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VINTAGE CAPITAL MODELS OF GROWTH AND FLUCTUATIONS

By

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C O N T E N T S

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F o r e w o r d

During a staff seminar on Growth and Capital theory at the University of New-South-Wales in January 1963 , Professor Murray Kemp suggested to me a research topic which has become the subject matter of this thesis. The entire investigation was carried out under his supervision and without his kind help and patient guidance this thesis could not have been written . My indebtedness and gratitude to Professor Kemp however go beyond what a student owes to his teacher : I am referring to the more important prolonged stimulations for many years which he has created and to his ready permission to include (in chapters 4 and 7) the substance of a joint article of ours which, as Professor Kemp himself stated in a letter dated August 22nd 1966, " was truly a joint product to which we contributed equally in effort and ideas".

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Chapter 1

I N T R O D U C T I O N

Growth is a very old theme in Economics : it is almost two hundred years since Adam Smith made his celebrated enquiry(1). Perhaps one must reach even further back in time to the writings of mercantilists and physiocrats in which the contours and elements of a theory of economic growth first appear(2).These pioneers who broke the path left for their progeny little more than an intriguing myth and the legacy has not always been particularly welcome.

The laws which determine the pattern of society's progress fascinate layman and economic theorist alike.The passion is evidenced by the prevalence of this theme in the vast literature of Economics.Yet,admittedly,there was a long period when economic analysis was,so to speak,"fobbed off with another bride— a theory of value"(3).However,the very nature of the affair was pedantic and political rather than academic.In the first place,when David Ricardo named Distribution ,the centre-piece of the theory of value, as "the principal problem in Political Economy"(4), he only tried to shift the emphasis from an area in which to him it was so extremely difficult to conduct an analysis.*Vide* his letter to Malthus of October 9th,1820 :"Political Economy you think is an enquiry into the nature and cause of wealth —I think it should be called an enquiry into the laws which determine the division of the produce of industry amongst the classes who concur in its formation.No law can be laid down respecting quantity but a tolerable correct one can be laid down respecting proportions"(5).

In the second place there was also a deep-seated political reason. Ricardo's days were days of unrest. In England modern industry was just emerging from the age of childhood but it already had made some impact upon the class structure of the society. Early English radicals read Ricardo in the 1820's and interpret his theory as describing the antagonism of class interests of wages and profits, of profits and rents. The succeding period witnessed a quarrel between industrial capital and aristocratic landed property and the growing discord between the government and the feudal aristocracy gathered around the Holy Alliance on the one hand and and the popular Mass on the other. The literature of Political Economy in England of this period, as Marx put it, "calls to mind the stormy forward movement in France after Dr. Quesney's death, but only as a Saint Martin's summer reminds us of spring" (6). Then with the emerge of Sismondi, Proudhon, Robertus, Lassalle, Marx and other continental socialists, economic writings suddenly became social messages and genuine scientific reseach was replaced by "the bad conscience and the evil intent of apologetic" [7]. The popularity of distribution theory was thus inevitable.

After this long period during which the new social structure finally took shape, attention was again focussed on the problem of growth and fluctuation. At first the interest was purely academic: the problem with all the rigour complexity and purity of its logic is extremely exciting and is good as an exercise for the mind. But it is not like the morning jam that one would see and touch each day. Until the late 'thirties the most socially important problem seems to have been that of unemployment and Keynes with his short-run comparative static tool had handled it masterly. Only after the second war did the problem of growth become also a real and urgent problem: on the one hand there was the task of reconstruction in countries which had been bankrupted by the war, on the other there was the task of development in countries which had long suffered from backwardness. As a result , planning for growth has become the national policy in most countries: it has become the common concern of every one, from philosophers, theologians, pamphleteers, special pleaders, reformers to the general public. This additional feature of the problem has helped to create a new literature which is so strikingly esoteric.

I am totally intrigued by the intellectual fascination that the problem offers, but I also share the interest of the so-called traditionally-minded, empirically- inclined, earthy economists. Most of the contributions to the literature so far have been criticised by them as being unrealistic: it is a bad show that after more than two decades of theorising we are still somewhat in the domain of one-sector world and its fable of a golden age. I think there is much truth in these criticisms and one must take them very seriously.

This thesis is an attempt to inject certain features of reality into an analysis of the kind so dominant in the literature, namely the neo-classical approach. We shall begin with a stocktaking by giving a brief survey of modern growth theory with special emphasis being placed on a synthesis of Leif Johansen's which had made the injection of these features into the neo-classical scheme possible. We shall then proceed to investigate models which incorporate these features in great detail.

It will soon be obvious that the additional features will immensely complicate the analysis. This seems to be the price of keeping a nose for reality. Aiming at this I shall offer no apology for the serious mathematics that will be employed.

Footnotes to chapter 1.

(1) Cf Adam Smith: *An Inquiry Into the Nature and Causes of the Wealth of Nations*, 1776.

(2) Cf. Joseph J. Spengler: "Mercantilist and Physiocratic Growth Theory", in *Theories of Economic Growth*, The Free Press of Glencoe, Illinois, 1960, pp. 3-64.

(3) Joan Robinson: *The Accumulation of Capital*, London: Mac Millan & Co Ltd, 1956, p.v.

(4) *Works of David Ricardo*, ed. P. Sraffa, Vol I, Preface to the Principles, p. 5.

(5) *Works of David Ricardo, ibid.*

(6) Karl Marx: *Das ~~C~~Kapital*, afterword to the Second German Edition, Hamburg, Verlag von Otto Meissner, 1873.

(7) Karl Marx, *op. cit.*

Chapter 2.

MODERN GROWTH THEORIES: A BRIEF SURVEY

When Harrod made a "tentative and preliminary attempt" in 1939 to "give the outline of a dynamic theory"(1) he did not have in mind a theory which would "determine the course of events in detail" but aimed only at providing a framework of concepts relevant to the study of change analogous to that provided by static theory for the study of rest"(2). With John von Neumann's paper on general economic equilibrium(3) still lying obscure and untranslated in the *Ergebnisse eine mathematischen Kolloquiums* (4), Harrod's essay had all the charm of a pathbreaking paper. It did not contribute much to modern dynamics in the sense of Ragnar Frisch (5). But its importance lies in the stimulus it gave to what we may term "traditional dynamics" - a dynamical theory of the kind introduced by John Stuart Mill (6), namely, the analysis of long run equilibrium. (7) An analysis of this kind calls for an explicit introduction of the time element.

That "the element of time is a chief cause of those difficulties in economic investigations"(8) is well known and yet up to 1939(9) economic literature had never gone beyond the "famous fiction of the Stationary State" (10). It is true that the "steady state" of economic progress was discussed as long ago as in 1826 by Johan Heinrich von Thünen(11) and that Gustav Cassel introduced, as a generalisation of the stationary state, the concept of an "evenly progressive economy where population grows at a constant rate and production is increased in proportion to population"(12) but they failed to make any use of their idea. It was Harrod's article that laid the foundation stone of modern growth theory. For, as Sir John Hicks has recently observed, "what used to be called the theory of long period equilibrium has turned, in modern economics, into the theory of growth"(13).

There is in effect not a unified body of thought that can appropriately be referred to as *the* theory of economic growth. If such a theory exists, it must be able to inform us about all the phenomena of a changing economy. These phenomena are so immensely complex that any theory about them is "bound to simplify, or at least in some way to over-simplify" (14). What really do exist are economic growth *models*. By growth theories therefore will be meant the set of hypotheses concerning the behaviour over time of certain economic variables in various simplified economic structures. In this sense perhaps one should consider growth theory as a post-war development. Since the war there have been two distinct lines of thought on the construction of simplified economic structures. The first has developed from the input-output model of Leontief and from the above mentioned article of vonNeumann, the english translation of which was made available by the *Review of Economic Studies* in 1945; its general character is sufficiently indicated perhaps by the caption Linear Economics. The second line of development had its origin in Harrod's 1939 essay.

Harrod's original theme was taken up by Evsey Domar in 1946(15) and again in 1947,(16) and a year later Harrod himself published his *Towards A Dynamic Economics*(17). It is in these writings that the first modern growth model of traditional type took shape.

The schematic form of the Harrod - Domar model is extremely simple and may be presented as follows:

(1) A constant proportion s of income (Y) is devoted to saving.

(2) A unit of output is produced by a given minimum amount of capital and labour - in other words, the capital-output ratio v is fixed.

(3) The labour force grows overtime at a constant rate n, fixed by non-economic, demographic factors.

Much of subsequent work stemming from this model is but a modification of the assumptions concerning these three key parameters (s,v and n).It is easy to see that in a state of steady growth income must grow at the same rate as the capital stock ("warranted rate") since the capital output ratio is assumed to be fixed; and that ,in the absence of technical progress,income must also grow as fast as the labour force ("natural rate") since a given amount of labour is required for each unit of output.In other words,in terms of the above notations,for steady growth it must be true that $n = s/v$ .Since n is determined by demographic factors,s by psychological factors and v by technological factors,it is only by pure chance – or,to use an expression of Joan Robinson, "by a fluke"(18) – that this equality holds. Unless it does hold,however, there is no possibility of steady growth.

A divergence of the warranted from the natural rate is not the only obstacle to full employment steady growth. By definition,the warranted rate of growth is " that rate of growth which,if it occurs,will leave all parties satisfied that they have produced neither more nor less than the right amount "(19). What happens then, if the economy is actually growing at a rate which is not warranted by present technological conditions ? The answer will depend of course on our assumptions about the behaviour of the economic agents who appear in the system ; in particular,it will depend on their expectations and on their adjustments to error. Harrod's original assumption led him to conclude that the warranted rate is highly unstable ,in the sense that a "departure from equilibrium (warranted rate) instead of being self-righting will be self-aggravating"(20) although other sets of assumptions could lead to the opposite result (21).

In general therefore,the implication of this analysis is that " even for the long run the economic system is at best balanced on a knife-edge(22) of equilibrium growth. Were the magnitudes of the key parameters –the saving ratio,the capital output ratio,the rate of increase of the labour force – to stop over so slightly from dead center,the consequence would be either growing unemployment or prolonged inflation "(23).

In view of the "stylised fact" , Kaldor-style or otherwise , this line of thought seems to have led to a far-too-strong conclusion . Several attempts were made to "systematize"(24) the relations between the warranted and the natural rate of growth in order to weaken the Harrod - Domar conclusion. Given the rate of growth of the labour force n variations in the saving ratio s or in the capital output ratio v, or in both these ratios, may eliminate any differences between n and s/v. These views were taken simultaneously by two schools of thought almost a decade after the publication of Harrod's and Domar's post-war articles. To oppose the assumption that the capital-output ratio is a constant is to take a position similar to that of the neo-classical economists against the classical: in the classical theory capital and labour were treated as if they were highly complementary factors whereas in neo-classical theories they are regarded as potentially substituable. In 1956, Solow [25] and Swan [26] independently presented growth models which incorporate this neo-classical assumption: the Harrod - Domar model was re-formulated to include the assumption that the capital-output ratio is a variable. The opposition to a constant saving ratio on the other hand was dominated by an explicit injection of income distribution theory into growth analysis with a special Keynesian flavour. This was undertaken by Kaldor [27] and Mrs. Joan Robinson. [28].

Explicitly introduced in the neo-classical scheme is a smooth production function relating output to inputs (capital and labour) and revealing the existence of a continuum of techniques. Flexibility of factor proportions ensures the equality of n and s/v and thus removes the first obstacle to steady full employment growth. However the knife edge problem does not necessarily disappear: to get rid of this second obstacle to steady growth we need to make futher restrictive assumption concerning saving and investment behaviour or the price mechanism in the factor market. [29].

The introduction of a saving function could also remove one of the obstacles to full employment steady growth. If the propensity to save differs from one group of income earners to another then the aggregate saving ratio depends on the distribution of income. If v is fixed factor rewards are not restricted to the rule of marginal productivity calculation, hence the distribution of income (and the value of s) may be assumed to be such that the equality of n and s/v is ensured. If v is a variable, the production function may be such that the substitution between capital and labour is unitary-elastic and the stability properties of the system will be the same as those of Swan's and Solow's neo-classical models. If the elasticity of substitution is not equal to unity the movement in profit"s share will be such that the stability of the steady growth solution is unimpaired.

These models are usually constructed so as to incorporate also the assumption of technical progress and/or the depreciation of capital equipment.

Technical change has two aspects which are important : the *nature* of the change and the *bias* of the change.Up to 1962 most treatments of technical progress were based on the assumption that it is an exogenous factor which may either be embodied or disembodied.Arrow[(30)] was the first* to advance the hypothesis that "technical change in general can be ascribed to experience,that it is the very activity of production which gives rise to problems for which favorable responses are selected over time"[(31)].With respect to depreciation there are also two kinds of treatment: the *sudden death* and the *radioactive decay*.In the former case machines are assumed to last for some T periods and then suddenly fall into pieces,whereas in the latter case it is assumed that a physical proportion of the machine "evaporates" at every moment of time : it is like a fall-out effect and every machine will suffer,hence the term "radioactive".

The neo-classical model with or without technical change has been generalised by Uzawa[(32)],Inada[(33)],Takayama[(34)],Dhrymes[(35)],Kurz[(36)], Solow[(37)] and by Drandakis[(38)] to cover the case in which consumption goods and capital goods are formally separated.

Up to the late'fifties one may say generally therefore that growth theory is dominated by the neo-classical scheme.

Footnotes to chapter 2.

(1) R.F. Harrod: "An Essay in Dynamic Theory", *Economic Journal*, vol XLIX, March 1939, reprinted as essay 13 in R.F. Harrod: *Economic Essays*, London: Macmillan & co. Ltd, 1952, p. 254.

(2) R.F. Harrod, *loc. cit.*

(3) von Neumann: "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes" (On an Economic Equilibrium System and a generalisation of Brouwer's Fix-point Theorem). English translation appeared under the title: "A Model of General Economic Equilibrium", *Review of Economic Studies*, 13 (1), 1 - 9 (1945 - 1946).

(4) Franz Denticke, Leipzig and Vienna, 1937.

(5) Ragnar Frisch: "On the Notion of Equilibrium and Disequilibrium", *Review of Economic Studies*, III (1935-36), pp. 100 - 106. According to Frisch, a system is dynamical if its behaviour over time is determined by functional equations in which variables at different points of time are involved in an essential way.

(6) Mill seems to have taken the concept from Auguste Comte, with whom he frequently corresponded. (cf. *Lettres d'Auguste Comte ā John Stuart Mill*, 1841 - 1846, Paris, 1877). In his *Cours de Philosphie Positive*, Comte separated the problems presented by social organism into two kinds: those that may be said to belong to the social evolution and those which do not. In order to emphasize a parallelism between this method of study and that of natural science, Comte borrowed the terminology of a zoologist, H. de Blainville, to call the former (social evolution) statics and the latter (non-evolutionary phenomena) Dynamics.

(7) *Vide* Mill's *Principles of Political Economy with Some of their Applications to Social Philosophy*, London: Longman, Green, Reader & Dyer, (1869) Book V, chapter 1 (p.421) We have still to consider the economical condition of mankind as liable to change, and indeed... as at all times undergoing

progressive changes. We have to consider what these changes are, what are their laws and what their ultimate tendenceies thereby adding a theory of motion to our theory of equilibrium - the Dynamics of Political Economy to the statics."

(8) Alfred Marshall: *Principles of Economics*Book V, chap. V, § 2. London Macmillan, 1891, p. 366.

(9) E. Lundberg's *Studies in the theory of Economic Expansion*, (1937) seems to have made no impact on the literature.

(10) Alfred Marshall, *loc. cit.*

(11) von Thünen : *Der isolierte Staat in Bezielung auf Landwirthschaft und Nationalokonomie, (1826)*

(12) Gustav Cassel: *Theoretische Sozialokonomie*, Stockholm, 1918, chap. 1. § 6. English translation by Joseph McCabe: *The Theory of Social Economy*, T. Fisher Unwin Ltd, London, 1923, Vol 1, Book 1, Chap. 1. § 6, p. 35.

(13) J. R. Hicks: *The Theory of Wages*, London: Macmillan, 2nd Edition, 1963

(14) J. R. Hicks: *Capital and Growth* Oxford at the Clarendon Press, 1965, p.v.

(15) E. Domar: "Capital Expansion, Rate of Growth and Employment". *Econometrica*, Vol 14, April 1946

(16) E. Domar: "Expansion and Unployment", *American Economic Review*, Vol 37, March 1947,

(17) London: Macmillan, 1948.

(18) Joan Robinson: *The Accumulation of Capital* *op. cit*, p. 405

(19) R. F. Harrod, *loc. cit.*, p. 256.

(20) R.F. Harrod, *ibid.*, *p. 264*.

(21) Cf. H. Rose: "The Possibility of Warranted Growth". *Economic Journal*, Vol LXIX, June 1959. D.W. Jorgenson: "On Stability in the Sense of Harrod", *Economica*, Vol XXVII, August 1960.

(22) The term *knife-edge*, as used frequently in the literature, refers to the possibility of a divergence between the *actual rate* and the *warranted rate* only and not to the general instability of Harrod's model which includes also the possibility of a divergence between the *warranted rate* and the *natural rate* of growth.

(23) R.M. solow: "A Contribution to the Theory of Economic Growth". *Quarterly Journal of Economics*, Vol. LXX, No. 1. February 1956 p. 65.

(24) T.W. Swan: "Economic Growth and Capital Accumulation", *Economic Record* Vol. XXXll, November 1956. pp. 334-61.

(25) R.M. Solow, *ibid.*

(26) T.W. Swan, *ibid* .

(27) Cf. Nicholas Kaldor: "Alternative Theories of Distribution". *Review of Economic Studies*, Vol. 21 1954. "A Model of Economic Growth", *Economic Journal*, Vol. LXXll, December 1957. "Capital Accumulation and Economic Growth" in *Theory of Capital*, eds. F.Lutz and D.C. Hague, London: Macmillan, 1961.

(28) Joan Robinson. *loc. cit.*

(29) Cf. F.H. Hahn: "The Stability of Growth Equilibrium", *Quarterly Journal of Economics*, Vol. LXXlV, May 1960, A.W. Phillips: "A Simple Model of Employment, Money and Price in A Growing Economy", *Economica*, Vol. XXVlll, Nov. 1961, and "Employment Inflation and Growth", *Economica*, Vol. 29, Feb. 1962, J.R. Sargent: "The Stability of Growth Equilibrium: Comment". *Quarterly Journal of Economics* Vol. LXXVl August 1962.

(30) K.J. Arrow: "The Economic Implications of Learning by Doing" , *The Review of Economic Studies* No 80. Vol XX1X (3), June 1962.

(31) K.J. Arrow, *ibid.* p. 156.

(32) H.Uzawa: "On a Two Sector Model of Economic Growth: I", *Review of Economic Studies*, Vol XX1X, October 1961. "On a Two-sector Model of Economic Growth: II", *Review of Economic Studies*, June 1963 (Vol XXX).

(33) K.I. Inada: "On Two Sector Model of Economic Growth: Comments and a Generalisation, *Review of Economic Studies*, Vol XXX, June 1963 "On the Stability of Growth Equilibria in Two Sector Models" *Review of Economic studies*, Vol 31, 1964.

(34) A. Takayama: "On a Two-sector Model of Economic Growth: A Comparative Static Analysis", *Review of Economic Studies*, Vol XXX, June 1963.

(35) P. Dhrymes: "A Multi-Sector Model of Economic Growth", *Quarterly Journal of Economics"*, June 1962.

(36) M. Kurz: "A Two Sector extension of Swan's Model of Economic Growth: The Case of No Technical Change". *International Economic Review*, Jan. 1963.

(37) R.M. Solow: "Note on Uzawa's Two Sector Model of Economic Growth", *Review of Economic Studies*, Vol XX1X, October 1961

(38) E. Drandakis: "Factor Substitution in the two-sector Growth Model" *Review of Economic Studies*. Vol XXX, October 1963

(*) Perhaps we should say that he is the first to treat technical change as being endogenous *in the context of growth models*. Similar treatment of technical change appeared long ago in the infant industry arguement.

Chapter 3.

L E I F J O H A N S E N'S C R I T I Q U E A N D A C R I T I Q U E O F J O H A N S E N.

There is much merit in classical economics and this can hardly be denied. However Meade would further insist that the preservation of the continuity of tradition and method is in itself "a positive virtue" (1). The neoclassical scheme in growth analysis no doubt has this virtue: it enjoys, so to speak, "the Ricardian vice" (2) while it supplies, in the most elegant manner, a description of the behaviour overtime of the simplest form of classical economic system. But the form of neo-classical theory as it stands is much too simple to be accepted as a good description of reality. In particular, its treatment of capital is unsatisfactoty in at least two respects.

(1) In the first place the re-employment of a general production function, which had been used frequently and unquestioningly by Böhm-Bawerk, J.B. Clark, Wicksteed, Marshall and even Wicksell, re-opens a number of conceptual problems that had never been raised properly before. In these days it is understandable: Wicksell for example was more interested in expounding a theory of distributive shares than a theory of growth and he obviously had wished to speak of a social marginal productivity. However in growth theory, one would also like to discuss the problem of capital accumulation . Thus to accept the neoclassical procedure is to argue that capital, as a concept, can be treated as a single homogeneous factor. Not being aware of this is simply being miseducated.(3)

There is of course a vital difference between the aggregation of consumption goods and that of capital goods. Since consuming has always been identified with purchasing, consumption goods are implicitly assumed to have no time dimension: the production *period* has no counterpart in consumption. Aggregation of consumption goods is therefore only the correlative of an aggregation of various commodities in *cross-section,* which involves - in comparative statics at least - simply the ordinary index number problem. This problem was almost settled in 1936 when Leontief [(4)] showed that an aggregate of two or more goods which are used in given proportions can be treated as a single good in equilibrium analysis. The quantity and price of the aggregate must be measured by indices, the quantity index varying proportionatley with the quantities of the separate goods and the price index varying proportionately with the ratio of total expenditure of the quantity index. Three years later, Hicks [(5)] showed that those goods whose prices vary proportionately can also be treated as a composite commodity. (Similar proof was also obtained, apparently independently, by Lange[(6)]. In these proofs only infinitesimal changes are considered. In 1952 Wold[(7)] supplied an alternative proof which does not suffer from this weakness.) While these results are important from a demand analysis view point, they play only a minor role in growth analysis, particularly in connection with the aggregation of capital[(8)]. The difficulty in capital aggregation is twofold: on the one hand, as Klein[(9)] has pointed out, if the production function is to retain its original meaning - namely the relationship between output and factor inputs, one of which being capital - it must be a singled-valued function of the factor inputs and it must be *purely technical* in character: it must not depend on prices or other purely economic variable. On the other hand, as have been agreed by all, capital goods have a time dimension. The aggregation of capital, as clearly recongnised by Swan[(10)], will involve therefore not only the ordinary index problem but also the "reduction of a very high order system of lagged equations.....to a more manageable system with fewer lags"[(11)]. To dodge this difficulty, at first one is asked to think in term of a very simplified world and imagine that "capital is made up of a large number of identical meccano sets, which never wear out and can be put together, taken apart, and reasesembled with negligible cost or delay in a great variey of models so as to work with various combinations of labour and land, to produce various products" .....that are "sold at constant price-ratios among themselves no matter how the rates of wage, rent and profits may vary."[(12)] Of course this is only a simplified picture and "it is not true that only one kind of capital good exists, but there is also more than one kind of labour"[(13)]. In other words if the heterogeneity of labour can be ignored then the heterogeneity of capital can very well be ignored "in the same way"[(14)].

Recently attempts [15] have been made to show that in some cases, there exists a meaningful aggregate capital stock that can be regarded as a single homogeneous factor in an aggregative surrogate production function. But this result does not seem to be very helpful: the surrogate production function formulated in terms of this aggregate capital stock do not imitate well the behaviour of the production process.[16] In models where technical change is assumed to be embodied only in new equipment, "there would be a tendency for the surrogate (production function) to underestimate the shares of wages .... because wages eat up such a large part of the output of the oldest capital in use"[17].

(2) In the second place there is also another unsatisfactory aspect in the neo classical treatment of capital which makes the defectiveness of the proceedure become more serious: "it insulates the analysis from contact with reality"[18]. This is the implicit assumption of *perfect malleability* of machinery.[19] Meade points out that if capital is homogeneous old capital goods must somehow be equatable to new capital goods - not merely equivalent in value terms but technically equivalent. And not only old capital goods must be made homogeneous with new; all capital goods must be made homogeneous *with each other*. This, to quote Hicks, "surely is the point at which the theory loses contact with reality in a way that really matters. Real machines are not malleable, the beating of swords into ploughshares is always a costly operation, if indeed it is not a figure of speech" [20]. When technical change is assumed to take place, the homogeneity assumption for capital also has another unrealistic connotation: it would imply that new and old equipments alike benefit from technical changes at equal rate. This is perhaps conceivable if technical changes are totally disembodied. Common observation however tells us a contradictory story: that "each unit of equipment represents the technological stage of development reached in its year of origin"[21] and that in reality, "technical specification of capital goods is constantly changing, new kinds of goods constantly appear and others disappear....."[22]. In other words, technical improvements in fact do not accrue indifferenly to equipment of all vintages.

Taking these facts into account, it seems that the homogeneity assumption for capital can no longer have a *raison d'être*. It is not altogether clear however ~~that~~ whether or not, once this assumption is dropped one must also abandon the neoclassical scheme. The answer to this question was provided by Leif Johansen in a masterly analysis.

Johansen's critique which was published in *Econometrica* in 1959[23] did not quite bring out into light these obscure implications of the neo-classical scheme. His aim was modest: he simply wanted to "propose a kind of synthesis" between different approaches in growth theory. His proposal to replace the Harrod-Domar-Leontief assumption of fixed factor proportions and the Solow-Swan assumption of variable factor proportions by the assumption that "substitution possibilities between capital and labour exist ex-ante but not ex-post" was made as a "compromise" (24). However besides this compromise there is also an explicit re-formulation of the technical progress factor to take account of the fact that "new production techniques can be introduced only by means of new capital equipment." (25) Equipments are now distinguishable by the proportions at which they are combined with labour. These proportions are co-determined by the state of technical knowledge and the rate of the real wage which prevail at the time when the equipment is being installed. In other words, different capital equipments may be differentiated by their *vintages*.

