Isomorphisms of BV(\sigma) and AC(\sigma) Spaces.

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ISOMORPHISMS OF $BV(\sigma)$ AND $AC(\sigma)$ SPACES

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Isomorphisms of BV(σ) and AC(σ) spaces

Abstract

The Banach algebras AC(σ) and BV (σ) were introduced by Ashton and Doust in 2005 in order to extend the theory of well-bounded operators. Here is a nonempty compact subset of the plane and the elements of these algebras are called absolutely continuous functions and functions of bounded variation respectively. The aim of this thesis is to investigate the extent to which analogues of the classical 1939 theorem of Gelfand and Kolmogoroff for C(K) spaces might hold in the context of these functions. That is, we study the relationship between the topological structure of the domain set and the structure of the Banach algebra AC(σ).

In 2015 Doust and Leinert had shown that if AC (σ_ 1) is isomorphic to AC (σ_ 2) then σ_1 must be homeomorphic to σ_ 2, providing one direction of a Gelfand-Kolmogoroff type theorem. They also showed that although the converse implication fails in general, if one restricts the class of sets considered, then positive theorems are possible. In particular they showed that if σ_ 1 and σ_ 2 are polygonal, then AC (σ_ 1) is isomorphic to AC (σ_ 2) if and only if σ_1 and σ_ 2 are homeomorphic.

In this thesis we consider the situation for three natural classes of compact sets: those that are the spectra of compact operators, those which are the union of a finite number of line segments, and a more general class of polygonally inscribed curve. Full analogues of the Gelfand-Kolmogoroff theorem are proven for the latter two classes. Considering the first class, it is shown that there are infinitely many homeomorphic sets in that class with mutually non-isomorphic AC (σ).

A study of isomorphisms of BV (σ) spaces is also initiated. The main result is that if AC(σ_1) is isomorphic to AC(σ_2) then necessarily BV (σ_1) is isomorphic to BV (σ_2). An example is given to show that the converse of this result is not true.

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Shaymaa Al-shakarchi

Abstract

The Banach algebras $AC(\sigma)$ and $BV(\sigma)$ were introduced by Ashton and Doust in 2005 in order to extend the theory of well-bounded operators. Here σ is a nonempty compact subset of the plane and the elements of these algebras are called absolutely continuous functions and functions of bounded variation respectively. The aim of this thesis is to investigate the extent to which analogues of the classical 1939 theorem of Gelfand and Kolmogorov for $C(K)$ spaces might hold in the context of these functions. That is, we study the relationship between the topological structure of the domain set σ and the structure of the Banach algebra $AC(\sigma)$.

In 2015 Doust and Leinert had shown that if $AC(\sigma_1)$ is isomorphic to $AC(\sigma_2)$ then σ_1 must be homeomorphic to σ_2, providing one direction of a Gelfand–Kolmogorov type theorem. They also showed that although the converse implication fails in general, if one restricts the class of sets σ considered, then positive theorems are possible. In particular they showed that if σ_1 and σ_2 are polygonal regions of finite genus, then $AC(\sigma_1)$ is isomorphic to $AC(\sigma_2)$ if and only if σ_1 and σ_2 are homeomorphic.

In this thesis we consider the situation for three natural classes of compact sets: those that are the spectra of compact operators, those which are the union of a finite number of line segments, and a more general class of 'polygonally inscribed curves'. Full analogues of the Gelfand–Kolmogorov theorem are proven for the latter two classes. Considering the first class, it is shown that there are infinitely many homeomorphic sets in that class with mututally non-isomorphic $AC(\sigma)$.

A study of isomorphisms of $BV(\sigma)$ spaces is also initiated. The main result is that if $AC(\sigma_1)$ is isomorphic to $AC(\sigma_2)$ then necessarily $BV(\sigma_1)$ is isomorphic to $BV(\sigma_2)$. An example is given to show that the converse of this result is not true.

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Chapter 1

Introduction

The main focus of this thesis is the structure of two families of function spaces, $AC(\sigma)$ spaces and $BV(\sigma)$ spaces, which arise in the spectral theory of linear operators acting on Banach spaces.

For the most part, the results in this thesis are independent of the operator theory material presented here, but it is important to be aware of the application that led to the development of these spaces. We shall return occasionally throughout the thesis to make connections back to the operator theory.

In this chapter we give a review of the history of the Banach-Stone theorem, the background knowledge concerning the spectral theory of Hilbert space and Banach space operators, and outline the overall structure of the thesis.

1.1 The Banach-Stone Theorems

The study of isometries of Banach algebras of functions dates back to the classical Banach-Stone theorem. This theorem is a classical result within the theory of the spaces $C_(K)$ of continuous real-valued maps on some compact Hausdorff space K, with the sup-norm $\|f\|_\infty = \sup\{|f(x)| : x \in K\}$. In 1932 Banach considered the problem of when two spaces of type $C(K)$ are isometric, in the case of compact metric spaces. After that, in 1937 Stone extended this result to general compact Hausdorff spaces K, in what is now known as the Banach-Stone theorem. This says that two compact Hausdorff spaces K_1 and K_2 are homeomorphic if and only if $C_(K_1)$ is isometric to $C_(K_2)$. Moreover, every surjective linear isometry $\Phi : C_(K_1) \to C_(K_2)$ is of the form $\Phi(f)(k) = m(k)f(h^{-1}(k))$, for $k \in K_2$ where $h : K_1 \to K_2$ is a homeomorphism and $m \in C_(K_2)$ is such that $|m(k)| = 1$ for all $k \in K_2$ (see [GJ1] and [Pie]). It is clear that any such map is a linear isometry. The hard direction is showing that every linear isometry is of this form. The Banach-Stone theorem is also valid for complex spaces. In the remainder of this thesis all spaces will be assumed to be over the complex field.

The Banach-Stone theorem has been generalized in various directions (see [GJ1]). One of these direction was proved in 1965-1966 independently by Cambern and Amir. They considered isomorphisms with certain bounds on the norm. Amir [Ami] proved that if Φ is a linear isomorphism from $C(K_1) \to C(K_2)$ with $\|\Phi\| \|\Phi^{-1}\| < 2$, then K_1 and K_2 are homeomorphic. Cambern ([Cam1], [Cam2]) showed the same for isomorphisms for the spaces $C_0(K)$, of all continuous functions vanishing at infinity on a locally compact space K. In the same paper Amir gave examples of non-homeomorphic compact Hausdorff spaces K_1 and K_2 with linear isomorphism $\Phi : C(K_1) \to C(K_2)$ with bound exactly 3 and he posed whether there exists two non-homeomorphic compact Hausdorff spaces K_1,K_2 and linear isomorphism Φ of $C(K_1)$ to $C(K_2)$ with $2 \leq \|\Phi\| \|\Phi^{-1}\| < 3$. This problem was still open for a number of years until 1975. It was finally solved by Cohen [Coh]. He gave an example of two non-homeomorphic compact metric spaces K_1 and K_2 and an isomorphism $\Phi : C(K_1) \to C(K_2)$ with Banach-Mazur distance equal to 2.

Another direction concerning the Banach-Stone theorem stresses the link between algebraic (as opposed to metric) properties of $C(K)$ and the topology of K. This line of research was addressed by Gelfand and Kolmogorov [GK] in 1939 for compact spaces. For that, they used the maximal ideal spaces to show that if $C(K_1)$ and $C(K_2)$ are isomorphic as Banach algebras then K_1 is homeomorphic to K_2 (see Gillman and Jerison [GJ2]). (Note that some references, such as [Sim], refer to this result as the Banach-Stone theorem).

A natural question is to ask whether similar results might hold for different families of function algebras, $\Lambda(\Omega)$ where Ω is some family of domains. That is, to what extent do the properties of the set Ω determine the properties of $\Lambda(\Omega)$? Conversely, to what extent are the properties of Ω determined by the properties of $\Lambda(\Omega)$.

The algebras $C(K)$ where K is a compact subset of $\mathbb{C}$ play a central role in the spectral theory of normal operators on a Hilbert space. One might naturally look at other families of function algebras that arise in spectral theory. The questions we shall be addressing concern the spaces $AC(\sigma)$ of absolutely continuous functions on a compact set $\sigma \subseteq \mathbb{C}$.

1.2 Spectral theorem

1.2.1 Spectral theory of Hilbert space operators

The spectral theorem for self-adjoint operators provides an important generalization of the familiar diagonalization theorem for self-adjoint matrices in linear algebra. It provides a representation of a self-adjoint operator T on a Hilbert space H with respect to a family of orthogonal projections on H. In its early form this was written as

$$T = \int_a^b \lambda \, d\xi_\lambda \tag{1.2.1}$$

where this integral is a Riemann-Stieltjes type and the set $\{\xi_\lambda\}_{\lambda \in \mathbb{R}}$ is an increasing family of projections. Later, an alternative approach was developed in extending this to the spectral theorem for normal operators on a Hilbert space. That theorem gives a projection valued measure $\xi(\cdot)$ defined on the Borel subsets of $\mathbb{C}$ and a Lebesgue type integral. The measure $\xi(\cdot)$ is strongly countably additive and a modern version of the spectral theorem might be expressed as

$$\begin{aligned} T^*T = TT^* &\iff \|\boldsymbol{p}(T, T^*)\| = \sup_{z \in \sigma(T)} |\boldsymbol{p}(z, \overline{z})| \text{ for all polynomial } \boldsymbol{p}(z, \overline{z}) \\ &\iff T = \int_{\sigma(T)} \lambda \, \xi(d\lambda) \end{aligned}$$

with a proof that uses the representation theory of C^*-algebras.

In 1943 Gelfand and Naimark [GN] showed that if T is a normal operator on a Hilbert space H, then there exists an isometric isomorphism between the C^*-algebra generated by T and the algebra $C(\sigma(T))$ of continuous functions on the spectrum of T. The Banach-Stone theorem then allows us to say when the C^*-algebras generated by two normal operators must be isomorphic.

A good deal of this theory generalizes to Banach space operators. For example, if a Banach space X does not contain a subspace isomorphic to c_0, then every operator T with a $C(\sigma(T))$ functional calculus has an integral representation of the above type. The precise connection between $C(\sigma(T))$ and the algebra of operators $\{f(T) : f \in C(T)\}$ is not as strong, but with some additional conditions one may still sometimes use the Banach-Stone type theorems as a tool to show that the algebras of operators generated by two such operators must be isomorphic.

1.2.2 Spectral theory of Banach space operators

Various analogues of self-adjointness have been developed for operators on Banach spaces, motivated by the basic properties of self-adjoint operators on Hilbert space. During the decade 1940-1950, N. Dunford initiated the theory of scalar-type spectral operators to extend the theory of self-adjoint and normal operators to the Banach space setting. He considered operators T having an integral representation of the form

$$T = \int_{\sigma(T)} \lambda \, d\xi(\lambda),$$

where $\xi(\cdot)$ is a projection-valued measure on the Borel sets of $\mathbb{C}$. The operator T is called scalar-type spectral and the projection-valued measure $\xi(\cdot)$ is called the resolution of the identity of T. In 1954, Dunford in [Dun1] considered the operators of the form $T_1 + T_2$, where T_2 is any operator commuting with T_1 and which is quasinilpotent, that is $\sigma(T_2) = \{0\}$. These operators $T_1 + T_2$ are called spectral, see [DS]. The main drawback of Dunford's theory is that many operators which are normal on L^2 fail to be scalar-type spectral operator when they are acting on other L^p spaces. This is connected to the fact that many important bases, such as the Fourier series bases of $L^2[0, 2\pi]$, while forming an orthogonal and unconditional basis of L^2, are only conditional bases in L^p for $p \neq 2$.

It has been shown that we can characterize the scalar-type spectral operators by their functional calculus ([Klu], Corollary 1) (see also [Spa1]). They proved that a bounded operator T is scalar-type spectral if and only if T has a weakly compact $C(\sigma(T))$ functional calculus. Moreover, Doust proved that if the Banach space X does not contain a subspace isomorphic to c_0, then T is scalar-type spectral if and only if T has a $C(\sigma(T))$ functional calculus [Dou2]. Moreover, Doust and DeLaubenfels ([Dd], Theorem 3.2) gave us examples of operators on spaces containing c_0 which have a $C(\sigma(T))$ functional calculus but which are not scalar-type spectral.

Other authors developed a range of other theories. For example Bonsall and Duncan and others developed a theory of Hermitian operators, that is, those bounded operators T satisfying $\left\|e^{izT}\right\| \leq 1, z \in \mathbb{R}$ (see [Dow]). It is known in finite dimensions that every Hermitian operator is real scalar.

Unfortunately some 'nice' operators fail to be spectral because of the lack of a family of projections with some of the required properties. Foiaş [Foi] developed a theory by replacing the projections with invariant subspaces with rich spectral properties. This theory covers operators which admit a functional calculus for certain classes of function algebras. Colojoară and Foiaş [CF] defined what they called an admissible algebra $\mathfrak{U}$ of functions, $\mathfrak{U}$-spectral functions (which are a type of functional calculus map) and the class of $\mathfrak{U}$-scalar operators on a Banach space. An operator T is $\mathfrak{U}$-scalar if there exists an algebraic homomorphisms $U : \mathfrak{U} \to B(X)$ verifying some algebraic and analytic properties such that $U_\lambda = T$ where λ is the identity function on $\mathbb{C}$.

1.2.3 Well-bounded operators

Another notion is that of well-boundedness. This concept was introduced by Smart [Sma] to provide a theory for Banach space operators which was similar to the theory of self adjoint operators on Hilbert space, but for operators whose eigenvalue expansions may only converge conditionally.

Well-bounded operators were initially defined as operators on a reflexive Banach space having a functional calculus based on the absolutely continuous functions on some compact interval of the real line. That is, an operator T is said to be well-bounded if there exist a real constant K and a compact interval $[a, b] \subset \mathbb{R}$ such that for all complex polynomials $\boldsymbol{p}$

$$\|\boldsymbol{p}(T)\| \leq K\Big(|\boldsymbol{p}(b)| + \int_a^b |\boldsymbol{p}'(t)|dt\Big).$$

Here $\boldsymbol{p}(T)$ has its natural meaning. That is, if $\boldsymbol{p}(\lambda) = \sum_{n=0}^{k} \alpha_n \lambda^n$ then

$$\boldsymbol{p}(T) = \sum_{n=0}^{k} \alpha_n T^n, \text{ where } T^0 = I. \tag{1.2.2}$$

The map $\boldsymbol{p} \to \boldsymbol{p}(T)$ is an algebra homomorphism from the space $\boldsymbol{P}[a, b]$ of polynomials to $B(X)$. The term $\displaystyle\int_a^b |\boldsymbol{p}'(t)|dt$ is equal to the variation of $\boldsymbol{p}$ over the interval $[a, b]$.

The set of functions of bounded variation is:

$$BV[a, b] = \{f : [a, b] \to \mathbb{C} : \operatorname*{var}_{[a,b]} f < \infty\}$$

which is a Banach algebra with the norm $\|f\|_{BV} = \sup_{[a,b]} |f| + \operatorname*{var}_{[a,b]} f$. (For more details see Chapter 2.)

Smart [Sma] defined an operator T to be well-bounded if there exists a positive real constant K such that

$$\|\boldsymbol{p}(T)\| \leq K \, \|\boldsymbol{p}\|_{BV}\,, \tag{$*$}$$

for all polynomials $\boldsymbol{p} \in \boldsymbol{P}[a, b]$. In other words, T is well-bounded if and only if the algebra homomorphism in equation (1.2.2) is bounded from $(\boldsymbol{P}[a, b], \|\cdot\|_{BV}) \to B(X)$.

The bound $(*)$ means that we can extend this map by continuity to the closure of the polynomials in $BV[a, b]$, which is the algebra $AC[a, b]$ of absolutely continuous functions. This provides an $AC[a, b]$ functional calculus for T.

On a general Banach space an operator is said to be well-bounded if it admits an $AC[a, b]$ functional calculus for some compact interval $[a, b] \subseteq \mathbb{R}$. That is there is a bounded Banach algebra homomorphism $\Phi : AC[a, b] \to B(X)$, such that $\Phi(\boldsymbol{p}) = \boldsymbol{p}(T)$ for all polynomials $\boldsymbol{p}$. We will write $f(T)$ for $\Phi(f)$.

Smart [Sma] and Ringrose [Rin1] showed that if T is a well-bounded operator on a reflexive Banach space then T has a representation as a Riemann-Stieltjes integral with respect to an increasing family of projections on Banach space of the form $T = \int \lambda \, dE(\lambda)$, similar to (1.2.1). Furthermore, every operator of the form $\int \lambda \, dE(\lambda)$ was shown to be well-bounded.

Sills [Sil] later used a new method for obtaining the spectral theorem for well-bounded operators. In particular, he showed that for a well-bounded operator on a reflexive Banach space, the functional calculus for the absolutely continuous functions can be extended by working in the second conjugate AC^{**} which is an algebra which contains a larger supply of idempotents.

In 1963, Ringrose [Rin2] extended the theory to study well-bounded operators acting on nonreflexive Banach spaces. Ringrose showed that such operators can be represented as an integral with respect to a family of projections known as a decomposition of the identity which consists of a uniformly bounded family of projections $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ on the dual Banach space X^* rather than X itself. He showed that when given a decomposition of identity $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ on $[a,b]$ there is a (unique) well-bounded operator T such that

$$\langle Tx, x^* \rangle = b\langle x, x^* \rangle - \int_a^b \langle x, E(\lambda)x^* \rangle d\lambda, \tag{1.2.3}$$

for all $x \in X, x^* \in X^*$. Every well-bounded operator has such a representation, but in general the decomposition of the identity is not uniquely determined by T (see [Dow], Example 15.25).

However, on general Banach spaces one would not expect to obtain the same sort of spectral theorem as on reflexive spaces. For example, the operator T on $L^\infty[0,1]$ given by $(Tf)(t) = tf(t)$, $t \in [0,1]$ has an $AC[0,1]$ functional calculus. The natural family of projections, $E(\lambda)f = \chi_{[0,\lambda]}f$ for $0 \le \lambda \le 1$ fails however to have the continuity properties required of a spectral family. Even more seriously, if we replace $L^\infty[0,1]$ by $C[0,1]$, there are not even any suitable projections from which to try to give a spectral representation.

Ringrose's result and others in the nonreflexive case are particularly unsatisfactory because the projections in the decomposition of the identity for T act on the dual space, and are not necessarily the adjoints of the projections on X.

To overcome these problems surrounding general well-bounded operators, several subclasses were studied in [Rin2] and [BD1]. Ringrose [Rin2] introduced two subclasses:

1. well-bounded operators with a unique decomposition of the identity are called uniquely decomposable;
2. well-bounded operators decomposable in X are those with a decomposition of the identity $\{E(\lambda)\}_{\lambda\in\mathbb{R}}$ which is formed from the adjoints of projections on $\{F(\lambda)\}_{\lambda\in\mathbb{R}}$ on X.

Also, he proved that a well-bounded operator decomposable in X is uniquely decomposable.

In 1970, Berkson and Dowson [BD1] introduced two smaller subclasses of well-bounded operators. They defined type (A) well-bounded operators which form a subclass of the well-bounded operators decomposable in X. If a well-bounded operator $T \in B(X)$ has a decomposition of the indentity $\{F(\lambda)^*\}_{\lambda\in\mathbb{R}}$ formed by the adjoints of projections on X, then T is said to be of type (A) if for all $\lambda \in \mathbb{R}$ and $x \in X$, $\lim_{\alpha\to\lambda^+} F(\alpha)x = F(\lambda)x$. Later (Theorem 3.2 in [ACD]) it has been shown that an operator T is well-bounded of type (A) on a Banach space X if and only if T is a well-bounded operator decomposable in X.

An operator T is well-bounded of type (B) if T is well-bounded of type (A) and, in addition, for each real λ, $\lim_{\alpha\to\lambda^-} F(\alpha)x$ for all $x \in X$. It was shown (see [BD1] and [Spa2]) that this is equivalent to the condition that for all x in the Banach space X, the map $f : AC \to X$ such that $f \to f(T)x$ is compact in the weak topology.

It follows that every well-bounded operator on a reflexive Banach space is of type (B). It is still an open question as to whether on every nonreflexive Banach space there is a well-bounded operator which is not of type (B). In 1994, Doust and deLaubenfels showed that if the Banach space X contain a subspace isomorphic to c_0 or a complemented subspace isomorphic to ℓ_1 then there exists a well-bounded operator on X which is not of type (B) ([Dd], Theorem 4.4).

In [Spa2], Spain characterized the well-bounded operators of type (B) as being those which possess a weakly compact functional calculus. This compactness property allows one to define a unique spectral family $\{E(\lambda)\}$ of projections on X such that for some compact interval $[a, b] \in \mathbb{R}$, $\{E(\lambda)\}$ is concentrated on $[a, b]$, and $T = \int_{[a,b]}^{\oplus} \lambda\, dE(\lambda)$, where the integral is a limit of Riemann-Stieltjes sums (see [Dow, Chapter 17]). In 1993, Doust and Qiu [DB] presented an alternative approach to the spectral theorems of [Rin2] and [BD1] for obtaining integral representations for both general well-bounded operators and the well-bounded operators of type (B).

Berkson and Dowson [BD1] initiated the study of the relationship between well-bounded and spectral operators. They showed that if an operator is both well-bounded and spectral, then it is scalar-type operator and a scalar-type operator with real spectrum is a well-bounded operator of type (B).

The distinction between scalar-type spectral and well-bounded operators is illustrated by the properties of compact operators in these classes. If T is either a compact scalar-type spectral or a compact well-bounded operator, then there exists a sequence of disjoint projections $\{\boldsymbol{P}_{\lambda_k}\}_{k=1}^{\infty}$ such that

$$T = \sum_{k=1}^{\infty} \lambda_k \boldsymbol{P}_{\lambda_k}, \tag{1.2.4}$$

where $\{\lambda_k\}$ is the countable set of non-zero eigenvalues of T. In the case of scalar-type spectral operators, this sum converges unconditionally in the norm topology but the convergence of that sum for compact well-bounded operators is conditional. That is, it may depend on how the eigenvalues are ordered (see [CD1]).

The reader can also see the papers by Doust [Dou1] and [Dou2] to get many results on well-bounded and scalar-type spectral operators on classical Banach spaces.

1.2.4 Non real spectrum

The theory of well-bounded operators provides an analogue of the theorem of self-adjoint operators on a Hilbert space. It was natural to try to provide a similar theory corresponding to normal operators.

There were various attempts to address the issue of extending the theory of well-bounded operators to cover operators whose spectrum lies in the plane. One of the main restrictions of the theory of well-bounded operator has been that if only deals with operators whose spectra are subsets of a compact interval $[a, b]$ in the real line.

[Rin2] proved that all the main properties remain valid if the closed interval $[a, b]$ is extended to a suitable curve in the complex plane. However, he also gave examples of operators on Hilbert spaces that show that if we replace the interval

$[a, b]$ by a closed curve then the well-boundedness's concept becomes too weak to be significant.

In [BG1], Berkson and Gillespie introduced the concept of an AC operator as an operator which possesses a functional calculus for the absolutely continuous functions on some rectangle in $\mathbb{C}$ in the sense of Hardy and Krause [CA] (a more detailed definition is given in Chapter 2). Berkson and Gillespie showed that these operators can be characterized by the fact they possess a splitting into real and imaginary parts, $T = U + iV$, where U and V are commuting well-bounded operators. They also showed that if U and V are well-bounded of type (B), this splitting is unique, and if $S \in B(X)$ commutes with $U + iV$ then S commute with U and V. However, Berkson, Doust and Gillespie [BDG] showed that neither result holds if the type (B) hypothesis is omitted.

In the positive direction, perhaps more successfully, Berkson and Gillespie also introduced the class of trigonometrically well-bounded operators, which are analogues in this context of unitary operators. An operator T is trigonometrically well-bounded if there exists a type (B) well-bounded operator $A \in B(X)$ such that $T = e^{iA}$ and $\sigma(A) \subset [0, 2\pi]$ (see [BBG]). These operators have a number of applications in harmonic analysis and differential equations (see [BG2], [BG3]). Importantly every trigonometrically well-bounded operator is also an AC operator.

Furthermore, Doust and Walden [DW] studied compact AC operators and they showed that compact AC operators have representations as combinations of disjoint projections of the form (1.2.4).

One of the main problems with well-bounded and AC operators is that they are based on an algebra of functions whose domain is either an interval in the real line axis or a rectangle in the plane rather than the spectrum of T. Another problem with the theory of AC was shown in [BDG] that the class of AC operators is not closed under scalar multiplication. That is, there exists an AC operator T on a Hilbert space such that $(1 + i)T$ fails to be an AC-operator. In operator theory this is undesirable since if one's theory provides a structure theorem for T, then it should also provide a structure theorem for $\alpha T + \beta I$ for any $\alpha, \beta \in \mathbb{C}$.

In his PhD thesis, Ashton [Ash] aimed to overcome some of these difficulties by adopting different concepts of variation and absolute continuity for functions of two real variables. To do this he sought to define spaces $BV(\sigma)$ and $AC(\sigma)$ such that

1. these spaces are defined for any nonempty compact subset $\sigma \subseteq \mathbb{C}$;
2. these spaces agree with the classical spaces in the case that σ is an interval in the real line;
3. $AC(\sigma)$ always contains all sufficiently well-behaved functions;
4. if $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ then the space $AC(\alpha\sigma + \beta)$ should be isometrically isomorphic to $AC(\sigma)$.

Although many different notions of absolutely continuity for functions of two variables existed, none satisfied all of these conditions. This led to the definition of a new class of spaces which was introduced in [AD1], and it is this class which forms the central object of study in this thesis.

In [AD4], Ashton and Doust looked at those operators which have an $AC(\sigma)$ functional calculus, which they called $AC(\sigma)$ operators. This class of $AC(\sigma)$ operators not only includes all normal Hilbert space operators, but also the classes of well-bounded operators, scalar-type spectral operators and trigonometrically well-bounded operators on any Banach space.

1.3 Summary

Many studies have been addressed to extend the Banach-Stone theorems to other classes of function algebras. The reader can see [Vie], [Bac] and [Kaw] for a sample of some of these studies. In this thesis we will investigate extensions of Gelfand–Kolmogorov type theory to the families of function spaces $\{BV(\sigma)\}$ and $\{AC(\sigma)\}$ for non-empty compact sets $\sigma \subseteq \mathbb{C}$.

In this thesis shall say that two Banach algebras are A and B are isomorphic, denoted $A \simeq B$, if there exists a linear and multiplication bijection $\Phi : A \to B$ such that Φ and Φ^{-1} are both continuous. In this setting, the continuity of Φ^{-1} is automatic by the Banach isomorphism theorem.

It was shown by Doust and Leinert [DL2] that if $AC(\sigma_1)$ is isomorphic to $AC(\sigma_2)$ then σ_1 and σ_2 must be homeomorphic. The main question addressed in this thesis is the extent to which the converse implication might hold. Doust and Leinert gave examples of two homeomorphic sets σ_1 and σ_2 for which $AC(\sigma_1)$ is not isomorphic to $AC(\sigma_2)$. In a positive direction, they showed that if one restricts σ_1 and σ_2 to be polygonal sets then the reverse implication does hold. Our aim in this thesis is to extend this study to other natural classes of compact subset of the plane.

In the next chapter, we provide the fundamental definitions and general background required for our study. We give the classical definition of functions of bounded variation and absolute continuity for functions defined on an interval. We then give Ashton and Doust's definition of variation over a compact subset of the plane and recall the main properties of AC and BV functions in this context.

Little has been said in the literature about the structure of the larger spaces $BV(\sigma)$. We begin such a study in Chapter 3. An important result here states that the isomorphisms of such spaces are always obtained by composition by a bijection between the domain sets, these bijections need not be homeomorphisms. There are many natural questions concernng the relationship between isomorphisms of $AC(\sigma)$ and $BV(\sigma)$ spaces. We shall give a number of examples answering many of these questions, although there will remain several unanswered questions at the end of the chapter.

In Chapter 4 we will examine the isomorphisms of $AC(\sigma)$ spaces where σ is the spectrum of a compact operator. In particular, we examine the case where σ is a compact countable subset of the plane with unique limit point. All these set are homeomorphic, but we can obtain infinitely many non-isomorphic $AC(\sigma)$ spaces. In particular we show that within this class of compact sets, if the spectrum is contained in a finite number k of rays from the origin, the isomorphism class of $AC(\sigma)$ is completely determined by k. In particular, there are exactly two isomorphisms classes of $AC(\sigma)$ spaces where σ is of this form and $\sigma \subseteq \mathbb{R}$.

In Chapter 5 we will look at the class of compact sets formed from a finite union of closed line segments. Each of these can be considered as a drawing of a planar graph, such as the set in Figure 1.1

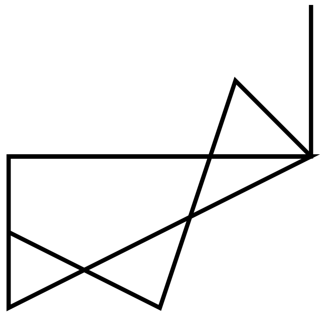


Figure 1.1: A "linear graph set".

The main aim in this chapter is to prove an analogue of the Gelfand–Kolmogorov theorem which covers all compact connected subsets which are made up of line segments.

In the final chapter we will extend this result to the case when σ is the union of convex edges instead of line segments, such as the set in Figure 1.2

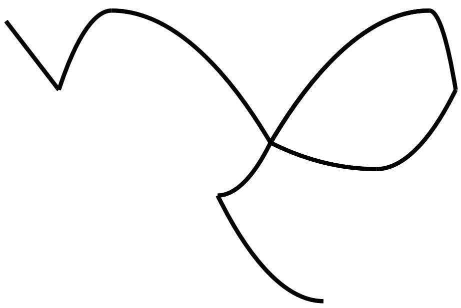


Figure 1.2: A 'polygonally inscribed curve'.

Again this leads to a positive result, but the proofs here are more complicated.

Throughout this thesis references are provided for the more basic and well-known results instead of proofs and much of our notation is standard. However, we have included a list of some of the notation at the end of the thesis.

We note here that the results in Chapters 4 and 5 have appeared in the publications [DAS] and [ASD].

Chapter 2

Preliminaries

In this chapter we introduce the background material that is fundamental to our study. Most of the material covered in this chapter is standard and therefore we do not include the proofs of all results. We have just included a proof for one that we believe contributes to a better understanding of it.

2.1 General definitions and standard results

Throughout this thesis $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers, respectively. Let X, Y be Banach spaces, i.e. complete normed vector spaces. The dual of X will be denoted by X^*. Most of this thesis is concerned with the structure of certain algebras of functions whose domain is a subset of the plane. We shall identify the plane as $\mathbb{C}$ or $\mathbb{R}^2$ as is most convenient. Throughout, we will use σ, σ_1 and σ_2 etc. to denote nonempty compact subsets of the plane.

In the following we will address the classical definitions of bounded variation and absolute continuity of functions. We refer the reader to references such as [Roy] and [KF] for more details.

2.1.1 $BV(\sigma)$ for $\sigma \subset \mathbb{R}$ compact

The classical concepts of functions of bounded variation and absolutely continuous functions depend on the variation of a function on a compact interval $I = [a, b] \subseteq \mathbb{R}$. Let f be a real or complex valued function with domain $[a, b]$ and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$. We denote the set of all partitions of I by $\Lambda(I)$.

We introduce the variation of f over the partition P, $V(P,f)$, by setting

$$V(P,f) = \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \geq 0. \tag{2.1.1}$$

Definition 2.1.1. A function f defined on an interval $[a,b]$ is said to be of bounded-variation if there is a constant $C > 0$ such that $V(P,f) \leq C$ for every partition P of $[a,b]$.

The collection of all functions of bounded variation on $[a,b]$ is denoted by $BV(I)$, or $BV_{\mathbb{R}}(I)$ or $BV_{\mathbb{C}}(I)$ if we need to specify the scalar field. In the later sections we will almost always be concerned with complex spaces. Let

$$\operatorname*{var}_{[a,b]} f = \sup\{V(P,f) \colon P \in \Lambda(I)\}.$$

Then $\operatorname*{var}_{[a,b]} f$ is called the **total variation** of f on $[a,b]$. It is clear that f is a function of bounded variation if and only if $\operatorname*{var}_{[a,b]} f < \infty$. Then $BV(I)$ is a Banach algebra with the norm

$$\|f\|_{BV(I)} = \|f\|_{\infty} + \operatorname*{var}_{[a,b]} f. \tag{2.1.2}$$

See [Nat] for properties of $BV(I)$. (The $\|f\|_{\infty}$ term could be replaced with $|f(c)|$ for some $c \in I$. For our generalization later, the $\|\cdot\|_{\infty}$ term is more natural).

It is also clear that a complex function is of bounded variation if and only if its real and imaginary parts are of bounded variation.

There are different classes of functions that are of bounded variation.

Theorem 2.1.2. *Suppose that $f : [a,b] \to \mathbb{R}$ satisfies any one of the following :*

1. *f is monotone on $[a,b]$,*
2. *f is piecewise monotone function on $[a,b]$,*
3. *f is Lipschitz on $[a,b]$, that is, there exists an $M \geq 0$ such that $|f(x) - f(y)| \leq M|x-y|$ for each $x, y \in [a,b]$,*
4. *f is differentiable on $[a,b]$ such that $f'(x)$ is bounded on $[a,b]$.*

Then f is of bounded variation on $[a,b]$.

