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APPROXIMATION OF THE BOUNDARY IN GALERKIN BOUNDARY ELEMENT METHODS

by

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DECLARATION

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of a university or other institute of higher learning, except where due acknowledgement is made in the text.

YI ZENG

ABSTRACT

The Dirichlet boundary value problem is reformulated as a first kind integral equation on the boundary by means of single layer and double layer potentials. A Galerkin boundary element method is used to solve the integral equation numerically. The aim of the thesis is to study the errors associated with approximation of the boundary.

If the trial functions are piecewise polynomials of degree at most $(r-1)$, and if an exact parametric representation of the boundary is used, then the Galerkin error in the energy norm is $O\left(h^{r+1/2}\right)$. In 1977, Le Roux showed that this rate of convergence is preserved when continuous piecewise polynomial interpolation of degree $(p-1)$ is used to approximate the boundary, provided $p \geq r+1$. We extend this result by allowing a very general class of boundary approximations accurate "to order p", that include, for instance, piecewise rational approximations.

Under appropriate conditions, the convergence is of a higher order when the error is measured in a more negative norm, resulting in a better error bound for the error in the potential. In the best case, this super-convergence effect leads to $O\left(h^{2r+1}\right)$ accuracy if an exact parametric representation of the boundary is used. We consider the super-convergence property for an approximation to the boundary of order p, and we show that the $O\left(h^{2r+1}\right)$ rate of convergence is maintained, provided $p \geq 2r+1$.

We confirm the theoretical results in numerical experiments with piecewise constant $(r=1)$ trial functions, and with piecewise linear $(p=2)$ or piecewise quadratic $(p=3)$ approximation to the boundary. We then carry out numerical studies of some problems in which the boundary has a corner. A mesh grading is used to refine around the corner, and a singularity subtraction method is used to weaken the singularity in the kernel of the double layer potential operator occurring on the right-hand side.

Contents

Chapter 1

INTRODUCTION

1.1 Brief Review

The theory and application of integral equations is an important subject within applied mathematics. Integral equations are used as mathematical models for many and varied physical and engineering situations, and also occur as reformulations of other mathematical problems. A number of boundary value problems traditionally cast as partial differential equations can be reformulated as boundary integral equations. Boundary element methods are a class of numerical techniques for solving such integral equations. The literature is copious, such as Fichera [19], Hsiao and MacCamy [20], Jaswon [27], Jaswon and Symm [28] and some others in recent years [3, 7, 8, 9, 30, 33, 43, 45, 49, 50].

Many natural problems arising from physics and engineering can be mathematically described by Laplace's equation which is a basic model linear elliptic equation. The Dirichlet boundary value problem of the Laplace equation is defined by

$$\Delta\phi = 0 \qquad \text{in } \Omega, \tag{1.1.1}$$

with the boundary condition

$$\phi = g \qquad \text{on } \Gamma, \tag{1.1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with boundary Γ. The solution ϕ is often some kind of potential function.

This Dirichlet boundary value problem can be reduced to a first kind integral equation on the boundary by means of single layer and double layer potentials. In the literature, the two standard approaches are known as the "direct" and "indirect" methods. In the direct method, the solution of the integral equation has a direct interpretation in terms of ϕ, namely, the solution of the integral equation is the normal derivative of ϕ on the boundary. This approach is based on Green's theorem and yields an integral representation of ϕ in terms of both a single and a double layer potential. In the indirect method, the solution of the boundary integral equation is the 'single-layer density' as one seeks a representation of ϕ in the form of a single layer potential alone. Details of mathematical formulations of boundary integral equations for several of the most important linear elliptic boundary value problems can be found by Jaswon and Symm [28], Kress [30] and Chen and Zhou [10].

Boundary element methods are various methods of discretization of the boundary integral equations by finite elements on the boundary. Their main advantages are that they reduce the computational dimension by one and give a simple discretization of exterior problems. For the numerical analysis of boundary integral methods and certain recent developments, we refer to, for example, [30, 38, 37, 39, 40, 46]. Also [43, 49] give an overview of the theory of strong ellipticity for pseudo-differential operators as it applies to boundary integral equations. The survey of boundary integral equation methods [3] lays particular stress on three dimensional problems.

