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The Two Dimensional Starting Plume

by

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The Two Dimensional Starting Plume.

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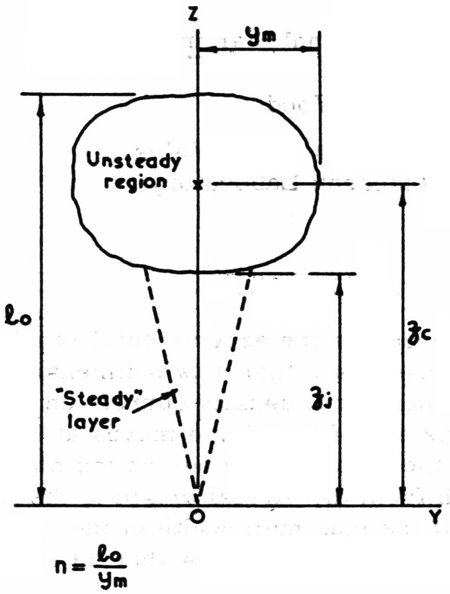
Abstract.

This paper presents the experimental results for the rate of advance and the spread of either a two-dimensional starting plume or a two dimensional density current on a vertical slope. It is shown that the cap of the plume moves at 0.38 times the maximum velocity of the steady layer behind the plume and that the ratio of the length from the virtual origin to the leading edge of the cap to one half the maximum width of the cap is 3. The virtual mass concept is used to show that these two ratios are connected.

Introduction

A considerable amount of work has recently been devoted to the study of unsteady growing turbulent flows. Scorer (1), Richards (2), Saunders (3), Woodward (4), Turner (5),(6) extensively studied the buoyant vortex ring or thermal. Richards (7), has extensively studied the buoyant vortex pair or the cylindrical thermal and Turner (8) has also recently examined the case of an axisymmetric plume. It is the object of this paper to extend the work and use a slightly different approach to cover the case of a two dimensional plume.

For this case, fluid containing a density excess is released from a line source and a two dimensional plume forms as in Figure 1. The plume consists of two distinct portions; a "steady" layer where the mean properties of the flow do not vary with time and an unsteady head or cap. As time progresses the cap grows in size, advances and lays down a further length of steady layer. As with all the flows mentioned above, only very small density differences will be used. This implies that although the buoyant force per unit mass is sufficiently large to produce vertical accelerations, the corresponding variations in the mass densities are small enough to be neglected. For these cases the motion of a rising buoyant fluid will appear the same as the motion of a falling denser fluid (provided the density excess in the first case equals the density deficit in the second) and two will be treated as if only one case existed.



NOMENCLATURE

FIGURE 1.

Theory

In the development of the theory, the positive z axis will be taken as vertical and the zx plane will be taken as the plane of symmetry. The length from the virtual origin of the line source to the leading edge of the vorticity containing region is defined as ℓ_o and we can immediately write that

ℓ_o is a function of the flux of density difference $[q \Delta\rho_1 g]$ and time (t) and the density of the ambient fluid ρ_1

where q is one half the volumetric flow rate per foot length of line source

ρ_1 is the density of the ambient fluid

$\rho_1 - \Delta\rho$ is the density of any element of fluid which has a density different from ρ_1

$\rho_1 - \Delta\rho_1$ is the density of the fluid released from the line source

g is the acceleration of gravity

Because of the absence of walls in the case of a plume in infinite surroundings, no force other than the buoyancy forces can act on the plume and it is therefore permissible to leave viscosity out of the variables.

Dimensional analysis then leads to

$$\frac{l_o}{t\left[q\frac{\Delta\rho_1}{\rho_1}g\right]^{\frac{1}{3}}} = C_1 \tag{1}$$

or

$$\frac{Dl_o}{Dt} = C_1\left[q\frac{\Delta\rho_1}{\rho_1}g\right]^{\frac{1}{3}}$$

where C_1 is a constant.