Johansen's article thus provides a neoclassical model in which capital is treated as being heterogeneous for the first time. The neoclassical results are shown to hold good still: when general (ex-ante) production functions are used and when equipments are assumed to last indefinitely while depreciating exponentially or not at all, output will grow asymtptotically to the path of steady growth. Johansen also considers a non-classical case and shows that if the physical life-time of equipments is fixed and production functions ex-ante are linear, then not only are fluctuations in income possible but also (even if we neglect complex solutions) the asymptotic rate of growth will depend on the rate of saving.

The most remarkable thing about Johansen's analysis is that it marks a departure from the neoclassical parable of one-capital-stock-and-one-production-function. One is now totally convinced, as Samuelson (26) has rightly insisted, that "capital theory can be rigously developed without using any Clark-like concept of "aggregate"capital" and that growth analysis can be undertaken in a system where there is a continuum of heterogeneous physical capital goods and processes through time. In this respect, beyond all doubt, Johansen's formulation is a landmark in both growth analysis and capital theory. His analysis in its original form is however incomplete in that it overlooked one non-trivial detail posed by the economic problem of obsolescence. In the framework of Johansen's system, because technical change is assumed to continuously take place, the normal situation is that of a rising real wage. If there are no substitution possibilities ex-post, then after some interval of time a piece of equipment may no longer be capable of earning a non-negative quasi rent and should be scrapped. This critical cut-off point is reached when the current real wage is such that the output produced by the equipment is just sufficient to cover the associated wage cost. The period during which an equipment can be operated profitably is its economic life. Johansen's model can be revised to solve for this critical vintage.

Also, in Johansen's original analysis the rôle of entrepreneurs' expectations was not at all clear and the distributional aspects of the model were not fully investigated. In the next chapter we shall explore these specific points in a context of steady growth. In chapter 5 and chapter 6 is studied the asymptotic behaviour of the system, with chapter 5 devoted to the discussion of the non-steady growth paths and chapter 6 to the stability conditions of the steady growth solution. To these chapters are appended the numerical results of several simultation runs on a discrete model of similar character. In chapter 7 and chapter 8 we offer a multi-sector generalisation of the model.

Footnotes to chapter 3

(1) J.E. Meade: *"A Neoclassical Theory of Economic Growth"*, London: Unwin University Books, 1962, p.1.

(2) T.W. Swan: "Economic Growth and Capital Accumulation", *Economic Record*, Vol XXXII, November 1956, p. 334.

(3) *Vide* Mrs. Joan Robinson's account of an English classroom in which this particular problem is discussed: "The Student of Economic theory is taught to write $O = f(L,C)$ where L is a quantity of labour, C is a quantity of capital and O a rate of output of commodities. He is instructed to assume all workers alike, and to measure L in man-hours of labour; he is told something about the index-number problem involved in choosing a unit of output, and then he is hurried on to the next question, in the hope that he will forget to ask in what units C is measured. Before ever he does ask, he has become a professor, and so sloppy habits of thought are handed on from one generation to the next."
C.f. Joan Robinson: "The Production Function and the Theory of Capital", *Review of Economic Studies*, Vol XXI, 1953-54, p.81.

(4) W. Leontief: "Composite Commodities and the Problem of Index Numbers", *Econometrica*, 4, 1936, pp.39-59.

(5) J.R. Hicks: *Value and Capital*, Oxford, 1939, p.312-3.

(6) Oskar Lange: *Price Flexibility and Employment*, Bloomington, 1944, p.103-6

(7) Herman Wold: *Demand Analysis*, Uppsala, 1952, p.329.

(8) See for example, N. Liviatan's "The Concept of Capital in Professor Solow's Model", *Econometrica*, Vol 34, No. 1, Jan. 1966. Liviatan shows that aggregation of capital is possible if all equipments have the same durability. For, if the cost of construction of equipments is dependent only on the equipment's labour intensity, then machines' rentals will be proportionate to each other provided they have the same physical life time. In this case the concept of "capital in general" is analogous to Hick's

composite commodity concept.

(9) Lawrence R. Klein: "Macroeconomics and the Theory of Rational Behaviour", *Econometrica,* Vol 14 (1946) pp. 93-108. "Remarks on the Theory Aggregation", *ibid,* pp. 303-312.

(10) T.W. Swan: *loc.cit.,* Appendix: Notes on Capital.

(11) T.W. Swan, *ibid,* p.345.

(12) T.W. Swan, *ibid,* p.344.

(13) Robert M. Solow: "The Production Function and the Theory of Capital", *Review of Economic Studies,* Vol XXII, 1955-56, p.101.

(14) Speech of R.M. Solow at Corfu as recorded by Douglas C. Hague in *The Theory of Capital,* eds F.Lutz and D.C. Hague, London: MacMillan, 1961, p.297.

(15) Cf. R.M. Solow: "Capital, Labour and Income in Manufacturing", Conference in Income and Wealth, *The Behaviour of Income Shares,* Princeton University Press, 1964, p. 101-128. F.M. Fisher: "Embodied Technical Change and the Existence of an Aggregate Capital Stock", *Review of Economic Studies,* Vol XXXII October 1965, p.263-288. N. Liviatan: *loc.cit.,* Pham Chi Thanh: "Production Processes with Heterogeneous Capital", *Economic Record,* September 1966. (to be published.) W.M. Gorman: "Capital Aggregation in Vintage Models", *Review of Economic Studies,* forthcoming,

(16) C.f. Pham Chi Thanh, *loc.cit.*

(17) R.M. Solow, *private correspondence.*

(18) Joan Robinson: "Equilibrium Growth Models" *The American Economic Review,* Vol LI, No. 3, June 1961, p. 360.

(19) J.E. Meade, *loc.cit.*

(20) J.R. Hicks: Review of J.E. Meade's A Neo-classical Theory of Economic Growth, *Economic Journal*, June 1962, Vol LXXII, p. 373.

(21) K. Maywald: "The Best and the Average in Productivity Studies and in Long Term Forecasting" *The Productivity Measurement Review*, No. 9. Reprint Series No. 132, University of Cambridge, Department of Applied Economics, Cf. also Ingvar Svennilson: *"Growth and Stagnation in the European Economy"*, Geneva ECE, 1954 and "Capital Accumulation and National Wealth in an Expanding Economy" in *25 Economic Essays in Honour of Erik Lindahl*, Stockholm: Ekonomisk Tidskrift, 1956. W.E.G. Salter: *Productivity and technical change*. Cambridge University Press, London, 1960.

(22) Nicholas Kaldor: "Capital Accumulation and Economic Growth" in *Theory of Capital*, eds. F. Lutz and D.C. Hag~~g~~ue, McMillan, London 1961, p.203.

(23) Leif Johansen: "Substitution Versus Fixed Production Coefficient in The Theory of Economic Growth: A Synthesis," *Econometrica*, Vol 27, No.2, April 1959, pp. 157-176.

(24) Leif Johansen, *loc.cit.* p.157.

(25) Leif Johansen, *loc.cit.*, p. 159.

(26) Cf. his claim in "Parable and Realism in Capital Theory: The Surrogate Production Function", *Review of Economic Studies*, No. 80, Vol XXIX (3) June 1962, p.193.

Chapter 4.

ONE SECTOR GROWTH MODEL WITH HETEROGENEOUS CAPITAL

In what follows we shall study a classical system in which output is produced by two competitive factors: labour and capital. The term "competitive" is used here in a rather restrictive sense: the two factors are competitive only at the *margin* so that the fundamental process of labour-capital substitution is the construction of different kinds of capital goods. However, once a machine is installed, no variation in the complementary labour required to operate it is possible. In other words, we follow Leif Johansen in assuming that substitution possibilities exist *ex-ante* but not *ex-post* . The characteristics of a machine are summarized by its labour requirement. The latter is, in turn, assumed to be determined by the wage rate which currently prevails, the pattern in which it is expected to vary and by the extent to which new techniques can be introduced. Improvements in technology are assumed to accure exponentially and to be of the embodiment type so that they affect output only to the extent that they are carried into practice either by net capital formation or by the replacement of old fashioned equipment by the latest models. It follows from these assumptions that capital equipment may be distinguished by its vintage. Since time will be treated as a continous variable these assumptions imply that the system under consideration is a system in which there is a continuum of production functions one function for each type of capital equipment. Expectations concerning the behaviour of real wages are of two types: static, in which case the real wage is expected to remain stationary, and non-static, in which case it is expected to increase exponentially at the "correct" golden age rate. On the steady growths path, where the wage rate does rise exponentially, the former is the case of zero foresight whereas the latter is the case of perfect foresight. As regards depreciation we shall also follow the two lines of treatment which are so familiar in the literature: the radioactive type and the sudden death type. Accordingly we shall consider two cases: the case where all machines are infinitely durable but depreciate steadily at a constant exponential rate and the case where all machines have a fixed physical life time independent of use.

In this chapter we shall show the existence of a steady growth solution of a one sector system and study the economic properties of such growth paths. The asymptotic behavior of the system will be studied in the next chapter where, strictly in the spirit of the old tradition, the fundamental mechanism which sets the system to work is assumed to be generated in the labour market. Also, to preserve the Neo-classical scheme in the spirit of Solow and Swan, we shall assume that the ex-ante production function for each type of equipment is of the Wicksell- Cobb- Douglas type so that technical improvements, as described above, may be regarded as being neutral in the sense of Hicks.

1.ZERO FORESIGHT : EXPONENTIAL DEPRECIATION

Let $K_v(t)dv$ stand for the amount of equipment of vintages v to v+dv surviving at time t ($t \geqq v$); and $L_v(t)dv$ for the labour required to work that equipment. For $t>v$, the ratio of $K_v(t)$ to $L_v(t)$ is, of course, fixed; for $t=v$, it is variable. Then for each vintage of equipment there is a production relationship of the form

$$(4.1) \qquad Q_v(t) = e^{\lambda v} K_v(t)^{\alpha} L_v(t)^{\beta}, \quad \alpha + \beta = 1,\ \alpha,\beta>0$$

where $Q_v(t)dv$ stands for the rate of output at time t from equipment of vintages v to v+dv, and λ is the rate at which technical improvements accrue. Since all equipment decays exponentially we have $K_v(t)=e^{-\delta(t-v)}K_v(v)$ where δ is the rate of depreciation and $K_v(v)$ is the rate of gross investment at time v. Since, ex-post, factor proportions are fixed, we have also $L_v(t) = e^{-\delta(t-v)}L_v(v)$. Hence equation (4.1) may be written as

$$(4.2) \qquad Q_v(t) = e^{-\delta t+(\lambda+\delta)v} K_v(v)^{\alpha} L_v(v)^{\beta}$$

Total output at time t, therefore, is

$$(4.3) \qquad Q(t) = e^{-\delta t}\int_{V(t,W)} e^{(\lambda+\delta)v} K_v(v)^{\alpha} L_v(v)^{\beta} dv$$

where V(t,W) is the set of profitable vintages at t. It is not difficult to see that along the steady growth path this set reduces to V* :

$$V^* = \{t-\theta \leqq v \leqq t\} \quad \text{for all } t$$

where θ is the constant (Golden Age) economic life time of machines. Since in this chapter we are primarily interested in studying the properties of the steady growth path, we shall treat the economic life time of machines as a constant. Thus (4.3) may be re-written as

$$(4.3') \qquad Q(t) = e^{-\delta t}\int_{t-\theta}^{t} e^{(\lambda+\delta)v} K_v(v)^{\alpha} L_v(v)^{\beta} dv$$

The labour force L(t) is spread over vintages ($t-\theta$) to t and is growing exponentially, say, at rate n. We therefore may write

$$(4.4)\qquad L(t)=L_o e^{nt} = \int_{t-\theta}^{t} L_v(t)dv = e^{-\delta t}\int_{t-\theta}^{t} e^{\delta v} L_v(v)dv$$

Let $L_t(t)$ stand for the flow of labour available to work with equipment newly installed at time t. $L_t(t)$ is composed of three parts: the rate of net addition to the labour force, $nL(t)$; the rate at which labour is freed by the steady decay of existing equipment, $\delta L(t)$; and the rate at which labour is freed by the scrapping of equipment just θ periods old. The third component may be calculated as an infinite weighted sum of the first two components, dated at $t-\theta$, $t-2\theta$, $t-3\theta$,... Thus

$$(4.5)\qquad L_t(t)=(n+\delta)\ L(t)+(n+\delta)\ L(t-\theta)e^{-\delta\theta}$$

$$+(n+\delta)\ L(t-2\theta)e^{-2\delta\theta}$$

$$+(n+\delta)\ L(t-3\theta)e^{-3\delta\theta}$$

$$+\ldots\ldots\ldots\ldots$$

But the labour force is growing exponentially. Hence (4.5) reduces to

$$(4.6)\qquad L_t(t)=(n+\delta)\ L_o e^{nt}\ \left[1+e^{-(n+\delta)\theta}+e^{-2(n+\delta)\theta}+e^{-3(n+\delta)\theta}+\ldots\ldots\right]$$

$$=\frac{(n+\delta)L_o e^{nt}}{1-e^{-\theta(n+\delta)}}\qquad \text{if } n+\delta > 0$$

The rate of gross capital formation at time t is given by

$$(4.7)\qquad K_t(t) = sQ(t)$$

where s is the constant average propensity to save. Note that both $K_t(t)$ and $Q(t)$ are gross of depreciation.

Differentiating (4.3) with respect to t, and noting (4.6) and (4.7), we obtain the nonlinear difference differential equation

(4.8) $$\dot{Q}(t)=aQ(t)+b\left[Q(t)^{\alpha}+cQ(t-\theta)^{\alpha}\right]e^{dt}$$

where

$$a=-\delta$$

$$b=s^{\alpha}\left[\frac{(n+\delta)L_{\varrho}}{1-e^{-\theta(n+\delta)}}\right]^{\beta}$$

$$c=-e^{-\theta(\lambda+\delta+\beta n)},$$

and

$$d=\lambda+\beta n.$$

Let the exponential path of total output be

(4.9) $$\dot{Q}(t)=\bar{Q}e^{\varrho t}.$$

Equation (4.8) then reduces to

(4.10) $$(\varrho-a)\bar{Q}^{\beta}=b(1+ce^{-\alpha\varrho\theta})\ e^{\left[\lambda+\beta(n-\varrho)\right]t}.$$

The coeffient of t is extinguished by choosing

(4.11) $$\varrho=n+\frac{\lambda}{\beta}$$

whence

(4.12) $$\bar{Q}=\left\{\frac{b}{\varrho-a}\left[1-e^{-\theta(\varrho+\delta)}\right]\right\}^{1/\beta}$$

and

(4.9) $$Q(t)=\left\{\frac{(n+\delta)L_{o}}{1-e^{-\theta(n+\delta)}}\right\}\left\{\frac{s^{\alpha}}{\varrho+\delta}\left[1-e^{-\theta(\varrho+\delta)}\right]\right\}^{1/\beta}e^{\varrho t}.$$

We have not yet solved for θ. But at this stage we may note that a steady path exists, and that φ, the rate of growth along that path is independent both of the proportion of income saved and of the rate of depreciation. The reason is that in steady growth, output, capital and the *effective* labour supply (that is the labour supply corrected for its growing efficiency) must all grow at the same rate; and the effective labour supply, $e^{(\lambda/\beta)t}L(t)$, evidently grows at the exogeneously given rate $\varphi = n+(\lambda/\beta)$, which is independent of δ and s. The latter determines the level of output on the steady growth path and the annual increment of output, but not the ratio of one to the other. Of special interest, perhaps, is the elasticity of steady growth output with respect to the savings ratio :

$$\frac{s}{Q(t)} \frac{\partial Q(t)}{\partial s} = \frac{s}{Q(t)} \left[\left.\frac{\partial Q(t)}{\partial s}\right|_{\theta \text{ constant}} + \frac{\partial Q(t)}{\partial \theta} . \frac{\partial \theta}{\partial s} \right]$$

$$\frac{s}{Q(t)} \left[\frac{\alpha}{\beta} + \frac{\partial Q(t)}{\partial \theta} . \frac{\partial \theta}{\partial s} \right]$$

But, as we shall soon discover (equation (4.20)), $\partial\theta/\partial s = 0$, hence

$$\frac{s}{Q(t)} . \frac{\partial Q(t)}{\partial s} = \frac{\alpha}{\beta}$$

The total rent earned at time t by equipment of vintage v is

$$(4.14) \qquad \Pi_v(t) = Q_v(t) - w(t)L_v(t)$$

where w(t) is the wage rate, in terms of product, prevailing at t. And, $(t-\theta)$ is the vintage of that equipment which at time t can just "pay its way". It is clear, therefore, that θ must satisfy the equation (1)

$$(4.15) \qquad \Pi_{t-\theta}(t) = Q_{t-\theta}(t) - w(t)L_{t-\theta}(t) = 0$$

Now $Q_{t-\theta}(t)$ may be found by substituting in (4.2), first for $L_v(v)$ and $K_v(v)$

(from (4.6) and (4.7), then for $Q(t)$ (from (4.9')))

$$(4.16) \qquad Q_{t-\theta}(t)=(n+\delta)L_o \left\{ \frac{e^{-\theta(\rho+\delta)}}{1-e^{-\theta(n+\delta)}} \right\} \left\{ \frac{s}{\rho+\delta} \left[1-e^{-\theta(\rho+\delta)}\right] \right\}^{\alpha/\beta} e^{\rho t} .$$

Futher, under our present assumption of zero foresight,

$$(4.17) \qquad w(t)=\frac{\delta Q_t(t)}{\delta L_t(t)} = \beta \left\{ \frac{s}{\rho+\delta} \left[1-e^{-\theta(\rho+\delta)}\right] \right\}^{\alpha/\beta} e^{(\lambda/\beta)t}.$$

Also,

$$(4.18) \qquad L_{t-\theta}(t)=e^{-\delta\theta}L_{t-\theta}(t-\theta) = \left[\frac{(n+\delta)L_o}{e^{\theta(n+\delta)}-1} \right] e^{nt} \qquad \text{from (4.6).}$$

Hence, substituting in (4.15) from (4.16) - (4.18) and simplifying,

$$(4.19) \qquad \beta=e^{\theta(n-\rho)} ,$$

since

$$\rho=n+\frac{\lambda}{\beta} ,$$

$$(4.20) \qquad \theta=-\frac{\beta}{\lambda} \ln\beta.$$

Thus the economic life of equipment depends only on the rate at which technical improvements accrue and on the labour index. It does *not* depend on the rate of depreciation; nor does it depend on the saving ratio. If $\lambda=0$, $\theta=\infty$ and steady growth is impossible. An increase in the rate of technical progress result in a shortening of the economic life:

(4.21) $$\frac{\partial \theta}{\partial \lambda} = (\beta/\lambda^2)\log\beta < 0$$

But the effect of an increase in the labour index is ambiguous, depending on the initial value of the index:

(4.22) $$\frac{\partial \theta}{\partial \beta} = -\frac{1}{\lambda}(1+\log\beta)$$

$$\frac{\partial^2 \theta}{\partial \beta^2} = -\frac{1}{\lambda}\cdot\frac{1}{\theta} < 0$$

Viewed as a function of β, θ reaches a maximum at $e^{-1} \approx 0.37$ (Figure (4.1). Thus for realistic values of β, β and θ are inversely related.

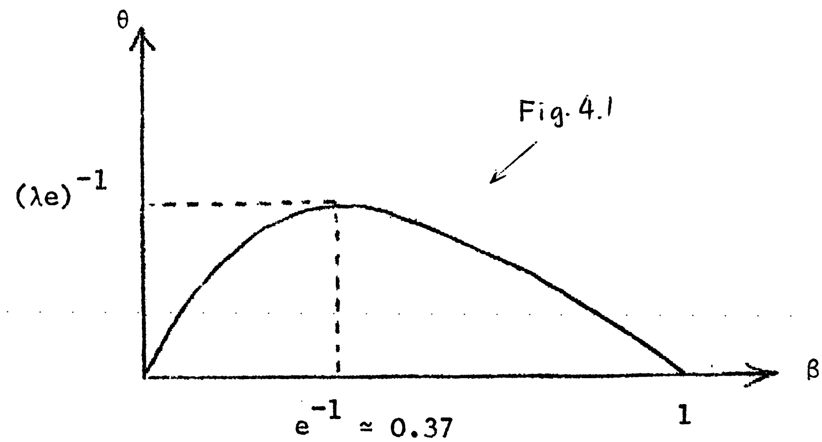
As an illustration, consider an economy with $\lambda = 0.01$ and $\beta = 0.75$. The economic life of equipment is then between twenty-one and twenty-two years. If the rate of progress were to double, the economic life of equipment on the new steady growth path would be halved.

We can now write down an expression for the distribution of income. Labour's share is, of course, L(t) w(t). Taking account of (4.4), (4.9'), and (4.17), this reduces to

(4.23) $$\frac{L(t)w(t)}{Q(t)} = \frac{\beta(\rho + \delta)\{1 - e^{-\theta(n+\delta)}\}}{(n + \delta)\{1 - e^{-\theta(\rho+\delta)}\}} \gtreqless \beta \quad \text{as} \quad \lambda \lesseqgtr 0$$

Thus, on a steady growth path labour's share is constant. It is independent of the saving ratio, varies inversely with the rate of technical progress, and varies directly both with the rate of depreciation and with the rate of population growth. Not surprisingly, it also varies directly with the labour exonent β. Note, however, that, except in the special unprogressive case in which $\lambda = 0$, the share is not equal to, but greater than, β. The explanation lies in the assumed lack of entrepreneurial foresight. On the assumption that the current wage rate will continue to prevail throughout the life of equipment, that is, for ever, the hiring of labour is pushed to the point where the marginal product of labour working with new equipment is equal to the wage rate. Labour's share of the output of the most recently installed equipment is, therefore, β. But the wage rate is steadily rising (equation(4.17)), and hence labour's share in the output of all older equipment exceeds β

Along the path of exponential growth the average rate of return on invest-



ment is constant, the steady increase in the capital to labour ratio being exactly offset by technical improvements. To see this we first calculate the rental at time t of a unit of equipment of vintage v:

$$(4.24) \qquad \Pi_v(t) = \begin{cases} \dfrac{Q_t(t)}{K_t(t)} = \dfrac{\alpha(\rho+\delta)}{s\{1-e^{-\theta(\rho+\delta)}\}} & \text{for } v = t \\[2ex] \dfrac{Q_v(t) - w(t)L_v(t)}{K_v(t)} & \text{for } v < t \end{cases}$$

The rate of return on investment in equipment of vintage v, say $r(v)$, is then given implicitly by the "present value" relation

$$(4.25) \qquad \int_{v}^{v+\theta} e^{-\{r(v)+\delta\}(t-v)}\Pi_v(t)\,dt = 1$$

Since $Q_V(t)$, $K_V(t)$, and $w(t)L_v(t)$ all grow at the same exponential rate, $\Pi_v(t)$ is function of $(t-v)$ only. It follows that $r(v)$ is stationary; and, since $\Pi_v(t)$ is a non-negative function of t, $r(v)$ is unique. Going a step further, by applying earlier results we can reduce (4.25) to a form which allows us to study the response of r to changes in each of the parameters s,λ,n,α, and δ:

$$(4.26) \qquad s\left[\frac{1-\beta^{\beta\{n+(\lambda/\beta)+\delta\}/\lambda}}{n+\frac{\lambda}{\beta}+\delta}\right] = \left[\frac{1-\beta^{\beta(r+\delta)/\lambda}}{r+\delta}\right] - \left[\frac{1-\beta^{\beta\{r+\delta-(\lambda/\beta)\}\lambda}}{n+\delta-\frac{\lambda}{\beta}}\right]$$

The response of the rate of return to an increase in the saving ratio, for example, is unambiguously negative: $\partial r/\partial s < 0$.

As a final exercise we note that the now familiar rule for maximum consumption on the steady growth path, viz., that $s=\alpha$, carries over to the present model. This may be confirmed by differentiating $(1-s)Q(t)$ with respect to s and equating to zero. It must be remembered, however, that in the present model α is not equal to the return to capital; as we have already noted (equation (4.23)), capital's share of total product is less than α

The case of perfectly durable equipment may be studied by equating δ to zero in the preceding derivations.

3. ZERO FORESIGHT : "SUDDEN DEATH" DEPRECIATION

We turn briefly to the case in which equipment has a fixed lifetime. each piece of equipment retains its efficiency during its lifetime, then disintegrates.

If $\bar{\theta} \geq -(\beta/\lambda)\ln\beta$, the limited physical lifetime of the equipment is of no consequence . All the results of Section 2 carry over with the simplification that δ may be equated to zero. In particular, there exists a path of steady growth along which the rate of growth is independent of the saving ratio . This is in contrast to Johansen's conclusion, based on the assumption of a linear production function .

If, however, $\bar{\theta} < -(\beta/\lambda)\ln\beta$, steady, full-employment growth is impossible . For then the wage rate can be read from (4.15) as

$$W(t) = \left\{\frac{-s}{\rho+\delta}\left[1-e^{-\bar{\theta}(\rho+\delta)}\right]\right\}^{\alpha/\beta} e^{(\alpha/\beta)(t-\bar{\theta})}$$

and from (4.17) as

$$W(t) = \beta\left\{\frac{s}{\rho+\delta}\left[1-e^{-\bar{\theta}(\rho+\delta)}\right]\right\}^{\alpha/\beta} e^{(\alpha/\beta)t}$$

The two wage rates coincide if and only if $\bar{\theta} = -(\beta/\lambda)\ln\beta$ which is contrary to assumption. It is worth noting perhaps that the barrier to steady growth in this case is not technological but derives from the competitive character of the labour market. If the wage rate were set administratively at a level consistent with (4.17), (4.15) could be dropped and there would remain no obstacles to steady growth .

