ millivolts and is independent of y.

Therefore 10% change in V also causes only 0.1% error in the switch. Similar considerations show that 10% change in V_1 also causes only 0.1% error.

The use of high stability resistors and regulated powers supplies make errors due to the above variations negligible.

Errors arising from ambient temperature variations should be quite small because changes in R will be compensated by changes in R_1 and changes in D_1 will be compensated by like changes in D_2. The magnitude of the changes in D_1 and D_2 may be estimated from the fact that the current through a germanium diode increases by 10% per degree centigrade change in ambient temperature [47].

A slightly higher accuracy could be achieved if R were adjusted, by means of a small potentiometer in series with it, until e = o when y = o. This was not necessary in this application.

8.2 Transient Response of the y Switches S_1 and S_2.

The transient performance of S_1 and S_2 may be analysed by considering their operation with a condenser connected from e to earth (fig. 8.1). During operation the switches will be on for 100 microseconds. This time has been chosen as a reasonable pulse width at which to operate the dx voltage discriminators which are simple cathode coupled monostable multivibrators. A 1000 micromicrofarad condenser can be charged to 50 volts with the 1 milliamp available from the y switch in 50 microseconds. This leaves a very wide safety margin to ensure that charging is complete.

The charging times may be calculated as follows:

(1) For y = +50 volts and e initially zero.

During the charging period D_3 and D_4 are cut off. D_1 is also cut off until the end of the charging period. Therefore C will charge exponentially towards a final voltage equal to that which it would reach if D_1 were absent, with a time constant of

$$C \frac{R_1 R}{R_1 + R} \equiv 103 \text{ microseconds.}$$

Therefore $$e = 112(1 - \exp - t/103) \qquad (8.12)$$

where t = time in microseconds.

Therefore $e = 50$ volts when $t = 61$ microseconds.

At 50 volts it will be caught and held by D_1.

When C has been charged to +50 volts it is then returned, after about 40 microseconds delay, to zero by conduction through R_1 until caught by D_3.

$$e = 50 - 350(1 - \exp - t/330) \qquad (8.13)$$

Therefore $\quad e = o \quad$ when $\quad t = 52$ microseconds.

When $y = -50$ volts and e is initially zero.

$$e = -300\ (1 - \exp - t/330) \qquad (8.14)$$

Therefore $\quad e = -50 \quad$ when $\quad t = 60$ microseconds.

When C has been charged to -50 volts it is then returned, after about 40 microseconds delay, to zero by conduction through R_1, D_3 and R.

$$e = -50 + 162(1 - \exp - t/103) \qquad (8.15)$$

Therefore $\quad e = o \quad$ when $\quad t = 37$ microseconds.

Plates 4, 5, 16 and 17 show the waveforms at the outputs of the y switches when $y = +50$ volts and -50 volts. In this case x was a linear function of time and so the switching PRF was constant.

By suitable rearrangement the times calculated for S_1 may be applied to S_2 and both are seen to be well correlated with the photographed waveforms.

9.0 THE ACCUMULATOR GRID SWITCH S_3.

Again the requirement is for an accurate changeover switch. The switches described in section 8 are not suitable because they have residual errors of about 10 millivolts when the output is ostensibly at earth. Such a voltage at the accumulator grid would overload it and cause serious errors. The same remarks apply to the switches described in reference 43. It was therefore necessary to devise a switch which would not apply a voltage, or feed a current into the accumulator grid during the time when the junction of C_6 and C_7 was earthed. On the other hand all the charge on C_6 or C_7 should flow through the switch when the junction of C_6 and C_7 is connected to the accumulator grid. Stating this in another way we may say that because of the low level of signal at this point a current switch rather than a voltage switch is needed.

Fig. 9.1 shows the switch finally selected. It was evolved by starting with the bridge DFCG with the switching pulses fed through D_1 and D_4. When the voltage at A is +2 volts and that at B is minus 2 volts D_1 and D_4 conduct and current flows through R_1 and R_2 raising point F to about +1.5 volts and lowering point G to about -1.5 volts. D_2 and D_3 are thus cut off and there is no connection from D to C, D being held approximately at earth. When A is switched to -2 volts and B to +2 volts D_1 and D_4 are cut off and current will flow in from D through R_2 D_3 to C, or out from C through R_1 D_2 to C according to the state of C_6 and C_7.

S_1 and S_2 are not able to supply more than 1 milliamp each to the other ends of the condensers and they are never switched simultaneously. Therefore the voltages of D and G do not exceed a magnitude of 1 volt and the suggested 2 volt switching pulses are adequate to keep D_1 and D_2 cut off. In order to ensure that D is accurately connected to earth when A is positive and B is negative R_3, R_4 and R_5 are connected from A to B and the centre tap of R_4 is earthed. The switch pulses to A and B are supplied from high impedance sources so that adjustment of R_5 centre tap will ensure that D is accurately switched to earth.

9.1 Errors in S_3

The sources of error in S_3 are:

(1) Innaccurate earthing of D in the quiescent position.

(2) Reverse current through D_3 and D_2 during the quiescent period.

(3) Reverse current through D_1 and D_4 during the "accumulate" period; i.e. when D is connected to C.

(4) Capacitive feed through across D_3 and D_2 to the accumulator.

There is no residual voltage error from D to C at the end of the "accumulate" period as all the charge has been conducted off C_6 and C_7 and there is no current flow through R_2 D_3 or R_1 D_2. The voltage drop across semiconductor diodes at zero current is zero and so the voltage between D and C is zero at the end of an accumulate period.

Error No. 1 may be corrected by adjustment of R_5.

Errors No. 2 and 3 are made low by the use of OA202 silicon diodes which have very low reverse currents. Approximately equal reverse voltages are applied to D_1 and D_4 and to D_2 and D_3 and hence the reverse currents of an OA202 at -2 volts is of the order of 10^{-9} amps (estimated from the published figure of 50×10^{-9} for an inverse voltage of 150 volts). The net current into the accumulator as a result of unbalance between pairs of diodes may be expected to be, say, 0.3×10^{-9} amps. This is an order smaller than the grid current of the accumulator amplifier and will therefore cause only very small errors.

Error No. 4 is made suitably small by using switch pulses of approximately equal amplitude and making them small, i.e. ± 2 volts.

It is difficult to measure the above errors by direct measurements on the switch and so tests were done with the complete integrator operating to prove that these errors were negligible. These tests are reported under the section on performance of the complete integrator.

10.0 THE POSITIVE dx DISCRIMINATOR

In the interest of simplicity monostable multivibrators [35] were chosen for the dx discriminators. They give a pulse of fixed width once the triggering voltage is reached by the input - otherwise they remain quiescent.

Fig. 10.1 shows the positive dx discriminator. The main source of error in discrimination level is drift in contact potential due to cathode temperature changes. This is reduced by using cathode coupling of the two valves. Errors from this source should not exceed 30 millivolts and a signal of 25 volts is available at the input. The error is thus of the order of 0.1%. Errors arising from supply voltage and component variations will be smaller than this because high stability resistors and closely regulated power supplies are used. The errors due to heater voltage changes and H.T. changes were measured and the results of these measurements are set out under the section on overall performance of the integrator.

100 microsecond pulse width was chosen because experience with this type of multivibrator indicates that 60 volt pulses with rise times of the order of several microseconds are possible. The choice of this pulse width was made with some care as this width is the main limiting factor to the integrator bandwidth. Its effect on bandwidth is examined in a later section.

The y switches S and S require $\pm$ 60 volt pulses for their operation. The width of these pulses is constant (100 microseconds) but their PRF is not constant, being dependent on dx/dt. Therefore the pulses must be direct coupled from the anodes of V_1, fig. 10.1. This figure shows the circuit used for the + ve dx discriminator.

In the early stages of design attempts were made to ensure that the switch pulses operated S_3 before S_1 and S_2 so that only the leading edge of the y pulse was fed to the accumulator (fig. 5.3). The use of lumped parameter delay lines in between the discriminator anodes and the y switches seems an obvious solution. However, the characteristic impedance of these would have to be quite high if they were not to need excessive driving currents. This would immediately lead to practical difficulties in achieving good pulse shape. Delays of 2 - 3 microseconds would be necessary and these would require a number of sections in each delay line with the attendant disadvantages of large physical size and cost.

The solution finally adopted was to provide the input grid switch S_3 with its own multivibrator and trigger this with a delayed pulse from the discriminator anodes. The accumulator thus has its charge altered by the trailing edges of the y pulses and the leading edges leave it unchanged.

The choice of valve type for the discriminators was governed by the following factors:

(1) $\pm$ 60 volt pulses are required at the anodes

(2) The anode loads should have a resistance not greater than 15k ohms to ensure fast rise and

fall times.

(3) The resultant cathode current will be about 12 milliamps when y switch currents are taken into account.

(4) It is desirable to keep the discriminator input switching level as near to earth as possible so that little gain will be lost in potential dividers from the output of the digitizer amplifier.

(5) The heater to cathode voltage rating should be quite high to avoid the provision of a separate heater winding for the discriminators.

The conducting valve will thus have about 12 milliamps flowing in it and its anode will be at -60 volts. Its cathode will therefore need to be at about 150 volts below earth if positive grid operation is to be avoided.

The 12BH7 double triode was chosen as it fulfils these requirements with the following ratings:

(1) Max. Heater Cathode volts -200.

(2) Anode current = 20 milliamps when anode voltage = 90 and grid bias is zero.

(3) Maximum anode wattage = 3.5.

The starting point for the design was the choice of a triggering voltage of approximately -150 volts and a cathode current of 12.5 milliamps. The +dx discriminator must be triggered by a negative going voltage as an increasing x will give a decreasing output from the Digitizer amplifier. The input valve must therefore be normally conducting and so that its anode voltage will not follow variations in x, except those near the trigger point, the diodes D_1 and D_2 are included. These diodes are arranged so that when the input is above -150 volts, D_2 is cut off and D_1 conducts and holds the input grid at -150 volts. When the input drops below -150 volts D_2 conducts and D_1 is cut off. The input grid then follows the input until slightly below -150 volts where the triggering point is reached. When triggering occurs the anode of the second valve falls to -60 volts and C takes the input grid down by approximately 100 volts and cutting off D_1, D_2 and V_{6A}. V_{6B} is then conducting with the cathode at approximately -165 volts. C_1 then charges through R_1 until the conduction point of V_{1A} is reached and the multivibrator returns to its quiescent position. The capacity of C_1 has been chosen so that the pulse width is 100 microseconds [35].

C_2 and the diodes D_5 and D_6 correspond to C_3, D_1 and D_2 of fig. 5.3 and their purpose is to reset the Digitizer output to zero. D_6 is a silicon diode type OA202 chosen for its high reverse resistance to prevent significant current flow to the digitizer amplifier while D_5 is conducting. Originally C_2 was connected directly to the anode of V_{6B} (fig. 10.1). Under these conditions the discriminator will not trigger when x is varying slowly. When the input to V_{6A} (fig. 10.1) falls below -150 volts it reaches a point where both V_{6A} and V_{6B} are both conducting but their loop gain has not yet reached unity. The valve V_6 and the Digitizer amplifier are then in a loop closed by C_2 (fig. 10.1). This closed loop tends to hold the output of the digitizer at a constant value and so temporarily stops the integrator from working. This defect has been overcome by inserting R_4, D_3 and D_4 between C_2 and the anode of V_{6B}. D_3 will not conduct until triggering has occurred and the anode of V_6 has fallen to earth. D_4 is then cut off and a -60 volt pulse is applied to C_2 to reset the digitizer output to zero via D_6. At the end of 100 microseconds the anode of V_{6B} returns to +60 volts and D_3 is cut off. C_3 is then discharged through R_4 and D_5 (while D_6 is cut off) until D_4 conducts when the discharge is complete. C_3 has been added to the anode of V_{6A} to slow down its rise and fall times to speeds approaching those of V_{6B}. If C_3 is not used the y switches exhibit undesirable spikes at their outputs.

The load resistors for V_{6A} and V_{6B} were calculated as follows.

Cathode current = 12.5 milliamps

$R_k = 150/12.5 = 12\text{k ohms}$

(1) V_{6A} anode negative:

current to $S_2 = 360/150 = 2.4$ milliamps

Trigger current to grid switch = 0

Current in $R_{L1} = 360/R_{L1}$

" " $R_{E1} = 60/R_{E1}$

$$\therefore \quad 360/R_{L1} + 60/R_{E1} + 2.4 = 12.5 \qquad (10.1)$$

(2) V_{6A} positive:

current to trigger grid switch $= 360/470$

$= 0.77$ milliamps

$$\therefore \quad 240/R_{L1} = 60/R_{E1} + 0.77 \qquad (10.2)$$

Solution of equations 10.1 and 10.2 gives:-

$$R_{L1} = 55.3\text{k ohms}$$

$$R_{E1} = 16.7\text{k ohms}$$

V_{6B} Anode positive:

$$240/R_{L2} = 60/R_{E2} \qquad (10.3)$$

V_{6B} Anode negative:

$$360/150 + 360/820 + 360/R_{L2} + 60/R_{E2} = 135/12 \qquad (10.4)$$

Solution of equations 10.3 and 10.4 gives:-

$$R_{L2} = 72\text{k ohms}$$

$$R_{E2} = 18\text{k ohms}$$

Plates 1, 2 and 3 show the waveforms at the anodes and the grid of V_6.

11.0 THE NEGATIVE dx DISCRIMINATOR

This discriminator is shown in fig. 11.1. The design is similar to that of the +dx discriminator except that the input valve V_{7A} is normally cut off so that the multivibrator will trigger from increasing voltages. Also the reset pulse to the Digitizer amplifier is necessarily of the opposite polarity to that from the +dx discriminator. The Diode D_1 and resistor R_1 on the input grid are included to ensure that C_1 always recharges to −180 volt rather than to a voltage dependent on x. This makes the pulse width independent of changes in x.

Fig. 11.1 shows the circuit, which will not be discussed further as all the design principles are outlined in section 10.0.

Plates 12, 13, 14 and 15 show anode grid and cathode waveforms of V_7.

12.0 THE ACCUMULATOR GRID SWITCH MULTIVIBRATOR

In section 10.0 it was pointed out that the grid switch was delayed so as to operate on the trailing edge of the y pulse. Fig. 12 shows the multivibrator with the delayed trigger circuit connected to its input grid. The two diodes D_1 and D_2 are provided to prevent the anodes of V_7 and V_8 interfering with one another during the triggering period. D_1 and D_2 are connected to the anode of V_6 and V_7 respectively which become positive after triggering.

In fig. 12.1 D_1 and D_2 are normally cut off and D_3 and D_4 are conducting and there is no charge on C_1. When V_6 or V_7 are triggered by the digitizer amplifier D_1 or D_2 take point A positive and so cut off D_3 and D_4. C_4 then commences to charge positively through R_2, R_3 and R_4 until the input grid of V_8 reaches its triggering potential. V_8 then changes state so that V_{8B} is cut off and V_{8A} is conducting. Shortly after this the appropriate dx multivibrator returns to its quiescent state and C_1 is rapidly discharged through D_4 and R_1 until caught by D_3 at earth.

The purpose of the above delay circuit is to ensure that the delay is independent of the amplitudes of the anode waveforms of V_6 and V_7 and is independent of the P.R.F. The timing waveform at C_1 is shown in plate 7. A delay of 80 microseconds is used and a pulse width of 120 microseconds is generated at the anodes of V_8.

D_5 (fig. 12.1) is provided to catch the timing grid at -50 volts and so prevent unwanted spikes from appearing at the anodes of V_8 and make the timing more definite than it would be if the grid resistor were simply taken to -50 volts and no diode was used. The anode loads of V_8 have been calculated in the same way as for V_6 (Section 10) to give $\pm$ 2 volt pulses at the points A and B, fig. 9.1.

R_9 (fig. 12.1) has been included to allow the anode of V_{8A} to generate a 20 volt pulse so that C_2 will have an ample voltage range to charge through before turning V_{8B} on again.

Plates 8, 9, 10 and 11 show the waveforms produced by the Accumulator grid switch multivibrator.

13.0 THE DIGITIZER AMPLIFIER

Except for the several microseconds reset period the amplifier will have the reverse resistance of the silicon diodes D_2 and D_4 (fig. 5.3) as its input resistance. The published data for these OA202 diodes gives a reverse current of 0.1 microamp at 150 volts so that input resistance will be of the order of 100 megohms if allowance be made for leakage resistance around the valve socket. However, in order to use the grid current compensation technique described in section 4.3.1.(c), a 10 megohm input resistor is connected from input grid to ground. The input condenser C, (fig. 5.3) is limited to a maximum value of 0.1 microfarad because of its effect on the stability of amplifiers feeding into it. This effect has been discussed in section 7.2. The gain of the digitizer amplifier from x input to dx output has been chosen at 100 so that ½ volt increments of x will result in 50 volt changes at its output. The 50 volt changes are desirable so that when their d.c. level is reduced by a factor of 2 to feed the voltage discriminators, a change of 25 volts is still available. This change will then be large enough to mask the drifts in the voltage discriminators. The choice of earth as the mean output level, rather than −150 volts, resulted from the requirement of being able to construct the integrator from standard computer components.

The minimum time interval for one increment dx of x is 100 microseconds and it will always be possible to arrange by scaling C_1 (fig. 5.3) that the interval does not usually exceed 10 milliseconds. There may, however, be periods where x may even be constant for some considerable time during a particular problem. For these prolonged periods it is not necessary for the digitizer amplifier to store dx to 0.1% accuracy because the error in the particular dx concerned would only be causing similar error in a quantity which would be of the order of 1% of the output z. Normally there would not be more than 2 or 3 such periods in any one problem, and it should suffice to choose an amplifier gain which will hold the value of dx for 10 milliseconds to an accuracy of 0.1%. The resultant ratio t/CR is therefore

$$0.01/(0.001 \times 10^{-6} \times 10 \times 10^{6}) = 1$$

From equation 4.46 the gain required to hold dx for 10 milliseconds to an accuracy of 0.1% is therefore 1000.

13.1 Circuit Design

Fig. 13.1 shows the circuit of the amplifier. One valve is insufficient to give a gain of 1000 with input and output at earth, so two valves were used and arranged for maximum gain. A 12AX7 was chosen for V_1 because it was known that quite small grid currents could be obtained with this valve. The 6BL8 was chosen for V_2 because it has in one envelope, a high gain pentode and a medium μ triode for use as a cathode follower. Drift is reduced by the use of a balanced input stage and the use of Martin's [38] heater circuit. This heater circuit has already been described, section 7.3. As for the accumulator amplifier, a short term (2 min.) drift of 0.25 millivolt was attained and a long term drift of 1 millivolt. 10% change in the +300 volt supply caused a drift of 7 millivolts and 10% change in the -300 volt supply caused a drift of 5 millivolts.

V_1 is operated at low anode potential (approximately 50 volts) and current (approximately 0.5 milliamp) so that grid current is low and gain is high. Grid current was measured as described in section 7.4 (the 10 megohm input resistor having been removed). The anode potential of the input valve was varied by altering its anode load, until a minimum of grid current was found. This grid current was 0.95×10^{-10} amperes. This would cause an error of 0.1 volt per second in dx. Replacement of the 10 megohm resistor and adjustment of the balance control enabled the digitizer amplifier to hold zero output with a change of less than 1 volt in 100 seconds. This is more than adequate for the purpose. The theory underlying this method of grid current compensation is outlined in section 4.3.1(c).

The gain of the first valve was calculated as follows.[35]

$$A = \frac{\mu \quad R_{L_2}}{(2r_a + R_{L_1} + R_{L_2})} \tag{13.1}$$

where

$$\mu = 100$$
$$R_{L_1} = 300\text{k ohms.}$$
$$R_{L_2} = 470\text{k ohms.}$$
$$r_a = 60\text{k ohms.}$$
$$\therefore \quad A = 53 \tag{13.2}$$

The triode is coupled to the pentode through a low current 50 volt neon tube which shifts the d.c. level without loss of gain.

The pentode is not operated under "starvation" conditions because of the current required to change the charge in the stray capacity at the anode during the rapid 80 volt changes which occur at the anode.

The gain of the pentode is given by [35]

$$A = gm\ R_L \qquad (13.3)$$

$$R_L = 132k \text{ ohms}$$

$$g_m = 1.0 \text{ milliamps per volt.}$$

$$\therefore \quad A = 132 \qquad (13.4)$$

The gain of the coupling network from the anode of the pentode to the grid of the cathode follower is 1.2/(1.2 + 0.75) = 0.615 and the cathode follower has a gain closely equal to 1.

The calculated gain is therefore

$$53 \times 132 \times 0.61 = 4300$$

The measured overall gain was 4000 and this figure has been used in subsequent calculations.

13.2 Stabilization Against Oscillation

Bode diagrams were used, as in section 7.2, for determining the stabilizing networks for the amplifier. Fig. 13.2 shows the results.

A B C is the frequency response from input grid to pentode grid before the addition of the lag lead condenser A D E is the uncorrected response of the pentode anode circuit. The coupling network from pentode anode to cathode follower grid has been adjusted so that its gain is independent of frequency. A.F.G. is the frequency response of the cathode follower with a 0.001 microfarad condenser on its output. This is the condition which exists when an 0.1 microfarad condenser is used on the input of the digitizer and is connected to a low impedance source.

Addition of these three responses results in an amplifier which is clearly unstable with a condenser from output to input. As in the case of the accumulator amplifier the amplifier stability is considered with the output directly connected to the input.

The open loop response has been modified by two networks, one in each active anode circuit, to produce the responses A H J K E and A L M N C which when added together give the open loop response A L O P Q R S. The phase response has been drawn on the same diagram and it can be seen that the phase margin at the point where the loop gain is unity is 39° which is an adequate margin for stability.

When the 0.1 microfarad input condenser to the digitizer is earthed the loop gain of the amplifier is reduced by 20 db. It is therefore essential to see that the phase margin at frequencies lower than 500,000 cycles per second is sufficient. Inspection of the diagram shows that the phase margin in this region does not fall below 41°.

Should the input condenser be left unconnected the point F will move about a decade higher in frequency resulting in an improvement in amplifier stability. The transfer functions of the stabilizing networks are given by equation 7.2 and in each case the ratio of resistances and the capacity have been calculated from the known position of the zeros on fig. 13.2.

The closed loop frequency response has been calculated using $M\alpha$ curves [39] and is shown in figure 13.3. This shows a peak gain 2.3 decibels above the low frequency gain and a bandwidth of 800 kilocycles per second. The rise time of the response to step of voltage at the input will be approximately [14]

$$\frac{0.35}{f_h} = \frac{0.35}{800000} = 0.44 \text{ microseconds.}$$

Since the pulse width used is 100 microseconds there will be adequate time for the amplifier to reach its final value before a succeeding pulse arrives at the input.

Comparison of figure 13.3 with the standard curves for a second order system whose transfer function is

$$A(S) = \frac{\zeta W_n^2}{S^2 + 2\zeta W_n S + W_n^2} \qquad (13.5)$$

shows that ζ the damping factor is 0.45 This value is approximate as the amplifier is not a simple second order system. Reference to curves giving the unit step response of a second order system indicates that there will be a 20% overshoot followed by a very slight undershoot. Since there is adequate time for the system to settle to final value these oscillations may be safely ignored. When the 100 ohm resistor mentioned in section 7.2 is inserted in series with the 0.1 microfarad input condenser the observed overshoot decreases to about 5% and there is no undershoot.

A root locus diagram (fig. 13.4) has been drawn for the amplifier to confirm the above estimates of transient response. This diagram shows reasonable correlation with the above results. If an improved transient response was needed the root locus diagram would provide an easy means of determining the necessary compensation networks.

Plate 19 shows the digitizer output when $x = 9.5 \sin 314t$. The damped transients after the reset pulse are clearly visible as changes in brightness.

14.0 THE COMPLETE INTEGRATOR

Fig. 14.1 shows the integrator with the necessary controls for its operation. Resistive networks are connected to the accumulator and digitizer amplifiers for setting in initial conditions. RL1A and RL1B are contacts on the "initiate" relay which are open during the problem. When they are closed Z is equal to Z_o and Δx is equal to Δx_o. Δx_o is normally zero.

This relay, the balance potentiometers in the amplifiers and the value of Z_o are the only things which need adjustment before each run. There are other controls which need periodic adjustment as follows:

(a) An oscilloscope is connected to the output of the digitizer amplifier and the switching levels of the two discriminators are adjusted (by means of RV1 and RV2) until switching occurs at $\pm$ 50 volts. The inputs in this case being obtained artificially by connecting a resistance of several megohms to the input grid of the digitizer amplifier and applying a small positive or negative voltage to the other end of the resistance. This is equivalent to an input of $x = kt$. The digitizer will run indefinitely at constant P.R.F. under these conditions and plenty of time is available for the adjustment.

(b) With the digitizer running as in (a) C_3 and C_4 are each adjusted until the reset pulses accurately return the digitizer output to zero.

(c) The resistance is removed from the digitizer input and 6.3 volts a.c. 50 cycles per second is fed into the x input with the y input at earth. A meter is connected to the output Z and RV3 is adjusted until Z has minimum drift.

14.1 Scaling the Integrator

Equation 5.5 shows that the required value of Z is k times the integrator output where

$$k = \triangle x . C_8/C_7 \qquad (14.1)$$

Where C_8 and C_7 are shown on figure 14.1. C_6 is chosen accurately equal to C_7.

$\triangle x$ may itself be scaled as

$$\triangle x = 50\ C_2/C_1 \qquad (14.2)$$

but it is usually made 0.5 volts.

The scaling process consists in choosing C_1 so that intervals $\triangle x$ of x are passed through in less than 200 microseconds and preferably not more than 10 milliseconds. C_1 is always chosen so that the digitizer switches a maximum number of times for the given problem.

C_8 is then chosen so that the estimated value of Z will not quite reach the maximum value of 50 volts allowed at the accumulator output.

C_6, C_7 and C_2 (fig. 14.1) are fixed in value because of the limited current available from the y switches and the digitizer amplifier. C_2 could, of course, be decreased but this would result in errors of storage of dx because of limited amplifier gain.

Reference 49 gives an excellent treatment of the process of scaling complete problems which involve many amplifiers and integrators.

14.2 Errors in the Integrator

The integrator is subject to two distinct forms of error. Those arising from the numerical approximation on which the method is based and those arising from deficiencies in the equipment forming the integrator. Both are discussed hereunder.

14.2.1 Approximation Errors

For $y = \sin x$ Table 1 shows the error from accumulating ordinates instead of using an ideal integrator. The values of ε_1 shown in column 3 of the table are the result of accumulating ordinates in steps of 0.9 degree (0.015708 radians) and subtracting the true value $(1 - \cos x)$. These results were calculated using 6 figure tables for sin x and cos x.

TABLE I

Degrees θ	Sin θ	$\Delta x \sum_{n=0}^{m} y_n + (\cos\theta - 1) = \varepsilon_1$	$\varepsilon = \frac{\Delta x}{2} \sin\theta$
0	0	0	0
9	0.156434	0.001228	0.00123
18	0.309017	0.002427	0.00242
27	0.453990	0.003564	0.00356
36	0.587785	0.004613	0.00461
45	0.707107	0.005548	0.00555
54	0.809017	0.006345	0.00635
63	0.891007	0.006987	0.00700
72	0.951057	0.007457	0.00740
81	0.987688	0.007741	0.00780
90	1.000000	0.007840	0.00790

This table may be extended to 360° by reflection about the 90° ordinate and then reflection about the x axis and translation of the reflected half cycle.

At any switching point i.e. where $x = n \Delta x$ we are representing the area under a sine curve between $x = (n-1)\Delta x$ and $x = n \Delta x$ by a rectangle sine $(n \Delta x)$ high and Δx wide. The error in doing this is approximately $(\Delta y)(\Delta x)(0.5)$

but
$$\Delta y = \frac{dy}{dx} \Delta x \tag{14.3}$$
and the error is thus $\mathcal{E} = 0.5 \Delta x^2 \, dy/dx$

So that at any point (x.y) the accumulated error is

$$\mathcal{E} = 0.5 \Delta x \sum_{n=0}^{m} (dy/dx)_n \Delta x \tag{14.4}$$

$$\mathcal{E} \doteq 0.5 \Delta x \int_{x_0}^{x} (dy/dx)\, dx \tag{14.5}$$

$$\mathcal{E} = 0.5 \Delta x (y - y_0) \tag{14.6}$$

$$\mathcal{E} = 0.5 \Delta x \cdot \sin x \tag{14.7}$$

Column 4, table 1, shows the results of using this to calculate the error. The correlation between these results and those from ordinate accumulation is so close that there appears to be no reason why equation 14.6 cannot be used to calculate the value of $\mathcal{E}$ for any known function of y. Provided always that the change in dy/dx over an interval Δx is small.

The above gives the error at the instant of accumulation of each ordinate and for $y = \text{sine } x$ curve A fig. (14.2) shows the results. Just before the accumulation of each ordinate there exists another error equal to Δx times the ordinate. This error is in the opposite direction to that of equation 14·6 above and is twice its magnitude. Therefore the error just before accumulation of an ordinate of $y = \sin x$ is shown by curve B of fig. (14.2). The instantaneous error is thus a triangular waveform whose envelope is the curves A and B fig. (14.2). The period of the triangular wave is equivalent to $1/(2\Delta x)$.

For $y = \sin x$ the error in Z never exceeds 0.8% of maximum amplitude.

In the above analysis each cycle of x has been split up into an integral number of intervals. If the number of intervals per cycle was not integral then the effect on the error curve would be to cause slight variations in the maximum error and its phase.

A better understanding of the errors may be had by considering the "trapezoidal" rule for integration. References 52 and 53 show this rule to be a particular case ($n = 1$) of the Newton-Cotes formulae of the closed type. The "trapezoidal" rule [53] is stated as:

$$\int_a^b f(x)\,dx = h(\tfrac{1}{2}f_o + f_1 + \text{---}\ f_{n-1} + \tfrac{1}{2}f_n) - \frac{nh^3}{12} f^{11}(\xi) \qquad (14.8)$$

where $f(o) = f(a)$, $f_k = (a + kh)$ and $f_n = f(b)$, and where ξ is somewhere in (a,b). The first term of the two in the above formula is equivalent to replacing $f(x)$ by a series of line segments joining the ends of adjacent ordinates. It will have an error closely equal to that remaining after the addition of $\mathcal{E}$ (equation 14.6) to the basic summation of ordinates done by the integrator.

An upper bound can be set to this remaining error by considering $|f''(\xi)| \equiv$ the maximum value of $f''(x)$ between a and b. Hence the maximum remaining error is

$$|E_n| = \frac{nh^3}{12}\,|f''(\xi)| = \frac{(b-a)^3}{12\,n^2}\,|f''(\xi)| \tag{14.9}$$

Thus if $f''(x)$ is continuous (and hence bounded) in (a,b) the remaining error will tend to zero as $1/n^2$ as n approaches ∞. For the case of $y = \sin x$, considered above $|f''(\xi)| = 1$

$h = 0.0157$ radians and $n = 0 - 400$ (from 0 to 1 cycle of x)

$\therefore$ $|E_n|$ will increase with n. The worst case (in one cycle of x) will be when $n = 400$. Then

$$|E_n| = 0.00013 = 0.013\%$$

By representing the various $y = f(x)$ which occur in practice by portions of a sine wave it can be shown that, in general, $|E_n|$ may be neglected in comparison with the main error (of equation 14.4).

Thus the integration error due to the inadequacy of simple ordinate summation may be calculated from equation (14.4) with acceptable accuracy.

A further comment on the difference between the simple ordinate summation corrected by equation (14.4) and the trapezoidal rule is pertinent. The above discussion assumed that $f(o) = 0$. If this were not the case the difference between the simple summation and the trapezoidal rule would be

$$\varepsilon_2 = h(fo/2 - fn/2) - \frac{nh^3}{12}\,f''(\xi) \tag{14.10}$$

since the summation is $h \sum_{r=1}^{n} f_r$. The last term in (14.10) is negligible and the first term h fo/2 may be allowed for by setting in an initial value of z on the accumulator. The remaining term is 0.5 h.fn which is equivalent to $\mathcal{E}$ of equation 14.6. Thus, if allowance is made for f_o, the trapezoidal rule (neglecting its error term) is equivalent to the simple summation of ordinates corrected by $\mathcal{E}$ of equation 14.6.

The recommended procedure for calculating the integrator approximation error is therefore

(1) Check that $|E_n|$ (equation 14.9) is negligible. If not include it in the error.

(2) See that $f_o/2$ is set in as an initial value of Z.

(3) Calculate the error from equation 14.6.

Should f(x) be known only in tabular form at intervals greater than the proposed Δ x an error analysis may be carried out by using a finite difference method of interpolation (Reference 53, Chapter 2) to produce a series of values of f(x) at intervals of x equal to Δ x. Equation 14.6 may then be applied to determine the error.

Alternatively the ordinates may be directly accumulated at intervals Δ x of x to get the expected output from the integrator. The true value of the integral may be found directly from the ordinates at the larger intervals by the use of one of the Newton-Cotes closed formulae [(53)] or by using the following "finite difference" method of integration [(51)].

$$\int_{x_0}^{x_n} f(x)dx = \tfrac{1}{2}\ x[f_0 + 2f_1 + 2f_2 + --- + 2f_{n-1} + f_n$$

$$- \frac{1}{12}(\mu\ \ f_n - \mu\ \ f_o) + \frac{11}{720}(\mu\ \ {}^3f_n - \mu\ \ {}^3f_o)$$

$$- \frac{191}{60480}(\mu\ \ {}^5f_n - \mu\ \ {}^5f_o)] \qquad (14.3)$$

where μ is the "averaging operator" defined by

$$\mu f(x) = \tfrac{1}{2}[f(x + \tfrac{1}{2}\ \ x) + f(x - \tfrac{1}{2}\ \ x)] \qquad (14.4)$$

and

$$ {}^nf_j = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}\ f_{j + \frac{1}{2}n - k} \qquad (14.5)$$

is a 'difference operator'.

The error is then calculated as the difference between the sum of the accumulated ordinates and the true value.

14.2.2 <u>Equipment Errors</u>

A number of measurements were made on the complete integrator to check the errors introduced by supply voltage changes, hum and noise. The following results were noted:

(a) A decrease of 35 volts in the +300 volt supply causes the discriminator switching levels to each move down by 4 volts (as measured at output of digitizer amplifier). This is equivalent to adding a small quantity to x at the beginning of the problem and will not result in serious errors. The actual error is, of course, very much smaller than indicated above because the supply is regulated to better than 1%. The amplitude of the + dx reset pulse changes by 6 volts under these conditions. This can cause serious errors unless well regulated supplies are used.

(b) A decrease of 35 volts in the -300 volt supply causes the discriminator switching levels to each move up by 2 volts and the - dx reset pulse to increase by 4 volts.

(c) A decrease of 35 volts in both the +300 volt and -300 volt supplies causes the negative dx discriminator switching level to rise by 4 volts and the positive dx discriminator switching level to fall by 4 volts. The reset pulse amplitudes remain fairly constant. No serious errors will result from these variations.

(d) The discriminator switching levels change by 0.15 volts for 8% change in heater voltage. This could cause 0.3% errors in dx. However the heater volts rarely change by more than 2 or 3% and the resultant dx error is 0.1%.

(e) The hum and noise from the accumulator amplifier is less than 10 millivolts peak to peak. The hum and noise from the digitizer amplifier is less than 100 millivolts peak to peak and appears to be mainly power supply ripple.

(f) With the digitizer running at a constant rate of 5000 pulses per second the accumulator output drift rate (for $y = 0$ and $C_8 = 0.1$ microfarad) between 5 and 10 millivolts per second and its direction depends on whether x is increasing or decreasing. However, if x changed at this rate it would reach its maximum value in 0.2 seconds and would then have to reverse its direction and the cumulative error would be small.

14.3 Measurement of Performance

The choice of a Δx of 1/100 of the peak amplitude of x and a period of approximately twice the pulse width (i.e. 200 microseconds) for the cycle time of the digitizer sets a limit to the maximum rate of rise of x. This limit is $0.5/(200 \times 10^{-6})$ volts per second = 2500 volts per second. This corresponds to the maximum slope of a 50 volt peak 8 cycle per second sine wave. The bandwidth of the digitizer is thus limited to 8 cycles per second. Since it is undesirable for the y input to change by more than 1% of maximum y during one Δx interval a similar bandwidth limit is set for the y input. If both x and y have simultaneous rates of change of 2500 volts per second then the error in z will be somewhere between 1 and 2% of maximum output. This latter condition will rarely occur in practice, and the bandwidth of the integrator is taken as 8 cycles per second. Hence the figure of merit for the dynamic performance of the integrator is M = 8/1 = 8.

The integrator was set up with a scale factor of 50. C_1 = 0.1 microfarad and C_8 = 0.1 microfarad, therefore Δx = 0.5 volt. In order to show that the integrator operated correctly over the full range of inputs and outputs a series of measurements of z were made, with a meter of 1% accuracy, for various constant values of y with x varying. The results of these measurements are shown in figure 14.3. To eliminate errors due to drift the integrator was reset to 0 before each reading. x was rapidly brought up to the desired value and z was read as quickly as possible.

It was impossible to tell whether the errors (of the order of 1%) were caused by the integrator or the meters so a potentiometric method of measuring the error was devised. A 0 to ± 1 volt centre zero meter was connected from x to z and with y = -50 volts x was quickly taken to the value of input for which the error was desired and the error (x - z) noted. The Integrator was zeroed between readings. Figure 14.4 shows the results obtained. This test was repeated using a direct coupled cathode ray oscilloscope with differential input and the error was seen to be only slightly in excess of 0.5 volt at any time during the changing of x from 0 to -50 or from 0 to +50.

Two equal high stability resistors were connected in series between x and z. The junction of these 2 resistors was then connected to a high impedance d.c. valve voltmeter, the other side of which was earthed. With y at +50 volts a series of measurements were taken for various values of x. The results are shown in fig. 14.4. The error in z is twice the meter reading in this case and figure 14.4 shows the error in z.

From figure 14.4 we see that the error in z does not exceed 1.0% and is therefore mainly due to the numerical approximation rather than errors arising in the equipment.

The dynamic performance of the integrator was measured by connecting the output of low frequency oscillator to the x input and a constant voltage (45 volts) to the y input. The integrator was scaled as for the previous measurements. The output z was measured by means of a graticule on a direct coupled oscilloscope. The peak value of z was found to vary by 1% or less as the input frequency was changed from 0.5 cycles per second to 8 cycles per second. At frequencies above 8 cycles per second the digitizer ceased to function correctly because the reset pulse condensers were not being completely discharged between each 0.5 volt interval of x. Plates 24 and 25 show the waveform at the integrator output for an x input of frequency 8 cycles per second. The equal increments of y can just be distinguished.