Further, if A is the area of the vorticity containing region in the positive yz plane and the characteristic vorticity, ζ_o and the dimensionless density difference times g are defined by

$$\zeta_o = \int_A \zeta_x \, dA \Big/ A \tag{2}$$

$$\left[\frac{\Delta\rho g}{\rho_1}\right]_o = \int_A \Delta\rho \cdot g \, dA \Big/ \rho_1 dA \tag{3}$$

and if y_m is taken as the maximum y coordinate of the vorticity containing region (i. e. the turbulent region of the plume) then dimensional analysis yields

$$\zeta_o = \frac{C_2}{t} \tag{4}$$

$$\left[\frac{\Delta\rho g}{\rho_1}\right]_o = C_3\left[q\frac{\Delta\rho_1}{\rho_1}g\right]^{\frac{1}{3}}\frac{1}{t} \tag{5}$$

$$\frac{l_o}{y_m} = n \tag{6}$$

where C_2 C_3 and n are constants.

Similar relationships hold for all lengths, velocities, vorticities at characteristic points in the flow.

These relationships imply that the dimensionless shape, dimensionless velocity distribution pattern, dimensionless density distribution pattern are self preserving (i. e. are the same at all times during the flow). It is also important to note that the above solutions are consistent with the generation of circulation by the buoyant force acting over the plane of symmetry. It is therefore possible to replace the plane of symmetry with a frictionless plane.

Some further information can be obtained if the force applied to the vorticity containing region is equated to the rate of change of impulse. The total force applied to the vorticity containing region in the positive zy plane at any time after the start of the experiment is $[q\Delta\rho_i g]\,t$. Lamb (9) gives the impulse of a vorticity containing region as $\iint(\rho\zeta_x)\,y\,dy\,dz$

Thus we get

$$\frac{D}{Dt}\iint_{Cap}\rho_i\zeta_x\,y\,dy\,dz + \frac{D}{Dt}\iint_{Steady\ layer}\rho_i\zeta_x\,y\,dy\,dz = [q\Delta\rho_i g]\,t \qquad (7)$$

If the flow around the cap at any time may be approximated to the flow around a solid of the same area as the vorticity containing region of the cap then the virtual mass concept may be used and equation (7) becomes

$$\frac{D}{Dt}[\rho_i A_{cp}(1 + C_v)\,\omega_c] + \frac{D}{Dt}\iint_{Steady\ Layer}\rho_i\zeta_x\,y\,dy\,dz = [q\Delta\rho_i g]\,t \qquad (8)$$

where A_{cp} = area of the cap

C_v = coefficient of virtual mass of the cap

ω_c = velocity of the cap

A more detailed analysis shows that this implies the neglect of terms which involve the integral of turbulent fluctuations over the plane of symmetry. The neglected term, however, would be of a relatively small magnitude.

Making use of the fact that dimensionless velocities, areas etc. are preserved and assuming that the cap is cylindrical $(A_{cp} = \frac{\pi}{2}y_m^2,\ C_v = 1)$ then the equation (8) may be written as

$$\frac{(n-1)}{n^3}\,\frac{\pi}{\left[q\frac{\Delta\rho_i}{\rho_i}g\right]t}\,\frac{D}{Dt}\left[b_0^2\frac{Db_0}{Dt}\right] - \frac{1}{\left[q\frac{\Delta\rho_i}{\rho_i}g\right]t}\,\frac{D}{Dt}\iint_{Steady\ layer}\zeta_x\,y\,dy\,dz - 1 = 0 \qquad (9)$$

It now remains to calculate the integral over the steady layer. This can be computed from the available measurements in steady plumes provided -

(1) the join between the steady layer and the head or cap can be identified, and

(2) the steady layer velocity and density distributions hold up to this join.

The steady layer of a jet or plume is confined to a region which is relatively long in the direction of the main flow and relatively narrow in the direction perpendicular to the main flow direction. Hence in the steady layer the variation of the mean quantities in the direction perpendicular to the plane of symmetry are an order of magnitude greater than the variation along the plane of symmetry. That is

$$\frac{\partial \bar{v}}{\partial z} \doteqdot \frac{1}{L} \frac{\partial \bar{v}}{\partial y}$$

and

$$\frac{\partial \bar{w}}{\partial z} \doteqdot \frac{1}{L} \frac{\partial \bar{w}}{\partial y}$$

