CONSTRUCTION OF BICLOSED CATEGORIES

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PREFACE

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ABSTRACT

The starting point is the concept of a monoidal category. By formulating the entire theory with respect to a suitable closed category $\mathcal{V}$, the tensor-product functor in a monoidal category may be supposed to carry a $\mathcal{V}$-bifunctor structure.

First we specialise to the well-known concept of a biclosed category which is a monoidal category whose tensor-product functor admits right adjoints to both variables. Since this biclosed property can be expressed in terms of the representability of certain functors, a biclosed category may be thought of as a "complete" monoidal category.

In the other direction, we generalise monoidal category to the concept of promonoidal category. A promonoidal category is less than a monoidal category in that its tensor product and identity can only be expressed as a "profunctor" and a "proobject" respectively. This is the case with many small categories which occur as model categories. While a monoidal structure is a special instance of a promonoidal one, there do exist promonoidal categories which are not monoidal.

Broadly speaking, the thesis provides conditions under which a promonoidal structure on a category $\mathcal{A}$ can be

extended, along a given dense functor $A^{op} \to B$, to produce a biclosed structure on B. The tensor product, internal homs, and so on, for B are then expressed as Kan extensions of the given structure on A. As described in detail in our Introduction, the actual proof of this general result is derived from the consideration of two special cases, namely, the functor category theorem and the reflection theorem.

Much of the thesis is concerned with examples and we divide these into various types. The first is the functor category type, including such familiar examples as the closed category of modules over a commutative ring and the closed category of algebras over a commutative theory. The second deals with biclosed structures obtained by reflection from larger biclosed categories; here we discuss several cartesian closed categories of topological spaces, including that of compactly generated spaces. Lastly, the general construction theorem is applied to the consideration of algebraic closed categories generated by commutative monads.

CONTENTS

INTRODUCTION

The thesis aims to present a theorem which underlies the construction of many well-known examples of both ordinary closed categories and enriched closed categories. We shall approach this theorem through the discussion of two special cases. Therefore, by way of introduction, it seems desirable to provide a brief outline of the development. Some notation and terminology are also introduced here, although we mainly follow that used in [3], [9], and [11].

Section 0.1. Terminology

A monoidal category consists of a category V together with an identity object $I \in V$, a tensor-product functor $\otimes : V \times V \to V$, and natural isomorphisms $\ell : I \otimes A \cong A$, $r : A \otimes I \cong A$, and $a : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$, satisfying the coherence axioms:

MC1
$$\begin{tikzcd} (A\otimes I)\otimes B \arrow[rr, "a"] \arrow[dr, "r\otimes 1"'] & & A\otimes(I\otimes B) \arrow[dl, "1\otimes\ell"] \\ & A\otimes B & \end{tikzcd}$$
commutes,

MC2
$$\begin{tikzcd}
((A\otimes B)\otimes C)\otimes D \arrow[r, "a"] \arrow[d, "a\otimes 1"'] & (A\otimes B)\otimes(C\otimes D) \arrow[r, "a"] & A\otimes(B\otimes(C\otimes D)) \\
(A\otimes(B\otimes C))\otimes D \arrow[rr, "a"'] & & A\otimes((B\otimes C)\otimes D) \arrow[u, "1\otimes a"']
\end{tikzcd}$$

commutes.

This entire structure is often denoted by the single letter $\mathcal{V}$.

A <u>symmetric monoidal category</u> is a monoidal category plus a natural isomorphism $c : A\otimes B \cong B\otimes A$ satisfying the coherence axioms:

MC3
$$\begin{tikzcd}
A\otimes B \arrow[r, "c"] \arrow[dr, "1"'] & B\otimes A \arrow[d, "c"] \\
 & A\otimes B
\end{tikzcd}$$

commutes,

MC4
$$\begin{tikzcd}
(A\otimes B)\otimes C \arrow[r, "a"] \arrow[d, "c\otimes 1"'] & A\otimes(B\otimes C) \arrow[r, "c"] & (B\otimes C)\otimes A \arrow[d, "a"] \\
(B\otimes A)\otimes C \arrow[r, "a"'] & B\otimes(A\otimes C) \arrow[r, "1\otimes c"'] & B\otimes(C\otimes A)
\end{tikzcd}$$

commutes.

A <u>biclosed category</u> is a monoidal category $\mathcal{V}$ for which the endofunctors $A\otimes -$ and $-\otimes B$ both have right adjoints:

$$\mathcal{V}(A\otimes B,C) \cong \mathcal{V}(A,C/B) \cong \mathcal{V}(B,A\backslash C).$$

These right adjoints are called the internal-hom functors of $\mathcal{V}$.

A <u>closed category</u> is a symmetric monoidal category for which the endofunctor $-\otimes B$ has a right adjoint:

$$\mathcal{V}(A\otimes B,C) \cong \mathcal{V}(A,[BC]).$$

Note that a closed category is essentially a symmetric biclosed category.

Only when $\mathcal{V}$ is closed do we get a really workable theory of categories over $\mathcal{V}$. In order to employ this theory efficiently, we may suppose that $\mathcal{V}$ is normalised; that is, that a functor $V : \mathcal{V} \to \mathcal{S}$, where $\mathcal{S}$ = small sets, is so chosen that $V[AB]$ is precisely $\mathcal{V}(AB)$, and not merely isomorphic to $\mathcal{V}(AB)$. Any closed category can be normalised, perhaps after replacing it with an isomorph. We will suppose that <u>given</u> closed categories are normalised, but will not bother to provide normalisations for the closed categories <u>constructed</u>.

If $\mathcal{W}$ is a given closed category, we obtain the concepts of monoidal category <u>over</u> $\mathcal{W}$, closed category <u>over</u> $\mathcal{W}$, etc., if we use "$\mathcal{W}$-category" in place of "category", "$\mathcal{W}$-functor" in place of "functor", $\mathcal{V}\otimes\mathcal{V}$ in place of $\mathcal{V}\times\mathcal{V}$, and so on, in the above definitions. "Coherence" remains unaltered.

Section 0.2. Closed categories of functors

Most familiar examples of closed categories can be non-trivially represented as categories of functors from A to B for suitable domain and codomain categories A and B. These functors may be either ordinary functors, or else V-functors for some closed category V. They may comprise the category $[A,B]$ of all functors (or V-functors) from A to B, or else a definite full subcategory of this. Clearly the latter alternative remains available even when A is large; that is, even when $[A,B]$ does not exist. The following table gives some typical examples of closed functor categories:

Functor category	A	B	Functors considered
simplicial sets	Δ^{op}, where Δ is the simplicial category	sets	all functors
small categories	Δ^{op}	sets	those satisfying the category axioms

Functor category	A	B	Functors considered
quasi-topological spaces	C^{op}, where C is compact hdf. spaces and cts. maps	sets	those of the form $C \rightsquigarrow Ad(C,X)$ for some quasi-space X, where Ad(C,X) is the set of admissible maps from C to X, $C \in C$
(real) Banach spaces	P^{op}, where P is the full subcategory of Banach spaces determined by the Euclidean plane R^2	sets	certain functors; determined by the fact that P is dense in Banach spaces
abelian groups	G^{op}, where G is the theory of abelian groups	sets	those preserving finite products

Functor category	$\mathcal{A}$	$\mathcal{B}$	Functors considered
$\mathbb{Z}$-graded abelian groups	the discrete category of integers $\mathbb{Z}$	abelian groups	all functors
differential graded abelian groups	the additive category generated by the totally ordered category $\mathbb{Z}$, with the relation $d^2 = 0$	abelian groups	all additive functors
modules over a commutative ring K	the additive category with one object whose endomorphism ring is K	abelian groups	all additive functors
sheaves of K-modules	T^{op}, where T is a topology	K-modules	those functors creating certain limits

In these and many other examples, the codomain $\mathcal{B}$ is closed. We therefore choose this as a starting point and replace the letter $\mathcal{B}$ by $\mathcal{V}$, a given closed category.

The above examples also indicate that sometimes $\mathcal{A}$ is an ordinary category and the functors $\mathcal{A} \to \mathcal{V}$ being considered are ordinary functors (e.g. sheaves of K-modules), while othertimes $\mathcal{A}$ is a $\mathcal{V}$-category and the functors $\mathcal{A} \to \mathcal{V}$ are $\mathcal{V}$-functors (e.g. differential graded abelian groups). However, provided $\mathcal{V}$ admits set-indexed copowers of its identity object I, the first case may be <u>included</u> in the second by simply replacing $\mathcal{A}$ with the <u>free</u> $\mathcal{V}$-category generated by $\mathcal{A}$. More precisely, $\mathcal{V}$ is usually complete enough for the representable functor $V : \mathcal{V} \to \mathcal{S}$ to admit the left adjoint $F : \mathcal{S} \to \mathcal{V}$ which sends a set X to the copower $FX = \sum_X I$ in $\mathcal{V}$. Then, by [11], the <u>closed</u> functor $V : \mathcal{V} \to \mathcal{S}$ has a <u>closed</u> left adjoint $F : \mathcal{S} \to \mathcal{V}$, and this induces a 2-functor $F_* : \mathcal{S}\text{-}\mathcal{C}at \to \mathcal{V}\text{-}\mathcal{C}at$. There then results a canonical bijection between functors $\mathcal{A} \to \mathcal{V}$ and $\mathcal{V}$-functors $F_*\mathcal{A} \to \mathcal{V}$.

Therefore we suppose throughout that $\mathcal{V}$ is a closed category and $\mathcal{A}$ is a $\mathcal{V}$-category and that the functors

$A \to V$ being considered are V-functors. This means that our general theory is developed <u>entirely over</u> V, so we stipulate that henceforth (save in specific examples) the unqualified words "category, functor, natural transformation, adjoint, monoidal category", etc., mean "V-category, V-functor, V-natural transformation, V-adjoint, monoidal category over V," etc. The prefix "V" is occasionally retained for emphasis.

Furthermore, it is convenient to use the same symbol for both a V-functor and its underlying S-functor. In the special case of left represented functors from A to V we shall use $A(A-)$ and LA <u>interchangeably</u>.

Section 0.3. Outline of development

The general construction theorem, formulated in Chapter 5, is derived essentially in terms of two special cases.

First we take the case where $\mathcal{A}$ is a small category and $\mathcal{V}$ admits small limits and colimits, and we consider the category $[\mathcal{A},\mathcal{V}]$ of <u>all</u> functors from $\mathcal{A}$ to $\mathcal{V}$ (as done in [5]). The structure of $[\mathcal{A},\mathcal{V}]$ as a $\mathcal{V}$-category is recalled in Chapter 1. Here we also recall the Yoneda full embedding $L : \mathcal{A}^{op} \to [\mathcal{A},\mathcal{V}]$ which is the canonical functor sending $A \in \mathcal{A}^{op}$ to the left represented functor $LA = \mathcal{A}(A-) : \mathcal{A} \to \mathcal{V}$. The functor L is <u>dense</u>, a fact which is essentially contained in the expression

$$T \cong \int^A TA \otimes LA$$

of each functor $T : \mathcal{A} \to \mathcal{V}$ as a colimit in $[\mathcal{A},\mathcal{V}]$ of left represented functors.

We now consider the possibility of enriching $[\mathcal{A},\mathcal{V}]$ to a <u>biclosed</u> category. For this we note that a tensor product

$$\bar{\otimes} : [\mathcal{A},\mathcal{V}]\otimes[\mathcal{A},\mathcal{V}] \to [\mathcal{A},\mathcal{V}],$$

for which $S\bar{\otimes}-$ and $-\bar{\otimes}T$ both admit right adjoints, is essentially determined by its values $LA\bar{\otimes}LA'$ on represented functors. This is so because $S\bar{\otimes}-$ and $-\bar{\otimes}T$ both preserve colimits:

$$S\bar{\otimes}T \cong (\int^{A}SA\otimes LA)\bar{\otimes}(\int^{A'}TA'\otimes LA')$$
$$\cong \int^{AA'}(SA\otimes TA')\otimes(LA\bar{\otimes}LA').$$

Writing $P(AA'-)$ for the functor $LA\bar{\otimes}LA'$, we obtain a functor $P : A^{op}\otimes A^{op}\otimes A \to V$. Conversely, given <u>any</u> functor $P : A^{op}\otimes A^{op}\otimes A \to V$, we can define a tensor product $\bar{\otimes}$ on $[A,V]$ by means of the expression

$$S\bar{\otimes}T = \int^{AA'}(SA\otimes TA')\otimes P(AA'-).$$

Moreover, this definition of $\bar{\otimes}$ is easily seen to provide a natural isomorphism $LA\bar{\otimes}LA' \cong P(AA'-)$ and right adjoints to each of $S\bar{\otimes}-$ and $-\bar{\otimes}T$. These facts simply express the correspondence, to within isomorphism, of functors $A^{op}\otimes A^{op} \to [A,V]$ to their Kan extensions $[A,V]\otimes[A,V] \to [A,V]$ along $L\otimes L : A^{op}\otimes A^{op} \to [A,V]\otimes[A,V]$.

An identity object $J \in [A,V]$ for $\bar{\otimes}$ is just a functor $J : A \to V$. Natural isomorphisms $\bar{\ell}$, $\bar{r}$, $\bar{a}$, completing $\bar{\otimes}$, J to a monoidal structure, are easily seen to translate into natural isomorphisms

$$\lambda : \int^{X}JX\otimes P(XA-) \cong LA$$
$$\rho : \int^{X}JX\otimes P(AX-) \cong LA$$
$$\alpha : \int^{X}P(AA'X)\otimes P(XA''-) \cong \int^{X}P(A'A''X)\otimes P(AX-).$$

In turn, the coherence conditions for $\bar{\ell}$, $\bar{r}$, $\bar{a}$ translate into corresponding "coherence conditions" for λ, ρ, α.

The main result of Chapter 3 is a bijection (at least to within isomorphism) between biclosed structures on the functor category $[A,V]$, and certain structures $(P,J,\lambda,\rho,\alpha)$ on A. Given the biclosed structure on $[A,V]$, its "trace" on $A^{op} \subset [A,V]$ yields the structure on A; given the structure on A, the biclosed structure on $[A,V]$ is obtained by Kan extension.

Such structures on A are called promonoidal, because $P : A^{op} \otimes A^{op} \otimes A \to V$ is what Bénabou has described as a "profunctor" from $A \otimes A$ to A. It then turns out that each monoidal structure $(\tilde{\otimes}, \tilde{I}, \ldots)$ on A can be identified with a corresponding promonoidal structure whose P and J are given by

$$P(AA'-) = A(A\tilde{\otimes}A',-)$$

$$J = A(\tilde{I},-).$$

In other words, monoidal structures on A are a special case of promonoidal ones. An important feature of the monoidal case is that the opposite category A^{op} admits a canonical monoidal structure (also denoted by A^{op}) and consequently is promonoidal (Bénabou has called A^{op} the monoidal "conjugate" of A).

With the idea in mind of using promonoidal structures as generalised monoidal ones, we introduce a definition of promonoidal functor. This is so done as to produce the usual concept of monoidal functor whenever the domain and codomain categories are actually monoidal. For any two promonoidal categories A and B, the promonoidal enrichments of a functor $T : A \to B$ correspond bijectively to the monoidal enrichments of the "restriction" functor $[T,1] : [B,V] \to [A\ V]$.

Moreover, the concept of promonoidal structure on a category A is readily seen to be independent of the smallness of A and the completeness of V. In defining a general promonoidal category we simply insist that the particular coends needed to write down the definition do exist in V. For instance, <u>any</u> monoidal category is promonoidal, the existence of the necessary coends being guaranteed by the representation theorem. Thus we may view the biclosed structure of $[A,V]$ as a large "completion" of the promonoidal structure on a small category A.

Many closed categories arise, not as total functor categories $[A,V]$, but as full reflective subcategories of these. For example, sheaves of abelian groups on a topology T arise as a reflective subcategory of $[T^{op},Ab]$, and abelian groups arise as a reflective subcategory of $[G^{op},S]$ where G denotes the theory of abelian groups. This brings us to the second construction.

Commencing with a biclosed structure $(\bar{\otimes},\bar{I},\bar{\ell},\bar{r},\bar{a},/,\backslash)$ on a category $\mathcal{B}$, let $\theta : \mathcal{C} \to \mathcal{B}$ be a full embedding functor with a left adjoint $\psi : \mathcal{B} \to \mathcal{C}$. Then there exists a biclosed structure $(\hat{\otimes},\hat{I},\ldots)$ on $\mathcal{C}$, for which ψ admits enrichment to a monoidal functor $\Psi = (\psi,\tilde{\psi},\psi^{o})$ with

$$\tilde{\psi} : \psi B \hat{\otimes} \psi B' \to \psi(B \bar{\otimes} B')$$
$$\psi^{o} : \hat{I} \to \psi \bar{I}$$

<u>isomorphisms</u>, if and only if:

(*) For all $B \in \mathcal{B}$ and $C \in \mathcal{C}$, the objects $\theta C/B$ and $B\backslash\theta C$ of $\mathcal{B}$ admit isomorphs in $\mathcal{C}$.

The condition (*) has numerous equivalent forms which we list in Chapter 4. In any given application one form may be more convenient to use than the others. In particular, if the original biclosed category $\mathcal{B}$ contains a <u>dense</u> subcategory $\mathcal{A}$ then the condition (*) becomes:

(**) For all $A \in \mathcal{A}$ and $C \in \mathcal{C}$, there exist objects $H(AC)$ and $K(AC)$ of $\mathcal{C}$ together with isomorphisms

$$\mathcal{C}(\psi(A'\bar{\otimes}A),C) \cong \mathcal{C}(\psi A',H(AC))$$
$$\mathcal{C}(\psi(A\bar{\otimes}A'),C) \cong \mathcal{C}(\psi A',K(AC))$$

which are natural in $A' \in \mathcal{A}$.

The new condition (**) applies non-trivially to the case where $\mathcal{B} = [\mathcal{A},\mathcal{V}]$ for a small promonoidal category $\mathcal{A}$. Here $\mathcal{B}$ contains the dense subcategory of left represented functors. But to say that a category $\mathcal{C}$ is a full reflective subcategory of $[\mathcal{A},\mathcal{V}]$ is precisely to say that there exists a dense functor $M : \mathcal{A}^{op} \to \mathcal{C}$, the reflection of a functor $T \in [\mathcal{A},\mathcal{V}]$ being given by the coend $\psi T = \int^A TA \otimes MA$ in $\mathcal{C}$. In terms of the functor M, the condition (**) reads:

(***) For all $A \in \mathcal{A}$ and $C \in \mathcal{C}$, there exist objects $H(AC)$ and $K(AC)$ of $\mathcal{C}$, together with isomorphisms

$$C(Q(A'A),C) \cong C(MA',H(AC))$$

$$C(Q(AA'),C) \cong C(MA',K(AC))$$

which are natural in $A' \in \mathcal{A}$, where

$$Q(AA') = \int^X P(AA'X) \otimes MX.$$

The last condition (***) makes no explicit reference to the functor category $[\mathcal{A},\mathcal{V}]$ as a whole, only to the reflections $Q(AA')$ in $\mathcal{C}$ of the functors $P(AA'-) : \mathcal{A} \to \mathcal{V}$. Thus we may ask whether satisfaction of (***) guarantees a biclosed structure on a category $\mathcal{C}$ when a dense functor $M : \mathcal{A}^{op} \to \mathcal{C}$ is given from the dual of an <u>arbitrary</u> promonoidal category $\mathcal{A}$ over an arbitrary ground category $\mathcal{V}$. The answer is yes, provided we postulate the existence in $\mathcal{C}$ of the coends $\int^A TA \otimes MA$ for <u>certain</u> functors $T : \mathcal{A} \to \mathcal{V}$.

More precisely, we require that the coends

$$Q(AA') = \int^X P(AA'X) \otimes MX$$

$$\hat{I} = \int^X JX \otimes MX$$

$$C \hat{\otimes} C' = \int^{XX'} (C(MX,C) \otimes C(MX',C')) \otimes Q(XX')$$

exist in C, together with the ends

$$C'/C = \int_X [C(MX,C), H(XC')]$$

$$C \backslash C' = \int_X [C(MX,C), K(XC')].$$

In Chapter 5 we establish that the satisfaction of condition (***), together with the existence of the above coends and ends, is sufficient for the existence of a biclosed structure on C having $\hat{\otimes}$ for tensor product, $\hat{I}$ for identity object, and / and \ for internal homs. Furthermore, when A is monoidal, this biclosed structure on C is characterised, uniquely to within isomorphism, by the existence of a monoidal enrichment $(\phi, \tilde{\phi}, \phi^o)$ of the functor $\phi = M^{op} : A \to C^{op}$ for which both $\tilde{\phi}$ and ϕ^o are isomorphisms; an analogous result is true for A an arbitrary promonoidal category. Conversely, given a biclosed structure on C, together with a suitable promonoidal enrichment of $M^{op} : A \to C^{op}$, the required coends and ends exist in C and condition (***) is satisfied. These results are collected to form the general construction theorem of Section 5.3.

The already mentioned biclosed structure of a total functor category $[A,V]$ and that of a reflective subcategory $C \subset [A,V]$ may be recovered from the construction theorem by equating M to $L : A^{op} \to [A,V]$ and to $\psi : [A,V] \to C$ respectively. An application lying outside the scope of these two special cases is outlined in Section 5.4 where we take M to be the inclusion $V_{\mathbb{T}} \subset V^{\mathbb{T}}$ of the category $V_{\mathbb{T}}$ of free algebras into the category $V^{\mathbb{T}}$ of algebras over a "commutative" monad $\mathbb{T}$ on V. Here the commutativity of $\mathbb{T}$ provides a canonical monoidal structure on the large category $A = V_{\mathbb{T}}{}^{op}$. The resulting closed structure on $V^{\mathbb{T}}$ is the one we would normally obtain if we took, say, V to be S and $\mathbb{T}$ to be the abelian group monad on S.

CHAPTER 1

PRELIMINARIES

Again we emphasise that, unless otherwise indicated, concepts are relative to the given normalised closed category $\mathcal{V}$. For each category $\mathcal{A}$, the underlying $\mathcal{S}$-category of $\mathcal{A}$ is denoted by $\mathcal{A}_0$, as in [9].

Section 1.1. Completeness concepts

We recall the basic aspects of completeness for $\mathcal{V}$-categories.

Definition 1.1.1 An end in $\mathcal{B}$ of a functor $T : A^{op} \otimes A \to \mathcal{B}$ is a natural family $\alpha_A : B \to T(AA)$ in $\mathcal{B}_0$ having the property that for each $B' \in \mathcal{B}$, any natural family $\beta_A : X \to \mathcal{B}(B',T(AA))$ in $\mathcal{V}_0$ admits a unique factorisation of the form

$$\begin{tikzcd}
X \arrow[rr, "\beta"] \arrow[dr, dashed] & & \mathcal{B}(B',T(AA)) \\
& \mathcal{B}(B'B) \arrow[ur, "\mathcal{B}(1,\alpha)"'] &
\end{tikzcd}$$

It is clear from this definition that an end in $\mathcal{V}$ of a functor $T : A^{op} \otimes A \to \mathcal{V}$ is simply a "universally natural" family $\alpha_A : X \to T(AA)$. Thus, by definition, representable functors preserve ends. The concept dual to end is called coend.

Whenever the end of $T : A^{op} \otimes A \to B$ exists, it is clearly unique to within a unique isomorphism, and consequently is usually referred to as "the" end of T in B. When defining a functor, say, using ends, we shall presume that a definite choice has been made of them; we often adopt the notation $s_A : \int_A T(AA) \to T(AA)$ for end, and $s^A : T(AA) \to \int^A T(AA)$ for coend, regarding $\int_\cdot$ and $\int^\cdot$ as well defined operations.

Definition 1.1.2 Let $T : A \to B$ be a functor. We say that B is T-tensored if the left represented functor $LTA = B(TA,-) : B \to V$ has a left adjoint, denoted $-\otimes TA : V \to B$, for each $A \in A$. We say that B is tensored if it is 1_B-tensored.

We note that, by [11] §3.5, the adjunction

$$p : B(X\otimes TA,B) \cong [X,B(TA,B)] \qquad (1.1.1)$$

endows $-\otimes T-$ with a canonical bifunctor structure $Ten_T : V\otimes A \to B$. Dually, B is T-cotensored if each functor $RTA = B(-,TA) : B^{op} \to V$ has a left adjoint. The dual of this adjoint is usually denoted $[-,TA] : V^{op} \to B$.

The existence of ends and of cotensoring in a particular category B are completeness properties of B. In addition, if K is an S-category then the (inverse) limit $\alpha_K : B \to SK$ in B_0 of an S-functor $S : K \to B_0$ is called the <u>$\mathcal{V}$-limit</u> of S in $\mathcal{B}$ if $\mathcal{B}(1,\alpha_K) : \mathcal{B}(B'B) \to \mathcal{B}(B',SK)$ is a limit of $\mathcal{B}(B',S-)$ in $\mathcal{V}_0$ for all $B' \in \mathcal{B}$.

In practice, these completeness concepts may overlap considerably. First, if the normalisation $V : \mathcal{V} \to \mathcal{S}$ admits the closed left adjoint $F : \mathcal{S} \to \mathcal{V}$ (see Section 0.2) then the $\mathcal{V}$-limit of $S : \mathcal{K} \to \mathcal{B}_0$ coincides with the <u>end</u> of the (canonical) composite

$$F_*\mathcal{K}^{op} \otimes F_*\mathcal{K} \xrightarrow{F_{**}P} F_*\mathcal{K} \xrightarrow{\bar{S}} \mathcal{B}$$

where $\bar{S}$ is the $\mathcal{V}$-functor lifting $S : \mathcal{K} \to \mathcal{B}_0$ and $P : \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}$ is projection onto the second factor (in other words, the first variable in this end is "dead"). Furthermore, when $\mathcal{V} = \mathcal{S}$, a category $\mathcal{B}$ is <u>T-cotensored</u> for a functor $T : \mathcal{A} \to \mathcal{B}$ precisely when it admits all <u>products</u> of the form $\prod_X TA$ where $A \in \mathcal{A}$ and $X \in \mathcal{S}$.

