Equations with Dirichlet Boundary Noise

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In 1993, Da Prato and Zabczyk showed that if one considers the heat equation on the interval (0,1) with white noise Dirichlet boundary conditions then a function-valued mild solution cannot be obtained. In this thesis, we revisit this 'Dirichlet boundary noise problem'. First, we return the deterministic situation and extend the approach of Lasiecka and Balakrishan to the Banach setting to provide a point of comparison for the stochastic case. We then construct an equivalent stochastic theory and show that we cannot obtain Lp-valued solutions. Next we extend to higher-dimensions the idea of Alos and Bonaccorsi: function-valued solutions can be obtained on the half-line if considered in an appropriate weighted Lp space. We also show that a dichotomy is obtained: either a process is obtained on the weighted space or boundary values are understood in terms of traces, but not both. We also study the properties of the Dirichlet heat semigroup on weighted spaces in an attempt to build a foundation for the semigroup approach. Next, from the desire to understand the 'Dirichlet map' of random boundary data we move to the unit disk and perform a stochastic extension of some classic harmonic analysis results. This leads us to work with harmonic Hardy spaces and to consider radial and non-tangential convergence towards the boundary, to obtain an interesting representation theorem, and to finally reconnect our results with weighted spaces. Returning to our problem, we conclude that harmonic Hardy spaces are 'too small' to consider the white noise boundary data. Therefore, what is the appropriate space? We consider this question from two angles. First we take a larger space, specifically the Bloch space, and show that it may be an appropriate place to consider these dynamics. We then extend Makarov's LIL theorem to our stochastic case which gives a rate of blow-up near the boundary of our harmonic random field. We conclude that although the random Bloch dynamics are very interesting, we still do not have clear relationship with the boundary data but only an "inside-out" viewpoint. This inspires a "outside-in" approach whereby, due to a simple representation, we start with the white-noise data on the boundary and moving inwards we show rates of blow-up near typical and exceptional points. Finally, we show how these concepts may be extended to more general situations to provide a framework for understanding the local spatial dynamics of SPDE.

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Preface

In this thesis, we study equations with boundary noise with a particular aim to explore new techniques and new questions for equations with Dirichlet boundary noise. Boundary noise arise naturally in physical problems where uncertainty arises on the boundary of a domain but also presents a number of interesting technical challenges. Let $U \subset \mathbb{R}^d$ be a domain with boundary ∂U and ξ a Gaussian random field on $\mathbb{R}_+ \times \partial U$. A prototypical example is the following problem: find a function $u := u(t,x)$ satisfying

$$\partial_t u = \Delta u \text{ on } \mathbb{R}_+ \times U, \quad \tau u = \xi \text{ on } \mathbb{R}_+ \times \partial U. \tag{1}$$

When $\tau u := \partial_\nu u|_{\partial U}$ we obtain the Neumann problem and when $\tau u := u|_{\partial U}$ the Dirichlet problem, in addition, there are four cases for the Gaussian boundary noise ξ that can be considered: ξ is a space-time correlated, ξ is a time white noise but spatially correlated, ξ is a space white noise, or ξ is a space-time white noise.

When $d \geq 2$, existence of "function-valued solutions" to (1), e.g., values in $L^2(U)$, is an open problem in the Dirichlet boundary noise case when the noise ξ is either time white noise or space-time white noise and, even for an elliptic problem, the space white noise case is unresolved.

In this thesis, we partially resolve the space-time noise case using weighted L^p spaces in dimension two and higher. Then, in the elliptic setting, we draw from ideas in harmonic analysis to resolve some structural question for the space white noise case.

This work is structured as follows. In Chapter 1, we provide a comprehensive survey on equations with boundary noise. We take a chronological perspective, which is well suited for the foundational developments in this area; then, as we move to more recent contributions, we classify the literature thematically. Our aim is to summarise, using a homogeneous notation, the main ideas and approaches to the problem of boundary noise in the current literature.

In Chapter 2, we collect a number of known definitions and theorems that we will make use of in the rest of the thesis. This material is necessary for the development in later chapters.

In Chapter 3, we start by considering boundary value problems for deterministic data and extend the abstract Hilbert space approach of Washburn, Balakrishnan and Lasiecka [1, 2, 3] to the Banach space setting. As a Banach space theory for boundary value problems has only been considered in a number of special cases [4, 5], there is a need for a unified theory here. In addition, these results provide both a precursor for Chapter 4 where we consider the stochastic case and a framework for transferring the elliptic results, obtained in Chapter 6 and Chapter 7, to the parabolic setting.

In Chapter 4, we extend the deterministic Banach space theory of Chapter 3 to the stochastic setting. This work grew out of Project 3[1] posed in March 2008 by Ben Goldys for the 11$^{\text{th}}$ TULKA Internet Seminar (ISEM) on 'Stochastic Evolution Equations'. Although a few results on Banach-space valued Stochastic Evolution Equations (SEE) had been obtained in the literature, in this seminar, Van Neerven [6] presented a unified approach using γ-radonifying operators, based on his recent results with Weis and Veraar [7, 8, 9, 10]. That is, a theory

[1] http://fa.its.tudelft.nl/isemwiki/moin.cgi/Phase_2.html

was presented for equations of the form

$$dX(t) = AX(t)\,dt + B\,dW(t), \quad t \in [0,T], \tag{2}$$

where A is a linear operator generating a semigroup $(S(t))_{t\geq 0}$ of bounded linear operators on a Banach space E, B is a bounded linear operator from a Hilbert space H to E, and $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on H. Project 3 aimed to develop an appropriate extension of SEE of the form (2) so that boundary noise problems could be considered in L^p spaces using the semigroup approach. In §4.2, we develop this extension, then in §4.3 and §4.4 we consider the Dirichlet and Neumann boundary noise problems for the heat equation in L^p spaces as examples.

Of course, it has been known since the seminal work of Da Prato and Zabczyk in 1993 [11] that one cannot obtain L^2-valued solutions for the heat equation with Dirichlet white noise boundary conditions even in dimension one. Remarkably, in 2002, Alòs and Bonaccorsi [12] showed that function-valued solutions to the Dirichlet white noise problem could be obtained on the half-line $\mathbb{R}_+$ provided solutions are considered in the space of real-valued functions f such that

$$\int_{\mathbb{R}_+} |f(x)|^p (x^{p-1+\gamma} \wedge 1)\,dx < \infty$$

where $0 < \gamma < 1$ and $p \geq 2$. Their approach is analytic and has not been extended to higher dimensions. Critically, two key questions arise out of their work. Can a similar result be obtained using the semigroup approach? Can one extend this idea to arbitrary domains in $\mathbb{R}^d$?

In Chapter 5, we address and answer these questions by obtaining a number of results which allow us to extend the approach of Chapter 4 to weighted L^p spaces. We consider examples of the Dirichlet boundary noise problem in the elliptic and parabolic settings and show that 'potential operators' which map the boundary data to a solution in the state space E are γ-radonifying when E

is a weighted L^p space of the form

$$E = L^p(U, \mathrm{dist}(x, \partial U)^\alpha), \tag{3}$$

for appropriate choices of the parameter $\alpha \in \mathbb{R}$. Further, we show that the Dirichlet heat semigroup on L^p is Hilbert-Schmidt in weighted spaces of the form (3) when $\alpha < 2$. Finally, we apply these results to the Dirichlet boundary noise problem for the heat equation.

Therefore, using our approach we partially resolve the question of obtaining function-valued solutions to (1) in the space-time white noise setting in the case $d \geq 2$. We claim only partial resolution as these solution are not well-posed: the trace relationship with the boundary data is lost. Well-posedness (in the sense of Hadamard) is a desired property if one wants to construct a stable approximation scheme for numerical solutions. Further, as our approach intertwines the question of existence of solutions and the question of γ-radonification, our approach does not provide a sharper result for the time white noise case. This motivates the results of the second half of this thesis whereby we apply new techniques to the Dirichlet boundary value problem.

In a broader context, weighted function spaces have also been used to handle elliptic equations on domains U whose boundary ∂U is rough (or exhibits various singularities like corners or edges) or to handle elliptic equations with degenerate or singular coefficients [13]. In fact, over the last two decades there has been considerable activity in the study of boundary value problems with minimal assumptions on the coefficients or on the boundary of the domain in question. When studying such problems, it has become apparent that replacing weighted function space techniques with a harmonic analysis approach has proven to be extremely useful [14]. As we are using weighted spaces to obtain existence of solutions to the boundary noise problem, this raises the question: could harmonic analysis techniques be useful for the boundary noise problem?

Consider the canonical example on the unit disk given by

$$\Delta u = 0 \text{ on } \mathbb{D}, \quad u = \xi \text{ on } \mathbb{T}, \tag{4}$$

where ξ is a space white noise on $\mathbb{T}$. Then, even in this simple setting, a $L^2(\mathbb{D})$-valued solution cannot be obtained and one might apply the weighted L^p space theory of Chapter 5 to obtain a solution.

In Chapter 6, we explore a harmonic analysis approach to the elliptic boundary noise problem (4). We develop a theory of “randomized harmonic analysis”. To do this, we explore the concept of Gaussian random variables in the Hardy spaces $\mathscr{H}^p(\mathbb{D})$. We show that an $\mathscr{H}^2(\mathbb{D})$-valued Gaussian random variable can not be obtained if the noise on boundary is spatially white. More promisingly, we show that the Poisson integral is γ-radonifying from $L^2(\mathbb{T})$ to the space of harmonic functions $\mathscr{H}(\mathbb{D})$ endowed with the norm of uniform convergence on compact sets $K \Subset \mathbb{D}$, a result which suggests that it might be possible to obtain solutions in a larger space. Next, under the assumption that the noise on the boundary is not spatially white, we relate these Hardy space results to the results obtained in Chapter 5.

As mentioned above, the Hardy spaces $\mathscr{H}^p$ are too small to allow us to consider white noise on the boundary. A key question is, thus, can we find a larger space that is suitable to handle spatial white noise on the boundary? We consider two approaches to this question: an ‘inside-out’ approach in Chapter 7 and an ‘outside-in’ approach in Chapter 8.

In Chapter 7, we start with a simple example of a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable that exhibits the correct type of ‘white noise behaviour’ as one approaches the boundary $\mathbb{T}$. This suggests that we might be able to find an appropriate space. To explore this idea, we start by studying the mean growth of circle moments (e.g., the variance of $\mathscr{H}(\mathbb{D})$-valued random variable over a circle of radius r). This motivates us to construct the space of random variables, which we call Bloch random variables[2], such that

$$\sup_{z\in\mathbb{D}} \|u'(z)\|_{L^2(\Omega)}(1-|z|^2) < \infty,$$

where $u'(z) = \partial_\theta u(re^{i\theta})$. We show that the norm of Bloch random variables is invariant under conformal transformations and then obtain a random variable extension of Makarov's law of iterated logarithms which yields a rate of blow-up for a Bloch random variable near the boundary $\mathbb{T}$.

Relatively little work has been published on the question of blow-up for a stochastic partial differential equations. Further, even though numerous papers study boundary blow-up for deterministic PDEs, we are unaware of any results on blow-up for PDEs with random noise terms on the boundary, so this is a critical contribution of this thesis.

The development of Chapter 7 suggests an alternative approach. We started with a 'candidate solution' for the elliptic problem

$$\Delta u = 0 \ \text{in}\ \mathbb{D}, \quad u = w\ \text{on}\ \mathbb{T},$$

by taking a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable and working outwards, showing that it had properties that suggested it might provide a solution to the problem. One could call this an 'inside-out' approach. This raises the question: can we start with a white noise on the boundary $\mathbb{T}$ and work our way inwards to characterise the rate of blow-up? That is, can we obtain a 'outside-in' result? Further, where is the blow-up occurring? Do we have blow-up at a point, on a subset of $\mathbb{T}$, or blow-up everywhere? To make matters even more interesting, as the data is random in our situation, this behaviour might change for every path of the stochastic process. In Chapter 8, we address and answer these questions for the unit disk $\mathbb{D}$. Our approach is to construct a Poisson-Wiener integral and relate the rates of blow-up with the fine behaviour of the white noise on the boundary.

[2] Natural random analogue of Bloch functions

Thus far, these results are only available for the unit disk $\mathbb{D}$. A natural question is whether these results could be extended to an arbitrary domain $U \subset \mathbb{R}^d$ with smooth boundary ∂U. In Chapter 9, we consider this question. For an arbitrary smooth domain $U \subset \mathbb{R}^d$, one typically considers the boundary behaviour by using a partition of unity and straightening of the boundary argument to transform the situation to the half-space $\mathbb{R}^d_+$. In this spirit, we consider the blow-up behaviour on the half-space $\mathbb{R}^d_+$ by drawing a connection between the recently-made definition of "thick points" of Gaussian random fields (in particular, the Gaussian free field) and the classic definition of "Lebesgue points". This allows us to draw from ideas in harmonic analysis and propose a new maximal function definition: a maximal thickness function. We then show that the 'ball averaging' operator is γ-radonifying. Finally, as an application of these results we show how we can use it to quantify the blow-up rate of the white-noise Poisson integral in higher-dimensions.

Remark

Just before submitting this thesis, we have become aware of the thesis of Khader [15] where the Poisson equation with Gaussian white noise on the boundary is considered. The main results of Khader's thesis are obtained in Chapter 7 and Chapter 8. In Chapter 7, the Poisson equation with Gaussian white noise on the boundary is considered and in Chapter 7 an extension to the nonlinear setting is performed.

We believe that our results in Chapter 6 complement Khader's results in the following ways: First, we obtain an L^p theory (as opposed to an L^2 theory in [15]). We both show that the solution to the Poisson equation with Gaussian white noise on the boundary is a well-defined Gaussian random variable in $\mathscr{H}(\mathbb{D})$. However, in this thesis we proceed to characterise when $\mathscr{H}^p$-valued random variables can be obtained. In return, Khader shows that the solution to the Poisson equation is a Markov random field in §7.1.3, then considers the Poisson equation for the d-dimensional ball and obtains pointwise estimates for the solution and its derivatives in §7.2 (as opposed to this thesis where we only consider the domain $\mathbb{D}$). Further, Khader considers the nonlinear analogues of his theory in Chapter 8. We finally note that the results used by Khader to prove his results are different to ours as he applies the theory of Steklov eigenfunctions.

However, we believe the existence of the thesis of Khader shows that there seems to be an interest (outside our thesis) in obtaining results for the elliptic problem with boundary noise.

1 Equations with Boundary Noise

Consider an infinitesimally thin piece of string of length ℓ clamped at its end points. A classic problem is to apply a force to this string and study how it vibrates. If $f(t,x)$ is the amount of pressure applied in the direction of the y-axis at time t and horizontal location $x \in [0,\ell]$, then physics tells us that the position $u(t,x)$ of the string solves the one-dimensional wave equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} = \kappa \frac{\partial^2 u(t,x)}{\partial x^2} + f(t,x), \tag{1.1}$$

for $(t,x) \in [0,\infty) \times [0,\ell]$ where κ is a physical constant that depends on the linear mass density and the tension of the string. As the string is clamped, we set the (homogeneous) Dirichlet boundary conditions $u(t,0) = u(t,\ell) = 0$. The field of stochastic partial differential equations addresses the question: what if f is random noise? In such a situation, Walsh [16] physically interpreted (1.1) as a model for a guitar string being struck by particles of sand.

Now consider the following small variation of (1.1) whereby the homogeneous boundary conditions are replaced by the inhomogeneous Dirichlet boundary conditions,

$$u(t,0) = g(t), \quad u(t,\ell) = h(t), \tag{1.2}$$

where $g(t)$ and $h(t)$ model the position of the string at time t at its end points. Again, it becomes natural to consider the stochastic analogue of the model (1.1)-(1.2) whereby g and h are replaced by random noise. Physically, we suggest that one could interpret this as a model for the position of a piece of string being shaken at its end points. Of course, one can imagine infinitely many variations on this theme: replace (1.1) by any partial differential equation (PDE) or stochastic partial differential equation (SPDE) and (1.2) with a large range of possible types of inhomogeneous random boundary conditions (Neumann, Robin, etc.). As such, we shall call this class of problems **equations with boundary noise**.

Stochastic partial differential equations arise naturally as models for dynamical systems[1] subject to random influences. Sometimes the noise affects a complex system not only inside the physical medium but also at the physical boundary. This occurs in a variety of difficult problems: air-sea interactions on the ocean surface, heat transfer in a solid in contact with a fluid, chemical reactor theory, and colloid and interface chemistry. In fact, we argue that randomness at the boundary of an object is even more natural than noise on the interior, especially in higher dimensions. Intuitively, one could draw an analogy with the divergence theorem which relates the flow of a vector field through a surface to the behaviour of the vector field inside the surface. Finally, as we shall show, not only are boundary noise problems of practical interest but they also present us with an fascinating setting where a number of areas of mathematics intersect.

In this chapter we survey the current literature on equations with boundary noise. We take a chronological perspective, which is well suited for the foundational developments in this area; then, as we move to more recent contributions, we shall attempt to classify the literature thematically. Our aim is to sketch, using a homogeneous notation, what we believe to be the main ideas and approaches to the problem of boundary noise. We have tried to not enter

[1] modelled by partial differential equations

into too much detail and refer the reader to the original publications if further details or clarifications are required.

Notation

We use the following notation. The relation $a \lesssim b$ means that a is bounded by some constant times b uniformly in all parameters on which a and b may depend. We write $a \sim b$ to mean that $a \lesssim b$ and $b \lesssim a$ holds. $L^p(U)$, $1 \le p \le \infty$, are the usual Lebesgue spaces on a domain $U \subset \mathbb{R}^d$ and $W^{k,p} \subset L^p(U)$, for $k \in \mathbb{N}$ are the Sobolev spaces of functions whose weak derivatives up to order k are bounded in $L^p(U)$. We express derivatives in a number of ways. First, the partial derivatives are expressed as $\partial u/\partial t$ or ∂_t and $\Delta u := \sum_{i=1}^d \partial_{x_i}^2 u$. We also use the notation $\dot{u}$ to be the formal time derivative of u with respect to the time variable t. This is especially relevant when we talk about white noise and Wiener processes (which are, almost surely, not differentiable). In that case, one should interpret an equation by formally multiplying by dt on both sides to obtain a stochastic differential notation. For example, if $(W(t))_{t\ge0}$ is a Wiener process then $\dot{W}(t)\,dt$ is interpreted as $dW(t)$. Sometimes, we write X_t for $X(t)$ to make the notation simpler. For Hilbert spaces H_1 and H_2, we denote by $\mathscr{L}(H_1, H_2)$ the space of bounded linear operators from H_1 into H_2 and $\mathscr{L}_2(H_1, H_2)$ the subspace of Hilbert-Schmidt operators.

1.1 Semigroup approach to boundary control

Between 1976 and 1984, it emerged that parabolic equations with boundary control could be described by a semigroup model [2, 17, 3] and thus problems in optimal control such as quadratic control, stabilizability, and boundary control where studied extensively in this framework [17, 18]. This was an interesting development, as although evolution equations of the form $u'(t) = A(t)u(t) + B(t)g(t)$ in Banach spaces had been studied for a while, the theory was only sufficient to handle the case of distributed parameter systems.

Therefore, the main achievement to that time was the development of an appropriate abstract Hilbert space theory to handle partial differential equations for which the control is applied at the boundary. To be precise, let E and H be Hilbert spaces and consider the simpler autonomous evolution equation

$$u'(t) = Au(t) + Bg(t), \quad u(0) = u_0 \in \mathscr{D}(A) \subset E. \tag{1.3}$$

We shall call $u : [0, T] \to E$ the **state** and $g : [0, T] \to H$ the **control** or **data**. If A is an infinitesimal generator of a strongly continuous semigroup $((S(t))_{t\geq 0}$ on E then one way to interpret a solution of (1.3) is to formally treat it as an ordinary differential equation (albeit infinite-dimensional) then setting $S(t) = e^{tA}$ one obtains the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)Bg(s)\,ds. \tag{1.4}$$

If $B \in \mathscr{L}(H, E)$ and $g \in L^2(0, T; H)$ then such a solution is known as a "mild solution" [17]. However, this situation is insufficient to handle the case of boundary controls and an extension to case where B is an unbounded operator from H to E was required. It was noticed that in this situation, sufficient conditions to obtain a mild solution are

$$\overline{\mathscr{D}(B)} = H,$$

$$\|S(t)Bg(t)\|_E \leq \frac{C}{t^\alpha}\|g(t)\|_H,$$

for $g \in \mathscr{D}(B)$, $t < T$, $\alpha < 1/2$ and some constant $C = C(\alpha) > 0$. Under these conditions, this abstract framework allows consideration of a large class of parabolic and hyperbolic partial differential equations on a manifold M whereby controls or data are limited to regions of the boundary or a submanifold $N \subset M$ of lower dimension [17, 18]. Further, all standard boundary conditions can be handled (Dirichlet, Neumann, Robin, etc.) and even feedback loops (i.e. dynamical conditions) can be incorporated.

1.2 Semigroup approach to boundary noise

Due to the close relationship between optimal control theory and stochastic evolution equations (e.g., see conditions (9.50) and (9.51) in [19]), the first papers on equations with boundary noise considered in a semigroup framework appeared shortly after, see [20, 21, 11]. Curtain [20] obtained a stochastic version of his optimal control theory [17] using the Green formula and the duality between control and observation. Ichikawa [21] proposed a semigroup model for parabolic equations with finite-dimensional boundary noise or pointwise noise and obtained existence, uniqueness, and regularity results. In 1993, Da Prato and Zabczyk [11] extended these results to the case of infinite-dimensional boundary noise (i.e., the white noise case). In particular, they studied the nonlinear evolution equation with white-noise boundary conditions on a Hilbert space H given by

$$\begin{cases} X'(t) = \underline{A}X(t) + F(X(t)), & t \in [0,T], \\ \tau X(t) = \dot{W}(t), & t \in (0,T], \\ X(0) = 0. \end{cases} \tag{1.5}$$

where $\underline{A} : \mathscr{D}(\underline{A}) \subset H \to H$, $F : \mathscr{D}(F) \subset H \to H$ is a nonlinear operator, and $\dot{W}$ represents a white noise process on $L^2(0,T;\partial H)$, i.e. a formal time derivative of a cylindrical Wiener process $(W(t))_{t\geq 0}$ taking values on ∂H. We recall that, as opposed to a Wiener process, a **cylindrical Wiener process** is a 'true' infinite-dimensional stochastic process (see §2.9). Finally, $\tau : \mathscr{D}(\tau) \subset H \to \partial H$ models the boundary condition and the relationship between the Hilbert spaces H and ∂H. For example, this abstract formulation may be used to study dynamics on a bounded domain $U \subset \mathbb{R}^d$ with C^∞ boundary ∂U by posing $H = L^2(U)$ and $\partial H = L^2(\partial U)$. To set Dirichlet boundary conditions one would choose $\tau u = u|_{\partial U}$ (in terms of trace) for $u \in L^2(U)$ and for Neumann boundary conditions one would choose $\tau u = (\partial_\nu u)|_{\partial U}$ for $u \in L^2(U)$. Finally, setting $\underline{A}u = \Delta u$ for $u \in \mathscr{D}(\underline{A})$ where $\mathscr{D}(\underline{A})$ is the Sobolev space $W^{2,2}(U)$, then (1.5) is the abstract formulation of a nonlinear heat equation with white-noise boundary conditions.

Da Prato and Zabczyk showed that if one defines the operator

$$Au := \underline{A}u, \quad \mathscr{D}(A) := \{u : \underline{A}u \in H, \tau u = 0\} \tag{1.6}$$

and assumes that A is the infinitesimal generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on H, then a continuous adapted H-valued process $(X(t))_{t\geq 0}$ is a mild solution for (1.5) if it satisfies the stochastic integral equation

$$X(t) = \int_0^t S(t-s)F(X(s))\,ds + (\lambda - A)\int_0^t S(t-s)\Lambda_\lambda\,dW(s), \tag{1.7}$$

where $\Lambda_\lambda : \partial H \to H$ is the **Dirichlet map** associated with $\underline{A}$ and τ. That is, assume that the stationary boundary value problem

$$(\lambda - \underline{A})u = 0, \quad \tau u = g \tag{1.8}$$

has a unique solution $\Lambda_\lambda g := u \in \mathscr{D}(\underline{A})$ for arbitrary $g \in \partial H$ (see §2.7). They suggest that due to the form of (1.7), one can formally view (1.5) as the stochastic evolution equation

$$dX(t) = [AX(t) + F(t, X(t))]\,dt + (\lambda - A)\Lambda_\lambda\,dW(s). \tag{1.9}$$

When $F = 0$, they show that a sufficient condition for existence of a mild solution to (1.9) is

$$\int_0^T \|AS(t)\Lambda_\lambda\|^2_{\mathscr{L}_2(\partial H, H)}\,dt < \infty, \tag{1.10}$$

where $\mathscr{L}_2(\partial H, H)$ is the space of Hilbert-Schmidt operators between ∂H and H. Under condition (1.10) and a Lipschitz assumption on F they show that a unique mild solution of (1.5) exists in H.

1.3 Dirichlet boundary noise: a negative result

In [11], Da Prato and Zabczyk illustrate their results by studying the one-dimensional heat equation

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x), \quad (t,x) \in [0,T] \times (0,1), \tag{1.11}$$

with $u(0,x) \equiv 0$. Let $\dot{W}_0$ and $\dot{W}_1$ be the (formal) time derivatives of the independent Wiener processes $(W_0(t))_{t\geq 0}$ and $(W_1(t))_{t\geq 0}$. The processes $\dot{W}_0$ and $\dot{W}_1$ model time white noise on the boundary of $(0,1)$. By explicit calculation, Da Prato and Zabczyk show that in the case of Neumann boundary conditions

$$\frac{\partial u}{\partial x}(t,x)\Big|_{x=0} = \dot{W}_0(t), \quad \frac{\partial u}{\partial x}(t,x)\Big|_{x=1} = \dot{W}_1(t), \tag{1.12}$$

then a $L^2(0,1)$-valued solution can be obtained. However, when the Dirichlet boundary conditions

$$u(t,0) = \dot{W}_0(t), \quad u(t,1) = \dot{W}_1(t),$$

are imposed then (1.11) does not have a solution in $H = L^2(0,1)$ but only in a larger space, i.e., one cannot obtain a function-valued solution using this approach. This is somewhat surprising for two reasons: first, this is not the case in the deterministic situation (i.e., (1.3) and (1.4)) and second, the one-dimensional stochastic heat equation with zero Dirichlet boundary conditions

$$\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + \dot{W}(t), \quad (t,x) \in [0,T] \times (0,1), \tag{1.13}$$

where $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on $L^2(0,1)$ does have a $L^2(0,1)$-valued solution owing to the regularity provided by the Dirichlet heat semigroup [19, Example 5.7].

1.4 Analytic approach to boundary noise

Independently, during 1992-1993, three new papers appeared on the topic of boundary noise [22, 23, 24]. These works used a different approach and were motivated by the study of dynamical systems subject to the influence of small random perturbations [25] which is typically concerned with the study of $\mathbb{R}^d$-valued stochastic differential equations (SDE) of the form

$$dX_t^\varepsilon = b(X_t^\varepsilon) + \varepsilon\sigma(X_t^\varepsilon)\,dw_t, \quad X_0^\varepsilon = x, \tag{1.14}$$

where b is a suitably smooth $\mathbb{R}^d$-valued function defined on a domain $U \subset \mathbb{R}^d$, σ is a $d \times d$ matrix-valued function, w_t is a d-dimensional Wiener process and ε is a small real parameter. Due to the close relationship between probability and analysis, one could also view this area as the study of the differential operator $\mathscr{A}^\varepsilon$ given by

$$(\mathscr{A}^\varepsilon u)(x) := \frac{\varepsilon^2}{2}\sum_{j,k=1}^{d} a_{jk}(x)\frac{\partial^2 u}{\partial x_j \partial x_k}(x) + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j}(x), \tag{1.15}$$

for $x \in U$ and sufficiently smooth u, where $a(x) := \sigma(x)\sigma^*(x)$. A variety of different questions then arise naturally about the behaviour of (1.14) when $\varepsilon \to 0$. For example: Do solutions of (1.14) approach, in a suitable sense, solutions of the Cauchy problem $u'(t) = b(u(t))$ with $u(0) = x$? Can one estimate the first exit time of solutions to (1.14) from a given domain in $\mathbb{R}^d$? Can one estimate, as $\varepsilon \to 0$, the principal eigenvalue of (1.15) with vanishing Dirichlet boundary data?

It was therefore natural, in 1992, for Freidlin and Wentzell to further explore these ideas and consider semilinear PDE with **fast oscillating boundary conditions** [22]. In other words, perturbations of the boundary conditions were examined in the following sense. For $t > 0$ and $x \in [-1, 1]$, consider the semilinear heat equations

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + f(x,u), \quad \frac{\partial u}{\partial x}(t,x)\Big|_{x=\pm 1} = 0 \tag{1.16}$$

and for a small real parameter ε,

$$\frac{\partial u^\varepsilon}{\partial t}(t,x) = \frac{1}{2}\Delta u^\varepsilon(t,x) + f(x,u^\varepsilon), \quad \frac{\partial u^\varepsilon}{\partial x}(t,\pm 1) = \dot{W}_\pm\left(\frac{t}{\varepsilon}\right) \tag{1.17}$$

where $(W_+(t))_{t\geq 0}$ and $(W_-(t))_{t\geq 0}$ are correlated $\mathbb{R}$-valued Wiener processes. For (1.16) and (1.17), we assume $u(0,x)=u_0(x)\in C([0,1])$. The main theorem of [22], is that for any $T>0$ and $\delta>0$,

$$\lim_{\varepsilon\downarrow 0}\mathbb{P}\left\{\sup_{\substack{0\leq t\leq T\\ |x|\leq 1}}|u^\varepsilon(t,x)-u(t,x)|>\delta\right\}=0, \tag{1.18}$$

and one should note, as opposed to the study of (1.14), that this is now an infinite-dimensional problem. Using purely analytic derivations (i.e., without relying on semigroup results), they then derived large deviation results and presented some interesting examples.

In 1992, Freidlin and Sowers [23] studied the nonlinear stochastic heat equation on the unit disk $\mathbb{D}=\{x\in\mathbb{R}^2:|x|<1\}$ with Neumann boundary conditions given on $\mathbb{T}=\{x\in\mathbb{R}^2:|x|=1\}$ by

$$\frac{\partial u^\varepsilon}{\partial t}(t,x)=\Delta u^\varepsilon(t,x)+f(x,u^\varepsilon),\quad \frac{\partial u^\varepsilon}{\partial x}(t,x)\bigg|_{\mathbb{T}}=\dot{W}^\varepsilon(t), \tag{1.19}$$

where $(W^\varepsilon(t))_{t\geq 0}$ is process given on $\mathbb{T}$ by

$$W^\varepsilon(t)(x)=\sum_{n=0}^{\infty}\lambda_n h_n(x)\frac{1}{\varepsilon}w_n\left(\frac{t}{\varepsilon^2}\right),\quad \text{s.t.}\sum_{n=0}^{\infty}|\lambda_n|<\infty, \tag{1.20}$$

and where $(h_n)_{n=0}^\infty$ is an orthonormal basis of $L^2(\mathbb{T})$ and $(w_n(t))_{t\geq 0}$ are $\mathbb{R}$-valued Wiener processes. Or in more modern language,

$$W^\varepsilon(t)=\frac{1}{\varepsilon}QW\left(\frac{t}{\varepsilon^2}\right)$$

where $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on a Hilbert space $L^2(\mathbb{T})$ and Q is the operator on $L^2(\mathbb{T})$, given by $Qh_n:=\lambda_n h_n$ and $\operatorname{Tr}Q<\infty$ by (1.20). They obtain an estimate of the form (1.18) and a central limit result for (1.19).

Two years later in [24], Sowers extends the ideas of Freidlin and Wentzell and considers the semilinear parabolic equation with white noise boundary perturbations given for $t>0$ and $x\in M$ by

$$\frac{\partial u}{\partial t}=\frac{1}{2}\Delta u+(b,\Delta u)+cu+f(x,u),\quad (\nu,\nabla u)+\beta(x)u|_{\mathbb{R}_+\times\partial M}=\sigma(x)\dot{w}, \tag{1.21}$$

where $M \subset \mathbb{R}^d$ is a Riemannian manifold with smooth boundary ∂M, $\dot{w}$ is a space-time white noise on $\mathbb{R}_+ \times \partial M$, and ν is the inward-pointing normal vector field on ∂M. Here, $(\cdot,\cdot)$ denotes the Riemannian metric tensor, ∇ is the gradient operator defined by $(\cdot,\cdot)$, and $\Delta := \operatorname{div}\nabla$ is the Laplace-Beltrami operator. It is assumed that b is a C^∞ vector field on M, c and β are some C^∞ functions on M and ∂M, respectively. The initial condition for (1.21) is given by $u(0,x) = u_0 \in C(M)$. Again using analytic methods, existence and uniqueness of a solution to (1.21) follow. Although Sowers claims that his work is "essentially an extension of the efforts of Freidlin and Wentzell", several novel ideas are presented in his work. In particular, (1.21) is studied in the following way. Let M° denote the interior of M and define the second-order differential operator

$$\mathscr{A}\varphi := \frac{1}{2}\Delta\varphi + (b,\nabla\varphi) + c\,\varphi, \quad \varphi \in C^\infty(M),$$

and the first-order differential operator

$$\mathscr{B}\varphi := (\nu,\nabla\varphi) + \beta\,\varphi, \quad \varphi \in C^\infty(M).$$

First, Sowers shows the existence of a unique **Robin kernel** R: for each $y \in M$, R_y is a C^∞ function of (t,x) in $\mathbb{R}_+ \times M^\circ \setminus \{(0,y)\}$ that satisfies

$$\frac{\partial R_y}{\partial t} = \mathscr{A} R_y, \quad \lim_{t\downarrow 0} R_y(t,\cdot) = \delta_y, \quad \mathscr{B} R_y\big|_{\mathbb{R}_+\times\partial M} = 0. \tag{1.22}$$

Using R, he shows that the linear version of (1.21), namely,

$$\frac{\partial u}{\partial t} = \mathscr{A} u, \quad u(0,\cdot) = 0, \quad \mathscr{B} u\big|_{\mathbb{R}_+\times\partial M} = \sigma(x)\dot{w}, \tag{1.23}$$

has a unique solution given by the stochastic integral

$$u(t,x) = -\frac{1}{2}\int_0^t \int_{\partial M} R_y(t-s,x)\sigma(y)w(ds,dy), \quad (t,x) \in \mathbb{R}_+ \times M^\circ. \tag{1.24}$$

This result is achieved by using a number of intricate kernel estimates for R that he carefully justifies and concludes that, under assumptions on the nonlinear potential f, a transformation to (1.24) gives the solution of (1.21). We note that Sowers definition of a solution is slightly non-standard. Let the distance function on M defined by $(\cdot,\cdot)$ be denoted by $d(\cdot,\cdot)$ and for any subset S of M and any point $x \in M$, we write $\operatorname{dist}(x,S) := \inf_{y\in S} d(x,y)$. To make sense of (1.24), one must shift the boundary in some orderly manner into M°. Specifically, for each $\varepsilon > 0$ we shrink M to

$$M_{\varepsilon} := \{x \in M : \operatorname{dist}(x,\partial M) > \varepsilon\},$$

and then replace M by M_{ε} and ∂M by ∂M_{ε}. For every $\varepsilon > 0$, let $u^{\varepsilon}(t,x)$ be the stochastic integral (1.24) when $(t,x) \in \mathbb{R}_{+} \times M_{\varepsilon}$ and $u^{\varepsilon}(t,x) = 0$ when $x \notin M_{\varepsilon}$. It is shown that for every $\varepsilon > 0$, u^{ε} is well-defined and a solution to (1.23) is defined as a (weak) limit of u^{ε} as $\varepsilon \to 0$. This fact follows from his insight that there exists a boundary layer degeneracy: if $M = \mathbb{R}^{d}_{+}$ and $\partial M = \mathbb{R}^{d-1}$ then asymptotically near the boundary

$$\mathbb{E}|u(t,(\overline{x},x_d))|^2 \sim \frac{1}{x_d^{d-1}},$$

where $\overline{x} \in \mathbb{R}^{d-1}$, $x_d > 0$, and $t > 0$. Or more precisely, he shows that for each $T > 0$ and $\gamma > (d-1)/2$,

$$\limsup_{\substack{x\to\partial M\\ 0\le t\le T}} \operatorname{dist}(x,\partial M)^{\gamma}|u(t,x)| = 0, \quad \mathbb{P}\text{-a.s.} \tag{1.25}$$

We conclude our discussion of Sowers results with a few remarks. First, due to his assumption on σ, it appears that the noise on the boundary is not space-time white noise but only time white noise as σ appears to be a Hilbert-Schmidt kernel (i.e., σ has a γ-radonifying effect [26]). Secondly, the nonlinearity in (1.21) is more complicated than in (1.5) and due to (1.25), in addition to (standard) Lipschitz assumptions of f, one must also impose a growth condition on f: for all $x \in M^{\circ}$ and $z \in \mathbb{R}$ such that $|z| > 1$,

$$|f(x,z)| \lesssim \operatorname{dist}(x,\partial M)^{\eta_1}|z|^{\eta_2},$$

where $\eta_1 - \eta_2\gamma > -1$ is satisfied. Third, due to the shrinking of the domain used in his methods, such results seem somewhat difficult to obtain using an operator theoretical or semigroup approach.

1.5 Stochastic evolution equations with boundary noise and ergodicity

In 1995, Maslowski [27] used the semigroup approach to study stochastic nonlinear boundary value problems with boundary or pointwise noise on a bounded domain $U \subset \mathbb{R}^d$. Let H and ∂H be Hilbert spaces. As we have already seen (e.g., (1.5) and (1.7)), it should come as no surprise that such problems may be treated abstractly in the framework of semilinear stochastic evolution equations of the form

$$\begin{aligned} dX(t) = {} & [AX(t) + f(X(t)) + \Pi h(X(t))]\, dt \\ & + g(X(t))dW_1(t) + \Pi k(X(t))\, dW_2(t), \end{aligned} \tag{1.26}$$

with $X(0) = x \in H$, where $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ are independent cylindrical Wiener processes on H and ∂H, respectively. The operators $A : H \to H$ and $\Pi : \partial H \to H$ are unbounded linear operators, while $f : H \to H$, $h : H \to \partial H$, $g : H \to \mathscr{L}(H)$, and $k : H \to \mathscr{L}(\partial H)$ are Lipschitz continuous. In essence, this is a slight generalisation of (1.5) and the representation (1.7), however after proving existence and uniqueness of solutions, Maslowski goes further and obtains some results on the asymptotic behaviour of solutions such as exponential stability in mean and the existence and uniqueness of an invariant measure. A number of examples are also presented. For example, it is also shown that (1.26) can also be used to study stochastic plate equations with structural damping. Although it is claimed in [27, Example 3.1] that the Dirichlet boundary problem can be considered, a Neumann condition is in fact assumed during calculations. Therefore we observe that again, similar to

(1.5), one still cannot obtain solutions in H when Dirichlet boundary noise is imposed.

1.6 Markovian dynamical systems

In 1996, Da Prato and Zabczyk dedicate a chapter of their monograph [28] to **systems perturbed through the boundary**. They recall their results from [19], extend the Neumann boundary condition example (1.11)-(1.12) to the domain $U = (0,\pi)^d \subset \mathbb{R}^d$ to show that for $d = 1,2,3$ solutions can be obtained, and conclude with some remarks on ergodicity of such systems. Further, as Neumann boundary conditions are a special case of Robin boundary conditions, we observe that the restriction $d = 1,2,3$ makes sense in terms of (1.25).

We should also remark that an important observation was made in [28] and [27]. Under appropriate assumptions on A and F in (1.7) then the H-valued stochastic process $(X(t))_{t\geq 0}$ is Markovian [19, Proposition 13.2.3]. Further, if $(X^x(t))_{t\geq 0}$ is the solution of (1.26) with $X(0) = x \in H$, then by [27, Proposition 1.3], $(X^x(t))_{t\geq 0}$ defines a H-valued homogeneous Feller Markov process with the transition probability function

$$P(t,y,A) = \mathbb{P}\{X^x(t) \in A\}, \quad A \in \mathscr{B}(H).$$

This fact is a strong motivation for studying white-noise boundary conditions as the Markov property is highly desirable from both a theoretical and applied point of view.

1.7 Optimal control of stochastic systems

Extending his earlier work [27], Maslowski in collaboration with Duncan and Pasik-Duncan, study boundary and point control of semilinear stochastic evolution equations [29]. In the case where the control and noise are distributed, the existence of an optimal control was proved in [30]. In the framework of

[29], let $H = L^2(0,\pi)$, $\mathscr{K} \subset \mathbb{R}^{k_1}$ and $\mathscr{C} \subset \mathbb{R}^{k_2}$ compactly with $k_1, k_2 \in \mathbb{N}$, then one may consider the one-dimensional stochastic heat equation with Neumann boundary conditions on $(0,\pi)$

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(u(t,x)) + \dot{W}(t) \quad \text{on } [t_0, T] \times (0,\pi) \\ \frac{\partial u}{\partial x}(t,0) = h_1(\alpha, u(t,\cdot), g(u(t,\cdot))) + \dot{w}_1(t), \\ \frac{\partial u}{\partial x}(t,\pi) = h_2(\alpha, u(t,\cdot), g(u(t,\cdot))) + \dot{w}_2(t), \end{aligned} \tag{1.27}$$

where $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on H, $(w_1(t))_{t\geq 0}$ and $(w_2(t))_{t\geq 0}$ are independent $\mathbb{R}$-valued Wiener processes, $h_i : \mathscr{C} \times H \times \mathbb{R} \to \mathbb{R}$, $\alpha \in \mathscr{C}$ represents a parameter, u is the unknown $\mathbb{R}$-valued process representing the state of the system, and the control is given by the $\mathbb{R}$-valued process g. We note that in our example, we do not need a Hilbert-Schmidt operator in front of the cylindrical process W as this is a one-dimensional situation, see (1.13).

The control problem considered in [29] is to minimize the ergodic cost functional

$$J(x, g, \alpha) := \limsup_{T\to\infty} \mathbb{E}\frac{1}{T}\int_0^T c(u(t), g(u(t)))\, dt$$

over the set of Markov controls $\mathscr{G} = \{g : H \to \mathscr{K}, g \text{ Borel measurable}\}$ where $c : H \times \mathscr{K} \to \mathbb{R}$. In contrast to [30], where the control is of distributed parameter type on H, the control and the noise act on the boundary of $(0,\pi)$ in (1.27). We note that boundary controls are more natural and realistic since, in practice, distributed parameter controls are hard to implement.

We conclude that the main development here is that the ideas of optimal control for (1.3) and stochastic evolution equations of the form (1.26) are combined, i.e., these are results on **optimal control of stochastic boundary systems**. We shall return to this thematic development later, but we continue chronologically here.

1.8 Dirichlet boundary noise: a conceptual breakthrough

Although the paper [24] is cited in [27, 28, 29], up to this time there seemed to be no attempt to interpret Sowers results, in particular the boundary layer degeneracy (1.25), from a semigroup perspective. Perhaps [24] was simply viewed (see e.g., [27]) as a "thorough analysis of the multi-dimensional Neumann problem"? In addition, due to the negative result obtained for the Dirichlet boundary noise case in [11], there had been no further attempts to study this difficult situation since 1993.

However in 2002, Alós and Bonaccorsi [12] made a number of insightful observations with regard to [24] and the Dirichlet boundary noise problem. First they noted that the qualitative arguments of Sowers for Robin boundary noise, such as the boundary layer degeneracy, could also be performed for the Dirichlet boundary noise case on the one-dimensional domain $U=\mathbb{R}_+$ with boundary $\partial U=\{0\}$. Second, they observed that this qualitative information could then be incorporated into the choice of function space where the process will take values. That is, one should replace the Hilbert space $H=L^2(0,\infty)$ by the weighted L^p-space of all functions $f:\mathbb{R}_+\to\mathbb{R}$ such that

$$\int_0^\infty |f(x)|(x^{p-1+\gamma}\wedge 1)\,dx<\infty, \tag{1.28}$$

for some choice of $0<\gamma<1$ and $p\geq 2$. Moreover, they continued to examine the problem, as their aim was to study the stochastic semilinear heat equation with white-noise boundary condition given for $x\in\mathbb{R}_+$ by

$$du_t=\partial_x^2 u_t\,dt+\sum_{j=1}^n[b_j(x)\partial_x u_t+F_j(t,x,u_t)]\,dw_j(t),\quad u(t,0)=\dot{w}_0(t), \tag{1.29}$$

where for $j=0,1,\ldots,n$, $(w_j(t))_{t\geq 0}$ are independent $\mathbb{R}$-valued Wiener processes, $b_j:\mathbb{R}_+\to\mathbb{R}$ are $C_b^3(\mathbb{R}_+)$-functions, $F_j(t,x,\cdot)$ are Lipschitz continuous uniformly in (t,x), using the short-hand notation $u_t:=u(t,x)$. $C_b^3(\mathbb{R}_+)$ is the space of thrice differentiable, continuous, and bounded functions on $\mathbb{R}_+$. We observe that the structure of (1.29) is more complicated than that of (1.21) as there is noise present on the interior of the domain and that the results of [24] rely fundamentally on obtaining a kernel (i.e. fundamental solution) of the homogeneous interior dynamics, e.g., (1.22) and deriving kernel estimates. Therefore, to use this approach for (1.29), Alós and Bonaccorsi used Malliavin calculus to obtain a **stochastic heat kernel** $p_D(s,t,y,x)$ for the stochastic heat equation with zero Dirichlet boundary conditions

$$du_t = \partial_x^2 u_t\,dt + \sum_{j=1}^n b_j(x)\partial_x u_t\,dw_j(t), \quad u(t,0)=0,$$

so that the solution of (1.29) yielded

$$\begin{aligned} u(t,x) = &\int_0^t \frac{\partial p_D}{\partial y}(s,t,0,x)\,dw_0(s) \\ &+ \sum_{j=1}^n \int_0^t \left(\int_{\mathbb{R}_+} p_D(s,t,y,x) F_j(s,y,u(s,y))\,dy \right) dw_j(s). \end{aligned}$$

Kernel estimates for $p_D(s,t,y,x)$ are then used to prove existence and uniqueness of solutions in the weighted L^p space given by (1.28). Then they show that $u(t,\cdot)$ is continuous on $[\delta,\infty)$ for every $\delta>0$ and

$$x^{1+\alpha}u(t,x) \to 0, \quad \mathbb{P}\text{-a.s.},$$

for every $\alpha>0$. In similar fashion to Sowers, they also define a concept of weak solution whereby the solution u is understood as a (weak) limit, as $\varepsilon\to 0$, of a sequence of solutions u^ε defined on the shrunk domains $[\varepsilon,\infty)$. Continuing their results in [31], Alós and Bonaccorsi proceed to study the asymptotic behaviour of the solutions to (1.29) and prove that they have a unique invariant measure that is exponentially mean-square stable.

We note that these results are fundamentally analytic in nature and cannot be placed in the framework of (1.5) or (1.26) for the following reasons. First, to obtain an optimal theory, the process must take values in a weighted L^p

space but the abstract semigroup approach typically used to model SPDE is a Hilbert space theory, and we would require a Banach space theory for stochastic evolution equations. As noted in the recent thesis of Veraar [32]: "The main problem for this is to find a 'good' stochastic integration theory for processes with values in a Banach space. In the 70's and 80's, several authors found negative results in this direction, and it turned out that the stochastic integration theory for Hilbert spaces does not extend to the Banach space setting". Second, in a semigroup approach, one relies on a number of assumptions such as analyticity of the C_0-semigroup $(S(t))_{t\geq 0}$ generated by the unbounded operator A on the Hilbert space H. Leaving Banach space results aside, if we simply take H to be a weighted L^2-space then to formulate the theory, we must show that $(S(t))_{t\geq 0}$ is analytic on the weighted L^2-space. Further, a good characterisations[2] of the fractional spaces $\mathscr{D}((-A)^{\alpha})\subset H$ and estimates for the Dirichlet map $\Lambda:\partial H\to H$ are also required. Unfortunately, these types of results do not seem to exist in the literature.

1.9 Hyperbolic equations with boundary noise

Although hyperbolic SPDEs can be handled within the semigroup framework of [11] or [27] by writing them as a system of two first order equations, then considering H as the tensor product $H_1\otimes H_2$ for appropriate choices of Hilbert spaces H_1 and H_2 (e.g., see [33, Chapter 13] or [6, Section 15.2]), in this section we review a number of specialised papers on hyperbolic equations with boundary noise.

In 1993, Mao and Markus [34] investigated the stochastic vibrations of a flexible string excited by a boundary force of white noise type. That is, the one-dimensional wave equation (1.1) with $f=0$ and boundary values

$$u(t,0)=0,\quad \frac{\partial u}{\partial x}(t,\ell)=\dot{w}(t) \tag{1.30}$$

[2]Possibly in terms of **weighted** Sobolev-Slobodeckij spaces.

where $\dot{w}$ is the formal time derivative of a $\mathbb{R}$-valued Wiener process $(w(t))_{t\geq 0}$. They also consider the case

$$u(t,0)=0, \quad \frac{\partial u}{\partial t}(t,\ell)=\dot{w}(t) \tag{1.31}$$

so that $u(t,\ell)=w(t)$. They start by showing existence, uniqueness and regularity theorems for the problems (1.1)-(1.30) and (1.1)-(1.31) with $f=0$. They then show that the amplitude $\|u(t,\cdot)\|:=\max_{0\leq x\leq \ell}|u(t,x)|$, for both types of boundary conditions, satisfies the asymptotic estimates

$$\frac{\pi}{2\sqrt{2}} \leq \liminf_{n\to\infty} \frac{\|u(n,\cdot)\|}{\sqrt{n/\log n}} \quad \mathbb{P}\text{-a.s.}, \qquad \limsup_{n\to\infty} \frac{\|u(n,\cdot)\|}{\sqrt{n\log n}} \leq \sqrt{2} \quad \mathbb{P}\text{-a.s.}$$

We note that this paper was received by the journal in 1991, hence these results could in fact be considered as some of the first on the topic of boundary noise.

In 2001, Lévêque completed his thesis on hyperbolic stochastic partial differential equations driven by boundary noise. Lévêque answered a fundamentally different question to the papers [34, 11, 27]. As the wave equation driven by a space-time Gaussian white noise admits a solution that takes its value in a space of distributions when the spatial dimension is greater than one, the aim of his thesis was to understand what assumptions need to be placed on the spatial correlation structure of the boundary noise if the solution is to be function-valued. Using Fourier analytic techniques, a number of results were obtained such as necessary and sufficient conditions on the spatial correlations of the noise for the existence of a square integrable solution to the linear hyperbolic SPDE

$$\frac{\partial^2 u}{\partial t^2}+2a\frac{\partial u}{\partial t}+bu-\Delta u=\dot{w}, \quad \left.\frac{\partial u}{\partial \nu}\right|_{\mathbb{R}_+\times\partial U}=0, \tag{1.32}$$

on a domain $U\subset\mathbb{R}^d$ with initial condition $u(0,x)=u_0(x)$ and $\partial_t u(0,x)=v_0(x)$ for $x\in U$. The noise $\dot{w}$ has covariance given by

$$\mathbb{E}\dot{w}(t,x)\dot{w}(s,y)=\delta_0(t-s)\Gamma(x,y),$$

where δ_0 is the Dirac measure on $\mathbb{R}$ and Γ is some negative definite distribution on $U \times U$. His results were published jointly with Dalang in the papers [35, 36].

In [37], Kim considers the one-dimensional wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(a(t,x)\frac{\partial u}{\partial x}\right) - b(t,x)\frac{\partial u}{\partial x} + c(t,x)u, \\ u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = u_1(x), \end{aligned} \tag{1.33}$$

on the interval $(0,\ell)$ with Neumann white noise boundary conditions

$$a(t,0)\frac{\partial u}{\partial x}(t,0) = g(t,u(t,0))\dot{B}(t), \quad u(t,\ell) = 0, \tag{1.34}$$

where $(B(t))_{t\geq 0}$ is a $\mathbb{R}$-valued Brownian motion, $a \in C^3(\mathbb{R}_+ \times [0,\ell])$, $b \in C^2(\mathbb{R}_+ \times [0,\ell])$, $c \in C^1(\mathbb{R}_+ \times [0,\ell])$, and $a(t,x) \geq \alpha > 0$ for all (t,x). Notice that the given functions a, b and g are time dependent, so they cannot be handled by (1.32) nor by the time-independent semigroup theory given by [11, 27] using the tensor product space approach. We believe it is due to the time dependence present in (1.33) that the semigroup theory cannot be applied, not due to the fact that hyperbolic equations cannot be handled in the framework of [11, 27] as claimed by Kim. Due to the time dependence, Kim constructs a Galerkin approximation scheme for the deterministic problem to justify manipulations and obtain the estimates required for the white noise boundary conditions. Let $H^k(0,\ell)$, $k \in \mathbb{Z}$, be the standard Sobolev spaces over the interval $(0,\ell)$ and define the self-adjoint positive operators

$$\begin{aligned} A(t) &:= -\frac{\partial}{\partial x}\left(a(t,x)\frac{\partial}{\partial x}\right), \\ \mathscr{D}(A(t)) &:= \left\{u \in H^2(0,\ell) : \frac{\partial u}{\partial x}(0) = 0, u(\ell) = 0\right\}. \end{aligned}$$

Let $(\lambda_k)_{k=1}^{\infty}$ and $(\varphi_k)_{k=1}^{\infty}$ be the sequence of all eigenvalues and corresponding eigenfunctions for $A(0)$ and for $\theta \in \mathbb{R}$ define

$$H_\dagger^\theta := \left\{f = \sum_{k=1}^{\infty} a_k \varphi_k : \sum_{k=1}^{\infty} \lambda_k^\theta |a_k|^2 < \infty\right\}.$$

First, Kim considers (1.33) combined with the deterministic boundary condition

$$a(t,0)\frac{\partial u}{\partial x}(t,0) = h(t), \quad u(t,\ell) = 0, \tag{1.35}$$

and shows that for $(u_0, u_1) \in L^2(0,\ell) \times H_\dagger^{-1}$ and $h \in H^{-1}(0,T)$ there exists a unique function u satisfying $(u, \dot{u}) \in C([0,T]; L^2(0,\ell) \times H_\dagger^{-1})$ and (1.33)-(1.35) in a weak sense. In particular, he shows this holds if $h = \dot{v}$ with $v \in C([0,T])$. Hence, there exists a continuous linear mapping

$$\Phi : (u_0, u_1, v) \mapsto (u, \dot{u}, u(\cdot, 0))$$

from $L^2(0,T) \times H_\dagger^{-1} \times C([0,T])$ into $C([0,T]; L^2(0,\ell) \times H_\dagger^{-1}) \times L^2(0,T)$. A weak solution for (1.33)-(1.34) is then obtained by noticing that if $g \in L^2(\Omega; L^2(0,T))$ and

$$v(t) = \int_0^t g(s)\, dW(s),$$

then v is a continuous martingale such that

$$\mathbb{E}\|v\|^2_{C([0,T])} \leq C\,\mathbb{E}\left(\int_0^T |g(s)|^2\, dt\right)$$

for some constant $C > 0$. Hence, if

$$(u_0, u_1, v) \in L^2(\Omega; L^2(0,\ell) \times H_\dagger^{-1} \times C([0,T])),$$

then the mapping Φ gives a unique solution to (1.33)-(1.34) such that

$$(u_0, u_1, u(\cdot, 0)) \in L^2(\Omega; C([0,T]; L^2(0,\ell) \times H_\dagger^{-1}) \times L^2(0,T)).$$

In effect, the equation is solved pathwise due to the absence of a stochastic integral. Kim's approach for handling time-dependent coefficients is quite powerful and could be applied for parabolic equations with boundary noise and higher-dimensional domains.

See also the recent paper by Bŕezniak and Peszat [38].

1.10 Dynamical boundary noise

When modelling physical phenomena, boundary conditions can be either static or dynamical. Static conditions, such as Dirichlet or Neumann boundary conditions, do not involve the time derivative of the system state variable. However, **dynamical boundary conditions** contain a dependence on time derivatives.