3. PERFECT FORESIGHT: EXPONETIAL DEPRECIATION

In deriving the fundamental growth equation (4.9') we did not find it necessary to make any specific assumptions about the formation of expectations. That equation therefore carries over to the present section. It should not be inferred from this, however, that the level of steady growth output is independent of expectations. For, as (4.9) makes clear, the level of output depends on θ, the economic lifetime of equipment. And, as we shall shortly demonstrate, θ does depend on expectations. Specifically, if $\lambda > 0$ the economic life of equipment is longer and the level of output higher when the foresight of businessmen is perfect than when the future is expected to be a repetition of the present. (It follows immediately that with zero foresight, steady-growth production is inefficient. But more of that later).

We begin by recalling to service equation (4.15), which requires that equipment be scrapped when it ceases to cover wage costs. Substituting for $Q_{t-\theta}(t)$ and $L_{t-\theta}(t)$ (from (4.16) and (4.18), respectively) we obtain

$$(4.27) \qquad w(t) = \left[\frac{s}{\rho + \delta}\{1 - e^{-\theta(\rho + \delta)}\}\right]^{\alpha/\beta} e^{(\lambda/\beta)(t-\theta)}$$

We note next that if businessmen expect wages to change they will take that into account in arriving at their investing and hiring decisions. In particular, if they expect the wage to rise they will not push the hiring of labour to the point where the marginal product of labour is equal to the wage rate (Cf. (4.17)). Rather they will stop at the point at which an extra worker makes no contribution to the present value of current investment. The latter may be expressed as

$$(4.28) \qquad V(t) = \int_t^{t+\theta} [Q_t(v) - w(v)L_t(v)] e^{-r(t)(v-t)} dv$$

where the expression in square brackets is the rental earned at time v by equipment of vintage t and where r (t) is to be interpreted as the market rate of interest at time t. (Under conditions of perfect foresight it is also equal to the internal rate of return. We note that this discounting formulation is valid only in the Goldern Age where r(t) is in fact a constant).

Recalling that both Q_t (v) and $L_t(v)$ viewed as functions of v, decay exponentially at the rate δ, and noting from (4.27) that the wage rate grows exponentially at the rate λ/β, we may rewrite (4.28) as

$$(4.29)\qquad V(t)=\int_t^{t+\theta}\{Q_t(t)-w(t)L_t(t)e^{\lambda(v-t)/\beta}\}\,e^{-[\delta+r(t)]\,[v-t]}dv$$

and express the above mentioned condition of equilibrium as

$$(4.30)\qquad 0=\frac{\delta V(t)}{\delta L_t(t)}=\frac{\delta Q_t(t)}{\delta L_t(t)}\int_t^{t+\theta}e^{-[\delta+r(t)]\,[v-t]}dv-w(t)\int_t^{t+\theta}e^{-[\delta+r(t)-(\lambda/\beta)]\,[v-t]}dv.$$

Interpreting this, the current marginal product is equated not to the wage rate but to the "effective" wage rate, defined as that annuity with the same present value as the wage payments expected during the economic life of the equipment. Calculating the partial derivative with the aid of (4.6), (4.7), and (4.9'), and evaluating the integrals, (4.30) reduces to

$$(4.31)\qquad \beta e^{(\lambda/\beta)t}\left\{\frac{\beta}{\varrho+\delta}\left[1-e^{-\theta(\varrho+\delta)}\right]\right\}^{\alpha/\beta}\text{x}\begin{Bmatrix}\dfrac{1-e^{-\theta(r+\delta)}}{r+\delta} & \text{if } r+\delta\neq 0\\ \theta & \text{if } r+\delta=0\end{Bmatrix}$$

$$-w(t)\begin{Bmatrix}\dfrac{1-e^{-\theta[r+\delta-(\lambda/\beta)]}}{r+\delta-(\lambda/\beta)} & \text{if } r+\delta-(\lambda/\beta)\neq 0.\\ \theta & \text{if } r+\delta-(\lambda/\beta)=0\end{Bmatrix}=0.$$

Thus, as in the case of zero foresight, the wage rate grows exponentially at the rate λ/β.

We note finally that in a competitive equilibrium the wage and interest rates settle at whatever level will ensure that net profits are zero:

$$(4.32)\qquad K_t(t)=V(t).$$

Differentiating partially with respect to $K_t(t)$, we obtain

$$(4.33) \qquad 1 = \frac{\delta Q_t(t)}{\delta K_t(t)} \int_t^{t+\theta} e^{-[\delta+(t)][v-t]} dv$$

which, after evaluation of the integral and rearrangement, reduces to

$$(4.34) \qquad \frac{\alpha}{s} = H[r(t), \theta]$$

where

$$H[r(t), \theta] \equiv \frac{(r+\delta)}{(\rho+\delta)} \cdot \left[\frac{1-e^{-\theta(\rho+\delta)}}{1-e^{-\theta(r+\delta)}}\right]$$

It is evident from (4.34) that the rate of interest is in fact a constant; in what follows, therefore, the functional notation $r(t)$ will be dropped. It is also clear that if $s > \alpha$, that is, if the saving ratio exceeds the capital coefficient, the rate of growth will exceed the rate of interest. Similarly, if the saving ratio falls short of the capital coefficient the rate of interest will exceed the rate of growth. Finally, if the saving ratio equals the capital exponent the rate of growth and the rate of interest will both equal $n+(\lambda/\beta)$.

If solutions exist for θ, r, and $w(t)$, they may be found by solving (4.27), (4.31), and (4.34). Thus, eliminating $w(t)$ from (4.1) and (4.5) we obtain

$$(4.35) \qquad \beta e^{(\lambda/\beta)\theta} = J(r, \theta)$$

where

$$J(r, \theta) = \frac{(r+\delta)}{r+\delta-\lambda/\beta} \; \frac{1-e^{-\theta(r+\delta-\lambda/\beta)}}{1-e^{-\theta(r+\delta)}}$$

The two equations (4.34) and (4.35) contain r and θ only. It is immediately obvious that in the present case θ, the economic life of equipment depends on all of the parameters, not merely on λ and β.

Explicit solutions are out of reach, except in limiting cases; but simple graphical analysis suffices to indicate the general nature of the solutions (if any), as well as their responses to changes in parameters, The graphs of (4.34) and (4.35) are sketched in Figure 4.2. Thus the graph of (4.35) is downward sloping with vertical asymptote $\theta=-(\beta/\lambda)\log\beta$ (whence the economic life of equipment is greater under conditions of perfect foresight than under conditions of zero foresight) and horizontal asymptote $r=\delta-\lambda/(1-\beta)$.

The graph of (4.34) on the other hand depends for its general shape on the relative values of α, and s (and, therefore, of r and ϱ). Thus consider a savings ratio s_1 smaller than α, so that $r>\varrho$. For a specific rate of interest r', the graph of $H_1(r',\theta)$ leaves the vertical axis at unity (de l'Hospital's theorem), increases with θ, and approaches $(r'+\delta)/(\varrho+\delta)$ asymptotically. It intersects α/s_1, which lies betwwen unity and $(r'+\delta)/(\varrho+\delta)$ and which is greater than unity, at θ'. If a higher rate of interest is considered, the graph of H_1 swings in a clockwise direction, pivoting on the vertical intercept $H_1 = 1$; it therefore intersects α/s_1 at a lower value of θ. As the assumed rate of interest falls and approaches $(\alpha/s_1)(\varrho+\delta)-\delta$, θ approaches infinity. As the rate of interest approaches infinity, θ approaches zero. The graph of (4.34) therefore has negative slope and asymptotes $\theta=0$ and $r=(\alpha/s_1)(\varrho+\delta)-\delta$, In Figure 4.2 it is labelled $H_1 = \alpha/s_1$.

Consider now a savings ratio s_2, larger than α, so that $r<\varrho$. For a specific rate of interest r'', the graph of $H_2(r'',\theta)$ leaves the vertical axis at unity but, unlike H_1, decreases with increasing θ and approaches $(r''+\delta)/(\varrho+\delta)$, which is of course less than one, asymptotically. It intersects α/s_2, which lies between unity and $(r''+\delta)/(\varrho+\delta)$, and is less than one. If a higher rate of interest is substituted, the graph swings clockwise and intersects α/s_2 at a *higher* value of θ. As r approaches $(\alpha/s_2)(\varrho+\delta)-\delta$, θ approaches infinity ; as r approaches minus infinity , θ approaches zero. Thus the graph of (4.34) has positive slope and asymptotes $\theta = 0$ and $r = (\alpha/s_2)(\varrho+\delta) - \delta$.

Finally , if $s = \alpha$ then $r = \varrho$ for all values of θ. The θ-solution is found by solving $J(\varrho,\theta) = \beta e^{(\lambda/\beta)\theta}$.

Leaving aside the limiting ,unprogressive case in which $\lambda = n = 0$,

$$\frac{\alpha}{s}(\varrho + \delta) - \delta > -\delta - \lambda/(1-\beta),$$

and,as Figure 4.2 makes clear,there is a unique finite solution for r and θ. A little tinkering with Figure 4.2 reveals the possibility,however, that the equilibrium rate of interest may be negative. This possibility emerges naturally when $s > \alpha$,for in that case $\varrho > r$ and a negative rate of interest is consistent with a positive rate of growth.

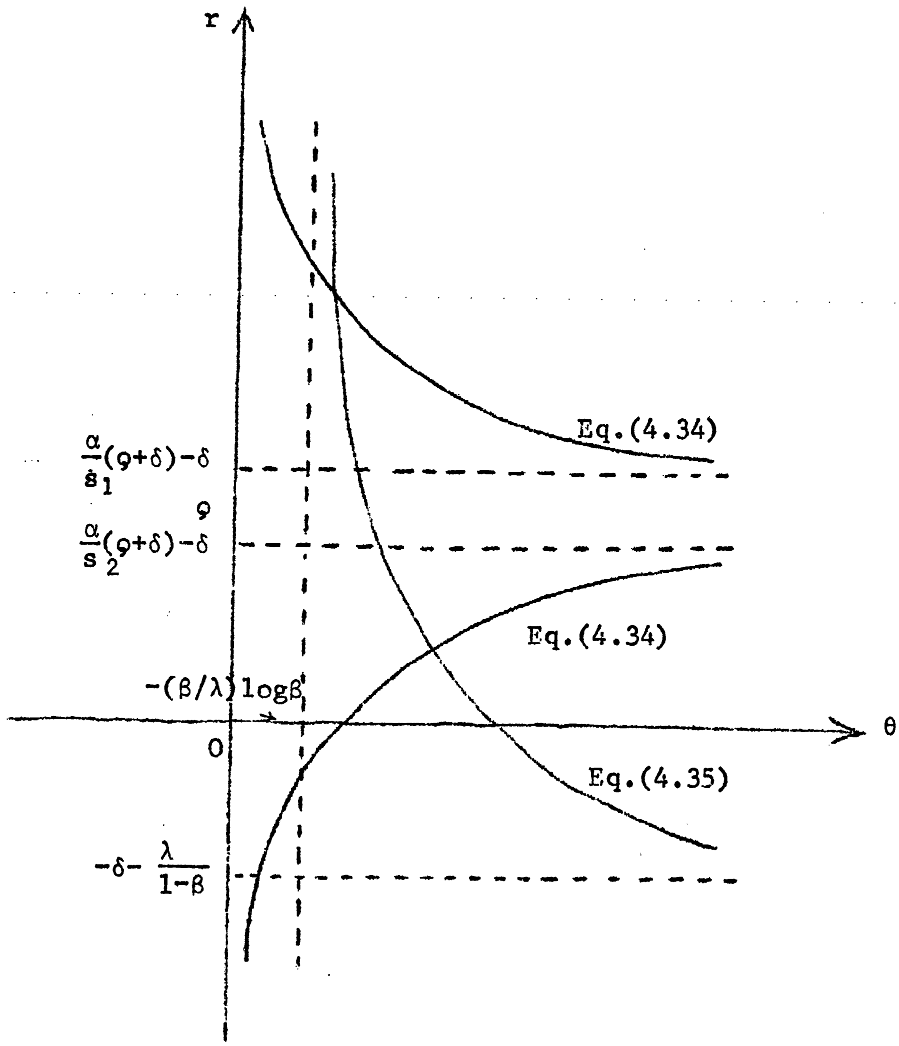

r
$\frac{\alpha}{s_1}(\rho+\delta)-\delta$
ρ
$\frac{\alpha}{s_2}(\rho+\delta)-\delta$
$-(\beta/\lambda)\log\beta$
0
$-\delta-\frac{\lambda}{1-\beta}$
Eq.(4.34)
Eq.(4.34)
Eq.(4.35)
θ

FIGURE 4.2

As an illustration consider the case where $\lambda = 0.02$, $\beta = 0.9$, $\delta = 0.05$ $s = 0.4$, $n = 0.04$ so that $\rho = 0.06$. Calculation of the values of r and θ on an IBM 1620 gives the following results :

	Along Eq. (4.34)	*Along Eq. (4.35)*
r	θ	θ
-0.06	9.84	30.04
-0.07	10.04	26.14
-0.08	10.24	23.24
-0.09	10.44	20.94
-0.10	10.64	19.14
-0.11	10.94	17.64
-0.12	11.14	16.34
-0.13	11.44	15.24
-0.14	11.84	14.24
-0.15	12.34	13.44
-0.16	*12.54*	*12.64*
-0.17	13.04	11.94
-0.18	13.64	11.44
-0.19	14.24	10.84
-0.20	15.04	10.34
-0.21	16.04	9.94
-0.22	17.44	9.54
-0.23	19.34	9.14

Clearly the equilibrium for r and θ are approximately -16% and $12\frac{1}{2}$ years.

That completes our graphical discussion of the solution to (4.34) and (4.35).

It was noted at the beginning of the present section that lack of foresight would give rise to an inefficient allocation of resources; in particular, the number of men working with any piece of newly installed equipment would be excessive, with the result that the equipment would be scrapped too soon. An appropriate measure of inefficiency is provided by the expression

$$(4.36) \qquad I_1 = 1 - \frac{Q(t)}{Q^*(t)} = 1 - \frac{\bar{Q}}{\bar{Q}^*}$$

Since $\bar{Q}^*$ is of the same form as $\bar{Q}$ (see (4.12) and (4.9')), differering only in the value of θ inserted, we are now in a position to evaluate I_1 for any particular set of parameter values. The following values were chosen with one eye to ralism, the other to manageability : $n = 0$, $\delta = 0.20$ $s = \lambda = 0.20$, $\beta = 0.60$. With zero foresight $\theta \approx 1.53$, with perfect foresight $\theta \approx 2.30$. Hence $I_1 = 4.6$ per cent so that the effect of imperfect foresight is to depress steady growth output to 95.4 per cent of its potential. At higher rates of saving the inefficiency would have been greater (for $s = 0.4$, other parameter values unchanged, $I_1 = 6.3$ per cent); if inductry had been more capital intensive, inefficiency would have been less.

It may be inferred from Figure 4.2, or proved by straightforward calculation, that an increase in the savings ratio will result in a lower rate of interest and a longer economic life of equipment. From this it is possible to infer the effect of an increase in thrift on the distribution of income. For, from (4.1), making use of (2.4) and (2.9'), we have

$$(4.11) \qquad \frac{w(t)L(t)}{Q(t)} = \begin{cases} \dfrac{\rho + \delta}{n + \delta} \cdot \left[\dfrac{e^{\theta(n+\delta)} - 1}{e^{\theta(\rho+\delta)} - 1}\right] & \text{if } \lambda > 0 \\[2ex] \beta & \text{if } \lambda = 0 \end{cases}$$

$$(4.38) \quad \frac{w(t)L(t)}{Q(t)} = \beta\left[\frac{\delta + r - \lambda/\beta}{\delta + r}\right]\cdot\left[\frac{\varrho + \delta}{n + \delta}\right]\cdot\left[\frac{1 - e^{-\theta(n+\delta)}}{1 - e^{-\theta(\varrho+r)}}\right]\cdot\left[\frac{1-e^{-\theta(\delta+r)}}{1-e^{-\theta(\delta+r-\lambda/\beta)}}\right]$$

It follow that if $s = \alpha$, so that $\varrho = r$, labour's share is equal to β, the labour exponent; the "young" machines that yield a share to labour less than β are balanced by the "old" machines from which labour derives a share in excess of β. If, however, $s > \alpha$, so that $\varrho > r$, labour's share is less than β; and if $s < \alpha$, so that $r > \varrho$, labour's share is greater than β.

The relation between thrift and steady growth output may be inferred from (4.9'). In the case of zero foresight, as we have seen, the only effect of a change in s is the direct one measured by the partial elasticity of Q(t) with respect to s :

$$\frac{\partial \log Q(t)}{\partial \log s} = \frac{\alpha}{\beta}$$

When foresight is perfect, however, allowance must be made also for the indirect effect of thrift, working through θ. We know already that increased thriftiness gives rise to a longer economic life of equipment. It remains to determine the effect of longer equipment life on output. This is measured by the partial elasticity of Q(t) with respect to θ :

$$(4.39) \quad \frac{\partial \log Q(t)}{\partial \log \theta} = \theta\left[\frac{\varrho+\delta}{\beta}\{e^{\theta(\varrho+\delta)} - 1\}^{-1} - (n+\delta)\{e^{\theta(n+\delta)} - 1\}^{-1}\right]$$

which is positive or negative as (7)

$$(4.40) \quad \frac{\varrho+\delta}{n+\delta}\;\frac{e^{\theta(n+\delta)} - 1}{e^{\theta(\varrho+\delta)} - 1} \quad > \text{ or } < \quad \beta$$

that is (from (4.37)), as the share of wages is greater or less than β. But we have just discovered that the share of wages is greater or less than β as s is less than or greater than α(that is, as r is greater or less than ϱ).

It follows that the indirect effect of greater thrift on productivity reinforces the direct effect if $s < \alpha$; if $s > \alpha$ the two pull against each other.

Finally, we note that the condition for maximum steady growth consumption under zero foresight , $s = \alpha$, carries over to the present model. For it is a necessary condition of maximum consumption that

$$(4.41) \qquad \frac{\partial}{\partial s}\{(1-s)Q(t)\} + \frac{\partial}{\partial \theta}\{(1-s)Q(t)\}\frac{d\theta}{ds} = 0$$

and it can be shown that , when $s = \alpha$, both terms vanish. Thus, in view of (4.9'), the first term can be written

$$(4.42) \qquad \frac{\partial}{\partial s}\{(1-s)Q(t)\} = -Q(t) + \{\frac{1-s}{s} \cdot \frac{\alpha}{\beta}\} Q(t) = 0 \quad \text{when } \alpha = s$$

On the other hand,

$$(4.43) \qquad \frac{\partial \log Q(t)}{\partial \theta} = Z\left[\frac{\rho + \delta}{n + \delta} \cdot \frac{e^{\theta(n+\delta)} - 1}{e^{\theta(\rho+\delta)} - 1} - \beta\right]$$

where $Z = \{(n+\delta)/\beta\}\{e^{\theta(n+\delta)} - 1\}^{-1}$. The first expression inside the braces, however, is simply labour's share (4.37) which, when $s = \alpha$, is equal to β.

The case of perfectly durable equipment may be studied by equating δ to zero in the above derivations.

4. PERFECT FORESIGHT : " SUDDEN DEATH " DEPRECIATION

Little need be said about this case. If the economic life of equipment does not exceed its physical life, $\bar{\theta}$, the analysis of section 4.3 (with $\delta = 0$) carries over in its entirety. Otherwise, with $\theta \equiv \bar{\theta}$, we are left with two mutually inconsistent equations in r, (4.34) and (4.35), and conclude that steady full employment growth is impossible.

5. THE ROLE OF EXPECTATIONS

We are now able to see in what respects "expectations matter" in the analysis of steady growth. First, it is clear from Figure 4.2 that the economic life of equipment is longer when foresight is perfect than when it is absent. Firm blessed with perfect foresight stop hiring labour short of the point where the rate is equal to the marginal product of labour. It therefore takes longer for technical improvements to eat into the quasi-rents earned by equipment. Second, the longer life of equipment gives rise to a higher level of output (but not to a higher rate of growth of output). The difference between the two steady-growth outputs, expressed as a proportion of output under conditions of perfect foresight, is a measure of the inefficiency with which resources are allocated under conditions of zero foresight. Third, the rate of wages is higher under conditions of perfect foresight (compare (4.17) and (4.31) but the share of wages is smaller. The reason, roughly stated, is as follows. With zero foresight, the share of labour working on the latest equipment is β; the share of all other labour is greater than β. Under conditions of perfect foresight, on the other hand, the share of labour working with recently installed equipment is less than β; only on equipment of at least a certain minimal age is labour's share greater than β.

An increase in thrift will push up the level of steadygrowth income, without affecting the rate of its growth. Its implications for other variables, however, depend crucially on the nature of expectations. If foresight is perfect, increased thrift will reduce the rate of return, increase the economic life of equipment, and reduce the income share of labour; with zero foresight, increased thrift affects none of these variables.

Footnotes to chapter 4.

(1) Johansen's equation (4.17), with his $c = 0$ and his $a + b = 1$, emerges from (4.9') as θ is made to approach infinity.

(2) It is assumed that discarded equipment has no scrap value.

(3) Not to be indentified with the market rate of interest. To establish a relationship between the rate of interest and the rate of return on investment one would need much more detailed assumptions about expectations and saving behaviour than those implicit in a constant ratio of saving to output.

(4) Cf. T.W. Swan: "of golden Ages and Production Functions". in K. Berrill, ed, *Economic Development with Special Preference to East Asia,* London: McMillan & Co. Ltd, 1964.

(5) These results are valid also in the more general "mixed" case in which equipment decays exponentially until age $\bar{\theta}$, when it falls apart.

(6) Leif Johansen, op. cit., p.173.

(7) Cf. Massell: "Investment Inovation and Growth", *Econometrica,* XXX, no.2, April 1962, 239-52.

Chapter 5.

ASYMPTOTIC BEHAVIOUR OF ONE SECTOR MODEL: I. OFF THE GOLDEN AGE PATH.

In the previous chapter we explored some of the properties of the steady growth solution generally known in the literature as the *Golden Age*. In this path output grows at exponential rate $\lambda/\beta+n$ and the real wage grows at rate λ/β. A proper golden age path is a steady growth path consistent with full employment of labour. A *bastard* golden age path is a steady growth path where there is persistent unemployment.

In order to study the asymptotic behaviour of the system and/or the stability properties of the golden age solution it is necessary to assume a certain mechanism which sets the system in motion from any particular arbitrary initial point. In this chapter we study some dynamical aspects of the one sector model when it is off the golden age path. We shall explicitly assume that the dynamic mechanism is generated from the instantaneous relations of supply and demand in the labour market. It will be assumed that there is a *minimum* (or subsistence) level below which the real wage cannot fall. Maximal demand for labour may thus be defined as the total demand for labour when all machines which are capable of earning non-negative quasi-rent at subsistence wage rate are operated. Labour supply is assumed to be perfectly inelastic so that this rate will prevail whenever the maximal demand for labour is smaller than the total supply of labour. When the maximal demand for labour is equal to the supply of labour the real wage may settle at the subsistence level or above it, depending on whether the set of machines which can be operated profitably at the subsistence wage level is such that all the least profitable machines earn zero quasi-rent. If this is not the case, then of course there exists a range where the real wage may rise above the subsistence level without affecting the total demand for labour. Such indeterminacy may exist also when the maximal demand for labour is greater than the total supply of labour: there may exist more than one (non subsistence) wage level which can generate the employment of this volume of total labour supply. The existence of this indeterminacy in vintage models is well known. To get rid of it we shall follow Inada[(1)] in assuming that in such situation the real wage will settle at the greatest level consistent with the full employment of labour, i.e. at the level which makes* the least profitable machines earn zero quasi-rent.(See Fig.5.1 and 5.2)

We shall consider two cases: the case where the subsistence wage is a constant and the case where it rises as fast as the rate of labour-saving technical progress. In this chapter we shall confine ourselves to the case of static expectation and investigate only the asymptotic behaviour of the system when it is off the steady growth path. Until section 4 labour is not assumed to be a non-redundant factor. The task of section 4 is to show that if the usual neo-classical assumption of full employment is adopted then the system will approach the Golden Age equilibrium asymptotically.

The case of constant subsistence wage level may be divided into two sub-cases: (1) it may be so low that all machines in existence can be operated profitably and (2) it may be so high that only a proportion of machines in existence can be operated profitably. In both cases it will be shown that unless labour is a non-redundant factor, the system will switch from a period of full employment to a period of unemployment or *vice versa*.

When the subsistence wage is assumed to rise as fast as the rate of labour saving technical progress, there are also possibilities of a *perpetual Bastard Golden Age* and of *cyclic fluctuations* in income. Labour non-redundancy is again sufficient to ensure steady growth. However, since subsistence real wage is *not* incompatible with full employment, there are in fact two kinds of Golden Age: one where the real wage is consistently at the subsistence level and the other where it always lies above the subsistence level. Conditions for for stability of the former may be derived without begging the question of full employment. To specify these conditions is a very complex task and we shall leave it for the next chapter.