No measurement was done for sinusoidal inputs to y because the low frequency oscillator used would not supply the 2 milliamps necessary to drive the y switches. The output amplitude for $x = kt$ and $y = A \sin wt$ would be inversely proportional to frequency because $\int \sin wt.d(kt) = -(k/w) \cos wt$. There would be no output for the case where x is constant. The operation under these conditions has been demonstrated, but not measured, by using the output of a valve heater transformer for y. Plate 22 shows the resultant output waveform z. Plate 21 shows the corresponding waveform at the output of the y switch S_2. Plate 18 shows the digitizer output waveform.

The waveforms corresponding to a constant y input and an x input of 50 cycles per second (of small amplitude) are shown in Plates 19, 20 and 23.

14.4 General Notes on Construction

The prototype integrator was built on two chassis each 15 inches high, 5 inches wide and 3 inches long, with the valves protruding from the top. The two high gain operational amplifiers were constructed on one chassis and the switching circuits and discriminators on the other.

The input and output terminals of each operational amplifier were each enclosed in holes in the earthed metal chassis so that no direct leakage from input to output was possible except through the computing condensers.

All resistors were generously rated so that their heat dissipation would not cause a large temperature rise. Polystyrene computing condensers were used because the leakage resistance of paper condensers was too low and resulted in serious errors.

Now that the circuit design is finalized a more compact layout could be devised if required.

The power consumption is

(a) Accumulator Amplifier:

Heaters			15	Watts
+300 volt d.c.	13.8	milliamps	3.8	"
-300 volt d.c.	11.23	"	3.4	"
TOTAL			22.2	Watts

(b) Digitizer Amplifier:

Heaters		7.6 Watts
+300 volt d.c.	11.4 milliamps	3.4 "
-300 volt d.c.	8.4 "	2.5 "
<u>TOTAL</u>		13.5 Watts

(c) Switching Circuits:

Heaters		9.4 Watts
+300 volt d.c.	36.5 milliamps	11.0 "
-300 volt d.c.	39.5 "	11.8 "
<u>TOTAL</u>		32.2 Watts

Making a GRAND TOTAL of .. .. .. 67.9 Watts

Therefore considerable care is necessary in ventilating the equipment to avoid excessive temperature rise.

15.0 CONCLUSION

The Electronic Analogue Integrator described herein has an accuracy of 1% of maximum output and a bandwidth of 8 cycles per second. It is capable of integration with respect to any argument for which a voltage analogue can be made available. The argument is not restricted to linear functions of time and the integrator accuracy is not dependent on precisely timed switching pulses.

Integrators of this type may be used for any of the problems normally done on mechanical differential analysers. They may also be used to generate functions in the same way as mechanical wheel and disc integrators. They are useful in conjunction with a conventional electronic analogue computer but are not intended for computations requiring great accuracy together with long term storage. Such computations are best done on a mechanical differential analyser or a digital differential analyser [50].

The accuracy could be improved, at the expense of bandwidth, by decreasing the increments of the argument and increasing the accumulator amplifier L.F. gain. The dynamic figure of merit of 8 is comparable with that of the fastest mechanical integrators.

In a future design considerable savings in heat dissipation could be had by using transistor operational amplifiers. It should be possible to reduce the digitizer amplifier output amplitude and use transistor discriminators without loss of accuracy and with perhaps a gain in speed. Some form of high voltage driver stage would still be required for the y switches.

The maximum problem duration can only be extended by the use of very high gain, very low input current, operational amplifiers. This is yet another application for an analogue storage unit with a fast response and long memory. The development of such a storage unit would make accuracies of 0.1% feasible with this type of integrator.

The individual units of the integrator are themselves useful for other purposes. The digitizer may be used as an analogue to digital converter by feeding its output to a binary counter chain. The digitizer allows changes in the input variable to go on while its storage condenser is being reset. This enhances its speed of response to changes in input variable. The accumulator and y switch may be used as a sample and hold circuit driven either by a separate source of switching pulses or by running the digitizer at a constant P.R.F. If the feedback condensers are replaced by resistors the accumulator may be used as an accurate sampling circuit. Alternatively, the y switches themselves may be used as a sampling circuit.

Thus, in addition to its use as an integrator, the device is useful for studying sampled data systems and systems using quantized information.

APPENDIX I

Dynamic Performance of some Typical Differential Analysers

Machine	Cambridge University (9,10)	C.S.I.R.O. Sydney (11)	G.E. at U.C.L.A. (12)
Maximum Time Shaft Speed in R.P.M.	400	300	1000
Gear Ratio from Time Shaft to Disc.	0.4	0.268	1
Wheel Diameter (Inches)	2.56	1	2
Useful Disc Diameter (Inches)	8	6	8
Gear Ratio from Wheel to Output Shaft for Maximum Output Speed.	1	1	1
Maximum Output Shaft Speed (R.P.M.)	500	300*	4000+
Resolution at Output (Revolutions)	0.01	0.042	0.01
Error %	0.1	0.1	0.1
Maximum Output Frequency (c.p.m.) for a Resolution ≡ Accuracy.	8	1.2	64
M	1.33	0.2	10.6

* Limited by maximum speed of M type motor.

+ Limited by slip of 0.03% at 1000 R.P.M.

APPENDIX II

Differentiation with a time delay:-

Consider the function

$$x = A \cos wt \qquad (1)$$

Equation 4.55 of the text gives

$$dx/dt = (1/\tau)[A \cos wt - A \cos w(t - \tau)] \qquad (2)$$

$$= (1/\tau)[A \cos wt - A \cos wt.\cos(w\tau) + A \sin wt.\sin(-w\tau)]$$

True value of dx/dt is $- wA \sin wt$ (3)

$\therefore$ Error is

$$\varepsilon = - wA \sin wt - (1/\tau)[A \cos wt\ (1 - \cos w\tau) - A \sin wt \sin w\tau]$$

$$= A \left[\cos wt \left(\frac{\cos w\tau - 1}{\tau}\right) + \sin wt \left(\frac{\sin w\tau}{\tau} - w\right)\right] \qquad (4)$$

but τ must be very much less than $1/w$ so that sufficient samples are taken and hence $w\tau$ is small $\therefore$ $\sin w\tau \doteqdot w\tau$

Hence $\qquad \varepsilon \doteqdot (A/\tau)(\cos w\tau - 1).\cos wt \qquad (5)$

which is in phase with the input, and the error amplitude expressed as a fraction of the output amplitude is $(1/w\tau)(\cos w\tau - 1)$

Thus if τ is chosen equal to 1/50 of $\frac{1}{w}$ then $w\tau = 0.02$ and the fractional error is

$$50(0.9998 - 1)$$

$$= - 0.01$$

Since we are considering amplitudes only we may omit the minus sign and say that the output error amplitude is 1% of the output amplitude.

The following table clearly shows the effect of $w\tau$ on error.

$w\tau$ = Delay (radians)	% Amplitude Error
0.2	10
0.02	1
0.002	0.1

APPENDIX III

Method suggested by Thomson [20] for realizing a maximally flat delay transfer function *: consider the third order maximally flat delay function:

$$\exp.\ (-\tau s) = \frac{-x^3 + 12x^2 - 60x + 120}{x^3 + 12x^2 + 60x + 120} \qquad (1)$$

where $x = \tau s$

A continued fraction expansion of this enables us to realize the function without factorizing the polynomials concerned.

Let
$$\exp.\ (-x) = -1 + \frac{2}{1 + G} \qquad (2)$$

Whence
$$G = \frac{12x^2 + 120}{x^3 + 60x} \qquad (3)$$

which expanded as a continued fraction gives

$$G = \cfrac{1}{\cfrac{12}{x} + \cfrac{1}{\cfrac{6x}{25} + \cfrac{1}{\cfrac{5x}{12}}}} \qquad (4)$$

$$G = \cfrac{\cfrac{12}{x}}{1 + \cfrac{\cfrac{12}{x}\cdot\cfrac{25}{6x}}{1 + \cfrac{25}{6x}\cdot\cfrac{12}{5x}}} \qquad (5)$$

Which is realized as shown in figure 4.20.

* Thomson points out that the maximally flat delay transfer function, arising in communication engineering, is equivalent to the Pade approximation to a delay [19,21,22,49].

Smith & Wood [49] give a simple curve fitting method yielding a more constant delay for a given bandwidth and number of amplifiers.

R E F E R E N C E S

(1) W. Thomson (Lord Kelvin).: 'Mechanical Integration of the Linear Differential Equations of the second order with Variable Coefficients', Proc. Roy. Soc. (London) 1875-76, Vol. 24, pp. 269-271.

(2) W. Thomson (Lord Kelvin).: 'Mechanical Integration of the General Linear Differential Equation of any Order with Variable Coefficients', Proc. Roy. Soc. (London) 1875-76, Vol. 24, pp. 271-275.

(3) V. Bush.: 'The Differential Analyser. A New Machine for Solving Differential Equations', J. Franklin Inst. 1931, Vol. 212, pp. 447-488.

(4) George A. Bekey.: 'Generalized Integration on an Analogue Computer', I.R.E. Trans. on Electronic Computers, June 1959, Vol. EC8, pp. 210-217.

(5) N.R. Bell.: 'The New Marconi Counter Type Frequency Meter Type OA1071', Marconi Instrumentation March 1956, Vol. 5 No. 5, pp. 113-120.

(6) Walter W. Soroka.: 'Analog Methods in Computation and Simulation' (McGraw-Hill 1954).

(7) 'Bethlehem Torque Amplifier', Amer. Machinist, 1927, Vol. 66, pp. 895-897.

(8) F.C. Williams and A. Uttley.: 'The Velodyne', J.I.E.E. (London) Pt IIIA, 1946, Vol. 93, pp. 317-318 and 1256 to 1274.

(9) J. Crank.: 'The Differential Analyser' (Longmans Green and Company, 1947).

(10) D.R. Hartree.: 'Calculating Instruments and Machines,' (University of Illinois Press 1949).

(11) D.M. Myers & W.R. Blunden.: 'The C.S.I.R.O. Differential Analyser' (Proceedings of Conference on Automatic Computing Machines. Held in the Department of Electrical Engineering, University of Sydney, August 1951) pp. 18-28.

(12) H.P. Kuehni & H.A. Peterson.: 'A New Differential Analyser,' Transactions of the American Institution of Electrical Engineers, May 1944, Vol. 63, pp. 221-228.

(13) D.M. Myers & W.R. Blunden.: 'The C.S.I.R.O. Differential Analyser,' Journal of the Institution of Engineers Australia, October/November, 1952, Vol. 24, pp. 195-204.

(14) I.A. Greenwood, Jr., J.V. Holdam, Jr., and D. MacRae, Jr.: 'Electronic Instruments', M.I.T. Radiation Laboratory Series, Vol. 21, (McGraw-Hill 1948).

(15) G.A. Korn and Theresa M. Korn.: 'Electronic Analog Computers' (McGraw-Hill 1952).

(16) C.P. Gilbert.: 'Lectures on Analogue Computers', given to Post Graduate Students in the School of Electrical Engineering, University of N.S.W. during 1956.

(17) F.C. Williams, F.J.U. Ritson & T. Kilburn.: 'Automatic Strobes and Recurrence - Frequency Selectors,' J.I.E.E. (London) Pt IIIA, 1946, Vol. 93, No. 7, pp. 1275-1300.

(18) E.A. Goldberg.: 'A High Accuracy Time-Division Multiplier,' R.C.A. Review, September 1952, pp. 265-274.

(19) W.J. Cunningham.: 'Time Delay Networks for an Analog Computer,' I.R.E. Transactions on Electronic Computers, December 1954, Vol. EC3, pp. 16-18.

(20) W.E. Thomson.: 'Time Delay Circuits' I.R.E. Transactions on Electronic Computers, June 1955, Vol. EC4, p. 74 (correspondence).

(21) C.D. Morrill.: 'A Sub-audio Time Delay Circuit,' I.R.E. Transactions on Electronic Computers, June 1954, Vol. EC3, pp. 45-49.

(22) W.E. Thomson.: 'Delay Networks Having Maximally Flat Frequency Characteristics,' Proc. I.E.E., Nov. 1949, Vol. 96 Pt III pp. 487.

(23) R.S. Stone & R.A. Dandl.: 'A Variable Function Delay for Analog Computers,' I.R.E. Transactions on Electronic Computers, Sept., 1957, Vol. EC6, No. 3, pp. 187-189.

(24) C.A.A. Wass.: 'Introduction to Electronic Analogue Computers,' (Pergamon Press Ltd. 1955).

(25) O. Amble.: 'On a Principle of Connection for Bush Integrators,' J. Sci. Inst. (London), Dec. 1956.

(26) P.E. Pfeiffer.: 'A Four Quadrant Multiplier using Triangular Waves, Diodes, Resistors, and Operational Amplifiers,' I.R.E. Transactions on Electronic Computers, June 1959, Vol. EC8, pp. 222-227.

(27) N. Diamantides.: 'Electronic Switch for Analog Computer Simulation,' I.R.E. Transactions on Electronic Computers, December, 1956, Vol. EC5, pp. 197-202.

(28) E.A. Goldberg.: 'Stabilization of Wide Band Direct-Current Amplifiers for Zero and Gain,' R.C.A. Review, June 1950, Vol. XI, pp. 296-300.

(29) W.E. Thomson.: 'Networks with Maximally Flat Delay,' Wireless Engineer, 1952, Vol. 29, p. 256.

(30) L.M. Graves.: 'Theory of Functions of Real Variables,' McGraw-Hill 1956, p. 56.

(31) T.M. Apostal.: 'Mathematical Analysis,' Addison Wesley, pp. 77 & 78.

(32) I.C. Hutcheon and D. Summers.: 'A Low-Drift Transistor Chopper-Type D.C. Amplifier with High Gain and Large Dynamic Range,' Proc. I.E.E., Sept. 1960, Vol. 107, Pt B, pp. 451-465.

(33) P.S.T. Buckerfield.: 'The Parallel Tee D.C. Amplifier,' A Low Drift Amplifier with Wide Frequency Response, J.I.E.E. Pt II 1952, pp. 497-506.

(34) W.A. Ferguson.: 'High Stage Gain at Low Frequencies,' Mullard Technical Communications, January 1954, Vol. 1, Number 6, pp. 137-141.

(35) J. Millman & H. Taub.: 'Pulse and Digital Circuits,' McGraw-Hill, 1956.

(36) J.B. Earnshaw.: 'Heater Voltage Compensation for D.C. Amplifiers,' Electronic Engineering, January 1957, pp. 31-35.

(37) R.E. Aitchison.: 'A New Circuit for Balancing the Characteristics of Pairs of Valves,' Electronic Engineering, May 1955, Vol. 27, p. 224.

(38) D.J.R. Martin.: 'Direct Coupled Amplifiers,' Electronic & Radio Engineers, Dec. 1957, pp. 438-441.

(39) H. Chesnut & R.W. Mayer.: 'Servo-mechanisms and Regulating System Design,' Wiley, New York, 1951.

(40) J.G. Truxal.: 'Automatic Feedback Control System Synthesis,' McGraw-Hill, 1955.

(41) H.W. Bode.: 'Network Analysis and Feedback Amplifier Design,' D. Van Nostrand, New York, 1945.

(42) W.R. Evans.: 'Graphical Analysis of Control Systems' Trans. A.I.E.E., 1948, Vol. 67, pp. 547-551.

(43) J. Millman & T.H. Puckett.: 'Accurate Linear Bidirectional Diode Gates,' Proc. I.R.E. Jan. 1955, Vol. 43, pp. 27-37.

(44) G. Sebestyen.: 'A Design Technique for Pedestal Free Switching Circuits,' I.R.E. Trans. on E.C., Sept. 1957, Vol. EC6, No. 3, pp. 162-166.

(45) H.J. Gray, Jr., M. Ribinoff & J. Tomkins.: 'A Diode Multiplexer for Analog Voltages, I.R.E. Trans. on E.C. June 1955, Vol. EC4, No. 2, pp. 64-66.

(46) D.M. McKay.: 'High-Speed Electronic Analogue Computing Techniques,' Proc. I.E.E., 1954, Measurements Section Paper No. M1738.

(47) R.F. Shea.: 'Principles of Transistor Circuits,' Wiley, New York, 1955, pp. 5-10.

(48) Philips Electrical Industries.: 'Miniwatt Transistors Germanium Diodes and Silicon Diodes,' "Miniwatt" June 1957, pp. 127 & 128.

(49) G.W. Smith & R.C. Wood.: 'Analog Computation,' McGraw-Hill, 1959.

(50) R.K. Richards.: 'Arithmetic Operations in Digital Computers' D. Van Nostrand, 1956, pp. 303-311.

(51) D.R. Hartree.: 'Numerical Analysis' Oxford University Press, 1952, p. 102.

(52) W.E. Milne.: 'Numerical Calculus,' Princeton University Press, 1949, p. 117.

(53) F.B. Hildebrand.: 'Introduction to Numerical Analysis,' McGraw-Hill, 1956, pp. 71-75.

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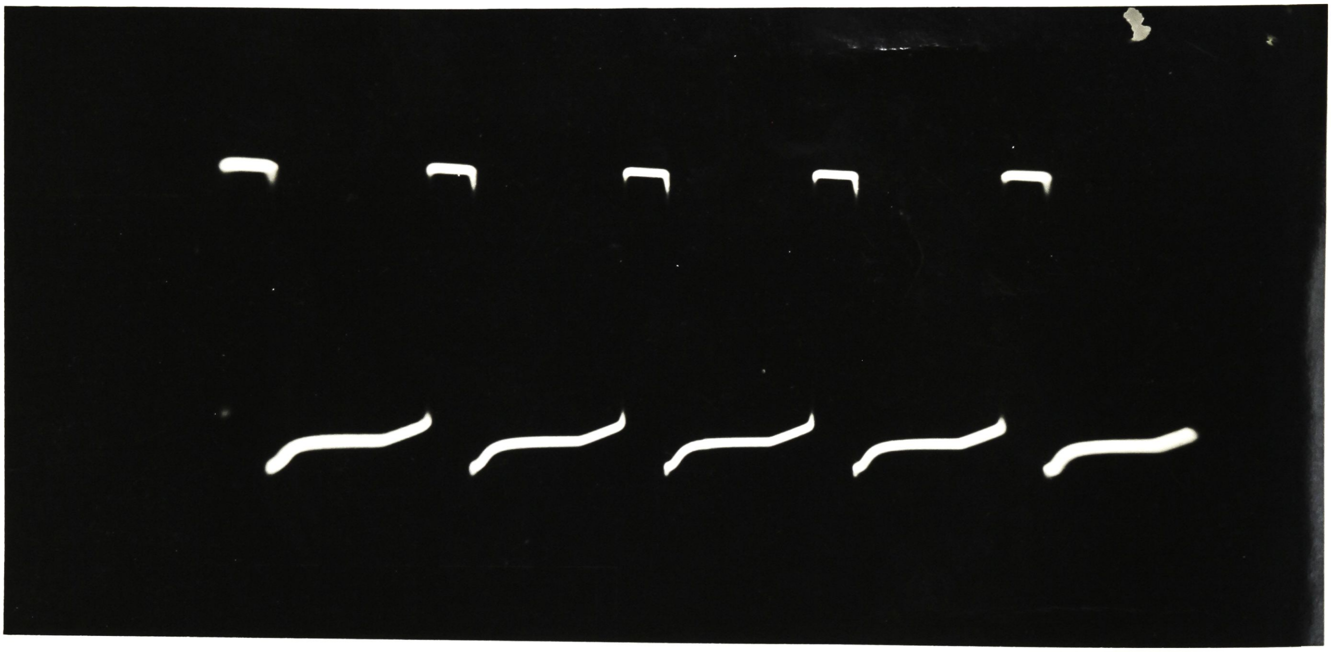
Plate 25 Plate 24 on an Expanded Time Scale.

Plate 26 The Prototype Integrator. Switching circuits in the foreground.

Accumulating Amplifier	-	Top left.
Digitizing Amplifier	-	Top right.



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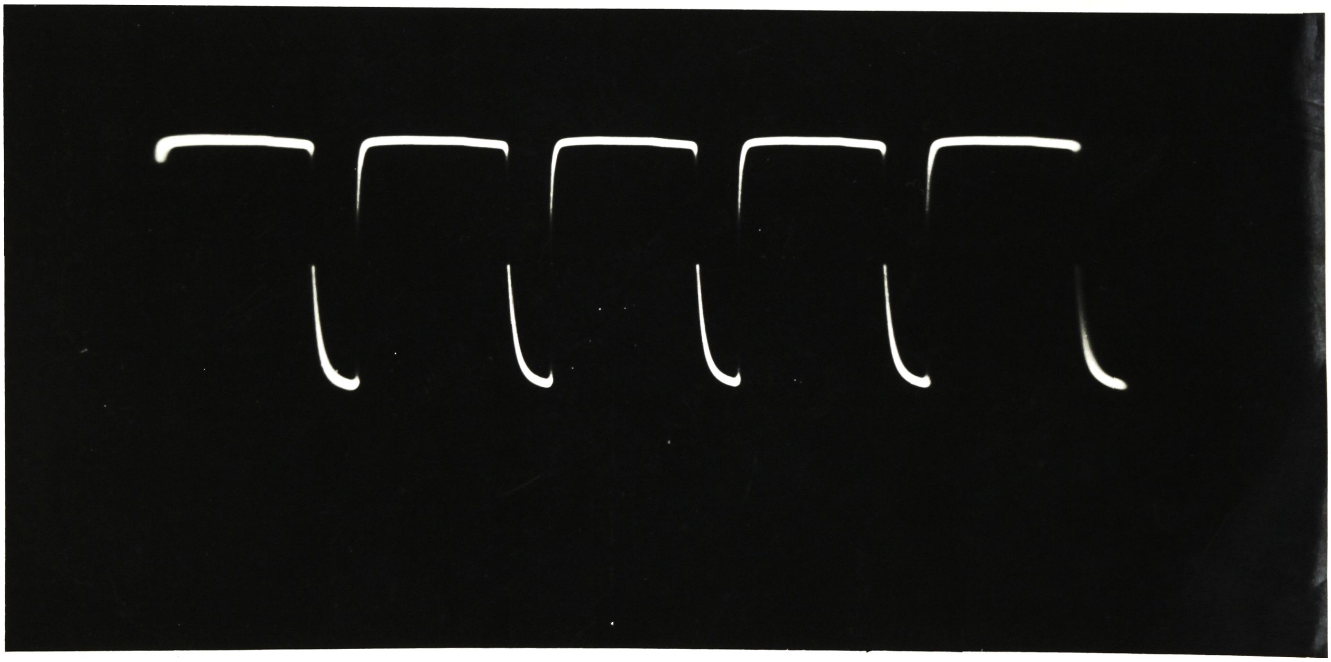

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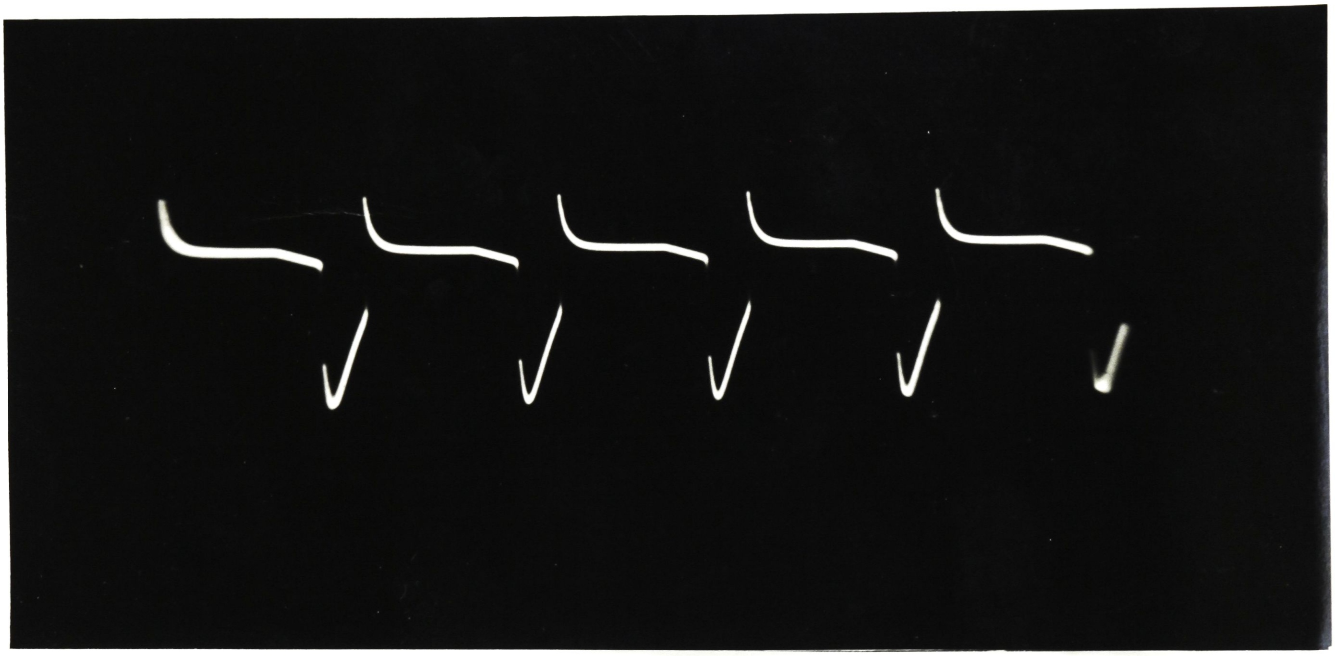

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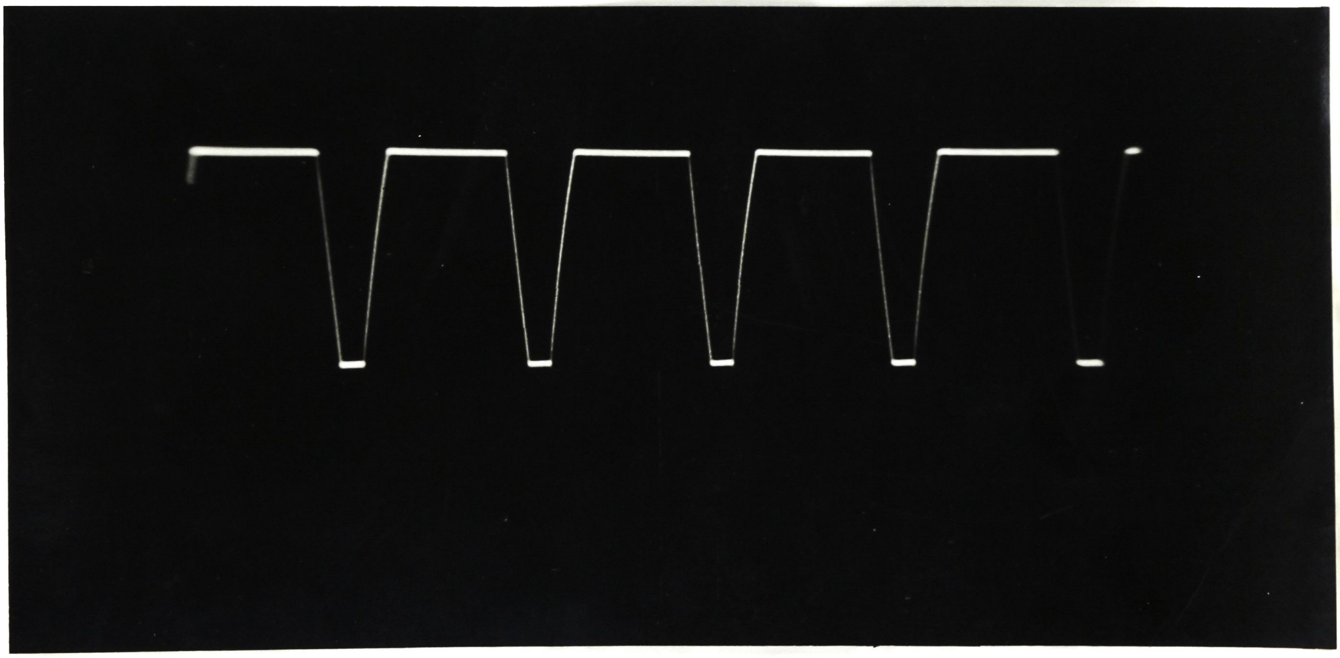

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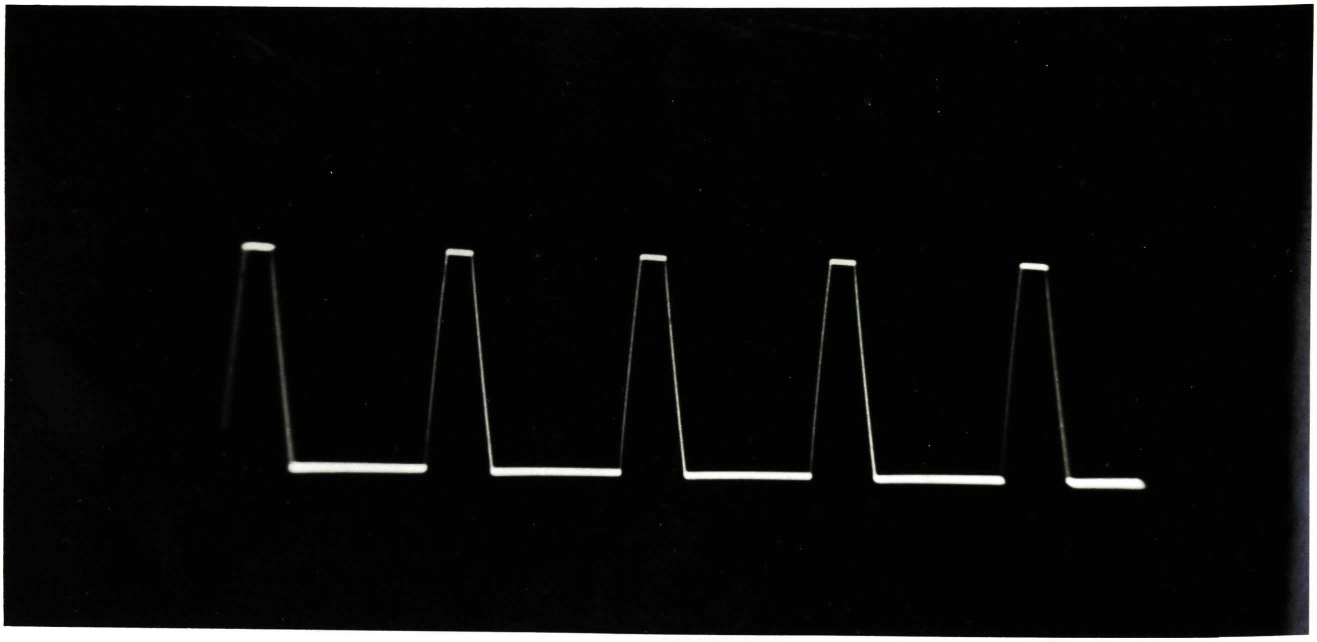

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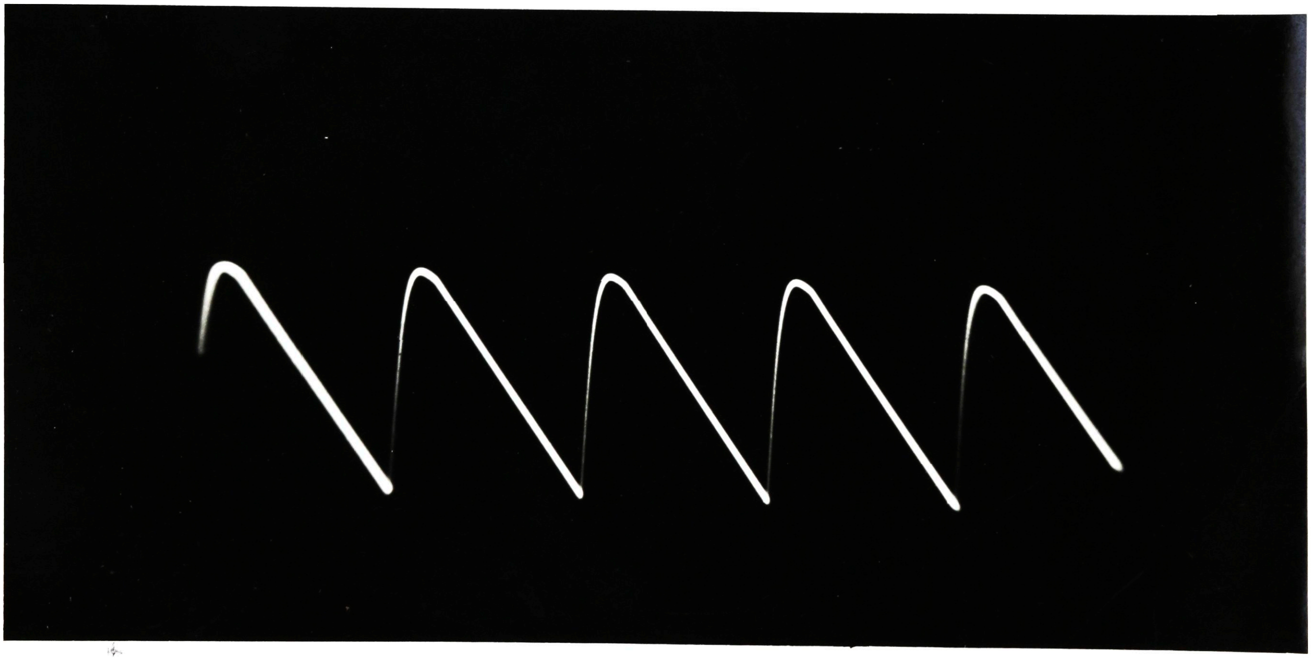

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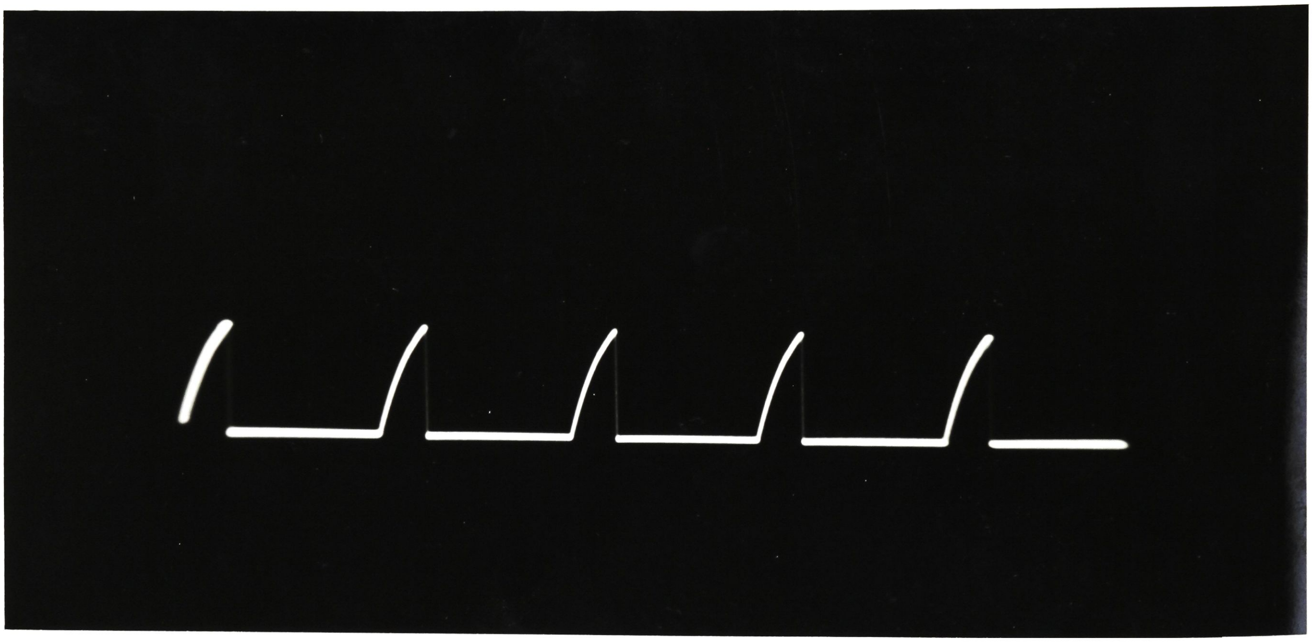

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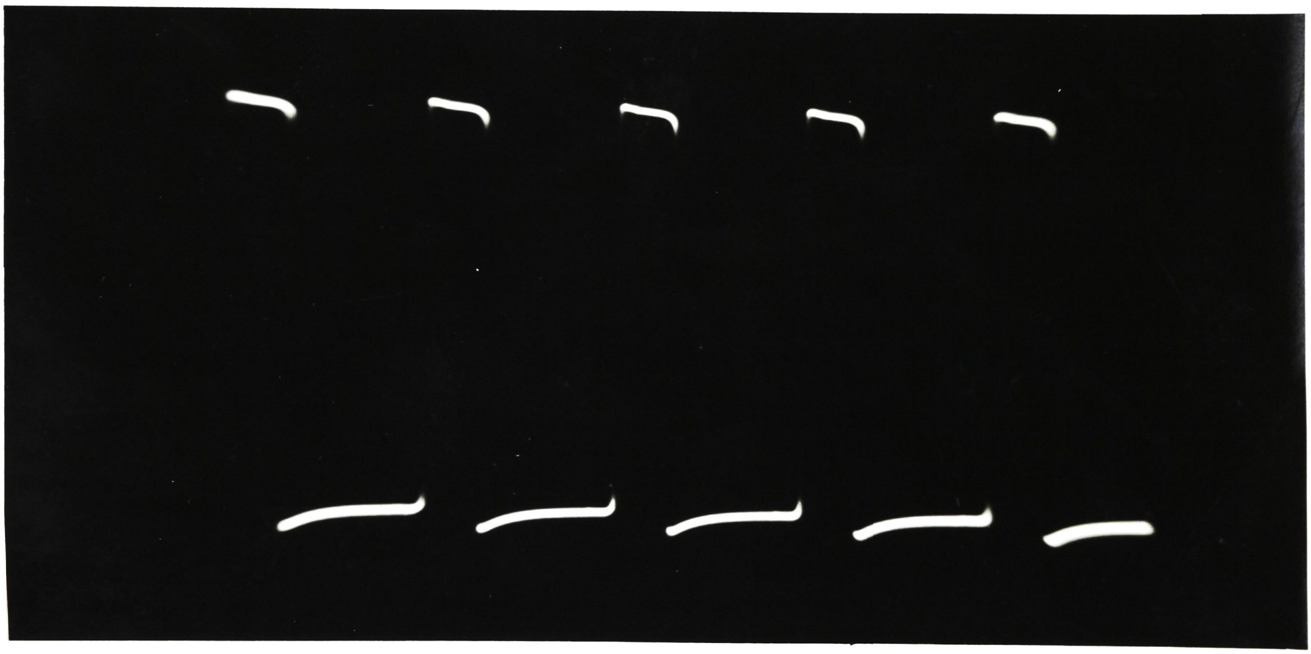