In 2004, Chueschov and Schmalfuß [39] considered the system of quasi-linear parabolic SPDE on a C^∞ bounded domain $U \subset \mathbb{R}^d$ with boundary $\partial U = \overline{\partial U_1} \cup \overline{\partial U_2}$,

$$
\begin{aligned}
du_t &= (\mathscr{A}(t)u_t + f(t,u_t,\nabla u_t))\,dt + g(t,u_t)\,dw_0(t) \text{ on } \mathbb{R}_+ \times U \\
\varepsilon^2 du_t &= (-\mathscr{B}_1(t)u_t + h(t,u_t))\,dt + \varepsilon\sigma(t,u_t)\,dw_1(t) \text{ on } \mathbb{R}_+ \times \partial U_1 \\
&\mathscr{B}_2(t)u_t = 0 \text{ on } \mathbb{R}_+ \times \partial U_2, \quad u(0,\cdot) = u_0,
\end{aligned}
\tag{1.36}
$$

where $u_t := u(t,\cdot)$, $\mathscr{A}$ is a normally elliptic second-order differential operator, the operators $\mathscr{B}_i$ satisfy a uniformly strong complementing condition with respect to $\mathscr{A}(t)$, $(w_i(t))_{t\geq 0}$ are independent $\mathbb{R}$-valued Wiener processes, and $\varepsilon \in (0,1]$ is a parameter. The boundary conditions are different from the standard ones and arise in physical models with dynamics on the boundary. Similar to (1.17), the parameter ε emphasizes that (1.36) is a perturbation of a deterministic equation. Their approach for studying (1.36) is, for any $\varepsilon \in (0,1]$, to first consider the linear deterministic PDE

$$
\begin{aligned}
\partial_t u + \mathscr{A}(t)u &= 0 \text{ on } (0,T] \times U, \\
\varepsilon^2 \partial_t (u|_{\partial U_1}) + \mathscr{B}_1(t)u &\text{ on } (0,T] \times \partial U_1, \\
\mathscr{B}_2(t)u = 0 \text{ on } (0,T] \times \partial U_2, &\quad u(0,\cdot) = u_0,
\end{aligned}
$$

and to formulate conditions so that the operators $A_\varepsilon(t) = (\mathscr{A}(t), \frac{1}{\varepsilon}\mathscr{B}_1(t))$ define a fundamental solution (or evolution operator) $S(t,s)$ that satisfies the evolution equation $u'(t) + A(t)u(t) = 0$ on a Hilbert space H. Similar to (1.5) and (1.26), a stochastic evolution equation for (1.36) is defined and they prove existence and uniqueness of a mild solution given by

$$u(t)=S(t,0)u_0+\int_0^t S(t,s)F(s,u(s))\,ds + \int_0^t S(t,s)G(s,u(s))\,dW(s).$$

Note that this is a non-autonomous (or time-dependent) stochastic evolution equation theory. They also study smoothness of the mild solution and compactness properties[3].

In 2007, Wang and Duan [40] also studied a SPDE with dynamical boundary conditions. However, the novelty in their work lies in the fact that the randomness enters the system at the boundary of small scale obstacles: the smooth bounded domain $U\subset\mathbb{R}^d$ is perforated with small holes (modelling obstacles or heterogeneities) and random dynamical boundary conditions are defined on the boundaries of these small holes. In particular, let U_ε be the domain given by removing O_ε, a collection of small holes of size ε, periodically distributed in the fixed domain $U\subset\mathbb{R}^d$, $d\geq 2$. When $\varepsilon\to 0$, the holes inside the domain become smaller and the number of holes increases to infinity. This process models the heterogeneities becoming finer and finer. In this perforated domain, they study the sequence of equations

$$\begin{aligned}\frac{\partial u^\varepsilon}{\partial t}&=\Delta u^\varepsilon+f(t,x,u^\varepsilon,\nabla u^\varepsilon)+g_1(t,x)Q_1\dot W_1 \text{ on } (0,T)\times U_\varepsilon,\\ \varepsilon^2\frac{\partial u^\varepsilon}{\partial t}&=-\frac{\partial u^\varepsilon}{\partial \nu_\varepsilon}-\varepsilon b u^\varepsilon+\varepsilon g_2(t,x)Q_2\dot W_2 \text{ on } (0,T)\times\partial O_\varepsilon,\end{aligned}\tag{1.37}$$

where $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ are cylindrical Wiener processes on $H_\varepsilon:=L^2(U_\varepsilon)$ and $\partial H_\varepsilon:=L^2(\partial O_\varepsilon)$, respectively. The holes are assumed to have no intersection with the boundary ∂U, this implies that $\partial U_\varepsilon=\partial U\cup\partial O_\varepsilon$. The positive symmetric operators $Q_1\in\mathscr{L}(H_\varepsilon)$ and $Q_2\in\mathscr{L}(\partial H_\varepsilon)$ and functions g_1 and g_2 satisfy

$$\|g_1(t,\cdot)\sqrt{Q_1}\|^2_{\mathscr{L}_2(H_\varepsilon)}\leq C_T,\quad \|g_2(t,\cdot)\sqrt{Q_2}\|^2_{\mathscr{L}_2(\partial H_\varepsilon)}\leq C_T,\quad t\in[0,T].$$

[3] In fact, this is the reason why the parameter ε was introduced.

The aim of their paper is to study the **homogenization problem** for (1.37). The aim of (deterministic) homogenization theory is to establish the macroscopic behaviour of a system that is microscopically heterogeneous in order to describe some characteristics of the heterogeneous medium (e.g., its thermal or electrical conductivity) [41]. As (1.37) is a stochastic partial differential equation modelling the microscopic heterogeneous system, their goal is to derive a homogenized effective equation (which is a new SPDE) by a probabilistic variation of homogenization techniques. That is, u^ε converges to a solution u of the homogenized problem as $\varepsilon \downarrow 0$ in the sense of probability distribution. Their approach is as follows. First, they show that (1.37) can be formulated as a stochastic evolution equation and that there exists a mild solution taking values in the product space $H_\varepsilon \times \partial H_\varepsilon$. Next, regarding the mild solution u^ε as a random variable in $L^2(0,T;L^2(U))$ by extending u^ε to the whole domain U, they obtain tightness of the distributions. Finally, they show that u^ε converges to the solution u of a homogenized equation in probability under different types of conditions on f: polynomial nonlinearity, sublinear nonlinearity, and nonlinearity containing a gradient term Δu_ε. Their results have applications in the study of composite materials but this problem also provides an illustration of how boundary noise could also appear on “inner boundaries”.

Also in 2007, Yang and Duan [42] studied the Cahn-Hilliard equation with random dynamical boundary conditions. The Cahn-Hilliard equation serves as a mathematical model for the description of phase separation phenomena in materials such as binary alloys. The concentration u of one of the two components of the binary alloy satisfies

$$\frac{\partial u}{\partial t} = \Delta\left(-\Delta u + f(u)\right) \quad \text{in } U, \tag{1.38}$$

where $U = \prod_{i=1}^{d}(0,\ell_i)$ with $\ell_i > 0$ and $d \in \{1,2,3\}$. As physicists started considering phase separation phenomena in confined systems where interactions with the wall were taken into account, it was a natural development to consider (1.38) with dynamical boundary conditions. Yang and Duan made the extension to **random** dynamical boundary conditions by considering a problem of the form

$$
\begin{aligned}
&\frac{\partial u}{\partial t}=\Delta\mu+Q_1\dot{W}_1(t), \quad \mu=-\Delta u+\varepsilon\frac{\partial u}{\partial t}+f(u) \quad \text{in } U,\\
&\frac{\partial\mu}{\partial\nu}=0, \quad \frac{\partial u}{\partial t}=\Delta_{\|}u-\lambda u-\frac{\partial u}{\partial\nu}u-g(u)+Q_2\dot{W}_2 \quad \text{on } \partial U,
\end{aligned}
\tag{1.39}
$$

with initial condition $u(0,\cdot)=u_0$. Here $\Delta_{\|}$ is the Laplace-Beltrami operator on ∂U, μ is the chemical potential, ν is the unit outer normal vector on ∂U, $\lambda>0$, f is a polynomial of odd degree, and $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ are cylindrical Wiener processes on $H=L^2_0(U)$ and $\partial H=L^2(\partial U)$, respectively. The space $L^2_0(U)$ is the subspace of functions $u\in L^2(U)$ with $\frac{1}{|U|}\int_U u(x)\,dx=0$. The operators Q_1 and Q_2 are assumed to be Hilbert-Schmidt and of trace class. Further, they assume Q_1 is orthogonal with respect to the orthonormal basis of eigenfunctions for Δ on U and Q_2 is orthogonal with respect to the orthonormal basis of eigenfunctions of $\Delta_{\|}$ on ∂U. They first obtain a solution for the interior dynamics of (1.39) with homogeneous Neumann boundary conditions given by

$$
\frac{\partial u}{\partial t}=-\Delta^2u-\Delta f(u)+Q_1\dot{W}(t) \text{ on } U, \quad \frac{\partial u}{\partial\nu}=0 \text{ on } \partial U. \tag{1.40}
$$

We note that setting $Au=\Delta u$ for $u\in\mathscr{D}(A)=W^{2,2}(U)\cap\{u:\partial_\nu=0\}$ and under the assumption that $\mathrm{Tr}(A^{-1+\delta}Q_1)<\infty$ for some $\delta>0$, a $L^2(U)$-valued mild solution to (1.40) exists by [43]. As it is “impossible to define a semigroup on the phase space” [42], they obtain a solution of (1.39) using a stochastic flow given by the solution of (1.40) instead of a semigroup approach. This stochastic flow satisfies the so-called cocycle property which allows them to consider (1.39) as a random dynamical system. As such, they study the long-term dynamics of (1.39) and the properties of the system as λ varies.

In 2009, Wang and Duan [44] studied the SPDE with dynamical boundary conditions given on a smooth domain $U\subset\mathbb{R}^d$, $1\leq d\leq 3$, by

$$
\begin{aligned}
\frac{\partial u^\varepsilon}{\partial t}&=\Delta u^\varepsilon+f(u^\varepsilon)+\sigma_1\dot{w}_1 \text{ on } (0,T)\times U_\varepsilon\\
\varepsilon\frac{\partial u^\varepsilon}{\partial t}&=-\frac{\partial u^\varepsilon}{\partial\nu_\varepsilon}-u^\varepsilon+\sqrt{\varepsilon}\sigma_2\dot{w}_2 \text{ on } (0,T)\times\partial O_\varepsilon,
\end{aligned}
\tag{1.41}
$$

where $\dot{w}_1$ and $\dot{w}_2$ are time white noises on U_ε and ∂O_ε, respectively. Note that (1.41) is an extension of (1.17) to dynamical boundary condition. Instead of using the analytic approach used by Freidlin and Wentzell, Wang and Duan consider a semigroup formulation of (1.41) to obtain a mild solution, then for dynamical boundary conditions, they answer similar questions as those answered for the static boundary conditions considered in [22]. First, they show tightness of the distributions of the solutions and then they obtain a SPDE with a simpler boundary condition when $\varepsilon \to 0$. Denoting by u a solution to this limiting equation, they then show that u^ε converges to u in the sense of (1.18). Next, they then proceed to study the normalized deviations between the solution u^ε of (1.41) and u. In particular, it is shown that $v_\varepsilon = (u^\varepsilon - u)/\sqrt{\varepsilon}$ converges to a process that solves a linear partial differential equation with random coefficients under a white noise **static** boundary condition. Finally, a large deviation result is proved for $(u^\varepsilon - u)/\varepsilon^\kappa$, where $0 < \kappa < 1/2$. Although it is not mentioned, we believe that the restriction to $d \le 3$ comes from the boundary layer degeneracy (1.25).

In 2006, Bonaccorsi and Ziglio [45] applied the technique of product spaces and operator matrices to solve stochastic evolution equations with randomly perturbed dynamic boundary conditions. This theory is largely a development of the equivalent deterministic theory obtained in [46, 47]. One can view these deterministic results as a more refined version of the developments presented in §1.1. As in (1.5), let H and ∂H be Hilbert spaces, $\underline{A} : \mathscr{D}(\underline{A}) \subset H \to H$, $F : H \to H$ is a nonlinear operator, and they consider the stochastic evolution equation on H given by

$$dX(t) = (\underline{A}X(t) + F(X(t)))\,dt + G(X(t))\,dW(t), \quad X(0) = x \in H, \tag{1.42}$$

where $(W(t))_{t\ge 0}$ is a cylindrical Wiener process on H. Now, on ∂H they consider another stochastic evolution equation

$$dZ(t) = (BZ(t) + \Phi X(t))\,dt + C(Z(t))\,dV(t), \quad Z(0) = z \in \partial H, \tag{1.43}$$

where $\Phi : \mathscr{D}(\Phi) \subset H \to \partial H$ is the **feedback** operator and it is assumed that $\mathscr{D}(\underline{A}) \subset \mathscr{D}(\Phi)$, $(V(t))_{t\geq 0}$ is another cylindrical Wiener process on ∂H taking values on ∂H, $B : \mathscr{D}(B) \subset \partial H \to \partial H$. The state space H and the boundary space ∂H are coupled together with the condition

$$Z(t) = \tau X(t), \quad t > 0, \tag{1.44}$$

where $\tau : \mathscr{D}(\underline{A}) \to \partial H$. Defining the operator $(A, \mathscr{D}(A))$ by (1.6) and under the assumptions that A generates a C_0-semigroup $(S(t))_{t\geq 0}$ on H and B generates a C_0-semigroup $(T(t))_{t\geq 0}$ on ∂H, then one can formulate the coupling (1.42)-(1.43)-(1.44) as a single stochastic abstract Cauchy problem on the product space $H \times \partial H$ in the following way. First, define the operator matrix

$$\mathbb{A} := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \mathscr{D}(\mathbb{A}) := \mathscr{D}(A) \times \mathscr{D}(B),$$

the Dirichlet map $\Lambda_\lambda : \partial H \to H$ by (1.8), and the operator matrix

$$\mathbb{L} := \begin{pmatrix} I_H & -\Lambda_\lambda \\ 0 & I_{\partial H} \end{pmatrix},$$

where I_H and $I_{\partial H}$ are the identity operators on H and ∂H, respectively. Then

$$d\mathbb{X}(t) = (\mathbb{A}\mathbb{X}(t) + \mathbb{F}(\mathbb{X}(t)))\,dt + \mathbb{G}(\mathbb{X}(t))\,d\mathbb{W}(t), \quad \mathbb{X}(0) = \mathbf{x} \in H \times \partial H, \tag{1.45}$$

where $\mathbb{W} := (W, V)$. They show that when $\Phi = 0$ (i.e. there is no boundary feedback), the mild solution of (1.45) is given by

$$\mathbb{X}(t) = \mathbb{S}(t)\mathbf{x} + \int_0^t \mathbb{S}(t-s)\mathbb{F}(\mathbb{X}(s))\,ds + \int_0^t \mathbb{S}(t-s)\mathbb{G}(s)\,d\mathbb{W}(s),$$

where $(\mathbb{S}(t))_{t\geq 0}$ is the C_0-semigroup generated by $\mathbb{A}$ on $H \times \partial H$. Although the deterministic framework for this construction is formulated in a Banach space setting [46, 47], the extension of these results to the stochastic setting of Bonnacorsi and Ziglio are only given in the Hilbert space case. More details of their construction can be found in Ziglio's thesis [48]. We also note that this formulation is not entirely appropriate when one wants to consider the white noise case as in (1.5). One can easily see this from the coupling condition (1.44) whereby if $B=0$, $\Phi=0$, $C=I$ in (1.43) then trivially solving for $Z(t)$ we obtain $\tau X(t)=Z(t)=V(t)$, and not $\tau X(t)=\dot{V}(t)$ as needed.

In 2008, Bonaccorsi in collaboration with Marinelli and Ziglio [49] considered the stochastic FitzHugh-Nagumo equations on networks with impulsive noise. The FitzHugh-Nagumo equations result from a simplification of the Hodgkin-Huxley model which describes how action potentials in neurons are initiated and propagated. In [49] they study a system of FitzHugh-Nagumo equations defined on a graph G described by a set of n vertices $v_1,\dots,v_n$ and m oriented edges $e_1,\dots,e_m$ which are assumed to be normalised (i.e., $e_j=[0,1]$ for $i=1,\dots,m$). The graph G is described by an incidence matrix $\Phi=\Phi^+-\Phi^-$ where $\Phi^+=(\varphi^+_{ij})_{n\times m}$ and $\Phi^-=(\varphi^-_{ij})_{n\times m}$ are given by

$$\varphi^-_{ij}=\begin{cases}1, & v_i=e_j(1)\\ 0, & \text{otherwise}\end{cases},\qquad \varphi^+_{ij}=\begin{cases}1, & v_i=e_j(0)\\ 0, & \text{otherwise.}\end{cases}$$

The electrical potential in the network denoted by $u(t,x)$ where $u\in(L^2(0,1))^m$ is the vector $(u_1(t,x),\dots,u_m(t,x))$ and $u_j(t,\cdot)$ is the electrical potential on the edge e_j. Let

$$\Gamma(v_i):=\{j\in\{1,\dots,m\}: e_j(0)=v_i \text{ or } e_j(1)=v_i\},$$

and it follows that the degree of a vertex v_i has cardinality $|\Gamma(v_i)|$. On every edge e_j, the potential u_j satisfies the parabolic PDE

$$\frac{\partial u_j}{\partial t}=\frac{\partial}{\partial x}\left(c_j(x)\frac{\partial u_j}{\partial x}u_j\right)+f_j(u_j), \tag{1.46}$$

on $\mathbb{R}_+\times(0,1)$, where $u_j=u_j(t,x)$, in combination with the initial conditions $u_j(0,x)=u_j^0(x)\in C([0,1])$, the continuity assumption on every node given by

$$p_i(t):=u_j(t,v_i)=u_k(t,v_i),\quad j,k\in\Gamma(v_i),\ i=1,\dots,n,$$

and the random dynamics at the nodes $v_1,\dots,v_n$ are given by

$$\frac{\partial p_i}{\partial t}(t) = -b_i p_i(t) + \sum_{j\in\Gamma(v_i)} \varphi_{ij}\mu_j c_j(v_i)\frac{\partial u_j}{\partial x}(t,v_i) + \sigma_i \dot{L}(t,v_i), \quad t>0, \tag{1.47}$$

where $\mu_j > 0$ for $j = 1,\dots,m$ and $\sigma_i > 0$ for $i = 1,\dots,n$. Here $\dot{L}(t,v_i)$ is the formal time derivative of a $\mathbb{R}$-valued Lévy process, see §1.11.

Of course, one could make the observation here that a Brownian motion is a Lévy process (see [50]) so that we could replace $\dot{L}(t,v_j)$ by $\dot{w}_j(t)$ where $(w_j(t))_{t\geq 0}$ are $\mathbb{R}$-valued Brownian motions, then notice that (1.47) is a dynamical Neumann condition so, in respect to §1.3 and the regularity provided by Neumann conditions, we should expect existence of a $L^2(0,1)$-valued solution for each stochastic boundary value problem (1.46)-(1.47) defined on each edge e_i in this Brownian motion case.

In [48, 49], it is shown that (1.46) can be formulated as a stochastic evolution equation of the form (1.42) by taking the state space $H = L^2(0,1)^m$ and (1.47) as a stochastic evolution equation of the form (1.43) on the boundary space $\partial H = \mathbb{R}^n$ where the processes W and V in (1.42) and (1.43) are replaced by Lévy processes. Using this semigroup approach, they then prove existence and uniqueness of a mild solution to (1.46)-(1.47).

In 2009, Brune, Duan and Schmalfuß studied a coupled system of the two-dimensional Navier–Stokes equations and the salinity transport equation [51]. The Navier-Stokes equations are often coupled with other equations, especially with the scalar transport equations for fluid density, salinity, or temperature. This coupling allows models to be developed for a variety of phenomena in environmental, geophysical, and climate systems. Let $U \subset \mathbb{R}^2$ be a bounded domain with C^1 boundary ∂U. In [51] they take random influences into account and consider a problem of the form

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = \left(c_1 \Delta u - \nabla p - u \cdot \nabla u - c_2 v e_2\right) + \dot{w}_1 \text{ on } \mathbb{R}_+ \times U \\
&\operatorname{div} u = 0 \text{ on } \mathbb{R}_+ \times U, \quad u = 0 \text{ on } \mathbb{R}_+ \times \partial U \\
&\frac{\partial v}{\partial t} = (c_3 \Delta v - u \cdot \nabla v) + \dot{w}_2 \text{ on } \mathbb{R}_+ \times U \\
&\frac{\partial v^\circ}{\partial t} = \frac{1}{\varepsilon}\left(-\frac{\partial v^\circ}{\partial \nu} - c_4 v^\circ + f(x)\right) + \dot{w}_3 \text{ on } \mathbb{R}_+ \times \partial U
\end{aligned}
\tag{1.48}
$$

where $v^\circ = v|_{\partial U}$ in terms of trace, $u = u(t,x) \in \mathbb{R}^2$ is the velocity, $v = v(t,x) \in \mathbb{R}$ is the salinity, p is the pressure, $f(x)$ is the mean salinity flux through the boundary, $e_1 = (0,1)^T \in \mathbb{R}^2$, c_i for $i \in \{1,2,3,4\}$ are constants, and $\dot{w}_i$ for $i \in \{1,2,3\}$ are independent white noises. When the positive constant ε becomes zero, then the dynamical boundary condition is interpreted as a static Robin boundary condition. Their approach to studying (1.48) is to embed the dynamical boundary condition into a stochastic evolution equation, called a stochastic Boussinesq equation, then taking the linear part of (1.48) with homogeneous boundary conditions they obtain a linear symmetric operator whose eigenfunctions form a complete orthonormal system. In this basis, they then construct a Galerkin approximation of the associated random system, then the solution map obtained defines a random dynamical system. We should note that this is the same idea employed by Kim for (1.33). Finally, they show that the random dynamical system obtained has a random attractor.

In 2010, Sun, Gao, Duan and Schmalfuß [52] considered rare events for the Boussinesq system with fluctuating dynamical boundary conditions given by (1.48). A large deviations principle is established and small probability events are studied in this context.

1.11 Lévy processes

In 2007, Peszat and Zabczyk [33, Chapter 15] consider the development of (1.5) to the case of Lévy white noise. We refer the reader to the monograph [33] for the definition of Hilbert space valued Lévy processes. We simply mention that Lévy processes serve as a tractable model for processes with "jumps".

Let H be a Hilbert space. In [33, Chapter 15], they consider the slight modification of (1.5) given by

$$X'(t) = \underline{A}X(t) + F(X(t)), \quad \tau X(t) = \dot{L}(t), \quad X(0) = x, \tag{1.49}$$

where $\dot{L}$ is the formal time-derivative of a Lévy process taking values in the Hilbert space H and the other operators are defined like (1.5). They additionally assume that (1.49) models dynamics on a C^∞ bounded domain $U \subset \mathbb{R}^d$ with boundary ∂U, H is embedded in the space of distributions on ∂U, and the operator A has domain $\mathscr{D}(A) := \{u \in W^{2,p}(U) : \tau u = 0\}$ and generates a C_0-semigroup on $L^p(U)$. Of course, following (1.6), this implies that $\underline{A}$ has domain $\mathscr{D}(\underline{A}) = W^{2,p}(U)$. They define two new types of weak solutions when $\underline{A} = \Delta$. First, for the Dirichlet case $\tau u := u|_{\partial U}$ (in terms of trace), they say that a $L^p(U)$-valued process $(X(t))_{t \geq 0}$ is a weak solution of (1.49) if

$$\langle X(t), \varphi \rangle = \langle x, \varphi \rangle + \int_0^t \langle X(s), \Delta\varphi \rangle \, ds + (L(t), \partial_\nu \varphi) \tag{1.50}$$

holds where ν is the exterior normal of ∂U, $\langle \cdot, \cdot \rangle$ is the canonical bilinear form on $\mathscr{D}(U) \times C^\infty(U)$, $(\cdot, \cdot)$ is the bilinear form on $\mathscr{D}(\partial U) \times C^\infty(\partial U)$, and $\varphi \in C^\infty(\overline{U})$ satisfying $\varphi = 0$ on ∂U. Second, for the Neumann case $\tau u := \partial_\nu u$, a process $(X(t))_{t \geq 0}$ is a weak solution of (1.49) if

$$\langle X(t), \varphi \rangle = \langle x, \varphi \rangle + \int_0^t \langle X(s), \Delta\varphi \rangle \, ds + (L(t), \varphi)$$

holds for $\varphi \in C^\infty(\overline{U})$ satisfying $\partial_\nu \varphi = 0$ on ∂U. These definitions follow naturally from Green's second formula. Similar to (1.7), they show that a unique mild solution to (1.49) exists in $L^2(U)$ and is given by

$$X(t) = \int_0^t S(t-s)F(X(s)) \, ds + \int_0^t (\lambda - A)S(t-s)\Lambda_\lambda \, dL(s)$$

if L is a square-integrable, centered Lévy process with reproducing kernel Hilbert space $\mathscr{H} \subset H = L^2(U)$, $F : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function and

$$\int_0^t \|(\lambda - A)S(s)\Lambda_\lambda\|^2_{\mathscr{L}_2(\mathscr{H}, L^2(U))}\, ds < \infty,$$

for $t > 0$. They also show that if $(X(t))_{t\geq 0}$ is a mild solution to (1.49) with $\underline{A} = \Delta$ then it is also a weak solution. Next, they present the example of §1.3 for the case of (1.49) and again conclude that no solutions exist for the Dirichlet case in $L^2(0,1)$. Finally, they conclude that "in order to study the Dirichlet problem with random perturbations of the Gaussian white noise type, one should introduce weighted L^2-spaces. Namely, the space $L^2(U, \kappa(x)dx)$ where k vanishes on the boundary ∂U". However, these arguments are not presented and the reader is referred to [24, 12, 31] and a preprint by Peszat and Russo[4].

As already mentioned in §1.10, the boundary noise problem studied in [54], in particular (1.46)-(1.47), is studied under the influence of Lévy boundary noise. As such, the operator matrix approach whereby the boundary value problem is formulated as the abstract Cauchy problem (1.45) on the product space $H \times \partial H$ is presented for Lévy processes in Ziglio's thesis [48].

1.12 Optimal control of stochastic systems: a continuing story

As briefly mentioned in §1.5, the first results on optimal control of stochastic boundary value problems were obtained in 1998 [29]. Recently a number of new results have appeared on this topic.

In 2007, Debussche, Fuhrman and Tessitore [55] studied the optimal control problem for a nonlinear stochastic heat equation on the interval $(0,\pi)$ where

[4]We believe this has become the preprint [53].

the boundary conditions are of Neumann type,

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,u(t,x)) \quad \text{on } [t_0,T]\times(0,\pi) \\ \frac{\partial u}{\partial x}(t,0) &= g_1(t) + \dot{w}_1(t), \quad \frac{\partial u}{\partial x}(t,\pi) = g_2(t) + \dot{g}_2(t). \end{aligned} \tag{1.51}$$

In (1.51), $(w_1(t))_{t\geq 0}$ and $(w_2(t))_{t\geq 0}$ are independent $\mathbb{R}$-valued Wiener processes, u is the unknown $\mathbb{R}$-valued process representing the state of the system, and the control is given by the $\mathbb{R}$-valued processes g_1 and g_2 acting at the points 0 and π, respectively.

Their approach is to use the framework of [11] and to formulate (1.51) as (1.9) resulting in a stochastic evolution equation on a Hilbert space H of the form

$$dX(t) = [AX(t) + F(t,X(t)) + (\lambda - A)\Lambda_\lambda G(t)]\,dt + (\lambda - A)\Lambda_\lambda\, dW(s),$$

where, as for (1.9) we have a boundary Hilbert space ∂H, $\Lambda_\lambda : H \to \partial H$ is the map associated with the Neumann boundary condition $\tau u = \partial_\nu u$, $G(t) = (g_1(t), g_2(t))$ and $W(t) = (w_1(t), w_2(t))$ take values on $\partial H = L^2(\{0,\pi\}) \simeq \mathbb{R}^2$, and A is defined by (1.6). The optimal control problem treated is the minimisation of the finite horizon cost

$$\begin{aligned} J(t,X(0),G(\cdot)) = \mathbb{E}\int_t^T\int_0^\pi c(s,z,X(s)(z),G(s))\,dz\,ds \\ + \mathbb{E}\int_0^\pi \varphi(z,X(T)(z))\,dz, \end{aligned} \tag{1.52}$$

where $c : [0,T]\times[0,\pi]\times\mathbb{R}\times\mathscr{K} \to \mathbb{R}$ and $\varphi : [0,\pi]\times\mathbb{R}\to\mathbb{R}$ and the set of control actions $\mathscr{K}$ is a bounded closed subset of $\mathbb{R}^2$. They obtain the existence and uniqueness of a C^1 mild solution of the Hamilton-Jacobi-Bellman equation. This enables them to find an optimal feedback for their problem.

Although the control problem (1.51) seems simpler than (1.27) due to the absence of noise on the interior of the domain, in fact the problem is surprisingly more difficult, as the presence of enough noise guarantees that the linear operator in the Hamilton-Jacobi equation is strongly elliptic. As a result, their method would apply equally well in the presence of interior noise. Due to the absence of interior noise, a number of technical difficulties arise which they surmount by formulating the problem as a system of forward-backward stochastic differential equations.

Finally we note that they mention “the case of Dirichlet boundary conditions is more complicated since the solutions are much less regular in space”. Further, they suggest that their method can be extended to this case by using the techniques of [24, 12] and that “the real structural restriction - beside technical complications - is that the ‘image of the noise operator’ is larger than the image of the control”. They also recall the negative result of §1.3 obtained in the semigroup framework of [11] and conclude that “the smoothing properties of the heat equation are not strong enough to regularize a rough term such as white noise”. These ideas where implemented in 2009 by Fabbri and Goldys [56], however in order to continue chronologically, we shall discuss their contributions later.

In 2008, Bonaccorsi, Confortola, and Mastrogiacomo [54] study the extension of (1.51) to the case of dynamical Neumann boundary conditions. That is, to dynamics of the form

$$
\begin{aligned}
\frac{\partial u}{\partial t}(t,x) = \frac{\partial^2 u}{\partial x^2}(t,x) + f(t,x,u) + g(t,x,u)\dot{W}(t) \quad \text{on } [t_0,T]\times(0,\pi)\\
z_1'(t) = -b_1 z_1(t) - \frac{\partial u}{\partial x}(t,0) + h_1(t)(g_1(t) + \dot{w}_1(t))\\
z_2'(t) = -b_2 z_2(t) - \frac{\partial u}{\partial x}(t,\pi) + h_2(t)(g_2(t) + \dot{w}_2(t)),
\end{aligned}
\tag{1.53}
$$

where $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on $H = L^2(0,\pi)$ and for $i = 1,2$: $b_i > 0$, $h_i(t)$ are bounded measurable functions, and $(w_i(t))_{t\geq 0}$ are independent $\mathbb{R}$-valued Wiener process. The control process $G(t) = (g_1(t), g_2(t))$ takes values in a compact $\mathcal{K} \subset \mathbb{R}^2$ and we set $V(t) = (w_1(t), w_2(t))$. Note that the boundary dynamics are given by $Z(t) = (z_1(t), z_2(t))$ which is a system of two $\mathbb{R}$-valued stochastic differential equations. Setting $X(t) = u(t,\cdot)$, their approach is to pose (1.53) in the form of (1.42)-(1.43)-(1.44) using the operator matrix theory presented in §1.10. This results in a Cauchy problem on the space $H \times \partial H = L^2(0,\pi) \times \mathbb{R}^2$ of the form

$$d\mathbb{X}(t) = [\mathbb{A}\mathbb{X}(t) + \mathbb{F}(t,\mathbb{X}(t))]dt + \mathbb{G}(t,\mathbb{X}(t))[\mathbb{I}Z(t)dt + d\mathbb{W}(t)],$$

with initial condition $X(0) = \mathbf{x}$. The operator $\mathbb{I} : \partial H \to H$ gives the immersion of the boundary space into $H \times \partial H$ by $\mathbb{I}z := (0,z)$. This formulation is then used to minimize a cost functional of the form

$$J(t, X(0), G) = \mathbb{E}\int_t^T c(s, X(s), Z(s), G(s))\,ds + \mathbb{E}\varphi(X(T), Z(T)), \tag{1.54}$$

where $c : [0,T] \times H \times \partial H \times \mathscr{K} \to \mathbb{R}$ and $\varphi : H \times \partial H \to \mathbb{R}$ and the set of control actions $\mathscr{K}$ is a bounded closed subset of $\mathbb{R}^2$. Due to the presence of interior noise in (1.53) and the structure of the cost functional in (1.54), this problem seems less technical than (1.52) but the presence of dynamical Neumann boundary conditions is an interesting feature that impacts the theory.

In 2009, Fabbri and Goldys [56] furthered the idea mentioned in [55]. That is, they apply the weighted space technique of [12] that we presented in §1.8, to handle the optimal control problem for the heat equation on the half-line $(0,\infty)$ with Dirichlet boundary noise and boundary control given by

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) \quad \text{on } [t_0, T] \times (0,\infty), \\ u(t,0) &= g(t) + \dot{w}(t), \end{aligned} \tag{1.55}$$

where $(w(t))_{t\geq 0}$ is a $\mathbb{R}$-valued Wiener process and g is a square integrable control. Setting $X(t) = u(t,\cdot)$, their approach is to formulate (1.55) as a stochastic evolution equation of the form (1.9) with $F = 0$ whereby the state space $H = L^2(0,\infty; \varrho(x)dx)$ is a weighted L^2 space with $\varrho(x) := \min(1, x^{1+\theta})$ such that $\theta \in (0,1)$ and the boundary space $\partial H \simeq \mathbb{R}$. In other words,

$$dX(t) = [AX(t) + (\lambda - A)\Lambda_\lambda g(t)]dt + (\lambda - A)\Lambda_\lambda\, dW(s), \tag{1.56}$$

where $Au := \partial_x^2 u$ with $\mathscr{D}(A) = W_0^{2,2}(0,\infty)$, Λ_λ is the Dirichlet map, and $W(t) = w(t)$. They indicate that the case of (1.55) with interior noise and interior control could be easily considered in the same framework. They study the linear quadratic problem given by the cost functional

$$J(t, X(0), g) = \mathbb{E}\int_t^T \left(\|BX(s)\|_{H_2}^2 + |g(s)|^2\right) ds + \mathbb{E}\langle QX(T), X(T)\rangle_H, \tag{1.57}$$

where H_2 is another Hilbert space, $B \in \mathscr{L}(H, H_2)$, and $Q \in \mathscr{L}(H)$ is a positive symmetric operator. Following a result by Krylov [57] which shows that A extends to a generator $\widetilde{A}$ of an analytic semigroup $(S(t))_{t\geq 0}$ on $H = L^2(0,\infty;\varrho(x)dx)$, they are able to obtain a unique mild solution of (1.56) using the classic methods of [11]. They solve the minimisation (1.57) by working directly with a solution of the associated deterministic problem $u'(t) = Au(t) + (\lambda - A)\Lambda_\lambda g(t)$ on H.

In 2010, Masiero [58] considered the nonlinear version of (1.55) given by

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + f(s, u(t,x)) \quad \text{on } [t_0, T] \times (0,\infty), \\ u(t,0) &= g(t) + \dot{w}(t), \end{aligned} \tag{1.58}$$

where $(w(t))_{t\geq 0}$ is a $\mathbb{R}$-valued Wiener process and the control g is given by the $\mathbb{R}$-valued process g. Similar to the approach of Fabbri and Goldys, Masiero formulates (1.58) as a stochastic evolution equation of the form

$$dX(t) = [AX(t) + F(s, X(t)) + (\lambda - A)\Lambda_\lambda g(t)]dt + (\lambda - A)\Lambda_\lambda\, dW(s),$$

where $Au := \partial_x^2 u$ with $\mathscr{D}(A) = W_0^{2,2}(0,\infty)$, Λ_λ is the Dirichlet map, and $W(t) = w(t)$. First, solutions are considered on the weighted space $H = L^2(0,\infty;\varrho(x)dx)$ using the results of [56] and extending them to nonlinear case. The optimal control problem considered is similar to (1.52) and of the form

$$\begin{aligned} J(t, X(0), g) = \mathbb{E}\int_t^T\int_0^\infty & c(s, z, X(s)(z), g(s))\, dz\, ds \\ &+ \mathbb{E}\int_0^\infty \varphi(z, X(T)(z))\, dz. \end{aligned}$$

where $c : [0,T] \times [0,\infty] \times \mathbb{R} \times \mathscr{K} \to \mathbb{R}$ and $\varphi : [0,\infty] \times \mathbb{R} \to \mathbb{R}$ and the set of control actions $\mathscr{K}$ is a bounded closed subset of $\mathbb{R}$. Existence and uniqueness of the Hamilton-Jacobi-Bellman equation is proved and it is shown how these results can be applied to perform the synthesis of the optimal control. The infinite horizon problem is also considered.

Again, we note that these results for the Dirichlet boundary noise case rely heavily on the weighted space idea obtained in [12] and the one-dimensional example can only be treated in [56, 58] as there is a lack of a higher-dimensional existence theory.

1.13 Qualitative theory

In 2007, Chueshov and Schmalfuß [59] continued their study of (1.36) and considered the qualitative properties of the dynamical system obtained. Under certain conditions on the nonlinear term f, they show that the solution of the system generates a monotone order-preserving random dynamical system with respect to the metric dynamical system generated by the underlying Wiener process. Under a coercitivity condition, it is shown that this system has a random pull-back attractor. They then apply these results to the case of a nonlinear stochastic heat equation with a polynomial nonlinearity and show that the attractor is a random equilibrium point.

In 2008, Ziglio [48] studied the existence of an attracting set at time $t = 0$ for a heat equation in a bounded domain $U \subset \mathbb{R}^3$ with smooth boundary ∂U under random dynamical Neumann boundary conditions.

1.14 Physics

Briefly looking at the physics literature, we notice that in a number of papers during 2008 and 2009, Sabelfeld and Shalimova considered different types of elliptic equations with random boundary excitations [60, 61, 62, 63]. Their papers all follow the same structure, hence we shall illustrate their results with the simple example

$$\Delta u = 0 \text{ in } U, \quad u = w \text{ on } \partial U$$

where w is a Gaussian (spatial) white noise on ∂U and U is the half-space $\mathbb{R}^2_+$ with boundary $\partial U \equiv \mathbb{R}$. Their "solutions" are defined by convolution against a Poisson kernel, however an existence and uniqueness result is not proved and it is simply stated that u is a Gaussian random field with zero mean. The main aim of their papers is to obtain analytic approximations to the "correlation tensor" of the u and to represent u as a Karhunen-Loève (KL) series expansion. By truncating the KL expansion at a desired term, they obtain analytic estimates for the correlation structure of u which they verify through numerical simulation.

Although they have not obtained an existence or uniqueness result for these elliptic problems, these papers show there is an interest from physicists to understand elliptic problems with boundary noise. Further, the noise on the boundary is **spatially white** which is a situation that has not been hitherto studied in the literature. This is due to the standard assumption that the noise is typically assumed to be of the form $B\dot{W}$ where B is a Hilbert-Schmidt operator or that $U \subset \mathbb{R}$ and ∂U is the endpoint(s) of an interval[5].

1.15 Recent developments

Recently, Cerrai and Freidlin have considered a nonlinear stochastic parabolic equation with Neumann boundary noise [64]. The interior noise is of multiplicative type and the uniformly elliptic second order differential operator $\mathscr{A}$ is multiplied by a parameter ε^{-1} such that $0 < \varepsilon \ll 1$, i.e., ε^{-1} is large. More

[5] Noise cannot be spatially white in this case.

precisely, they consider

$$\begin{aligned}\frac{\partial u^\varepsilon}{\partial t}(t,x) &= \frac{1}{\varepsilon}\mathscr{A} u^\varepsilon(t,x) + f(t,x,u^\varepsilon(t,x)) \\ &\quad + g(t,x,u^\varepsilon(t,x))Q\dot{W}_1(t) \text{ on } \mathbb{R}_+ \times U, \\ \frac{1}{\varepsilon}\frac{\partial u^\varepsilon}{\partial \nu}(t,x) &= \sigma(t,x)B\dot{W}_2(t) \text{ on } \mathbb{R}_+ \times \partial U,\end{aligned} \tag{1.59}$$

where $U \subset \mathbb{R}^d$ is bounded with C^∞ boundary ∂U. The functions $f, g : [0,\infty)\times U\times\mathbb{R} \to \mathbb{R}$ are assumed to be measurable and satisfy a Lipschitz condition with respect to the third variable, uniformly with respect to the first two variables, and the mapping $\sigma : [0,\infty)\times\partial U \to \mathbb{R}$ is bounded with respect to the space variable. The operators $Q \in \mathscr{L}(L^2(U))$ and $B \in \mathscr{L}(L^2(\partial U))$ are assumed to be nonnegative and symmetric. The processes $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ are independent cylindrical Wiener processes on $H := L^2(U)$ and $\partial H := L^2(\partial U)$, respectively. Let A be the unbounded operator on H defined by $Au := \mathscr{A} u$ for $u \in \mathscr{D}(A) := \{u : \mathscr{A} u \in H, \partial_\nu u|_{\partial U} = 0\}$. Different from previous results in the literature, they assume that the strongly continuous semigroup $(S(t))_{t\geq 0}$ in H generated by A admits a unique invariant measure μ and spectral gap occurs. That is, there exists some $\gamma > 0$ such that, for any $h \in L^2(U,\mu)$,

$$\left\| S(t)h - \int_U h(x)\mu(dx)\right\|_{L^2(U,\mu)} \lesssim e^{-\gamma t}\|h\|_{L^2(U,\mu)}, \quad t \geq 0. \tag{1.60}$$

We shall use the notation $H_\mu := L^2(U,\mu)$ and $\langle h,\mu\rangle := \int_U h(x)\mu(dx)$ and $(\cdot,\cdot)$ for the inner product in $L^2(U)$.

Their aim is to study the limiting behaviour of (1.59) as the parameter ε goes to zero and to show that the spatial average of (1.59) can be approximated by a $\mathbb{R}$-valued stochastic differential equation of the form

$$dv(t) = \tilde{F}(t,v(t))\,dt + \tilde{G}(t,v(t))Q\,dW_1(t) + \tilde{\Sigma}(t)B\,dW_2(t), \tag{1.61}$$

with initial condition $v(0) = \langle u_0,\mu\rangle$ where for $\varphi \in L^2(U,\mu)$, $\psi \in L^2(U)$, $\kappa \in L^2(\partial U)$, and $t \geq 0$, they defined the $\mathbb{R}$-valued mappings $\tilde{F}$, $\tilde{G}$, and $\tilde{\Sigma}$ by the spatially averaging

$$\tilde{F}(t,\varphi) := \int_U f(t,x,\varphi(x))\mu(dx),$$
$$\tilde{G}(t,\varphi)\psi := \int_U g(t,x,\varphi(x))\psi(x)\mu(dx),$$
$$\tilde{\Sigma}(t)\kappa := \lambda \int_U \Lambda_\lambda[\sigma(t,\cdot)\kappa](x)\mu(dx).$$

Here, Λ_λ is the Neumann map associated with $(\mathscr{A}, \partial_\nu)$ and the solution v of (1.61) approximates the solution u^ε of (1.59) in the sense of

$$\lim_{\varepsilon\downarrow 0} \mathbb{E} \sup_{t\in[\delta,T]} \left| \int_U |u^\varepsilon(t,x) - v(t)|^2 \mu(dx) \right|^p = 0,$$

for any fixed $0 < \delta < T$ and $p \geq 1/2$.

To obtain this approximation result, they first show that there exists a unique solution to (1.59) in the following way. First, they assume that

$$QW_1(t) = \sum_{n=0}^{\infty} \lambda_n e_n w_n(t), \quad BW_2(t) = \sum_{n=0}^{\infty} \theta_n \tilde{e}_n \tilde{w}_n(t),$$

where $(e_n)_{n=0}^\infty$ is an orthonormal basis of H that diagonalizes Q with eigenvalues $(\lambda_n)_{n=0}^\infty$, $(\tilde{e}_n)_{n=0}^\infty$ is an orthonormal basis of ∂H that diagonalizes B with eigenvalues $(\theta_n)_{n=0}^\infty$, and $(w_n(t))_{t\geq 0}$ and $(\tilde{w}_n(t))_{t\geq 0}$ are two sequences of independent $\mathbb{R}$-valued Wiener processes. They stress that they are “not imposing the Hilbert-Schmidt condition on the operators Q and B” and instead assume that if $d \geq 2$ then there exists $\varrho < 2d/(d-2)$ and $\beta < 2d/(d-1)$ such that

$$\sum_{n=0}^{\infty} \lambda_n^\varrho \|e_n\|^2_{L^\infty(U)} < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \theta_n^\beta < \infty. \tag{1.62}$$

We recall however that if an operator $T \in \mathscr{L}(L^2(U))$ satisfies $TL^2(U) \subset L^\infty(U)$ then T is Hilbert-Schmidt so (1.62) is equivalent to saying that, depending on d, that the cylindrical Wiener processes W_1 and W_2 take values in $\overline{H} \subset H$ and

$\overline{\partial H} \subset \partial H$ respectively, and we can identify Q and B with $\overline{Q} \in \mathscr{L}_2(\overline{H}, H)$ and $\overline{B} \in \mathscr{L}_2(\overline{\partial H}, \partial H)$.

Under the conditions (1.62) and posing for $h_1, h_2 \in H$,

$$F(t, h_1)(x) := f(t, x, h_1(x)), \quad [G(t, h_1)h_2](x) := g(t, x, h_1(x))h_2(x),$$

for all $x \in U$, they prove that a H-valued mild solution of (1.59) is given by

$$\begin{aligned} u^\varepsilon(t) = S^\varepsilon(t)u_0 &+ \int_0^t S^\varepsilon((t-s))F(s, u^\varepsilon(s))\,ds \\ &+ \int_0^t S^\varepsilon(t-s)G(s, u^\varepsilon(s))Q\,dW_1(s) \\ &\quad + \underbrace{(\lambda - A)\int_0^t S^\varepsilon(t-s)\Lambda_\lambda B\,dW_2(s)}_{L^\varepsilon(t)} \end{aligned}$$

where $S^\varepsilon(t) := S(t/\varepsilon)$, Λ_λ is the Neumann map, and $u^\varepsilon(t)(x) = u^\varepsilon(t, x)$. Further, for any $T > 0$ and $p \geq 1$,

$$\sup_{\varepsilon \in (0,1]} \mathbb{E}\|L^\varepsilon\|^p_{C([0,T];H)} < \infty.$$

Finally, once a solution is obtained, they show that for any $u_0 \in H$, $p \geq 1$, $\theta < 1$ and $\delta > 0$ it holds that

$$\mathbb{E} \sup_{t \in [\delta, T]} \|u^\varepsilon(t) - v(t)\|^p_{H_\mu} \lesssim (\varepsilon + \varepsilon^{p\theta/2})$$

where v solves (1.61).

Let us consider (1.60) for the case where $U = (0, \ell) \subset \mathbb{R}$ and A is the Neumann Laplacian on $L^2(0, \ell)$, i.e., $Au = u''$ with

$$\mathscr{D}(A) = \{u \in W^{2,2}(0, \ell) : u'(0) = u'(\ell) = 0\}.$$

The trigonometric functions $h_n(x) = -\sqrt{2}\sin(\frac{n\pi}{\ell}x)$ form an orthonormal basis on $L^2(0, \ell)$ such that $Ah_n = -\lambda_n h_n$ with $\lambda_n = (\frac{n\pi}{\ell})^2$ for $n = 1, 2, \dots$ and $h_0(x) =$

$1/\sqrt{\ell}$ with $\lambda_0 = 0$. As $S(t) = e^{tA}$, it follows that $S(t)h_n = e^{-t\lambda_n}h_n$. Using the fact that

$$S(t)h = \int_0^\ell h(x)\frac{1}{\ell}\,dx + \sum_{n=1}^{\infty} e^{-t\lambda_n}(h, h_n)h_n,$$

it follows that $\mu(dx) = \ell^{-1}\,dx$ is an invariant measure of $(S(t))_{t\geq 0}$. In fact, for a bounded domain $U \subset \mathbb{R}^d$, in the same way it follows that $\mu(dx) = |U|^{-1}\,dx$ where $|U|$ is the volume of U.

In the preprint [65], Veraar and Schnaubelt have considered the well-posedness and pathwise regularity of semilinear non-autonomous parabolic equations with boundary and interior noise in an L^p setting. Let H and ∂H be separable Hilbert spaces and let E and ∂E be Banach spaces. On E, they consider the stochastic evolution equation

$$\begin{aligned} dX(t) = {} & [A(t)X(t) + F(t, X(t)) + \Pi_1(t)G(t, X(t))]\,dt \\ & + B(t, X(t))\,dW_1(t) + \Pi_2(t)C(t, X(t))\,dW_2(t) \end{aligned} \tag{1.63}$$

with initial condition $X(0) = x \in E$ where $(A(t))_{t\in[0,T]}$ is a family of closed operators on E. The processes $(W_1(t))_{t\geq 0}$ and $(W_2(t))_{t\geq 0}$ are cylindrical Wiener processes on H and ∂H, respectively. The operators $\Pi_i : \mathscr{D}(\Pi_i) \subset \partial E \to E$ for $i \in \{1, 2\}$ are used to treat the boundary conditions.

Their approach to handling the family of operators $(A(t), \mathscr{D}(A(t)))_{t\geq 0}$ is through the non-autonomous theory developed by Acquistapace and Terrini [66]. That is, the family $(A(t), \mathscr{D}(A(t)))_{t\geq 0}$ is assumed to satisfy the following conditions:

- $A(t)$ are densely defined, closed linear operators on a Banach space E and there are constants $\sigma \in \mathbb{R}$, $K \geq 0$, and $\theta \in (\frac{\pi}{2}, \pi)$ such that $\Sigma(\theta, \sigma) \subset \varrho(A(t))$ and $\|R(\lambda, A(t))\| \leq K/(1 + |\lambda - \sigma|)$ holds for all $\lambda \in \Sigma(\theta, \sigma)$ and $t \in [0, T]$,

- There are constants $L \geq 0$ and $\mu, \nu \in (0, 1]$ such that $\mu + \nu > 1$ and

$$\|A_\sigma(t)R(\lambda, A_\sigma(t))(A_\sigma(t)^{-1} - A_\sigma(s)^{-1})\| \leq L|t - s|^\mu(|\lambda| + 1)^{-\nu}$$

holds for all $\lambda \in \Sigma(\theta, 0)$ and $s, t \in [0, T]$, where $A_\sigma(t) := A(t) - \sigma$.

Under these conditions, the non-autonomous Cauchy problem $u'(t) = A(t)u(t)$ for $t \in [s, T]$ with $u(s) = x \in E$, has a solution $u \in C([s, T]; E) \cap C^1((s, T]; E)$ with $u(t) \in \mathscr{D}(A(t))$ for all $t \in (s, T]$. This gives rise to a **strongly continuous evolution family** of bounded operators $(S(t, s))_{0 \le s \le t \le T}$ on E with properties:

- $S(s, s) = I$ for all $s \in [0, T]$;
- $S(t, s) = S(t, r)S(r, s)$ for all $0 \le s \le r \le t \le T$;
- the map $(t, s) \mapsto S(t, s)$ is strongly continuous.

This evolution family allows Schnaubelt and Veraar to prove existence and uniqueness of a mild solution of (1.63) under linear growth and uniform Lipschitz conditions (in space) on the mappings F, G, B, and C. The conditions are technical, therefore let us consider an example where $F \equiv 0$, $G \equiv 0$, $B \equiv 0$. In this simple case, a candidate mild solution of (1.63) is given by

$$X(t) = S(t, 0)x + \int_0^t S(t, s)\Pi_2 C(s, X(s))\, dW_2(s). \tag{1.64}$$

Setting $\Pi_2 = (\lambda - A)\Lambda_\lambda$, one can see (1.64) is a non-autonomous version of (1.7) and under similar conditions on Λ_λ, a mild solution in E can be obtained. We note that their results can be viewed as an extension of Chapter 4 to the non-autonomous setting by applying the Acquistapace-Terrini conditions.

In the preprint [67], Bonnacorsi and Ziglio consider a nonlinear stochastic partial differential equation with dynamical boundary conditions given by

$$\begin{aligned} &\frac{\partial u}{\partial t}(t, x) = \operatorname{div} \mathbf{a}(x, \nabla u) + Q\dot{W}_1(t) \text{ in } \mathbb{R}_+ \times U, \\ &\frac{\partial u^\circ}{\partial t}(t, x) = -u^\circ(t, x)|u^\circ(t, x)|^{p-2} - \mathbf{a}(x, \nabla u) \cdot \nu + B\dot{W}_2(t) \text{ on } \mathbb{R}_+ \times \partial U \end{aligned} \tag{1.65}$$

where $u^\circ = u|_{\partial U}$ in terms of trace. Here, $(W_1(t))_{t \ge 0}$ and $(W_2(t))_{t \ge 0}$ are cylindrical Wiener processes on $H = L^2(U)$ and $\partial H = L^2(\partial U)$ respectively, $Q \in \mathscr{L}_2(H)$, $B \in \mathscr{L}_2(\partial H)$, and $\mathbf{a} : U \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function satisfying the conditions:

- there exists $\alpha > 0$ such that $\mathbf{a}(x,\xi)\cdot\xi \geq \alpha|\xi|^p$ for almost every $x \in U$ and for all $\xi \in \mathbb{R}^d$,
- there exists $\sigma > 0$ and $f \in L^q(U)$ such that $|\mathbf{a}(x,\xi)| \leq \sigma(f(x) + |\xi|^{p-1})$ for almost every $x \in U$ and for all $\xi \in \mathbb{R}^d$ where $q = p/(p-1)$,
- for almost every $x \in U$ and for all $\xi \neq \eta \in \mathbb{R}^d$, $(\mathbf{a}(x,\xi) - \mathbf{a}(x,\eta))\cdot(\xi - \eta) > 0$.

Notice than when $p = 2$ and $\mathbf{a}(x,\nabla u) = \nabla u$ then $\operatorname{div}\mathbf{a}(x,\nabla u) = \Delta u$ and (1.65) reduces to the stochastic heat equation with dynamical Robin boundary conditions of the form

$$\begin{aligned}
&\frac{\partial u}{\partial t}(t,x) = \Delta u + Q\dot{W}_1(t) \text{ in } \mathbb{R}_+ \times U,\\
&\frac{\partial u^\circ}{\partial t}(t,x) = -u^\circ(t,x) - \frac{\partial u^\circ}{\partial \nu}(t,x) + B\dot{W}_2(t) \text{ on } \mathbb{R}_+ \times \partial U,
\end{aligned}$$

and it may be posed as a problem of type (1.42)-(1.43)-(1.44) taking values in H. However, when $p \neq 2$, $\mathbf{a}$ can no longer be extended to a linear operator on $H \times \partial H$ and the framework of [45] no longer applies. This is the case, for example, when $\mathbf{a}(x,\xi) = |\xi|^{p-2}\xi$ which gives the p-Laplacian. As such, to consider (1.65), Bonaccorsi and Ziglio use the pivot space approach, a standard PDE technique to handle nonlinearities like $\mathbf{a}$ (e.g., see §II.3 in [68]) which was extended to the case of SPDEs (with interior noise) in [69]. Using the product space ideas of [45], Bonaccorsi and Zigio extend [69] to the case of stochastic dynamical boundary conditions by defining $E = L^p(U)$ and $\partial E = W^{1-1/p,p}(\partial U)$, where $W^{1-1/p}(\partial U)$ is the space of all $u \in L^p(\partial U)$ such that

$$\|u\|_{\partial E}^p := \int_{\partial U}\int_{\partial U} \frac{|u(x) - u(y)|^p}{|x-y|^{p+d-2}}\, d\sigma(x) d\sigma(y) < \infty,$$

then taking the pivot (or Gelfand triple)

$$E \times \partial E \hookrightarrow H \times \partial H (\equiv H^* \times \partial H^*) \hookrightarrow (E \times \partial E)^*,$$

where $H \times \partial H$ is identified with its dual $H^* \times \partial H^*$ by the Riesz isomorphism and $\hookrightarrow$ denotes a continuous and dense embedding. The nonlinear operator $\mathbf{a}$

is then considered by introducing the matrix operator

$$\mathbb{A}(u,u^\circ)^T := \begin{pmatrix} \operatorname{div}\mathbf{a}(\cdot,\nabla u) & 0 \\ -u^\circ|u^\circ|^{p-2} & -\mathbf{a}(\cdot,\nabla u)\cdot\nu \end{pmatrix}$$

for all $u \in C^\infty(\overline{U})$ and $u^\circ \in C^\infty(\partial U)$, then showing that $\mathbb{A}$ extends to a bounded nonlinear operator from $E\times\partial E$ into $(E\times\partial E)^*$, and finally posing (1.65) as

$$d\mathbb{X}(t) = \mathbb{A}(\mathbb{X}(t))\,dt + \mathbb{B}\,d\mathbb{W}(t), \quad \mathbb{X}(0) = \mathbf{x} \in H\times\partial H, \tag{1.66}$$

where $\mathbb{B} = (Q,B)$ and $\mathbb{W} = (W_1, W_2)$. Existence and uniqueness of a **variational solution** to (1.66) is obtained by showing that $\mathbb{A}$ satisfies the necessary assumptions (i.e., hemicontinuity, weak monotocity, coercivity, and boundedness) and $\mathbb{B} \in \mathscr{L}_2(H\times\partial H)$ so that the framework of [69] applies. Here, a $(\mathscr{F}_t)$-adapted process $(\mathbb{X}(t))_{t\geq 0}$ is a variational solution to (1.66) if $\mathbb{X}$ is $H\times\partial H$-valued and there exists a process $\widetilde{\mathbb{X}}$ that is $dt\otimes\mathbb{P}$-equivalent to $\mathbb{X}$ such that $\widetilde{\mathbb{X}}$ belongs to $L^\alpha([0,T]\times\Omega; E\times\partial E)\cap L^2([0,T]\times\Omega; H\times\partial H)$ with $\alpha > 1$ (dependent on $\mathbb{A}$) and for $t\in[0,T]$,

$$\mathbb{X}(t) = \mathbf{x} + \int_0^t \mathbb{A}(\widetilde{\widetilde{\mathbb{X}}}(s))\,ds + \int_0^t \mathbb{B}\,d\mathbb{W}(s), \quad \mathbb{P}\text{-a.s.},$$

where $\widetilde{\widetilde{\mathbb{X}}}$ is a $E\times\partial E$-valued progressively measurable $dt\otimes\mathbb{P}$-version of $\widetilde{\mathbb{X}}$.