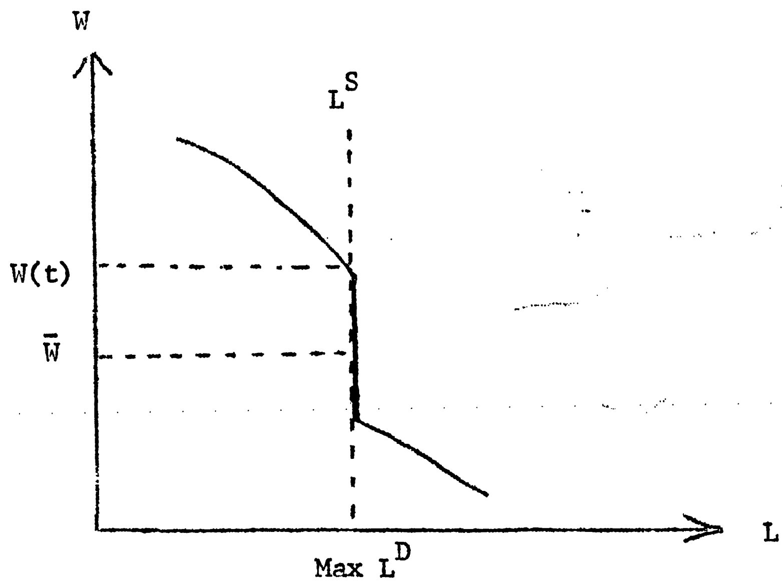


FIGURE 5.1

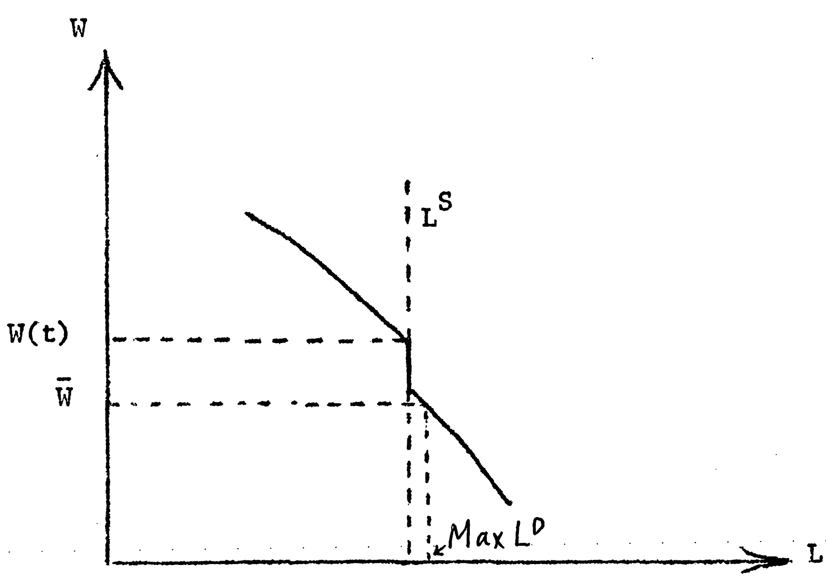


FIGURE 5.2

1.LOW CONSTANT SUBSISTENCE WAGE

If the constant subsistence wage level is so low that all machines in existence can be operated profitably then the maximal demand for labour, Max L^D, is the total demand for labour when every machine in existence is operated, viz.,

$$(5.1) \qquad \text{Max } L^D(t) = \int_T^t L_v(t)\,dv$$

where T is the age of the oldest machine.

We denote the labour:capital ratio (labour measured in efficiency units) at time t by b(t), i.e.,

$$(5.2) \qquad b(t) = \frac{e^{(\lambda/\beta)t} L_t(t)}{K_t(t)}$$

and define

$$(5.3) \qquad g[b(t)] = [b(t)]^\beta = \left[\frac{e^{(\lambda/\beta)t} L_t(t)}{K_t(t)}\right]^\beta = \frac{Q_t(t)}{K_t(t)}$$

Let W(t) be the real wage rate which prevails at t, then the static expectation assumption implies that the following marginal condition must be satisfied :

$$(5.4) \qquad W(t) = \frac{\partial Q_t(t)}{\partial L_t(t)} = e^{(\lambda/\beta)t} g'[b(t)] = e^{(\lambda/\beta)t} \beta \frac{[b(t)]^{\beta-1}}{b(t)}$$

and (5.1) may be re-written as

$$(5.5) \qquad \text{Max } L^D(t) = e^{-\delta t} \int_T^t e^{[\delta-(\lambda/\beta)]v} b(v) K_v(v)\,dv$$

Let $L^S(t)$ be the total supply of labour at t, then

$$(5.6) \quad L^S(t) = L_o e^{nt}$$

where L_o is the labour force at t = 0 and n is the natural growth rate of labour.

Dividing (5.5) by (5.6) and denoting the ratio $K_v(v)/L^S(v)$ by k(v) we obtain

$$(5.7) \quad e^{-(n+\delta)t}\int_T^t e^{[n+\delta-(\lambda/\beta)]v} b(v)k(v)\,dv = \frac{\text{Max } L^D(t)}{L^S(t)} \gtreqless 1$$

If at t,v is a profitable vintage, then we must have

$$Q_v(t) > W(t)L_v(t)$$

or

$$\frac{Q_v(t)}{L_v(t)} = \frac{Q_v(v)}{L_v(v)} > W(t)$$

and if v is the vintage of marginal machines, (i.e.,machines which earn zero quasi-rent) then

$$\frac{Q_v(v)}{L_v(v)} = W(t)$$

$$= \frac{g[b(v)]e^{(\lambda/\beta)v}}{b(v)}$$

Thus the description of wage determination may be expressed as

$$(5.8) \quad W(t) = \text{Max}\left[\bar{W},\ \sup\{W : e^{-(n+\delta)t}\int_{V(t,W)} e^{(n+\delta-\lambda/\beta)v} b(v)k(v) \geqq 1\}\right]$$

where $\bar{W}$ is the subsistence wage and V(t,W) is the set of profitable vintages :

$$V(t,W) = \{\, v \mid \frac{g[b(v)]e^{(\lambda/\beta)v}}{b(v)} - W(t) \geqq 0\}$$

(i) Case 1 : Max $L^D(t) > L^S(t)$

If at t_0 the maximal demand for labour is smaller than the total supply of labour, i.e., if

$$(5.9)\qquad e^{-(n+\delta)t_0}\int_T^{t_0} e^{(n+\delta-\lambda/\beta)v} b(v)k(v)\,dv < 1$$

then there will be unemployment and the real wage, as described by (5.8), will settle at the subsistence level :

$$(5.10)\qquad W(t_0) = \bar{W}, \text{ a constant.}$$

We shall show presently that (5.9) and (5.10) cannot hold for all $t \geqq t_0$. If (5.9) and (5.10) can hold for all $t \geqq t_0$ then from (5.4) it follows that

$$(5.11)\qquad b(t) = (\beta/\bar{W})^{1/\alpha} e^{(\lambda/\alpha\beta)t} \qquad \text{for all } t \geqq t_0$$

Since gross investment is a fixed proportion of gross output we write

$$(5.12)\qquad K_t(t) = sQ(t)$$

$$= se^{-\delta t}\int_T^{t} e^{(\delta+\lambda)v} K_v(v)^{\alpha} L_v(v)^{\beta}\,dv$$

where s is the marginal propensity to save.

(5.12) may be re-written as

$$(5.13)\qquad K_t(t) = se^{-\delta t}\int_T^{t} e^{\delta v} \left[e^{(\lambda/\beta)v} L_v(v)/K_v(v)\right]^{\beta} K_v(v)\,dv$$

$$= se^{-\delta t}\int_T^{t} e^{\delta v}\left[b(v)\right]^{\beta} K_v(v)\,dv$$

Dividing both sides of (5.13) by $L^S(t)$ gives

$$(5.14) \qquad k(t) = se^{-(\delta+n)t}\int_T^t e^{(\delta+n)v}[b(v)]^{\beta}k(v)\,dv$$

Differentiating (5.14) with respect to t and then solving for k(t) for $t \geqq t_0$ yields

$$(5.15) \qquad k(t) = e^{s(\beta/\bar{W})^{\beta/\alpha}(\alpha/\lambda)e^{(\lambda/\alpha)t}-(\delta+n)t + K_0}$$

where K_0 is an arbitrary constant of integration.

It follows that

$$(5.16) \qquad \lim_{t\to\infty} k(t) = \infty$$

Hence (5.9) and (5.10) cannot hold for all $t \geqq t_0$. In other words, for some t_1 sufficiently large, we must have

$$e^{-(n+\delta)t_1}\int_T^{t_1} e^{(n+\delta-\lambda/\beta)v}b(v)k(v)\,dv \geqq 1$$

and unemployment will then disappear.

(ii) Case 2 : Max $L^D(t) = L^S(t)$

If at t_1 the maximal demand for labour is equal to the total supply of labour, i.e., if

$$(5.17) \qquad e^{-(n+\delta)t_1}\int_T^{t_1} e^{(n+\delta-\lambda/\beta)v}b(v)k(v)\,dv = 1$$

then the real wage will settle at a level which, as described by (5.8), makes the least profitable machines earn zero quasi rent. If u is the vintage of marginal machines at t_1 ,then

$$(5.18) \qquad W(t_1) = \inf_{T \leq v \leq t_1} \frac{g\{b(v)\}e^{(\lambda/\beta)v}}{b(v)}$$

$$= \frac{g\{b(u)\}e^{(\lambda/\beta)u}}{b(u)}$$

$$= W_u \text{ ,say.}$$

We shall show that (5.17) and (5.18) cannot hold for all $t \geq t_1$.

It can be easily seen that if (5.17) holds for all $t \geq t_1$ then $W(t)$ will remain at W_u for all $t \geq t_1$. In other words,if (5.17) holds, then

$$\inf_{T \leq v \leq \infty} \frac{g\{b(v)\}e^{(\lambda/\beta)v}}{b(v)} = W_u$$

This can be seen in the following way. If (5.17) holds for all $t \geq t_1$, then ,since

$$W(t) = g'\{b(t)\}e^{(\lambda/\beta)t}$$

$$= \frac{\beta g\{b(t)\}e^{(\lambda/\beta)t}}{b(t)}$$

It follows that

$$\frac{g\{b(t)\}e^{(\lambda/\beta)t}}{b(t)} = \frac{1}{\beta} \cdot \frac{g\{b(u)\}e^{(\lambda/\beta)u}}{b(u)} > \frac{g\{b(u)\}e^{(\lambda/\beta)u}}{b(u)}$$

Thus for all $t \geq t_1$,

$$W(t) = \frac{g\{b(u)\}e^{(\lambda/\beta)u}}{b(u)} = \inf_{T \leq v \leq \infty} \frac{g\{b(v)\}e^{(\lambda/\beta)v}}{b(v)} = W_u$$

Q.E.D.

Thus if (5.17) and (5.18) can hold for all $t \geq t_1$ then from (5.4) it follows that

$$(5.19) \qquad b(t) = (\beta/W_u)^{1/\alpha} e^{(\lambda/\alpha\beta)t} \qquad \text{for all } t \geq t_1 .$$

Again we write

$$(5.20) \qquad k(t) = se^{-(\delta+n)t} \int_T^t e^{(\delta+n)v} b(v)^{\beta} k(v)\,dv$$

Differentiating (5.20) with respect to t and then solving for k(t) for $t \geq t_1$ yields :

$$(5.21) \qquad k(t) = e^{s(\beta/W_u)^{\beta/\alpha}(\alpha/\lambda)e^{(\lambda/\alpha)t} - (\delta+n)t + K_1}$$

where K_1 is an arbitrary constant of integration. It follows that

$$(5.22) \qquad \lim_{t\to\infty} k(t) = \infty$$

Hence (5.17) and (5.18) cannot hold for all $t \geq t_1$; in other words, for some t_2 sufficiently large we must have

$$e^{-(n+\delta)t_2} \int_T^{t_2} e^{(n+\delta-\lambda/\beta)v} b(v)k(v) > 1$$

(iii) <u>Case 3 : Max $L^D(t) > L^s(t)$</u>

If at t_2 the maximal demand for labour is greater than the total supply of labour,i.e.,if

$$(5.23) \qquad e^{-(n+\delta)t_2} \int_T^{t} e^{(n+\delta-\lambda/\beta)v} b(v)k(v)\,dv > 1$$

then the real wage will settle at a level which,as described by (5.8), makes the least profitable machines earn zero quasi rent .If q is the vintage of marginal machines at t_2 then

$$(5.24) \qquad W(t_2) = \inf_{v \varepsilon V(t_2, W)} \frac{g\{b(v)\}e^{(\lambda/\beta)v}}{b(v)}$$

$$= \frac{g\{b(q)\}e^{(\lambda/\beta)q}}{b(q)}$$

$$= W_q \text{ , say.}$$

We shall show presently that W(t) cannot remain at W_q for all t.

Suppose first that (5.23) and (5.24) can hold simultaneously for all $t \geqq t_2$; then ,since (5.23) implies full employment of labour and (5.24) implies stationary real wage,we have

$$(5.25) \qquad e^{-(n+\delta)t}\left[\int_q^t e^{(n+\delta)v}\frac{L_v(v)}{L^S(v)}dv - \int_{\tilde{V}(t_2,W_q)} e^{(n+\delta)v}\frac{L_v(v)}{L^S(v)}dv\right] = 1$$

where $\tilde{V}(t_2,W_q)$ is the set of unprofitable vintages at t :

$$\tilde{V}(t_2,W_q) = \{v | q \leqq v \leqq t_2; \frac{g\{b(v)\}e^{(\lambda/\beta)v}}{b(v)} - W(t_2) < 0\}$$

Differentiating (5.25) with respect to t yields :

$$(5.26) \qquad L_t(t) = (n + \delta)L_0e^{nt}$$

Again we write

$$(5.27) \qquad K_t(t) = sQ(t)$$

$$= s\{(n+\delta)L_0\}^{\beta}e^{-\delta t}\int_q^t e^{(\delta+\lambda+\beta n)v}\{K_v(v)\}^{\alpha}dv$$

$$- s\{(n+\delta)L_0\}^{\beta}e^{-\delta t}\int_{\tilde{V}(t,W_q)} e^{(\delta+\lambda+\beta n)v}\{K_v(v)\}^{\alpha}dv$$

Differentiating (5.27) with respect to t we obtain

$$(5.28) \qquad \dot{K}_t(t) = -\delta K_t(t) + s\{(n+\delta)L_0\}^{\beta} e^{(\lambda+\beta n)t} K_t^{\alpha}(t)$$

(5.28) is a Bernouilli-type non-linear differential equation in $K_t(t)$. It can be reduced to a linear form by the substitution :

$$(5.29) \qquad \chi(t) = \{K_t(t)\}^{1-\alpha} = \{K_t(t)\}^{\beta}$$

so that

$$(5.30) \qquad \dot{\chi}(t) = \beta\{ K_t(t)\}^{-\alpha}\dot{K}_t(t)$$

(5.28) may now be re-written as

$$(5.31) \qquad \dot{\chi}(t) + \delta\beta\chi(t) = s\beta\{(n+\delta)L_0\}^{\beta} e^{(\lambda+\beta n)t}$$

The general solution of (5.31) may be written as

$$(5.32) \qquad \chi(t) = \frac{s\beta\{(n+\delta)L_0\}^{\beta}}{\beta(n+\delta)+\lambda} \cdot e^{(\lambda+\beta n)t} + \chi^* e^{-\beta\delta t}$$

where χ^* is an arbitrary constant of integration. From (5.29) it follows that

$$(5.33) \qquad K_t(t) = \left[\frac{s\beta}{\beta(n+\delta)+\lambda}\right]^{1/\beta} (n+\delta)L_0 e^{(n+\lambda/\beta)t} \quad \text{,for large t.}$$

Since, by assumption, the production functions are Cobb-Douglas, (5.26) and (5.33) imply that the real wage must rise. The supposition that (5.23) and (5.24) can hold simultaneously for all t is therefore false. It follows that (5.23) can only be compatible with a situation where the real wage varies with t .

If W(t) is a function of t with a (constant) greatest lower bound at $\bar{W}$, then for any value of t where $W(t) = \bar{W}$, eq. (5.23) will of course breaks down and unemployment will be inevitable. Such unemployment, as we have seen however, cannot be permanent, *provided* that the subsistence wage is a constant. The case of non-constant subsistence wage will be dealt with in Section 3 of this chapter and in Chapter 6 where it will be shown that permanent unemployment is, under certain circumstances, unavoidable.

If the greatest lower bound of $W(t)$ is greater than $\bar{W}$ for all $t \geq t_2$ — so that labour supply can never be redundant — then $W(t)$ must converge to an exponential path and the system approaches its Golden Age equilibrium. Since this result applies also to the case of non-constant subsistence wage and since the proof of this proposition is rather elaborate, we shall defer our discussion of this case untill Section 4.

So long as the subsistence wage is a constant, therefore, the system will either approach the Golden Age or will switch from one period of unemployment to a period of full employment or *vice versa*. This vicious cycle may repeat itself continuously for any length of time, during which output exhibits no specific behaviour pattern.

2. HIGH CONSTANT SUBSISTENCE WAGE

If the constant subsistence wage level is so high that only a proportion of machines in existence can be operated profitably then the maximal demand for labour ,Max L^D, is the total demand for labour which is needed to operate these machines,i.e.,

$$(5.34) \qquad \text{Max } L^D(t) = \int_{V(t,\bar{\bar{W}})} L_v(t)dv$$

where $V(t,\bar{\bar{W}})$ is the set of profitable vintages at time t,the real wage being at the subsistence level $\bar{\bar{W}}$, i.e.,

$$(5.35) \qquad V(t,\bar{\bar{W}}) = \{v \mid \frac{g\{b(v)\}e^{(\lambda/\beta)v}}{b(v)} - \bar{\bar{W}} \geqq 0\}$$

Again,the wage determination may be expressed as

$$(5.36) \qquad W(t) = \text{Max}\left[\bar{\bar{W}},\ \sup\{W: e^{-(n+\delta)t} \int_{V(t,W)} e^{(n+\delta-\lambda/\beta)v} b(v)k(v)dv \geqq 1\}\right]$$

It is easy to see that the assumption of a high constant subsistence wage level does not in any essential way alter the result of the analysis presented in the previous section.

Suppose that at t_0 the maximal demand for labour is smaller than the total supply of labour ,i.e.,

$$(5.37) \qquad e^{-(n+\delta)t_0} \int_{V(t,\bar{\bar{W}})} e^{(n+\delta-\lambda/\beta)v} b(v)k(v)dv < 1$$

then the real wage will settle at the subsistence level $\bar{\bar{W}}$:

$$(5.38) \qquad W(t_0) = \bar{\bar{W}}$$

which implies that

$$(5.39) \qquad b(t) = (\beta/\bar{\bar{W}})^{1/\alpha} e^{(\lambda/\alpha\beta)t}$$

If (5.37) and (5.38) can hold for all $t \geqq t_0$,then,it follows that

$$(5.40) \qquad k(t) = se^{-(n+\delta)t}\int_p^t e^{(n+\delta)v}\{b(v)\}^{\beta}k(v)dv \\ -se^{-(n+\delta)t}\int_{\tilde{V}(t_0,\bar{\bar{W}})} e^{(n+\delta)v}\{b(v)\}^{\beta}k(v)dv$$

where p is the age of the oldest machines employed and $\tilde{V}(t_0,\bar{\bar{W}})$ is the set of unprofitable vintages at t_0 :

$$\tilde{V}(t_0,\bar{\bar{W}}) = \{v|p \leqq v \leqq t_0, \frac{g\{b(v)\}e^{(\lambda/\beta)v}}{b(v)} - \bar{\bar{W}} < 0\}$$

Differentiating (5.40) with respect to t then solving for k(t) for $t \geqq t_0$, yields :

$$(5.41) \qquad k(t) = e^{s(\beta/\bar{\bar{W}})^{\beta/\alpha}(\alpha/\lambda)e^{(\lambda/\alpha)t} - (n+\delta)t + K_0} \quad ,\text{for } t \geqq t_0$$

where K_0 is an arbitrary constant of integration. Since

$$(5.42) \qquad \lim_{t\to\infty} k(t) = \infty \quad ,$$

it follows that there exists some t_1, sufficiently large, such that

$$e^{-(n+\delta)t_1}\int_{V(t,\bar{\bar{W}})} e^{(n+\delta-\lambda/\beta)v}b(v)k(v)dv \geqq 1$$

Thus, as in the case of low subsistence wage, unemployment can only exist temporarily.

For other cases where initially the maximal demand for labour is greater than or equal to the total supply of labour, the results obtained in the previous section carry over in their entirety .

3. EXPONENTIAL SUBSISTENCE WAGE

We now turn to the case where the subsistence wage is assumed to rise continuously as fast as the labour-saving technical progress. As it will be evident, this case is more difficult but is also more interesting. Among other things, there are possibilities of a *perpetual bastard Golden Age* and of *cyclic fluctuations in income*.

If we denote the subsistence wage by $\bar{W}(t)$ then the maximal demand for labour may be written as

$$(5.43) \qquad \text{Max } L^D(t) \; = \; \int_{V(t,\bar{W}(t))} L_v(t)\,dv$$

where $V(t,\bar{W}(t))$ is the set of profitable vintages, the real wage being at the subsistence level, i.e.,

$$V(t,\bar{W}(t)) \; = \; \{v \mid \frac{g\{b(v)\}e^{(\lambda/\beta)v}}{b(v)} \; - \; \bar{W}(t) \; \geqq \; 0\}$$

The process of wage determination may accordingly be expressed as

$$(5.44) \qquad W(t) \; = \; \text{Max}\left[\bar{W}(o)e^{(\lambda/\beta)t}, \sup\{W: e^{-(n+\delta)t}\int_{V(t,\bar{W}(t))} e^{(n+\delta-\lambda/\beta)v}b(v)k(v)dv \geqq 1\}\right]$$

(i) Case 1: Max $L^D(t) < L^S(t)$

If at t_0 the maximal demand for labour is smaller than the supply of labour, i.e., if

$$(5.45) \qquad e^{-(n+\delta)t_0}\int_{V(t,\bar{W}(t))} e^{(n+\delta-\lambda/\beta)v}b(v)k(v)dv \; < \; 1$$

then unemployment (of *both* labour and machines) is inevitable and the real wage ,as described by (5.44),will settle at the subsistence level:

$$(5.46) \qquad W(t_0) = \bar{W}(t_0) = \bar{W}(0)e^{(\lambda/\beta)t_0}$$

It is easy to show that if (5.45) holds for all $t \geqq t_0$,then there exists a $t_1 > t_0$ such that for all $t \geqq t_1$, (5.45) may be written simply as

$$(5.47) \qquad e^{-(n+\delta)t}\int_{t-\theta}^{t} e^{(n+\delta-\lambda/\beta)v} b(v)k(v)dv < 1$$

where θ is defined by the equation

$$Q_{t-\theta}(t) - \bar{W}(t)L_{t-\theta}(t) = 0$$

This can be seen in the following way :

a) Since for all $t-\theta \geqq t_0$,

$$\bar{W}(t-\theta) = \frac{\beta Q_{t-\theta}(t-\theta)}{L_{t-\theta}(t-\theta)}$$

and

$$\bar{W}(t) = e^{(\lambda/\beta)\theta}\bar{W}(t-\theta)$$

we have

$$\theta = -(\beta/\lambda)\log\beta$$

In other words,if (5.45) holds for all $t \geqq t_0$ then the economic life of all machines installed *after* t_0 is a constant θ.

b) For those machines which are installed *before* t_0,the more profitable they are at t_0 the longer they will last, *economically*. Let z be the age of the most profitable machines at t_0 and write

$$W_z = \sup_{T \leq v \leq t_0} \frac{g\{b(v)\}e^{(\lambda/\beta)v}}{b(v)} > \bar{W}(t_0)$$

$$= \frac{g\{b(z)\}e^{(\lambda/\beta)z}}{b(z)}$$

If (5.45) holds for all $t \geq t_0$ then these machines will become marginal machines at t_1 where

$$W_z = \bar{W}(0)e^{(\lambda/\beta)t_1}$$

and will be scrapped. It follows that after ϕ periods where

$$\phi = t_1 - t_0$$

$$= (\beta/\lambda)\log\{W_z/\bar{W}(0)\} - t_0$$

every machine which was installed *before* t_0 will cease to exist economically. Hence for all $t \geq t_1$, (5.45) reduces to (5.47). Q.E.D.

From (5.4) and (5.46) we have:

$$(5.48) \qquad b(t) = \{\beta/\bar{W}(0)\}^{1/\alpha} \text{ ,aconstant.}$$

We now write

$$(5.49) \qquad K_t(t) = s\{\beta/\bar{W}(0)\}^{\beta/\alpha}e^{-\delta t}\int_{t-\theta}^{t} e^{\delta v}K_v(v)dv$$

Dividing both sides of (5.49) by $L^S(t)$ yields

$$(5.50) \qquad k(t) = s\{\beta/\bar{W}(0)\}^{\beta/\alpha}e^{-(n+\delta)t}\int_{t-\theta}^{t} e^{(n+\delta)v}k(v)dv$$

whence by differentiating (5.50) with respect to t we obtain

$$(5.51) \qquad \dot{k}(t) + b_0 k(t) + b_1 k(t-\theta) = 0$$

where

$$b_1 = s\{\beta/\bar{W}(0)\}^{\beta/\alpha} e^{-(n+\delta)\theta}$$

$$b_0 = n+\delta - s\{\beta/\bar{W}(0)\}^{\beta/\alpha}$$

(5.51) is a difference - differential equation whose asymptotic solutions are of a very great variety. Because of the complexity of the problem we shall defer our discussion of the solution of (5.51) to the next chapter. It suffices at this stage to point out that

I. The asymptotic solution of (5.51) may take the form

$$k(t) = k(o)e^{rt}$$

where r is real.

1) If $r > \lambda/\beta$ then obviously (5.47) cannot hold for all $t \geq t_1$ and unemployment will disappear.

2) If $r = \lambda/\beta$ then

$$e^{-(n+\delta)t} \int_{t-\theta}^{t} e^{(n+\delta-\lambda/\beta)v} b(v)k(v)dv = \frac{k(o)\{\beta/\bar{W}(0)\}\{1-e^{-\theta(n+\delta)}\}}{n+\delta}$$

$$= R^*, \text{say.}$$

In this case, there are two possibilities:

(a) If $R^* \geq 1$ then (5.47) cannot hold for all $t \geq t_1$ and unemployment will disappear.

(b) If $R^* < 1$ then (5.47) can hold for all $t \geq t_1$. With $k(t)$

grows exponentially at rate λ/β , output will grow steadily at rate $\lambda/\beta+n$ and unemployment of both machines and labour will always persist. This is the case of *perpetual bastard Golden Age.*

3) If $r < \lambda/\beta$ then obviously (5.47) will hold for all $t \geqq t_1$ and unemployment will be permanent.