Conversely, it has been shown by G.M. Kelly (see [3] and [11]) that an <u>end</u> in a $\mathcal{V}$-category can be constructed as the <u>$\mathcal{V}$-limit</u> of a certain diagram involving <u>cotensor products</u>. Briefly, let $T : A^{op} \otimes A \to \mathcal{B}$ be a functor into a cotensored category $\mathcal{B}$. Then the diagram

$$\begin{tikzcd}
\mathcal{A}(AA') \arrow[rr, "T(A-)"] \arrow[d, "T(-A')"'] & & \mathcal{B}(T(AA),T(AA')) \arrow[d, "\mathcal{B}(\alpha_A,1)"] \\
\mathcal{B}(T(A'A'),T(AA')) \arrow[rr, "\mathcal{B}(\alpha_{A'},1)"'] & & \mathcal{B}(B,T(AA'))
\end{tikzcd}$$

expressing the naturality of a family $\alpha_A : B \to T(AA)$, transforms under the cotensor adjunction

$$\sigma : \mathcal{V}_0(X,\mathcal{B}(BB')) \cong \mathcal{B}_0(B,[XB'])$$

into the diagram

$$\begin{tikzcd}
B \arrow[r, "\alpha_A"] \arrow[d, "\alpha_{A'}"'] & T(AA) \arrow[d, "\sigma(T(A-))"] \\
T(A'A') \arrow[r, "\sigma(T(-A'))"'] & {[\mathcal{A}(AA'),T(AA')]}
\end{tikzcd}$$

in $\mathcal{B}_0$. Hence the end of T can be obtained as the $\mathcal{V}$-limit in $\mathcal{B}$ of a connected diagram of the form

$$\begin{tikzcd}
 & \cdots \\
T(AA) \arrow[ur] \arrow[dr] & \\
 & {[\mathcal{A}(AA'),T(AA')]} \\
T(A'A') \arrow[ur] \arrow[dr] & \\
 & {[\mathcal{A}(A'A''),T(A'A'')]} \\
T(A''A'') \arrow[ur] \arrow[dr] & \\
 & \cdots
\end{tikzcd}$$

.

This construction shows, in particular, that small <u>ends exist</u> in any V-category that is cotensored and admits small V-limits.

The possibility of completing a category with respect to these concepts is discussed by E. Dubuc in [7].

We recall, also from [3] and [11], that a functor admitting a left adjoint preserves any ends and cotensor products which happen to exist in its domain.

Section 1.2. Functor categories

A primary use of ends is in the construction of functor categories relative to $\mathcal{V}$.

Suppose that $\mathcal{A}$ and $\mathcal{B}$ are categories with the property that the end

$$E^A{}_{ST} = s_A : \int_A \mathcal{B}(SA,TA) \to \mathcal{B}(SA,TA) \qquad (1.2.1)$$

exists in $\mathcal{V}$ for each pair of functors $S,T : \mathcal{A} \to \mathcal{B}$. Then, as verified in [3], there exists an essentially unique category $[\mathcal{A},\mathcal{B}]$ whose objects are the functors $S,T,\ldots : \mathcal{A} \to \mathcal{B}$, and whose hom-objects are given by

$$[\mathcal{A},\mathcal{B}](S,T) = \int_A \mathcal{B}(SA,TA).$$

Thus, by construction, we obtain an evaluation functor $E^A : [\mathcal{A},\mathcal{B}] \to \mathcal{B}$ for each $A \in \mathcal{A}$, given by $E^A S = SA$ and $E^A{}_{ST}$ is (1.2.1).

An element α of the set $[\mathcal{A},\mathcal{B}]_0(S,T) = V\int_A \mathcal{B}(SA,TA)$ is seen to correspond, via the projections $Vs_A : V\int_A \mathcal{B}(SA,TA) \to V\mathcal{B}(SA,TA) = \mathcal{B}_0(SA,TA)$, to a natural family of morphisms $\alpha_A : SA \to TA$ in the sense of [9]. Hence the underlying $\mathcal{S}$-category $[\mathcal{A},\mathcal{B}]_0$ is precisely the $\mathcal{S}$-category of all functors from $\mathcal{A}$ to $\mathcal{B}$ and natural transformations between them.

Many properties of the codomain $\mathcal{B}$ carry over to the functor category $[\mathcal{A},\mathcal{B}]$. In particular, ends and cotensoring in $[\mathcal{A},\mathcal{B}]$ are always computed evaluationwise, so that any choice of these made in B fixes a choice in $[A,B]$.

Section 1.3. Lemmas on induced naturality

In this section we record some of the "computational" aspects of ends and coends. It is assumed that the reader is familiar with the rules governing the composition of natural transformations (as generalised in Eilenberg-Kelly [8]). The results are stated in terms of coends because they will be used chiefly in this form.

Lemma 1.3.1 Let $T : \mathcal{A}^{op} \otimes \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$ be a functor and let $\alpha_{AB} : T(AAB) \to SB$ be a coend over A for each $B \in \mathcal{B}$. Then there exists a unique functor $S : \mathcal{B} \to \mathcal{C}$ making the family α_{AB} natural in B.

Proof For each pair $B, B' \in \mathcal{B}$ consider the diagram

$$\begin{tikzcd}
\mathcal{B}(BB') \arrow[rr, dashed, "S_{BB'}"] \arrow[d, "T(AA-)_{BB'}"'] & & \mathcal{C}(SB,SB') \arrow[d, "\mathcal{C}(\alpha,1)"] \\
\mathcal{C}(T(AAB),T(AAB')) \arrow[rr, "\mathcal{C}(1,\alpha)"'] & & \mathcal{C}(T(AAB),SB')
\end{tikzcd} .$$

Because $\mathcal{C}(\alpha,1)$ is an end and $\mathcal{C}(1,\alpha).T(AA-)_{BB'}$ is natural in A, we can define $S_{BB'}$ to be the unique morphism making this diagram commute. The functor axioms VF1' and VF2' of [9] are easily verified for this definition of S by using the fact that $\mathcal{C}(\alpha,1)$ is an end. S is then the unique functor making α_{AB} natural in B.

In a typical calculation with ends we have to determine whether naturality in any extra variables survives the various factorisations. All our requirements may be extracted from the following lemma.

Lemma 1.3.2 Let $T : \mathcal{A}^{op} \otimes \mathcal{A} \otimes \mathcal{B}^{op} \otimes \mathcal{B} \to \mathcal{V}$ and $S : \mathcal{B}^{op} \otimes \mathcal{B} \to \mathcal{V}$ be functors, let $\alpha_{ABB'} : S(BB') \to T(AABB')$ be an end over A, natural in B and B', and let $\beta_{AB} : I \to T(AABB)$ be natural in A and B. Then the induced family $\gamma_B : I \to S(BB)$ is natural in B.

Proof For each pair $B, B' \in \mathcal{B}$, consider the diagram

$$\begin{tikzcd}
 & {[T(AABB),T(AABB')]} \arrow[drr, "{[\alpha,1]}"] & & & \\
 & {[S(BB),S(BB')]} \arrow[u, phantom, "2"] \arrow[rr, "{[1,\alpha]}"] \arrow[dr, "{[\gamma,1]}"] & & {[S(BB),T(AABB')]} \arrow[dr, "{[\gamma,1]}"] \arrow[d, phantom, "3"] & \\
\mathcal{B}(BB') \arrow[uur, "T(AAB-)"] \arrow[ur, "S(B-)"] \arrow[dr, "S(-B')"'] \arrow[ddr, "T(AA-B')"'] \arrow[rr, phantom, "1"] & & {[I,S(BB')]} \arrow[rr, "{[1,\alpha]}"] & & {[I,T(AABB')]} \\
 & {[S(B'B'),S(BB')]} \arrow[ur, "{[\gamma,1]}"'] \arrow[rr, "{[1,\alpha]}"'] \arrow[d, phantom, "5"] & & {[S(B'B'),T(AABB')]} \arrow[ur, "{[\gamma,1]}"'] \arrow[u, phantom, "4"] & \\
 & {[T(AAB'B'),T(AABB')]} \arrow[urr, "{[\alpha,1]}"'] & & &
\end{tikzcd}$$

.

Regions 3 and 4 commute trivially. Regions 5 and 2 commute by the naturality of $\alpha_{ABB'}$ in B and B' respectively. The exterior commutes by naturality of $\beta_{AB} = \alpha_{ABB} \cdot \gamma_B$ in B. Then, because $[1,\alpha]$ is an end, region 1 commutes as required.

<u>Lemma 1.3.3</u> Let $T : \mathcal{A}^{op} \otimes \mathcal{A} \otimes \mathcal{B}^{op} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathcal{E}$, $S : \mathcal{B}^{op} \otimes \mathcal{B} \otimes \mathcal{C} \to \mathcal{E}$, and $R : \mathcal{D}^{op} \otimes \mathcal{D} \otimes \mathcal{C} \to \mathcal{E}$ be functors, let $\alpha_{ABB'C} : T(AABB'C) \to S(BB'C)$ be a coend over A, natural in B, B', and C, and let $\beta_{ABCD} : T(AABBC) \to R(DDC)$ be natural in A, B, C, and D. Then the induced family $\gamma_{BCD} : S(BBC) \to R(DDC)$ is natural in B, C, and D.

<u>Proof</u> This is a straightforward consequence of the preceding Lemma 1.3.2 combined with [9] Lemma III.7.8; the latter result states that the three types of naturality may be expressed in terms of one, namely, the $I \to T(AA)$ form in $\mathcal{V}$.

The next lemma asserts that coends are preserved by coends and tensor products. The verifications are straightforward and shall be omitted.

<u>Lemma 1.3.4</u>

a) Let $T : \mathcal{A}^{op} \otimes \mathcal{A} \otimes \mathcal{B}^{op} \otimes \mathcal{B} \to \mathcal{C}$ and $S : \mathcal{B}^{op} \otimes \mathcal{B} \to \mathcal{C}$ be functors, let $\alpha_{ABB'} : T(AABB') \to S(BB')$ be a coend over A, natural in B and B', and let $\beta_{AB} : T(AABB) \to C$ be natural in A and B. Then the induced family $\gamma_B : S(BB) \to C$ is a coend over B if and only if β_{AB} is a coend over A and B.

b) Let $T : A^{op} \otimes A \to \mathcal{C}$ be a functor into a tensored category $\mathcal{C}$, and let $\alpha_A : T(AA) \to C$ be natural in A. Then α_A is a coend over A if and only if $1 \otimes \alpha_A : X \otimes T(AA) \to X \otimes C$ is a coend over A for all $X \in \mathcal{V}$.

In some circumstances it is desirable to use a simplified notation for coends. Let $\mathcal{A}$ be a category and let $T(AA-)$ be a functor into $\mathcal{V}$, whose coend $s^A : T(AA-) \to \int^A T(AA-)$ over $A \in \mathcal{A}$ exists for all values of the extra variables "-". Then, if $T(AA-) = S(A-) \otimes R(A-)$ for functors S and R into $\mathcal{V}$ (with different variances in A), we frequently abbreviate the notation to $s^A : S(A-) \otimes R(A-) \to S(A-) \underline{\otimes} R(A-)$, leaving the repeated dummy variable A to indicate the domain of integration. By Lemma 1.3.1, $S(A-) \underline{\otimes} R(A-)$ is (canonically) functorial in its extra variables.

The following considerations are introduced in order to handle expressions formed entirely by the repeated use of $\underline{\otimes}$. To each expression $\underline{N}$ which is formed by one or more uses of $\underline{\otimes}$, there corresponds an expression

N in which each $\underline{\otimes}$ is replaced by $\otimes$, the dummy variables in $\underline{N}$ becoming repeated variables in N; for example, if $\underline{N}$ is $(RA\underline{\otimes}S(AB))\underline{\otimes}T(BC)$ for functors $R : \mathcal{A} \to \mathcal{V}$, $S : \mathcal{A}^{op}\otimes\mathcal{B} \to \mathcal{V}$, and $T : \mathcal{B}^{op}\otimes\mathcal{C} \to \mathcal{V}$, then N is $(RA\otimes S(AB))\otimes T(BC)$. Furthermore, there is a canonical natural transformation $q = q_N : N \to \underline{N}$ defined, as follows, by induction on the number of occurrences of $\underline{\otimes}$ in $\underline{N}$: If $\underline{N}$ contains no occurrence of $\underline{\otimes}$ then $N = \underline{N}$ and $q_N = 1$; otherwise $\underline{N} = \underline{N}'\underline{\otimes}\underline{N}''$ and q_N is the composite

$$N'\otimes N'' \xrightarrow[q'\otimes q'']{} \underline{N}'\otimes\underline{N}'' \xrightarrow[s]{} \underline{N}'\underline{\otimes}\underline{N}''.$$

In the above example, q would be the composite

$$(RA\otimes S(AB))\otimes T(BC) \xrightarrow[s\otimes 1]{} (RA\underline{\otimes}S(AB))\otimes T(BC) \xrightarrow[s]{} (RA\underline{\otimes}S(AB))\underline{\otimes}T(BC)$$

and this is natural in A, B, and C; we say that the variables A and B are "summed out" by q.

In fact the path $q_N : N \to \underline{N}$ is a multiple coend over all those variables in N which are summed out by q_N.

Lemma 1.3.5 Let T be a functor into $\mathcal{V}$ and let $f : N \to T$ be a natural transformation which is, in particular, natural in all the repeated variables in N which are summed out by $q_N : N \to \underline{N}$. Then f factors as $g.q_N$ for a unique natural transformation $g : \underline{N} \to T$.

<u>Proof</u> By induction on the number of occurrences of $\otimes$ in N. If $\underline{\otimes}$ does not occur in $\underline{N}$ the result is trivial. Otherwise $\underline{N} = \underline{N}'\underline{\otimes}\underline{N}''$ and we can, using Lemma 1.3.4, factor f in three steps:

$$\begin{tikzcd}
N = N'\otimes N'' \arrow[r, "q'\otimes 1"] \arrow[dr, "f"'] & \underline{N}'\otimes N'' \arrow[r, "1\otimes q''"] \arrow[d, "f'"] & \underline{N}'\otimes \underline{N}'' \arrow[r, "s"] \arrow[dl, "f''"'] & \underline{N}'\underline{\otimes N''} = \underline{N} \arrow[dll, "g"] \\
 & T & &
\end{tikzcd}$$

The naturality of g follows from Lemma 1.3.3.

When the transformation f in the preceding Lemma 1.3.5 is of the form $q'.n$ for a path $q' : N' \to \underline{N}'$, the induced $g : \underline{N} \to \underline{N}'$ is, for obvious reasons, denoted by $\underline{n}$. Induced transformations of this form are an essential feature of the definition of promonoidal category; we make two important observations in this regard.

First, if $n : N \to N'$ is a natural isomorphism constructed from the coherent data isomorphisms a, r, ℓ, c of $\mathcal{V}$ then $\underline{n} : \underline{N} \to \underline{N}'$ is a natural isomorphism and is called an <u>induced coherence isomorphism</u>. In view of the uniqueness assertion of Lemma 1.3.5, and the original coherence of a, r, ℓ, c, it is clear that induced coherence isomorphisms are coherent. In other words, the induced coherence isomorphism $\underline{n} : \underline{N} \to \underline{N}'$ is completely determined by the positions of $\underline{\otimes}$ in the expressions $\underline{N}$ and $\underline{N}'$; consequently, such isomorphisms need not be labelled.

Secondly, when $n = h\otimes k : S(A-)\otimes R(A-) \to S'(A-)\otimes R'(A-)$ for natural transformations $h : S \to S'$ and $k : R \to R'$, we write $h\underline{\otimes}k$ for $\underline{h\otimes k}$. This not only makes the symbol $\underline{\otimes}$ $\mathcal{S}$-functorial insofar as it is defined, but also makes the coend $s^A : S(A-)\otimes R(A-) \to S(A-)\underline{\otimes}R(A-)$ $\mathcal{S}$-natural in S and R.

Section 1.4 The representation theorem

Let $M : \mathcal{A} \to \mathcal{B}$ and $T : \mathcal{B} \to \mathcal{C}$ be functors. If $\mathcal{C}$ is TM-tensored then

$$T_{MA,B} : \mathcal{B}(MA,B) \to \mathcal{C}(TMA,TB)$$

transforms, under the tensoring adjunction isomorphism (1.1.1), to a natural transformation

$$\zeta = \zeta_{T,B}^{M,A} : \mathcal{B}(MA,B)\otimes TMA \to TB.$$

When the coend over A of $\mathcal{B}(MA,B)\otimes TMA$ exists in $\mathcal{B}$, there results an induced morphism

$$z = z_{T,B}^{M} : \int^{A}\mathcal{B}(MA,B)\otimes TMA \to TB \qquad (1.4.1)$$

which is natural in $B \in \mathcal{B}$ by Lemma 1.3.3. The letter z will be reserved for this morphism.

For this section we consider the case where M is the identity functor $1 : \mathcal{B} \to \mathcal{B}$.

<u>Theorem</u> (<u>the higher representation theorem</u>) If $T : \mathcal{B} \to \mathcal{C}$ is a functor into a T-tensored category $\mathcal{C}$ then the transformation

$$\zeta_B : \mathcal{B}(BB')\otimes TB \to TB',$$

obtained by adjunction from $T_{BB'} : \mathcal{B}(BB') \to \mathcal{C}(TB,TB')$, is a coend over B.

<u>Proof</u> This is exactly as in [3] §3.5 where the codomain was assumed to be tensored. For each $C \in \mathcal{C}$, the transformation

$$\mathcal{C}(\zeta,1) : \mathcal{C}(TB',C) \to \mathcal{C}(\mathcal{B}(BB')\otimes TB,C)$$

is an end over $B \in \mathcal{B}$. This follows from the correspondence of transformations

$$\alpha_B : X \to \mathcal{C}(\mathcal{B}(BB')\otimes TB,C)$$

to transformations

$$\beta_B : X \to [\mathcal{B}(BB'),\mathcal{C}(TB,C)],$$

by the tensor adjunction, to transformations

$$\gamma_B : \mathcal{B}(BB') \to [X,\mathcal{C}(TB,C)],$$

by symmetry in $\mathcal{V}$, to morphisms

$$f : X \to \mathcal{C}(TB',C),$$

by the representation theorem (below).

In our notation, the higher representation theorem states that

$$y = y_{T,B'} = z_{T,B'}^{1} : \int^{B}\mathcal{B}(BB')\otimes TB \to TB' \qquad (1.4.2)$$

is an isomorphism which we call the <u>Yoneda isomorphism</u>. The letter y will be reserved for this isomorphism.

<u>Theorem</u> (<u>the representation theorem</u>) Let $T : \mathcal{B} \to \mathcal{V}$ be a functor and let $B \in \mathcal{B}$ and $X \in \mathcal{V}$. Then there is a bijection b between the class of natural transformations $\alpha : \mathcal{B}(B-)\otimes X \to T$ and the elements $f \in \mathcal{V}_0(X,TB)$.

This result is established by Eilenberg and Kelly in [9] Proposition II.7.4. The bijection b is given by

$$b(\alpha) = X \xrightarrow{\ell^{-1}} I\otimes X \xrightarrow{j\otimes 1} \mathcal{B}(BB)\otimes X \xrightarrow{\alpha_B} TB,$$

$$b^{-1}(f) = \mathcal{B}(B-)\otimes X \xrightarrow{T\otimes 1} [TB,T-]\otimes X \xrightarrow{[f,1]\otimes 1} [X,T-]\otimes X \xrightarrow{e} T,$$

where e is "evaluation" in $\mathcal{V}$, that is, the transform of $1 : [TB,T-] \to [TB,T-]$ under the tensoring adjunction for $\mathcal{V}$. We shall refer to b as the <u>Yoneda correspondence</u>.

Section 1.5 Dense functors and strongly generating classes

<u>Definition 1.5.1</u> A functor $M : \mathcal{A} \to \mathcal{B}$ is <u>dense</u> if the natural transformation

$$LMA_{BB'} : \mathcal{B}(BB') \to [\mathcal{B}(MA,B),\mathcal{B}(MA,B')] \tag{1.5.1}$$

is an end over A for all $B,B' \in \mathcal{B}$.

The terminology (introduced by Ulmer [17] for $\mathcal{V}$ = abelian groups) is best explained by the case where $\mathcal{B}$ is M-tensored.

<u>Lemma 1.5.2</u> Let $M : \mathcal{A} \to \mathcal{B}$ be a functor into an M-tensored category $\mathcal{B}$. Then M is dense if and only if the transformation

$$\zeta : \mathcal{B}(MA,B)\otimes MA \to B, \tag{1.5.2}$$

obtained by adjunction from $1 : \mathcal{B}(MA,B) \to \mathcal{B}(MA,B)$, is a coend over A for each $B \in \mathcal{B}$.

<u>Proof</u> On applying the representation theorem, the diagram

$$\begin{tikzcd}
\mathcal{B}(BB') \arrow[r, "LMA"] \arrow[dr, "\mathcal{B}(\zeta,1)"'] & {[\mathcal{B}(MA,B),\mathcal{B}(MA,B')]} \\
& \mathcal{B}(\mathcal{B}(MA,B)\otimes MA,B') \arrow[u, "p", "\cong"']
\end{tikzcd}$$

is seen to commute for all $B,B' \in \mathcal{B}$. Then the result follows from the Definition 1.5.1 of "dense" and the definition of "coend".

To say that the transformation (1.5.2) is a coend is, of course, equivalent to saying that $z : \int^A (MA,B)\otimes MA \to B$ is an isomorphism.

A functor $T : \mathcal{B} \to \mathcal{C}$ is called a full embedding if $T_{BB'} : \mathcal{B}(BB') \to \mathcal{C}(TB,TB')$ is an isomorphism for all $B,B' \in \mathcal{B}$. An important instance of a dense functor is the Yoneda (full) embedding:

$$L : \mathcal{A}^{op} \to [\mathcal{A},\mathcal{V}], \quad A \rightsquigarrow LA.$$

Here we have

$$z : \int^A [\mathcal{A},\mathcal{V}](LA,T)\otimes LA \xrightarrow{\cong} \int^A TA\otimes LA \xrightarrow{\cong} T$$

by the opposite forms of the higher representation theorem.

The left adjoint to a full embedding provides another well-known example of a dense functor. Briefly, if a full embedding $T : \mathcal{B} \to \mathcal{C}$ has a left adjoint $S : \mathcal{C} \to \mathcal{B}$ then the adjunction counit $\varepsilon : ST \to 1$ is an isomorphism, whence the composite

$$\begin{aligned} \mathcal{B}(BB') &\cong \mathcal{B}(STB,B') \\ &\cong \int_C [\mathcal{B}(C,TB),\mathcal{B}(SC,B')] \text{ by the higher repn. thm.,} \\ &\cong \int_C [\mathcal{B}(SC,B),\mathcal{B}(SC,B')] \text{ by the adjunction,} \end{aligned}$$

is an isomorphism; by using the representation theorem, this composite is easily seen to be induced by LSC.

Returning to the Definition 1.5.1, the functor $M : \mathcal{A} \to \mathcal{B}$ is dense precisely when the resulting functor

$$\mathcal{B} \to [\mathcal{A}^{op},\mathcal{V}], \quad B \rightsquigarrow \mathcal{B}(M-,B)$$

is a full embedding. This makes sense even when the functor category $[\mathcal{A}^{op},\mathcal{V}]$ does not exist. At the underlying-sets level, we have:

<u>Lemma 1.5.3</u> If $M : \mathcal{A} \to \mathcal{B}$ is dense then each natural transformation $\alpha_A : \mathcal{B}(MA,B) \to \mathcal{B}(MA,B')$ is of the form $\mathcal{B}(1,f)$ for a unique $f \in \mathcal{B}_0(BB')$.

<u>Proof</u> This is the result of applying $V : \mathcal{V} \to \mathcal{S}$ to the end (1.5.1) in $\mathcal{V}$.

Thus, on taking $\mathcal{V} = \mathcal{S}$, the concept of dense functor is seen to be equivalent to the original idea of "adequate functor" introduced by Isbell [10].

<u>Lemma 1.5.4</u> Suppose we have functors

$$\begin{tikzcd} \mathcal{A} \arrow[r, "M"] & \mathcal{B} \arrow[r, shift left, "S"] \arrow[r, shift right, "T"'] & \mathcal{C} \arrow[l, bend right=60, "R"'] \end{tikzcd}$$

where M is dense and R is right adjoint to S. Then each natural transformation $\alpha : SM \to TM : \mathcal{A} \to \mathcal{C}$ admits a unique extension to a natural transformation $\bar{\alpha} : S \to T : \mathcal{B} \to \mathcal{C}$.

<u>Proof</u> Let $\eta : 1 \to RS : \mathcal{B} \to \mathcal{B}$ be the unit of the adjunction. Then the mapping which sends a natural transformation $\beta = \beta_B : SB \to TB$ to the composite natural transformation

$$B \xrightarrow{\eta_B} RSB \xrightarrow{R\beta_B} RTB,$$

is a bijection. Furthermore, by Lemma 1.5.3 and the density of M, there is a bijection between transformations $\gamma = \gamma_B : B \to RTB$ and transformations $\delta_{AB} : \mathcal{B}(MA,B) \to \mathcal{B}(MA,RTB)$. Hence, given a natural transformation $\alpha : SM \to TM : \mathcal{A} \to \mathcal{C}$, we define $\bar{\alpha} : S \to T : \mathcal{B} \to \mathcal{C}$ to be the unique natural transformation making the following diagram commute:

$$\begin{tikzcd}
\mathcal{B}(MA,B) \arrow[r, "\mathcal{B}(1{,}\eta_B)"] \arrow[d, "RT_{MA,B}"'] & \mathcal{B}(MA,RSB) \arrow[r, "\mathcal{B}(1{,}R\bar{\alpha}_B)"] & \mathcal{B}(MA,RTB) \\
\mathcal{B}(RTMA,RTB) \arrow[rr, "\mathcal{B}(R\alpha_A{,}1)"'] & & \mathcal{B}(RSMA,RTB). \arrow[u, "\mathcal{B}(\eta_{MA}{,}1)"']
\end{tikzcd}$$

On applying $V : \mathcal{V} \to \mathcal{S}$ to the diagram, putting $B = MA$, and evaluating both legs at $1_{MA} \in \mathcal{B}_0(MA,MA)$, we obtain

$$\alpha = \bar{\alpha}M$$

as required.

<u>Definition 1.5.5</u> (Kelly) A class $\mathcal{A}$ of objects in a category $\mathcal{B}$ is <u>strongly generating</u> if $f \in \mathcal{B}_0(BB')$ is an isomorphism whenever $\mathcal{B}(1,f) : \mathcal{B}(AB) \to \mathcal{B}(AB')$ is an isomorphism in $\mathcal{V}_0$ for all $A \in \mathcal{A}$.

This concept is closely related to that of a dense functor. If $M : \mathcal{A} \to \mathcal{B}$ is a dense functor then the class $\{MA; A \in \mathcal{A}\}$ is strongly generating in $\mathcal{B}$ by Lemma 1.5.3. In the other direction we have:

<u>Proposition 1.5.6</u> If $M : \mathcal{A} \subset \mathcal{B}$ is the inclusion of a strongly generating class $\mathcal{A}$ into an M-tensored category $\mathcal{B}$, and $\int^{A'} \mathcal{B}(MA',B) \otimes MA'$ exists in $\mathcal{B}$ and is preserved by $\mathcal{B}(MA-)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then M is dense.