Let $(\mathbb{X}^{\mathbf{x}}(t))_{t\geq 0}$ denote a solution to (1.66) with initial condition $\mathbb{X}(0) = \mathbf{x} \in H\times\partial H$ and let $C_b(H\times\partial H)$ be the space of bounded continuous functions on $H\times\partial H$. In the second part of [67], it is shown that $(\mathbb{X}^{\mathbf{x}}(t))_{t\geq 0}$ is a Markov process that satisfies

$$\mathbb{E}\sup_{t\in[0,T]} \|\mathbb{X}^{\mathbf{x}}(t)\|^2_{H\times\partial H} < \infty,$$

where $\|\overline{u}\|_{H\times\partial H}$ is defined by the inner product $\langle\overline{u},\overline{v}\rangle_{H\times\partial H} := \langle u,v\rangle_H + \langle u^\circ, v^\circ\rangle_{\partial H}$ with $\overline{u} := (u,u^\circ)$, and there the exists an ergodic invariant measure μ for the transition semigroup defined by $P_t\varphi(\mathbf{x}) := \mathbb{E}\varphi(\mathbb{X}^{\mathbf{x}}(t))$ for $\varphi\in C_b(H\times\partial H)$. As $E\times\partial E \hookrightarrow H\times\partial H$ is a compact embedding, μ is concentrated on $E\times\partial E$ and, under an additional "superlinearity" assumption on $\mathbb{A}$, μ is unique.

In the preprint [53], Brzeźniak, Fabbri, Goldys, Peszat and Russo consider the Poisson and heat equation with white noise Dirichlet boundary conditions on a (possibly unbounded) domain $U \subset \mathbb{R}^d$ with boundary ∂U (of class C^∞ when $d > 1$). That is, for $\lambda \geq 0$ they consider the elliptic equation

$$\Delta u = \lambda u \text{ on } U, \quad \tau u = \xi \text{ on } \partial U, \tag{1.67}$$

where $\tau u := u|_{\partial U}$ in terms of trace and ξ is a $\mathscr{S}'(\partial U)$-valued Gaussian random variable, and the heat equation

$$\frac{\partial u}{\partial t} = \Delta u \text{ on } \mathbb{R}_+ \times U, \quad \tau u = \dot{\xi} \text{ on } \mathbb{R}_+ \times \partial U, \tag{1.68}$$

with initial condition $u(0,\cdot) = u_0(\cdot)$ where $(\xi(t))_{t\geq 0}$ is a $\mathscr{S}'(\partial U)$-valued stochastic process with continuous paths. Here, $\mathscr{S}'(\mathbb{R}^d)$ is the space of tempered distributions on $\mathbb{R}^d$ which is the (topological) dual of the space of functions with rapid decrease $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(U)$ denotes distributions in $\mathscr{S}'(\mathbb{R}^d)$ with support on U [70, §V.3]. When U is unbounded, it is assumed that $\lambda > 0$ in (1.67). First, they show that a $\mathscr{S}'(U)$-valued mild solution of (1.68) is given by

$$u(x) = S(t)u_0 + \int_0^t S(t-s)(\lambda - A)\Lambda_\lambda \, d\xi(s), \tag{1.69}$$

where $(S(t))_{t\geq 0}$ is the C_0-semigroup on $L^2(U)$ generated by the Dirichlet Laplacian A with domain $\mathscr{D}(A) := W_0^{1,2}(U)$. Then they show equivalence between a weak solution defined in terms of (1.50) and the mild solution (1.69). Let $\delta(x) := \min(\operatorname{dist}(x, \partial U), 1)$ and $\varrho_{\alpha,\beta}(x) := \delta(x)^{1+\alpha}(1+|x|^2)^{-\beta}$ for $x \in \mathbb{R}^d$ and let (h_n) be an orthonormal basis of $\partial H := L^2(\partial U)$, their two main theorems are as follows.

First, there exists a unique solution u to (1.67) with $\xi = \sum_n \gamma_n h_n$ where (γ_n) is a sequence of independent standard $\mathbb{R}$-valued Gaussians. Moreover, $\mathbb{P}$-almost surely $u \in C^\infty(U)$, and for all $\beta > d/2$, $p \geq 1$, and $\alpha \in \mathbb{R}$ such that

$$\beta > -1, \, p \geq 1 \text{ if } d = 1, \quad \frac{d+1+\alpha}{d-1} > p \geq 1 \text{ if } d > 1,$$

u is a Gaussian random variable in the weighted space $L^p(U, \varrho_{\alpha,\beta}(x)\,dx)$.

Second, there exists a unique solution u to (1.68) with $\xi(t) = \sum_n w_n(t) h_n$ where $(w_n(t))_{t\geq 0}$ is a sequence of independent standard $\mathbb{R}$-valued Wiener processes such that $u \in L^q(\Omega; C^m([0,T] \times K))$ for any compact $K \subset U$ and all $q \geq 1$, $T > 0$, $m \in \mathbb{N}$. Further, for all $\beta > d/2$, $p \geq 1$, and $\alpha \in \mathbb{R}$ such that

$$\frac{d+1+\alpha}{d} > p \geq 1$$

u is a Markov process with values in $C([0,\infty); L^p(U, \varrho_{\alpha,\beta}(x)\,dx))$.

Finally, they demonstrate their results for the Dirichlet problem (1.68) for the case where $U = (0,1) \subset \mathbb{R}$ and show that solutions exist in $L^2(0,1; \delta(x)^{1+\alpha}\,dx)$ for $\alpha > 1$.

We note that the case where the sequence of Wiener processes $(w_n(t))_{t\geq 0}$ is replaced by fractional Brownian motions is also considered in this preprint.

1.16 Conclusion

As we have shown in this survey, since the first papers appeared in the 1990s many advances have been made on the topic of equations with boundary noise. These advances have largely been motivated by the need of more accurate physical models whereby noise enters through the boundary but also from a mathematical desire to extend existing results to more abstract and difficult situations. Although the boundary noise theory originally grew out of results from optimal control theory (i.e., the boundary control problem), it has taken a life of its own due to the significant new difficulties and differences that arise when the boundary noise is of white noise type. We recall that the white noise assumption is required to obtain solutions with the Markov property which is a highly desirable feature from both the theoretical and applied point of view. To date, a number of interesting results have been obtained by asking the natural question: can standard results for the interior noise problem be extended to the boundary noise case? This question has lead to numerous developments such as optimal control of systems perturbed by boundary noise, ergodicity for boundary noise problems, and the extension to the Lévy process case. A large number of new results have also appeared from a desire to consider the dynamical boundary noise situation. We note however that although such a large body of theory already exists, a number of results are lacking:

- Apart from a few examples, most results have been obtained for the Neumann or Robin boundary noise problem due to the inability to obtain solutions in $L^2(U)$ in the Dirichlet case.

- The only successful approach for the Dirichlet boundary noise problem has been to consider solutions in weighted L^p spaces however these results have all been for the one-dimensional case where $U \subset \mathbb{R}$. That is, no examples of Dirichlet boundary noise have been explicitly consider for $d \geq 2$.

- Apart from some recent preprints, the existing boundary noise theory has been largely considered in a Hilbert space framework. That is, there lacks an extension of these results into the Banach space setting. Such a theory would be useful to be able to consider the boundary noise problem in a (weighted) L^p space from a semigroup perspective allowing one to "tune" the parameter p.

- As only the one-dimensional situation has been consider for the Dirichlet boundary noise problem, the space-time white noise case has not been studied so far.

2

Background Material

In this section we recall some well-known results and definitions. The reader is referred to the lecture notes [6], which was our main reference, if more details or further references are required. For the Hilbert space or L^p setting, the reader may also refer to [19, 33].

Notation

For $[a,b] \subset \mathbb{R}$ and Banach space E, we denote by $B([a,b];E)$ and $C([a,b];E)$ the Banach spaces of all bounded (respectively, continuous) functions from $[a,b]$ to E endowed with the supremum norm $\|f\|_\infty := \sup_{a\le t\le b}\|f(t)\|_E$. We denote by $C^\alpha([a,b];E)$ the Banach space of all α-Hölder continuous functions from $[a,b]$ to E, endowed with the norm $\|f\|_{C^\alpha([a,b];E)} := \|f\|_\infty + [f]_{C^\alpha([a,b];E)}$ where $[f]_{C^\alpha([a,b];E)} := \sup_{a\le s,t\le b}\|f(t)-f(s)\|_E/(t-s)^\alpha$.

2.1 Semigroups

Let E be a Banach space. A family of operators $S(t)\in\mathscr{L}(E)$ with $t>0$ is a one-parameter **semigroup** if $S(t_1+t_2)=S(t_1)S(t_2)$ for $t_1,t_2>0$. We only deal with

strongly continuous semigroups (sometimes called C_0-semigroups) which means that $\lim_{t\to 0^+} S(t)x = x$ for all $x \in E$. Every $A \in \mathscr{L}(E)$ generates the semigroup $S(t) = e^{tA}$ where $e^{tA} := I + tA + \frac{1}{2}t^2A^2 + \cdots$. We define the **infinitesimal generator** of a semigroup $(S(t))_{t\geq 0}$ the operator $Ax := \lim_{h\to 0^+}(S(h)x - x)/h$ whose **domain** $\mathscr{D}(A)$ consists of all elements $x \in E$ for which the right-hand limit exists. The **resolvent set** of A is the set $\varrho(A)$ consists of all $\lambda \in \mathbb{C}$ for which there exists a unique bounded linear operator $R(\lambda, A)$ on E such that (i) $R(\lambda, T)(\lambda - A)x = x$ for all $x \in \mathscr{D}(A)$, (ii) $R(\lambda, A)x \in \mathscr{D}(A)$ and $(\lambda - A)R(\lambda, A)x = x$ for all $x \in E$. The **spectrum** of A is the complement $\sigma(A) := \mathbb{C} \setminus \varrho(A)$. The operator $R(\lambda, A) = (\lambda - A)^{-1}$ is called the **resolvent** of A at λ.

For $\sigma \in (0, \pi]$ define the open sector

$$\Sigma_\sigma = \left\{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \sigma\right\},$$

where the argument is taken in $(-\pi, \pi]$. A strongly continuous semigroup $(S(t))_{t\geq 0}$ is called **analytic** on Σ_σ if for all $x \in E$ the function $t \mapsto S(t)x$ extends analytically to Σ_σ and satisfies

$$\lim_{z\in\Sigma_\sigma, z\to 0} S(z)x = x.$$

First, let us recall, from [71] and [72], some equivalent characterizations of analytic semigroups and their generators.

Theorem 2.1. *For a densely defined closed operator A on a Banach space E, the following are equivalent:*

1. $\{\lambda R(\lambda, A) : \lambda \in \Sigma_\sigma\}$ *is bounded for some* $\sigma > \pi/2$.

2. *There is an analytic semigroup* $(S(t))_{t\geq 0}$ *on a sector* Σ_δ, $\delta > 0$, *such that*

$$\frac{d}{dz}S(z) = AS(z), \quad z \in \Sigma_{\sigma-\pi/2} \tag{2.1}$$

   holds and $(S(z))_{z\in\Sigma(\delta)}$ *is bounded.*

3. *There is a strongly continuous semigroup* $(S(t))_{t\geq 0}$ *such that* (2.1) *holds for* $t \in \mathbb{R}_+$ *and* $\{tAS(t) : t > 0\}$ *is bounded.*

2.2 $\mathscr{R}$-Bounded and γ-Bounded operators

$\mathscr{R}$-Boundedness and γ-Boundedness are generalisations, to a Banach space setting, of uniform boundedness of families of operators in Hilbert spaces.

Let $(\varrho_n)_{n=1}^N$ be a sequence of independent Rachemacher random variables. Let E_1 and E_2 be Banach spaces. An operator family $\mathscr{T} \subset \mathscr{L}(E_1, E_2)$ is said to be $\mathscr{R}$**-bounded** if there exists a constant $M \geq 0$ such that

$$\left(\mathbb{E}\left\|\sum_{n=1}^{N} \varrho_n T_n x_n\right\|^2\right)^{1/2} \leq M\left(\mathbb{E}\left\|\sum_{n=1}^{N} \varrho_n x_n\right\|^2\right)^{1/2}, \tag{2.2}$$

for all $N \geq 1$, all $T_1, \ldots, T_N \in \mathscr{T}$, and all $x_1, \ldots, x_N \in E_1$. If (2.2) holds with the sequence (ϱ_n) is replaced by a sequence of independent Gaussian random variables $(\gamma_n)_{n=1}^N$ the operator family $\mathscr{T}$ is called γ**-bounded**. The least admissible constant M is called the $\mathscr{R}$**-bound** (respectively, γ**-bound**) of $\mathscr{T}$.

Theorem 2.2 (Kalton & Weis)**.** *Let E_1, E_2 be Banach spaces. Suppose that $M : (0,T) \to \mathscr{L}(E_1, E_2)$ is strongly measurable and has γ-bounded range $\{M(t) : t \in (0,T)\} := \mathscr{M}$. Then for every finite rank simple function $\Phi : (0,T) \to \gamma(H, E_1)$ the operator $R_{M\Phi}$ belongs to $\gamma(L^2(0,T;H), E_2$ and*

$$\|R_{M\Phi}\|_{\gamma(L^2(0,T;H),E_2)} \leq \gamma(\mathscr{M})\|R_\Phi\|_{\gamma_p(L^2(0,T;H),E_1)}.$$

As a result, the map $\widetilde{M} : R_\Phi \to R_{M\Phi}$ has a unique extension to a bounded operator

$$\widetilde{M} : \gamma_p(L^2(0,T;H), E_1) \to \gamma_p(L^2(0,T;H), E_2)$$

of norm $\|\widetilde{M}\| \leq \gamma(\mathscr{M})$.

Lemma 2.3 (Lemma 10.17 [6])**.** *For all real numbers $\alpha, \beta, \eta \geq 0$ satisfying $0 \leq \alpha + \eta < \beta < 1$, the set*

$$\{t^\beta(-A)^\eta S(t) : t \in (0,T)\}$$

is $\mathscr{R}$-bounded (and hence γ-bounded) in $\mathscr{L}(E, E_\alpha)$ with γ-bound $O(T^{\beta-\alpha-\eta})$.

2.3 Abstract Cauchy problems

We shall now recall some results (e.g., see [71, 5]) on the inhomogeneous abstract Cauchy problem

$$u'(t) = Au(t) + f(t), \quad u(0) = u_0, \quad t \in [0, T], \tag{2.3}$$

taking values in a Banach space E where the unbounded linear operator $A : \mathscr{D}(A) \subset E \to E$ generates a strongly continuous semigroup of operators $(e^{tA})_{t\geq 0}$ on E and $f : [0, T] \to E$.

A function $u : [0, T] \to E$ is called a **classical solution** of (2.3) in the interval $[0, T]$ if

$$u \in C^1((0, T]; E) \cap C((0, T]; \mathscr{D}(A)) \cap C([0, T]; E),$$

and satisfies $u'(t) = Au(t) + f(t)$ for every $t \in (0, T]$, and $u(0) = u_0$. Further, if $f \in L^1(0, T; E) \cap C((0, T]; E)$, $u_0 \in \overline{\mathscr{D}(A)}$, and u is a classical solution of (2.3) then u satisfies the **variation of constants formula**

$$u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} f(s)\, ds, \quad 0 \leq t \leq T. \tag{2.4}$$

As the integral form (2.4) makes sense whenever $f \in L^1(0, T; E)$ and $u_0 \in E$, in this situation it is customary to call such a function u given by (2.4) a **mild solution** of (2.3). Thanks to this representation, questions about existence of classical solutions may be reduced to questions about the regularity of mild solutions. Although every classical solution has the representation (2.4), not every mild solution is a classical solution therefore it is often convenient to work with alternative solution concepts.

Definition 2.4. *For $1 \leq p < \infty$, a **strong solution** of (2.3) is a function $u \in L^1(0, T; E)$ such that for all $t \in [0, T]$ we have $\int_0^t u(s)\, ds \in \mathscr{D}(A)$ and*

$$u(t) = e^{tA} u_0 + A \int_0^t u(s)\, ds + \int_0^t f(s)\, ds.$$

*A **weak solution** of* (2.3) *is a function $u \in L^1(0,T;E)$ such that for all $t \in [0,T]$ and $v \in \mathscr{D}(A^*)$ we have*

$$\langle u(t), v\rangle = \langle u_0, v\rangle + \int_0^t \langle u(s), A^* v\rangle\, ds + \int_0^t \langle f(s), v\rangle\, ds.$$

One may show (e.g., Proposition 7.16 in [6]) the following useful equivalence between weak and strong solutions for (2.3).

Theorem 2.5. *Every weak solution of* (3.1) *is a strong solution, and vice-versa.*

Further, a strong (hence weak) solution satisfies a variation of constants formulation (e.g., Theorem 7.17 in [6]).

Theorem 2.6. *Fix $1 \le p < \infty$, then for all $u_0 \in E$ and $f \in L^1(0,T;E)$ the problem* (2.3) *admits a strong solution u, which is given by the convolution formula*

$$u(t) = e^{tA} u_0 + \int_0^t e^{(t-s)A} f(s)\, ds.$$

If $f \in L^p(0,T;E)$ with $1 \le q < \infty$, then $u \in L^p(0,T;E)$.

Let $A : \mathscr{D}(A) \subset E \to E$ be a sectorial operator, then one can introduce the intermediate Banach spaces between $\mathscr{D}(A)$ and E given, for $0 < \alpha < 1$, by all elements $x \in E$ such that

$$[x]_\alpha := \sup_{0<t\le 1} \|t^{1-\alpha} A e^{tA} x\| < \infty.$$

We denote these spaces by $\mathscr{D}_A(\alpha,\infty)$ and endow them with the norms

$$\|x\|_{\mathscr{D}_A(\alpha,\infty)} := \|x\|_E + [x]_\alpha.$$

We also introduce the space $\mathbb{B}_\alpha := B([0,T];\mathscr{D}_A(\alpha,\infty))$ and write $\dot{u}(t) := \frac{d}{dt} u(t)$.

Theorem 2.7. *Let $0 < \alpha < 1$, $A : \mathscr{D}(A) \subset E \to E$ be a sectorial operator, and let $f \in C([0,T];E) \cap \mathbb{B}_\alpha$. Then*

$$v(t) = \int_0^t e^{(t-s)A} f(s)\, ds,$$

solves (2.3) *with* $u_0 = 0$ *and belongs to* $C([0,T];\mathscr{D}(A)) \cap C^1([0,T];E)$*. Moreover,* $\dot{v}$ *and* Av *belong to* $\mathbb{B}_\alpha$ *and* Av *to* $C^\alpha([0,T];E)$*, and we have the estimate*

$$\|\dot{v}\|_{\mathbb{B}_\alpha} + \|Av\|_{\mathbb{B}_\alpha} + \|Av\|_{C^\alpha([0,T];E)} \lesssim \|f\|_{\mathbb{B}_\alpha}.$$

Let $A : \mathscr{D}(A) \subset E \to E$ be a sectorial operator on the Banach space E. There are a number of equivalent definitions for the intermediate spaces $\mathscr{D}_A(\alpha, p)$, $0 < \alpha < 1$, between $\mathscr{D}(A)$ and E given by

$$\mathscr{D}_A(\alpha, p) = \left\{ x \in E : t \mapsto v(t) = \|t^{1-\alpha-1/p} A e^{tA} x\| \in L^p(0,1) \right\}$$

with norm given by $\|x\|_{\mathscr{D}_A(\alpha,p)} := \|x\| + \|v\|_{L^p(0,1)}$. In particular, for $0 < \alpha < 1$ and $1 \le p \le \infty$ and for $(\alpha, p) = (1, \infty)$, $\mathscr{D}_A(\alpha, p)$ is a real interpolation space between E and $\mathscr{D}(A)$. i.e., $\mathscr{D}_A(\alpha, p) = (E, \mathscr{D}(A))_{\alpha,p}$.

2.4 Parabolic Hölder spaces

Let $J \subset \mathbb{R}^d$ be an open set and $0 < \theta < 1$, then the Banach space $C^\theta(\overline{J})$ is the space of all continuous functions $f : \overline{J} \to \mathbb{C}$ such that

$$[f]_{C^\theta(\overline{J})} := \sup_{x,y \in \overline{J}, x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\theta} < \infty.$$

For $U \subset \mathbb{R}^d$ and $T > 0$, the parabolic Hölder space $C^{\theta/2,\theta}([0,T] \times U)$ is the space of continuous functions $f : [0,T] \times U \to \mathbb{C}$ such that

$$\|u\|_{C^{\theta/2,\theta}(I \times U)} := \|u\|_{L^\infty([0,T]\times U)} + \sup_{x \in \overline{U}} [f(\cdot, x)]_{C^{\theta/2}([0,T])} + \sup_{t \in [0,T]} [f(t, \cdot)]_{C^\theta(\overline{U})} < \infty.$$

Remark 2.8. We recall [73, p.59] that $f \in C^{\theta/2,\theta}([0,T] \times \overline{U})$ if and only if

$$f \in C^{\theta/2}([0,T]; C(\overline{U})) \cap B([0,T]; C^\theta(\overline{U})).$$

2.5 The Dirichlet and Neumann Laplacians

In this section we recall the definition of the Dirichlet and Neumann Laplacians and their associated semigroups on $L^p(U)$. We refer to the books of Pazy [71] and Haroske and Triebel [74] for further details.

Let $U \subset \mathbb{R}^d$ be an open set. We shall denote by $C_c(U)$ the space of all continuous functions $u : U \to \mathbb{R}$ such that the support of u is a compact subset[1] of U, i.e.,

$$\operatorname{supp} u := \overline{\{x \in U : u(x) \neq 0\}} \Subset U,$$

and by $\mathscr{D}(U) := C^\infty(U) \cap C_c(U)$ the space of all **test functions**. For $u \in \mathscr{D}(U)$, we define the **Laplacian** $-\Delta$ by

$$(-\Delta u)(x) := -\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(x).$$

Let $L^p(U)$, $1 \leq p \leq \infty$, be the Lebesgue spaces and $L^1_{\text{loc}}(U)$ be the space of measurable functions $u : U \to \mathbb{R}$ such that $\int_K |u(x)|\, dx < \infty$ for all $K \Subset U$. Note that $L^p(U) \subset L^1_{\text{loc}}(U)$ for all $1 \leq p \leq \infty$. We define the **weak Laplacian** A by integration by parts. That is, for $u \in L^1_{\text{loc}}(U)$ and $v \in L^1_{\text{loc}}(U)$ we say $Au := -\Delta u = v$ weakly if

$$\int_U (-\Delta \varphi)\, u\, dx = \int_U \varphi\, v\, dx$$

holds for all $\varphi \in \mathscr{D}(U)$. Of course, this means that weak Laplacian Au is only unique up to a set of measure zero. For obvious reasons, the weak Laplacian is also sometimes called the **distributional Laplacian**. We define the **Sobolev space** $W^{1,p}(U)$ as the space of functions $u \in L^p(U)$ for which there exists functions $v_i \in L^p(U)$ that satisfy

$$\int_U u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_U v_i \varphi\, dx$$

[1] we use the notation $K \Subset U$ to mean that K is a compact subset of U.

for all $\varphi \in \mathscr{D}(U)$. The Sobolev spaces $W^{k,p}(U)$ with $k \geq 2$ are defined inductively as

$$W^{k,p}(U) := \{u \in W^{1,p}(U) : \frac{\partial u}{\partial x_i} \in W^{k-1,p}(U), 1 \leq i \leq d\},$$

and under the norm

$$\|u\|_{W^{k,p}(U)} := \|u\|_{L^p(U)} + \sum_{i=1}^{d} \|\frac{\partial u}{\partial x_i}\|_{W^{k-1,p}(U)},$$

the spaces $W^{k,p}(U)$ are Banach spaces. Finally, $W_0^{k,p}(U)$ is the closure of $\mathscr{D}(U)$ in the $W^{k,p}(U)$ norm.

On the space $L^2(U)$ with inner product $\langle\cdot,\cdot\rangle$, the **Dirichlet Laplacian** is the operator A_D defined by

$$A_D f := -\Delta f; \quad \mathscr{D}(A_D) := W^{2,2}(U) \cap W_0^{1,2}(U).$$

That is, A_D is the weak Laplacian on $L^2(U)$ subject to homogeneous (i.e., zero) Dirichlet boundary conditions. Notice that $\langle A_D u, u\rangle \geq 0$ for $u \in \mathscr{D}(A_D)$ so A_D is positive on $L^2(U)$. We recall that A_D is a self-adjoint dissipative operator on $L^2(U)$ that generates a contractive strongly continuous semigroup $(S_2(t))_{t\geq 0}$ on $L^2(U)$. We shall sometimes use the symbolic notation

$$e^{tA_D} f := S_2(t) f,$$

for $f \in L^2(U)$. For $t > 0$, the space $L^1 \cap L^\infty(U)$ is invariant under $S_2(t)$ and $(S_2(t))_{t\geq 0}$ may be extended from $L^1 \cap L^\infty(U)$ to a positive contraction semigroup $(S_p(t))_{t\geq 0}$ on $L^p(U)$ for each $1 \leq p \leq \infty$. As such, we shall use the notation $(S(t))_{t\geq 0}$ for all these semigroups (i.e. as p varies) as the choice of p will be clear from the context. We call $(S(t))_{t\geq 0}$ the **Dirichlet heat semigroup** on $L^p(U)$ for any $1 \leq p \leq \infty$, and write $e^{tA_D} f := S(t) f$ for $f \in L^p(U)$.

In a similar way, we can define the **Neumann Laplacian** A_N on $L^2(U)$ by setting $\mathscr{D}(A_N)$ to be all functions $u \in W^{1,2}(U)$ for which there exists $v \in L^2(U)$ so that

$$-\int_U \nabla u \cdot \nabla w \, dx = \int_U v w \, dx$$

holds for all $w \in W^{1,2}(U)$. Then $A_N u = v$ for $u \in \mathscr{D}(A_N)$ and A_N generates a strongly continuous semigroup $(S_2(t))_{t\geq 0}$ on $L^2(U)$. Again, $(S_2(t))_{t\geq 0}$ may be extended to $L^p(U)$ and we write $e^{tA_N} f := S(t)f$ for $f \in L^p(U)$ and call $(e^{tA_N})_{t\geq 0}$ the **Neumann heat semigroup** on $L^p(U)$ for any $1 \leq p \leq \infty$. Finally, let $G_U : (0,\infty) \times U \times U \to \mathbb{R}$ be the **Dirichlet heat kernel**, that is, the positive $C_0^\infty(U)$ function such that

$$(S(t)f)(x) = \int_U G_U(t,x,y) f(y)\, dy$$

for any $f \in L^p(U)$, $1 \leq p \leq \infty$.

2.6 Trace of a function

In this section we recall the concept of "trace" for functions in $L^p(U)$. The books of Adams [75] and Haroske and Triebel [74] are our main references.

If $u \in C(\overline{U})$ then the function given by taking the restriction of u to ∂U, written $u|_{\partial U}$, gives a function $u|_{\partial U} \in C(\partial U)$. Hence, for $f \in C(\overline{U})$, we can define the **(pointwise) trace** of f by $\tau f := f|_{\partial U}$. This definition does not extend immediately to functions $u \in L^p(U)$, $1 \leq p < \infty$ as $u|_{\partial U}$ need not have sense in general.

Let $L^p(\partial U)$, $1 \leq p < \infty$, be the Lebesgue spaces on the boundary ∂U of U normed by

$$\|f\|_{L^p(\partial U)} = \left(\int_{\partial U} |f(x)|^p \, d\sigma(x) \right)^{1/p},$$

where σ is the $(d-1)$-dimensional surface measure[2] on ∂U and let $C_0^\infty(\mathbb{R}^d)$ be the space of functions $u \in C^\infty(\mathbb{R}^d)$ such that $\lim_{|x|\to\infty} u(x) = 0$. If $U \subset \mathbb{R}^d$ is an open subset with C^1 boundary ∂U (or $U = \mathbb{R}^d_+$), then we have

$$\|\tau\varphi\|_{L^p(\partial U)} \lesssim \|\varphi\|_{W^{1,p}(U)}, \quad \forall \varphi \in C_0^\infty(\mathbb{R}^d).$$

[2]Naïvely defined in terms of surface patches.

As $\{\varphi|_{\partial U} : \varphi \in C_0^\infty(\mathbb{R}^d)\}$ is dense in $W^{1,p}(U)$ when ∂U is of class C^1, it becomes natural to define a trace of a function in $W^{1,p}(U)$ as a limit of pointwise traces of functions in $C_0^\infty(\mathbb{R}^d)$.

Definition 2.9. *Let U be of class C^1 with compact boundary ∂U (or $U = \mathbb{R}_+^d$) and $u \in W^{1,p}(U)$. Then $\tau u := \lim_{n\to\infty} \tau u_n$, in $L^p(\partial U)$ where $\{u_n\} \in C_0^\infty(\mathbb{R}^d)$ is such that $u_n \to u$ in the $W^{1,p}(U)$ norm. We call τu the **trace** of u.*

This definition is consistent as it is independent of the sequence (u_n) chosen. For $k \in \mathbb{N}$ and $1 \le p < \infty$ it holds that the linear operators

$$\begin{aligned} \tau : W^{k,p}(U) &\to L^p(U), \quad k \ge 1, \\ \tau \frac{\partial}{\partial \nu} : W^{k,p}(U) &\to L^p(U), \quad k \ge 2, \end{aligned}$$

are bounded and continuous. For $0 < s < 1$, we can define the fractional Sobolev spaces[3] $W^{s,2}(U)$ to be functions $f \in L^p(\partial U)$ such that $\|f\|_{W^{s,p}(\partial U)} < \infty$ where

$$\|f\|_{W^{s,p}(\partial U)} := \|f\|_{L^p(\partial U)} + \left(\int_{\partial U} \int_{\partial U} \frac{|f(x) - f(y)|^p}{|x-y|^{d-1+ps}} \, d\sigma(x) d\sigma(y) \right)^{1/p}.$$

Using this definition, we can say more about the trace τ and the **Neumann trace** $\tau\partial/\partial\nu$ where ν is the unit (outward) normal. That is, if $U \subset \mathbb{R}^d$ is a bounded domain with C^∞ boundary ∂U, then τ is a linear and bounded map of $W^{s,p}(U)$ onto $W^{s-\frac{1}{p},p}(\partial U)$ for $s > \frac{1}{2}$ and $\tau \frac{\partial}{\partial \nu}$ is a linear and bounded map of $W^{s,p}(U)$ onto $W^{s-1-\frac{1}{p},p}(\partial U)$ for $s > 1 + 1/p$.

2.7 The Dirichlet and Neumann maps

In this section we recall results on solutions to the Dirichlet and Neumann problems and, as such, we introduce the Dirichlet and Neumann maps. We

[3] Sobolev-Slobodeckii spaces

refer to the books of Wu, Yin and Wang [76] or Chen and Wu [77] for details in the L^p setting and Haroske and Triebel [74] in the L^2 setting.

Consider the inhomogeneous Dirichlet problem

$$-\Delta u = f, \qquad x \in U, \tag{2.5}$$

$$u = g, \qquad x \in \partial U. \tag{2.6}$$

where $g : \overline{U} \to \mathbb{R}$ such that $g \in W^{2,p}(U)$. If $u \in W^{2,p}(U)$, $u - g \in W_0^{1,p}(U)$ and satisfies (2.5) almost everywhere, then u is said to be a **strong solution** to the Dirichlet problem (2.5)-(2.6).

Theorem 2.10. *Let $f \in L^p(U)$ and $g \in W^{2,p}(U)$, $1 < p < \infty$. Then the Dirichlet problem* (2.5)-(2.6) *has a unique strong solution $u \in W^{2,p}(U)$ satisfying $u - g \in W_0^{1,p}(U)$.*

Definition 2.11. *The **Dirichlet map** is the continuous operator*

$$\Lambda_\lambda : W^{2-1/p,p}(\partial U) \to W^{2,p}(U)$$

such that for $g \in W^{2-1/p,p}(\partial U)$ we have the equivalence

$$\Lambda_\lambda g := u \iff \begin{cases} \Delta u = \lambda u, & \text{in } U, \\ \tau u = g, & \text{on } \partial U. \end{cases}$$

2.8 Gaussian random variables

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and we shall use $\mathbb{E}$ to denote the expectation operator $\mathbb{E}X := \int_\Omega X(\omega)\, d\mathbb{P}(\omega)$.

We recall that a $\mathbb{R}$-valued random variable ξ is called **Gaussian** if there exists a number $\sigma \geq 0$ such that

$$\mathbb{E}e^{-i\theta\xi} = e^{-\frac{1}{2}\sigma^2\theta^2}, \quad \theta \in \mathbb{R},$$

and we call ξ a **standard Gaussian** if $\mathbb{E}\xi = 0$ and $\mathbb{E}\xi^2 = \sigma^2 = 1$.

Let E be a Banach space with norm $\|\cdot\|_E$. A random variable $X:\Omega\to E$ is called **strongly measurable** if it is the pointwise limit of a sequence of simple random variables. We write $L^0(\Omega;X)$ as the vector space of strongly measurable random variables $X:\Omega\to E$ with the usual identification of variables equal $\mathbb{P}$-almost surely. Endowed with the topology induced by convergence in probability, $L^0(\Omega;E)$ is a complete metric space.

For random variables $X,X_1,X_2,\ldots\in L^0(\Omega;E)$ we have $X=\lim_{n\to\infty}X_n$ if and only if $X=\lim_{n\to\infty}X_n$ in probability. Let $X^*:=\mathscr{L}(E,\mathbb{R})$ and $\langle x,x^*\rangle:=x^*(x)$.

Definition 2.12. *A random variable X is called* ***Gaussian*** *if it is strongly measurable and for all $x^*\in X^*$, $\langle X,x^*\rangle$ is a $\mathbb{R}$-valued Gaussian random variable.*

2.9 Cylindrical Wiener process

Let $\mathscr{H}$ and H be Hilbert spaces with inner products $[\cdot,\cdot]$ and $(\cdot,\cdot)$, respectively. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space with associated expectation operator $\mathbb{E}$ and the space $L^2(\Omega)$ is endowed with the inner product $\mathbb{E}(XY)$ for $X,Y\in L^2(\Omega)$. We shall denote by $\mathbf{1}_A$ the indicator function of a set A.

Definition 2.13. *A $\mathscr{H}$*-***isonormal process*** *on Ω is a mapping $\mathscr{W}:\mathscr{H}\to L^2(\Omega)$ with the following properties:*

- *For all $h\in\mathscr{H}$ the random variable $\mathscr{W}h$ is Gaussian,*
- *For all $h_1,h_2\in\mathscr{H}$ we have $\mathbb{E}(\mathscr{W}h_1\mathscr{W}h_2)=[h_1,h_2]$.*

Example 2.14. A classic example of a $\mathscr{H}$-isonormal process is given in the case $\mathscr{H}=L^2(0,T)$, then $w(t):=\mathscr{W}\mathbf{1}_{[0,t]}$ defines a $\mathbb{R}$-valued Brownian motion on $[0,T]$.

The following definition will be of fundamental nature in this thesis as it represents a 'true' infinite-dimensional process.

Definition 2.15. *A H-**cylindrical Wiener process** on $[0,T]$ is a family $(W(t))_{t\in[0,T]}$ of mappings from H to $L^2(\Omega)$ with the following properties:*

- *$(W(t)h)_{t\in[0,T]}$ is a Brownian motion for all $h\in H$,*
- *for all $0\le t_1,t_2\le T$ and $h_1,h_2\in H$ we have*

$$\mathbb{E}(W(t_1)h_1W(t_2)h_2)=\min\{t_1,t_2\}(h_1,h_2).$$

The following two characterisations of a H-cylindrical Wiener process $W := (W(t))_{t\in[0,T]}$ are sometimes useful:

- If we choose $\mathscr{H}$ such that $\mathscr{H}=L^2(0,T;H)$ where H is a Hilbert space, then we can define W in terms of a $\mathscr{H}$-isonormal process $\mathscr{W}$ by

  $$W(t)h := \mathscr{W}(\mathbf{1}_{[0,t]}\otimes h),$$

  where $\otimes$ denotes the tensor product, e.g., if $h\in L^2(\mathbb{R})$ then

  $$\left(\mathbf{1}_{[0,t]}\otimes h\right)(s,x)=\mathbf{1}_{[0,t]}(s)h(x)$$

  for $s,x\in\mathbb{R}$.

- If (w_n) is a sequence of independent $\mathbb{R}$-valued Brownian motions $w_n := (w_n(t))_{t\ge0}$ and H is a separable Hilbert space with orthonormal basis (h_n), then

  $$W(t)h := \sum_{n=1}^{\infty} w_n(t)(h,h_n), \quad t\ge 0,$$

  defines a H-cylindrical Wiener process.

It should be noticed from the second characterisation that the H-cylindrical Wiener process $(W(t))_{t\in[0,T]}$ formally given by

$$W(t)=\sum_{n=1}^{\infty} w_n(t)h_n$$

is $\mathbb{P}$-almost surely divergent for $t \in (0, T]$ since

$$\mathbb{E}\|W(t)\|_H^2 = \sum_{n=1}^{\infty} t = \infty.$$

However, as we shall see in §2.11, if E is a Banach space and $B \in \mathscr{L}(H, E)$ is chosen correctly then we may have $\mathbb{E}\|BW(t)\|_E^2 < \infty$.

2.10 White noise

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. The following definition provides a mathematical formulation of idealised randomness which is independent between different (disconnected) locations and has large fluctuations at any location. It provides a fundamental building block for modelling random systems.

Definition 2.16. *Let $(M, \mathscr{M}, \mu)$ be a σ-finite measure space and denote by $\mathscr{M}_0$ the collection of all $B \in \mathscr{M}$ such that $\mu(B) < \infty$. A* ***Gaussian white noise*** *on $(M, \mathscr{M}, \mu)$ is a mapping $w : \mathscr{M}_0 \to L^2(\Omega)$ such that:*

- *each $w(B)$ is a centered Gaussian with*

$$\mathbb{E}w(B)^2 = \mu(B),$$

- *if $B_1 \cap B_2 = \emptyset$, then $w(B_1)$ and $w(B_2)$ are independent and*

$$w(B_1 \cup B_2) = w(B_1) + w(B_2).$$

Remark 2.17. As we only consider Gaussian white noise in this thesis, we simply refer to Gaussian white noise as **white noise**.

Definition 2.18. *If $U \subset \mathbb{R}^d$ and we set $M = U$ in Definition 2.16, then we call w a* ***space white noise****.*

Canonically associated with a space white noise w is a $L^2(U)$-cylindrical Gaussian random variable X, defined by

$$X1_B := W(1)1_B := w(B), \quad B \in \mathcal{M}_0(U).$$

where $(W(t))_{t\geq 0}$ is a $L^2(U)$-cylindrical Wiener process.

Definition 2.19. *A white noise on* $[0,T]\times U$*, where* $U \subset \mathbb{R}^d$ *and* $T > 0$*, will be called a* ***space-time white noise*** *on* U*.*

Canonically associated with a space-time white noise w is a $L^2(U)$-cylindrical Wiener process W, defined by

$$W(t)1_B := w([0,t]\times B), \quad B \in \mathcal{M}_0(U).$$

2.11 γ-Radonifying operators

When $h \in H$ and $x \in E$, we denote by $h \otimes x$ the operator in $\mathcal{L}(H,E)$ defined by

$$(h \otimes x)h' := (h, h')x, \quad h' \in H. \tag{2.7}$$

An operator in $\mathcal{L}(H,E)$ is said to be of **finite rank** if it is a linear combination of operators of the form (2.7). Even further, every finite rank operator $T : H \to E$ can be represented in the form $T = \sum_{n=1}^N h_n \otimes x_n$ where $(h_n)_{n=1}^N$ is an orthonormal sequence in H and $(x_n)_{n=1}^N$ is a sequence in E. For a such an operator we define the norm

$$\|T\|^2_{\gamma(H,E)} := \mathbb{E}\left\|\sum_{n=1}^N \gamma_n x_n\right\|^2.$$

where (γ_n) is a sequence of independent standard $\mathbb{R}$-valued Gaussian random variables. This formula is independent of the particular representation of T and defines a norm on the space of finite rank operators from H to E which is stronger than the uniform operator norm.

Definition 2.20. *The space $\gamma(H,E)$ is defined as the closure of all finite rank operators in the norm $\|\cdot\|_{\gamma(H,E)}$. The operators in $\gamma(H,E)$ are called γ**-radonifying**.*

Since convergence in $\gamma(H,E)$ implies convergence in $\mathscr{L}(H,E)$, every operator $T\in\gamma(H,E)$, being the operator norm limit of a sequence of finite rank operators from H to E, is compact.

Theorem 2.21. *If H is separable, then for an operator $T\in\mathscr{L}(H,E)$ the following assertions are equivalent:*

- *$T\in\gamma(H,E)$,*
- *for all orthonormal bases (h_n) in H and all $1\le p<\infty$ the sum $\sum_{n=1}^{\infty}\gamma_n T h_n$ converges in $L^p(\Omega;E)$,*
- *for some orthonormal bases (h_n) in H and some $1\le p<\infty$ the sum $\sum_{n=1}^{\infty}\gamma_n T h_n$ converges in $L^p(\Omega;E)$.*

If H is separable and $B\in\gamma(H,E)$ then the sum $\sum_{n=1}^{\infty}\gamma_n B h_n$ converges $\mathbb{P}$-almost surely and defines an E-valued Gaussian random variable with covariance operator BB^*. In particular, if $(W(t))_{t\in[0,T]}$ is a cylindrical Wiener process on H, then for all $t\in[0,T]$,

$$\mathbb{E}\|BW(t)\|_E^2<\infty,$$

so $(BW(t))_{t\in[0,T]}$ is an E-valued Wiener process (that is no longer cylindrical!).

We denote by $\mathscr{L}_2(H_1,H_2)$ the space of all **Hilbert-Schmidt operators** between the Hilbert spaces H_1 and H_2. The following characterisation when E is a Hilbert space is useful:

Theorem 2.22. *If E is a Hilbert space, then $R\in\gamma(H,E)$ if and only if $R\in\mathscr{L}_2(H,E)$, and in this case we have*

$$\|R\|_{\gamma(H,E)}=\|R\|_{\mathscr{L}_2(H,E)}.$$

When H is a finite-dimensional space, for example $H = \mathbb{R}^d$, then the following theorem states that even the identity operator $I \in \mathscr{L}(H)$ is a Hilbert-Schmidt operator.

Theorem 2.23. *Let H be a Hilbert space and $R \in \mathscr{L}(H)$. If $\dim(H) < \infty$, then $R \in \mathscr{L}_2(H)$.*

2.12 Wiener process

Definition 2.24. *An E-valued process $((W(t))_{t\geq 0}$ is called an **E-valued Brownian motion** if it has the following properties:*

- *$W(0) = 0$ almost surely,*
- *$W(t-s)$ and $W(t) - W(s)$ are identically distributed Gaussian random variables for all $0 \leq s \leq t \leq T$,*
- *$W(t) - W(s)$ is independent of $\{W(r) : 0 \leq r \leq s\}$ for all $0 \leq s \leq t \leq T$.*

The next proposition shows that given $B \in \gamma(H, E)$ and a cylindrical Wiener process $(W(t))_{t\in[0,T]}$ we can construct a E-valued Wiener process $(W^B(t))_{t\in[0,T]}$. In fact, the converse holds too: every E-valued Brownian motion is of the form W^B for canonical choices of H and $B \in \gamma(H, E)$, see [6].

Proposition 2.25. *Let $(W(t))_{t\in[0,T]}$ be an H-cylindrical Brownian motion and let $B \in \gamma(H, E)$. If $(h_n)_{n=1}^{\infty}$ is an orthnormal basis of $(\ker(B))\perp$, then:*

- *the sum*
$$W^B(t) := \sum_{n=1}^{\infty} W(t)h_n \otimes Bh_n$$
converges almost surely and in $L^p(\Omega; E)$, $1 \leq p < \infty$, for all $t \in [0, T]$,
- *up to a null set, $W^B(t)$ is independent of the basis $(h_n)_{n=1}^{\infty}$,*
- *the process $(W^B(t))_{t\in[0,T]}$ defines an E-valued Brownian motion.*

2.13 Stochastic integration

We shall now quickly recall how to define a stochastic integral of a function $\Phi : (0,T) \to \mathscr{L}(H,E)$ with respect to W.

For an $\mathscr{L}(H,E)$-valued step function of the form $\Phi = \mathbf{1}_{(a,b)} \otimes (h \otimes x)$ with $0 \le a < b \le T$ and $h \in H$, $x \in E$, we define the random variable $\int_0^T \Phi\, dW \in L^2(\Omega; E)$ by

$$\int_0^T \Phi\, dW := (W(b)h - W(a)h) \otimes x$$

and extend this definition by linearity to step functions with values in the space of the finite rank operators. Any step function $\Phi : (0,T) \to \mathscr{L}(H,E)$ uniquely defines a bounded operator $R_\Phi \in \mathscr{L}(L^2(0,T;H),E)$ by the formula

$$R_\Phi f := \int_0^T \Phi(t) f(t)\, dt, \quad f \in L^2(0,T;H).$$

Theorem 2.26 (Itô isometry)**.** *For all finite rank step functions $\Phi : (0,T) \to \mathscr{L}(H,E)$ we have $R_\Phi \in \gamma(L^2(0,T;H),E)$, the stochastic integral $\int_0^T \Phi\, dW$ is a Gaussian random variable, and*

$$\mathbb{E}\Big\| \int_0^T \Phi\, dW \Big\|^2 = \|R_\Phi\|^2_{\gamma(L^2(0,T;H),E)}.$$

Definition 2.27. *A function $\Phi : (0,T) \to \mathscr{L}(H,E)$ is said to be **stochastically integrable** with respect to W if there exists a sequence of finite rank step functions $\Phi_n : (0,T) \to \mathscr{L}(H,E)$ such that*

- *for all $h \in H$ we have $\lim_{n\to\infty} \Phi_n h = \Phi h$ in measure,*
- *there exists an E-valued random variable X such that*

  $$\lim_{n\to\infty} \int_0^T \Phi_n\, dW = X$$

  in probability.

*The **stochastic integral** of a stochastically integrable function $\Phi : (0,T) \to \mathscr{L}(H,E)$ is then defined as the limit in probability*

$$\int_0^T \Phi\, dW := \lim_{n\to\infty} \int_0^T \Phi_n\, dW.$$

Remark 2.28. Convergence of $\Phi_n h \to \Phi h$ in measure means

$$\lim_{n\to\infty} \mathrm{Leb}(\{t : \|\Phi_n(t)h - \Phi(t)h\| > r\}) = 0$$

for all $h \in H$, $t \in (0,T)$, and $r > 0$.

For a function $\Phi : (0,T) \to \mathscr{L}(H,E)$ and elements $h \in H$ and $x^* \in E^*$ we define $\Phi h : (0,T) \to E$ and $\Phi^* x^* : (0,T) \to H$ by $(\Phi h)(t) := \Phi(t)h$ and $(\Phi^* x^*)(t) := \Phi^*(t)x^*$. A function $\Phi : (0,T) \to \mathscr{L}(H,E)$ is called H**-strongly measurable** if for each $h \in H$ the function $\Phi h : (0,T) \to E$ is strongly measurable.

Theorem 2.29. *For an H-strongly measurable function $\Phi : (0,T) \to \mathscr{L}(H,E)$ the following assertions are equivalent:*

- *Φ is stochastically integrable with respect to W*
- *$\Phi^* x^* \in L^2(0,T;H)$ for all $x^* \in E^*$, and there exists an E-valued random variable X such that for all $x^* \in E^*$, almost surely we have*

  $$\langle X, x^* \rangle = \int_0^T \Phi^* x^*\, dW,$$

- *$\Phi^* x^* \in L^2(0,T;H)$ for all $x^* \in E^*$, and there exists an operator*

  $$R \in \gamma(L^2(0,T;H), E)$$

  such that for all $f \in L^2(0,T;H)$ and $x^* \in E^*$ we have

  $$\langle Rf, x^* \rangle = \int_0^T \langle \Phi(t) f(t), x^* \rangle\, dt.$$

If these equivalent conditions are satisfied, the random variable X and the operator R are uniquely determined, we have $X = \int_0^T \Phi\, dW$ almost surely, and

$$\mathbb{E}\Big\| \int_0^T \Phi\, dW \Big\|^2 = \|R\|^2_{\gamma(L^2(0,T;H),E)}.$$

Theorem 2.30. *Let $(A, \mathscr{A}, \mu)$ be a σ-finite measure space.*

- *If E has type 2, then the mapping $(f \otimes h) \otimes x \mapsto f \otimes (h \otimes x)$ has a unique extension to a continuous embedding*

  $$L^2(A; \gamma(H,E)) \hookrightarrow \gamma(L^2(A;H), E)$$

  of norm at most $T_2(E)$. Conversely, if the identity mapping $f \otimes x \mapsto f \otimes x$ extends to a bounded operator from $L^\infty(0,1;E)$ to $\gamma(L^2(0,1),E)$, the E has type 2.

- *If E has cotype 2, then the mapping $f \otimes (h \otimes x) \mapsto (f \otimes h) \otimes x$ has a unique extension to a continuous embedding*

  $$\gamma(L^2(A;H), E) \hookrightarrow L^2(A; \gamma(H,E))$$

  of norm at most $C_2(E)$. Conversely, if the identity mapping $f \otimes x \mapsto f \otimes x$ extends to a bounded operator from $\gamma(L^2(0,1),E)$ to $L^1(0,1;E)$, then E has cotype 2.

2.14 Stochastic abstract Cauchy problems

We now recall existence and uniqueness results for the **stochastic abstract Cauchy problem**

$$dX(t) = AX(t)\, dt + G\, dW(t), \quad t \in [0,T] \qquad \text{(SACP)}$$

with $X(0) = x \in E$. Here A is the generator of a strongly semigroup $(S(t))_{t\geq 0}$ on E and $G \in \mathscr{L}(H,E)$ is a given bounded operator. We call an E-valued process $(X(t))_{t\in[0,T]}$ **strongly measurable** if it has a version which is strongly $\mathscr{B}([0,T]) \times \mathscr{F}$-measurable on $[0,T] \times \Omega$.

Definition 2.31. *A **weak solution** of the problem* (SACP) *is an E-valued process $(X^x(t))_{t\in[0,T]}$ which has a strongly measurable version with the following properties:*

- *almost surely, the paths $t \mapsto X^x(t)$ are integrable,*
- *for all $t \in [0,T]$ and $x^* \in \mathscr{D}(A^*)$ we have, almost surely,*

$$\langle X^x(t), x^*\rangle = \langle x, x^*\rangle + \int_0^t \langle X^x(s), A^*x^*\rangle \, ds + W(t)G^*x^*.$$

Remark 2.32. We do not distinguish between $(X^x(t))_{t\in[0,T]}$ and its version $(\widetilde{X^x(t)})_{t\in[0,T]}$ that satisfies the previous definition.

Theorem 2.33. *The following assertions are equivalent:*

- *the problem* (SACP) *has a weak solution $(X^x(t))_{t\in[0,T]}$,*
- *$t \mapsto S(t)G$ is stochastically integrable on $(0,T)$ with respect to W.*

If one holds, then for every $t \in (0,T)$ the function $s \mapsto S(t-s)G$ is stochastically integrable on $(0,t)$ with respect to W and almost surely we have

$$X^x(t) = S(t)x + \int_0^t S(t-s)G \, dW(s). \tag{2.8}$$

If we assume that $G \in \gamma(H,E)$ then the term '$G\,dW$' may be replaced by dW^G where W^G is an E-valued Wiener process canonically associated with G: if (h_n) is an orthonormal basis of $(\ker(G))^\perp$ then

$$W^G(t) := \sum_{n=1}^{\infty} W(t)h_n \otimes Gh_n$$

converges almost surely and in $L^p(\Omega; E)$, $1 \le p < \infty$, for all $t \in [0,T]$.

Definition 2.34. *Let $G \in \gamma(H,E)$. A **strong solution** of* (SACP) *is a strongly measurable E-valued process $(X^x(t))_{t\in[0,T]}$ with the following properties:*

- *the trajectories of X^x are integrable almost surely,*
- *for all $t \in [0,T]$, almost surely we have $\int_0^t X^x(s)\,ds \in \mathscr{D}(A)$ and*

$$X^x(t) = x + A\int_0^t X^x(s)\,ds + W^G(t).$$

Theorem 2.35. *Let $G \in \gamma(H,E)$. The following assertions are equivalent:*

- *the problem* (SACP) *has a strong solution,*
- *the problem* (SACP) *has a weak solution.*

In this situation, the weak and strong solutions are versions of each other and both are given by (2.8).

2.15 Weighted Sobolev spaces

Let $U \subset \mathbb{R}^d$ be a domain with boundary ∂U (of codimension 1) and let ϱ be a vector of nonnegative (positive almost everywhere) measurable functions on U, that will be called a **weight**, i.e.

$$\varrho := \{\varrho_s = \varrho_s(x) : x \in U, |\alpha| \le k\},$$

where s is a multi-index $s := (s_1, s_2, \ldots, s_d)$ with $s_n \in \mathbb{N} \cup \{0\}$ and $|s| := s_1 + \ldots + s_d$.

Definition 2.36. *The space $W^{k,p}(U,\varrho)$, $k \in \mathbb{N} \cup \{0\}$ and $1 \le p < \infty$, is defined as the set of all functions u that are defined a.e. on U and whose (distributional) derivatives $D^s u$ for orders $|s| \le k$ satisfy*

$$\int_U |D^s u(x)|^p \varrho_s(x)\,dx < \infty.$$

We call these ***weighted Sobolev spaces****.*

A weighted Sobolev space is a normed linear space if equipped with the norm

$$\|u\|_{k,p,\varrho} := \left(\sum_{|s|\leq k}\int_U |D^s u(x)|^p \varrho_s(x)\,dx\right)^{1/p}.$$

For $k=0$, by convention, $W^{0,p}(U,\varrho)=L^p(U,\varrho)$ and we call $L^p(U,\varrho)$ a **weighted L^p space**. If $\varrho_s(x)\equiv 1$ for $|s|\leq k$, then $W^{k,p}(U,\varrho)=W^{k,p}(U)$, i.e. we retrieve the "classical" Sobolev space. We shall mostly assume that the components ϱ_s coincide, that is,

$$\varrho_s(x)=\varrho(x), \quad \forall s,\ |s|\leq k.$$

We note that if a weight function ϱ satisfies

$$0<c_1\leq\varrho(x)\leq c_2, \quad \forall x\in U$$

for fixed $c_1, c_2 \in \mathbb{R}_+$ then the space $W^{k,p}(U,\varrho)$ is equivalent to the space $W^{k,p}(U)$.

Weighted Sobolev spaces have been used in many places in the PDE literature but also in the harmonic analysis literature. We shall now present two types of weights, the associated weighted Sobolev spaces, and their properties.

We now recall some facts about the class of Muckenhoupt weights that sometimes make an appearance in the harmonic analysis literature.

We call a $Q\subset\mathbb{R}^d$ a **cube** if it is of form $Q=\prod_{j=1}^n I_j$ where $I_1,\dots I_n\subset\mathbb{R}$ are bounded intervals of the same length and Q has sides parallel to the axes.

Definition 2.37. *Let* $1<q<\infty$. *A function* $0\leq\varrho\in L^1_{\text{loc}}(\mathbb{R}^d)$ *is called an* A_p*-**weight** if*

$$\sup_Q\left(\frac{1}{|Q|}\int_Q \varrho\,dx\right)\left(\frac{1}{|Q|}\int_Q \varrho^{-1/(p-1)}\,dx\right)^{p-1}<\infty$$

where the supremum is taken over all cubes $Q\subset\mathbb{R}^d$ *and* $|Q|$ *assigns the Lebesgue measure of* Q.

We now recall the concept of a maximal function so that we may present some useful characterisations of the class of A_p weights.

Definition 2.38. *For a nonnegative Borel measure μ on $\mathbb{R}^d$, we call*

$$M\mu(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q d\mu,$$

the ***Hardy-Littlewood maximal function****. Here, the supremum is taken over all cubes Q containing x and for a measurable function $w \geq 0$ defining the measure $d\mu(y) = w(y)\,dy$ we write $M\mu(x) = Mw(x)$.*

We now collect a number of characterisation of the class of A_p weights, found in [78], in the next lemma.