where L is a number of order 10. The continuity equation for the mean flow is

$$\frac{\partial \bar{v}}{\partial y} = -\frac{\partial \bar{w}}{\partial z}$$

Hence

$$\begin{aligned} \zeta_x &= \frac{\partial \bar{w}}{\partial y} - \frac{\partial \bar{v}}{\partial z} \\ &\doteqdot \frac{\partial \bar{w}}{\partial y} - \frac{1}{L}\frac{\partial \bar{v}}{\partial z} \end{aligned}$$

from the equation of continuity

$$\begin{aligned} \zeta_x &\doteqdot \frac{\partial \bar{w}}{\partial y} + \frac{1}{L}\frac{\partial \bar{w}}{\partial z} \\ &\doteqdot \frac{\partial \bar{w}}{\partial y} + \left[\frac{1}{L}\right]^2 \frac{\partial \bar{w}}{\partial y} \end{aligned}$$

Thus to a reasonable approximation $\zeta_x = \frac{\partial \bar{w}}{\partial y}$

and hence

$$\frac{D}{Dt}\int_0^{z_j}\int_0^{\infty} \zeta_x \, y \, dy \, dz = \frac{D}{Dt}\int_0^{z_j}\int_0^{\infty} \frac{\partial \bar{w}}{\partial y} \, y \, dy \, dz \qquad (10)$$

where z_j is the level of the join between the steady layer and the head or cap.

Now the numerous experiments with free turbulent flow in stationary surroundings have shown that the velocity and density difference distributions at different distances from the virtual origin may be plotted in a dimensionless form. Thus these distributions are preserved at all distances from the virtual origin and the velocity distribution may be written as

$$w = w_z \, f(\eta) \qquad (11)$$

where w_z is the characteristic velocity of the steady layer at a particular distance z from the virtual origin. Throughout this work this will be taken as the maximum velocity at the level z and

$$\eta = y/b_z$$

where b_z is the characteristic width of the steady layer at the level z and in this study this will be defined by

$$b_z = \int_0^\infty w\,dy / w_z \tag{12}$$

The dimensionless density difference times g distribution may be written as

$$\left[\frac{\Delta\rho g}{\rho_i}\right] = \left[\frac{\Delta\rho g}{\rho_i}\right]_z \phi(\eta) \tag{13}$$

where $\left(\frac{\Delta\rho g}{\rho_i}\right)_z$ is the maximum dimensionless density difference times g at the level z. Using the expression for the velocity distribution equation (10) becomes

$$\frac{D}{Dt}\int_0^{z_i}\int_0^\infty \frac{\partial \bar{w}}{\partial y}\, y\,dy\,dz = H_5 \frac{D}{Dt}\int_0^{z_i} w_z\, b_z\, dz \tag{14}$$

where $H_5 = \int_0^\infty f'(\eta)\,\eta\,d\eta$

The form of the variation of w_z, b_z and $\left[\frac{\Delta\rho g}{\rho_i}\right]_z$ with z has been derived using mixing length theories and a similarity approach (10). However, it is convenient to derive these relationships again using the entrainment concept introduced by Morton et al (11). At the boundary between the turbulent and non-turbulent fluid, there exists a small mean entrainment velocity in the ambient non-turbulent fluid. This velocity is in the direction perpendicular to the main flow.

The velocity distribution at all distances from the virtual origin may be plotted in the same dimensionless form and thus the velocity at any point in a section at a level z will depend only on the characteristic width and velocity at this section. Thus the inflow velocity at a particular section is likely to depend only on the characteristic velocity and width at that section. Dimensional analysis then leads to an entrainment velocity proportional to the characteristic velocity at the particular section. The constant of proportionality is called the entrainment constant E. Using this constant the equation of continuity for the steady layer becomes

$$\frac{D}{Dz}\int_0^\infty w_z\, b_z\, f(\eta)\,d\eta = E\, w_z \tag{15}$$

The equation of continuity of density deficit may be written as

$$\frac{D}{Dz}\int_0^\infty w_z \left[\frac{\Delta\rho g}{\rho_i}\right]_z b_z\, f(\eta)\,\phi(\eta)\,d\eta = 0 \tag{16}$$