II. The asymptotic solution of (5.51) may also take the form

$$k(t) = Ae^{zt} + Be^{\bar{z}t}$$

where z is a complex number, $\bar{z}$ its conjugate and A and B are arbitiary numbers. Since $k(t)$ takes on real values, A and B must be equal if they are real. If they are complex numbers, one must be the conjugate of the other. In this case we write.

$$A = \eta(\cos\varepsilon + i\sin\varepsilon)$$

$$B = \eta(\cos\varepsilon - i\sin\varepsilon)$$

$$e^z = e^x(\cos x + i\sin x)$$

Accordingly $k(t)$ may be written as

$$k(t) = 2\eta e^{xt}\cos(xt+\varepsilon)$$

clearly if $x < \lambda/\beta$ or if $x = \lambda/\beta$ but

$$\sup_t \{2\eta\left[\beta/\bar{W}(0)\right]e^{-(n+\delta)t}\int_{t-\theta}^{t} e^{(n+\delta)v}\cos(xv+\varepsilon)dv\} < 1$$

then unemployment will persist and income will *fluctuate periodically.*

If, on the other hand, $x > \lambda/\beta$ or if $x = \lambda/\beta$ but

$$\sup_t \{2\eta[\beta/\bar{W}(0)]e^{-(n+\delta)t}\int_{t-\theta}^{t} e^{(n+\delta)v}\cos(xv+\varepsilon)dv\} \geq 1$$

then unemployment will disappear.

III The asymptotic solution of (5.51) may be such that

$$\lim_{t\to\infty} k(t) = 0$$

In this case (5.47) will always be satisfied. With $k(t)$ approaching zero, output will either reach a stationary value $\bar{Q}$ or grow at a rate less than n or decline toward zero.

Thus unless the asymptotic solution of (5.51) be such that (5.47) can hold for all $t \geq t_1$ unemployment must disappear. In such case there exist a t_2, sufficiently large, such that

$$e^{-(n+\delta)t_2}\int_{V(t,\bar{W}(t))} e^{(n+\delta-\lambda/\beta)v}b(v)k(v)dv = 1$$

(ii) Case 2: Max $L^D(t) = L^S(t)$

If at t_2 the maximal demand for labour is equal to the total supply of labour, i.e., if

$$e^{-(n+\delta)t_2}\int_{V(t,\bar{W}(t))} e^{(n+\delta-\lambda/\beta)v}b(v)k(v)dv = 1 \tag{5.52}$$

then labour will be fully employed and the real wage will settle at a level which, as described by (5.54), makes the least profitable machines earn zero quasi-rent. If u is the vintage of the marginal machines, then

$$(5.53) \qquad W(t_2) = W_u = \frac{g\{b(u)\}e^{(\lambda/\beta)u}}{b(u)} \text{, a constant}$$

$$\geqq \bar{W}(t_2) = \bar{W}(0)e^{(\lambda/\beta)t_2}$$

Clearly (5.53) cannot hold for all $t \geqq t_2$: if $W_u = \bar{W}(t_2)$ then $W(t)$ must change as t increases; if $W_u > W(t_2)$ then there exists a t_3 such that

$$W_u = \bar{W}(t_3) = \bar{W}(0)e^{(\lambda/\beta)t_3}$$

and $W(t)$ must change as $t(t \geqq t_3)$ increases.

In any case, whether (5.52) can hold for all $t \geqq t_3$ depends on the asymptotic solution of (5.51). Again without going into the discussion of this problem which is the subject of the next chapter, we may point out that there are two broad categories of the asymptotic behaviour of $k(t)$

1) It may be such that (5.52) can hold for all $t \geqq t_3$; In such circumstance , labour will be fully employed and output will either grow steadily (the case of unique real solution) or fluctuate periodically (the case of complex solutions)

2) It may be such that (5.52) cannot hold for all $t \geqq t_3$:

a) either because there exists some $\bar{t}_4$, sufficiently large, such that

$$e^{-(n+\delta)\bar{t}_4}\int_{V(t,\bar{W}(0))} e^{(n+\delta)v}b(v)k(v)dv < 1$$

(In such circumstance the system will switch back to case 1 where Max $L^D(t) < L^S(t)$)

b) or because there exists some t_4, sufficiently large, such that

$$e^{-(n+\delta)t_4}\int_{V(t,\bar{W}(t))} e^{(n+\delta)v} b(v)k(v)dv > 1$$

<u>(iii) Case 3. Max $L^D(t) > L^S(t)$</u>

If at t_4 the maximal demand for labour is greater than the total supply of labour, i.e. if

$$e^{-(n+\delta)t_4}\int_{V(t,\bar{W}(t))} e^{(n+\delta)v} b(v)k(v)dv > 1 \tag{5.54}$$

then labour will be fully employed and the real wage will settle at a level which is above the subsistence level and which makes the least profitable machines earn zero quasi-rent. If q is the vintage of marginal machines, then we may write

$$\begin{aligned} W(t_4) &= \frac{g\{b(q)\}e^{(\lambda/\beta)q}}{b(q)} > \bar{W}(t_4) \\ &= W_q\text{, say, a constant.} \end{aligned} \tag{5.55}$$

Clearly since $\bar{W}(t)$ grows exponentially, $W(t)$ cannot remain at W_q for all $t \geq t_4$. $W(t)$ must therefore be regarded as a function of t with greatest lower bound $\bar{W}(t)$; the later is itself an exponential function of t. If the behaviour of $W(t)$ is such that for some t_5 ,

$$W(t_5) = \bar{W}(t_5)$$

then the system will switch back to case 1 or case 2 with all their subsequent consequences.

If the behaviour of W(t) is such that

$$W(t) > \bar{W}(t) \quad \text{for all } t$$

then this would implies that labour is always a non-redundant factor. In such case W(t) will converge to an exponential path and the system will asymptotically approach its Golden Age equilibrium. We now turn to prove this proposition .

4. THE IMPLICATION OF LABOUR NON-REDUNDANCY

In most of the recent literature of economic growth, attention was focussed on the asymptotic behaviour of the full employment paths rather than on the general behaviour of a given system. Thus it is usually assumed that labour is a non redundant factor.(2) In the one sector neo-classical model, the full employment paths, as shown by Solow and others, are asymptotically steady growth paths. Under certain circumstances this is also true for the multi-sector neo-classical models.(3)

Does such property hold for a vintage model? If there is no substitution, *ex-ante* and *ex-post*, Solow, Tobin, Weizsäcker and Yaari(4) have shown that the answer is affirmative. In our model which is complicated by the assumption of ex-ante substitutability, it can also be shown that this result holds. In fact with a slight modification the ingeneous method of Solow, Tobin, Weizsäcker and Yaari can also be employed for the purpose.

In order to use their method we need first define a new variable $\kappa(v)$ where

$$\kappa(v) = \frac{K_v(v)}{e^{(\lambda/\beta)}L^s(v)} = e^{-(\lambda/\beta)v}k(v)$$

The full employment equilibrium equation may now be written as

$$\int_{V(t,W)} e^{-(n+\delta)(t-v)}b(v)\kappa(v)dv = 1 \quad , \quad V(t,W) \in S^t$$

where $V(t,W)$ is the set of profitable vintages and S^t the set of surviving vintages at t. Accordingly, since new investment is a fixed proportion of total output we can write

$$\kappa(v) = s\int_{V(t,W)} e^{-(\delta+\lambda/\beta+n)(t-v)}\{b(v)\}^{\beta}\kappa(v)dv$$

Let $\xi(t)$ be the real wage at t,measured in terms of labour efficiency units ,i.e.,

$$\xi(t) = e^{-(\lambda/\beta)t} W(t)$$

In terms of this new notation, the set of profitable vintage at t may be defined as

$$V(t,W) = \{t - T \leqq v \leqq t \mid \frac{g[b(v)]e^{-(\lambda/\beta)(t-v)}}{b(v)} - \xi(t) \geqq 0\}$$

The economic life time of machines at t, $\theta(t)$,is thus defined by the equation

$$\frac{g\{b(\theta(t))\}e^{-(\lambda/\beta)\theta(t)}}{b(\theta(t))} = \xi(t)$$

We note that along the Golden Age path, $b(t)$, $\kappa(t)$ and $\xi(t)$ are all constants . We shall refer to the Golden Age solution as the triple (b^*,κ^*,ξ^*). Along this path ,the economic life of machines is also a constant θ^*,where θ^* is given by

$$g'(b^*) = \frac{g(b^*)e^{-(\lambda/\beta)\theta^*}}{b^*}$$

In general $\kappa(t)$,$b(t)$ and $\xi(t)$ are regarded as bounded continuous function of t. Using the method of Solow,Tobin,Weizsäcker and Yaari , we can show that

<u>*Theorem :*</u> *If labour is a non-redundant factor for all t ,then*

$$\lim_{t\to\infty} b(t) = b^*$$

$$\lim_{t\to\infty} \kappa(t) = \kappa^*$$

$$\lim_{t\to\infty} \xi(t) = \xi^*$$

To establish the above Theorem we shall assume that at the starting point $t = 0$ the economy is endowed with an arbitrary capital profile but that all the $b(v)$ and $\kappa(v)$ are bounded for all $v \leq 0$.

Let

$$\inf_{v \leq 0} b(v) = \underline{b}_0 , \qquad \sup_{v \leq 0} b(v) = \bar{b}_0$$

$$\inf_{v \leq 0} \kappa(v) = \underline{\kappa}_0 \qquad \sup_{v \leq 0} \kappa(v) = \bar{\kappa}_0$$

We shall assume that $\underline{b}_0 < b^* < \bar{b}_0$ and $\underline{\kappa}_0 < \kappa^* < \bar{\kappa}_0$.

Let $\underline{\theta}$ and $\bar{\theta}$ be defined by

$$\underline{b}_0 \bar{\kappa}_0 \int_0^{\underline{\theta}} e^{-(n+\delta)v} dv = 1$$

$$\bar{b}_0 \underline{\kappa}_0 \int_0^{\bar{\theta}} e^{-(n+\delta)v} dv = 1$$

Since the production functions are Cobb-Douglas ,it is readily verified that

$$\underline{\theta} \leq \theta^* \leq \bar{\theta}$$

Following Solow *et als.*,we shall develop the theorem in a series of lemmas.

Lemma 1 : $\underline{\kappa}_0 \leq \kappa(v) \leq \bar{\kappa}_0$ *and* $\underline{b}_0 \leq b(v) \leq \bar{b}_0$ *for all* $v \geq 0$.

Proof : Take any $t_1 \geq 0$ and suppose that $\underline{\kappa}_0 \leq \kappa(t) \leq \bar{\kappa}_0$ and $\underline{b}_0 \leq b(v) \leq \bar{b}_0$ for all $t < t_1$.

The upper bound of $\kappa(t)$ at t_1 may be found by considering the following variational problem :

$$\text{Maximize} \quad J(\kappa,b) = s \int_{\tau} e^{-(\delta+\lambda/\beta+n)v} \{b(v)\}^{\beta} \kappa(v) dv$$

Subject to

$$\int_{\tau} e^{-(n+\delta)v} b(v)\kappa(v)\,dv = 1$$

$$\underline{b}_0 \leq b(v) \leq \bar{b}_0 \ , \quad \underline{\kappa}_0 \leq \kappa(v) \leq \bar{\kappa}_0$$

where τ is any subset of $S : S = (0,\infty]$ and b and κ are integrable over S. The maximizing functions $\hat{\kappa}$ and $\hat{b}$ are seen to be $\hat{\kappa}(v) = \bar{\kappa}_0$ and $\hat{b}(v) = \underline{b}_0$ for $\tau = \{v|0 < v < \underline{\theta}\}$ and zero otherwise .

Since $\underline{b}_0 < b^*$ and $\underline{\theta} \leq \theta^*$,

$$\text{Max } J = \hat{\kappa}_0(\underline{b}_0)^{\beta} s \int_0^{\bar{\theta}} e^{-(\delta+\lambda/\beta+n)v} dv < \bar{\kappa}_0$$

It follows that

$$\kappa(t_1) \leq \text{Max } J < \bar{\kappa}_0$$

By using similar arguments we can show that

$$\kappa(t_1) > \underline{\kappa}_0$$

and

$$\underline{b}_0 < b(t_1) < \bar{b}_0$$

Since t_1 is arbitrary and $\kappa(v)$ and $b(v)$ are continuous, Lemma 1 follows .

Lemma 2 : $\lim\limits_{t\to\infty}$ *sup* $\kappa(t) \leq \kappa^*$ *and* $\lim\limits_{t\to\infty}$ *inf* $b(t) \geq b^*$

Proof : From Lemma 1, the maximum possible value of $\theta(t)$ for $t \geq 0$, which we shall denote by $\hat{\theta}$, is given by

$$\underline{\kappa}_0\underline{b}_0 \int_0^{\hat{\theta}} e^{-(\delta+n)v} dv = 1$$

Thus for all $t \geqq 0$, $V(t,W) \in [t-\hat{\theta},t)$. It is convenient therefore to divide $(0,\infty]$ into the sucessive intervals of the form $[t_n, t_{n+1}]$, $n = 1,2,\ldots$ and $t_k = t_{k-1} + \hat{\theta}$

Suppose at t_n we can find numbers θ_n, κ_n and b_n such that $\kappa(t) \leqq \kappa_n$ and $b(t) \geqq b_n$ for all $t \geqq t_n$ where κ_n and b_n satisfy the conditions

$$\kappa_n b_n \int_0^{\theta_n} e^{-(n+\delta)v} dv = 1$$

$$g'(b_n) = \frac{g(b_n)e^{-(\lambda/\beta)\theta_n}}{b_n}$$

Let κ_{n+1} and b_{n+1} be the numbers such that $\kappa(t) \leqq \kappa_{n+1}$ and $b(t) \geqq b_{n+1}$ for all $t \geqq t_{n+1}$. Obviously

$$\kappa_{n+1} \leqq \hat{\kappa}_{n+1} \quad \text{and} \quad b_{n+1} \geqq \hat{b}_{n+1}$$

where $\hat{\kappa}_{n+1}$ is the maximum possible value of $\kappa(t)$ for all $t \geqq t_{n+1}$ and $\hat{b}_{n+1}$ is the minimum possible value of $b(t)$ for al $t \geqq t_{n+1}$. But we already proved in Lemma 1 that $\hat{\kappa}_{n+1}$ is given by

$$\hat{\kappa}_{n+1} = \kappa_n (b_n)^{\beta} \int_0^{\theta_n} e^{-(\delta+\lambda/\beta+n)v} dv$$

Clearly $\kappa_n > \kappa^*$ and $b_n < b^*$ (otherwise Lemma 2 would follow automatically). Consequently $\theta_n \leqq \theta^*$. Hence $\hat{\kappa}_{n+1} < \kappa_n$. In other words, $\kappa_{n+1} < \kappa_n$. It also follows that $b_{n+1} > b_n$.

Thus $\{\kappa_n\}$ is a decreasing sequence while $\{b_n\}$ is an increasing sequence . Lemma 2 therefore follows.

<u>*Lemma 3* :</u> $\lim_{t\to\infty} \inf \kappa(t) \geqq \kappa^*$ *and* $\lim_{t\to\infty} \sup b(t) \leqq b^*$

The proof to this lemma is symetrical to that of Lemma 2 and will not be given here.

On combining the results of Lemmas 1,2 and 3 the Theorem follows.

Footnotes to chapter 5.

(1) Ken-ichi Inada: "Economic Growth and Factor Substitution", *International Economic Review*, Vol 5., September 1964, pp. 318-27

(2) Cf. Robert M. Solow: "A Contribution to the Theory of Economic Growth", *Quarterly Journal of Economics*, Vol 32 (1956), pp. 65-94. Hirofumi Uzawa: "On a Two Sector Model of Economic Growth", *Review of Economic Studies*, Vol 29 (1961-62) pp. 40-47, and, *id.bid.*, Vol 30 (1962-63) pp. 105-118

(3) Hirofumi Uzawa, *op. cit.*, Ken-ichi Inada: "On the Stability of Growth Equilibria in Two Sector Models" *Review of Economic Studies*, Vol 31 (1964) pp. 127-142.

(4) Robert M. Solow, James Tobin, Christian von Weizsäcker and Menahem Yaari: "A Model of Fixed Capital without Substitution", *Cowles Foundation Discussion Paper* No. 188, August 1965.

Chapter 6.

ASYMTOTIC BEHAVIOUR OF ONE SECTOR MODEL: II. CONDITIONS FOR GOLDEN AGE STABILITY.

In chapter 5 it was pointed out that when the subsistence wage is assumed to grow as fast as the rate of labour-saving technical progress, then the asymptotic behaviour of $k(t)$, as defined by the difference - differential equation

$$(6.1) \qquad \dot{k}(t) + b_0 k(t) + b_1 k(t-\theta) = 0$$

where

$$b_0 = (n+\delta) - s(\beta/\bar{W}(0))^{\beta/\alpha}$$
$$b_1 = s(\beta/\bar{W}(0))^{\beta/\alpha} e^{-(n+\delta)\theta}$$
$$\theta = -(\lambda/\beta)\log\beta$$

will determine whether or not the system will approach its Golden Age equilibrium. If the conditions for stability of the Golden Age ~~is~~ are not met, then the behaviour of $k(t)$ will also determine whether unemployment is permanent or whether income will fluctuate periodically.

It was also pointed out that ,in order that the system reach its Golden Age equilibrium from an initial position where the maximal demand for labour is either smaller than or equal to the total supply of labour,it must be true (i) that the asymptotic solution of (6.1) be

$$k(t) = k(o)e^{(\lambda/\beta)t}$$

and (ii) that

$$\frac{k(o)\{\beta/\bar{W}(o)\}\{1-e^{-\theta(n+\delta)}\}}{n+\delta} = 1$$

We shall assumed that (ii) is satisfied and proceed to derive the conditions which will ensure (i).This would amount to the determinating of conditions which will allocate the characteristic roots of (6.1) so as to ensure that one of these roots be real,unique and equal to λ/β and that all other complex roots have non-positive real parts.We shall refer to these conditions as the stability conditions.

To use a classification due to A.D.Myškic[1],(6.1) is a homogeneous-first-order linear difference-differential equation of retarded type with constant coefficients.

Let $\Psi\{k\}$ be the linear operator :

$$(6.2) \qquad \Psi\{k\} = \dot{k}(t) + b_0 k(t) + b_1 k(t-\theta) = 0$$

and let h(r) be the characteristic function of Ψ,i.e.,

$$(6.3) \qquad h(r) = r + b_0 + b_1 e^{-\theta r}$$

Then the roots of the transcendental function

$$(6.4) \qquad h(r) = r + b_0 + b_1 e^{-\theta r} = 0$$

are called the characteristic roots of Ψ. It is known that the solution of (6.1) may be expressed as

$$k(t) = \Sigma p_j e^{r_j t}$$

where $\{r_j\}$ is any sequence of characteristic roots of Ψ, p_j is a polynomial of degree less than the multiplicity of r_j and the sum is either finite or infinite.

The term $k(t)$ in equations of type similar to (6.1) are, in the mathematical literature, often thought of as representing the displacement or derivation of some quantity from an undisturbed state, usually in mechanical or electrical context. In our model, however, the equilibrium state is a *steady* state. Therefore in order to avoid confusion about the terms we are going to employ we shall first give some definition of stability. It will be obvious that "stability" is used here in a sense which differs significantly from that employed by such writer as L.S. Pontrjagin [(2)], S. Lefschetz [(3)], N. Minorsky [(4)], E.M. Wright [(5)] or Ricard Bellman [(6)].

<u>Definition 6.1</u>

The solution of Equation (6.1) is said to be stable in the weak sense as $t \to \infty$ *if*

a) there is a real solution which can be written in the form

$$k(t) = k_o e^{(\lambda/\beta)t}$$

b) given a positive number ε*, there exists* $\delta = \delta(\varepsilon)$ *such that every continous solution* $k(t)$ *of* (6.1) *which satisfies*

$$\text{(6.5)} \qquad \max_{t_o \leq t \leq t_o + \theta} \left| k(t) - k_o e^{(\lambda/\beta)t} \right| \leqq \delta$$

will also satisfy

$$\text{(6.6)} \qquad \max_{t_o < t < \infty} \left| k(t) - k_o e^{(\lambda/\beta)t} \right| \leqq \varepsilon$$

The solution of equation (6.1) *is said to be asymptotically stable in the weak sence, if for each* $t_o \geqq 0$, *there is a* $\delta = \delta(t_o)$ *such that every solution which satisfies* (6.5) *will also satisfy the relation*

$$\text{(6.7)} \qquad \lim_{t \to \infty} \left| k(t) - k_o e^{(\lambda/\beta)t} \right| = 0$$

<u>Definition 6.2</u>

The solution of equation (6.1) is said to be stable in the strong sense, or asymptotically stable in the large, or stable for arbitrary perturbations, if

a) it is weak stable

b) every solution $k(t)$ either satisfies the relation (6.7) or satisfies

$$\lim_{t\to\infty} k(t) = 0$$

In economic terms, a growth path of gross investment per capita $k(t)$ is weak stable if small initial disturbances from the golden age path remain small for all t. It is asymptotically weak stable if small initial disturbances die out and is asymptotically stable in the large if every initial disturbance, no matter how large, dies out.

Theorem 1.

A necessary and sufficient condition for the solution of equation (6.1) to be stable in the weak sense as $t \to \infty$ is that

a) there exist a real characteristic root which is equal to λ/β.

b) all the other characteristic roots have non-positive real parts.

c) if r_j is a root with zero real part, the residue of $e^{rt}h^{-1}(r)$ at r_j is bounded as $t \to \infty$. In other words, we require that each root with zero real part be simple.

Theorem 2.

A necessary and sufficient condition for the solution of equation (6.1) to be stable in the strong sense is that

a) there exists a real characteristic root which is equal to λ/β.

b) all the other characteristic roots have negative real parts.

The proves are almost trivial.

We shall now proceed to describe the distribution of the characteristic roots of (6.4) whereby the stability conditions of the golden age equilibrium can be established.

We shall first transform (6.4) into a more suitable form [7].

Let

(6.8) $$f(r) = e^{\theta r}h(r)$$

and let

(6.9) $$z = \theta r$$

We establish the function H(z) is the following way:

(6.10) $$H(z) = \theta f(r) = ze^{z} - a_1 e^{z} - a_2 = 0$$

where

$$a_1 = -\theta b_0$$

$$a_2 = -\theta b_1$$

Hence if in (6.10) we write

(6.10a) $$a_2 = -Ce^{a_1}$$

(6.10b) $$z = a_1 - z_1$$

then

(6.11) $$z_1 = Ce^{z_1}$$

Obviously there is an infinite number of roots of equation (6.11). We shall show presently that all the complex roots are symmetrical with respect to the real axis and that, when $C > 0$, they lie one in each strip

$$2p\Pi < y < (2p + 1)\Pi \qquad \begin{cases} p = 0, \pm 1, \pm 2, \ldots \\ z = x + iy \end{cases}$$

at the only intersection of the appropriate branches of the curves

(6.12) $$y = \pm (C^2 e^{2x} - x^2)^{\frac{1}{2}}$$

(6.13) $$x = y \cot y$$

except that, when $0 \leq C \leq e^{-1}$, the strip corresponding to $p = 0$ contains no intersection and no root.

When $C < 0$ the complex roots of (6.11) lie one in each strip

(6.14) $$(2p+1)\Pi < y < 2(p+1)\Pi, \; p = 0, \pm 1, \pm 2, \ldots$$

at the intersection of (6.12) and (6.13).

When $C = e^{-1}$ there is a double real root $x = 1$; when $0 < C < e^{-1}$ there are two real roots, one at each intersection of (6.12) with the positive real axis and when $C < 0$ there is one real root at the only intersection of (6.12) with the negative real axis.

If $u + iv$ is a root then so also $u - iv$. It suffices therefore to consider roots in the upper half plane only.

By writing $z_1 = \rho e^{i\gamma} = x + iy$, $C > 0$ and substituting in (6.11) we have

$$\rho = Ce^{x}$$

$$\gamma = y$$

$$y = \pm (C^2 e^{2x} - x^2)^{\frac{1}{2}}$$

also

$$\tan\gamma = y/x \quad \text{so that} \quad x = y \cot y.$$

In the strip (6.14) y and $\sin y$ are of opposite sign and since $y = \rho \sin\gamma$, $\gamma \neq y$. Therefore when $C > 0$ there are no roots of equation (6.11) on the branches of (6.13) lying in these strips. When $C < 0$, on the other hand, we have

$$\rho = -Ce^{x}$$

$$\gamma = y + \Pi$$

and the roots of (6.11) lie at the intersection of (6.12) and (6.13) in the strip (6.14).

Now from (6.12),

$$(6.15) \qquad \frac{dy}{dx} = \pm \frac{C^2 e^{2x} - x}{(C^2 e^{2x} - x^2)^{\frac{1}{2}}}$$

which is finite except at the root of $x = \pm Ce^{x}$. When $x = 0, y = \pm C$. Thus for a given value of C, the graph of (6.12) cuts the imaginary axis at two symmetrical points $+C$ and $-C$. The graph passes through the same points if C has the positive sign. When $y = 0$, the graph of (6.12) cuts the real axis at one or three points, depending on the value of C : these points are the roots of the equation $x = Ce^{x}$. This can be seen from Fig.(6.1) : when $C = e^{-1}$ the curve $f(x) = Ce^{x}$ is tangent to the line $f(x) = x$ at $x = 1$; if $C > e^{-1}$ there is no intersection; if $0 < C < e^{-1}$ there are two points of intersection at positive values of x and when $C < 0$ there is one point of intersection at negative values of x. In Fig. (6.2) we draw the graph of (6.12) for four different values of C.