Lemma 2.39. *A_p weights have the following properties and relationships:*

- *Let μ be a nonnegative Borel measure such that $M\mu(x) < \infty$ almost everywhere, then $(M\mu)^\beta$ is an A_1 weight for all $\beta \in [0,1)$.*
- *$A_p \subset A_q$, $1 \leq p < q$.*
- *$\varrho \in A_p$ if and only if $\varrho^{1-p'} \in A_{p'}$.*
- *If $\varrho_0, \varrho_1 \in A_1$ then $\varrho_0 \varrho_1^{1-p} \in A_p$.*

When we take the weight ϱ in the definition of weighted Sobolev spaces to be of the class A_p then the spaces $W^{k,p}(U,\varrho)$ have a few properties that we shall now recall.

To avoid confusion and to formalise the correspondance between the index p in the definition of an A_p weight and the index p of the Sobolev spaces we give

Definition 2.40. *If ϱ is an A_p weight and $U \subset \mathbb{R}^d$ an open set we define*

$$L^p(U,\varrho) := \left\{ f \in L^1_{\mathrm{loc}}(\overline{U}) : \|f\|_{p,\varrho} := \left(\int_U |f|^p \varrho \, dx \right)^{1/p} < \infty \right\}.$$

As noted before, if $\varrho \equiv 1$ then $L^p(U,\varrho) = L^p(U)$ and if $\sup_{x\in U} |\varrho(x)| < \infty$ then $L^p(U,\varrho) \simeq L^p(U)$. We also have that

$$(L^p(U,\varrho))^* = L^q(U,\varrho^*) \text{ where } \frac{1}{p} + \frac{1}{q} = 1 \text{ and } \varrho^* := \varrho^{-\frac{1}{p-1}} = \varrho^{-q/p}.$$

For every $1 < p < \infty$ and A_p weight ϱ, there exists $r_1, r_2 \in \mathbb{R}$ such that $r_2 < p < r_1$ and the following imbeddings hold [79]:

$$L^{r_1}(U) \hookrightarrow L^p(U,\varrho) \hookrightarrow L^{r_2}(U).$$

or more precisely,

Lemma 2.41. *If* $1 \le s$, ϱ *is an* A_s *weight, and* $1 \le q < \infty$, *then for* $p \ge sq$ *one has*

$$L^p(U,\varrho) \hookrightarrow L^q(U).$$

For an A_p weight, the Sobolev space definitions follow from our initial definition or equivalently by

$$W^{k,p}(U,\varrho) := \left\{ f \in L^p(U,\varrho) : \|f\|_{k,p,\varrho} := \sum_{|s|\le k} \|D^s f\|_{p,\varrho} < \infty \right\}$$

and the **dual space** is given by

$$W^{-k,p}(U,\varrho) = \left(W^{k,q}(U,\varrho^*)\right)^*, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1,\ \varrho^* = \varrho^{-\frac{1}{p-1}}.$$

For $k \in \mathbb{N}$, $p \in (1,\infty)$ and ϱ an A_p weight, we define the **trace space** as

$$T^{k,p}(\partial U,\varrho) := (W^{k,p}(U,\varrho))|_{\partial U}$$

equipped with the norm

$$\|g\|_{T^{k,p}(\partial U,\varrho)} := \inf\left\{\|u\|_{W^{k,p}(U,\varrho)} : u \in W^{k,p}(U,\varrho),\ u|_{\partial U} = g\right\}.$$

It it well-known that in case $\varrho \equiv 1$ that for $1 < p < \infty$ and $k \in \mathbb{N}$ that

$$T^{k,p}(\partial U) = W^{k-1/p,p}(\partial U).$$

Finally, as in the unweighted case, the following relationships hold when $1 < p < \infty$:

- For $k \in \mathbb{N}$, the trace restriction

$$\tau : u \mapsto u|_{\partial U} : W^{k,p}(U,\varrho) \to T^{k,p}(U,\varrho),$$

is continuous (with norm 1).

- A Green's formula holds: for $u \in W^{1,p}(U,\varrho)$ and $v \in W^{1,q}(U,\varrho^*)$ where $1/p + 1/q = 1$ and $\varrho^* = \varrho^{-1/(p-1)}$,

$$\int_U u \nabla v \, dx = \int_{\partial U} u \partial_\nu v \, d\sigma - \int_U \nabla u \, v \, dx,$$

where σ is the surface measure on ∂U.

- There exists a continuous extension operator

$$\Lambda : T^{1,p}(\partial U,\varrho) \to W^{1,p}(U,\varrho)$$

with $\tau\Lambda g = g$ for every $g \in T^{1,p}(\partial U,\varrho)$. We call Λ the **Dirichlet map**.

- For $\varphi \in C^{k-1}(U)$ the multiplication operator

$$u \mapsto u\varphi, \; W^{k,p}(U,\varrho) \to W^{k,p}(U,\varrho),$$

is continuous.

The class of A_p weights is less familiar in the PDE literature where the concept of weighted Sobolev spaces is generally taken directly to be a space weighted by $\varrho(x) := \mathrm{dist}(x,\partial U)^\alpha$. Here, $\mathrm{dist}(x,\partial U)$ stands for the distance of the point $x \in U$ from the boundary ∂U, i.e.

$$\mathrm{dist}(x,\partial U) := \inf_{z \in \partial U} |x - z|,$$

and $\alpha \in \mathbb{R}$. We shall often abbreviate $\delta(x) := \mathrm{dist}(x,\partial U)$. Setting $\varrho(x) = \delta(x)^\alpha$ for $\alpha \in \mathbb{R}$ in the definition of our weighted Sobolev spaces gives the spaces $W^{k,p}(U,\delta^\alpha)$.

Lemma 2.42 (Theorem 3.6 in [13])**.** *The spaces $W^{k,p}(U,\delta^{\alpha})$ are separable Banach spaces.*

Let $U \subset \mathbb{R}^d$ be a bounded domain. The Sobolev spaces $W^{k,p}(U,\delta^{\alpha})$ have a number of imbedding properties.

Lemma 2.43 (Lemma 6.2 in [13])**.** *Let $\alpha, \beta \in \mathbb{R}$. Then*

$$L^p(U,\delta^{\alpha}) \hookrightarrow L^p(U,\delta^{\beta})$$

for $\alpha \leq \beta$.

Theorem 2.44 (Theorem 6.3 in [13])**.**

$$\begin{aligned} W^{k,p}(U) \hookrightarrow W^{k,p}(U,\delta^{\alpha}), &\quad \alpha \geq 0, \\ W^{k,p}(U,\delta^{\alpha}) \hookrightarrow W^{k,p}(U), &\quad \alpha \leq 0. \end{aligned}$$

As a consequence, the weighted Sobolev spaces $W^{k,p}(U,\delta^{\alpha})$ for $\alpha > 0$ are richer and for $\alpha < 0$ poorer than the corresponding classical Sobolev spaces $W^{k,p}(U)$. Nevertheless, for **some** $\alpha > 0$ it is possible to imbed the spaces $W^{k,p}(U,\delta^{\alpha})$ into a certain Sobolev space.

Theorem 2.45 (Prop. 6.5 & Cor. 6.7 in [13])**.** *Let $p > 1$ and $1 \leq q < p$. Then*

$$W^{k,p}(U,\delta^{\alpha}) \hookrightarrow W^{k,q}(U),$$

if α and q satisfy

$$0 \leq \alpha < p-1, \quad 1 \leq q < \frac{p}{\alpha+1}.$$

Let $U \subset \mathbb{R}^d$ have a C^2 boundary ∂U. The following theorem shows that the trace map τ is bounded from $W^{1,p}(U,\delta^{\alpha})$ to $L^p(\partial U)$.

Theorem 2.46 (Theorem 9.15 in [13])**.** *Let $1 < p < \infty$ and $u \in W^{1,p}(U,\delta^{\alpha})$ for $0 \leq \alpha < p-1$. Then*

$$\|\tau u\|_{L^p(\partial U)} \lesssim \|u\|_{W^{1,p}(U,\delta^{\alpha})}.$$

3

Deterministic Boundary Data

In this chapter, we extend the abstract Hilbert space approach for boundary value problems [1, 2, 3] to the abstract Banach space setting. As a Banach space theory seems largely folklore, there seems to be a case for providing a unified theory here. In addition, these results to provide a point of departure for Chapter 4 where we consider the stochastic case and a framework for transferring elliptic results to parabolic results on "nonstandard" spaces (see §6.3 and §7.3).

Let E, ∂E, and Y be Banach spaces, let $\underline{A} : \mathscr{D}(\underline{A}) \subset E \to E$ be a closed and densely defined linear operator, and let $T > 0$ be some finite time horizon. In this chapter we study the inhomogeneous abstract boundary value problem

$$u'(t) = \underline{A}u(t) + f(t), \quad \tau u(t) = Bg(t), \quad u(0) = x, \tag{3.1}$$

where $g : [0, T] \to Y$, $B \in \mathscr{L}(Y, \partial E)$, and $\tau : \mathscr{D}(\tau) \subset E \to \partial E$.

In §3.1 we make use of the assumptions and decomposition from [46] to obtain a Banach space theory then, in §3.2, we weaken these assumptions to allow less regular data.

The cases $E = C(\overline{U})$ and $E = L^p(U)$, $1 < p < \infty$, for some domain $U \subset \mathbb{R}^d$ for Dirichlet and Neumann boundary conditions have been studied before.

For this theory, we refer to Lunardi's monograph [5] and Amann's paper [4]. We sketch how these cases can be obtained as examples in §3.3, as such, this chapter provides an abstraction of their ideas by applying results of Greiner [46]. We also show how the abstract formulation can be related to parabolic layer potentials in §3.4, thus relating the abstract approach to the well-known PDE approach (e.g., see [80, 81]).

3.1 Abstract boundary value problems

We now consider the abstract boundary value problem (3.1) in a Banach space setting. To initiate this construction, we introduce another linear operator $A : \mathscr{D}(A) \subset E \to E$ that is defined by $Au := \underline{A}u$ for $u \in \mathscr{D}(A)$ where

$$\mathscr{D}(A) := \mathscr{D}(\underline{A}) \cap \ker \tau.$$

We call $\underline{A}$ the **maximal operator** and A the **restricted** or **constrained operator**. This naming convention follows from the fact that A is the operator $\underline{A}$ with domain constrained to functions with zero boundary conditions. Of course, this implies $\mathscr{D}(A) \subseteq \mathscr{D}(\underline{A})$.

Example 3.1. Let $U \subset \mathbb{R}^d$ be a bounded domain with C^∞ boundary ∂U and set $E = L^2(U)$. On the space E, we define the operator $\underline{A}$ by

$$\begin{aligned} \mathscr{D}(\underline{A}) &:= W^{2,2}(U), \\ \underline{A}u &:= -\Delta u \quad \text{for } u \in \mathscr{D}(\underline{A}). \end{aligned}$$

Let $\tau u := u|_{\partial U}$ in terms of trace (see §2.6) then

$$\ker \tau = \{u \in E : u|_{\partial U} = 0 \text{ in trace}\},$$

and it follows that the operator A is given by

$$\mathscr{D}(A) = W_0^{2,2}(U), \quad Au := -\Delta u \text{ for } u \in \mathscr{D}(A).$$

In other words, A is the Dirichlet Laplacian on E (see §2.5).

Example 3.2. Let $U \subset \mathbb{R}^d$ with smooth boundary ∂U. Other common boundary conditions include $\tau u := \partial_\nu u|_{\partial U}$ (Neumann), $\tau u := u|_{\partial U} + \partial_\nu u|_{\partial U}$ (Robin) and $\tau u := u|_{\partial U} + \partial_\nu|_{\partial U} + \Delta|_{\partial U} u$ (Wentzell), where $\Delta|_{\partial U}$ is the Laplace-Beltrami operator on the manifold ∂U.

In [46], Greiner studied perturbations of the domain $\mathscr{D}(\underline{A})$ of an unbounded operator $\underline{A}$ on a Banach space E. As part of his study, Greiner developed a number of useful lemmas that we shall make use of in this chapter. As such, we need to introduce a number of assumptions into our abstract framework.

3.1.1 Assumptions

We shall consider (3.1) under the following assumptions:

- $\mathscr{D}(\underline{A}) \equiv \mathscr{D}(\tau)$ and $\tau : \mathscr{D}(\underline{A}) \to \partial E$ is surjective, i.e., $\mathscr{R}(\tau) = \partial E$,
- The operator $(\underline{A}, \tau) : \mathscr{D}(\underline{A}) \subset E \to E \times \partial E$ is closed,
- The operator A generates a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ on E,

Remark 3.3. Analyticity of the semigroup $(e^{tA})_{t\geq 0}$ on E is not assumed in this section.

To clarify these assumptions, we present the following example.

Example 3.4. We continue Example 3.1, therefore to satisfy the surjectivity assumption on τ, we choose ∂E to be the Sobolev-Slobodeckii space $W^{3/2,2}(\partial U)$. Next, as A is the Dirichlet Laplacian on E, it is well-known that A generates a strongly continuous semigroup $(e^{tA})_{t\geq 0}$ on E (see §2.5).

Remark 3.5. We make two remarks. First, since trace theorems hold for $u \in W^{1,p}(U)$ for $p \geq 1$ (see §2.6), the assumption $\mathscr{D}(\underline{A}) \equiv \mathscr{D}(\tau)$ is quite strong. Second, our approach is to choose the space ∂E so that $\mathscr{R}(\tau) = \partial E$ holds.

3.1.2 Greiner's decomposition

Under the assumptions of §3.1.1, we may apply the abstract decomposition of $\mathscr{D}(\underline{A})$ obtained by Greiner [46]. First, the domain of the operator $\underline{A}$ can be decomposed and related to A.

Lemma 3.6 (Greiner [46])**.** *Let* $\lambda \in \varrho(A)$*, then* $\mathscr{D}(\underline{A}) = \mathscr{D}(A) \oplus \ker(\lambda - \underline{A})$*.*

Next, as the map τ is surjective and $(\underline{A}, \tau)$ is closed, a restriction of τ properly maps the points in $\mathscr{D}(\underline{A}) \setminus \mathscr{D}(A)$ to the boundary space ∂E.

Lemma 3.7 (Greiner [46])**.** *If* $\tau : \mathscr{D}(\underline{A}) \to \partial E$ *is surjective and the operator*

$$(\underline{A}, \tau) : \mathscr{D}(\underline{A}) \subset E \to E \times \partial E$$

is closed, then the restriction $\tau_\lambda := \tau|_{\ker(\lambda - \underline{A})} : \ker(\lambda - \underline{A}) \to \partial E$ *is invertible and its inverse is bounded.*

Thanks to the previous lemma, it makes sense to define the linear operator $\Lambda_\lambda \in \mathscr{L}(\partial E, \ker(\lambda - \underline{A}))$ given by

$$\Lambda_\lambda := \tau_\lambda^{-1}.$$

Lemma 3.8 (Greiner [46])**.** *The operator* Λ_λ *has the following properties:*

- $(\lambda - \underline{A})\Lambda_\lambda \equiv 0$,
- $\tau\Lambda_\lambda = I_{\partial E}$ *where* $I_{\partial E}$ *is the identity operator on* ∂E,
- $\Lambda_\lambda \tau$ *is the projection in* $\mathscr{D}(\underline{A})$ *onto* $\ker(\lambda - \underline{A})$ *along* $\mathscr{D}(A)$,
- $R(\mu, A)\Lambda_\lambda = R(\lambda, A)\Lambda_\lambda$,
- $\Lambda_\lambda = (I - (\lambda - \mu)R(\lambda, A))\Lambda_\mu$,

where $\lambda, \mu \in \varrho(A)$ *and* $R(\lambda, A) := (\lambda - A)^{-1}$*.*

By applying Greiner's lemmas, we can obtain the following useful result.

Lemma 3.9. *For $u \in \mathscr{D}(\underline{A})$, we have*

$$\underline{A}u = A(I - \Pi_\lambda)u + \lambda \Pi_\lambda u. \tag{3.2}$$

Proof. The previous lemmas imply that the projection $\Pi_\lambda := \Lambda_\lambda \tau$ gives for $u \in \mathscr{D}(\underline{A})$ the decomposition

$$u = \underbrace{(I - \Pi_\lambda)u}_{\mathscr{D}(A)} + \underbrace{\Pi_\lambda u}_{\ker(\lambda - \underline{A})},$$

hence as $\mathscr{D}(A) \subseteq \mathscr{D}(\underline{A})$ it follows that

$$\underline{A}u = A(I - \Pi_\lambda)u + \lambda \Pi_\lambda u.$$

■

3.1.3 Classical, strong, and weak solutions

Under the assumptions presented in §3.1.1, in this section we construct a Banach space theory for solutions to (3.1). First, we must make clear what we mean by "solution".

Definition 3.10. *A function u is a* ***classical solution*** *of* (3.1) *in* $[0, T]$ *if:*

- $u \in C^1((0, T]; E) \cap C((0, T]; \mathscr{D}(\underline{A})) \cap C([0, T]; E)$,
- $u'(t) = \underline{A}u(t) + f(t)$ *for every* $t \in (0, T]$,
- $\tau u(t) = Bg(t)$ *for every* $t \in (0, T]$, *and*
- $u(0) = u_0$.

We now show that if u is a classical solution to (3.1) and $f \equiv 0$, then the solution is given by the variation of constants formula

$$u(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A}(\lambda - A)\Lambda_\lambda Bg(s)\, ds, \quad 0 \le t \le T, \tag{3.3}$$

for $\lambda \in \varrho(A)$, where $(e^{tA})_{t\geq 0}$ is the strongly continuous semigroup generated by the operator A on E.

Remark 3.11. One should notice that the boundary value problem (3.1) is given in terms of the operator $\underline{A}$ but the integral formula (3.3) only contains the operator A.

Proposition 3.12. *Let $g \in C((0,T];Y)$ be such that $t \mapsto \|g(t)\| \in L^1(0,T)$, let $f \equiv 0$, and let $u_0 \in \overline{\mathscr{D}(\underline{A})}$ be given. If u is a classical solution of* (3.1)*, then it is given by formula* (3.3)*.*

Proof. Let u be a classical solution of (3.1) with $f \equiv 0$, let $(e^{tA})_{t\geq 0}$ be the strongly continuous semigroup generated on E by A, and fix $t \in (0,T]$. By definition, it follows that $u \in C^1((0,T];E) \cap C((0,T];\mathscr{D}(\underline{A})) \cap C([0,T];E)$ and thus the function

$$z(t) := e^{(t-s)A}u(t), \quad 0 \leq s \leq t,$$

is in $C([0,t];E) \cap C^1((0,t);E)$ and

$$z(0) = e^{tA}u_0, \quad z(t) = u(t).$$

Further, for $0 < s < t$,

$$z'(s) = -Ae^{(t-s)A}u(s) + e^{(t-s)A}u'(s) \tag{3.4}$$

Now using the fact that u is a classical solution and the decomposition (3.2), we get

$$\begin{aligned} u'(s) &= \underline{A}u(s) \\ &= \underline{A}((I-\Pi_\lambda)u(s) + \Pi_\lambda u(s)) \\ &= A\left(u(s) - \Lambda_\lambda Bg(s)\right) + \lambda\Lambda_\lambda Bg(s). \end{aligned}$$

Substituting this expression for $u'(s)$ into (3.4) and simplifying,

$$z'(s) = e^{(t-s)A}(\lambda - A)\Lambda_\lambda Bg(s).$$

Now, for $0 < 2\varepsilon < t$, we get

$$z(t-\varepsilon) - z(\varepsilon) = \int_{\varepsilon}^{t-\varepsilon} e^{(t-s)A}(\lambda - A)\Lambda_\lambda Bg(s)\,ds,$$

and taking $\varepsilon \downarrow 0$, the variations of constants formula given by (3.3) follows. ∎

Remark 3.13. Due to the representation (3.3), we see that (3.1) with $f \equiv 0$ can be formally related to the abstract Cauchy problem

$$u'(t) = Au(t) + (\lambda - A)\Lambda_\lambda Bg(t), \quad 0 \le t \le T.$$

Hence, we can write (3.1) as the abstract Cauchy problem

$$u'(t) = Au(t) + f(t) + (\lambda - A)\Lambda_\lambda Bg(t), \quad u(0) = u_0, \quad t \in (0, T] \tag{3.5}$$

and now a solution u of (3.1) is given by a superposition of the solution u_1 to (2.3) and the solution u_2 of (3.1) with $f \equiv 0$ and $u_0 \equiv 0$.

The previous theorem implies that existence of a classical solution can be viewed as a problem about determining the regularity of a function u given by the variation of constants formula (3.3). It should be clear that even assuming the boundary data g is continuous is not sufficient to ensure u has enough regularity to be a classical solution. However, similar to definition of a strong solution for the abstract Cauchy problem (2.3) where an integrated version of the equation is shown to be satisfied, we propose a similar definition for the abstract boundary value problem (3.1).

Definition 3.14. *We call a **strong solution** of* (3.1) *the function* $u \in L^1(0, T; E)$ *such that for all* $t \in [0, T]$ *we have* $\int_0^t u(s)\,ds \in \mathscr{D}(\underline{A})$,

$$u(t) = u_0 + \underline{A}\int_0^t u(s)\,ds + \int_0^t f(s)\,ds,$$

and $\tau \int_0^t u(s)\,ds = \int_0^t Bg(s)\,ds$, *or equivalently,* $\int_0^t u(s)\,ds - \int_0^t Bg(s)\,ds \in \mathscr{D}(A)$.

If u is defined by the variation of constants formula (3.3) and the time-derivative of the data $g : [0,T] \to Y$ is continuous and bounded on $(0,T)$ then we can show that u is a strong solution to (3.1).

Proposition 3.15. *Let $g \in C^1_b((0,T);Y)$. If u is defined by* (3.3) *then u is a strong solution to* (3.1) *with $f \equiv 0$.*

Proof. Assume $u(t)$ is given by (3.3), then $u(t) = u_1(t) + u_2(t) + u_3(t)$ where

$$\begin{aligned}
u_1(t) &:= e^{tA}u_0,\\
u_2(t) &:= \lambda \int_0^t e^{(t-s)A}\Lambda_\lambda Bg(s)\,ds,\\
u_3(t) &:= \int_0^t (-A)e^{(t-s)A}\Lambda_\lambda Bg(s)\,ds,
\end{aligned}$$

As $\mathscr{D}(A) \subseteq \mathscr{D}(\underline{A})$, we have by the properties of strongly continuous semigroups that

$$\underline{A}\int_0^t u_1(s)\,ds = A\int_0^t e^{sA}u_0\,ds = e^{tA}u_0 - u_0, \quad \tau\int_0^t u_1(s)\,ds = 0.$$

Next, by Fubini's theorem

$$\begin{aligned}
\int_0^t u_2(s)\,ds &= \lambda \int_0^t \int_0^s e^{(s-r)A}\Lambda_\lambda Bg(r)\,dr\,ds\\
&= \lambda \int_0^t \int_r^t e^{(s-r)A}\Lambda_\lambda Bg(r)\,ds\,dr
\end{aligned}$$

and

$$\int_r^t e^{(s-r)A}\Lambda_\lambda Bg(r)\,ds = \int_0^{t-r} e^{\tau A}\Lambda_\lambda Bg(r)\,d\tau \in \mathscr{D}(A) \subseteq \mathscr{D}(\underline{A}),$$

so as $\underline{A}u = Au$ for $u \in \mathscr{D}(A)$, it follows that

$$\underline{A}\int_0^t u_2(s)\,ds = \lambda\int_0^t (e^{(t-r)A} - I)\Lambda_\lambda Bg(r)\,dr, \quad \tau\int_0^t u_2(s)\,ds = 0.$$

Finally, we consider u_3 and notice that by our assumption and integration by parts,

$$\int_0^t (-A)e^{(t-s)A}\Lambda_\lambda Bg(s)\,ds = \Lambda_\lambda Bg(t) - e^{tA}\Lambda_\lambda Bg(0) - \int_0^t e^{(t-s)A}\Lambda_\lambda Bg'(s)\,ds.$$

which allows us to calculate

$$\begin{aligned}\int_0^t u_3(s)\,ds &= \int_0^t \int_0^s (-A)e^{(s-r)A}\Lambda_\lambda Bg(r)\,dr\,ds \\ &= I_1 + I_2 - \int_0^t \int_0^s e^{(s-r)A}\Lambda_\lambda Bg'(r)\,dr\,ds\end{aligned}$$

where

$$I_1 := \int_0^t \Lambda_\lambda Bg(s)\,ds, \quad I_2 := -\int_0^t e^{sA}\Lambda_\lambda Bg(0)\,ds.$$

Considering the terms I_1 and I_2 first, we have $I_1 \in \ker(\lambda - \underline{A})$ and $I_2 \in \mathscr{D}(A) \subset \mathscr{D}(\underline{A})$ so that $\tau I_1 = \int_0^t Bg(s)\,ds$, $\tau I_2 = 0$,

$$\underline{A}I_1 = \lambda \int_0^t \Lambda_\lambda Bg(s)\,ds, \quad \underline{A}I_2 = -e^{tA}\Lambda_\lambda Bg(0) + \Lambda_\lambda Bg(0).$$

Now similar to the u_2 term, we have

$$J := -\int_0^t \int_0^s e^{(s-r)A}\Lambda_\lambda Bg'(r)\,dr\,ds \in \mathscr{D}(A) \subseteq \mathscr{D}(\underline{A}),$$

$\tau J = 0$, and

$$\begin{aligned}\underline{A}J &= -\int_0^t (e^{(t-r)A} - I)\Lambda_\lambda Bg'(r)\,dr \\ &= -\int_0^t e^{(t-r)A}\Lambda_\lambda Bg'(r)\,dr + \int_0^t \Lambda_\lambda Bg'(r)\,dr \\ &= -\Lambda_\lambda Bg(t) + e^{tA}\Lambda_\lambda Bg(0) + \int_0^t e^{(t-r)A}(-A)\Lambda_\lambda Bg(r)\,dr \\ &\qquad + \int_0^t \Lambda_\lambda Bg'(r)\,dr\end{aligned}$$

Collecting terms we get $\tau \int_0^t u(s)\,ds = \int_0^t Bg(s)\,ds$ and

$$\underline{A}\int_0^t u(s)\,ds = e^{tA}u_0 - u_0 + \int_0^t e^{(t-s)A}(\lambda - A)\Lambda_\lambda Bg(r)\,dr.$$

Hence, u is a strong solution to (3.1). ∎

In the previous proof, we saw that by assuming $g \in C^1([0,T];Y)$ gives an alternative representation of the solution u. We formalise this in the next lemma and give a self-contained proof of this fact.

Lemma 3.16. *If u is a classical solution of* (3.1) *with $g \in C^1([0,T];Y)$ and $f \equiv 0$, then u is given by*

$$u(t) = e^{tA}(u_0 - \Lambda_\lambda Bg(0)) + \Lambda_\lambda Bg(t) + \int_0^t e^{(t-s)A}\Lambda_\lambda B(\lambda g(s) - g'(s))\,ds. \quad (3.6)$$

Proof. Assume u is a classical solution of (3.1). From the form of (3.6), we guess that we seek a solution in the form

$$u(t) = v(t) + \Lambda_\lambda Bg(t) \quad (3.7)$$

where $v(t)$ is a classical solution of the abstract Cauchy problem

$$v'(t) = Av(t) + w(t) \quad (3.8)$$

and the form of $w(t)$ is to be determined. Differentiating u and using the fact that $v(t) = u(t) - \Lambda_\lambda Bg(t)$ by definition and $\mathscr{D}(A) \subseteq \mathscr{D}(\underline{A})$, we get

$$\begin{aligned} u'(t) &= v'(t) + \Lambda_\lambda Bg'(t) \\ &= (Av(t) + w(t)) + \Lambda_\lambda Bg'(t) \\ &= \underline{A}(u(t) - \Lambda_\lambda Bg(t)) + w(t) + \Lambda_\lambda Bg'(t) \\ &= \underline{A}u(t) - \lambda\Lambda_\lambda Bg(t) + w(t) + \Lambda_\lambda Bg'(t). \end{aligned}$$

We set $w(t) = \Lambda_\lambda B(\lambda g(t) - g'(t))$ and since $v(t) \in \mathscr{D}(A)$ it follows that $\tau v(t) = 0$ and

$$\begin{cases} u'(t) = \underline{A}u(t), \\ \tau u(t) = \tau(v(t) + \Lambda_\lambda B g(t)) = g(t) \end{cases}$$

so (3.1) is formally satisfied. As $g \in C^1([0,T], Y)$ by assumption, it follows that $w \in C([0,T]; E)$ so by Theorem 2.6 the abstract Cauchy problem (3.8) has a unique strong solution given by

$$v(t) = S(t)v(0) - \int_0^t S(t-s)w(t)\,ds$$

and substituting $v(t)$ into (3.7) gives the representation (3.6). ■

Similar to the theory for abstract Cauchy problems, we can also define the concept a weak solution for the abstract boundary value problem (3.1) when $f \equiv 0$ and $u_0 \equiv 0$.

Definition 3.17. *A* ***weak solution*** *of* (3.1) *when* $u_0 \equiv 0$ *and* $f \equiv 0$ *is a function* $u \in L^1(0,T;E)$ *such that for all* $t \in [0,T]$ *and* $v \in \mathscr{D}(A^*)$ *we have*

$$\langle u(t), v\rangle = \langle -\Lambda_\lambda B g_0, v\rangle + \int_0^t \langle u(s), \underline{A}^* v\rangle\,ds \tag{3.9}$$

and for all $t \in [0,T]$, $[\tau u(t), w] = [Bg(t), w]$ *for* $w \in (\partial E)^*$ *where* $[\cdot,\cdot]$ *is the dual-pairing between* ∂E *and* $(\partial E)^*$.

Remark 3.18. In the previous definition, one should note the subtle choice of $v \in \mathscr{D}(A^*)$ but that the operator in (3.9) is the adjoint of $\underline{A}$.

Proposition 3.19. *Every weak solution of* (3.1) *is a strong solution, and vice-versa.*

Let E_1 and E_2 be Banach spaces. To prove Proposition 3.19 we shall make use of the following lemma which ‘dualises’ the definition of $\mathscr{D}(A^*)$, see [6, Proposition 7.14].

Lemma 3.20. *Let $(A, \mathscr{D}(A))$ be a closed and densely defined linear operator from E_1 to E_2. If $x_1 \in E_1$ and $x_2 \in E_2$ are such that $\langle x_2, x_2^* \rangle = \langle x_1, A^* x_2^* \rangle$ for all $x_2^* \in \mathscr{D}(A^*)$, then $x_1 \in \mathscr{D}(A)$ and $Ax_1 = x_2$.*

Proof of Proposition 3.19. Choose $v = (e^{tA})^* w$ for $w \in E^*$ then $v \in \mathscr{D}(A^*)$. The proof follows from Lemma 3.20. ∎

The equivalence between weak and strong solutions gives us an alternative method of proving existence. We also obtain uniqueness of the solution.

Theorem 3.21. *For $u_0 \equiv 0$, $f \equiv 0$, and $g \in C_b^1([0,T];Y)$ the problem* (3.1) *admits a unique strong (or weak) solution.*

Proof. By Proposition 3.19, we only need to check this is a weak solution. Notice that u is a weak solution with initial value $u_0 - \Lambda_\lambda B g_0$ if and only if $t \mapsto u(t) - S(t)(u_0 - \Lambda_\lambda B g_0)$ is a weak solution corresponding to the initial value 0. Therefore, without loss of generality, we assume $u_0 = 0$ and $\Lambda_\lambda B g_0 = 0$.

Let u be given by (3.6) and set $z(t) := \Lambda_\lambda B(\lambda g(t) - g'(t))$. As $z(t) \in C([0,T];E)$, it is clear that $u \in L^1(0,T;E)$. Let $v \in \mathscr{D}(A^*)$ then $\underline{A}^* v = A^* v$ so for all $t \in [0,T]$ using Fubini's theorem and defining $S^*(t) := (e^{tA})^*$,

$$\begin{aligned}
\int_0^t \langle u(t), \underline{A}^* v \rangle \, ds &= \int_0^t \int_0^s \langle z(r), S^*(s-r) \underline{A}^* v \rangle \, dr \, ds \\
&\quad + \int_0^t \langle \Lambda_\lambda B g(s), \underline{A}^* v \rangle \, ds \\
&= \int_0^t \int_r^t \langle z(r), S^*(s-r) A^* v \rangle \, ds \, dr \\
&\quad + \int_0^t \langle \Lambda_\lambda B g(s), \underline{A}^* v \rangle \, ds \\
&= \int_0^t \langle z(r), S^*(t-r) v - v \rangle \, dr \\
&\quad + \int_0^t \langle \Lambda_\lambda B g(s), \underline{A}^* v \rangle \, ds
\end{aligned}$$

$$
\begin{aligned}
&= \langle \int_0^t e^{(t-r)A} z(r)\,dr - \int_0^t z(r)\,dr, v\rangle + \\
&\qquad \langle \Lambda_\lambda B g(t), v\rangle + \langle \int_0^t \lambda \Lambda_\lambda B g(s)\,ds, v\rangle \\
&= \langle u(t), v\rangle.
\end{aligned}
$$

By Lemma 3.20, it follows that $\int_0^t u(s)\,ds \in \mathscr{D}(\underline{A})$. The condition $[\tau u(t), v] = [g(t), v]$ for $v \in (\partial E)^*$ follows readily as $z(t) \in E$ for all $t \in [0, T]$,

$$h(t) := \int_0^t e^{(t-s)A} z(s)\,ds \in \mathscr{D}(A)$$

so $\tau h(t) = 0$ and, by definition, $\tau \Lambda_\lambda B g(t) = B g(t)$ so

$$[\tau u(t), v] = [\tau(h(t) + \Lambda_\lambda B g(t)), v] = [B g(t), v].$$

We now prove uniqueness. Suppose $\tilde{u}$ and u are strong solutions of (3.1), then $v := \tilde{u} - u$ is integrable and satisfies $v(t) = \underline{A}\int_0^t v(s)\,ds$ for all $t \in [0, T]$ and also $\tau v(t) = \tau \tilde{u}(t) - \tau u(t) = 0$ for all $t \in [0, T]$. Set

$$w(t) := \int_0^t \int_0^s v(r)\,dr\,ds,$$

then it also follows that $\tau w(t) = 0$. By the fundamental theorem of calculus, w is continuously differentiable on $[0, T]$, and using Hille's theorem (e.g. [6, Theorem 1.19]) we see that $w(t) \in \mathscr{D}(\underline{A})$ and since $\tau w(t) = 0$ it follows that $w(t) \in \mathscr{D}(A)$ and

$$w'(t) = \int_0^t v(s)\,ds = \int_0^t \underline{A} \int_0^s v(r)\,dr\,ds = A w(t).$$

Fix $t \in [0, T]$ and put $h(s) := S(t-s)w(s)$. Then h is continuously differentiable on $[0, t]$ with derivative

$$h'(s) = -AS(t-s)w(s) + S(t-s)w'(s) = 0.$$

It follows that h is constant on $[0,t]$. Hence,

$$w(t) = h(t) = h(0) = S(t)w(0) = 0.$$

As we have shown that $\int_0^t \int_0^s v(r)\,dr\,ds = 0$ for all $t \in [0,T]$, it follows that $v = 0$ almost everywhere so we must have $u = \tilde{u}$ almost everywhere. ■

We now give a sufficient condition for a mild solution to be a classical solution.

Lemma 3.22. *Let $g \in C_b^1((0,T];Y)$ and let u be a mild solution of* (3.1) *with $u_0 \equiv 0$ and $f \equiv 0$. Then the following conditions are equivalent:*

- $u \in C((0,T];\mathscr{D}(\underline{A}))$,
- $u \in C^1((0,T];E)$,
- *u is a classical solution of* (3.1) *with $u_0 \equiv 0$ and $f \equiv 0$.*

3.2 Under an analyticity assumption

In this section, we change the assumption given in §3.1.1 slightly and now assume:

- The operator $A : \mathscr{D}(A) \subset E \to E$ is sectorial and generates an analytic semigroup $(e^{tA})_{t\geq 0}$ on E,
- $\tau : \mathscr{D}(\tau) \subset E \to \partial E$ has range given by $\mathscr{R}(\tau) = \partial E$,

As before, the $\underline{A} : \mathscr{D}(\underline{A}) \subset E \to E$ is the maximal operator and the relationship with the constrained operator A is given by

$$\mathscr{D}(A) := \mathscr{D}(\underline{A}) \cap \ker \tau.$$

Writing $\mathbb{B}_\alpha := B([0,T];\mathscr{D}_A(\alpha,\infty))$, we now obtain a regularity theorem for the variation of constants formula (3.3).

Theorem 3.23. *Let $g \in C([0,T];Y)$, and assume $\Lambda_\lambda \in \mathscr{L}(\partial E, \mathscr{D}_A(\alpha,\infty))$ for some $0 < \alpha \leq 1$. Then the variations of constants formula v given by*

$$v(t) = (\lambda - A)\int_0^t e^{(t-s)A}\Lambda_\lambda B g(s)\,ds, \quad 0 \leq t \leq T, \tag{3.10}$$

belongs to $C((0,T];\mathscr{D}_A(\alpha,\infty)) \cap \mathbb{B}_\alpha$, and

$$\|v\|_{\mathbb{B}_\alpha} + \|v\|_{C^\alpha([0,T];E)} \lesssim \|g\|_{C([0,T];\partial E)}. \tag{3.11}$$

Proof. As $B \in \mathscr{L}(Y, \partial E)$ and $\Lambda_\lambda \in \mathscr{L}(\partial E, \mathscr{D}_A(\alpha,\infty))$ it follows that if we define $f(t) := \Lambda_\lambda B g(t)$ then $f \in C([0,T];\mathscr{D}_A(\alpha,\infty))$ and

$$\|f\|_{C([0,T];\mathscr{D}_A(\alpha,\infty))} \lesssim \|g\|_{C([0,T];Y)}.$$

The result now follows by applying Theorem 2.7 to the function

$$z(t) := \int_0^t e^{(t-s)A} f(s)\,ds, \quad 0 \leq t \leq T,$$

which gives that $v = (\lambda - A)z$ belongs to $C([0,T];\mathscr{D}(A))$, to $\mathbb{B}_\alpha$, and to $C^\alpha([0,T];E)$. Further, estimate (3.11) holds. ■

For (3.10) to make sense as a "solution" to (3.1) with $f \equiv 0$ we need to make sure that $\tau v(t) = Bg(t)$ for all $t \in (0,T]$. This follows readily if

$$\mathscr{D}(\tau) = \mathscr{D}_A(\alpha,\infty)$$

for the same α as in Theorem 3.23, as Theorem 3.23 then ensures that $v \in \mathscr{D}(\tau)$.

3.3 Application to parabolic equations

In this section, we shall apply the abstract theory presented in the last section to study a parabolic partial differential equation on a domain $U \subset \mathbb{R}^d$ subject to inhomogeneous Dirichlet or Neumann boundary conditions on the boundary ∂U.

3.3.1 Neumann boundary conditions

Let U be either the half-space $\mathbb{R}^d_+$, or an open bounded subset of $\mathbb{R}^d$ with uniformly C^2 boundary ∂U. We shall denote by $\nu(x)$ the exterior unit normal vector to ∂U at the point $x \in \partial U$. We first consider the inhomogeneous Neumann boundary value problem

$$\begin{cases} u_t(t,x) = \mathscr{A}u(t,x), & (t,x) \in [0,T] \times U, \\ u(0,x) = 0, & x \in U, \\ \partial_\nu u(t,x) = g(t,x), & t \in [0,T] \times \partial U, \end{cases}$$

where $\mathscr{A} := \mathscr{A}(x,D)$ is the second order differential operator

$$\mathscr{A}(x,D) := \sum_{i,j=1}^{d} a_{ij}(x) D_{ij} + \sum_{i=1}^{d} b_i(x) D_i + c(x) I$$

with real uniformly continuous and bounded coefficients a_{ij}, b_i, c. We assume that the matrix $[a_{ij}]$ is symmetric and satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \kappa|\xi|^2, \quad x \in \overline{U},\, \xi \in \mathbb{R}^d,$$

for some $\kappa > 0$.

Case $E = C(\overline{U})$

In this section we consider the case $E = C(\overline{U})$ and apply our Banach space theory to obtain the same results as those found in Chapter 5 of [5]. As such, we pose $E = C(\overline{U})$ and $\partial E = C^1(\partial U)$ and define the operators

- $\underline{A}u = \mathscr{A}u$ for $u \in \mathscr{D}(\underline{A})$ where

$$\mathscr{D}(\underline{A}) := \left\{ u \in \bigcap_{p \geq 1} W^{2,p}_{\text{loc}}(U) : u, \mathscr{A}u \in C(U) \right\}$$

- $\tau : \mathscr{D}(\tau) \subset E \to \partial E$ where

$$\tau u := \frac{\partial u}{\partial \nu}\Big|_{\partial U}$$

- As $\mathscr{D}(A) := \mathscr{D}(\underline{A}) \cap \ker \tau$, it follows that

$$\mathscr{D}(A) := \left\{ u \in \bigcap_{p \geq 1} W^{2,p}_{\text{loc}}(U) : u, \mathscr{A} u \in C(U), \partial_\nu u|_{\partial U} = 0 \right\}$$

Applying Corollary 3.1.24 of [5], it follows that the resolvent set of the operator $A : \mathscr{D}(A) \subset E \to E$ contains the halfplane $\{\lambda \in \mathbb{C} : \Re\lambda > l_1\}$ and A is sectorial[1]. Hence, A generates an analytic semigroup $(e^{tA})_{t \geq 0}$ on E. By Theorems 3.1.30 and 3.1.31 in [5], we also have the characterisation

$$\mathscr{D}_A(\alpha, \infty) = \begin{cases} C^{2\alpha}(\overline{U}), & \text{if } \alpha \leq 1/2, \\ C^1_\tau(\overline{U}), & \text{if } \alpha = 1/2, \\ C^{2\alpha}_\tau(\overline{U}), & \text{if } \alpha \geq 1/2, \end{cases} \tag{3.12}$$

where the subscript τ means that the function space is only comprised of functions u such that $u \in \ker \tau$. The space $C^1_\tau(\overline{U})$ is defined [5, p. 109] as

$$C^1_\tau(\overline{U}) = \left\{ u \in C^1(\overline{U}) : \sup_{\mathscr{T}} \frac{|u(x - h\beta(x)) - u(x)|}{h} < \infty \right\},$$

where $\mathscr{T} := \{x \in \partial U, h \in \mathbb{R}, x - h\beta(x) \in \overline{U}\}$ with $\beta(x) := (\beta_1(x), \ldots, \beta_d(x))$. Let Y be another Banach space and $B \in \mathscr{L}(Y, \partial E)$ or let $Y = \partial E$ with $B = I$ (i.e., the identity operator). Finally, we define Λ_λ as the Neumann map which is given by the solution of the elliptic Neumann problem on $E = C(\overline{U})$ with $g \in \partial E = C^1(\partial U)$ given by

$$(\lambda - \mathscr{A})u = 0, \quad \partial_\nu u|_{\partial U} = g.$$

[1] See Corollary 3.1.24 [5] for the definition of the constant l_1

That is, the solution u defines $\Lambda_\lambda \in \mathscr{L}(\partial E, E)$ by $\Lambda_\lambda g = u$. By regularity theory for the Neumann problem, we have that $\Lambda_\lambda \in \mathscr{L}(C^1(\partial U), C^2(\overline{U}))$. Further by (3.12), it follows that $C^2(\overline{U})$ is continuously embedded in $\mathscr{D}_A(1/2,\infty)$ and we have

$$\Lambda_\lambda \in \mathscr{L}(\partial E, \mathscr{D}_A(1/2,\infty)).$$

By the theory of this chapter, a mild solution is given by

$$u(t) = (\lambda - A)\int_0^t e^{(t-s)A}\Lambda_\lambda B g(s)\,ds, \quad 0 \le t \le T, \tag{3.13}$$

and as all the assumptions of Theorem 3.23 are satisfied, the following result (given also by Theorem 5.1.17 in [5]) is obtained as an example of our theory.

Theorem 3.24. *Let* $g \in C([0,T]; C^1(\partial U))$. *Then* u *given by* (3.13) *belongs to* $C^\alpha([0,T]; C(\overline{U}))$ *and* $B([0,T]; C^1(\overline{U}))$.

Case $E = L^p(U)$

We now consider the case $E = L^p(U)$ by defining the operators

- $\underline{A}u = \mathscr{A}u$ for $u \in \mathscr{D}(\underline{A})$ where

$$\mathscr{D}(\underline{A}) := W^{2,p}(U)$$

- $\tau : \mathscr{D}(\tau) \subset E \to \partial E$ where

$$\tau u := \frac{\partial u}{\partial \nu}\Big|_{\partial U}$$

- As $\mathscr{D}(A) := \mathscr{D}(\underline{A}) \cap \ker\tau$, it follows that

$$\mathscr{D}(A) := \left\{ u \in W^{2,p}(U) : \partial_\nu u|_{\partial U} = 0 \right\}$$

Assume U is an open set in $\mathbb{R}^d$ with uniformly C^2 boundary and fix $p \in (1,\infty)$. Then (e.g., see Theorem 3.1.2 in [5]) there exists $\kappa_1 \in \mathbb{R}$ such that if $\Re\lambda \ge \kappa_1$, then for every $f \in L^p(U)$ and $g \in W^{1,p}(U)$, the elliptic problem

$$(\lambda - \mathscr{A})u = f \text{ in } U, \quad \partial_\nu u = g \text{ in } \partial U, \tag{3.14}$$

has a unique solution $u \in W^{2,p}(U)$ depending continuously on f and g. By taking $g \equiv 0$ it follows that $\{\lambda \in \mathbb{C} : \Re\lambda \geq \kappa_1\} \subset \varrho(A)$. If U is unbounded then the constant κ_1 may depend on p. It is known (e.g., see Theorem 3.1.3 in [5]) that the following bounds on the norm of the resolvent operator of A holds: there exists $\kappa_p \geq \kappa_1$, $M_p > 0$ such that if $\Re\lambda \geq \kappa_p$, then for every $u \in W^{2,p}(U)$ we have

$$|\lambda| \|u\| + |\lambda|^{1/2} \|Du\| + \|D^2 u\| \\ \leq M_p \left(\|\lambda u - \mathscr{A} u\| + |\lambda|^{1/2} \|\tilde{g}\| + \|D\tilde{g}\| \right)$$

where $\tilde{g}$ is any extension of g belonging to $W^{1,p}(U)$, $\|Du\| = \sum_{i=1}^{d} \|D_i u\|$, $\|D^2 u\| = \sum_{i,j=1}^{d} \|D_{ij} u\|$. Henceforth, A is sectorial on $E = L^p(U)$ and generates an analytic semigroup $(e^{tA})_{t\geq 0}$ on E.

Let u be the weak solution to (3.14) where $\partial E = W^{s-1-1/p,p}(\partial U)$ for $s \in (1, 1+1/p)$, by regularity theory for the Neumann problem it follows that $u \in W^{s,p}(U)$. Hence, defining $\Lambda_\lambda g = u$ we get

$$\Lambda_\lambda \in \mathscr{L}(W^{s-1-1/p,p}(\partial U), W^{s,p}(U)).$$

For $1 < p < \infty$ and $0 < \alpha < 1$ where 2α and $2\alpha - 1/p$ are not integers, we have the characterisation (e.g., see Theorem 15.5 in [33] or Theorem 3.2.3 in [5]),

$$\mathscr{D}_A(\alpha, p) = \begin{cases} W^{2\alpha,p}(U), & \text{if } 2\alpha < 1 + 1/p, \\ \{u \in W^{2\alpha,p}(U) : \partial_\nu u|_{\partial U} = 0\}, & \text{if } 2\alpha > 1 + 1/p. \end{cases}$$

Hence to apply our framework, fix $1/p < s < 1 + 1/p$, let $Y = L^p(\partial U)$, and set $\partial E = W^{s-1-1/p,p}(\partial U)$. Notice that since $s < 1 + 1/p$ we have $Y \subset \partial E$. Finally, assume $B \in \mathscr{L}(Y, \partial E)$ and set $\alpha = s/2$. By Proposition 2.2.15 in [5], we have

$$\mathscr{D}_A(\alpha, 1) \subset \mathscr{D}((-A)^\alpha) \subset \mathscr{D}_A(\alpha, \infty), \quad 0 < \alpha < 1.$$

Hence, it follows that $B\Lambda_\lambda \in \mathscr{L}(Y, \mathscr{D}_A(\alpha, \infty))$ with $2\alpha \in (1/p, 1 + 1/p)$ and since all the assumptions of Theorem 3.23 are satisfied, the following result is obtained as an example.

Theorem 3.25. *Let* $g \in C([0,T];Y)$ *and* $B \in \mathscr{L}(Y, W^{s-1-1/p,p}(\partial U))$ *for some* $s \in (1/p, 1+1/p)$. *Then* u *given by* (3.13) *belongs to* $C^\alpha([0,T];L^p(U))$ *and* $B([0,T];W^{s,p}(U))$. *In particular, we can take* $Y = L^p(\partial U)$ *and* $B = I$.

3.3.2 Dirichlet boundary conditions

Let U be either the half-space $\mathbb{R}^d_+$, or an open bounded subset of $\mathbb{R}^d$ with uniformly C^2 boundary ∂U. We shall denote by $\nu(x)$ the exterior unit normal vector to ∂U at the point $x \in \partial U$. We shall now consider the inhomogeneous Dirichlet boundary value problem

$$\begin{cases} u_t(t,x) = \mathscr{A} u(t,x) + f(t,x), & (t,x) \in [0,T] \times U, \\ u(0,x) = u_0(x), & x \in U, \\ u(t,x) = g(t,x), & t \in [0,T] \times \partial U, \end{cases}$$

where $\mathscr{A} := \mathscr{A}(x,D)$ is the second order differential operator

$$\mathscr{A}(x,D) := \sum_{i,j=1}^d a_{ij}(x) D_{ij} + \sum_{i=1}^d b_i(x) D_i + c(x) I$$

with real uniformly continuous and bounded coefficients a_{ij}, b_i, c. We assume that the matrix $[a_{ij}]$ is symmetric and satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^d a_{ij}(x)\xi_i\xi_j \geq \kappa|\xi|^2, \quad x \in \overline{U},\, \xi \in \mathbb{R}^d,$$

for some $\kappa > 0$.

Case $E = C(\overline{U})$

Let us first consider the case $E = C(\overline{U})$ and $\partial E = C^\theta(\partial U)$ for some $\theta \geq 0$. We define the operators

- $\underline{A}u = \mathscr{A}u$ for $u \in \mathscr{D}(\underline{A})$ where

$$\mathscr{D}(\underline{A}) := \left\{ u \in \bigcap_{p\geq 1} W^{2,p}_{\mathrm{loc}}(U) : u, \mathscr{A}u \in C(U) \right\}$$

- $\tau : \mathscr{D}(\tau) \subset E \to \partial E$ where

$$\tau u := u|_{\partial U}$$

- As $\mathscr{D}(A) := \mathscr{D}(\underline{A}) \cap \ker\tau$, it follows that

$$\mathscr{D}(A) := \left\{ u \in \bigcap_{p\geq 1} W^{2,p}_{\mathrm{loc}}(U) : u, \mathscr{A}u \in C(U), u|_{\partial U} = 0 \right\}$$

It is known that the operator A is sectorial on E (e.g., see Corollary 3.1.21 in [5]) and generates an analytic semigroup $(e^{tA})_{t\geq 0}$ on E. Next, for $\Re\lambda$ large, the elliptic boundary value problem

$$(\lambda - \mathscr{A})u = f \ \text{ in } U, \quad u = g \ \text{ in } \partial U, \tag{3.15}$$

with $f \in L^p_{\mathrm{loc}}(U)$ and $g \in W^{2,p}_{\mathrm{loc}}(U) \cap C^1(\overline{U})$ is solvable (e.g., Theorem 3.1.19 in [5]). The solution is unique and satisfies $u \in W^{2,p}_{\mathrm{loc}}(U) \cap C^1(\overline{U})$. Assuming U has a uniformly $C^{2+\beta}$ boundary ∂U where $0 \leq \beta < 1$, then (e.g. Theorem 0.3.2 in [5]) there exists an extension operator $\mathscr{E} \in \mathscr{L}(C^\theta(\partial U), C^\theta(\overline{U}))$ for each $\theta \in [0, 2+\beta]$ such that

$$\mathscr{E}g|_{\partial U} = g, \quad \forall g \in C(\partial U).$$

Hence, defining the operator $\Lambda_\lambda g = u$ where u is the solution to (3.15) gives $\Lambda_\lambda \in \mathscr{L}(\partial E, \mathscr{D}(\underline{A}))$. As for the Neumann case, we need Λ_λ to map into $\mathscr{D}_A(\alpha, \infty)$ for some $0 < \alpha < 1$. However, by [5, Theorem 3.1.29], for $0 < \alpha < 1$,

$$\mathscr{D}_{A_D}(\alpha, \infty) = \begin{cases} C_0^{2\alpha}(\overline{U}), & \alpha \neq 1/2, \\ C_0^1(\overline{U}), & \alpha = 1/2, \end{cases}$$

This causes a problem with our theory as Λ_λ does not vanish on the boundary, hence unlike the Neumann case we cannot find $\alpha \in (0,1)$ for the condition on zero boundary data to disappear. We conclude that Theorem 3.23 is insufficient to handle such a case.

Remark 3.26. The inability to apply this theory to the space $E = C(\overline{U})$ is another motivation for our results in Chapter 6.

Case $E = L^p(U)$

We now consider the case $E = L^p(U)$, $1 < p < \infty$. We define the operators

- $\underline{A}u = \mathscr{A}u$ for $u \in \mathscr{D}(\underline{A})$ where
$$\mathscr{D}(\underline{A}) := W^{2,p}(U)$$
- $\tau : \mathscr{D}(\tau) \subset E \to \partial E$ where
$$\tau u := u|_{\partial U} \text{ in trace}$$
- As $\mathscr{D}(A) := \mathscr{D}(\underline{A}) \cap \ker \tau$, it follows that
$$\mathscr{D}(A) := W_0^{1,p}(U) \cap W^{2,p}(U).$$

It is known that the operator A is sectorial on E (e.g., see Theorem 3.1.3 in [5]) and generates an analytic semigroup $(e^{tA})_{t\geq 0}$ on E. Also, (e.g., Theorem 3.1.2 in [5]) if U has uniformly C^2 boundary, then there exists $\kappa_1 \in \mathbb{R}$ such that if $\Re\lambda \geq \kappa_1$, then for every $f \in L^p(U)$ and $g \in W^{2,p}(U)$ the problem

$$(\lambda - \mathscr{A})u = f \text{ in } U, \quad u = g \text{ in } \partial U, \tag{3.16}$$

has a unique solution $u \in W^{2,p}(U)$, depending continuously on f and g. Let $s > 1/p$ then $\tau \in \mathscr{L}(W^{s,p}(U), W^{s-1/p,p}(\partial U))$ and we can define $\Lambda_\lambda g = u$ where u solve (3.16). Then $\Lambda_\lambda \in \mathscr{L}(W^{\alpha-1/p,p}(\partial U), W^{\alpha,p}(U))$ for $\alpha > 1/p$. For $1 < p <$

∞ and $0 < \alpha < 1$ where 2α and $2\alpha - 1/p$ are not integers and $\mathscr{A} = \Delta$, we have the characterisation (e.g., see Theorem 15.5 in [33]),

$$\mathscr{D}_A(\alpha, p) = \begin{cases} W^{2\alpha,p}(U), & \text{if } 2\alpha < 1/p, \\ \{u \in W^{2\alpha,p}(U) : \tau u = 0\}, & \text{if } 2\alpha > 1/p. \end{cases}$$

Hence, choosing $0 < \alpha < 1$ such that $2\alpha < 1/p$ and setting $\partial E = W^{\alpha-1/p,p}(\partial U)$ the assumptions for Theorem 3.23 are satisfied and we obtain a similar result to [4, Theorem 11.2].

Theorem 3.27. *Let $g \in C([0,T];Y)$ and $B \in \mathscr{L}(Y, W^{2\beta-1/p,p}(\partial U))$ for $1/p < 2\beta < 1+1/p$. Then u given by* (3.13) *belongs to $C^\alpha([0,T];L^p(U))$ and $B([0,T];W^{2\beta,p}(U))$.*

3.4 Parabolic layer potentials

We shall make use of Green's formula (e.g., see [81]) to obtain some explicit representations for the abstract approach presented in the previous sections in the case where $E = L^p(U)$, $1 < p < \infty$ and $U \subset \mathbb{R}^d$ is a bounded domain with smooth boundary ∂U.

The results of this section are known for the L^2 case (e.g., see [82]) however we could not find a reference for the (straight-forward) extension to the L^p setting. We have included this characterisation as we believe it illustrates the connection between the abstract approach and the classic double-layer potential approach.

Theorem 3.28 (Green's second formula)**.** *Let $U \subset \mathbb{R}^d$ be a bounded open set with boundary ∂U of class C^1 and $u, v \in C^2(\overline{U})$. Then*

$$\int_U (v(x)\Delta u(x) - u(x)\Delta v(x))\,dx = \int_{\partial U} \left(v(z)\frac{\partial u}{\partial \nu_z}(z) - u(z)\frac{\partial v}{\partial \nu_z}(z) \right) \sigma(dz)$$

where σ is the surface measure on $(\partial U, \mathscr{B}(\partial U))$ and ν_z is the exterior normal at $z \in \partial U$.