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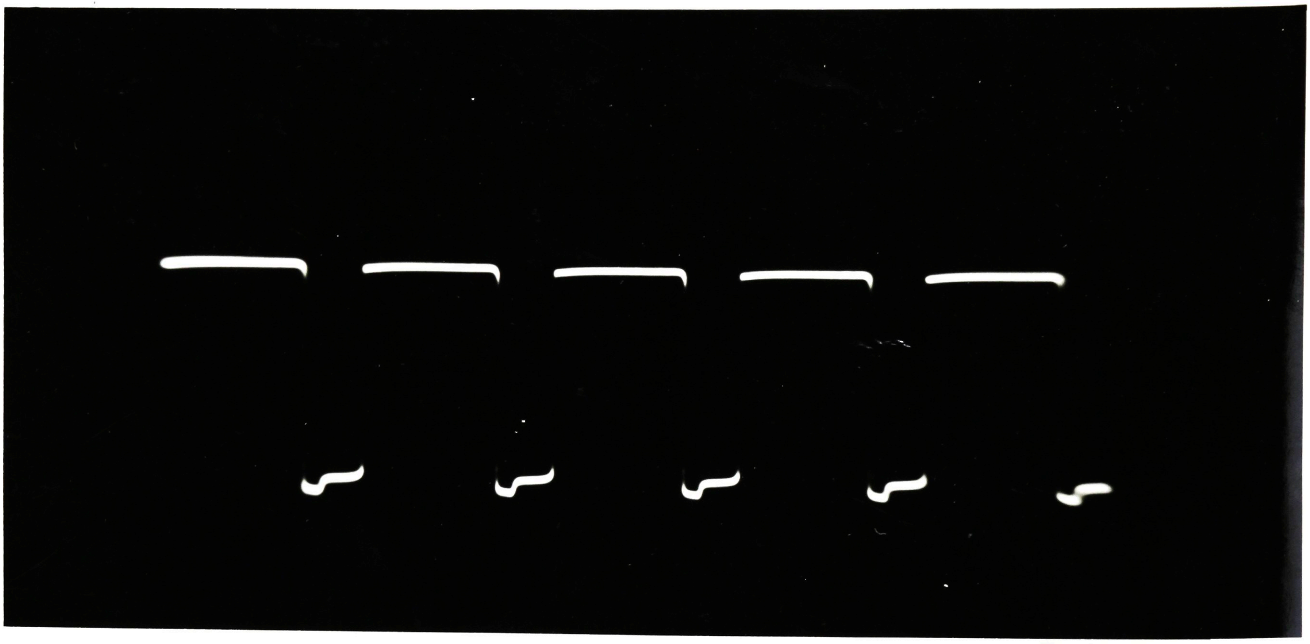

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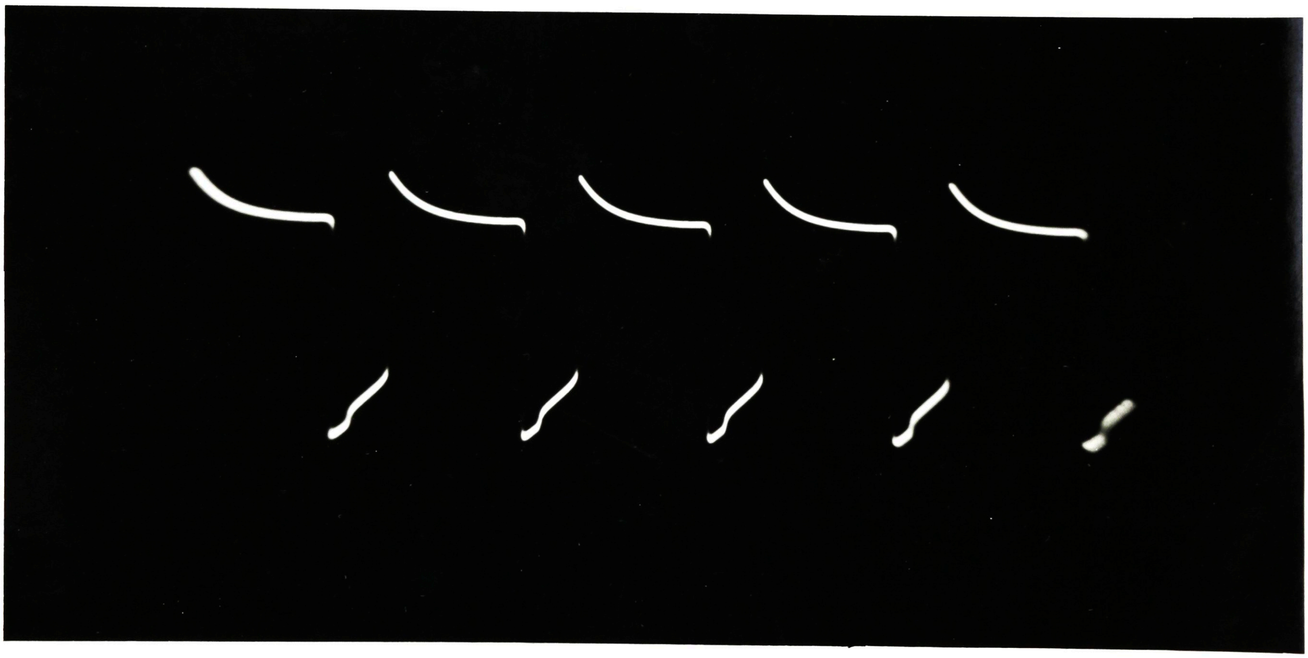

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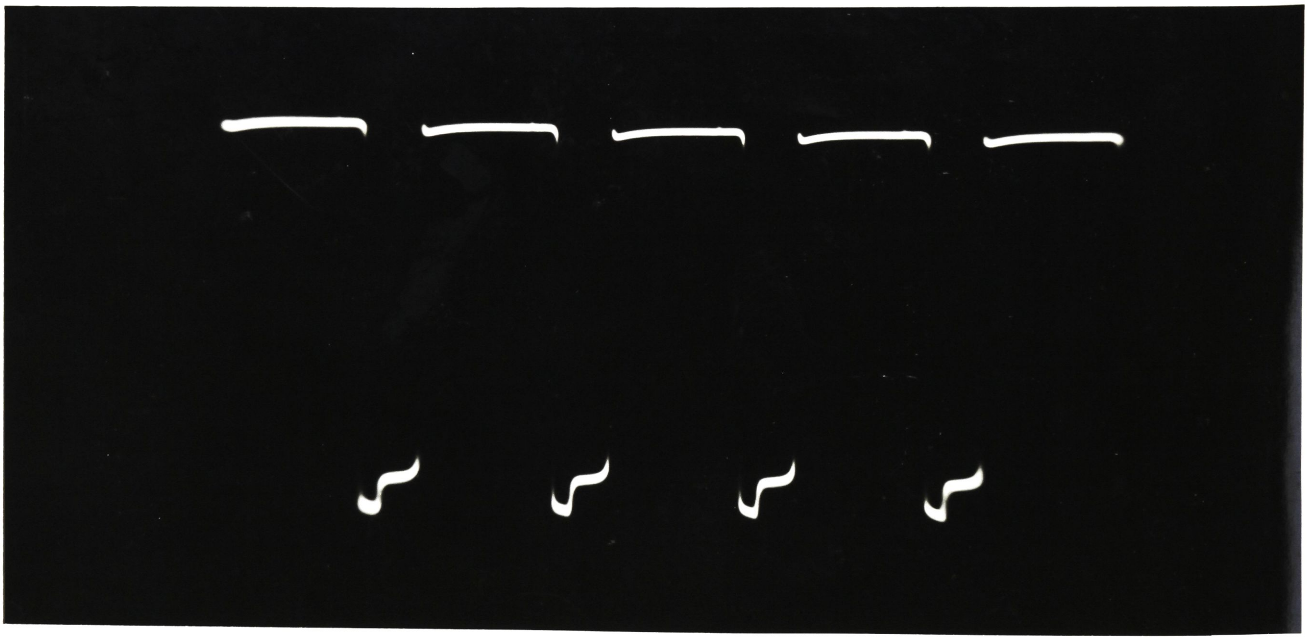

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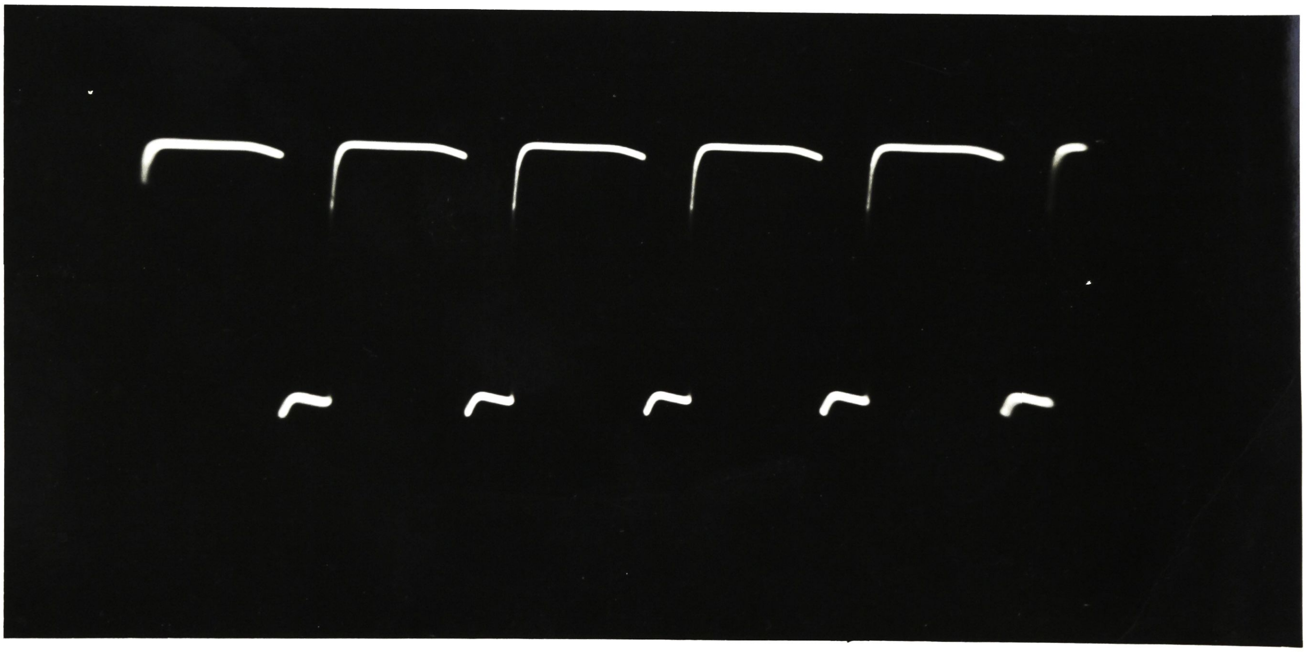

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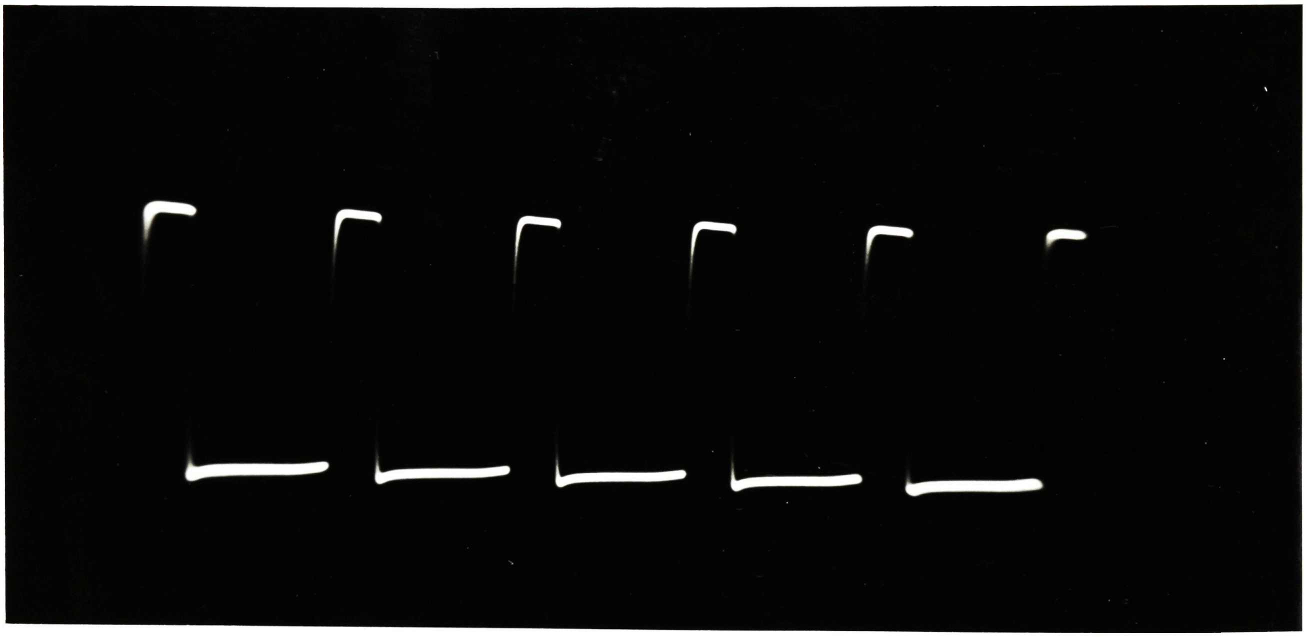

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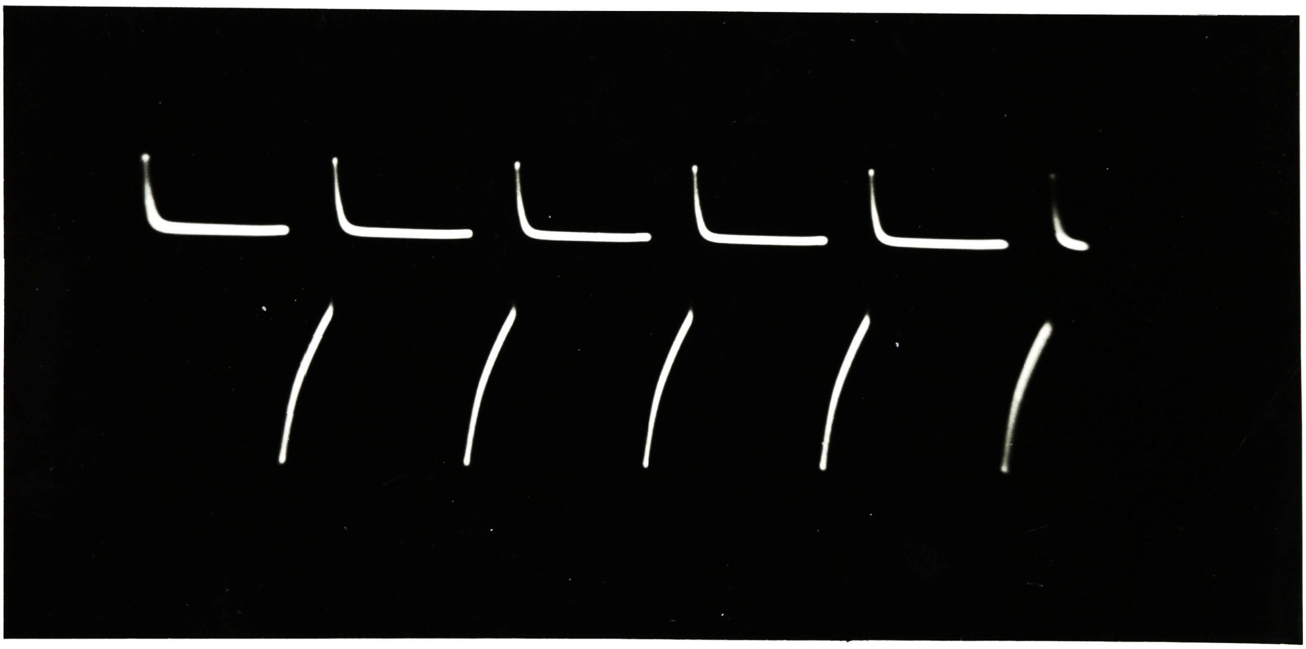

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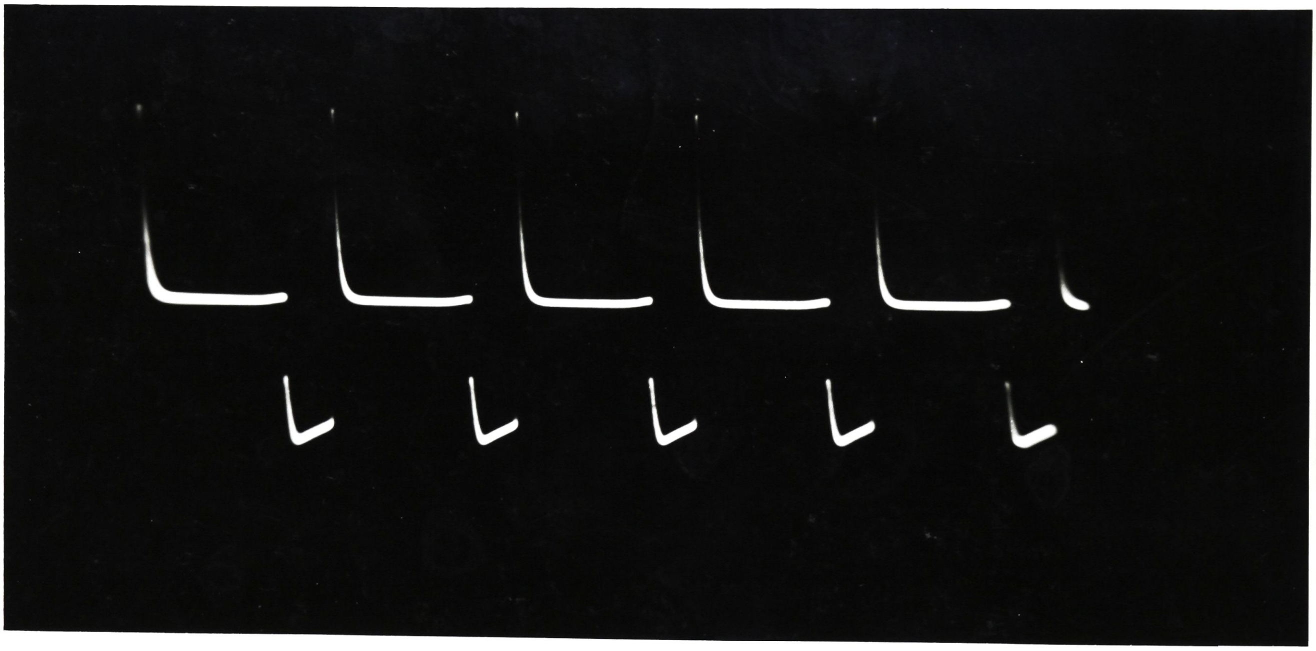

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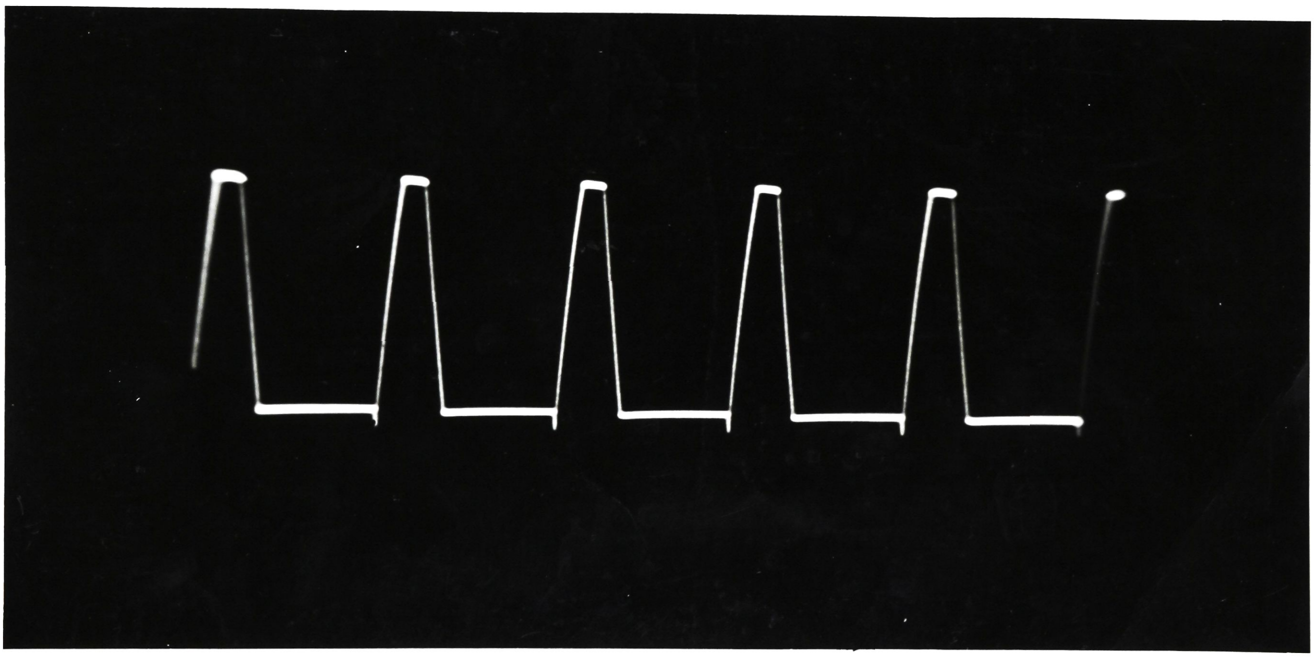

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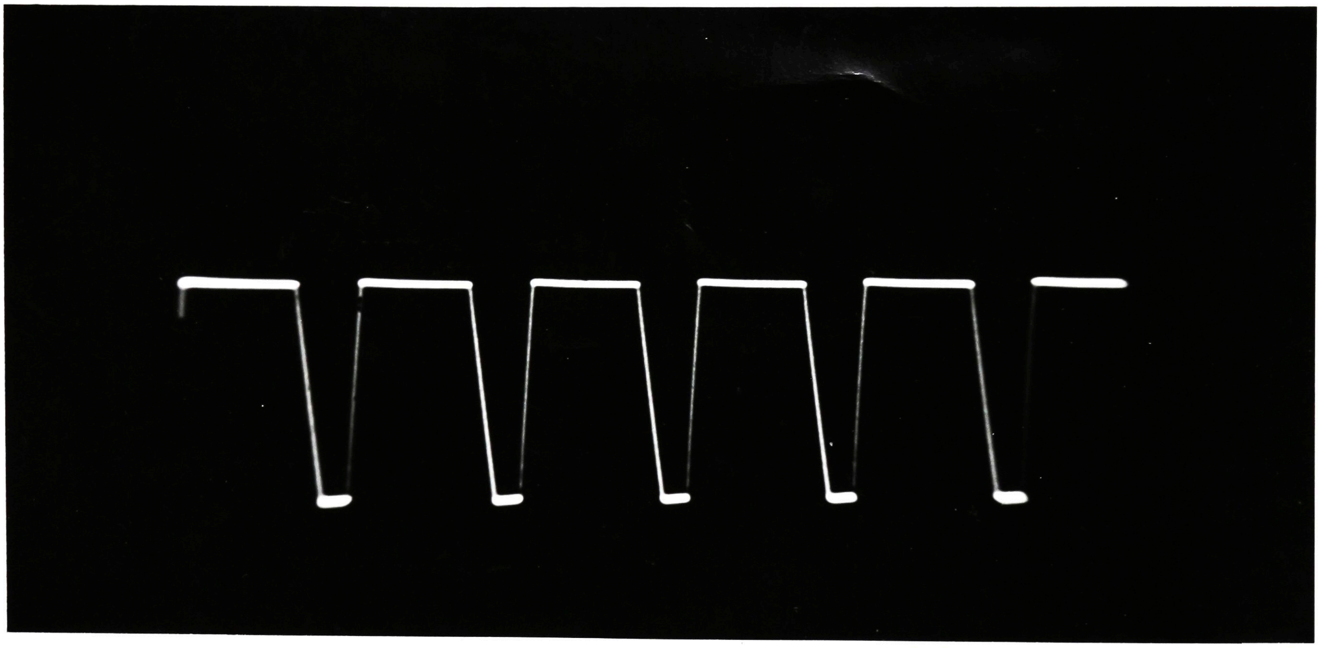

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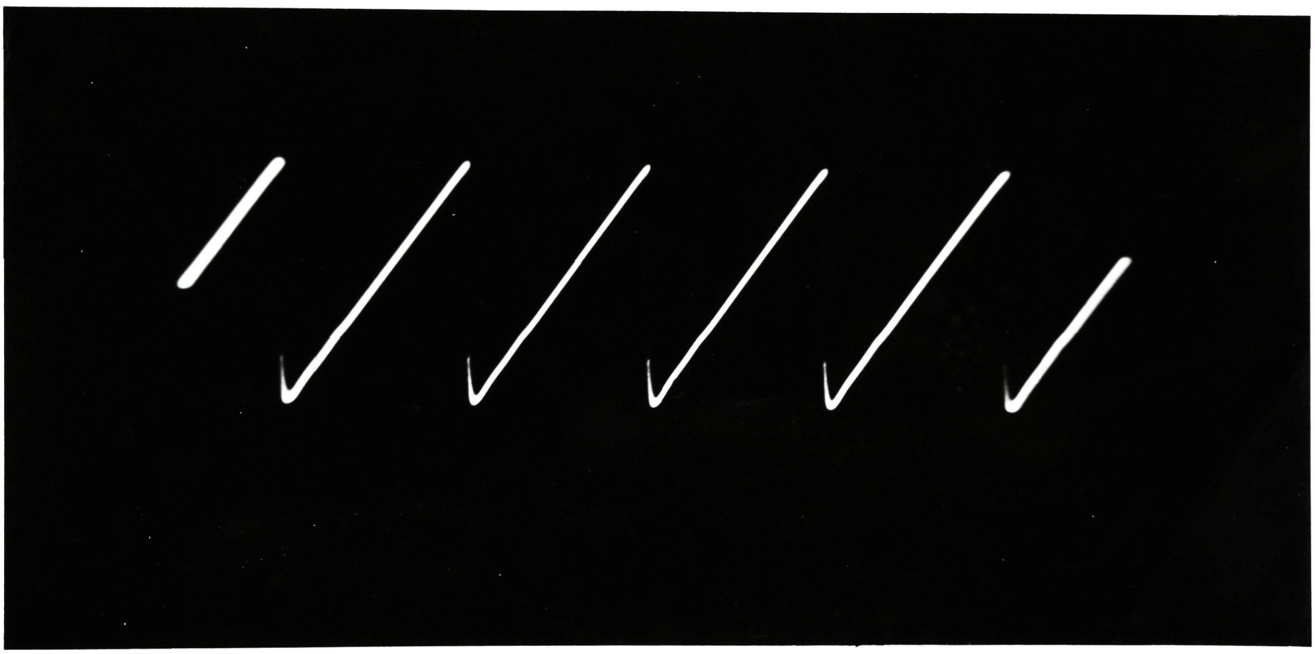

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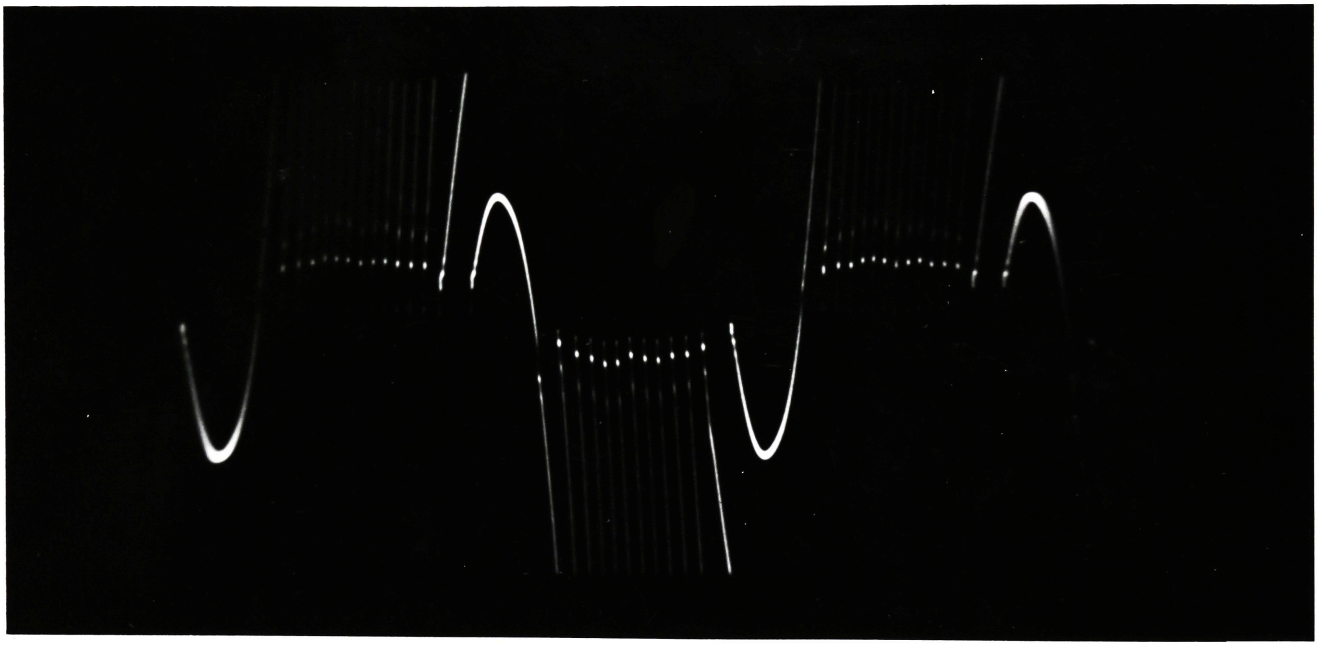

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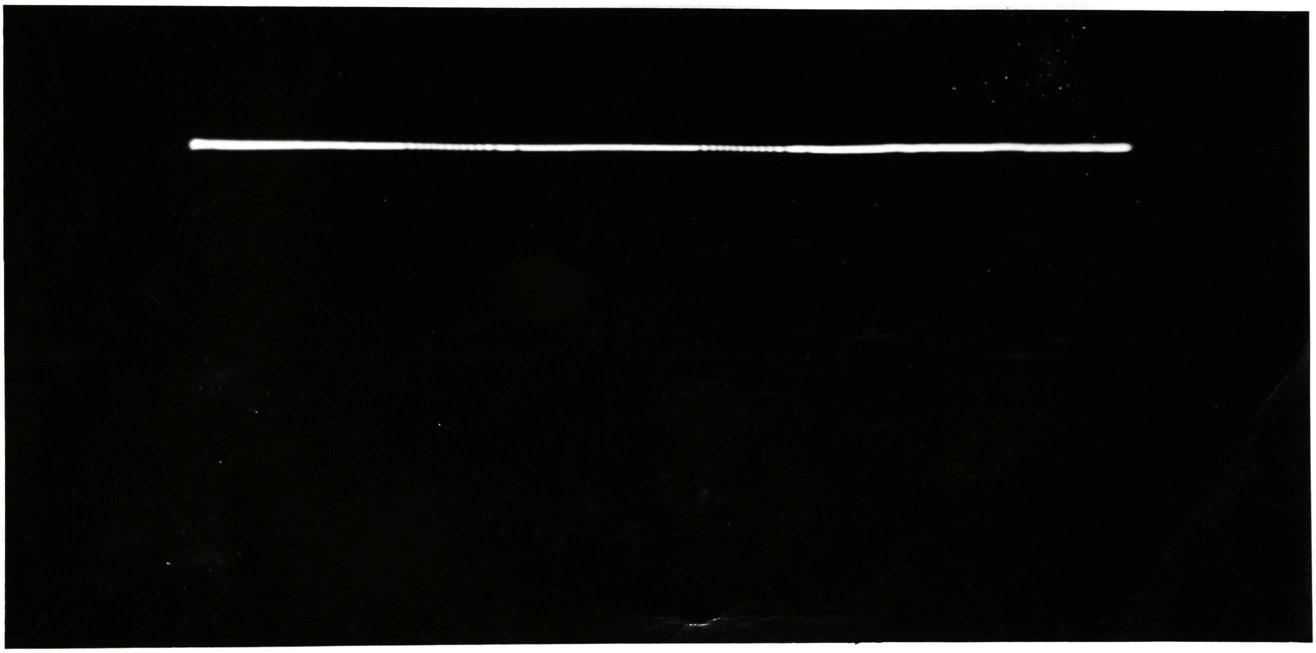

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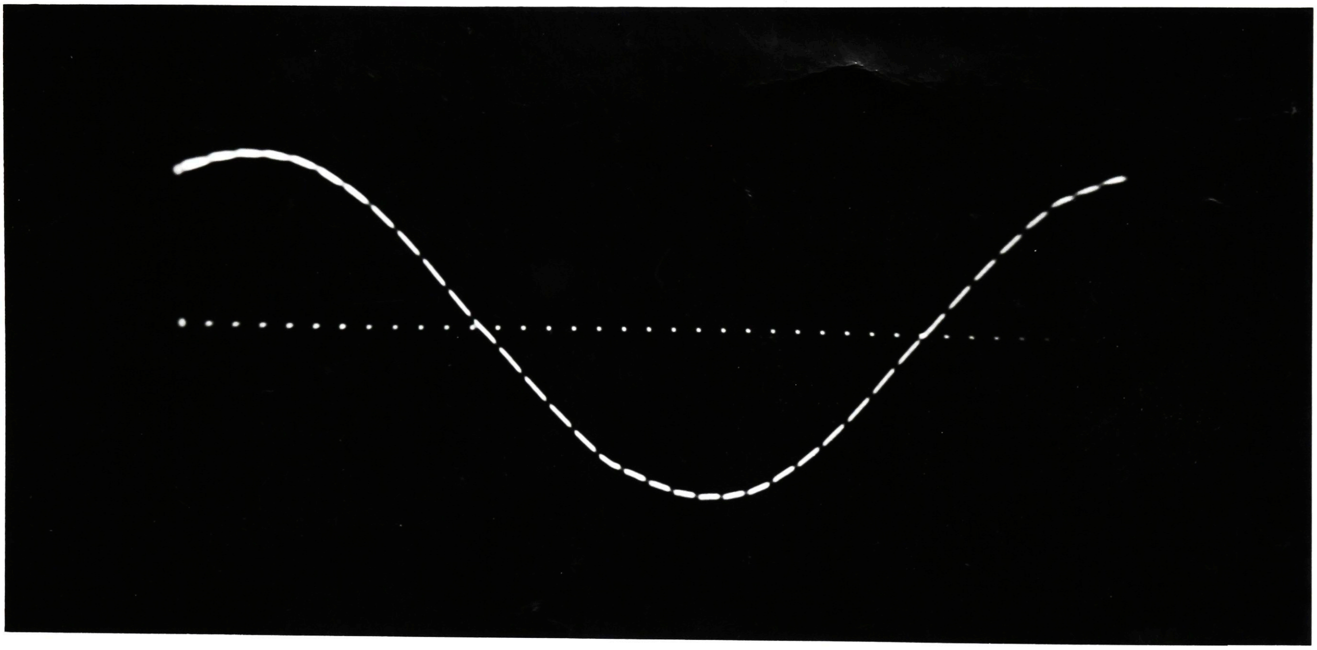

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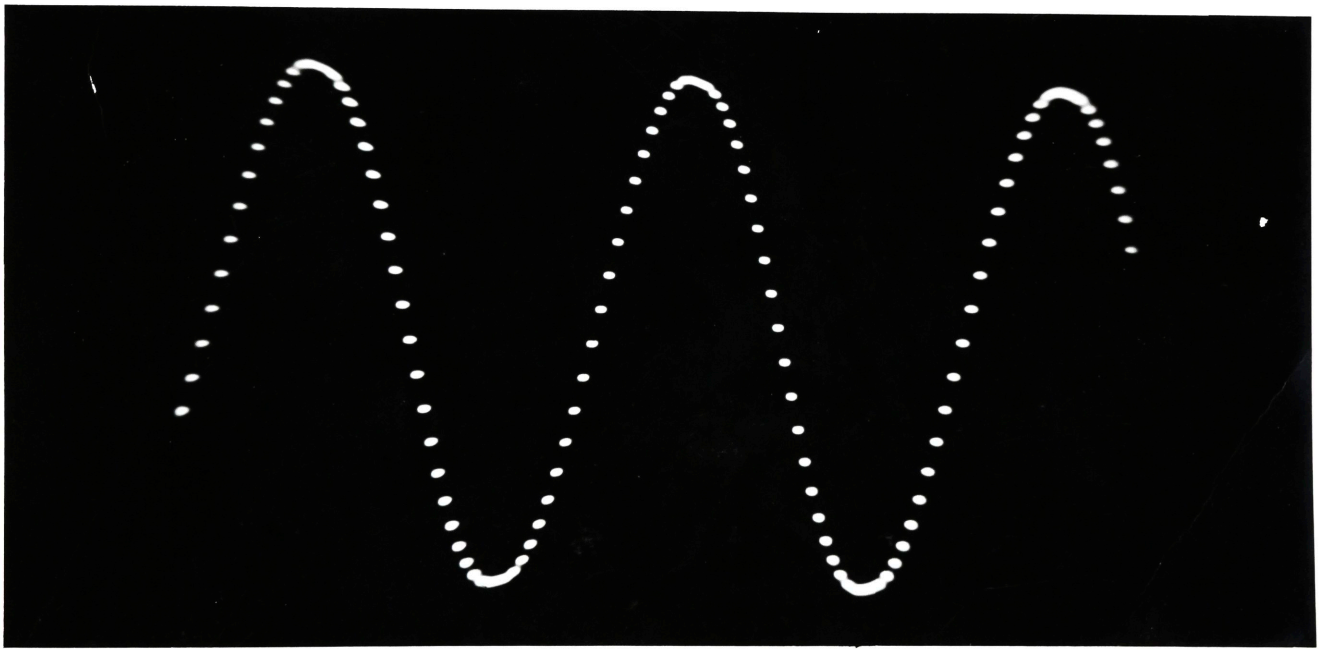