On the other hand, from (6.13), in the strip $0 < y < \pm \Pi$ we have

$$x = 0 \text{ if } y = \pm \Pi/2$$

$$y = 0 \text{ if } x = 1$$

$$x \to -\infty \text{ when } y \to \pm \Pi$$

In the strip $\Pi < y < \pm 2\Pi$ we have

$$x = 0 \quad \text{if } y = \pm 3\Pi/2$$

$$x \to -\infty \quad \text{when } y \to \pm 2\Pi$$

$$x \to +\infty \quad \text{when } y \to \pm \Pi$$

In fig. (6.1) the graph of (6.13) is drawn against the graph of (6.12). The positions of the complex roots of (6.11) are marked by letter z with subscripts referring to the sign of C. The real roots of (6.11) are marked by letter x, also with subscripts referring to the sign of C. It can easily be seen that (8)

a) the roots of $z_1 = Ce^{z_1}$ all lie to the right of a real line $R(z_1) = K$ if and only if $K < 1$, and $Ke^{-K} < C < e^{-K}(Y^2 + K^2)^{\frac{1}{2}}$ where $Y = Y(K)$ is the unique root of $Y \cot Y = K$ for $0 < Y < \Pi$;

b) one root of $z_1 = Ce^{z_1}$ lies on $R(z_1) = K$ and all the other roots to the right if and only if $K \leq 1$, and $C = Ke^{-K}$;

c) two roots of $z_1 = Ce^{z_1}$ lie on $R(z_1) = K$ and all the other roots to the right if and only if $K \leq 1$ and $C = e^{-K}(Y^2 + K^2)^{\frac{1}{2}}$

Since

$$z = a_1 - z_1$$

the preceeding statement may be re-written as

a) the roots of $ze^z - a_1e^z - a_2 = 0$ *lie to the left of a real line* $R(z) = k$ *if and only if* $a_1 - k < 1$ *and* $(a_1 - k)e^k < -a_2 < e^k[Y^2 + (a_1 - k)^2]^{\frac{1}{2}}$ *where* Y *is the unique root of* $Y \cot Y = a_1 - k$ *such that* $0 < Y < \Pi$;

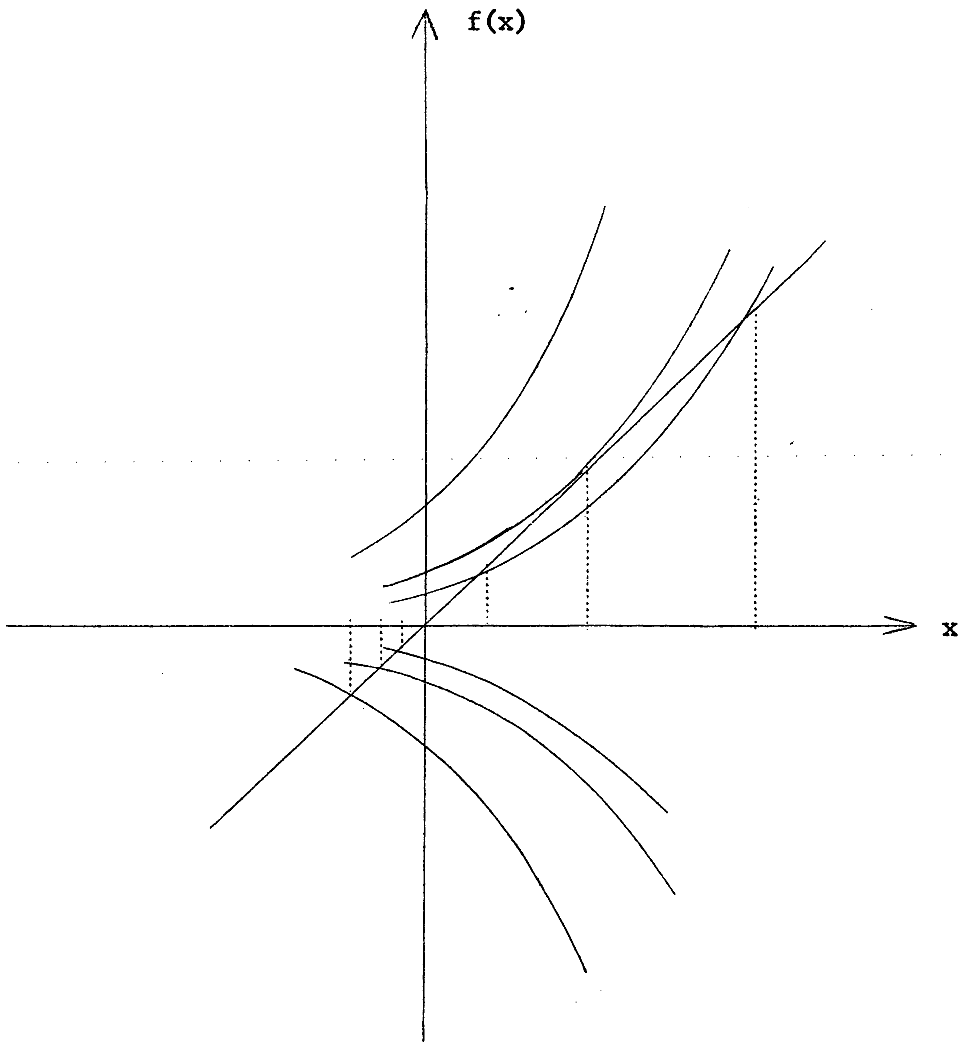
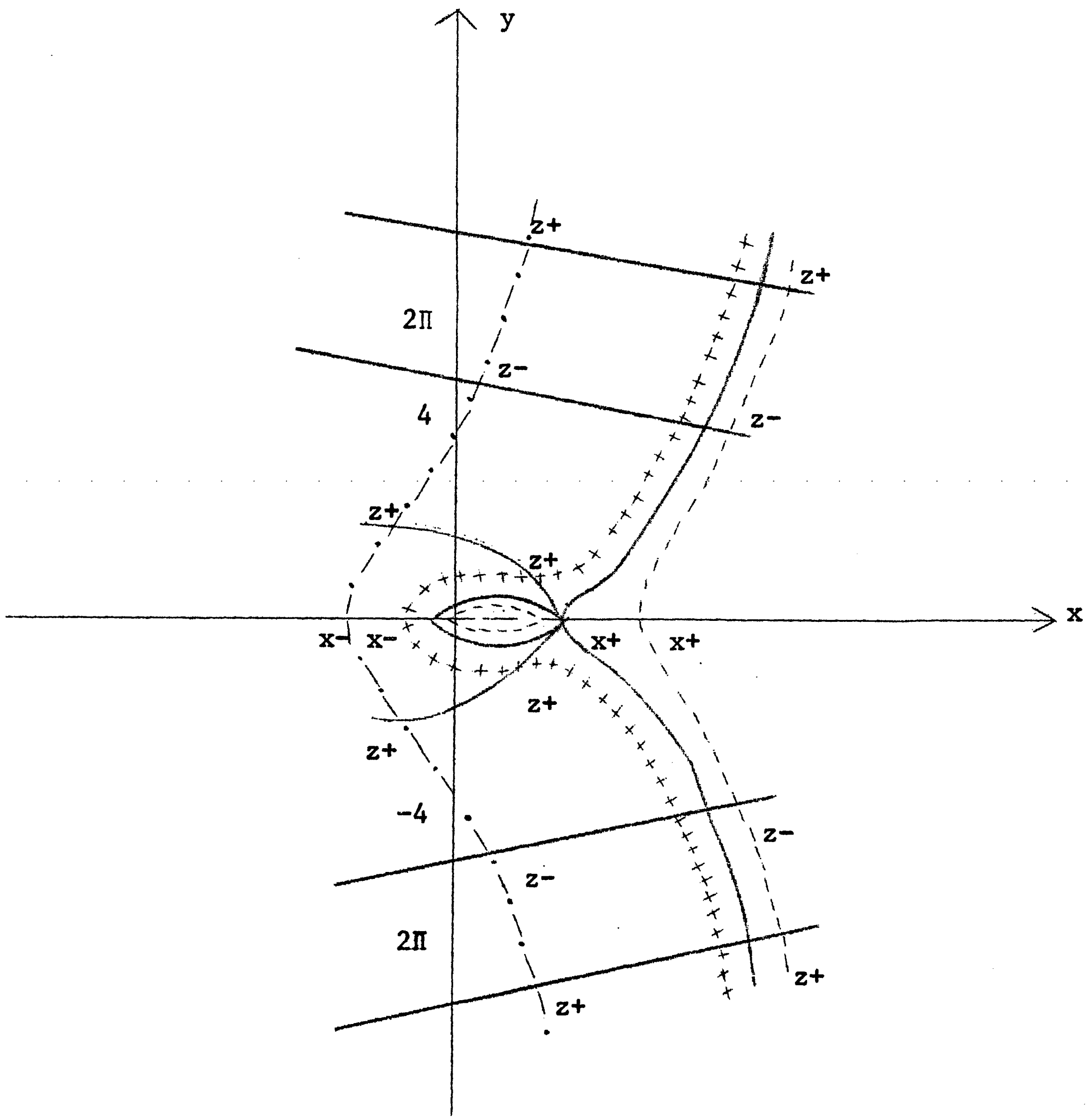


Fig. 6.1



$x = y \cot y$

$c = \pm 4$

$c = \pm 0.8$

$c = \pm e^{-1}$

$c = \pm 0.3$

$$y = \pm (c^2 e^{2x} - x^2)^{\frac{1}{2}}$$

Fig. 6.2

b) one root of $ze^z - a_1e^z - a_2 = 0$ *lies on* $R(z) = k$ *and all the other roots to the left if and only if* $a_1 - k \leq 1$ *and* $- a_2 = (a_1 - k)e^k$*;*

c) two roots of $ze^z - a_1e^z - a_2 = 0$ *lie on* $R(z) = k$ *and all the other roots to the left if and only if* $a_1 - k \leq 1$ *and* $- a_2 = e^k[Y^2 - (a_1 - k)^2]^{\frac{1}{2}}$.

From our definition of stability it is (b) that will interest us most. For weak stability we require that

(i) (6.4) has a real root $R(r) = \lambda/\beta$ and that (ii) all the other roots lie on or to the left of the imaginary axis. For strong stability, the second requirement is replaced by the requirement that (iia) all the other roots lie to the left of the imaginary axis.

Since

$$z = \theta r = \theta\frac{\lambda}{\beta} = - \log \beta$$

for weak stability we require that one (real) root of $ze^z - a_1e^z - a_2 = 0$ lies on the real line

$$R(z) = k = - \log \beta > 0$$

and all the other roots lie *sufficiently* to the left of this line to ensure that they also lie on or to the left of the imaginary axis. For strong stability we require that all the other roots lie to the left of the imaginary axis.

The requirement that one real root of $ze^z - a_1e^z - a_2 = 0$ lies on the real line

$$R(z) = k = - \log \beta > 0$$

is equivalent to the requirement that one (real) root of

$$z_1 = Ce^{z_1}$$

lies on the real line $Re(z_1) = K = a_1 - k$. In other words, in order that r have a real root $Re(r) = \lambda/\beta$ we must have

(6.16) $\quad a_1 + \log \beta \leq 1$

and

(6.17) $\quad -a_2 e^{-a_1} = (a_1 + \log \beta) e^{(a_1 + \log \beta)}$

Since

$$a_2 = -\theta b_1 < 0$$

from (6.10a):

$$C = -a_2 e^{-a_1} > 0$$

it follows that we have to consider only the case when the real roots of $z_1 = Ce^{z_1}$ are positive. In other words, the real line K lies between the origin and $x = 1$. We note that in this case, real roots exist if and only if

$$0 < C \leq e^{-1}$$

However, if $0 < K < 1$ then $0 < a_1 + \log \beta \leq 1$ it follows that $a_1 > 0$

Thus

1) when $0 < C < e^{-1}$, there are two real roots x_1 and x_2. Let x_1 be the smaller root, i.e.,

$$0 < x_1 < 1$$

and let us assume that (6.16) and (6.17) are satisfied so that x_1 lies on the real line $Re(z_1) = K$. Corresponding to x_1 is of course $Re(r) = \lambda/\beta$. Since all the other (complex) roots of $z_1 = Ce^{z_1}$ lie to the right of x_2 and since $a_1 > 0$, for weak stability we require only that the value of a_1 be such that the second real root of (6.10) corresponding to x_2 be zero or negative, viz.,

(6.18) $\quad a_1 \leq x_2$

For a strong stability, the strict inequality is required.

ii) When $C = e^{-1}$ there is a double real root at $x = 1$. With C positive the complex root z_1^* with smallest real part lies in the strip $2\Pi \leq y \leq 3\Pi$ and to the right of the point of intersection between the graph of (6.12) and the line $y = 2\Pi$. In other words,

$$Re(z_1^*) > x^*$$

where x^* is the unique solution of the equation

$$2\,\Pi = (e^{2(x-1)} - x^2)^{\frac{1}{2}}$$

Hence strong stability is ensured if

(6.18b) $\quad a_1 \leqq x^*$

It is a simple matter to extend the above analysis to cover the case where

$$\lim_{t\to\infty} k(t) = 0$$

Under such circumstance, unemployment will become inevitable for some large t. This can be done by making the real line K pass through the origin, i.e., $\mathrm{Re}(z_1) = K = 0$.

On the other hand, if (6.18) and (6.18b) are not satisfied then the characteristic function of (6.1) will have complex roots with *positive* real parts. This, as we have pointed out in the previous chapter, will lead to cyclic fluctuations in income.

Thus we have established the following theorem :

Theorem :

If the maximal demand for labour is smaller than or equal to the total supply of labour, then the real wage will settle at the subsistence level. If the subsistence wage rises as fast as the rate of labour-saving technical progress, output will converge to a Golden Age path if and only if the following conditions are satisfied :

(1) $\quad a_1 + \log\beta \leqq 1$

(2) $\quad -a_2 e^{-a_1} = (a_1 + \log\beta)e^{(a_1 + \log\beta)}$

(3) $\quad a_1 \leqq x_2 \text{ if } 0 < -a_2 e^{-a_1} < e^{-1}$

$\quad a_1 \leqq x^* \text{ if } -a_2 e^{-a_1} = e^{-1}$

where

$$a_1 = (\beta/\lambda)\log\beta\{(n+\delta) - s(\beta/\bar{W}(o))^{\beta/\alpha}\}$$

$$a_2 = (\beta/\lambda)s(\beta/\bar{W}(o))^{\beta/\alpha} e^{-(n+\delta)\theta_{\log\beta}}$$

x_2 is the large root if the equation

$$x = -a_2 e^{-a_1 e^x}$$

and x^* is the unique solution of the equation

$$2\Pi = (e^{2(x-1)} - x^2)^{\frac{1}{2}}$$

Footnotes to Chapter 6

(1) A.D. Myskic: *"Lineare Differentialgleichungen mit nacheilendem Argument"* Berlin, 1955 (translation of the 1951 Russian Edition).

(2) L. S. Pontryagin: "O Nulyakh Elementarnykh Transtsendentnykh Funktsiy", (on The Zeros of Some Elememtary Transcendental Functions) *Izv. Akad. Nauk S.S.S.R., Ser. Mat.*, Vol 6, 1942.

(3) S. Lefschetz: *"Differential Equations: Geometric Theory"*, New York, Interscience, 1957.

(4) N. Minorsky: *"Introduction to Nonlinear Mechanics"*, An Arbor, Mich: Edward Bros., 1947.

(5) R. Bellman: *"Stability Theory of Differential Equations"*, New York, McGraw Hill, 1953.

(6) It will be obvious that the method we are going to employ serves our purposes very nicely. However we may note that the *direct method* of Liapunov may also be used to derive the conditions for $k(t) \to 0$. To do this we define the functional:

$$V(t,v) = v^2(t) + b_0 \int_{t-\theta}^{t} v^2(s)\,ds \quad , \quad t \geqq t_0 \ , v \equiv v(t)$$

for any continuous real valued function $v(s)$ defined on $t_0 - \theta \leqq s$.
Clearly if $b_0 > 0$,

$$v^2(t) \leqq V(t,v) \leqq (1 + b_0\theta)\left[\sup_{t-\theta \leqq s \leqq t} v^2(s)\right]$$

Moreover if $k(t)$ is a solution of (6.1) then ($k \equiv k(t)$)

$$\frac{d}{dt}V(t,k) = -b_0 k^2(t) - 2b_1 k(t)k(t-\theta) - b_0 k^2(t-\theta)$$

$$\leqq -(b_0 - |b_1|)\{k^2(t) + k^2(t-\theta)\}$$

If $|b_1| \leq b_0$ this is non-positive and $V(t,k)$ is non increasing. Therefore for all $t \geq t_0$,

$$k^2(t) \leq V(t,k) \leq (1+b_0\theta)\left[\sup_{t_0-\theta \leq s \leq t_0} k^2(s)\right]$$

and this proves that $\lim k(t) \to 0$ as $t \to \infty$.

(8)Note that the roots cannot all lie to the right of $R(z_1) = K$ for $K \geq 1$. For, when $C < 0$, there is a real root $x_3 < 0$, when $0 < C \leq e^{-1}$ there is a real root x_1 , $0 < x_1 \leq 1$ and for other values of C there is a complex root on the branch of $x = y \cot y$ in $0 < y < \Pi$ and on this branch $x \leq 1$.

We consider first the greatest value C can take such that the complex roots in $-\Pi < v < \Pi$ lie to the right of $R(z_1) = K$. Suppose these roots lie on $R(z_1) = K$, let these values be $K + iY$. Then

$$Y \cot Y = K$$

and

$$Y = (C^2 e^{2K} - K^2)^{\frac{1}{2}}$$

so that

$$C = e^{-K} (Y^2 + K^2)^{\frac{1}{2}}$$

and for C less than this value the roots lie to the right of $R(z_1) = K$.

If $K < 0$ the smallest value of C must be great enough for the negative real root x_3 to fall to the right of $R(z_1) = K$ since all other roots lie to the right of this root. Then $z_1 - Ce^{z_1} \lesseqgtr 0$ according as $R(z_1) \gtreqless x_3$.

Thus $x_3 > K$ if $K - Ce^K < 0$ so that $C > Ke^{-K}$.

On combining these results and including the degenerate case $C = 0$ giving a root at the origin, we obtain the desired result .

$Y(K)$ is unique, for

$$\frac{d}{dy}(y \cot y) = \frac{\sin(y)\cos(y) - y}{\sin^2 y} < 0$$

When $0 < y < \Pi$ so that $\phi(y) = y \cot y$ decreases steadily as y increases from 0 to Π and since $\phi(\Pi) = -\infty$ and $\phi(0) = 1$, there is one root only of $\phi(y) = K$ in the interval

Appendix to Chapters 5 and 6 :

A N   I L L U S T R A T I O N   B Y   S I M U L A T I O N

This appendix provides a numerical illustration of the mathematical analysis of chapters 5 and 6. The following charts give the numerical values of output (denoted by Y), wage rate (denoted by W), and the economic life of machines (denoted by theta). The system was simulated on IBM digital computers. Calculations reported here were obtained from a series of FORTRAN IV programmes designed for the IBM 7040 at the computer Room of the Australian Atomic Energy Commission's Research Establishment in Lucas Heights, N.S.W. and from a series of FORTRAN II b programmes designed for the IBM 1620 at the DUCHESS laboratory of the Faculty of Applied Science in the University of N.S.W.

As is well known, it in necessary that time be treated as a discrete variable in the digital simulation process. This does not alter our system in any essential way. We note however that, on the Golden Age path, the discrete forms for Q(t), W(t), and L(t) are

$$Y(t) = L_0 \left[\frac{s^{\alpha/\beta}\{e^{(n+\delta)} - 1\}}{1 - e^{-\theta(n+\delta)}} \right] \left[\frac{e^{-(n+\delta+\lambda/\beta)\theta} - 1}{1 - e^{(n+\delta+\lambda/\beta)}} \right]^{1/\beta} e^{gt}$$

$$W(t) = \beta \left[\frac{s\{1 - e^{-\theta(n+\delta+\lambda/\beta)}\}}{e^{(n+\delta+\lambda/\beta)} - 1} \right]^{\alpha/\beta} e^{(\lambda/\beta)t}$$

$$L_t(t) = \frac{\{e^{(n+\delta)} - 1\}}{1-e^{-\theta(n+\delta)}} L_0 e^{nt}$$

The computer is instructed to start the system from a given set of initial conditions.(Here $\delta = 0.02$,$n = 0.04$,$\alpha = 0.25$,$\lambda = 0.05$,$s = 0.2$,$\bar{W} = 1.$,and $L_0 = 10$; the Golden Age value of theta is thus $\theta = 4.3$.). We consider three cases according as the maximal demand for labour is greater than, equal to, or smaller than the supply of labour. In each case a wage rate is established and output is produced from machines which can be operated profitably. Past output and machine stocks are recorded in the computer's memory and then we move to the next period. In each period, the computer is instructed to print out the values of output, wage rate, the calendar period, and the economic life of machine. It is instructed to print out also the Golden Age solution and to compare the simulated values with those of the steady path by printing out the deviation. The time interval is chosen so that theta has an integral value. In the exponential solution, this value is equal to 10 periods. Table 1 gives the numerical results of a trial programme in which output is actually on a Golden Age path. We verify that theta remains at the value 10 and that the deviation is quite small,allowing for rounding errors. The machine records data for 1000 periods.

In the following charts we only re-produce the simulated values at 10 period intervals. Thus, for example, when $T = 0.86$ it would mean that we are in the second period (since theta = 10 periods = 4.3, a period has a numerical value of 0.43) and when $T = 5.6$ (printed as 0.56E+01) it would mean that we are in the 13th period, etc.....

Table 2 describes the behaviour of the system when it starts from a situation where the demand for labour is smaller than the supply of labour and the inital real wage has settled at the subsistence level. We see that this kind of situation can persist only for 13 periods. After 13 periods have elapsed, the demand for labour is greater than the supply of labour and the real wage starts to rise above the subsistence level: at the 14th period (i.e., at $T = 65.6$) $W(t) = 1.3 > \bar{W} = 1$.The past capital profile in this case is assumed to be on an exponential path. If we begin at period 14, then, the system may be regarded as having an opposite kind of initial condition: it starts from a situation where the demand for labour is greater than the supply of labour. No separate program therefore is needed for this case.

Table 3 provides a similar set of initial conditions except that the past capital endowment is assumed to be on a *non*-exponential path.The subsistence wage level prevails ,in this case,only for two periods and the system quickly moves to the second situation where the demand for labour is greater than the supply of labour.We note that labour is a *non-redundant* factor at the starting point (the maximal demand for labour being equal to the total supply of labour)and that output does *not* lie on an exponential path. Yet,from $t = 38.8$ on,the real wage starts to converge to an exponential path with the consequence that the economic life of machines becomes a constant at a value approximately equal to 10 periods, the Golden Age value (in other words,the real wage is rising exponentially at rate λ/β).Output,as shown in the simulated colum ,accordingly converges to a steady growth path ,very close indeed to the Golden Age values.Note that this state of affairs can only prevail up to $t = 337$. After this point,the real wage starts to diverge from the exponential path and the economic life of machines begins to fluctuate (it is equal to 3 periods at $t = 428$ and at $t = 336$ it is equal to 146 periods).It is readily verified that the set of parameters chosen does *not* satisfy the stability conditions. In fact it is chosen so as to ensure that $\lim k(t) = 0$ as $t \to \infty$. (We recall,from footnote 7 of Chapter 6, that if we are to employ the *direct method* of Liapunov ,the condition for $k(t)$,as given by equation (6.1), to have an asymptotic zero solution is that $|b_1| \leq b_0$. The set of parameters chosen satisfies this *instability* condition , for here $b_1 = 0.0025$ and $b_0 = 0.056$).This is confirmed by Table 3 where it is shown that after $t = 366$,output begins to fall.