Fix $1 < p < \infty$ and $1/p < 2\alpha < 1+1/p$. Let $\langle\cdot,\cdot\rangle$ be the dual pairing between $L^p(U)$ and $(L^p(U))^*$ and $[\cdot,\cdot]$ the dual pairing between $W^{2\alpha-1/p,p}(\partial U)$ and $(W^{2\alpha-1/p,p}(\partial U))^*$. Let us define the **normal derivative** operator $\mathscr{N} : C^1(\overline{U}) \to \mathbb{R}$ by

$$(\mathscr{N} u)(x) := (\nabla u)(x) \cdot \nu(x),$$

and, as before, we define

- $\underline{A}u := \Delta u$ for $u \in \mathscr{D}(\underline{A}) = W^{2,p}(U)$,
- $Au := \Delta u$ for $u \in \mathscr{D}(A) = W_0^{1,p}(U) \cap W^{2,p}(U)$,
- $\Lambda : W^{2\alpha-1/p,p}(\partial U) \to W^{2\alpha,p}(U)$ be the Dirichlet map, i.e., Λg solves the elliptic Dirichlet problem

$$\Delta u = 0 \text{ in } U, \quad u|_{\partial U} = g \text{ on } \partial U \text{ (in trace)},$$

- $(e^{tA})_{t\geq 0}$ be the Dirichlet heat semigroup on $L^p(U)$.

The following lemma extends the characterisation of $((-A)\Lambda)^*$ in [82] to the L^p setting.

Lemma 3.29. *For $v \in \mathscr{D}(A^*)$,*

$$((-A)\Lambda)^* v = \mathscr{N} v.$$

Proof. Take $g \in \mathscr{D}(\Lambda)$ and $v \in \mathscr{D}(A^*)$ and apply Theorem 3.28 to get

$$\begin{aligned}
\langle (-A)\Lambda g, v\rangle &= \langle \Lambda g, (-A)^* v\rangle \\
&= \langle \Lambda g, (-\underline{A})^* v\rangle \\
&= \langle (-\underline{A})\Lambda g, v\rangle + [\Lambda g|_{\partial U}, \partial_\nu v|_{\partial U}] - [\partial_\nu(\Lambda g)|_{\partial U}, v|_{\partial U}] \\
&= [g, \partial_\nu v]
\end{aligned}$$

as $\Lambda g|_{\partial U} = g$ by definition, $v|_{\partial U} = 0$ as $v \in \mathscr{D}(A^*)$, and $\langle (-\underline{A})\Lambda g, v\rangle = 0$ by the definition of Λ. ■

The next characterisation connects the semigroup approach with the double-layer potential approach to boundary value problems. We recall there exists a positive C^∞-function $G_U : (0,\infty)\times U \times U \to \mathbb{R}$ called the **Dirichlet heat kernel** such that

$$(e^{tA} f)(x) = \int_U G_U(t,x,y) f(y)\, dy$$

for any $f \in L^p(U)$, $1 \le p \le \infty$.

Lemma 3.30. *For $g \in L^q(0,T;L^p(\partial U))$, we have*

$$\left(\int_0^T (-A)e^{tA}\Lambda f(t)\,dt\right)(x) = -\int_0^T \int_{\partial U} \frac{\partial G_U}{\partial \nu_y}(t,x,y) f(t,y)\sigma(dy)dt,$$

where σ is the surface measure on $(\partial U, \mathscr{B}(\partial U))$.

Proof. Write $S(t) := e^{tA}$ and let $g \in L^q(0,T;C(\partial U))$ then $\Lambda g \in C^2(U) \cap L^p(U)$ and as $(S(t))_{t\ge 0}$ is analytic on $L^p(U)$, $S(t)$ maps into $\mathscr{D}(A)$ for $t > 0$. As $Af = \Delta f$ for $f \in \mathscr{D}(A)$ and $G_U(t,x,y) = G_U(t,y,x)$ for all $x, y \in U$ and $t > 0$ we have

$$\begin{aligned}
(-A)S(t)\Lambda g(t,x) &= (-\Delta)\int_U G_U(t,x,y)\Lambda g(t,y)\,dy \\
&= \int_U (-\Delta) G_U(t,x,y)\Lambda g(t,y)\,dy \\
&= \int_U (-\Delta_y) G_U(t,x,y)\Lambda g(t,y)\,dy \\
&= -\int_U G_U(t,x,y)\Delta_y \Lambda g(t,y)\,dy \\
&\quad + \int_{\partial U} G_U(t,x,z)\partial_{\nu_z}(\Lambda g(t,z))\sigma(dz) \\
&\quad - \int_{\partial U} \partial_{\nu_z} G_U(t,x,z) g(t,z)\sigma(dz) \\
&= -\int_{\partial U} \partial_{\nu_z} G_U(t,x,z) g(t,z)\sigma(dz)
\end{aligned}$$

as $G_U(t,x,z)=0$ for $z\in\partial U$ and $\Delta_y\Lambda g(t,y)=0$ for $y\in U$. Therefore, by density of $C(\partial U)$ in $L^p(\partial U)$ we get the identity by approximation and integrating over time. ■

Using the explicit representations, we can derive the following alternative definition of a weak solution.

Definition 3.31. *A **weak solution** of* (3.1) *is a function* $u\in L^1(0,T;L^p(U))$ *such that for all* $t\in[0,T]$ *and* $v\in\mathscr{D}(\underline{A}^*)\cap\{v:\tau v=0\}$ *we have*

$$\langle u(t),v\rangle=\langle u_0-\Lambda g_0,v\rangle+\int_0^t\langle u(s),\underline{A}^*v\rangle\,ds-\int_0^t[g(s),\mathcal{N}v]\,ds.$$

Lemma 3.32. *If a function u satisfies Definition 3.31 then it satisfies Definition 3.17, and vice-versa.*

Proof. Follows by applying Green's theorem (i.e. Theorem 3.28). ■

Remark 3.33. Definition 3.31 forms the basis of the definition of a weak solution given by [33, Definition 15.1] (also see [53]) for the stochastic setting.

4

Stochastic Boundary Data

This chapter[1] introduces the theory for evolution equations driven by stochastic boundary data. Our theory extends (1.5), given by Da Prato and Zabczyk in [11], to the Banach space setting.

Let E and ∂E be Banach spaces and H and ∂H be Hilbert spaces. Let $\underline{A} : \mathscr{D}(\underline{A}) \subset E \to E$ be a closed and densely defined linear operator and let $T > 0$ be some finite time horizon. We now consider the stochastic version of (3.1) given by

$$X'(t) = \underline{A}X(t), \quad \tau X(t) = B\dot{W}(t), \quad X(0) = x, \tag{4.1}$$

where $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on ∂H, $B \in \mathscr{L}(\partial H, \partial E)$ and $\tau : \mathscr{D}(\tau) \subset E \to \partial E$.

[1] Most results of this chapter were presented during the meeting held from June 16 to 19 at the Heinrich-Fabri Institute in Blaubeuren (Germany) for the 11th TULKA Internet Seminar titled "Stochastic Evolution Equations" and form part of project 3: "L^p theory of the heat equation driven by boundary noise" posed by Ben Goldys.

4.1 Strong and weak solutions

Following the results of Chapter 3, one could suggest several types of solutions to (4.1). Assume there exists another linear operator $A : \mathscr{D}(A) \subset E \to E$ that is defined by $Ax := \underline{A}x$ for $x \in \mathscr{D}(A)$ where

$$\mathscr{D}(A) := \mathscr{D}(\underline{A}) \cap \ker \tau.$$

If $B \in \gamma(\partial H, \partial E)$ and $(W(t))_{t\geq 0}$ then (see §2.12) we can identify $(BW(t))_{t\geq 0}$ with a Wiener process $(W^B(t))_{t\geq 0}$. Let $[\cdot,\cdot]$ be the dual pairing between ∂E and $(\partial E)^*$.

The following definitions seem the most natural stochastic extensions of Definition 3.14 and Definition 3.17.

Definition 4.1. *Let $B \in \gamma(\partial H, \partial E)$. A **strong solution** to* (4.1) *is a strongly measurable E-valued stochastic process $(X^x(t))_{t\in[0,T]}$ such that*

- *$t \mapsto X^x(t)$ is integrable $\mathbb{P}$-almost surely,*
- *for all $t \in [0,T]$, $\mathbb{P}$-almost surely, we have*

$$\int_0^t X^x(s)\,ds \in \mathscr{D}(\underline{A}), \qquad \int_0^t X^x(s)\,ds \in \mathscr{D}(\tau),$$

- *for all $t \in [0,T]$, $\mathbb{P}$-almost surely,*

$$X^x(t) = x + \underline{A}\int_0^t X^x(s)\,ds, \qquad \tau \int_0^t X^x(s)\,ds = W^B(t).$$

Definition 4.2. *A **weak solution** to* (4.1) *is a E-valued process $(X^x(t))_{t\in[0,T]}$ which has a strongly measurable version with the following properties:*

- *$\mathbb{P}$-almost surely, the paths $t \mapsto X^x(t)$ are integrable,*
- *for all $t \in [0,T]$ and $x^* \in \mathscr{D}(A^*)$ we have $\mathbb{P}$-almost surely,*

$$\langle X(t), x^*\rangle = \langle x, x^*\rangle + \int_0^t \langle X^x(s), \underline{A}^* x^*\rangle\,ds,$$

- *for all* $t \in [0, T]$ *and* $z^* \in (\partial E)^*$*, we have* $\mathbb{P}$*-almost surely,*

$$[\tau X^x(t), z^*] = [BW(t), z^*].$$

Although these definitions seem like appropriate extensions of their deterministic counterparts as they are given in terms of the operator $\underline{A}$ (and not A), they are difficult to work with in the stochastic setting. Therefore, in the next section, we take the standard approach of formulating the boundary value problem (4.1) as an abstract Cauchy problem.

4.2 Mild solutions

In this section we follow the well-known methodology [11, 27, 28, 33] whereby the boundary value problem (4.1) is formulated as an abstract Cauchy problem.

Assume there exists another linear operator $A : \mathscr{D}(A) \subset E \to E$ that is defined by $Ax := \underline{A}x$ for $x \in \mathscr{D}(A)$ where

$$\mathscr{D}(A) := \mathscr{D}(\underline{A}) \cap \ker \tau.$$

We recall that $\underline{A}$ is called the maximal operator and A is called the restricted operator. Similar to Chapter 3, we assume that:

- $A : \mathscr{D}(A) \subset E$ generates an analytic semigroup $(e^{tA})_{t \geq 0}$ on E,
- $\Lambda_\lambda : \partial E \to E$ continuously for some $\lambda \geq 0$,
- $B : \partial H \to \partial E$ continuously.

In this chapter, the Hilbert space ∂H replaces the use of the Banach space Y in Chapter 3. Using these assumptions and Remark 3.13, we can formally rewrite (4.1) as the abstract Cauchy problem

$$\begin{cases} dX(t) = AX(t)\,dt - (\lambda - A)\Lambda_\lambda B\,dW(t), & t \in [0, T], \\ X(0) = x \in E, \end{cases} \tag{4.2}$$

by formally setting $g = \dot{W}(t)$ in (3.5), multiplying both sides of (3.5) by dt, and writing $\dot{W}(t)\,dt = dW(t)$. Alternatively, one could follow the derivation of [33, Section 15.1] to obtain the integral version of (4.1) given by

$$X(t) = e^{tA}x + \int_0^t (\lambda - A)e^{tA}\Lambda_\lambda B\,dW(t), \quad t \in [0,T]. \tag{4.3}$$

As the second term in (4.3) is a stochastic convolution the following definition of a solution has become customary (e.g., see [11, 27, 33]).

Definition 4.3. *Let $x \in E$. The process $(X^x(t))_{t\geq 0}$ (given by (4.3)) is called a **mild solution** of (4.1) if*

$$\sup_{t\in[0,T]} \mathbb{E}\|X^x(t)\|_E^p < \infty, \quad p \geq 2.$$

However, we suggest that for (4.3) to make sense as a solution to the boundary value problem (4.1) (and not simply as a mild solution of the formal abstract Cauchy problem (4.2)) we propose the following slight modification of the definition of a mild solution.

Definition 4.4. *Let $x \in E$. The process $(X^x(t))_{t\geq 0}$ is called the **well-posed mild solution** of (4.1) if*

$$\sup_{t\in[0,T]} \mathbb{E}\|X^x(t)\|_E^p < \infty, \quad p \geq 2.$$

and $X^x(t) \in \mathscr{D}(\tau)$ for $t \in (0,T]$.

The following example explains why we suggest this definition.

Example 4.5. Consider the case $E = L^p(U)$ for some bounded domain $U \subset \mathbb{R}^d$ with smooth boundary ∂U. Suppose $\tau u := u|_{\partial U}$ in terms of trace, then it is well-known τ only makes sense for $u \in \mathscr{D}(\tau) = W^{1,p}(U)$. However, suppose that a mild solution $X^x := (X^x(t))_{t\geq 0}$ (in the sense of Definition 4.3) is obtained such that $X^x(t) \notin \mathscr{D}(\tau)$ for $t \in (0,T]$, then X^x does not satisfy the boundary condition of (4.1) in any meaningful way.

Remark 4.6. Example 4.5 suggests either working in a space where the concept of trace holds for a larger class of functions or modifying the definition of τ so that the relationship between the solution and the boundary data is understood in a different way (e.g., pointwise instead of in trace). This motivates the weighted L^p approach we present in Chapter 5 and the Harmonic analysis approach we present in Chapter 6.

By Theorem 2.33, a sufficient condition for the existence of a mild solution to (4.2) is that $\Phi(t) := AS(t)\Lambda B$ is stochastically integrable with respect to W, or equivalently, that the operator

$$R_\Phi f := \int_0^T \Phi(t) f(t)\, dt, \quad f \in L^2(0,T;\partial H),$$

is γ-radonifying from $L^2(0,T;\partial H)$ to E. In [11] and [33, Chapter 13, Equation 15.3], in the case where $E = L^2(U)$, it is stated that a necessary and sufficient condition for

$$t \mapsto -\int_0^t AS(t-s)\Lambda B\, dW(s), \quad t \in [0,T],$$

to be a well-defined square integrable process taking values in $L^2(U)$ is given by

$$\int_0^T \|AS(t)\Lambda B\|^2_{\mathscr{L}_2(\partial H, L^2(U))}\, dt < \infty.$$

Therefore, one could suggest that a Banach space extension of this sufficient condition is

$$\int_0^T \|AS(t)\Lambda B\|^2_{\gamma(\partial H, E)}\, dt < \infty,$$

instead of $R_\Phi \in \gamma(L^2(0,T;\partial H), E)$. This raises the question: which condition implies the other? In the Banach space setting this depends on the space E.

Theorem 4.7. *Assume $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on ∂H. If E has type 2 and for $\lambda \in \varrho(A)$,*

$$\int_0^T \|(\lambda - A)e^{tA}\Lambda_\lambda B\|^2_{\gamma(\partial H, E)}\, dt < \infty.$$

Then for any $x \in E$ there exists a unique process $(X^x(t))_{t\in[0,T]}$ such that

$$\sup_{t\in[0,T]} \left(\mathbb{E}\|X(t)\|_E^2\right)^{1/2} < \infty.$$

Proof. Follows by taking $A = (0,T)$ and $\mu = dt$ (Lebesgue measure) in Theorem 2.30 and then Theorem 2.33. ■

Let $U \subset \mathbb{R}^d$ be a bounded domain with smooth boundary ∂U. Spaces of type 2 include $L^p(U)$ for $p \geq 2$. Peszat and Zabczyk obtain the following necessary and sufficient condition for existence of a mild solution in the case $E = L^2(U)$. Recall that $\gamma(\partial H, E) = \mathscr{L}_2(\partial H, E)$ when E is a Hilbert space.

Theorem 4.8 (Peszat/Zabczyk)**.** *Assume $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on ∂H. If $E = L^2(U)$ for a bounded domain $U \subset \mathbb{R}^d$ with smooth boundary ∂U and $x \in L^2(U)$. Then*

$$\int_0^t \|(\lambda - A)e^{tA}\Lambda_\lambda\|^2_{\mathscr{L}_2(\partial H, L^2(U))}\, ds < \infty, \quad \textit{for } t > 0,$$

is a necessary and sufficient condition for (4.3) *to be a mild solution to* (4.2)*.*

Theorem 4.9. *Assume $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on ∂H. If E has cotype 2 and for $\lambda \in \varrho(A)$,*

$$Rg := \int_0^T (\lambda - A)e^{tA}\Lambda_\lambda B g(t)\, dt, \quad g \in L^2(0,T;\partial H),$$

is γ-radonifying from $L^2(0,T;\partial H)$ to E. Then for any $x \in E$ there exists a unique process $(X^x(t))_{t\in[0,T]}$ such that

$$\sup_{t\in[0,T]} \left(\mathbb{E}\|X(t)\|_E^2\right)^{1/2} < \infty.$$

Proof. Follows by taking $A = (0,T)$ and $\mu = dt$ (Lebesgue measure) in Theorem 2.30 and then Theorem 2.33. ■

We now provide a refined sufficient condition for the existence of a mild solution to (4.2). This condition is an abstraction of [33, Theorem 15.4] obtained in the case $E = L^p(U)$ where $U \subset \mathbb{R}^d$ with smooth boundary ∂U.

Theorem 4.10. *If* $(-A)^\kappa \Lambda B \in \gamma(\partial H, E)$ *for some* κ *satisfying* $1/2 < \kappa \leq 1$*, then* $t \mapsto (-A)S(t)\Lambda B$ *is stochastically integrable with respect to* W.

Proof. By Theorem 2.33 it is sufficient to check that $\Phi(t) = (-A)S(t)\Lambda B$ is stochastically integrable on $(0, T)$ or equivalently that

$$R_\Phi f := \int_0^T \Phi(t) f(t)\, dt, \quad f \in L^2(0, T; \partial H),$$

is γ-radonifying from $L^2(0, T; \partial H)$ to E. Choose $\beta \in (0, 1/2)$ and $0 \leq \eta < \beta$ such that $\kappa = 1 - \eta$. We now factorise $\Phi(t)$ as

$$\begin{aligned} \Phi(t) &= t^\beta (-A)^\eta S(t) t^{-\beta} (-A)^{1-\eta} \Lambda B \\ &= t^\beta (-A)^\eta S(t) \Psi(t) \end{aligned}$$

where $\Psi(t) := t^{-\beta}(-A)^{1-\eta}\Lambda B$. By Lemma 2.3, the set $\{t^\beta(-A)^\eta S(t) : t \in (0, T)\}$ is γ-bounded in $\mathscr{L}(E, E)$. Hence, by Theorem 2.2, R_Φ belongs to $\gamma(L^2(0, T; \partial H), E)$ once $R_\Psi \in \gamma(L^2(0, T; \partial H), E)$. By assumption,

$$(-A)^{1-\eta}\Lambda B = (-A)^\kappa \Lambda B \in \gamma(\partial H, E),$$

and as $t^{-\beta} \in L^2(0, T)$, $R_\Psi \in \gamma(L^2(0, T; \partial H), E)$ with norm

$$\|R_\Psi\|_{\gamma(L^2(0,T;\partial H),E)} = \|t^{-\beta}\|_{L^2(0,T)} \|(-A)^\kappa \Lambda B\|_{\gamma(\partial H, E)}.$$

■

Corollary 4.11. *If* $(-A)^\kappa \Lambda B \in \gamma(H, E)$ *for some* κ *satisfying* $1/2 < \kappa \leq 1$*, then* (4.2) *has a weak solution.*

4.3 Dirichlet boundary noise problem

In this section we recall the classic example of [11] (see also [33, Theorem 15.6]) which shows that L^2-valued solutions for the Dirichlet boundary noise problem for the heat equation cannot be obtained (even in dimension one).

Fix $2 \geq p < \infty$. Let $U = (0,1) \subset \mathbb{R}$ and $E = L^p(U)$. We set A to be the Dirichlet Laplacian on $L^p(U)$ which generates an analytic semigroup $(e^{tA})_{t\geq 0}$. As $\partial U = \{0,1\}$, we see that any function space on the two-point boundary ∂U (e.g., $W^{s,p}(\partial U)$ for any $s \geq 0$) can be identified with $\mathbb{R}^2$. Hence, we take $\partial H = \partial E = \mathbb{R}^2$. As U is bounded we can take $\lambda = 0$ and consider the elliptic boundary value problem

$$-\Delta u = 0 \text{ in } (0,1), \quad u(0) = a, \quad u(1) = b,$$

to obtain the Dirichlet map $\Lambda : \mathbb{R}^2 \to L^2(U)$ given by

$$\Lambda : (a,b)^T \mapsto a(1-\xi) + b\xi, \quad \xi \in (0,1). \tag{4.4}$$

Let $(W(t))_{t\geq 0}$ by a cylindrical Wiener process on $\partial H = \mathbb{R}^2$. It follows automatically that $B = I \in \mathscr{L}_2(\partial H)$, hence $(W(t))_{t\geq 0}$ is a Wiener process (i.e., not cylindrical) given explicitly by $W(t) = (w_0(t), w_1(t))$ where $(w_i(t))_{t\geq 0}$ for $i = 1,2$ are independent $\mathbb{R}$-valued Wiener processes. This setup models the boundary value problem

$$\partial_t u(t,\xi) = \partial_{\xi\xi} u(t,\xi) \text{ on } (0,1), \quad u(t,0) = \dot{w}_0(t), \quad u(t,1) = \dot{w}_1(t),$$

with $u(0,\xi) = x(\xi)$ by the relation $X^x(t)(\xi) = u(t,\xi)$ and, in this case, (4.3) is given by

$$X^x(t) = e^{tA}x + \int_0^t (-A)e^{tA}\Lambda \, dW(s), \quad t \in [0,T].$$

By Theorem 4.10, it is sufficient to check that

$$(-A)^\kappa \Lambda \in \gamma(\partial H, L^p(0,1))$$

for some $1/2 < \kappa \leq 1$. However, since $\partial H = \mathbb{R}^2$ we have

$$\gamma(\partial H, L^p(0,1)) = \mathscr{L}(\mathbb{R}^2, L^p(0,1))$$

and we only need to check that $\Lambda \in \mathscr{D}((-A)^\kappa)$ for some $1/2 < \kappa \leq 1$. That is,

$$a(1-\xi) + b\xi \in \mathscr{D}((-A)^\kappa), \quad \kappa \in (1/2, 1].$$

However, setting $\alpha = \kappa$ in the characterisation

$$\mathscr{D}_A(\alpha, p) = \begin{cases} W^{2\alpha,p}(U), & \text{if } 2\alpha < 1/p, \\ \{u \in W^{2\alpha,p}(U) : \tau u = 0\}, & \text{if } 2\alpha > 1/p, \end{cases}$$

we see that for $p \geq 2$ we have $2\alpha = 2\kappa > 1 > 1/p$. Hence, the function $a(1-\xi) + b\xi$ must vanish at $\xi = 0$ and $\xi = 1$. Clearly, this is not the case, hence Theorem 4.10 is insufficient to handle this problem.

Specialising to the case $p = 2$, one may check the necessary and sufficient condition given in Theorem 4.8 directly and obtain the following result that was originally noticed in [11].

Theorem 4.12 (Theorem 15.6 in [33]). *Let $E := L^2(0,1)$, $\partial E := \{0,1\} \simeq \mathbb{R}^2$, $B := I$, then* (4.1) *does not have a $L^2(0,1)$-valued solution when Λ is given by* (4.4).

4.4 Neumann boundary noise problem

In this section we consider the example of the Neumann boundary noise problem on a bounded domain $U \subset \mathbb{R}^d$ with C^2 boundary ∂U when $E = L^p(U)$ for some $2 \leq p < \infty$. This example has been considered previously in [11, 33] for the one-dimensional case, in [28, Section 13.3] for the case $p = 2$ and $U = [0,\pi]^d$ then extended to $p \geq 1$ by embeddings. We refer the reader to Chapter 1 for a more comprehensive survey. The novelty of this section is a 'direct' Banach space approach to this well-studied example.

We consider the Neumann boundary value problem

$$\partial_t u = \mathscr{A} u \ \text{ in } [0,T]\times U, \quad \partial_\nu u = \dot{W}^B \ \text{ on } [0,T]\times \partial U, \tag{4.5}$$

with initial condition $u(0,x) = u_0(x)$. The second order differential operator $\mathscr{A}$ is given by

$$\mathscr{A}(x,D) := \sum_{i,j=1}^{d} a_{ij}(x) D_{ij} + \sum_{i=1}^{d} b_i(x) D_i + c(x) I$$

with real uniformly continuous and bounded coefficients a_{ij}, b_i, c. We assume that the matrix $[a_{ij}]$ is symmetric and satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i \xi_j \geq \kappa |\xi|^2, \quad x \in \overline{U},\, \xi \in \mathbb{R}^d,$$

for some $\kappa > 0$. The process $(W(t))_{t\geq 0}$ is a cylindrical Wiener process and $B \in \gamma(\partial H, \partial E)$, hence we identify $BW(t)$ as a Wiener process $(W^B(t))_{t\geq 0}$. We apply the setup and results given in §3.3.1 whereby

- $\underline{A} u = \mathscr{A} u$ for $u \in \mathscr{D}(\underline{A})$ where

$$\mathscr{D}(\underline{A}) := W^{2,p}(U)$$

- $\tau : \mathscr{D}(\tau) \subset E \to \partial E$ where

$$\tau u := \frac{\partial u}{\partial \nu}\bigg|_{\partial U}$$

- As $\mathscr{D}(A) := \mathscr{D}(\underline{A}) \cap \ker \tau$, it follows that

$$\mathscr{D}(A) := \left\{ u \in W^{2,p}(U) : \partial_\nu u|_{\partial U} = 0 \right\}$$

The operator A is sectorial and generates an analytic semigroup $(e^{tA})_{t\geq 0}$ on E. The Neumann map $\Lambda_\lambda : \partial E \to E$ is defined by the unique solution u to the elliptic Neumann boundary value problem

$$(\lambda - \mathscr{A})u = 0 \ \text{ in } U, \quad \partial_\nu u = g \ \text{ in } \partial U,$$

by setting $\Lambda_\lambda g = u$. We set $\partial E = W^{s,p}(\partial U)$ for $s \geq 0$ and by regularity theory for the elliptic problem we get

$$\Lambda_\lambda : \partial E \to W^{s+1+1/p,p}(U) \text{ continuously.}$$

We now choose the Hilbert space ∂H where the cylindrical Wiener process $(W(t))_{t\geq 0}$ takes values to be $\partial H = W^{2a,2}(\partial U)$. As U is a bounded domain we have for $p \in (1,2)$ the natural embedding $I_p : W^{2a,2}(\partial U) \hookrightarrow W^{2a,p}(U)$ and by the Sobolev embedding theorem for $p \geq 2$,

$$I_p : W^{2a,2}(\partial U) \hookrightarrow W^{r,p}(\partial U),$$

where $r = 2a - d(1/2 - 1/p)$. Setting $X(t)(\xi) = u(t,\xi)$, a mild solution is obtained if

$$X(t) = e^{tA}u_0 + \int_0^t (\lambda - A)e^{tA}\Lambda_\lambda \, dW^B(t), \quad t \in [0,T],$$

makes sense as an E-valued process.

The following theorem gives conditions on when one can obtain a mild solution to (4.5) depending on the regularity of the noise on the boundary (controlled by the parameter a), the space $L^p(U)$, and the ambient spatial dimension d.

Theorem 4.13. *The function $t \mapsto (\lambda - A)e^{tA}\Lambda_\lambda I_p$ is stochastically integrable on $(0,T)$ with respect to W^B when one of the following conditions hold:*

- *$d \in \{1,2,3\}$, $p \geq 2$, and $a \geq d/4$,*
- *$a \geq 0$, $4a < d < 4a+1$, and $2 \leq p < 2/(d-4a)$,*
- *$d \geq 2$, $0 < a < (d-1)/2$, and $(d-1)/2a < p < 2$,*
- *$d \geq 2$, $(d-1)/2 \leq a$, and $1 < p < 2$.*

As a consequence, the stochastic Cauchy problem (4.1) admits a unique mild solution when these conditions hold.

Proof. By [6, Theorem 8.6] and [6, Theorem 8.10], it suffices to check that the function $\Phi(t) = S(t)(\lambda - A)\Lambda_\lambda I_p B$ is stochastic integrable with respect to W, or equivalently, that the operator

$$R_\Phi f := \int_0^T \Phi(t) f(t)\, dt, \quad f \in L^2(0,T;\partial H),$$

is γ-radonifying from $L^2(0,T;\partial H)$ to E. Choose a small $\varepsilon > 0$ and β so that $0 < 1/2 - \varepsilon < \beta < 1/2$ to ensure that

$$\left\{ t^\beta (\lambda - A)^{1/2-\varepsilon} e^{tA} : t \in (0,T) \right\}$$

is γ-bounded in $\mathscr{L}(L^p(U))$. This holds as $1/2 - \varepsilon < \beta$. We write

$$\Phi(t) = t^\beta (\lambda - A)^{1/2-\varepsilon} e^{tA} t^{-\beta} (\lambda - A)^{1/2+\varepsilon} \Lambda_\lambda I_p =: t^\beta (\lambda - A)^{1/2-\varepsilon} e^{tA} \Psi(t),$$

where $\Psi(t) := t^{-\beta}(\lambda - A)^{1/2+\varepsilon}\Lambda_\lambda I_p$. By [6, Lemma 10.17] and the γ-multiplier theorem [6, Theorem 9.14], the operator R_Φ belongs to $\gamma(L^2(0,T;\partial H), E)$ once we know that

$$R_\Psi \in \gamma(L^2(0,T;\partial H), E).$$

As $\beta < 1/2$, it is clear that $t \mapsto t^{-\beta} \in L^2(0,T)$ so all we need to check is

$$(\lambda - A)^{1/2+\varepsilon} \Lambda_\lambda I_p \in \gamma(\partial H, E).$$

We know that $\Lambda_\lambda \in \mathscr{L}(W^{r,p}(\partial U), W^{r+1+1/p,p}(U))$ by [83]. Next, we know that

$$(\lambda - A)^{1/2+\varepsilon} : W^{r,p}(U) \to W^{r-1-2\varepsilon,p}(U)$$

continuously, so we have that

$$(\lambda - A)^{1/2+\varepsilon} D_\lambda : W^{r,p}(\partial U) \to W^{r+1/p-2\varepsilon,p}(U)$$

continuously. Thus, $(\lambda - A)^{1/2+\varepsilon}\Lambda_\lambda \in \mathscr{L}(W^{r,p}(\partial U), E)$ if and only if $r + 1/p - 2\varepsilon \geq 0$. As $0 < \varepsilon < 1/2$ it is clear that this holds for $p > 1$ and $r \geq 0$. By Sobolev embedding, one has $W^{r+1/p-2\varepsilon,p}(U) \hookrightarrow C_b(\overline{U})$ when $r + 1/p - 2\varepsilon > d/p$.

If this held, then the embedding into $L^p(U)$ would be γ–radonifying by [10, Lemma 2.1]. Assuming $\varepsilon \downarrow 0$, we see that this holds (for $p \geq 2$ by setting $r = 2a - d(1/2 - 1/p)$ and $r = 2a$ when $p \in (1,2)$) when one of the following holds:

- $a \geq 0$, $0 < d \leq 4a$, and $p \geq 2$,
- $a \geq 0$, $4a < d < 4a+1$, and $2 \leq p < 2/(d-4a)$
- $a > 0$, $d > 2a+1$, and $(d-1)/2a < p < 2$,
- $a > 0$, $0 < d \leq 2a+1$, and $1 < p < 2$.

■

The next theorem extends [11, Proposition 3.2] to the case $d = 1,2,3$ which showed that in the case that $U = (0,\pi)$ (i.e. $d = 1$) that the solution $(X(t))_{t\in[0,T]}$ of (4.1) is an E_α-valued process if and only if $\alpha < 1/4$ and $(X(t))_{t\in[0,T]}$ has an $L^2(0,\pi)$-valued continuous version. Using a different method, a similar result was obtained for a $L^2((0,\pi)^d)$-valued solution in [28, Theorem 13.3.6].

Theorem 4.14. *Under the assumptions of Theorem 4.13, for all* $0 \leq \alpha < 1/4$ *and* $\beta \geq 0$ *satisfying* $\alpha + \beta < 1/2$ *and* $1 \leq p < \infty$ *the mild solution* $(X(t))_{t\geq 0}$ *belongs to* $L^p(\Omega; E_\alpha)$ *and there exists and constant* $C \geq 0$ *such that for all* $0 \leq s, t \leq T$,

$$\left(\mathbb{E}\|X(t) - X(s)\|_{E_\alpha}^p\right)^{1/p} \leq C|t-s|^\beta.$$

As a consequence, for all $0 \leq \alpha < 1/4$ *and* $\beta \geq 0$ *satisfying* $\alpha + \beta < 1/2$ *the process* $(X(t))_{t\in[0,T]}$ *has a version with trajectories in* $C^\beta([0,T]; E_\alpha)$.

Proof. By the Kahane-Khintchine inequality [6, Theorem 3.12] it suffices to prove the estimate for $p = 2$. We fix $0 \leq \alpha < \alpha' < \min(1/2, s + 1/(2p))$ and choose $\beta \geq 0$ such that $\alpha + \beta < 1/2$. We first prove that for all $t \in [0,T]$ the random variable $X(t)$ takes it values in E_α almost surely.

By [6, Theorem 10.17], we know that

$$\left\{t^{\alpha'}(\lambda-A)^{1/2-s-1/(2p)}e^{tA} : t\in(0,T)\right\}$$

is γ-bounded in $\mathscr{L}(E,E_\alpha)$ as $\alpha+1/2-s-1/(2p)<1/2$. This implies that

$$\|R_\Phi\|_{\gamma(L^2(0,T;\partial H),E_\alpha)} \lesssim \|(\lambda-A)^{s+1/2+1/(2p)}i_p\Lambda_\lambda B\|_{\mathscr{L}(\partial H,E)}.$$

Fix $0\le s\le t\le T$. By the triangle inequality in $L^2(\Omega;E)$,

$$\begin{aligned}
&\left(\mathbb{E}\|X(t)-X(s)\|_{E_\alpha}^2\right)^{1/2}\\
&\quad\le\left(\mathbb{E}\left\|\int_0^s[e^{(t-r)A}-e^{(s-r)A}](\lambda-A)^{s+1/2+1/(2p)}i_p\Lambda_\lambda B\,dW(r)\right\|_{E_\alpha}^2\right)^{1/2}\\
&\qquad\qquad+\left(\mathbb{E}\left\|\int_s^t e^{(t-r)A}(\lambda-A)^{s+1/2+1/(2p)}i_p\Lambda_\lambda B\,dW(r)\right\|_{E_\alpha}^2\right)^{1/2}.
\end{aligned}$$

Choose $\lambda\in\mathbb{R}$ sufficiently large so that the fractional powers of $\lambda-A$ exist. For the first term we have, for any choice of $\eta,\theta\ge0$ satisfying $\alpha+\beta<\eta+\alpha<\theta<1/2$ with $\eta\ge1/4$, and using [6, Lemma 10.8] and [6, Lemma 10.15] that the first term is estimated as

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^s[e^{(t-r)A}-e^{(s-r)A}](\lambda-A)^{1-\eta}\Lambda_\lambda B\,dW(r)\right\|_{E_\alpha}^2\\
&\simeq\mathbb{E}\left\|\int_0^s(s-r)^\theta(\lambda-A)^{\eta+\alpha}e^{(s-r)A}\right.\\
&\qquad\left.\times(s-r)^{-\theta}[e^{(t-s)A}-I](\lambda-A)^{-\eta}(\lambda-A)^{1-\eta}\Lambda_\lambda B\,dW(r)\right\|^2\\
&\lesssim\mathbb{E}\left\|\int_0^s(s-r)^\theta[e^{(t-s)A}-I](\lambda-A)^{1-2\eta}\Lambda_\lambda B\,dW(r)\right\|^2\\
&=\left\|[e^{(t-s)A}-I](\lambda-A)^{1-2\eta}\Lambda_\lambda B\right\|_{\mathscr{L}_2(L^2(\partial U),E)}^2\int_0^s(s-r)^{-2\theta}\,dr\\
&\lesssim s^{1-2\theta}\left\|\left[e^{(t-s)A}-I\right](\lambda-A)^{-\eta}\right\|_{\mathscr{L}(E)}^2\left\|(\lambda-A)^{1-\eta}\Lambda_\lambda B\right\|_{\mathscr{L}_2(L^2(\partial U),E)}^2
\end{aligned}$$

$$\lesssim_T (t-s)^{2\beta} \left\|(\lambda-A)^{1-\eta}\Lambda_\lambda B\right\|^2_{\mathscr{L}_2(L^2(\partial U),E)}$$

Now for any $\eta \geq 1/4$ satisfying $1/2-\beta<\eta<1/2$ we have $\alpha<\eta$ and the second term is estimated as

$$\begin{aligned}
&\mathbb{E}\left\|\int_s^t e^{(t-r)A}(\lambda-A)^{1-\eta}\Lambda_\lambda B\,dW(r)\right\|^2_{E_\alpha} \\
&\simeq \mathbb{E}\left\|\int_s^t (t-r)^{\eta}(\lambda-A)^{\alpha}e^{(t-r)A}(t-r)^{-\eta}(\lambda-A)^{1-\eta}\Lambda_\lambda B\,dW(r)\right\|^2_{E} \\
&\lesssim \mathbb{E}\left\|\int_s^t (t-r)^{-\eta}(\lambda-A)^{1-\eta}\Lambda_\lambda B\,dW(r)\right\|^2_{E} \\
&\lesssim \mathbb{E}\left\|(\lambda-A)^{1-\eta}\Lambda_\lambda B\right\|^2_{\mathscr{L}_2(L^2(U),E)}\int_s^t (t-r)^{-2\eta}\,dr \\
&\lesssim_T (t-s)^{2\beta}\left\|(\lambda-A)^{1-\eta}\Lambda_\lambda B\right\|^2_{\mathscr{L}_2(L^2(U),E)}.
\end{aligned}$$

Now combining these estimates and extending to all $1<p\leq\infty$ we get

$$\left(\mathbb{E}\|X(t)-X(s)\|^p_{E_\alpha}\right)^{1/p} \lesssim_T (t-s)^{\beta}\left\|(\lambda-A)^{1-\eta}\Lambda_\lambda B\right\|_{\mathscr{L}_2(L^2(U),E)},$$

so we have shown the first part of the theorem. Now pick $\beta<\beta'<1/2-\alpha$. Given $p\geq 1$, from the above estimate we can find a constant C such that for all $0\leq s,t\leq T$,

$$\mathbb{E}|X(t)-X(s)|^p_{E_\alpha} \leq C^p|t-s|^{\beta' p}.$$

For p large enough the existence of a version with β-Hölder continuous trajectories now follows from Kolmogorov's theorem [6, Theorem 6.9]. ■

5

Weighted L^p Theory for White Noise Data

In this chapter we extend the weighted L^p space approach of [12] to higher dimensions, to elliptic problems, and to space-time white noise.

In §5.1, we ask whether there are weights μ for which it is possible to apply the theory of Chapter 4 in the case $E = L^p(U,\mu)$. Our approach makes use of a theorem by Brzeźniak and van Neerven [84] and in contradistinction to the standard applications of the theorem, the weight μ is not a priori given. We show that if μ is chosen appropriately the Poisson kernel and the Dirichlet heat kernel are γ-radonifying from $L^2(\partial U)$ to $L^p(U,\mu)$. This approach gives an alternative proof of Theorem 4 and Theorem 5 in the preprint [53].

In §5.1, we show that certain integral operators related to the solution of boundary value problems are γ-radonifying from the boundary space to the state space. In particular, in §5.1.2 we consider the elliptic case and in §5.1.3 we consider the parabolic case. In §5.2, we consider the Dirichlet heat semigroup taking values in weighted L^p spaces and in §5.3 we apply this theory to the stochastic heat equation taking values in a weighted L^p space.

5.1 γ-Radonifying mappings into weighted spaces

5.1.1 γ-Radonifying operators into L^p spaces

Let $(U, \mathscr{U})$ be a measurable space and let K be an integral operator associated with a kernel function $k(x, y)$ by

$$(Kf)(x) := \int_U k(x,y) f(y)\, dy.$$

It is well-known (e.g., see [85]) that the condition $k \in L^2(U \times U)$ characterises K as a Hilbert-Schmidt operator from $L^2(U)$ into $L^2(U)$. Theorem 5.1 (below) extends this characterisation to the γ-radonifying operator setting and is based on the following fact. Let H be a separable Hilbert space and let $L^p(U; H)$ denote the H-valued L^p space[1]. Every $f \in L^p(U; H)$ defines a bounded operator $R_f \in \mathscr{L}(H, L^p(U))$ by posing

$$(R_f h)(x) := [f(x), h], \quad x \in U,\ h \in H,$$

where $[\cdot,\cdot]$ is the inner product on H. It holds that $R_f \in \gamma(H, L^p(U))$ and every $R \in \gamma(H, L^p(U))$ is of this form [6, Theorem 5.22].

Theorem 5.1 (Bźezniak/van Neerven [84])**.** *Let $(U, \mathscr{U}, \mu)$ be a σ-finite measure space and H a separable Hilbert space with inner product $[\cdot,\cdot]$. For $K \in \mathscr{L}(H, L^p(U, \mu))$, $1 \le p < \infty$, the following are equivalent:*

1. *K is γ-radonifying,*
2. *There exists a μ-measurable function $k \in L^p(U, \mu; H)$ such that for μ-almost all $x \in U$ we have*

$$(Kh)(x) = [k(x), h], \qquad x \in U,\ h \in H.$$

We recall the proof from [84] for convenience of the reader.

Proof. Let (h_n) be an orthonormal basis for H and (γ_n) a sequence of i.i.d. standard Gaussian random variables.

[1] in the sense of Bochner.

(1 ⇒ 2) By assumption, K is γ-radonifying from H to $L^p(U,\mu)$ so

$$\mathbb{E}\Big\|\sum_{n=1}^{\infty}\gamma_n K g_n\Big\|_{L^p(U,\mu)}^p < \infty.$$

and $(\omega,x)\mapsto \sum_{n=1}^{\infty}\gamma_n(\omega)(Kh_n)(x)$ is measurable from $\Omega\times U$ to $\mathbb{R}$. Hence by Fubini's theorem,

$$\begin{aligned}\mathbb{E}\Big\|\sum_{n=1}^{\infty}\gamma_n K g_n\Big\|_{L^p(U,\mu)}^p &= \int_U \mathbb{E}\Big|\sum_{n=1}^{\infty}\gamma_n (Kh_n)(x)\Big|^p \mu(dx)\\ &\simeq_p \int_U \left(\sum_{n=1}^{\infty}|Kh_n|(x)|^2\right)^{p/2} \mu(dx)\end{aligned}$$

In particular,

$$\sum_{n=1}^{\infty}|(Kh_n)(x)|^2 < \infty$$

for μ-almost all $x\in U$. It follows that there exists a measurable $\tilde{U}\subset U$ with $\mu(U\setminus\tilde{U})=0$ such that for all $x\in\tilde{U}$ the map $k_x : H\to\mathbb{R}$,

$$k_x h := (Kh)(x)$$

is Hilbert-Schmidt, hence bounded. By the Riesz representation theorem, we obtain a function $k:\tilde{U}\to H$ such that

$$k_x h = [k(x),h]_H, \quad h\in H,\ x\in\tilde{U}.$$

Noting that

$$[k,h_n]_H = Kh_n|_{\tilde{U}},$$

we see that $x\mapsto [k(x),h_n]_H$ is measurable for each n and therefore $x\mapsto k(x)$ is measurable by Pettis's measurability theorem and the separability of H. By the Parseval formula,

$$\sum_{n=1}^{\infty}|(Kh_n)(x)|^2 = \sum_{n=1}^{\infty}|[k(x),h_n]|^2 = \|k(x)\|_H^2, \quad x\in\tilde{U}.$$

We extend k to a function on U by extending it identically zero on $U \setminus \tilde{U}$. Combining everything, we find

$$\int_U \|k(x)\|_H^p \mu(dx) \simeq_p \mathbb{E}\Big\|\sum_{j=1}^{\infty} \gamma_n K h_n\Big\|_{L^p(U,\mu)}^p < \infty.$$

(2 ⇒ 1) Using the Kahane-Khintchine inequality, for all $1 \leq M \leq N$ we have

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{n=M}^{N} \gamma_n K h_n\Big\|_{L^p(U,\mu)}^2\Big)^{p/2} &\lesssim_p \mathbb{E}\Big\|\sum_{n=M}^{N} \gamma_n K h_n\Big\|_{L^p(U,\mu)}^p \\ &= \mathbb{E}\int_U \Big\|\sum_{n=M}^{N} \gamma_n [k(x), h_n]\Big\|^p \mu(dx) \\ &\lesssim_2 \int_U \Big(\sum_{n=M}^{N} [k(x), h_n]^2\Big)^{p/2} \mu(dx).\end{aligned}$$

By assumption, the right hand side tends to 0 as $M, N \to \infty$. Thus the series $\sum_{n=1}^{\infty} \gamma_n K h_n$ converges in $L^2(\Omega; L^p(U,\mu))$ and, by the Itō-Nisio theorem, almost surely. This means that K is γ-radonifying. ■

Due to the equivalence $\gamma(H,E) = \mathscr{L}_2(H,E)$ when E is a Hilbert space, one can obtain the familiar Hilbert-Schmidt setting with an appropriate choice of H and setting $p = 2$.

Corollary 5.2. *Let $(U_1, \mathscr{U}_1, \mu_1)$ and $(U_2, \mathscr{U}_2, \mu_2)$ be a σ-finite measure spaces. For $K \in \mathscr{L}(L^2(U_1,\mu_1), L^2(U_2,\mu_2))$, the following are equivalent:*

1. *K is a Hilbert-Schmidt operator,*
2. *There exists a $\mu_1 \otimes \mu_2$-measurable function $k \in L^2(U_1 \times U_2, \mu_1 \otimes \mu_2)$ such that for μ_2-almost all $x \in U$ we have*

$$(Kf)(x) = \int_{U_1} k(x,y) f(y) \mu_1(dy), \qquad x \in U,\ f \in L^2(U_1,\mu_1).$$

Let $U \subset \mathbb{R}^d$ and $L^p(U)$ be the L^p given in terms of d-dimensional Lebesgue measure ℓ_d on $(U, \mathscr{U})$ and we recall that we call a measure μ a weight if $\mu \ll \ell_d$, i.e., we have

$$\mu(dx) = \varrho(x)dx,$$

for some ϱ. To simplify notation, we often write $\varrho(x)$ instead of $\varrho(x)dx$ and call ϱ the weight.

The typical use of Theorem 5.1 and Corollary 5.2 is: Given a fixed space $L^p(U,\mu)$ and an operator $K \in \mathscr{L}(H, L^p(U,\mu))$ one checks the condition

$$k \in L^p(U,\mu; H)$$

to conclude that K is a γ-radonifying operator from H to $L^p(U,\mu)$, or equivalently (in the context of Corollary 5.2) that

$$\int_{U_2}\int_{U_1} |k(x,y)|^2 \mu_1(dy)\mu_2(dx) < \infty,$$

for K to be a Hilbert-Schmidt operator from $L^2(U_1,\mu_1)$ to $L^2(U_2,\mu_2)$.

In the next section we will take an integral operator K such that $K \notin \gamma(H, L^p(U))$ and ask whether there exists a class of weights $\{\varrho_\alpha : \alpha \in \mathbb{R}\}$ such the operator $K \in \gamma(H, L^p(U, \varrho_\alpha))$ for admissible $\alpha \in A \subset \mathbb{R}$. Therefore, in contradistinction to the typical use of Theorem 5.1 and Corollary 5.2, the space $L^p(U,\mu)$ is not a priori given.

5.1.2 Elliptic case

Let $U \subset \mathbb{R}^d$ be a domain with boundary ∂U and $\lambda \in \mathbb{R}$. In this section, we consider the white-noise elliptic problem

$$\Delta u = 0 \text{ on } U, \quad u|_{\partial U} = w \text{ on } \partial U \text{ (in trace)}, \tag{5.1}$$

where w is a space white noise on ∂U. Recall that the solution of the Dirichlet problem with continuous boundary data $f \in C(\partial U)$ is given by the Poisson integral

$$u_f(x) = \int_{\partial U} P(x,z) f(z) \sigma(dz), \tag{5.2}$$

where σ is the surface measure on ∂U. Hence, we can view the Poisson integral as an operator P with kernel defined by

$$(Ph)(x) = [P(x,\cdot), h]$$

where $[\cdot,\cdot]$ is the inner product on $L^2(\partial U)$. The following estimates on the Poisson kernel are known (e.g., [86]).

Lemma 5.3. *If $U \subset \mathbb{R}^d$ with piecewise smooth boundary ∂U and $d \geq 2$, then for all $x \in U$ and $z \in \partial U$,*

$$|P(x,z)| \lesssim |x-z|^{1-d}.$$

Therefore, by applying the Poisson kernel estimates and choosing the measure μ in Theorem 5.1 so that (5.3) holds we show in the next theorem that $P \in \gamma(L^2(\partial U), L^p(U,\mu))$.

Remark 5.4. Note that since $\gamma(H,E) \subset \mathscr{L}(H,E)$ once we obtain $P \in \gamma(H,E)$ then automatically P extends to a linear operator from H to E as well. This is relevant as we have only defined (5.2) for $f \in C(\partial U)$.

Theorem 5.5. *The operator $P \in \mathscr{L}(C(\partial U), C(U))$ given by*

$$(Pf)(x) = \int_{\partial U} P(x,z) f(z) \sigma(dz)$$

is γ-radonifying from $L^2(\partial U)$ to $L^p(U,\mu)$ if μ and p are chosen so that

$$\int_U \operatorname{dist}(x,\partial U)^{p(1-d)} \mu(dx) < \infty. \tag{5.3}$$

Proof. Setting $H = L^2(\partial U)$, $A = U$, $K = P$, and $k(x) = P(x,\cdot)$ in Theorem 5.1, all one needs is to choose the measure μ and $1 \le p < \infty$ such that

$$\int_U \left(\int_{\partial U} |P(x,z)|^2 \sigma(dz)\right)^{p/2} \mu(dx) < \infty,$$

for P to be γ-radonifying from $L^2(\partial U, \sigma)$ to $L^p(U,\mu)$. Using Lemma 5.3, we approximate

$$\int_{\partial U} |P(x,z)|^2 \sigma(dz) \lesssim \operatorname{dist}(x,\partial U)^{2(1-d)}.$$

and the result follows. ■

Therefore, for an appropriate choice of μ this implies that the Poisson integral is γ-radonifying from $L^2(\partial U)$ to $L^p(U,\mu)$ and, as such, we can consider white noise on the boundary. We write $\delta(x) := \operatorname{dist}(x,\partial U)$ and introduce the class of weights

$$\{\delta^\alpha(x) : \alpha \in \mathbb{R}\}.$$

Notice that when $\alpha = 0$ we have $L^p(U,\delta^\alpha) = L^p(U)$ and we have the scale of spaces

$$L^p(U,\delta^\alpha) \subset L^p(U,\delta^\beta), \quad \alpha < \beta.$$

Theorem 5.6. *The random variable*

$$X = \int_{\partial U} P(\cdot,z)\, dw(z)$$

is well-defined and takes values in $L^p(U,\delta^\alpha)$ *if* $d \ge 2$ *and* $\alpha > p(d-1)-1$.

Proof. Follows from Theorem 5.5 by choosing $\mu(dx) = \delta^\alpha(x)\,dx$ which implies that the operator P is γ-radonifying from H to $L^p(U,\delta^\alpha)$ for $\alpha > p(d-1)-1$ and thus maps a cylindrical Gaussian random variable on H of the form $\sum_{k=1}^\infty \gamma_k h_k$ where (h_k) is an orthonormal basis of H and (γ_k) is a sequence of independent Gaussian random variables on $(\Omega,\mathscr{F},\mathbb{P})$ to a well-defined Gaussian random variable on the space $L^p(U,\delta^\alpha)$. ■

Example 5.7. Let $\mathbb{D}$ be the unit disk in $\mathbb{R}^2 \simeq \mathbb{C}$ with boundary $\mathbb{T}$. Then

$$u(x) = \int_{\mathbb{T}} P(x,z)\,dw(z), \quad x \in \mathbb{D},$$

is a well-defined Gaussian random variable in $L^p(\mathbb{D},\delta^\alpha)$ if $\alpha > p-1$. In particular, if $p=2$ then we must take $\alpha > 1$.

Recall that given $f \in C(\partial U)$ and the Poisson integral of f given for every $x \in U$ by

$$u(x) = \int_{\partial U} P(x,y) f(y)\sigma(dy)$$

where σ is the surface measure on ∂U, it follows (e.g., [87]) that we have

$$D^\kappa u(x) = \int_{\partial U} D^\kappa P(x,y) f(y)\sigma(dy), \quad |\kappa| \geq 0,$$

hence we can obtain the following theorem.

Theorem 5.8. *The random variable*

$$X = \int_{\partial U} P(\cdot,z)\,dw(z)$$

takes values in $W^{1,p}(U,\delta^\alpha)$ if $d \geq 2$ and $\alpha > pd-1$.

5.1.3 Parabolic case

We now consider the white-noise parabolic problem

$$\partial_t u = \Delta u \text{ on } [0,T]\times U, \quad u(t,\cdot)|_{\partial U} = w(t,\cdot) \text{ on } \partial U \text{ (in trace)}, \tag{5.4}$$

with initial condition $u(0,x) = 0$ where w is a space-time white noise on $[0,T]\times\partial U$.

In this case the application of Theorem 5.1 is a bit more subtle as one needs to determine the correct choice for the Hilbert space H.

Let A be the Dirichlet Laplacian on $L^p(U)$ and $(S(t))_{t\in[0,T]}$ be the analytic C_0-semigroup generated by A. We recall that there exists a positive C^∞-function $G_U:(0,\infty)\times U\times U\to\mathbb{R}$ called the Dirichlet heat kernel such that

$$(S(t)f)(x)=\int_U G_U(t,x,y)f(y)\,dy, \qquad f\in L^p(U).$$

and by Lemma 3.30 we have that for $f\in C([0,T]\times\partial U)$, the solution to the inhomogeneous Dirichlet problem for the heat equation (with zero initial condition) is given by

$$u_f(t,x)=\int_0^t\int_{\partial U}\partial_{\nu_y}G_U(t-s,x,y)f(t,y)\,dy\,ds.$$

See for example [80] or, in connection to our abstract approach, see §3.4. We will make use of the following estimates on the kernel G_U.

Lemma 5.9 (e.g., see [80])**.** *If $U\subset\mathbb{R}^d$ is of class C^2, there are constants $C_1,C_2>0$ dependent on U such that*

$$\left|\frac{\partial^{m+k+\ell}G_U}{\partial t^\ell\partial y^k\partial x^m}(t,x,y)\right|\le C_1t^{-(d+m+k+2\ell)/2}\exp\left(-\frac{|x-y|^2}{C_2t}\right).$$

Mirroring our elliptic result, our γ-radonifying result for the parabolic case is obtained by taking $H=L^2(0,T;L^2(\partial U))$ and using the estimates for the kernel G_U.

Theorem 5.10. *If $U\subset\mathbb{R}^d$ is a bounded domain then $R:L^2(0,T;L^2(\partial U))\to L^p(U,\mu)$ given by*

$$(Rf)(x)=\int_0^T\int_{\partial U}\partial_{\nu_y}G_U(t,x,y)f(t,y)\,dy\,dt.$$

is γ-radonifying if the measure μ and $p\ge1$ are chosen so that

$$\int_U \operatorname{dist}(x,\partial U)^{-pd}\,\mu(dx)<\infty.$$

Proof. Setting $H = L^2(0,T;L^2(\partial U))$ in Theorem 5.1 and using the characterisation of R given by Lemma 3.30, we only need to check that

$$\frac{\partial G_U}{\partial \nu}(\cdot,x,\cdot) \in L^p(U;L^2(0,T;L^2(\partial U))).$$

Using Lemma 5.9, we can estimate

$$\int_{\partial U}\left|\frac{\partial G_U}{\partial \nu_z}(t,x,z)\right|^2 \sigma(dz) \lesssim t^{-(d+1)}\exp\left(-\frac{\operatorname{dist}(x,\partial U)^2}{ct}\right).$$

and from the estimate $\int_0^T t^{-a}\exp(-b/(ct))\,dt \lesssim b^{1-a}$ we get

$$\begin{aligned}\int_0^T\int_{\partial U}\left|\frac{\partial G_U}{\partial \nu_z}(t,x,z)\right|^2 dz\,dt &\lesssim \int_0^T t^{-(d+1)}\exp\left(-\frac{\operatorname{dist}(x,\partial U)^2}{ct}\right)dt\\ &\lesssim \operatorname{dist}(x,\partial U)^{-2d}\end{aligned}$$

Therefore,

$$\int_U\left\|\frac{\partial G_U}{\partial \nu}(\cdot,x,\cdot)\right\|_H^p \mu(dx) < \infty \quad \text{if} \quad \int_U \operatorname{dist}(x,\partial U)^{-pd}\,\mu(dx) < \infty.$$

■

As in the elliptic case by choosing $\mu(dx) := \delta(x)^\alpha\,dx$, we can now obtain.