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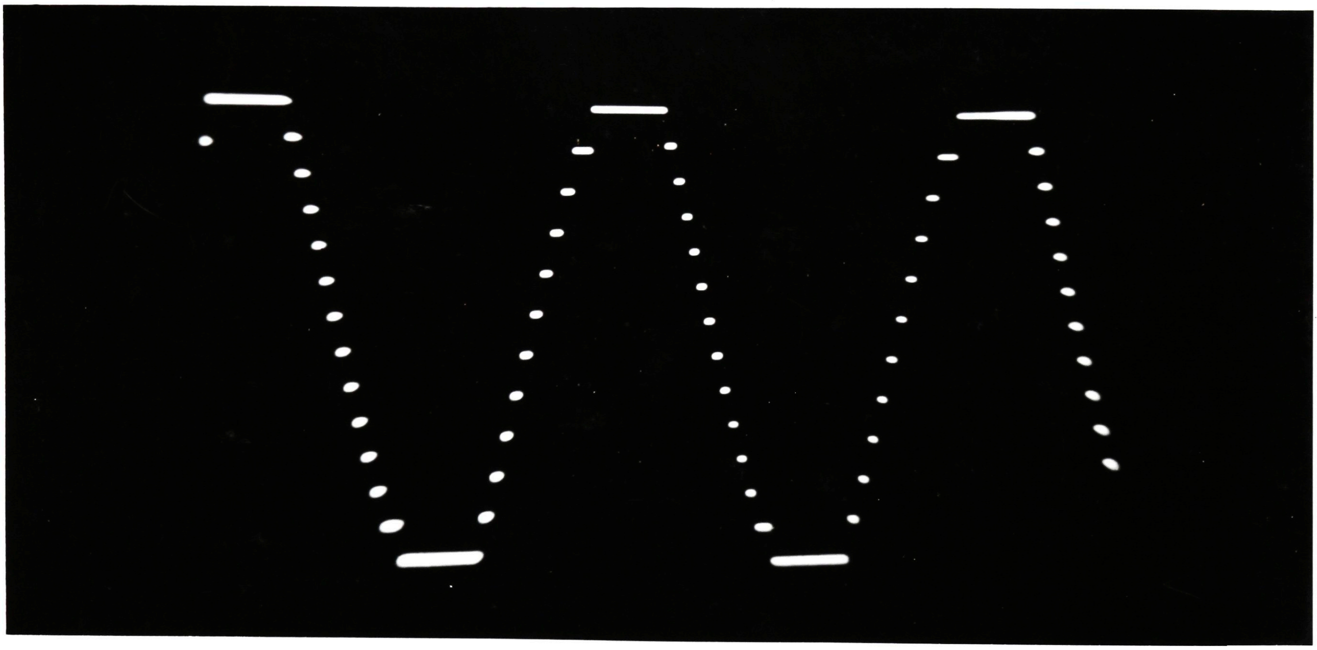

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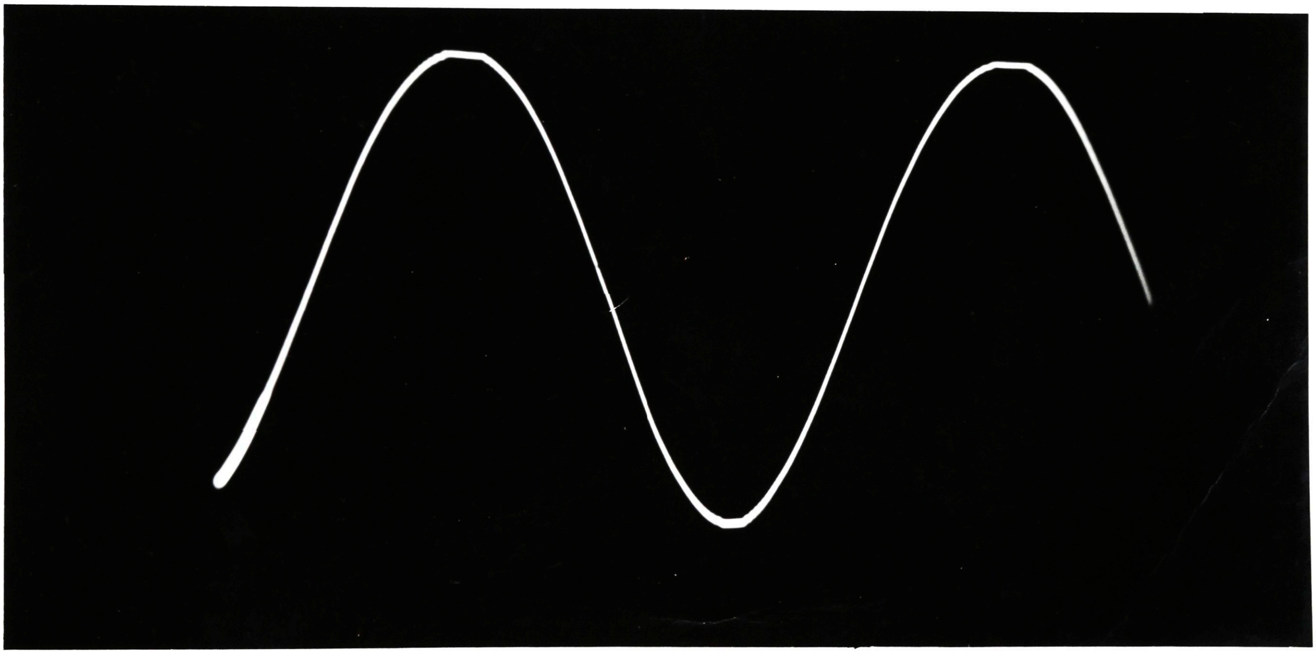

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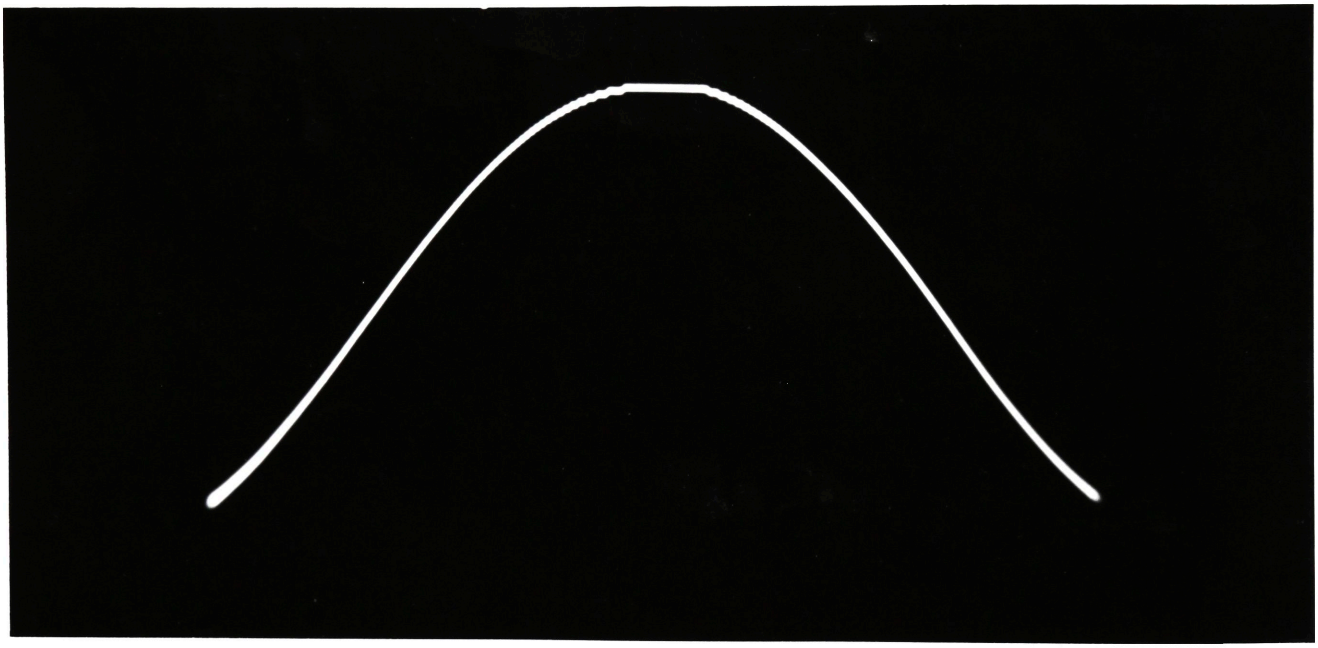
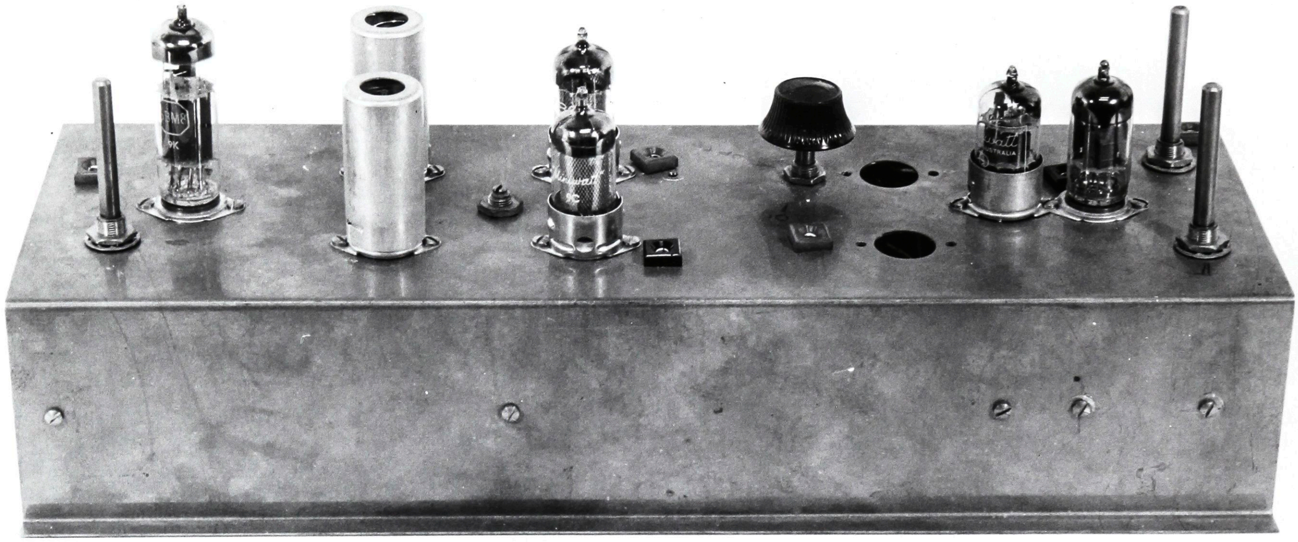
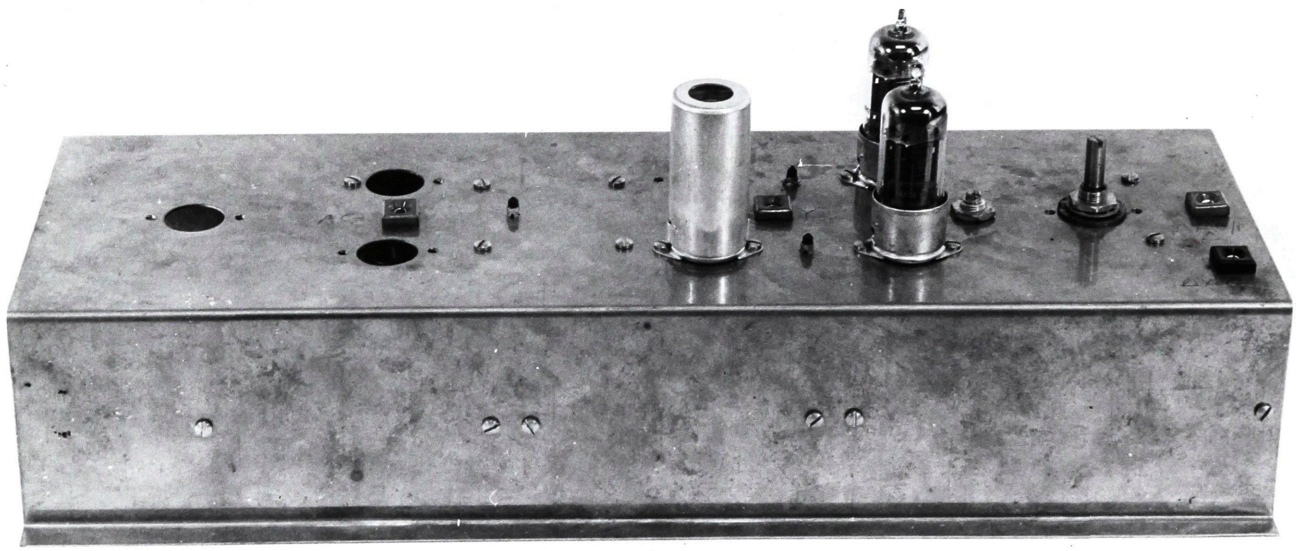






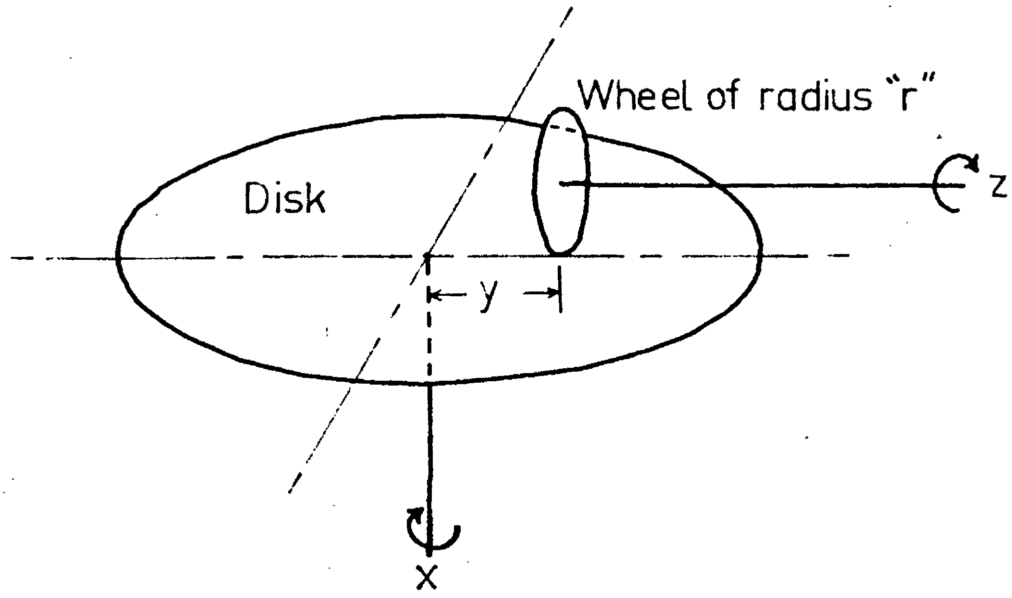


FIG. 4·1a. WHEEL AND DISK INTEGRATOR

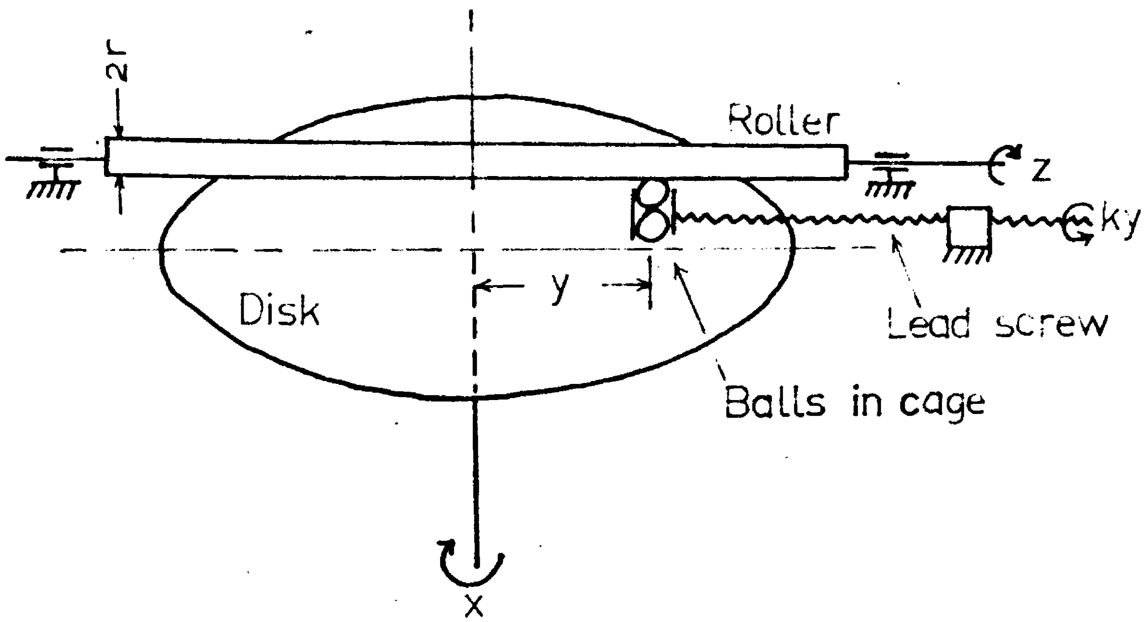


FIG. 4·1b. BALL AND DISK INTEGRATOR

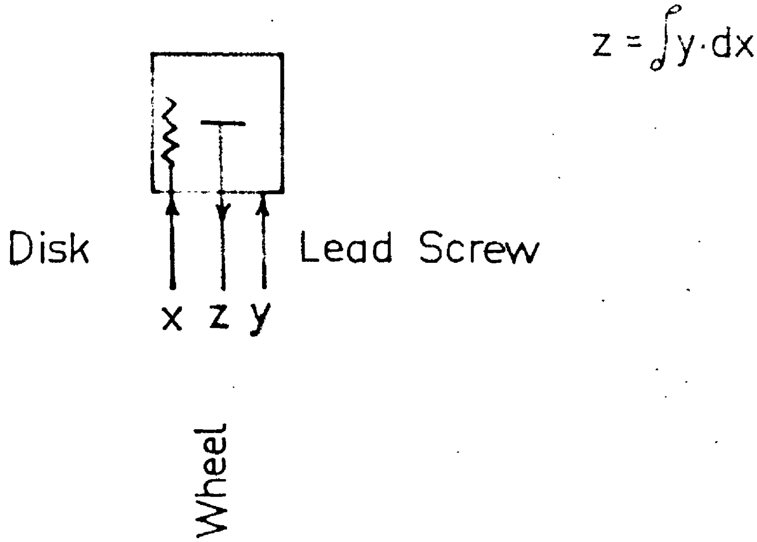


FIG. 4·2. REPRESENTATION OF MECHANICAL INTEGRATOR

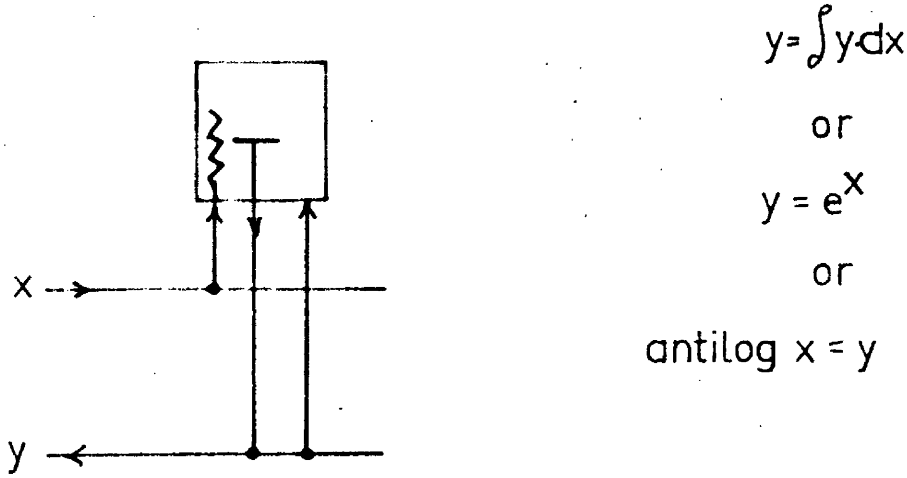


FIG. 4·3. SOLVING $y = dy/dx$

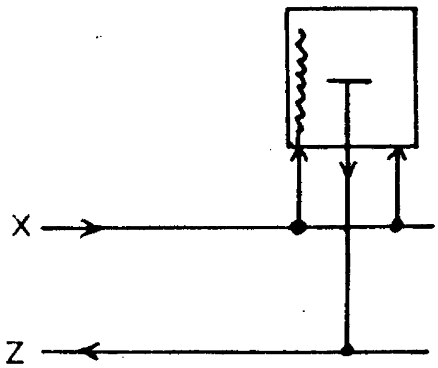


FIG. 4·4 $z = \int x \cdot dx = x^2/2$

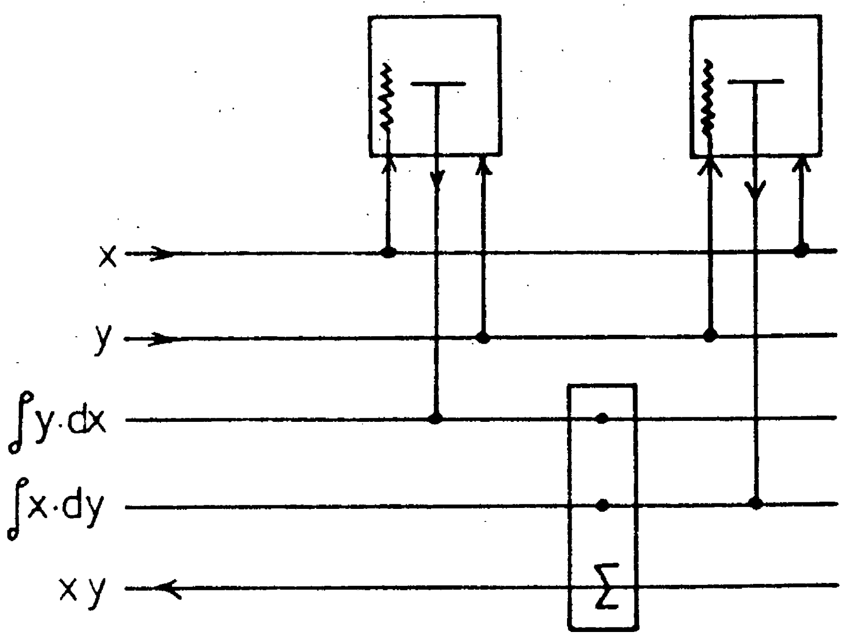


FIG. 4·5. $yx = \int x \cdot dy + \int y \cdot dx$

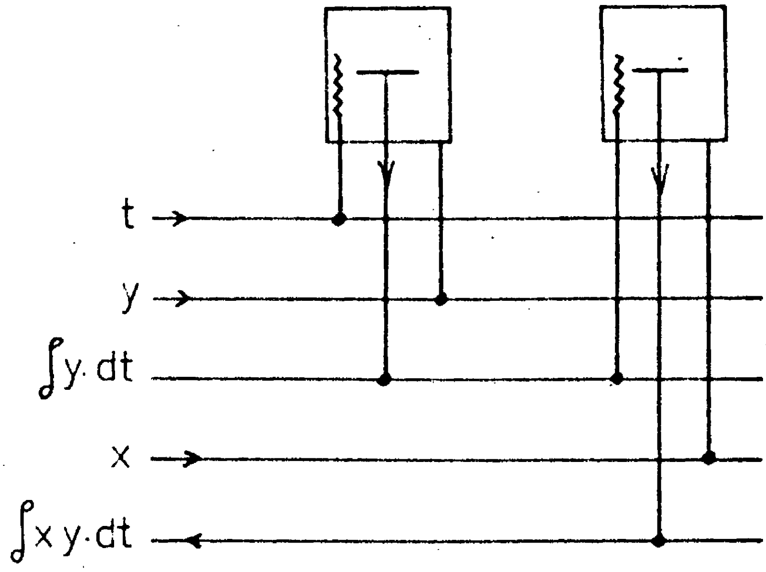


FIG. 4·6. $\int x\,y\cdot dt = \int x\cdot d\left[\int y\cdot dt\right]$

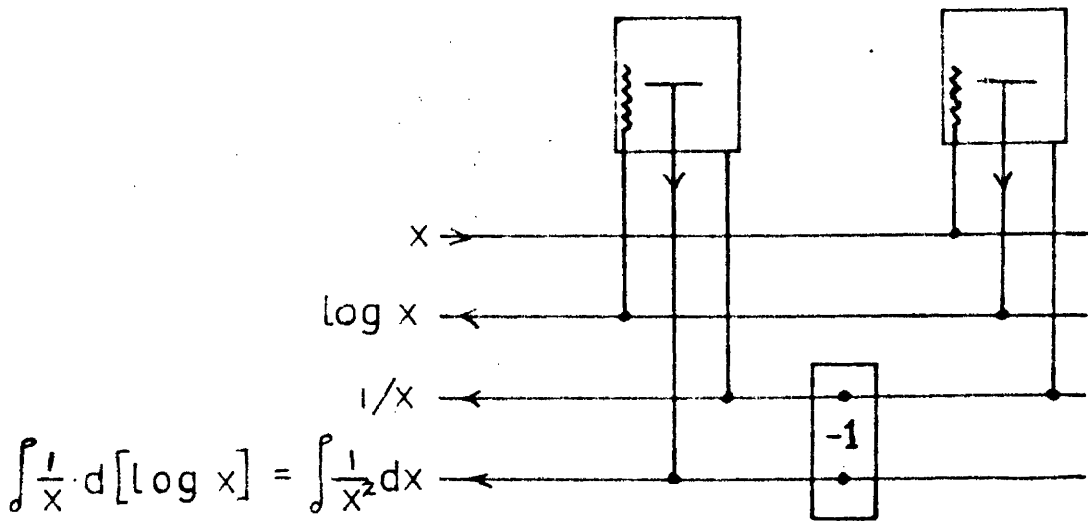


FIG. 4·7. $\text{Log}\, x = \int\left[\int (1/x)\cdot d(\log x)\right] dx$

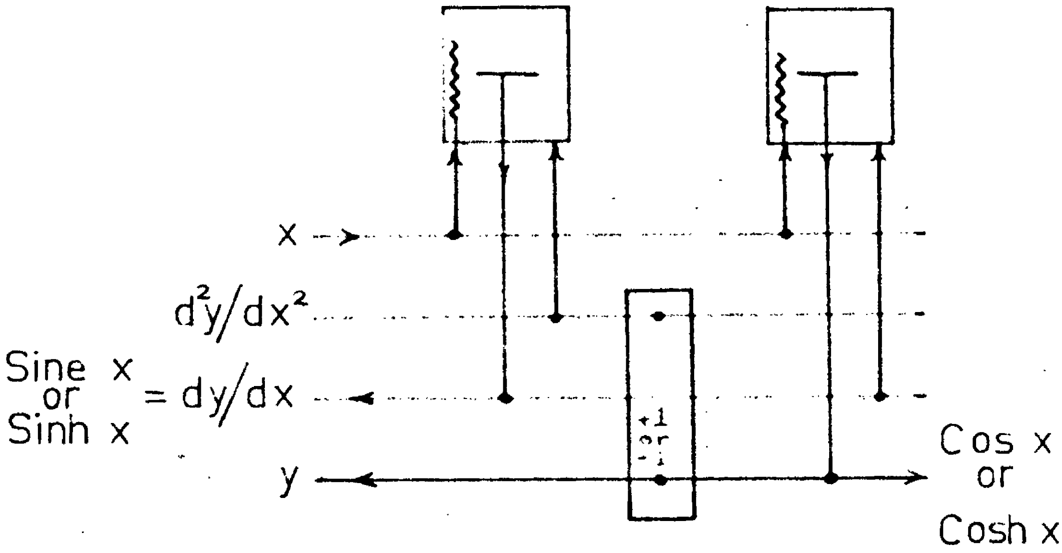


FIG. 4·8. $d^2y/dx^2 = y$ or $-y$

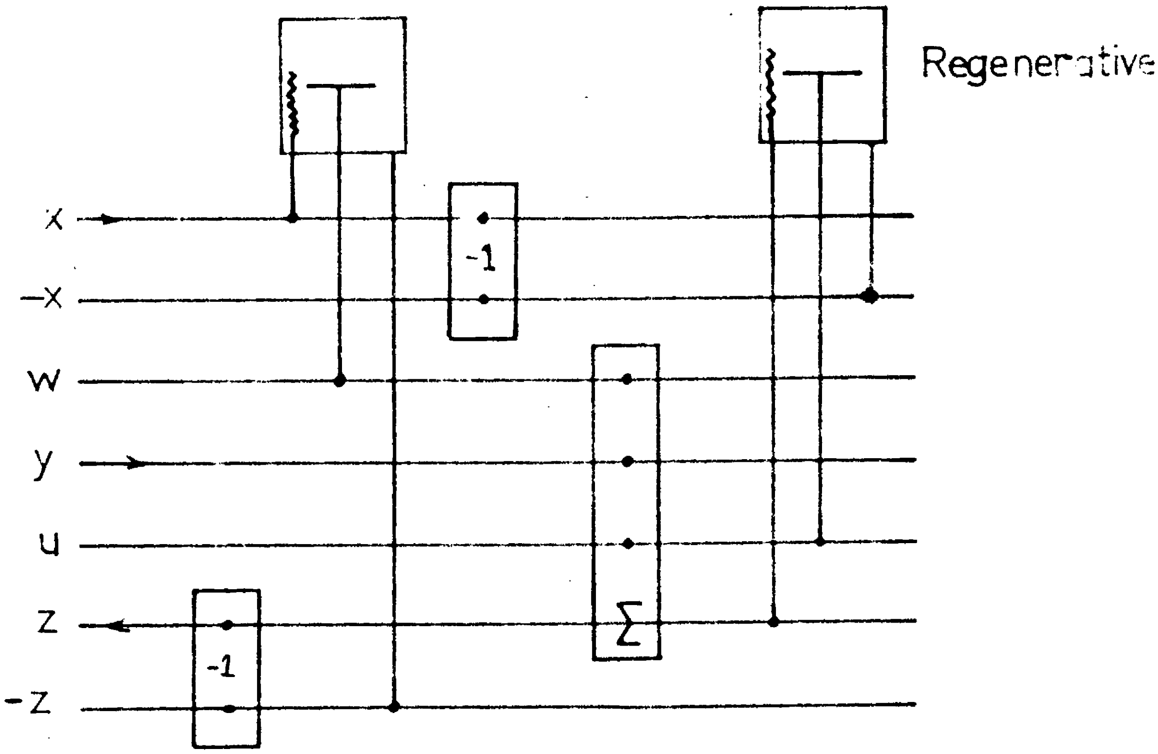


FIG. 4·9. $z = y/x$

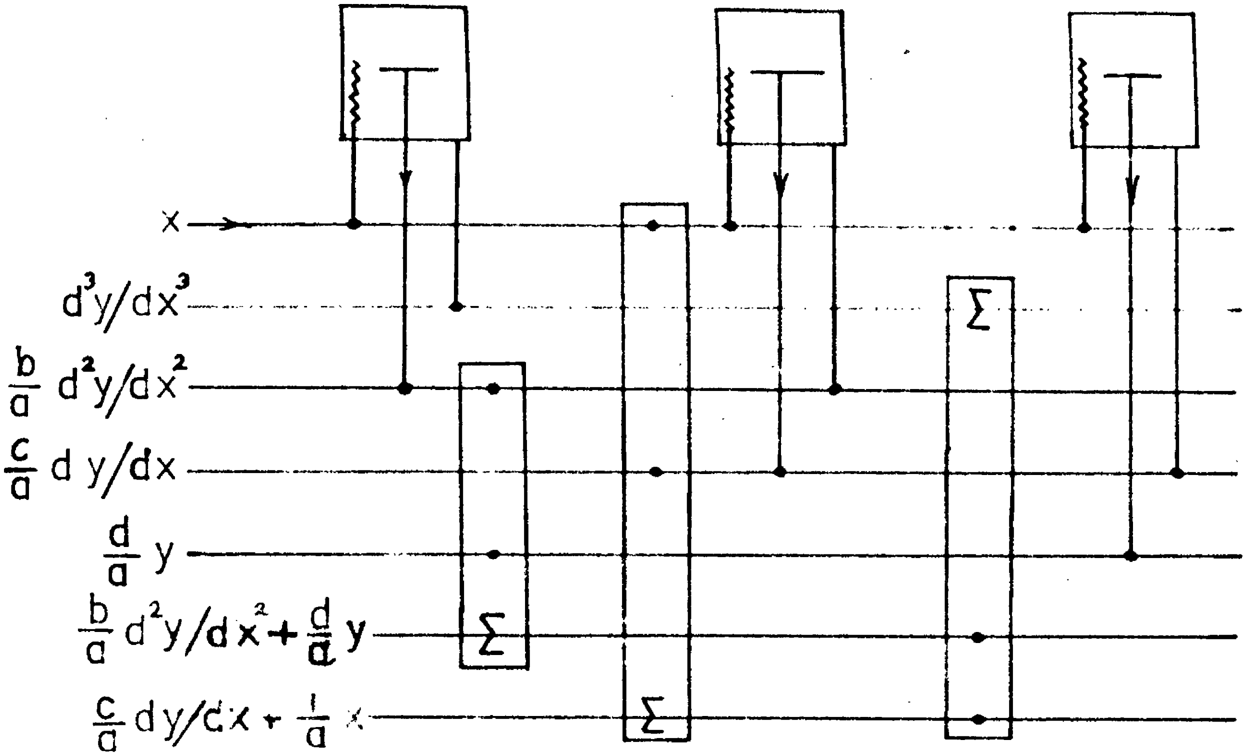


FIG. 4·10. Solution of the linear differential equation $a\cdot d^3y/dx^3 + b\cdot d^2y/dx^2 + c\cdot dy/dx + d\cdot y + x = 0$

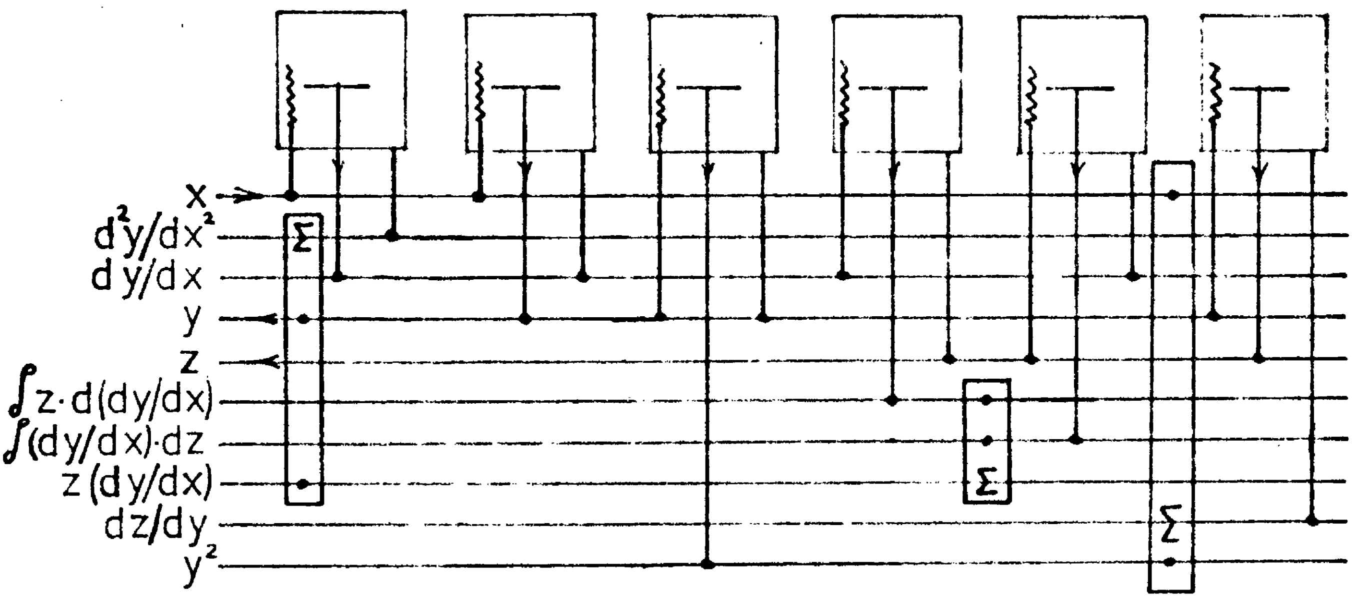


FIG. 4·11 Solution of the non-linear simultaneous differential equations: $d^2y/dx^2 = z\cdot dy/dx + y$
$dz/dy = y^2 + x$

x already known

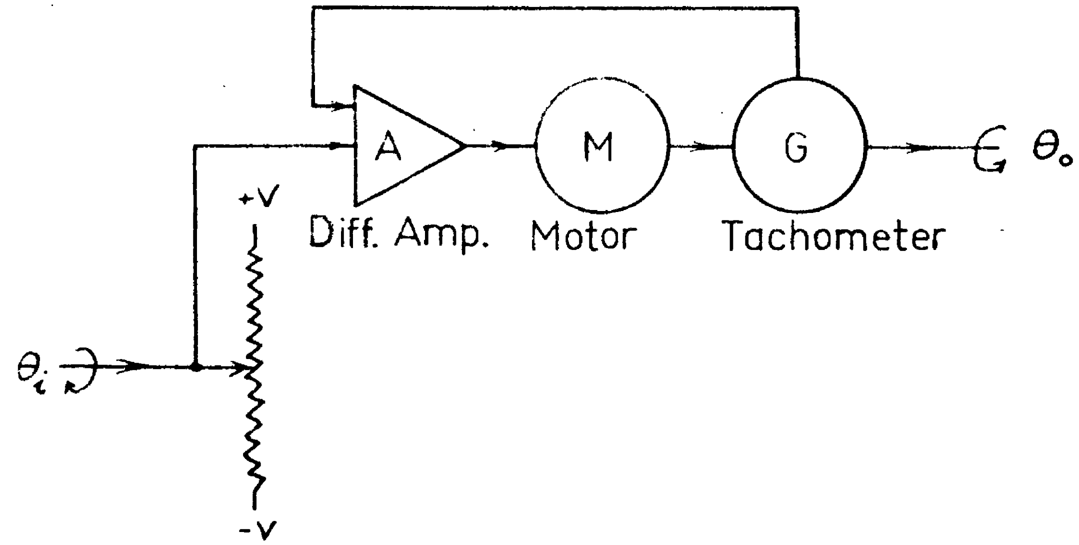


FIG. 4·12. Electromechanical integrator

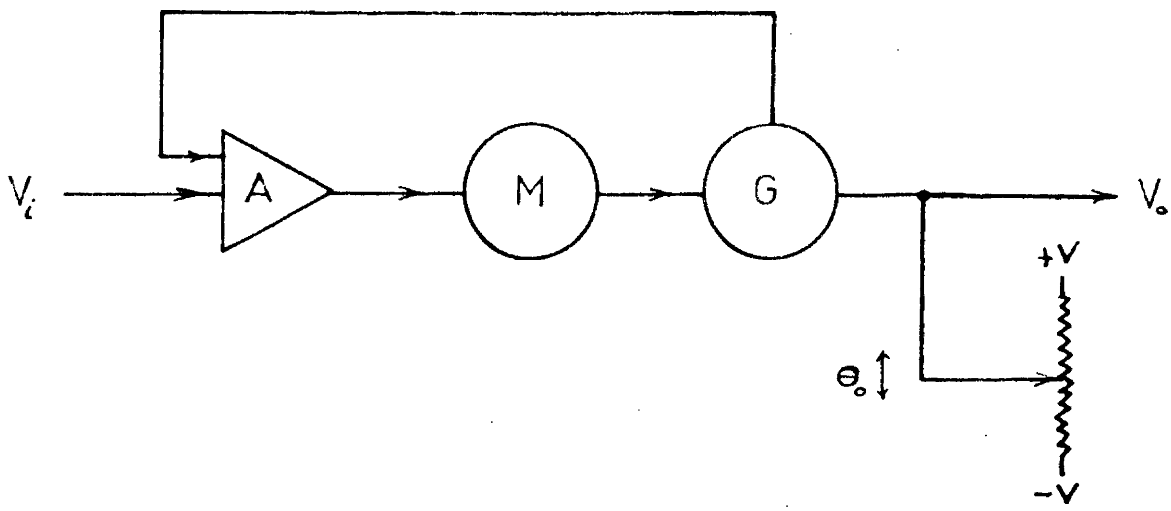


FIG. 4·13. Electromechanical integrator

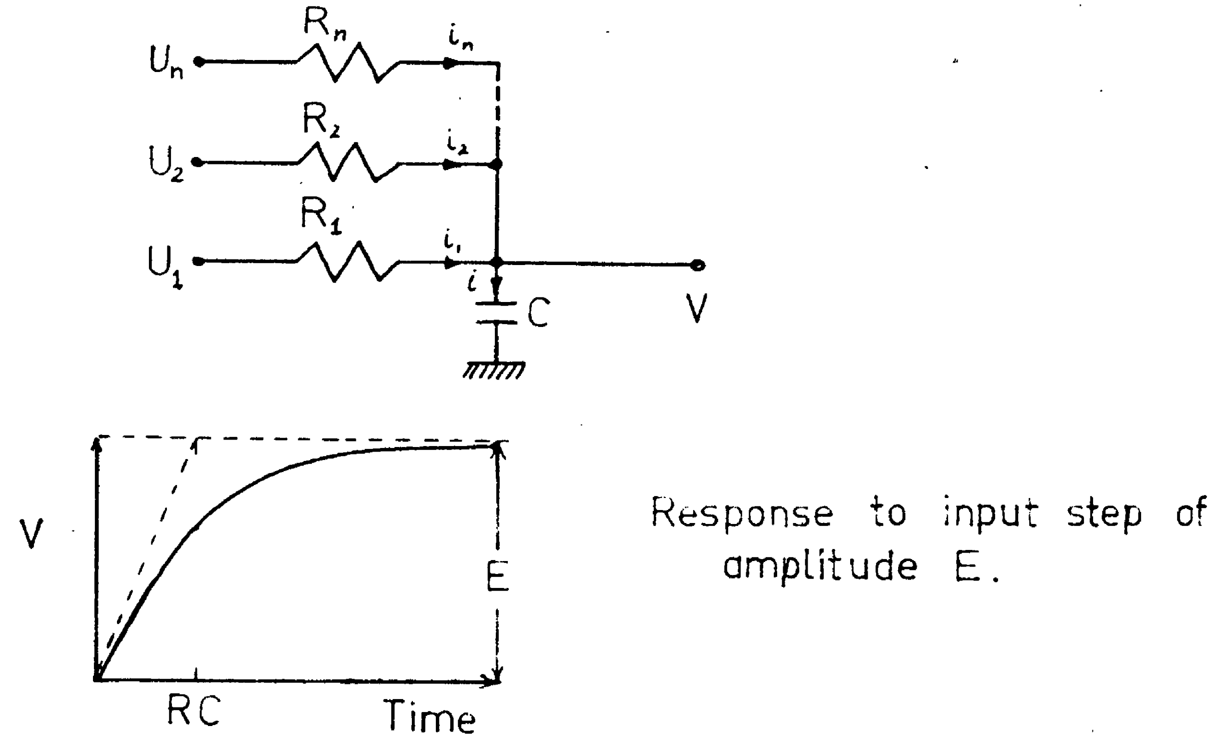


FIG. 4·14. Simple RC integrator.