The momentum equation may be written as

$$\frac{D}{Dz}\int_0^\infty \omega_z^2 b_z \,[f(\eta)]^2 d\eta = \int_0^\infty \left[\frac{\Delta\rho g}{\rho_i}\right]_z b_z \,\phi(\eta)\, d\eta \tag{17}$$

Defining $H_1 = \int_0^\infty f(\eta)\, d\eta \qquad H_2 = \int_0^\infty f(\eta)\,\phi(\eta)\, d\eta$

$H_3 = \int_0^\infty [f(\eta)]^2 d\eta \qquad H_4 = \int_0^\infty \phi(\eta)\, d(\eta)$

The solution of equations (15), (16), (17) is

$$\begin{aligned} \frac{Db_z}{Dz} &= \frac{E}{H_1} \\ \omega_z &= \left[\frac{H_1 H_4}{H_2 H_3 E}\right]^{\frac{1}{3}} \left[\frac{q \Delta\rho_1 g}{\rho_1}\right]^{\frac{1}{3}} \end{aligned} \tag{18}$$

Thus for a two-dimensional plume

$$\frac{D}{Dt}\int_0^{z_j}\int_0^\infty \zeta_x \, y \, dy\, dz = \left[\frac{H_1 H_4}{H_2 H_3}\right]^{\frac{1}{3}} \frac{H_5}{H_1} (E)^{\frac{2}{3}} \left[\frac{q \Delta\rho_1 g}{\rho_1}\right]^{\frac{1}{3}} z_j \frac{dz_j}{dt} \tag{19}$$

Now the results of Rouse, et al.(9) experiments may be written as

$$\begin{aligned} \omega &= \omega_z \, e^{-\frac{\pi}{4}\frac{y^2}{b_z^2}} \\ \left[\frac{\Delta\rho g}{\rho_1}\right] &= \left[\frac{\Delta\rho g}{\rho_1}\right]_z e^{-\frac{\pi}{4}\frac{y^2}{[0{\cdot}88\, b_z]^2}} \\ \frac{Db_z}{Dz} &= 0{\cdot}156 \end{aligned} \tag{20}$$

This yields a value of E of 0.156. Ellison and Turner (12) in a series of experiments described in the next section measured E directly and obtained a value of 0.06. They suggest that the results of the earlier experiments were affected by wall effects. Ellison and Turner's value of E is much closer to the values obtained in other free turbulent flows (12),(14), and will be adopted. Further, in view of the small difference in the characteristic widths obtained in the velocity and density difference distributions and the doubt in the value of E, it seems reasonable to assume that the characteristic widths for the velocity and density difference distributions are the same. Making this assumption, equation (19) becomes

$$\frac{D}{Dt}\int_0^{z_j}\int_0^\infty \zeta_x \, y \, dy\, dz = -\,[2]^{\frac{1}{3}}[0{\cdot}06]^{\frac{2}{3}}\left[q\frac{\Delta\rho_1 g}{\rho_1}\right]^{\frac{1}{3}} z_j \frac{dz_j}{dt} \tag{21}$$

Hence using the geometry in Figure 1 and the above equation, (9) becomes

$$2\pi\frac{(n-1)}{n^3}\left[\frac{\rho_1}{q\Delta\rho_1 g}\right]\left(\frac{D\ell_0}{Dt}\right)^3 + [2]^{\frac{1}{3}}[0{\cdot}06]^{\frac{2}{3}}\left[\frac{\rho_1}{q\Delta\rho_1 g}\right]^{\frac{2}{3}}\left[\frac{\ell_0 - 2y_m}{t}\right]\frac{d(\ell_0 - 2y_m)}{dt} - 1{\cdot}0 \tag{22}$$

but since all characteristic points move at a constant rate

$$\frac{\ell_0 - 2y_m}{t} \; \frac{d(\ell_0 - 2y_m)}{dt} = \left[1 - \frac{2}{n}\right]^2 \left[\frac{D\ell_0}{Dt}\right]^2$$