TABLE I

EXPONENTIAL HISTORY CASE WITH THETA TREATED AS A VARIABLE

ZERO FORESIGHT AND EXPONENTIAL DEPRECIATION

DEMAND FOR LABOUR IS EQUAL TO SUPPLY OF LABOUR AT T=0

DELTA= 0.20000E-01 ALPHA= 0.25000E+00

N= 0.40000E-01 LAMDA= 0.50000E-01

S= 0.20000E+00 L(0)= 0.10000E+02 THETA= 0.43152E+01

T	THETA	W(T)	Y(T)	Y(T) AS IN EXP SOLUTION	DEVIATION
0.863E+00	0.100E+02	0.883E+00	0.103E+02	0.103E+02	0.113E-01
0.560E+01	0.100E+02	0.121E+02	0.171E+02	0.171E+02	0.555E-01
0.103E+02	0.100E+02	0.166E+01	0.285E+02	0.284E+02	0.159E+00
0.151E+02	0.100E+02	0.228E+01	0.475E+02	0.471E+02	0.378E+00
0.198E+02	0.100E+02	0.313E+01	0.790E+02	0.782E+02	0.815E+00
0.245E+02	0.100E+02	0.429E+01	0.131E+03	0.129E+03	0.166E+01
0.293E+02	0.100E+02	0.589E+01	0.218E+03	0.215E+03	0.328E+01
0.340E+02	0.100E+02	0.809E+01	0.363E+03	0.357E+03	0.631E+01
0.388E+02	0.100E+02	0.111E+02	0.605E+03	0.593E+03	0.119E+02
0.435E+02	0.100E+02	0.152E+02	0.100E+04	0.984E+03	0.221E+02
0.483E+02	0.100E+02	0.209E+02	0.167E+04	0.163E+04	0.407E+02
0.530E+02	0.100E+02	0.287E+02	0.278E+04	0.270E+04	0.742E+02
0.578E+02	0.100E+02	0.393E+02	0.462E+04	0.449E+04	0.134E+03
0.625E+02	0.100E+02	0.540E+02	0.769E+04	0.745E+04	0.240E+03
0.673E+02	0.100E+02	0.741E+02	0.128E+05	0.123E+05	0.430E+03
0.720E+02	0.100E+02	0.101E+03	0.212E+05	0.205E+05	0.764E+03
0.768E+02	0.100E+02	0.139E+03	0.354E+05	0.340E+05	0.135E+04
0.815E+02	0.100E+02	0.191E+03	0.588E+05	0.565E+05	0.238E+04
0.863E+02	0.100E+02	0.262E+03	0.979E+05	0.937E+05	0.418E+04
0.910E+02	0.100E+02	0.360E+03	0.162E+06	0.155E+06	0.732E+04
0.957E+02	0.100E+02	0.495E+03	0.270E+06	0.258E+06	0.127E+05
0.100E+03	0.100E+02	0.679E+03	0.450E+06	0.428E+06	0.222E+05
0.105E+03	0.100E+02	0.932E+03	0.749E+06	0.710E+06	0.387E+05
0.110E+03	0.100E+02	0.127E+04	0.124E+07	0.177E+07	0.671E+05
0.114E+03	0.100E+02	0.175E+04	0.207E+07	0.195E+07	0.116E+06
0.119E+03	0.100E+02	0.240E+04	0.344E+07	0.324E+07	0.200E+06
0.124E+03	0.100E+02	0.330E+04	0.573E+07	0.538E+07	0.346E+06

0.129E+03	0.100E+02	0.453E+04	0.953E+07	0.893E+07	0.597E+06
0.133E+03	0.100E+02	0.622E+04	0.158E+08	0.148E+08	0.102E+07
0.138E+03	0.100E+02	0.854E+04	0.263E+08	0.245E+08	0.176E+07
0.143E+03	0.100E+02	0.117E+05	0.438E+08	0.408E+08	0.303E+07
0.148E+03	0.100E+02	0.160E+05	0.729E+08	0.677E+08	0.520E+07
0.152E+03	0.100E+02	0.220E+05	0.121E+09	0.112E+09	0.890E+07
0.157E+03	0.100E+02	0.302E+05	0.201E+09	0.186E+09	0.152E+08
0.162E+03	0.100E+02	0.415E+05	0.335E+09	0.309E+09	0.260E+08
0.166E+03	0.100E+02	0.570E+05	0.557E+09	0.513E+09	0.445E+08
0.171E+03	0.100E+02	0.782E+05	0.927E+09	0.851E+09	0.759E+08
0.176E+03	0.100E+02	0.107E+06	0.154E+10	0.141E+10	0.129E+09
0.181E+03	0.100E+02	0.147E+06	0.256E+10	0.234E+10	0.220E+09
0.185E+03	0.100E+02	0.202E+06	0.426E+10	0.388E+10	0.376E+09
0.190E+03	0.100E+02	0.277E+06	0.686E+10	0.645E+10	0.415E+09
0.195E+03	0.100E+02	0.380E+06	0.112E+11	0.107E+11	0.517E+09
0.200E+03	0.100E+02	0.538E+06	0.183E+11	0.177E+11	0.577E+09
0.204E+03	0.100E+02	0.717E+06	0.298E+11	0.294E+11	0.361E+09
0.209E+03	0.100E+02	0.984E+06	0.491E+11	0.488E+11	0.271E+09
0.214E+03	0.100E+02	0.135E+07	0.813E+11	0.811E+11	0.272E+09
0.219E+03	0.100E+02	0.185E+07	0.138E+12	0.134E+12	0.350E+10
0.223E+03	0.100E+02	0.254E+07	0.231E+12	0.223E+12	0.832E+10
0.228E+03	0.100E+02	0.349E+07	0.386E+12	0.370E+12	0.155E+11
0.233E+03	0.100E+02	0.479E+07	0.643E+12	0.614E+12	0.282E+11
0.238E+03	0.100E+02	0.657E+07	0.107E+13	0.102E+13	0.508E+11
0.242E+03	0.100E+02	0.902E+07	0.178E+13	0.169E+13	0.910E+11
0.247E+03	0.100E+02	0.127E+08	0.297E+13	0.280E+13	0.162E+12
0.252E+03	0.100E+02	0.169E+08	0.492E+13	0.465E+13	0.265E+12
0.257E+03	0.100E+02	0.226E+08	0.820E+13	0.772E+13	0.471E+12
0.261E+03	0.100E+02	0.319E+08	0.137E+14	0.128E+14	0.899E+12
0.266E+03	0.100E+02	0.438E+08	0.228E+14	0.212E+14	0.157E+13
0.271E+03	0.100E+02	0.602E+08	0.380E+14	0.353E+14	0.275E+13
0.276E+03	0.100E+02	0.826E+08	0.633E+14	0.585E+14	0.480E+13
0.280E+03	0.100E+02	0.113E+09	0.105E+15	0.971E+14	0.836E+13
0.285E+03	0.100E+02	0.155E+09	0.173E+15	0.161E+15	0.121E+14
0.290E+03	0.100E+02	0.213E+09	0.288E+15	0.267E+15	0.213E+14
0.295E+03	0.100E+02	0.391E+09	0.481E+15	0.443E+15	0.373E+14
0.299E+03	0.100E+02	0.402E+09	0.791E+15	0.736E+15	0.546E+14

0.304E+03	0.100E+02	0.536E+09	0.131E+16	0.122E+16	0.904E+14
0.309E+03	0.100E+02	0.757E+09	0.219E+16	0.202E+16	0.167E+15
0.314E+03	0.100E+02	0.103E+10	0.365E+16	0.336E+16	0.290E+15

TABLE 2

EXPONENTIAL HISTORY CASE WITH THETA TREATED AS A VARIABLE

ZERO FORESIGHT AND EXPONENTIAL DEPRECIATION

DEMAND FOR LABOUR IS SMALLER THAN SUPPLY OF LABOUR AT T=0

DELTA= 0.20000E-01 ALPHA= 0.25000E+00

N= 0.40000E-01 LAMDA= 0.50000E-01

S= 0.20000E+00 L(0)= 0.10000E+02 THETA= 0.43152E+01

T	THETA	W(T)	Y(T) AS SIMULATED	Y(T) AS IN EXP SOLUTION	DEVIATION
0.863E+00	0.699E+01	0.100E+01	0.686E+01	0.103E+02	-0.345E+01
0.560E+01	0.110E+02	0.133E+02	0.175E+02	0.171E+02	0.453E+00
0.103E+02	0.100E+02	0.177E+01	0.300E+02	0.284E+02	0.165E+01
0.151E+02	0.100E+02	0.237E+01	0.512E+02	0.471E+02	0.409E+01
0.198E+02	0.100E+02	0.316E+01	0.853E+02	0.782E+02	0.703E+01
0.245E+02	0.100E+02	0.421E+01	0.145E+03	0.129E+03	0.160E+02
0.293E+02	0.100E+02	0.561E+01	0.261E+03	0.215E+03	0.455E+02
0.340E+02	0.100E+02	0.742E+01	0.473E+03	0.357E+03	0.115E+03
0.388E+02	0.899E+01	0.133E+02	0.778E+03	0.593E+03	0.185E+03
0.435E+02	0.899E+01	0.177E+02	0.121E+04	0.984E+03	0.229E+03
0.483E+02	0.799E+01	0.236E+02	0.199E+04	0.163E+04	0.365E+03
0.530E+02	0.799E+01	0.315E+02	0.338E+04	0.270E+04	0.680E+03
0.578E+02	0.100E+02	0.420E+02	0.623E+04	0.449E+04	0.174E+04
0.625E+02	0.899E+01	0.561E+02	0.107E+05	0.745E+04	0.330E+04
0.673E+02	0.899E+01	0.741E+02	0.205E+05	0.123E+05	0.815E+04
0.720E+02	0.100E+02	0.988E+02	0.404E+05	0.205E+05	0.198E+05
0.768E+02	0.100E+02	0.177E+03	0.540E+05	0.340E+05	0.200E+05
0.815E+02	0.899E+01	0.236E+03	0.886E+05	0.565E+05	0.321E+05
0.863E+02	0.799E+01	0.315E+03	0.158E+06	0.937E+05	0.648E+05
0.910E+02	0.100E+02	0.420E+03	0.291E+06	0.155E+06	0.135E+06
0.957E+02	0.899E+01	0.555E+03	0.558E+06	0.258E+06	0.300E+06
0.100E+03	0.100E+02	0.740E+03	0.112E+07	0.428E+06	0.694E+06
0.105E+03	0.899E+01	0.132E+04	0.236E+07	0.710E+06	0.165E+07
0.110E+03	0.699E+01	0.177E+04	0.306E+07	0.117E+07	0.188E+07
0.114E+03	0.899E+01	0.236E+04	0.444E+07	0.195E+07	0.248E+07

0.119E+03	0.799E+01	0.314E+04	0.848E+07	0.324E+07	0.524E+07
0.124E+03	0.899E+01	0.416E+04	0.167E+08	0.538E+07	0.113E+08
0.129E+03	0.799E+01	0.555E+04	0.376E+08	0.893E+07	0.287E+08
0.133E+03	0.799E+01	0.986E+04	0.923E+08	0.148E+08	0.775E+08
0.138E+03	0.599E+01	0.131E+05	0.125E+09	0.245E+08	0.101E+09
0.143E+03	0.699E+01	0.175E+05	0.159E+09	0.408E+08	0.188E+09
0.148E+03	0.899E+01	0.233E+05	0.283E+09	0.677E+08	0.261E+09
0.152E+03	0.799E+01	0.311E+05	0.671E+09	0.112E+09	0.558E+09
0.157E+03	0.110E+02	0.415E+05	0.150E+10	0.186E+09	0.131E+10
0.162E+03	0.899E+01	0.739E+05	0.219E+10	0.309E+09	0.188E+10
0.166E+03	0.699E+01	0.985E+05	0.383E+10	0.513E+09	0.332E+10
0.171E+03	0.799E+01	0.131E+06	0.592E+10	0.851E+09	0.506E+10
0.176E+03	0.899E+01	0.175E+06	0.114E+11	0.141E+10	0.100E+11
0.181E+03	0.100E+02	0.233E+06	0.270E+11	0.234E+10	0.246E+11
0.185E+03	0.899E+01	0.415E+06	0.451E+11	0.388E+10	0.412E+11
0.190E+03	0.500E+01	0.553E+06	0.142E+12	0.645E+10	0.136E+12
0.195E+03	0.100E+02	0.738E+06	0.889E+11	0.107E+11	0.782E+11
0.200E+03	0.699E+01	0.984E+06	0.279E+12	0.177E+11	0.262E+12
0.204E+03	0.100E+02	0.131E+07	0.536E+12	0.294E+11	0.507E+12
0.209E+03	0.799E+01	0.233E+07	0.203E+13	0.488E+11	0.198E+13
0.214E+03	0.799E+01	0.311E+07	0.178E+13	0.811E+11	0.170E+13
0.219E+03	0.100E+02	0.414E+07	0.228E+13	0.134E+12	0.214E+13
0.223E+03	0.699E+01	0.553E+07	0.677E+13	0.223E+12	0.655E+13
0.228E+03	0.100E+02	0.737E+07	0.120E+14	0.370E+12	0.116E+14
0.233E+03	0.699E+01	0.131E+08	0.499E+14	0.614E+12	0.492E+14
0.238E+03	0.100E+02	0.174E+08	0.228E+14	0.102E+13	0.218E+14
0.242E+03	0.699E+01	0.233E+08	0.818E+14	0.169E+13	0.802E+14
0.247E+03	0.799E+01	0.310E+08	0.124E+15	0.280E+13	0.121E+15
0.252E+03	0.110E+02	0.414E+08	0.267E+15	0.465E+13	0.263E+15
0.257E+03	0.799E+01	0.736E+08	0.600E+15	0.772E+13	0.592E+15
0.261E+03	0.500E+01	0.982E+08	0.197E+16	0.128E+14	0.196E+16
0.266E+03	0.100E+02	0.130E+09	0.673E+15	0.212E+14	0.652E+15
0.271E+03	0.400E+01	0.232E+09	0.789E+16	0.353E+14	0.785E+16
0.276E+03	0.140E+02	0.232E+09	0.333E+16	0.585E+14	0.328E+16
0.280E+03	0.599E+01	0.413E+09	0.464E+16	0.971E+14	0.454E+16
0.285E+03	0.599E+01	0.551E+09	0.717E+17	0.161E+15	0.715E+17

0.290E+03	0.899E+01	0.735E+09	0.142E+17	0.267E+15	0.139E+17
0.295E+03	0.699E+01	0.981E+09	0.103E+18	0.443E+15	0.103E+18
0.299E+03	0.100E+02	0.130E+10	0.169E+18	0.736E+15	0.168E+18
0.304E+03	0.899E+01	0.232E+10	0.344E+18	0.122E+16	0.343E+18
0.309E+03	0.100E+02	0.310E+10	0.386E+18	0.202E+16	0.384E+18
0.314E+03	0.130E+02	0.413E+10	0.123E+17	0.336E+16	0.894E+16
0.318E+03	0.210E+02	0.413E+10	0.142E+17	0.558E+16	0.862E+16
0.323E+03	0.280E+02	0.413E+10	0.182E+17	0.926E+16	0.895E+16
0.328E+03	0.450E+02	0.413E+10	0.206E+17	0.153E+17	0.532E+16
0.333E+03	0.559E+02	0.413E+10	0.252E+17	0.254E+17	-0.254E+15
0.337E+03	0.669E+02	0.413E+10	0.302E+17	0.422E+17	-0.120E+17
0.342E+03	0.460E+02	0.413E+10	0.377E+17	0.701E+17	-0.324E+17
0.347E+03	0.909E+02	0.413E+10	0.443E+17	0.116E+18	-0.721E+17
0.352E+03	0.100E+02	0.413E+10	0.172E+18	0.193E+18	-0.205E+17
0.356E+03	0.210E+02	0.413E+10	0.157E+18	0.320E+18	-0.163E+18
0.361E+03	0.320E+02	0.413E+10	0.142E+18	0.531E+18	-0.389E+18
0.366E+03	0.430E+02	0.413E+10	0.129E+18	0.882E+18	-0.752E+18
0.371E+03	0.540E+02	0.413E+10	0.118E+18	0.146E+19	-0.134E+19
0.375E+03	0.100E+01	0.413E+10	0.257E+24	0.242E+19	0.257E+24
0.380E+03	0.120E+02	0.413E+10	0.128E+21	0.403E+19	0.124E+21
0.385E+03	0.230E+02	0.413E+10	0.116E+21	0.668E+19	0.109E+21
0.390E+03	0.340E+02	0.413E+10	0.106E+21	0.110E+20	0.949E+20
0.394E+03	0.450E+02	0.413E+10	0.964E+20	0.184E+20	0.780E+20
0.399E+03	0.559E+02	0.413E+10	0.877E+20	0.305E+20	0.571E+20
0.404E+03	0.669E+02	0.413E+10	0.797E+20	0.506E+20	0.291E+20
0.409E+03	0.779E+02	0.413E+10	0.725E+20	0.840E+20	-0.115E+20
0.413E+03	0.889E+02	0.413E+10	0.659E+20	0.139E+21	-0.735E+20
0.418E+03	0.100E+03	0.413E+10	0.600E+20	0.231E+21	-0.171E+21
0.423E+03	0.111E+03	0.413E+10	0.545E+20	0.384E+21	-0.329E+21
0.428E+03	0.122E+03	0.413E+10	0.496E+20	0.637E+21	-0.587E+21

END-OF-DATA ENCOUNTERED ON SYSTEM INPUT FILE.

TABLE 3

NON-EXPONENTIAL HISTORY CASE WITH THETA TREATED AS A VARIABLE

ZERO FORESIGHT AND EXPONENTIAL DEPRECIATION

DEMAND FOR LABOUR IS EQUAL TO SUPPLY OF LABOUR AT T=0

DELTA= 0.20000E-01 ALPHA= 0.25000E+00

N= 0.40000E-01 LAMDA= 0.50000E-01

S= 0.20000E+00 L(O)= 0.10000E+02 THETA= 0.43152E+01

T	THETA	W(T)	Y(T) AS SIMULATED	Y(T) AS IN EXP. SOLUTION	DEVIATION
0.863E+00	0.110E+02	0.999E-01	0.473E+01	0.103E+02	-0.559E+01
0.560E+01	0.100E+01	0.236E+01	0.528E+04	0.171E+02	0.526E+04
0.103E+02	0.110E+02	0.315E+01	0.974E+02	0.284E+02	0.689E+02
0.151E+02	0.220E+02	0.315E+01	0.124E+03	0.471E+02	0.769E+02
0.198E+02	0.260E+02	0.420E+01	0.971E+02	0.782E+02	0.188E+02
0.245E+02	0.190E+02	0.420E+01	0.147E+03	0.129E+03	0.176E+02
0.293E+02	0.130E+02	0.561E+01	0.228E+03	0.215E+03	0.130E+02
0.340E+02	0.110E+02	0.748E+01	0.370E+03	0.357E+03	0.128E+02
0.388E+02	0.100E+02	0.997E+01	0.605E+03	0.593E+03	0.125E+02
0.435E+02	0.100E+02	0.177E+02	0.102E+04	0.984E+03	0.362E+02
0.483E+02	0.100E+02	0.236E+02	0.168E+04	0.163E+04	0.509E+02
0.530E+02	0.110E+02	0.315E+02	0.283E+04	0.270E+04	0.126E+03
0.578E+02	0.110E+02	0.420E+02	0.470E+04	0.449E+04	0.208E+03
0.625E+02	0.110E+02	0.560E+02	0.778E+04	0.745E+04	0.327E+03
0.673E+02	0.110E+02	0.747E+02	0.129E+05	0.123E+05	0.563E+03
0.720E+02	0.110E+02	0.996E+02	0.217E+05	0.205E+05	0.123E+04
0.768E+02	0.110E+02	0.132E+03	0.365E+05	0.340E+05	0.253E+04
0.815E+02	0.100E+02	0.177E+03	0.567E+05	0.565E+05	0.226E+03
0.863E+02	0.100E+02	0.236E+03	0.953E+05	0.937E+05	0.153E+04
0.910E+02	0.100E+02	0.419E+03	0.163E+06	0.155E+06	0.802E+04
0.957E+02	0.100E+02	0.559E+03	0.261E+06	0.258E+06	0.307E+04
0.100E+03	0.100E+02	0.746E+03	0.429E+06	0.428E+06	0.163E+04
0.105E+03	0.110E+02	0.995E+03	0.741E+06	0.710E+06	0.306E+05
0.110E+03	0.110E+02	0.132E+04	0.122E+07	0.117E+07	0.494E+05
0.114E+03	0.110E+02	0.176E+04	0.205E+07	0.195E+07	0.986E+05

0.119E+03	0.110E+02	0.235E+04	0.343E+07	0.324E+07	0.190E+06
0.124E+03	0.110E+02	0.314E+04	0.577E+07	0.538E+07	0.390E+06
0.129E+03	0.100E+02	0.419E+04	0.895E+07	0.893E+07	0.174E+05
0.133E+03	0.100E+02	0.559E+04	0.152E+08	0.148E+08	0.429E+06
0.138E+03	0.100E+02	0.994E+04	0.254E+08	0.245E+08	0.814E+06
0.143E+03	0.100E+02	0.132E+05	0.416E+08	0.408E+08	0.798E+06
0.148E+03	0.100E+02	0.176E+05	0.677E+08	0.677E+08	0.297E+05
0.152E+03	0.100E+02	0.235E+05	0.112E+09	0.112E+09	0.418E+06
0.157E+03	0.110E+02	0.314E+05	0.194E+09	0.186E+09	0.815E+07
0.162E+03	0.110E+02	0.419E+05	0.325E+09	0.309E+09	0.162E+08
0.166E+03	0.110E+02	0.558E+05	0.544E+09	0.513E+09	0.308E+08
0.171E+03	0.110E+02	0.744E+05	0.913E+09	0.851E+09	0.625E+08
0.176E+03	0.100E+02	0.993E+05	0.141E+10	0.141E+10	0.401E+07
0.181E+03	0.100E+02	0.132E+06	0.241E+10	0.234E+10	0.700E+08
0.185E+03	0.100E+02	0.235E+06	0.402E+10	0.388E+10	0.137E+09
0.190E+03	0.100E+02	0.313E+06	0.660E+10	0.645E+10	0.150E+09
0.195E+03	0.100E+02	0.418E+06	0.107E+11	0.107E+11	0.477E+08
0.200E+03	0.110E+02	0.558E+06	0.185E+11	0.177E+11	0.798E+09
0.204E+03	0.110E+02	0.744E+06	0.307E+11	0.294E+11	0.129E+10
0.209E+03	0.110E+02	0.992E+06	0.511E+11	0.488E+11	0.221E+10
0.214E+03	0.110E+02	0.132E+07	0.859E+11	0.811E+11	0.484E+10
0.219E+03	0.110E+02	0.176E+07	0.144E+12	0.134E+12	0.993E+10
0.223E+03	0.100E+02	0.235E+07	0.224E+12	0.223E+12	0.771E+09
0.228E+03	0.100E+02	0.313E+07	0.381E+12	0.370E+12	0.112E+11
0.233E+03	0.100E+02	0.557E+07	0.636E+12	0.614E+12	0.220E+11
0.238E+03	0.100E+02	0.743E+07	0.104E+13	0.102E+13	0.243E+11
0.242E+03	0.100E+02	0.991E+07	0.170E+13	0.169E+13	0.828E+10
0.247E+03	0.110E+02	0.132E+08	0.293E+13	0.280E+13	0.127E+12
0.252E+03	0.110E+02	0.176E+08	0.486E+13	0.465E+13	0.205E+12
0.257E+03	0.110E+02	0.234E+08	0.808E+13	0.772E+13	0.352E+12
0.261E+03	0.110E+02	0.313E+08	0.135E+14	0.128E+14	0.769E+12
0.266E+03	0.110E+02	0.417E+08	0.228E+14	0.212E+14	0.157E+13
0.271E+03	0.100E+02	0.556E+08	0.354E+14	0.353E+14	0.129E+12
0.276E+03	0.100E+02	0.742E+08	0.595E+14	0.585E+14	0.933E+12
0.280E+03	0.100E+02	0.131E+09	0.102E+15	0.971E+14	0.495E+13
0.285E+03	0.100E+02	0.175E+09	0.163E+15	0.161E+15	0.179E+13
0.290E+03	0.100E+02	0.234E+09	0.268E+15	0.267E+15	0.781E+12

0.295E+03	0.110E+02	0.312E+09	0.462E+15	0.443E+15	0.186E+14
0.299E+03	0.100E+02	0.417E+09	0.730E+15	0.736E+15	-0.637E+13
0.304E+03	0.110E+02	0.556E+09	0.128E+16	0.122E+16	0.594E+14
0.309E+03	0.110E+02	0.741E+09	0.215E+16	0.202E+16	0.123E+15
0.314E+03	0.110E+02	0.988E+09	0.361E+16	0.336E+16	0.250E+15
0.318E+03	0.100E+02	0.131E+10	0.560E+16	0.558E+16	0.193E+14
0.323E+03	0.100E+02	0.175E+10	0.954E+16	0.926E+16	0.281E+15
0.328E+03	0.100E+02	0.312E+10	0.159E+17	0.153E+17	0.549E+15
0.333E+03	0.110E+02	0.416E+10	0.251E+17	0.254E+17	-0.323E+15
0.337E+03	0.150E+02	0.416E+10	0.326E+17	0.422E+17	-0.962E+16
0.342E+03	0.300E+02	0.416E+10	0.374E+17	0.701E+17	-0.327E+17
0.347E+03	0.599E+01	0.416E+10	0.879E+17	0.116E+18	-0.285E+17
0.352E+03	0.170E+02	0.416E+10	0.799E+17	0.193E+18	-0.113E+18
0.356E+03	0.280E+02	0.416E+10	0.727E+17	0.320E+18	-0.247E+18
0.361E+03	0.650E+02	0.416E+10	0.799E+17	0.531E+18	-0.451E+18
0.366E+03	0.300E+01	0.416E+10	0.114E+20	0.882E+18	0.105E+20
0.371E+03	0.140E+02	0.416E+10	0.104E+20	0.146E+19	0.894E+19
0.375E+03	0.250E+02	0.416E+10	0.946E+19	0.242E+19	0.703E+19
0.380E+03	0.360E+02	0.416E+10	0.860E+19	0.403E+19	0.457E+19
0.385E+03	0.470E+02	0.416E+10	0.782E+19	0.668E+19	0.113E+19
0.390E+03	0.580E+02	0.416E+10	0.711E+19	0.110E+20	0.397E+19
0.394E+03	0.689E+02	0.416E+10	0.647E+19	0.184E+20	-0.119E+20
0.399E+03	0.799E+02	0.416E+10	0.588E+19	0.305E+20	-0.246E+20
0.404E+03	0.909E+02	0.416E+10	0.535E+19	0.506E+20	-0.453E+20
0.409E+03	0.102E+03	0.416E+10	0.486E+19	0.840E+20	-0.792E+20
0.413E+03	0.113E+03	0.416E+10	0.442E+19	0.139E+21	-0.135E+21
0.418E+03	0.124E+03	0.416E+10	0.402E+19	0.231E+21	-0.227E+21
0.423E+03	0.135E+03	0.416E+10	0.366E+19	0.384E+21	-0.380E+21
0.428E+03	0.146E+03	0.416E+10	0.332E+19	0.637E+21	-0.633E+21

END-OF-DATA ENCOUNTERED ON SYSTEM INPUT FILE:

Chapter 7

MULTISECTOR GROWTH MODEL WITH HETEROGENEOUS CAPITAL

So far we have assumed that there is only one good which can both be consumed or accumulated. However in our discussion of economic obsolescence in Chapter 4 there is an implicit distinction between a capital good and a consumption good: once the good is used as capital it cannot be consumed. For, if the contrary were true all the scrapped capital will be consumed and the system may end up producing only goods for accumulation and consumption will be entirely composed of θ-old goods being scrapped from industry.

In this chapter we generalise the system by considering a multisectoral economy. Our purpose in undertaking this generalisation is to determine which of our one sector results carry over to more general models and to see if multisectoral growth has any special features not visible in the one sector model.