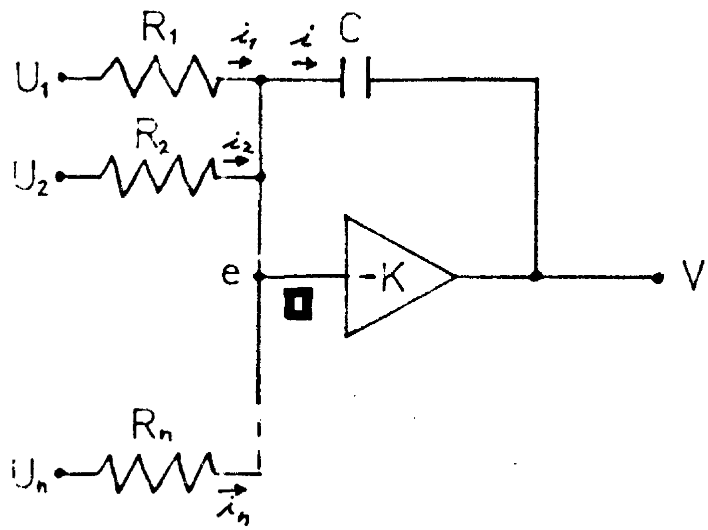


FIG. 4·15. Feedback integrator.

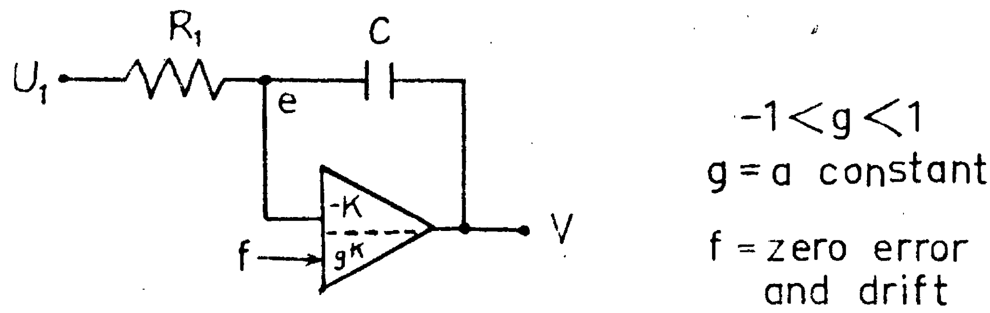


FIG. 4·16. Zero error and drift.

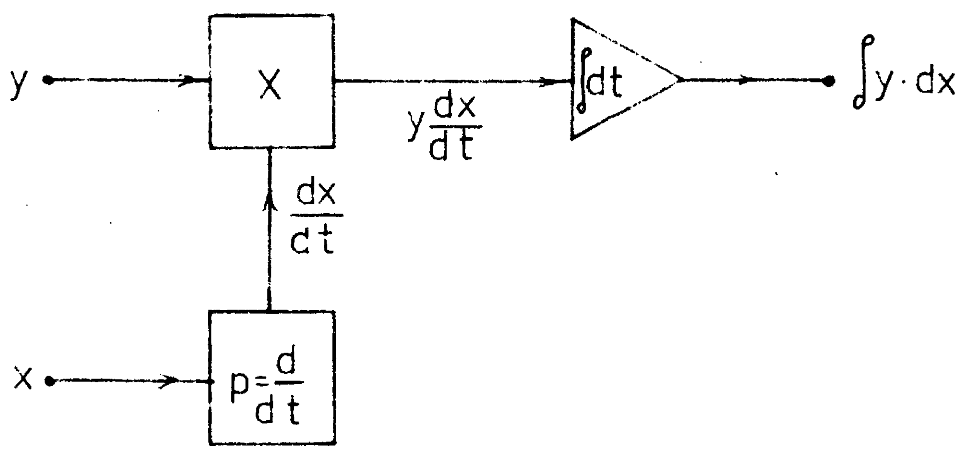


FIG. 4·17 Generalized integration.

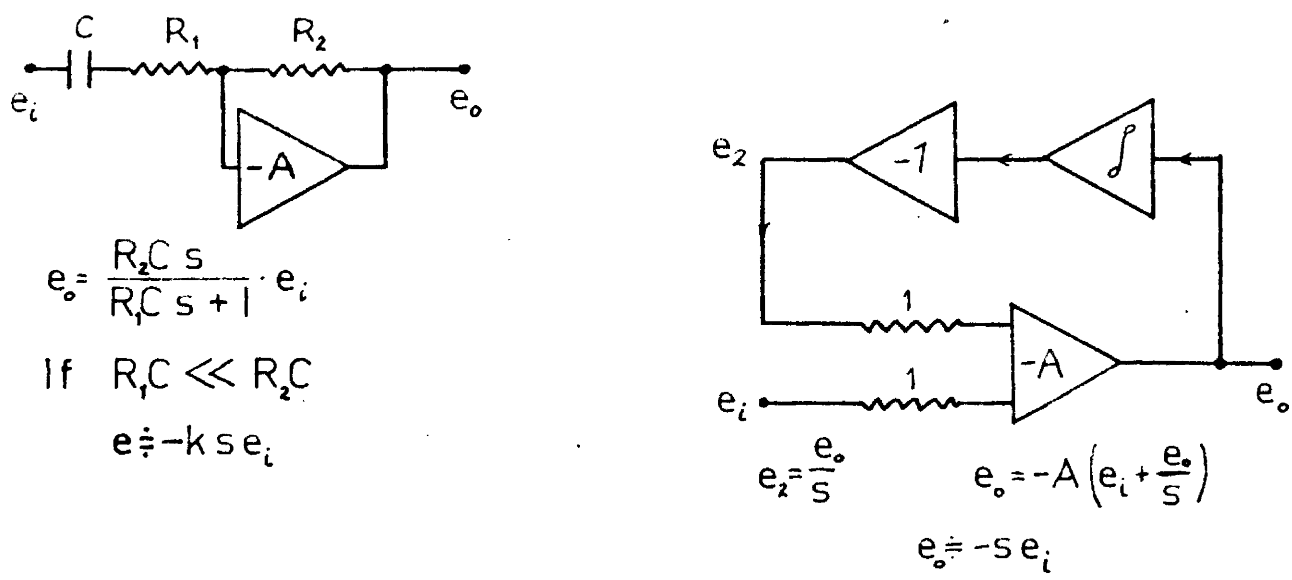


FIG. 4·18. Approximate differentiation

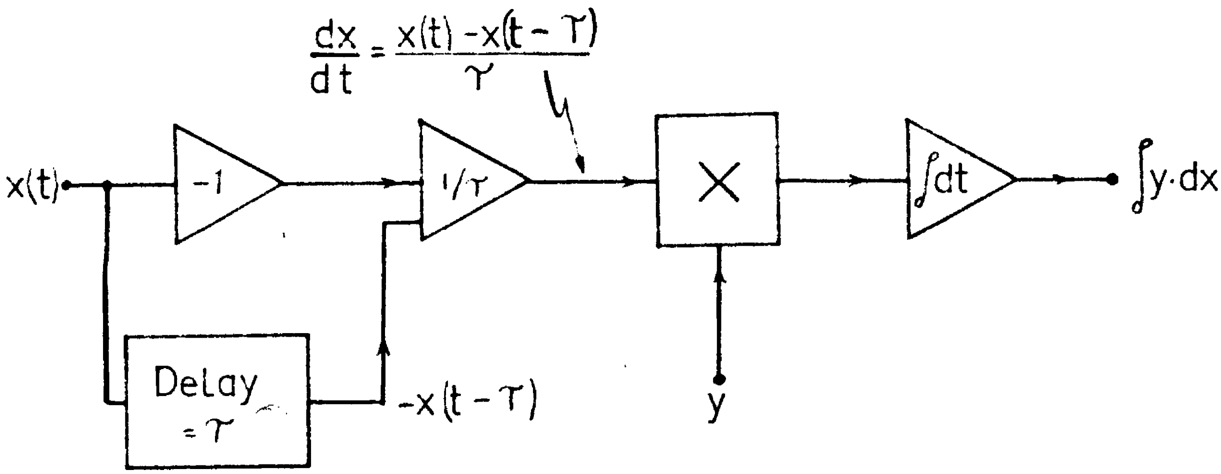


FIG. 4·19. Integration with a time delay

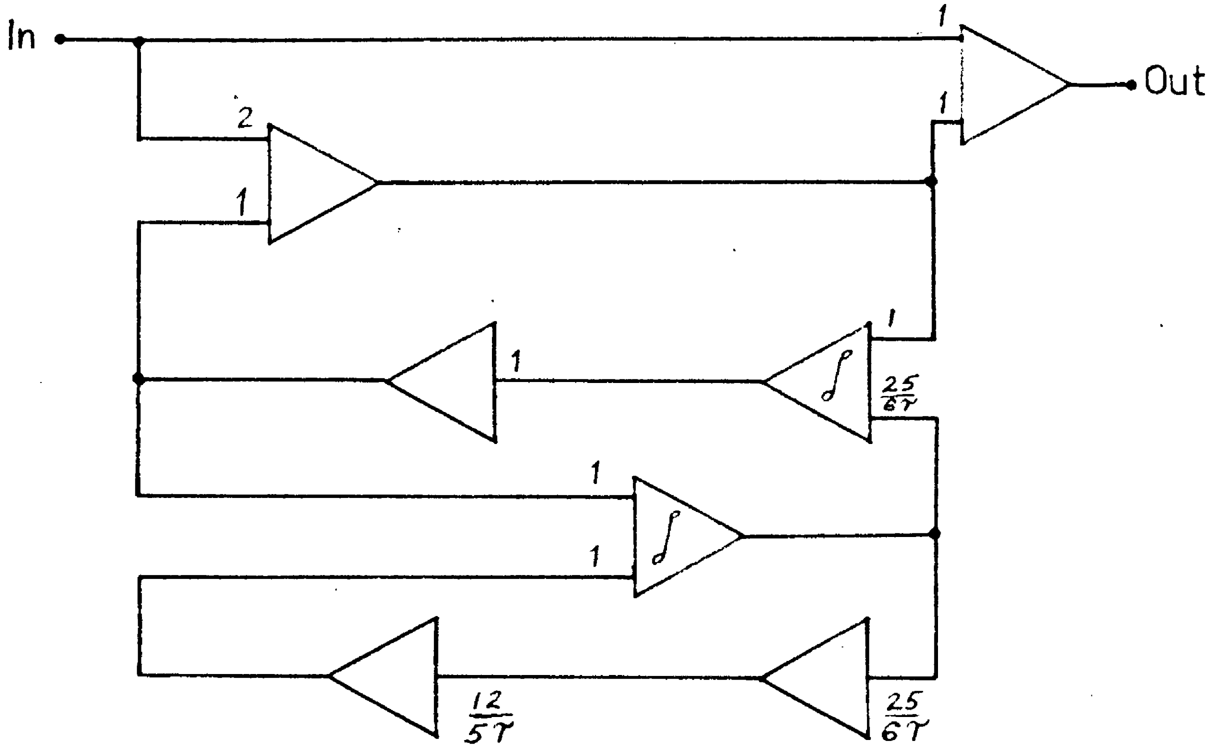
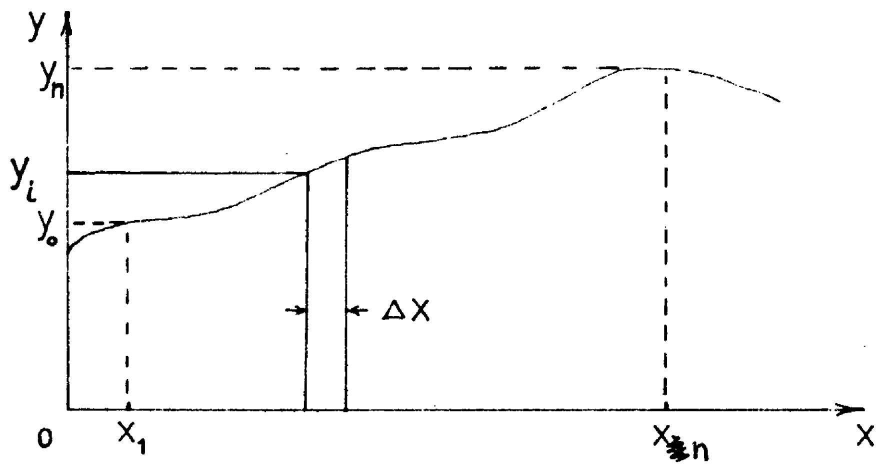


FIG. 4·20. Analogue realization of a maximally flat delay function



$$F = \int_{x_1}^{x_n} y \cdot dx \doteq \Delta x \cdot \sum_{i=0}^{n} y_i$$

FIG. 4·21. Basis of Bekey's method

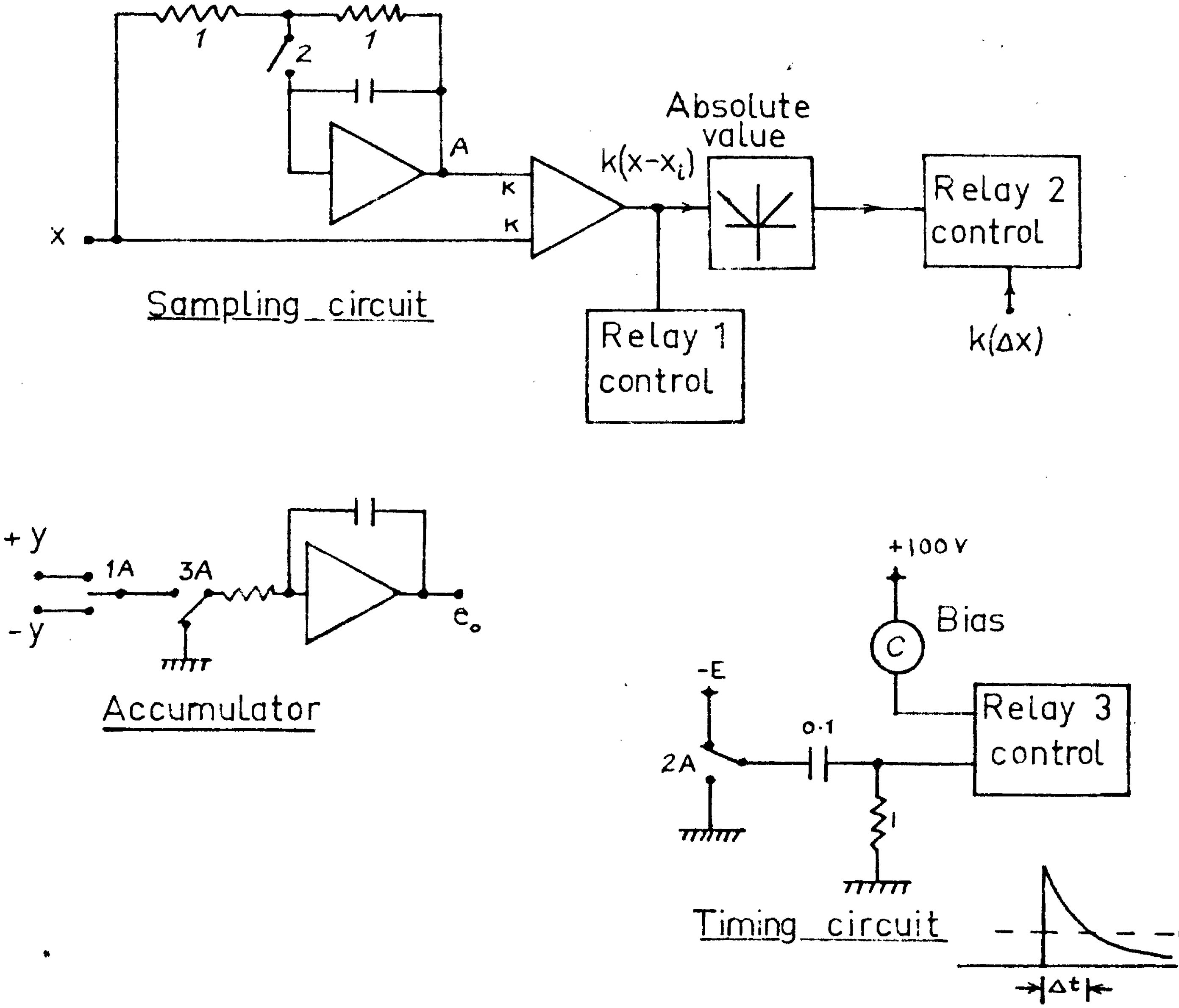


FIG. 4·22 Bekeys generalized integrator

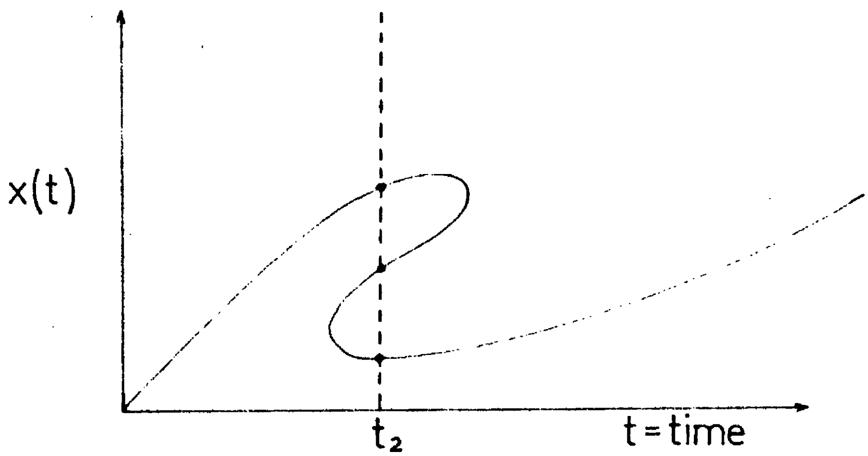


FIG. 4·23. An "impossible" function of time

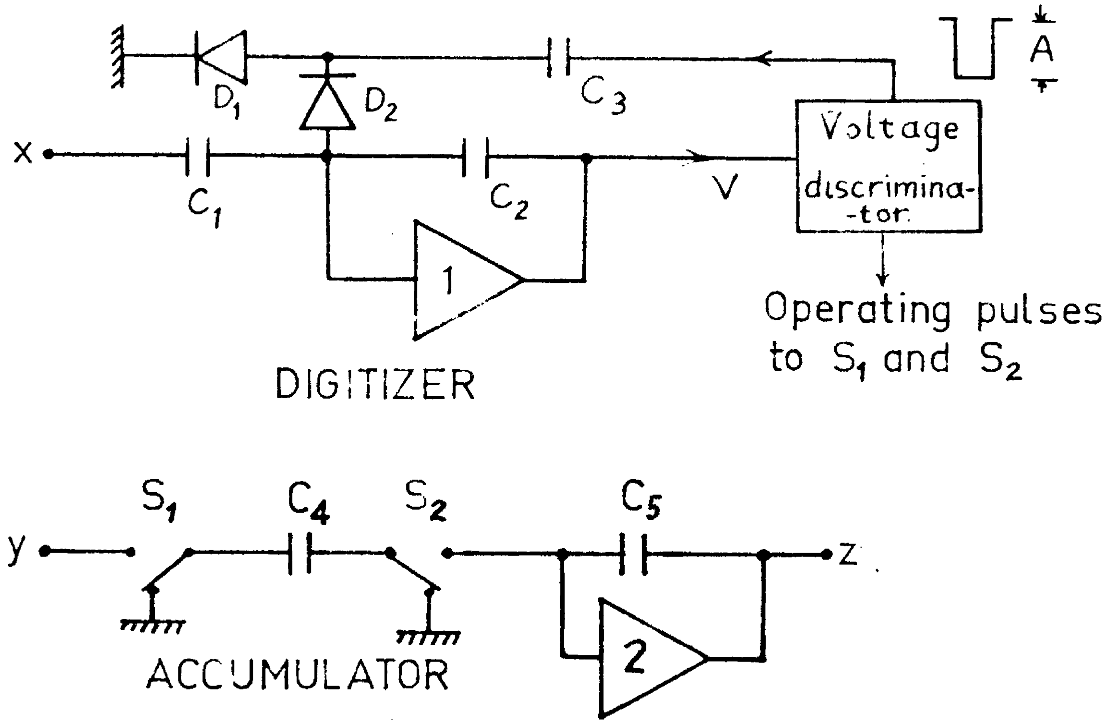


FIG. 5·1 The integrator for positive values only of x and y

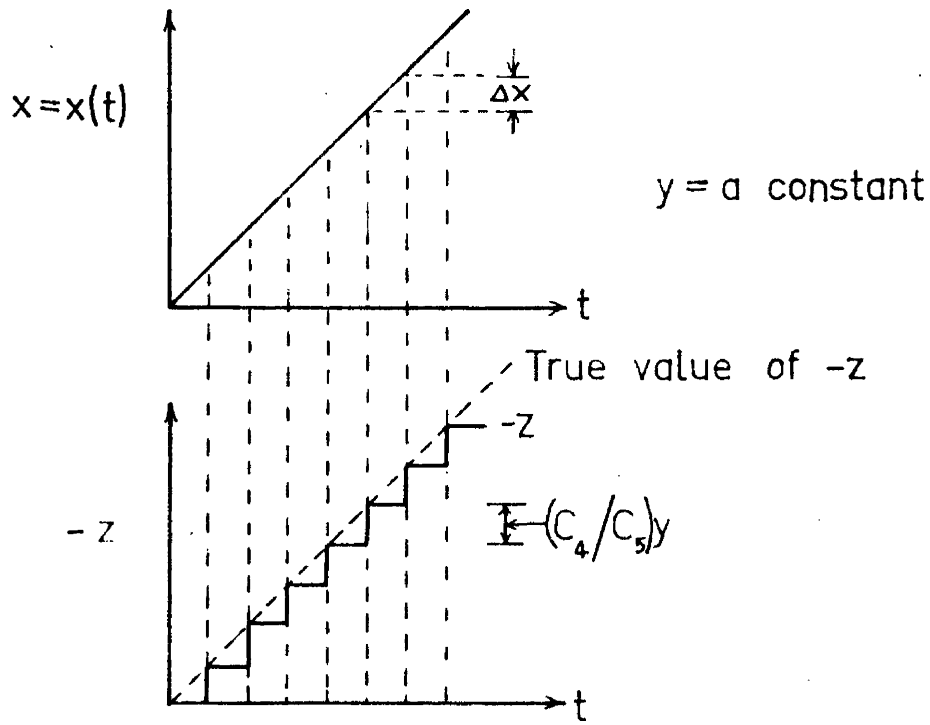
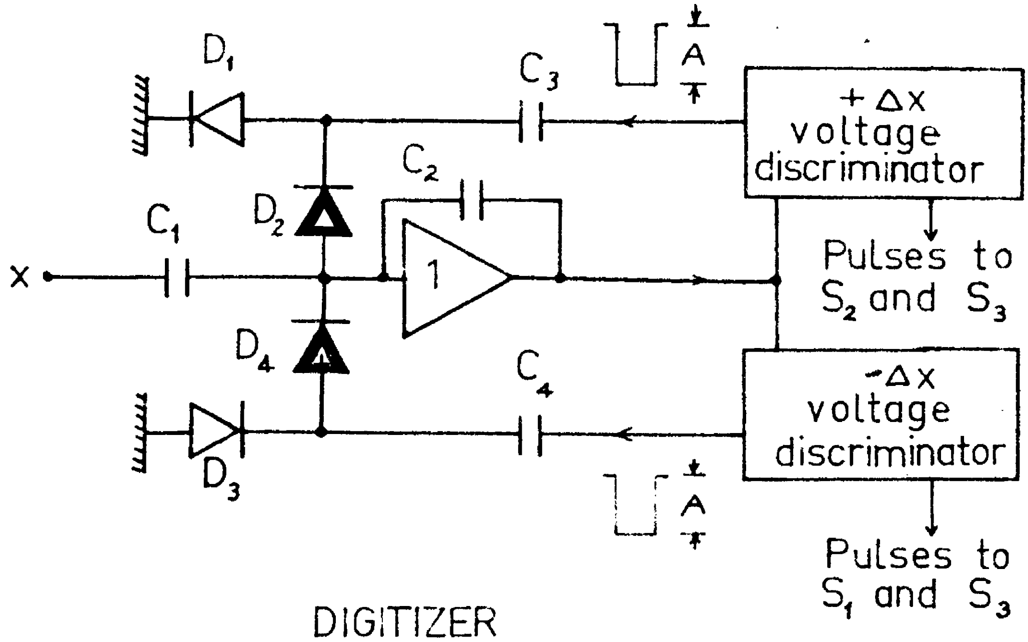
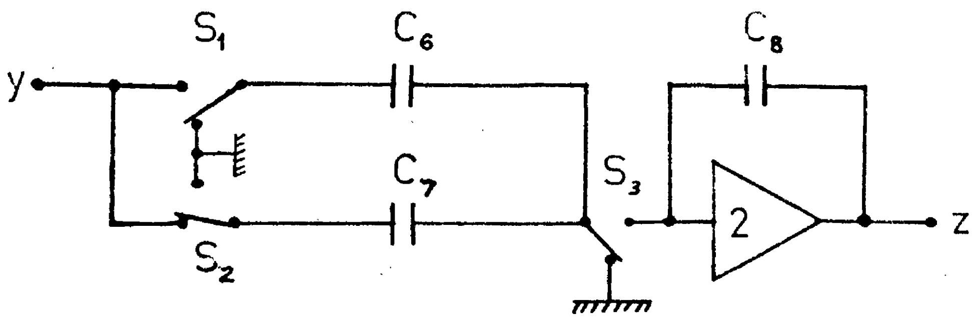