and the equation (22) becomes

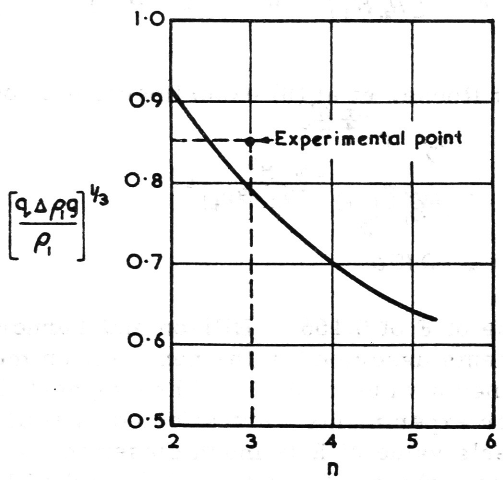
$$2\pi\frac{(n-1)}{n^3}\,\frac{\rho_i}{q\Delta\rho_i g}\left[\frac{D\ell_0}{Dt}\right]^3 + 2^{\frac{1}{3}}[0.06]^{\frac{2}{3}}\left[\frac{\rho_i}{q\Delta\rho_i g}\right]^{\frac{2}{3}}\left[1-\frac{2}{n}\right]^2\left[\frac{D\ell_0}{Dt}\right]^2 - 1 = 0 \qquad (23)$$

Now multiplying throughout by $-\left[q\,\frac{\Delta\rho_i g}{\rho_i}\right] \Big/ \left[\frac{D\ell_0}{Dt}\right]^3$

the equation may be written as

$$\left[\frac{q\Delta\rho_i g}{\rho_i}\right] \Big/ \left[\frac{D\ell_0}{Dt}\right]^3 - 0.19\left(1-\frac{2}{n}\right)^2\left[\frac{q\Delta\rho_i g}{\rho_i}\right]^{\frac{1}{3}} \Big/ \frac{D\ell_0}{Dt} - \frac{2\pi}{n^2}\left(1-\frac{1}{n}\right) = 0 \qquad (24)$$

This equation is plotted in Figure 2



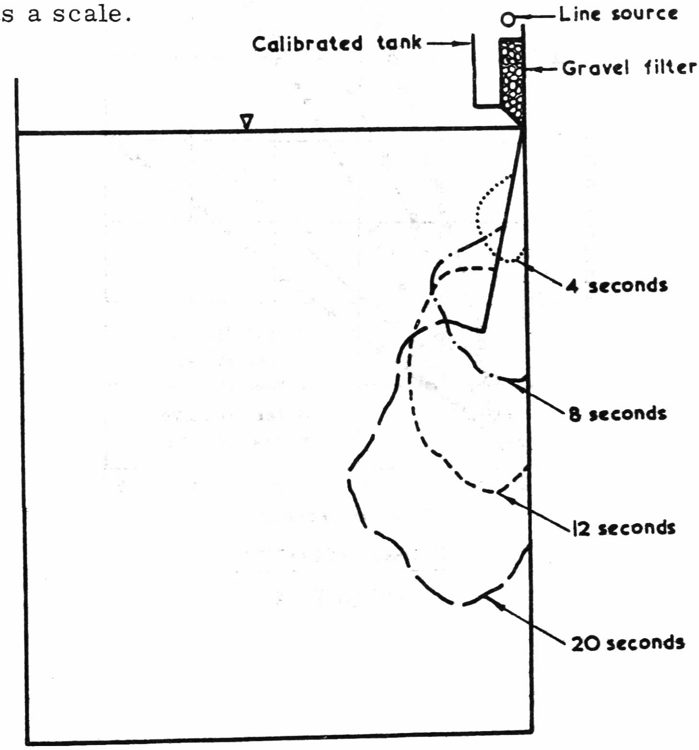
The Two Dimensional Plume

THEORETICAL CURVE

FIGURE 2.