A continuous function need not be of bounded variation.

Example 2.1.3. The standard continuous function with infinite total variation is $f : [0, 1] \to \mathbb{R}$:

$$f(x) = \begin{cases} x\sin(\frac{\pi}{2x}), & 0 < x \leq 1, \\ 0, & x = 0. \end{cases}$$

The following result gives more properties of the bounded variation functions.

Theorem 2.1.4. *Let $f : [a, b] \to \mathbb{R}$ be a function of bounded variation on the interval $[a, b]$. Then we have the following:*

1. *f is bounded on $[a, b]$,*
2. *$|f|$ is a function of bounded variation on $[a, b]$.*

The converse implications in this theorem are not true.

It is more natural for spectral theory to consider function algebras containing functions defined on the spectrum of an operator, that is, a non-empty compact subset of $\mathbb{C}$. The space $AC(\sigma)$ for a non-empty compact subset of $\mathbb{R}$ was considered in the 1930's by [Sak, Chapter VII], and has been used more recently in [AD1]. Suppose then that $\sigma \subseteq \mathbb{R}$ is compact and nonempty, and that $f : \sigma \to \mathbb{C}$. The **variation of f** over σ is defined as

$$\operatorname{var}(f, \sigma) = \sup \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|, \tag{2.1.3}$$

where the supremum is taken over all finite increasing subsets $x_0 < x_1 < \cdots < x_n$ in σ.

If the bounded variation norm of f is

$$\|f\|_{BV(\sigma)} = \|f\|_\infty + \operatorname{var}(f, \sigma), \tag{2.1.4}$$

then the set of functions of bounded variation on σ is

$$BV(\sigma) = \{f : \sigma \to \mathbb{C} \,:\, \|f\|_{BV(\sigma)} < \infty\}. \tag{2.1.5}$$

It is easy to use the properties of variation to show that $\|\cdot\|_{BV(\sigma)}$ is an algebra norm on $BV(\sigma)$ (see [AD1]). The space $BV(\sigma)$ with $\|\cdot\|_{BV(\sigma)}$ is a Banach algebra.

2.1.2 $AC(\sigma)$ for $\sigma \subset \mathbb{R}$ compact

The algebra of absolutely continuous functions is the most important subalgebra of $BV([a,b])$. Absolutely continuous functions arose originally in the context of classical analysis, and there are now several equivalent definitions. For the classical situation we refer the reader to [Roy] for more details.

Definition 2.1.5. A real or complex valued function defined on $[a,b]$ is said to be **absolutely continuous** on $[a,b]$ if for every $\epsilon > 0$, there is $\lambda > 0$ such that for every finite system of pairwise disjoint subintervals $(a_j, b_j) \subset [a,b]$ $(j = 1, \dots, n)$ with $\sum_{j=1}^{n}(b_j - a_j) \le \lambda$ we have $\sum_{j=1}^{n} |f(b_j) - f(a_j)| \le \epsilon$.

It is clear we have the following:

- Every Lipschitz function is absolutely continuous.
- Taking $n = 1$, we see that every absolutely continuous function is uniformly continuous, and hence continuous.
- Every absolutely continuous functions on $[a,b]$ is of bounded variation.

Theorem 2.1.6. *Suppose that $f : [a,b] \to \mathbb{R}$ (or $\mathbb{C}$). Then the following conditions are equivalent*

1. *f is absolutely continuous,*
2. *f is differentiable almost everywhere on $[a,b]$ and $f(x) = f(a) + \int_a^x f'(t)dt$, for all $x \in [a,b]$*
3. *for all $\epsilon > 0$ there exists a polynomial p such that $\|f - p\|_{BV} < \epsilon$.*

It is known that there are many continuous functions of bounded variation which are not absolutely continuous. The polynomials, the Lipschitz functions and the continuous piecewise linear functions are dense subsets of $AC[a,b]$.

One can extend the definition of absolute continuity to a compact subset $\sigma \subseteq \mathbb{R}$ in much the same way as was employed in the previous section (see [Sak, Section 7.5] or [AD1, Section 2]). The set of all absolutely continuous functions on σ is denoted $AC(\sigma)$. Although it may be difficult to make sense of condition 2 in the above theorem, it is still the case that the polynomials forms a dense subset of $AC(\sigma)$.

2.2 $BV(\sigma)$ for $\sigma \subset \mathbb{C}$ compact

There have many definitions of the variation of a function of two real variables. In 1933 Clarkson and Adams [CA] had collected seven variants. These variations applied to a range of different conditions on the underlying domain σ. Some, for example require that the domain is open.

In [BG1], Berkson and Gillespie aimed to generalize the theory of well-bounded operators to give an analogue of normal operators. They worked to develop a suitable definition of absolutely continuity for a function of two real variables (or equivalently of one complex variable) by using the definition of bounded variation for a function in two variables given by Hardy and Krause. This definition applies to functions defined on a rectangle in $\mathbb{R}^2$ whose sides are parallel to the coordinate axes.

Let $J = [a, b]$ and $K = [c, d]$ be two compact intervals in $\mathbb{R}$. Let Λ be a rectangular partition of $J \times K$, we mean two finite sets $\{x_i\}$, $\{y_j\}$ with

$$a = x_0 \leq x_1 \leq \cdots \leq x_n = b, \qquad c = y_0 \leq y_1 \leq \cdots \leq y_m = d.$$

Suppose that $f : J \times K \to \mathbb{C}$. Define

$$V_\Lambda = \sum_{i=1}^{n} \sum_{j=1}^{m} |f(x_i, y_j) - f(x_i, y_{j-1}) - f(x_{i-1}, y_j) + f(x_{i-1}, y_{j-1})| \,.$$

The variation of f in the sense of Hardy and Krause is defined to be

$$\operatorname*{var}_{J\times K} f = \sup\{V_\Lambda \colon \Lambda \text{ is a rectangular partition of } J \times K\}.$$

The function f is of bounded variation if $\operatorname*{var}_{J\times K} f$, $\operatorname*{var}_{J} f(\cdot, d)$ and $\operatorname*{var}_{K} f(b, \cdot)$ are all finite. The set $BV_{HK}(J \times K)$ of all functions $f : J \times K \to \mathbb{C}$ of bounded variation is a Banach algebra under the norm

$$\|f\|_{BV_{HK}} = |f(b, d)| + \operatorname*{var}_{J} f(\cdot, d) + \operatorname*{var}_{K} f(b, \cdot) + \operatorname*{var}_{J\times K} f.$$

In this context, a function $f : J \times K \to \mathbb{C}$ is said to be absolutely continuous if

1. For all $\epsilon \geq 0$, there exists δ such that $\sum_{R \in \varDelta} \operatorname{var}_R f \leq \epsilon$ whenever $\varDelta$ is a finite collection of non-overlapping subrectangles of $J \times K$ with $\sum_{R \in \varDelta} m(R) \leq \delta$, where m denotes the Lebesgue measure on $\mathbb{R}^2$.

2. The marginal functions $f(\cdot, d)$ and $f(b, \cdot)$ are absolutely continuous functions on J and K respectively.

The set $AC_{HK}(J \times K)$ of all absolutely continuous functions $f : J \times K \to \mathbb{C}$ is a Banach subalgebra of $BV_{HK}(J \times K)$, and is the closure in $BV_{HK}(J \times K)$ of the polynomials in two real variables on $J \times K$. In [BG1], Berkson and Gillespie developed a class of operators, the AC operators, which are a generalization in the context of well-boundedness of normal operators on Hilbert space. These are the Banach space operators which have $AC_{HK}(J \times K)$ functional calculus for some rectangle $J \times K$.

They characterized AC operators as being those of the form $T = A + iB$, where A and B are commuting well-bounded operators on X. Doust and Walden [DW] showed that every compact AC operator has a representation in the form $T = \sum_{i=1}^{\infty} \lambda_i P_i$ by taking the eigenvalues λ_i in an appropriate order.

As we commented in the introduction the theory of AC operators had some drawbacks. In [BDG] it is shown that the real and imaginary parts are not necessarily unique and that the class of AC operators is not closed under scalar multiplication. That is, there exists an AC operator T and $\alpha \in \mathbb{C}$ such that αT is not an AC operator. In terms of spectral theory, this is somewhat undesirable since the structure of T and αT should be more or less the same.

2.3 The Ashton-Doust definition

The concept of two-dimensional variation introduced in [Ash] and [AD1] was developed to have specific properties to overcome the issues with the Berkson-Gillespie theory. This variation in two variables was defined in terms of the variation along continuous parametrized curves in the plane. Ashton and Doust in [AD1] worked with the piecewise linear curves determined by a finite ordered list of points instead of general continuous curves. In [DL1] Doust and Leinert presented a development to the original definitions given in [AD1]. Here we will follow [DL1].

Our first step in defining the function space $BV(\sigma)$ will be to give an appropriate definition for the variation of a general complex-valued function f defined on a compact subset $\sigma \subseteq \mathbb{C}$. Let σ be a nonempty compact subset of the plane and suppose that $f : \sigma \to \mathbb{C}$. The main obstacle to be overcome is to deal with the fact that one no longer has a natural ordering for finite subsets of the domain σ. Any measure of variation then needs to account for not just the change in values of the function between such points, but also their arrangement and ordering in the plane.

Suppose that $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ is a finite ordered list of elements of σ. These elements do not need to be different although we assume that $\boldsymbol{x}_k \neq \boldsymbol{x}_{k+1}$ for each k and we assume $n \geq 1$. Let γ_S denote the piecewise linear curve joining the points of S in order.

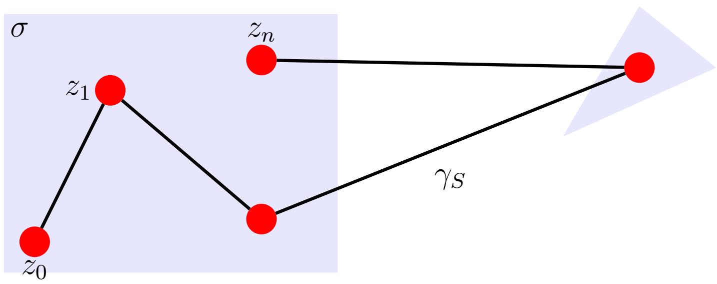


Figure 2.1: Example of a piecewise linear curve γ_S between the points of S in order.

We define the **curve variation** of f on S to be

$$\operatorname{cvar}(f, S) = \sum_{i=1}^{n} |f(\boldsymbol{x}_i) - f(\boldsymbol{x}_{i-1})| .$$

Unless f is constant, the quantity $\operatorname{cvar}(f, S)$ is clearly unbounded over all possible lists S. To overcome this we introduce the variation factor.

The variation factor $\operatorname{vf}(S)$ associated to the set S takes into account the variation which is due to the curve γ_S. Loosely speaking $\operatorname{vf}(S)$ is the greatest number of times that γ_S crosses any line in the plane. To be more precise we need to consider the concept of a crossing segment.

Definition 2.3.1. Suppose that ℓ is a line in the plane. For $0 \leq i < n$ we say that $\overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}$ is a **crossing segment** of S on ℓ if any one of the following holds:

(i) $\boldsymbol{x}_i$ and $\boldsymbol{x}_{i+1}$ lie on (strictly) opposite sides of ℓ.

(ii) $i = 0$ and $\boldsymbol{x}_i \in \ell$.

(iii) $\boldsymbol{x}_i \notin \ell$ and $\boldsymbol{x}_{i+1} \in \ell$.

Let $\mathrm{vf}(S, \ell)$ denote the number of crossing segments of S on ℓ.

Example 2.3.2.

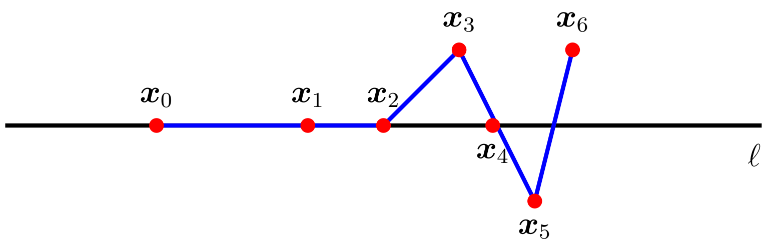


Figure 2.2: For this set of points there are three crossing segments on ℓ.

In Figure 2.2 we consider $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5, \boldsymbol{x}_6]$ and label the segments $\ell_i = \overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}$. Then the crossing segments for S on ℓ are ℓ_0, ℓ_3 and ℓ_5 but the remaining segments are not crossing segments of S on ℓ. Thus $\mathrm{vf}(S, \ell) = 3$.

Definition 2.3.3. The ***variation factor*** of S is defined to be

$$\mathrm{vf}(S) = \max\{\mathrm{vf}(S, \ell) : \ell \text{ is a line in } \mathbb{C}\}.$$

For instance, in the following the variation factor $\mathrm{vf}(S) = 3$.

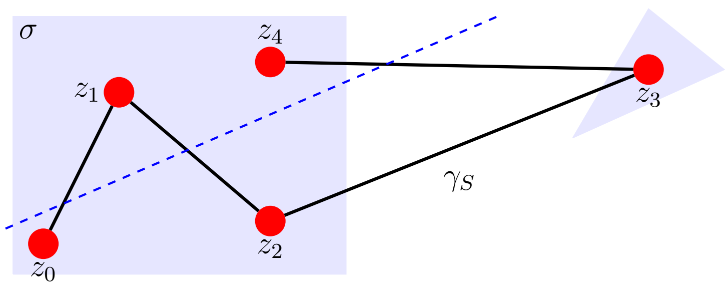


Figure 2.3: The variation factor $\mathrm{vf}(\gamma_S) = 3$.

Clearly $1 \leq \mathrm{vf}(S) \leq n$. For completeness, in the case that $S = [\boldsymbol{x}_0]$ we set $\mathrm{cvar}(f, [\boldsymbol{x}_0]) = 0$ and let $\mathrm{vf}([\boldsymbol{x}_0], \ell) = 1$ whenever $\boldsymbol{x}_0 \in \ell$.

The following lemma and examples show the value of $\mathrm{vf}(S)$ by considering the concept of crossing segment.

Lemma 2.3.4. *Let $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ with $\boldsymbol{x}_j \neq \boldsymbol{x}_{j+1}$ for all j. Suppose that for some $t \in \mathbb{C}$, $x_j = t$ for k different values of j. Then $\mathrm{vf}(S) \geq k$.*

Proof. Choose a line ℓ in the plane which is going through t, but no other points in S. Now $\mathrm{vf}(S, \ell)$ is equal to the number of crossing segments of S on ℓ, so we need to consider the definition of a crossing segment.

- if $\boldsymbol{x}_0 = t$, then by condition (ii), $\overline{\boldsymbol{x}_0 \boldsymbol{x}_1}$ is a crossing segment.
- if $\boldsymbol{x}_i = t$ for some $i > 0$ then by condition (iii), $\boldsymbol{x}_{i-1} \notin \ell$ then $\overline{\boldsymbol{x}_{i-1} \boldsymbol{x}_i}$ is a crossing segment.

Hence the number of crossing segments of S on ℓ is at least equal to the number of $\boldsymbol{x}_i's$ equal to t. □

Example 2.3.5. Suppose that $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n$ are the vertices of a convex polygon σ (taken in anticlockwise order) and $S = [\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n]$. Then $\mathrm{vf}(S) = 2$. (see Figure 2.4).

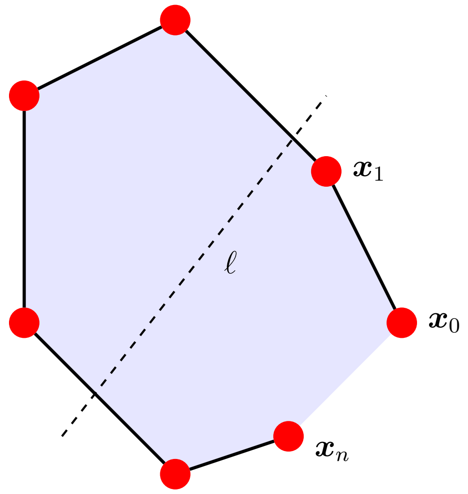


Figure 2.4: $\mathrm{vf}(S) = 2$.

Example 2.3.6. Let σ be as above and $S = [\boldsymbol{x}_0, \ldots, \boldsymbol{x}_n, \boldsymbol{x}_0]$ and $\ell_i = \overline{\boldsymbol{x}_i \boldsymbol{x}_{i+1}}$ for $0 \leq i < n$.

Let ℓ be a line passing through $\boldsymbol{x}_0$ and a side of the polygon as in Figure 2.5. Then the crossing segments for the set S on ℓ are ℓ_0 (rule (ii)) and ℓ_n (rule (iii)) in definition 2.3.1 and ℓ_k (rule (i)). By considering lines through two sides, or two vertices, one can see that $\mathrm{vf}(S, \ell) = 3$. Thus $\mathrm{vf}(S) = 3$.

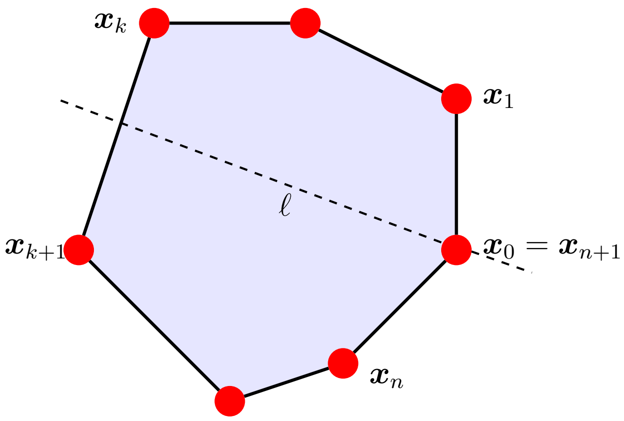


Figure 2.5: $\mathrm{vf}(S) = 3$.

Definition 2.3.7. The ***two-dimensional variation*** of f is defined to be

$$\mathrm{var}(f, \sigma) = \sup_S \frac{\mathrm{cvar}(f, S)}{\mathrm{vf}(S)},$$

where the supremum is taken over all finite ordered lists S of elements of σ.

It is not immediately clear that the definition of two dimensional variation is a reasonable generalization of the one dimensional concept! It is quite easy to see that if $\sigma = [a, b] \subseteq \mathbb{R}$ then $\operatorname{var}_{[a,b]} f \leq \mathrm{var}(f, [a, b])$. The reverse inequality is however also true [AD1, Proposition 3.6], and so for such sets, the two concepts of variation agree.

Ashton and Doust also showed that, for any compact set σ, their concept of variation always has algebraic properties which match those of the classical concept.

Proposition 2.3.8. *[AD1, Proposition 3.7] Let $\sigma_1 \subset \sigma \subset \mathbb{C}$ both be compact. Let $f, g : \sigma \to \mathbb{C}$, $k \in \mathbb{C}$. Then*

1. $\mathrm{var}(f + g, \sigma) \leq \mathrm{var}(f, \sigma) + \mathrm{var}(g, \sigma)$,
2. $\mathrm{var}(fg, \sigma) \leq \|f\|_\infty \mathrm{var}(g, \sigma) + \|g\|_\infty \mathrm{var}(f, \sigma)$,
3. $\mathrm{var}(kf, \sigma) = |k| \mathrm{var}(f, \sigma)$,
4. $\mathrm{var}(f, \sigma_1) \leq \mathrm{var}(f, \sigma)$.

One property which is easy to see is that the definition is invariant under invertible affine transformations. Suppose that $h : \mathbb{C} \to \mathbb{C}$ is an invertible affine transformation $h(z) = az + b$ and that $\sigma_h = h(\sigma)$. Then every function $f : \sigma \to \mathbb{C}$ has an associated function $f_h : \sigma_h \to \mathbb{C}$ given by $f_h(z) = f(h^{-1}(z))$. Since h maps lines to lines, one has that $\operatorname{var}(f_h, \sigma_h) = \operatorname{var}(f, \sigma)$. (This is of course still true if one considers the plane as being $\mathbb{R}^2$ and one used real affine transformations $H\boldsymbol{x} = A\boldsymbol{x} + \boldsymbol{b}$ for some invertible matrix A.)

Despite these nice properties, it is often not easy to actually calculate the variation of a given function. An important fact which we shall use often throughout the thesis is that functions of bounded variation in one real variable extend to functions of bounded variation on subsets of the plane.

Theorem 2.3.9. *[AD1, Lemma 3.12] Suppose that σ is a nonempty compact subset of $\mathbb{C}$ and that $\sigma_{\mathbb{R}} = \{\operatorname{Re} z \,:\, z \in \sigma\}$. Suppose that $f : \sigma_{\mathbb{R}} \to \mathbb{C}$ and that $\hat{f} : \sigma \to \mathbb{C}$ is defined by $\hat{f}(x + iy) = f(x)$. Then*

$$\operatorname{var}(\hat{f}, \sigma) \le \operatorname{var}(f, \sigma_{\mathbb{R}}).$$

It might be noted that the inequality here may be strict.

Example 2.3.10. Let $\sigma = \{0, 1 + i, 2\} \subseteq \mathbb{C}$, so $\sigma_{\mathbb{R}} = \{0, 1, 2\}$. Let $f(x) = |x - 1|$. Then $\operatorname{var}(f, \sigma_{\mathbb{R}}) = 2$. On the other hand $\hat{f}(x+iy) = 1-y$ which is just the extension to σ of a function defined on the imaginary axis, so applying the theorem (and affine invariance) $\operatorname{var}(\hat{f}, \sigma) \le 1$ (and indeed is equal to 1).

For $f : \sigma \to \mathbb{C}$ where σ is a non-empty compact subset of $\mathbb{C}$, let

$$\|f\|_{BV(\sigma)} = \sup_{z \in \sigma} |f(z)| + \operatorname{var}(f, \sigma).$$

It is clear from Proposition 2.3.8 that $\|\cdot\|_{BV(\sigma)}$ is a norm on the set of functions of bounded variation on σ,

$$BV(\sigma) = \{f : \sigma \to \mathbb{C} \,:\, \|f\|_{BV(\sigma)} < \infty\}.$$

Note that if $f, g \in BV(\sigma)$ then, by Proposition 2.3.8(2)

$$\begin{aligned}
\|fg\|_{BV(\sigma)} &= \|fg\|_\infty + \operatorname{var}(fg, \sigma) \\
&\le \|f\|_\infty \|g\|_\infty + \|f\|_\infty \operatorname{var}(g, \sigma) + \|g\|_\infty \operatorname{var}(f, \sigma) \\
&\le (\|f\|_\infty + \operatorname{var}(f, \sigma)) (\|g\|_\infty + \operatorname{var}(g, \sigma)) \\
&= \|f\|_{BV(\sigma)} \|g\|_{BV(\sigma)} .
\end{aligned}$$

It follows that $BV(\sigma)$ is not just a normed space, it is also a normed algebra. As was shown in [AD1, Theorem 3.8], this algebra is always complete, so $BV(\sigma)$ is a Banach algebra.

Theorem 2.3.9 and affine invariance allows one to show that many useful functions must lie in $BV(\sigma)$. For example the restriction to σ of the characteristic function of any half-plane must always be of bounded variation. Multiplying several of these together shows that the restriction to σ of the characteristic function of any convex polygon is also in $BV(\sigma)$. A fact we will need later is that the characteristic function of any single point in σ is in $BV(\sigma)$.

We will now discuss the most important subalgebra of $BV(\sigma)$.

2.4 $AC(\sigma)$ for $\sigma \subseteq \mathbb{C}$

As we noted in Section 2.1, there are several equivalent conditions for a function $f : [a, b] \to \mathbb{C}$ to be absolutely continuous in the classical sense. Given that the primary motivation for the $AC(\sigma)$ spaces is in terms of functional calculus, it is most natural here to proceed from the fact that $AC[a, b]$ is the closure of the polynomials inside $BV[a, b]$.

Proofs of the spectral theorem for normal operators typically use the fact that the C^*-algebra generated by a normal operator T is isometrically isomorphic to $C(\sigma(T))$, and this depends on the fact that the polynomials in z and $\overline{z}$ are dense in this space of functions. This space of polynomials is the starting point for the definition of the $AC(\sigma)$ spaces.

Writing $z = x + iy$, it is clear that every polynomial $p(z, \overline{z}) = \sum_{n,m} b_{nm} z^n \overline{z}^m$ for some $b_{nm} \in \mathbb{C}$ could also be written in the form $\sum_{n,m} c_{nm} x^n y^m$ (and vice-versa), and as we will mostly be using the $\mathbb{R}^2$ model of the plane, it will be slightly easier to work with these polynomials in two real variables.

Let $\mathcal{P}_2$ denote the space of complex polynomials in two real variables of the form $p(x, y) = \sum_{n,m} c_{nm} x^n y^m$, and let $\mathcal{P}_2(\sigma)$ denote the restrictions of elements of $\mathcal{P}_2$ to σ. By the last theorem and Proposition 2.3.8, the algebra $\mathcal{P}_2(\sigma)$ is always a subalgebra of $BV(\sigma)$.

The first important fact is that polynomials do in fact always lie inside $BV(\sigma)$. This is a consequence of the following theorem, which itself is just a corollary of Theorem 2.3.9.

Theorem 2.4.1. *Suppose that σ is a nonempty compact subset of $\mathbb{R}^2$. Define $u, v : \sigma \to \mathbb{C}$ by $u(x, y) = x$ and $v(x, y) = y$. Then $u, v \in BV(\sigma)$.*

Definition 2.4.2. The set of **absolutely continuous functions** on σ, denoted $AC(\sigma)$, is the closure of $\mathcal{P}_2(\sigma)$ in $BV(\sigma)$.

Thus $AC(\sigma)$ always forms a closed subalgebra of $BV(\sigma)$. Since the BV norm agrees with the classical one if $\sigma \subseteq \mathbb{R}$, and $\overline{z} = z$ on such sets, this definition must give the classical space in these cases.

Supose that $h : \mathbb{C} \to \mathbb{C}$ is an invertible affine transformation as in Section 2.3. For any $p \in \mathcal{P}_2$, $p_h = p \circ h^{-1}$ is also in $\mathcal{P}_2$. Thus, the affine invariance of the BV norm immediately gives the following first result on isomorphisms of these spaces.

Theorem 2.4.3. *If $h : \mathbb{C} \to \mathbb{C}$ is an invertible affine transformation of the plane, then $AC(h(\sigma))$ is isometrically isomorphic to $AC(\sigma)$, via the algebra isomorphism $\Phi(f) = f \circ h^{-1}$.*

Much of the thesis will be concerned with determining broader classes of homeomorphisms h for which this result still holds. Theorem 2.4.3 will be used frequently in the later proofs to transform problems to a simpler setting.

The other easy consequence of the properties of the BV norm is that if $\sigma_1 \subseteq \sigma$ and $f \in AC(\sigma)$, then $f|\sigma_1 \in AC(\sigma_1)$.

Note that if $\{p_n\}_{n=1}^{\infty}$ is a sequence of polynomials and $\|p_n - f\|_{BV(\sigma)} \to 0$ then certainly $\|p_n - f\|_\infty \to 0$ and so f must be continuous. Thus, every absolutely continuous function is indeed continuous. The Cantor function is the standard example of a continuous function on $[a, b]$ which is not absolutely continuous, so in general the inclusion $AC(\sigma) \subseteq C(\sigma) \cap BV(\sigma)$ is strict. On the other hand, there are some spaces where one gets equality here. It appears to be an open problem to characterize all sets σ for which this is the case (see [DL2, Question 2.5]).

It is useful to note that the corresponding result to Theorem 2.3.9 works for absolutely continuous functions.

Theorem 2.4.4. *[AD1, Proposition 4.4] Suppose that σ is a non-empty compact subset of $\mathbb{C}$ and that $\sigma_{\mathbb{R}} = \{\operatorname{Re} z : z \in \sigma\}$. Suppose that $f : \sigma_{\mathbb{R}} \to \mathbb{C}$ and that $\hat{f} : \sigma \to \mathbb{C}$ is defined by $\hat{f}(x + iy) = f(x)$. If $f \in AC(\sigma_{\mathbb{R}})$ then $\hat{f} \in AC(\sigma)$ with $\|\hat{f}\|_{BV(\sigma)} \leq \|f\|_{BV(\sigma_{\mathbb{R}})}$.*

Said another way, the map $f \mapsto \hat{f}$ provide a Banach algebra homomorphism of norm at most one from $AC(\sigma_{\mathbb{R}})$ to $AC(\sigma)$. It should be noted that the converse to Theorem 2.4.4 is false.

Example 2.4.5. Let $\sigma \subseteq \mathbb{R}^2$ be the graph of the Cantor function on $[0, 1]$, so $\sigma_{\mathbb{R}} = [0, 1]$. If we let $f : [0, 1] \to \mathbb{C}$ be the Cantor function, then $\hat{f} : \sigma \to \mathbb{C}$ is given by $\hat{f}(x, y) = y$. That is, $\hat{f}$ is the extension onto σ of an absolutely continuous function on the y-axis, and hence by Theorem 2.4.4, $\hat{f} \in AC(\sigma)$. But of course $f \notin AC(\sigma_{\mathbb{R}})$.

It is of course desirable to have sufficient conditions for a function to be absolutely continuous which do not depend on approximating it with polynomials. One of the fundamental results about variation proven in [AD1] concerns its relationship with Lipschitz continuity.

Let $\lambda(z) = z = x + iy$ be the identity function on $\sigma \subseteq \mathbb{C}$. Clearly, $\lambda \in AC(\sigma)$. Let $C_\sigma = \operatorname{var}(\lambda, \sigma)$. Then, writing $\sigma_{\mathbb{I}}$ for the projection of σ onto the imaginary axis, and letting u and v be as in Theorem 2.4.1,

$$C_\sigma \leq \operatorname{var}(u, \sigma) + \operatorname{var}(iv, \sigma) \leq \operatorname{var}(u, \sigma_{\mathbb{R}}) + \operatorname{var}(v, \sigma_{\mathbb{I}}) \leq \sqrt{2}\operatorname{diam}(\sigma).$$

Recall that the Lipschitz constant of a function $f : \sigma \to \mathbb{C}$ is defined to be

$$L(f, \sigma) = \inf\{k : |f(z) - f(w)| \leq k|z - w| \text{ for all } z, w \in \sigma\}.$$

Let $\mathrm{Lip}(\sigma)$ denote the Banach space of all Lipschitz continuous functions on σ, that is, all functions f with $L(f,\sigma) < \infty$, equipped with the norm $\|f\|_{\mathrm{Lip}(\sigma)} = \|f\|_\infty + L(f,\sigma)$.

Theorem 2.4.6. *[AD1, Lemma 3.15]* $\mathrm{Lip}(\sigma) \subseteq BV(\sigma)$. *Indeed, for all* $f \in \mathrm{Lip}(\sigma)$,

$$\mathrm{var}(f,\sigma) \le L(f,\sigma)\, C_\sigma.$$

Proof. Let $S = [\boldsymbol{x}_0, \dots, \boldsymbol{x}_n]$ be a finite list of points in σ. Then

$$\begin{aligned}
\mathrm{cvar}(f,S) &= \sum_{i=1}^{n} |f(\boldsymbol{x}_i) - f(\boldsymbol{x}_{i-1})| \\
&\le \sum_{i=1}^{n} L(f,\sigma)|\boldsymbol{x}_i - \boldsymbol{x}_{i-1}| \\
&= L(f,\sigma)\,\mathrm{cvar}(\lambda, S).
\end{aligned}$$

Dividing through by $\mathrm{vf}(S)$ and then taking the supremum gives the required result. $\square$

A consequence of this theorem is that any sequence of functions which approximates f in Lipschitz norm, also approximates it in $BV(\sigma)$ norm. In the case that $\sigma = [a,b]$, every Lipschitz function is absolutely continuous. One of the important differences in the two variable case is that this is not longer true in the two variable setting. In [AD1, Example 4.13] Ashton and Doust give a Lipschitz function on the square $[0,1] \times [0,1]$ which is not absolutely continuous.

All sufficiently smooth functions however are always absolutely continuous. We shall write $C^k(\sigma)$ for the vector space of all functions $f : \sigma \to \mathbb{C}$ which admit a k-times differentiable (as a function of two real variables) extension to some open set containing σ. In the unpublished note [DL2] it is shown that $C^1(\sigma) \subseteq AC(\sigma)$, although we will not need that fact in our proofs.

Since the set $C^\infty(\sigma)$ is a dense subset of $AC(\sigma)$ by Proposition 4.7 in [AD1], this allows us to prove the following lemma that absolutely continuous functions are stable under simple operations.

Lemma 2.4.7. *If* $f \in AC(\sigma)$ *then* $\mathrm{Re}(f), \mathrm{Im}(f) \in AC(\sigma)$.