FIG. 5·2. Operation of the integrator



DIGITIZER



S_3 operates before S_1 or S_2

ACCUMULATOR

FIG. 5·3. The integrator arranged for 4 quadrant operation

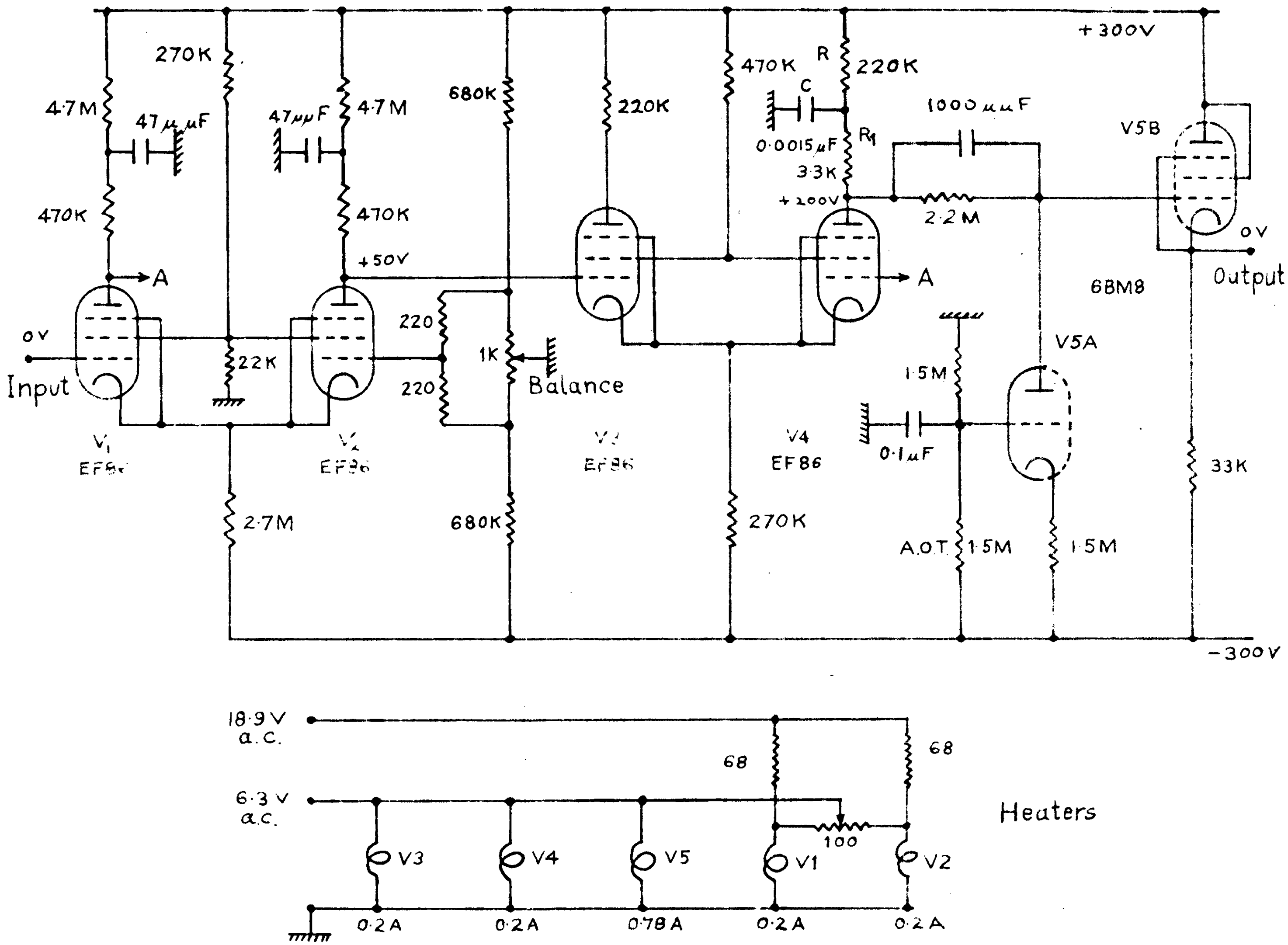


FIG.7·1 Accumulator amplifier

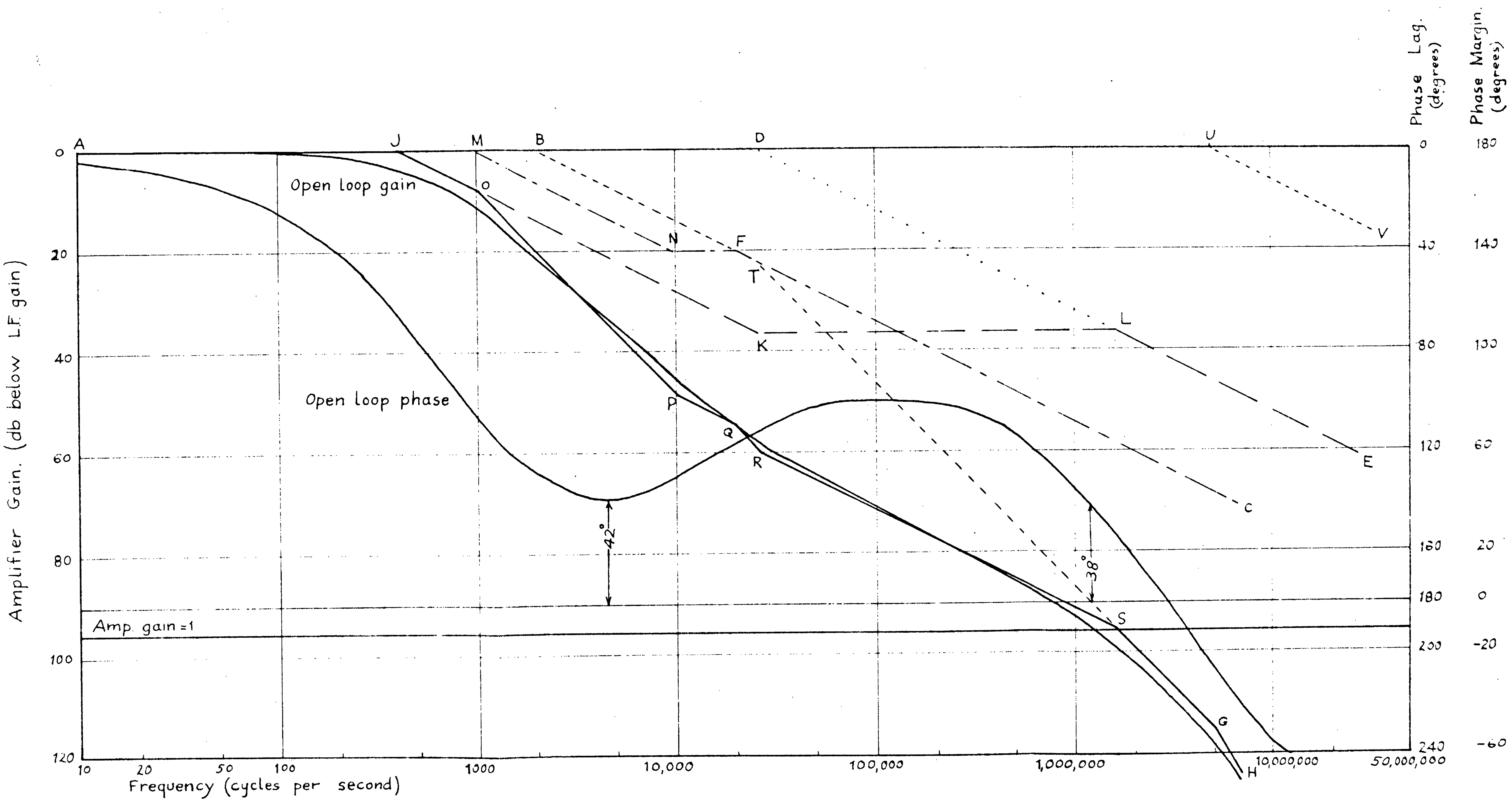


FIG. 7·2. Bode diagram for accumulator amplifier

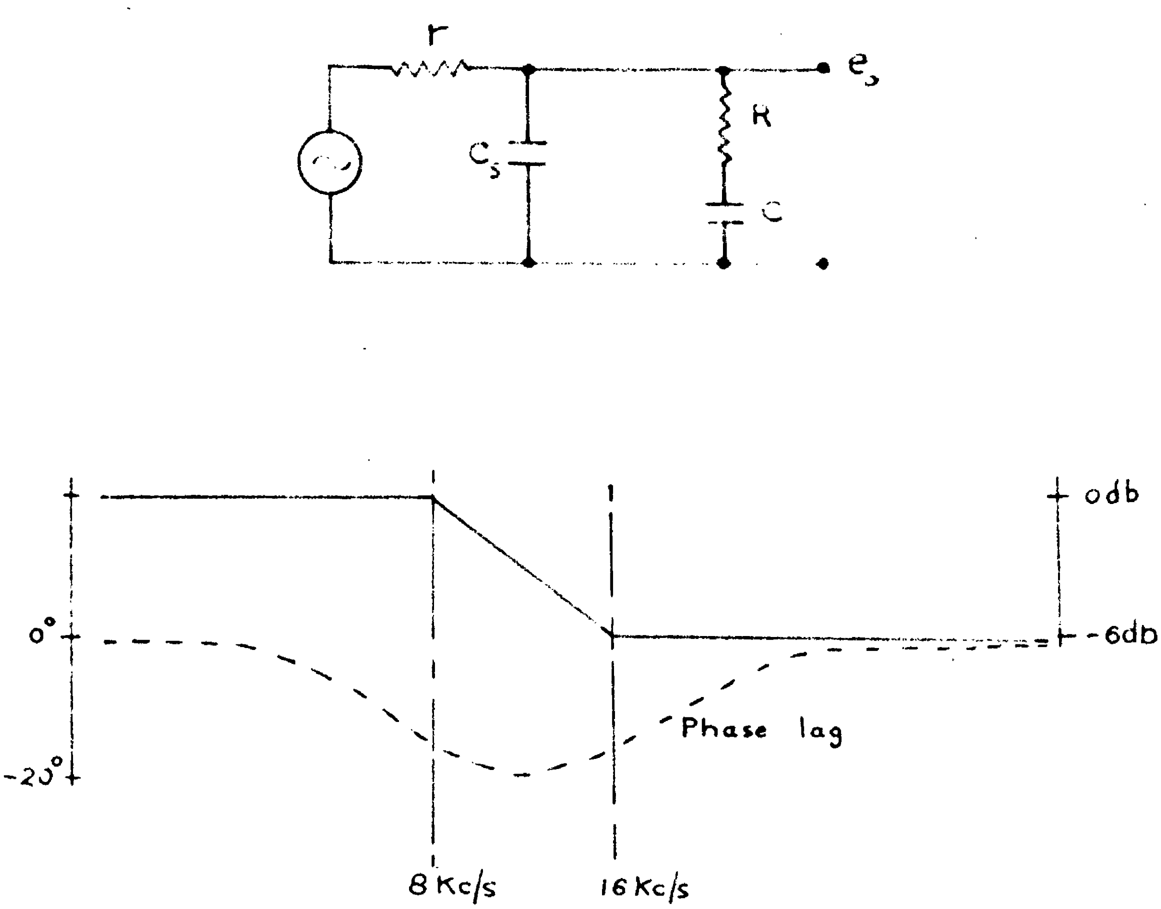


FIG. 7-3. Response of cathode follower with RC load

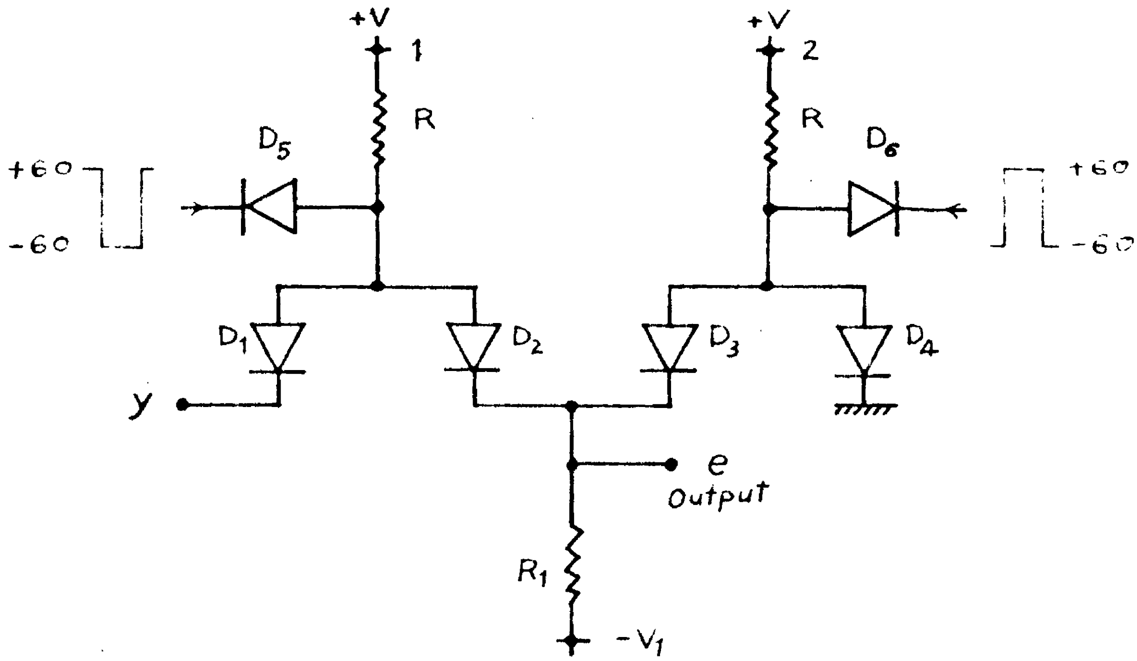


FIG. 8·1. y switch

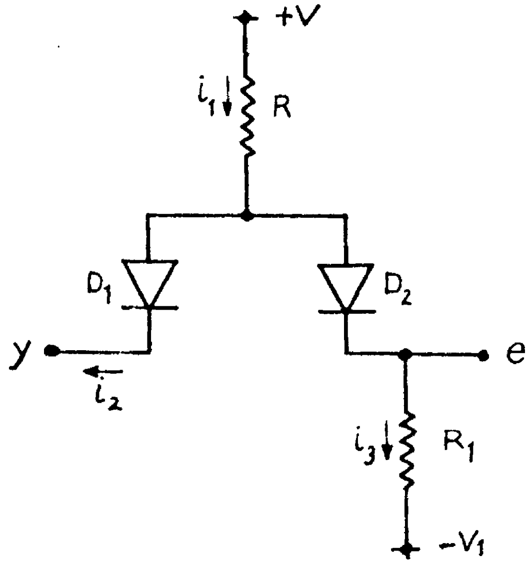


FIG. 8·2. y switch in y=e position

FIG. 8·3. P.T.O.

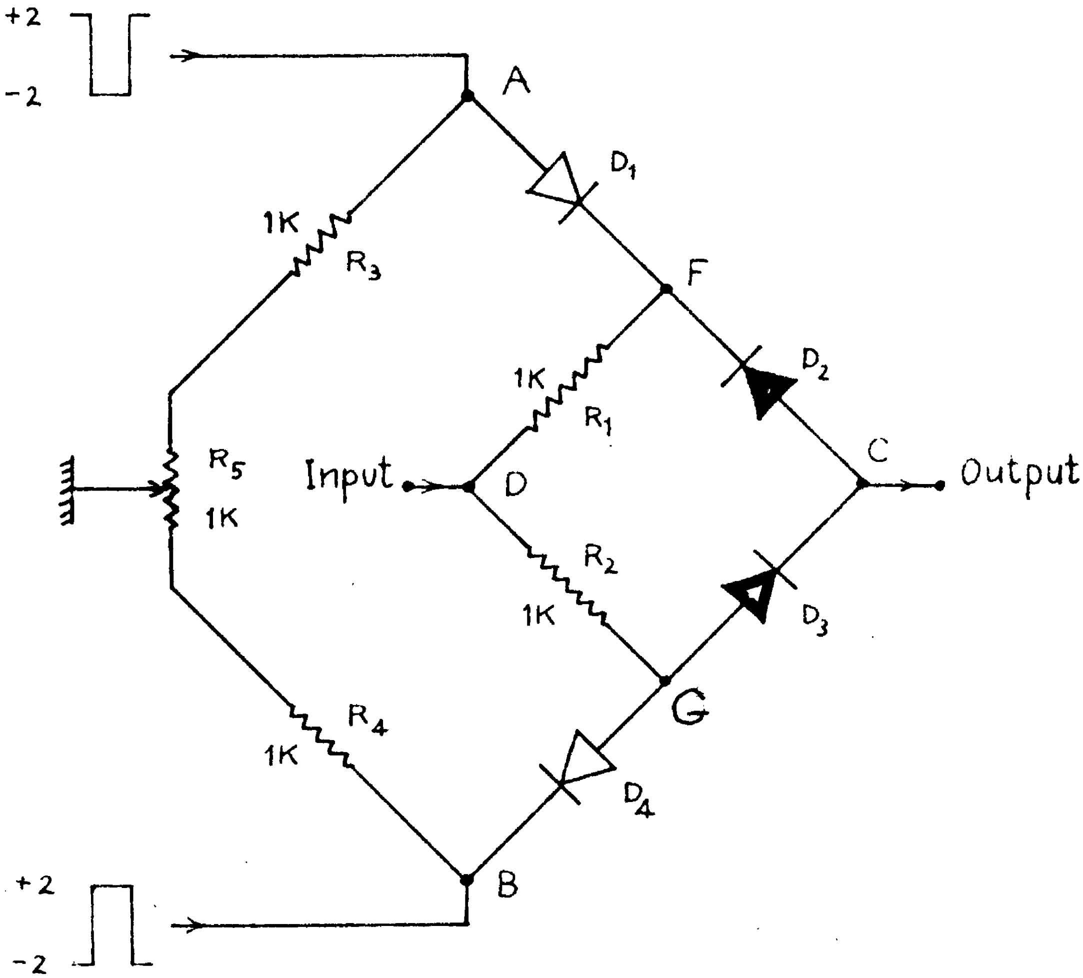


FIG. 9·1 The accumulator grid switch S_3

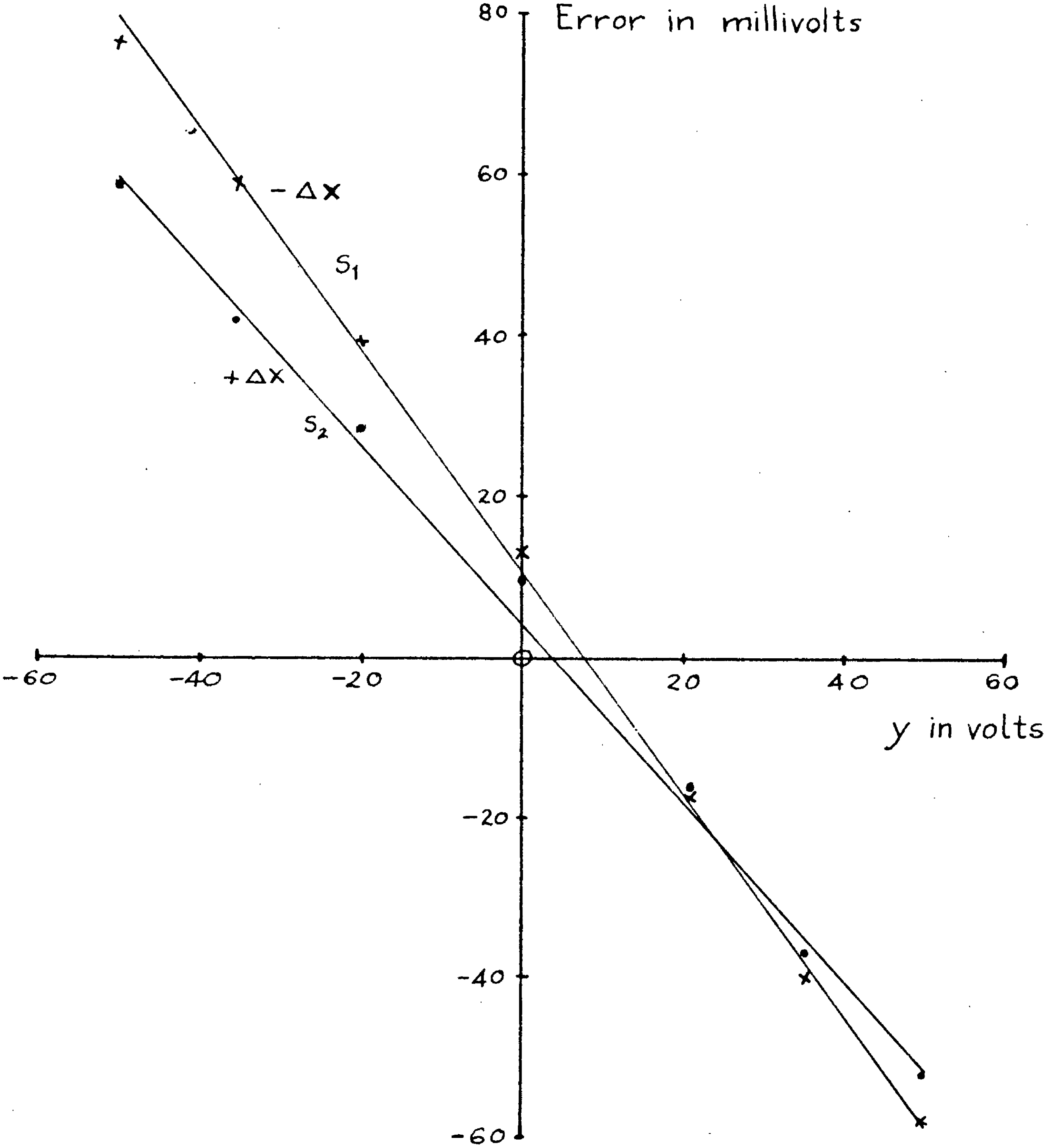


FIG. 8·3. Measured errors in y switches S_1 and S_2

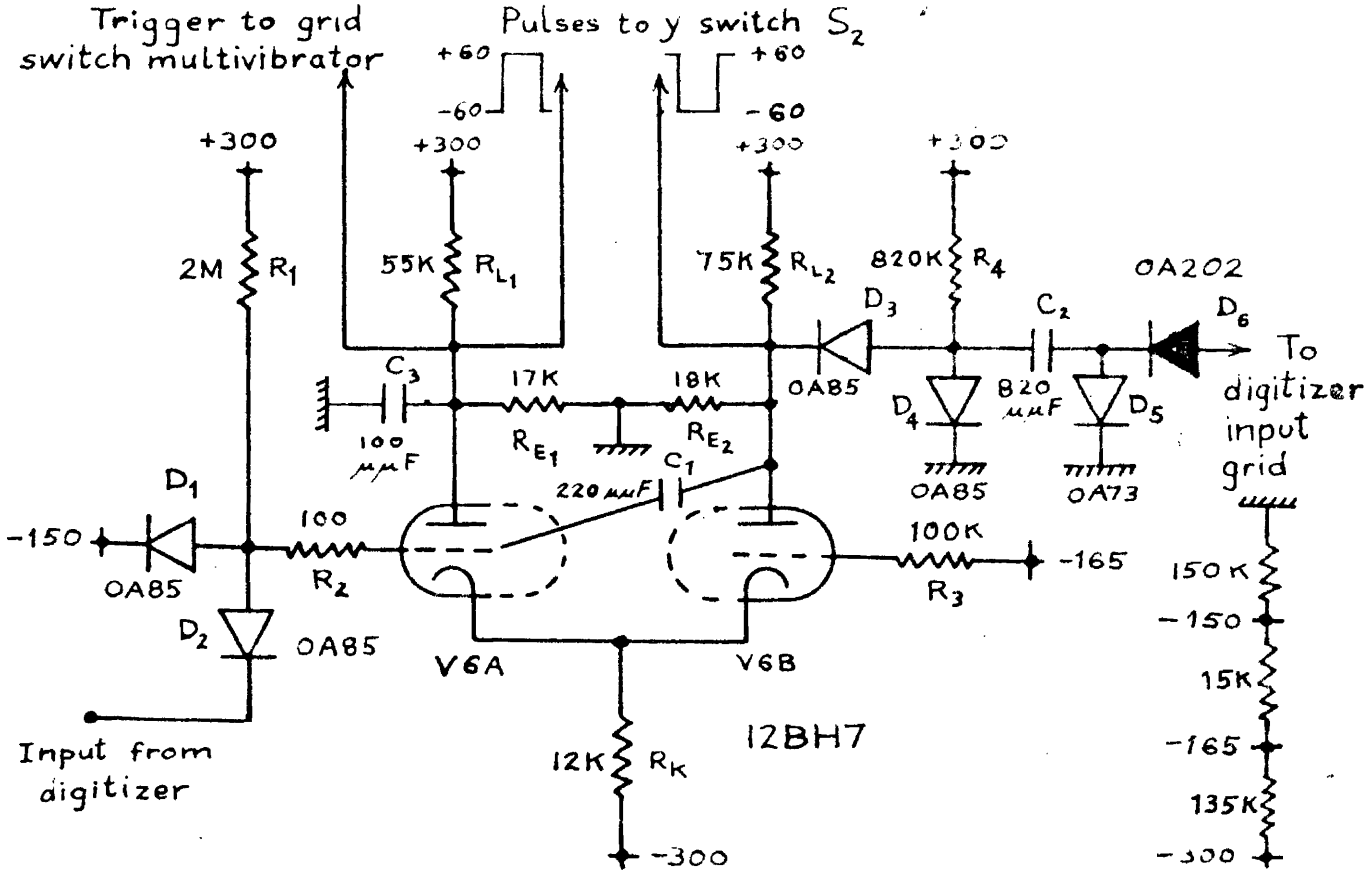


FIG. 10·1 +dx voltage discriminator

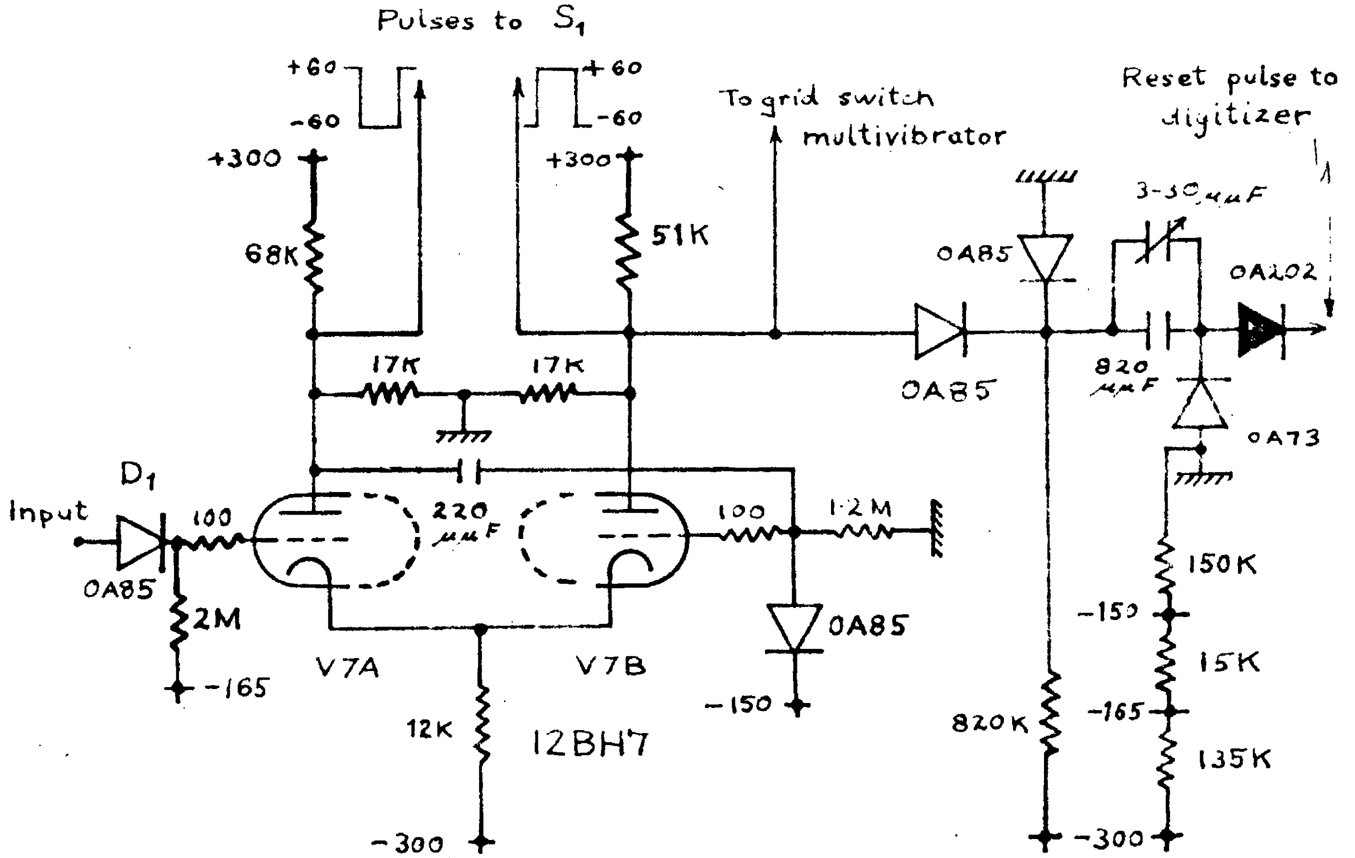


FIG. 11·1 −dx voltage discriminator

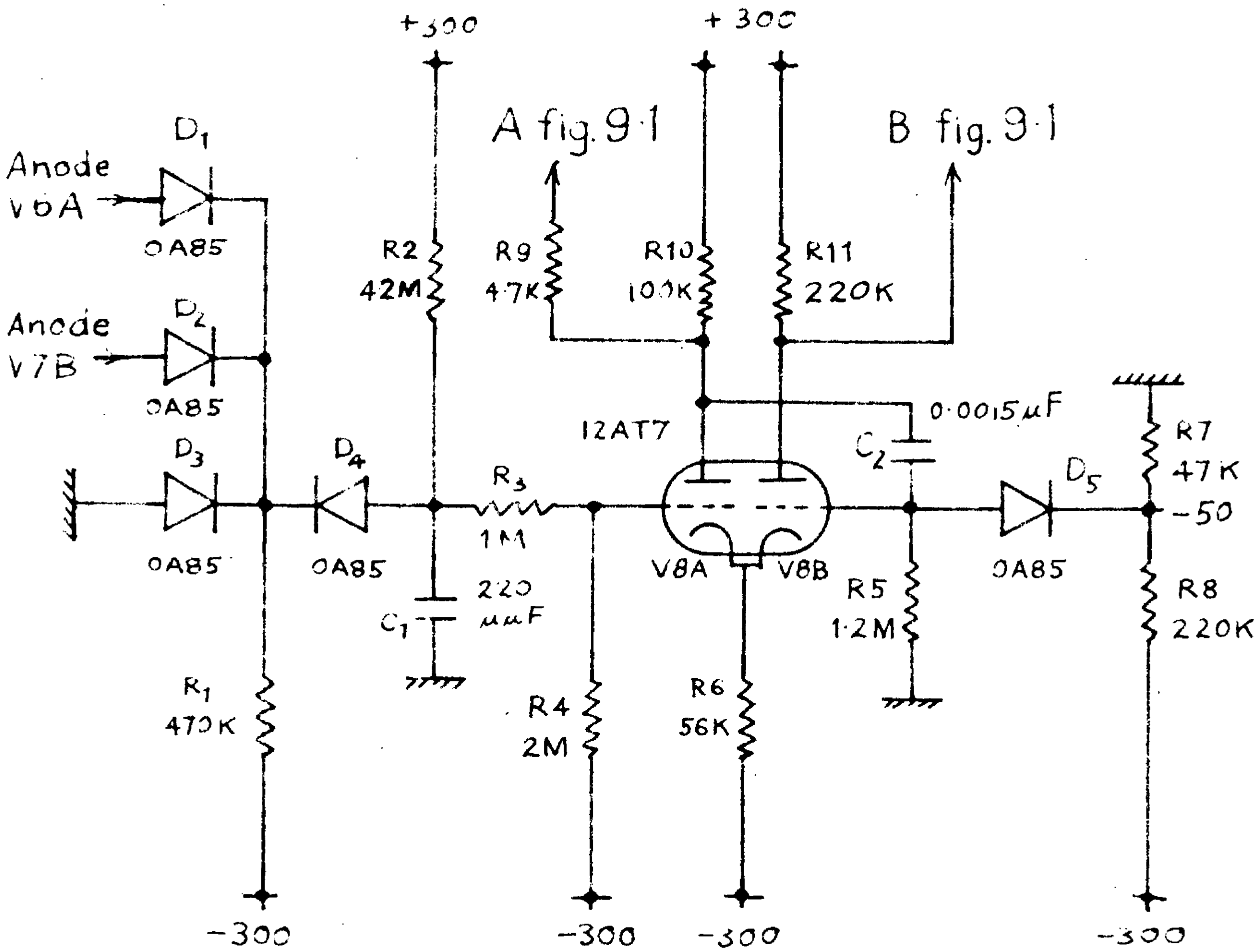


FIG. 12·1. The accumulator grid switch multivibrator

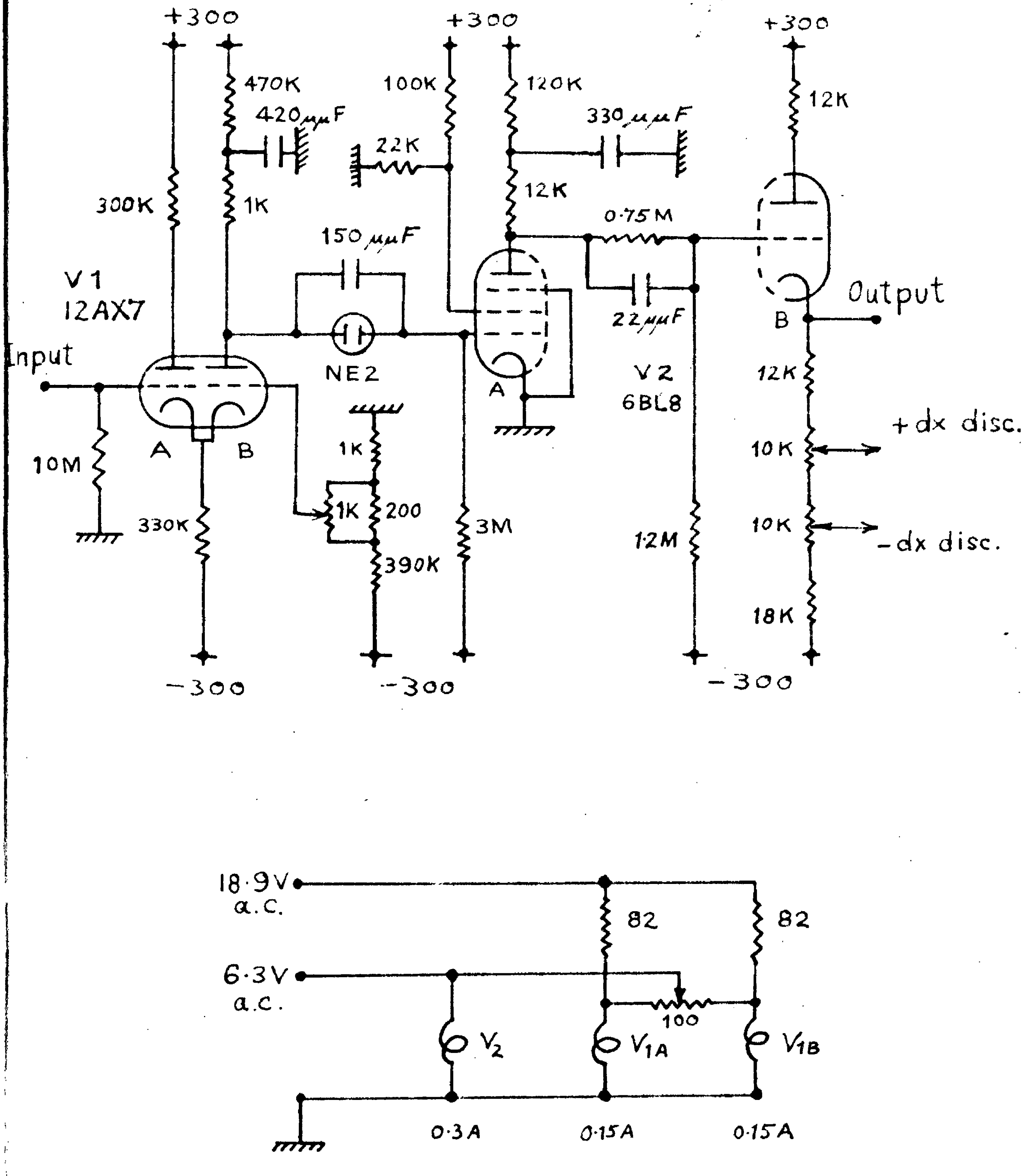


FIG. 13·1 Digitizer amplifier

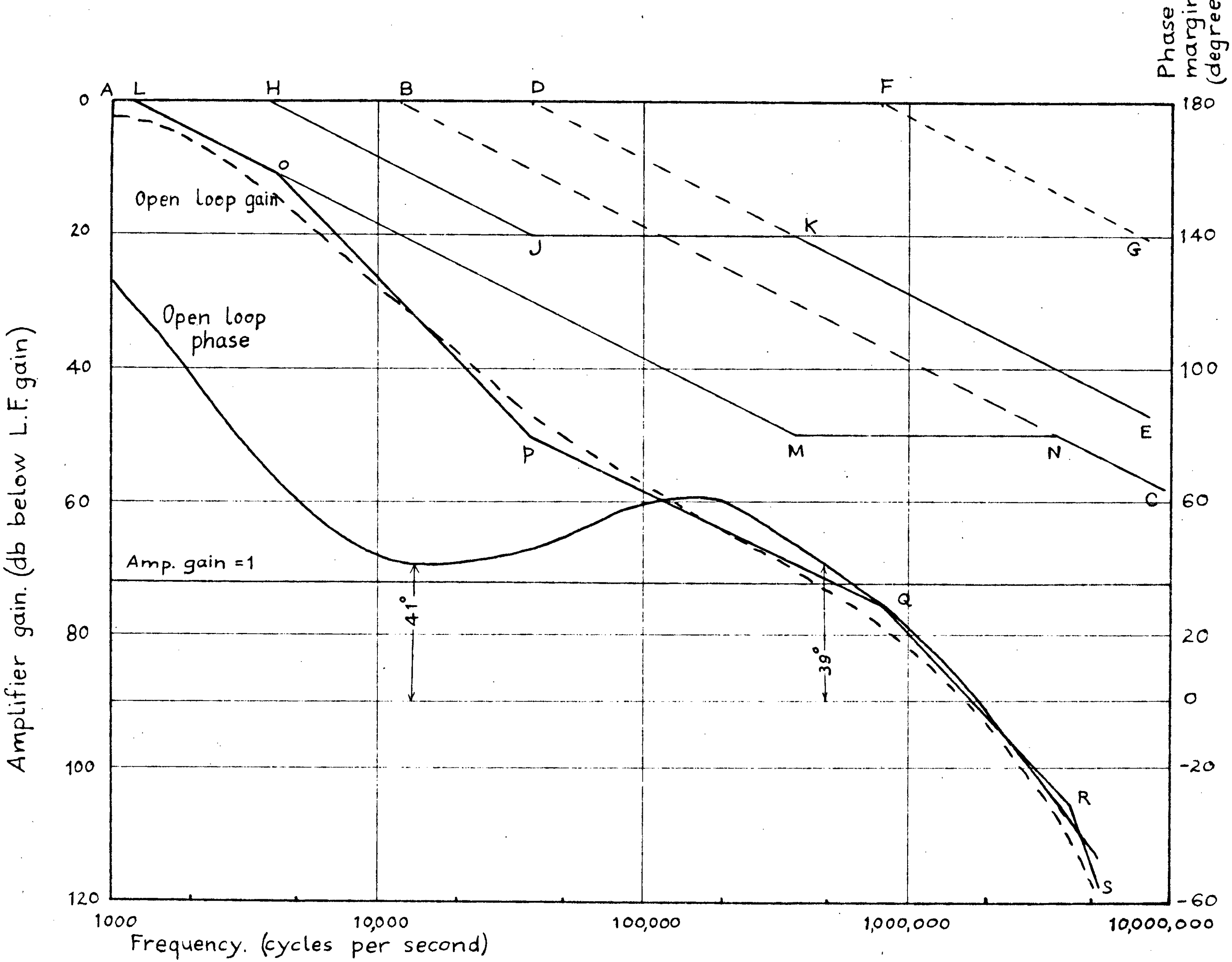


FIG. 13·2 Bode diagram for digitizer amplifier

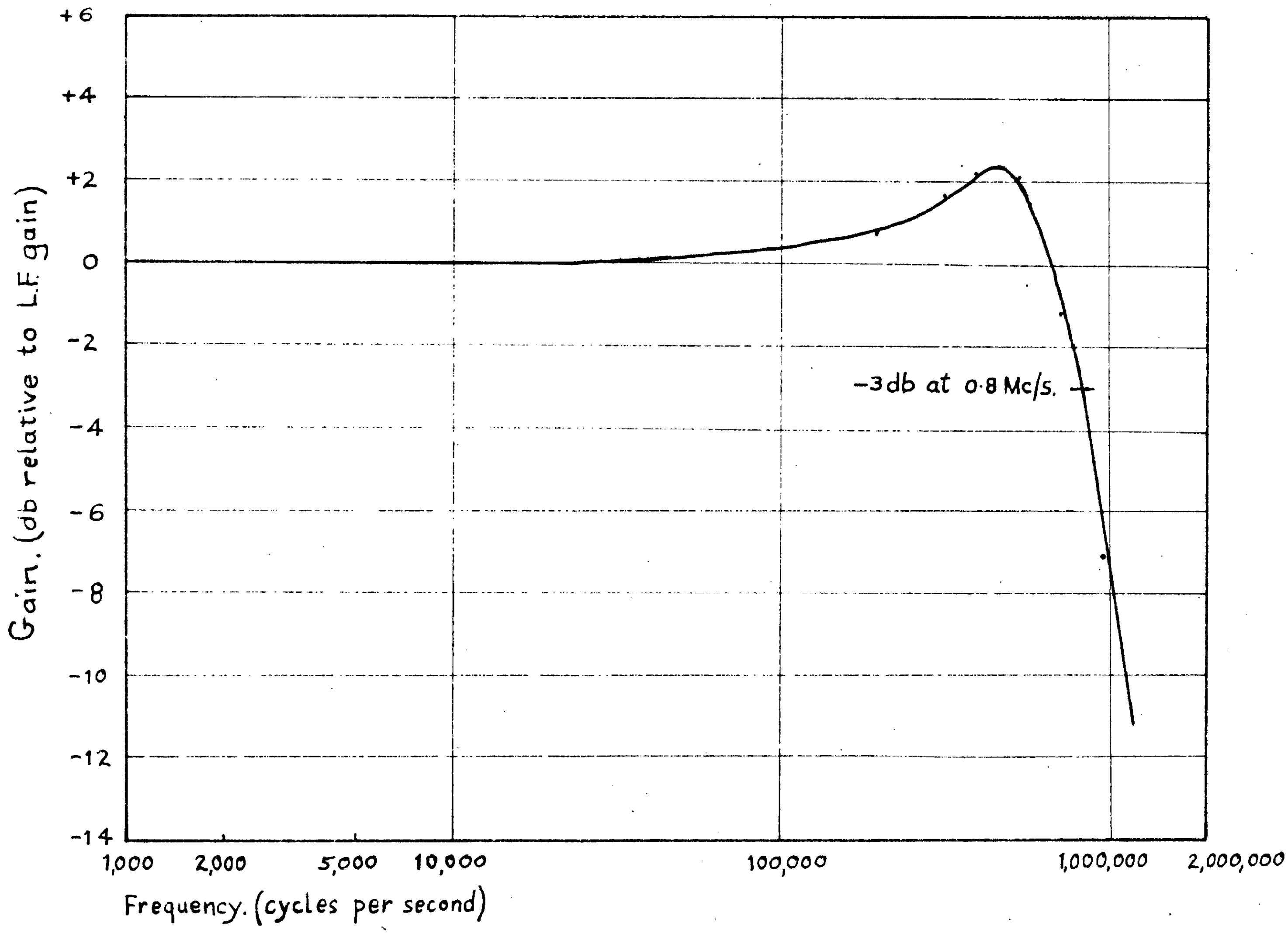
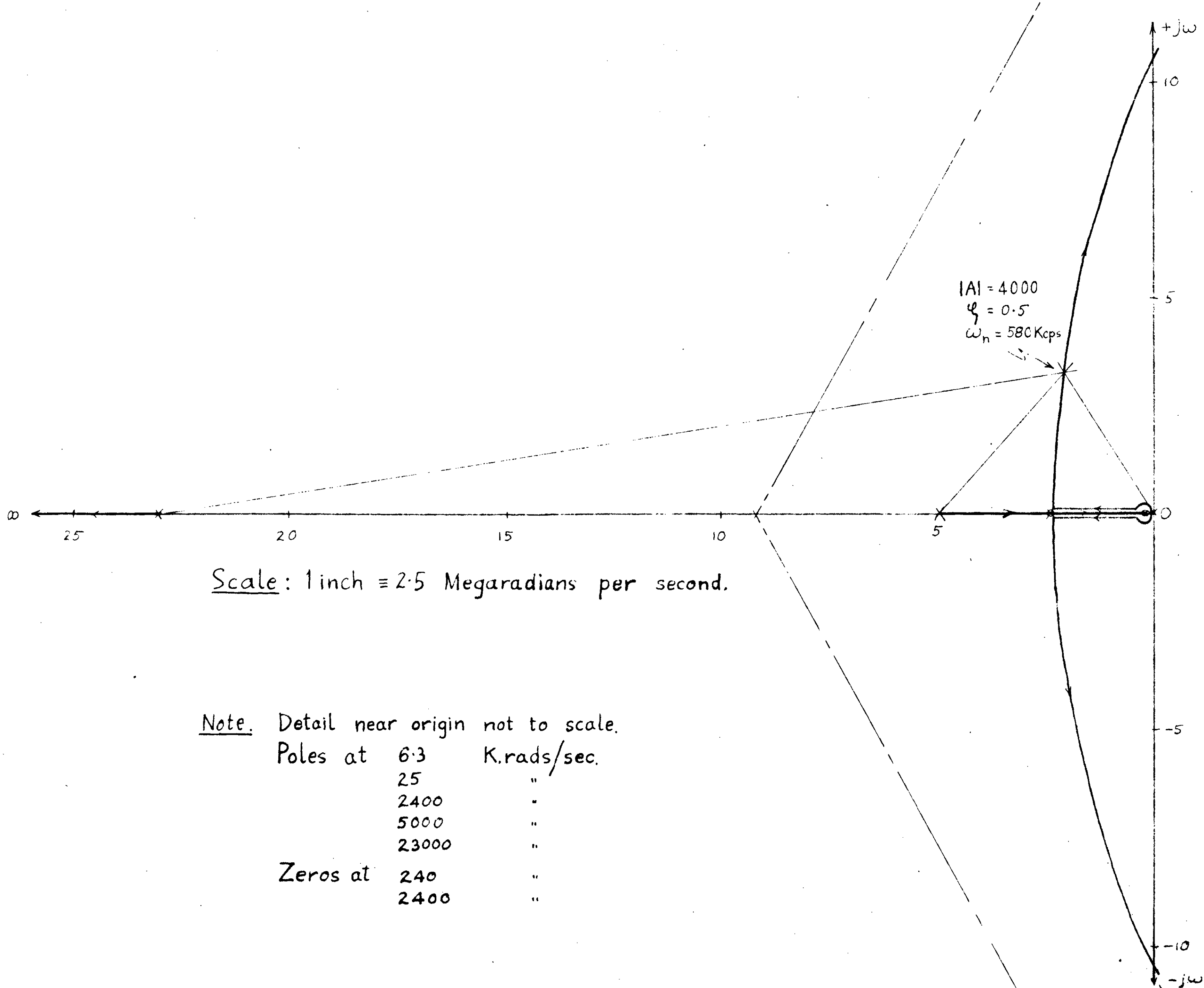


FIG. 13·3. Closed loop frequency response of digitizer amplifier



Scale: 1 inch ≡ 2·5 Megaradians per second.