Experiments

The experiments were performed in a 1.5 x 3.0 x 6.0 foot tank with a 3.0 x 3.5 foot perspex window. The tank was filled with fresh water and a line source of denser fluid (salt water) was introduced through a 0.125" x 1.5 foot long slot. A fine gravel filter was placed above the slot to ensure that the fluid entered the tank with the minimum of momentum. Before carrying out the series of experiments the variation of discharge along the slot was checked and found to be negligible. The discharge was measured by timing between points in a calibrated tank and then the discharge tubes were moved above the slot and the experiment was commenced. The heavier salt solution was marked with a fine suspension of graphite and the plume was photographed at 0.5 second intervals using a Shackman automatic camera. Relative density measurements were made by weighing samples of the fresh water collected immediately prior to the test and of the salt water collected from the discharge line immediately after the test in the same specific gravity bottle. The trace from a typical series of photographs is shown in Figure 3. The flux of density deficit was varied over a fortyfold range and this gave a threefold range in velocity. Two sets of experiments were carried out, one with the slot over the centre of the tank and one with wall plumes produced with the slot at the edge of the tank. The films were analysed by projecting the 1" x 1" negatives on a microfilm reader, tracing the plume outlines and using a marker rod placed alongside the tank as a scale.

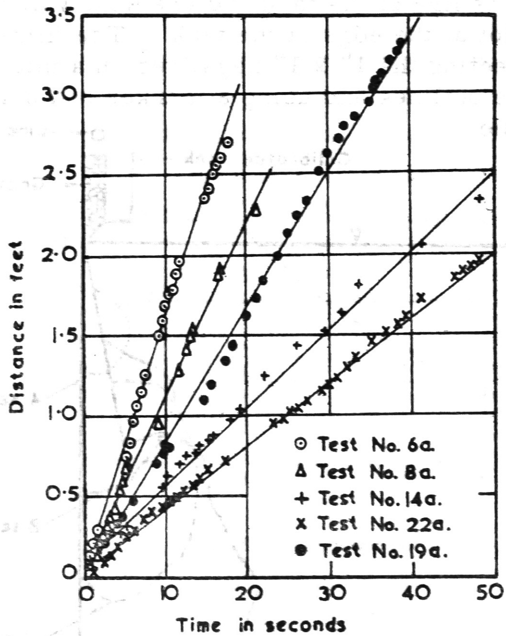


TRACE OF THE PLUME

Outlines at 4 second intervals

FIGURE 3.

For each experiment a plot of ℓ against time was obtained (Figure 4) and from the line of best fit the velocity of the cap was deduced. In addition, values of n were taken from about one half of the negatives. In spite of the cap velocity being approximately constant from the beginning of the observations, the values of n for the first five seconds were always considerably higher than the later values. The earlier values were therefore disregarded. It was also noted that in almost every case the velocity of the cap exhibited a periodic variation about the mean velocity and it was thought at first that this might be correlated with the irregular rate of growth of width of the cap. However, no definite correlation has been established.

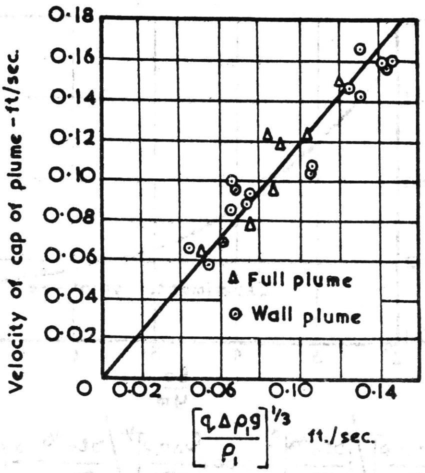


PLUME VELOCITY

FIGURE 4.

Results

The rate of advance of the cap was plotted against $\left[\frac{Q\,\Delta\rho_1 g}{\rho_1}\right]^{\frac{1}{3}}$ (Figure 5) and the mean line gave $\frac{D\ell}{Dt} = 1{\cdot}20\left(\frac{Q\,\Delta\rho\, g}{\rho}\right)^{\frac{1}{3}}$ where the constant of 1.20 has a standard deviation of $\pm$ 0.10. If Ellison and Turner's (9) figures for the entrainment coefficient into the steady plume are accepted, this leads to the ratio of the velocity of