<u>Proof</u> For each $A \in \mathcal{A}$ and $B \in \mathcal{B}$, consider the diagram

$$\begin{tikzcd}
\int^{A'} \mathcal{B}(MA',B) \otimes \mathcal{A}(AA') \arrow[r, "\int 1 \otimes M", "\cong"'] \arrow[d, phantom, "\cong"] & \int^{A'} \mathcal{B}(MA',B) \otimes \mathcal{B}(MA,MA') \arrow[dd, "\kappa", "\cong"'] \\
\int^{A'} \mathcal{A}(AA') \otimes \mathcal{B}(MA',B) \arrow[d, "y"', "\cong"] & \\
\mathcal{B}(MA,B) & \mathcal{B}(MA, \int^{A'} \mathcal{B}(MA',B) \otimes MA') \arrow[l, "\mathcal{B}(1,z)"]
\end{tikzcd}$$

in which κ is the isomorphism asserting that $\mathcal{B}(MA,-)$ preserves the given coend. This diagram is verified to commute by applying the representation theorem to B.

Thus $\mathcal{B}(1,z)$ is an isomorphism. Hence, because $\mathcal{A}\subset\mathcal{B}$ is strongly generating, z is an isomorphism as required in Lemma 1.5.2 for M : $\mathcal{A}\subset\mathcal{B}$ to be dense.

CHAPTER 2

PROMONOIDAL STRUCTURES

Section 2.1 Promonoidal categories

The concept of a promonoidal category shall be introduced by considering the outcome of restricting a monoidal structure on a category $\mathcal{B}$ to a full subcategory $\mathcal{A}$ of $\mathcal{B}$. In brief, we seek sufficient conditions on such an embedding in order for the resulting structure on $\mathcal{A}$ to admit a formulation which makes no explicit reference to the monoidal structure on $\mathcal{B}$.

First, let $\mathcal{B}$ be an arbitrary category. Then each choice of a functor $\bar{\otimes} : \mathcal{B}\otimes\mathcal{B} \to \mathcal{B}$, together with an object $\bar{I}\in\mathcal{B}$, provides canonical functors

$$\bar{P} : \mathcal{B}^{op}\otimes\mathcal{B}^{op}\otimes\mathcal{B} \to \mathcal{V}$$

$$\bar{J} : \mathcal{B} \to \mathcal{V}$$

where $\bar{P}(BB'B") = \mathcal{B}(B\otimes B',B")$ and $\bar{J}B = \mathcal{B}(IB)$; when there is no danger of confusion we denote $\bar{\otimes}$ and $\bar{I}$ by $\otimes$ and I respectively. Next, let $M : \mathcal{A} \to \mathcal{B}$ be an arbitrary functor into $\mathcal{B}$. This enables us to define, by "restriction", functors

$$P : \mathcal{A}^{op}\otimes\mathcal{A}^{op}\otimes\mathcal{A} \xrightarrow{M^{op}\otimes M^{op}\otimes M} \mathcal{B}^{op}\otimes\mathcal{B}^{op}\otimes\mathcal{B} \xrightarrow{\bar{P}} \mathcal{V}$$

$$J : \mathcal{A} \xrightarrow{M} \mathcal{B} \xrightarrow{\bar{J}} \mathcal{V}.$$

Now consider the <u>exteriors</u> of the following three diagrams (where $z = z^{M^{op}}$):

$$\begin{tikzcd}
JX\underline{\otimes}P(XA-) \arrow[rr, "\lambda"] \arrow[d, equal] & & A(A-) \arrow[ddd, "M"] \\
\mathcal{B}(I,MX)\underline{\otimes}\mathcal{B}(MX\otimes MA,M-) \arrow[dr, dashed] \arrow[dd, "z_1"'] & & \\
& \mathcal{B}(IY)\underline{\otimes}\mathcal{B}(Y\otimes MA,M-) \arrow[dl, "y"'] \arrow[dr, "\overline{\lambda}"] & \\
\mathcal{B}(I\otimes MA,M-) & & \mathcal{B}(MA,M-) \arrow[ll, "\mathcal{B}(\overline{\ell},1)"]
\end{tikzcd} \tag{2.1.1}$$

$$\begin{tikzcd}
JX\underline{\otimes}P(AX-) \arrow[rr, "\rho"] \arrow[d, equal] & & A(A-) \arrow[ddd, "M"] \\
\mathcal{B}(I,MX)\underline{\otimes}\mathcal{B}(MA\otimes MX,M-) \arrow[dr, dashed] \arrow[dd, "z_2"'] & & \\
& \mathcal{B}(IY)\underline{\otimes}\mathcal{B}(MA\otimes Y,M-) \arrow[dl, "y"'] \arrow[dr, "\overline{\rho}"] & \\
\mathcal{B}(MA\otimes I,M-) & & \mathcal{B}(MA,M-) \arrow[ll, "\mathcal{B}(\overline{r},1)"]
\end{tikzcd} \tag{2.1.2}$$

$$\begin{tikzcd}[column sep=small]
& P(AA'X)\underline{\otimes}P(XA''-) \arrow[r, "\alpha"] \arrow[dl, equal] & P(A'A''X)\underline{\otimes}P(AX-) \arrow[dr, equal] & \\
\mathcal{B}(MA\otimes MA',MX)\underline{\otimes}\mathcal{B}(MX\otimes MA'',M-) \arrow[dr, dashed] \arrow[dd, "z_3"'] & & & \mathcal{B}(MA'\otimes MA'',MX)\underline{\otimes}\mathcal{B}(MA\otimes MX,M-) \arrow[dl, dashed] \arrow[dd, "z_4"] \\
& \mathcal{B}(MA\otimes MA',Y)\underline{\otimes}\mathcal{B}(Y\otimes MA'',M-) \arrow[r, "\bar{\alpha}"] \arrow[dl, "y"] & \mathcal{B}(MA'\otimes MA'',Y)\underline{\otimes}\mathcal{B}(MA\otimes Y,M-) \arrow[dr, "y"] & \\
\mathcal{B}((MA\otimes MA')\otimes MA'',M-) & & & \mathcal{B}(MA\otimes(MA'\otimes MA''),M-) \arrow[lll, "\mathcal{B}(\bar{a},1)"]
\end{tikzcd} \qquad (2.1.3)$$

If natural isomorphisms $\bar{\ell} : I\otimes B \cong B$, $\bar{r} : B\otimes I \cong B$, and $\bar{a} : (B\otimes B')\otimes B'' \cong B\otimes(B'\otimes B'')$ are provided for $\bar{\otimes}$, $\bar{I}$ then, in order that the above diagrams should define natural isomorphisms λ, ρ, and α respectively, it is clearly <u>sufficient</u> that $M : \mathcal{A} \to \mathcal{B}$ be a <u>full embedding</u> and that the transformations

$$z_1 : \mathcal{B}(I,MX)\underline{\otimes}\mathcal{B}(MX\otimes MA,M-) \to \mathcal{B}(I\otimes MA,M-)$$
$$z_2 : \mathcal{B}(I,MX)\underline{\otimes}\mathcal{B}(MA\otimes MX,M-) \to \mathcal{B}(MA\otimes I,M-)$$
$$z_3 : \mathcal{B}(MA\otimes MA',MX)\underline{\otimes}\mathcal{B}(MX\otimes MA'',M-) \to \mathcal{B}((MA\otimes MA')\otimes MA'',M-)$$
$$z_4 : \mathcal{B}(MA'\otimes MA'',MX)\underline{\otimes}\mathcal{B}(MA\otimes MX,M-) \to \mathcal{B}(MA\otimes(MA'\otimes MA''),M-)$$

be <u>isomorphisms</u> for all $A,A',A''\in\mathcal{A}$.

Assuming that these conditions are satisfied, consider the diagrams (2.1.4) and (2.1.5); in order to simplify these, we have suppressed the symbol M as much as possible and have assumed that the dummy variables X in each expression are paired off from the left unless otherwise indicated. If axiom MC1 holds for $(\bar{\ell},\bar{r},\bar{a})$ then the center region of (2.1.4) commutes. Assuming all the exhibited coends exist in this diagram, regions 1 and 4 commute by the lemma in Appendix 1; regions 2, 5, and 10 commute by the respective definitions of α, λ, and ρ; regions 3, 6, 7, 8, and 9 commute by the naturality of z, $\bar{a}$, and M. Similarly, if axiom MC2 holds for $\bar{a}$ then the center region of (2.1.5) commutes. Regions 1, 4, 7, 10, and 13 commute by the lemma in Appendix 1; regions 2, 5, 8, 11, and 14 commute by definition of α; regions 3, 6, 9, 12, and 15 commute by the naturality of z and $\bar{a}$. Thus, if axioms MC1 and MC2 hold for $(\bar{\ell},\bar{r},\bar{a})$ then the exteriors of (2.1.4) and (2.1.5) commute.

To summarise, we have that a full embedding $M : A \to B$ into a monoidal category B induces, whenever $z_1, \ldots, z_4$ are isomorphisms, a trace of B on A in the sense that the relationships among the P, J, λ, ρ, α are expressible in terms of A alone. This provokes the

Diagram (2.1.4)

$$
\begin{tikzcd}[column sep=small]
 & JX\underline{\otimes}(P(AXX)\underline{\otimes}P(XA'-)) \arrow[rrrr, "1\underline{\otimes}\alpha", "(2)"'] \arrow[dl, leftrightarrow, "\cong"'] \arrow[dr, "1\underline{\otimes}z"] & & & & JX\underline{\otimes}(P(XA'X)\underline{\otimes}P(AX-)) \arrow[dl, "1\underline{\otimes}z"'] \arrow[dr, leftrightarrow, "\cong"] & \\
 & & JX\underline{\otimes}\mathcal{B}((A\otimes X)\otimes A',-) \arrow[dd, "z"'] & & JX\underline{\otimes}\mathcal{B}(A\otimes(X\otimes A'),-) \arrow[ll, "1\underline{\otimes}\mathcal{B}(\overline{a},1)"'] \arrow[dd, "z"] & & \\
(JX\underline{\otimes}P(AXX))\underline{\otimes}P(XA'-) \arrow[dddd, "\rho\underline{\otimes}1"'] \arrow[ddr, "z\underline{\otimes}1"] & & & (3) & & & (JX\underline{\otimes}P(XA'X))\underline{\otimes}P(AX-) \arrow[ddl, "z\underline{\otimes}1"'] \arrow[dddd, "\lambda\underline{\otimes}1"] \\
 & (1) & \mathcal{B}((A\otimes I)\otimes A',-) & & \mathcal{B}(A\otimes(I\otimes A'),-) \arrow[ll, "\mathcal{B}(\overline{a},1)"'] & (4) & \\
 & \mathcal{B}(A\otimes I,X)\underline{\otimes}P(XA'-) \arrow[ur, "z"] & (9) & & (6) & \mathcal{B}(I\otimes A',X)\underline{\otimes}P(AX-) \arrow[ul, "z"'] & \\
 & \mathcal{B}(AX)\underline{\otimes}P(XA'-) \arrow[u, "\mathcal{B}(\overline{r},1)\underline{\otimes}1"', "(10)"] \arrow[rr, "z"'] & & \mathcal{B}(A\otimes A',-) \arrow[uul, "\mathcal{B}(\overline{r}\otimes 1,1)"] \arrow[uur, "\mathcal{B}(1\otimes\overline{\ell},1)"'] \arrow[dd, equal] & & \mathcal{B}(A'X)\underline{\otimes}P(AX-) \arrow[u, "\mathcal{B}(\overline{\ell},1)\underline{\otimes}1", "(5)"'] \arrow[ll, "z"] & \\
\mathcal{A}(AX)\underline{\otimes}P(XA'-) \arrow[ur, "M\underline{\otimes}1"'] \arrow[drrr, "y"'] & & (8) & & (7) & & \mathcal{A}(A'X)\underline{\otimes}P(AX-) \arrow[ul, "M\underline{\otimes}1"] \arrow[dlll, "y"] \\
 & & & P(AA'-) & & &
\end{tikzcd}
$$

Diagram (2.1.5)

$$
\begin{tikzcd}[column sep=tiny]
 & & P(AA'X)\underline{\otimes}(P(A''A'''X')\underline{\otimes}P(XX'-)) \arrow[rr, leftrightarrow, "\cong"] \arrow[d, "1\underline{\otimes}z"] & & P(A''A'''X')\underline{\otimes}(P(AA'X)\underline{\otimes}P(XX'-)) \arrow[d, "1\underline{\otimes}z"'] \arrow[ddrr, "1\underline{\otimes}\alpha"] & & \\
 & (2) & P(AA'X)\underline{\otimes}\mathcal{B}(X\otimes(A''\otimes A'''),-) \arrow[dddl, "1\underline{\otimes}\mathcal{B}(\bar{a},1)"'] \arrow[ddr, "z"] & (4) & P(A''A'''X')\underline{\otimes}\mathcal{B}((A\otimes A')\otimes X,-) \arrow[ddl, "z"'] & (5) & \\
P(AA'X)\underline{\otimes}(P(XA''X)\underline{\otimes}P(XA'''-)) \arrow[uurr, "1\underline{\otimes}\alpha"] \arrow[ddr, "1\underline{\otimes}z"] \arrow[ddd, leftrightarrow, "\cong"'] & & & & & & P(A''A'''X)\underline{\otimes}(P(A'XX)\underline{\otimes}P(AX-)) \arrow[ddl, "1\underline{\otimes}z"'] \arrow[ddd, leftrightarrow, "\cong"] \\
 & & & \mathcal{B}((A\otimes A')\otimes(A''\otimes A'''),-) \arrow[dddl, "\mathcal{B}(\bar{a},1)"] & & & \\
(1) & P(AA'X)\otimes\mathcal{B}((X\otimes A'')\otimes A''',-) \arrow[ddr, "z"] & & & & P(A''A'''X)\underline{\otimes}\mathcal{B}(A\otimes(A'\otimes X),-) \arrow[uuul, "1\underline{\otimes}\mathcal{B}(\bar{a},1)"'] \arrow[ddl, "z"'] & (7) \\
(P(AA'X)\underline{\otimes}P(XA''X))\underline{\otimes}P(XA'''-) \arrow[ddr, "z\underline{\otimes}1"] \arrow[ddddd, "\alpha\underline{\otimes}1"'] & & (3) & & (6) & & (P(A''A'''X)\underline{\otimes}P(A'XX))\underline{\otimes}P(AX-) \arrow[ddl, "z\underline{\otimes}1"'] \\
 & & \mathcal{B}(((A\otimes A')\otimes A'')\otimes A''',-) & & \mathcal{B}(A\otimes(A'\otimes(A''\otimes A''')),-) \arrow[uuul, "\mathcal{B}(\bar{a},1)"'] \arrow[ddd, "\mathcal{B}(1\otimes\bar{a},1)"'] & & \\
(14) & \mathcal{B}(((A\otimes A')\otimes A'',X)\underline{\otimes}P(XA'''-) \arrow[ur, "z"] & (15) & & (9) & \mathcal{B}(A'\otimes(A''\otimes A'''),X)\underline{\otimes}P(AX-) \arrow[ul, "z"'] \arrow[d, "\mathcal{B}(\bar{a},1)\underline{\otimes}1"] & (8) \\
 & \mathcal{B}(A\otimes(A'\otimes A''),X)\underline{\otimes}P(XA'''-) \arrow[u, "\mathcal{B}(\bar{a},1)\underline{\otimes}1"'] \arrow[dr, "z"] & & & & \mathcal{B}((A'\otimes A'')\otimes A'''),X)\underline{\otimes}P(AX-) \arrow[dl, "z"'] & \\
 & & \mathcal{B}((A\otimes(A'\otimes A''))\otimes A''',-) \arrow[uuu, "\mathcal{B}(\bar{a}\otimes 1,1)"'] & & \mathcal{B}(A\otimes((A'\otimes A'')\otimes A'''),-) \arrow[ll, "\mathcal{B}(\bar{a},1)"'] & & \\
(P(A'A''X)\underline{\otimes}P(AXX))\underline{\otimes}P(XA'''-) \arrow[uur, "z\underline{\otimes}1"] \arrow[ddrr, leftrightarrow, "\cong"'] & & & (12) & & & (P(A'A''X)\underline{\otimes}P(XA'''X))\underline{\otimes}P(AX-) \arrow[uul, "z\underline{\otimes}1"'] \arrow[uuuuu, "\alpha\underline{\otimes}1"'] \arrow[ddll, leftrightarrow, "\cong"] \\
 & (13) & P(A'A''X)\underline{\otimes}\mathcal{B}((A\otimes X)\otimes A''',-) \arrow[uu, "z"] & (11) & P(A'A''X)\underline{\otimes}\mathcal{B}(A\otimes(X\otimes A'''),-) \arrow[uu, "z"'] \arrow[ll, "1\otimes\mathcal{B}(\bar{a},1)"'] & & (10) \\
 & & P(A'A''X)\underline{\otimes}(P(AXX)\underline{\otimes}P(XA'''-)) \arrow[u, "1\underline{\otimes}z"] \arrow[rr, "1\underline{\otimes}\alpha"] & & P(A'A''X)\underline{\otimes}(P(XA'''X)\underline{\otimes}P(AX-)) \arrow[u, "1\underline{\otimes}z"'] & &
\end{tikzcd}
$$

following definition:

Definition 2.1.1 A promonoidal category $\mathcal{A} = (A,P,J,\lambda,\rho,\alpha)$ over $\mathcal{V}$ consists of

a category A

a functor $P : A^{op}\otimes A^{op}\otimes A \to \mathcal{V}$

a functor $J : A \to \mathcal{V}$

and natural isomorphisms

$\lambda = \lambda_A : JX\underline{\otimes}P(XA-) \to LA$

$\rho = \rho_A : JX\underline{\otimes}P(AX-) \to LA$

$\alpha = \alpha_{AA'A''} : P(AA'X)\underline{\otimes}P(XA''-) \to P(A'A''X)\underline{\otimes}P(AX-)$

satisfying the following two axioms:

PC1. The exterior of diagram (2.1.4) commutes.

PC2. The exterior of diagram (2.1.5) commutes.

The existence of the required coends in $\mathcal{V}$ is taken as part of the definition.

It is possible that A is the trace of a symmetric monoidal category $\mathcal{B}$. In this event we define a symmetry σ on $P : A^{op}\otimes A^{op}\otimes A \to \mathcal{V}$ by

$$\begin{tikzcd}
P(AA'-) \arrow[r, "\sigma"] \arrow[d, equal] & P(A'A-) \arrow[d, equal] \\
B(A\otimes A',-) & B(A'\otimes A,-) \arrow[l, "B(\bar{c},1)"]
\end{tikzcd} \tag{2.1.6}$$

commutes, where $\bar{c}$ is the symmetry on B. To obtain axioms, we consider the following two diagrams.

$$\begin{tikzcd}
& P(A'A-) \arrow[dr, "\sigma"] \arrow[d, equal] & \\
P(AA'-) \arrow[ur, "\sigma"] \arrow[d, equal] \arrow[r, phantom, "1"] & B(A'\otimes A,-) \arrow[dl, "B(\bar{c},1)"'] \arrow[r, phantom, "2"] & P(AA'-) \arrow[d, equal] \\
B(A\otimes A',-) \arrow[rr, equal] & & B(A\otimes A',-) \arrow[ul, "B(\bar{c},1)"']
\end{tikzcd} \tag{2.1.7}$$

The regions 1 and 2 of diagram (2.1.7) commute by definition of σ. In diagram (2.1.8), regions 1, 3, and 5 commute by definition of α; regions 2, 4, and 6 commute by definition of σ, together with the naturality of z and $\bar{c}$. If axioms MC3 and MC4 hold for $(\bar{c},\bar{a})$ then the center regions of both diagrams commute whence the exteriors commute. Thus σ satisfies conditions which depend only on the trace A.

Diagram (2.1.8)

$$
\begin{tikzcd}
P(AA'X)\underline{\otimes}P(XA''-) \arrow[rr, "\alpha"] \arrow[dr, "z"'] \arrow[ddd, "\sigma\underline{\otimes}1"'] & & P(A'A''X)\underline{\otimes}P(AX-) \arrow[rr, "1\underline{\otimes}\sigma"] \arrow[d, "z"] & & P(A'A''X)\underline{\otimes}P(XA-) \arrow[dl, "z"] \arrow[ddd, "\alpha"] \\
 & \mathcal{B}((A\otimes A')\otimes A'',-) \arrow[ur, phantom, "1"] & \mathcal{B}(A\otimes(A'\otimes A''),-) \arrow[l, "\mathcal{B}(\overline{a},1)"'] & \mathcal{B}((A'\otimes A'')\otimes A,-) \arrow[l, "\mathcal{B}(\overline{c},1)"'] \arrow[ul, phantom, "2"] \arrow[dr, phantom, "3"] & \\
 & \mathcal{B}((A'\otimes A)\otimes A'',-) \arrow[u, "\mathcal{B}(\overline{c}\otimes 1,1)"] \arrow[ul, phantom, "6"] \arrow[dr, phantom, "5"] & \mathcal{B}(A'\otimes(A\otimes A''),-) \arrow[l, "\mathcal{B}(\overline{a},1)"] & \mathcal{B}(A'\otimes(A''\otimes A),-) \arrow[l, "\mathcal{B}(1\otimes\overline{c},1)"] \arrow[u, "\mathcal{B}(\overline{a},1)"'] \arrow[dl, phantom, "4"] & \\
P(A'AX)\underline{\otimes}P(XA''-) \arrow[ur, "z"'] \arrow[rr, "\alpha"'] & & P(AA''X)\underline{\otimes}P(A'X-) \arrow[u, "z"'] \arrow[rr, "\sigma\underline{\otimes}1"'] & & P(A''AX)\underline{\otimes}P(A'X-) \arrow[ul, "z"']
\end{tikzcd}
$$

Definition 2.1.2 A symmetry for a promonoidal category $\mathcal{A} = (A,P,J,\lambda,\rho,\alpha)$ is a natural isomorphism

$$\sigma = \sigma_{AA'} : P(AA'-) \to P(A'A-)$$

satisfying the following two axioms:

PC3. The exterior of diagram (2.1.7) commutes (i.e. $\sigma^2 = 1$).

PC4. The exterior of diagram (2.1.8) commutes.

A particular consequence of the above arguments is that every monoidal category may be considered to be a promonoidal category. More precisely, we have:

Lemma 2.1.3 Taking $\mathcal{A} = \mathcal{B}$ and M to be the identity, the diagrams (2.1.1) to (2.1.5) establish a bijection between monoidal completions of the data $(\overline{\otimes},\overline{I})$ on $\mathcal{B}$, and promonoidal completions of the data (P,J) on $\mathcal{B}$. Furthermore, diagrams (2.1.6) to (2.1.8) provide a bijection between monoidal symmetries and promonoidal symmetries on the respective resulting structures.

Proof In each of the diagrams (2.1.1) to (2.1.8) the transformations z become Yoneda isomorphisms. Hence each diagram becomes a diagram of isomorphisms. The results follow immediately from the representation theorem.

We shall see in Section 2.4, however, that not all promonoidal categories are monoidal.

Section 2.2 Promonoidal functors, natural transformations

Let $\mathcal{A} = (\mathcal{A},P,J,\lambda,\rho,\alpha)$ and $\mathcal{B} = (\mathcal{B},\overline{P},\overline{J},\overline{\lambda},\overline{\rho},\overline{\alpha})$ denote arbitrary promonoidal categories.

<u>Definition 2.2.1</u> A <u>promonoidal functor</u> $\Phi = (\phi,\hat{\phi},\phi^{\cdot}) : \mathcal{A} \to \mathcal{B}$ consists of

a functor $\phi : \mathcal{A} \to \mathcal{B}$

and natural transformations

$$\hat{\phi} = \hat{\phi}_{AA'} : P(AA'-) \to \overline{P}(\phi A,\phi A',\phi -)$$

$$\phi^{\cdot} : J \to \overline{J}\phi$$

satisfying the following three axioms (in which the uncommented diagrams commute by construction):

PF1. The diagram $*$ commutes:

$$\begin{tikzcd}
JX \otimes P(XA-) \arrow[r, "s"] \arrow[d, "\phi^{\cdot}\otimes\hat{\phi}"'] & JX \underline{\otimes} P(XA-) \arrow[r, "\lambda"] \arrow[d, dashed] \arrow[dr, phantom, "*"] & \mathcal{A}(A-) \arrow[d, "\phi"] \\
\overline{J}\phi X \otimes \overline{P}(\phi X,\phi A,\phi -) \arrow[r, "s\phi"'] & \overline{J}Y \underline{\otimes} \overline{P}(Y,\phi A,\phi -) \arrow[r, "\overline{\lambda}"'] & \mathcal{B}(\phi A,\phi -)
\end{tikzcd} .$$

PF2. The diagram $*$ commutes:

$$\begin{tikzcd}
JX \otimes P(AX-) \arrow[r, "s"] \arrow[d, "\phi^{\cdot}\otimes\hat{\phi}"'] & JX \underline{\otimes} P(AX-) \arrow[r, "\rho"] \arrow[d, dashed] \arrow[dr, phantom, "*"] & \mathcal{A}(A-) \arrow[d, "\phi"] \\
\overline{J}\phi X \otimes \overline{P}(\phi A,\phi X,\phi -) \arrow[r, "s\phi"'] & \overline{J}Y \underline{\otimes} \overline{P}(\phi A,Y,\phi -) \arrow[r, "\overline{\rho}"'] & \mathcal{B}(\phi A,\phi -)
\end{tikzcd} .$$

PF3. The diagram * commutes

$$\begin{tikzcd}
 & P(AA'X)\underline{\otimes}P(XA''-) \arrow[r, "\alpha"] \arrow[ddd, dashed] & P(A'A''X)\underline{\otimes}P(AX-) \arrow[ddd, dashed] & \\
P(AA'X)\otimes P(XA''-) \arrow[ur, "s"] \arrow[d, "\hat{\phi}\otimes\hat{\phi}"'] & & & P(A'A''X)\otimes P(AX-) \arrow[ul, "s"'] \arrow[d, "\hat{\phi}\otimes\hat{\phi}"] \\
\overline{P}(\phi A\phi A'\phi X)\otimes\overline{P}(\phi X\phi A''\phi -) \arrow[dr, "s\phi"'] & \arrow[r, phantom, "*"] & & \overline{P}(\phi A'\phi A''\,X)\otimes\overline{P}(\phi A\phi X\phi -) \arrow[dl, "s\phi"] \\
 & \overline{P}(\phi A\phi A'Y)\underline{\otimes}\overline{P}(Y\phi A''\phi -) \arrow[r, "\overline{\alpha}"'] & \overline{P}(\phi A'\phi A''Y)\underline{\otimes}\overline{P}(\phi AY\phi -) &
\end{tikzcd} .$$

<u>Definition 2.2.2</u> If promonoidal categories A and B are equipped with symmetries σ and $\overline{\sigma}$ respectively then a promonoidal functor $\Phi = (\phi, \hat{\phi}, \phi^{\cdot}) : A \to B$ is <u>symmetric</u> if it satisfies the axiom:

PF4. The following diagram commutes:

$$\begin{tikzcd}
P(AA'-) \arrow[r, "\sigma"] \arrow[d, "\hat{\phi}"'] & P(A'A-) \arrow[d, "\hat{\phi}"] \\
\overline{P}(\phi A\phi A'\phi -) \arrow[r, "\overline{\sigma}"'] & \overline{P}(\phi A'\phi A\phi -)
\end{tikzcd} .$$

<u>Proposition 2.2.3</u> If $M : \mathcal{A} \to \mathcal{B}$ is a full embedding into a monoidal category $\mathcal{B}$ and $\mathcal{A}$ is the trace of $\mathcal{B}$ along M then $(M,1,1) : \mathcal{A} \to \mathcal{B}$ is promonoidal.