Note. Detail near origin not to scale.

Poles at	6·3	K.rads/sec.
	25	"
	2400	"
	5000	"
	23000	"
Zeros at	240	"
	2400	"

FIG. 13·4. Root locus diagram for digitizer amplifier

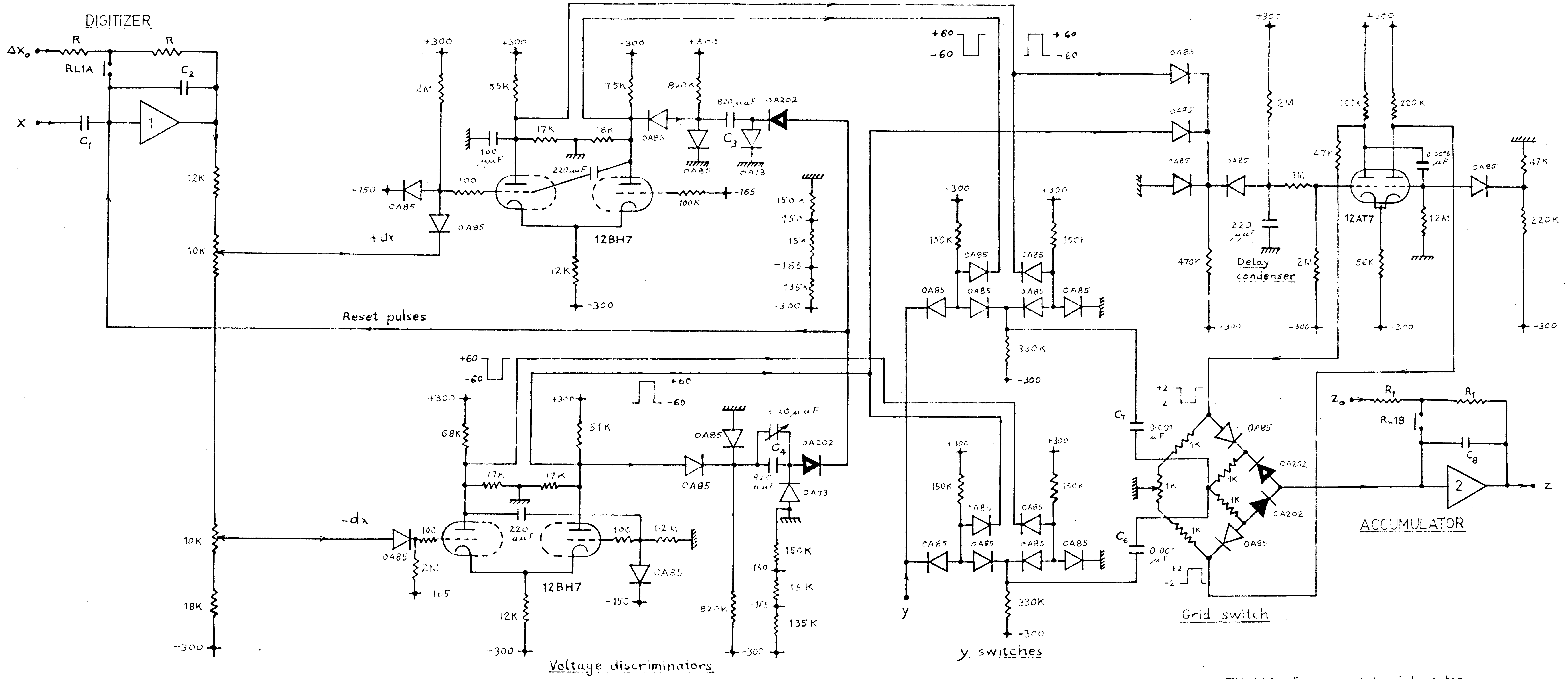


FIG.14.1. The complete integrator

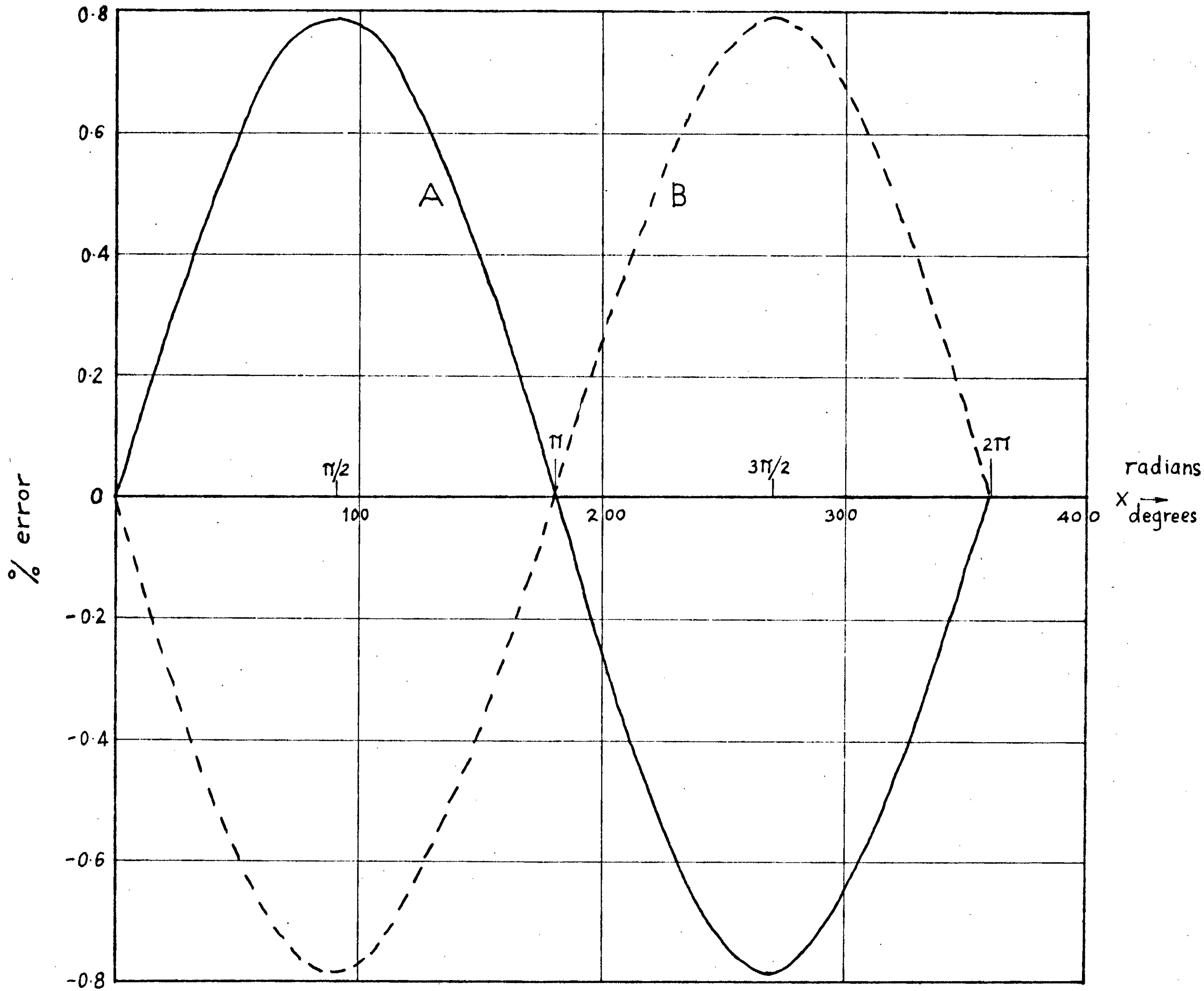


FIG. 14·2. Calculated errors for y = sine·x

$$z = \int_0^x y \cdot dx$$

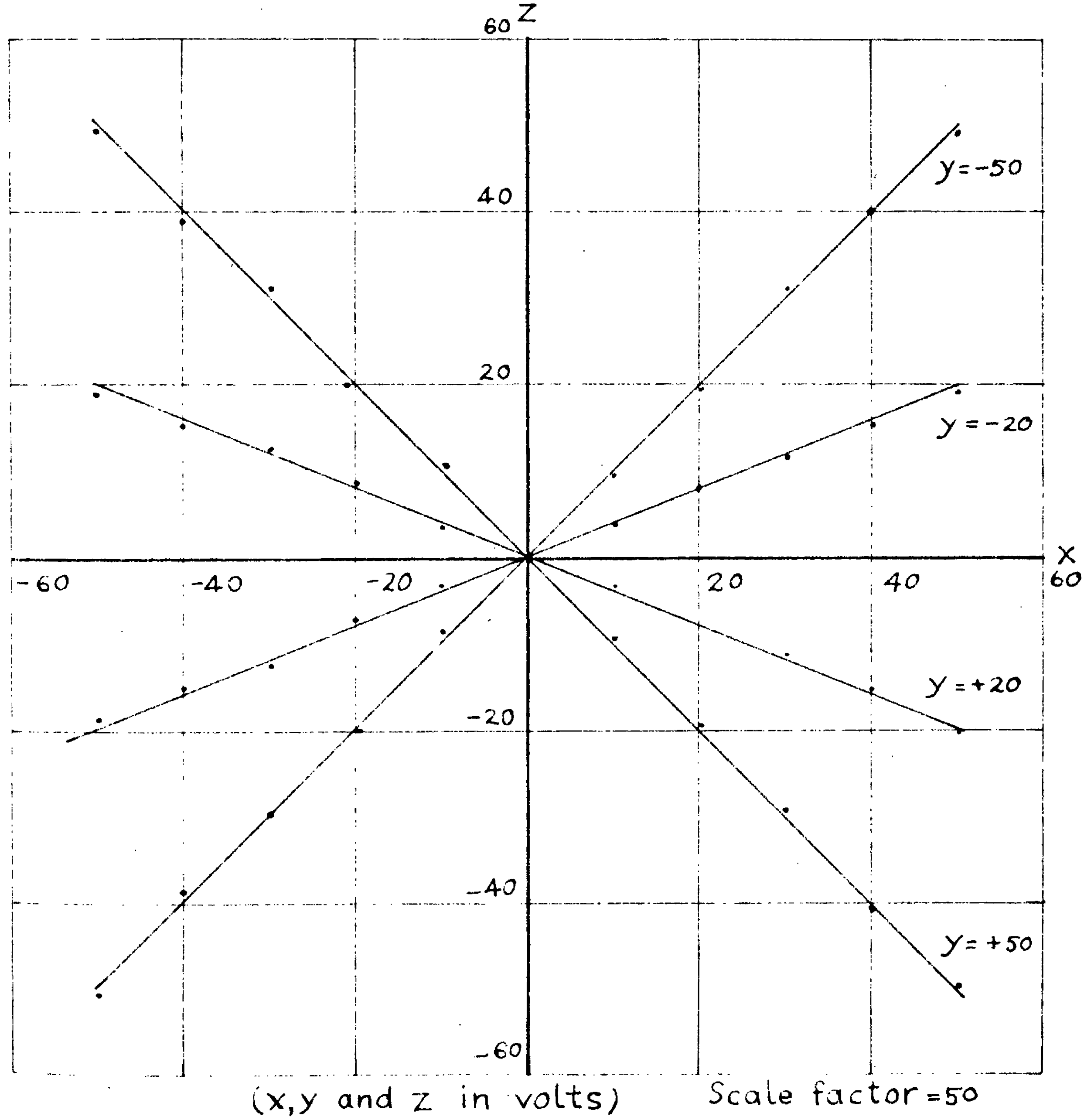
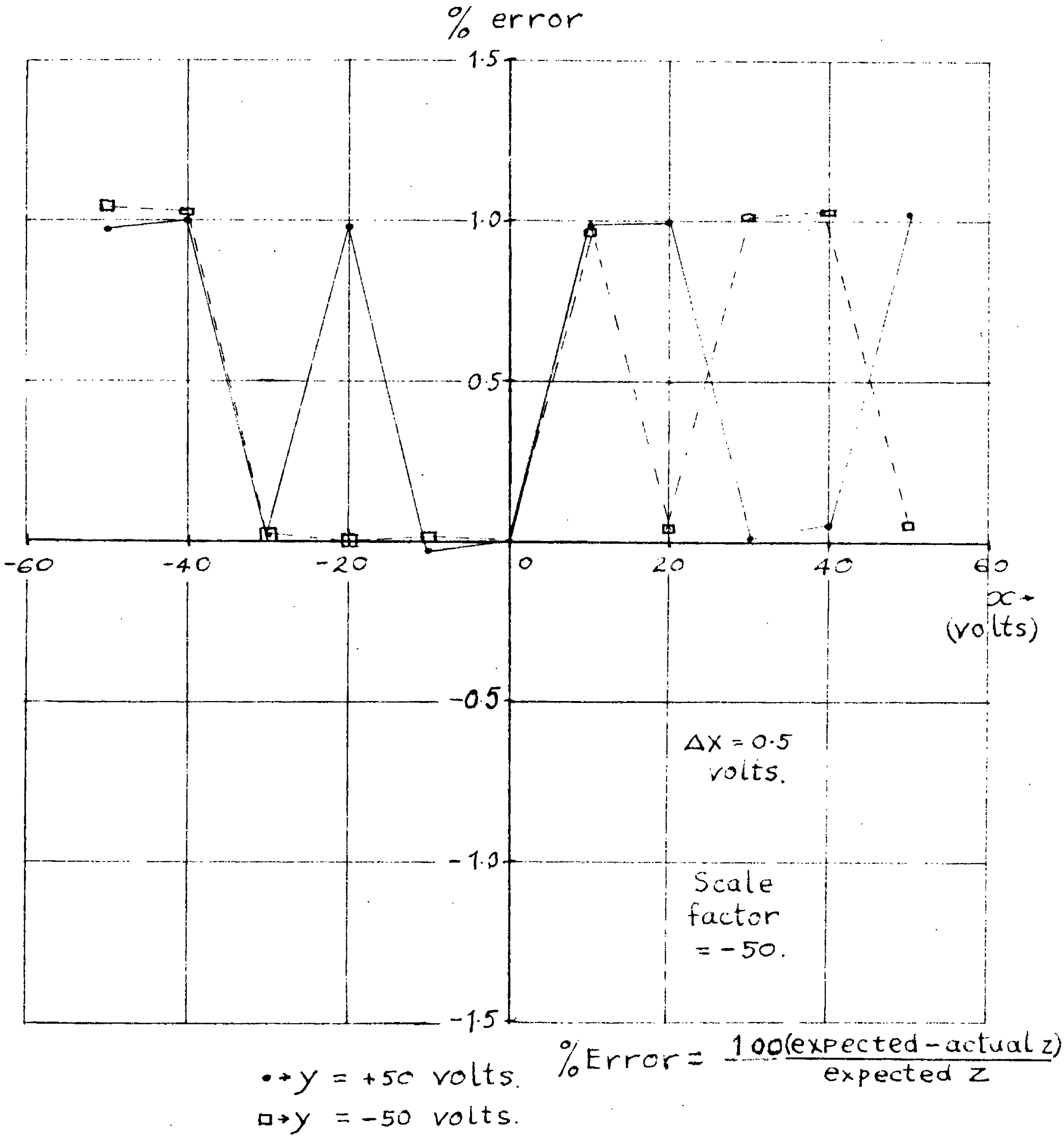


FIG. 14·3. Integrator performance



Error is ≈ 0 or ≈ 1% according to whether a switching cycle has just taken place or not.

FIG. 14·4. Errors of integration

First Conf. on A.C. & D.P. in Aust. May 1960.

AN ELECTRONIC METHOD OF INTEGRATION WITH RESPECT TO VARIABLES OTHER THAN TIME *

A.F. Smith, B.E.E., A.M.I.E.E., A.M.I.E. (Aust.) **

One of the main restrictions to the class of problems which can be done on all Electronic Analogue Computer arises from the inability to integrate with respect to variables other than time.

Several electronic methods of integration with respect to variables other than time have been described, the two most important being:-

(1) Using the relationship -

$$z = \int y \, dx = \int y \frac{dx}{dt} \, dt$$

Here the differentiation of X with respect to time is difficult. The simple RC differentiator leads to a greatly restricted bandwidth if the noise content is to be kept low. Differentiation using a time delay is an improvement on the simple RC differentiator but also suffers from bandwidth restriction because of noise and the minimum delay of 0.01 sec. available from standard Padé delays. Typically an 0.02 sec. time delay is required for a 1% differentiation of a 1 radian per second sine wave.

(2) Bekey's method

$$\int y \, d\,x = K\Delta x \sum_{i=1}^{n} y_i \Delta t$$

where Δt is a constant interval of time and Δx is a constant increment of X. This method also suffers from a bandwidth restriction, due to the finite time required to charge the memory circuit. Errors may also arise from variations in Δt during a particular problem. Bekey quotes a maximum frequency of 1 radian per second using standard Electronic Associates components.

The new method to be described consists essentially of summing a series of equally spaced ordinates under a curve. This sum is an approximation to the area under the curve and hence to the integral of the ordinate variable with respect to the abscissa variable. Neither the inputs nor the outputs of the integrator are restricted as to sign, that is the integrator operates in all four quadrants, i.e.

$$z = \int y \, d\,x \doteq k \sum_{n=1}^{m} y_n \Delta x$$

Figure 1 shows the general arrangement of the integrator. The initial value of z is set in at A while S_1 and S_2 are closed. When the problem is to start S_1 and S_2 are opened. Any change in X will appear at point B and when this reaches a certain value equivalent to $\pm \Delta x$ then either the positive or the negative discriminator will supply an output pulse which throws the S.P.D.T. diode switch S_5 and the appropriate one of S_3 and S_4 so as to add $+ y_i$ or $- y_i$ to the output Z according to whether Δx was positive or negative.

While this is going on the discriminator also supplies a pulse (via C_6 or C_7) so as to alter the charge on C_5 by an amount equivalent to Δx - in effect returning point B towards zero again.

It will be noticed that while point B is returning towards zero the charge on C_5 is also changing as a result of changes in X. Thus the process of recharging C_5 does not cause the loss of any information about changes in X - this considerably increases the bandwidth of the integrator beyond that of integrators which directly charge the memory condenser to an instantaneous value of X.

A further advantage of this integrator is that y_i is added to Z directly and does not depend on maintaining a fixed time interval. This integrator does not need a sign reversing amplifier for y its place being taken by S_3 and C_1. All the switches (except S_1 and S_2 which are relay contacts) are constructed of four semi-conductor diodes.

Slides will be shown illustrating the performance of the integrator.

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*. This paper is based on the work done at the University of New South Wales by the author for a Master of Engineering Thesis and is published by kind permission of Professor R. E. Vowells.

** E.M.I. (Australia) Ltd., Sydney.

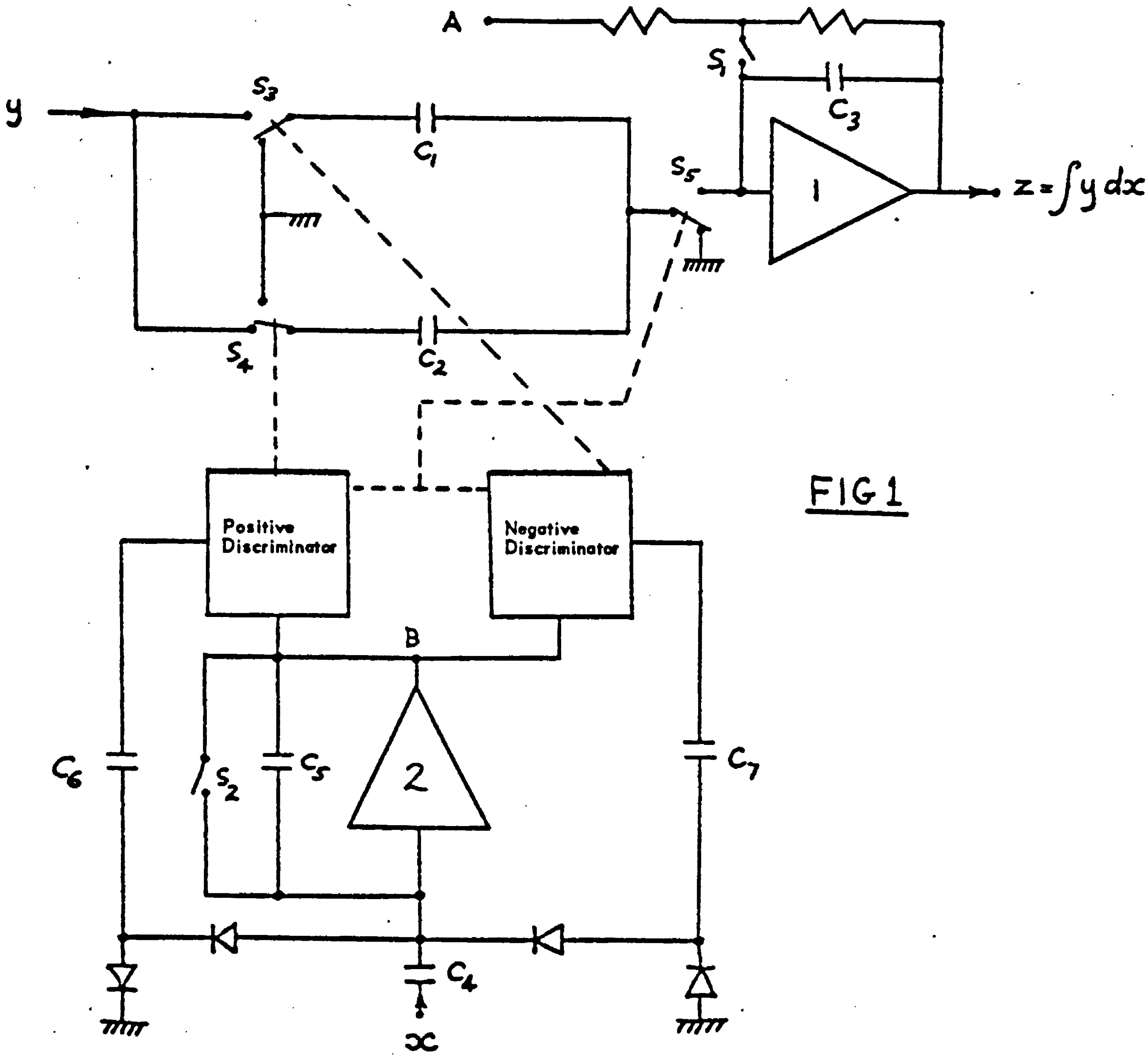

A
S1
C3
y
S3
C1
S5
1
z = ∫y dx
S4
C2
Positive Discriminator
Negative Discriminator
B
C6
S2
C5
2
C7
C4
x

FIG 1

First Conf. on A.C. & D.P. in Aust.
May 1960.

THE SOLID STATE DATA PROCESSING COMPUTER EMIDEC 1100

A. F. Smith, B.E.E., A.M.I.E.E., A.M.I.E. (Aust.)*

EMIDEC 1100 is a solid state data processing system primarily suited for commercial work but capable of being used for scientific work.

The computer is a two address non-synchronous parallel mode machine of 36 bit word length in which all arithmetic is carried out in the binary code at a nominal clock rate of 100 KC/s. Automatic conversions are wired into the machine so that the user may communicate with it in ordinary decimal, alphanumeric or sterling quantities without the use of special program subroutines. Magnetic cores and transistors are used for all logical functions of the machine. An unusually high standard of reliability is produced by operating these components with wide safety margins. Marginal checking facilities are provided to ensure that a component failure is detected before it can cause errors in computation.

The computer (see block diagram) consists of six (6) main units:

(1) Arithmetic Unit:
This contains a 36 stage accumulator capable of accumulating positively or negatively and of shifting left or right. Double length working is also catered for by provision of another shifting register. The use of these two registers enables the programmer to use the whole of the immediate access store as accumulating registers.

(2) Control Unit:
Each unit of EMIDEC is controlled by the application at the appropriate time of single pulses to the control lines entering that unit. The control unit's purpose is to supply to the computer the sequence of signals appropriate to the function being performed. Its main component is the Control Matrix - a matrix of magnetic cores which has a large number of "laces", one for each operation of the machine. An operation is performed by first passing a current through all the cores of a particular lace thus "setting" all its cores. Each column of the matrix is then driven in turn by a counter - thus producing a series of control pulses on the horizontal co-ordinate wires. Any series of computer operations may be produced by lacing a wire through the correct cores in the control matrix.
The correct lace for a particular function is selected by the "Control Decoder" which decodes the instruction function number and selects the corresponding lace from the total of 128. The end of an operation selected by the control unit may in fact initiate a subsequent operation (in this case called a microprogram) required, for example, in multiplication which is a repetitive process.
After an instruction has been carried out the control unit brings down the next instruction from the immediate access store unless a "halt" or other special signal has been given. Thus the machine works at full speed taking the instructions one at a time in their proper sequence.
Modification of instructions is carried out in the Arithmetic Unit before the instruction goes to the Control Unit.

(3) Storage Facilities:
There are three separate storage facilities:
(a) The immediate access working store of 1024 words capacity is constructed from magnetic cores and is used to hold instructions specifying the operations to be performed and also contains data which is being currently used.

(b) A magnetic drum store of 16384 words is provided as a backing store with slow access time which is used to "feed" the working store. Up to four drums can be provided each with 256 tracks of 64 words. Transfers to and from the drum are made in blocks of four words and up to one complete track may be transferred at any one time.

(c) Magnetic tape storage is provided on 1" tape and is probably more properly treated as an input/output device rather than internal machine storage. Magnetic tapes are used for permanent files and as large temporary storage during, for example, production control problems.
Data is recorded on magnetic tape in 2" blocks, each carrying 32 words. The tape speed is 200 inches per second and the tape is used in a bi-directional manner - there being, in effect, one complete block space between adjacent blocks recorded in one direction. Start/stop times are less than 5 milliseconds.

*E.M.I. (Australia) Ltd., Sydney

(4) Input Units:
  (a) Elliott card reader with a speed of 400 ~~characters~~ Cards per minute.
  (b) Ferranti punched paper tape reader with a speed of 300 characters per second.
  (c) Magnetic tape decks. Several of these may be fitted and each can be used as either input or output. Storage capacity of each tape is approximately 360,000 words.

(5) Output Units:
  (a) The Creed High Speed Re-perforator operating at 33 characters per second can be fitted to punch output on to paper tape.
  (b) A Samastronic line printer operating at 300 lines of 140 characters per minute.
  (c) Other devices such as teleprinter units and card punches can be fitted. The maximum number of peripheral units is limited to 16.

Off-Line Working:

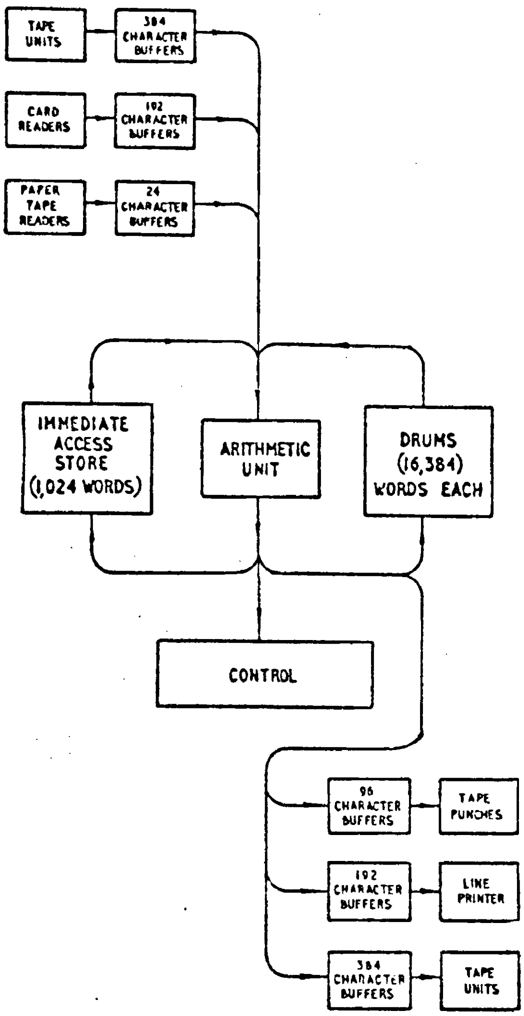
A magnetic tape unit may be used to operate the printer directly while the main computer goes on with another job.

General:

All input and output units are fitted with split buffer storage between the unit and the computer. The buffer allows the computer to run at its normal speed while input and output transfers are proceeding.

During the lecture slides will be shown illustrating the machine and giving the order code together with operating times.

A block diagram, showing how a typical commercial data processing job would be done on the machine, will also be shown.



BLOCK DIAGRAM OF EMIDEC

APPLICATIONS OF AN ANALOGUE COMPUTER TO INDUSTRIAL PROBLEMS

By

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To be delivered at I.R.E. Radio & Electronic Engineering Convention - March 20th - 25th, 1961.

By courtesy of E.M.I. (Australia) Limited.

SUMMARY

This paper is intended to acquaint designers working in fields other than electronics with electronic analogue computers. An attempt is made to present the subject in such a way that mechanical and structural engineers can readily appreciate the facilities which are available to them on a typical computer. The use of an analogue computer for design studies is illustrated by examples from Automobile Engineering, Chemical Engineering and Aeronautical Engineering.

-oOo-

1. INTRODUCTION

An electronic analogue computer [1,2] is essentially a device for rapidly solving linear differential equations by the construction of a model or "analogue" of the physical system which the equations represent. Some machines include accurate four quadrant multipliers, diode limiters and function generators. These latter machines are capable of solving a wide range of non-linear problems.

The machines may be used in either of two ways. Firstly as "simulators", in which each component of the system being studied

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1. Smith & Wood., "Analog Computation", McGraw Hill, 1959.
2. Korn & Korn., "Electronic Analog Computers", McGraw Hill, 1952.

appears as a distinct unit in the simulator. The machine is usually arranged to work at a rate corresponding to "real time" so that the simulator units may be replaced by actual system "hardware" where the necessary transducers are fitted to convert the system variable to a voltage. Secondly the machine may be used as a "computer". In this case the differential equations representing the system are rearranged so as to use the minimum number of computer components and all the physical system variables may not be explicitly available in the computer. The time scale is usually chosen to give the greatest possible computer accuracy without regard to the possibility of using real system "hardware" with the computer.

THE BASIC UNITS

The two most important units in an analogue computer are direct coupled feedback amplifiers of very accurate gain and integrators.

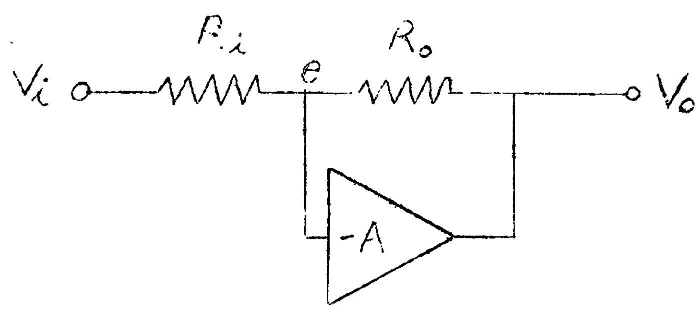

Fig. 1. HIGH GAIN AMPLIFIER WITH FEEDBACK COMPONENTS

D.C. Amplifiers

These consist of direct coupled high gain amplifiers, the gain of which is accurately controlled by precise resistors in the feedback circuit. Figure 1 shows the arrangement used. The amplifier has a large gain A, usually of the order of 100,000, and is arranged so that the output V_o is of opposite sign to the input e. The input impedance of the amplifier is very high and hence little or no current flows in the amplifier from the junction of R_i and R_o. Also the maximum value of V_o is usually 100 volts and therefore the maximum value of e is typically 1 mV. Thus from the point of view of the input V_i to the complete feedback amplifier the junction of R_i and R_o may be regarded as a "virtual earth".[3] Hence the current in R_i is $\frac{V_i}{R_i}$ and equals the current in R_o which is $-\frac{V_o}{R_o}$.

Therefore $$V_o = - V_i \frac{R_o}{R_i} \qquad (1)$$

It is usual to make $\frac{R_o}{R_i}$ accurate to 0.1% by the use of precision resistors which are readily adjustable or replaceable so that various values of gain $\frac{R_o}{R_i}$ may be obtained.

The amplifier may also be used as an adder by the addition of further input resistors as in Fig. 2. Then $i_o = i_1 + i_3 + --- i_n$ (2)

and so $$\frac{V_1}{R_1} + \frac{V_2}{R_2} + ---- + \frac{V_n}{R_n} = - V_o/R_o \qquad (3)$$

---

3. Williams & Ritson., "Electronic Servo Simulators," J.I.E.E., Vol. 94, Pt. IIA, December 1947.

hence $V_o = -V_1\ (R_o/R_1)\ -V_2\ (R_o/R_2)\ - - - - - - V_n\ (R_o/R_n)$ (4)

Thus $-V_o$ may be regarded as the sum of the inputs V_1 to V_n with appropriate scaling.

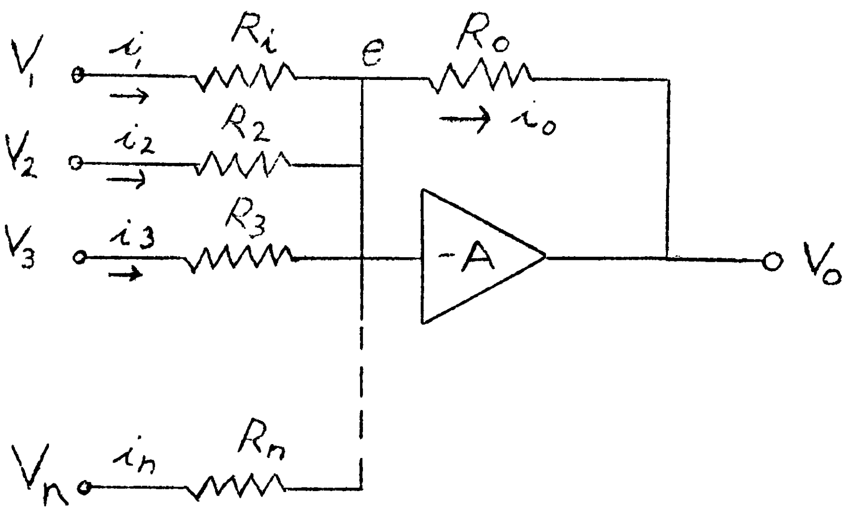

Fig. 2. AN ADDER

Integrators

Figure 3. shows the general arrangement used. As in the adder e is very small and hence the junction of the condenser and resistors is a virtual earth.

Thus $$i_o = i_1 + i_2 + - - - i_n \qquad (5)$$

and $$V_o = 1/c \int_0^t i_o.dt \qquad (6)$$

hence $$V_o = - \int_0^t (V_1/CR_1 + V_2/CR_2 + - - - + V_n/CR_n)\ dt \qquad (7)$$

Therefore $-V_o$ may be regarded as the time integral of the sum of the inputs with appropriate scaling.

A SIMPLE EXAMPLE

Fig. 4. shows a simple mass M supported by a spring, whose

force is K times its displacement. The mass moves in guides which oppose the movement of M with a force F times its velocity.