Proof. Let $\{p_n\}_{n=1}^{\infty}$ be a sequence of polynomials such that $\lim_{n\to\infty} \|f - p_n\|_{BV(\sigma)} = 0$. Then $\{\mathrm{Re}(p_n)\}_{n=1}^{\infty} \subseteq C^{\infty}(\sigma)$. By [AD1, Proposition 3.18], $\lim_{n\to\infty} \|\mathrm{Re}(f) - \mathrm{Re}(p_n)\|_{BV(\sigma)} \le \lim_{n\to\infty} \|f - p_n\|_{BV(\sigma)} = 0$. Hence $\mathrm{Re}(f) \in AC(\sigma)$. Similarly $\mathrm{Im}(f) \in AC(\sigma)$. □

2.5 Isomorphisms of $AC(\sigma)$ spaces

In this section we will recall the main known results related to analogues of the Gelfand–Kolmogorov theorem in the context of the spaces of absolutely continuous functions, addressing the corresponding relationship between the structures of the algebra $AC(\sigma)$ and the domain of functions σ. We note here that all the results in this section are from [DL2].

In [DL2, Theorem 2.6], Doust and Leinert proved one direction of Gelfand and Kolmogorov's result always holds in this setting.

It was noted by Doust and Leinert that any algebra isomorphism between $AC(\sigma)$ spaces must be continuous. Their proof depends upon the following lemma, which is a consequence of the fact that if $f \in AC(\sigma)$ and $f(z)$ is never zero on σ, then $\frac{1}{f} \in AC(\sigma)$. Note that this lemma implies that the only quasinilpotent element of $AC(\sigma)$ is the zero function, and hence $AC(\sigma)$ is a semisimple Banach algebra.

Lemma 2.5.1. *[DL2, Lemma 2.5] Suppose that $f \in AC(\sigma)$. Then the spectrum of f is $\sigma(f) = f(\sigma)$ and hence the spectral radius of f is $r(f) = \|f\|_{\infty}$.*

Any homomorphisms between commutative semisimple Banach algebras are automatically continuous. Doust and Leinert's proof does not use this fact directly, although it essentially depends on the same ideas.

Theorem 2.5.2. *Suppose that σ_1 and σ_2 are nonempty compact subsets of the plane. If $\Phi : AC(\sigma_1) \to AC(\sigma_2)$ is an isomorphism then there exists a homeomorphism $h : \sigma_1 \to \sigma_2$ such that $\Phi(f) = f \circ h^{-1}$ for all $f \in AC(\sigma_1)$.* ***Furthermore, Φ is continuous.***

Proof. Since Φ preserves the identity element, it preserves the spectrum of elements. Thus using the above lemma,

$$\|f\|_{\infty} = r(f) = r(\Phi(f)) = \|\Phi(f)\|_{\infty}$$

for all $f \in AC(\sigma)$.

Since $AC(\sigma_1)$ is dense in $C(\sigma_1)$, this implies that Φ extends to an isometric isomorphism $\Phi : C(\sigma_1) \to C(\sigma_2)$ and hence, by the Banach-Stone theorem, σ_1 is homeomorphic to σ_2. Indeed there exists a homeomorphism $h : \sigma_1 \to \sigma_2$ such that $\Phi(f) = f \circ h^{-1}$ for all $f \in C(\sigma_1)$. This is clearly still true if we restrict this to $AC(\sigma_1)$.

Suppose that $f_n \to f \in AC(\sigma_1)$. Then certainly $f_n \to f$ uniformly and hence pointwise. Suppose that $\Phi(f_n) \to g \in AC(\sigma_2)$. Then for all $x \in \sigma_2$,

$$g(x) = \lim_n \Phi(f_n)(x) = \lim_n f_n(h^{-1}(x)) = f(h^{-1}(x)) = \Phi(f)(x)$$

and hence $\Phi(f) = g$. Thus, by the Closed Graph Theorem Φ is continuous. $\square$

Doust and Leinert showed that the converse of Theorem 2.5.2 is not true.

Theorem 2.5.3. *$AC(\sigma_1)$ and $AC(\sigma_2)$ are not isomorphic, where σ_1 is the closed unit disk and σ_2 is the closed unit square.*

They then deduced that if Φ is an isometric Banach algebra isomorphism which preserves the supremum norm $\|f\|_\infty = \|\Phi(f)\|_\infty$ for all $f \in AC(\sigma_1)$ and that $\operatorname{var}(f, \sigma_1) = \operatorname{var}(\Phi(f), \sigma_2)$ for all $f \in AC(\sigma_1)$.

In the other direction, there are some positive results that can be obtained about when $AC(\sigma_1)$ and $AC(\sigma_2)$ are isomorphic.

The main content of [DL2] involves determining families of homeomorphisms h which preserve the isomorphism class of $AC(\sigma)$ spaces. They introduced a type of homeomorphism which they called a locally piecewise affine map.

Let $\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible affine map, and let C be a convex n-gon. Then $\alpha(C)$ is also a convex n-gon. Denote the sides of C by $s_1, \dots, s_n$. Suppose that $\boldsymbol{x}_0 \in \operatorname{int}(C)$. The point $\boldsymbol{x}_0$ determines a triangulation $T_1, \dots, T_n$ of C, where T_j is the (closed) triangle with side s_j and vertex $\boldsymbol{x}_0$. A point $\boldsymbol{y}_0 \in \operatorname{int}(\alpha(C))$ determines a similar triangulation $\hat{T}_1, \dots, \hat{T}_n$ of $\alpha(C)$, where the numbering is such that $\alpha(s_j) \subseteq \hat{T}_j$.

Lemma 2.5.4. *With the notation as above, there is a unique map $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

- $h(\boldsymbol{x}) = \alpha(\boldsymbol{x})$ *for* $\boldsymbol{x} \notin \mathrm{int}(C)$.
- h *maps* T_j *onto* $\hat{T}_j$, *for* $1 \le j \le n$.
- $\alpha_j = h|T_j$ *is affine, for* $1 \le j \le n$.
- $h(\boldsymbol{x}_0) = \boldsymbol{y}_0$.

We shall say that h is the ***locally piecewise affine map*** *determined by* $(C, \alpha, \boldsymbol{x}_0, \boldsymbol{y}_0)$.

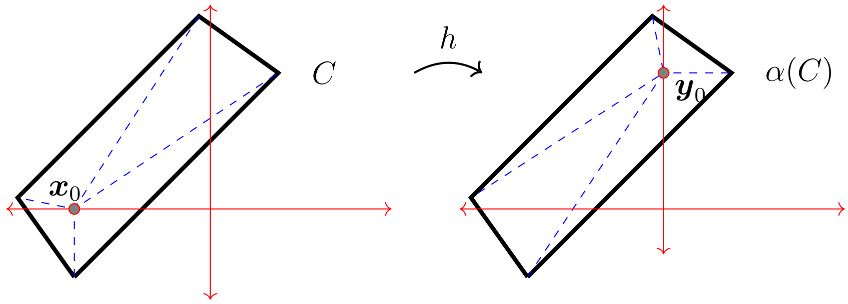


Figure 2.6: A locally piecewise affine map h moving $\boldsymbol{x}_0$ to $\boldsymbol{y}_0$.

It is clear that h is necessarily continuous and invertible. That is, any locally piecewise affine map is a homeomorphism.

Lemma 2.5.5. *[DL2, Lemma 5.2] Let h be the locally piecewise affine map determined by $(C, \alpha, \boldsymbol{x}_0, \boldsymbol{y}_0)$. Then h^{-1} is the locally piecewise affine map determined by $(h(C), \alpha^{-1}, \boldsymbol{y}_0, \boldsymbol{x}_0)$.*

The important property of locally piecewise affine maps is that they preserve the isomorphism class of $AC(\sigma)$ spaces.

Theorem 2.5.6. *[DL2, Theorem 5.5] Suppose that σ is a nonempty compact subset of the plane, and that h is locally piecewise affine map. Then we have $AC(\sigma) \simeq AC(h(\sigma))$ and $BV(\sigma) \simeq BV(h(\sigma))$.*

The proof of this theorem is based on the following lemma:

Lemma 2.5.7. *[DL2, Lemma 5.4] Suppose that h is a locally piecewise affine map determined by (C, α, x_0, y_0) where C is a convex n-gon. Let $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_m]$ be a list of elements of $\mathbb{R}^2$ and let $\hat{S} = [h(\boldsymbol{x}_0), h(\boldsymbol{x}_1), \dots, h(\boldsymbol{x}_m)]$. Then*

$$\frac{1}{c_n} \mathrm{vf}(S) \le \mathrm{vf}(\hat{S}) \le c_n \, \mathrm{vf}(S)$$

for some positive constant c_n which is independent of S.

A consequence of the above results is a Gelfand–Kolmogorov type theorem for $AC(\sigma)$ spaces when σ belong to the class of polygonal subsets of the plane. A set $P \subseteq \mathbb{R}^2$ is a **polygon** if it is a compact set whose boundary consists of finite number of line segments, and which is homeomorphic to a closed disk.

Theorem 2.5.8. *Suppose that P_1 and P_2 are polygons. Then $AC(P_1) \simeq AC(P_2)$.*

Doust and Leinert extended the above result to cover more general sets based on polygons with finitely many polygonal holes. A compact set $\sigma \subseteq \mathbb{R}^2$ defines a **polygonal region of genus** n if there exists a polygon P with n non-intersecting windows $W_1, \ldots, W_n$ such that

$$\sigma = P \backslash (W_1 \cup \cdots \cup W_n)$$

and write $G(\sigma) = n$ for the genus of σ.

Theorem 2.5.9. *[DL2, Theorem 7.2] Suppose that σ_1 and σ_2 are polygonal regions of genus n_1 and n_2. Then $AC(\sigma_1) \simeq AC(\sigma_2)$ if and only if $n_1 = n_2$.*

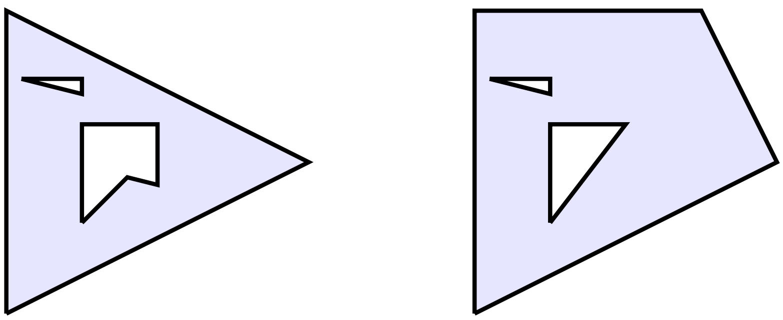


Figure 2.7: Two polygonal regions of genus 2.

The aim of our work in the next chapters is to extend this result to consider the different classes of compact sets. For example we shall consider in Chapter 5 the class of sets which are the union of a finite number of line segments.

Chapter 3

Isomorphisms results for $BV(\sigma)$ spaces

We saw in the last chapter that if $AC(\sigma_1)$ is isomorphic as a Banach algebra to $AC(\sigma_2)$, then this isomorphism is of the form $\Phi_h(f) = f \circ h^{-1}$ for some homeomorphism $h : \sigma_1 \to \sigma_2$. A natural question to ask is whether there is also some connection between the algebraic structure of the $BV(\sigma)$ spaces and the properties of the domain sets σ. In this chapter we will show that there is a somewhat weaker connection, and provide many examples which show that things are rather less well determined is this setting. We will also examine the relationship between $AC(\sigma)$ space isomorphisms and $BV(\sigma)$ space isomorphisms.

Throughout this chapter, isomorphism will mean a Banach algebra isomorphism, that is, a continuous algebra isomorphism with a continuous inverse, and $\sigma, \sigma_1, \sigma_2$ etc will denote nonempty compact subsets of the plane.

3.1 Isomorphisms of $BV(\sigma)$ spaces

It is easy to check that the analogue of the Theorem 2.5.2 for $AC(\sigma)$ spaces does not extend directly to $BV(\sigma)$ spaces.

Example 3.1.1. Let $\sigma_1 = \sigma_2 = [0,1]$ and define $h : \sigma_1 \to \sigma_2$,

$$h(x) = \begin{cases} \frac{1}{2} - x, & \text{if } 0 \le x \le \frac{1}{2}, \\ x, & \text{if } \frac{1}{2} < x \le 1. \end{cases}$$

A simple rearrangement of the variation of $\Phi_h(f)$ over any partition P of $[0,1]$ shows that

$$V(P, \Phi_h(f)) \le 2 \operatorname{var}(f, [0,1])$$

and so (noting that $\Phi_h^{-1} = \Phi_h$),

$$\frac{1}{2} \|f\|_{BV[0,1]} \leq \|\Phi_h(f)\|_{BV[0,1]} \leq 2 \|f\|_{BV[0,1]} .$$

Thus Φ_h is a Banach algebra isomorphism from $BV(\sigma_1)$ to $BV(\sigma_2)$. But of course the map h is not a homeomorphism in this case.

In a positive direction we have the following.

Theorem 3.1.2. *Suppose that σ_1 and σ_2 are non-empty compact subsets of the plane. If $\Phi : BV(\sigma_1) \to BV(\sigma_2)$ is an isomorphism then there exists a bijection $h : \sigma_1 \to \sigma_2$ such that $\Phi(f) = f \circ h^{-1}$ for all $f \in BV(\sigma_1)$.*

The proof of the theorem depends on the following lemma.

Lemma 3.1.3. *Suppose that σ is a non-empty compact subset of the plane. For all $\boldsymbol{z} \in \sigma$, $\chi_{\{\boldsymbol{z}\}} \in BV(\sigma)$.*

Proof. If H_1, H_2 and H_3 are three distinct closed half-planes as shown in Figure 3.1, then $\chi_{\{\boldsymbol{z}\}} = \chi_{H_1}\chi_{H_2}\chi_{H_3}$ and so $\chi_{\{\boldsymbol{z}\}} \in BV(\sigma)$. □

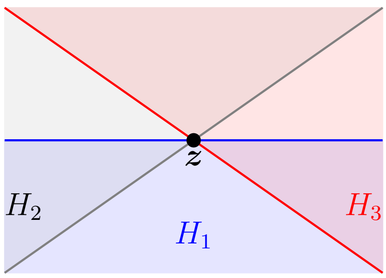


Figure 3.1: The characteristic function of a point is the product of the characteristic function of three half-planes.

Now we will go to the proof of Theorem 3.1.2.

Proof. Since Φ is an algebra isomorphism, it must map idempotents to idempotents. Note that by Lemma 3.1.3 for all $z \in \sigma_1$, the function $f_z = \chi_{\{z\}}$ lies in $BV(\sigma_1)$ and hence $g_z = \Phi(f_z)$ is an idempotent in $BV(\sigma_2)$. Since Φ is one-to-one, g_z is not the zero function and hence the support of g_z is a nonempty set $\tau \subseteq \sigma_2$. If τ is more than a singleton then we can choose $w \in \tau$ and write $g_z = \chi_{\{w\}} + \chi_{\tau \setminus \{w\}}$ as a sum of two nonzero idempotents in $BV(\sigma_2)$. But then $f_z = \Phi^{-1}(\chi_{\{w\}}) + \Phi^{-1}(\chi_{\tau\setminus\{w\}})$ is the sum of two nonzero idempotents in $BV(\sigma_1)$ which is impossible. It follows that g_z is the characteristic function of a singleton set and this clearly induces a map $h : \sigma_1 \to \sigma_2$ so that $\Phi(f_z) = \chi_{\{h(z)\}}$. Indeed, by considering Φ^{-1} it is clear that h must be a bijection between the two sets. $\square$

There are several questions one might ask concerning possible strengthening of Theorem 3.1.2.

Question 3.1.4. Suppose that $h : \sigma_1 \to \sigma_2$ is a bijection. Does Φ_h map $BV(\sigma_1)$ to $BV(\sigma_2)$?

Question 3.1.5. Suppose that σ_1 and σ_2 have the same cardinality. Is $BV(\sigma_1) \simeq BV(\sigma_2)$?

Question 3.1.6. Suppose that σ_1 and σ_2 are homeomorphic. Is $BV(\sigma_1) \simeq BV(\sigma_2)$?

Question 3.1.7. Suppose that $BV(\sigma_1) \simeq BV(\sigma_2)$. Is σ_1 homeomorphic to σ_2?

Questions 3.1.4 and 3.1.5 are easily disposed of.

Example 3.1.8. Let $\sigma_1 = \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \dots\}$ and let $\sigma_2 = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$. Define $h : \sigma_1 \to \sigma_2$ by

$$h(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{2n-1}, & \text{if } x = -\frac{1}{n} < 0, \\ \frac{1}{2n}, & \text{if } x = \frac{1}{n} > 0. \end{cases}$$

It is readily checked that h is a bijection (indeed a homeomorphism). If f is the characteristic function of the positive elements of σ_1 then $f \in BV(\sigma_1)$ but $\Phi_h(f)$ is not in $BV(\sigma_2)$.

We will see in Chapter 4 that in fact no bijection between the two sets in Example 3.1.8 determines an isomorphism of the spaces of functions of bounded variation. For the moment we can give a simpler example which answers Question 3.1.5 negatively.

Example 3.1.9. Let $\sigma_1 = [0, 1] \times [0, 1]$ and $\sigma_2 = [0, 1]$, which of course have the same cardinality. Suppose that $h : \sigma_1 \to \sigma_2$ is a bijection which induces an isomorphism $\Phi_h : BV(\sigma_1) \to BV(\sigma_2)$.

For $t \in [0,1]$ let $V_t = \{(x,y) \in \sigma_1 : x = t\}$. By Proposition 3.20 in [AD1], $\chi_{V_t} \in BV(\sigma_1)$ for all t and so $g_t = \Phi_h(\chi_{V_t}) \in BV(\sigma_2)$. Now g_t is an idempotent function in $BV(\sigma_2)$, so it must be the characteristic function of an uncountable set $S_t = h(V_t) \subseteq [0,1]$. Since g_t can only contain finitely many jumps, any such set must contain at least one interval, say I_t. Of course the sets I_t must all be disjoint since h is a bijection and so $\{I_t\}_{t\in[0,1]}$ would form an uncountable collection of disjoint subsets of $[0,1]$ each with positive measure. This is impossible so no such bijection h can exist.

An examination of the proof of Theorem 3.1 in [DL] shows that if h is any homeomorphism from the unit square to the closed unit disk, the map Φ_h must be unbounded with respect to the BV norms, and hence the answer to Question 3.1.6 is also no. Note that our work in the later chapters will show that within certain classes of compact sets, if σ_1 and σ_2 are homeomorphic then $BV(\sigma_1) \simeq BV(\sigma_2)$.

Showing that the answer to Question 3.1.7 is 'no' will need to wait until Example 5.6.2 in Chapter 5. The following example shows that there are isomorphisms of $BV(\sigma)$ spaces which are not induced by homeomorphisms of the domains σ_1 and σ_2.

Example 3.1.10 ([DAS], Example 3.4)**.** Let $\sigma_1 = \sigma_2 = \{0\} \bigcup \{\frac{1}{n}\}_{n=1}^{\infty}$. Define $h : \sigma_1 \to \sigma_2$ by

$$h(x) = \begin{cases} 1, & \boldsymbol{x} = 0, \\ 0, & \boldsymbol{x} = 1, \\ \boldsymbol{x}, & \text{otherwise.} \end{cases}$$

and for $f \in BV(\sigma_1)$ let $\Phi(f) : \sigma_2 \to \mathbb{C}$ be $\Phi(f) = f \circ h^{-1}$. A simple calculation shows that $\frac{1}{3}\operatorname{var}(f,\sigma_1) \le \operatorname{var}(\Phi(f),\sigma_2) \le 3\operatorname{var}(f,\sigma_1)$ and so Φ is a Banach algebra isomorphism from $BV(\sigma_1) \to BV(\sigma_2)$. The map h is of course not a homeomorphism.

If $h : \sigma_1 \to \sigma_2$ is a bijection then $\Phi_h(f) = f \circ h^{-1}$ is always an algebra isomorphism from the algebra of all complex-valued functions on σ_1 to the algebra of all functions on σ_2, with inverse $\Phi_h^{-1} = \Phi_{h^{-1}}$. Our next aim is to characterize those h for which this map restricts to a Banach algebra isomorphism from $BV(\sigma_1)$ to $BV(\sigma_2)$.

For every $\boldsymbol{x} \in \sigma$, the map $f \mapsto f(\boldsymbol{x})$ is a non-zero multiplicative linear functional on $BV(\sigma)$. A consequence of this is that the radical of $BV(\sigma)$ is $\{0\}$ and so $BV(\sigma)$ is semisimple. This means that if Φ_h is an algebra isomorphism from $BV(\sigma_1)$ to $BV(\sigma_2)$ then necessarily Φ_h and Φ_h^{-1} are continuous.

What really determines whether a bijection defines an isomorphism of the $BV(\sigma)$ spaces is what it does to the variation factors of lists of points.

Definition 3.1.11. Suppose that $h : \sigma_1 \to \sigma_2$ is a bijection.

1. If $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n]$ is a finite list of elements of σ_1, we denote by $h(S) = [h(\boldsymbol{x}_0), h(\boldsymbol{x}_1), \dots, h(\boldsymbol{x}_n)]$ the corresponding list of elements in σ_1.
2. The **variation factor of** h is

$$\operatorname{vf}(h) = \sup_S \frac{\operatorname{vf}(S)}{\operatorname{vf}(h(S))}.$$

Note that $\operatorname{vf}(h) \geq 1$, and that $\operatorname{vf}(h)$ may be infinite.

Lemma 3.1.12. *If* $\operatorname{vf}(h) = K < \infty$ *then* $\Phi_h(f) \in BV(\sigma_2)$ *for all* $f \in BV(\sigma_1)$ *and*

$$\|\Phi_h(f)\|_{BV(\sigma_2)} \leq K \|f\|_{BV(\sigma_1)} .$$

Proof. Let $\hat{S}$ be a finite list of points in σ_2. As h is a bijection, there exists a finite list S in σ_1 such that $\hat{S} = h(S)$. Then

$$\frac{\operatorname{cvar}(\Phi_h(f), \hat{S})}{\operatorname{vf}(\hat{S})} = \frac{\operatorname{cvar}(f, S)}{\operatorname{vf}(h(S))} \leq \frac{K \operatorname{cvar}(f, S)}{\operatorname{vf}(S)} \leq K \operatorname{var}(f, \sigma_1).$$

Since $\|\Phi_h(f)\|_\infty = \|f\|_\infty$, the result follows. □

From the comments above, if Φ_h maps $BV(\sigma_1)$ into $BV(\sigma_2)$, then it must be bounded.

Lemma 3.1.13. *If* $\operatorname{vf}(h) = \infty$ *then* Φ_h *is not bounded on* $BV(\sigma_1)$ *and hence* Φ_h *does not map* $BV(\sigma_1)$ *into* $BV(\sigma_2)$.

Proof. Suppose that $\operatorname{vf}(h) = \infty$ and that $K > 1$. Then there exists a finite list $S = [\boldsymbol{x}_0, \dots, \boldsymbol{x}_n]$ in $BV(\sigma_1)$ such that $\frac{\operatorname{vf}(S)}{\operatorname{vf}(h(S))} > K$. Suppose that $\operatorname{vf}(S) = n$, and note that $n > 1$. Choose any line ℓ such that $\operatorname{vf}(S, \ell) = \operatorname{vf}(S)$. This line determines two closed half-planes whose intersection is ℓ, and, by Proposition 3.20 in [AD1], the characteristic functions of these half planes χ_1 and χ_2 are of bounded variation on σ_1, with $\|\chi_i\|_{BV(\sigma_1)} \le 2$.

By definition n of the segments in S are crossing segments of S on ℓ. Of these, at least $n-1$ must satisfy either rule (ii) or rule (iii). If $\overline{\boldsymbol{x}_{j-1}\,\boldsymbol{x}_j}$ satisfies rule (iii) then $|\chi_i(\boldsymbol{x}_j) - \chi_i(\boldsymbol{x}_{j-1})| = 1$ for each i. If $\overline{\boldsymbol{x}_{j-1}\,\boldsymbol{x}_j}$ satisfies rule (ii) then $|\chi_i(\boldsymbol{x}_j) - \chi_i(\boldsymbol{x}_{j-1})| = 1$ for one of $i = 1$ or $i = 2$. Combining these facts shows that for at least one of the values $i = 1$ or $i = 2$,

$$\sum_{j=1}^{n} |\chi_i(\boldsymbol{x}_j) - \chi_i(\boldsymbol{x}_{j-1})| \ge \frac{n-1}{2}.$$

Let $f = \chi_i$ for such a value of i. Then (as $n > 1$)

$$\operatorname{var}(\Phi_h(f)) \ge \frac{\operatorname{cvar}(\Phi_h(f), h(S))}{\operatorname{vf}(h(S))} = \frac{\operatorname{cvar}(f, S)}{\operatorname{vf}(h(S))} \ge \frac{n-1}{2} \cdot \frac{K}{n} \ge \frac{K}{4}.$$

Thus

$$\|\Phi_h\| \ge \frac{\|\Phi_h(f)\|_{BV(\sigma_2)}}{\|f\|_{BV(\sigma_1)}} \ge \frac{K+1}{8}.$$

Since K was arbitrary, Φ_h is not bounded on $BV(\sigma_1)$. $\square$

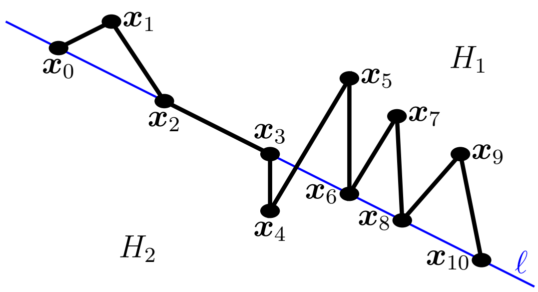


Figure 3.2: Choosing a BV function in the proof of Lemma 3.1.13. In this case $n = \operatorname{vf}(S, \ell) = 6$, $\operatorname{cvar}(\chi_{H_1}, S) = 2 < \frac{n-1}{2}$ and $\operatorname{cvar}(\chi_{H_2}, S) = 8 \ge \frac{n-1}{2}$.

The following is an immediate consequence of the two lemmas.

Theorem 3.1.14. *Suppose that $h : \sigma_1 \to \sigma_2$ is a bijection. Then Φ_h is a (Banach algebra) isomorphism from $BV(\sigma_1)$ to $BV(\sigma_2)$ if and only if $\operatorname{vf}(h)$ and $\operatorname{vf}(h^{-1})$ are both finite.*

Example 3.1.15. The conditions on $\mathrm{vf}(h)$ and $\mathrm{vf}(h^{-1})$ are both necessary in the above theorem. To see this let $\sigma_2 = [0,1]$ and let $g : [0,1] \to \mathbb{C}$ be the function

$$g(x) = \begin{cases} x\sin(1/x), & \text{for } 0 < x \le 1, \\ 0, & \text{for } x = 0. \end{cases}$$

Let σ_2 be the graph of g, and let $h : \sigma_1 \to \sigma_2$ be the obvious bijection $h(x) = (x, g(x))$. It is not hard to verify that $\mathrm{vf}(h(S)) \ge \mathrm{vf}(S)$ for any list $S \subseteq [0,1]$ and hence $\mathrm{vf}(h) = 1$. This implies that $\Phi_h(f) \in BV(\sigma_2)$ for all $f \in BV(\sigma_1)$. This is of course just a special case of Theorem 2.3.9. On the other hand Φ_h does not map $BV(\sigma_1)$ **onto** $BV(\sigma_2)$. For example $v : \sigma_2 \to \mathbb{C}$, $v(x,y) = y$ is in $BV(\sigma_2)$, but $\Phi_h^{-1}(v) \notin BV(\sigma_1)$. Again, it is not hard to verify directly that $\mathrm{vf}(h^{-1}) = \infty$.

3.2 $AC(\sigma)$ and $BV(\sigma)$ spaces

From the last section we know that Banach algebra isomorphisms of $AC(\sigma)$ and $BV(\sigma)$ spaces are always of the form $\Phi_h(f) = f \circ h^{-1}$ for some bijection $h : \sigma_1 \to \sigma_2$. For the $AC(\sigma)$ spaces the bijections are necessarily homeomorphisms, but this need not be the case for the $BV(\sigma)$ spaces. A consequence of this is that any such isomorphism of $AC(\sigma)$ spaces or $BV(\sigma)$ spaces always extends to an isomorphism of the algebras of complex-valued functions on the corresponding sets.

To avoid the notation becoming too cumbersome, we shall not distinguish between the restrictions of Φ_h to different subalgebras of functions on σ_1. Thus, for example, if we say that Φ_h is an isomorphism from $AC(\sigma_1)$ to $AC(\sigma_2)$ we are really referring to the map $\Phi_h | AC(\sigma_1)$.

Several natural questions arise as to under what conditions such maps preserve the properties of the functions on each of the spaces.

Question 3.2.1. Suppose that $BV(\sigma_1) \simeq BV(\sigma_2)$ via the isomorphism $\Phi_h(f) = f \circ h^{-1}$ where $h : \sigma_1 \to \sigma_2$ is a bijection. Does Φ_h map $AC(\sigma_1)$ to $AC(\sigma_2)$? That is, is $\Phi_h(f) \in AC(\sigma_2)$ for all $f \in AC(\sigma_1)$?

Question 3.2.2. Suppose that $BV(\sigma_1) \simeq BV(\sigma_2)$ (via any isomorphism). Is $AC(\sigma_1) \simeq AC(\sigma_2)$?

Question 3.2.3. Suppose that $BV(\sigma_1) \simeq BV(\sigma_2)$ via the isomorphism $\Phi_h(f) = f \circ h^{-1}$ where $h : \sigma_1 \to \sigma_2$ is a homeomorphism. Does Φ_h map $AC(\sigma_1)$ to $AC(\sigma_2)$?

Question 3.2.4. Suppose that $h : \sigma_1 \to \sigma_2$ and that Φ_h is an isomorphism from $AC(\sigma_1)$ to $AC(\sigma_2)$. Must Φ_h also be an isomorphism from $BV(\sigma_1)$ to $BV(\sigma_2)$?

The answer to Question 3.2.1 is obviously 'no'. As in Example 3.1.10 there are discontinuous bijections h which generate isomorphisms of the $BV(\sigma)$ spaces, but which (by Theorem 2.5.2) can not be isomorphisms of the $AC(\sigma)$ spaces. The negative answer to Question 3.1.7 foreshadowed above will also give a negative answer to Question 3.2.2. We will give a second example below.

Before answering the remaining two questions it is worth recording a few lemmas.

For the next results we assume that σ_1 and σ_2 are non-empty and compact sets subsets of $\mathbb{C}$.

Lemma 3.2.5. *Suppose that $h : \sigma_1 \to \sigma_2$ is a homeomorphism. If Φ_h is an isomorphism from $AC(\sigma_1)$ to $AC(\sigma_2)$ then $h \in AC(\sigma_1)$ and $h^{-1} \in AC(\sigma_2)$.*

Proof. The identity map $f(\boldsymbol{x}) = \boldsymbol{x}$ is clearly in $AC(\sigma_1)$ and so $\Phi_h(f) = h^{-1} \in AC(\sigma_2)$. The same argument applied to Φ^{-1} shows that $h \in AC(\sigma_1)$. □

Example 3.2.6. Let $\sigma_1 = \sigma_2 = [0,1]$. Let $h : \sigma_1 \to \sigma_2$ be an increasing (necessarily continuous) bijection which is not absolutely continuous. (For example, $h(x) = \frac{1}{2}(x + C(x))$ where C is the Cantor function.) Then Φ_h is an isomorphism from $BV(\sigma_1)$ to $BV(\sigma_2)$. However, as in the lemma Φ_h^{-1} will map the identity function on σ_2 to h and so Φ_h does not restrict to an isomorphism from $AC(\sigma_1)$ to $AC(\sigma_2)$. It follows that the answer to Question 3.2.3 is 'no'.

The following fact will be needed in the proof of the main result, Theorem 3.2.8.

Theorem 3.2.7. *Suppose that $h : \sigma_1 \to \sigma_2$ is a homeomorphism such that $h^{-1} \in AC(\sigma_2)$. Then for any $p \in P_2(\sigma_1)$, $\Phi_h(p) \in AC(\sigma_2)$.*

Proof. By Theorem 2.4.7, $\operatorname{Re} h^{-1}, \operatorname{Im} h^{-1} \in AC(\sigma_2)$. Now $\Phi_h(p)$ is just a polynomial in $\operatorname{Re} h^{-1}$ and $\operatorname{Im} h^{-1}$ and hence $\Phi_h(p) \in AC(\sigma_2)$ too. □

Theorem 3.2.8. *Suppose that $AC(\sigma_1) \simeq AC(\sigma_2)$. Then $BV(\sigma_1) \simeq BV(\sigma_2)$.*

Proof. As before, there exists a homeomorphism $h : \sigma_1 \to \sigma_2$ so that $\Phi_h(f) = f \circ h^{-1}$ is a bounded invertible map from $AC(\sigma_1)$ to $AC(\sigma_2)$, with $\|\Phi_h(f)\|_{BV(\sigma_2)} \leq K \|f\|_{BV(\sigma_1)}$ say. Suppose then that Φ_h is not an isomorphism from $BV(\sigma_1)$ to $BV(\sigma_2)$. By Theorem 3.1.14 this means that $\mathrm{vf}(h) = \infty$. We can therefore choose a list of points $S \subseteq \sigma_1$ so that $\mathrm{vf}(S) > 8K\,\mathrm{vf}(h(S))$. As in the proof of Lemma 3.1.13, we can choose a line ℓ such that $\mathrm{vf}(S, \ell) = \mathrm{vf}(S) = n$ say, and which forms the boundary of a half-plane H whose characteristic function χ_H satisfies $\mathrm{cvar}(\chi_H, S) \geq \frac{n-1}{2}$.