<u>Proof</u> Return to diagrams (2.1.1), (2.1.2), and (2.1.3). Regarding $\mathcal{B}$ as a promonoidal category (by Lemma 2.1.3), fill in the appropriate $\overline{\lambda}$, $\overline{\rho}$, and $\overline{\alpha}$ (as shown). By Definition 2.2.1, the upper regions of these diagrams now assert that $(M,1,1) : \mathcal{A} \to \mathcal{B}$ is a promonoidal functor. Furthermore, if $\mathcal{B}$ is symmetric monoidal then the diagram (2.1.6) asserts that $(M,1,1) : \mathcal{A} \to \mathcal{B}$ is symmetric (by Definition 2.2.2).

For <u>any</u> functor $\phi : \mathcal{A} \to \mathcal{B}$, the representation theorem establishes a bijection between the natural transformations $\gamma : LA \to LB.\phi$ and the elements $f \epsilon \mathcal{B}_0(B,\phi A)$, by means of the diagram

$$\begin{tikzcd}
\mathcal{A}(A-) \arrow[rr, "\gamma"] \arrow[d, "\phi"'] & & \mathcal{B}(B,\phi-) \\
\mathcal{B}(\phi A,\phi-) \arrow[urr, "\mathcal{B}(f,1)"'] & &
\end{tikzcd} .$$

Now suppose that $\phi : \mathcal{A} \to \mathcal{B}$ is a functor between <u>monoidal</u> categories. Then the diagrams

$$\begin{tikzcd}
\mathcal{A}(A\otimes A',-) \arrow[r, "\hat{\phi}"] \arrow[d, "\phi"'] & \mathcal{B}(\phi A\otimes\phi A',\phi-) \\
\mathcal{B}(\phi(A\otimes A'),\phi-) \arrow[ur, "\mathcal{B}(\tilde{\phi},1)"'] &
\end{tikzcd} \tag{2.2.1}$$

$$\begin{tikzcd}
\mathcal{A}(I-) \arrow[r, "\phi^{\cdot}"] \arrow[d, "\phi"'] & \mathcal{B}(I,\phi-) \\
\mathcal{B}(\phi I,\phi-) \arrow[ur, "\mathcal{B}(\phi^{o},1)"'] &
\end{tikzcd} \tag{2.2.2}$$

set up a bijection between promonoidal (respt. symmetric promonoidal) functor structures $(\phi,\hat{\phi},\phi^{\cdot})$ and monoidal (respt. symmetric monoidal) functor structures $(\phi,\tilde{\phi},\phi^{o})$ on ϕ. The monoidal functor axioms (taken from [9])

MF1.

$$\begin{tikzcd}
\phi I\otimes\phi A \arrow[r, "\tilde{\phi}"] & \phi(I\otimes A) \arrow[d, "\phi\ell"] \\
I\otimes\phi A \arrow[u, "\phi^{o}\otimes 1"] \arrow[r, "\ell"'] & \phi A
\end{tikzcd} \tag{2.2.3}$$

commutes,

MF2.

$$\begin{tikzcd}
\phi A\otimes\phi I \arrow[r, "\tilde{\phi}"] & \phi(A\otimes I) \arrow[d, "\phi r"] \\
\phi A\otimes I \arrow[u, "1\otimes\phi^{o}"] \arrow[r, "r"'] & \phi A
\end{tikzcd} \tag{2.2.4}$$

commutes,

MF3.

$$\begin{tikzcd}
\phi((A\otimes A')\otimes A'') \arrow[r, "\phi a"] & \phi(A\otimes(A'\otimes A'')) \\
\phi(A\otimes A')\otimes\phi A'' \arrow[u, "\tilde{\phi}"] & \phi A\otimes\phi(A'\otimes A'') \arrow[u, "\tilde{\phi}"'] \\
(\phi A\otimes\phi A')\otimes\phi A'' \arrow[u, "\tilde{\phi}\otimes 1"] \arrow[r, "a"'] & \phi A\otimes(\phi A'\otimes\phi A'') \arrow[u, "1\otimes\tilde{\phi}"']
\end{tikzcd} \qquad (2.2.5)$$

commutes,

MF4. (for symmetry)

$$\begin{tikzcd}
\phi(A\otimes A') \arrow[r, "\phi c"] & \phi(A'\otimes A) \\
\phi A\otimes\phi A' \arrow[u, "\tilde{\phi}"] \arrow[r, "c"'] & \phi A'\otimes\phi A \arrow[u, "\tilde{\phi}"']
\end{tikzcd}$$

commutes,

are precisely the result of substituting (2.2.1) and (2.2.2) into axioms PF1-4 and then applying the representation theorem.

Finally, returning to general promonoidal considerations, there is an appropriate concept of "2-cell".

<u>Definition 2.2.4</u> A <u>promonoidal natural transformation</u> $\eta : \Phi \to \Psi : \mathcal{A} \to \mathcal{B}$ is a natural transformation $\eta : \phi \to \psi$ satisfying the following two axioms:

PN1. The following diagram commutes:

$$\begin{tikzcd} JA \arrow[r, "\phi^{\cdot}"] \arrow[dr, "\psi^{\cdot}"'] & \bar{J}\phi A \arrow[d, "\bar{J}\eta"] \\ & \bar{J}\psi A \end{tikzcd} .$$

PN2. The following diagram commutes:

$$\begin{tikzcd} P(AA'-) \arrow[r, "\hat{\phi}"] \arrow[d, "\hat{\psi}"'] & \bar{P}(\phi A\phi A'\phi -) \arrow[d, "\bar{P}(1,1,\eta)"] \\ \bar{P}(\psi A\psi A'\psi -) \arrow[r, "\bar{P}(\eta,\eta,1)"'] & \bar{P}(\phi A\phi A'\psi -) \end{tikzcd} .$$

The net result is the "2-category" $\mathcal{P}rom$. Promonoidal functors $\Phi : \mathcal{A} \to \mathcal{B}$ and $\Psi : \mathcal{B} \to \mathcal{C}$ are composed in the obvious manner, the data of the composite promonoidal functor $\Psi.\Phi$ being $(\psi\phi, \hat{\psi}\hat{\phi}, \psi^{\cdot}\phi^{\cdot})$. Similarly, one can compose promonoidal natural transformations with promonoidal functors, and with one another. An <u>isomorphism</u> in $\mathcal{P}rom$ is readily seen to be a promonoidal functor $\Phi = (\phi, \hat{\phi}, \phi^{\cdot})$ whose data ϕ, $\hat{\phi}$, and $\phi^{\cdot}$ are all isomorphisms. A <u>natural isomorphism</u> in $\mathcal{P}rom$ is a natural transformation all of whose components are isomorphisms.

The following lemmas allow us to replace given promonoidal structures by suitable isomorphs when required. The verifications are straightforward computations from the definitions and shall be omitted.

<u>Lemma 2.2.5</u> Given categories A and B and an isomorphism $\phi : A \to B$ together with functors

$$P : A^{op} \otimes A^{op} \otimes A \to V, \quad \overline{P} : B^{op} \otimes B^{op} \otimes B \to V$$

$$J : A \to V, \quad \overline{J} : B \to V$$

and natural isomorphisms

$$\hat{\phi} = \hat{\phi}_{AA'} : P(AA'-) \to \overline{P}(\phi A \phi A' \phi -)$$

$$\phi^{\cdot} : J \to \overline{J}\phi,$$

the axioms PF1-3 for $\Phi = (\phi, \hat{\phi}, \phi^{\cdot})$ establish a bijection between promonoidal completions of the data (P,J) on A and the data $(\overline{P},\overline{J})$ on B.

<u>Lemma 2.2.6</u> Given promonoidal categories A and B together with functors $\phi, \psi : A \to B$ and a natural isomorphism $\eta : \phi \to \psi : A \to B$, the axioms PN1 and PN2 for η establish a bijection between promonoidal functor completions of ϕ and of ψ.

Section 2.3 Dualities

Of the two basic dualities available for monoidal categories (Bénabou [2], §3) only "transpose" remains available in the general promonoidal setting. Nevertheless, we shall make much use of the "conjugate" of a monoidal category.

Proposition and Definition 2.3.1 (Bénabou) If $\mathcal{A} = (A,\otimes,I,\ell,r,a)$ is a monoidal category then so is its conjugate $\mathcal{A}^{op} = (A^{op},\otimes^{op},I,\ell^{-1},r^{-1},a^{-1})$.

This duality is a special attribute of the "category" of monoidal categories and strong monoidal functors.

Definition 2.3.2 A monoidal functor $\Phi = (\phi,\tilde{\phi},\phi^{o}) : \mathcal{A} \to \mathcal{B}$ is strong if the transformations

$$\tilde{\phi} : \phi A\otimes\phi A' \to \phi(A\otimes A')$$
$$\phi^{o} : I \to \phi I$$

are isomorphisms.

Clearly a strong monoidal functor $\Phi = (\phi,\tilde{\phi},\phi^{o}) : \mathcal{A} \to \mathcal{B}$ admits a (strong) conjugate

$\Phi^{op} = (\phi^{op}, \tilde{\phi}^{-1}, \phi^{o^{-1}}) : A^{op} \to B^{op}$. Strong monoidal functors occur quite widely; as shown in Kelly [11], the closed left adjoint of a normal closed functor is always strong.

<u>Proposition and Definition 2.3.3</u> If $A = (A,P,J,\lambda,\rho,\alpha)$ is a promonoidal category then so is its <u>transpose</u> $A^* = (A,P^*,J,\rho,\lambda,\alpha^{-1})$, where P^* is the composite

$$A^{op} \otimes A^{op} \otimes A \xrightarrow{c \otimes 1} A^{op} \otimes A^{op} \otimes A \xrightarrow{P} V.$$

It is a simple matter to construct the transpose of a promonoidal functor. Moreover, if a promonoidal category or functor is <u>symmetric</u> then it is (canonically) isomorphic to its own transpose.

Section 2.4 Comonoid categories

Suppose, for the moment, that the ground category $\mathcal{V}$ is a cartesian closed category. Then any category $\mathcal{B}$ which admits finite $\mathcal{V}$-coproducts (including an initial object 0) becomes a monoidal category if we set

$$B \otimes B' = B + B' \text{ and } I = 0$$

and take $\bar{l}, \bar{r}$, and $\bar{a}$ to be the canonical isomorphisms

$$0 + B \cong B \cong B + 0$$

$$(B + B') + B'' \cong B + (B' + B'').$$

Such a structure $\mathcal{B}$ might well be called a cocartesian monoidal category (over $\mathcal{V}$).

Theorem 2.4.1 ($\mathcal{V}$ cartesian closed) The trace of a cocartesian monoidal category $\mathcal{B}$ exists on any full subcategory of $\mathcal{B}$.

Proof Let $\mathcal{A}$ be an arbitrary full subcategory of $\mathcal{B}$. For all $A, A' \in \mathcal{A}$ and $B \in \mathcal{B}$, consider the composite

$$\begin{aligned}
\mathcal{B}(BX)\underline{\otimes}\mathcal{B}(X + A, A') &= \int^{X\in\mathcal{A}} \mathcal{B}(BX)\times\mathcal{B}(X + A, A') \\
&\cong \int^{X\in\mathcal{A}} \mathcal{B}(BX)\times(\mathcal{A}(XA')\times\mathcal{A}(AA')) \\
&\cong \Big(\int^{X\in\mathcal{A}} (\mathcal{B}(BX)\times\mathcal{A}(XA'))\Big)\times\mathcal{A}(AA')
\end{aligned}$$

because $-\times\mathcal{A}(AA')$ has a right adjoint,

$$\cong \mathcal{B}(BA')\times\mathcal{A}(AA')$$

by the higher representation theorem,

$$\cong \mathcal{B}(B + A,A').$$

This composite isomorphism is verified to be

$$z : \mathcal{B}(BX)\underline{\otimes}\mathcal{B}(X\otimes A,A') \to \mathcal{B}(B\otimes A,A')$$

by applying the representation theorem to $B\in\mathcal{B}$. In particular, $z_1 \ldots z_4$ are isomorphisms, whence the trace of $\mathcal{B}$ exists on $\mathcal{A}$.

Thus, because a full subcategory $\mathcal{A}$ of $\mathcal{B}$ might not admit finite coproducts (in $\mathcal{A}$), there do exist examples of promonoidal categories which are <u>not</u> monoidal. To generalise this new type of promonoidal structure, observe that we have

$$\begin{aligned} P(A'A''A) &= \mathcal{B}(A' + A'',A) \\ &\cong \mathcal{B}(A'A)\times\mathcal{B}(A''A) \\ &= \mathcal{A}(A'A)\times\mathcal{A}(A''A) \\ &= \mathcal{A}(A'A)\otimes\mathcal{A}(A''A), \end{aligned} \tag{2.4.1}$$

the final expression being functorial in $A\in\mathcal{A}$ by virtue of the <u>diagonal functor</u> $\delta : \mathcal{A} \to \mathcal{A}\times\mathcal{A}$. Similarly,

$$JA = \mathcal{B}(0,A) \cong I \tag{2.4.2}$$

is functorial by virtue of the <u>constant functor</u> $\varepsilon : \mathcal{A} \to \mathcal{I}$.

If we wish to <u>remove</u> the cartesian restriction on $\mathcal{V}$ we must ensure that (2.4.1) and (2.4.2) remain functorial in $A\in\mathcal{A}$. This is so if $\mathcal{A}$ admits the structure of a <u>comonoid</u> in the "monoidal category" $V\text{-}Cat$. Such a comonoid comprises a comultiplication functor $\delta : A \to A \otimes A$ and a counit functor $\varepsilon : A \to I$ satisfying the following coassociative and left and right counit laws:

$$\begin{tikzcd} & A\otimes A \arrow[r, "\delta\otimes 1"] & (A\otimes A)\otimes A \arrow[d, "\cong"] \\ A \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & & \\ & A\otimes A \arrow[r, "1\otimes\delta"'] & A\otimes(A\otimes A) \end{tikzcd}$$

$$\begin{tikzcd} I\otimes A \arrow[d, "\cong"'] & A\otimes A \arrow[l, "\varepsilon\otimes 1"'] \arrow[r, "1\otimes\varepsilon"] & A\otimes I \arrow[d, "\cong"] \\ A \arrow[r, equal] & A \arrow[u, "\delta"] \arrow[r, equal] & A \end{tikzcd} .$$

Note that commutativity of these diagrams implies that δ maps an object $A \epsilon A$ to the pair $(A,A) \epsilon A\otimes A$, and that the morphisms

$$\delta = \delta_{AA'} : A(AA') \to A(AA') \otimes A(AA')$$
$$\varepsilon = \varepsilon_{AA'} : A(AA') \to I$$

provide a $\otimes$-comonoid (that is, an associative $\otimes$-coalgebra with a counit) structure on <u>each</u> hom-object $A(AA')$.

A promonoidal structure is then defined on A by the following data:

$$P : (A^{op}\otimes A^{op})\otimes A \xrightarrow{1\otimes\delta} (A^{op}\otimes A^{op})\otimes A\otimes A \xrightarrow{\mathrm{Hom}(A\otimes A)} V,$$

$$J : A \xrightarrow{\varepsilon} I \xrightarrow{I} V,$$

$$\begin{tikzcd}
JX\underline{\otimes}P(XA'A) \arrow[rr, "\lambda"] \arrow[d, equal] & & A(A'A) \\
JX\underline{\otimes}(A(XA)\otimes A(A'A)) \arrow[d, equal, "\wr"'] & & \\
(A(XA)\underline{\otimes}JX)\otimes A(A'A) \arrow[rr, "y\otimes 1"'] & & JA\otimes A(A'A), \arrow[uu, "\ell"']
\end{tikzcd}$$

$$\begin{tikzcd}
JX\underline{\otimes}P(A'XA) \arrow[rr, "\rho"] \arrow[d, equal] & & A(A'A) \\
JX\underline{\otimes}(A(A'A)\otimes A(XA)) \arrow[d, equal, "\wr"'] & & \\
A(A'A)\otimes(A(XA)\underline{\otimes}JX) \arrow[rr, "1\otimes y"'] & & A(A'A)\otimes JA, \arrow[uu, "r"']
\end{tikzcd}$$

$$\begin{tikzcd}
P(A'A''X)\underline{\otimes}P(XA'''A) \arrow[r, "\alpha"] \arrow[d, equal] & P(A''A'''X)\underline{\otimes}P(A'XA) \arrow[d, equal] \\
P(A'A''X)\underline{\otimes}(A(XA)\otimes A(A'''A)) \arrow[d, equal, "\wr"'] & P(A''A'''X)\underline{\otimes}(A(A'A)\otimes A(XA)) \arrow[d, equal, "\wr"] \\
(A(XA)\underline{\otimes}P(A'A''X))\otimes A(A'''A) \arrow[d, "y\otimes 1"'] & A(A'A)\otimes(A(XA)\underline{\otimes}P(A''A'''X)) \arrow[d, "1\otimes y"] \\
P(A'A''A)\otimes A(A'''A) \arrow[d, equal] & A(A'A)\otimes P(A''A'''A) \arrow[d, equal] \\
(A(A'A)\otimes A(A''A))\otimes A(A'''A) \arrow[r, "a"'] & A(A'A)\otimes(A(A''A)\otimes A(A'''A)),
\end{tikzcd}$$

where the definitions of λ, ρ, and α implicitly involve the comonoid axioms for ε and δ. Furthermore, if the comultiplication δ is commutative, we can define a symmetry $\sigma = \sigma_{A'A''A}$ for A as

$$P(A'A''A) = A(A'A)\otimes A(A''A) \xrightarrow[c]{} A(A''A)\otimes A(A'A) = P(A''A'A).$$

This σ clearly satisfies PC3; the other axioms are too long to verify here but they are essentially a result of the "coherence" of the Yoneda isomorphism y.

<u>Remark on symmetry</u> To each assertion that is made in the sequel, there is a corresponding assertion-with-symmetry. To avoid restatement, we have omitted it, noting that the corresponding proof-with-symmetry requires nothing that is essentially new.

CHAPTER 3

THE FUNCTOR CATEGORY THEOREM

Section 3.1 The *Hom* construction

From the considerations of the preceding section, not every promonoidal category is monoidal. However we still can ask whether every promonoidal category arises as the trace of a monoidal category. For "small" promonoidal categories this can be answered in the affirmative by use of the Yoneda embedding.

Let A be a category for which the functor category $F = [A,V]$ exists; in practice this will mean that (the set of objects of) A is small and V is complete. Further, suppose that there is given a functor

$$\bar{\otimes} : F \otimes F \to F$$

together with an object

$$\bar{I} \in F.$$

Then, in the discussion of Section 2.1, we set $B = F^{op}$ and $M = L^{op} : A \to F^{op}$, thus obtaining functors

$$P : A^{op} \otimes A^{op} \otimes A \to V$$

$$J : A \to V$$

where $P(AA'-) = F(L-, LA \bar{\otimes} LA')$ and $J = F(L-, \bar{I})$.

<u>Lemma 3.1.1</u> If the given functor $\bar{\otimes} : F \otimes F \to F$ admits a right adjoint to each variable then diagrams (2.1.1) to

(2.1.5) establish a bijection between biclosed completions of the data $(\bar{\otimes},\bar{I})$ on F and promonoidal completions of the constructed data (P,J) on A.

<u>Proof</u> First we have that $z_1 \ldots z_4$ are isomorphisms because the transformations

$$\left.\begin{array}{l} z : F(LX,S)\underline{\otimes}F(LA,LX\otimes LA') \to F(LA,S\otimes LA') \\ z : F(LX,S)\underline{\otimes}F(LA,LA'\otimes LX) \to F(LA,LA'\otimes S) \end{array}\right\} \qquad (3.1.1)$$

are isomorphisms for <u>all</u> $S\in F$. To see this, combine the following three facts:

(1) For each $A\in A$, the functor $F(LA,-) : F \to V$ has a right adjoint $Z \rightsquigarrow [Z,LA]$.

(2) By hypothesis, the functor $\bar{\otimes}$ has a right adjoint to each variable.

(3) The functor $L : A^{op} \to F$ is dense (Section 1.5).

In other words, the functors $F(LA,-\otimes LA')$ and $F(LA,LA'\otimes -)$ preserve the expression

$$S \cong F(LX,S)\underline{\otimes}LX$$

for all $A,A'\in A$ and $S\in F$.

Then, because $z_1 \ldots z_4$ are isomorphisms, the diagrams (2.1.1), (2.1.2), and (2.1.3) provide a bijection between natural isomorphisms λ, ρ, and α for (P,J) and natural isomorphisms $\bar{l} : I\otimes LA \cong LA$, $\bar{r} : LA\otimes I \cong LA$, and $\bar{a} : (LA\otimes LA')\otimes LA'' \cong LA\otimes(LA'\otimes LA'')$ for $(\bar{\otimes},\bar{I})$, by taking the inverses of the $\bar{\ell}$, $\bar{r}$, $\bar{a}$ actually shown in these diagrams; in other words, we are looking at the conjugate of $\mathcal{B}$. Because (3.1.1) are isomorphisms for all $S \in \mathcal{F}$, the diagrams (2.1.4) and (2.1.5) are also diagrams of isomorphisms. Consequently, axioms PC1 and PC2 hold for λ, ρ, and α if and only if the center regions of (2.1.4) and (2.1.5), respectively, commute. But, by the higher representation theorem, $\mathcal{F}(L-,S) \cong S$ for all $S \in \mathcal{F}$, hence these center regions commute if and only if the diagrams

$$\begin{tikzcd}
(LA\otimes I)\otimes LA' \arrow[rr, "\bar{a}"] \arrow[dr, "\bar{r}\otimes 1"'] & & LA\otimes(I\otimes LA') \arrow[dl, "1\otimes\bar{\ell}"] \\
& LA\otimes LA' &
\end{tikzcd} \tag{3.1.2}$$

and

$$\begin{tikzcd}
((LA\otimes LA')\otimes LA'')\otimes LA''' \arrow[r, "\bar{a}"] \arrow[d, "\bar{a}\otimes 1"'] & (LA\otimes LA')\otimes(LA''\otimes LA''') \arrow[r, "\bar{a}"] & LA\otimes(LA'\otimes(LA''\otimes LA''')) \\
(LA\otimes(LA'\otimes LA''))\otimes LA''' \arrow[rr, "\bar{a}"'] & & LA\otimes((LA'\otimes LA'')\otimes LA''') \arrow[u, "1\otimes\bar{a}"']
\end{tikzcd} \tag{3.1.3}$$

commute for all A, A', A'', A''' in $\mathcal{A}$.

Finally, because $\bar{\otimes}$ has a right adjoint to each variable and $L : \mathcal{A}^{op} \to \mathcal{F}$ is dense, we can use Lemma 1.5.4 to deduce that the transformations $\bar{\ell}$, $\bar{r}$, $\bar{a}$ admit unique extensions to natural isomorphisms

$\bar{l}_S : I \otimes S \cong S$, $\bar{r}_S : S \otimes I \cong S$, and

$\bar{a}_{STR}: (S \otimes T) \otimes R \cong S \otimes (T \otimes R)$,

satisfying axioms MC1 and MC2 if and only if (3.1.2) and (3.1.3) commute. This completes the proof.

Theorem 3.1.2 (the functor category theorem) Let A be a category for which the functor category $F = [A, V]$ exists, let $P : A^{op} \otimes A^{op} \otimes A \to V$ be a functor for which the coend

$$S \bar{\otimes} T = \int^{AA'} (SA \otimes TA') \otimes P(AA'-) \qquad (3.1.4)$$

exists for all $S, T \in F$, and let $J \in F$. Then there exists a bijection between promonoidal completions of the data (P, J) on A and biclosed completions of $(\bar{\otimes}, J)$ on F.

Proof The expression (3.1.4) defines a (canonical) functor $\bar{\otimes} : F \otimes F \to F$ by Lemma 1.3.1. Furthermore, $\bar{\otimes}$ has a right adjoint to each variable:

$$\begin{aligned} F(S \bar{\otimes} T, R) &= \int_{A''} [(S \otimes T)A'', RA''] \\ &= \int_{A''} [\int^{AA'} (SA \otimes TA') \otimes P(AA'A''), RA''] \\ &\cong \int_{AA'A''} [(SA \otimes TA') \otimes P(AA'A''), RA''] \\ &\cong \int_{AA'A''} [SA, [P(AA'A''), [TA', RA'']]] \\ &\cong \int_{A} [SA, \int_{A'A''} [P(AA'A''), [TA', RA'']]] \\ &= \int_{A} [SA, (R/T)A] \text{ say,} \\ &= F(S, R/T) \end{aligned}$$

where the isomorphisms are the canonical ones. Assuming that each of the ends involved is made functorial in its extra variables using Lemma 1.3.1, we have that the composite isomorphism is natural in S, T, and R. Thus $-\bar{\otimes}T$ admits a right adjoint $-/T$ defined by

$$R/T = \int_{A'A''}[P(-A'A''),[TA',RA'']] \qquad (3.1.5)$$

for all $R \in \mathcal{F}$. Similarly, we have

$$\begin{aligned} \mathcal{F}(T\bar{\otimes}S,R) &= \int_{A''}[\textstyle\int^{AA'}(TA\otimes SA')\otimes P(AA'A''),RA''] \\ &\cong \int_{A'}[SA',\int_{AA''}[P(AA'A''),[TA,RA'']]] \end{aligned}$$

whence $T\bar{\otimes}-$ admits a right adjoint $T\backslash -$ defined by

$$T\backslash R = \int_{AA''}[P(A-A''),[TA,RA'']] \qquad (3.1.6)$$

for all $R \in \mathcal{F}$.

Now define functors $P' : \mathcal{A}^{op}\otimes\mathcal{A}^{op}\otimes\mathcal{A} \to \mathcal{V}$ and $J' : \mathcal{A} \to \mathcal{V}$ by

$$P'(AA'-) = \mathcal{F}(L-,LA\bar{\otimes}LA')$$

$$J' = \mathcal{F}(L-,J).$$

Then the Lemma 3.1.1 provides a bijection between promonoidal completions of (P',J') on $\mathcal{A}$ and biclosed completions of $(\bar{\otimes},J)$ on $\mathcal{F}$. Furthermore, the higher representation theorem provides isomorphisms

$$\hat{\chi} : P(AA'-) \xrightarrow{\cong} P'(AA'-)$$

$$\chi^{\cdot} : J \xrightarrow{\cong} J'.$$

By Lemma 2.2.5, the promonoidal functor axioms for $(1,\hat{\chi},\chi^{\cdot})$ establish a bijection between promonoidal completions of the data (P,J) and of the data (P',J'). The proof is completed by composing the two bijections.