Theorem 5.11. *The stochastic process $(X(t))_{t\in[0,T]}$ given by*

$$X(t) := \int_{\partial U}\int_0^t \partial_{\nu_y} G_U(t-s,\cdot,y)\,w(ds,dy), \quad t\in[0,T],$$

takes values in $L^p(U,\delta^\alpha)$ if $\alpha > dp-1$.

Proof. The proof follows by taking a partition of unity of U and locally mapping each ball to the half-plane. Then $\operatorname{dist}(x,\partial U) \asymp x_d$ as $x \to \partial\mathbb{R}^d_+ \equiv \mathbb{R}^{d-1}$ where $x = (x_1,\dots,x_d)$ we have

$$\int_0^M x_d^{-pd} x_d^\alpha\,dx_d < \infty$$

if $dp < \alpha + 1$. ■

Remark 5.12. We note that the same conditions on d, p, and α are obtained in the main theorem of the preprint [53] for (5.1) and (5.4).

Again, similar to the elliptic case, we can use the kernel estimates to obtain existence of a well-defined stochastic process on the weighted Sobolev space.

Theorem 5.13. *The stochastic process* $(X(t))_{t\in[0,T]}$ *given by*

$$X(t) := \int_{\partial U} \int_0^t \partial_{\nu_y} G_U(t-s,\cdot,y)\, w(ds,dy), \quad t \in [0,T],$$

takes values in $W^{1,p}(U,\delta^\alpha)$ *if* $\alpha > dp + p - 1$.

Remark 5.14. We have been careful **not** to say that our result gives a solution to (5.1) and (5.4) (compare to [53]). In particular, we spoke only of existence of a "γ-radonifying map".

5.2 Dirichlet heat semigroup on weighted L^p spaces

We note that the results of the last section are analytic in nature and fall outside the abstract approach developed in Chapter 4. Recall Theorem 4.7 which states that a mild solution to (4.1) may be found once we know that

$$\int_0^T \|(\lambda - A)e^{tA}\Lambda_\lambda B\|^2_{\gamma(\partial H,E)}\, dt < \infty. \tag{5.5}$$

If $E = L^p(U,\delta^\alpha)$ where $\delta(x) = \mathrm{dist}(x,\partial U)$ and $\alpha > 0$ then to check condition (5.5) in the case where A is the Dirichlet Laplacian on $L^p(U)$ and $(e^{tA})_{t\ge0}$ is the Dirichlet heat semigroup on $L^p(U)$ then it is desirable to understand the properties of $(e^{tA})_{t\ge0}$ taking values in $L^p(U,\delta^\alpha)$. In this section, we provide some results in this direction.

We assume $(e^{tA})_{t\ge0}$ is the Dirichlet heat semigroup on $L^p(U)$ where $U \subset \mathbb{R}^d$ is a bounded domain with C^2 boundary ∂U and $\delta(x) := \mathrm{dist}(x,\partial U)$. We shall make frequent use of the representation

$$(e^{tA}f)(x) = \int_U G_U(t,x,y)f(y)\,dy, \quad f \in L^p(U), \tag{5.6}$$

where $G_U : (0,\infty)\times U \times U \to \mathbb{R}$ is the Dirichlet heat kernel.

5.2.1 γ-Radonifying properties

It is well-known that the Dirichlet heat semigroup $(e^{tA})_{t\geq 0}$ on $L^p(U)$ is γ-radonifying in dimension one, i.e., $U \subset \mathbb{R}$. This property ensures that the stochastic convolution

$$u(t) := \int_0^t e^{(t-s)A}\,dW(s), \quad t \in [0,T],$$

is well-defined in $L^p(U)$, $p \geq 2$, even when $(W(t))_{t\geq 0}$ is a cylindrical Wiener process taking values in $L^2(U)$. We recall that this is why the stochastic heat equation has function-valued solutions in dimension one.

We shall now proceed to study the γ-radonifying properties of the Dirichlet heat semigroup $(e^{tA})_{t\geq 0}$ on $L^p(U)$ acting the weighted space $L^p(U,\delta(x)^\alpha)$ for $\alpha > 0$. In the next lemma we determine an explicit representation of the semigroup $(e^{tA})_{t\geq 0}$ acting on the weighted space $L^p(U,\delta^\alpha)$.

Lemma 5.15. *For $\alpha \geq 0$, the semigroup $(e^{tA})_{t\geq 0}$ is given by*

$$(e^{tA}f)(x) = \int_U K(t,x,y)f(y)\delta(y)^\alpha\,dy$$

for any $f \in L^p(U,\delta^\alpha)$ where the kernel $K : (0,\infty)\times U \times U \to \mathbb{R}$ is given by

$$K(t,x,y) := G_U(t,x,y)\delta(y)^{-\alpha}.$$

Proof. For $\alpha \geq 0$, this follows from (5.6) and identifying

$$(e^{tA}f)(x) = \int_U G_U(t,x,y)f(y)\,dy$$

$$= \int_U G_U(t,x,y) f(y) \delta^{-\alpha}(y) \delta^{\alpha}(y)\, dy$$
$$=: \int_U K(t,x,y) f(y) \delta^{\alpha}(y) dy$$

■

Let G be the **Heat kernel** on $\mathbb{R}^d$ given by

$$G(t,x) := (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

We shall make use of the following estimate on the Dirichlet heat kernel.

Theorem 5.16 ([88])**.** *For $C > 0$ we have the estimates*

$$m(t,x,y) G(t, C(x-y)) \lesssim G_U(t,x,y) \lesssim m(t,x,y) G(t,(x-y)/C)$$

where

$$m(t,x,y) := \left(1 \wedge \frac{\delta(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta(y)}{\sqrt{t}}\right),$$

We recall that if H_1, H_2 are Hilbert spaces then the space of γ-radonifying operators $\gamma(H_1, H_2)$ is equivalent to the space of Hilbert-Schmidt operators $\mathscr{L}_2(H_1, H_2)$. As such, we have the following result.

Theorem 5.17. *For $t > 0$ and any fixed $0 \le \alpha < 2$,*

$$e^{tA} \in \mathscr{L}_2(L^2(U, \delta^{\alpha})).$$

Proof. Suppose the conditions on α are satisfied, then for $f \in L^p(U, \delta^{\alpha})$ we have from Lemma 5.15 that

$$(S(t)f)(x) = \int_U K(t,x,y) f(y) \delta(y)^{\alpha} dy.$$

We fix $t > 0$, and start by checking the square integrability of the kernel K on the weighted space $L^p(U, \delta^{\alpha})$ by computing

$$\iint_U |K(t,x,y)|^2 \delta^{\alpha}(x) \delta^{\alpha}(y) dx dy = \sum_{i=1}^{4} \iint_{U_i} |K(t,x,y)|^2 \delta^{\alpha}(x) \delta^{\alpha}(y) dx dy$$

where we take the sub-domains:

$$
\begin{aligned}
U_1 &:= \{(x,y) : 0 \le \delta(x) < \sqrt{t}, 0 \le \delta(y) < \sqrt{t}\},\\
U_2 &:= \{(x,y) : \delta(x) \ge \sqrt{t}, \delta(y) \ge \sqrt{t}\},\\
U_3 &:= \{(x,y) : 0 \le \delta(x) < \sqrt{t}, \delta(y) \ge \sqrt{t}\},\\
U_4 &:= \{(x,y) : \delta(x) \ge \sqrt{t}, 0 \le \delta(y) < \sqrt{t}\}.
\end{aligned}
$$

Consider the domain U_1 then assuming $\alpha < 2$ and using Theorem 5.16,

$$
\begin{aligned}
&\iint_{U_1} K^2(t,x,y)\delta^\alpha(x)\delta^\alpha(y)dx\,dy\\
&\quad\lesssim \iint_{U_1} G_U^2(t,x,y)\delta^{-\alpha}(y)\delta^\alpha(x)\,dx\,dy\\
&\quad\lesssim t^{-2}\iint_{U_1} \delta^{2+\alpha}(x)\delta^{2-\alpha}(y)G^2(t,(x-y)/C)\,dx\,dy\\
&\quad\lesssim t^{-2}\iint_{U_1} \delta^{2+\alpha}(x)\delta^{2-\alpha}(y)G^2(t,(x-y)/C)\,dx\,dy\\
&\quad\lesssim t^{-2}(\sqrt{t})^{2+\alpha}(\sqrt{t})^{2-\alpha}\iint_{U_1} G^2(t,(x-y)/C)\,dx\,dy\\
&\quad\lesssim \iint_{U_1} G^2(t,(x-y)/C)\,dx\,dy\\
&\quad=: I_1.
\end{aligned}
$$

Using on-diagonal estimates for the Heat kernel G we have

$$
\iint_{U_1} G^2(t,(x-y)/C)dx\,dy \lesssim \iint_{U_1} G^2(t,0)dx\,dy \lesssim \frac{|U_1|^2}{(4\pi t)^d},
$$

and as U_1 is a slice of U of height $O(t)$, we have that $|U_1|^2 \lesssim t^2|U|^2$. So we conclude that $I_1 \lesssim t^{2-d}$. Now we consider the sub-domain U_2,

$$
\int_{U_2} K^2(t,x,y)\delta^\alpha(x)\delta^\alpha(y)\,dx\,dy
$$

$$\begin{aligned}
&= \iint_{U_2} G_U^2(t,x,y)\left(\frac{\delta(x)}{\delta(y)}\right)^{\alpha} dx\,dy \\
&= \iint_{U_2} G_U^2(t,x,y)\left(\frac{\delta(x)-\delta(y)}{\delta(y)}+1\right)^{\alpha} dx\,dy \\
&\lesssim \iint_{U_2} G(t,(x-y)/C)\left(\frac{\delta(x)-\delta(y)}{\delta(y)}+1\right)^{\alpha} dx dy \\
&=: I_2.
\end{aligned}$$

For some $\varepsilon > 0$, slice the domain U_2 into $M_\varepsilon \in \mathbb{N}$ sets of the form

$$S_k := \{(x,y) : k\varepsilon\sqrt{t} < |x-y| < (k+1)\varepsilon\sqrt{t}\},$$

with $k = 0,1,\ldots,M_\varepsilon - 1$. As U is a bounded domain, $M_\varepsilon < \infty$. Estimating $\delta(x)-\delta(y) \le |x-y|$ we get

$$\begin{aligned}
I_2 &\lesssim t^{-d}\sum_{k=1}^{M_\varepsilon}\iint_{U_2\cap S_k} e^{-c|x-y|^2/t}\left(\frac{|x-y|}{\delta(y)}+1\right)^{\alpha} dx dy \\
&\lesssim t^{-d}\sum_{k=1}^{M_\varepsilon} e^{-c\varepsilon^2k^2}\iint_{U_2\cap S_k}\left(\frac{(k+1)\varepsilon\sqrt{t}}{\sqrt{t}}+1\right)^{\alpha} dx dy \\
&\lesssim t^{-d}\sum_{k=1}^{M_\varepsilon} e^{-c\varepsilon^2k^2}\iint_{U_2\cap S_k} dx dy
\end{aligned}$$

We rotate the coordinate system so that the diagonal slices S_k are now parallel to one of the d coordinate axes, then each slice has height $O(\sqrt{t})$, so we have

$$\iint_{U_2\cap S_k} dy\,dx \lesssim t|U|^2.$$

Therefore, $I_2 \lesssim t^{1-d}$. We now consider the sub-domain U_3 using the same slicing procedure to obtain

$$\begin{aligned}
&\iint_{U_3} K^2(t,x,y)\delta^{\alpha}(x)\delta^{\alpha}(y)\,dx dy \\
&\qquad\lesssim \iint_{U_3} G_U^2(t,x,y)\delta^{\alpha}(x)\delta^{-\alpha}(y)\,dx\,dy
\end{aligned}$$

$$
\begin{aligned}
&\lesssim t^{-1}\iint_{U_3}\delta^{2+\alpha}(x)\delta^{-\alpha}(y)G^2(t,(x-y)/C)\,dx\,dy\\
&\lesssim t^{-1-d+1+\alpha/2-\alpha/2}\iint_{U_3} e^{-C|x-y|^2/t}\,dx\,dy\\
&\lesssim t^{-d}\iint_{U_3} e^{-C|x-y|^2/t}\,dx\,dy\\
&\lesssim t^{-d}\sum_{k=1}^{M_\varepsilon} e^{-C\varepsilon^2k^2}\iint_{U_3\cap S_k} dx\,dy\\
&\lesssim t^{1-d}.
\end{aligned}
$$

Finally, we consider the sub-domain U_4, then if $2-\alpha>0$, it follows using the slicing procedure that

$$
\begin{aligned}
&\iint_{U_4} K^2(t,x,y)\delta^{\alpha}(x)\delta^{\alpha}(y)\,dx\,dy\\
&\qquad\lesssim \iint_{U_4} G_U^2(t,x,y)\delta^{\alpha}(x)\delta^{-\alpha}(y)\,dx\,dy\\
&\qquad\lesssim t^{-1}\iint_{U_4}\delta^{\alpha}(x)\delta^{2-\alpha}(y)G^2(t,(x-y)/C)\,dx\,dy\\
&\qquad\lesssim t^{-1+1-\alpha/2}\iint_{U_4} G^2(t,(x-y)/C)\,dx\,dy\\
&\qquad\lesssim t^{-\alpha/2-d}\iint_{U_4} e^{-C|x-y|^2/t}\,dx\,dy\\
&\qquad\lesssim t^{-\alpha/2-d}\sum_{k=1}^{M_\varepsilon} e^{-C\varepsilon^2k^2}\iint_{U_4\cap S_k} dx\,dy\\
&\qquad =: I_4
\end{aligned}
$$

Now considering the double integral over $U_4\cap S_k$,

$$
\iint_{U_4\cap S_k} dx\,dy \lesssim \int_0^{\sqrt{t}}\int_{y+\varepsilon k\sqrt{t}}^{y+\varepsilon(k+1)\sqrt{t}} dx\,dy \lesssim \sqrt{t}\sqrt{t}=t,
$$

so we have $I_4 \lesssim t^{1-\alpha/2-d}$. Finally, recombining the pieces we see that we have

$$\iint_U |K(t,x,y)|^2 \delta^\alpha(x)\delta^\alpha(y)\,dx\,dy \lesssim 2t^{1-d} + t^{2-d} + t^{1-\alpha/2-d},$$

so the conclusion follows. ■

Corollary 5.18. *For all $0<\alpha<2$ and $t>0$ the operator $S(t)$ is bounded and we have*

$$\sup_{t>0} t^{d+\alpha/2-1} \|S(t)\|_{L^p(U,\delta^\alpha)} < \infty.$$

5.3 Stochastic heat equation on weighted spaces

Theorem 5.19. *Let U be an open domain in $\mathbb{R}^d$ with boundary ∂U. Let $X := L^2(U,\mu)$ with $\mu(d\xi) \simeq \operatorname{dist}(\xi,\partial U)^\alpha$ for $\alpha<2$. Denote*

$$U(t) := S(t)x + \int_0^t S(t-s)(-A)D\,dW(s)$$

where $(S(t))_{t\geq 0}$ is the Dirichlet Heat semigroup on $L^2(U)$ generated by the Dirichlet Laplacian A, D is the Dirichlet map, and W is a $L^2(U)$-cylindrical Brownian motion. $x \in X$. Then U is mean square continuous in $(0,T)$.

Proof. Take $x \in X$ and $B := (-A)D$. Then for $h>0$,

$$\begin{aligned} E\|U(t+h)-U(t)\|^2 &\lesssim \|S(t+h)x - S(t)x\|^2 \\ &\quad + E\left\|\int_0^{t+h} S(t+h-s)B\,dW(s) - \int_0^t S(t-s)B\,dW(s)\right\|^2 =: I_1 + I_2 \end{aligned}$$

By strong continuity of $(S(t))$ on X, we have that $I_1 \to 0$ as $h \downarrow 0$. Breaking the integral I_2 into two parts and using independence of the two stochastic integrals, we get

$$I_2 = E\left\|\int_0^{t+h} S(t+h-s)B\,dW(s) - \int_0^t S(t-s)B\,dW(s)\right\|^2$$

$$
\begin{aligned}
&= E\left\|(S(h)-I)\int_t^{t+h} S(t-s)B\,dW(s) - \int_t^{t+h} S(t+h-s)B\,dW(s)\right\|^2 \\
&\lesssim \|(S(h)-I)\|^2_{\mathscr{L}(X)}\int_t^{t+h} \|S(t-s)B\|^2_{\mathscr{L}_2}\,ds + \int_t^{t+h} \|S(t+h-s)B\|^2_{\mathscr{L}_2}\,ds \\
&=: I_3 + I_4.
\end{aligned}
$$

We know that $B \in \mathscr{L}(L^2(\partial U), X)$ and $\|S(t)\|_{\mathscr{L}_2} \lesssim t^{1/2-\alpha/2-1}$ for $t \in [0,T]$. Therefore, $\|S(t)\|^2_{\mathscr{L}_2}$ is integrable as $t \downarrow 0$ so the integrals part of I_3 and I_4 are bounded for $t \in [0, T-h]$. Further, as $\|S(h)x - x\|_X \downarrow 0$ as $h \downarrow 0$ for every $x \in X$, we have that $I_3 \downarrow 0$ by the dominated convergence theorem. We can conclude that for $t \in [0,T)$

$$
\lim_{h\downarrow 0} E\|U(t+h) - U(t)\|^2 = 0,
$$

so we have right-continuity. We shall now show left-continuity. Take $h > 0$, then

$$
\begin{aligned}
E\|U(t-h) - U(t)\|^2 &\lesssim \|S(t-h)x - S(t)x\|^2 \\
&\quad + E\left\|\int_0^{t-h} S(t-h-s)B\,dW(s) - \int_0^t S(t-s)B\,dW(s)\right\|^2 =: I_5 + I_6.
\end{aligned}
$$

For $t > 0$ and $x \in X$, we have

$$
\begin{aligned}
I_5 &= \|S(t-h)x - S(t)x\| \\
&= \|S(t-h)x - S(t-h)S(h)x\| \\
&= \|S(t-h)\|_{\mathscr{L}(X)}\|x - S(h)x\|
\end{aligned}
$$

As $(S(t))_{t\geq 0}$ is strongly continuous, there exists $\delta > 0$ and $M \geq 1$ such that $\|S(t)\| \leq M$ for all $t \in [0,\delta]$. This can be extended to any $t \in [h,T]$ using the semigroup property. Thus, we can find $M \geq 1$ such that $\|S(t-h)\|_{\mathscr{L}(X)} \leq M$. So $I_5 \to 0$ as $h \downarrow 0$. Now,

$$
I_6 = E\left\|\int_0^{t-h} S(t-h-s)B\,dW(s) - \int_0^t S(t-s)B\,dW(s)\right\|^2
$$

$$
\begin{aligned}
&= E\left\|(I-S(h))\int_0^{t-h} S(t-h-s)B\,dW(s) - \int_{t-h}^t S(t-s)B\,dW(s)\right\|^2\\
&\lesssim \|(I-S(h))\|^2_{\mathscr{L}(X)}\int_0^{t-h} \|S(t-h-s)B\|^2_{\mathscr{L}_2}\,ds - \int_{t-h}^t \|S(t-s)B\|^2_{\mathscr{L}_2}\,ds\\
&=: I_7 + I_8.
\end{aligned}
$$

where the third line is obtained by independence of the two stochastic integrals. As before, it follows that the integral part of I_7 and I_8 are bounded for $t \in [h, T]$. Also, $\|(I-S(h))\|^2_{\mathscr{L}(X)} \to 0$ as $h \downarrow 0$ so it follows by dominated convergence that $I_7 \to 0$ as $h \downarrow 0$ and we can conclude that

$$
\lim_{h\downarrow 0} E\|U(t-h) - U(t)\|^2 = 0,
$$

so we have left-continuity. ■

Theorem 5.20. *Let U be an open domain of $\mathbb{R}$ with boundary ∂U. Let $X := L^2(U,\mu)$ with $\mu(d\xi) \simeq \operatorname{dist}(\xi, \partial U)^\alpha$ for $\alpha < 2$. Denote*

$$
U(t) := S(t)x + \int_0^t S(t-s)(-A)D\,dW(s)
$$

where $(S(t))_{t\geq 0}$ is the Dirichlet Heat semigroup on $L^2(U)$ generated by the Dirichlet Laplacian A, D is the Dirichlet map, and W is a $L^2(\partial U)$-cylindrical Brownian motion. $x \in X$. Then U is Gaussian and has a predictable version.

Proof. By the definition of stochastic integrals, we have that $U(t)$ is Gaussian with covariance

$$
\int_0^t S(r)(-A)DD^*(-A)^*S^*(r)\,dr
$$

for $t \in [0, T]$. Further, by Theorem 5.19 we know that U is mean square continuous in $(0, T)$. Stochastic continuity of U follows from the Chebychev inequality. As the integrand of the stochastic convolution U is deterministic, U is adapted to the filtration $(\mathscr{F}_t)_{t\geq 0}$ generated by $(W(t))_{t\geq 0}$. By Proposition 3.6 in [11], there exists a predictable version of U. ■

Theorem 5.21. *Let U be an open interval in $\mathbb{R}$ with two-point boundary ∂U. Let $X := L^2(U,\mu)$ with $\mu(d\xi) \simeq \operatorname{dist}(\xi, \partial U)^\alpha$ for $\alpha < 2$. Denote*

$$U(t) := S(t)x + \int_0^t S(t-s)(-A)D\, dW(s)$$

where $(S(t))_{t\geq 0}$ is the Dirichlet Heat semigroup on $L^2(U)$ generated by the Dirichlet Laplacian A, D is the Dirichlet map, and W is a $L^2(\partial U)$-cylindrical Brownian motion. $x \in X$. Then U has a continuous version.

Proof. (Sketch) We have $E\|U(t) - U(s)\|^2 \lesssim t^{-\alpha}(t-s)$, so using Kolmogorov's theorem, U has a continuous version. ■

6

Harmonic Extensions to the Unit Disk

In this chapter we consider solutions to the elliptic boundary value problem

$$\Delta u = 0 \text{ in } \mathbb{D}, \quad u = \xi \text{ on } \mathbb{T}, \tag{6.1}$$

where ξ is a Gaussian noise on $\mathbb{T}$. We believe this problem is interesting for a number of reasons. First, the solution of (6.1) provides an explicit example of the Dirichlet map Λ that has been used in previous chapters. Second, our aim is to move away from classic PDE methods and to apply harmonic analysis techniques to understand the case where ξ is a space white noise.

We start by studying the harmonic extension of random measures of the form $\sum_{n\geq 1} \gamma_n \mu_n$ where (γ_n) is a sequence of independent Gaussian random variables and (μ_n) is a sequence of measures on $\mathbb{T}$ to identify a sufficient condition for the extension to be a well-defined Gaussian random variable taking values in the space of harmonic functions on $\mathbb{D}$. Then, in §6.2, we apply the theory of γ-radonifying operators to study the case where $\xi = BW_H$ where W_H is a (cylindrical) Gaussian on a separable Hilbert space H and $B \in \mathscr{L}(H, L^p(\mathbb{T}))$. This leads us in §6.3 and §6.4 to consider the Hardy spaces $\mathscr{H}^p(\mathbb{D})$ and obtain a representation theorem that allows us to conclude that the solution to (6.1) takes values in $\mathscr{H}^p(\mathbb{D})$ if and only if $\mathbb{E}\|\xi\|^p_{L^p(\mathbb{T})} < \infty$. This of course implies that the Hardy spaces $\mathscr{H}^p(\mathbb{D})$ are too small to consider space white noise on the boundary $\mathbb{T}$. In order to study the boundary behaviour of $\mathscr{H}^p$-valued Gaussian random variables, we obtain pointwise growth bounds of their moments in §6.5. Next, in §6.6, we show that the Poisson operator is γ-radonifying from $L^2(\mathbb{D})$ into $\mathscr{H}(\mathbb{D})$. This implies that although the Hardy spaces are too small for the spatial white noise setting, we might be able to find a slightly larger space of harmonic functions where it might be possible to consider this case and that the blow-up is concentrated on a small set of positive Lebesgue measure near the boundary. To tie our result back with Chapter 5, we relate our Hardy space results with weighted Sobolev space results in §6.7 and in §6.8 we show that the embedding $\mathscr{H}^2 \hookrightarrow L^2(\mathbb{D}, (1-|z|^2)^\alpha)$ is Hilbert-Schmidt for $\alpha > 0$. Finally, in §6.9, we sketch how these results may be extended to the parabolic setting.

6.1 Harmonic extension to the unit disk

Suppose that the boundary space ∂E is chosen to be the space of all continuous functions on $\mathbb{T}$ which we denote by $C(\mathbb{T})$. Then the Dirichlet map $\Lambda : C(\mathbb{T}) \to C(\mathbb{D})$ can be defined by the solution to the Dirichlet problem. That is, given a function $f \in C(\mathbb{T})$, the **Dirichlet problem** for f on $\mathbb{D}$ is to find a function $u \in C(\overline{\mathbb{D}})$ such that $\Delta u = 0$ on $\mathbb{D}$ and $u|_{\mathbb{T}} = f$. Therefore, if u is a solution to the Dirichlet problem, the Dirichlet map Λ is given by $u = \Lambda f$. It is well-known fact (e.g., see [87]) that if we define the **Poisson kernel**

$$P(z) := \frac{1-|z|^2}{|1-z|^2}, \quad z \in \mathbb{D},$$

and write $P_r(\theta) := P(re^{i\theta})$ and

$$u_f(re^{i\theta}) := \int_{-\pi}^{\pi} P_r(\theta - t) f(t) \frac{dt}{2\pi}$$

then

$$u(re^{i\theta}) = \begin{cases} u_f(z), & z \in \mathbb{D} \\ f(z), & z \in \mathbb{T} \end{cases}$$

is the solution to the Dirichlet problem and for any point $e^{i\theta_0} \in \mathbb{T}$,

$$\lim_{\mathbb{D} \ni z \to e^{i\theta_0}} u(z) = f(e^{i\theta_0}).$$

The function u_f is called the **Poisson integral** of f and u_f has a number of special properties, in particular, it is harmonic on $\mathbb{D}$. We recall that a complex-valued function $u \in C^2(\mathbb{D})$ is called **harmonic** if $\Delta u = 0$ in $\mathbb{D}$ and denote by $\mathscr{H}(\mathbb{D})$ the space of all harmonic functions on $\mathbb{D}$ endowed with the topology of uniform convergence on compact subset of $\mathbb{D}$. The space $\mathscr{H}(\mathbb{D})$ is complete and has the Heine-Borel property: if $u_n \in \mathscr{H}(\mathbb{D})$ is a sequence of functions that are uniformly bounded on compact subsets, then there is a subsequence tending uniformly on compact subsets to a harmonic function. Further, we recall that if $B(z,r)$ is an open ball centred at z and of radius r in $\mathbb{D}$, then

- if u is harmonic on $\mathbb{D}$ and $u = 0$ in $B(z,r)$ then $u = 0$ in $\mathbb{D}$,
- if u is harmonic on $\overline{B(z,r)}$ then

  $$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z + re^{i\theta})\, d\theta,$$

  i.e. $u(z)$ equals the average of u over $\partial B(z,r)$.

Finally, as $\|u_f\|_{\mathscr{H}(\mathbb{D})} \le C\|f\|_{C(\mathbb{T})}$, it follows that Λ is a bounded linear operator from $C(\mathbb{T})$ to $\mathscr{H}(\mathbb{D})$ and if $(W(t))_{t\ge 0}$ is a cylindrical Wiener process on some Hilbert space H and $B \in \gamma(H, C(\mathbb{T}))$ then

$$\mathscr{W} := \Lambda B W(1)$$

is a Gaussian random variable taking values in $\mathscr{H}(\mathbb{D})$.

The knowledge that $\mathscr{W}$ is harmonic gives the Gaussian random variable $\mathscr{W}$ a number of interesting properties that we have not exploited in previous chapters. Exploiting the assumption of harmonic or analytic data in the theory of deterministic PDE is not an uncommon in the literature, the Cauchy–Kowalewski theorem is an example of such a concept (e.g., see [86]).

6.1.1 Random Fourier series

By the results of §2.10, a spatial white noise on the boundary is obtained when the cylindrical Wiener process $(W(t))_{t\geq 0}$ takes values in $\partial E = L^2(\partial U)$. This motivates us to understand the situation where ∂E is a larger space than $C(\mathbb{T})$.

We recall that by the Karhunen-Loève expansion (e.g., Theorem 4.12 in [6]), any ∂E-valued Gaussian random variable W can be represented as a Gaussian sum of the form

$$\sum_{n\geq 1} \gamma_n x_n$$

where $(\gamma_n)_{n\geq 1}$ is a Gaussian sequence and $(x_n)_{n\geq 1}$ is a (finite or infinite) sequence in ∂E. Therefore, suppose we take our boundary space ∂E to be the space of all complex Borel measures on $\mathbb{T}$ denoted by $\mathscr{M}(\mathbb{T})$. This space is equipped with the norm

$$||\mu|| = |\mu|(\mathbb{T}),$$

where $|\mu|$ denotes the total variation of μ and is a Banach space. Recall that the total variation $|\mu|$ is the smallest positive Borel measure satisfying

$$|\mu(B)| \leq |\mu|(B)$$

for all Borel sets $B \subset \mathbb{T}$. Now let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, $(\gamma_n)_{n=1}^{\infty}$ a sequence of standard Gaussian random variables, and $(\mu_n)_{n=1}^{\infty}$ a sequence of measures in $\mathscr{M}(\mathbb{T})$. From these sequences we define the $\mathscr{M}(\mathbb{T})$-valued Gaussian random variable

$$W := \sum_{n=1}^{\infty} \gamma_n \mu_n.$$

and as the definition of Fourier transform $\mathscr{F}$ can be extended to Borel measures on $\mathbb{T}$; i.e., the k-th Fourier coefficient of $\mu \in \mathscr{M}(T)$ is defined by

$$\widehat{\mu}(k) := \int_{-\pi}^{\pi} e^{-ikt} \frac{d\mu(e^{it})}{2\pi}, \quad k \in \mathbb{Z},$$

we notice the following relationship between the sequence of measures $(\mu_n) \in \mathscr{M}(\mathbb{T})$ and the Fourier coefficients of the measure-valued Gaussian random variable W.

Lemma 6.1. *If $\sum_{k=1}^{m} \|\mu_k\|^2$ converges as $m \to \infty$ then almost surely we have $\widehat{W} \in \ell^\infty(\mathbb{Z})$ and $\mathbb{E}\|\widehat{W}\|_\infty^2 \le \sum_k \|\mu_k\|^2$.*

Proof. For $m \in \mathbb{N}$ we define $W_m = \sum_{k=1}^{m} \gamma_k \mu_k$, then for each $n \in \mathbb{Z}$,

$$\begin{aligned} |\widehat{W_m}(n)| &= \left| \int_{\mathbb{T}} e^{-int} \, dW_m(e^{it}) \right| \\ &= \left| \sum_{k=0}^{m} \gamma_k \int_{\mathbb{T}} e^{-int} \, d\mu_k(e^{it}) \right| \\ &= \left| \sum_{k=0}^{m} \gamma_k \widehat{\mu}_k(n) \right| \end{aligned}$$

so by Chebyshev's inequality, we have for $\varepsilon > 0$ that

$$\begin{aligned} \mathbb{P}\{|\widehat{W_m}(n)| > \varepsilon\} &\le \varepsilon^{-2} \sum_{k=0}^{m} |\widehat{\mu}_k|^2 \\ &\le \varepsilon^{-2} \sum_{k=0}^{m} \int_T |e^{-int}|^2 d|\mu_k|^2(e^{it}) \\ &= \varepsilon^{-2} \sum_{k=0}^{m} |\mu_k|^2(\mathbb{T}) \\ &= \varepsilon^{-2} \sum_{k=0}^{m} \|\mu_k\|^2. \end{aligned}$$

which is independent of n. If $\sum_k \|\mu_k\|^2$ converges, then by the Kolmogorov convergence criterion, taking $m \to \infty$ we have $\widehat{W_m}(n) \to \widehat{W}(n)$ $\mathbb{P}$-almost surely hence the result holds. ■

For any $\mu \in \mathscr{M}(\mathbb{T})$, we denote the **Poisson integral** of μ by

$$P[\mu](re^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t)\,\frac{d\mu(e^{it})}{2\pi},$$

and, as $P_r \in C(\mathbb{T})$ for $0 \le r < 1$, $P[\mu](re^{i\theta})$ is well defined for all $\theta \in (-\pi, \pi)$.

Due to Lemma 6.1, we assume that $\sum_k \|\mu_k\|^2$ converges, then using the Fourier series representation of P_r it follows that the random Fourier coefficients of $P_r[W]$ are given by

$$\widehat{P_r[W]}(n) = r^{|n|}\widehat{W}(n), \quad n \in \mathbb{Z},$$

and the (formal) random Fourier series of $P_r[W]$ is given by

$$\sum_{n=-\infty}^{\infty} \widehat{W}(n) r^{|n|} e^{in\theta}.$$

Following standard convention, we call this series the **Abel-Poisson means** of the Fourier series of W that is given by

$$\sum_{n=-\infty}^{\infty} \widehat{W}(n) e^{in\theta}.$$

Surprisingly, by considering the boundary noise problem on $\mathbb{D}$, we have now found a connection with the classic area concerned with the study of random Fourier series of the form

$$\sum_{n=-\infty}^{\infty} \varepsilon_n m_n e^{in\theta}$$

where (ε_n) is a sequence of Rademacher or Gaussian random variables and (m_n) is a sequence of constants was originally considered by Paley, Zygmund, and Wiener in the 1930s. A large collection of results may also be found in the monograph of Kahane [89].

6.1.2 Extending $\mathscr{M}(\mathbb{T})$-valued Gaussians to $\mathbb{D}$

Although the Fourier series of W may not be necessarily pointwise convergent, we shall now show that the Abel-Poisson means of W behaves much better.

Let R be the $\mathscr{M}(\mathbb{T})$-valued random variable given by

$$R = \sum_{n=1}^{\infty} \varepsilon_n m_n \delta_{\theta_n}$$

where (ε_n) is a sequence of independent Rademacher random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, (θ_j) is a sequence such that $\theta_j \in \mathbb{Q} \cap [0, 2\pi)$, (m_n) is a sequence of constants such that $\sum_{n=1}^{\infty} m_n^2 < \infty$, and δ_θ is a Dirac measure at $e^{i\theta} \in \mathbb{T}$. In [89] considered the harmonic extension of R to $\mathbb{D}$ given by

$$u(z) := P_r[R](z) = \sum_{n=1}^{\infty} \varepsilon_n m_n P(z e^{-i\theta_n}), \quad z \in \mathbb{D},$$

and showed that u is harmonic on $\mathbb{D}$ and the series converges $\mathbb{P}$-a.s. uniformly on every compact subset of $\mathbb{D}$. Our next theorem is a Gaussian extension of this result where the sequence of measures (μ_k) are not necessarily Dirac measures.

Theorem 6.2. *Assume $\sum_k \|\mu_k\|^2$ converges and let*

$$u(re^{i\theta}) = P_r[W](e^{i\theta})$$

then

$$u(re^{i\theta}) = \sum_{n=-\infty}^{\infty} \widehat{W}(n) r^{|n|} e^{in\theta},$$

for $re^{i\theta} \in \mathbb{D}$ and the series is, almost surely, absolutely and uniformly convergent on compact subsets of $\mathbb{D}$ and u is harmonic on $\mathbb{D}$.

Proof. From the elementary observation

$$\begin{aligned} \frac{1-r^2}{1+r^2-2r\cos\theta} &= \frac{1-r^2}{1+r^2-r(e^{i\theta}-e^{-i\theta})} \\ &= \frac{1-r^2}{(1-r(e^{i\theta})(1-re^{-i\theta})} \end{aligned}$$

$$= \frac{1}{1-re^{i\theta}} + \frac{1}{1-re^{-i\theta}} - 1$$

and, as $1+x+x^2+\cdots = \frac{1}{1-x}$ for $|x|<1$, we have for $|re^{i\theta}|<1$ that

$$\begin{aligned}
&= \sum_{n=0}^{\infty}(re^{i\theta})^n + \sum_{n=0}^{\infty}(re^{-i\theta})^n - 1 \\
&= \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}
\end{aligned}$$

so if we fix $0 \le r < 1$ and θ then

$$\begin{aligned}
u(re^{i\theta}) &= \int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} \frac{dW(e^{it})}{2\pi} \\
&= \int_{\mathbb{T}} \left(\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-t)} \right) \frac{dW(e^{it})}{2\pi}
\end{aligned}$$

as the series is uniformly convergent with respect to e^{it} and $W \in \mathscr{M}(\mathbb{T})$ we can interchange integration and summation to get

$$\begin{aligned}
u(re^{i\theta}) &= \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta} \int_{-\pi}^{\pi} e^{-int} \frac{dW(e^{it})}{2\pi} \\
&= \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}\widehat{W}(n).
\end{aligned}$$

By Lemma 6.1 and our assumption that $\sum_k \|\mu_k\|^2$ converges,

$$\mathbb{E}|\widehat{W}(n)r^{|n|}e^{in\theta}|^2 \le \mathbb{E}\|\widehat{W}\|_\infty^2 r^{2|n|}$$

so the series $\sum_n \widehat{W}(n)r^{|n|}e^{in\theta}$ is $\mathbb{P}$-almost surely, absolutely and uniformly convergent on compact subsets of $\mathbb{D}$ by the Kolmogorov convergence criterion. This allows us to interchange summation and any linear differential operator, in particular,

$$\Delta u(re^{i\theta}) = \Delta\left(\sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}\widehat{W}(n) \right)$$

$$
\begin{aligned}
&= \sum_{n=-\infty}^{\infty} \widehat{W}(n)\Delta r^{|n|} e^{in\theta} \\
&= 0
\end{aligned}
$$

as $r^{|n|}e^{in\theta}$ is harmonic. ■

Therefore the convergence of the partial sum $\sum_k \|\mu_k\|^2$ is a sufficient condition to obtain a well-defined $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable.

6.2 Random L^p boundary data

Given $f : \mathbb{T} \to \mathbb{C}$, we define the norms

$$\|f\|_p := \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^p \, d\theta \right)^{1/p}, \quad 1 \le p < \infty,$$

and

$$\|f\|_\infty = \inf_{M>0} \{M : |\{e^{i\theta} : |f(e^{i\theta})| > M\}| = 0\},$$

where $|\{e^{i\theta} : a \le \theta \le b\}|$ denotes the Lebesgue measure of the set $[a,b] \subset \mathbb{R}$. The complex Lebesgue spaces $L^p(\mathbb{T})$, $1 < p \le \infty$ are defined[1] by

$$L^p(\mathbb{T}) = \{f : \|f\|_p < \infty\}.$$

If $1 \le p \le \infty$, $L^p(\mathbb{T})$ is a Banach space. In particular, $L^2(\mathbb{T})$ equipped with the inner product

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \, d\theta,$$

is a Hilbert space. As measurability of $f : \mathbb{T} \to \mathbb{C}$ is determined in terms of the measurability of $\Re f$ and $\Im f$, these definitions and standard L^p space results follow by decomposing functions into their real and imaginary parts and applying the standard definitions of L^p spaces (e.g., [75]).

[1] modulo equivalence classes of functions equal almost everywhere

The main outcome of the last section is identification of a sufficient condition for existence of a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable. The sufficient condition hints that we should start with the assumption that our random boundary data W is a well-defined $L^p(\mathbb{T})$-valued Gaussian random variable (i.e., W is not cylindrical).

Let H be a Hilbert space with orthonormal basis $(h_n)_{n=1}^{\infty}$ and let W_H be the (cylindrical) Gaussian random variable defined by

$$W_H := \sum_{n=1}^{\infty} \gamma_n h_n,$$

where $(\gamma_n)_{n=1}^{\infty}$ is a sequence of independent standard (complex-valued) Gaussian random random variables on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and let B be a linear operator from H to $L^p(\mathbb{T})$. To ensure that BW_H is a well-defined $L^p(\mathbb{T})$-valued Gaussian random variable, in this section we shall assume that

$$B \in \gamma(H, L^p(\mathbb{T})).$$

Then it follows that BW_H is a $L^p(\mathbb{T})$-valued Gaussian random variable with covariance operator BB^*. In other words, BW_H is a $L^p(\mathbb{T})$-valued strongly $\mathbb{P}$-measurable function on $(\Omega, \mathscr{F}, \mathbb{P})$.

For $f \in C(\mathbb{T})$, we can write for $0 \le r < 1$,

$$Pf(re^{i\theta}) = P_r f(e^{i\theta}) = \int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it}) \frac{dt}{2\pi}.$$

and we interpret P in two different ways:

- P as an operator $C(\mathbb{T}) \to \mathscr{H}(\mathbb{D})$,

- P_r as an operator $C(\mathbb{T}) \to C(\mathbb{T})$ for $0 \le r < 1$.

By considering P as an operator, we can now move from the Fourier series approach used in the last section to an operator theoretical approach and study how the Poisson operator P maps the Gaussian random variables BW_H under the assumption

$$\mathbb{E}\|BW_H\|^p_{L^p(\mathbb{T})} < \infty.$$

We shall often make use of the following well-known lemma (e.g., see [87]).

Lemma 6.3. *For each $r \in [0,1)$ and $f \in C(\mathbb{T})$, $P_r f \in C(\mathbb{T})$ with $\|P_r f\|_\infty \lesssim \|f\|_\infty$. Further, $\lim_{r\to 1^-} \|P_r f - f\|_\infty = 0$. In other words, $P_r f$ converges uniformly to f on $\mathbb{T}$.*

6.2.1 Case $1 < p < \infty$

We first consider the case where BW_H is a Gaussian $L^p(\mathbb{T})$-valued random variable and study how PBW_H approaches BW_H in a $L^p(\Omega)$ sense as $r \to 1^-$ by applying the theory of γ-radonifying operators.

Theorem 6.4. *Let $B \in \gamma(H, L^p(\mathbb{T}))$ for some fixed $1 < p < \infty$. Then*

(i) PBW_H *is harmonic in* $\mathbb{D}$,

(ii) $\sup_{r<1} \mathbb{E}\|P_r BW_H\|^p_p = \lim_{r\to 1} \mathbb{E}\|P_r BW_H\|^p_p = \mathbb{E}\|BW_H\|^p_p$,

(iii) $\lim_{r\to 1} \mathbb{E}\|P_r BW_H - BW_H\|^p_p = 0.$

Proof. (1) follows from Theorem 6.2. Let BW_H be a Gaussian $L^p(\mathbb{T})$-valued random variable. Then as

$$\mathbb{E}\|P_r BW_H\|^p_p \le \mathbb{E}\|P_r\|^p_1 \|BW_H\|^p_p \le C\mathbb{E}\|BW_H\|^p_p < \infty,$$

for $0 \le r < 1$, it follows that $P_r BW_H$ is a $L^p(\mathbb{T})$-valued random variable. Further,

$$\begin{aligned}
\mathbb{E}\|P_r BW_H - BW_H\|^p_p &= \mathbb{E}\|(P_r B - B)W_H\|^p_p \\
&= \|P_r B - B\|^p_{\gamma(H, L^p(\mathbb{T}))} \\
&\le \|P_r - I\|^p_{\mathscr{L}(L^p(\mathbb{T}))} \|B\|^p_{\gamma(H, L^p(\mathbb{T}))}.
\end{aligned}$$

Take $r > 0$, $f \in L^p(\mathbb{T})$ with $\|f\|_p = 1$, and choose some $\varepsilon > 0$. As $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$ we can find $\varphi \in C(\mathbb{T})$ such that $\|f - \varphi\|^p < \varepsilon$. Then

$$\begin{aligned}
\|(P_r - I)f\|_p^p &= \|P_r f - P_r\varphi + P_r\varphi - \varphi + \varphi - f\|_p^p \\
&\leq \|P_r(f - \varphi)\|_p^p + \|P_r\varphi - \varphi\|_p^p + \|\varphi - f\|_p^p \\
&\leq (C^p + 1)\|f - \varphi\|_p^p + \|P_r\varphi - \varphi\|_p^p \\
&\leq (C^p + 1)\varepsilon + \|P_r\varphi - \varphi\|_\infty^p.
\end{aligned}$$

Now by Lemma 6.3, we can find r_ε such that $\|P_r\varphi - \varphi\|_\infty^p < \varepsilon$ for $r > r_\varepsilon$. Hence,

$$\mathbb{E}\|P_r BW_H - BW_H\|_p^p \leq \varepsilon(2 + C^p)\|B\|_{\gamma(H, L^p(\mathbb{T}))}^p$$

As ε was chosen arbitrarily, (iii) is obtained and (ii) follows from (iii). ■

To conclude, we have shown that if we use the Poisson kernel to extend a Gaussian random variable $BW_H \in L^p(\mathbb{T})$ to the open unit disk we obtain a harmonic random variable $u = PBW_H$ whose (p-th moment) mean values $r \mapsto \mathbb{E}\|u_r\|_p^p$ are uniformly bounded.

6.2.2 Pointwise blow-up is permitted

As we have seen in previous chapters, one of the main issues with the standard approach to boundary noise is that one must assume BW is the trace of a function $\Lambda BW \in W^{1,2}(U)$ on the boundary ∂U, that is,

$$\mathbb{E}\|BW_H\|_{W^{1/2,2}(\partial U)}^2 < \infty. \tag{6.2}$$

However, if there is a positive probability that BW_H has a singularity at a point $x_0 \in \partial U$ or we had the white noise case $B = I$ and $H = L^2(\partial U)$ then a fortiori the condition (6.2) is not fulfilled and $\mathbb{E}\|\Lambda BW\|^2 = \infty$. In this section, we explore the behaviour of $P_r BW_H$ as $r \to 1^-$.

As $\mathbb{T}$ is a compact space, for $p \in [1, \infty]$ we have the scale of spaces

$$L^\infty(\mathbb{T}) \subset L^p(\mathbb{T}) \subset L^1(\mathbb{T}),$$

and any continuous function on $\mathbb{T}$ is necessarily bounded. As such, the space of all continuous functions on the unit circle $C(\mathbb{T})$ can be considered as a subspace of $L^\infty(\mathbb{T})$ and $L^1(\mathbb{T})$ can be considered as the largest L^p space. Therefore, without loss of generality, we now assume $B \in \gamma(H, L^1(\mathbb{T}))$. Of course, a natural example for our boundary noise problem would be the case $H = L^2(\mathbb{T})$.

We first consider the pointwise convergence of $P_r BW_H$ as $r \to 1^-$ under the assumption that $(BW_H(\omega))(e^{i\theta})$ is a point of continuity for $BW_H(\omega)$.

Theorem 6.5. *Let $B \in \gamma(H, L^1(\mathbb{T}))$ and suppose $\omega \mapsto BW_H(\omega)$ is continuous at $e^{i\theta_0} \in \mathbb{T}$. Then $PBW_H(\omega)$ is harmonic in $\mathbb{D}$ and*

$$\lim_{r\to 1^-} PBW_H(\omega)(re^{i\theta_0}) = BW_H(\omega)(e^{i\theta_0}).$$

Proof. As $P_r \in L^\infty(\mathbb{T})$ and $BW_H(\omega) \in L^1(\mathbb{T})$, then $P_r BW_H(\omega)(e^{i\theta})$ is well-defined for all $e^{i\theta} \in \mathbb{T}$ by an application of Young's inequality. By assumption, given $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$|BW_H(\omega)(e^{i\theta}) - BW_H(\omega)(e^{i\theta_0})| < \varepsilon$$

as soon as $|\theta - \theta_0| < 2\delta$. Using the estimate

$$\int_{-\delta}^{\delta} |P_r(e^{it})| \frac{dt}{2\pi} < C,$$

and taking $|\theta - \theta_0| < \delta$, we get

$$\begin{aligned}
&\left|P_r BW_H(\omega)(e^{i\theta}) - P_r BW_H(\omega)(e^{i\theta_0})\right| \\
&= \left|\int_{-\pi}^{\pi} P_r(e^{it}) \left(BW_H(\omega)(e^{i(\theta-t)}) - BW_H(\omega)(e^{i\theta_0})\right) \frac{dt}{2\pi}\right| \\
&\le \left(\int_{-\delta}^{\delta} + \int_{\delta\le|t|\le\pi}\right) |P_r(e^{it})||BW_H(\omega)(e^{i(\theta-t)}) - BW_H(\omega)(e^{i\theta_0})| \frac{dt}{2\pi} \\
&\le \varepsilon C + \int_{\delta\le|t|\le\pi} |P_r(e^{it})|(|BW_H(\omega)(e^{i(\theta-t)})| + |BW_H(\omega)(e^{i\theta_0})|) \frac{dt}{2\pi}
\end{aligned}$$

$$\leq \varepsilon C + \sup_{\delta \leq |t| \leq \pi} |P_r(e^{i\theta})| \left(\|BW_H(\omega)\|_1 + BW_H(\omega)(e^{i\theta_0}) \right)$$

Then choosing $R < 1$ such that $\sup_{\delta \leq |t| \leq \pi} |P_r(e^{i\theta})| < \varepsilon$ when $r > R$, we get for $r > R$ and $|\theta - \theta_0| < \delta$, that

$$|P_r BW_H(\omega)(e^{i\theta}) - BW_H(e^{i\theta_0})| < \varepsilon(\|BW_H(\omega)\|_1 + |BW_H(\omega)(e^{i\theta_0})| + C).$$

As ε can be chosen arbitrarily, we are done. ■

Of course, a given realisation $\omega \mapsto BW_H(\omega)$ may also exhibit blow-up at a point on the boundary. We now study the case when

$$\lim_{\theta \to \theta_0} BW_H(\omega)(e^{i\theta}) = \infty,$$

for some point $e^{i\theta_0} \in \mathbb{T}$.

Theorem 6.6. *Let $B \in \gamma(H, L^1(\mathbb{T}))$ and suppose that*

$$\lim_{\theta \to \theta_0} BW_H(\omega)(e^{i\theta}) = \infty.$$

Then $\lim_{r \to 1^-} PBW_H(\omega)(re^{i\theta_0}) = \infty$.

Proof. As $P_r \in L^\infty(\mathbb{T})$ and $BW_H(\omega) \in L^1(\mathbb{T})$, then $P_r BW_H(\omega)(e^{i\theta})$ is well-defined for all $e^{i\theta} \in \mathbb{T}$ by an application of Young's inequality. By assumption, given any $M > 0$ we can find $\delta > 0$ such that $BW_H(\omega)(e^{i\theta}) > 2M$ whenever $|\theta - \theta_0| < 2\delta$. So taking $|\theta - \theta_0| < 2\delta$, we have

$$\begin{aligned}
P_r BW_H(\omega)(e^{i\theta}) &= \int_{-\pi}^{\pi} P_r(t) BW_H(\omega)(e^{i(\theta - t)}) \frac{dt}{2\pi} \\
&= \left(\int_{-\delta}^{\delta} + \int_{\delta \leq |t| \leq \pi} \right) P_r(t) BW_H(\omega)(e^{i(\theta - t)}) \frac{dt}{2\pi} \\
&\geq 2M \int_{-\delta}^{\delta} P_r(e^{it})\, dt
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\delta \le |t| \le \pi} P_r(t) |BW_H(\omega)(e^{i(\theta - t)})| \frac{dt}{2\pi} \\
&\ge 2M - \int_{\delta \le |t| \le \pi} P_r(t)(2M + |BW_H(e^{i(\theta-t)})|) \frac{dt}{2\pi} \\
&\ge 2M - \sup_{\delta \le |t| \le \pi} P_r(t)(2M + \|BW_H(\omega)\|_1).
\end{aligned}
$$

Now choosing an $R < 1$ such that

$$
\sup_{\delta \le |t| \le \pi} P_r(t)(2M + \|BW_H(\omega)\|_1) < M
$$

as soon as $r > R$ we have for $r > R$ and $|\theta - \theta_0| < \delta$ that

$$
P_r BW_H(\omega) > M.
$$

■

6.3 Hardy spaces

Recall that the family of complex harmonic functions on the open unit disk $\mathbb{D}$ is denoted by $\mathscr{H}(\mathbb{D})$. If $u \in \mathscr{H}(\mathbb{D})$ we write $u_r(e^{i\theta}) := u(re^{i\theta})$ and define the norms

$$
\|u\|_{\mathscr{H}^p(\mathbb{D})} := \sup_{0 \le r < 1} \|u_r\|_{L^p(\mathbb{T})} = \sup_{0 \le r < 1} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^p \, d\theta \right)^{1/p}
$$

for $0 < p < \infty$, the norm

$$
\|u\|_{\mathscr{H}^\infty(\mathbb{D})} := \sup_{z \in \mathbb{D}} |u(z)|,
$$

and the set of functions

$$
\mathscr{H}^p(\mathbb{D}) := \{u \in \mathscr{H}(\mathbb{D}) : \|u\|_{\mathscr{H}^p} < \infty\},
$$

where $0 < p \leq \infty$. To simplify notation we shall write $\mathscr{H}^p := \mathscr{H}^p(\mathbb{D})$. Due to the definition of $\|\cdot\|_{\mathscr{H}^p}$, it follows that $\mathscr{H}^p$ is a linear vector space. Further, by Hölder's inequality and as $\mathbb{D}$ is compact, we have

$$\mathscr{H}^\infty \subset \mathscr{H}^q \subset \mathscr{H}^p,$$

for $0 < p < q < \infty$. The spaces $\mathscr{H}^p$, $1 \leq p \leq \infty$ are called the **Hardy spaces** on $\mathbb{D}$ and where introduced by Riesz in 1923.

One should notice that if $u \in \mathscr{H}(\mathbb{D})$ and $r \mapsto \|u_r\|_{L^p(\mathbb{T})}$ behaves like $1/(1-r)$ as $r \to 1^-$ then $u \in \mathscr{H}^p$ but $u \neq L^p(\mathbb{D})$. Hence, the $\mathscr{H}^p$ norms allow a certain amount of growth near the boundary of $\mathbb{D}$.

6.4 A representation theorem

We shall now make use of the definition of harmonic Hardy spaces to further understand the relationship between the stochastic boundary data and the existence or non-existence of $L^p(\mathbb{D})$-valued Gaussians.

6.4.1 Sufficient condition

A sufficient condition to obtain a $\mathscr{H}^p$-valued Gaussian random variable is that the boundary data is spatially regular, i.e. $B \in \gamma(H, L^p(\mathbb{T}))$.

Theorem 6.7. *Let $1 \leq p \leq \infty$. If $B \in \gamma(H, L^p(\mathbb{T}))$ then $u := PBW_H$ is a $\mathscr{H}^p$-valued random variable and*

$$\mathbb{E}\|u\|^2_{\mathscr{H}^p} = \mathbb{E}\|BW_H\|^2_{L^p(\mathbb{T})}.$$

Proof. Follows from previous theorems. ■

6.4.2 Necessary condition

The next theorem shows that we can only obtain $\mathscr{H}^p$-valued random variables if the boundary data is spatially regular.

Theorem 6.8. *If u is a $\mathscr{H}^p$-valued Gaussian random variable for $1 < p \leq \infty$ then there exists a unique $L^p(\mathbb{T})$-valued Gaussian random variable ξ such that*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i(\theta-t)})\xi(e^{it})\,dt, \quad re^{i\theta} \in \mathbb{D},$$

and $\mathbb{E}\|u\|_p^2 = \mathbb{E}\|\xi\|_p^2$. Moreover, there exists a separable Hilbert space H_ξ and an operator $B \in \mathscr{L}(H_\xi, L^p(\mathbb{T}))$ such that

$$\mathbb{E}\|u\|_{\mathscr{H}^q}^p = \|B\|_{\gamma_p(H_\xi, L^q(\mathbb{T}))}^p$$

for $1 \leq p < \infty$ and $1 < q < \infty$.

Proof. Let u be a $\mathscr{H}^p$-valued Gaussian random variable and for $n \geq 2$ define $u_n(z) := u((1-1/n)z)$. As u_n is defined on the disk $\{z : |z| < n/(n-1)\}$ we have that $u_n \in \mathscr{H}(\overline{\mathbb{D}})$ and it follows from the reproducing property that for all $z = re^{i\theta} \in \mathbb{D}$ we have

$$u_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i(\theta-t)})u_n(e^{it})\,dt.$$

On the left hand side, it is clear that $u_n(z) \to u(z)$ as $n \to \infty$. Hence we need to show that when $n \to \infty$ the right-hand side expression has an integral representation. Writing for $v \in L^q(\mathbb{T})$,

$$\langle u_n, v\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} u_n(e^{it})v(e^{it})\,dt.$$

$\langle u_n, v\rangle$ is a $\mathbb{R}$-valued Gaussian and

$$\mathbb{E}|\langle u_n, v\rangle|^2 \leq \mathbb{E}\|u_n\|_p^2\|v\|_q \leq \mathbb{E}\|u\|_p^2\|v\|_q,$$

for $v \in L^q(\mathbb{T})$. Hence, passing to a subsequence, there exists a $\mathbb{R}$-valued Gaussian random variable ξ_v such that

$$\lim_{k\to\infty} \langle u_{n_k}, v\rangle = \xi_v$$

in probability for $v \in L^q(\mathbb{T})$. Now, for a fixed $z = re^{i\theta}$, we write $v_z(e^{it}) := P(re^{i(\theta-t)})$ and it follows that $v_z \in L^q(\mathbb{T})$. Hence,

$$\begin{aligned}
\xi_{v_z} &= \lim_{k\to\infty} \langle u_{n_k}, v_z\rangle \\
&= \lim_{k\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i(\theta-t)}) u_{n_k}(e^{it})\, dt \\
&= \lim_{k\to\infty} u((1-1/n_k)re^{i\theta}) \\
&= u(re^{i\theta})
\end{aligned}$$

But we also have by the Riesz representation theorem that there exists a Gaussian random variable $\xi \in L^p(\mathbb{T})$ such that

$$\xi_v = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(e^{it})\xi(e^{it})\, dt$$

for all $v \in L^p(\mathbb{T})$. So by choosing $v = v_z$ we get

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i(\theta-t)})\xi(e^{it})\, dt.$$

■

Remark 6.9. The previous theorem does not hold for $\mathscr{H}^1(\mathbb{D})$-valued Gaussian random variables as $L^1(\mathbb{T})$ is not the dual of any space.