Given $\delta > 0$ define $f_{H,\delta} : \sigma_1 \to \mathbb{C}$ by $f_{H,\delta}(\boldsymbol{x}) = \max\{1 - \frac{1}{\delta}\}\mathrm{dist}(\boldsymbol{x}, H), 0\}$. By Theorem 2.4.4, $f_{H,\delta} \in AC(\sigma_1)$ with $\|f_{H,\delta}\|_{BV(\sigma_1)} \leq 2$. Note $f_{H,\delta}$ which agrees with χ_H except on a small strip of width δ along the boundary of H (see Figure 3.3). Since S is a finite set, if δ is chosen small enough, then

$$\mathrm{cvar}(\Phi_h(f_{H,\delta}), h(S)) = \mathrm{cvar}(f_{H,\delta}, S) = \mathrm{cvar}(\chi_H, S) \geq \frac{n-1}{2}.$$

Thus

$$\begin{aligned}
\mathrm{var}(\Phi_h(f_{H,\delta}), \sigma_2) &\geq \frac{\mathrm{cvar}(\Phi_h(f_{H,\delta}), h(S))}{\mathrm{vf}(h(S))} \\
&> \frac{n-1}{2} \cdot \frac{8K}{\mathrm{vf}(S)} \\
&= 4\frac{(n-1)K}{n} \geq 2K.
\end{aligned}$$

We then have that

$$\|\Phi_h(f_{H,\delta})\|_{BV(\sigma_2)} > 2K \geq \|\Phi_h\|\ \|f_{H,\delta}\|_{BV(\sigma_1)}$$

which is impossible. Therefore $\mathrm{vf}(h)$ must be finite, and Φ_h must be bounded on $BV(\sigma_2)$.

An analogous argument using the relationship between the boundedness of Φ_h^{-1} and the finiteness of $\mathrm{vf}(h^{-1})$ completes the proof. $\square$

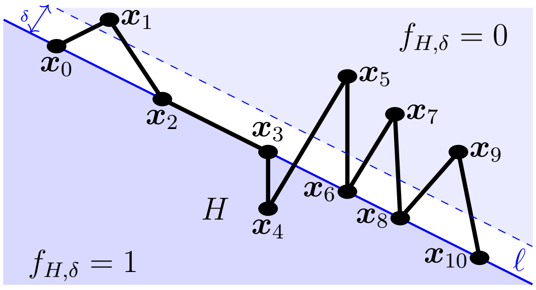


Figure 3.3: Choosing the function $f_{H,\delta} \in AC(\sigma_1)$ in the proof of Theorem 3.2.8. For δ small enough, $f_{H,\delta}(\boldsymbol{x}_j) = \chi_H(\boldsymbol{x}_j)$ for all j.

Combining the above results gives the following Theorem.

Theorem 3.2.9. *If $BV(\sigma_1) \simeq BV(\sigma_2)$ and $h : \sigma_1 \to \sigma_2$ is a homeomorphism, then Φ_h is an isomorphism from $AC(\sigma_1)$ to $AC(\sigma_2)$ if and only if h and h^{-1} are absolutely continuous functions.*

Proof. ($\Rightarrow$) Suppose that Φ_h is an isomorphism from $AC(\sigma_1)$ to $AC(\sigma_2)$. By the previous theorem, Φ_h extends to an isomorphisms from $BV(\sigma_1)$ to $BV(\sigma_2)$. The facts about h and h^{-1} follow from Lemma 3.2.5.

($\Leftarrow$) Suppose that the other side holds. By Theorem 3.2.7, $\Phi_h(p) \in AC(\sigma_2)$ for all $p \in P_2(\sigma_1)$. Since Φ_h is BV norm bounded this implies that $\Phi_h(f) \in AC(\sigma_1)$. Similarly Φ_h^{-1} maps $AC(\sigma_2)$ into $AC(\sigma_1)$ and hence Φ_h is an isomorphism of the spaces of absolutely continuous functions. $\square$

Chapter 4

Isomorphisms of $AC(\sigma)$ for countable sets

Compact linear operators form a very important special class in many applications. For instance, they play an important role in the numerical approximation of solutions to operator equations and in various problems in mathematical physics.

One of the important properties of compact operators is that they have a countable spectrum with at most one limit point at zero. In [AD2] Ashton and Doust studied the structure of compact $AC(\sigma)$ operators. All compact $AC(\sigma)$ operators have a representation like that for compact normal operators.

Our aim in this chapter is to address the isomorphisms of $AC(\sigma)$ where σ is the spectrum of a compact operator.

4.1 Isolated points

Before examining the case of countable sets, we shall look at what happens if one moves a single isolated point. That is, suppose that $\sigma = \sigma_1 \cup \{\boldsymbol{x}\}$ and $\tau = \sigma_1 \cup \{\boldsymbol{y}\}$ where σ_1 is a nonempty compact subset of $\mathbb{C}$ and $\boldsymbol{x}$ and $\boldsymbol{y}$ are points in the complement of σ_1 as in Figure 4.1. It is natural to ask whether we must have that $AC(\sigma) \simeq AC(\tau)$.

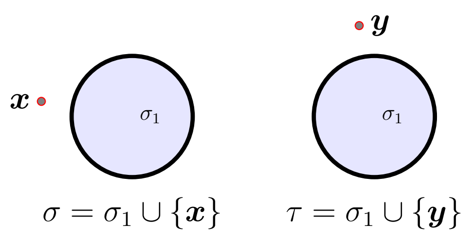


Figure 4.1: A single isolated point (case 1).

An important property of locally piecewise affine maps is that they preserve the isomorphism class of $AC(\sigma)$ spaces. (Explicit bounds on the norms of the isomorphisms are given in [DL2], but we shall not need these here. In any case, the known bounds are unlikely to be sharp.)

Theorem 4.1.1. *[DL2, Theorem 5.5] Suppose that σ is a nonempty compact subset of the plane, and that h is a locally piecewise affine map. Then $BV(\sigma) \simeq BV(h(\sigma))$ and $AC(\sigma) \simeq AC(h(\sigma))$.*

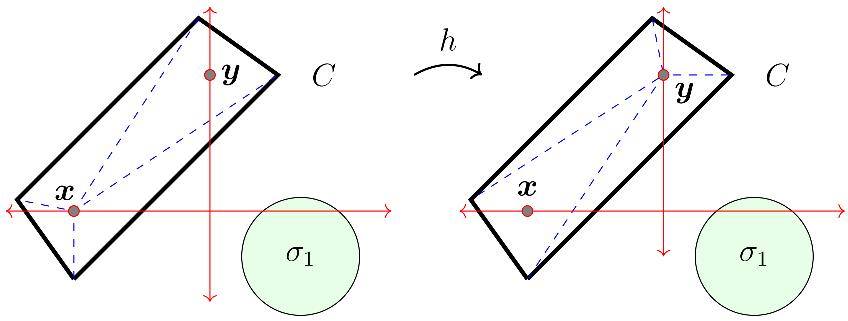
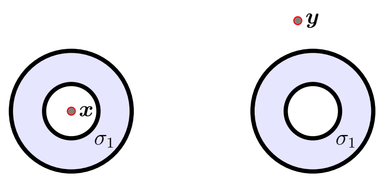


Figure 4.2: A locally piecewise affine map h moving $\boldsymbol{x}$ to $\boldsymbol{y}$.

If, as in Figure 4.1, there exists a convex polygon C which is disjoint from σ_1 and which includes $\boldsymbol{x}$ and $\boldsymbol{y}$ in its interior, then one can certain define a locally piecewise affine map $h : \mathbb{C} \to \mathbb{C}$ which fixes σ_1 and 'moves' $\boldsymbol{x}$ to $\boldsymbol{y}$, and consequently $BV(\sigma) \simeq BV(\tau)$ and $AC(\sigma) \simeq AC(\tau)$. One can of course compose several such maps, and so it is easy to see that if $\boldsymbol{x}$ and $\boldsymbol{y}$ can be joined by a polygonal path which avoids σ_1, the we have the same isomorphism result.

Such an approach does of course not work if $\boldsymbol{x}$ and $\boldsymbol{y}$ are as in Figure 4.3. In that case (and indeed in much of the later chapters), a homeomorphism from σ to τ can not be the restriction to σ of a homeomorphism of the entire plane. But this leaves the question of what sort of homeomorphisms of compact sets one can apply which preserve the isomorphism class of the $AC(\sigma)$ spaces.



$\sigma = \sigma_1 \cup \{\boldsymbol{x}\}$ $\quad$ $\tau = \sigma_1 \cup \{\boldsymbol{y}\}$

Figure 4.3: A single isolated point (case 2).

In general, if one can write σ as a disjoint union of compact sets σ_1 and σ_2 then one essentially has $AC(\sigma) = AC(\sigma_1) \oplus AC(\sigma_2)$, and so it is enough to deal with homeomorphisms of the component sets. To formally make sense of this one needs to identify $AC(\sigma_1)$ with the set $\{f \in AC(\sigma) : \operatorname{supp}(f) \subseteq \sigma_1\}$. This requires that if one extends a function $g \in AC(\sigma_1)$ to all of σ by making it zero on σ_2, then the extended function is absolutely continuous. This is more complicated than it might first appear since there are examples (see [DL1, Example 3.3]) where σ is a nondisjoint union of compact sets σ_1 and σ_2 and functions where $f|\sigma_1 \in AC(\sigma_1)$ and $f|\sigma_2 \in AC(\sigma_2)$, but f is not even of bounded variation on σ. We shall come back to these general issues at the end of the thesis.

For the moment we shall just deal with the case that σ_2 is a singleton set. That is σ is the union of a nonempty compact set σ_1 and an isolated singleton point $\{\boldsymbol{z}\}$. It is worth noting (using Proposition 4.4 of [AD1] for example) that $\chi_{\{\boldsymbol{z}\}}$ is always an element of $AC(\sigma)$. For $f \in BV(\sigma)$ let

$$\|f\|_D = \|f\|_{D(\sigma_1,\boldsymbol{z})} = \|f|\sigma_1\|_{BV(\sigma_1)} + |f(\boldsymbol{z})|.$$

Note that it is clear from the definition of variation that if $f \in BV(\sigma)$ then $f|\sigma_1 \in BV(\sigma_1)$. (To prevent the notation from becoming too cumbersome we will usually just write $\|f\|_{BV(\sigma_1)}$ rather than $\|f|\sigma_1\|_{BV(\sigma_1)}$ unless there is some risk of confusion.)

Proposition 4.1.2. *The norm $\|\cdot\|_D$ is equivalent to the usual norm $\|\cdot\|_{BV(\sigma)}$ on $BV(\sigma)$.*

Proof. We first remark that it is clear that $\|\cdot\|_D$ is a norm on $BV(\sigma)$. Also, noting the above remarks, $\|f\|_D \le 2\|f\|_{BV(\sigma)}$ so we just need to find a suitable lower bound for $\|f\|_D$.

Suppose then that $f \in BV(\sigma)$. Let $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ be an ordered list of points in σ. Form a sublist $S' = [\boldsymbol{y}_0, \ldots, \boldsymbol{y}_m]$ of S by first omitting every occurrence of the point $\boldsymbol{z}$, and then removing elements to ensure that consecutive elements of S' are distinct. (For example, if $S = [\boldsymbol{x}, \boldsymbol{z}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{x}]$, then $S' = [\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}]$.)

Our aim is to compare $\operatorname{cvar}(f, S)$ with $\operatorname{cvar}(f, S')$. In calculating $\operatorname{cvar}(f, S)$ we may assume (as usual) that no two consecutive points in this list are both equal to

$\boldsymbol{z}$, and that S' is nonempty. Let N be the number of times that the point $\boldsymbol{z}$ occurs in the list S.

Now if $\boldsymbol{x}_k = \boldsymbol{z}$ for some $0 < k < n$ then

$$
\begin{aligned}
|f(\boldsymbol{x}_k) - f(\boldsymbol{x}_{k-1})| + |f(\boldsymbol{x}_{k+1}) - f(\boldsymbol{x}_k)| \\
&\leq 2 \left\| f|\sigma_1 \right\|_\infty + 2|f(\boldsymbol{z})| \\
&\qquad \leq |f(\boldsymbol{x}_{k+1}) - f(\boldsymbol{x}_{k-1})| + 2 \left\| f|\sigma_1 \right\|_\infty + 2|f(\boldsymbol{z})|
\end{aligned}
$$

If $\boldsymbol{x}_0 = \boldsymbol{z}$ then $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_0)| \leq \|f|\sigma_1\|_\infty + |f(\boldsymbol{z})|$ and a similar estimate applies if $\boldsymbol{x}_n = \boldsymbol{z}$.

$$
\begin{aligned}
\operatorname{cvar}(f, S) &= \sum_{k=1}^{n} |f(\boldsymbol{x}_k) - f(\boldsymbol{x}_{k-1})| \\
&\leq \sum_{k=1}^{m} |f(\boldsymbol{y}_k) - f(\boldsymbol{y}_{k-1})| + 2N(\|f|\sigma_1\|_\infty + |f(\boldsymbol{z})|).
\end{aligned}
$$

Let ℓ be any line through $\boldsymbol{z}$ which doesn't intersect any other points of S. Checking Definition 2.3.1, one sees that we get a crossing segment of S on ℓ for each time that $\boldsymbol{x}_k = \boldsymbol{z}$ and so $\operatorname{vf}(S) \geq \operatorname{vf}(S, \ell) \geq N$. By [DL1, Proposition 3.5] we also have that $\operatorname{vf}(S) \geq \operatorname{vf}(S')$. Thus

$$
\begin{aligned}
\frac{\operatorname{cvar}(f, S)}{\operatorname{vf}(S)} &\leq \frac{\operatorname{cvar}(f, S') + 2N(\|f|\sigma_1\|_\infty + |f(\boldsymbol{z})|)}{\operatorname{vf}(S)} \\
&\leq \frac{\operatorname{cvar}(f, S')}{\operatorname{vf}(S')} + \frac{2N(\|f|\sigma_1\|_\infty + |f(\boldsymbol{z})|)}{N} \\
&\leq \operatorname{var}(f, \sigma_1) + 2(\|f|\sigma_1\|_\infty + |f(\boldsymbol{z})|) \\
&\leq 2 \|f\|_D .
\end{aligned}
$$

Taking the supremum over all lists S then shows that $\operatorname{var}(f, \sigma) \leq 2 \|f\|_D$ and hence that

$$
\|f\|_{BV(\sigma)} = \|f\|_\infty + \operatorname{var}(f, \sigma) \leq 3 \|f\|_D
$$

which completes the proof. $\square$

Note that the proof shows that in this situation $f \in BV(\sigma)$ if and only if $f|\sigma_1 \in BV(\sigma_1)$.

The constants obtained in the proof of Proposition 4.1.2 are in fact sharp. Suppose that $\sigma_1 = \{-1, 1\}$, $\boldsymbol{z} = 0$ and $\sigma = \sigma_1 \cup \{z\}$. Then $\left\|\chi_{\{0\}}\right\|_D = 1$ while $\left\|\chi_{\{0\}}\right\|_{BV(\sigma)} = 3$. On the other hand, if f is the constant function 1, then $\|f\|_D = 2$ while $\|f\|_{BV(\sigma)} = 1$.

Proposition 4.1.3. *Suppose that $f : \sigma \to \mathbb{C}$. Then $f \in AC(\sigma)$ if and only if $f|\sigma_1 \in AC(\sigma_1)$.*

Proof. Rather than use the heavy machinery of [DL1, Section 5], we give a more direct proof using the definition of absolute continuity. As noted above, one just needs to show that if $f|\sigma_1 \in AC(\sigma_1)$, then $f \in AC(\sigma)$. Suppose then that $f|\sigma_1 \in AC(\sigma_1)$. Given $\epsilon > 0$, there exists a polynomial $p \in P_2$ such that $\|f - p\|_{BV(\sigma_1)} < \epsilon/3$. Define $g : \sigma \to \mathbb{C}$ by $g = p + (f(\boldsymbol{z}) - p(\boldsymbol{z}))\chi_{\{z\}}$. Since $\chi_{\{z\}} \in AC(\sigma)$, we have that $g \in AC(\sigma)$ and $\|f - g\|_{BV(\sigma)} \leq 3\,\|f - g\|_D = 3\,\|f - p\|_{BV(\sigma_1)} < \epsilon$. Since $AC(\sigma)$ is closed, this shows that $f \in AC(\sigma)$. □

Corollary 4.1.4. *Suppose that σ_1 is a nonempty compact subset of $\mathbb{C}$ and that $\boldsymbol{x}$ and $\boldsymbol{y}$ are points in the complement of σ_1. Then $BV(\sigma_1 \cup \{\boldsymbol{x}\}) \simeq BV(\sigma_1 \cup \{\boldsymbol{y}\})$ and $AC(\sigma_1 \cup \{\boldsymbol{x}\}) \simeq AC(\sigma_1 \cup \{\boldsymbol{y}\})$.*

Proof. Let $h : \sigma_1 \cup \{\boldsymbol{x}\} \to \sigma_1 \cup \{\boldsymbol{y}\}$ be the natural homeomorphism which is the identity on σ_1 and which maps $\boldsymbol{x}$ to $\boldsymbol{y}$ and for $f \in BV(\sigma_1 \cup \{\boldsymbol{x}\})$ let $\Phi(f) = f \circ h^{-1}$. Then Φ is an algebra isomorphism of $BV(\sigma_1 \cup \{\boldsymbol{x}\})$ onto $BV(\sigma_1 \cup \{\boldsymbol{y}\})$ which is isometric under the norms $\|\cdot\|_{D(\sigma_1,\boldsymbol{x})}$ and $\|\cdot\|_{D(\sigma_1,\boldsymbol{y})}$, and hence it is certainly bicontinuous under the respective BV norms. It follows immediately from Proposition 4.1.3 that Φ preserves absolute continuity as well. □

More generally of course, this result says that one can move any finite number of isolated points around the complex plane without altering the isomorphism class of these spaces.

4.2 C-sets

The spectrum of a compact operator is either finite or else a countable set with limit point 0. If σ has n elements then $AC(\sigma)$ is an n-dimensional algebra and consequently for finite sets, one has a trivial Banach–Stone type theorem: $AC(\sigma_1) \simeq AC(\sigma_2)$ if and only if σ_1 and σ_2 have the same number of elements. (Of course the same result is also true for the $BV(\sigma)$ spaces.)

The case where σ is a countable set is more complicated however.

Definition 4.2.1. We shall say that a subset $\sigma \subseteq \mathbb{C}$ is a ***C-set*** if it is a countably infinite compact set with unique limit point 0. If further $\sigma \subseteq \mathbb{R}$ we shall say that σ is a *real C-set.*

Any two C-sets are homeomorphic, but as we shall see, they can produce an infinite number of non-isomorphic spaces of absolutely continuous functions. In most of what follows, it is not particularly important that the limit point of the set is 0 since one can apply a simple translation of the domain σ to achieve this and any such translation induces an isometric isomorphism of the corresponding function spaces.

The easiest C-sets to deal with are what were called spoke sets in [AD3], that is, sets which are contained in a finite number of rays emanating from the origin. To state our main theorem, we shall need a slight variant of this concept. For $\theta \in [0, 2\pi)$ let R_θ denote the ray $\{te^{i\theta}$ such that $t \geq 0\}$.

Definition 4.2.2. Suppose that k is a positive integer. We shall say that a C-set σ is a ***k-ray set*** if there are k distinct rays $R_{\theta_1}, \dots, R_{\theta_k}$ such that

1. $\sigma_j := \sigma \cap R_{\theta_j}$ is infinite for each j,
2. $\sigma_0 := \sigma \setminus (\sigma_1 \cup \cdots \cup \sigma_k)$ is finite.

If the set σ_0 of exceptional points is empty then we shall say that σ is a *strict k-ray set.*

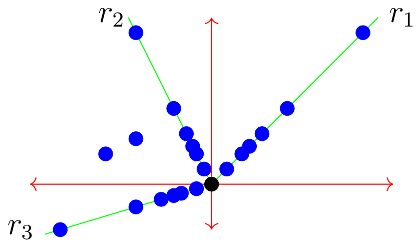


Figure 4.4: A 3-ray set with 2 exceptional points

Although in general the calculation of norms in $BV(\sigma)$ can be difficult, if σ is a strict k-ray set, then we can pass to a much more tractable equivalent norm, called the spoke norm in [AD2].

Definition 4.2.3. Suppose that σ is a strict k-ray set. The ***k-spoke norm*** on $BV(\sigma)$ is (using the notation of Definition 4.2.2)

$$\|f\|_{Sp(k)} = |f(0)| + \sum_{j=1}^{k} \|f - f(0)\|_{BV(\sigma_j)}.$$

Since each of the subsets σ_j is contained in a straight line, the calculation of the variation over these sets is straightforward. If we write $\sigma_j = \{0\} \cup \{\lambda_{j,i}\}_{i=1}^{\infty}$ with $|\lambda_{j,1}| > |\lambda_{j,2}| > \dots$ then

$$\|f - f(0)\|_{BV(\sigma_j)} = \sup_i |f(\lambda_{j,i}) - f(0)| + \sum_{i=1}^{\infty} |f(\lambda_{j,i}) - f(\lambda_{j,i+1})|.$$

Proposition 4.2.4 ([AD3, Proposition 4.3])**.** *Suppose that σ is a strict k-ray set. Then for all $f \in BV(\sigma)$,*

$$\frac{1}{2k+1} \|f\|_{Sp(k)} \le \|f\|_{BV(\sigma)} \le 3 \|f\|_{Sp(k)}.$$

One property which significantly simplifies the analysis for such spaces is that for a k-ray set σ, one always has that $AC(\sigma) = BV(\sigma) \cap C(\sigma)$. In particular a function of bounded variation on such a set σ is absolutely continuous if and only if it is continuous at the origin.

Proposition 4.2.5. *If σ is a k-ray set then $AC(\sigma) = BV(\sigma) \cap C(\sigma)$.*

Proof. Since $AC(\sigma)$ is always a subset of $BV(\sigma) \cap C(\sigma)$ we just need to prove the reverse inclusion.

Suppose first that σ is a strict k-ray set, and suppose that $f \in BV(\sigma) \cap C(\sigma)$. For $n = 1, 2, \dots$, define $g_n : \sigma \to \mathbb{C}$ by

$$g_n(\boldsymbol{z}) = \begin{cases} f(\boldsymbol{z}), & \text{if } |\boldsymbol{z}| \geq \frac{1}{n}, \\ f(0), & \text{if } |\boldsymbol{z}| < \frac{1}{n} \end{cases} = f(0) + \sum_{|\boldsymbol{z}| \geq 1/n} (f(\boldsymbol{z}) - f(0))\chi_{\{\boldsymbol{z}\}}.$$

Since $\chi_{\{\boldsymbol{z}\}} \in AC(\sigma)$ for all nonzero points in σ, $g_n \in AC(\sigma)$. Now

$$\|f - g_n\|_{Sp(k)} = \sum_{j=1}^{k} \|f - g_n\|_{BV(\sigma_j)} . \tag{4.2.1}$$

Fix j and label the elements of σ_j as above. Then, for all n there exists an index $I_{j,n}$ such that $|\lambda_{j,i}| < \frac{1}{n}$ if and only if $i \geq I_{j,n}$. Thus

$$\|f - g_n\|_{BV(\sigma_j)} = \sup_{i \geq I_{j,n}} |f(\lambda_{j,i}) - f(0)| + \sum_{i \geq I_{j,n}} |f(\lambda_{j,i}) - f(\lambda_{j,i+1})| + |f(\lambda_{j,I_{j,n}} - f(0)|.$$

The first and last of these terms converge to zero since $f \in C(\sigma)$. The middle term also converges to zero since it is the tail of a convergent sum.

Since we can make each of the k terms in (4.2.1) as small as we like, $\|f - g_n\|_{Sp(k)} \to 0$ and hence $g_n \to f$ in $BV(\sigma)$. Thus $f \in AC(\sigma)$.

Suppose finally that σ is not a strict k-ray set, that is that $\sigma_0 \neq \emptyset$. Let $\sigma' = \sigma \setminus \sigma_0$. If $f \in BV(\sigma) \cap C(\sigma)$, then $f|\sigma' \in BV(\sigma') \cap C(\sigma')$. By the above, $f|\sigma' \in AC(\sigma')$. Repeated use of Proposition 4.1.3 then shows that $f \in AC(\sigma)$. □

It would be interesting to know whether Proposition 4.2.5 holds for more general C-sets.

Corollary 4.2.6. *Suppose that σ is a strict k-ray set and that $f : \sigma \to \mathbb{C}$. For $j = 1, \dots, k$, let f_j denote the restriction of f to σ_j. Then $f \in AC(\sigma)$ if and only if $f_j \in AC(\sigma_j)$ for all j.*

Proof. By Lemma 4.5 of [AD1] if $f \in AC(\sigma)$ then the restriction of f to any compact subset is also absolutely continuous. If each $f_j \in AC(\sigma_j)$, then certainly $f \in C(\sigma)$. Furthermore $\|f\|_{Sp(k)}$ is finite and hence $f \in BV(\sigma)$. Thus, by Proposition 4.2.5, $f \in AC(\sigma)$. □

The following classification theorem is the main result in this chapter.

Theorem 4.2.7. *Suppose that σ is a k-ray set and that τ is an ℓ-ray set. Then $AC(\sigma) \simeq AC(\tau)$ if and only if $k = \ell$.*

Proof. Write $\sigma = \cup_{j=0}^{k} \sigma_j$ and $\tau = \cup_{j=0}^{\ell} \tau_j$ as in Definition 4.2.2. It follows from Corollary 4.1.4 that by moving the finite number of points in σ_0 onto one of the rays containing a set σ_j, that $AC(\sigma)$ is isomorphic to $AC(\sigma')$ for some strict k-ray set. To prove the theorem then, it suffices therefore to assume that σ and τ are strict k and ℓ-ray sets.

Suppose first that $k > \ell$ and that there is a Banach algebra isomorphism Φ from $AC(\sigma)$ to $AC(\tau)$. By Theorem 2.5.2, $\Phi(f) = f \circ h^{-1}$ for some homeomorphism $h : \sigma \to \tau$.

By the pigeonhole principle there exists $L \in \{1, \dots, \ell\}$ such that $h(\sigma_j) \cap \tau_L$ is infinite for (at least) two distinct values of j. Without loss of generality we will assume that this is true for $j = 1$ and $j = 2$. Indeed, since rotations produce isometric isomorphisms of these spaces, we may also assume that $\tau_L \subset [0, \infty)$. Let $\sigma_j = \{0\} \cup \{\lambda_{j,i}\}_{i=1}^{\infty}$, where the points are labelled so that $|\lambda_{j,1}| > |\lambda_{j,2}| > \dots$. There must then be two increasing sequences $i_1 < i_2 < \dots$ and $k_1 < k_2 < \dots$ such that

$$h(\lambda_{1,i_1}) > h(\lambda_{2,k_1}) > h(\lambda_{1,i_2}) > h(\lambda_{2,k_2}) > \dots .$$

For $n = 1, 2, \dots$ define $f_n \in AC(\sigma)$ by

$$f_n(\boldsymbol{z}) = \begin{cases} 1, & \boldsymbol{z} \in \{\lambda_{1,1}, \dots, \lambda_{1,n}\} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|f_n\|_{Sp(k)} = 2$ for all n, but $\|\Phi(f_n)\|_{Sp(\ell)} \geq 2n$. Using Proposition 4.2.4, this means that Φ must be unbounded which is impossible. Hence no such isomorphism can exist.

Finally, suppose that $k = \ell$. For each $j = 1, 2, \ldots, k$ order the elements of σ_j and τ_j by decreasing modulus and let h_j be the unique homeomorphism from σ_j to τ_j which preserves this ordering. Let h be the homeomorphism whose restriction to each σ_j is h_j and let $\Phi(f) = f \circ h^{-1}$. Then Φ is an isometric isomorphism from $(BV(\sigma), \|\cdot\|_{Sp(k)})$ to $(BV(\tau), \|\cdot\|_{Sp(k)})$, and hence is a Banach algebra isomorphism between these spaces under their usual BV norms. Since Φ is also an isomorphism from $C(\sigma)$ to $C(\tau)$, the result now follows from Proposition 4.2.5. $\square$

Corollary 4.2.8. *There are infinitely many mutually non-isomorphic $AC(\sigma)$ spaces with σ a C-set.*

The corresponding result for the $BV(\sigma)$ spaces also holds.

Corollary 4.2.9. *Suppose that σ is a k-ray set and that τ is an ℓ-ray set. Then $BV(\sigma) \simeq BV(\tau)$ if and only if $k = \ell$.*

Proof. The proof is more or less identical to that of Theorem 4.2.7. In showing that if $k \neq \ell$ then $AC(\sigma) \not\simeq AC(\tau)$ we used the fact that any isomorphism between these spaces is of the form $\Phi(f) = f \circ h^{-1}$. In showing that such a map cannot be bounded, the continuity of h was not used, only the fact that h must be a bijection, and so one may use Theorem 3.1.2 in place of Theorem 2.5.2 in this case.

The fact that if $k = \ell$ then $BV(\sigma) \simeq BV(\tau)$ is already noted in the above proof. $\square$

Clearly any real C-set is either a 1-ray set, or a 2-ray set.

Proposition 4.2.10. *There are exactly two isomorphism classes of $AC(\sigma)$ spaces with σ a real C-set.*

Example 4.2.11. Let $\sigma_1 = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$, $\sigma_2 = \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}, \ldots\}$ and $\sigma_3 = \{0, 1, -1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Then $AC(\sigma_1) \simeq AC(\sigma_3)$ since σ_1 and σ_3 are both 1-ray sets.

Indeed it is easy to check that a suitable homeomorphism to implement the isomorphism of the function spaces is $h : \sigma_1 \to \sigma_3$

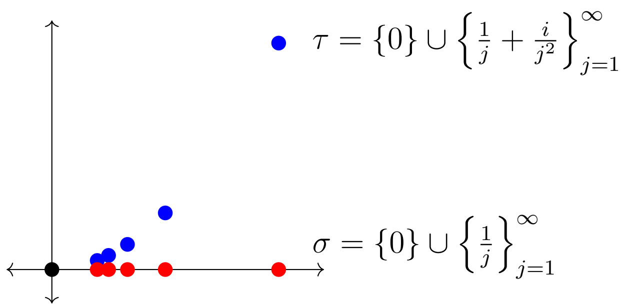
$$h(t) = \begin{cases} 0, & t = 0, \\ -1, & t = 1, \\ \frac{1}{k-1}, & t = \frac{1}{k},\ k = 2, 3, \dots . \end{cases}$$

On the other hand $AC(\sigma_1) \not\simeq AC(\sigma_2)$ since σ_2 is a 2-ray set. Indeed suppose that h is any bijection from σ_1 to σ_2. The characteristic function χ of σ_1 has norm 2 in $BV(\sigma_2)$. One can argue as in the proof of Theorem 4.2.7 that the variation of $\Phi_h(\chi) = \chi \circ h^{-1}$ must have infinite variation.

We should point out at this point that Theorem 4.2.7 is far from a characterization of the sets τ for which $AC(\tau)$ is isomorphic to $AC(\sigma)$ where σ is some k-ray set.

There is a C-set τ which is not a 1-ray set such that $AC(\tau)$ is isomorphic to $AC(\sigma)$ where σ is a 1-ray set.

Example 4.2.12. Let $\tau = \{0\} \cup \left\{\frac{1}{j} + \frac{i}{j^2}\right\}_{j=1}^{\infty}$ and let $\sigma = \{0\} \cup \left\{\frac{1}{j}\right\}_{j=1}^{\infty}$. Clearly τ is not a k-ray set for any k.



For $f \in BV(\sigma)$ let $\Phi(f)(t+it^2) = f(t)$, $t \in \sigma$. It follows from Theorem 2.3.9 that $\|\Phi(f)\|_{BV(\tau)} \le \|f\|_{BV(\sigma)}$. For the other direction, suppose that $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_n$ are points in σ and let $S = [\lambda_0 + i\lambda_0^2, \dots, \lambda_n + i\lambda_n^2]$ be the corresponding list of points in τ. It is easy to see that $\mathrm{vf}(S)$ is 2 if $n > 1$ (and is 1 if $n = 1$). Then

$$\begin{aligned}\sum_{j=1}^{n} |f(\lambda_j) - f(\lambda_{j-1})| &= \sum_{j=1}^{n} |\Phi(f)(\lambda_j + i\lambda_j^2) - \Phi(f)(\lambda_{j-1} + i\lambda_j - 1^2)| \\ &\le 2\frac{\mathrm{cvar}(\Phi(f), S)}{\mathrm{vf}(S)} \\ &\le 2\,\mathrm{var}(\Phi(f), \tau).\end{aligned}$$

Since the variation of f is given by the supremum of such sums over all such ordered subsets of σ, $\mathrm{var}(f, \sigma) \le 2\,\mathrm{var}(\Phi(f), \tau)$ and hence $\|f\|_{BV(\sigma)} \le 2\,\|\Phi(f)\|_{BV(\tau)}$. This shows that $BV(\sigma) \simeq BV(\tau)$.