We shall call a promonoidal category "small" if its data satisfy the hypothesis of the Theorem 3.1.2.

Definition 3.1.3 When A is a "small" promonoidal category the biclosed structure provided by Theorem 3.1.2 on $[A,V]$ shall be called the *Hom* of A and V and denoted by $\{A,V\}$.

The justification for this terminology lies in the observation that

$$\{A,V\}(S\bar{\otimes}T,R) \cong \int_{AA'A''}[P(AA'A''),[SA\otimes TA',RA'']]$$

$$\{A,V\}(S,T) \cong \int_{AA'}[A(AA'),[SA,TA']]$$

$$\{A,V\}(J,S) \cong \int_{A}[JA,[I,SA]].$$

These formulas display the P, Hom, and J of $\{A,V\}$ as the respective "inner products" of those of A and V.

Corollary 3.1.4 Any "small" promonoidal category A is isomorphic to the trace of a monoidal structure, namely $\{A,V\}^{op}$.

Proof By the construction of $\{A,V\}$ in Theorem 3.1.2.

<u>Corollary 3.1.5</u> Each biclosed structure on $[A,V]$ is isomorphic to one of the form $\{A,V\}$ for some promonoidal structure on A.

<u>Proof</u> Let $(\bar{\otimes},J,\ldots)$ be a biclosed structure on $F = [A,V]$. Define a functor $P : A^{op}\otimes A^{op}\otimes A \to V$ with

$$P(AA'-) = LA\bar{\otimes}LA' \tag{3.1.7}$$

Define $\bar{\otimes}' : F\otimes F \to F$ and $\tilde{\psi}_{ST} : S\bar{\otimes}T \cong S\bar{\otimes}'T$ by the composite isomorphism

$$\begin{aligned}
S\bar{\otimes}T &\cong (\textstyle\int^{A}SA\otimes LA)\bar{\otimes}(\textstyle\int^{A'}TA'\otimes LA') \\
&\qquad\qquad \text{by the higher repn. thrm.,} \\
&\cong \textstyle\int^{AA'}(SA\otimes TA')\otimes(LA\bar{\otimes}LA') \\
&\qquad\qquad \text{because } \bar{\otimes} \text{ has right adjoints,} \\
&= \textstyle\int^{AA'}(SA\otimes TA')\otimes P(AA'-) \\
&\qquad\qquad \text{by (3.1.7)} \\
&= S\bar{\otimes}'T.
\end{aligned}$$

By Lemma 2.2.5, there exists a unique (biclosed) monoidal completion of $(\bar{\otimes}',J)$ making $(1,\tilde{\psi},1)$ a monoidal isomorphism. Moreover, the Theorem 3.1.2 asserts that this is $\{A,V\}$ for a unique promonoidal completion of the data (P,J) for A.

The net result is a correspondence to within isomorphism, between "small" promonoidal structures on a category A and biclosed structures on $[A,V]$.

The *Hom* construction has the interesting, though not unexpected, property that it turns certain <u>promonoidal functors</u> into <u>algebras</u>: If A is a "small" promonoidal category then a promonoidal functor $\Phi : A \to V$ is precisely a $\otimes$-monoid (by which we mean an associative $\otimes$-algebra with a unit) in $\{A,V\}$, and a promonoidal natural transformation between two such functors corresponds to a monoid homomorphism. One immediate consequence of this fact is that each functor $T : A \to V$ generates a <u>free promonoidal functor</u>

$\sum_{n=0}^{\infty} T_n : A \to V$ where

$$T_0 = J$$

$$T_1 = T$$

$$T_n = \int^{A_1 \ldots A_n} (TA_1 \otimes \ldots \otimes TA_n) \otimes (P(A_1A_2X)\underline{\otimes}P(XA_3X)\underline{\otimes}\ldots\underline{\otimes}P(XA_n-)) \qquad \text{for } n > 1,$$

provided V admits the required colimits. This result follows on combining our definition of $\overline{\otimes}$ in $\{A,V\}$ with the free $\otimes$-monoid construction for a biclosed category which is essentially provided by M. Barr in [1] §2.

Section 3.2 Monoidal examples

For special types of promonoidal category A the formulation of the biclosed structure $\{A,V\}$ may be simplified by application of the higher representation theorem. In this section we shall suppose that $A = (A,\tilde{\otimes},\tilde{I},\ldots)$ is a "small" monoidal category. Then the internal-hom formulas (3.1.5) and (3.1.6) reduce to

$$\begin{aligned}(T/S)A &= \int_{BC}[P(ABC),[SB,TC]] \\ &= \int_{BC}[A(A\otimes B,C),[SB,TC]] \\ &\cong \int_{B}[SB,T(A\otimes B)]\end{aligned}$$

and

$$\begin{aligned}(S\backslash T)A &= \int_{BC}[P(BAC),[SB,TC]] \\ &= \int_{BC}[A(B\otimes A,C),[SB,TC]] \\ &\cong \int_{B}[SB,T(B\otimes A)]\end{aligned}$$

respectively.

If, in addition, the monoidal category A is biclosed with

$$A(A\otimes B,C) \cong A(A,C/B) \cong A(B,A\backslash C)$$

then the tensor-product (3.1.4) reduces to either

$$\begin{aligned}(S\bar{\otimes}T)C &= \int^{AB}(SA\otimes TB)\otimes P(ABC) \\ &\cong \int^{AB}(SA\otimes TB)\otimes A(A,C/B) \\ &\cong \int^{B}S(C/B)\otimes TB\end{aligned}$$

or

$$(S\bar{\otimes}T)C \cong \int^{AB}(SA\otimes TB)\otimes A(B,A\backslash C)$$
$$\cong \int^{A}SA\otimes T(A\backslash C).$$

These formulas present $\{A,V\}$ as a <u>convolution</u> of A and V.

The following are typical examples of $\{A,V\}$ with A monoidal and possibly biclosed. For convenience, we suppose in each example that A is small as a category and that V admits all small limits and colimits.

<u>Example 3.2.1</u> Let A be a category with a single object $\tilde{I}$. Then the endomorphism-object $M = A(\tilde{I},\tilde{I})$ admits a canonical $\otimes$-monoid structure in V, the multiplication $\mu : M\otimes M \to M$ and identity $\eta : I \to M$ being described by composition and identity in A respectively. Moreover, every $\otimes$-monoid in V may be so obtained. By Eilenberg-Kelly [9] Propn III.4.2, the data

$$\tilde{I}\tilde{\otimes}\tilde{I} = \tilde{I}$$
$$\mu : M\otimes M \to M$$

define a bifunctor

$$\tilde{\otimes} : A\otimes A \to A$$

if and only if $M = (M,\mu,\eta)$ is a <u>commutative</u> $\otimes$-monoid. Thus, on taking each of $\tilde{l}$, $\tilde{r}$, $\tilde{a}$, and $\tilde{c}$ to be the identity transformation of the identity functor on A, we obtain a symmetric monoidal category $(A, \tilde{\otimes}, \tilde{I}, \tilde{\ell}, \tilde{r}, \tilde{a}, \tilde{c})$ whenever M is commutative. The resulting closed category $\{A, V\}$ is the category of M-modules with the usual tensor-product and internal-hom; to see this, note that

$$[S,T](\tilde{I}) = \int_{I}[S(\tilde{I}),T(\tilde{I})]$$
$$= \text{"natural transformations" from } S \text{ to } T.$$

Example 3.2.2 Suppose that A is a symmetric promonoidal category. Then $\{A, V\}$ is closed. Moreover, a commutative $\otimes$-monoid in $\{A, V\}$ is precisely a symmetric promonoidal functor $\Phi : A \to V$. Thus, from the preceding example, the category Φ-*Mod* of Φ-modules is closed.

Example 3.2.3 (V cartesian closed) Take $V = S$ and let A be a (finitary) commutative theory in the sense of Linton [14]. Recall that A is commutative if, for each m-ary operation $\mu \in A(m,1)$ and n-ary operation $\nu \in A(n,1)$, the following diagram commutes:

$$\begin{tikzcd}
(1^n)^m = n^m \arrow[r, "\nu^m"] \arrow[dd, "\cong"'] & 1^m = m \arrow[dr, "\mu"] & \\
& & 1 \\
(1^m)^n = m^n \arrow[r, "\mu^n"'] & 1^n = n \arrow[ur, "\nu"'] &
\end{tikzcd}$$

where the isomorphism is the canonical one. By [9] Propn III.4.2, this is precisely the condition that the rules

$$m\tilde{\otimes}n = n^m$$

$$\mu\tilde{\otimes}\nu = \mu.\nu^m$$

should define a functor

$$\tilde{\otimes} : A\times A \to A.$$

With this in mind, let us replace S by an arbitrary <u>cartesian closed</u> category V. We now define a finitary <u>V-theory</u> to be a V-category A having for objects the non-negative integers $N = \{0, 1, \ldots , n, \ldots\}$ and having the property that

$$n = 1^n \text{ (the n-fold product of 1) in } A_0,$$

and $A(m,n) \cong A(m,1)^n$ in V_0, for all $m,n \in A$. The <u>category of A-algebras</u> is the full subcategory of $[A,V]$ determined by those functors from A to V which preserve finite products.

For each $m,n \in A$, the morphism

$$A(n,1)^m \to A(n^m,1^m),$$

defined by the $(i=1,\ldots,m)$ diagrams

$$\begin{tikzcd}
A(n,1)^m \arrow[r, dashed] \arrow[d, "p_i"'] & A(n^m,1^m) \arrow[d, "A(1,p_i)"] \\
A(n,1) \arrow[r, "A(p_i,1)"'] & A(n^m,1)
\end{tikzcd}$$

of projections, may be composed with the diagonal function $\delta : A(n,1) \to A(n,1)^m$ to yield a morphism

$$t_{mn} : A(n,1) \to A(n^m,1^m).$$

This in turn enables us to define canonical functors

$$m\tilde{\otimes}- : A \to A \text{ and } -\tilde{\otimes}n : A \to A$$

with $m\tilde{\otimes}n = n^m$ for each $m,n \in A$.

By <u>definition</u>, the V-theory A is <u>commutative</u> if $m\tilde{\otimes}-$ and $-\tilde{\otimes}n$ are the partial functors of a bifunctor

$$\tilde{\otimes} : A \times A \to A.$$

Assuming this is so, let $\tilde{I} = 1$, let $\tilde{\ell}$, $\tilde{r}$, and $\tilde{a}$ be the appropriate identity isomorphisms, and let $\tilde{c} : n^m \cong m^n$ be the canonical non-identity isomorphism. These data provide A with the structure of a symmetric monoidal category. The resulting closed structure $\{A,V\}$ on the category $[A,V]$ of all functors from A to V, can be <u>restricted</u> to the category of A-algebras. First, it is easily verified that the internal hom

$$[S,T] = \int_n [Sn, T(n\tilde{\otimes}-)]$$

of S and T preserves finite products whenever T does. More importantly, if S and T are both A-algebras then so is the tensor product

$$S\tilde{\otimes}T = \int^{mn} (Sm \times Tn) \times A(m\tilde{\otimes}n,-)$$
$$= \int^{mn} (S\times T)(m,n) \times A(m\tilde{\otimes}n,-).$$

This result follows from combining the Kan extension theorem given in Appendix 2, with the fact that

$S \times T : A \times A \to V$ preserves finite products whenever S and T both do.

When $V = S$, we obtain the usual closed category of algebras over a commutative theory (Linton [14]). Other straightforward applications are obtained by taking V to be the cartesian closed category of compactly generated spaces and continuous set maps (discussed in Section 4.3). For instance, the ordinary (commutative) S-theory of abelian groups provides a (commutative) V-theory if we take the same sets of operations but give them the discrete topology. The resulting category of compactly generated abelian groups and continuous group homomorphisms is closed by the above procedure. Less "trivially", we could consider the category of compactly generated modules over, say, the field of real numbers with the usual topology; in this case the sets of operations of the corresponding theory are non-trivially topologised. Moreover, the theory is commutative and thus yields a monoidal closed category of topological vector spaces.

<u>Example 3.2.4</u> Any (multiplicative) group $G = \{g,h,k,...\}$ may be viewed as a discrete biclosed category over S by taking

$$g \otimes h = gh, \quad g/h = gh^{-1}, \quad g \backslash h = g^{-1}h.$$

The associated free V-category $A = F_*G$ then has an induced biclosed structure over V. Because G is a discrete category, $\int^{\cdot}$ reduces to $\sum$ and $\int_{\cdot}$ to Π in V_0 so that the resulting biclosed structure $\{A, V\}$ on the category of <u>G-graded objects of</u> V is given by the familiar formulas

$$\begin{aligned}
(X \bar{\otimes} Y)_k &= \sum_{gh=k} X_g \otimes Y_h \\
&\cong \sum_h X_{kh^{-1}} \otimes Y_h \\
&\cong \sum_g X_g \otimes Y_{g^{-1}k} \\
(Y/X)_g &= \prod_h [X_h, Y_{gh}] \\
(X\backslash Y)_g &= \prod_h [X_h, Y_{hg}]
\end{aligned}$$

for all $X = \{X_g\}$ and $Y = \{Y_g\}$ in $[F_*G, V]$. Generally this biclosed structure is <u>non-symmetric</u>; it is symmetric whenever G is an abelian group.

<u>Example 3.2.5</u> Another non-symmetric example of $\{A, V\}$ is obtained by considering the simplicial category Δ whose objects are the finite totally ordered sets $\{0, 1, \ldots, n\}$ and whose morphisms are non-decreasing set maps. This category has a <u>non-symmetric</u> monoidal structure obtained by

concatenation:

$$\{0, 1, \ldots, m\}\tilde{\otimes}\{0, 1, \ldots, n\} = \{0, 1, \ldots, m, 1, \ldots, n\}.$$

The resulting $\mathcal{V}$-monoidal structure on $F_*\Delta$ extends to a non-symmetric biclosed structure on the category $[F_*\Delta^{op},\mathcal{V}]$ of simplicial objects in $\mathcal{V}$. This differs from the usual symmetric closed structure on simplicial objects; the latter is treated by Example 3.3.3.

Example 3.2.6 Lastly, let us take $\mathcal{A}$ to be the free $\mathcal{V}$-category on the arrow category $0 \to 1$, together with the symmetric monoidal structure given by finite products in $0 \to 1$. In this simple case, the tensor product and internal hom in $\{\mathcal{A},\mathcal{V}\}$ can be easily computed from the $\mathcal{V}$-limit construction for coends and ends (outlined in Section 1.1). The resulting closed structure $\{\mathcal{A},\mathcal{V}\}$ on the category $[\mathcal{A},\mathcal{V}]$ of morphisms in $\mathcal{V}$, may be described as follows. Let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms in $\mathcal{V}_0$. Then $f\bar{\otimes}f'$ is defined by the diagram

$$\begin{tikzcd}
 & Y\otimes X' \arrow[dr] \arrow[drr, "1\otimes f'"] & & \\
X\otimes X' \arrow[ur, "f\otimes 1"] \arrow[dr, "1\otimes f'"'] & & Q \arrow[r, dashed, "f\bar{\otimes}f'"] & Y\otimes Y' \\
 & X\otimes Y' \arrow[ur] \arrow[urr, "f\otimes 1"'] & &
\end{tikzcd}$$

where Q is the push-out of $(f\otimes 1, 1\otimes f')$ in $\mathcal{V}_0$, and $[f,f']$ is defined by the diagram

$$\begin{tikzcd}
 & & {[XX']} \arrow[dr, "{[1,f']}"] & \\
{[YX']} \arrow[urr, "{[f,1]}"] \arrow[drr, "{[1,f']}"'] \arrow[r, dashed, "{[f,f']}"] & P \arrow[ur] \arrow[dr] & & {[XY']} \\
 & & {[YY']} \arrow[ur, "{[f,1]}"'] &
\end{tikzcd}$$

where P is the pull-back of $([f,1],[1,f'])$ in $\mathcal{V}_0$. The identity $\overline{I}$ is the unique morphism $0 \to I$, where 0 denotes the initial object of $\mathcal{V}_0$. A second closed structure on this category will be described in the following section.

<u>Section 3.3 Comonoid examples and general remarks</u>

When A has the structure of a comonoid in V-*Cat*, the tensor product (3.1.4) reduces to a <u>pointwise</u> expression:

$$\begin{aligned}(S\bar{\otimes}T)C &= \int^{AB}(SA\otimes TB)\otimes P(ABC)\\ &= \int^{AB}(SA\otimes TB)\otimes(A(AC)\otimes A(BC))\\ &\cong (\int^{A}A(AC)\otimes SA)\otimes(\int^{B}A(BC)\otimes TB)\\ &\cong SC\otimes TC.\end{aligned}$$

The internal-hom formulas (3.1.5) and (3.1.6) become

$$\begin{aligned}(T/S)A &= \int_{BC}[P(ABC),[SB,TC]]\\ &= \int_{BC}[A(AC)\otimes A(BC),[SB,TC]]\\ &\cong \int_{BC}[A(AC)\otimes A(BC)\otimes SB,TC]\\ &\cong \int_{C}[A(AC)\otimes\int^{B}A(BC)\otimes SB,TC]\\ &\cong \int_{C}[A(AC)\otimes SC,TC]\end{aligned}$$

and

$$\begin{aligned}(S\backslash T)A &= \int_{BC}[P(BAC),[SB,TC]]\\ &= \int_{BC}[A(BC)\otimes A(AC),[SB,TC]]\\ &\cong \int_{C}[SC\otimes A(AC),TC]\end{aligned}$$

respectively.

This type of biclosed functor category arises frequently in practice and some examples are listed below. Again, we shall suppose that A is a small category and that V is complete and cocomplete.

Example 3.3.1 If A is a comonoid in V-Cat with only one object then its endomorphism-object is a hcpf monoid in V, and $\{A,V\}$ is the usual biclosed category of modules over this monoid (cf. [9] IV §5).

Example 3.3.2 (V cartesian closed) If V is a cartesian closed category then V-Cat is a cartesian monoidal "category", hence every V-category A admits a unique (commutative) comonoid structure in V-Cat, with the diagonal functor $A \to A \times A$ as comultiplication and the unique functor $A \to I$ as counit. The reduced tensor-product formula obtained above shows that $\{A,V\}$ is cartesian closed.

Example 3.3.3 If A is an S-category then the comonoid structure on A induces a (commutative) comonoid structure in V-Cat on the free V-category F_*A generated by A. Thus the category $[F_*A,V]$, whose underlying S-category is isomorphic to the S-category $[A,V_0]$ of ordinary S-functors from A to V_0 and S-natural transformations between them, always admits a symmetric monoidal closed structure over V.

A given functor category $[A,V]$ may, of course, admit several distinct biclosed structures. Moreover, by the functor category theorem, these will correspond to non-isomorphic promonoidal structures on A. To illustrate, take the S-category in Example 3.3.3 to be the dual of the simplicial category Δ. The resulting symmetric closed structure on the category $[F_*\Delta^{op},V]$ of simplicial objects in V, differs from the non-symmetric biclosed structure discussed in Example 3.2.5.

Again, in Example 3.3.3, take A to be the free V-category on the arrow category $0 \to 1$. The category $[A,V]$ of morphisms in V then admits a closed structure with a pointwise tensor product; that is, with

$$(X \xrightarrow{f} Y)\,\bar{\otimes}\,(X' \xrightarrow{f'} Y') = X \otimes X' \xrightarrow{f \otimes f'} Y \otimes Y'.$$

This differs from the tensor product, discussed in Example 3.2.6, which was constructed from finite products in $0 \to 1$. In fact, the pointwise structure could equally be regarded as a "monoidal example" by taking the monoidal structure on A to be that arising from finite coproducts in $0 \to 1$.

While a large number of promonoidal categories can be viewed either as monoidal categories or as comonoids (or both), there do exist other types which we have not bothered to elaborate here. For example, consider the category $[A^{op}\otimes A,V]$ of "bimodules" over a small category A. This functor category admits a canonical biclosed structure for which

$$S\bar{\otimes}T(AB) = \int^{C} S(AC)\otimes T(CB) = S(AC)\underline{\otimes}T(CB)$$

$$T/S(AB) = \int_{C}[S(BC),T(AC)]$$

$$S\backslash T(AB) = \int_{C}[S(CA),T(CB)]$$

$$\bar{I}(AB) = A(AB)$$

$$\bar{\ell} : A(AC)\underline{\otimes}T(CB) \xrightarrow[y]{} T(AB)$$

$$\bar{r} : T(AC)\underline{\otimes}A(CB) \cong A(CB)\underline{\otimes}T(AC) \xrightarrow[y]{} T(AB)$$

$$\bar{a} : (R(AC)\underline{\otimes}S(CD))\underline{\otimes}T(DB) \cong R(AC)\underline{\otimes}(S(CD)\underline{\otimes}T(DB))$$

where $R,S,T\in[A^{op}\otimes A,V]$ and $(AB)\in A^{op}\otimes A$. This biclosed structure corresponds, by Corollary 3.1.5, to a promonoidal structure on $A^{op}\otimes A$ which, in general, is neither monoidal nor a comonoid.

CHAPTER 4

THE REFLECTION THEOREM

Section 4.1 Reflection of biclosed structures

Many closed categories arise as full reflective subcategories of others that are more "freely" constructed. For instance, the cartesian closed category of compactly generated spaces arises as a full reflective subcategory of the category of quasi-topological spaces introduced by E. Spanier [16], the latter category being cartesian closed in a particularly simple manner. Again, the closed category of sheaves of abelian groups, on a topology T, is a full reflective subcategory of the closed functor category $\{T^{op},Ab\}$.

In general, we seek properties of a reflection which enable us to conclude that the reflective subcategory is closed. The properties we shall discuss here are each equivalent to the requirement that the reflecting functor admit enrichment to a strong monoidal functor. This result is compatible with G.M. Kelly's observation (referred to in Section 2.3) that the closed left adjoint to any normal closed functor, is strong.

<u>Theorem 4.1.1</u> (<u>the reflection theorem</u>) Let $\mathcal{B} = (\mathcal{B}, \overline{\otimes}, \overline{I}, \ldots)$ be a biclosed category and let $\theta : \mathcal{C} \to \mathcal{B}$ be a full embedding with left adjoint ψ (we shall omit the symbol θ and denote the unit of the adjunction by $\eta : 1 \to \psi : \mathcal{B} \to \mathcal{B}$). Further, let $\mathcal{A} \subset \mathcal{B}$ be a strongly generating class of objects in $\mathcal{B}$. Then, in order that there exist a biclosed structure on $\mathcal{C}$ for which $\psi : \mathcal{B} \to \mathcal{C}$ admits enrichment to a strong monoidal functor, it is necessary and sufficient that <u>one</u> of the following pairs of morphisms be a pair of isomorphisms for all $A \in \mathcal{A}$, $B, B' \in \mathcal{B}$, and $C \in \mathcal{C}$:

1a) $\eta : C/B \to \psi(C/B)$

b) $\eta : B\backslash C \to \psi(B\backslash C)$

2a) $\eta : C/A \to \psi(C/A)$

b) $\eta : A\backslash C \to \psi(A\backslash C)$

3a) $\eta\backslash 1 : \psi B\backslash C \to B\backslash C$

b) $1/\eta : C/\psi B \to C/B$

4a) $\psi(\eta\otimes 1) : \psi(B\otimes B') \to \psi(\psi B\otimes B')$

b) $\psi(1\otimes\eta) : \psi(B'\otimes B) \to \psi(B'\otimes\psi B)$

5a) $\psi(\eta\otimes 1) : \psi(B\otimes A) \to \psi(\psi B\otimes A)$

b) $\psi(1\otimes\eta) : \psi(A\otimes B) \to \psi(A\otimes\psi B)$

6 $\psi(\eta\otimes\eta) : \psi(B\otimes B') \to \psi(\psi B\otimes\psi B')$.

The biclosed structure on $\mathcal{C}$ is then unique to within isomorphism.

<u>Proof</u> First we establish the equivalence of the six numbered conditions. These are arranged in transpose pairs (the last being self-transpose) so that m $\Rightarrow$ n whenever ma) $\Rightarrow$ na). We prove

1a) $\Rightarrow$ 2a) $\Rightarrow$ 3a) $\Rightarrow$ 4a) $\Rightarrow$ 5a) $\Rightarrow$ 3a)

4 $\Rightarrow$ 6 $\Rightarrow$ 1a).

1a) $\Rightarrow$ 2a) because $\mathcal{A} \subset \mathcal{B}$.

2a) $\Rightarrow$ 3a). Consider the following commuting diagram

$$\begin{tikzcd}
\mathcal{B}(A,\psi B\backslash C) \arrow[rr, "\mathcal{B}(1{,}\eta\backslash 1)"] \arrow[d, phantom, "\wr\|"] & & \mathcal{B}(A,B\backslash C) \arrow[d, phantom, "\|\wr"] \\
\mathcal{B}(\psi B,C/A) \arrow[rr, "\mathcal{B}(\eta 1)"] \arrow[d, "\mathcal{B}(1\eta)"'] & & \mathcal{B}(B,C/A) \arrow[d, "\mathcal{B}(1\eta)"] \\
\mathcal{B}(\psi B,\psi(C/A)) \arrow[rr, "\mathcal{B}(\eta 1)"'] & & \mathcal{B}(B,\psi(C/A))
\end{tikzcd}$$

where the vertical arrows are isomorphisms by 2a) and the bottom arrow is an isomorphism by the adjunction $\psi \dashv \theta$. Thus the top arrow is an isomorphism for all $A \in \mathcal{A}$. This is sufficient for 3a) to be an isomorphism because $\mathcal{A}$ is strongly generating.

3a) $\Rightarrow$ 4a). Consider the commuting diagram

$$\begin{array}{ccc} \mathcal{C}(\psi(\psi B\otimes B'),C) & \xrightarrow{\mathcal{C}(\psi(\eta\otimes 1),1)} & \mathcal{C}(\psi(B\otimes B'),C) \\ \downarrow & & \downarrow \\ \mathcal{B}(\psi B\otimes B',C) & \xrightarrow{\mathcal{B}(\eta\otimes 1,1)} & \mathcal{B}(B\otimes B',C) \\ \wr\Vert & & \Vert\wr \\ \mathcal{B}(B',\psi B\backslash C) & \xrightarrow[\mathcal{B}(1,\eta\backslash 1)]{} & \mathcal{B}(B',B\backslash C) \end{array} \qquad (4.1.1)$$

where the vertical arrows are isomorphisms from the adjunction $\psi \dashv \theta$, and the bottom arrow is an isomorphism by 3a). Thus, applying the representation theorem to C,

$$\psi(\eta\otimes 1) : \psi(B\otimes B') \to \psi(\psi B\otimes B')$$

is an isomorphism, as required.