6.4.3 Non-existence in the white-noise case

Our next theorem shows that we cannot obtain a $\mathscr{H}^2$-valued Gaussian random variable in the white-noise case.

Theorem 6.10. *If $H = L^2(\mathbb{T})$, $B = I$, and $u = PW$ then*

$$\mathbb{E}\|u\|^2_{\mathscr{H}^2} = \infty.$$

Proof. Let (h_n) be an orthonormal basis of $L^2(\mathbb{T})$ and (γ_n) a sequence of standard Gaussians. As we can represent W by

$$W = \sum_{n=1}^{\infty} \gamma_n h_n$$

and $\mathbb{E}\|W\|^2_{L^2(\mathbb{T})} = \infty$ it follows from Theorem 6.7 that

$$\mathbb{E}\|u\|^2_{\mathscr{H}^2} = \infty.$$

■

6.5 Pointwise growth bounds

We have the following pointwise growth estimate for $\mathscr{H}^p$-valued Gaussian random variables. It is a straight-forward extension of a standard deterministic estimate (e.g., Proposition 6.16 in [87]).

Theorem 6.11. *For $1 \le p < \infty$, if u is a $\mathscr{H}^p$-valued Gaussian random variable then*

$$\mathbb{E}|u(z)|^p \le \left(\frac{1+|z|}{1-|z|}\right) \mathbb{E}\|u\|^p_{\mathscr{H}^p}, \quad \forall z \in \mathbb{D}.$$

Proof. Fix $1 < p < \infty$, $z \in \mathbb{D}$, and let u be a $\mathscr{H}^p$-valued Gaussian random variable. It follows that $u \in L^p(\Omega; \mathscr{H}^p)$ and there exists a Gaussian random variable $f \in L^p(\Omega; L^p(\mathbb{T}))$ such that $u(z) = (Pf)(z)$ and $\mathbb{E}\|u\|^p_{\mathscr{H}^p} = \mathbb{E}\|f\|^p_{L^p(\mathbb{T})}$. Let $1/q + 1/p = 1$ and $z = re^{i\theta}$, then by Hölder's inequality

$$\mathbb{E}|u(re^{i\theta})|^p = \mathbb{E}\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i(\theta-\tau)}) f(e^{i\tau})\, d\tau\right|^p$$

$$\leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P(re^{i(\theta-\tau)}) \right|^q \, d\tau \right)^{p/q} \mathbb{E} \|f\|_{L^p(\mathbb{T})}^p$$
$$= \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \left| P(re^{i(\theta-\tau)}) \right|^q \, d\tau \right)^{p/q} \mathbb{E} \|u\|_{\mathscr{H}^p}^p$$

and since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P(re^{i(\theta-\tau)})^q \, d\tau \leq \sup_{\tau} P(re^{i(\theta-\tau)})^{q-1} \frac{1}{2\pi} \int_{-\pi}^{\pi} P(re^{i(\theta-\tau)}) \, d\tau$$
$$= \left(\frac{1-|z|^2}{|1-z|^2} \right)^{q-1}$$
$$= \left(\frac{1+|z|}{1-|z|} \right)^{q-1}$$

the result follows as $(q-1)p/q = 1$. The case $p = 1$ follows in a similar way. ■

We shall now extend the ideas seen in Proposition 6.23 of [87] to obtain a sharper estimate in the case $p = 2$. First, notice that for $z \in \mathbb{T}$ and $x \in \mathbb{D}$ we can write the Poisson kernel as

$$P_x(z) = \frac{1-|x|^2}{|x-z|^2} = \frac{1-|x|^2}{1-2x\cdot z+|x|^2}$$

Let $[\cdot,\cdot]$ be the inner product on $L^2(\mathbb{T})$ then as $f \mapsto Pf$ is a linear isometry of $L^2(\mathbb{T})$ onto $\mathscr{H}^2$ we can transfer the Hilbert space structure to $\mathscr{H}^2$ as

$$(Pf, Pg) := [f, g], \quad f, g \in L^2(\mathbb{T}),$$

and for $f \in L^2(\mathbb{T})$ and $x \in \mathbb{D}$ we have $(Pf)(x) = [P_x, f]$. To extend this pointwise representation to $\mathscr{H}^2$ we extend the domain of P_x by defining

$$P_x(z) = \frac{1-|x|^2|z|^2}{1-2x\cdot z+|x|^2|z|^2}$$

for all $x, z \in \mathbb{C}$ for which the denominator is not zero. If $z \in \mathbb{T}$ then this agrees with the previous definition of P_x. Further, $P_x(z) = P_z(x)$ and $P_x(z) = P_{|x|z}(x/|x|)$. Therefore, for $u \in \mathscr{H}^2$ we have

$$u(x) = (P_x, u).$$

This allows us to get a sharper estimate in the $\mathscr{H}^2$ case.

Theorem 6.12. *If u is a $\mathscr{H}^2$-valued Gaussian random variable then*

$$\mathbb{E}|u(z)|^2 \le \left(\frac{1+|z|^2}{1-|z|^2}\right)\mathbb{E}\|u\|^2_{\mathscr{H}^2}, \quad \forall z \in \mathbb{D}.$$

Proof. Let u be a $\mathscr{H}^2$-valued Gaussian random variable and $z \in \mathbb{D}$ then

$$\mathbb{E}|u(z)|^2 = \mathbb{E}|(P_z, u)|^2$$

and by the Cauchy-Schwarz inequality we get

$$\mathbb{E}|u(z)|^2 \le \|P_z\|^2_{\mathscr{H}^2}\mathbb{E}\|u\|^2_{\mathscr{H}^2}.$$

Finally, as $\|P_z\|^2_{\mathscr{H}^2} = (P_z, P_z) = P_z(z)$ the estimate follows from the final calculation

$$P_z(z) = \frac{1-|z|^2|z|^2}{1-2z\cdot z+|z|^2|z|^2} = \frac{1+|z|^2}{1-|z|^2}.$$

■

6.6 γ-Radonifying property of Poisson kernel

In the previous sections we have shown that $B \in \gamma(H, L^p(\mathbb{T}))$ is a necessary and sufficient conditions to obtain a well-defined $\mathscr{H}^p$-valued Gaussian random variable that is given by PBW_H. However, we notice that Theorem 6.2 suggests that the Poisson operator P also has a certain amount of 'radonifying' behaviour irrespective of the operator B.

This concept is well-known for the stochastic heat equation

$$\partial_t u(t) = \Delta u(t) + \dot{W}(t), \quad x \in (a,b) \subset \mathbb{R} \tag{6.3}$$

with $u(t,a) = u(t,b) = 0$ for $t > 0$ and $u(0,\cdot) = 0$ whereby, for $H = L^2(a,b)$ and due to the fact that the Dirichlet heat semigroup $(e^{tA})_{t\ge 0}$ on H is radonifying as it satisfies for $T>0$,

$$\int_0^T \|e^{tA}\|^2_{\mathscr{L}_2(H)}\,dt<\infty,$$

ensures that the stochastic convolution

$$u(t)=\int_0^t e^{(t-s)A}\,dW(s),\quad t\le T,$$

is a mild solution to (6.3) and $\mathbb{E}\|u(t)\|_H^2<\infty$ for $t>0$, even in the case when $(W(t))_{t\ge 0}$ is a cylindrical Wiener process on H! See Example 5.7 in [19].

In this section we shall show a similar result whereby the Poisson operator P is radonifying and, in the white noise case $B=I$ and $H=L^2(\mathbb{T})$, we can obtain a well-defined Gaussian random variable on $\mathscr{H}(\mathbb{D})$.

For $f\in L^2(\mathbb{T})$ recall our notation

$$(Pf)(re^{i\theta})=(P_rf)(e^{i\theta})=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)f(e^{it})\,dt.$$

Theorem 6.13. *For $0\le r<1$ we have $P_r\in\gamma(L^2(\mathbb{T}),C(\mathbb{T}))$. Or equivalently,*

$$P\in\gamma(L^2(\mathbb{T}),\mathscr{H}(\mathbb{D})).$$

To prove this result we start with a simple estimate.

Lemma 6.14. *Let $(\gamma_n)_{n=1}^\infty$ be a sequence of standard $\mathbb{R}$-valued Gaussian random variables. Then for any $\alpha>1$, almost surely we have*

$$|\gamma_n|\le\sqrt{2\alpha\log(n+1)}$$

for all but at most finitely many $n\ge 1$.

Proof. Let $(\gamma_n)_{n=1}^\infty$ be a sequence of $\mathbb{R}$-valued standard Gaussians and $\alpha>1$. As we can bound for any $t>0$

$$\mathbb{P}\{|\gamma_n|>t\}\le\frac{2}{t\sqrt{2\pi}}e^{-t^2/2},$$

for each n, we get that

$$\begin{aligned}\mathbb{P}\{|\gamma_n| > \sqrt{2\alpha\log(n+1)}\} &\le \frac{2}{\sqrt{4\pi\alpha\log(n+1)}}\exp(-\alpha\log(n+1)) \\ &= \frac{1}{\sqrt{\pi\alpha\log(n+1)}}\frac{1}{(n+1)^\alpha}.\end{aligned}$$

As it follows that

$$\sum_{n=1}^{\infty}\mathbb{P}\{|\gamma_n| > \sqrt{2\alpha\log(n+1)}\} \lesssim \sum_{n=1}^{\infty}\frac{1}{(n+1)^\alpha} < \infty$$

as $\alpha > 1$, so by an appeal to the Borel-Cantelli lemma

$$\mathbb{P}(\{|\gamma_n| > \sqrt{2\alpha\log(n+1)}\} \text{ i.o.}) = 0.$$

Hence we can conclude that, almost surely, $|\gamma_n| \le \sqrt{2\alpha\log(n+1)}$ for all but at most finitely many $n \ge 1$. ■

We now construct a dyadic decomposition of $\mathbb{T}$ by taking the dyadic intervals $I := [-\pi+2\pi k 2^{-j}, -\pi+2\pi(k+1)2^{-j})$ and defining $I_L := [-\pi+2\pi k 2^{-j}, -\pi+2\pi(k+\frac{1}{2})2^{-j})$ and $I_R := [-\pi+2\pi(k+\frac{1}{2})2^{-j}, -\pi+2\pi(k+1)2^{-j})$ to be the left and right parts of I, respectively. The function

$$h_I(e^{it}) := 2^{j/2}\chi_{I_L}(e^{it}) - 2^{j/2}\chi_{I_R}(e^{it}),$$

is called the **Haar function associated with the interval** I, where we have defined the indicator function χ_I so that $\chi_I(e^{it}) = 1$ if $t \in I$ and 0 otherwise. Notice that we have defined the Haar functions so that their $L^2(\mathbb{T})$ norm with respect to the uniform measure $\frac{1}{2\pi}dt$ is 1. That is,

$$\int_{\mathbb{T}} h_I(e^{it})\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} h_I(e^{it})\,dt = 1.$$

Lemma 6.15. *On $L^2(\mathbb{T})$, the Haar functions have the orthogonality property:*

$$\int_{\mathbb{T}} h_I(e^{it})h_{I'}(e^{it})\,dt = \begin{cases} 0, & \text{when } I \neq I' \\ 1, & \text{when } I = I'. \end{cases}$$

The collection of all dyadic intervals I of $\mathbb{T}$ is denoted by $\mathscr{D}$ and $\mathscr{D}_j$ denotes all dyadic intervals I such that $|I| = 2^{-j}$, also called the j-th level. It is clear that each $\mathscr{D}_j$ provides a partition of $\mathbb{T}$ and

$$\mathscr{D} = \bigcup_{j\in\mathbb{Z}} \mathscr{D}_j.$$

Lemma 6.16. *The sequence $\{h_I\}_{I\in\mathscr{D}}$ forms an orthonormal basis of $L^2(\mathbb{T})$.*

Proof. We can see $\int_{\mathbb{T}} h_I = 0$, $\|h_I\|_2 = 1$ and $[h_I, h_{I'}] = \delta_{I,I'}$ for $I, I' \in \mathscr{D}$. Finally, if $[f, h_I] = 0$ for all $I \in \mathscr{D}$ then $f = 0$ in $L^2(\mathbb{T})$. Hence, the conclusion follows. ■

For each $r \in [0,1)$ we define the operator

$$(P_r f)(e^{it}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\tau}) f(e^{i(t-\tau)})\, d\tau, \quad f \in C(\mathbb{T}),\ t \in [-\pi, \pi].$$

We recall that the Poisson kernel satisfies

- for $r \in [0,1)$, we have $z \mapsto P(rz) \in L^\infty(\mathbb{T})$,
- $\lim_{r\to 1^-}\Big(\sup_{0<\delta\le|\theta|\le\pi} |P(re^{i\theta})|\Big) = 0.$

By density of continuous functions in $L^2(\mathbb{T})$ we can extend this operator to $L^2(\mathbb{T})$ functions such that $P_r \in \mathscr{L}(L^2(\mathbb{T}), L^2(\mathbb{T}))$.

Proof of Theorem 6.13. Let $\{h_n\}_{n=1}^\infty$ be defined by $h_n = h_I$ with the dyadic interval I chosen so that $n = 2^j + k$ with $j = 0, 1, \dots$ and $k = 0, \dots, 2^j$. Then $\{h_n\}$ is an orthonormal basis of $L^2(\mathbb{T})$ by Lemma 6.16. Fix $r \in [0,1)$, then we see that for $n = 2^j + k$,

$$\begin{aligned}|(P_r h_n)(e^{it})| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} P(re^{i\tau}) h_n(e^{i(t-\tau)})\, d\tau\right| \\ &\le \frac{1}{2\pi}\left(\int_{-\pi}^{-\delta} + \int_{-\delta}^{\delta} + \int_{-\delta}^{\pi}\right) |P(re^{i\tau})|\,|h_n(e^{i(t-\tau)})|\, d\tau\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2\pi}\int_{\delta\leq|\tau|\leq\pi} |P(re^{i\tau})||h_n(e^{i(t-\tau)})|\,d\tau \\
&\qquad + \frac{1}{2\pi}\int_{-\delta}^{\delta} |P(re^{i\tau})||h_n(e^{i(t-\tau)})|\,d\tau \\
&\leq \sup_{\delta\leq|\tau|\leq\pi} |P(re^{i\tau})| \frac{1}{2\pi}\int_{-\pi}^{\pi} |h_n(e^{i(t-\tau)})|\,d\tau \\
&\qquad + C_P \frac{1}{2\pi}\int_{-\pi}^{\pi} |h_n(e^{i(t-\tau)})|\,d\tau \\
&\leq \left(\sup_{\delta\leq|\tau|\leq\pi} |P(re^{i\tau})| + C_P\right) 2^{-j/2} \\
&=: C_1 2^{-j/2}
\end{aligned}
$$

and $C_1 < \infty$ by the properties of the Poisson kernel.

By Lemma 6.14, it holds $\mathbb{P}$-almost surely that there exists some $N > 0$ such that we can estimate

$$
\begin{aligned}
\sum_{n=1}^{\infty} |\gamma_n(P_r h_n)(e^{it})| &\leq \sum_{n=1}^{N} |\gamma_n(P_r h_n)(e^{it})| \\
&\qquad + \sum_{n=N+1}^{\infty} \sqrt{2\alpha\log(n+1)} |P_r h_n(e^{it})| \\
&\leq C + \sum_{j=1}^{\infty}\sum_{k=1}^{2^j} \sqrt{2\alpha\log(2^j+k+1)} |P_r h_n(e^{it})| \\
&\leq C_0 + C_1 \sum_{j=1}^{\infty} \sqrt{\alpha\log(2^j+k'+1)} 2^{-j/2}
\end{aligned}
$$

by estimating $2^j + k' + 1 \leq 2^{j+1} + 1 \leq 2^{j+2}$,

$$
\leq C_0 + C_1\sqrt{\alpha\log(2)} \sum_{j=1}^{\infty} 2^{-j/2}\sqrt{j+2}
$$

by estimating $\sqrt{j+1} \leq C_2 2^{(j+1)/4}$,

$$
\leq C_1 + C_1 C_2 \sqrt{\alpha\log(2)} \sum_{j=1}^{\infty} 2^{-(j+1)/4}
$$

$$< \infty,$$

for all $t \in [0, 2\pi)$. Hence $\mathbb{P}$-almost surely, the sum converges absolutely and uniformly for all $e^{it} \in \mathbb{T}$.

As $|\mathbb{T}| < \infty$, we have $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$ hence by Young's inequality and the fact that $z \mapsto P_r = P(rz) \in L^\infty(\mathbb{T})$ for $r \in [0,1)$, $(P_r h_n)(e^{it})$ is well-defined for all $e^{it} \in \mathbb{T}$ and $P_r h_n \in C(\mathbb{T})$ and $\|Ph_n\|_\infty \leq \|P_r\|_\infty \|h_n\|_1 \lesssim \|P_r\|_\infty \|h_n\|_2$.

Therefore, for $\mathbb{P}$-almost every $\omega \in \Omega$,

$$e^{it} \mapsto \sum_{n=1}^{\infty} \gamma_n(\omega)(P_r h_n)(e^{it})$$

belongs to $C(\mathbb{T})$. By uniform convergence, we have

$$S_N := \sum_{n=1}^{N} \gamma_n (P_r h_n)(e^{it})$$

converging to

$$S := \sum_{n=1}^{\infty} \gamma_n (P_r h_n)(e^{it})$$

as $N \to \infty$, $\mathbb{P}$-almost surely and since

$$\mathbb{E}\left(\sup_{e^{it}} S\right)^2 < \infty.$$

The Ito-Nisio theorem gives that

$$\lim_{N \to \infty} \mathbb{E}\|S - S_N\|^2_{C(\mathbb{T})} = 0.$$

We can then conclude that $P_r \in \gamma(L^2(\mathbb{T}), C(\mathbb{T}))$ for $r \in [0,1)$. ■

Combined with the results of the previous sections, this shows that if we consider the white noise case $u = PW$ (i.e. $H = L^2(\mathbb{T})$, $B = I$) then, almost surely,

$$u \notin \mathscr{H}^2 \quad \text{but} \quad u \in \mathscr{H}(\mathbb{D}).$$

Hence, for any compact $K \subset \mathbb{D}$,

$$\sup_{z \in K} \mathbb{E}|u(z)|^2 < \infty.$$

For some small $\varepsilon > 0$, take a compact $K \subset \mathbb{D}$ such that $|\mathbb{D} \setminus K| < \varepsilon$ then our results show that the blow-up behaviour that restricts us from taking white noise on $\mathbb{T}$ is concentrated on a very small set of positive Lebesgue measure near the boundary.

6.7 Relationship with weighted Sobolev spaces

In this section is to answer the following question: if u is a well-defined $\mathscr{H}^p$-valued Gaussian random variable, is it a well-defined Gaussian random variable in some weighted Sobolev space of the form $W^{k,p}(\mathbb{D}, \delta^\alpha)$ where $\delta(z) := \operatorname{dist}(z, \mathbb{T})$ and $\alpha > 0$?

We achieve this by proving a **randomized** version of the classic equivalence between harmonic Hardy spaces and the weighted Sobolev spaces. As such, let u be a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable and for $p \geq 2$ we define

$$S_p(u) := \int_{\mathbb{D}} |u(z)|^{p-2} |\nabla u(z)|^2 (1 - |z|^2)\, d\Theta(z),$$

where Θ is the normalized Lebesgue measure on $\mathbb{D}$ (i.e. $\Theta(\mathbb{D}) = 1$).

Remark 6.17. In the case $p = 2$, harmonic analysts may recognize $S_2(u)$ in terms of "square functions" i.e. estimates of Littlewood-Paley type of the form

$$S_2(u) \simeq \left\| \left(\int_0^1 |(1 - r^2) \nabla u(rz)|^2 \frac{dr}{(1 - r^2)} \right)^{1/2} \right\|_{L^2(\mathbb{T})}^2.$$

From the definition of $S_p(u)$ we then prove the following

Theorem 6.18. *We have the equivalence*

$$\mathbb{E}\|u\|_{\mathscr{H}^p}^p \simeq \mathbb{E}|u(0)|^p + \frac{p(p-1)}{2} \mathbb{E} S_p(u).$$

Then considering the case $p=2$ and noticing that $(1-|z|^2)\simeq \operatorname{dist}(z,\mathbb{T})$ we get the following answer to our question.

Corollary 6.19. *We have the equivalence*

$$\mathbb{E}\|u\|^2_{\mathcal{H}^2}\simeq \mathbb{E}\|u\|^2_{W^{1,2}(\mathbb{D},\delta)}.$$

Comparing with the results in the last chapter, we see that we can obtain solution in the weighted space $W^{1,2}(\mathbb{D},\delta^\alpha)$ at the critical power $\alpha=1$ where the traces theorems break down. Hence, we have solutions where the relationship with the boundary data is now understood in terms of radial or non-tangential convergence towards the boundary.

We now quickly recall some known facts that will be useful for working on $\mathbb{D}$. If $f\in C^2(\mathbb{R}^2)$ we can define a function $F(r,\theta):=f(r\cos(\theta),r\sin(\theta))$ with $r\in\mathbb{R}_+$ and $\theta\in[-\pi,\pi)$, then

$$\nabla f=\left(\frac{\partial F}{\partial r},\frac{1}{r}\frac{\partial F}{\partial \theta}\right)^T \qquad \Delta f=\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}+\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2}.$$

Let $B_R:=\{z:|z|<R\}\subset\mathbb{C}$ and let $\nu(Re^{i\theta})$ be the outer unit normal to ∂B_R at the point $Re^{i\theta}$, then

$$\frac{\partial f}{\partial \nu}(Re^{i\theta})=(\nabla f\cdot\nu)(Re^{i\theta})=\frac{\partial F}{\partial r}(r,\theta).$$

By direct computation, it can be shown that $\log(\varrho/|z|)$ is harmonic in $\mathbb{C}\setminus\{0\}$ if $0<\varrho<|z|$. Identifying $F(z)=F(re^{i\theta})=F(r,\theta)$ for $z:=re^{i\theta}\in\mathbb{C}$, by the divergence theorem,

$$\frac{1}{2\pi r}\int_{|z|<r}\Delta F(z)\,dz=\frac{1}{2\pi}\frac{d}{dr}\int_{-\pi}^{\pi}F(re^{i\theta})\,d\theta$$

where dz is Lebesgue measure in $\mathbb{C}$. Let Θ be Lebesgue measure on $\mathbb{D}$ such that $\Theta(\mathbb{D})=1$. We also have,

$$\int_{|z|=\varrho}F(z)\,d\sigma(z)-F(0)=\int_{|z|<\varrho}\Delta F(z)\log(\varrho/|z|)\,d\Theta(z).$$

The following lemma provides a standard result that is heavily used in the literature (e.g., see Ex. 7, Chap. 1 in [87]).

Lemma 6.20. *If u is a positive function in $C^2(\mathbb{D})$ and p is a constant, then*

$$\Delta(u^p) = pu^{p-1}\Delta u + p(p-1)u^{p-2}|\nabla u|^2.$$

Moreover, if u is harmonic and strictly positive, then

$$\Delta(u^p) = p(p-1)u^{p-2}|\nabla u|^2.$$

Proof. Writing $z = x + iy$ and directly differentiating,

$$\Delta(u^p) = (p-1)pu^{p-2}\left(\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) + pu^{p-1}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right),$$

then using the fact that u is harmonic the result follows. ■

Posing

$$M_p(r,u) := \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p \, d\theta\right)^{1/p}$$

the following theorem is standard folklore: it is given as an exercise in the book by Garnett and Marshall [90]. As a proof could not be found in the literature, we provide one here.

Theorem 6.21 (Hardy identity on $\mathbb{D}$)**.** *If $u \in \mathscr{H}(\mathbb{D})$, then the function $r \mapsto M_p(r,u)$, $0 < r < 1$, is of class C^1 and*

$$\frac{d}{dr}M_p^p(r,u) = \frac{p(p-1)}{2r}\int_{|z|<r} |u(z)|^{p-2}|\nabla u(z)|^2 \, d\Theta(z),$$

where Θ is a Lebesgue measure such that $\Theta(\mathbb{D}) = 1$.

Proof. Let $\varepsilon > 0$ and choose a fixed $r \in (\varepsilon, 1)$. Since $\log(r/|z|)$ is harmonic in $\mathbb{C} \setminus \{0\}$ we can use Green's theorem on the annulus $\circledcirc(\varepsilon, r) := \{z : \varepsilon < |z| < r\}$ to obtain

$$\int_{\circledcirc} \Delta(|u|^p)\log\frac{r}{|z|}\, dz = \int_{|z|=r}\left(\partial_\nu |u|^p \log\frac{r}{|z|} - |u|^p \partial_\nu \log\frac{r}{|z|}\right) d\sigma_r$$

$$-\int_{|z|=\varepsilon}\left(\partial_\nu |u|^p \log\frac{r}{|z|} - |u|^p \partial_\nu \log\frac{r}{|z|}\right)\, d\sigma_\varepsilon,$$

where σ_r is the surface measure on the circle of radius r. On the circle $\{z : |z| = r\}$ we have that $\log(r/|z|) = 0$, $\partial_\nu \log(r/|z|) = -1/r$, and $d\sigma_r = r\, d\sigma_1$. As $\varepsilon \log(1/\varepsilon) \to 0$ as $\varepsilon \to 0$ and $u \in \mathscr{H}(\mathbb{D})$,

$$\lim_{\varepsilon\to 0}\int_{|z|=\varepsilon}\left(\partial_\nu |u|^p \log\frac{r}{|z|} - |u|^p \partial_\nu \log\frac{r}{|z|}\right)\, d\sigma_\varepsilon = |u(0)|^p.$$

So using the identity $\Delta|u|^p = p(p-1)|u|^{p-2}|\nabla u|^2$ and taking $\varepsilon \to 0$, we have

$$p(p-1)\int_{|z|<r} |u|^{p-2}|\nabla u|^2 \log\frac{r}{|z|}\, dz = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p\, d\theta - |u(0)|^p.$$

By continuity we see that this identity holds for all $r \in (0,1)$. Without loss of generality (as we shall see below), assume $u(0) = 0$. Now using polar coordinates on the left-hand side,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p\, d\theta \\
&\qquad = \frac{p(p-1)}{2\pi}\int_0^r s\log\frac{r}{s}\int_{-\pi}^{\pi} |u(se^{i\theta})|^{p-2}|\nabla u(se^{i\theta})|^2\, d\theta\, ds \\
&\qquad =: \frac{p(p-1)}{2\pi}\int_0^r s\log\frac{r}{s}\, G(s)\, ds \\
&\qquad = \frac{p(p-1)}{2\pi}\left(\int_0^r s\log r\, G(s)\, ds + \int_0^r s\log\frac{1}{s}\, G(s)\, ds\right)
\end{aligned}$$

Now differentiating both sides by r (this justifies $u(0) = 0$ assumed above), we get

$$\begin{aligned}
&\frac{d}{dr}\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^p\, d\theta \\
&\qquad = \frac{p(p-1)}{2\pi}\left(\frac{1}{r}\int_0^r G(s)s\, ds + rG(r)\log(r) + rG(r)\log\frac{1}{r}\right)
\end{aligned}$$

$$= \frac{p(p-1)}{2\pi}\frac{1}{r}\int_0^r G(s)s\,ds$$
$$= \frac{p(p-1)}{2\pi}\frac{1}{r}\int_0^r\int_{-\pi}^{\pi}|u(se^{i\theta})|^{p-2}|\nabla u(se^{i\theta})|^2 s\,d\theta\,ds$$
$$= \frac{p(p-1)}{2r}\int_{|z|<r}|u(z)|^{p-2}|\nabla u(z)|^2\,d\Theta(z),$$

where $\Theta(\mathbb{D}) = 1$ (i.e. $\Theta(dz) = \frac{s}{\pi}\,ds\,d\theta$). ■

Corollary 6.22. *For $u \in \mathscr{H}(\mathbb{D})$, we have the identity*

$$M_p^p(r,u) = |u(0)|^p + p(p-1)\int_{|z|<r}|u(z)|^{p-2}|\nabla u(z)|^2\log\frac{r}{|z|}\,dz.$$

Proof of Theorem 6.18. Fix $0 \le r < 1$, then from the identity $\Delta(u^p) = p(p-1)u^{p-2}|\nabla u|^2$ and Green's theorem,

$$\begin{aligned}\int_{|z|<r}|u|^{p-2}|\nabla u|^2\,d\Theta &= \frac{1}{p(p-1)}\int_{|z|<r}\Delta(u^p)\,d\Theta\\ &= \frac{2\pi r}{p(p-1)}\frac{1}{2\pi r}\int_{|z|<r}\Delta(u^p)\,d\Theta\\ &= \frac{2\pi r}{p(p-1)}\frac{d}{dr}\frac{1}{2\pi}\int_{-\pi}^{\pi}u^p\,d\theta\\ &= \frac{r}{p(p-1)}\frac{d}{dr}\int_{-\pi}^{\pi}u^p\,d\theta\end{aligned}$$

Multiplying both sides by $p(p-1)$,

$$p(p-1)\int_{|z|<r}|u|^{p-2}|\nabla u|^2\,d\Theta = r\frac{d}{dr}\int_{-\pi}^{\pi}u^p\,d\theta$$

We now integrate r over the interval $[0,1)$ on both sides. On the left-hand side we get through the use of Fubini's theorem that

$$\int_0^1 p(p-1)\int_{|z|<r}|u|^{p-2}|\nabla u|^2\,d\Theta\,dr$$

$$
\begin{aligned}
&= p(p-1)\int_{|z|<1}\int_{|z|}^{1} r|u|^{p-2}|\nabla u|^2\,dr\,d\Theta \\
&= \frac{p(p-1)}{2}\int_{|z|<1} |u|^{p-2}|\nabla u|^2(1-|z|^2)\,d\Theta.
\end{aligned}
$$

and on the right-hand side we get by integration by parts that

$$
\begin{aligned}
\int_0^1 r\frac{d}{dr}\int_{-\pi}^{\pi} u^p\,d\theta\,dr &= \int_{-\pi}^{\pi} u^p\,d\theta - \int_0^1\int_{-\pi}^{\pi} u^p\,d\theta\,dr \\
&= \int_{-\pi}^{\pi} u^p\,d\theta - \int_{\mathbb{D}} u^p\,d\Theta(z)
\end{aligned}
$$

Hence, reassembling the left and right hand sides it follows that

$$
\frac{p(p-1)}{2}\int_{|z|<1} |u|^{p-2}|\nabla u|^2(1-|z|^2)\,d\Theta = \int_{-\pi}^{\pi} u^p\,d\theta - \int_{\mathbb{D}} u^p\,d\Theta(z)
$$

Rearranging then taking expectations both sides, we get

$$
\begin{aligned}
\mathbb{E}\|u\|_{\mathscr{H}^p}^p = \mathbb{E}\int_{\mathbb{D}} |u|^p\,d\Theta& \\
&+ \frac{p(p-1)}{2}\mathbb{E}\int_{|z|<1} |u|^{p-2}|\nabla u|^2(1-|z|^2)\,d\Theta.
\end{aligned}
$$

■

6.8 γ-Radonifying embeddings

Let $U \subset \mathbb{R}^d$ be a bounded domain. In this section we show that the Hardy space $\mathscr{H}^2$ is Hilbert-Schmidt embedded into the weighted space $L^2(\mathbb{D},(1-|z|^2)^\alpha)$. The idea of obtaining such a result comes from Maurin's theorem (see [75]) which gives conditions for embeddings between Sobolev spaces to be Hilbert-Schmidt.

Theorem 6.23 (Maurin)**.** *If $m \in \mathbb{N} \cup \{0\}$ and $k > d/2$, then the imbedding*

$$W_0^{m+k,2}(U) \hookrightarrow W_0^{m,2}(U)$$

is Hilbert-Schmidt.

Remark 6.24. Taking $m = 0$ and $d = 1$ in Maurin's theorem, we see why we can obtain function-valued solutions to the stochastic heat equation on the interval $(0,1)$.

Similar to Maurin's theorem, we have obtained the following result which also shows why the harmonic Hardy spaces are a nice space to work in.

Theorem 6.25. *The imbedding*

$$\mathscr{H}^2 \hookrightarrow L^2(\mathbb{D},(1-|z|)^\alpha)$$

is γ-radonifying for $\alpha > 0$.

Proof. Given $z \in \mathbb{D}$, we define the point evaluation functional $\Pi_z : \mathscr{H}^2 \to \mathbb{R}$ defined by $\Pi_z u := u(z)$. It follows from Theorem 6.11 that

$$|u(z)| \le \left(\frac{1+|z|}{1-|z|}\right)^{1/2} \|u\|_{\mathscr{H}^2},$$

so Π_z is continuous. Let $(\cdot,\cdot)$ be the inner product on $\mathscr{H}^2$. By the Riesz representation theorem, there exists a unique function k_z such that

$$\Pi_z u = (k_z, u)$$

and it follows that $\|k_z\|^2_{\mathscr{H}^2} \le (1+|z|)/(1-|z|)$. In fact, one can show that k_z is the Poisson kernel. If $(e_n)_{n=1}^\infty$ is an orthonormal basis of $\mathscr{H}^2$ then

$$\|k_z\|^2_{\mathscr{H}^2} = \sum_{n=1}^\infty |(e_n, k_z)|^2 = \sum_{n=1}^\infty |e_n(z)|^2.$$

Hence, if $\alpha > 0$ we have that

$$\mathbb{E}\left\|\sum_{n=M}^N \gamma_n e_n\right\|^2_{L^2(\mathbb{D},(1-|z|)^\alpha)} = \sum_{n=M}^N \|e_n\|^2_{L^2(\mathbb{D},(1-|z|)^\alpha)}$$

$$
\begin{aligned}
&= \sum_{n=M}^{N} \int_{\mathbb{D}} |e_n(z)|^2 (1-|z|)^\alpha \, dz \\
&\leq \int_{\mathbb{D}} ||k_z||^2_{\mathscr{H}^2} (1-|z|)^\alpha \, dz \\
&\leq \int_{\mathbb{D}} \frac{1+|z|}{1-|z|} (1-|z|)^\alpha \, dz \\
&\leq \int_0^1 \frac{1+r}{1-r} (1-r)^\alpha \, dr \\
&< \infty
\end{aligned}
$$

Hence, we can conclude that the imbedding is γ-radonifying. ■

In fact, we can get a sharper embedding by applying the same estimates as Theorem 6.12.

Theorem 6.26. *The imbedding*

$$\mathscr{H}^2 \hookrightarrow L^2(\mathbb{D}, (1-|z|^2)^\alpha)$$

is γ-radonifying for $\alpha > 0$.

Combined with the equivalence shown in the last section, we obtain the following corollary.

Corollary 6.27. *For $\alpha > 0$, the embedding*

$$W^{1,2}(\mathbb{D}, (1-|z|)) \hookrightarrow L^2(\mathbb{D}, (1-|z|)^\alpha)$$

is γ-radonifying.

6.9 Parabolic case

After the negative result of the last section, one may wonder if existence of a solution might be obtained using an “heat kernel” approach, in particular, the characterisation given by Lemma 3.30. In this section, we shall assume:

- $B \in \gamma(H, L^p(\partial U))$

so that the term 'BdW' may be replaced by dW^B where W^B is an $L^p(\partial U)$-valued Wiener process, in other words, the noise on the boundary is regular.

Let us first consider the half-space case $U = \mathbb{R}^d_+$ with boundary $\partial U = \mathbb{R}^{d-1}$. By Lemma 3.30, we have for $g \in L^q(0, T; L^p(\mathbb{R}^{d-1}))$, we have

$$\left(\int_0^T AS(t)\Lambda f(t)\,dt\right)(x) = \int_0^T \int_{\mathbb{R}^{d-1}} \frac{\partial G_U}{\partial \nu_z}(t,x,z) f(t,z)\,dz\,dt,$$

where G_U is the Dirichlet heat kernel. In this half-space case, G_U is obtained by reflection of the (free) heat kernel G in $\mathbb{R}^d$ given by $G(t,x) := (4\pi t)^{-d/2}\exp(-x^2/4t)$, that is

$$G_U(t,x,z) = G(t,x-z) - G(t,x-\mathscr{R}z)$$

where $\mathscr{R}z := (z_1, z_2, \ldots, z_{d-1}, -z_d)$ is the **reflection operator**. For $(x,y) \in \mathbb{R}^d_+$ with $x \in \mathbb{R}^{d-1}$ and $y > 0$, we shall write

$$P(t,x,y) := \frac{\partial G_U}{\partial \nu_y}(t,(x,y),0)$$

and note that $P : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}_+ \to \mathbb{R}$ is given explicitly by

$$P(t,x,y) = \begin{cases} \dfrac{y t^{-1}}{(4\pi t)^{d/2}} \exp\left(-\dfrac{|x|^2 + y^2}{4t}\right), & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where $x \in \mathbb{R}^{d-1}$ and $y > 0$. Here $|x|$ is the Euclidean norm of the vector $x := (x_1, \ldots, x_{d-1})$. One should think of P as the **parabolic Poisson kernel** of the boundary value problem

$$\begin{cases} \partial_t u(t,x,y) = \Delta_x u(t,x,y) + \partial^2_{yy} u(t,x,y), & (t,x,y) \in \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}_+, \\ \quad u(t,x,0) = g(t,x), & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}. \end{cases}$$

To ease notation, let $U := \mathbb{R}^d_+$, $\partial U := \mathbb{R}^{d-1}$, $U_T := (0,T] \times U$ and $\partial U_T := (0,T] \times \partial U$. It is known that if $g \in C(\partial U_T)$ then this boundary value problem is uniquely solvable and is given by the **parabolic Poisson integral**

$$u(t,x,y) := \iint_{\partial U_T} P(t-s, x-z, y) g(s,z)\, dz\, ds.$$

Further, $u \in C^{1,2}(U_T)$. We denote $(Pg)(t,x,y) := u(t,x,y)$ so that P can be viewed as the operator

$$P : C(\partial U_T) \to C(U_T).$$

One may now ask when P extends to a bounded operator from $L^p(\partial U_T)$ to $L^q(U_T)$ with $1 \le p, q \le \infty$. If such an extension is possible then a possible solution to (4.2) is given by

$$u(t,x,y) = (PW^B)(t,x,y).$$

In [91], the following parabolic Littlewood-Paley estimate was shown

Theorem 6.28. *If $1 < p < \infty$ and $f \in L^p(\partial U_T)$ then*

$$\left\| \left(\int_0^\infty |y \partial_y Pf|^2 \frac{dy}{y} \right) \right\|_{L^p(\partial U_T)} \simeq_p \|f\|_{L^p(\partial U_T)}.$$

This allows us to prove the following

Theorem 6.29. *P can be extended to $P \in \mathscr{L}(L^2(0,T; L^2(\partial U)), L^2(0,T; L^2(U)))$.*

Proof.

Considering the integral term on the left-hand side, we have

$$\int_0^\infty |y \partial_y Pf|^2 \frac{dy}{y} = \int_0^\infty |\partial_y Pf|^2 y\, dy,$$

and taking the case $p = 2$,

$$\left\| \left(\int_0^\infty |y \partial_y Pf|^2 \frac{dy}{y} \right) \right\|_{L^p(\partial U_T)} = \iint_{\partial U_T} \int_0^\infty |\partial_y Pf|^2 y\, dy\, dx\, dt$$

$$
\begin{aligned}
&= \int_0^T \int_U |\partial_{z_d} Pf|^2 z_d \, dz \, dt \\
&\simeq \int_0^T \|Pf\|^2_{W^{1,2}(U,z_d)} \, dt
\end{aligned}
$$

as $Pf \in C^2(K)$ for compact K away from boundary ∂U so $\nabla_x Pf \in L^2(K)$. Writing $\delta(z) := \mathrm{dist}(z, \partial U)$ we have

$$Pf \in L^2(0,T; W^{1,2}(U,\delta)),$$

where $W^{1,2}(U,\delta)$ is the weighted Sobolev space and as

$$W^{1,2}(U,\delta) \simeq W^{1/2,2}(U),$$

it follows that P can be extended to $\widetilde{P} \in \mathscr{L}(L^2(0,T;L^2(\partial U)), L^2(0,T;L^2(U)))$. ■

7

Blow-up for White Noise Data

We saw in the last chapter that the space $\mathscr{H}^p$ with $p \geq 2$ is too small to handle white-noise data on $\mathbb{T}$. In this chapter we consider a larger space of harmonic functions that seem to exhibit the correct blow-up near the boundary $\mathbb{T}$ to consider spatial white noise on $\mathbb{T}$. We call this an "inside-out" approach.

Relatively few papers exist on the question of blow-up for a stochastic partial differential equations. The first results can be found in [92, 93] for the stochastic heat equation on the interval $(0, \ell)$ given by

$$\partial_t u = \Delta u + u^\gamma \dot{W}(t), \quad u(t,0) = u(t,\ell) = 0, \quad u(0,\cdot) = u_0(\cdot), \tag{7.1}$$

where $(W(t))_{t \geq 0}$ is a cylindrical Wiener process taking values in $L^2(0,\ell)$, $\gamma > 1$, and the initial data u_0 is continuous and positive. In [92] it was shown that the solution $u(t,x)$ to (7.1) exists for all time if $1 \leq \gamma < 3/2$ and in [93] it is shown that if $\gamma \gg 3/2$ then the solution to (7.1) blows-up in finite time with positive probability. These results were extended in [94, 95] to cover the case

$$\partial_t u = \Delta u + \varphi(u) \dot{W}(t), \quad u(t,0) = u(t,\ell) = 0, \quad u(0,\cdot) = u_0(\cdot),$$

where the nonlinearity φ is locally Lipschitz, $\varphi(0) = 0$, and $\varphi(u) \geq cu^{\gamma}$ for some $c > 0$ and $\gamma > 3/2$. The question of blow-up has also been considered for wave equations in a few papers. In [96], long-time existence for a stochastic wave equation was studied. More recently in [97], a nonlinear stochastic wave equation in a domain $U \subset \mathbb{R}^d$ with $d \leq 3$ was studied and solutions were shown to blow-up in the L^2 norm under appropriate conditions on the initial data and the nonlinear term.

As the study of blow-up for nonlinear partial differential equations has been (and still is) a very active area of research, it is somewhat surprising that such few papers exist on blow-up for stochastic problems. Further, even though numerous papers study boundary blow-up for deterministic PDEs, there have been no results on blow-up for PDEs with random noise terms on the boundary.

In the next section, we give an example of a random harmonic function u that gives the expected behaviour if u satisfied the Dirichlet problem

$$\Delta u = 0 \text{ on } \mathbb{D}, \quad u = \xi \text{ on } \mathbb{T},$$

where ξ is a space white noise. In §7.2 we derive growth estimates for the moments of $\mathscr{H}(\mathbb{D})$-valued Gaussians taken over concentric circles of radius $r < 1$. This motivates the definition given in §7.3 of a space of Gaussian random variables that satisfy

$$\sup_{z\in\mathbb{D}} \|u'(z)\|_{L^2(\Omega)}(1-|z|^2) < \infty, \tag{7.2}$$

and in §7.4 we show that the norm of this space is invariant under conformal transformations. Finally, in §7.5, we derive a law of iterated logarithms for random variables satisfying (7.2).

7.1 White noise behaviour near the boundary

As before, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and $(\gamma_n)_{n=1}^{\infty}$ a sequence of standard Gaussian random variables. This chapter is related to understanding $\mathscr{H}(\mathbb{D})$-

valued Gaussian random variables exhibiting the following behaviour.

Example 7.1. Consider the random function

$$u(z)=\sum_{n=0}^{\infty}\gamma_n z^n, \quad z\in\mathbb{D}.$$

As z^n is harmonic in $\mathbb{D}$ it follows readily that u is a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable. Further, as

$$\mathbb{E}\|u\|^2_{L^2(\mathbb{T})}=\sum_{n=0}^{\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{in\theta}e^{-in\theta}\,d\theta=\infty.$$

but $u_N := \sum_{n=0}^{N}\gamma_n z^n$ is such that $\mathbb{E}\|u_N\|^2_{L^2(\mathbb{T})}<\infty$. We may consider u as a prototypical example of a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable generated by a white-noise on $\mathbb{T}$.

7.2 Mean growth of circle moments

Let $L^2(\Omega)$ be the Hilbert space of random variables with second moments that we endow with the norm $\|\cdot\|_{L^2(\Omega)}$ and $L^p(\Omega)$ the Banach space with norm $\|\cdot\|_{L^p(\Omega)}$ given by

$$\|X\|_{L^2(\Omega)} := (\mathbb{E}|X|^2)^{1/2}, \quad \|X\|_{L^p(\Omega)} := (\mathbb{E}|X|^p)^{1/p}.$$

Mean growth estimates for harmonic functions on $\mathbb{D}$ are well known (for example, the book by Duren [98]). We now derive an extension of these classic estimates to the random variable setting. Let

$$M_{p,q}(r,u) := \frac{1}{2\pi}\int_{-\pi}^{\pi}\|u(re^{i\theta})\|^p_{L^q(\Omega)}\,d\theta$$

and we write $M_p(r,u) := M_{p,p}(r,u)$. In particular, if u is a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable with $\mathbb{E}u(z)=0$ for $z\in\overline{\mathbb{D}}$ then

$$M_2(r,u)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbb{E}|u(re^{i\theta})|^2\,d\theta.$$

measures the average variance (i.e. the average second moment) of u over concentric circles of radius $r<1$.

Lemma 7.2. *Let u be a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable. Then for $r<1$,*

$$M_{p,q}(r,u) \leq \|u(0)\|_{L^q(\Omega)} + \int_0^r M_{p,q}(s,u')\,ds. \tag{7.3}$$

and

$$r^2 M_{p,q}(r^2,u') \leq \frac{2}{1-r^2} M_{p,q}(r,u). \tag{7.4}$$

Proof. Estimate (7.3) follows by writing for $re^{i\theta}\in\mathbb{D}$

$$u(re^{i\theta}) = u(0) + \int_0^r u'(se^{i\theta})\,ds.$$

Without loss of generality, we assume $u(0)=0$. Then,

$$\|u(re^{i\theta})\|_{L^q(\Omega)} \leq \int_0^r \|u'(se^{i\theta})\|_{L^q(\Omega)}\,ds$$

and an application of the Minkowski integral inequality gives

$$\begin{aligned} M_{p,q}(r,u) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \|u(re^{i\theta})\|^p_{L^q(\Omega)}\,d\theta \\ &\leq \int_0^r \int_{-\pi}^{\pi} \|u'(re^{i\theta})\|^p_{L^q(\Omega)}\,d\theta\,ds \\ &= \int_0^r M_{p,q}(s,u')\,ds. \end{aligned}$$

We shall now prove (7.4). Let P_x be the kernel

$$P_x(z) = \frac{1-|x|^2|z|^2}{1-2x\cdot z-|x|^2|z|^2} = \frac{z+x}{z-x}$$

if $z\in\mathbb{T}$ then P_x is the standard Poisson kernel and due to the reproducing property we have for $x=\varrho e^{i\theta}$ and $\varrho<r<1$ that

$$u(x) = \frac{1}{2\pi r}\int_{|z|=r} P_x(z)u(z)\,dz.$$

Now notice that for $f \in \mathscr{H}(\mathbb{D})$ we have

$$\frac{\partial}{\partial \theta} f(re^{i\theta}) = f'(re^{i\theta})re^{i\theta}i$$

so that

$$u'(x) = \frac{1}{ix}\frac{\partial}{\partial \theta}\frac{1}{2\pi r}\int_{|z|=r} P_x(z)u(z)\,dz.$$

We also have for $x = \varrho e^{i\theta}$ that

$$\begin{aligned}
\frac{\partial}{\partial \theta} P_x(z) &= \frac{\partial}{\partial \theta}\left(\frac{z+\varrho e^{i\theta}}{z-\varrho e^{i\theta}}\right) \\
&= \frac{(z-\varrho e^{i\theta})re^{i\theta}i + (z+\varrho e^{i\theta})re^{i\theta}i}{(z-\varrho e^{i\theta})^2} \\
&= i\varrho e^{i\theta}\frac{2z}{(z-\varrho e^{i\theta})^2} \\
&= ix\frac{2z}{(z-x)^2}.
\end{aligned}$$

Hence, taking $\partial/\partial\theta$ under the integral we have for $x = \varrho e^{i\theta}$ that

$$u'(x) = \frac{1}{\pi r}\int_{|z|=r}\frac{z}{(z-x)^2}u(z)\,dz$$

Writing $\|\cdot\|_q := \|\cdot\|_{L^q(\Omega)}$, it now follows that

$$\begin{aligned}
\|u'(\varrho e^{i\theta})\|_q &\le \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\|u(re^{it})\|_q}{(re^{it}-\varrho e^{i\theta})^2}\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\|u(re^{i(s+\theta)})\|_q}{r^2 - 2r\varrho\cos(s) + \varrho^2}\,ds
\end{aligned}$$

so integrating both sides and using Minkowski's integral inequality,

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\|u'(\varrho e^{i\theta})\|_q^p\,d\theta \\
&\qquad = \frac{1}{2\pi}\int_{\pi}^{\pi}\left(\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\|u(re^{i(s+\theta)})\|_q}{r^2-2r\varrho\cos(s)+\varrho^2}\,ds\right)^p\,d\theta
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{M_{p,q}(r,u)}{r^2-2r\varrho\cos(s)+\varrho^2}\,ds\\
&=\frac{2}{r^2-\varrho^2}M_{p,q}(r,u).
\end{aligned}
$$

As $r<1$ we can set $\varrho=r^2$ so that $\varrho<r$ to get

$$
M_{p,q}(r^2,u')\leq\frac{2}{r^2-(r^2)^2}M_{p,q}(r,u)=\frac{1}{r^2}\frac{2}{1-r^2}M_{p,q}(r,u).
$$

Multiplying both sides by r^2 gives our estimate. ■

Let u be a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable such that $\mathbb{E}u(z)=0$ for $z\in\overline{\mathbb{D}}$ then considering the special case of $q=p=2$, these estimates show that for $r<1$,

$$
r^2\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbb{E}|u'(re^{i\theta})|^2\,d\theta\leq\frac{2}{1-r^2}\frac{1}{2\pi}\int_{-\pi}^{\pi}\mathbb{E}|u(re^{i\theta})|^2\,d\theta.
$$

Hence, the average variance of u' on concentric circles of radius r grows faster than the average variance of u by a factor of $(1-r^2)^{-1}$.

This is one of the motivations to look at the space of $\mathscr{H}(\mathbb{D})$-valued Gaussian random variables u such that

$$
\sup_{z\in\mathbb{D}}\|u'(z)\|_{L^2(\Omega)}(1-|z|^2)<\infty.
$$

7.3 Bloch random variables

We recall that an analytic function $g:\mathbb{D}\to\mathbb{C}$ is called a **Bloch function** if

$$
\sup_{z\in\mathbb{D}}|g'(z)|(1-|z|^2)<\infty.
$$

Readers interested in this scalar setting may refer to Chapter VII in the book by Garnett and Marshall [90] for more information and further references.

We shall now extend this concept by replacing the absolute value in this definition by the norm $\|\cdot\|_{L^q(\Omega)}$ to suit our random variable situation. That is, if u is a $\mathscr{H}(\mathbb{D})$-valued random variable we define

$$\|u\|_{\mathfrak{B}^p(\mathbb{D})} := \sup_{z\in\mathbb{D}} \|u'(z)\|_{L^p(\Omega)}(1-|z|^2),$$

and, without loss of generality, if u is Gaussian we can simply consider

$$\|u\|_{\mathfrak{B}} := \sup_{z\in\mathbb{D}} \|u'(z)\|_{L^2(\Omega)}(1-|z|^2).$$

We denote by $\mathfrak{B}$ the space of all $\mathscr{H}(\mathbb{D})$-valued Gaussian random variables u such that $\|u\|_{\mathfrak{B}} < \infty$. We call $u \in \mathfrak{B}$ a **Bloch random variable**.

7.4 Conformal invariance

Let $\mathscr{S} = \mathscr{S}(\mathbb{R}^d;\mathbb{C})$ be the Schwartz space of complex-valued rapidly decreasing functions on $\mathbb{R}^d$ and let $\mathscr{S}'$ be its topological dual: the space of complex tempered distributions on $\mathbb{R}^d$. Every Borel complex measure μ on $\mathbb{R}^d$, that is a measure whose variation $\|\mu\|$ satisfies

$$\int_{\mathbb{R}^d} \frac{1}{(1+|x|^2)^n} \|\mu\|(dx) < \infty,$$

for a certain $n \in \mathbb{N}$ can be identified with the distribution

$$\langle \mu, \varphi \rangle = \int_{\mathbb{R}^d} \varphi(x)\mu(dx),$$

where φ is a test function in $\mathscr{S}$. Define the group of translations $(\tau_x)_{x\in\mathbb{R}^d}$ on $\mathscr{S}$ by the formula

$$\tau_x\varphi(y) = \varphi(x+y), \quad \varphi \in \mathscr{S},\ x, y \in \mathbb{R}^d.$$

The group (τ_x) can be extended to $\mathscr{S}'$ by the relationship

$$\langle \tau_x\mu, \varphi \rangle = \langle \mu, \tau_{-x}\varphi \rangle, \quad \varphi \in \mathscr{S}.$$

We recall that a Wiener process $(W(t))_{t\geq 0}$ taking values in $\mathscr{S}'$ is called **spatially homogeneous** if for every $t \geq 0$ the law $\mathscr{L}[W(t)]$ of $W(t)$ in $\mathscr{S}'$ in invariant with respect to the group of translations (τ_x), that is, for any Borel set $B \subset \mathscr{S}'$,

$$\mathscr{L}[W(t)](B) := \mathbb{P}[W(t) \in B] = \mathbb{P}[W(t) \in \tau_x^{-1} B] =: \tau_x \star \mathscr{L}[W(t)](B).$$

Stochastic evolution equation in $\mathscr{S}'$ driven by spatially homogeneous noise of the form

$$dX(t) = AX(t)\,dt + dW(t), \quad X(0) = 0, \tag{7.5}$$

where A is a pseudodifferential operator in $\mathscr{S}'$ and $(W(t))_{t\geq 0}$ is a spatially homogeneous Wiener process have been considered in a number of papers [99, 100, 101, 84, 102] and in the monographs [11, 28, 33]. A concrete example of a SPDE that can be considered in the framework of (7.5) is the stochastic heat equation in $\mathbb{R}^2$ given by

$$\partial_t u(t) = \Delta u(t) + \dot{W}(t), \quad u(0) = 0. \tag{7.6}$$

where $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on $L^2(\mathbb{R}^2, \mu)$, i.e., so that W models a space-time white noise. Unlike the one-dimensional case, this equation is known not to have function-valued solutions.

Clearly, the concept of spatially homogeneous noise is not appropriate for noise on a bounded domain $U \in \mathbb{R}^d$ as a translation may take points inside U to $\mathbb{R}^d \setminus U$. Therefore, one may pose the question: what might be an interesting replacement for spatially homogeneous noise on a bounded domain?

When (7.6) is considered on some bounded domain $U \subset \mathbb{R}^2$ with zero Dirichlet boundary conditions and $(W(t))_{t\geq 0}$ is a cylindrical Wiener process on $L^2(U)$, the same problem occurs: solutions are $\mathscr{S}'$ valued and not $L^2(U)$-valued. This situation is particularly interesting as the stationary solution is given by the Gaussian free field (e.g., see Remark 2.1 in [103]) which has been an object of intense study over the last years due to its connection to Schramm-Loewner evolutions (SLE). It is also known that the law of the Gaussian free field (GFF) on the plane is invariant under a conformal transformation [104]. This suggests that an interesting replacement for spatially homogeneous noise for a domain $U \subset \mathbb{R}^2$ is perhaps to consider noise whose law is invariant under conformal transformations. That is, stochastic evolution equations in the plane $\mathbb{R}^2 \simeq \mathbb{C}$ of the form

$$dX(t) = AX(t)\,dt + d\mathscr{W}(t), \quad X(0) = 0,$$

where $(\mathscr{W}(t))_{t\geq 0}$ is a conformally invariant noise.

In the spirit of this idea, let CSM($\mathbb{D}$) be the set of conformal self maps of $\mathbb{D}$ which are given by transformations of the form

$$T(z) = \lambda \frac{z+a}{1+\overline{a}z},$$

with $a \in \mathbb{D}$ and $|\lambda| = 1$. These maps form a group under composition. We shall now show the $\mathfrak{B}$-norm of a Gaussian Bloch random variable is invariant under CSM transformations.