Proposition 2.4.4 ensures that if $f \in AC(\sigma)$ then $\Phi(f) \in AC(\tau)$. Conversely, if $g = \Phi(f) \in AC(\tau)$ then certainly $g \in C(\tau)$ and consequently $f \in C(\sigma)$. By the previous paragraph $f \in BV(\sigma)$ too and hence, by Proposition 4.2.5, $f \in AC(\sigma)$. Thus $AC(\sigma) \simeq AC(\tau)$.

Example 4.2.13. Let $\sigma = \{0\} \cup \left\{\frac{e^{i/m}}{n} \,:\, n, m \in \mathbb{Z}^+\right\} \cup \left\{\frac{1}{n} \,:\, n \in \mathbb{Z}^+\right\}$ (where $\mathbb{Z}^+$ denotes the set of positive integers) and let τ be an ℓ-ray set. Repeating the proof of Theorem 4.2.7, one sees that there can be no Banach algebra isomorphism from $AC(\sigma)$ to $AC(\tau)$ so even among C-sets there are more isomorphism classes than those captured by Theorem 4.2.7.

4.3 Operator algebras

If $\sigma = \{0\} \cup \left\{\frac{1}{n}\right\}_{n=1}^{\infty}$, the map $\Psi : AC(\sigma) \to \ell^1$,

$$\Psi(f) = (f(1), f(\tfrac{1}{2}) - f(1), f(\tfrac{1}{3}) - f(\tfrac{1}{2}), \dots)$$

is a Banach space isomorphism. Indeed it is not hard to see that Proposition 4.2.4 implies that if σ is a strict k-ray set, then, as Banach spaces, $AC(\sigma)$ is isomorphic to $\oplus_{j=1}^{k} \ell^1$ which in turn is isomorphic to ℓ^1, and consequently all such $AC(\sigma)$ spaces are Banach space isomorphic.

Given any nonempty compact set $\sigma \subseteq \mathbb{C}$, the operator $Tg(z) = zg(z)$ acting on $AC(\sigma)$ is an $AC(\sigma)$ operator. Indeed the functional calculus for T is given by $f(T)g = fg$ for $f \in AC(\sigma)$ from which one can deduce that $\|f(T)\| = \|f\|_{BV(\sigma)}$, and therefore the Banach algebra generated by the functional calculus for T is isomorphic to $AC(\sigma)$. Proposition 6.1 of [AD3] shows that if σ is a C-set then any such operator T is a compact $AC(\sigma)$ operator.

Combining these observations, together with Corollary 4.2.8, shows that on ℓ^1 there are infinitely many nonisomorphic Banach subalgebras of $B(\ell^1)$ which are generated by (non-finite rank) compact $AC(\sigma)$ operators on ℓ^1, so things are rather different to the situation for compact normal operators on ℓ^2.

Chapter 5

Isomorphisms of $AC(\sigma)$ for linear graphs

Although we have seen that one cannot obtain analogues of the Gelfand–Kolmogorov theorem in general for $AC(\sigma)$ spaces, positive results have been obtained by restricting the class of compact sets. For example, Doust and Leinert [DL1] showed that the isomorphism class of the $AC(\sigma)$ spaces is completely determined by the homeomorphism class of the set σ if one only considers polygonal regions with polygonal holes. Another natural class of sets to examine are those which are in some sense one-dimensional. It is straightforward to construct examples of operators whose spectrum is composed of one dimensional pieces but which are topologically more complicated than a line segment or a loop. For example, if H denotes the discrete Hilbert transform on $\ell^p(\mathbb{Z})$ (with $1 < p < \infty$) then it follows from [DS] that the operator $T \in B(\ell^p(\mathbb{Z}) \oplus \ell^p(\mathbb{Z}))$ defined by $T(x, y) = (Hx, iHy)$ has a cross-shaped spectrum $\sigma(T) = \sigma = [-\pi, \pi] \cup i[-\pi, \pi]$. Furthermore, T is an $AC(\sigma)$ operator which is not a scalar-type spectral operator unless $p = 2$.

The aim of this chapter is to prove a Gelfand–Kolmogorov type theorem which covers all compact subsets which are made up of line segments. Any such set can be thought of as a drawing of a planar graph, and indeed we will make use of several ideas from graph theory in this chapter. The collection of such sets will be denoted by LG and we will call the sets linear graphs.

We will start with some definitions and results related to graph theory. Then we will prove that within the class LG we do get a Gelfand–Kolmogorov type theorem.

5.1 The class LG of linear graphs

Definition 5.1.1. The class LG of 'linear graphs' consists of those compact connected subsets of the plane which are the union of a finite number of (compact) line segments.

Our main result, which is proved in Section 5.7, is the following.

Theorem 5.1.2. *Suppose that* $\sigma, \tau \in \mathrm{LG}$. *Then* $AC(\sigma) \simeq AC(\tau)$ *if and only if* σ *is homeomorphic to* τ.

To prove Theorem 5.1.2 one needs to show that if $\sigma, \tau \in \mathrm{LG}$ are homeomorphic then there exists at least one homeomorphism $\phi : \sigma \to \tau$ such that $f \in AC(\sigma)$ if and only if $\Phi(f) = f \circ \phi^{-1} \in AC(\tau)$. In the proof of Theorem 2.5.8 the corresponding map ϕ could be chosen to be the restriction of a homeomorphism of the whole plane. This homeomorphism of the plane was constructed as a composition of a finite number of simple homeomorphisms called locally piecewise affine maps and the main step was to show that each of these maps generated an isomorphism between the appropriate spaces of absolutely continuous functions. It is not too hard to see that such a strategy is not possible in the current setting since it may be that none of the homeomorphisms between LG sets σ and τ are the restrictions of homeomorphisms of the plane. To get around this problem we shall use some concepts from graph theory which provide suitable tools for dealing with homeomorphic pairs of sets in LG.

By definition, any $\sigma \in \mathrm{LG}$ can be written as a finite union of line segments, $\sigma = \bigcup_{j=1}^{m} s_j$. We shall always impose the condition that any two distinct line segments s_i and s_j can only intersect at the endpoints of the line segments. Such a representation will be called proper. (It is a simple matter to replace a decomposition of σ which does not satisfy this condition with one which does.) The set σ can then be thought of as providing a drawing of a graph whose vertex set consists of the endpoints of these line segments. It might be noted, and indeed it will be useful to us below, that any such set σ will admit many proper representations and hence many different graphs.

Throughout we shall frequently need to consider the restriction of a function $f : \sigma_1 \to \mathbb{C}$ to a subset $\sigma \subseteq \sigma_1$. Unless there is a risk of confusion we shall not notationally distinguish the function from its restriction. Where appropriate we shall use $\|f\|_{\infty,\sigma}$ to denote $\sup_{\boldsymbol{z}\in\sigma} |f(\boldsymbol{z})|$. We shall denote the number of elements of a set A by $|A|$.

5.2 Graphs and graph drawings

Our proof of Theorem 5.1.2 uses a number of ideas from graph theory. In doing so we shall need to switch between our analyst's view of a set σ sitting as a subset of the plane, and a more graph theoretic viewpoint, where a graph G consists of a (possibly abstract) collection of vertices and points. Roughly speaking, a set $\sigma \subseteq \mathbb{R}^2$ comes with topological information inherited as a subset of the plane, but without graph theoretic properties such as vertices and edges. On the other hand, a general graph may lack the topological structure suitable for our purposes. By restricting the class of objects to linear graphs, we can apply ideas from both areas. The challenge is to match up the different senses of 'isomorphism'.

We shall generally stick with the standard graph theoretic notation and terminology, as may be found in references such as [Die]. To simplify matters, we shall generally restrict ourselves to the setting of planar graphs. Thus an undirected, simple graph $G = (V, E)$ consists of a set V of vertices, and a set E comprising two element subsets of V which are the edges of G. We shall only consider finite, undirected, simple graphs here, so the reader should assume these properties unless otherwise indicated.

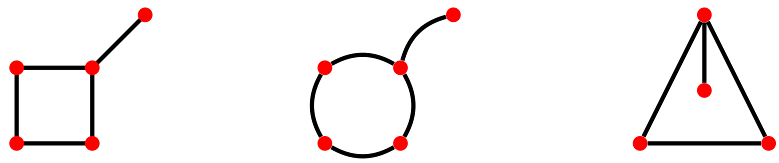
A graph $G = \bigl(\{v_i\}_{i=1}^n, \{e_j\}_{j=1}^m\bigr)$ is said to be **planar** if there exists a set of points $\hat{V} = \{\boldsymbol{x}_i\}_{i=1}^n$ in $\mathbb{R}^2$ and a set of smooth curves $\hat{E} = \{\gamma_j\}_{j=1}^m$ in $\mathbb{R}^2$ such that

- for $1 \le i < j \le n$, there is a curve joining $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ if and only if there is an edge joining v_i and v_j,
- distinct curves in $\hat{E}$ do not intersect, except possibly at the endpoints.

The pair $\hat{G} = \bigl(\hat{V}, \hat{E}\bigr)$ will be referred to as a **drawing** of G, and say that $\hat{V}$ and $\hat{E}$ represent the vertices and edges of G. The distinction between a graph and its drawing is often blurred.

The distinction between the graph theoretic and the analyst senses of 'isomorphism' is illustrated in Figure 5.1. The graph theorist considers graphs σ_1 and σ_2

to be different drawings of the same planar graph, while σ_3 represents a different graph. From an analyst's point of view (now ignoring the vertices shown), the sets σ_1 and σ_3 are homeomorphic linear graphs, but dealing with the curved edges in σ_2 leads to complications in generalizing Theorem 5.1.2.



σ_1 σ_2 σ_3

Figure 5.1: 'Isomorphic' graphs in different senses.

Fáry's Theorem (see, for example, [CLZ, Theorem 6.36]) provides a link between connected linear graphs and the class of simple planar graphs.

Theorem 5.2.1 (Fáry's Theorem)**.** *Every simple planar graph has a drawing* $(\hat{V}, \hat{E})$ *where the edges are represented by straight line segments.*

For our application in proving Theorem 5.1.2, we start with an element $\sigma \in \mathrm{LG}$ and need to impose some graph theoretic structure on the set. By definition, any $\sigma \in \mathrm{LG}$ can be written as $\sigma = \bigcup_{j=1}^{m} \overline{\boldsymbol{x}_j, \boldsymbol{x}'_j}$, and so σ can be thought of as providing a drawing of a graph whose vertex set consists of the endpoints of these line segments $\{\boldsymbol{x}_j, \boldsymbol{x}'_j\}_{j=1}^{m}$. Unfortunately, any such set σ can be represented as a union of line segments in many different ways. There is however always a uniquely determined minimal set of vertex points $V(\sigma) = \{\boldsymbol{x}_i\}_{i=1}^{n} \subseteq \sigma$ such that σ can be expressed as the union of line segments between the elements of this set. For example, for the set σ in Figure 5.1, the minimal set of vertex points is $V(\sigma) = \{\boldsymbol{x}_i\}_{i=1}^{5}$ as shown in Figure 5.2, and

$$\sigma = \overline{\boldsymbol{x}_1, \boldsymbol{x}_2} \cup \overline{\boldsymbol{x}_2, \boldsymbol{x}_3} \cup \overline{\boldsymbol{x}_3, \boldsymbol{x}_4} \cup \overline{\boldsymbol{x}_4, \boldsymbol{x}_5} \cup \overline{\boldsymbol{x}_5, \boldsymbol{x}_2}.$$

Our next step is to match the topological notion of homeomorphism between subsets of the plane and the appropriate graph theoretic notions.

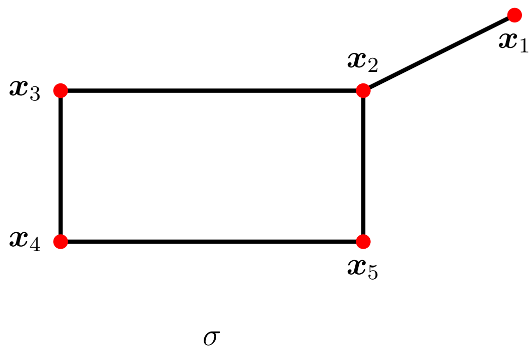


Figure 5.2: The minimal vertex set.

5.3 Graph isomorphisms

Firstly we will explain the essential definitions of graph homeomorphic and graph isomorphic.

Definition 5.3.1. Graph isomorphism: Two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are called (graph) isomorphic if there exists a bijective mapping, $f : V_1 \to V_2$ such that there is a edge between v_1 and $\hat{v}_1$ in G_1, if and only if there exists an edge between $f(v_1)$ and $f(\hat{v}_1)$ in G_2.

In the case when the graph is simple, we just need to check if there is a bijection $f : V(G_1) \to V(G_2)$ which preserves adjacent vertices (i.e. if v_1, v_2 are adjacent in graph G_1, then $f(v_1), f(v_2)$ must be adjacent in graph G_2).

In the following figure, there does not exist such a bijection from $V(G_1) \to V(G_2)$ so these graphs are not isomorphic.

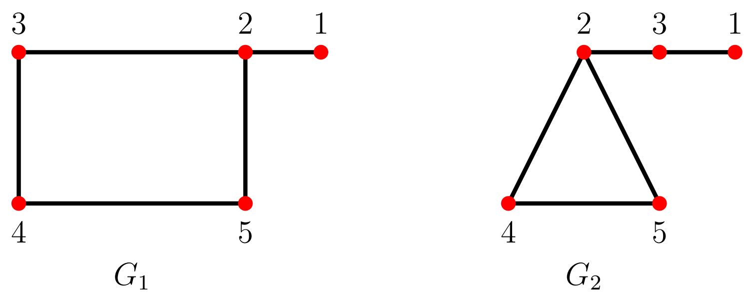


Figure 5.3: Two graphs which are not isomorphic

Definition 5.3.2. Subdivision: A *subdivision* of an edge $\{u, v\}$ of a graph G comprises forming a new graph with an additional vertex w, and replacing the edge $\{u, v\}$ with the two edges $\{u, w\}$ and $\{w, v\}$. A *subdivision* of G is a graph formed by starting with G and performing a finite sequence of subdivisions of its edges.

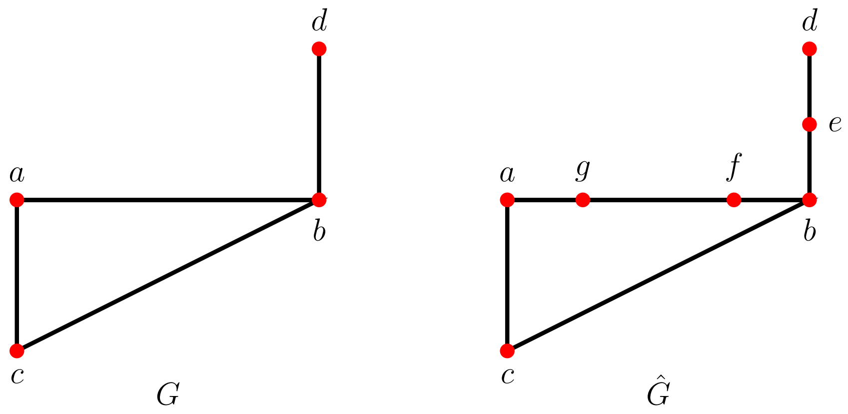
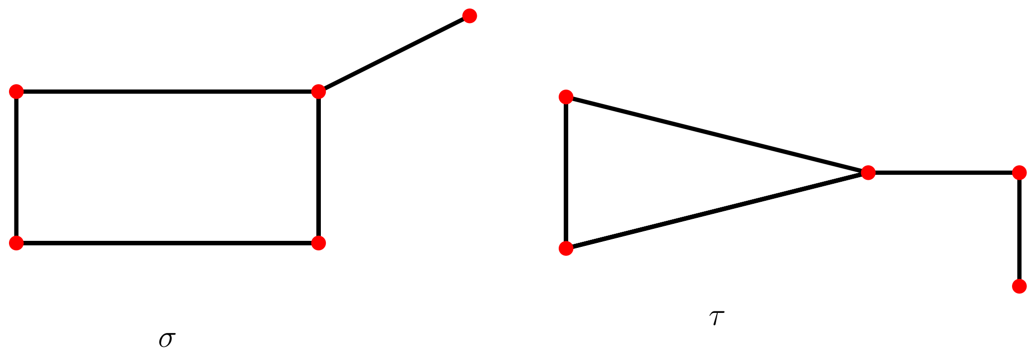


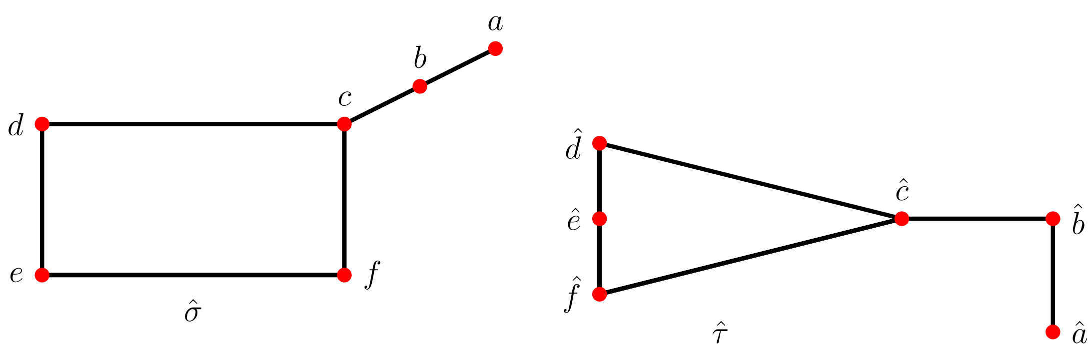
Figure 5.4: A subdivision of G.

Definition 5.3.3. Two graphs G_1 and G_2 are **graph homeomorphic** if there is a graph isomorphism from some subdivision of G_1 to some subdivision of G_2.

For example, the graphs



are graph homeomorphic via the following subdivisions.



Fortunately, at least amongst the graphs which concern us, the two notions of homeomorphism agree.

Theorem 5.3.4. *[GT, P.18] Suppose that G_1 and G_2 are planar graphs with drawings $\hat{G}_1$ and $\hat{G}_2$ in the plane. Then G_1 and G_2 are graph homeomorphic if and only if $\hat{G}_1$ and $\hat{G}_2$ are topologically homeomorphic.*

If fact we shall only use one direction of this result. For completeness, we include a proof of this implication.

Corollary 5.3.5. *Suppose that $\sigma, \tau \in LG$ are drawings of graphs $G(\sigma) = (V(\sigma), E(\sigma))$ and $G(\tau) = (V(\tau), E(\tau))$. If σ and τ are topologically homeomorphic then $G(\sigma)$ and $G(\tau)$ are graph homeomorphic.*

Proof. Suppose that $h : \sigma \to \tau$ is a homeomorphism. Define sets

$$\hat{V}(\sigma) = V(\sigma) \cup h^{-1}(V(\tau)), \qquad \hat{V}(\tau) = h(V(\sigma)) \cup V(\tau).$$

As the following pictures:

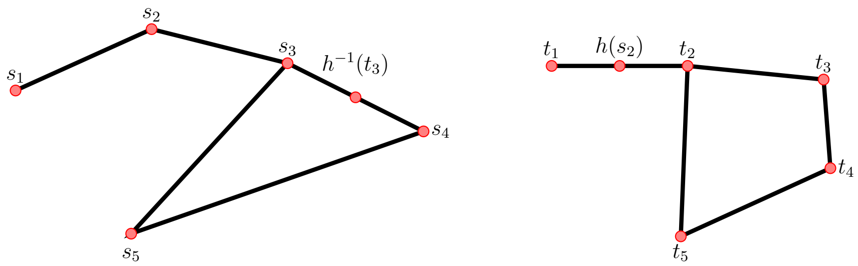


Figure 5.5: $\hat{V}(\sigma) = V(\sigma) \cup h^{-1}(V(\tau))$, $\hat{V}(\tau) = h(V(\sigma)) \cup V(\tau)$

It is clear that h maps $\hat{V}(\sigma)$ bijectively to $\hat{V}(\tau)$.

If $\boldsymbol{u}, \boldsymbol{v} \in \hat{V}(\sigma)$, then we shall say that $(\boldsymbol{u}, \boldsymbol{v}) \in \hat{E}(\sigma)$ if there exists a continuous curve $\gamma : [0, 1] \to \sigma$ such that $\gamma(0) = \boldsymbol{u}, \gamma(1) = \boldsymbol{v}$ and $\gamma(t) \notin \hat{V}(\sigma)$ for $0 < t < 1$. The edge set $\hat{E}(\tau)$ is defined , and so we can consider the two graphs $\hat{G}(\sigma) = (\hat{V}(\sigma), \hat{E}(\sigma))$ and $\hat{G}(\tau) = (\hat{V}(\tau), \hat{E}(\tau))$. Clearly γ is a curve joining $\boldsymbol{u}, \boldsymbol{v} \in \hat{V}(\sigma)$ if and only if $h \circ \gamma$ is a curve joining $h(\boldsymbol{u}), h(\boldsymbol{v}) \in \hat{V}(\tau)$. It follows that the graphs $\hat{G}(\sigma)$ and $\hat{G}(\tau)$ are isomorphic and hence that $G(\sigma)$ and $G(\tau)$ are graph homeomorphic. □

For completeness, we note that there is a distinct graph theoretic notion of topological isomorphism. Let S^2 denote the unit sphere in $\mathbb{R}^3$ and let S^* denote the punctured sphere, that is, S^2 with the north pole removed. Then S^* is homeomorphic to $\mathbb{R}^2$ via the stereographic projection map $\pi : S^* \to \mathbb{R}^2$, while S^2 is homeomorphic to the extended plane.

Definition 5.3.6. Topological isomorphism: Suppose that G_1 and G_2 are planar graphs with drawings $\hat{G}_1$ and $\hat{G}_2$ in the plane. We say that G_1 and G_2 are

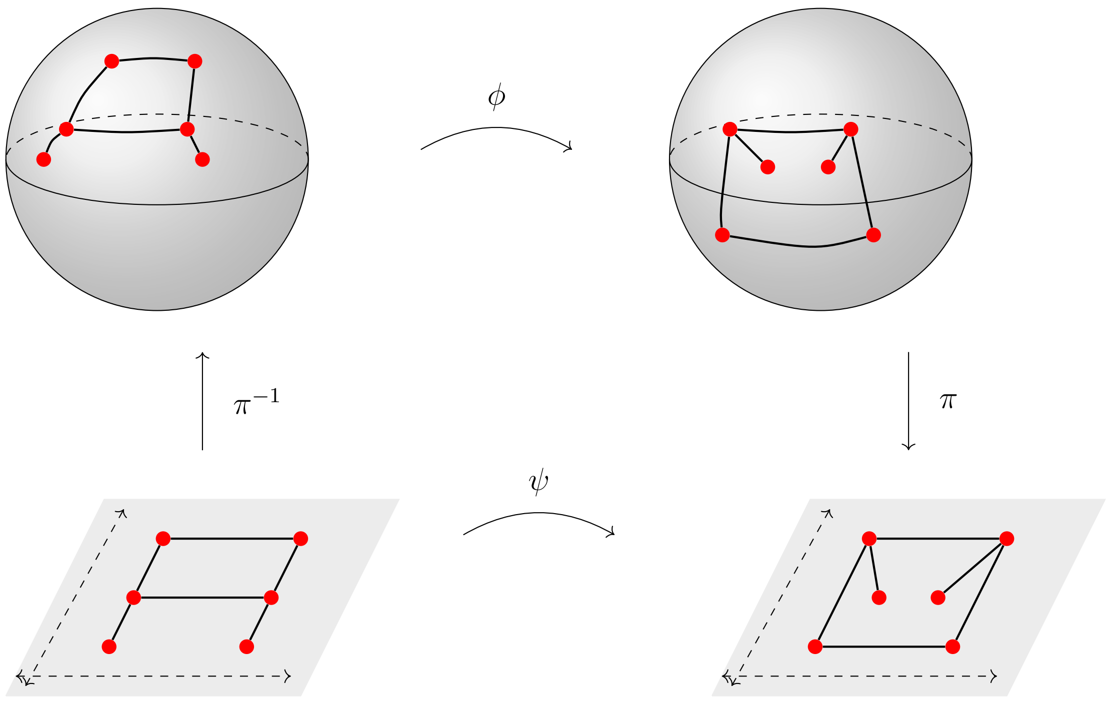


Figure 5.6: Two topologically isomorphic planar graphs. Each graph has two faces, one with four edges and one with 8 edges.

topologically isomorphic if there exists a homeomorphism $\phi : S^2 \to S^2$ so that if $\psi = \pi \circ \phi \circ \pi^{-1}$, then $\hat{G}_2 = \psi(\hat{G}_1)$.

This definition is illustrated in Figure 5.6. The main distinction between topological isomorphism and graph homeomorphism is that topological isomorphism is concerned with preserving the relationship of the faces as well as the edges and vertices, whilst homeomorphisms are are only concerned with the points forming the drawing of the graph.

Homeomorphic planar graphs need not be topologically isomorphic. An example is shown in Figure 5.7. To see that these graphs are not topologically isomorphic, note that the graph on the right has a face with 4 edges, while none of the faces in the graph on the left have this number of edges.

5.4 Subgraphs

Definition 5.4.1. Subgraphs: A graph $\hat{G}$ is ***a subgraph*** of a graph G if each of its vertices belongs to $V(G)$ and each of its edges belongs to $E(G)$.

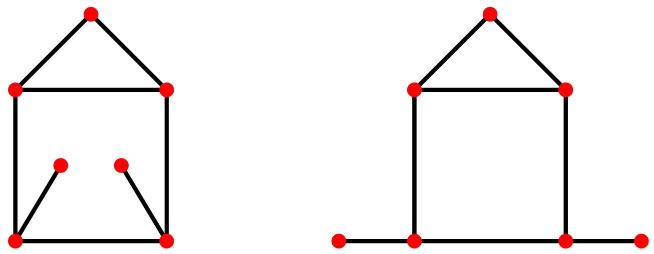


Figure 5.7: Two homeomorphic graphs which are not topologically isomorphic.

We can obtain subgraphs of a graph by deleting edges and vertices. If e is an edge of a graph G, we denote by $G-e$ the graph with vertex set $V(G)$ and edge set $E(G)\setminus\{e\}$. Generally, if E is any set of edges in G, we denote by $G-E$ the graph obtained by deleting the edges in E . Similarly, if v is a vertex of G, we denote by $G-v\subseteq G$ the graph obtained from G by deleting the vertex v together with the edges incident with v such that $G-v\subseteq G$. Generally, if V is any set of vertices in G, we denote by $G-V$ the graph obtained by deleting the vertices in V and all edges incident with any of them.

Lemma 5.4.2. *If G is a connected graph with n edges then there exist subgraphs $G_1\subseteq G_2\subseteq\cdots\subseteq G_n=G$, when each G_k is a connected graph with k edges.*

Proof. The steps in the algorithm are as follows.

Step 1 Form G_1 by choosing one edge

Step 2 Build up the other by adding one edge at each step. Suppose we have G_k

Case 1 Not every vertex is in G_k. Since G is connected then there exists at least one edge between G_k and the remaining vertices. Add any such edge to form G_{k+1}.

Case 2 G_k contains all the vertices. Add any missing edge to form G_{k+1}.

Repeat until all edges are in G_n. □

Definition 5.4.3. A sequence $\{G_k\}_{k=1}^{n}$ of subgraphs satisfying the conclusion of Lemma 5.4.2 is called an ***edge by edge decomposition*** of G.

Since graph isomorphisms preserve connectivity, we have the following lemma.

Lemma 5.4.4. *Suppose that G and $\hat{G}$ are isomorphic graphs with n edges with graph isomorphism f. If $\{G_k\}_{k=1}^{n}$ is an edge by edge decomposition of G then $\{f(G_k)\}_{k=1}^{n}$ is an edge by edge decomposition of $\hat{G}$.*

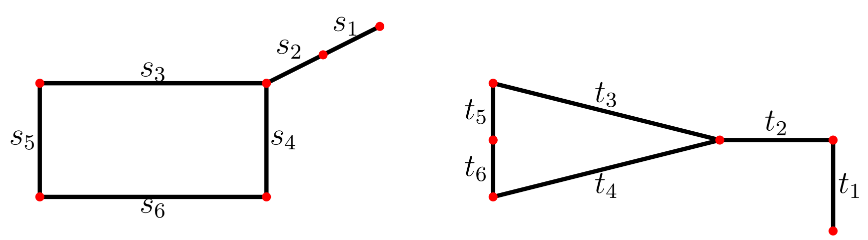
These terms and results extend naturally to the sets which are drawings of simple planar graphs.

A consequence of these results is that if $\sigma, \tau \in \mathrm{LG}$ are homeomorphic then there exist line segments $\{s_k\}_{k=1}^{m}$ and $\{t_k\}_{k=1}^{m}$ so that if we set

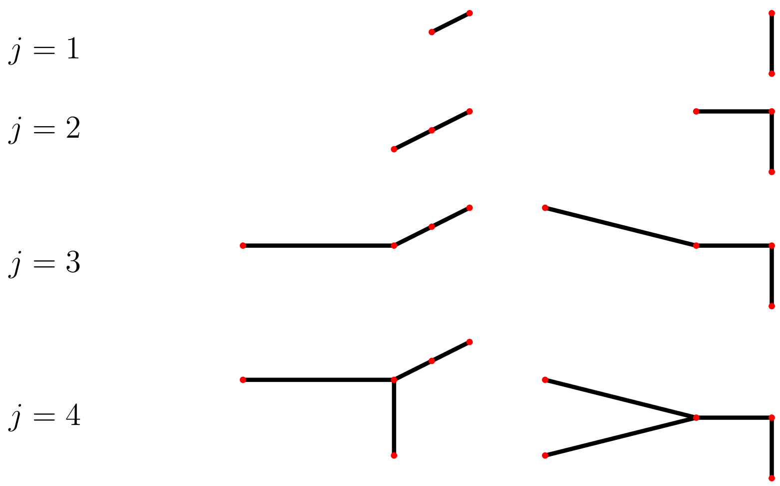
$$\sigma_j = \bigcup_{k=1}^{j} s_j, \qquad \tau_j = \bigcup_{k=1}^{j} t_k, \qquad j = 1, 2, \ldots, m,$$

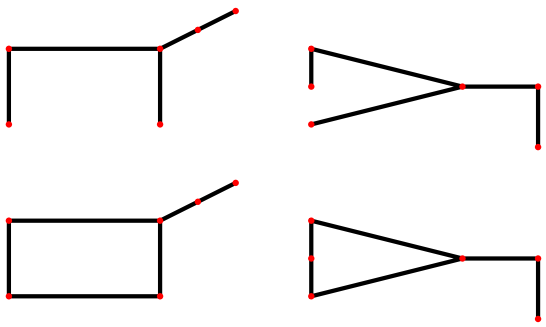
then $\{\sigma_j\}_{j=1}^{m}$ and $\{\tau_j\}_{j=1}^{m}$ are edge by edge decompositions of σ and τ such that for each j, σ_j is homeomorphic and graph isomorphic to τ_j.

Example 5.4.5. Consider the two homeomorphic sets σ and τ. By adding the vertices as in the previous example, we could label the edges in the generated subdivisions as $s_1, \ldots, s_6$ and $t_1, \ldots, t_6$ as shown.



This generates a sequence of pairs of homeomorphic sets representing pairs of isomorphic graphs.





5.5 The LG norm

Suppose that $\sigma = \bigcup_{j=1}^m s_j \in \mathrm{LG}$ is comprised of line segments $s_j = \overline{\boldsymbol{x}_j, \boldsymbol{x}'_j}$ which do not intersect except at some of the endpoints. For each j, $BV(s_j) \simeq BV[0,1]$ and $AC(s_j) \simeq AC[0,1]$, so it is relatively easy to deal with these regions. Rather than working with the norm $\|\cdot\|_{BV(\sigma)}$ it is easier to use the following equivalent norm.

Definition 5.5.1. For $f : \sigma \to \mathbb{C}$ let

$$\|f\|_{LG(\sigma)} = \|f\|_\infty + \sum_{j=1}^m \mathrm{var}(f, s_j).$$

Let $LG(\sigma)$ denote the set of all f such that $\|f\|_{LG(\sigma)} < \infty$.

Theorem 5.5.2. *$\|\cdot\|_{LG(\sigma)}$ is a norm on $LG(\sigma)$ and is equivalent to $\|\cdot\|_{BV(\sigma)}$.*

Proof. Checking that $\|\cdot\|_{LG(\sigma)}$ is a norm is straightforward. Suppose that $f \in LG(\sigma)$. Note that since $\mathrm{var}(f, s_j) \le \mathrm{var}(f, \sigma)$ for each j, it is clear that

$$\|f\|_{LG(\sigma)} \le \|f\|_\infty + m\,\mathrm{var}(f,\sigma) \le m\,\|f\|_{BV(\sigma)}\,.$$

The reverse inequality is more difficult. Let LG_m denote the family of all $\sigma \in \mathrm{LG}$ which can be written as a union of m line segments. Our aim is to prove that for all m there exists a constant C_m such that for all $\sigma \in \mathrm{LG}_m$ and all $f \in BV(\sigma)$

$$\|f\|_{BV(\sigma)} \le C_m\,\|f\|_{\mathrm{LG}(\sigma)}\,, \tag{5.5.1}$$

We will proceed by induction on m. If $m = 1$ then the two norms are identical so we may take $C_m = 1$. Suppose now that we have shown that 5.5.1 holds for some integer $m \geq 1$. Let $\sigma_1 \in \mathrm{LG}_{m+1}$. We can write $\sigma_1 = \sigma \cup s$ where $\sigma \in LG_m$ and $s = \overline{\boldsymbol{x}\,\boldsymbol{y}}$ is a line segment such that $\sigma \cap s$ is either $\{\boldsymbol{x}\}$ or $\{\boldsymbol{x}, \boldsymbol{y}\}$.