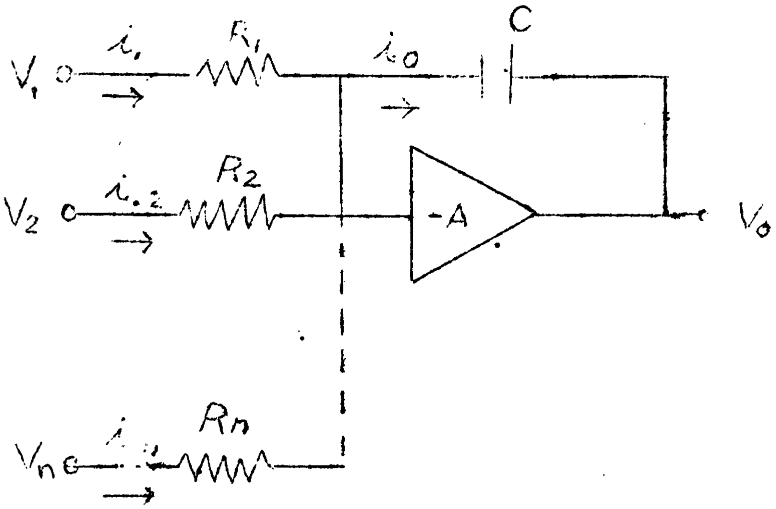

Fig. 3. AN INTEGRATOR

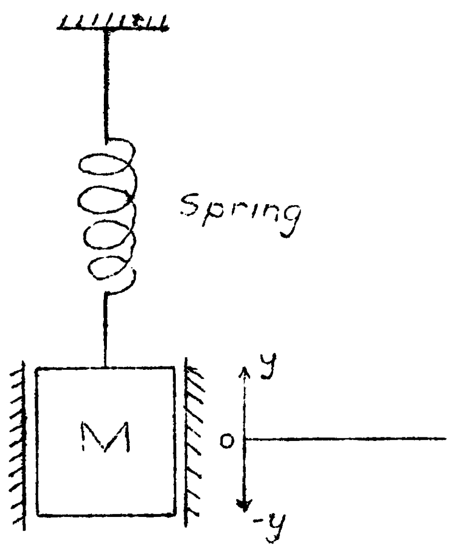

Fig. 4. DAMPED HARMONIC OSCILLATOR

The equation of motion is

$$M\ddot{y} + F\dot{y} + Ky = o \tag{8}$$

where y may have initial value other than zero. For solution on the analogue computer the equation is transposed as follows

$$\ddot{y} = -F\dot{y}/M - Ky/M \qquad (9)$$

$\ddot{y}$ is assumed to be known and $\dot{y}$ and y are obtained by successive integration as shown in Fig. 5. Fig. 6. is obtained from Fig. 5 by substituting $-F\dot{y}/M - Ky/M$ for $\ddot{y}$ as in equation (9). 3 is simply a sign reversing amplifier. R_4 is chosen so that $1/(R_4C_1) = F/M$ and R_5 is chosen so that $1/(R_5C_1) = K/M$. $1/(R_2C_2)$ is made = 1. The computer has switches built in to it to allow C_1 and C_2 to be changed to the initial values of $\ddot{y}$ and $\dot{y}$ before the solution is commenced.

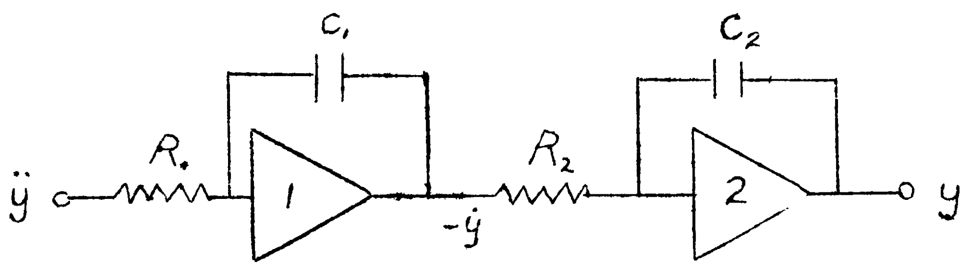

Fig. 5. $\dot{y}$ AND y FROM $\ddot{y}$

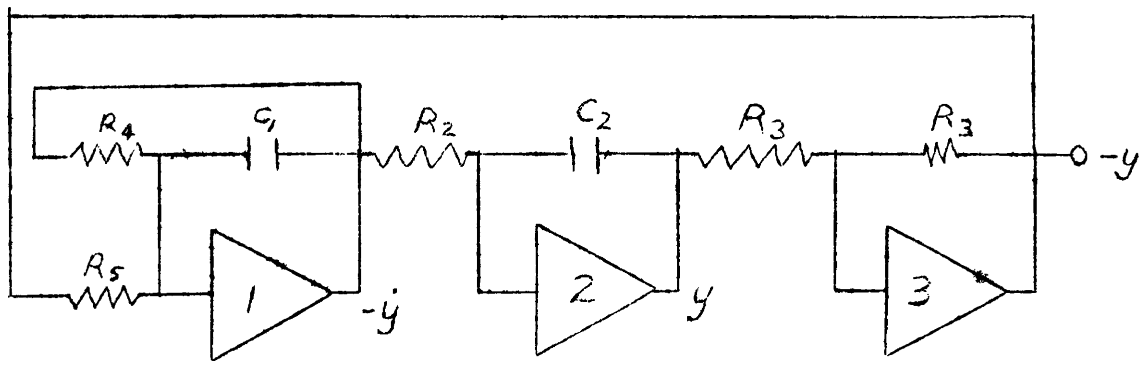

Fig. 6. ANALOGUE OF FIG. 4.

Numerical Example

Suppose $M = 60 \text{ lb sec}^2/\text{ft.}$

$K = 6000 \text{ lb/ft.}$

$F = 120 \text{ lb sec/ft.}$

Thus in numerical form equation 9 becomes

$$\ddot{y} = -2\dot{y} - 100y \tag{10}$$

If the mass is initially displaced 1 ft. from rest the maximum amplitude of y will be 1 ft. The maximum amplitude of $\dot{y}$ will be 10 ft./sec. and that of $\ddot{y}$ 10 ft./sec^2. Now the maximum output voltage of each amplifier is known to be 60 volts, so y may be scaled at 50 volts/ft. and $\dot{y}$ may be scaled as 5 volts per foot per second. Thus we expect the output of amplifier 2, of Fig. 6, to be 50y and that of amplifier 1 to be 5 $\dot{y}$ and hence $1/(R_2C_2) = 10$. A convenient value of C is 1 μF and therefore $R_2 = 0.1$ megohm. Amplifier No. 3 is simply a reversing amplifier and it is convenient to make $R_3 =$ 1 megohm. Now the $\dot{y}$ component of $\ddot{y}$ (see equation 10) is $2\dot{y}$ and therefore $1/(R_4C_1) = 2$ and once again C_1 is chosen as 1 μF for convenience. Hence $R_4 = 0.5$ megohms.

The y component of $\ddot{y}$ is 100 y and so $1/(R_5C_1) = \frac{100 \times 5}{50} = 10$

$$\therefore R_5 = 0.1 \text{ megohm}$$

Figure 7 shows these resistors and condensers in position and also shows the method of setting in the initial conditions of y = + 1 ft. and $\dot{y} = 0$. Before the solution is commenced S_1 and S_2 are closed making $\dot{y} = 0$ and $y = 1$ foot; they are released to start the computer which then produces y as a function of time.

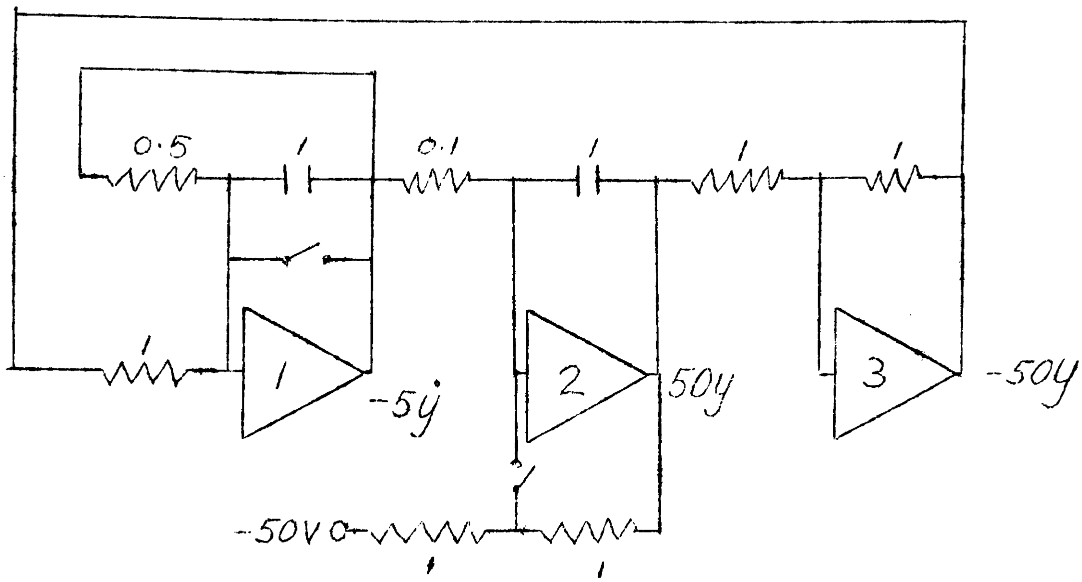

Fig. 7. SCALED ANALOG OF MECHANICAL OSCILLATOR

The solution to equation 10 is a damped sinusoid of frequency 10 radians per second, with a damping factor of 0.1 (see Fig. 8).

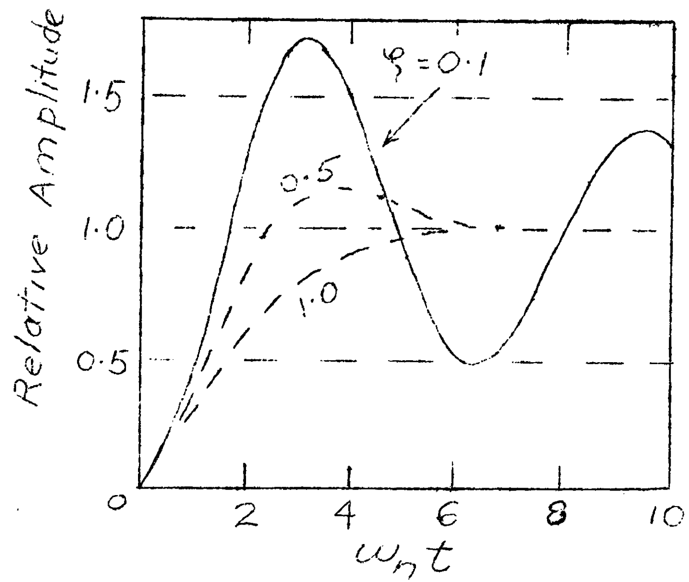

Fig. 8. RESPONSE OF MECHANICAL OSCILLATOR TO UNIT STEP DISTURBANCE

It may be displayed by means of connecting a pen recorder to the output of amplifier number 2. Arrangements may easily be made to vary K, M and F so that their individual effects on the oscillation may be studied. The solution to equation (9), even for various values of damping factor, spring constant and mass, can be obtained easily by analytical methods. It is therefore not worth while putting such a problem on a computer and the above should be viewed simply as an introduction to computer technique.

Change of Time Scale

Generally most analogue computers have an upper and a lower limit to the length of time which may be taken for a problem. The upper limit results from errors arising from amplifier grid current, condenser leakage and the finite gain of integrators. It is usually of the order of 2-5 minutes. The lower limit is set by the bandwidth of the computing amplifiers and the degree of synchronisation of the "initiate" relays. Solutions should not involve frequencies greater than 10 to 50 cycles per second depending on the computer being used.

It should also be noted that there are a class of "repetitive" analogue computers which, at the completion of the solution, automatically reset the initial conditions and repeat the solution as many times as required. These machines are generally of lower accuracy than the "single shot" machines and may repeat the solution once every fiftieth of a second. Their bandwidth at full accuracy may be as great as 20,000 cycles per second and is not usually a restriction on speed of solution. They are used where the designer wishes to quickly study the effect of many parameter changes in a system and is not aiming at the highest possible accuracy. Repetitive machines will not be further discussed in this paper and all that follows refers to "single shot" machines.

As a result of the above upper and lower limits to the speed of solution it is frequently necessary to scale the problem so that the machine works faster or slower than the actual system being studied. If t is the independent variable of the problem and T is the computer time scale we may make the time transformation T = at. If a is less than 1 the solution is speeded up 1/a times. If a is greater than 1 the solution is slowed down a times.

If
$$t = \frac{T}{a} \tag{11}$$

$$\frac{d}{dt} = a\,\frac{d}{dT} \tag{12}$$

and
$$\frac{d^n}{dt^n} = a^n\,\frac{d^n}{dt^n} \tag{13}$$

From 13 we see that a change in time constant of each integrator by a factor a will produce the necessary change in time scale for the whole operation. Since the n^{th} derivative will go through n integrators before the final variable is obtained it will be multiplied by a^n by this process.

It is usual to aim at time scales such that the computer takes one to two minutes to obtain the whole solution.

AN AUTOMOBILE SUSPENSION SYSTEM

Figure 9 shows the dynamic system where

M_1 = One quarter of mass of vehicle

M_2 = Mass of wheel and axle

K_1 & C_1 = Constants of main spring and shock absorber

K_2 = Linearized spring constant of the pneumatic tyre

x = displacement of M_1

y = " " M_2

z = road surface contour.

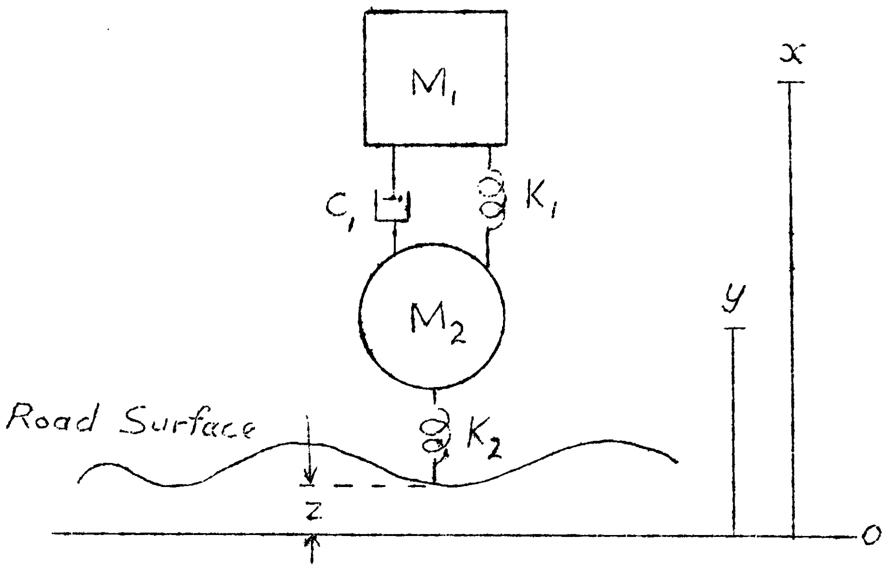

Fig. 9. SCHEMATIC OF AN AUTOMOBILE SUSPENSION

The equations describing the system motion are

$$M_1\ddot{x} + C_1(\dot{x} - \dot{y}) + K_1(x - y) = 0 \quad (14)$$

$$M_2\ddot{y} + C_1(\dot{y} - \dot{x}) + K_1(y - x) + K_2(y - z) = 0 \quad (15)$$

$$z = z(t) \quad (16)$$

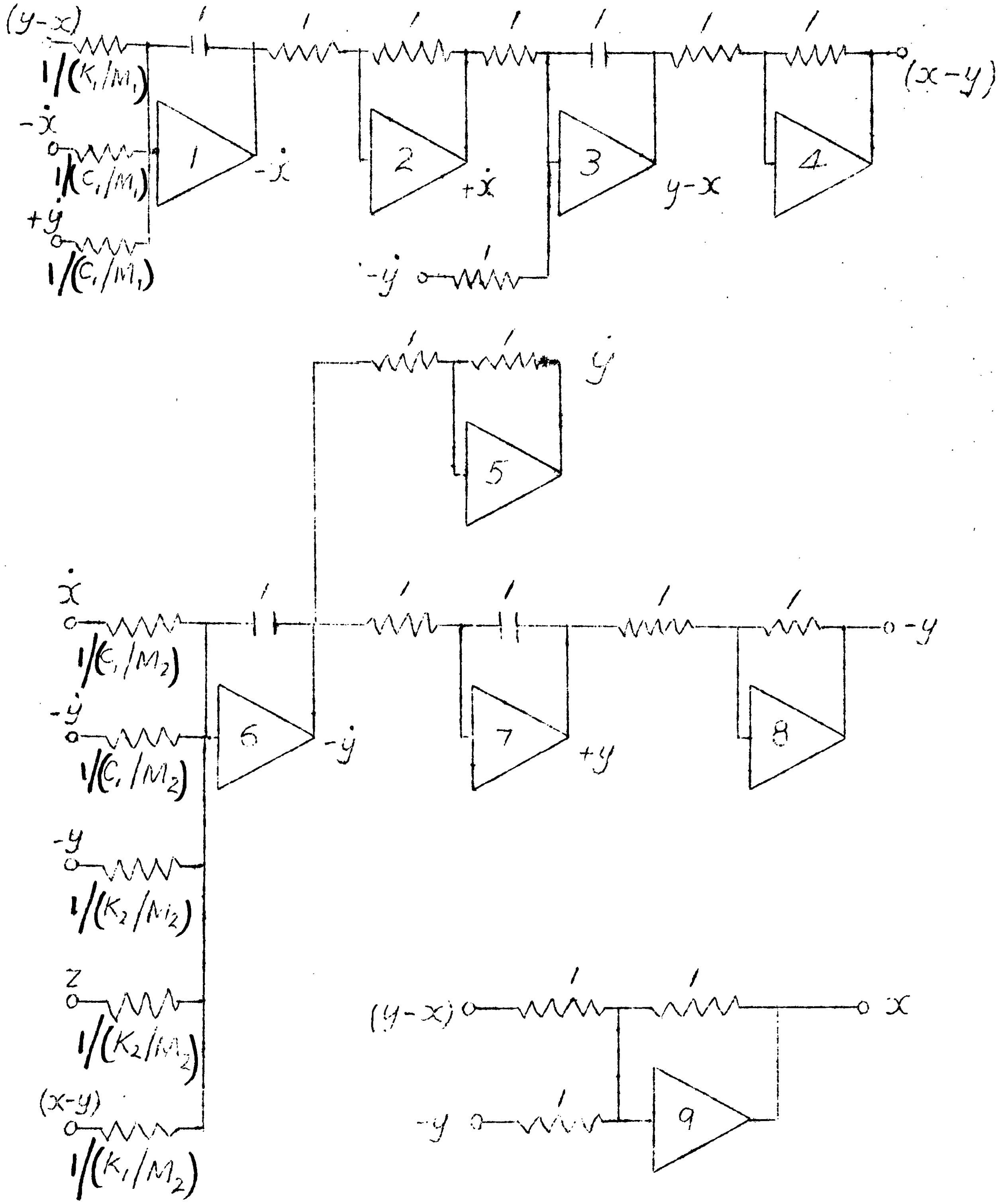

Fig. 10. <u>COMPUTER ARRANGEMENT FOR AUTOMOBILE SUSPENSION OF FIG. 9.</u>

Figure 10 shows a method of solving these simultaneous linear differential equations on an analogue computer. As before $\ddot{x}$ & $\ddot{y}$ are assumed and integrated to obtain $\dot{x}$, $\dot{y}$, x and y. Equations 17 and 18 below then specify how the interconnections are made in the computer

$$\ddot{x} = (C_1/M_1)(\dot{y} - \dot{x}) + (K_1/M_1)(y - x) \tag{17}$$

$$\ddot{y} = (C_1/M_2)(\dot{x} - \dot{y}) + (K_1/M_2)(x - y) + (K_2/M_2)(z - y) \tag{18}$$

Not all interconnections are shown in Fig. 10 but the inputs are labelled with the output to which they are connected. It is obvious from the diagram that equations 17 and 18 are satisfied by the inputs to integrators 1 and 4. Amplifier 8 is provided for monitoring purposes only and is shown connected to have an output representing the movement of the car body with respect to earth. It could equally well be arranged to have an output $(\dot{x} - \dot{y})$, the velocity of the wheel with respect to the car. Figure 10 has not been numerically scaled. Scaling may be carried out in a manner similar to the previous example.

Having set up the problem on the computer the designer will then inject various disturbances z typical of road surfaces which the vehicle will encounter. Among the simpler functions used will be an instantaneous jump in z. This will give the transient response of the system and the shock absorber and spring system can then be adjusted to give the required type of response. It may also be useful to simulate a "corrugated" road by feeding in

$$z = A \sin wt \tag{19}$$

and varying w over a wide range, noting and correcting any excessive resonances which may arise.

Normally there will be a practical limit to the size of $(y - x)$ beyond which structural damage will occur. This may be simulated by putting a diode limiter on integrator number 3 as shown in figure 11. The system is now nonlinear and the response is dependent on the amplitude of the input disturbance z, which should be varied over the range expected to occur in practice. Any undesirable responses can then be noted and corrected by changing the constants C, K, and possibly K_2 in the computer without costly alternations to prototype equipment. The designer will readily see that other non-linear circuits will be required to fully represent the actual suspension. For example, the tyre force on the road will not be a simple linear function of $(y - z)$ and a diode function generator will be needed for its representation in the computer. Such devices are supplied with most analogue computers and are simple to set up and operate.

LIQUID FLOW IN A VALVE CONTROLLED PIPE WITH A PUMP

Figure 12 shows portion of a chemical plant consisting of a pump and pipe system whose flow is controlled by a remote level controller.

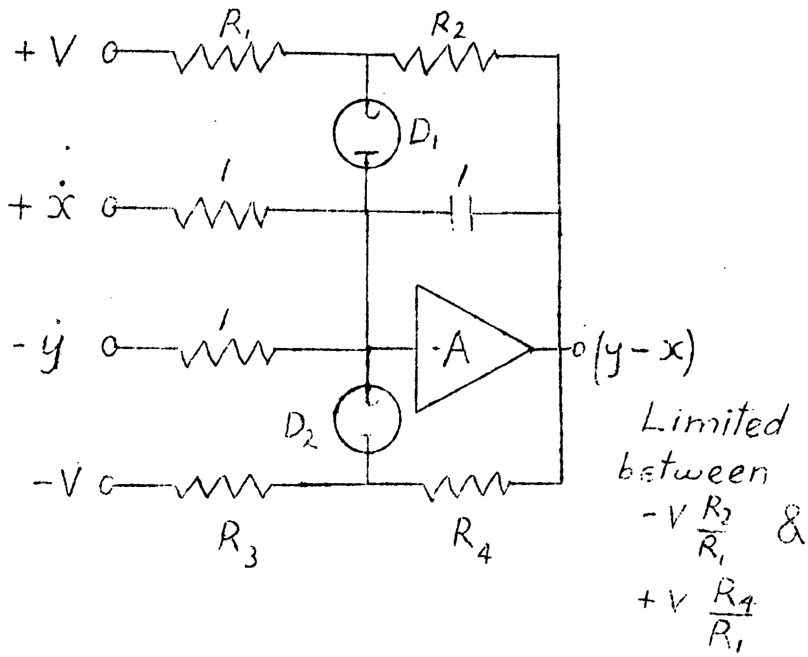

Fig. 11. AN INTEGRATOR WITH LIMITS

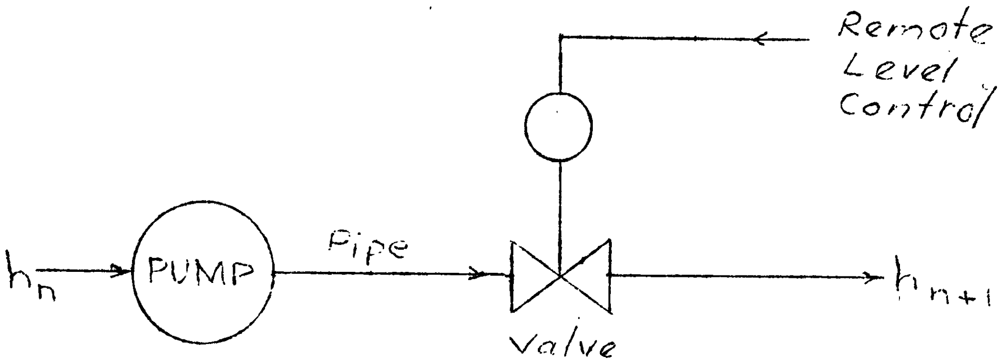

Fig. 12. PUMP AND CONTROL VALVE SYSTEM FROM A CHEMICAL PLANT

f = Flow rate

h_n = Pressure head at a point in the system

p(f) = Pump head as a function of flow through pump

K_1 & K_2 = Constants corresponding to viscous and kinetic head drops.

C_v = Valve flow coefficient

h_v = " head drop

a = a constant associated with the level controller

e = error in controlled level

r = total range of level control

Flow in the controlled valve is given by

$$f = C_v \sqrt{h_v} \tag{20}$$

and hence

$$h_v = f^2/C^2_v \tag{21}$$

where

$$C_v = C_{v\,max} \quad \exp a(1 + 2e/r) \tag{22}$$

shows the manner in which C_v is controlled from the level error e.

Equating pressure drops in the line we have:

$$(h_n - h_{n+1}) + p(f) - (K_1 + 1/C_v^2)f^2 + K_2 f = 0 \tag{23}$$

or

$$K_2 f = (h_{n+1} - h_n) - p(f) + (K_1 + 1/C_v^2)f^2 \tag{24}$$

Figure 13 shows how this may be set up on the computer.

The top line shows the calculation of $(K_1 + 1/C_v^2)$ from the level error e $1/C_v^2$ is formed from e in a diode function generator.

If we then assume that f is known we may form f^2 in another function generator and use the multiplier M to form

$$(K_1 + 1/C_v^2)f^2$$

The pump head p(f) is formed from f in yet another function generator; it is then subtracted from $(h_{n+1} - h_n)$ in amplifier 3 to give $(h_{n+1} - h_n - p(f))$. This is added to $(K_1 + 1/C_v^2)f^2$ in amplifier 4, which is scaled suitably to give f as the output according to equation 24. The inputs to the system are h_n, h_{n+1} and e.

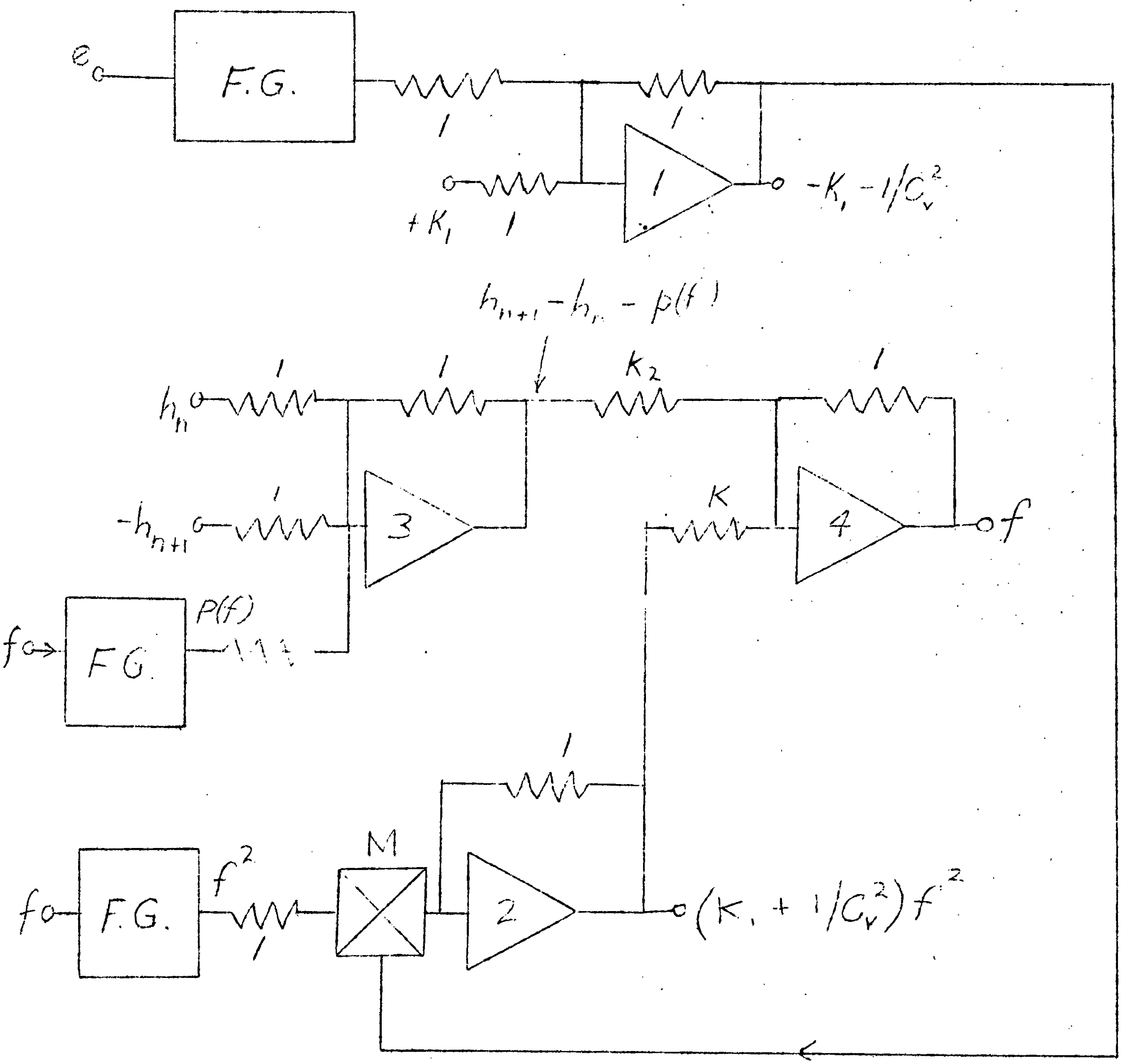

Fig. 13. COMPUTER REPRESENTATION OF FIG. 12.

The above illustrates the way in which a small portion of a chemical plant may be represented on an analogue computer. Complete chemical plants may be represented by suitably interconnecting such blocks.[4]

In addition to the design uses of such a computer plant operators may become familiar with the plants response to alterations in its manual controls before plant construction is completed, and without danger to the plant.

A 3 AXIS TRANSFORMATION WHICH OCCURS FREQUENTLY IN AERODYNAMICS

References to Figure 14 shows two orthogonal reference systems with the direction cosines

$$\begin{matrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{matrix}$$

between them. One reference system being stationary.

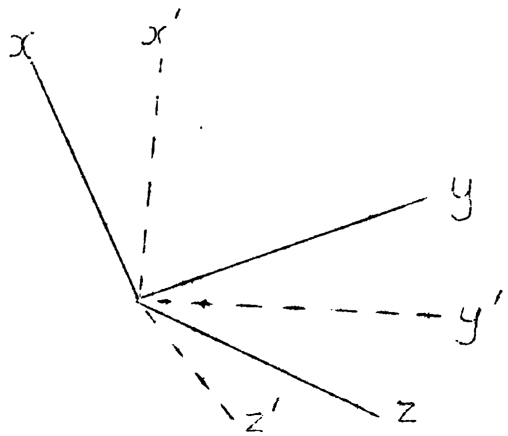

Fig. 14. A MOVING FRAME OF REFERENCE

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4. A.H. Doveton., "The Analogue Computer as an Aid to the Design and Operation of a Continuous Process". Proceedings First Conference on Automatic Computing and Data Processing in Australia. Australian National Committee on Computation and Automatic Control, May 1960.

If p, q and r are the angular rates of the moving axes about themselves then

$$\left.\begin{array}{lll} \dot{l}_1 = rl_2 - ql_3 & \dot{m}_1 = rm_2 - qm_3 & \dot{n}_1 = rn_2 - qn_3 \\ \dot{l}_2 = pl_3 - rl_1 & \dot{m}_2 = pm_3 - rm_1 & \dot{n}_2 = pn_3 - rn_1 \\ \dot{l}_3 = ql_1 - pl_2 & \dot{m}_3 = qm_1 - pm_2 & \dot{n}_3 = qn_1 - pn_2 \end{array}\right\} \tag{25}$$

If u, v and w are the components of a vector in the rotating system and u^1, v^1, w^1 the components of the vector referred to the other frame then

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} \begin{pmatrix} u^1 & v^1 & w^1 \end{pmatrix} \tag{26}$$

If the computer has an accurate four quadrant multiplier[5] available then the l's may be evaluated from equation 25 as in Fig. 15. The initial valves of l_1, l_2 and l_3 are set in on the integrators and subsequent variations arise from p q r. A similar computer section is required for the evaluation of the m's and n's.

Fig. 16 then shows how the u, v and w component is resolved using equation 26. This method is particularly powerful where the multiplier is capable of doing multiplications of the form

$$z = x(y_1 + y_2 + \text{---} + y_n) \tag{27}$$

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5. "EMIAC II Technical Specification," Ref. No. CP134, E.M.I. Electronics Ltd.

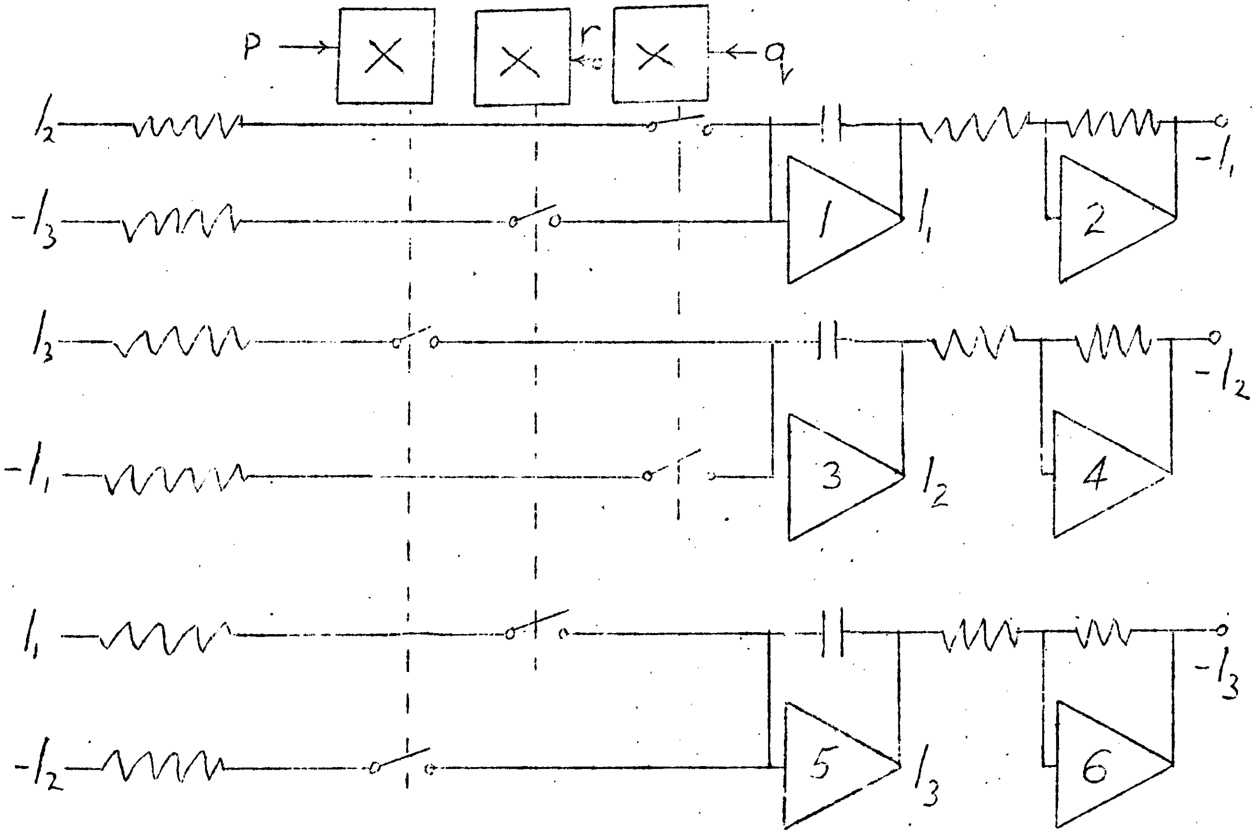

Fig. 15. EVALUATION OF l's

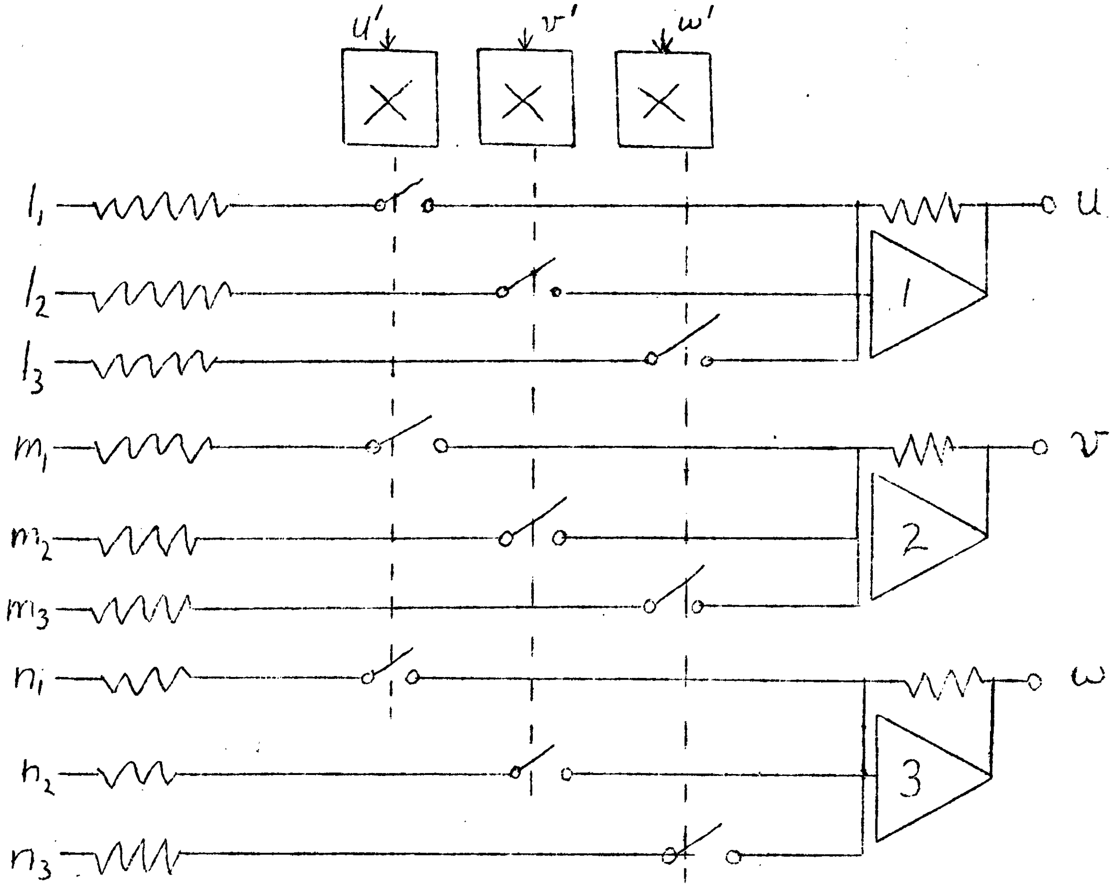

Fig. 16. RESOLVING THE VECTOR COMPONENTS

Conclusion:

An analogue computer can, in a few hours, solve problems which may take system designers many months of work.

It is hoped that the description of basic analogue computer components and the detailed treatment of a simple example has enabled designers of engineering systems to visualize ways in which analogue computers can be applied to their problems.

The three more difficult examples, although not scaled in detail, illustrate the range of linear and nonlinear problems which a typical modern analogue computer can handle.

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