4a) $\Rightarrow$ 5a) because $A \subset \mathcal{B}$.

5a) $\Rightarrow$ 3a). Consider diagram (4.1.1) with $B' = A\epsilon \mathcal{A}$. The top arrow is now an isomorphism by 5a), hence

$$\mathcal{B}(1,\eta\backslash 1) : \mathcal{B}(A,\psi B\backslash C) \to \mathcal{B}(A,B\backslash C)$$

is an isomorphism for all $A\epsilon\mathcal{A}$. Because $\mathcal{A}$ is strongly generating, this implies that 3a) is an isomorphism.

4 $\Rightarrow$ 6. This is immediate from the commutativity of

$$\begin{tikzcd} \psi(B\otimes B') \arrow[rr, "\psi(\eta\otimes\eta)"] \arrow[dr, "\psi(\eta\otimes 1)"'] & & \psi(\psi B\otimes\psi B') \\ & \psi(\psi B\otimes B') \arrow[ur, "\psi(1\otimes\eta)"'] & \end{tikzcd}$$

.

$6 \Rightarrow 1a)$. It suffices to produce a <u>left</u> inverse $\nu : \psi(C/B) \to C/B$ to $\eta : C/B \to \psi(C/B)$. Such a ν will automatically be <u>right</u> inverse to η because $\eta\nu : \psi(C/B) \to \psi(C/B)$, being in C_0, is uniquely determined by composition with the unit η; but $(\eta\nu)\eta = \eta(\nu\eta) = \eta$ hence $\eta\nu = 1$.

By the appropriate tensor-hom adjunction, it suffices to find a $\bar{\nu} : \psi(C/B)\otimes B \to C$ making

$$\begin{tikzcd}
(C/B)\otimes B \arrow[r, "e"] \arrow[d, "\eta\otimes 1"'] & C \\
\psi(C/B)\otimes B \arrow[ur, "\bar{\nu}"'] &
\end{tikzcd}$$

commute, where e is the appropriate evaluation morphism. This is achieved by taking $\bar{\nu} = \bar{e}.\psi(\eta\otimes\eta)^{-1}.\eta.1\otimes\eta$ from the diagram

$$\begin{tikzcd}
(C/B)\otimes B \arrow[rr, "e"] \arrow[drr, "\eta"] \arrow[d, "\eta\otimes 1"'] \arrow[dr, "\eta\otimes\eta"] & & C \\
 & & \psi((C/B)\otimes B) \arrow[u, dashed, "\bar{e}"'] \arrow[d, "\psi(\eta\otimes\eta)"] \\
\psi(C/B)\otimes B \arrow[r, "1\otimes\eta"'] & \psi(C/B)\otimes\psi B \arrow[r, "\eta"'] & \psi(\psi(C/B)\otimes\psi B)
\end{tikzcd}$$

where $\psi(\eta\otimes\eta)$ is an isomorphism by 6, and $\bar{e}$ is the unique factorisation of e through the adjunction unit η.

When these conditions are satisfied they produce a biclosed structure on $\mathcal{C}$ as follows. First set

$$C\hat{\otimes}C' = \psi(C\otimes C')$$

and

$$\tilde{\psi} = \tilde{\psi}_{BB'} : \psi B\hat{\otimes}\psi B' = \psi(\psi B\otimes\psi B') \xrightarrow{\psi(\eta\otimes\eta)^{-1}} \psi(B\otimes B')$$

$$\psi^{o} : \hat{I} = \psi I.$$

This $\hat{\otimes}$ will be biclosed because

$$\begin{aligned} \mathcal{C}(C\hat{\otimes}C',D) &= \mathcal{C}(\psi(C\otimes C'),D) \\ &\cong \mathcal{B}(C\otimes C',D) \text{ by the adjunction } \psi \dashv \theta \\ &\cong \mathcal{B}(C',C\backslash D) \\ &\cong \mathcal{B}(C',\psi(C\backslash D)) \text{ by 1a)} \\ &\cong \mathcal{C}(C',\psi(C\backslash D)) \end{aligned}$$

and, similarly,

$$\mathcal{C}(C\hat{\otimes}C',D) \cong \mathcal{C}(C,\psi(D/C')) \text{ by 1b).}$$

But ψ is dense, being left adjoint to a full embedding, hence, by Lemma 1.5.4, the monoidal functor <u>axioms</u> (2.2.3), (2.2.4), and (2.2.5) for $\Psi = (\psi,\tilde{\psi},\psi^{o})$ actually <u>define</u> isomorphisms

$$\hat{\ell} : \psi I\hat{\otimes}C \cong C, \quad \hat{r} : C\hat{\otimes}\psi I \cong C, \text{ and } \hat{a} : (C\hat{\otimes}C')\hat{\otimes}C'' \cong C\hat{\otimes}(C'\hat{\otimes}C'')$$

respectively. Furthermore, by Lemma 1.5.4, these isomorphisms satisfy MC1 and MC2 iff the centers of the diagrams (4.1.2) and (4.1.3) commute. But the exteriors of these diagrams commute by axioms MC1 and MC2 for $\mathcal{B}$, and each of the remaining subregions commutes either by definition of $\hat{\ell}$, $\hat{r}$, or $\hat{a}$, or by the $\mathcal{S}$-naturality of $\tilde{\psi}$ or $\hat{a}$. Hence $\mathcal{C} = (\mathcal{C},\hat{\otimes},\hat{I},\ldots)$ is a biclosed category and $\Psi = (\psi,\tilde{\psi},\psi^{o})$ is a strong monoidal functor.

To complete the proof, suppose that $(\hat{\otimes},\hat{I},\ldots)$ is a <u>given</u> biclosed structure on $\mathcal{C}$, and that $\Psi = (\psi,\tilde{\psi},\psi^{o})$ is a strong monoidal enrichment of ψ. Then, by the representation theorem, the following diagram (of isomorphisms) is completed for a unique isomorphism $f : C/B \to C\hat{/}\psi B$.

$$\begin{tikzcd}
\mathcal{B}(\psi(B'\otimes B),C) \arrow[rr, "\mathcal{B}(\eta,1)"] \arrow[d, "\mathcal{B}(\tilde{\psi},1)"'] & & \mathcal{B}(B'\otimes B,C) \arrow[d, phantom, "\cong"] \\
\mathcal{B}(\psi B'\,\hat{\otimes}\psi B,C) \arrow[d, phantom, "\cong"] & & \mathcal{B}(B',C/B) \arrow[d, dashed, "\mathcal{B}(1,f)"] \\
\mathcal{B}(\psi B',C\hat{/}\psi B) \arrow[rr, "\mathcal{B}(\eta,1)"'] & & \mathcal{B}(B',C\hat{/}\psi B)
\end{tikzcd} .$$

Hence $\eta : C/B \to \psi(C/B)$ is an isomorphism for all $B\epsilon\mathcal{B}$, $C\epsilon\mathcal{C}$, by naturality of η. Similarly, $\eta : B\backslash C \to \psi(B\backslash C)$ is always an isomorphism. Thus condition 1 is satisfied.

Diagram (4.1.2) (primes omitted)

$$
\begin{tikzcd}
\psi((B\otimes I)\otimes B) \arrow[rrrrrr, "\psi(a)"] \arrow[ddddrrr, "\psi(r\otimes 1)"'] & & & & & & \psi(B\otimes(I\otimes B)) \arrow[ddddlll, "\psi(1\otimes \ell)"] \\
& \psi(B\otimes I)\otimes\psi B \arrow[ul, "\tilde{\psi}"'] \arrow[ddrr, "\psi r\otimes 1"] & & & & \psi B\otimes\psi(I\otimes B) \arrow[ur, "\tilde{\psi}"] \arrow[ddll, "1\otimes\psi\ell"'] & \\
& & (\psi B\otimes\psi I)\otimes\psi B \arrow[ul, "\tilde{\psi}\otimes 1"'] \arrow[rr, "a"] \arrow[dr, "r\otimes 1"] & & \psi B\otimes(\psi I\otimes\psi B) \arrow[ur, "1\otimes\tilde{\psi}"] \arrow[dl, "1\otimes\ell"'] & & \\
& & & \psi B\otimes\psi B \arrow[d, "\tilde{\psi}"] & & & \\
& & & \psi(B\otimes B) & & &
\end{tikzcd}
$$

Diagram (4.1.3) (primes omitted)

$$
\begin{tikzcd}
\psi(((B\otimes B)\otimes B)\otimes B) \arrow[rrrr, "\psi(a)"] \arrow[ddddddd, "\psi(a\otimes 1)"'] & & & & \psi((B\otimes B)\otimes(B\otimes B)) \arrow[rrrr, "\psi(a)"] & & & & \psi(B\otimes(B\otimes(B\otimes B))) \\
 & \psi((B\otimes B)\otimes B)\otimes\psi B \arrow[ul, "\tilde{\psi}"'] \arrow[ddddd, "\psi a\otimes 1"] & & & \psi(B\otimes B)\otimes\psi(B\otimes B) \arrow[u, "\tilde{\psi}"'] & & & \psi B\otimes\psi(B\otimes(B\otimes B)) \arrow[ur, "\tilde{\psi}"] & \\
 & & (\psi(B\otimes B)\otimes\psi B)\otimes\psi B \arrow[ul, "\tilde{\psi}\otimes 1"'] \arrow[r, "a"] & \psi(B\otimes B)\otimes(\psi B\otimes\psi B) \arrow[ur, "1\otimes\tilde{\psi}"] & & (\psi B\otimes\psi B)\otimes\psi(B\otimes B) \arrow[ul, "\tilde{\psi}\otimes 1"'] \arrow[r, "a"] & \psi B\otimes(\psi B\otimes\psi(B\otimes B)) \arrow[ur, "1\otimes\tilde{\psi}"] & & \\
 & & ((\psi B\otimes\psi B)\otimes\psi B)\otimes\psi B \arrow[u, "(\tilde{\psi}\otimes 1)\otimes 1"'] \arrow[rr, "a"] \arrow[d, "a\otimes 1"'] & & (\psi B\otimes\psi B)\otimes(\psi B\otimes\psi B) \arrow[ul, "\tilde{\psi}\otimes 1"'] \arrow[ur, "1\otimes\tilde{\psi}"] \arrow[rr, "a"] & & \psi B\otimes(\psi B\otimes(\psi B\otimes\psi B)) \arrow[u, "1\otimes(1\otimes\tilde{\psi})"'] & & \\
 & & (\psi B\otimes(\psi B\otimes\psi B))\otimes\psi B \arrow[rrrr, "a"'] \arrow[d, "(1\otimes\tilde{\psi})\otimes 1"] & & & & \psi B\otimes((\psi B\otimes\psi B)\otimes\psi B) \arrow[u, "1\otimes a"'] \arrow[d, "1\otimes(\tilde{\psi}\otimes 1)"'] & & \\
 & & (\psi B\otimes\psi(B\otimes B))\otimes\psi B \arrow[rrrr, "a"'] \arrow[dl, "\tilde{\psi}\otimes 1"] & & & & \psi B\otimes(\psi(B\otimes B)\otimes\psi B) \arrow[dr, "1\otimes\tilde{\psi}"'] & & \\
 & \psi(B\otimes(B\otimes B))\otimes\psi B \arrow[dl, "\tilde{\psi}"] & & & & & & \psi B\otimes\psi((B\otimes B)\otimes B) \arrow[uuuuu, "1\otimes\psi a"'] \arrow[dr, "\tilde{\psi}"'] & \\
\psi((B\otimes(B\otimes B))\otimes B) \arrow[rrrrrrrr, "\psi(a)"'] & & & & & & & & \psi(B\otimes((B\otimes B)\otimes B)) \arrow[uuuuuuu, "\psi(1\otimes a)"']
\end{tikzcd}
$$

Section 4.2 Additional conditions

Throughout this section we suppose that $\mathcal{A}\subset\mathcal{B}$, $\mathcal{C}\subset\mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ are as given in the hypotheses of Theorem 4.1.1. Furthermore, we take $\mathcal{D}\subset\mathcal{C}$ to be a strongly cogenerating class in the subcategory $\mathcal{C}$.

Corollary 4.2.1 Conditions 1 to 6 are (each) equivalent to:

7a) $\eta : D/A \to \psi(D/A)$

b) $\eta : A\backslash D \to \psi(A\backslash D)$

are isomorphisms for all $A\in\mathcal{A}$ and $D\in\mathcal{D}$.

Proof 2a) $\Rightarrow$ 7a) because $\mathcal{D}\subset\mathcal{C}$.

7a) $\Rightarrow$ 5a). Consider the commuting diagram

$$\begin{tikzcd}
\mathcal{C}(\psi(\psi B\otimes A),D) \arrow[r, "\mathcal{C}(\psi(\eta\otimes 1)\text{,}1)"] \arrow[d] & \mathcal{C}(\psi(B\otimes A),D) \arrow[d] \\
\mathcal{B}(\psi B\otimes A,D) \arrow[r, "\mathcal{B}(\eta\otimes 1\text{,}1)"] \arrow[d, phantom, "\cong"] & \mathcal{B}(B\otimes A,D) \arrow[d, phantom, "\cong"] \\
\mathcal{B}(\psi B,D/A) \arrow[r, "\mathcal{B}(\eta\text{,}1)"] \arrow[d, "\mathcal{B}(1\text{,}\eta)"'] & \mathcal{B}(B,D/A) \arrow[d, "\mathcal{B}(1\text{,}\eta)"] \\
\mathcal{B}(\psi B,\psi(D/A)) \arrow[r, "\mathcal{B}(\eta\text{,}1)"] & \mathcal{B}(B,\psi(D/A))
\end{tikzcd}$$

where the bottom and unlabelled vertical arrows are isomorphisms by the adjunction $\psi \dashv \theta$, and the arrows $\mathcal{B}(1,\eta)$ are isomorphisms by 7a). Then the top arrow is an isomorphism, whence, because $\mathcal{D}$ is strongly cogenerating in $\mathcal{C}$, $\psi(\eta\otimes 1) : \psi(B\otimes A) \to \psi(\psi B\otimes A)$ is an isomorphism for all $A\in\mathcal{A}$ and $B\in\mathcal{B}$, as required.

Corollary 4.2.2 If $\mathcal{A}\subset\mathcal{B}$ is dense then conditions 1 to 7 are equivalent to:

8. For each $A\in\mathcal{A}$ and $D\in\mathcal{D}$ there exist objects $H(AD)$ and $K(AD)$ in $\mathcal{C}$, and natural isomorphisms

a) $\mathcal{C}(\psi(-\otimes A),D) \cong \mathcal{C}(\psi-,H(AD))$

b) $\mathcal{C}(\psi(A\otimes -),D) \cong \mathcal{C}(\psi-,K(AD))$.

Proof 7a) ⇒ 8a). Take $H(AD) = \psi(D/A)$ and the isomorphism to be

$$\begin{aligned}\mathcal{C}(\psi(A'\otimes A),D) &\cong \mathcal{B}(A'\otimes A,D) \text{ by adjunction}\\ &\cong \mathcal{B}(A',D/A)\\ &\cong \mathcal{B}(A',\psi(D/A)) \text{ by 7a)}\\ &\cong \mathcal{C}(\psi A',\psi(D/A)) \text{ by adjunction,}\end{aligned}$$

which is natural in $A'\in\mathcal{A}$.

8a) ⇒ 7a). Because $\mathcal{A}\subset\mathcal{B}$ is dense, the composite

$$\begin{aligned}\mathcal{B}(A',D/A) &\cong \mathcal{B}(A'\otimes A,D)\\ &\cong \mathcal{C}(\psi(A'\otimes A),D) \text{ by adjunction}\\ &\cong \mathcal{C}(\psi A',H(AD)) \text{ by 8a)}\\ &\cong \mathcal{B}(A',H(AD)) \text{ by adjunction,}\end{aligned}$$

is of the form $\mathcal{B}(1,f)$ for a unique isomorphism $f : D/A \cong H(AD)$.

Then, because $H(AD)$ is in $\mathcal{C}$, 7a) is an isomorphism by the naturality of η.

Now suppose that the given $\mathcal{B}$ is a cartesian closed category; this requires us to suppose also that the ground category $\mathcal{V}$ is cartesian closed, otherwise finite products in $\mathcal{B}$ do not yield a bifunctor $\bar{\otimes}$ on $\mathcal{B}$. If

$$B \xleftarrow{\ p\ } B\times B' \xrightarrow{\ p'\ } B'$$

denotes a typical product in $\mathcal{B}$, the diagram

$$\begin{tikzcd}
\psi(B\times B') \arrow[r, "\psi(\eta\times\eta)"] \arrow[dr, "\begin{pmatrix}\psi p\\ \psi p'\end{pmatrix}"'] & \psi(\psi B\times\psi B') \\
 & \psi B\times\psi B' \arrow[u, "\eta"']
\end{tikzcd}$$

may be verified to commute by composing both legs with the unit $\eta : B\times B' \to \psi(B\times B')$. This procedure is valid because $\mathcal{C}$, being reflective in $\mathcal{B}$, is "closed" under the formation of finite limits in $\mathcal{B}$; in other words

$$\eta : \psi B\times\psi B' \to \psi(\psi B\times\psi B')$$

is an isomorphism. Thus $\psi(\eta\times\eta)$ is an isomorphism (condition 6) if and only if $\begin{pmatrix}\psi p\\ \psi p'\end{pmatrix}$ is an isomorphism.

Corollary 4.2.3 ($\mathcal{V}$ cartesian closed) If $\mathcal{B}$ is cartesian closed then conditions 1 to 8 are equivalent to:

9. $\psi : \mathcal{B} \to \mathcal{C}$ preserves finite products.

Section 4.3 A topological application

For this section we suppose that $\mathcal{V} = \mathcal{S}$ and let $\mathcal{Top}$ denote the category of all topological spaces and continuous maps. It is well known that $\mathcal{Top}$ itself is not cartesian closed. However there exists a general process for generating cartesian closed subcategories of $\mathcal{Top}$. We shall describe this below, making use of the reflection theorem.

Let $\mathcal{A}$ be a full subcategory of $\mathcal{Top}$, containing the one point space "*". Construct categories $\mathcal{B}$ and $\mathcal{C}$ as follows.

$\mathcal{B}$ = "$\mathcal{A}$-simplicial" bases: an object of $\mathcal{B}$ is a set B together with, for each $A \in \mathcal{A}$, a set Ad(AB) of admissible set maps from A to B; these may be thought of as the "simplices" of type A in B. The sets Ad(AB) are subject to the axioms:

A1. All constant maps are admissible

A2. If $g \in \mathcal{A}(AA')$ and $f \in \mathrm{Ad}(A'B)$ then $fg \in \mathrm{Ad}(AB)$.

A morphism $f : B \to B'$ of bases is a set map having the property that $fg \in \mathrm{Ad}(AB')$ whenever $g \in \mathrm{Ad}(AB)$, $A \in \mathcal{A}$.

It is straightforward to verify that $\mathcal{B}$ is complete and cocomplete, cartesian closed, and (canonically) contains $\mathcal{A}$ as a dense full subcategory. The internal-hom of bases B and B' in $\mathcal{B}$ is obtained by taking [BB'] to be the set $\mathcal{B}$(BB') with $f \in Ad(A,[BB'])$ if and only if the map $A \times B \to B'$, $(a,b) \rightsquigarrow f(a)(b)$, is a morphism in $\mathcal{B}$. The category $\mathcal{B}$ is thus constructed directly along the lines of the category of quasi-topological spaces introduced by E. Spanier [16]; however, we do not require Spanier's third and fourth axioms.

<u>$\mathcal{C}$ = $\mathcal{A}$-generated topological spaces</u>: this is the full subcategory of *Top* comprising those spaces X having the property:

A subset V is open in X if (and only if) $f^{-1}V$ is open in A for all $f \in Top(AX)$ and $A \in \mathcal{A}$.

The category $\mathcal{C}$ coincides with the full subcategory of *Top* determined by the spaces that are <u>direct limits</u> in *Top* of objects of $\mathcal{A}$. The embedding $\mathcal{C} \subset Top$ admits an evident <u>right adjoint</u> $W : Top \to \mathcal{C}$

$$Top(C,X) \cong \mathcal{C}(C,WX) \qquad (4.3.1)$$

where WX has the same underlying set as X, but a new (finer) topology given by:

A subset V is open in WX if and only if $f^{-1}V$ is open in A for all $f \in Top(AX)$ and $A \in \mathcal{A}$.

In particular, $\mathcal{C}$ is <u>complete</u> and <u>cocomplete</u>; however, one must be aware that a <u>product</u> in $\mathcal{C}$ is <u>not</u> a topological product unless the latter already lies in C.

The categories $\mathcal{B}$ and $\mathcal{C}$ are related by a <u>full embedding</u> $\theta : \mathcal{C} \subset \mathcal{B}$, given by

θC = the underlying set of C, with $\mathrm{Ad}(AC) = \mathit{Top}(AC)$ for each $A \in \mathcal{A}$,

with <u>left adjoint</u> $\psi : \mathcal{B} \to \mathcal{C}$ given by

ψB = the underlying set of B with $V \subset \psi B$ open if and only if $f^{-1}V$ is open in A for all $f \in \mathrm{Ad}(AB)$ and $A \in \mathcal{A}$.

In brief, ψB is the topological <u>realisation</u> of the base B.

We are now in a position to ask whether the cartesian closed structure on $\mathcal{B}$ is reflected into $\mathcal{C}$ by ψ. Before supplying <u>sufficient</u> conditions on $\mathcal{A}$ for this to be so, we recall two important points from Day-Kelly [4].

Firstly, we may define a <u>topology</u> $\Omega(X)$, on the set $\Omega(X)$ of open subsets of a space X, by taking $H \subset \Omega(X)$ to be open precisely when it satisfies the conditions

O1. If $V, V' \in \Omega(X)$ with $V \subset V'$ and if $V \in H$ then $V' \in H$.

O2. If $V_\lambda \in \Omega(X)$ for $\lambda \in \Lambda$ and if $\cup_\lambda V_\lambda \in H$, then there exists a finite subset $\{\lambda_1, \ldots, \lambda_n\} \subset \Lambda$ such that $V_{\lambda_1} \cup \ldots \cup V_{\lambda_n} \in H$.

Secondly, if a space X has the property that $-\times X : \mathit{Top} \to \mathit{Top}$ preserves topological quotient maps, then a map $f : Y \to \Omega(X)$ is continuous if and only if the set

$\{(y,x)\in Y\times X \mid x\in f(y)\}$ is open in $Y\times X$. We shall call such an X an <u>Ω-compact space</u>. These spaces are topologically characterised in [4] Theorem 3, and it is shown in [4] Proposition 5 that a hausdorff space is Ω-compact if and only if it is locally compact.

Now let $D = \Omega(*)$; this is the topological space of two points, one of which is open and the other not open. Then, using the obvious bijection between the open subsets of a space X and the elements of $\mathit{Top}(XD)$, we have that X is Ω-compact if and only if

$$\mathit{Top}(Y\times X,D) \cong \mathit{Top}(Y,\Omega(X)) \qquad (4.3.2)$$

for all $Y\in\mathit{Top}$.

<u>Theorem 4.3.1</u> If $\mathcal{A}\subset\mathit{Top}$ is a class of Ω-compact spaces with the property that each functor $-\times A : \mathit{Top} \to \mathit{Top}$, $A\in\mathcal{A}$, maps $\mathcal{A}$ into the category $\mathcal{C}$ of $\mathcal{A}$-generated spaces, then $\mathcal{C}$ is cartesian closed.

<u>Proof</u> Because each $A\in\mathcal{A}$ is Ω-compact, we have

$$\mathit{Top}(A'\times A,D) \cong \mathit{Top}(A',\Omega(A))$$

for all $A',A\in\mathcal{A}$, by (4.3.2). Moreover, the hypothesis that each $-\times A : \mathit{Top} \to \mathit{Top}$ maps $\mathcal{A}$ to $\mathcal{C}$ ensures that the topological product $A'\times A$ is the product of A' and A <u>in</u> $\mathcal{C}$. Thus we obtain

$$C(A' \times A, WD) \cong C(A', W\Omega(A))$$

for all $A', A \in \mathcal{A}$, by (4.3.1). Because, in this instance, the dense subcategory $\mathcal{A} \subset \mathcal{B}$ actually lies in $\mathcal{C}$, we have just established condition 8 in Corollary 4.2.2. Thus conditions 1 to 9 hold provided WD is a strong cogenerator for $\mathcal{C}$. But it isn't; it only detects open subsets. However, the adjunction unit $\eta : 1 \to \theta\psi$ is a bijection since neither θ nor ψ alters underlying sets. Thus the map

$$\psi(\eta \times 1) : \psi(B \times A) \to \psi(\psi B \times A),$$

being a bijection for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, is a homeomorphism if and only if

$$C(\psi(\eta \times 1), 1) : C(\psi(\psi B \times A), WD) \to C(\psi(B \times A), WD)$$

is a bijection of "open sets". In other words, we still obtain 7a) ⇒ 5a) in the proof of Corollary 4.2.1. Thus the chain of conditions is complete and the result follows from Theorem 4.1.1.

The above proof tells us that

a) the internal-hom $[BC]$ in $\mathcal{B}$ is an $\mathcal{A}$-generated topological space for any base B and $\mathcal{A}$-generated space C,

b) the realisation functor $\psi : \mathcal{B} \to \mathcal{C}$ is finite product preserving

c) the adjunction $\psi \dashv \theta$ lifts to a homeomorphism

$$[\psi B, C] \cong [B, C].$$

To apply the theorem, choose <u>any</u> class of Ω-compact topological spaces and close it up under finite products <u>if necessary</u>. Then the category of all direct limits of these spaces in *Top*, is cartesian closed. For example, we could take A to be <u>all</u> Ω-compact spaces. This A is already closed under finite products and arbitrary coproducts in *Top*, thus:

<u>Corollary 4.3.2</u> The category of all topological quotients of Ω-compact spaces, and continuous maps, is a cartesian closed category.

Similarly:

<u>Corollary 4.3.3</u> The category of all topological quotients of locally compact hausdorff spaces, and continuous maps, is cartesian closed.

The latter category is that of <u>compactly generated spaces</u> (examined in a previous thesis [6]). Another interesting example, which I indirectly owe to J. Moore, is obtained by taking the objects of A to be the affine simplices Δ^n, $n \in N$; although this A is not closed under finite topological products, each $\Delta^m \times \Delta^n$ <u>is</u> a topological quotient of Δ's. The resulting category C, of all topological quotients of CW-complexes, is cartesian closed.