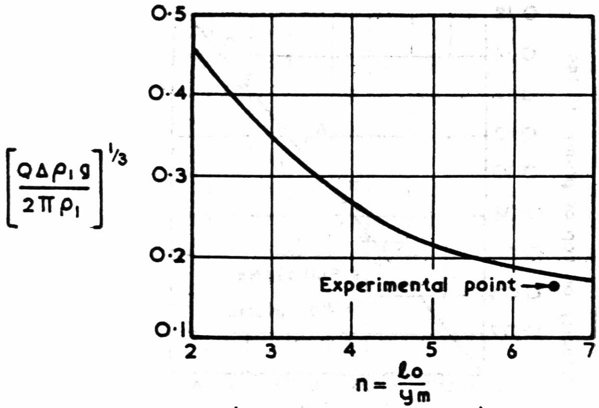
The Two Dimensional Plume

VELOCITY OF CAP AGAINST $\left[\frac{q \Delta \rho_1 g}{\rho_1}\right]^{1/3}$

FIGURE 5.

the cap to the maximum velocity of steady layer at level of cap of 0.38 $\pm$ 0.05. The maximum and minimum value as well as the mean value of n were recorded for each experiment. The spread in these values was considerably greater than those obtained in Turner's (8) three dimensional plume experiments. However, if it is accepted that the vorticity containing region extends to the maximum spread of the coloured region then the appropriate value of n (the average minimum value) 3.0 $\pm$ 0.30 when plotted against $\left[q \frac{\Delta\rho_1}{\rho_1} g\right]^{\frac{1}{3}} / \frac{D l_0}{Dt}$ lies close to the theoretical curve.

The method used to obtain the relationship between n and the combination of variables of l_0, t and $\left[q \frac{\Delta\rho_1}{\rho_1} g\right]^{\frac{1}{3}}$ is completely general. It has been applied for an axisymmetric plume and yields the relationship plotted in Figure 6. Turner's experimental points are also plotted in the figure and it can be seen that they are extremely close to the theoretical curve. It must be emphasised that further information is necessary to obtain a complete solution and that use of the virtual mass concept can only yield a relationship between the variables.



$$\left[\frac{Q\Delta\rho_1 g}{2\pi\rho_1}\right] \Bigg/ \left[\frac{Dl_o{}^{4/3}}{Dt}\right]^3 - 0.002\left[\frac{Q\Delta\rho_1 g}{2\pi\rho_1}\right]^{1/3} \Bigg/ \frac{Dl_o{}^{4/3}}{Dt} - \frac{3}{2}\left(\frac{n-1}{n^4}\right) = 0$$

Where Q is the volumetric flow from the point source

The Axisymmetric Plume

THEORETICAL CURVE

FIGURE 6.

Conclusions

Dimensional analysis predicts the form of the variation of l_o with time and the use of the virtual mass assumption yields a relationship between $\frac{Dl_o}{Dt} \Big/ \left[\frac{Q\Delta\rho g}{\rho}\right]^{\frac{1}{3}}$ and n. These relationships are satisfied by the experimental results provided that n is taken as l_o divided by maximum spread of the vorticity containing region.

Acknowledgments

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List of Symbols.

A = area of the cap

b_z = the characteristic width of the steady layer at the level z

$C_1 C_2 C_3$ = constants

C_v = coefficient of virtual mass

E = the entrainment constant

f = a function

g = the acceleration of gravity

$H_1 H_2 H_3 H_4$ and H_5 = integral constants connected with the steady layer

L = a number of order of magnitude 10

l_o = the length from the virtual origin to the leading edge of the vorticity oontaining region

n = l_o/y_m

Q	=	the volumetric flow rate from a point source
q	=	the volume flow rate per foot length of line source
t	=	time
v, w	=	velocities in the y and z directions respectively
xyz	=	cartesian coordinate
y_m	=	the maximum y coordinate of the vorticity containing region
z_j	=	the z coordinate of the join between the cap and the steady layer
z_c	=	the z coordinate of the centre of the cap
ζ_x	=	the x component of the vorticity
ζ_o	=	the characteristic vorticity
η	=	$\frac{y}{b_z}$
ρ_1	=	the density of the surrounding fluid
$\Delta\rho$	=	the density difference between an element of fluid and the surrounding fluid
$\Delta\rho_1$	=	the density difference between the inflowing fluid and the surrounding fluid
$\left[\frac{\Delta\rho g}{\rho_1}\right]_o$, $\left[\frac{\Delta\rho g}{\rho_1}\right]_z$	=	the characteristic dimensionless density difference times g for the vorticity containing region and the steady layer at the level z respectively
ϕ	=	a function