1. A Multisectoral Model: Zero foresight.

We consider an economy with k industries. The first industry produces equipment, the other (k-1) industries produce consumer goods. Technical improvements accrue exponentially but at rates which may vary from industry to industry. It follows that the economic life of equipment also may vary from industry to industry. The first commodity serves as *numéraire*. Quantity units are chosen so that initially, at t=0, all prices are unity.

In the present section zero foresight is assumed. The case of perfect foresight is examined in the next section. We begin by assuming that equipment decays exponentially, but at a rate which may vary from industry to industry. The possibility of sudden death depreciation is considered towards the end of the section.

Let $K_{iv}(t)$ represent the amount of equipment of vintage v surviving at time t in the *i*th industry $(t \geqslant v)$, and $L_{iv}(t)$ the labour required to work that equipment. Then for each vintage there is a production relationship of the form

$$(7.1) \qquad Q_{iv}(t) = e^{\lambda/v} K_{iv}(t)^{\alpha_i} L_{iv}(t)^{\beta_i}, \ \alpha_i+\beta_i=1; \ \alpha_i,\beta_i > 0.$$

Since in the *i*th industry equipment decays exponentially we have $K_{iv}(t)=e^{-\delta_i(t-v)} K_{iv}(v)$, where δ_i is the rate of depreciation and $K_{iv}(v)$ is the rate of gross investment in the *i*th industry at time v. Since, *ex post*, factor proportions are fixed, we have also

$$L_{iv}(t)=e^{-\delta_i(t-v)} L_{iv}(v)$$

Hence (7.1) may be rewritten as

$$(7.2) \qquad Q_{iv}(t)=e^{-\delta_i(t-v)} Q_{iv}(v).$$

Total output of the *i*th commodity at time t is, therefore,

$$(7.3) \qquad Q_i(t) = \int_{t-\theta_i(t)}^{t} Q_{iv}(t)dv$$

where $\theta_i(t)$ is the economic life of equipment in the *i*th industry at time *t*; and aggregate output, in terms of the first commodity, may be expressed as

$$(7.4) \qquad Y(t) = \sum_i p_i(t) Q_i(t)$$

where $p_i(t)$ is the price of the *i*th commodity at time *t* in terms of the first commodity. $(p_i(0) = p_1(t) = 1)$.

As in earlier sections, we seek paths of exponential growth

$$(7.5) \qquad Q_i(t) = \bar{Q}_i e^{\varphi_i t}$$

consistent with constancy of the economic life of equipment. In what follows, therefore, θ_i is treated as constant.

We complete the display of new notation by writing

$$(7.6) \qquad p_i(t) Q_i(t) = c_i Y(t)$$

where c_1 is the marginal propensity to save, $c_i (i > 1)$ is the marginal propensity to consume the *i*th commodity, and $\Sigma\, c_i = 1$.

It follows immediately from (7.4)-(7.6) that aggregate output must grow at the same rate as the output of the first industry:

$$(7.7) \qquad Y(t) = \bar{Q} e^{\varphi_1 t} \qquad (\bar{Q} = \bar{Q}_1 / c_1).$$

It also follows, from (7.4)-(7.6), that all prices must grow or decline exponentially:

$$(7.8) \qquad p_i(t) = e^{(\varphi_1 - \varphi_i)t} .$$

Our remaining conclusion fall out rather less easily.

We seek expressions for the rates of growth φ_i in terms of the structual parameters of the system. We proceed by a slightly roundabout route. Since θ_i is constant and since

$$(7.10) \qquad Q_i(t) = \int_{t-\theta_i}^{t} e^{-\delta_i (t-v)} Q_{iv}(v) dv$$

is assumed to grow exponentially, so must $Q_{iv}(v)$, and at the same rate ϱ_i. Thus, substituting in (7.3) from (7.2), and defferentiating with respect to t, we obtain

$$(7.11) \qquad \dot{Q}_i(t) = -\delta_i Q_i(t) + Q_{it}(t) - e^{-\delta_i\theta_i} Q_{i,t-\theta_i}(t-\theta_i)$$

which, in view of (7.5), reduces to

$$(7.12) \qquad Q_{it}(t) - e^{-\delta_i\theta_i} Q_{i,t-\theta_i}(t-\theta_i) = (\varrho+\delta_i)\bar{Q}_i e^{\varrho_i t},$$

a first order defference equation in $Q_{it}(t)$, with steady-growth solution

$$(7.13) \qquad Q_{it}(t) = \frac{(\varrho_i+\delta_i)\bar{Q}_i e^{\varrho_i t}}{1-e^{-\theta_i(\varrho_i+\delta_i)}} .$$

We note next that

$$(7.14) \qquad w(t) = p_i(t)\frac{\delta Q_{it}(t)}{\delta L_{it}(t)} = \beta_i p_i(t) Q_{it}(t)/L_{it}(t)$$

and that therefore

$$(7.15) \qquad L_{it}(t) = \left(\frac{\beta_i p_i(t) Q_{it}(t)}{\beta_1 Q_{1t}(t)}\right) L_{1t}(t) = \left(\frac{\beta_i c_i(\delta_i+\varrho_i)}{\beta_1 c_1(\delta_1+\varrho_1)} \cdot \frac{1-e^{-\theta_1(\delta_1+\varrho_1)}}{1-e^{-\theta_i(\delta+\varrho_i)}}\right) L_{1t}(t)$$

$$= N_i L_{1t}(t), \text{ say.}$$

A similar relationship may be obtained for capital. With zero foresight, a unit of new equipment will earn the same rental in all industries:

$$(7.16) \qquad \frac{\Pi_{it}(t)}{K_{it}(t)} = \frac{\Pi_{1t}(t)}{K_{1t}(t)}$$

where

$$\Pi_{it}(t)=p_i(t)Q_{it}(t)-w(t)L_{it}(t) \tag{7.17}$$

is the total rental earned by new equipment in the ith industry. It follows from (7.14)-(7.17) that

$$K_{it}(t) = \left(\frac{p_i(t)Q_{it}(t)-w(t)L_{it}(t)}{Q_{1t}(t)-w(t)L_{1t}(t)}\right)K_{1t}(t) = \left(\frac{\alpha_i p_i(t)Q_{it}(t)}{\alpha_1 Q_{1t}(t)}\right)K_{1t}(t) \tag{7.18}$$

$$= \left(\frac{\alpha_i \beta_1}{\alpha_1 \beta_i}\right) N_i K_{1t}(t) = M_i K_{1t}(t), \text{ say.}$$

But $Q_1(t)=\Sigma K_{it}(t)$ grows at the exponential rate φ_1. It follows from (7.18) that all $K_{it}(t)$ grow at the same rate:

$$K_{1t}(t) = \frac{\bar{Q}_1 e^{\varphi_1 t}}{\Sigma M_i}, \tag{7.19}$$

$$K_{it}(t) = M_i \left(\frac{\bar{Q}_1 e^{\varphi_1 t}}{\Sigma M_i}\right).$$

We have shown that $Q_{it}(t)$ grows exponentially at the rate φ_i and that $K_{it}(t)$ grows at the rate φ_1. It follows, from (7.1), that $L_{it}(t)$ and therefore $L_i(t)$ grow at the exponential rate

$$n_i = (\varphi-\lambda_i-\alpha_i\varphi_1)/\beta_i. \tag{7.20}$$

Now $L(t) = \Sigma L_i(t)$ grows at the exponential rate n; therefore all $L_i(t)$ and $L_{it}(t)$ grow at the same rate:

$$n_i = n. \tag{7.21}$$

It follows finally, from (7.20) and (7.21), that the output of the first or capital goods industry, and therefore aggregate output, grows exponentially as the rate

$$(7.22)\qquad \rho_1 = n+\lambda_1/\beta_1$$

and that the rate of growth of the *i*th consumer goods industry is

$$(7.23)\qquad \rho_i = n+\lambda_i+\alpha_i\lambda_1/\beta_1 .$$

(Evidently (7.22) is a special case of (7.23)). Thus, as in the corresponding onesector model studied in chapter 4, the rates of growth are independent both of the saving ratio and of the rates of depreciation.

It is now a simple matter to obtain the solutions for $\bar{Q}$ and the $\bar{Q}_i$, and therefore the levels of steady-growth outputs. By reasoning similar to that employed in deriving (4.6) bearing in mind the constancy of each industry's share of the labour force (equation (7.21)),

$$(7.24)\qquad L_{it}(t) = \left(\frac{n+\delta_i}{1-e^{-\theta_i(n+\delta_i)}}\right)L_i(t) = E_iL_i(t), \text{ say.}$$

From (7.15) and (7.24)

$$(7.25)\qquad L_{it}(t) = \frac{N_iL(t)}{\sum_s N_s/E_s}$$

Finally, from (7.1), (7.13), (7.19), and (7.25),

$$(7.26)\qquad \frac{(\rho_i+\delta_i)\bar{Q}_i}{1-e^{-\theta_i(\rho_i+\delta_i)}} = \left(\frac{M_ic_1\bar{Q}_i}{\Sigma M_s}\right)^{\alpha_i}\left(\frac{N_iL_o}{\Sigma N_s/M_s}\right)^{\beta_i} \qquad (i=1,\ldots\ldots, k)$$

which, together with (7.7), can be solved for $\bar{Q}$ and $\bar{Q}_i$.

We turn now to the calculation of θ_i, the economic life of equipment in the *i*th industry. The defining equation is derived from the requirement that equipment be scrapped when it ceases to earn a positive rent:

$$(7.27)\qquad \Pi_{it}(t+\theta_i) = p_i(t+\theta_i)Q_{it}(t+\theta_i) - w(t+\theta_i)L_{it}(t+\theta_i)$$

$$= p_i(t+\theta_i)\left(Q_{it}(t+\theta_i) - \frac{\delta Q_{i,t+\theta_i}(t+\theta_i)}{\delta L_{i,t+\theta_i}(t+\theta_i)}L_{it}(t+\theta_i)\right) = 0.$$

Substituting from (7.1), and (7.14), and (7.23), this reduces to

$$\beta_i = e^{-\theta_i(\lambda_i+\alpha_i(\lambda_1/\beta_1))}$$

whence

$$\theta_i = -\frac{\ln\beta_i}{\lambda_i+\alpha_i(\lambda_1/\beta_1)} \tag{7.28}$$

(For the first industry this reduces to $\theta_1 = -\beta_1/\lambda_1 \ln\beta_1$, in parity with (4.20). An interesting limiting case is that in which $\lambda_1 = \lambda_i = 0$, that is, when neither the equipment producing industry nor the capital using industries are progressive, the economic life of equipment is infinite, and steady growth is impossible.

As in the corresponding one-sector case, the economic life of equipmement is independent of the saving ratio and of rates of depreciation. In the first industry the life of equipment depends only on the rate at which improvements accrue and on the Cobb-Douglas exponent for that industry. The effect of variations in λ_1 and β_1 on θ_1 are, with the addition of appropriate subscripts, as set out in (4.21) and (4.22). In each consumer goods industry the economic life of equipment depends on the rate of technical improvement and on the Cobb-Douglas exponent for that industry, as well as on the corresponding parameters for the first or equipment-producing industry. An increase in the rate at which improvements accrue, whether in the first or *i*th industry, results in a shortening of the economic life of equipment in the *i*th industry:

$$\frac{\delta\theta_i}{\delta\lambda_1} = \frac{\alpha_i \ln\beta_i}{\beta_1(\lambda_1+\alpha_i\lambda_1/\beta_1)^2} < 0, \tag{7.29}$$

$$\frac{\delta\theta_i}{\delta\lambda_i} = \frac{\ln\beta_i}{(\lambda_i+\alpha_i\lambda_1/\beta_1)^2} < 0, \qquad i > 1.$$

An increase in labour-intensity in the equipment-producing industry results in a lengthening of the economic life of equipment in all other industries:

$$\frac{\delta\theta_i}{\delta\beta_1} = \frac{\alpha_i\lambda_1 \ln\beta_i}{(\beta_1\lambda_i+\alpha_i\lambda_1)^2} > 0; \qquad i > 1, \tag{7.30}$$

but the effect of an increase in labour-intensity in a consumer goods industry is ambiguous, depending on the initial value of β_i;

$$(7.31) \qquad \frac{\delta\theta_i}{\delta\beta_i} = -\lambda_1(\beta_i \ln\beta_i + \alpha_i + \lambda_i\beta_1/\lambda_1)/\beta_1\beta_i(\lambda_i + \alpha_i\lambda_1/\beta_1)^2$$

$$= 0 \text{ when } -\beta_i \ln\beta_i = \beta_1(\lambda_i + \alpha_i\lambda_1/\beta_1)/\lambda_1,$$

$$(7.32) \qquad \frac{\delta^2\theta_i}{\delta\beta_i^2} < 0.$$

The distributional properties of the model can be disposed of quickly. The wage rate is of course the same in all industries and, from (7.8), (7.13)-(7.15), and (7.23), grows exponentially at the rate λ_1/β_1. It is perhaps paradoxical that the rate of real wages, as defined, fails to grow when the capital goods sector is unprogressive, no matter how progressive are the consumer goods sectors. To resolve the paradox, however, it is necessary to recall only that wages are measured in terms of capital goods. In terms of the product of any progressive sector, say the *i*th, real wages grow at the rate λ_i.

The rate of return on capital differs from one industry to another-the joint result of imperfect foresight and differing lives of equipment. For any particular industry, however, the rate of return is constant. To see this we first calculate the rental at time t of a unit of equipment of vintage v employed in the *i*th industry:

$$(7.33) \qquad \Pi_{iv}(t) = \begin{cases} p_i(t)\dfrac{\delta Q_{it}(t)}{\delta K_{it}(t)} & \text{for } v=t, \\[2ex] \left[p_i(t)Q_{iv}(t) - w(t)L_{iv}(t)\right]/K_{iv}(t) & \text{for } v<t. \end{cases}$$

The rate of return on investment in equipment of vintage v, say $r_i(v)$, is then given implicity by the "present value" relation

$$(7.34) \qquad \int_v^{v+\theta_i} e^{-\left[r_i(v)+\delta_i\right]\left[t-v\right]} \Pi_{iv}(t)dt = 1.$$

It can be shown that, for given $(t-v)$, $\Pi_{iv}(t)$ is a constant. It follows that

$r_i(v)$ is stationary; it is also unique.

Turning to distributive shares we note, when making use of (7.13), (7.14), and (7.24)-(7.26), that labour's share in gross aggregate output is

$$(7.35)\qquad S_L(t) = \frac{w(t)L(t)}{Y(t)} = \sum_{i=1}^{k} \frac{\beta_i c_i(\delta_i+\varphi_i)\left[1-e^{-\theta_i(n+\delta_i)}\right]}{(n+\delta_i)\left[1-e^{-\theta_i(\delta_i+\varphi_i)}\right]} .$$

It is a simple matter to determine the effect on shares of variations in the parameters of the system: $\delta S_L / \delta n > 0$, etc. Of special interest, perhaps, is the ambiguous relationship between thrift and the share of labour:

$$\frac{dS_L(t)}{dc_1} = \sum_i \frac{\delta S_L(t)}{\delta c_i} \cdot \frac{dc_i}{dc_1} \gtreqless 0 \quad \left(\sum \frac{dc_i}{dc_1} = 0 \right).$$

This is in contrast to the conslusion derived from the corresponding one-sector model: that factor shares are independent of the saving ratio.

It remains to note the alternative possibility, that depreciation is by "sudden death". If in no industry does the economic life of equipment exceed its physical life (which may, of course, differ from industry to industry), the above analysis (with $\delta_i=0$) carries over. Otherwise, with $\theta_i \equiv \bar{\theta}_i$ where necessary, (7.14) and (7.27) are inconsistent, and we conclude that steady, full-employment growth is impossible. (1)

2. A MULTISECTORAL MODEL: PERFECT FORESIGHT

We proceed to an exploration of some of the implications of perfect foresight. In particular we derive solutions for each of the main variables on the assumption that sectoral outputs and the wage rate grow exponentially. We revert, for the time being, to the assumption of exponential depreciation.

Much of Section 1, including (7.1)-(7.13), is independent of assumption about expectations and carries over to the present section. Our changed assumptions about expectations bear mainly on the hiring and investing decisions of firms. Thus, when the wage rate is expected to rise, entrepreneurs will not push the employment of labour to the point where the value of its marginal product is equal to the wage rate. Hiring will stop when the marginal worker makes no contribution to the present value of current investment. The latter, for the *i*th industry, is

$$(7.36) \qquad V_i(t) = \int_t^{t+\theta_i} \left[p_i(v)Q_{it}(v) - w(v)L_{it}(v) \right] e^{-r(t)(v-t)} dv.$$

Hence, assuming that the wage rate grows at the rate ω, and recalling (7.8),

$$(7.37) \qquad 0 = \frac{\delta V_i(t)}{\delta L_{it}(t)} = p_i(t)\frac{\delta Q_{it}(t)}{\delta L_{it}(t)} \int_t^{t-\theta_i} e^{[\varrho_1-\varrho_i-\delta_i-r(t)][v-t]} dv$$

$$-w(t)\int_t^{t+\theta_i} e^{[\omega-\delta_i-r(t)][v-t]} dv,$$

that is,

$$(7.38) \qquad w(t)=p_i(t)\frac{\delta Q_{it}(t)}{\delta L_{it}(t)} \frac{\int_t^{t+\theta_i} e^{[\varrho_1-\varrho_i-\delta_i-r(t)][v-t]} dv}{\int_t^{t+\theta_i} e^{[\omega-\delta_i-r(t)][v-t]} dv}.$$

On the other hand, the present value of the earnings of a unit of new equipment are, in equilibrium, equal to the price of equipment:

$$(7.39) \qquad 1 = \frac{V_i(t)}{K_{it}(t)} = \frac{1}{K_{it}(t)} \int_t^{t+\theta_i} [p_i(v)Q_{it}(v)-w(v)L_{it}(v)]e^{-r(t)(v-t)}\, dv.$$

It follows that the rate of interest is constant. In what follows, therefore, the functional notation $r(t)$ will be dropped.

Equations (7.38) and (7.39) plays roles analogous to those of (7.14) and (7.16) in the preceding section.

From (7.38) and (7.13),

$$(7.40) \qquad L_{it}(t) = N_i^{*}L_{1t}(t) \qquad (N_i^{*} \text{ a constant}).$$

Similarly, from (7.39), (7.40), and (7.13),

$$(7.41) \qquad K_{it}(t) = M_i^{*}K_{1t}(t) \qquad (M_i^{*} \text{ a constant}).$$

and, since $\Sigma K_{it}(t) = Q_1(t)$, all $K_{it}(t)$ grow at the same exponential rate g_1:

$$(7.42) \qquad K_{1t}(t) = \bar{Q}_1^{*}e^{g_1 t}/\Sigma M_s,$$

$$K_{it}(t) = M_i^{*}\bar{Q}_1^{*}e^{g_1 t}/\Sigma M_s$$

From (7.41) and (7.42) it follows that, as with zero foresight,

$$(7.43) \qquad g_1 = n+\lambda_1/\beta_1$$

and

$$(7.44) \qquad g_i = n+\lambda_i+\alpha_i\lambda_1/\beta_1.$$

It follows also, from (7.39), that

$$(7.45) \qquad \omega = \lambda_1/\beta_1 ,$$

that is, the wage rate grows exponentially at the rate λ_1/β_1, as before. The *levels* of aggregate and sectoral output, however, are different. While the Q_i^* have the same form as the Q_i (see (7.26)), they differ in value from the latter bacause they are evaluated at different θ_i. The underlying reason for this descrepancy is, of course, the allocative inefficiency associated with imperfect foresight. In the one-sector model inefficiency revealed itself in the over-staffing of equipment. In a multisectoral world, with rates of depreciation and technical improvement which differ from industry to indystry, there is an additional source of inefficiency: over-investment in industries with high rates of improvement and under-investment in industries with low rates of improvement. In principle, one could evaluate a measure

$$(7.46) \qquad I_k = 1 - \frac{Y(t)}{Y^*(t)} = 1 - \frac{\bar{Q}_1}{\bar{Q}_1^*}$$

of the over-all inefficiency associated with lack of foresight. As will soon become clear, however, the calculation would be messy. Moreover, the measure is not independent of the choice of *numéraire*.

We turn now to the calculation of r and the θ_i. Substituting for $w(t+\theta_i)$ (equation (7.38)) into the no-rent condition (equation (7.27)), we obtain

$$(7.47) \qquad 1 = \frac{\beta_i\left[\delta_i+r-(\lambda_1/\beta_1)\right]\left[1-e^{-\theta_i(\delta_i+r+\varphi_i-\varphi_1)}\right]e^{\theta_i\left[\varphi_1-\varphi_i-(\lambda_1/\beta_1)\right]}}{(\delta_i+r+\varphi_i-\varphi_1)\{1-e^{-\theta_i\left[\delta_i+r-(\lambda_1/\beta_1)\right]}\}} \qquad i=1,\ldots,k.$$

And, differentiating (7.39) with respect to $K_{it}(t)$ and summing over i, we obtain

$$(7.48) \qquad c_1 = \sum_i \frac{c_i\alpha_i(\delta+\varphi_i)\left[1-e^{-\theta_i(\delta_i+r+\varphi_i-\varphi_1)}\right]}{(\delta_i+r+\varphi_i-\varphi_1)\left[1-e^{-\theta_i(\delta_i+\varphi_i)}\right]}$$

Equations (7.47) and (7.48) constitute a system of (k+1) equations in the (k+1) unknowns: $\theta_1,\ldots\ldots\theta_k$, r, It is not possible to solve this system explicitly. If a solution exists, one may calculate the effect on any variable of a change in any parameter; but such calculations are too complicated to reproduce.

We can confirm without difficulty, however, that perhaps the most interesting result of Chapter 4 generalizes to the multisectoral case. The income share of capital is

$$(7.49) \qquad S_K^* = 1 - \frac{w^*(t)L(t)}{Y^*(t)} .$$

In view of (7.38), (7.40), and (7.44), this expression may be written in the more explicit form

$$(7.50) \qquad S_K^* =$$

$$1 - \sum_i \frac{\beta_i c_i(\delta_i+\varphi_i)\left[\delta_i+r-(\lambda_1/\beta_1)\right]\left[1-e^{-\theta_i(\delta_i+r+\varphi_i-\varphi_1)}\right]\left[1-e^{-\theta_i(n+\delta_i)}\right]}{(n+\delta_i)(\delta_i+r+\varphi_i-\varphi_1)\left[1-e^{-\theta_i(\delta_i+\varphi_i)}\right]\left[1-e^{-\theta_i(\delta_i+r-(\lambda_1/\beta_1))}\right]}$$

Comparison of (7.48) and (7.50) reveals that

$$(7.51) \qquad S_K^* = c_1 \quad \text{if and only if} \quad \varphi_1 = r.$$

On the other hand, it is no longer possible to say anything about the relation between thrift and distributive shares. It will be recalled that in the corresponding one-sector model (Chapter 4), an increase in thrift, operating indirectly through the economic life of equipment, resulted in a decline in labour's share. As (7.50) makes clear, however, we must now allow for the additional direct effects of an increase in c_1 and for the effects of the change in the $c_i(i\neq1)$ implicit in the change in thrift. The result of the change is extraordinarily complicated and has defied our attempts to attach a sign to it.

Finally, if depreciation is by "sudden death" there are two familiar possibilities. If in no industry does the economic life of equipment exceed its physical life, the above analysis (with $\delta_i=0$) carries over. Otherwise, with $\theta_i\equiv\bar\theta_i$ wherever necessary, (7.47) and (7.48) are mutually inconsistent and we conclude that steady, fullemployment growth is impossible.

Footnotes to chapter 7.

(1) The offending industry (in which $\theta > \bar{\theta}$) may be quite insignificant - Lerner's peanut industry or Chipman's hairpin industry. In such cases steady growth, while strictly impossible, is yet attainable to a sufficiently close approximation.

Chapter 8

C O N C L U S I O N

What conclusion can one draw from the above chapters which are too often clouded with mathematics?

First of all it seems fairly clear that once the neo-classical aggregative production function is dropped, one must forgo also the simplicity of the neo-classical scheme. Yet, if attention is to be placed only on the full employment paths, very little is gained. Because of this , it is natural to be tempted to infer that one has reached "the point of diminishing returns" and that it is useless therefore to advance a step further. This opinion is in fact shared by many economists but with which I cannot find myself in agreement.

In the first place, there seems to be a confusion between the dim light of the morning with that of the evening - if one may use an analogy. The esoteric literature of growth economics of the last two decades has by no means brought growth economics to an end. For despite all the hard works, what has been achieved is but a reasonable description of reality: it is still a long way to reach an explanation of its formation. I believe, in other words, that in growth economics we are only at the *beginning*, not at the end. Any reasonable assessment of the literature will reaffirm us this state of affairs. Nevertheless this is not entirely discouraging if one is to remember only that the problem in question is of fantastic complexity, it requires hence something more than a high degree of abstraction.

What I tried to advocate is not a general theory of economic growth. The whole exercise is to demonstrate what kind of results one would obtain if emphasis is to be placed more on paths which are off the golden age. For, it is in the non-golden age that we have the hope of having some contact with reality.

This conclusion is therefore not a true ending: it opens a question rather than closing one. There is reason to suggest that there will be hard works to be done before an acceptable theory of growth can be established. Even within the neo-classical framework, one would expect more illuminating results, for example, if general micro economic production functions are employed although I believe that Cobb-Douglas functions can do almost anything that well-behaved functions can. The next step would naturally be to incorporate the wealth effect with the saving behaviour to free the analysis from its short-run Keynesian habits. Then there will also be numerous ways to treat technical progress: Arrow's *learning-by-doing* or Kennedy-Weiszäcker's *choice between inventions*. And finally there remains the majestic step of treating labour as a heterogeneous factor before we can hope to deal with the problem of wage differentials, changes in the distribution of incomes etc. ... Until this is done the legacy we are going to leave for our progeny will be as much an intriguing myth as the one we ourself have inherited.

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