Section 4.4 Reflection of *Hom*

On applying the reflection theorem of Section 4.1 to the *Hom* construction of Section 3.1, we immediately obtain a set of conditions under which a "small" promonoidal category A generates a biclosed structure on a reflective class of functors $C \subset [A, V]$. In particular, consider the second condition provided by Theorem 4.1.1 when $B = \{A, V\}$ and the strongly generating class in B is taken to be the class of left represented functors $LA : A \to V$. From the higher representation theorem applied to the internal-hom formulas (3.1.5) and (3.1.6), together with the naturality of the adjunction unit η, we need only establish that

$$S/LA = \int_{A'A''}[P(-A'A''),[A(AA'),SA'']] \cong \int_{A''}[P(-AA''),SA'']$$

$$LA\backslash S = \int_{A'A''}[P(A'-A''),[A(AA'),SA'']] \cong \int_{A''}[P(A-A''),SA''],$$

as functors from A to V, admit isomorphs in C whenever $A \in A$ and $S \in C$. Furthermore, if the category A is monoidal then the higher representation theorem yields

$$S/LA \cong S(-\otimes A)$$

$$LA\backslash S \cong S(A\otimes -).$$

To apply the preceding criterion to a familiar situation, let V be suitably complete and let A be the free V-category on the dual of the S-category of open subsets of a topological space X, the latter being given the cartesian closed structure:

$I = X$, $V \otimes V' = V \cap V'$, $[VV'] = \mathrm{int}\,((\mathrm{compl}\ V) \cup V')$ for open subsets V, V' of X.

Then the $\mathcal{V}$-category of <u>sheaves</u> over X with values in $\mathcal{V}$, is a closed category provided the functor $S(-\cap V) : A \to \mathcal{V}$ is a sheaf whenever S is one; but this is immediate from the definition of sheaf. Thus conditions 1 to 8 are satisfied, and 9 also holds if $\mathcal{V}$ is cartesian closed.

A second application is obtained by taking $\mathcal{V}$ <u>cartesian closed</u> and letting A be a commutative finitary $\mathcal{V}$-theory with the monoidal structure described in Example 3.2.3. <u>Assume</u> that the category C of A-algebras is reflective in $[A,\mathcal{V}]$; the reflection $\psi : [A,\mathcal{V}] \to C$ <u>can</u> be constructed if $\mathcal{V}$ has small colimits. However, we have already constructed the tensor product in $C \subset \{A,\mathcal{V}\}$, regardless of cocompleteness considerations for C (see Example 3.2.3). The reflection theorem now tells us that $\psi : \{A,\mathcal{V}\} \to C$ admits enrichment to a strong monoidal functor (that is, preserves tensor products) because each functor $-\tilde{\otimes} n : A \to A$ preserves finite products, hence $S(-\otimes n) : A \to \mathcal{V}$ is an A-algebra whenever S is one.

CHAPTER 5

THE CONSTRUCTION THEOREM

Section 5.1 Preliminaries

We wish to examine the extent to which procedures of the previous chapters can be applied to the consideration of an arbitrary promonoidal category A. In these circumstances the total functor category $[A,V]$ is no longer available because the end $[A,V](ST) = \int_A [SA,TA]$ may not exist in V for all functors $S,T : A \to V$. Nevertheless, certain "reflective subcategories" C of $[A,V]$ do exist in the sense that there is given a dense functor $M : A^{op} \to C$. In view of this, the promonoidal structure of A may yet determine a biclosed structure on C, without reference to $[A,V]$. Before formulating a theorem to this effect, in Section 5.3, we need two generalised concepts.

The concept of V-natural transformation may be extended to describe certain families of morphisms which occur between Kan extensions. First we note that, for functors $F,G : B \to C$ between tensored categories B and C, an S-natural transformation $\beta : F \to G$ is V-natural if and only if the canonical diagram

$$\begin{array}{ccc} X\otimes FB & \xrightarrow{\tau} & F(X\otimes B) \\ {\scriptstyle 1\otimes\beta_B}\downarrow & & \downarrow{\scriptstyle \beta_{X\otimes B}} \\ X\otimes GB & \xrightarrow[\tau]{} & G(X\otimes B) \end{array}$$

commutes for all $X\epsilon V$ and $B\epsilon \mathcal{B}$. This criterion for V-naturality is established in Appendix 3 (where we also recall the definition of τ).

To generalise, let $F,G : A^{op} \to C$ be functors into a suitably tensored category C, and let $SA\otimes FA \to \overline{F}S$ and $SA\otimes GA \to \overline{G}S$ be a pair of coends over A for each functor $S : A \to V$ admitting such; we think of $\overline{F}S$ and $\overline{G}S$ as being "functorial" in S. Now suppose that we are given a family $\beta_S : \overline{F}S \to \overline{G}S$ of morphisms in C_0, indexed by the class of functors $S : A \to V$ for which $\overline{F}S$ and $\overline{G}S$ both exist.

<u>Definition 5.1.1</u> The given family $\beta_S : \overline{F}S \to \overline{G}S$ is called <u>neonatural</u> in S if, for all natural transformations $\gamma : S \to T$ and objects $X\epsilon V$, the canonical diagrams

$$\begin{array}{ccc} \overline{F}S & \xrightarrow{\beta_S} & \overline{G}S \\ {\scriptstyle \overline{F}\gamma}\downarrow & & \downarrow{\scriptstyle \overline{G}\gamma} \\ \overline{F}T & \xrightarrow[\beta_T]{} & \overline{G}T \end{array} \qquad \begin{array}{ccc} X\otimes\overline{F}S & \cong & \overline{F}(X\otimes S) \\ {\scriptstyle 1\otimes\beta_S}\downarrow & & \downarrow{\scriptstyle \beta_{X\otimes S}} \\ X\otimes\overline{G}S & \cong & \overline{G}(X\otimes S) \end{array}$$

commute whenever they are defined.

In circumstances where $[A,V]$ can be constructed, and the coends $\overline{F}S$ and $\overline{G}S$ exist in C for all functors $S : A \to V$, we obtain canonical functors $\overline{F},\overline{G} : [A,V] \to C$ by Lemma 1.3.1. Then a neonatural transformation $\beta : \overline{F} \to \overline{G}$ is precisely a natural transformation from $\overline{F}$ to $\overline{G}$. But such a transformation is uniquely determined by its represented-functor components. This is true of neonatural transformations in general:

Lemma 5.1.2 There is a canonical bijection between neonatural transformations from $\overline{F}$ to $\overline{G}$ and natural transformations from $\overline{F}(L-)$ to $\overline{G}(L-)$.

Proof By the higher representation theorem, the coends $\overline{F}(LA) \cong FA$ and $\overline{G}(LA) \cong GA$ always exist; they are canonically functorial in A by Lemma 1.3.1. Again, the higher representation theorem provides a coend

$$\gamma : SA \otimes LA \to S$$

for each functor $S : A \to V$. If $\overline{F}S$ and $\overline{G}S$ both exist we obtain induced coends

$$\overline{F}\gamma : \overline{F}(SA \otimes LA) \to \overline{F}S$$

$$\overline{G}\gamma : \overline{G}(SA \otimes LA) \to \overline{G}S$$

by Lemma 1.3.4. Then, given a natural transformation $\beta : \overline{F}(L-) \to \overline{G}(L-)$, we define $\overline{\beta}_S : \overline{F}S \to \overline{G}S$ to be the unique morphism making the following diagram commute:

$$\begin{tikzcd}
SA\otimes\overline{F}(LA) \arrow[r, "1\otimes\beta_A"] \arrow[d, phantom, "\wr\|"] & SA\otimes\overline{G}(LA) \arrow[d, phantom, "\|\wr"] \\
\overline{F}(SA\otimes LA) \arrow[d, "\overline{F}\gamma"'] & \overline{G}(SA\otimes LA) \arrow[d, "\overline{G}\gamma"] \\
\overline{F}S \arrow[r, dashed, "\overline{\beta}_S"'] & \overline{G}S
\end{tikzcd} \qquad (5.1.1)$$

It is straightforward to verify that the resulting family $\overline{\beta}$ is neonatural. Moreover, the diagram

$$\begin{tikzcd}
LA'(A)\otimes\overline{F}(LA) \arrow[r, "1\otimes\beta_A"] \arrow[d, phantom, "\wr\|"] & LA'(A)\otimes\overline{G}(LA) \arrow[d, phantom, "\|\wr"] \\
\overline{F}(LA'(A)\otimes LA) \arrow[d, "\overline{F}\gamma"'] & \overline{G}(LA'(A)\otimes LA) \arrow[d, "\overline{G}\gamma"] \\
\overline{F}(LA') \arrow[r, "\beta_{A'}"'] & \overline{G}(LA')
\end{tikzcd} \qquad (5.1.2)$$

commutes for all $A,A'\in\mathcal{A}$ on applying the representation theorem to A'. Thus $\beta_A = \overline{\beta}_{LA}$ for all $A\in\mathcal{A}$. It is now immediate that the diagram (5.1.1), which defines $\overline{\beta}$, defines the unique neonatural transformation whose restriction is β. Conversely, given an arbitrary neonatural transformation $\overline{\beta} : \overline{F} \to \overline{G}$, the diagram (5.1.2), with β_A replaced by $\overline{\beta}_{LA}$ throughout, commutes by neonaturality. Thus $\overline{\beta}_{LA} : \overline{F}(LA) \to \overline{G}(LA)$ is natural in A by the Lemma 1.3.3 on induced naturality.

In practice a neonatural transformation may arise through the use of possibly distinct choices of coend representations for $\overline{F}S$ and $\overline{G}S$. Lemma 5.1.1 and 1.3.1 guarantee that this choice is irrelevant provided the functoriality of $\overline{F}(LA)$ and $\overline{G}(LA)$ in A is <u>fixed</u>, and that the chosen coends are <u>natural</u> in this extra variable.

An important example of a neonatural transformation is the isomorphism

$$z = z_S : \int^A [A,\mathcal{V}](LA,S)\otimes\overline{F}(LA) \cong \overline{F}S$$

which is defined to be the extension of the composite

$$\begin{array}{ccc} \int^A [A,\mathcal{V}](LA,LA')\otimes\overline{F}(LA) & \dashrightarrow & \overline{F}(LA') \\ \wr\Vert & & \Vert\wr \\ \int^A A(A'A)\otimes FA' & \xrightarrow[y]{\hspace{3em}} & FA' \end{array} .$$

Using this definition of z it is clear that the "coherence" diagrams in Appendix 1 still commute.

The second concept to be generalised is that of strong monoidal functor. We do this by observing that, if $\Phi = (\phi,\hat{\phi},\phi^{\bullet}) : \mathcal{A} \to \mathcal{B}$ is a promonoidal functor from a promonoidal category $\mathcal{A}$ to a ϕ-cotensored <u>monoidal</u> category $\mathcal{B}$, then the natural transformations

$$\hat{\phi} : P(AA'X) \to \mathcal{B}(\phi A\otimes\phi A',\phi X)$$

$$\phi^{\bullet} : JX \to \mathcal{B}(I,\phi X)$$

transform to

$$\phi A \otimes \phi A' \to [P(AA'X), \phi X]$$
$$I \to [JX, \phi X]$$

under the cotensoring adjunction. These in turn provide morphisms

$$\begin{aligned} &\phi A \otimes \phi A' \to \int_X [P(AA'X), \phi X] \\ &I \to \int_X [JX, \phi X] \end{aligned} \qquad (5.1.3)$$

<u>Definition 5.1.3</u> The promonoidal functor $\Phi : A \to B$ is <u>strong</u> if the morphisms (5.1.3) are isomorphisms.

Section 5.2 The "monoidal" presentation of a promonoidal category

Let A be a promonoidal category and, for functors $S,T : A \to V$, let

$$S\bar{\otimes}T = \int^{AA'}(SA\otimes TA')\otimes P(AA'-)$$

$$F(ST) = \int_A [SA,TA]$$

whenever they exist. The very definition of promonoidal category ensures that certain $\bar{\otimes}$-products do exist; for example, $J\bar{\otimes}LA$, $LA\bar{\otimes}LA'$, $(LA\bar{\otimes}J)\bar{\otimes}LA'$, $((LA\bar{\otimes}LA')\bar{\otimes}LA'')\bar{\otimes}LA'''$. Moreover, these expressions are canonically functorial in A, A', ... by Lemma 1.3.1.

Replace the promonoidal structure on A by the isomorphic structure determined by the isomorphisms

$$\hat{\chi} : P(AA'-) \to F(L-,LA\bar{\otimes}LA')$$

$$\chi^{\bullet} : J \to F(L-,J)$$

and Lemma 2.2.5. This modification of A enables us to produce natural isomorphisms $J\bar{\otimes}LA \cong LA$, $LA\bar{\otimes}J \cong LA$, and $(LA\bar{\otimes}LA')\bar{\otimes}LA'' \cong LA\bar{\otimes}(LA'\bar{\otimes}LA'')$ directly from diagrams (2.1.1), (2.1.2), and (2.1.3) respectively. Then, using Lemma 5.1.2 and the coend definition of $\bar{\otimes}$, these isomorphisms admit neonatural extensions:

$$\bar{\ell}_S : J\bar{\otimes}S = S$$

$$\bar{r}_S : S\bar{\otimes}J = S$$

$$\bar{a}_{SAA'} : (S\bar{\otimes}LA)\bar{\otimes}LA' = S\bar{\otimes}(LA\bar{\otimes}LA')$$

$$\bar{a}_{ASA'} : (LA\bar{\otimes}S)\bar{\otimes}LA' = LA\bar{\otimes}(S\bar{\otimes}LA')$$

$$\bar{a}_{AA'S} : (LA\bar{\otimes}LA')\bar{\otimes}S = LA\bar{\otimes}(LA'\bar{\otimes}S).$$

Now we are able to write down the "axioms":

$$\begin{tikzcd}
(LA\bar{\otimes}J)\bar{\otimes}LA' \arrow[rr, "\bar{a}"] \arrow[dr, "\bar{r}\otimes 1"'] & & LA\bar{\otimes}(J\bar{\otimes}LA') \arrow[dl, "1\otimes\bar{\ell}"] \\
& LA\bar{\otimes}LA' &
\end{tikzcd}$$

$$\begin{tikzcd}
((LA\bar{\otimes}LA')\bar{\otimes}LA'')\bar{\otimes}LA''' \arrow[r, "\bar{a}"] \arrow[d, "\bar{a}\otimes 1"'] & (LA\bar{\otimes}LA')\bar{\otimes}(LA''\bar{\otimes}LA''') \arrow[r, "\bar{a}"] & LA\bar{\otimes}(LA'\bar{\otimes}(LA''\bar{\otimes}LA''')) \\
(LA\bar{\otimes}(LA'\bar{\otimes}LA''))\bar{\otimes}LA''' \arrow[rr, "\bar{a}"'] & & LA\bar{\otimes}((LA'\bar{\otimes}LA'')\bar{\otimes}LA''') \arrow[u, "1\otimes\bar{a}"']
\end{tikzcd}$$

The commutativity of these diagrams is equivalent to the validity of axioms PC1 and PC2 for A. To see this, return to diagrams (2.1.4) and (2.1.5) and observe that the center regions commute if and only if the above diagrams do; the remaining subregions of (2.1.4) and (2.1.5) commute for the "same" reasons they did so before, namely the definitions of $\bar{\ell}$, $\bar{r}$, $\bar{a}$, the neonaturality of $\bar{\ell}$, $\bar{r}$, $\bar{a}$, z (using Lemma 5.1.2), and the "coherence" of z (Appendix 1).

Thus, the promonoidal structure on A may be formally represented by a "partial monoidal" structure on the left-represented functors from A to V.

Section 5.3 The construction theorem

Theorem 5.3.1 Let A be a promonoidal category and let $M : A^{op} \to C$ be a dense functor into a cotensored and M-tensored category C. Then, in order for there to exist a biclosed structure on C for which $M^{op} : A \to C^{op}$ admits enrichment to a strong promonoidal functor, it is necessary and sufficient that the coends and ends

(1) $Q(AA') = \int^X P(AA'X)\otimes MX$

(2) $\int^X JX\otimes MX$

(3) $\int^{XX'} (C(MX,C)\otimes C(MX',C'))\otimes Q(XX')$

(4) $H(AC) = \int^X C(Q(XA),C)\otimes MX$

(5) $K(AC) = \int^X C(Q(AX),C)\otimes MX$

(6) $\int_X [C(MX,C),H(XC')]$

(7) $\int_X [C(MX,C),K(XC')]$

exist in C for all $A,A'\in A$ and $C,C'\in C$, and the resulting morphisms

(8) $C(Q(XA),C) \to C(MX,H(AC))$

(9) $C(Q(AX),C) \to C(MX,K(AC))$

be isomorphisms. The biclosed structure on C is then unique to within isomorphism.

Proof of necessity Let $C = (C,\hat{\otimes},\hat{I},\ldots)$ be a biclosed structure on C for which $M^{op} : A \to C^{op}$ admits enrichment to a strong promonoidal functor. Then, by Definition

5.1.3, the coends (1) and (2) are required to exist with isomorphisms

$$Q(AA') = \int^{X} P(AA'X) \otimes MX \cong MA \hat{\otimes} MA'$$

$$\int^{X} JX \otimes MX \cong \hat{I}.$$

Coend (3) exists because $\hat{\otimes}$ admits right adjoints to each variable, hence preserves the density expression

$$\int^{X} C(MX,C) \otimes MX \cong C$$

in each variable. The existence of the coends (4) and (5), and the isomorphisms (8) and (9), follows from the isomorphisms

$$C(Q(XA),C) \cong C(MX \hat{\otimes} MA,C) \cong C(MX,C/MA)$$

$$C(Q(AX),C) \cong C(MA \hat{\otimes} MX,C) \cong C(MX,MA \backslash C)$$

together with the density of M. Finally, the ends (6) and (7) exist because the <u>opposites</u> of the functors $C/-,-\backslash C : C^{op} \to C$ admit right adjoints, namely $-\backslash C$ and $C/-$, hence preserve the density expression.

<u>Proof of sufficiency</u> By Lemma 1.3.1, a canonical functor $\hat{\otimes}$ from $C \otimes C$ to C is obtained on setting

$$C \hat{\otimes} C' = \int^{XX'} (C(MX,C) \otimes C(MX',C')) \otimes Q(XX')$$

$$\hat{I} = \int^{X} JX \otimes MX.$$

This $\hat{\otimes}$ is biclosed:

$$\begin{aligned}
C(C\hat{\otimes}C',D) &= C(\textstyle\int^{XX'}(C(MX,C)\otimes C(MX',C'))\otimes Q(XX'),D)\\
&\cong \textstyle\int_{XX'}[C(MX,C)\otimes C(MX',C'),C(Q(XX'),D)]\\
&\cong \textstyle\int_{XX'}[C(MX,C)\otimes C(MX',C'),C(MX,H(X'D))]\\
&\qquad\qquad \text{by the isomorphism (8),}\\
&\cong \textstyle\int_X[C(MX,C),\int_{X'}[C(MX',C'),C(MX,H(X'D))]]\\
&\cong \textstyle\int_X[C(MX,C),C(MX,\int_{X'}[C(MX',C'),H(X'D)])]\\
&\cong C(C,\textstyle\int_X[C(MX,C'),H(XD)])\\
&\qquad\qquad \text{by the density of } M,\\
&= C(C,D/C') \text{ say.}
\end{aligned}$$

Similarly $C(C\hat{\otimes}C',D) \cong C(C',\int_X[C(MX,C),K(XD)])$

$$= C(C',C\backslash D) \text{ say.}$$

Furthermore, there exists a natural isomorphism

$$MA\hat{\otimes}MA' \cong Q(AA') \tag{5.3.1}$$

which can be obtained as follows. The representation theorem applied to $C\in C$ in the composite

$$\begin{aligned}
C(\textstyle\int^X C(MX,MA)\otimes Q(XA'),C) &\cong \textstyle\int_X[C(MX,MA),C(Q(XA'),C)]\\
&\cong \textstyle\int_X[C(MX,MA),C(MX,H(A'C))]\\
&\qquad \text{by the isomorphism (8),}\\
&\cong C(MA,H(A'C))\\
&\qquad \text{by density of } M,\\
&\cong C(Q(AA'),C)\\
&\qquad \text{by isomorphism (8),}
\end{aligned}$$

yields an isomorphism

$$\int^X C(MX,MA)\otimes Q(XA') \cong Q(AA').$$

The transpose argument yields an isomorphism

$$\int^X C(MX,MA')\otimes Q(AX) \cong Q(AA').$$

Combining these yields

$$\begin{aligned} MA\hat{\otimes}MA' &= \int^{XX'}(C(MX,MA)\otimes C(MX',MA'))\otimes Q(XX') \\ &\cong Q(AA'). \end{aligned}$$

If the "reflection" of a functor $S : A \to V$ exists in C then it is denoted by

$$\psi S = \int^A SA\otimes MA.$$

Similarly, the reflection of a natural transformation $\alpha : S \to T : A \to V$ is denoted by

$$\psi(\alpha) = \int\alpha\otimes 1 : \int^A SA\otimes MA \to \int^A TA\otimes MA.$$

Furthermore, a transformation

$$\psi_S : F(LA,S) \to C(\psi LA,\psi S)$$

is defined to be the transform, under $\int^\cdot$ and tensoring in C, of the neonatural transformation $z = z_S : \int^A F(LA,S)\otimes\psi LA \to \psi S$. Then $V(\psi_S)(\alpha) = \psi(\alpha)$ for each natural transformation $\alpha : LA \to S : A \to V$.

From the definition of ψ, we have natural isomorphisms

$\psi LA = \int^X LA(X)\otimes MX \cong MA$ by the higher repn. thm.,

$$\begin{aligned}\tilde{\psi} : \psi LA \hat{\otimes} \psi LA' &\cong MA\hat{\otimes}MA' \\ &\cong Q(AA') \text{ by } (5.3.1), \\ &= \psi(P(AA'-)) \\ &\cong \psi(LA\bar{\otimes}LA'),\end{aligned}$$

where $\bar{\otimes}$ is as constructed in Section 5.2. Using Lemma 5.1.2 and the coend definition of $\hat{\otimes}$, $\tilde{\psi}$ admits neonatural extensions

$$\tilde{\psi}_{SA} : \psi S\hat{\otimes}\psi LA \to \psi(S\bar{\otimes}LA)$$
$$\tilde{\psi}_{AS} : \psi LA\hat{\otimes}\psi S \to \psi(LA\bar{\otimes}S).$$

We already have $\psi^o : \hat{I} = \psi J$ by definition of $\hat{I}$. Thus we are able to write down "axioms" for $(\psi,\tilde{\psi},\psi^o)$ to be a "monoidal functor":

$$\begin{tikzcd} \psi J\hat{\otimes}\psi LA \arrow[r, "\hat{\ell}"] \arrow[d, "\tilde{\psi}"'] & \psi LA \\ \psi(J\bar{\otimes}LA) \arrow[ur, "\psi\bar{\ell}"'] & \end{tikzcd} \tag{5.3.2}$$

$$\begin{tikzcd} \psi LA\hat{\otimes}\psi J \arrow[r, "\hat{r}"] \arrow[d, "\tilde{\psi}"'] & \psi LA \\ \psi(LA\bar{\otimes}J) \arrow[ur, "\psi\bar{r}"'] & \end{tikzcd} \tag{5.3.3}$$

$$\begin{tikzcd}
(\psi LA \hat{\otimes} \psi LA')\hat{\otimes}\psi LA'' \arrow[r, "\hat{a}"] \arrow[d, "\tilde{\psi}\otimes 1"'] & \psi LA\hat{\otimes}(\psi LA'\hat{\otimes}\psi LA'') \arrow[d, "1\otimes\tilde{\psi}"] \\
\psi(LA\bar{\otimes}LA')\hat{\otimes}\psi LA'' \arrow[d, "\tilde{\psi}"'] & \psi LA\hat{\otimes}\psi(LA'\bar{\otimes}LA'') \arrow[d, "\tilde{\psi}"] \\
\psi((LA\bar{\otimes}LA')\bar{\otimes}LA'') \arrow[r, "\psi\bar{a}"'] & \psi(LA\bar{\otimes}(LA'\bar{\otimes}LA''))
\end{tikzcd} \quad . \tag{5.3.4}$$

Because $\psi L \cong M : A^{op} \to C$ is dense and $\hat{\otimes}$ admits right adjoints to each variable, the isomorphisms $\hat{\ell}$, $\hat{r}$, $\hat{a}$ defined by the above three diagrams, admit unique extensions to natural isomorphisms $\hat{\ell} : \hat{I}\otimes C \cong C$, $\hat{r} : C\otimes\hat{I} \cong C$, $\hat{a} : (C\otimes C')\otimes C'' \cong C\otimes(C'\otimes C'')$, by Lemma 1.5.4. For similar reasons, these isomorphisms satisfy the monoidal functor axioms MC1 and MC2. To see this, note first that each of the diagrams (5.3.2), (5.3.3), and (5.3.4) still commutes if one of the variables ψLA is replaced by ψS throughout; this follows from the obvious neonaturality argument using Lemma 5.1.2. Then, substituting the results into diagrams (4.1.2) and (4.1.3), we see that MC1 and MC2 hold for $\hat{\ell}$, $\hat{r}$, and $\hat{a}$ whenever the vertices contain variables all of the form $\psi LA \cong MA$. Thus, because M is dense, axioms MC1 and MC2 hold all the time by Lemma 1.5.4.

Having established the biclosed structure on C, it remains to show that $M^{op} : A \to C^{op}$ admits enrichment to a strong promonoidal functor. Using Lemma 2.2.6, it suffices to consider strong enrichment of the isomorphic functor $\phi : A \to C^{op}$, $\phi A = \psi LA$, with respect to the isomorphic promonoidal structure on A introduced in Section 5.2.