A variety of numerical methods has been devised to find numerical solutions of the boundary integral equations, such as the Galerkin method, the collocation method and the qualocation method plus some of their modifications. Most theoretical treatments of the boundary element method give great attention to the Galerkin method. Very complete discussions of the Galerkin method exist, see, for example, [21, 23, 39, 40, 44, 46, 48, 49]. The Galerkin method is the only one which is in a reasonably satisfactory condition for a wide class of boundary integral equations. The convergence of the Galerkin scheme is valid for a large class of boundary pseudo-differential equations in any space dimension. Many papers discuss this method for the first-kind boundary integral equations, such as [22, 24, 25, 41, 42].

In the computation of an approximate solution of a boundary integral equation, it is often convenient to approximate the boundary of the domain, for instance using an approximate parametric representation based on curved finite elements. In 1976, Nedelec first analysed a construction of an approximate boundary Γ in $\mathbb{R}^3$, see [36], with the help of curvilinear triangles. Around the same time, Le Roux in [32] used arcs of polynomial curves to interpolate boundaries in $\mathbb{R}^2$. The paper [4], given by Atkinson in 1985, proposed a framework for the analysis of collocation methods using quadratic isoparametric interpolation for second kind integral equations in three dimensions. The main purpose of this thesis is to extend the theoretical results of [32] by allowing more general types of boundary approximation and by considering super-convergence. We remark that our technical formulation of the boundary approximation differs from the one in [32] in the following sense. In [32], the approximate solution of the integral equation is defined on the approximate boundary and is then mapped onto the exact boundary. In the approach we use in this thesis, the approximate solution is thought of as being defined directly on the real boundary. (See section 2.3).

Let Γ be the smooth boundary of a domain in $\mathbb{R}^2$. We consider Symm's first kind integral equation

$$\frac{1}{2\pi}\int_\Gamma \log\frac{b}{|\mathbf{x}-\mathbf{y}|}\,u(\mathbf{y})\,d\sigma_{\mathbf{y}} = f(\mathbf{x}), \qquad \mathbf{x}\in\Gamma, \tag{1.1.3}$$

where $d\sigma_{\mathbf{y}}$ is the element of arc length on Γ, and b is a constant. The boundary integral equation (1.1.3) is equivalent to the Dirichlet problem (1.1.1) and (1.1.2). For the indirect method, $f = g$; for the direct method, $f = -\frac{1}{2}\,g + Tg$, where T is a certain integral operator (see 2.1.8). We let u_h be an approximate solution generated by a Galerkin method and u_h^* be an approximation of u_h involving the normal projection onto the exact boundary; see (3.3.9).

We show that if the curved boundary is approximated "to order p", and if the boundary element spaces are spaces of piecewise polynomials of order r (i.e., of degree $\leq r-1$), then

$$\|u-u_h^*\|_{H^{-1/2}(\Gamma)} \leq C\left(\|f_h - R_h f\|_{H^{1/2}(\Gamma)} + h^{p-1/2}\,\|u\|_{H^0(\Gamma)} + h^{r+1/2}\,\|u\|_{H^r(\Gamma)}\right),$$

where $\|\cdot\|_{H^s(\Gamma)}$ is the norm in the Sobolev space $H^s(\Gamma)$, f_h is an appropriate approximation to f, and the linear operator R_h depends on the way Γ is approximated, see (4.1.1).

Thus, if $p = r+1$ and provided $\|f_h - R_h f\|_{H^{1/2}(\Gamma)} = O\left(h^{p-1/2}\right)$, then

$$\|u-u_h^*\|_{H^{-1/2}(\Gamma)} = O\left(h^{r+1/2}\right),$$

and as a consequence

$$\|u-u_h\|_{H^0(\Gamma)} = O\left(h^r\right),$$

a result similar to that of Le Roux [32].