Lemma 7.3. *Let u be a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable such that $\|u\|_{\mathfrak{B}} < \infty$, then the norm $\|u\|_{\mathfrak{B}}$ is invariant under $T \in$ CSM($\mathbb{D}$).*

Proof. As

$$\frac{\partial}{\partial z} u\left(\frac{z+a}{1+\overline{a}z}\right)\bigg|_{z=0} = u'(a)(1-|a|^2),$$

it follows that

$$\begin{aligned}\sup_{T\in \mathrm{CSM}(\mathbb{D})} \|(u\circ T)'(0)\|_{L^2(\Omega)} &= \sup_{a\in\mathbb{D}} \|u'(a)\|_{L^2(\Omega)}(1-|a|^2)\\ &= \|u\|_{\mathfrak{B}},\end{aligned}$$

and $\|u\circ T\|_{\mathfrak{B}} = \|u\|_{\mathfrak{B}}$. ■

7.5 A law of iterated logarithm

Our second motivation for defining the space of Bloch random variables is the well-known observation that Bloch functions of the form

$$\sum_{n=0}^{\infty} z^{2^n}$$

behave like a random series as $z \to \mathbb{T}$ [90]. This lead to the development by Makarov of a law of iterated logarithm result that gives a quantitative estimate of the rate of blow-up near the boundary of $\mathbb{D}$ for Bloch functions [105]. Due to this connection, it seems obvious that one should attempt a similar result for our Bloch random variables.

If u is a $\mathscr{H}(\mathbb{D})$-valued Gaussian random variable we write for $p \geq 0$,

$$I_p(r) := \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^{2p}\, d\theta.$$

Notice that, as opposed to $M_p(r,u)$, the evaluation $I_p(r)$ does not contain a $L^2(\Omega)$-norm hence it is a random variable generated by taking the $2p$-th (spatial) moment of u on a circle of radius r. In the special case of $p = 1$, we have the relationship

$$\mathbb{E} I_1(r) = M_2(r,u).$$

The following estimate will be used to obtain a growth rate for $I_p(r)$.

Lemma 7.4. *For $r < 1$ we have*

$$\frac{d}{dr}\left(r I_p'(r)\right) = \frac{2p(2p-1)r}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^{2p-2} |\nabla u(re^{i\theta})|^2\, d\theta.$$

Proof. As $u \in \mathscr{H}(\mathbb{D})$ and $r < 1$, we have by following the proof of Theorem 6.21 that

$$\frac{d}{dr} \frac{1}{2\pi} \int_{-\pi}^{\pi} |u(re^{i\theta})|^{2p}\, d\theta$$

$$= \frac{2p(2p-1)}{2\pi}\frac{1}{r}\int_0^r\int_{-\pi}^{\pi}|u(se^{i\theta})|^{2p-2}|\nabla u(se^{i\theta})|^2 s\,d\theta\,ds.$$

The conclusion now follows by multiplying both sides by r and differentiating with respect to r. ■

To extend the results of Makarov to the random variable (i.e. random field) setting, we need an almost sure pointwise bound of the gradient of the field u over $\mathbb{D}$. Our next lemma provides such a bound under the assumption that $\|u\|_{\mathfrak{B}} \leq 1$. We lose no generality by bounding by 1 as we can always scale our field u.

Lemma 7.5. *If $\|u\|_{\mathfrak{B}} \leq 1$ and $r < 1$ then, almost surely,*

$$\sup_{z\in\mathbb{D}}|\nabla u(rz)|^2 \leq \frac{1}{(1-r^2)^2}.$$

Proof. For $z \in \mathbb{D}$ we have

$$\begin{aligned}\mathbb{P}\{|\nabla u(rz)|(1-r^2) > n\} &\leq n^{-2}\mathbb{E}|\nabla u(rz)|^2(1-r^2)^2\\ &= n^{-2}\|u'(rz)\|^2_{L^2(\Omega)}(1-r^2)^2\\ &\leq n^{-2}\|u\|^2_{\mathfrak{B}}.\end{aligned}$$

So by a Borel-Cantelli argument, it holds almost surely that

$$|\nabla u(rz)| \leq \frac{1}{(1-r^2)}$$

Squaring both sides and taking the supremum over $z \in \mathbb{D}$ gives the claim. ■

By applying our previous two lemmas, we may now recursively iterate our definition of $I_p(r)$ to obtain an almost sure growth rate for $I_p(r)$.

Lemma 7.6. *For any $p \geq 0$, if $I_p(0) = 0$ and $\|u\|_B \leq 1$ then we have, almost surely for $r < 1$,*

$$I_p(r) \leq p!\left(\log\frac{1}{1-r^2}\right)^p.$$

Proof. By Lemma 7.4 we have

$$\frac{d}{dr}\left(rI_p'(r)\right) = \frac{2p(2p-1)r}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^{2p-2}|\nabla u(re^{i\theta})|^2\,d\theta$$

then by Lemma 7.5 and our assumption,

$$\begin{aligned}
&\leq \frac{2p(2p-1)r}{2\pi(1-r^2)^2}\int_{-\pi}^{\pi} |u(re^{i\theta})|^{2p-2}\,d\theta \\
&= \frac{2p(2p-1)r}{(1-r^2)^2} I_{p-1}(r)
\end{aligned}$$

Now notice that $I_0(r) = 1$ so our claim holds for $p = 0$. Considering $p = 1$, we get

$$\begin{aligned}
\frac{d}{dr}\left(rI_1'(r)\right) &\leq \frac{2r}{(1-r^2)^2} \\
&= \frac{1}{2}\frac{4r}{(1-r^2)^2} \\
&= \frac{1}{2}\frac{d}{dr}\left(r\frac{d}{dr}\left(\log\frac{1}{1-r^2}\right)\right)
\end{aligned}$$

As $I_p(0) = 0$ by assumption, we integrate both sides with respect to r, cancel r on both sides, then integrate again, to get

$$I_1(r) \leq \frac{1}{2}\log\frac{1}{1-r^2}.$$

Now for $p = 2$, we get

$$\begin{aligned}
\frac{d}{dr}\left(rI_2'(r)\right) &\leq \frac{12r}{(1-r^2)^2} I_1(r) \\
&= \frac{6r}{(1-r^2)^2}\log\frac{1}{1-r^2} \\
&\leq \frac{3}{4}\frac{8r}{(1-r^2)^2}\left(\log\frac{1}{1-r^2} + r^2\right) \\
&= \frac{3}{4}\frac{d}{dr}\left(r\frac{d}{dr}\left(\log\frac{1}{1-r^2}\right)^2\right)
\end{aligned}$$

and it follows that $I_2(r) \le \frac{3}{4}\log(\frac{1}{1-r^2})^2$. Now as

$$\frac{d}{dr}\left(r\frac{d}{dr}\left(\log\frac{1}{1-r^2}\right)^p\right)$$
$$= \frac{4pr}{(1-r^2)^2}\left(\log\frac{1}{1-r^2}\right)^{p-2}\left((p-1)r^2+\log\frac{1}{1-r^2}\right)$$

we see that for $p \ge 1$,

$$\frac{4pr}{(1-r^2)^2}\left(\log\frac{1}{1-r^2}\right)^{p-1} \le \frac{d}{dr}\left(r\frac{d}{dr}\left(\log\frac{1}{1-r^2}\right)^p\right).$$

With a bit of thought the proceeding calculations allow is to guess that at each iteration we pick up the constant $(2p-1)/2$ and since $\prod_{k=1}^{p}(2k-1)/2 = \Gamma(p+1/2)/\sqrt{\pi}$, we conjecture that

$$I_p(r) \le p!\left(\log\frac{1}{1-r^2}\right)^p.$$

Assume that this inequality holds for $p-1$, then

$$\begin{aligned}
\frac{d}{dr}\left(rI_p'(r)\right) &\le \frac{2p(2p-1)r}{(1-r^2)^2}I_{p-1}(r)\\
&\le \frac{(2p-1)(p-1)!}{2}\frac{4pr}{(1-r^2)^2}\left(\log\frac{1}{1-r^2}\right)^{p-1}\\
&\le \frac{(2p-1)(p-1)!}{2}\frac{d}{dr}\left(r\frac{d}{dr}\left(\log\frac{1}{1-r^2}\right)^p\right)\\
&\le p!\frac{d}{dr}\left(r\frac{d}{dr}\left(\log\frac{1}{1-r^2}\right)^p\right)
\end{aligned}$$

Now integrating both sides, cancelling r both sides, and integrating again we get

$$I_p(r) \le p!\left(\log\frac{1}{1-r^2}\right)^p.$$

Hence the result holds by induction. ■

Recall that for a function $u \in \mathscr{H}(\mathbb{D})$, the **radial maximal function** of u is the function $M_{\text{rad}}\, u$ defined for $e^{i\theta} \in \mathbb{T}$ by

$$(M_{\text{rad}}\, u)(e^{i\theta}) := \sup_{0<r<1} |u(re^{i\theta})|.$$

Our next result is an extension of Makarov's law of iterated logarithm for Bloch functions to Bloch random variables (i.e. random fields). Instead of Makarov's original, we have followed the approach found in [90] which is attributed to Carleson and Pommerenke.

If $u \in \mathscr{H}(\mathbb{D}) \cap \mathfrak{B}$ and $\mathbb{E}|u(z)|^2 = \infty$ for almost every point $z \in \mathbb{T}$, then the next result may be understood as an almost sure rate of blow-up for u near the boundary $\mathbb{T}$.

Theorem 7.7. *If $u \in \mathfrak{B}$ satisfies $\mathbb{E}u(0)^2 = 0$ and $\|u\|_{\mathfrak{B}} \leq 1$ then we have, almost surely,*

$$\limsup_{r\to 1^-} \frac{|u(re^{i\theta})|}{\sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}}} \leq 1,$$

for almost every $e^{i\theta} \in \mathbb{T}$.

Proof. We first fix $r < 1$ and write $u_r(e^{i\theta}) := u(re^{i\theta})$ and $u_r^*(e^{i\theta}) := M_{\text{rad}}\, u_r(e^{i\theta})$ so that

$$(u_r^*)(e^{i\theta}) = \sup_{0<\varrho<1} |u_r(\varrho e^{i\theta})| = \sup_{0<\varrho<r} |u(\varrho e^{i\theta})|.$$

By our assumptions, it follows that $u_r \in L^2(\mathbb{T})$ almost surely as $I_p(r) < \infty$ almost surely. Further, by the Hardy-Littlewood maximal theorem,

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |u_r^*(e^{i\theta})|^{2p}\, d\theta &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|^{2p}\, d\theta \\
&\lesssim p!\left(\log\frac{1}{1-r^2}\right)^p \\
&\lesssim p!\left(\log\frac{1}{1-r}\right)^p.
\end{aligned}$$

Now choose a fixed $\alpha > 1$. As we can bound

$$\frac{1}{(1-r)\log\frac{1}{1-r}} > 1$$

for $0 < r < 1$, it follows that

$$\begin{aligned}
\frac{1}{\left(\log\frac{1}{1-r}\right)^p}\frac{1}{\left(\log\log\frac{1}{1-r}\right)^\alpha} &< \frac{1}{1-r}\frac{1}{\left(\log\frac{1}{1-r}\right)^{p+1}}\frac{1}{\left(\log\log\frac{1}{1-r}\right)^\alpha} \\
&< \int_r^1 \frac{1}{1-s}\frac{1}{\left(\log\frac{1}{1-s}\right)^{p+1}}\frac{1}{\left(\log\log\frac{1}{1-s}\right)^\alpha}\,ds \\
&=: \int_r^1 L_p(s)\,ds
\end{aligned}$$

and then

$$\frac{|u_r^*(e^{i\theta})|^{2p}}{\left(\log\frac{1}{1-r}\right)^p\left(\log\log\frac{1}{1-r}\right)^\alpha} \lesssim \int_r^1 |u_s^*(e^{i\theta})|^{2p} L_p(s)\,ds$$

Now the bound on the right-hand side depends θ, p and our choice of α. We shall now derive a bound that holds for almost every $\theta \in [-\pi, \pi]$ in terms of p. Let us define the sets

$$Q_p := \{\theta : \int_r^1 |u_s^*(e^{i\theta}) L_p(s)\,ds > p^2 p!\}.$$

By an application of Chebyshev's inequality and Fubini's theorem,

$$\begin{aligned}
|Q_p| &\le \frac{1}{p^2 p!}\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_r^1 |u_s^*(e^{i\theta})|^{2p} L_p(s)\,ds \\
&= \frac{1}{p^2 p!}\int_r^1 L_p(s)\frac{1}{2\pi}\int_{-\pi}^{\pi} |u_s^*(e^{i\theta})|^{2p}\,d\theta\,ds \\
&= \frac{1}{p^2 p!}\int_r^1 L_p(s) p! \left(\log\frac{1}{1-r}\right)^p ds \\
&\le \frac{1}{p^2}\int_r^1 \frac{1}{1-s}\frac{1}{\log\frac{1}{1-s}}\frac{1}{\left(\log\log\frac{1}{1-s}\right)^\alpha}\,ds
\end{aligned}$$

$$\lesssim_\alpha \frac{1}{p^2}$$

As $\sum_{p=1}^{\infty} |Q_p| < \infty$ it follows by the Borel-Cantelli lemma that there exists a p_0 such that if $\theta \neq \bigcup_{p=p0}^{\infty} Q_p$ then we can bound

$$\frac{|u_r^*(e^{i\theta})|^{2p}}{\left(\log\frac{1}{1-r}\right)^p \left(\log\log\frac{1}{1-r}\right)^\alpha} \lesssim p^2 p!$$

Multiplying by $(\log\log(1/(1-r)))^\alpha$ and using the definition of u_r^* we get

$$\frac{|u(re^{i\theta})|^{2p}}{\left(\log\frac{1}{1-r}\right)^p} \lesssim p^2 p! \left(\log\log\frac{1}{1-r}\right)^\alpha$$

Taking powers of $1/2p$ both sides,

$$\frac{|u(re^{i\theta})|}{\sqrt{\left(\log\frac{1}{1-r}\right)}} \lesssim (p^2)^{\frac{1}{2p}} (p!)^{\frac{1}{2p}} \left(\log\log\frac{1}{1-r}\right)^{\frac{\alpha}{2p}} \tag{7.7}$$

We now using Stirling's formula: $m! \sim \sqrt{2\pi} m^{m+1/2} e^{-m}$ and take $p = \log\log\log\frac{1}{1-r}$ to bound

$$(p^2)^{\frac{1}{2p}} (p!)^{\frac{1}{2p}} \left(\log\log\frac{1}{1-r}\right)^{\frac{\alpha}{2p}} \lesssim p^{\frac{5}{4p}} \sqrt{p}$$

We now divide both sides of (7.7) by $\sqrt{p} = \sqrt{\log\log\log\frac{1}{1-r}}$ and notice that if $\lim_{r\to 1}$ then $p \to \infty$, so the final result follows by $\lim_{p\to\infty} p^{\frac{5}{4p}} = 1$. ■

8

Outside-in Approach

Although the Bloch random variable approach of Chapter 7 gives us a deeper understanding of the white noise situation on $\mathbb{D}$ and provides us with quantitative estimates of the rate of growth near the boundary, we believe that it would be even better if we could work from the outside inwards. That is, start from the definition of the boundary data and work our way inwards to obtain an estimate. This would provide a sharp connection between the boundary data and the dynamics on $\mathbb{D}$. In this chapter, we perform this "outside-in" approach.

In §8.1 we construct a spatial white noise on $\mathbb{T}$ using a one-dimensional Brownian motion. Then, in §8.2, we construct a Poisson Wiener integral which will form the base of our approach. In §8.3 and §8.4 we discuss the consequences of this characterisation and the relationship with tangential derivatives. In §8.5 we recall some special results on the paths of Brownian motion that we make use of in §8.6 to estimate the rate of blow-up near typical points and slow points of the Brownian motion on $\mathbb{T}$.

8.1 White noise on $\mathbb{T}$ and Brownian motion

Let $\mathscr{T}$ be a Borel σ-algebra of $\mathbb{T}$ and let λ be the Lebesgue measure on $(\mathbb{T}, \mathscr{T})$ normalised so that $\lambda(\mathbb{T}) = 1$. We call the white noise w on $(\mathbb{T}, \mathscr{T}, \lambda)$ a **space white noise** and model it in the following way.

Let $(\tilde{B}(t))_{t\geq 0}$ denote a standard Brownian motion with $\tilde{B}(0) = 0$ and $(\mathscr{F}_t)_{t\geq 0}$ being its natural filtration. We now shift $(\tilde{B}(t))_{t\geq 0}$ to obtain a standard Brownian motion $(B(t))_{t\geq 0}$ starting at $t = -\pi$ given by

$$B(t) := \tilde{B}(t+\pi),$$

so that $B(-\pi) = 0$. We model our space white noise w by the paths of $(B(t))_{t\geq 0}$ by defining for $\theta_1, \theta_2 \in [-\pi, \pi)$,

$$w(\{e^{i\theta} : \theta_1 \leq \theta < \theta_2\}) := \frac{B(\theta_2) - B(\theta_1)}{\sqrt{2\pi}}.$$

We can now check that w satisfies Definition 2.16. From the properties of Brownian motion it is clear that if $A_1 = \{e^{i\theta} : \theta_1 \leq \theta < \theta_2\}$ and $A_2 = \{e^{i\theta} : \theta_2 \leq \theta < \theta_3\}$ with $\theta_1 < \theta_2 < \theta_3$ that $w(A_1)$ is a centered Gaussian and

$$\mathbb{E}(w(A_1))^2 = \mathbb{E}\left(\frac{B(\theta_2) - B(\theta_1)}{\sqrt{2\pi}}\right)^2 = \frac{|\theta_2 - \theta_1|}{2\pi} = \lambda([\theta_1, \theta_2)).$$

It follows from the definition of A_1 and A_2 that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = \{e^{i\theta} : \theta_1 \leq \theta < \theta_3\}$ so thanks to the independent increment property of Brownian motion it follows that $w(A_1)$ and $w(A_2)$ are independent and

$$\begin{aligned} w(A_1 \cup A_2) &= B(\theta_3) - B(\theta_1) \\ &= B(\theta_3) - B(\theta_2) + B(\theta_2) - B(\theta_1) \\ &= w(A_2) + w(A_1). \end{aligned}$$

Hence, we can conclude that w is a white noise. We recall that by Kolmogorov's theorem, every Brownian motion has a version with Hölder continuous trajectories for any exponent $\gamma < 1/2$. We shall henceforth always assume that we have taken a version such that $\theta \mapsto B(\theta)$ is continuous almost surely.

We note that it is also possible to define a **complex-valued space white noise** by setting

$$B(t) = \frac{B_1(t) + iB_2(t)}{\sqrt{2}}$$

where B_1, B_2 are independent real Brownian motions and B becomes a standard complex Brownian motion scaled so that $\mathbb{E}B(1)\overline{B(1)} = 1$.

Alternatively, we know that $(B(\theta), 0 \leq \theta < 2\pi)$ can be given explicitly in terms of an orthonormal basis (h_n) of $L^2(\mathbb{T})$ by

$$B(\theta) = \sum_{n=1}^{\infty} \gamma_n (1_{[0,\theta]}, h_n),$$

where $(\gamma_n)_{n=1}^{\infty}$ is a Gaussian sequence and $(h_n)_{n=1}^{\infty}$ is an orthonormal basis for $L^2(\mathbb{T})$. Therefore, as $h_n(z) = z^n$ where $n \in -\mathbb{Z} \cup \mathbb{Z}$, we have

$$B(\theta) = \sum_{n=-\infty}^{\infty} \gamma_n \frac{1}{2\pi} \int_0^{\theta} e^{-in\sigma}\, d\sigma.$$

Intuitively, it should be clear that we have formally (and neglecting constants)

$$dw(e^{i\theta}) \simeq dB(\theta) \simeq \dot{B}(\theta)\, d\theta$$

so that space white noise on $\mathbb{T}$ can be seen pointwise as the time derivative of a standard Brownian motion that is “wrapped” around the boundary of $\mathbb{D}$.

8.2 Poisson Wiener integral

Using the representation of the space white noise w in terms of a Brownian motion, we shall now show that the Poisson integral $P[w](e^{i\theta})$ can be written in terms of a complex-valued Wiener integral.

As is customary, we start with indicator functions of the form $f := 1_A$ where $A = \{e^{i\theta} : \theta_1 \leq \theta < \theta_2\}$ and define the random variable

$$I(f) := w(A).$$

It should be clear that this definition extends by linearity to all step functions $f:\mathbb{T}\to\mathbb{C}$ and the following lemma is straight forward.

Lemma 8.1. *For all step functions $f:\mathbb{T}\to\mathbb{C}$, $I(f)$ is a complex Gaussian random variable and*

$$\mathbb{E}|I(f)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2\, d\theta$$

Proof. By the definition of the space white noise w for f of the form

$$f = \sum_{i=1}^{n} a_i 1_{\{e^{i\theta}:\theta\in[\theta_{i-1},\theta_i)\}}$$

with $\theta_0 = -\pi$ and $\theta_n = \pi$ with $a_i \in \mathbb{C}$, we have

$$I(f) = \sum_{i=1}^{n} a_i w([\theta_{i-1},\theta_i)) = \sum_{i=1}^{n} a_i \frac{1}{\sqrt{2\pi}}(B(\theta_i) - B(\theta_{i-1}))$$

which is a complex-valued Gaussian. By the independent increment and Gaussianity properties of Brownian motion it follows, by treating the complex terms carefully, that

$$\mathbb{E}|I(f)|^2 = \sum_{i=1}^{n} a_i \overline{a_i} \frac{1}{2\pi}(\theta_i - \theta_{i-1}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^2\, d\theta = \|f\|_2^2$$

■

Let $L^2(\Omega)$ denote the Hilbert space of square-integrable complex-valued random variables with inner product $[X,Y] := \mathbb{E}X\overline{Y}$. Let $f \in L^2(\mathbb{T})$ and take a sequence $(f_n)_{n=1}^{\infty}$ of step functions $f_n:\mathbb{T}\to\mathbb{C}$ such that $f_n \to f$ in $L^2(\mathbb{T})$. It follows from Lemma 8.1 that $(I(f_n))_{n=1}^{\infty}$ is a Cauchy sequence in $L^2(\Omega)$. We set

$$I(f) := \lim_{n\to\infty} I(f_n), \quad \text{in } L^2(\Omega),$$

and as it can be shown that this limit is independent of the sequence $(f_n)_{n=1}^{\infty}$ chosen, the random variable $I(f)$ is well-defined. We denote $I(f)$ by

$$I(f)(\omega) = \left(\int_0^{2\pi} f(e^{i\theta})\, dw(e^{i\theta})\right)(\omega), \quad \omega\in\Omega, \text{ almost surely,}$$

and it should also be clear from Lemma 8.1 that we have the alternative representation in terms of a Wiener integral

$$I(f) = \int_{-\pi}^{\pi} f(e^{i\theta})\, dw(e^{i\theta}) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(e^{i\theta})\, dB(\theta),$$

and is a complex-valued Gaussian random variable with mean 0 and variance $\|f\|_2^2$.

In particular, taking $f = P_x$ where

$$P_x(z) = \frac{1-|x|^2}{|x-z|^2}, \quad x \in \mathbb{D},\, z \in \mathbb{T}.$$

is the Poisson kernel gives us the **Poisson-Wiener integral**:

$$P_W[w](x) := \int_{\mathbb{T}} P_x(z) dw(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} P_x(e^{i\theta})\, dB(\theta), \quad x \in \mathbb{D}.$$

It should be noted that since we are only concerned with the case where $f = P_x$ and P_x is of bounded variation we can use path-by-path integration,

$$\left(\int_{\mathbb{T}} P_x(z)\, dw(z) \right)(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} P_x(e^{i\theta})\, dB(\theta, \omega).$$

Remark 8.2. A stochastic integral of this form was studied in §5.4.2 of [106] in connection with Gaussian Analytic functions (GAF) and their zeros.

8.3 Consequences of characterisation

We believe that this characterisation provides some interesting insight into the dynamics of the harmonic extension of white noise to the disk $\mathbb{D}$. For example, we recall that

Theorem 8.3. *A positive harmonic function is equal to the Poisson integral of an increasing function.*

but we also know that

Theorem 8.4. *Brownian motion has, almost surely, no local points of increase.*

Therefore, we can deduce that $P[w]$ is, almost surely, **not** a positive harmonic function.

8.4 Tangential derivatives

Let BV($\mathbb{T}$) denote the set of all functions of bounded variation on $\mathbb{T}$. We recall (e.g., see Chapter 1 of [98]) that the **Poisson-Stieljes integral** of a function $v \in \mathrm{BV}(\mathbb{T})$ is defined to be

$$P_S[v](x) := \int_{\mathbb{T}} P_x(z)\, dv(z). \tag{8.1}$$

Due to the connection between BV($\mathbb{T}$) and $\mathscr{M}(\mathbb{T})$, a function $u \in \mathscr{H}(\mathbb{D})$ belongs to $\mathscr{H}^1$ if and only if u is equal to the Poisson Stieljes integral of a function $v \in \mathrm{BV}(\mathbb{T})$. Further, we can rewrite (8.1) as

$$P_S[v](x) = \frac{v(\pi) - v(-\pi)}{2\pi} P_x(e^{i\pi}) + \int_{\mathbb{T}} P_x'(z) v(z) \frac{dz}{2\pi},$$

where for $x = re^{i\theta}$ and $z = e^{it}$ we have $P_x(z) = P(re^{i(\theta-t)})$ and

$$P_x'(z) = \frac{\partial P}{\partial t}(re^{i(\theta-t)}) = \frac{-2r\sin(\theta - t)}{1 - 2r\cos(\theta - t) + r^2} P(re^{i(\theta-t)}).$$

Alternatively, we can also write

$$P_S[v](re^{i\theta}) = \frac{v(\pi) - v(-\pi)}{2\pi} P(re^{i(\theta+\pi)}) + \frac{\partial}{\partial \theta} P[v](re^{i\theta})$$

where $P[v]$ is the (standard) Poisson integral of a function v. This representation is interesting as we can interpret the term

$$\frac{\partial}{\partial \theta} P[v](re^{i\theta})$$

as a **tangential derivative** of $P[v]$ at the point $re^{i\theta} \in \mathbb{D}$. Further, for a fixed θ we can estimate

$$|P_S[v](re^{i\theta})| \lesssim \left| \frac{\partial}{\partial \theta} P[v](re^{i\theta}) \right|$$

as $r \to 1$. We recall (e.g., see Theorem 1.1 in [98]) that the following three classes of functions are equivalent:

- Poisson-Stieljes integrals,
- differences of two positive harmonic functions,
- $\mathscr{H}^1$.

Further, Theorem 1.2 in [98] shows that if a function $v \in \mathrm{BV}(\mathbb{T})$ has a finite derivative at $e^{i\theta} \in \mathbb{T}$, then

$$\lim_{r \to 1} P_S[v](re^{i\theta}) = v'(e^{i\theta}).$$

It is clear that it would be nice to be able to use these concepts and representations for the Poisson-Wiener integral $P_W[w]$ that we defined in the last section. We note however the following complications:

- Brownian paths $t \mapsto B(t)$ are of **unbounded variation** (e.g., see Theorem 1.35 in [107]) hence $P_W[v]$ does not fall into the class of Poisson-Stieljes integrals,
- Almost surely, Brownian motion is nowhere differentiable (e.g., see Theorem 1.30 in [107]) hence naïvely we would obtain, almost surely, that

  $$\lim_{r \to 1} P_W[w](re^{i\theta}) = B'(\theta) = \infty$$

  at almost every point $e^{i\theta} \in \mathbb{T}$.

Therefore, at first glance it seems that the representations in this section are useless for our purpose but we take delight in the following:

- As $P_W[w] \notin P_S \equiv \mathscr{H}^1$ and $\mathscr{H}^2 \subset \mathscr{H}^1$, we reconfirm the results of last chapter that showed we could not obtain a well-defined $\mathscr{H}^2$-valued Gaussian random in the white noise case. In fact, this now shows that we can not even obtain a $\mathscr{H}^1$-valued Gaussian random variable (i.e. by imbedding $\mathscr{H}^2$ into the larger space $\mathscr{H}^1$) in the white noise case.

- The limit $P_W[w](re^{i\theta}) \to B'(\theta)$ makes "physical" sense, as white noise is often heuristically understood by physicists and engineers as the time derivative of Brownian motion.

We finish this section by stating that, as in the case of the Poisson-Stieljes integral, due to the nice properties of the Poisson kernel it is valid to write

$$P_W[w](re^{i\theta}) = \frac{B(\pi) - B(-\pi)}{\sqrt{2\pi}} P(re^{i(\theta+\pi)}) + \frac{\partial}{\partial \theta} P[B](re^{i\theta})$$

where

$$P[B](re^{i\theta}) := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} P(re^{i(\theta-t)}) B(t)\, dt.$$

Hence, the term

$$\frac{\partial}{\partial \theta} P[B](re^{i\theta})$$

is a random variable giving the tangential derivative of $P[B]$ at $re^{i\theta} \in \mathbb{D}$. By combining this characterisation with some special properties of Brownian motion we shall obtain, in the coming sections, a sharper upper bound for the behaviour of $P_W[w]$ near the boundary of $\mathbb{D}$.

8.5 Typical and Exceptional points of Brownian motion

In 1963, Dvoretzky improved this and established

Theorem 8.5 (Dvoretzky)**.** *Let* $(B_t, t \geq 0)$ *be a standard Brownian motion. Then*

$$\mathbb{P}\left(\limsup_{h \to 0^+} \frac{|B_{t+h} - B_t|}{h^{1/2}} > c_0, \forall t \right) = 1, \tag{8.2}$$

for a positive constant c_0*.*

The question whether (8.2) holds for all constants was settled by Kahane in 1974. The answer is no. He showed that

Theorem 8.6 (Kahane)**.** *Let* $(B_t, t \geq 0)$ *be a standard Brownian motion. Then for each* $\varepsilon > 0$*,*

$$\mathbb{P}\left(\exists t : \limsup_{h\to 0^+} \frac{|B_{t+h} - B_t|}{h^{1/2}} < c_1\right) = 1,$$

for a positive constant $c_1 < \infty$*.*

Kahane calls those t which satisfy

$$\limsup_{h\to 0} \frac{|B_{t+h} - B_t|}{\sqrt{|h|}} < \infty$$

the **slow points** of $(B_t, t \geq 0)$. Due the law of iterated logarithm, the slow points almost surely have Lebesgue measure 0, but Kahane proved that their Hausdorff dimension a.s. equals 1.

8.6 Blow-up near the boundary

At almost every point on the boundary we have the following upper estimate of the blow-up.

Theorem 8.7. *Almost surely, for almost all* $re^{i\theta} \in \mathbb{T}$

$$\limsup_{r\to 1^-} |P_W[w](re^{i\theta})| \lesssim \frac{\log\log \dfrac{1}{1-r}}{\sqrt{1-r}}.$$

Proof. We take $r < 1$ and write

$$P_W[w](re^{i\theta}) = \frac{B(\pi) - B(-\pi)}{2\pi} P_r(\theta + \pi) + \frac{\partial}{\partial\theta} \int_{-\pi}^{\pi} P_r(\theta - t) B(t) \frac{dt}{2\pi}.$$

So we estimate that the first term is almost surely bounded by some constant and now proceed to estimate the second term. We set

$$f(\theta) = \int_{-\pi}^{\pi} P_r(\theta - t) B(t) \frac{dt}{2\pi}$$

and assume that we have taken a continuous version of $t \mapsto B(t)$ (which exists almost surely). Then as f is harmonic it is twice differentiable and using Taylor's theorem, for $h > 0$ we can estimate

$$f(\theta + 2h) = f(\theta) + 2hf'(\theta) + 2h^2 f''(\theta) + O(h^3)$$

so that

$$|f'(\theta)| \leq \frac{|f(\theta + 2h) - f(\theta)|}{2h} + h|f''(\theta)|.$$

As $f(\theta + 2h) - f(\theta)$ is harmonic and $B(t)$ is continuous on $\mathbb{T}$ almost surely, it follows that

$$\lim_{r \to 1} |f(\theta + 2h) - f(\theta)| = |B(\theta + 2h) - B(\theta)|$$

for all $e^{i\theta} \in \mathbb{T}$. This allows us to estimate

$$|f'(\theta)| \leq \frac{|B(\theta + 2h) - B(\theta)|}{2h} + h|f''(\theta)|.$$

The law of iterated logarithm for Brownian motion says that, almost surely,

$$\limsup_{h \to 0^+} \frac{|B(h)|}{\sqrt{2h \log\log(1/h)}} = 1.$$

Setting $\varphi(h) = \sqrt{2h \log\log(1/h)}$, we have

$$|f'(\theta)| \leq \frac{\varphi(2h)}{2h} + h|f''(\theta)|.$$

Writing $M(r, f) = \sup_\theta |f(\theta)|$ we have

$$M(r, f') \leq \frac{\varphi(2h)}{2h} + hM(r, f''). \tag{8.3}$$

We now need to estimate $M(r, f'')$ and to relate an increment of θ to an increment of r.

We notice the following. Let $B_\varrho(z)$ of radius ϱ centered at z. If u is continuous on $\overline{B_\varrho(z)}$ and harmonic in $B_\varrho(z)$, then by the reproducing property for the Poisson kernel we have for $z = re^{i\theta}$ and $\varrho = (1 + r)/2$ that

$$u(re^{i\theta}) = \int_{-\pi}^{\pi} P_\varrho(\theta - \eta) u(\varrho e^{i\eta}) \frac{d\eta}{2\pi}.$$

As

$$\frac{d}{d\theta}P_\varrho(\theta-\eta)=\frac{2r\varrho\sin(t-\theta)}{r^2\varrho^2-2r\varrho\cos(t-\theta)+1}P_\varrho(\theta-\eta)$$

It follows that there exists $C>0$ such that

$$\begin{aligned}\sup_\theta\left|\frac{d}{d\theta}u(re^{i\theta})\right| &\leq \sup_\eta|u(\varrho e^{i\eta})|\int_{-\pi}^{\pi}\left|\frac{d}{d\theta}P_\varrho(\theta-\eta)\right|\frac{d\eta}{2\pi}\\ &\leq C\sup_\eta|u(\varrho e^{i\eta})|(\varrho^2-r^2)^{-1}\\ &\leq C\sup_\eta|u(\varrho e^{i\eta})|(1-r)^{-1}\end{aligned}$$

Therefore, setting $u=f'$ we get

$$M(r,f'')\leq C(1-r)^{-1}M((1+r)/2,f').$$

and it follows from (8.3) that

$$M(r,f')\leq\frac{\varphi(2h)}{2h}+Ch(1-r)^{-1}M((1+r)/2,f')$$

Setting $A(r)=(1-r)M(r,f')$ for $0<r<1$ we get

$$A(r)\leq(2h)^{-1}(1-r)\varphi(2h)+C2t(1-r)^{-1}A((1+r)/2)$$

Hence taking $h=(1-r)/(8C)$ we get

$$A(r)\leq 4C\varphi(1-r)+\frac{1}{4}A((1-r)/2)$$

For $0<\varrho<1$, we have

$$\int_\varrho^1 A(r)\,dr\leq 4C\int_\varrho^1\varphi(1-r)\,dr+\frac{1}{4}\int_\varrho^1 A((1+r)/2)\,dr$$

then performing a change of variables in the integrals on the right-hand side,

$$\int_\varrho^1 A(r)\,dr\leq 4C\int_0^{1-\varrho}\varphi(s)\,ds+\frac{1}{2}\int_{(1+\varrho)/2}^1 A(s)\,ds.$$

Taking the last term on the right-hand side to the left-hand side, we get

$$\frac{1}{2}\int_{\varrho}^{1} A(r)\,dr \le 4C\int_{0}^{1-\varrho} \varphi(s)\,ds$$

and as φ is increasing on $(0,1-\varrho)$ we estimate

$$\frac{1}{2}\int_{\varrho}^{1} A(r)\,dr \le 4C(1-\varrho)\varphi(1-\varrho).$$

As

$$M(\varrho, f')(1-\varrho)^2 \le 2\int_{\varrho}^{1} A(r)\,dr$$

we get

$$M(\varrho, f')(1-\varrho)^2 \le 16C(1-\varrho)\varphi(1-\varrho)$$

Giving

$$M(\varrho, f') \le 16C(1-\varrho)^{-1}\varphi(1-\varrho)$$

Then taking $\varrho = r$ and $f' = X$, we get

$$M(r, X) \le 16C\frac{\varphi(1-r)}{1-r}.$$

■

It is known that Brownian motion has certain exceptional behaviour. We call $z \in \mathbb{T}$ an a**-slow point** if $\arg z$ is an a**-slow time** for B, that is,

$$\limsup_{h\to 0^+} \frac{|B(\arg z + h) - B(\arg z)|}{\sqrt{h}} \le a.$$

a-slow times exist for $a > 1$ but not for $a < 1$.

Theorem 8.8. *If $z \in \mathbb{T}$ is an a-slow point then*

$$\limsup_{r\to 1^-} |P[\dot{B}](rz)| \lesssim \frac{1}{\sqrt{1-r}}.$$

Proof. Without loss of generality, assume that $z = 1$ is an a-slow point of B. Then by integration by parts,

$$\begin{aligned}
2\pi P[\dot{B}](r) &= P_r(t)B(t)|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} P_r'(t)B(t)\,dt \\
&= \frac{1-r}{1+r}(B(\pi) - B(-\pi)) - \int_{-\pi}^{\pi} P_r'(t)B(t)\,dt
\end{aligned}$$

As $r \to 1^-$ the first term goes to zero so we now estimate the second term. As $|P_r'(t)| \leq C_\delta(1-r)$ for $|t| > \delta$ for every small $\delta > 0$,

$$\limsup_{r\to 1^-} \int_{|t|>\delta} |P_r'(t)||B(t)|\,dt = 0.$$

Therefore, using $|t| \leq c(1-r)$ for some constant $c > 0$,

$$\begin{aligned}
2\pi\sqrt{1-r}|P[\dot{B}](r)| &\leq \left| \int_{|t|\leq\delta} \sqrt{1-r}\sqrt{t}P_r'(t)\frac{B(t)}{\sqrt{t}}\,dt \right| \\
&\leq c\int_{|t|\leq\delta} \left|(1-r)P_r'(t)\right| \left|\frac{B(t)}{\sqrt{t}}\right|\,dt \\
&\leq 2c\int_{-\pi}^{\pi} P_r(t)\left|\frac{B(t)}{\sqrt{t}}\right|\,dt \\
&\leq 2ca
\end{aligned}$$

The result follows taking $r \to 1^-$. ■

9

Blow-up in higher dimension

In the previous chapter we saw that in the two-dimensional setting where the domain is $\mathbb{D}$ with boundary $\mathbb{T}$, we can obtain a white noise on $\mathbb{T}$ by "wrapping" a one-dimensional Brownian motion. Once this construction was obtained we associated rates of blow-up with the local fine properties of Brownian motion. In this chapter we extend these concepts to higher dimensions whereby the noise on the boundary is given by a random field and we associate the rate of blow-up with the local behaviour of the random field.

In §9.1, we consider the operator which takes a sphere or ball average of a random field and show that it is a γ-radonifying operator. In §9.2, we relate the concept of thick points of Gaussian random fields with the notion of Lebesgue points. This allows us to define a new type of maximal function that we call the "maximal thickness function". In §9.3, apply this definition to derive a rate of blow-up for the Poisson integral of a random field on the boundary of the half-space $\mathbb{R}^d_+$, thus extending the results of Chapter 8 to higher dimension.

9.1 Sphere and ball averaging

We take inspiration from the notation found in [108] but generalize it in a number of ways. Let $U \subset \mathbb{R}^{d+1}$ be a domain with boundary ∂U of co-dimension one. One may think of two canonical cases: either U is a bounded domain and ∂U is its smooth boundary or U is the half-space $\mathbb{R}^{d+1}_+$ and ∂U is the hyperplane $\mathbb{R}^d$.

Let (f_n) be an orthonormal basis of a real-valued separable Hilbert space H. The Gaussian field $F = F_{\partial U}$ on the boundary ∂U is given formally as a random linear combination

$$F = \sum_n \gamma_n f_n, \tag{9.1}$$

where (γ_n) is an i.i.d. Gaussian sequence. In this paper we often have in mind the extreme case where $H = L^2(\partial U)$ so that F is a Gaussian white-noise on ∂U, as opposed to [108] where the noise is smoother as H is chosen to be a Sobolev space. One typically cannot make sense of F as a function unless H is chosen to be a sufficiently small space to allow an application of a Sobolev embedding theorem from H to $C(\partial U)$. As this embedding depends on the dimension d, obtaining a function-valued version of F becomes harder as the ambient space dimension d increases. Although F is not a function, one can make sense of F by average over sufficiently nice Borel sets such as spheres or balls.

9.1.1 In two dimensions

We start with the case where $\partial U = \mathbb{R}^2$ as we can quickly adapt the two-dimensional definitions and approach to averaging found in [108], [109], and [104].

Let $D(z,r)$ denote the disk of radius r centered at $z \in \partial U$, then one can make sense of the **circle average process**

$$F(z,r) = \frac{1}{2\pi r} \int_{\partial D(z,r)} F(x)\sigma(dx),$$

where $\sigma(dx)$ is the length measure. Further, it holds almost surely, that for all (z,r)

$$\int_0^r 2\pi s F(z,s)\,ds = \int_{D(z,r)} F(x)\,dx$$

where the right-hand side can be taken as the definition of a **disk average process**. It was shown in Proposition 2.1 of [108] that when F is a two-dimensional continuum Gaussian Free Field (GFF), its circle average process $F(z,r)$ possesses a modification $\tilde{F}(z,r)$ where $(z,r)\mapsto F(z,r)$ is locally Hölder continuous of order strictly less than 1/2. This begs the questions: how much regularity does the circle and disk averaging introduce? and can this process be extended to higher dimensions? Before attempting to answer these questions one may draw analogy to some one dimensional results.

9.1.2 In one dimension

Without loss of generality, as we are interested in local averaging, we can consider the case that $\partial U = (0,T)$. Or equivalently, identify $(0,T)$ with the boundary of the unit disk $\mathbb{D}\subset\mathbb{C}$ through the identification $t \longleftrightarrow e^{it}$ with $T = 2\pi$.

It is well-known by specialists in stochastic evolution equations and stochastic partial differential equations (see [19], [33], [6] and the references therein) that weak and mild solutions of equations perturbed by Gaussian white-noise can be obtained if the deterministic dynamics (i.e. sans noise) maps cylindrical measures to radon measures. In the Hilbert space literature one searches for Hilbert-Schmidt operators and in the Banach space literature, γ-radonifying operators. The following operator is important in the theory of Brownian motion and was proved by Ciesielski (see Exercise 5.5 in [6]).

Theorem 9.1. *The operator $I_T : L^2(0,T)\to C[0,T]$ defined by*

$$(I_T f)(t) = \int_0^t f(s)\,ds, \qquad f\in L^2(0,T),\ t\in[0,T],$$

is γ-radonifying.

One can clearly see that I_T is an averaging operator and thus conjecture that averaging in higher dimensions is also a radonifying process. Further, from the fundamental theorem of calculus, one can also conjecture that I_T increases the differentiability of f by an order of one.

9.1.3 In d-dimensions

In analogy with our previous discussion, we propose the following d-dimensional definitions.

Definition 9.2. *The* ***sphere average process*** *of radius r centered at $z \in \partial U$ is given by*

$$F(z,r) = \int_{\mathbb{S}^{d-1}} F(r\theta - z)\sigma(d\theta)$$

where σ is the surface measure on $\mathbb{S}^{d-1}$ and the ***ball average process*** *of radius r centered at $z \in \partial U$ is given by*

$$G(z,r) = \int_{B(r,z)} F(x)\,dx$$

where $B(z,r)$ is the ball of radius r centered at $z \in \partial U$. We write $G(r) = G(0,r)$.

Let us consider the case where the boundary is compact. We can take the canonical case where $\partial U = Q$ with Q an d-dimensional cube. Without loss of generality, we can scale and shift Q and assume $Q = [0,1)^d$.

Theorem 9.3. *The operator $G : L^2(Q) \to C(Q \times [0,1])$ defined by*

$$(Gf)(z,r) = \int_{B(z,r)} f(x)\,dx, \quad f \in L^2(Q),$$

is γ-radonifying.

To prove this theorem, we shall make use of the following result mentioned in [26].

Theorem 9.4 (Chevet-Carmona)**.** *For all $T_1 \in \gamma(H_1, E_1)$ and $T_2 \in \gamma(H_2, E_2)$ we have*

$$T_1 \otimes T_2 \in \gamma(H_1 \hat{\otimes} H_2, E_1 \hat{\otimes}_\varepsilon E_2),$$

where $H_1 \hat{\otimes} H_2$ denotes the Hilbert space completion of $H_1 \otimes H_2$ and $E_1 \hat{\otimes}_\varepsilon E_2$ denotes the injective tensor product of E_1 and E_2.

We now have the following result that applies this theorem to a situation that will be useful for the proof of Theorem 9.3.

Lemma 9.5. *If $K_1, K_2, \dots, K_n$ are compact sets in $\mathbb{R}^d$ and we have operators $T_i \in \gamma(L^2(K_i), C(K_i))$ for $i = 1, \dots, n$ then*

$$T_1 \otimes \ldots \otimes T_n \in \gamma(L^2(K_1 \times \cdots \times K_n), C(K_1 \times \cdots \times K_n)).$$

Proof. We first notice that for compact sets K_1 and K_2, we have $C(K_1) \hat{\otimes}_\varepsilon C(K_2) = C(K_1 \times K_2)$ (see [110, §5.7.2.10]) and that $L^2(K_1) \hat{\otimes} L^2(K_2) = L^2(K_1 \times K_2)$ (see [70, Section II.4]). Then the result follows by iterating Theorem 9.4. ■

We note that Theorem 9.3 can be proved from scratch by adapting the concepts found in the proof of Theorem 9.1 (see [26] for a refined proof) to the multidimensional setting through the construction of a multidimensional Haar basis. As this becomes quite messy, we propose the following short proof that uses Theorem 9.1 and Lemma 9.5.

Proof of Theorem 9.3. We first notice that for $d = 2$, we have for $h \in L^2([0,1]^2)$ the tensor decomposition $h = h_1 \otimes h_2$ with $h_i \in L^2([0,1])$ for $i = 1, 2$. Then for any point $z = (z_1, z_2) \in [0,1]^2$ and $r \in [0,1]$ we get

$$\begin{aligned}
(Gh)(z,r) &= \int_{B(z,r)} h(x)\,dx \\
&= \int_{z_1-r}^{z_1+r} \int_{z_2-\sqrt{r^2-x_2^2}}^{z_2+\sqrt{r^2-x_2^2}} h_1(x_1) h_2(x_2)\,dx_1\,dx_2
\end{aligned}$$

$$= \int_{z_1-r}^{z_1+r} h_1(x_1)\,dx_1 \int_{z_2-\sqrt{r^2-x_2^2}}^{z_2+\sqrt{r^2-x_2^2}} h_2(x_2)\,dx_2$$

Then we notice that for the first integral, we can estimate

$$\begin{aligned}\sup_{(z_1,r)}\left|\int_{z_1-r}^{z_1+r} h_1(x_1)\,dx_1\right| &= \sup_{(z_1,r)}\left|\int_0^{z_1+r} h_1(x_1)\,dx_1 - \int_0^{z_1-r} h_1(x_1)\,dx_1\right| \\ &\le 2\sup_t |(I_1 h_1)(t)|\end{aligned}$$

where I_1 is the indefinite integration operator (in the first coordinate) that was given in Theorem 9.1. We can estimate the second integral in the same way. It follows for arbitrary space dimension d that

$$\sup_{(z,r)} |(Gh)(z,r)| \le C_n \sup_t |(I_1 h_1)(t)| \cdots \sup_t |(I_d h_d)(t)|,$$

where $C_d > 0$ is a constant dependent on the dimension d. By Theorem 9.1, $I_i \in \gamma(L^2[0,1], C[0,1])$ for $i = 1, \dots, d$. Hence, by Lemma 9.5, we have

$$\begin{aligned}||G||_{\gamma(L^2(Q),C([0,1]^d\times[0,1]))} &\le ||I_1 \otimes \cdots \otimes I_n||_{\gamma(L^2(Q),C([0,1]^d))} \\ &< \infty.\end{aligned}$$

■

Corollary 9.6. *If F is a white noise on Q, i.e. given by* (9.1) *with* $(f_n) \in L^2(Q)$, *then the ball average process* $(z,r) \mapsto G(z,r)$ *has a continuous version almost surely.*

9.2 A thickness function

In connection with the study of extremes of the occupation measure of stochastic processes, the concept of a “thick point” is starting to become standard terminology. In particular, the following definition in [108] caught our attention: If $U \subset \mathbb{R}^2$ is a bounded domain, then a point z is an **a-thick point** provided

$$\lim_{r\to 0} \frac{\mu(D(z,r))}{\pi r^2 \log 1/r} = \sqrt{\frac{a}{\pi}}, \tag{9.2}$$

where $D(z,r)$ denotes the disk of radius r centered at $z \in U$ and $\mu(A) = \int_A F(x)\,dx$. In their case, F is a Gaussian Free Field. That is, $F = \sum_n \gamma_n f_n$ where $(f_n) \in W_0^{1,2}(U)$.

We find this definition interesting for the following reasons. In our attempt to derive a "Fatou theorem" for the harmonic random field generated by white-noise on the boundary it became apparent that we need to obtain a quantitative estimate on the local behaviour of the stochastic data around a point as is done in classic harmonic analysis by various "maximal functions". What needs to be defined suddenly becomes clear if we quote the first paragraph of Section 1.1 in [111]:

> According to the fundamental theorem of Lebesgue, the relation
>
> $$\lim_{r\to 0} \frac{1}{|B(z,r)|} \int_{B(z,r)} f(x)\,dx = f(z). \tag{9.3}$$
>
> holds for almost every z, whenever f is locally integrable function defined on $\mathbb{R}^d$. Here, $B(z,r)$ is the ball of radius r, centered at z, and $|B(z,r)|$ denotes its Lebesgue measure. In order to study the limit (9.3) we consider its quantitative analogue, where "$\lim_{r\to 0}$" is replaced by "$\sup_{r\to 0}$"; this is the **maximal function**.

One can now clearly see the analogy between the **Lebesgue point** of f given by (9.3) and the two dimensional thick point of F given by (9.2), all that is now needed is just a careful choice of definition that extends (9.2) to a higher dimensional and quantitative analogue. We propose the following:

Definition 9.7. *For every $z \in \mathbb{R}^d$, we call*

$$T_\varphi(F)(z) = \sup_{r>0} \frac{\mu(B(z,r))}{|B(z,r)|\varphi(|B(z,r)|^{1/d})},$$

the ***(maximal) thickness function*** *of the random field F with* ***gauge function*** *φ where $\mu(A) = \int_A F(x)\,dx$ measures the signed mass that F associates with the Borel set A whenever it is defined, $|\cdot|$ is d-dimensional Lebesgue measure, and $B(z,r)$ is a ball of radius r centered at z.*

We now proceed to justify this choice of definition. We have chosen the term 'gauge' function for φ to bring connection to the refinement of the notions of Hausdorff dimension and Hausdorff dimension that are used in the study of random fractals, see [107]. In regards to (9.2), the choice $\varphi(r) = \log 1/r$ would be an appropriate example of a gauge. Natural extensions of the classic Hardy-Littlewood maximal function to capture regularity of the boundary data have been proposed for when the data is in the space of Lipschitz continuous functions $\Lambda_\alpha(\mathbb{R}^d)$ or the Besov space $\Lambda_\alpha^{p,q}(\mathbb{R}^d)$. One can refer to Chapter V, §4 and §5 of [111] for the definitions of these spaces and to [112] or [113] for some new maximal function definitions associated with these spaces. After a few calculations, the choice of scaling $|B(z,r)|^{1/d}$ seemed appropriate.

Remark 9.8. We recently found a paper by Kolyada [114] where Calderon's maximal function definition is abstractly extended and a number Sobolev type embeddings are proved. A similar scaling to ours is present in his definition.

9.3 Blow-up near points on the boundary

Theorem 9.9. *Let F be a random field on $\partial U = \mathbb{R}^d$ and $u(x,t)$ be its Poisson integral on $U = \mathbb{R}^{d+1}_+$. Then almost surely*

$$\sup_{t>0} |u(x,t)| \lesssim (T_\varphi F)(x)\varphi(b_d^{1/d}t)/t,$$

where b_d is the volume of the unit ball in $\mathbb{R}^d$ and φ is an increasing gauge function.

Proof. Without loss of generality, we take $x = 0$ and use polar coordinates to get

$$u(0,t) = \int_{\mathbb{R}^d} P_t(-y)F(y)\,dy = \int_0^\infty \int_{\mathbb{S}^{d-1}} P_t(re_1)F(r\theta)r^{d-1}\,\sigma(d\theta)\,dr,$$

where $e_1 = (1,0,\ldots,0) \in \mathbb{S}^{d-1}$ and σ is the surface measure on $\mathbb{S}^{d-1}$. Let $G(r)$ be the ball average process of F centered at 0, assumed to be continuous almost surely. Then by integration by parts, for some large enough $R > 0$,

$$\begin{aligned}
\int_{\mathbb{R}^d} P_t(-y)F(y)\,dy &= G(tR)P_t(tRe_1) - G(0)P_t(0) - \int_0^{tR} \frac{\partial P_t}{\partial r}(re_1)G(r)\,dr \\
&= -\int_0^{tR} \frac{\partial P_t}{\partial r}(re_1)G(r)\,dr
\end{aligned}$$

as $G(0) = 0$, $P_t(0) < \infty$, and $P_t(tRe_1) = 0$. Setting b_d to be the volume of the unit ball in $\mathbb{R}^d$ and $B(r) = B(0,r)$, we get that

$$\begin{aligned}
G(r) &= \int_{B(r)} F(x)\,dx \\
&= \frac{|B(r)|\varphi(|B(r)|^{1/d})}{|B(r)|\varphi(|B(r)|^{1/d})}\mu(B(r)) \\
&= (T_\varphi F)(0)|B_r|\varphi(|B(r)|^{1/d}) \\
&= (T_\varphi F)(0) b_d r^d \varphi(b_d^{1/d} r)
\end{aligned}$$

Therefore, using the fact that $d b_d = \int_{\mathbb{S}^{d-1}} \sigma(d\theta)$,

$$\begin{aligned}
\left| \int_0^{tR} G(r)\frac{\partial P_t}{\partial r}(re_1)\,dr \right| &\le (T_\varphi F)(0) \int_0^\infty b_d r^d \varphi(b_d^{1/d} r) \left| \frac{\partial P_t}{\partial r}(re_1) \right| dr \\
&= (T_\varphi F)(0) \frac{1}{d} \int_0^\infty \int_{\mathbb{S}^{d-1}} r\varphi(b_d^{1/d} r) \left| \frac{\partial P_t}{\partial r}(re_1) \right| r^{d-1}\,\sigma(d\theta)\,dr \\
&= (T_\varphi F)(0) \frac{1}{d} \int_{\mathbb{R}^d} |y|\varphi(b_d^{1/d}|y|) \left| \frac{\partial P_t}{\partial y}(-y) \right| dy
\end{aligned}$$

$$= (T_\varphi F)(0)\frac{1}{d}\left(\int_{|y|<t} + \int_{|y|\geq t}\right)|y|\varphi(b_d^{1/d}|y|)\left|\frac{\partial P_t}{\partial y}(-y)\right| dy.$$

Using the kernel bounds

$$\left|\frac{\partial P_t}{\partial y}\right| \lesssim |y|^{-d-1}, \qquad \left|\frac{\partial P_t}{\partial y}\right| \lesssim t^{-d-1},$$

we estimate

$$\begin{aligned}
\int_{|y|<t} |y|\varphi(b_d^{1/d}|y|)\left|\frac{\partial P_t}{\partial y}(-y)\right| dy &\lesssim \int_{|y|<t} t\varphi(b_d^{1/d}|y|)t^{-d-1}\, dy \\
&= t^{-d}\int_{|y|<t} \varphi(b_d^{1/d}|y|)\, dy \\
&= t^{-d}\int_0^t \varphi(b_d^{1/d} r) r^{d-1}\, dr \\
&\lesssim d^{-1}t^{-1}\varphi(b_d^{1/d} t),
\end{aligned}$$

by the observation that φ is increasing and the change of variable $y = r\xi$ with $|\xi| = 1$ and $r = |y|$ so that $dy = d\xi\, r^{d-1}\, dr$. In a similar way,

$$\begin{aligned}
\int_{|y|\geq t} |y|\varphi(b_d^{1/d}|y|)\left|\frac{\partial P_t}{\partial y}(-y)\right| dy &\lesssim \int_{|y|\geq t} |y|\varphi(b_d^{1/d}|y|)|y|^{-d-1}\, dy \\
&= \int_r^\infty \varphi(b_d^{1/d} r)\, dr
\end{aligned}$$

and the conclusion follows. ■

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