To complete the structure, define

$$\hat{\phi} : F(LB, LA\bar{\otimes}LA') \xrightarrow{\psi} C(\psi LB, \psi(LA\bar{\otimes}LA')) \xrightarrow{C(1,\tilde{\psi}^{-1})} C(\psi LB, \psi LA\hat{\otimes}\psi LA')$$

$$\phi^{\cdot} : F(LB, J) \xrightarrow{\psi} C(\psi LB, \psi J).$$

Then, from the definition of ψ, $\Phi = (\phi, \hat{\phi}, \phi^{\cdot})$ will be strong. The promonoidal functor axioms PC1, PC2, and PC3 for Φ are established by transforming the respective "monoidal axioms" (5.3.2), (5.3.3), and (5.3.4) for $(\psi, \tilde{\psi}, \psi^{o})$. Briefly, the diagram (5.3.5) (and its transpose form) can be checked to commute by first transforming it under the tensoring adjunction of C, then using the neonaturality of $\tilde{\psi}$:

$$\begin{tikzcd}[column sep=small]
 & F(LX,S)\underline{\otimes}F(LB,LX\bar{\otimes}LA) \arrow[dr, "z"] & \\
F(LX,S)\otimes F(LB,LX\bar{\otimes}LA) \arrow[ur, "s"] \arrow[d, "\psi\otimes\psi"'] & & F(LB,S\bar{\otimes}LA) \arrow[d, "\psi"] \\
C(\psi LX,\psi S)\otimes C(\psi LB,\psi(LX\bar{\otimes}LA)) \arrow[d, "1\otimes C(1,\tilde{\psi}^{-1})"'] & & C(\psi LB,\psi(S\bar{\otimes}LA)) \arrow[d, "C(1,\tilde{\psi}^{-1})"] \\
C(\psi LX,\psi S)\otimes C(\psi LB,\psi LX\hat{\otimes}\psi LA) \arrow[dr, "s\phi"'] & & C(\psi LB,\psi S\hat{\otimes}\psi LA) \\
 & C(Y,\psi S)\underline{\otimes}C(\psi LB,Y\hat{\otimes}\psi LB) \arrow[ur, "y"'] &
\end{tikzcd} \qquad (5.3.5)$$

This diagram reduces axiom PC1 for Φ to the following:

$$\begin{tikzcd}
F(LX,J)\underline{\otimes}F(LB,LX\bar{\otimes}LA) \arrow[r, "\lambda"] \arrow[d, "z"'] \arrow[dr, phantom, "(1)"] & A(AB) \arrow[d, "L"] \\
F(LB,J\bar{\otimes}LA) \arrow[r, "F(1,\bar{\ell})"] \arrow[d, "\psi"'] \arrow[dr, phantom, "(2)"] & F(LB,LA) \arrow[d, "\psi"] \\
C(\psi LB,\psi(J\bar{\otimes}LA)) \arrow[r, "C(1,\psi\bar{\ell})"] \arrow[d, "C(1,\tilde{\psi}^{-1})"', "(3)"] & C(\psi LB,\psi LA) \\
C(\psi LB,\psi J\hat{\otimes}\psi LA) \arrow[ur, "C(1,\hat{\ell})"] & C(Y,\psi S)\underline{\otimes}C(\psi LB,Y\hat{\otimes}\psi LB). \arrow[l, "y"] \arrow[u, "\hat{\lambda}"', "(4)"]
\end{tikzcd}$$

Here the subregions (1), (3), and (4) commute by definition of $\overline{\ell}$, $\hat{\ell}$, and $\hat{\lambda}$ respectively, while the subregion (2) commutes because $\psi_S : F(LA,S) \to C(\psi LA,\psi S)$ is S-natural in S. The remaining axioms PC2 and PC3 for Φ are established by the same procedure.

Section 5.4 Commutative monads

In conclusion, we outline an application of Theorem 5.3.1 to the theory of commutative monads. Some familiarity with the usual constructions of $\mathcal{V}$-monad theory (as given, for example, in Dubuc [7] or Kock [13]) is assumed. Moreover, to permit these constructions, we suppose throughout this section that $\mathcal{V}$ has equalisers.

Notation:

$\mathbb{T} = (T, \mu, \eta)$ is a $\mathcal{V}$-monad on $\mathcal{V}$.

$\mathcal{C}$ is the category of $\mathbb{T}$-algebras $(C, \xi : TC \to C)$, whose hom objects are defined by the equaliser diagrams

$$\begin{tikzcd} \mathcal{C}(CD) \arrow[r] & {[CD]} \arrow[rr, "{[\xi,1]}"] \arrow[dr, "T_{CD}"'] & & {[TC,D]} \\ & & {[TC,TD]} \arrow[ur, "{[1,\xi]}"'] & \end{tikzcd}$$

$F \dashv U : \mathcal{C} \to \mathcal{V}$ is the associated free-algebra adjunction.

$\mathcal{K}$ is the Kleisli category of "free algebras", where obj $\mathcal{K}$ = obj $\mathcal{V}$ and $\mathcal{K}(XY) = [X, TY]$.

$J : \mathcal{V} \to \mathcal{K}$ is the canonical functor that is the identity on objects.

$M : \mathcal{K} \to \mathcal{C}$ is the functor that fully embeds "free algebras" into the category of algebras.

$\tau : X\otimes TY \to T(X\otimes Y)$ is the canonical natural transformation associated to T, and $\tau' : TX\otimes Y \cong Y\otimes TX \xrightarrow{\tau} T(Y\otimes X) \cong T(X\otimes Y)$ is its transpose.

The symbols U and J are usually omitted.

<u>Definition 5.4.1</u> (Kock) The monad $\mathbb{T} = (T,\mu,\eta)$ on V is <u>commutative</u> if the legs of the diagram

$$\begin{tikzcd}
TX\otimes TY \arrow[r, "\tau"] \arrow[d, "\tau'"'] & T(TX\otimes Y) \arrow[d, "T\tau'"] & \\
T(X\otimes TY) \arrow[r, "T\tau"'] & T^2(X\otimes Y) \arrow[r, "\mu"'] & T(X\otimes Y)
\end{tikzcd}$$

are equal; we write $\tilde{T} = \tilde{T}_{XY} : TX\otimes TY \to T(X\otimes Y)$ for their common value.

The commutativity condition on $\mathbb{T}$ is <u>equivalent</u> to the condition that the functors

$$X\tilde{\otimes}- : K \to K \text{ and } -\tilde{\otimes}Y : K \to K,$$

defined by

$$X\tilde{\otimes}Y = X\otimes Y,$$

$$\begin{tikzcd}
\mathcal{K}(YZ) \arrow[rr, dashed, "X\tilde{\otimes}-"] \arrow[d, equal] & & \mathcal{K}(X\otimes Y, X\otimes Z) \arrow[d, equal] \\
{[Y,TZ]} \arrow[r, "X\otimes-"'] & {[X\otimes Y, X\otimes TZ]} \arrow[r, "{[1,\tau]}"'] & {[X\otimes Y, T(X\otimes Z)],}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{K}(XZ) \arrow[rr, dashed, "-\tilde{\otimes}Y"] \arrow[d, equal] & & \mathcal{K}(X\otimes Y, Z\otimes Y) \arrow[d, equal] \\
{[X,TZ]} \arrow[r, "-\otimes Y"'] & {[X\otimes Y, TZ\otimes Y]} \arrow[r, "{[1,\tau']}"'] & {[X\otimes Y, T(Z\otimes Y)],}
\end{tikzcd}$$

should be the partial functors of a bifunctor

$$\tilde{\otimes} : \mathcal{K}\otimes\mathcal{K} \to \mathcal{K}.$$

Thus, if T is commutative then $\mathcal{K}$ assumes a (symmetric) monoidal structure for which $J : \mathcal{V} \to \mathcal{K}$ is a strong monoidal functor.

Theorem 5.4.2 If $\mathbb{T} = (T,\mu,\eta)$ is a commutative monad on $\mathcal{V}$ then, in order for there to exist a closed structure on $\mathcal{C}$ for which $M : \mathcal{K} \to \mathcal{C}$ admits enrichment to a strong monoidal functor, it is necessary and sufficient that the coequaliser of the pair

$$\begin{tikzcd}
T^2(C\otimes D) \arrow[rr, "\mu"] \arrow[dr, "T(\tilde{T})"'] & & T(C\otimes D) \\
& T(TC\otimes TD) \arrow[ur, "T(\xi\otimes\xi)"'] &
\end{tikzcd}$$

exist in $\mathcal{C}$ for all $\mathbb{T}$-algebras C and D.

<u>Proof</u> In Theorem 5.3.1, take $A^{op} = K$ and $M : K \to \mathcal{C}$ as given. It is easy to verify that $\mathcal{C}$ is cotensored and M-tensored. Furthermore, M is dense by the corollary to the monad representation theorem given in Appendix 4. By the higher representation theorem, the coends (1) and (2) in Theorem 5.3.1 become

$$Q(XY) = \int^{Z} K(X\otimes Y,Z)\otimes MZ \cong M(X\otimes Y)$$

$$\bar{I} = \int^{X} K(IX)\otimes MX \cong MI$$

respectively. On applying the monad representation theorem to the definitions of $X\tilde{\otimes}-$ and $-\tilde{\otimes}Y$, the coend (3)

$$C\bar{\otimes}D = \int^{XY} (\mathcal{C}(MX,C)\otimes\mathcal{C}(MY,D))\otimes M(X\otimes Y)$$

reduces to the <u>joint</u> coequaliser in $\mathcal{C}$ of the pairs

$$\begin{tikzcd}
T(C\otimes TD) \arrow[d, "T\tau"'] \arrow[dr, "T(1\otimes\xi)"] & & T(TC\otimes D) \arrow[dl, "T(\xi\otimes 1)"'] \arrow[d, "T\tau'"] \\
T^2(C\otimes D) \arrow[r, "\mu"'] & T(C\otimes D) & T^2(C\otimes D) \arrow[l, "\mu"]
\end{tikzcd} .$$

Moreover, this coincides with the coequaliser in $\mathcal{C}$ of the single pair

$$\begin{tikzcd}
T^2(C\otimes D) \arrow[rr, "\mu"] \arrow[dr, "T(\tilde{T})"'] & & T(C\otimes D) \\
& T(TC\otimes TD) \arrow[ur, "T(\xi\otimes\xi)"'] &
\end{tikzcd}$$

Thus, the codomain of this coequaliser is the tensor product of the algebras C and D in $\mathcal{C}$. To construct the internal hom in $\mathcal{C}$, we must first verify that the isomorphism

$$\mathcal{C}(FX,[YC]) \cong [X[YC]] \cong [X\otimes Y,C] \cong \mathcal{C}(F(X\otimes Y),C)$$

actually provides an isomorphism

$$\mathcal{C}(MX,[YC]) \cong \mathcal{C}(M(X\otimes Y),C) \qquad (5.4.1)$$

that is natural in $X\epsilon\mathcal{K}$. This is done simply by applying the monad representation theorem. Coend (4) now becomes

$$\begin{aligned} H(YC) &= \int^X \mathcal{C}(M(X\otimes Y),C)\otimes MX \\ &\cong \int^X \mathcal{C}(MX,[YC])\otimes MX \text{ by } (5.4.1) \\ &\cong [YC] \text{ by the density of M,} \end{aligned}$$

whence (8) is an isomorphism by (5.4.1). Finally, the end (6) always exists because $U : \mathcal{C} \to \mathcal{V}$ <u>creates</u> limits and we already know that

$$\begin{aligned} U(\int_X[\mathcal{C}(MX,C),H(XD)]) &\cong \int_X[\mathcal{C}(MX,C),\mathcal{C}(MX,D)] \\ &\cong \mathcal{C}(CD) \end{aligned}$$

exists in $\mathcal{V}$, by the density of M. This completes the proof.

<u>Remarks</u> The condition of commutativity on a monad $\mathbb{T}$ was first formulated by Kock [12] who established, in [13], that a commutative monad generates a category $\mathcal{C}$ with an <u>internal hom</u> (in the original Eilenberg-Kelly [9] sense).

The formula we provide in Theorem 5.4.2 for the extra monoidal structure of $\mathcal{C}$ has also been suggested by Linton [15]. It is not difficult to check that this closed structure on $\mathcal{C}$ coincides with the one obtained in Example 3.2.3 whenever $\mathbb{T}$ is a commutative "theoretical" monad, that is, a monad obtained by Kan extension from a commutative theory.

APPENDICES

Appendix 1 On the iterated use of z

We require that the transformation z, introduced in Section 4.1, be "coherent". In the absence of a general theorem to this effect, we verify the following:

Lemma Let $M : \mathcal{A} \to \mathcal{B}$, $S : \mathcal{B} \to \mathcal{B}$, $T : \mathcal{B} \to \mathcal{V}$, and $R : \mathcal{B}\otimes\mathcal{B} \to \mathcal{V}$ be functors for which the required coends exist. Then the exteriors of the following diagrams commute for all $B,C \in \mathcal{B}$.

1.
$$\begin{tikzcd}
\mathcal{B}(MX,B)\underline{\otimes}(\mathcal{B}(MY,SMX)\underline{\otimes}TMY) \arrow[d, "1\underline{\otimes}z"'] \arrow[rrd, dashed, "z"] \arrow[rr, phantom, "\cong"] & & (\mathcal{B}(MX,B)\underline{\otimes}\mathcal{B}(MY,SMX))\underline{\otimes}TMY \arrow[d, "z\underline{\otimes}1"] \\
\mathcal{B}(MX,B)\underline{\otimes}TSMX \arrow[dr, "z"'] & & \mathcal{B}(MY,SB)\underline{\otimes}TMY \arrow[dl, "z"] \\
& TSB &
\end{tikzcd}$$

2.
$$\begin{tikzcd}
\mathcal{B}(MX,B)\underline{\otimes}(\mathcal{B}(MY,C)\underline{\otimes}R(MX,MY)) \arrow[d, "1\underline{\otimes}z"'] \arrow[rrd, dashed, "z"] \arrow[rr, phantom, "\cong"] & & \mathcal{B}(MY,C)\underline{\otimes}(\mathcal{B}(MX,B)\underline{\otimes}R(MX,MY)) \arrow[d, "1\underline{\otimes}z"] \\
\mathcal{B}(MX,B)\underline{\otimes}R(MX,C) \arrow[dr, "z"'] & & \mathcal{B}(MY,C)\underline{\otimes}R(B,MY) \arrow[dl, "z"] \\
& R(B,C) &
\end{tikzcd}$$

<u>Proof</u> The lower subregions of both diagrams commute on applying the representation theorem to $B\epsilon\mathcal{B}$, and using the definition (1.4.1) of z. To verify that the upper region of diagram 1 commutes, we "expand" the coends present:

$$\begin{tikzcd}
\mathcal{B}(MX,B)\otimes(\mathcal{B}(MY,SMX)\otimes TMY) \arrow[r, "a^{-1}"] \arrow[d, "1\otimes s"'] & (\mathcal{B}(MX,B)\otimes\mathcal{B}(MY,SMX))\otimes TMY \arrow[d, "s\otimes 1"] \arrow[ddd, bend left=60, "\zeta\otimes 1"] \\
\mathcal{B}(MX,B)\otimes(\mathcal{B}(MY,SMX)\underline{\otimes}TMY \arrow[d, "s"'] \arrow[dd, bend right=60, "\zeta"'] & (\mathcal{B}(MX,B)\underline{\otimes}\mathcal{B}(MY,SMX))\otimes TMY \arrow[d, "s"] \arrow[dd, bend left=40, "z\otimes 1"'] \\
\mathcal{B}(MX,B)\underline{\otimes}(\mathcal{B}(MY,SMX)\underline{\otimes}TMY \arrow[r, phantom, "\cong"] \arrow[d, "z"'] \arrow[dr, phantom, "(*)"] & (\mathcal{B}(MX,B)\underline{\otimes}\mathcal{B}(MY,SMX))\underline{\otimes}TMY \arrow[dl, "z\underline{\otimes}1"] \\
\mathcal{B}(MY,SB)\underline{\otimes}TMY & \mathcal{B}(MY,SB)\otimes TMY \arrow[l, "s"]
\end{tikzcd}$$

Commutativity of the exterior of this diagram follows easily from the representation theorem applied to $B\epsilon\mathcal{B}$, together with coherence of a, r, ℓ in $\mathcal{V}$. The uncommented subregions commute by definitions. Thus the diagram (*) commutes because s(1⊗s) is a coend over X and Y. The upper region of diagram 2 commutes for similar reasons.

Appendix 2 A theorem on Kan extensions

Let $\mathcal{V}$ be a cartesian closed category and let $\mathcal{A}$ be a $\mathcal{V}$-category with finite $\mathcal{V}$-products, including a terminal object I. Then we say that a functor $T : \mathcal{A} \to \mathcal{V}$ preserves finite products if the canonical morphisms $T(A \times A') \to TA \times TA'$ and $TI \to I$ are isomorphisms for all $A, A' \in \mathcal{A}$.

Theorem ($\mathcal{V}$ cartesian closed) Let $M : \mathcal{A} \to \mathcal{B}$ be a functor between categories $\mathcal{A}$ and $\mathcal{B}$ which admit finite products. Then the Kan extension

$$\overline{T} = \int^A \mathcal{B}(MA,-) \times TA \;:\; \mathcal{A} \to \mathcal{V},$$

of a finite-product-preserving functor $T : \mathcal{A} \to \mathcal{V}$ along the functor M, is finite-product preserving.

Proof From the definition of terminal, the composite

$$I \cong \mathcal{A}(AI) \xrightarrow[M]{} \mathcal{B}(MA,MI) \xrightarrow[\mathcal{B}(1,u)]{} \mathcal{B}(MA,I) \cong I$$

is the identity isomorphism for all $A \in \mathcal{A}$. Thus

$$\begin{aligned}\overline{T}I &= \int^A \mathcal{B}(MA,I) \times TA \\ &\cong \int^A \mathcal{A}(AI) \times TA \\ &\cong TI \text{ by the higher repn. thm.,} \\ &\cong I \text{ because } T \text{ preserves } I \text{ by hypothesis.}\end{aligned}$$

Hence $\overline{T}$ preserves terminal objects. Secondly, if $B \leftarrow B \times C \to C$ is a product of B and C in $\mathcal{B}$, then the resulting morphism $\overline{T}(B \times C) \to \overline{T}B \times \overline{T}C$ is easily shown to be left inverse to the composite isomorphism

$$\begin{aligned}
\overline{T}B\times\overline{T}C &= (\textstyle\int^{X}\mathcal{B}(MX,B)\times TX)\times(\int^{Y}\mathcal{B}(MY,C)\times TY) \\
&\cong \textstyle\int^{XY}(\mathcal{B}(MX,B)\times\mathcal{B}(MY,C))\times(TX\times TY) \\
&\qquad \text{because } \mathcal{V} \text{ is cartesian closed,} \\
&\cong \textstyle\int^{XY}(\mathcal{B}(MX,B)\times\mathcal{B}(MY,C))\times T(X\times Y) \\
&\qquad \text{because } T \text{ preserves finite products,} \\
&\cong \textstyle\int^{XY}(\mathcal{B}(MX,B)\times\mathcal{B}(MY,C))\times\int^{Z}A(Z,X\times Y)\times TZ \\
&\qquad \text{by the higher repn. thm.,} \\
&\cong \textstyle\int^{XY}(\mathcal{B}(MX,B)\times\mathcal{B}(MY,C))\times\int^{Z}(A(ZX)\times A(ZY))\times TZ \\
&\qquad \text{because } A(Z-) \text{ preserves } \mathcal{V}\text{-limits,} \\
&\cong \textstyle\int^{Z}((\int^{X}\mathcal{B}(MX,B)\times A(ZX))\times(\int^{Y}\mathcal{B}(MY,C)\times A(ZY)))\times TZ \\
&\qquad \text{because } \mathcal{V} \text{ is cartesian closed,} \\
&\cong \textstyle\int^{Z}(\mathcal{B}(MZ,B)\times\mathcal{B}(MZ,C))\times TZ \\
&\qquad \text{by the higher repn. thm.,} \\
&\cong \textstyle\int^{Z}\mathcal{B}(MZ,B\times C)\times TZ \\
&\qquad \text{because } \mathcal{B}(MZ,-) \text{ preserves } \mathcal{V}\text{-limits,} \\
&= \overline{T}(B\times C).
\end{aligned}$$

<u>Remark</u> Special cases of this theorem have appeared elsewhere; in Ulmer [18] for example.

Appendix 3 A criterion for $\mathcal{V}$-naturality

Theorem Let $F,G : \mathcal{B} \to \mathcal{C}$ be functors between tensored categories $\mathcal{B}$ and $\mathcal{C}$, and let $\alpha : F \to G$ be an $\mathcal{S}$-natural transformation. Then α is $\mathcal{V}$-natural if and only if the canonical diagram

$$\begin{tikzcd}
X\otimes FB \arrow[r, "\tau"] \arrow[d, "1\otimes\alpha_B"'] \arrow[dr, phantom, "*"] & F(X\otimes B) \arrow[d, "\alpha_{X\otimes B}"] \\
X\otimes GB \arrow[r, "\tau"'] & G(X\otimes B)
\end{tikzcd}$$

commutes for all $X\in\mathcal{V}$ and $B\in\mathcal{B}$.

Proof By definition, the family $\alpha_B : FB \to GB$ is $\mathcal{V}$-natural if and only if

$$\begin{tikzcd}
\mathcal{B}(BB') \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{C}(FB,FB') \arrow[d, "\mathcal{C}(1,\alpha)"] \\
\mathcal{C}(GB,GB') \arrow[r, "\mathcal{C}(\alpha,1)"'] & \mathcal{C}(FB,GB')
\end{tikzcd}$$

commutes for all $B,B'\in\mathcal{B}$. By the representation theorem applied to $X\in\mathcal{V}$, this is so if and only if the center region of the following diagram commutes for all $B,B'\in\mathcal{B}$ and $X\in\mathcal{V}$.

$$\begin{tikzcd}
 & \mathcal{C}_0(F(X\otimes B),FB') \arrow[rr, "\mathcal{C}_0(\tau,1)"] \arrow[dd, phantom, "(1)"] & & \mathcal{C}_0(X\otimes FB,FB') \arrow[dl, phantom, "\cong"] \arrow[dr, "\mathcal{C}_0(1,\alpha)"] \arrow[dd, phantom, "(4)"] & \\
 & & \mathcal{V}_0(X,\mathcal{C}(FB,FB')) \arrow[dr, "\mathcal{V}_0(1,\mathcal{C}(1,\alpha))"] & & \\
\mathcal{B}_0(X\otimes B,B') \arrow[uur, "VF"] \arrow[ddr, "VG"'] \arrow[r, phantom, "\cong"] & \mathcal{V}_0(X,\mathcal{C}(BB')) \arrow[ur, "\mathcal{V}_0(1,F)"] \arrow[dr, "\mathcal{V}_0(1,G)"'] \arrow[dd, phantom, "(2)"] & & \mathcal{V}_0(X,\mathcal{C}(FB,GB')) \arrow[r, phantom, "\cong"] \arrow[dd, phantom, "(3)"] & \mathcal{C}_0(X\otimes FB,GB') \\
 & & \mathcal{V}_0(X,\mathcal{C}(GB,GB')) \arrow[ur, "\mathcal{V}_0(1,\mathcal{C}(\alpha,1))"'] \arrow[dr, phantom, "\cong"] & & \\
 & \mathcal{C}_0(G(X\otimes B),GB') \arrow[rr, "\mathcal{C}_0(\tau,1)"'] & & \mathcal{C}_0(X\otimes GB,GB') \arrow[uur, "\mathcal{C}_0(1\otimes\alpha,1)"'] &
\end{tikzcd}$$

In this diagram the bijections are those underlying tensoring adjunctions hence subregions (3) and (4) commute by the naturality of these. Subregions (1) and (2) commute by the definition of τ (recalled from [11] §4). Thus the center commutes if and only if the exterior commutes. But, by the representation theorem at the <u>$\mathcal{S}$-level</u> applied to $B'\in\mathcal{B}_0$, the exterior commutes if and only if the diagram * commutes, as required.

Appendix 4 The monad representation theorem

In this appendix we may as well suppose that $\mathbb{T} = (T,\mu,\eta)$ is a V-monad on an arbitrary V-category; in all other respects we shall use the notation given in Section 5.4. For ease of recognition, morphisms in the Kleisli category K shall be represented by their images under the full embedding $M : K \to C$.

<u>Theorem</u> Let $S : K^{op} \to V$ be a functor and $(C,\xi : TC \to C)$ be a $\mathbb{T}$-algebra. Then there is a (canonical) bijection between natural transformations $\alpha_X : C(MX,C) \to SX$ and elements in the equaliser of

$$VSC \underset{VS(T\xi)}{\overset{VS\underline{\mu}}{\rightrightarrows}} VSTC \quad .$$

<u>Proof</u> We shall establish the (equivalent) higher form of this assertion. First note that the fork on the right hand side of the diagram

$$\begin{tikzcd}[column sep=large]
 & \int_{\cdot}[C(MX,C),SX] \arrow[dl, dashed] \arrow[dr, "\int_{\cdot}[C(1,\xi),1]"] & \\
SC \arrow[r, "\cong", "y"'] \arrow[d, shift right, "S\mu"'] \arrow[d, shift left, "ST\xi"] & \int_{\cdot}[K(XC),SX] & \int_{\cdot}[C(MX,MC),SX] \arrow[l, "\cong"', "\int_{\cdot}[M,1]"] \arrow[d, shift right, "\int_{\cdot}[C(1,\mu),1]"'] \arrow[d, shift left, "\int_{\cdot}[C(1,T\xi),1]"] \\
STC \arrow[r, "\cong", "y"'] & \int_{\cdot}[K(X,TC),SX] & \int_{\cdot}[C(MX,MTC),SX] \arrow[l, "\cong"', "\int_{\cdot}[M,1]"]
\end{tikzcd}$$

is an equaliser diagram in V_0; this follows from the well-known characteristic of C that the fork

$$\begin{tikzcd}[column sep=large]
C(MX,MTC) \arrow[r, shift left, "C(1,\mu)"] \arrow[r, shift right, "C(1,T\xi)"'] & C(MX,MC) \arrow[r, "C(1,\xi)"] & C(MX,C)
\end{tikzcd}$$

is a coequaliser diagram in V_C for all $X \in K$. Because the two lower regions commute by naturality, we obtain an equaliser diagram on the left hand side (the dotted arrow is easily seen to be the composite

$$\begin{tikzcd}[column sep=large]
\int_{\cdot}[C(MX,C),SX] \arrow[r, dashed] \arrow[dd, "\int_{\cdot}[C(1,\xi),1]"'] & SC \\
 & {[I,SC]} \arrow[u, "\cong"'] \\
\int_{\cdot}[C(MX,MC),SX] \arrow[r, "s_C"'] & {[C(MC,MC),SC]} \arrow[u, "{[j,1]}"']
\end{tikzcd}$$

).

The theorem follows on applying $V : \mathcal{V} \to \mathcal{S}$ to this equaliser.

<u>Corollary</u> The functor $M : \mathcal{K} \to \mathcal{C}$ is dense.

<u>Proof</u> In the higher form of the monad representation theorem, take S to be $\mathcal{C}(M-,D) : \mathcal{K}^{op} \to \mathcal{V}$ for some algebra $D \in \mathcal{C}$. We thus obtain an equaliser diagram

$$\int_X [\mathcal{C}(MX,C),\mathcal{C}(MX,D)] \dashrightarrow \mathcal{C}(MC,D) \underset{\mathcal{C}(T\xi,1)}{\overset{\mathcal{C}(\mu,1)}{\rightrightarrows}} \mathcal{C}(MTC,D)$$

for each pair of algebras $C,D \in \mathcal{C}$. But the morphism $\mathcal{C}(\xi,1) : \mathcal{C}(CD) \to \mathcal{C}(MC,D)$ <u>is</u> the equaliser of $\mathcal{C}(\mu,1)$ and $\mathcal{C}(T\xi,1)$. The resulting isomorphism

$$\mathcal{C}(CD) \cong \int_X [\mathcal{C}(MX,C),\mathcal{C}(MX,D)]$$

makes $M : \mathcal{K} \to \mathcal{C}$ a dense functor by Definition 1.5.1.

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