Under appropriate conditions, the convergence is of a higher order when the error is measured in a more negative norm, resulting in a better error bound for the error of the potential, i.e., the method has a super-convergence property. In fact,

$$\begin{aligned}\|u-u_h^*\|_{H^{-r-1}(\Gamma)} &\leq Ch^{r+1/2}\|f_h - R_h f\|_{H^{1/2}(\Gamma)} + Ch^{r+p}\|u\|_{H^0(\Gamma)} \\ &\quad + Ch^{2r+1}\|u\|_{H^r(\Gamma)} + Ch^p\|u\|_{H^1(\Gamma)} + \|f_h - R_h f\|_{H^0(\Gamma)},\end{aligned}$$

and as a consequence, for the solution of the Dirichlet problem (1.1.1) and (1.1.2) we can achieve

$$\phi(\mathbf{z}) - \phi_h(\mathbf{z}) = O\left(h^{\min(2r+1,p)}\right), \qquad \mathbf{z} \in \Omega,$$

where the approximate potential ϕ_h is computed using u_h in place of u in the integral representation for ϕ.

Boundary integral equations for domains with corners or edges, and methods for their approximate solution, have attracted more and more attention in recent years both in the theoretical and practical aspects. Many mathematicians have obtained satisfactory numerical results for problems on non-smooth boundaries using different boundary element methods. Costabel and Stephan [12] established error estimates for the Galerkin approximation of boundary integral equations on a polygon; Costabel and Stephan [13] and Elschner and Graham [15] discuss collocation methods for boundary integral equations on polygons; Elschner and Stephan [16] and Kress [31] are devoted to the boundary integral equation on curves with corners by using a discrete collocation and a Nyström method respectively; Elschner and Graham [18] treat quadrature methods; Elschner, Prössdorf and Sloan [17] deal with a qualocation method on the non-smooth boundary; and Stephan and Wendland [45] is concerned with mixed boundary value problems.

If the boundary is not smooth, then singularities in the solution u will generally be produced at the corners so that the rate of convergence will be degraded when a boundary element method is applied with a uniform mesh. In order to maintain the order of convergence, the mesh should be refined around the corners. These meshes are called graded meshes. Note that when the direct method is used in the formulation of the boundary integral equation, fixed singularities arise in the kernel of the double layer potential operator T which appears in the right-hand side of the integral equation. In this thesis, we address this problem in our numerical experiments. We use a singularity subtraction method that weakens the singularities in the kernel of T, making the integrals easier to deal with.

1.2 Outline of the Thesis

This thesis is organised as follows.

In Chapter 2, we give an introduction to the direct boundary integral method for the Dirichlet problem. We state existence and uniqueness results, and review the classical error estimates of the Galerkin method, as well as describing in some detail its practical implementation.

In Chapter 3, we introduce a general approximation scheme for curved boundaries that includes as special cases piecewise polynomial or piecewise rational approximations. Under the boundary approximation, we study the error between the bilinear form in the Galerkin equation and the approximate bilinear form in the perturbed Galerkin equation, and show that this error is $O(h^{p-1})$. After that, we adopt the approach in [32] to show a sharper bound of $O(h^p)$ for a related quantity.

The main concerns in Chapter 4 are the theoretical analysis of the stability property for the perturbed Galerkin method, the error estimates of the perturbed Galerkin method in both the $H^{-1/2}$ and H^{-r-1} norms.

Chapter 5 discusses the implementation of the Galerkin method in more detail and presents numerical results. Firstly we consider an integral equation with a smooth boundary. Next we try an integral equation with a non-smooth boundary. We consider the case when the boundary data g is the restriction to Γ of a smooth function on $\mathbb{R}^2$, as well as the case when g is singular. For each problem, we compare the results for the exact boundary with those for piecewise linear and piecewise quadratic approximation to the boundary. We always use piecewise constant trial functions in the Galerkin procedure.

Finally, there is an appendix containing formulas for some integrals used in evaluating the stiffness matrix in Chapter 5.

Chapter 2

BOUNDARY ELEMENT METHOD

2.1 Boundary Integral Equations

The solution of boundary value problems for many linear partial differential equations can be reduced to the application of boundary integral equations.

The reformulation of elliptic boundary value problems as boundary integral equation has been discussed by many mathematicians, such as [27, 28] for potential theory, [29] for elasticity, [11] for the Helmholtz equation and [20, 26] concentrating on formulations of integral equation of the first kind. The classical mathematical formulations are studied thoroughly in [35].