Suppose that $f : \sigma_1 \to \mathbb{C}$ and that $S = [\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_n]$ is a list[1] of points in σ_1. Our aim is to bound $\operatorname{cvar}(f, S)/\operatorname{vf}(S)$.

For $j = 1, \ldots, n$, let $\ell_j = [\boldsymbol{z}_{j-1}, \boldsymbol{z}_j]$. Define subsets $I_1, I_2, I_3 \subseteq \{1, 2, \ldots, n\}$ by

$$I_1 = \{j : \boldsymbol{z}_j, \boldsymbol{z}_{j-1} \in \sigma\},$$
$$I_2 = \{j : \boldsymbol{z}_j, \boldsymbol{z}_{j-1} \in s\},$$
$$I_3 = \{1, 2, \ldots, n\} \setminus (I_1 \cup I_2).$$

Note that if $j \in I_3$ then one endpoint of ℓ_j is in $\sigma \setminus s$ and the other is in $s \setminus \sigma$.

Illustration. *Consider the list S in Figure 5.8.*

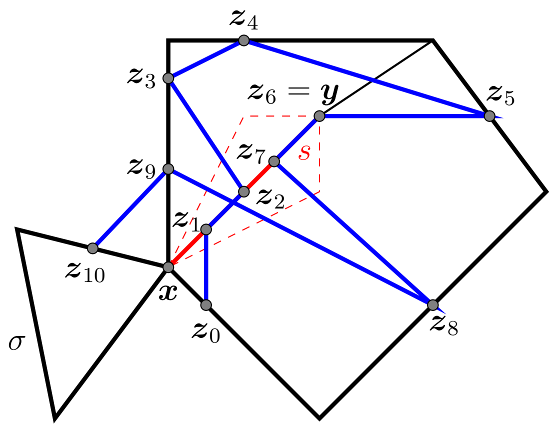


Figure 5.8: Example of a list S of points of $\sigma_1 = \sigma \cup s$.

In this example, $I_1 = \{4, 5, 6, 9, 10\}$, $I_2 = \{2, 7\}$ and $I_3 = \{1, 3, 8\}$. The sublists are $S_1 = [\boldsymbol{z}_3, \boldsymbol{z}_4, \boldsymbol{z}_5, \boldsymbol{z}_6, \boldsymbol{z}_8, \boldsymbol{z}_9, \boldsymbol{z}_{10}]$ and $S_2 = [\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{z}_6, \boldsymbol{z}_7]$.

Noting that $I_1 \cap I_2$ may be nonempty,

$$\operatorname{cvar}(f, S) \leq \sum_{i=1}^{3} \sum_{j \in I_i} |f(\boldsymbol{z}_j) - f(\boldsymbol{z}_{j-1})|. \tag{5.5.2}$$

[1] We can of course assume that no two consecutive points are equal.

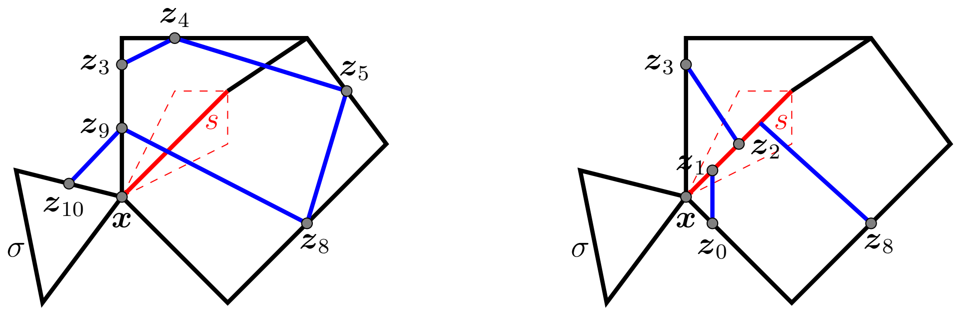


Figure 5.9: The sublist S_1 and the segments in $j \in I_3$.

By [DL2, Proposition 3.5] (see also [AD4]), $\mathrm{vf}(S_1) \le \mathrm{vf}(S)$, and so

$$\frac{\sum_{j \in I_1} |f(\boldsymbol{z}_j) - f(\boldsymbol{z}_{j-1})|}{\mathrm{vf}(S)} \le \frac{\mathrm{cvar}(f, S_1)}{\mathrm{vf}(S_1)} \le \mathrm{var}(f, \sigma). \tag{5.5.3}$$

Similarly, if S_2 is the sublist of S including all points which are endpoints of line segments ℓ_j with $j \in I_2$ then

$$\frac{\sum_{j \in I_2} |f(\boldsymbol{z}_j) - f(\boldsymbol{z}_{j-1})|}{\mathrm{vf}(S)} \le \frac{\mathrm{cvar}(f, S_2)}{\mathrm{vf}(S_2)} \le \mathrm{var}(f, s). \tag{5.5.4}$$

Finally, one may draw a quadrilateral Q with vertices at $\boldsymbol{x}$ and $\boldsymbol{y}$ which contains $s \setminus \sigma$ in its interior and $\sigma \setminus s$ in its exterior. If $j \in I_3$, then ℓ_j is a crossing segment on one of the four lines which determine Q. In particular, at least one of the four lines must have at least $\frac{1}{4}|I_3|$ crossing segments, and hence $\mathrm{vf}(S) \ge \frac{1}{4}|I_3|$. By a simple triangle inequality estimate

$$\frac{\sum_{j \in I_3} |f(\boldsymbol{z}_j) - f(\boldsymbol{z}_{j-1})|}{\mathrm{vf}(S)} \le \frac{2|I_3| \, \|f\|_{\infty,\sigma_1}}{\frac{1}{4}|I_3|} = 8 \, \|f\|_{\infty,\sigma_1}. \tag{5.5.5}$$

Combining the three estimates we see that

$$\mathrm{var}(f, \sigma_1) \le \mathrm{var}(f, \sigma) + \mathrm{var}(f, s) + 8 \, \|f\|_{\infty,\sigma_1}$$

and hence (using the induction hypothesis)

$$\begin{aligned}
\|f\|_{BV(\sigma_1)} &= \|f\|_{\infty,\sigma_1} + \operatorname{var}(f,\sigma_1) \\
&\le \|f\|_{\infty,\sigma} + \|f\|_{\infty,s} + \operatorname{var}(f,\sigma) + \operatorname{var}(f,s) + 8\,\|f\|_{\infty,\sigma_1} \\
&\le C_m\Big(\|f\|_{\infty,\sigma} + \sum_{j=1}^{m} \operatorname{var}(f,s_j)\Big) + 9\,\|f\|_{\infty,\sigma_1} + \operatorname{var}(f,s) \\
&\le (C_m+9)\,\|f\|_{LG(\sigma_1)}\,.
\end{aligned}$$

This completes the induction proof. $\square$

At first glance it might appear that $\|f\|_{\mathrm{LG}(\sigma)}$ depends on the representation of σ as a union of line segments. Note however that because variation is additive on adjacent intervals, taking a subdivision does not change the value of the LG norm. Since every representation is a subdivision of the one which comes from the minimal vertex set, this shows that $\|f\|_{\mathrm{LG}(\sigma)}$ depends only on σ and not on how it is represented.

We end this section with a proposition whose proof is an immediate consequence of the definition of the LG norm.

Proposition 5.5.3. *Suppose that $\sigma = \bigcup_{j=1}^{m} s_j \in \mathrm{LG}$, and that $\sigma_1 = \bigcup_{j=1}^{k} s_j$ and $\sigma_2 = \bigcup_{j=k+1}^{m} s_j$ are also in* LG*. Then for all $f \in BV(\sigma)$*

$$\max(\|f\|_{LG(\sigma_1)}, \|f\|_{LG(\sigma_2)}) \le \|f\|_{LG(\sigma)} \le \|f\|_{LG(\sigma_1)} + \|f\|_{LG(\sigma_2)}\,.$$

5.6 Algebra homomorphisms

We can now show that the Banach algebra of the BV spaces is invariant under homeomorphisms.

Given any two line segments s and t there is obviously a wide range of possible homeomorphisms between them. If we write

$$\begin{aligned}
s &= \{\boldsymbol{x} + \lambda \boldsymbol{v} \,:\, 0 \le \lambda \le 1\}, \\
t &= \{\boldsymbol{x}' + \lambda \boldsymbol{v}' \,:\, 0 \le \lambda \le 1\},
\end{aligned}$$

then the homeomorphism $h : s \to t$, $h(\boldsymbol{x} + \lambda \boldsymbol{v}) = \boldsymbol{x}' + \lambda \boldsymbol{v}'$ will be called an affine homeomorphism. There are two such homeomorphisms between any two line segments, depending on the orientation of the line segments.

Theorem 5.6.1. *Suppose that $\sigma, \tau \in \mathrm{LG}$ are homeomorphic. Then $BV(\sigma)$ is isomorphic to $BV(\tau)$ (as Banach algebras).*

Proof. By Corollary 5.3.5 (and the earlier comments) we may consider σ and τ to be graph drawings of isomorphic graphs with each edge represented by a line segment. By Lemma 5.4.4 then, there are line segments $\{s_k\}_{k=1}^m$ and $\{t_k\}_{k=1}^m$ so that if $\sigma_j = \bigcup_{k=1}^j s_k$ and $\tau_j = \bigcup_{k=1}^j t_k$ then $\{\sigma_j\}_{j=1}^m$ and $\{\tau_j\}_{j=1}^m$ are edge-by-edge decompositions of σ and τ, and such that each σ_j is homeomorphic and graph isomorphic to τ_j.

There are two affine homeomorphisms from s_1 to t_1. Fix one of these and call it ϕ_1. There is now a unique homeomorphism ϕ_2 from σ_2 to τ_2, extending ϕ_1 which is also affine when restricted to s_2. Continuing in this way, one can construct a homeomorphism $\phi : \sigma \to \tau$ which is affine on each of the component line segments.

For $f : \sigma \to \mathbb{C}$, define $\Phi(f) : \tau \to \mathbb{C}$ by $\Phi(f)(\boldsymbol{z}) = f(\phi^{-1}(\boldsymbol{z}))$. Clearly Φ is linear and multiplicative. Since the sets are just line segments, for each k,

$$\operatorname{var}(f, s_k) = \operatorname{var}(\Phi(f), t_k).$$

Consequently, Φ maps $\left(BV(\sigma), \|\cdot\|_{LG(\sigma)}\right)$ isometrically to $\left(BV(\tau), \|\cdot\|_{LG(\tau)}\right)$. By Theorem 5.5.2, this implies that the spaces $BV(\sigma)$ and $BV(\tau)$ are isomorphic as Banach algebras. $\square$

The reverse direction of Theorem 5.6.1 is not true as the following example shows.

Example 5.6.2. Consider the two sets σ and τ shown in Figure 5.10. These sets are clearly not homeomorphic. Let h be the bijection which maps the blue path from a to b onto the closed line segment $[\alpha, \beta]$ and the half-open line segment from c to d onto the half-open line segment from β to δ. A consequence of our results will be that Φ_h is a bounded map from $BV(\sigma)$ to $BV(\tau)$.

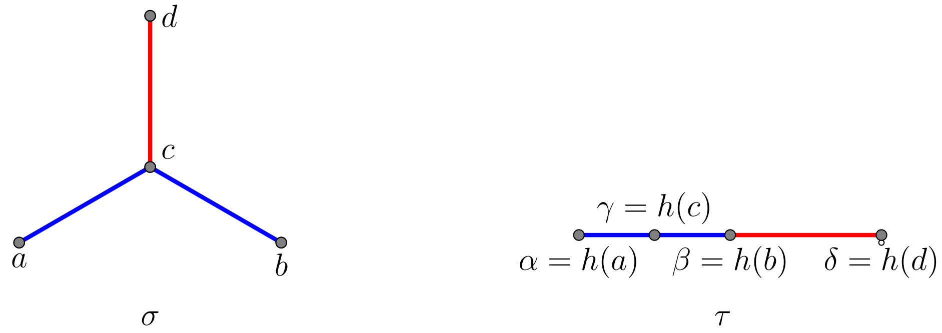


Figure 5.10: Two non-homeomorphic compact sets $BV(\sigma) \simeq BV(\tau)$.

Suppose that $f \in BV(\sigma)$ and that $\hat{f} = \Phi_h(f) = f \circ h^{-1} \in BV(\tau)$. Note first that

$$\operatorname{var}(\hat{f}, \overline{\alpha\,\beta}) = \operatorname{var}(f, \overline{a\,c}) + \operatorname{var}(f, \overline{c\,b}).$$

Suppose that $\{\lambda_j\}_{j=0}^n$ is an ordered partition of $\overline{\beta\,\delta}$. We may assume that $\lambda_0 = \beta$. Then

$$\begin{aligned}
\sum_{j=1}^{n} |\hat{f}(\lambda_j) - \hat{f}(\lambda_{j-1})| &= |\hat{f}(\beta) - \hat{f}(\lambda_1)| + \sum_{j=2}^{n} |\hat{f}(\lambda_j) - \hat{f}(\lambda_{j-1}) \\
&\le |\hat{f}(\beta) - \hat{f}(\gamma)| + |\hat{f}(\gamma) - \hat{f}(\lambda_1)| + \sum_{j=2}^{n} |\hat{f}(\lambda_j) - \hat{f}(\lambda_{j-1})| \\
&\le |f(b) - f(c)| + \operatorname{var}(f, \overline{c\,d}) \\
&\le \operatorname{var}(f, \overline{c\,b}) + \operatorname{var}(f, \overline{c\,d}).
\end{aligned}$$

Thus

$$\begin{aligned}
\|\hat{f}\|_{\mathrm{LG}(\tau)} &= \|\hat{f}\|_\infty + \operatorname{var}(\hat{f}, \overline{\alpha\,\beta}) + \operatorname{var}(\hat{f}, \overline{\beta\,\gamma}) \\
&\le \|f\|_\infty + \operatorname{var}(f, \overline{a\,c}) + \operatorname{var}(f, \overline{c\,b}) + \operatorname{var}(f, \overline{c\,b}) + \operatorname{var}(f, \overline{c\,d}) \\
&\le 2\|f\|_{\mathrm{LG}(\sigma)}.
\end{aligned}$$

An exactly analogous calculation shows that

$$\|f\|_{\mathrm{LG}(\sigma)} \le 2\|\hat{f}\|_{\mathrm{LG}(\tau)}$$

and hence Φ_h is a continuous linear map with continuous inverse from $(BV(\sigma), \|\cdot\|_{\mathrm{LG}(\sigma)})$ to $(BV(\tau), \|\cdot\|_{\mathrm{LG}(\tau)})$. Combining this with Theorem 5.5.2, this shows that Φ_h is a

Banach algebra isomorphism from $BV(\sigma)$ to $BV(\tau)$. On the other hand, since σ is not homeomorphic to τ we know that $AC(\sigma) \not\simeq AC(\tau)$.

5.7 AC functions

In this section we can finally complete the proof of the Gelfand–Kolmogorov theorem for LG subsets of the plane. Our first step will be to prove that a function is absolutely continuous on $\sigma \in \mathrm{LG}$ if and only if it is continuous and it is absolutely continuous on each line segment component of σ.

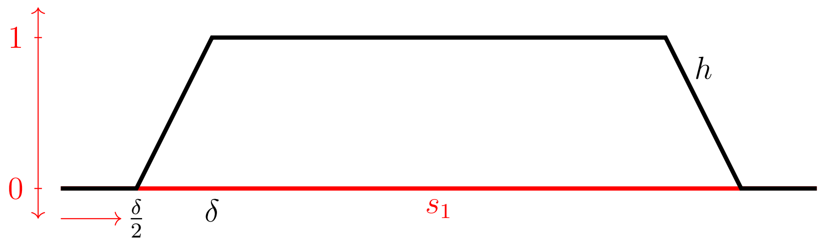
In the proof we will need an absolutely continuous function which is almost the indicator function of one of the component line segments of $\sigma \in \mathrm{LG}$. For $0 < \delta < \frac{1}{2}$ let $k_\delta \in AC[0,1]$ be the function which is 1 on $[\delta, 1-\delta]$, 0 on $[0,\delta/2] \cup [1-\delta/2, 1]$ and which linearly interpolates the remaining values.

Lemma 5.7.1. *Suppose that* $\sigma = \bigcup_{k=1}^m s_k \in \mathrm{LG}$ *with* $s_1 = [0,1]$. *Let* $0 < \delta < \frac{1}{2}$ *and define* $h : \sigma \to \mathbb{R}$ *by*

$$h(\boldsymbol{z}) = \begin{cases} 0, & \boldsymbol{z} \notin s_1, \\ k_\delta(\boldsymbol{z}), & \boldsymbol{z} \in s_1. \end{cases}$$

Then $h \in AC(\sigma)$.

Restricted to s_1, the graph of h is



The restriction on δ is just to ensure that the range of the function is $[0,1]$.

Proof. Suppose that $\alpha > 0$. By [AD1], proposition 4.4, the functions $h_1(x+iy) = k_\delta(x)$ and $h_2(x+iy) = \max(0, 1-\alpha\,|y|)$ are in $AC(\sigma)$ and hence $h_1 h_2 \in AC(\sigma)$. But if α is sufficiently large, $h_1 h_2 = h$, so we are done. $\square$

Theorem 5.7.2. *Suppose that* $\sigma_1 = \sigma \cup s \in \mathrm{LG}$ *where the line segment* s *intersects* σ *at one or two of its endpoints. Suppose that* $f \in BV(\sigma_1)$. *Then* $f \in AC(\sigma_1)$ *if and only if*

1. $f|\sigma \in AC(\sigma)$,
2. $f|s \in AC(s)$

Proof. It is clear that if $f \in AC(\sigma_1)$ then (1) and (2) hold.

Suppose then that f satisfies (1) and (2). To simplify things, we may apply an affine transformation to the plane so that $s = [0, 1]$. (This is fine since affine transformations preserve absolute continuity.) Define $f_s : \sigma_1 \to \mathbb{C}$ by

$$f_s(\boldsymbol{z}) = \begin{cases} f(0), & \text{if } \operatorname{Re} \boldsymbol{z} < 0, \\ f(\operatorname{Re} \boldsymbol{z}), & \text{if } 0 \le \operatorname{Re} \boldsymbol{z} \le 1, \\ f(1), & \text{if } \operatorname{Re} \boldsymbol{z} > 1. \end{cases}$$

By Theorem 2.4.4, $f_s \in AC(\sigma_1)$ since it is the extension to a subset of the plane of a function which is absolutely continuous on an interval in $\mathbb{R}$. Let $g = f - f_s \in BV(\sigma_1)$. Note that

- $g|\sigma \in AC(\sigma)$, and
- $g|s = 0$.

Fix $\epsilon > 0$. Using Theorem 5.5.2, it suffices now to show that there exists $q \in AC(\sigma_1)$ with $\|g - q\|_{LG(\sigma_1)} < \epsilon$. This will imply that $g \in AC(\sigma_1)$ and hence that $f \in AC(\sigma_1)$ too.

By definition there exists a polynomial $p \in \mathcal{P}_2$ such that $\|g - p\|_{LG(\sigma)} \le \frac{\epsilon}{5}$.

Suppose first that both 0 and 1 are elements of σ. Then

$$|p(0)| = |p(0) - g(0)| \le \|p - g\|_\infty \le \|p - g\|_{BV(\sigma)} < \frac{\epsilon}{7},$$

with a similar bound on $|p(1)|$. Since p is absolutely continuous on $[0, 1]$, there exists $\delta > 0$ such that $\operatorname{var}(p, [0, \delta]) < \frac{\epsilon}{7}$ and $\operatorname{var}(p, [1 - \delta, 1]) < \frac{\epsilon}{7}$. It follows that $|p(\boldsymbol{z})| < \frac{2\epsilon}{7}$ for all $\boldsymbol{z} \in [0, \delta] \cup [1 - \delta, 1]$. Use Lemma 5.7.1 to define $h \in AC(\sigma_1)$ which is supported on $[\delta/2, 1 - \delta/2]$, and let $q = p(1 - h)$. Then certainly $q \in AC(\sigma_1)$. Furthermore, by Corollary 5.5.3

$$\|g - q\|_{LG(\sigma_1)} \le \|g - q\|_{LG(\sigma)} + \|g - q\|_{LG(s)} = \|g - p\|_{LG(\sigma)} + \|g - q\|_{BV(s)} .$$

Now, using Proposition 2.3.8

$$\begin{aligned}
\|g-q\|_{BV(s)} &= \|q\|_{BV(s)} \\
&= \|q\|_{\infty,s} + \operatorname{var}(p(1-h),[0,1]) \\
&< \frac{2\epsilon}{7} + \operatorname{var}(p(1-h),[0,\delta]) + \operatorname{var}(p(1-h),[\delta,1-\delta]) \\
&\qquad + \operatorname{var}(p(1-h),[1-\delta,1]) \\
&\le \frac{2\epsilon}{7} + \|1-h\|_\infty \operatorname{var}(p,[0,\delta]) + \operatorname{var}(h,[0,\delta])\,\|p\|_{\infty,[0,\delta]} + 0 \\
&\qquad + \|1-h\|_\infty \operatorname{var}(p,[1-\delta,1]) + \operatorname{var}(h,[1-\delta,1])\,\|p\|_{\infty,[1-\delta,1]} \\
&< \frac{6\epsilon}{7}.
\end{aligned}$$

Thus $\|g-q\|_{LG(\sigma_1)} < \frac{\epsilon}{7} + \frac{6\epsilon}{7} = \epsilon$. The case where σ and s meet at just one point, say 0, is essentially the same, except that in this case one uses a function h which is zero on all of $[\delta,1]$ rather than just on $[\delta,1-\delta]$. ☐

We can now complete the proof of the main theorem.

Theorem 5.7.3. *Suppose that $\sigma,\tau \in LG$. Then $AC(\sigma)$ is isomorphic to $AC(\tau)$ if and only if σ is homemorphic to τ.*

Proof. The fact that if $AC(\sigma)$ is isomorphic to $AC(\tau)$ then σ is homeomorphic to τ is just Theorem 2.5.2.

Suppose conversely that σ is homeomorphic to τ. As in the proof of Theorem 5.6.1, fix corresponding edge-by-edge decompositions $\{\sigma_j\}_{j=1}^m$ and $\{\tau_j\}_{j=1}^m$ of σ and τ, and let ϕ be a homeomorphism from σ to τ which is affine on each of component line segments. Let $\Phi : BV(\sigma) \to BV(\tau)$ be the Banach algebra isomorphism $\Phi(f)(\boldsymbol{z}) = f(\phi^{-1}(\boldsymbol{z}))$.

Suppose that $f \in AC(\sigma)$. Then $f \mid \sigma_1 \in AC(\sigma_1)$ and since ϕ^{-1} is an affine map on τ_1, it is clear that $f \circ \phi^{-1}$ is absolutely continuous on τ_1. Repeated use of Theorem 5.7.2 now allows one to deduce that $f \circ \phi^{-1} \mid \tau_j \in AC(\tau_j)$ for each j, and in particular, that $\Phi(f) \in AC(\tau)$.

Since $\Phi^{-1}(g)(\boldsymbol{z}) = g(\phi(\boldsymbol{z}))$, the same proof shows that the image of $AC(\sigma)$ under Φ is all of $AC(\tau)$ and hence that $AC(\sigma)$ is isomorphic to $AC(\tau)$. ☐

5.8 A final example

One might naturally ask the extent to which the one-dimensional structure of sets in LG is vital in the results above. As we shall see in the next chapter, the linear structure can be relaxed to some extent. The following example gives an indication of some of the restrictions which will be required in generalizing these results.

Example 5.8.1. Let $\sigma = [0,1] = \{(t,0) : 0 \le t \le 1\}$ and let $\tau = \{(0,0)\} \cup \left\{\left(t, t\sin\frac{1}{t}\right) : 0 < t \le 1\right\}$. Then σ and τ are homeomorphic curves. Suppose that $AC(\sigma)$ is isomorphic to $AC(\tau)$ via an isomorphism Φ. Then there exists a homeomorphism $h : \sigma \to \tau$ such that $\Phi(f)(\boldsymbol{x}) = f(h^{-1}(\boldsymbol{x}))$.

The function $g : \tau \to \mathbb{C}$, $g(x,y) = y$ is in $\mathcal{P}_2(\tau)$ and hence $g \in AC(\tau)$. Let $f = \Phi^{-1}(g) = g \circ h$. For $j = 1,2,3,\dots$ let $t_j = 2/((2j-1)\pi)$, and let $\boldsymbol{x}_j = \left(t_j, t_j \sin\frac{1}{t_j}\right) \in \tau$. Let $\boldsymbol{z}_j = h^{-1}(\boldsymbol{x}_j)$ so that $f(\boldsymbol{z}_j) = (-1)^j t_j$. Since h is a homeomorphism it is clear that the sequence $\{\boldsymbol{z}_j\}$ must be monotone in $[0,1]$. Since σ is a line segment in $\mathbb{R}$ we can use the classical definition of variation and so for all n

$$\operatorname{var}(f,\sigma) \ge \sum_{j=2}^{n} |f(\boldsymbol{z}_j) - f(\boldsymbol{z}_{j-1})| = \frac{2}{\pi}\sum_{j=2}^{n}\left|\frac{1}{2j-1} + \frac{1}{2j-3}\right|$$

and hence f is not even of bounded variation on σ.

It follows that no such isomorphism Φ can exist. That is, σ and τ are homeomorphic curves for which the function spaces $AC(\sigma)$ and $AC(\tau)$ are not isomorphic.

Chapter 6

Isomorphisms of $AC(\sigma)$ spaces for convex edges

A central point in the proof of the Gelfand–Kolmogorov Theorem for Linear Graphs in the last chapter is that the variation of a function essentially is given by the sum of the variations along each of the edges of the graph. It is natural to ask whether this really depends on the edges being line segments, or whether one might be able to adapt the proofs to deal with a wider range of edge geometry. The aim of this chapter is to generalize the results of Chapter 5 to cover a much wider class of compact sets, denoted PIC, which are the unions of a finite number of one dimensional pieces which lie inside disjoint polygons. While it is unlikely that the class PIC is in any way optimal, it is large enough to include most standard 'nice' curves. The results will show in particular that if σ is a circle, an ellipse or a square, then the variation is again essentially just what one gets from computing the variation as one makes a circuit of the curve. Furthermore, the $AC(\sigma)$ spaces for these sets are all mutually isomorphic as well as being isomorphic to the algebra $AC_{HK}(\mathbb{T})$ used by Berkson and Gillespie in their definition of trigonometrically well-bounded operators.

Roughly, the sets we consider are ones which are unions of a finite number of convex curves. For technical reasons we need to be slightly more restrictive, and the next few sections carefully introduce the definitions we will use. As well as requiring that the variation over σ is essentially the sum of the variation along the individual one dimensional curves that make up σ, one also needs to ensure that there is an appropriate correspondence between the absolute continuity of a function on all of σ and the absolute continuity of the restriction of the function to each of the curves.

6.1 Overview of the proof

The aim of this chapter is to prove the following result, which generalizes our Gelfand–Kolmogorov Theorem for Linear Graphs.

Theorem 6.1.1. *Suppose that $\sigma, \tau \in \mathrm{PIC}$. Then $AC(\sigma) \simeq AC(\tau)$ if and only if σ is homeomorphic to τ.*

The general idea for the proof is quite similar. The $\Leftarrow$ direction is again the only one that really needs proving. The brief overview of the plan then is:

1. Given $\sigma = \cup_i c_i \in \mathrm{PIC}$, associate to it a planar graph G_σ (which comes from a 'polygonal mosaic'). The endpoints of the curves c_i are the vertices, and the curves are the edges. We then show that by 'refining the polygonal mosaic' we can reduce to the case that
   (a) the graph is simple, and
   (b) each curve c_i is 'projectable'.
2. If σ and τ are topologically homeomorphic, then G_σ and G_τ are graph homeomorphic, and so we can find subdivisions of these graphs which are graph isomorphic. (This is the same as in the LG case.)
3. Decompose σ and τ according to these subdivisions, $\sigma = \cup_{i=1}^n c_i$ and $\tau = \cup_{i=1}^n c_i'$.
4. Show that $AC(c) \simeq AC[0,1]$ for any convex curve c. Hence $AC(c_i)$ is isomorphic to $AC(c_i')$ for all i. One then takes a particular choice of homeomorphism $h_i : c_i \to c_i'$ and corresponding homomorphism. $\Phi_i(f) = f \circ h_i^{-1}$.
5. Note that you can choose the homeomorphisms h_i so that they are consistent at the endpoints and hence patch these together to produce a homeomorphism $h : \sigma \to \tau$. This generates an invertible algebra homomorphism $\Phi(f) = f \circ h^{-1}$ from functions on σ to functions on τ.
6. Define a new norm $\|\cdot\|_{\mathrm{PIC}(\sigma,\mathcal{P})}$ and show that this is equivalent to $\|\cdot\|_{BV(\sigma)}$. Note that the $\|\Phi(f)\|_{\mathrm{PIC}(\tau)} = \|f\|_{\mathrm{PIC}(\sigma)}$ and hence Φ is a continuous isomorphism from $BV(\sigma)$ to $BV(\tau)$.
7. Show that $f \in AC(\sigma)$ if and only if $f|c_i \in AC(c_i)$ for all i.

8. Combine (4) and (7) to show that

$$f \in AC(\sigma) \iff f|c_i \in AC(c_i) \iff \Phi_i(f|c_i) \in AC(c_i') \iff \Phi(f) \in AC(\tau)$$

   and so Φ is the required Banach algebra isomorphism from $AC(\sigma)$ to $AC(\tau)$.

We begin by examining the simplest case of a single convex curve. This will lead as to the definition of the class PIC in Section 6.3.

6.2 Convex curves

Let C denote a finite length curve with parametrization $\gamma(t)$, $0 \le t \le L$ and endpoints $\boldsymbol{x} = \gamma(0)$ and $\boldsymbol{y} = \gamma(L)$. We shall say that C is convex if it has a supporting line through each point of the curve. (A line L is a supporting line for of a curve C if L contains a point of C and C lies in one of the closed half-planes defined by L.) Convex curves are differentiable (that is, have a well-defined tangent) almost everywhere. To simplify matters we shall actually assume that each of our curves is differentiable, except at its endpoints. (Since we interested in sets which can be written as a union of such curves, this is only a minor restriction.) If C is a convex curve from $\boldsymbol{x}$ to $\boldsymbol{y}$ then C together with the line segment joining $\boldsymbol{x}$ and $\boldsymbol{y}$ forms the boundary of a compact convex set.

We shall generally use arc-length parameterizations, so that $\|\gamma'(t)\| = 1$ for $0 < t < L$.

Let $\mathcal{C}$ denote the set of differentiable finite length convex curves in the plane with distinct endpoints.

Definition 6.2.1. Suppose that $C \in \mathcal{C}$ has endpoints $\boldsymbol{x}$ and $\boldsymbol{y}$. We shall say that C is **projectable** if the orthogonal projection of C onto the line through $\boldsymbol{x}$ and $\boldsymbol{y}$ is precisely the line segment $\overline{\boldsymbol{x}\,\boldsymbol{y}}$.

This is illustrated in Figure 6.1.

Figure 6.1: C_1 is a projectable convex curve, while C_2 is a nonprojectable convex curve.

Given a nonprojectable convex curve C, one may always split it into projectable curves as shown in Figure 6.2.

Proposition 6.2.2. *Suppose that C is a differentiable convex curve. Then C can be split into a finite number of projectable curves.*

Proof. Let $\gamma(t)$, $t \in [0, L]$ be an arc-length parameterization of C in the complex plane. Let C_t and C'_t denote the convex curves $\{\gamma(u) : 0 \le u \le t\}$ and $\{\gamma(u) : t \le u \le L\}$. By a suitable affine transformation of the plane we can reduce to the case that $\gamma(0) = 1$ and $\gamma(L) = 0$. Noting the earlier remark about convex curves, $C \cup [0, 1]$ forms the boundary of a compact convex set and so, by reflecting if necessary, we can assume that C lies in the closed upper half plane. That is, that $\operatorname{Im} \gamma(t) \ge 0$ for all t.

For $0 < t < L$, let $r(t) = \operatorname{Arg}(\gamma(t) - 1)$ and $s(t) = \operatorname{Arg} \gamma'(t)$ where the argument is chosen (continuously) in $[0, 2\pi]$. Note that (using the convexity) both r and s are non-decreasing functions, that $0 \le r(t) \le \pi$ and that $0 \le s(t) \le 2\pi$, and hence we can take limits to give values to r and s at the endpoints. Indeed $r(0) = s(0)$.