In the classical method of I. Fredholm, layer potentials are used to reformulate the Dirichlet and Neumann problems for the Laplace equation as Fredholm integral equations of the second kind over the boundary. We will now give alternative reformulations of the Dirichlet problem which lead instead to Fredholm integral equations of the first kind over the boundary. This gives an emphasis to the relationship between the Dirichlet bilinear form associated with the Laplace operator and the bilinear form associated with the boundary integral operator.

Proofs of Theorems 2.1.1 - 2.1.5 can be found in the paper of Hsiao and Wendland [21] for smooth curves, or Costabel [14] for Lipschitz curves.

Let $\Omega^+ \in \mathbb{R}^n$ ($n = 2$ or 3) be a bounded Lipschitz domain with boundary Γ, and let Ω^- be the complement of $\Omega^+ \dot{\cup} \Gamma$ in $\mathbb{R}^n$, so that

$$\mathbb{R}^n = \Omega^+ \dot{\cup} \Gamma \dot{\cup} \Omega^- \quad \text{and} \qquad \partial\Omega^+ = \Gamma = \partial\Omega^-,$$

where the dot over $\cup$ indicates a disjoint union. Denote by ν the unit inward normal to Ω^+, let $d\sigma_{\mathrm{y}}$ denote the element of arc length or the surface element on Γ. If ϕ is a function defined on $\Omega^+ \dot{\cup} \Omega^-$, then $\phi^\pm$ denotes the one-sided trace of ϕ on Γ from $\Omega^\pm$.

All the Sobolev spaces involved in the following will be defined in Section 2.2.

Let Δ be the Laplacian operator, and suppose $\phi \in H^1(\Omega^+)$ is the weak solution of the Dirichlet problem

$$\Delta\phi = 0 \qquad \text{in } \Omega^+, \tag{2.1.1}$$

$$\phi^+ = g \qquad \text{on } \Gamma, \tag{2.1.2}$$

i.e., ϕ satisfies the essential boundary condition (2.1.2) in the sense of traces, and

$$\int_{\Omega^+} \nabla\phi \cdot \nabla\psi \, d\mathbf{x} = 0 \qquad \forall \psi \in H^1(\Omega^+)$$

with $\psi^+ = 0$ on Γ.

There is a standard method of reformulating the Dirichlet problem as a boundary integral equation of the form

$$Au = f \qquad \text{on } \Gamma. \tag{2.1.3}$$

For all $\mathbf{z} \in \mathbb{R}^n \setminus \{0\}$, let $K(\mathbf{z})$ be the fundamental solution of the Laplace equation, i.e.,

$$K(\mathbf{z}) = \frac{1}{2\pi} \log \frac{b}{|\mathbf{z}|} \qquad \text{if } n = 2,$$

or

$$K(\mathbf{z}) = \frac{1}{4\pi} \frac{1}{|\mathbf{z}|} \qquad \text{if } n = 3,$$

where b is an arbitrary constant. Observe that K is harmonic in $\mathbb{R}^n \setminus \{0\}$.

Remark: The scaling parameter b contained in the logarithmic kernel $K(\mathbf{z})$ should be chosen larger than $\mathrm{cap}(\Gamma)$ which is the logarithmic capacity (or transfinite diameter) of the boundary Γ. The characterisation of logarithmic capacity is:

$$\log\frac{1}{\operatorname{cap}(\Gamma)} = \min_{u\in H^{-1/2}(\Gamma)} \int_\Gamma\int_\Gamma \log\frac{1}{|\mathbf{x}-\mathbf{y}|}u(\mathbf{y})u(\mathbf{x})d\sigma_{\mathbf{x}}d\sigma_{\mathbf{y}}. \tag{2.1.4}$$

See [34], [41] or [47] for details of the role $\operatorname{cap}(\Gamma)$ plays.

The direct boundary integral formulation of the Dirichlet problem begins with the following representation formula (third Green identity), for a proof, see Costabel [14].