It is easy to see that if $r(t) < s(0) + \frac{\pi}{2}$ and $s(t) < r(t) + \frac{\pi}{2}$ then C_t is projectable. If these inequalities do hold for all $t \in [0, L]$ then we are done. From the above, the inequalities do hold for small values of t. Indeed if we choose t_1 so that these inequalities hold for $t \in [0, t_1]$, we can inductively reduce the problem to dealing with the smaller curve C'_{t_1}. By choosing t_1 close enough to where one of the inequalities fails we must have that either

$$s(t_1) \ge r(t_1) > s(0) + \frac{3}{2} \ge \frac{3}{2} \qquad \text{or} \qquad s(t_1) \ge r(t_1) + \frac{\pi}{2} > \frac{3}{2}.$$

That is we can ensure that $s(t_1) > \frac{3}{2}$. In repeating the argument with C'_{t_1} we now start the lower bound that $s(t_1) > \frac{3}{2}$. One can then again split off a suitable initial part of the curve and be left with a remaining part C'_{t_2} with now $s(t_2) > 3$. Clearly this procedure can only be done a finite number of times without violating the bound $s(t) < 2\pi$. $\square$

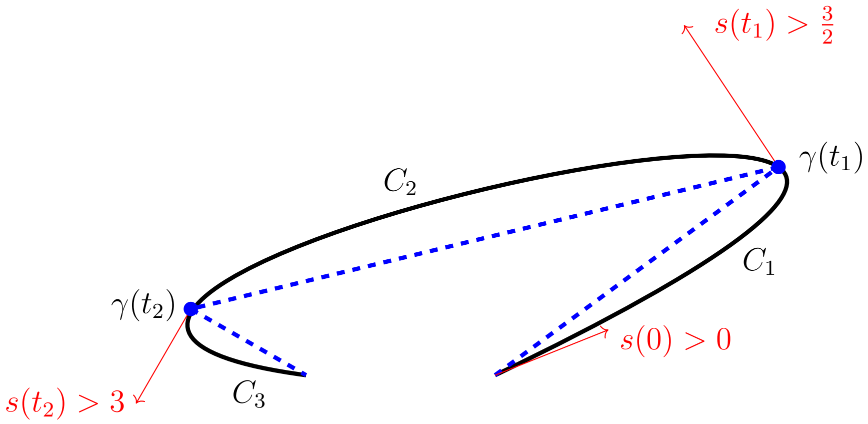


Figure 6.2: C is a nonprojectable convex curve, which can be split into the union of projectable convex curves C_1, C_2 and C_3 as in the proof of Proposition 6.2.2.

6.3 Polygonally inscribed curves

Roughly speaking the class PIC consists of connected compact subsets of the plane which are finite unions of smooth convex curves. However, for technical reasons, we need to introduce some mild conditions on the curves.

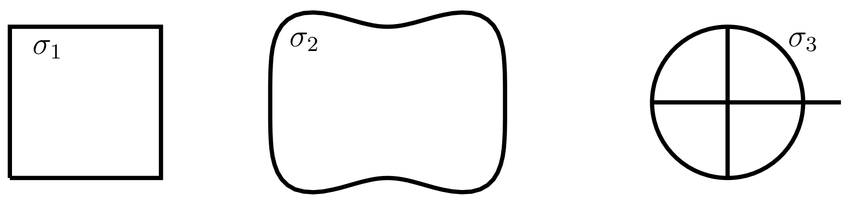


Figure 6.3: Three polygonally inscribed curves

To be definite, the term polygon will mean a simple closed polygon including its interior, and so all polygons are homeomorphic to the closed disk.

Definition 6.3.1. A (convex) **polygonal mosaic** in the plane is a finite collection $\mathcal{P}$ of convex polygons such that

1. $\bigcup_{P\in\mathcal{P}} P$ is connected.

2. if $P \neq Q \in \mathcal{P}$ intersect, then $P \cap Q$ is either
   - a single point which is a vertex of both P and Q, or
   - a line segment, which forms a full side of both P and Q.

Some of our estimate will depend on the nature of the polygons which are elements of the mosaic.

Definition 6.3.2. For a polygon P let $S(P)$ denote the number of sides of P. For a polygonal mosaic $\mathcal{P} = \{P_i\}_{i=1}^n$, let $S(\mathcal{P}) = max_i\{S(P_i)\}$.

Definition 6.3.3. A nonempty compact connected set σ is a **polygonally inscribed curve** if there exists a polygonal mosaic $\mathcal{P} = \{P_i\}_{i=1}^n$ such that for each i, $\sigma \cap P_i$ is a convex curve $c_i \in \mathbb{C}$ joining two vertices of P_i which only touches the boundary of P_i at those points. The curves c_i will be called the components of σ (with respect to the mosaic $\mathcal{P}$).

We shall denote the collection of all polygonally inscribed curves as PIC.

If we write below that $\sigma = \cup_{i=1}^n c_i$, then we will be implicitly assuming that there is a corresponding underlying polygonal mosaic $\mathcal{P}$.

It is worth noting that not every connected finite union of convex curves lies in PIC. Let $c_1 = \{(x, x^2) : 0 \leq x \leq 1\}$, $c_2 = \{(x, x^3) : 0 \leq x \leq 1\}$, and let $\sigma = c_1 \cup c_2$. The curves c_1 and c_2 are smooth convex curves, but it is impossible to find a polgonal mosaic $\{P_1, P_2\}$ so that $c_1 = \sigma \cap P_1$ and $c_2 = \sigma \cap P_2$. Indeed, the fact that c_1 and c_2 meet tangentially at $(0, 0)$ with the same convexity means that one cannot get around this by splitting σ into smaller pieces.

Any $\sigma \in$ PIC can be considered the drawing of a planar graph, with vertices at the endpoints of the curves c_i. Note that the graph is not uniquely determined by σ, since different choices of polygonal mosaic may produce different graphs.

Definition 6.3.4. Let $\sigma = \cup_{i=1}^n c_i$ be a polygonally inscribed curve. Let $G(\sigma, \{c_i\})$ denote the graph whose vertices are the endpoints of the curves c_i and whose edges are represented by these curves.

Thus σ is a drawing of the planar graph $G(\sigma, \{c_i\})$.

In general, the graph $G(\sigma, \{c_i\})$ need not be simple. To avoid this complication we first show that if $\sigma \in$ PIC then one may always choose a polygonal mosaic which does produce a simple graph. More generally, one can always add additional vertices to the graph by suitably partitioning the original mosaic. This also means that we will be able to assume that the convex curves that are the components of σ are all projectable.

Definition 6.3.5. Call $\mathcal{P}$ is ***a simple polygonal mosaic*** for σ if no two components $\boldsymbol{c}_i, \boldsymbol{c}_j$ of σ have the same pair of endpoints.

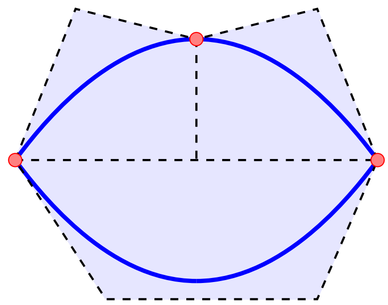


Figure 6.4: A simple polygonal mosaic

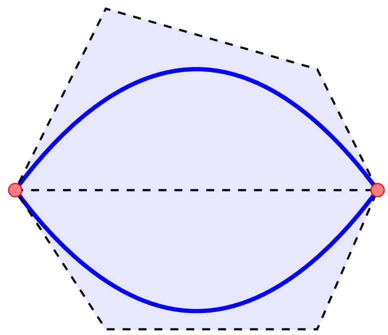


Figure 6.5: Not a simple polygonal mosaic

In the following, we shall see that any polygonal mosaic can be replaced by a simple polygonal mosaic by partitioning the polygons such that no two components of $\boldsymbol{c}$ have the same pair of endpoints.

An important factor in our ability to apply graph theoretic ideas to prove isomorphism theorems is the ability to add additional vertices within the edges of the graphs associated to PIC sets. Our primary tool will be the following Partition Lemma.

Lemma 6.3.6. ***Partition Lemma.*** *Suppose that P is a convex polygon and that $\boldsymbol{c}$ is a convex curve joining one vertex $\boldsymbol{v}_1$ of P to another vertex $\boldsymbol{v}_2$. Suppose that $\boldsymbol{v}$ is a point on $\boldsymbol{c}$ in the interior of P. Then there exist convex polygons $P_1, P_2 \subseteq P$ which only intersect at their boundaries and such that for $j = 1, 2$*

1. $\boldsymbol{v}_j$ *and* $\boldsymbol{v}$ *are vertices of* P_j*, and*

2. $c \cap P_j$ *is a convex curve joining* $\boldsymbol{v}_j$ *and* $\boldsymbol{v}$*.*

In any specific example it is generally easy to do such a partitioning. Showing that this is always possible requires a general fact about convex curves and sets.

Lemma 6.3.7. *Suppose that* K *is a closed convex subset of the plane and that* ℓ *is a line in the plane which intersects the boundary of* K *in at least three places. Then the line segment joining any two such points lies inside* $\ell \cap \partial K$*.*

Proof. Suppose that $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ are three distinct point in $\ell \cap \partial K$, labelled so that $\boldsymbol{y}$ in the line segment $\overline{\boldsymbol{x}\,\boldsymbol{z}}$. As K is closed and convex $\overline{\boldsymbol{x}\,\boldsymbol{y}} \subseteq K$. By the convexity of K, there exists a closed half-plane H bounded by a tangent line to ∂K at $\boldsymbol{y}$ with $K \subseteq H$. Since $\boldsymbol{x}, \boldsymbol{z} \in K$, this can only happen in the tangent line is ℓ. Every point of $\overline{\boldsymbol{x}\,\boldsymbol{z}}$ then has neighbours both in the complement of H and hence in the complement of K. Thus $\overline{\boldsymbol{x}\,\boldsymbol{z}} \subseteq \partial K$. □

Proof of Lemma 6.3.6. Suppose then that P is a convex polygon and that $\boldsymbol{c}$ is a convex curve joining one vertex $\boldsymbol{v}_1$ of P to another vertex $\boldsymbol{v}_2$. At each point $\boldsymbol{w} \in \boldsymbol{c}$ there is a closed tangent half-plane which contains all of $\boldsymbol{c}$. The intersection of all these half-planes and the polygon P is therefore a closed convex set $R_{\boldsymbol{c}}$ whose boundary consists of the curve $\boldsymbol{c}$ and one or more of the sides of P.

Let $\boldsymbol{m}$ be any point on the boundary of $R_{\boldsymbol{c}} \setminus \boldsymbol{c}$ and let $\ell_{\boldsymbol{m}}$ be the line through $\boldsymbol{v}$ and $\boldsymbol{m}$. Note that by the lemma, $\ell_{\boldsymbol{m}}$ cannot intersect the boundary of $R_{\boldsymbol{c}}$ at any other points (since $\boldsymbol{v}$ does not lie on any line segment in the boundary of $R_{\boldsymbol{c}}$ which contains $\boldsymbol{m}$). It follows that $\ell_{\boldsymbol{m}}$ cuts $\boldsymbol{c}$ into two parts at $\boldsymbol{v}$. Indeed if we intersect P with the two closed half-planes bounded by $\ell_{\boldsymbol{m}}$, then we obtain two closed convex polygons Q_1, chosen to contain the part of $\boldsymbol{c}$ containing $\boldsymbol{v}_1$ and $\boldsymbol{v}$, and Q_2, containing the part of $\boldsymbol{c}$ containing $\boldsymbol{v}_2$ and $\boldsymbol{v}$.

Applying this construction to a different point $\boldsymbol{m}' \in R_{\boldsymbol{c}} \setminus \boldsymbol{c}$ produces two polygons Q_1' and Q_2' with the same properties. Note that (again by the lemma), $\ell_{\boldsymbol{m}}$ and $\ell_{\boldsymbol{m}'}$ are distinct, and meet at $\boldsymbol{v}$. This means that if we set $P_1 = Q_1 \cap Q_1'$ and $P_2 = Q_2 \cap Q_2'$ then these polygons have a vertex at $\boldsymbol{v}$ and so satisfy the conclusions of the lemma.

□

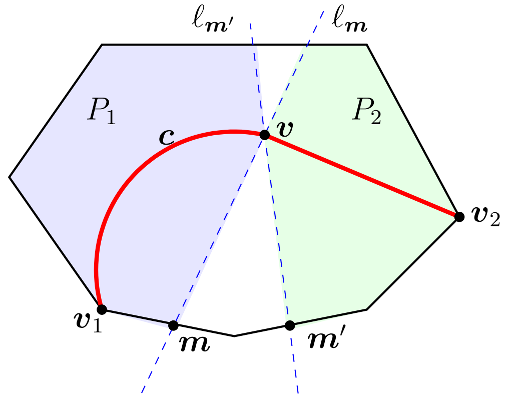


Figure 6.6: The construction in the proof of Lemma 6.3.6

Theorem 6.3.8. *If $\sigma \in \mathrm{PIC}$, then there is a simple polygonal mosaic for σ.*

Proof. Given $\sigma \in \mathrm{PIC}$ with mosaic $\mathcal{P} = \{P_j\}$, it could be that two curves $\boldsymbol{c}_j = P_j \cap \sigma$ and $\boldsymbol{c}_i = P_i \cap \sigma$ share both endpoints. The partition lemma allows to divide σ into a larger number of curves in a way that avoids this problem and provides a simple polygonal mosaic for σ. □

Combining this with Proposition 6.2.2 gives the following.

Theorem 6.3.9. *If $\sigma \in \mathrm{PIC}$ then there exists a simple polygonal mosaic $\mathcal{P} = \{P_i\}_{i=1}^n$ such that each component curve $c_i = \sigma \cap P_i$ is a projectable convex curve.*

Note that if one has a simple polygonal mosaic $\mathcal{P}$ for which each curve $c_i = \sigma \cap P_i$ is projectable, then any further partitioning of the curve will retain these properties.

Some of our estimate will depend on the nature of the polygons which are elements of the mosaic.

Definition 6.3.10. For a polygon P let $S(P)$, denote the number of sides of P. For a polygonal mosaic $\mathcal{P} = \{P_i\}_{i=1}^n$, let $S(\mathcal{P}) = \max_i\{S(P_i)\}$.

6.4 PIC sets and graph isomorphisms

Suppose that $\sigma, \tau \in \mathrm{PIC}$ are homeomorphic subsets of the plane. By Theorem 6.3.9 there are simple polygonal mosaics $\mathcal{M}_\sigma$ and $\mathcal{M}_\tau$ for each of these sets and hence the corresponding graphs $G_\sigma = G(\sigma, \mathcal{M}_\sigma)$ and $G_\tau = G(\tau, \mathcal{M}_\tau)$ are simple graphs with drawings σ and τ.

By Theorem 5.3.4, G_σ and G_τ are homeomorphic graphs, and hence they admit subdivisions $\hat{G}_\sigma$ and $\hat{G}_\tau$ which are isomorphic graphs. Let $H; \hat{G}_\sigma \to \hat{G}_\tau$ denote the graph isomorphism. By repeatedly applying the Partition Lemma, we can produce new simple polygonal mosaics $\widehat{\mathcal{M}}_\sigma = \{P_i\}_{i=1}^n$ and $\widehat{\mathcal{M}}_\tau = \{P'_i\}_{i=1}^n$ ordered in such a way that that for all i the edge $c'_i = P'_i \cap \tau$ in $\hat{G}_\tau$ is the image under H of the edge $c_i = P_i \cap \sigma$ in $\hat{G}_\sigma$.

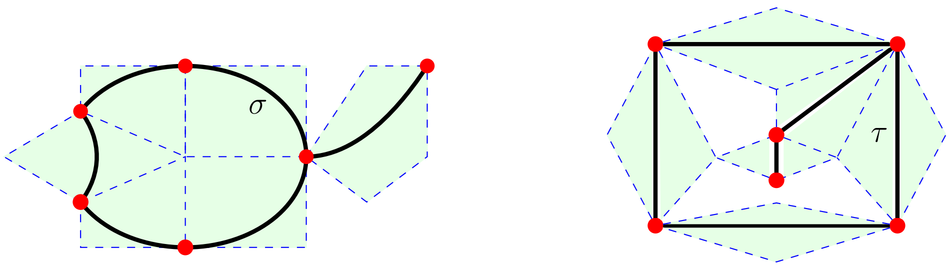


Figure 6.7: Two homeomorphic sets $\sigma, \tau \in \mathrm{PIC}$ with simple polygonal mosaics.

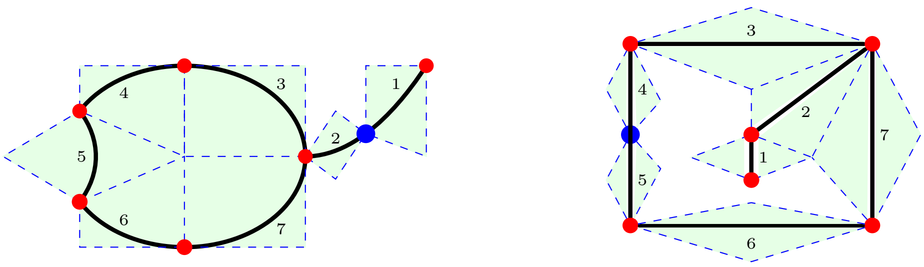


Figure 6.8: The refined mosaics $\widehat{\mathcal{M}}_\sigma$ and $\widehat{\mathcal{M}}_\tau$ with the corresponding curves labelled so that the graphs $\hat{G}_\sigma$ and $\hat{G}_\tau$ are graph isomorphic.

The arc-length parameterizations of the curves generate a natural homeomorphism $h_i : c_i \to c'_i$. If the directions of these parameterizations are chosen appropriately we can ensure that if $\boldsymbol{x}$ is an endpoint of both c_i and c_j then $h_i(\boldsymbol{x}) = h_j(\boldsymbol{x})$, and hence there exists a homeomorphism $h : \sigma \to \tau$ determined by $h(\boldsymbol{z}) = h_i(\boldsymbol{z})$ for $z \in c_i$.

6.5 The PIC norm

In general it is much easier to calculate the variation along a nice curve than to compute the two dimensional variation defined in Chapter 2.

Let C denote a finite length curve with parametrization $\gamma(t)$, $0 \le t \le L$ and endpoints $\boldsymbol{x} = \gamma(0)$ and $\boldsymbol{y} = \gamma(L)$. Suppose that $f : C \to \mathbb{C}$. The parameterized variation of f on C is

$$\operatorname{pvar}(f, C) = \operatorname*{var}_{[0,L]}(f \circ \gamma) = \sup \sum_{i=1}^{n} |f(\gamma(t_i)) - f(\gamma(t_{i-1}))|$$

where the supremum is taken over all finite partitions $0 \le t_0 < t_1 < \cdots < t_n \le L$ of the parameter set. Note that $\operatorname{pvar}(f, C)$ does not depend on the actual parameterization — any continuous one-to-one function γ mapping an interval to C will do.

The following important fact is essentially contained in Proposition 3.11 of [AD1].

Theorem 6.5.1. *Suppose that C is a projectable convex finite length curve. Then for any $f : C \to \mathbb{C}$,*

$$\operatorname{var}(f, C) \le \operatorname{pvar}(f, C) \le 2 \operatorname{var}(f, C).$$

Proof. To simplify the argument for the first inequality, note that by taking an appropriate affine transformation of the plane we can assume that C is of the form $\{(t, h(t)) : 0 \le t \le 1\}$ for some suitable function h and we may parameterize C by $\gamma(t) = (t, h(t), 0 \le t \le 1$.

Consider any (ordered) partition $\{t_i\}_{i=0}^{n}$ of the parameter set $[0, 1]$ and let $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ where $\boldsymbol{x}_i = \gamma(t_i)$. Since C is convex, $1 \le \operatorname{vf}(S) \le 2$ and hence

$$\sum_{i=1}^{n} |f(\gamma(t_i)) - f(\gamma(t_{i-1}))| \le 2 \frac{\operatorname{cvar}(f, S)}{\operatorname{vf}(S)} \le 2 \operatorname{var}(f, C)$$

which proves the right hand inequality.

Suppose now that $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$ is an arbitrary list of elements of C. There exist $t_0 < t_1 < \cdots < t_m$ so that every point $\boldsymbol{x}_j$ in S is $\gamma(t_i)$ for some i. Our aim is to bound $\operatorname{cvar}(f, S) / \operatorname{vf}(S)$. For $i = 1, 2, \ldots, m$ let $I_i = [t_{i-1}, t_i]$ (see Example 6.5.2).

Consider the term $|f(\boldsymbol{x}_j) - f(\boldsymbol{x}_{j-1})|$, where $\boldsymbol{x}_{j-1} = \gamma(t_{i_1})$ and $\boldsymbol{x}_j = \gamma(t_{i_2})$. Suppose first that $i_i < i_2$. Then by the triangle inequality

$$|f(\boldsymbol{x}_j) - f(\boldsymbol{x}_{j-1})| \leq \sum_{i=i_1+1}^{i_2} |f(\gamma(t_i)) - f(\gamma(t_{i-1}))| \leq \sum_{i=i_1+1}^{i_2} \operatorname*{var}_{I_i}(f \circ \gamma).$$

A similar argument applies if $i_1 > i_2$. Adding all the terms gives that

$$\sum_{j=1}^{n} |f(\boldsymbol{x}_j) - f(\boldsymbol{x}_{j-1})| \leq \sum_{i=1}^{m} k_i \operatorname*{var}_{I_i}(f \circ \gamma) \tag{6.5.1}$$

where k_i is the number of times that interval I_i lies between $\boldsymbol{x}_{j-1}$ and $\boldsymbol{x}_j$ (so $1 \leq k_i \leq m$). Let $k = \max\{k_1, k_2, \ldots, k_m\}$ and choose a value i_0 so that $k = k_{i_0}$.

If t is any number in the interior of I_{i_0} and ℓ is the vertical line through t, then S necessarily has k crossing segments on ℓ, and hence $\operatorname{vf}(S) \geq k$.

But (6.5.1), and the additivity of variation over contiguous intervals, implies that

$$\operatorname{cvar}(f, S) \leq k \sum_{i=1}^{m} \operatorname*{var}_{I_i}(f \circ \gamma) = k \operatorname*{var}_{[t_0, t_m]}(f \circ \gamma) \leq k \operatorname*{var}_{[0,1]}(f \circ \gamma) = k \operatorname{pvar}(f, C). \tag{6.5.2}$$

Combining these shows that

$$\frac{\operatorname{cvar}(f, S)}{\operatorname{vf}(S)} \leq \frac{k \operatorname{pvar}(f, C)}{k} = \operatorname{pvar}(f, C)$$

which completes the proof $\square$

Example 6.5.2. The argument in the proof above is best illustrated by an example. Consider the convex curve C is Figure 6.9 and the ordered list $S = [\boldsymbol{x}_0, \ldots, \boldsymbol{x}_5] \subseteq C$.

Given $f : C \to \mathbb{C}$,

$$\begin{aligned}
\sum_{j=1}^{5} |f(\boldsymbol{x}_j) - f(\boldsymbol{x}_{j-1})| \le (\operatorname*{var}_{I_1}(f \circ \gamma) + \operatorname*{var}_{I_2}(f \circ \gamma) + \operatorname*{var}_{I_3}(f \circ \gamma)) \\
+ (\operatorname*{var}_{I_2}(f \circ \gamma) + \operatorname*{var}_{I_3}(f \circ \gamma)) + \operatorname*{var}_{I_1}(f \circ \gamma) \\
+ (\operatorname*{var}_{I_1}(f \circ \gamma) + \operatorname*{var}_{I_2}(f \circ \gamma)) + \operatorname*{var}_{I_2}(f \circ \gamma) \\
= 3 \operatorname*{var}_{I_1}(f \circ \gamma) + 4 \operatorname*{var}_{I_2}(f \circ \gamma) + 2 \operatorname*{var}_{I_3}(f \circ \gamma) \\
\le 4 \operatorname*{var}_{[t_0,t_3]}(f \circ \gamma) \le 4 \operatorname{pvar}(f, C).
\end{aligned}$$

In the notation of the proof, $k = 4$ and we would take $i_0 = 2$. There are 4 crossing segments of S on ℓ as the interval I_2 is traversed 4 times by γ_S, so $\operatorname{vf}(S) \ge 4$.

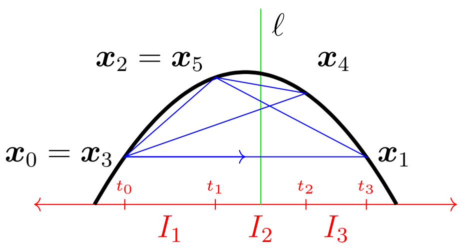


Figure 6.9: Example of the estimate in the proof of Theorem 6.5.1.

Definition 6.5.3. Suppose that $\sigma \in \mathrm{PIC}$ and that $\mathcal{P}$ is a simple polygonal mosaic such that $c_i = P_i \cap \sigma$ is projectable for each i. For $f : \sigma \to \mathbb{C}$ let

$$\|f\|_{\mathrm{PIC}(\sigma,\mathcal{P})} = \|f\|_\infty + \sum_{i=1}^{n} \operatorname{pvar}(f, c_i).$$

We will denote the set of all functions f such that $\|f\|_{\mathrm{PIC}(\sigma,\mathcal{P})} < \infty$ as $\mathrm{PIC}(\sigma, \mathcal{P})$.

Since $\|\cdot\|_\infty$ and each of the terms $\operatorname{pvar}(\cdot, c_i)$ have the appropriate homogeneity and subadditivity properties it is easy to verify that $\|\cdot\|_{\mathrm{PIC}(\sigma,\mathcal{P})}$ is a norm and that $\mathrm{PIC}(\sigma, \mathcal{P})$ is a vector space.

Our first aim is to show that it is equivalent to the $BV(\sigma)$ norm.

Theorem 6.5.4. *With σ as above, there exists a positive constant $K_\mathcal{P}$ such that*

$$\frac{1}{K_\mathcal{P}} \|f\|_{BV(\sigma)} \le \|f\|_{\mathrm{PIC}(\sigma,\mathcal{P})} \le K_\mathcal{P} \|f\|_{BV(\sigma)}$$

for all $f \in BV(\sigma)$.

Note that by Theorem 6.5.1,

$$\|f\|_{\mathrm{PIC}(\sigma,\mathcal{P})} \leq \|f\|_\infty + \sum_{i=1}^n 2\,\mathrm{var}(f, c_i) \leq \|f\|_\infty + 2n\,\mathrm{var}(f,\sigma) \leq 2n\,\|f\|_{BV(\sigma)}.$$

Proving an inequality in the reverse direction is more difficult. The proof in the following lemma is similar to the proof of Theorem 5.5.2.

Lemma 6.5.5. *Suppose that* $\sigma \in \mathrm{PIC}$. *Then there exists a constant* $L_{\sigma,\mathcal{P}}$ *such that* $\|f\|_{BV(\sigma)} \leq L_{\sigma,\mathcal{P}}\,\|f\|_{\mathrm{PIC}(\sigma,\mathcal{P})}$ *for all* $f \in \mathrm{PIC}(\sigma,\mathcal{P})$.

Proof. Suppose that $\sigma = \cup_{i=1}^n \boldsymbol{c}_i$ is a decomposition of σ into projectable convex curves coming from a simple polygonal mosaic $\mathcal{P}$. For $k = 1, 2, \ldots, n$, let $\sigma_k = \cup_{j=1}^k \boldsymbol{c}_j$. We shall assume the curves $\boldsymbol{c}_i$ have been ordered so that each set σ_k is connected.

Suppose that $f : \sigma \to \mathbb{C}$. We shall proceed by induction to show that

$$\|f\|_{BV(\sigma_m)} \leq (m + 2(m-1)S(\mathcal{P}))\,\|f\|_{PIC(\sigma_m,\mathcal{P})} \tag{6.5.3}$$

for m from 1 to n.

Since $\sigma_1 = \boldsymbol{c}_1$ is a projectable convex curve, it follows from Theorem 6.5.1 that

$$\|f\|_{BV(\sigma_1)} = \sup_{x\in\sigma_1} |f(z)| + \mathrm{var}(f,\sigma_1) \leq \|f\|_{PIC(\sigma_1)} \tag{6.5.4}$$

so (6.5.3) holds when $m = 1$.

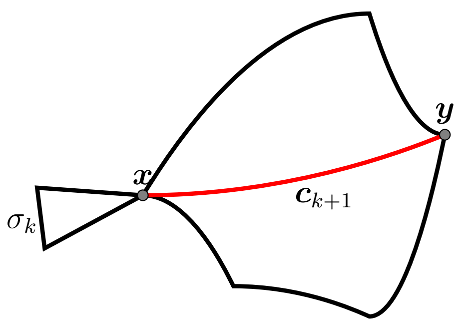


Figure 6.10: $\sigma_{k+1} = \sigma_k \cup \boldsymbol{c}_{k+1}$.

Suppose that $1 \leq k < n$, and that (6.5.3) holds if $m = k$. Let $S = [\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_n]$ be a list of points in $\sigma_{k+1} = \sigma_k \cup \boldsymbol{c}_{k+1}$. Denote the endpoints of $\boldsymbol{c}_{k+1}$ by $\boldsymbol{x}$ and $\boldsymbol{y}$. For the moment assume that both $\boldsymbol{x}$ and $\boldsymbol{y}$ are elements of σ_k.

For $j = 1, \ldots, n$, let $\ell_j = \overline{\boldsymbol{z}_{j-1}\,\boldsymbol{z}_j}$. Define subsets $I_1, I_2, I_3 \subseteq \{1, 2, \ldots, n\}$ by

$$\begin{aligned} I_1 &= \{j \,:\, \boldsymbol{z}_j, \boldsymbol{z}_{j-1} \in \sigma_k\}, \\ I_2 &= \{j \,:\, \boldsymbol{z}_j, \boldsymbol{z}_{j-1} \in \boldsymbol{c}_{k+1}\}, \\ I_3 &= \{1, 2, \ldots, n\} \setminus (I_1 \cup I_2). \end{aligned}$$

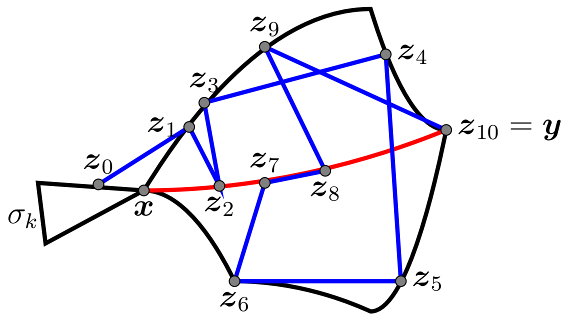


Figure 6.11: The segments for the sublist $S = \{\boldsymbol{z}_0, \boldsymbol{z}_1, \ldots, \boldsymbol{z}_{10}\}$

In the example $I_1 = \{1, 4, 5, 6, 10\}$, $I_2 = \{8\}$ and $I_3 = \{2, 3, 7, 9\}$. The sublist S_1 is $[\boldsymbol{z}_0, \boldsymbol{z}_1, \boldsymbol{z}_3, \boldsymbol{z}_4, \boldsymbol{z}_5, \boldsymbol{z}_6, \boldsymbol{z}_9, \boldsymbol{z}_{10}]$ and the sublist $S_2 = [\boldsymbol{z}_7, \boldsymbol{z}_8]$.

Note that if $j \in I_3$ then one end of ℓ_j must lie in $\sigma_k \setminus \boldsymbol{c}_{k+1}$ and the other must lie in $\boldsymbol{c}_{k+1} \setminus \sigma_k$.

Noting that $I_1 \cap I_2$ may be nonempty,

$$\operatorname{cvar}(f, S) \leq \sum_{i=1}^{3} \sum_{j \in I_i} |f(\boldsymbol{z}_j) - f(\boldsymbol{z}_{j-1})|. \tag{6.5.5}$$

Form the sublist S_1 of S by including all points which are endpoints of line segments ℓ_j with $j \in I_1$. By [DL1, Proposition 3.5] and [AD2], $\operatorname{vf}(S_1) \leq \operatorname{vf}(S)$, we have

$$\frac{\sum_{j \in I_1} |f(\boldsymbol{z}_j) - f(\boldsymbol{z}_{j-1})|}{\operatorname{vf}(S)} \leq \frac{\operatorname{cvar}(f, S_1)}{\operatorname{vf}(S_1)} \leq \operatorname{var}(f, \sigma_k). \tag{6.5.6}$$

Similarly, if S_2 is the sublist of S including all points which are endpoints of line segments ℓ_j with $j \in I_2$ then

$$\frac{\sum_{j\in I_2} |f(\boldsymbol{z}_j) - f(\boldsymbol{z}_{j-1})|}{\mathrm{vf}(S)} \le \frac{\mathrm{cvar}(f, S_2)}{\mathrm{vf}(S_2)} \le \mathrm{var}(f, \boldsymbol{c}_{k+1}). \tag{6.5.7}$$

Consider now the set σ and the list S as in the Figure 6.11. If $j \in I_3$, then ℓ_j is a crossing segment on one of the lines which form the boundary of P_{k+1}. In particular, at least one of the $S(P_{k+1})$ lines must have at least $\frac{1}{S(P_{k+1})}|I_3|$ crossing segments, and hence $\mathrm{vf}(S) \ge \frac{1}{S(P_{k+1})}|I_3|$. By a simple triangle inequality estimate

$$\frac{\sum_{j\in I_3} |f(\boldsymbol{z}_j - f(\boldsymbol{z}_{j-1})|}{\mathrm{vf}(S)} \le \frac{2|I_3| \, \|f\|_{\infty,\sigma_{k+1}}}{\frac{1}{S(P_{k+1})}|I_3|} = 2S(P_{k+1}) \, \|f\|_{\infty,\sigma_{k+1}} \tag{6.5.8}$$

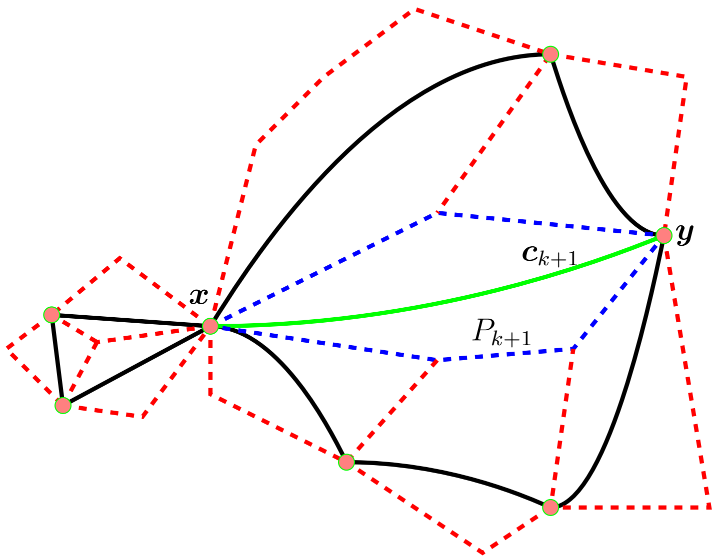


Figure 6.12: $\mathrm{PIC}(\sigma_{k+1}, \mathcal{P})$

.

Combining the three estimates we see that

$$\mathrm{var}(f, \sigma_{k+1}) \le \mathrm{var}(f, \sigma_k) + \mathrm{var}(f, \boldsymbol{c}_{k+1}) + 2S(P_{k+1}) \, \|f\|_{\infty,\sigma_{k+1}}$$

and hence

$$\begin{aligned}
\|f\|_{BV(\sigma_{k+1})} &= \|f\|_{\infty,\sigma_{k+1}} + \operatorname{var}(f,\sigma_{k+1}) \\
&\le \|f\|_{\infty,\sigma_k} + \|f\|_{\infty,\boldsymbol{c}_{k+1}} + \operatorname{var}(f,\sigma_k) + \operatorname{var}(f,\boldsymbol{c}_{k+1}) + 2S(P_{k+1})\,\|f\|_{\infty,\sigma_{k+1}} \\
&\le C_k(\|f\|_{\infty,\sigma_k} + \sum_{j=1}^{n} \operatorname{var}(f,\boldsymbol{c}_j) + (1+2S(P_{k+1}))\,\|f\|_{\infty,\sigma_{k+1}} + \operatorname{var}(f,\boldsymbol{c}_{k+1}) \\
&\le (C_k + 1 + 2S(P_{k+1}))\,\|f\|_{PIC(\sigma_{k+1},\mathcal{P})}\,.
\end{aligned}$$

This completes the proof. □

This proof and Theorem 6.5.1 show that the PIC norm is equivalent to the BV norm.

Corollary 6.5.6. *Suppose that $\sigma,\tau \in \mathrm{PIC}$. If σ is homeomorphic to τ then $BV(\sigma)$ is isomorphic to $BV(\tau)$.*

Proof. Using the construction in Section 6.3, we can write $\sigma = \cup_{i=1}^n c_i$ and $\tau = \cup_{i=1}^n c_i'$ with a homeomorphism $h : \sigma \to \tau$ which maps $c_i \to c_i'$ via the arc-length parameterization. For $f : \sigma \to \mathbb{C}$ define

$$\Phi(f)(\boldsymbol{z}) = f(h^{-1}(\boldsymbol{z})), \qquad \boldsymbol{z} \in \tau.$$

Then $\operatorname{pvar}(f,c_i) = \operatorname{pvar}(\Phi(f),c_i')$ and so $\|f\|_{\mathrm{PIC}(\sigma)} = \|\Phi(f)\|_{\mathrm{PIC}(\tau)}$. By Theorem 6.5.4 this means that Φ is a continuous map from $BV(\sigma)$ to $BV(\tau)$. Since $\Phi^{-1}(g) = g \circ h$, it follows that Φ is actually an isomorphism. □

6.6 Absolutely continuous functions

The isomorphism $\Phi : BV(\sigma) \to BV(\tau)$ defined in the proof of Corollary 6.5.6 is of the form $\Phi(f) = f \circ h^{-1}$ for a particular homeomorphism $h : \sigma \to \tau$. In fact the choice of homeomorphism here is not particularly important. One just wants a function which maps each component curve $c_i \subseteq \sigma$ continuously onto the corresponding curve $c_i' \subseteq \tau$. As discussed in Chapter 3 however, a badly chosen homeomorphism may not send $AC(\sigma)$ functions to $AC(\tau)$ functions. Our aim now is to show that one may choose a homeomorphism which preserves these subalgebras.

Let $c = k([0, L])$ be the graph of a convex function k. For $g : [0, L] \to \mathbb{C}$, define $\Psi(g) : c \to \mathbb{C}$ by $\Psi(g)(x, h(x)) = g(x)$. Then, by Theorem 6.5.1,

$$\begin{aligned}
\|g\|_{BV[0,L]} &= \|g\|_\infty + \operatorname{var}(g, [0, L]) \\
&= \|\Psi(g)\|_\infty + \operatorname{pvar}(\Psi(g), c) \\
&\le 2\left(\|\Psi(g)\|_\infty + \operatorname{var}(\Psi(g), c)\right) \\
&= 2\,\|\Psi(g)\|_{BV(c)} \\
&\le 2\left(\|g\|_\infty + \operatorname{pvar}(\Psi(g), c)\right) \\
&= 2\|g\|_{BV[0,L]}.
\end{aligned}$$

So $BV(c)$ is isomorphic to $BV[0, L]$ (and hence also isomorphic to $BV[0, 1]$). The more delicate thing is to check that Ψ preserves absolute continuity.

Proposition 6.6.1. *With Ψ as above, $f \in AC[0, L] \iff \Psi(f) \in AC(c)$.*

Proof. It is known (see [AD1, Proposition 4.4]) that Ψ is a norm-decreasing algebra homomorphism from $AC[0, L]$ to $AC(c)$.

Suppose first that $p \in AC(c)$ is a polynomial in two variables. Then $p_r(x) = p(x, k(x))$, is differentiable on $(0, L)$ with

$$p_r'(x) = \nabla p(x, k(x)) \cdot (1, k'(x)).$$

By the Fundamental Theorem for Line Integrals

$$p(x, k(x)) - p(0, k(0)) = \int_0^x \nabla p(s, k(s)) \cdot (1, k'(s))\, ds$$

or

$$p_r(x) = p_r(0) + \int_0^x p_r'(s)\, ds$$

and hence p_r is absolutely continuous. Of course $p_r = \Psi^{-1}(p)$.

Suppose now that $g \in AC(c)$ and that $\epsilon > 0$. Then there exists a polynomial in two variables $p \in \mathcal{P}_2$ such that $\|g - p\|_{BV(c)} < \frac{\epsilon}{2}$. Then

$$\begin{aligned}
\left\|\Psi^{-1}(g) - \Psi^{-1}(p)\right\|_{BV[0,L]} &= \left\|\Psi^{-1}(g) - \Psi^{-1}(p)\right\|_{\infty} + \operatorname{var}(\Psi^{-1}(g) - \Psi^{-1}(p), [0, L]) \\
&= \|g - p\|_{\infty} + \operatorname{pvar}(g - p, c) \\
&\le 2(\|g - p\|_{\infty} + \operatorname{var}(g - p, c) \\
&= 2\,\|g - p\|_{BV(c)} < \epsilon.
\end{aligned}$$

Since $AC[0, L]$ is a closed subalgebra of $BV[0, L]$ it follows that $\Psi^{-1}(g) \in AC[0, L]$. Thus Ψ is a continuous Banach algebra isomorphism. □

Of course the map $h : [0, L] \to c$, $h(x) = (x, k(x))$ is a homeomorphism, and it is an easy consequence of the proposition that $\Phi(f) = f \circ h^{-1}$ is a Banach algebra isomorphism from $AC[0, L]$ to $AC(c)$. By a suitable rotation and rescaling we an conclude the following.

Theorem 6.6.2. *Suppose that c is a projectable convex curve in $\mathbb{R}^2$. Then $AC(c) \simeq AC[0, 1]$.*

6.7 A cut-off function lemma

Every closed half-plane in $\mathbb{R}^2$ can be written as $H = \{\boldsymbol{x} : (\boldsymbol{x} - \boldsymbol{u}) \cdot \boldsymbol{v} \ge 0\}$ for some $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$, with $\|\boldsymbol{v}\| = 1$. For $\epsilon > 0$ let

$$g_\epsilon(t) = \begin{cases} 0, & \text{if } t \le \frac{\epsilon}{2}, \\ \frac{2t-\epsilon}{\epsilon}, & \text{if } \frac{\epsilon}{2} < t < \epsilon, \\ 1, & \text{if } t \ge \epsilon. \end{cases}$$

If, as usual σ is a nonempty compact subset of the plane and we define $h_{H,\epsilon} : \sigma \to \mathbb{C}$ by

$$h_{H,\epsilon}(\boldsymbol{x}) = g_\epsilon((\boldsymbol{x} - \boldsymbol{u}) \cdot \boldsymbol{v})$$

then $h_{H,\epsilon} \in AC(\sigma)$.

Suppose that P is a closed convex polygon. Then P can be written as the intersection of closed half-planes, $P = \bigcap_{i=1}^n H_i$. Given $\epsilon > 0$ we can define corresponding 'cut-off' functions $h_{i,\epsilon}$ as above corresponding to these half planes. For ϵ sufficiently small the function $h_\epsilon = \prod_{i=1}^n h_{i,\epsilon}$ is then an $AC(\sigma)$ function which is zero on an open neighbourhood of the complement of P and which is 1 on a smaller convex polygon in the interior of P (see Figure 6.13).

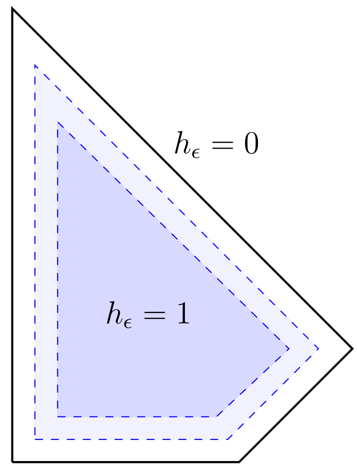


Figure 6.13: A polygonal cut-off function

Recall that $\operatorname{var}(fg, P) \le \|f\|_\infty \operatorname{var}(g, P) + \|g\|_\infty \operatorname{var}(f, P)$. Since for each i, $\|h_i\|_\infty = 1$ and $\operatorname{var}(h_i, P) = 1$, a simple induction proof shows that $\operatorname{var}(h_\epsilon, P) \le n$. Consequently, if c is any convex curve in P, then

$$\operatorname{pvar}(h_\epsilon, c) \le 2 \operatorname{var}(h_\epsilon, c) \le 2 \operatorname{var}(h_\epsilon, P) \le 2n. \tag{6.7.1}$$

6.8 The AC joining lemma

We shall now give the analogue for PIC sets of Theorem 5.7.2.

Theorem 6.8.1. *Suppose that $\sigma = \cup_{k=1}^n c_k \in$ PIC is represented as a union of projectable convex curves. Let $\sigma_0 = \cup_{k=1}^{n-1} c_k$ be connected and suppose that $f \in BV(\sigma)$. Then $f \in AC(\sigma)$ if and only if $f|\sigma_0 \in AC(\sigma_0)$ and $f|c_n \in AC(c_n)$.*

Proof. The forward implication follows from the general results about restricting AC functions.

For the reverse direction, by scaling, rotating and reflecting as necessary we can assume that the endpoints of $c = c_n$ are 0 and 1, and that c lies in the closed upper half-plane. We shall assume first that both 0 and 1 lie in σ_0; the case where c joins to σ_0 at just one endpoint is similar. Let P denote the polygon containing c from a suitable polygonal mosaic for σ, and let m denote the number of sides of P.

By Proposition 6.6.1 the function $f_r(\operatorname{Re} \boldsymbol{z}) = f(\boldsymbol{z})$, $\boldsymbol{z} \in c$ lies in $AC[0,1]$. We can therefore define $f_c : \sigma \to \mathbb{C}$ by

$$f_c(\boldsymbol{z}) = \begin{cases} f(0), & \text{if } \operatorname{Re} \boldsymbol{z} < 0, \\ f_r(\operatorname{Re} \boldsymbol{z}), & \text{if } 0 \le \operatorname{Re} \boldsymbol{z} \le 1, \\ f(1), & \text{if } \operatorname{Re} \boldsymbol{z} > 1. \end{cases}$$

By [AD1, Proposition 4.4], $f_c \in AC(\sigma)$. It follows then that $g = f - f_c$ is in $BV(\sigma)$. Our aim is to show that $g \in AC(\sigma)$ and hence that $f = g + f_c$ is in $AC(\sigma)$.

Since by hypothesis $f|\sigma_0 \in AC(\sigma_0)$ and, by restriction, $f_c|\sigma_0 \in AC(\sigma_0)$ we have that $g|\sigma_0 \in AC(\sigma_0)$. Clearly $g|c$ is identically zero.

It will suffice now to show that there are $q \in AC(\sigma)$ arbitrarily close to g, since this will imply that $g \in AC(\sigma)$. Fix $\epsilon > 0$.

Using the equivalence of the norms, there exists a polynomial $p \in \mathcal{P}_{\in}$ such that $\|g - p\|_{\mathrm{PIC}(\sigma_0)} < \epsilon$. Then

$$|p(0)| = |p(0) - g(0)| \le \|p - g\|_\infty \le \|p - g\|_{\mathrm{PIC}(\sigma_0)} < \epsilon.$$

Similarly $|p(1)| < \epsilon$. Let $p_r : [0,1] \to \mathbb{C}$ be defined by $p_r(\operatorname{Re} \boldsymbol{z}) = f(\boldsymbol{z})$, for $\boldsymbol{z} \in c$. Then $p_r \in AC[0,1]$ so there exists $\delta > 0$ such that $\operatorname{var}(p_r, [0,\delta]) < \epsilon$ and $\operatorname{var}(p_r, [1-\delta, 1]) < \epsilon$. It follows that $|p_r(t)| < 2\epsilon$ for $t \in [0,\delta] \cup [1-\delta, 1]$.

Consider the curves

$$\begin{aligned} c_L &= \{\boldsymbol{z} \in c \,:\, 0 \le \operatorname{Re} \boldsymbol{z} \le \delta\}, \\ c_\delta &= \{\boldsymbol{z} \in c \,:\, \delta \le \operatorname{Re} \boldsymbol{z} \le 1 - \delta\}, \\ c_R &= \{\boldsymbol{z} \in c \,:\, 1 - \delta \le \operatorname{Re} \boldsymbol{z} \le 1\}. \end{aligned}$$

Since c_δ is a compact set, there is a positive minimum distance η from this set to the boundary of P. By the results of the previous section there exists a cut-off function

$h \in AC(\sigma)$ such that $h(\boldsymbol{z}) = 1$ for $\boldsymbol{z} \in c_\delta$, and $h(\boldsymbol{z}) = 0$ for $\boldsymbol{z} \in \sigma_0$. Let $q = p(1-h)$. Then certainly $q \in AC(\sigma)$. Then

$$\begin{aligned}
\|g-q\|_{BV(\sigma)} &\le K_{\mathcal{P}} \|g-q\|_{\mathrm{PIC}(\sigma)} \\
&\le K_{\mathcal{P}} \left(\|g-q\|_\infty + \sum_{k=1}^{n} \mathrm{pvar}(g-q, c_k) \right) \\
&= K_{\mathcal{P}} \Big(\|g-q\|_{c,\infty} + \mathrm{pvar}(g-q, c) \Big) \\
&= K_{\mathcal{P}} \Big(\|q\|_{c,\infty} + \mathrm{pvar}(p(1-h), c) \Big) \\
&\le K_{\mathcal{P}} \big(2\epsilon + \mathrm{pvar}(p(1-h), c_L) \\
&\qquad\qquad + \mathrm{pvar}(p(1-h), c_\delta) + \mathrm{pvar}(p(1-h), c_R) \big) \\
&\le K_{\mathcal{P}} \big(2\epsilon + \|p\|_{c_L,\infty} \mathrm{pvar}(1-h, c_L) + \mathrm{pvar}(p, c_L) \|1-h\|_{c_L,\infty} \\
&\qquad\qquad + \|p\|_{c_R,\infty} \mathrm{pvar}(1-h, c_R) + \mathrm{pvar}(p, c_R) \|1-h\|_{c_R,\infty} \big) \\
&\le K_{\mathcal{P}} \big(2\epsilon + \epsilon\, \mathrm{pvar}(1-h, c_L) + \epsilon + \epsilon\, \mathrm{pvar}(1-h, c_R) + \epsilon \big) \\
&\le K_{\mathcal{P}} \big(4 + 4m \big) \epsilon
\end{aligned}$$

using (6.7.1). Since this can be made arbitrarily small by a suitable choice of ϵ, we are done.

The case where c joins σ_0 at just a single point is similar. $\square$

An easy induction proof then shows the following.

Corollary 6.8.2. *Suppose that $\sigma = \cup_{k=1}^{n} c_k \in \mathrm{PIC}$ is represented as a union of projectable convex curves. Suppose that $f \in BV(\sigma)$. Then $f \in AC(\sigma)$ if and only if $f|c_i \in AC(c_i)$ for all i.*

It should be noted that one cannot in general just patch two AC functions together, even if the resulting function is continuous.

Example 6.8.3. Let $\sigma_1 = \{(x, \sin\frac{1}{x} : 0 < x \le 1\} \cup \{(0,0)\}$ and let $\sigma_2 = \{(x, \sin\frac{1}{x} - x^2 : 0 < x \le 1\} \cup \{(0,0)\}$. Let $\sigma = \sigma_1 \cup \sigma_2$ and

$$f(x,y) = \begin{cases} x, & \text{if } (x,y) \in \sigma_1, \\ 0, & \text{if } (x,y) \in \sigma_2. \end{cases}$$

The $f|\sigma_1 \in AC(\sigma_1)$ and $f|\sigma_2 \in AC(\sigma_2)$, but although f is continuous, it is not even of bounded variation on σ. Of course, σ here is not in PIC.

6.9 Gelfand–Kolmogorov theorem for PIC sets

We can now give our Gelfand–Kolmogorov type theorem for PIC sets.

Theorem 6.9.1. *Suppose that $\sigma, \tau \in \mathrm{PIC}$. Then $AC(\sigma)$ is isomorphic to $AC(\tau)$ if and only if σ is homeomorphic to τ.*

Proof. As before, we only need to show the reverse implication. Suppose then that $\sigma, \tau \in \mathrm{PIC}$ and that σ is homeomorphic to τ.

We saw in Section 6.4 that we can find simple polygonal mosaics for σ and τ which split these sets up as drawings of isomorphic graphs. In particular we can write

$$\sigma = \bigcup_{i=1}^{n} c_i, \qquad \tau = \bigcup_{i=1}^{n} c_i'$$

where c_i and c_i' are matching edges of the associated graphs. If necessary one could use Proposition 6.2.2 and the Partition Lemma to further refine this decomposition so that each curve is projectable, so we will assume that all the curves have this property.

By Theorem 6.6.2, for each i, $AC(c_i)$ and $AC(c_i')$ are both isomorphic to $AC[0, 1]$ via the homeomorphisms which projects these curves onto the line segments joining their endpoints and then re-scales the interval. This in turn generates a homeomorphism $h_i : c_i \to c_i'$ whose orientation can be chosen to be consistent with the graph isomorphism in the way it maps endpoints (that is, graph vertices) from one set to another (see Figure 6.14).

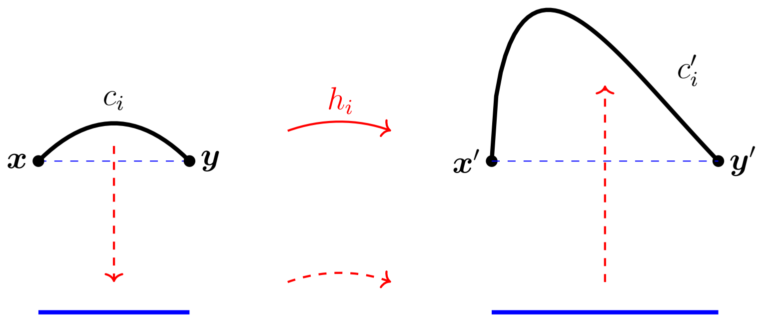


Figure 6.14: Each homeomorphism h_i is a composition of projections and rescaling. One can choose the orientation in the middle step to make sure that if $\boldsymbol{x}'$ is the point in τ which corresponds under the graph isomorphism to $\boldsymbol{x} \in \sigma$ then h_i maps $\boldsymbol{x}$ to $\boldsymbol{x}'$.

It follows that the map $h : \sigma \to \tau$ determined by $h|c_i = h_i$ is well-defined, and is a homeomorphism from σ to τ.

For $f : \sigma \to \mathbb{C}$, let $\Phi(f) = f \circ h^{-1}$ be the corresponding function defined on τ. As we have seen, Φ is an Banach algebra isomorphism from $BV(\sigma)$ to $BV(\tau)$. By Corollary 6.8.2,

$$\begin{aligned} f \in AC(\sigma) &\iff f|c_i \in AC(c_i) \text{ for all } i \\ &\iff \Phi(f)|c_i' \in AC(c_i') \text{ for all } i \\ &\iff \Phi(f) \in AC(\tau). \end{aligned}$$

Thus, $AC(\sigma)$ is isomorphic to $AC(\tau)$. □

6.10 A final remark

Most of the results in the thesis concern families of connected compact subsets of the plane. In this section we discuss how to deal with finite disjoint unions of such sets. The main step is proving Proposition 6.10.2. First we need an extension lemma which is essentially given in the unpublished note [DL1] (see Lemma 5.2).

Lemma 6.10.1. *Suppose that $\sigma_1, \sigma_2 \subseteq \mathbb{C}$ is are nonempty compact sets with $\operatorname{Re} \boldsymbol{x} < 0$ for all $\boldsymbol{x} \in \sigma_1$ and $\operatorname{Re} \boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \sigma_2$. Let $\sigma = \sigma_1 \cup \sigma_2$. Suppose that $g : \sigma \to \mathbb{C}$ and that $g|\sigma_2 \equiv 0$. Then*

1. $\|g\|_{BV(\sigma)} \le 2\,\|g\|_{BV(\sigma_1)}$.
2. *If $g|\sigma_1 \in AC(\sigma_1)$ then $g \in AC(\sigma)$.*

It is worth remarking that the separation of the two components is vital here. One cannot in general extend an absolutely continuous function on a compact set σ_1 to an absolutely continuous function on a superset by setting it to be zero off σ_1.

Proof. 1) Let $S = [\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_n]$ be an ordered list of elements in σ. Partition the indices $1, \dots, n$ into disjoint sets

$$\begin{aligned}
J_1 &= \{j \,:\, \boldsymbol{x}_{j-1}, \boldsymbol{x}_j \in \sigma_1\}, \\
J_2 &= \{j \,:\, \boldsymbol{x}_{j-1}, \boldsymbol{x}_j \in \sigma_2\}, \\
J_3 &= \{1, \dots, n\} \setminus (J_1 \cup J_2).
\end{aligned}$$

Note that $\mathrm{vf}(S)$ must be at least as large as the number of elements in J_3. Then, noting that g is identically zero on σ_2, and taking empty sums to be zero,

$$\begin{aligned}
\sum_{j=1}^{n} |g(\boldsymbol{x}_j) - g(\boldsymbol{x}_{j-1})| &= \sum_{i=1}^{3} \sum_{j \in J_1} |g(\boldsymbol{x}_j) - g(\boldsymbol{x}_{j-1})| \\
&\le \sum_{j \in J_1} |g(\boldsymbol{x}_j) - g(\boldsymbol{x}_{j-1})| + |J_3| \, \|g\|_\infty .
\end{aligned}$$

Now

$$\frac{|J_3| \, \|g\|_\infty}{\mathrm{vf}(S)} \le \|g\|_\infty = \|g\|_{\sigma_1,\infty} .$$

If $J_1 \ne \emptyset$, let $S_1 = [\boldsymbol{x}_{j_0}, \dots, \boldsymbol{x}_{j_\ell}]$ be the sublist of S containing all the x_j such that $x_j \in \sigma_1$ and at least one of x_{j-1} or x_{j+1} also lie in σ_1. Since omitting points from a list can only decrease the variation, $\mathrm{vf}(S_1) \le \mathrm{vf}(S)$. Thus

$$\begin{aligned}
\frac{\sum_{j \in J_1} |g(\boldsymbol{x}_j) - g(\boldsymbol{x}_{j-1})|}{\mathrm{vf}(S)} &\le \frac{\sum_{i=1}^{\ell} |g(\boldsymbol{x}_{j_i}) - g(\boldsymbol{x}_{j_{i-1}})|}{\mathrm{vf}(S_1)} \\
&\le \mathrm{var}(g, \sigma_1).
\end{aligned}$$

Thus (whether $J_1 = \emptyset$ or not)

$$\frac{\mathrm{cvar}(g, S)}{\mathrm{vf}(S)} \le \mathrm{var}(g, \sigma_1) + \|g\|_{\sigma_1,\infty}$$

and hence $\|g\|_{BV(\sigma)} \le 2 \, \|g\|_{BV(\sigma_1)}$.

2) Fix $\epsilon > 0$. Since $g|\sigma_1 \in AC(\sigma_1)$ there exists a polynomial $p \in \mathcal{P}_2$ such that $\|g - p\|_{BV(\sigma_1)} < \frac{\epsilon}{2}$. Let χ denote the characteristic function of the left half-plane, restricted to σ. Then $\chi \in AC(\sigma)$ and so $\hat{p} = \chi p \in AC(\sigma)$ too. By (1)

$$\|g - \hat{p}\|_{BV(\sigma)} \leq 2\,\|g - p\|_{BV(\sigma_1)} < \epsilon.$$

Since $AC(\sigma)$ is closed this is enough to ensure that $g \in AC(\sigma)$. □

The factor 2 in (1) is necessary, since if g is the characteristic function of σ_1 then $\|g\|_{BV(\sigma_1)} = 1$ while $\|g\|_{BV(\sigma)} = 2$.

Recall that if $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras then $\mathcal{A} \oplus \mathcal{B}$ is a Banach algebra under componentwise operations and the norm $\|(a,b)\| = \max\{\|a\|_{\mathcal{A}}, \|b\|_{\mathcal{B}}\}$.

Proposition 6.10.2. *Suppose that P and Q are disjoint polygons and that $\sigma \subseteq P \cup Q$ is a compact set such that $\sigma_P = \sigma \cap P$ and $\sigma_Q = \sigma \cap Q$ are both nonempty (and necessarily compact). Then $AC(\sigma)$ is isomorphic to $AC(\sigma_P) \oplus AC(\sigma_Q)$.*

Proof. For $f : \sigma \to \mathbb{C}$, let $J(f) = (f|\sigma_P, f|\sigma_Q)$. By the general restriction theorems, if $f \in AC(\sigma)$ then $J(f) \in AC(\sigma_P) \oplus AC(\sigma_Q)$. Indeed J is a norm 1 Banach algebra homomorphism. To complete the proof we need to show that J is onto and that it has a continuous inverse.

Suppose then that $f_P \in AC(\sigma_P)$. Define $\hat{f}_P : \sigma \to \mathbb{C}$ by

$$\hat{f}_P(\boldsymbol{x}) = \begin{cases} f_P(\boldsymbol{x}), & \text{if } \boldsymbol{x} \in \sigma_P, \\ 0, & \text{if } \boldsymbol{x} \in \sigma_Q. \end{cases}$$

Let S be a large polygon which includes both P and Q in its interior. Following the algorithm in Section 7 of [DL2], there exists a finite sequence of locally piecewise affine maps whose composition h is a homeomorphism of the plane which maps S to a triangle and P and Q to triangles with disjoint projections on the real axis. By shifting if necessary, we may assume that $h(\sigma_P)$ lies in the open left half-plane, and $h(\sigma_Q)$ lies in the open right half-plane.

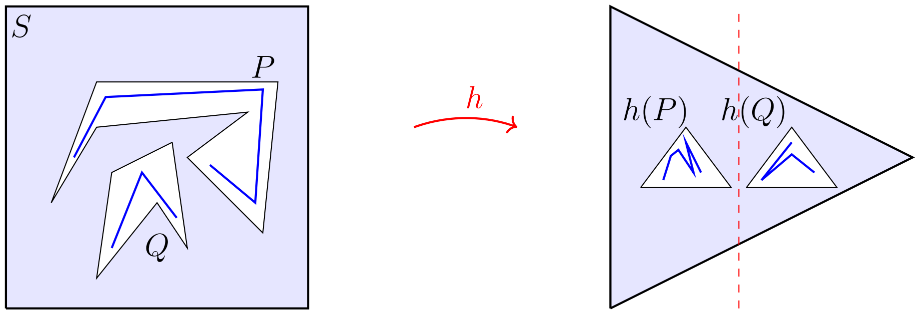


Figure 6.15: Transforming polygons to triangles via a sequence of locally piecewise affine maps.

Since h is a composition of locally piecewise affine maps, it generates an isomorphism $\Phi : AC(\sigma) \to AC(h(\sigma))$ via $\Phi(f) = f \circ h^{-1}$ (which of course restricts to an isomorphism from $AC(\sigma_P)$ to $AC(h(\sigma_P))$).

Define $g : h(\sigma) \to \mathbb{C}$ by $g = \hat{f}_P \circ h^{-1}$. Then certainly $g|h(\sigma_P) \in AC(\sigma_P)$ and $g|h(\sigma_Q) \equiv 0$, so by the lemma, $g \in AC(h(\sigma))$. Thus $\hat{f}_P = \Phi^{-1}(g) \in AC(\sigma)$. Furthermore, if we let $K_h = 2\,\|\Phi\|\,\|\Phi^{-1}\|$ then, on composing the linear maps we see that $\|\hat{f}_P\|_{BV(\sigma)} \leq K_h\,\|f_P\|_{BV(\sigma_P)}$.

Suppose now that $(f_P, f_Q) \in AC(\sigma_P) \oplus AC(\sigma_Q)$. It follows from the above that there exist $\hat{f}_P, \hat{f}_Q \in AC(\sigma)$ such that $f = \hat{f}_P + \hat{f}_Q$ restricts to f_P on σ_P and f_Q on σ_Q. That is $J(f) = (f_P, f_Q)$, and so J is onto. Finally

$$\begin{aligned}
\|f\|_{BV(\sigma)} &\leq \|\hat{f}_P\|_{BV(\sigma)} + \|\hat{f}_Q\|_{BV(\sigma)} \\
&\leq K_h\,\|f_P\|_{BV(\sigma_P)} + K_h\,\|f_Q\|_{BV(\sigma_Q)} \\
&\leq 2K_h\,\|(f_P, f_Q)\|_{AC(\sigma_1)\oplus AC(\sigma_2)}
\end{aligned}$$

and so J^{-1} is continuous. Thus J is a Banach algebra isomorphism from $AC(\sigma)$ to $AC(\sigma_1) \oplus AC(\sigma_2)$. □

Obviously, by induction, one can extend this result to deal with sets which sit inside any finite number of polygons.

Let UPIC denote the family of compact subsets of the plane which are finite unions of pairwise disjoint sets $\sigma_m \in \mathrm{PIC}$, $m = 1, \dots, M$.

Theorem 6.10.3. *Suppose that $\sigma, \tau \in \mathrm{UPIC}$. Then $AC(\sigma)$ is isomorphic to $AC(\tau)$ if and only if σ is homeomorphic to τ.*

Proof. Suppose that $\sigma, \tau \in \mathrm{UPIC}$ are homeomorphic. Then they must have the same number of connected components, say $\sigma = \cup_{m=1}^{M} \sigma_m$ and $\tau = \cup_{m=1}^{M} \tau_m$. Furthermore, theses sets can be ordered so that, for each m, σ_m is homeomorphic to τ_m, and hence $AC(\sigma_m)$ is isomorphic to $AC(\tau_m)$.

Since these subsets are all compact, one can find disjoint polygons $P_1, \ldots, P_M$ so that σ_m lies in the interior of P_m. Hence by the last proposition

$$AC(\sigma) \simeq \bigoplus_{m=1}^{M} AC(\sigma_m) \simeq \bigoplus_{m=1}^{M} AC(\tau_m) \simeq AC(\tau).$$

□

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