Theorem 2.1.1 *Let $g \in H^{1/2}(\Gamma)$. If $\phi \in H^1(\Omega^+)$ is the weak solution of* (2.1.1) *and* (2.1.2)*, then $\partial\phi/\partial\nu \in H^{-1/2}(\Gamma)$ and*

$$\phi(\mathbf{x}) = \int_\Gamma g(\mathbf{y})\frac{\partial}{\partial\nu_{\mathbf{y}}}K(\mathbf{x}-\mathbf{y})\,d\sigma_{\mathbf{y}} - \int_\Gamma \frac{\partial\phi}{\partial\nu}(\mathbf{y})\,K(\mathbf{x}-\mathbf{y})\,d\sigma_{\mathbf{y}}, \qquad \mathbf{x}\in\Omega^+,$$

where $\nu_{\mathbf{y}}$ is a unit inward normal to Ω^+ at the point $\mathbf{y}\in\Gamma$.

Given a function u defined on Γ, the single layer potential $\mathcal{V}u$ and the double layer potential $\mathcal{W}u$ are defined by

$$\mathcal{V}u(\mathbf{z}) \equiv \int_\Gamma K(\mathbf{z}-\mathbf{y})\,u(\mathbf{y})\,d\sigma_{\mathbf{y}}, \qquad \mathbf{z}\in\Omega^+\,\dot\cup\,\Omega^-, \tag{2.1.5}$$

$$\mathcal{W}u(\mathbf{z}) \equiv \int_\Gamma \frac{\partial}{\partial\nu_{\mathbf{y}}}K(\mathbf{z}-\mathbf{y})\,u(\mathbf{y})\,d\sigma_{\mathbf{y}}, \qquad \mathbf{z}\in\Omega^+\,\dot\cup\,\Omega^-. \tag{2.1.6}$$

Notice that

$$\frac{\partial}{\partial\nu_{\mathbf{y}}}K(\mathbf{z}-\mathbf{y}) = \frac{1}{2\pi}\frac{\nu(\mathbf{y})\cdot(\mathbf{z}-\mathbf{y})}{|\mathbf{z}-\mathbf{y}|^2} \qquad \text{if } n=2,$$

or

$$\frac{\partial}{\partial\nu_{\mathbf{y}}}K(\mathbf{z}-\mathbf{y}) = \frac{1}{4\pi}\frac{\nu(\mathbf{y})\cdot(\mathbf{z}-\mathbf{y})}{|\mathbf{z}-\mathbf{y}|^3} \qquad \text{if } n=3,$$

where $|\mathbf{z}-\mathbf{y}|$ is the Euclidean distance between $\mathbf{z}$ and $\mathbf{y}$.

We then define the linear operators A and T by

$$Au \equiv (\mathcal{V}u)^+ = (\mathcal{V}u)^-, \tag{2.1.7}$$

$$Tu \equiv (\mathcal{W}u)^+ + (\mathcal{W}u)^-. \tag{2.1.8}$$

It is not difficult to show that the single layer boundary integral operator A is given by

$$(Au)(\mathbf{x}) \equiv \int_{\Gamma} K(\mathbf{x}-\mathbf{y})\, u(\mathbf{y})\, d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Gamma. \tag{2.1.9}$$

Letting $u = 1$, from Theorem 2.1.1 we have

$$\mathcal{W}1 = \begin{cases} 1 & \text{on } \Omega^+ \\ 0 & \text{on } \Omega^-, \end{cases} \tag{2.1.10}$$

so

$$(\mathcal{W}1)^+ = 1, \qquad (\mathcal{W}1)^- = 0,$$

and from (2.1.8),

$$T1 = 1. \tag{2.1.11}$$

For a continuous function u defined on Γ, it can be shown that

$$\begin{aligned} &\lim_{\substack{\mathbf{z} \to \mathbf{x} \\ \mathbf{z} \in \Omega^{\pm}}} \int_{\Gamma} [u(\mathbf{y}) - u(\mathbf{x})]\, \frac{\partial}{\partial \nu_{\mathbf{y}}} K(\mathbf{z}-\mathbf{y})\, d\sigma_{\mathbf{y}} \\ &= \int_{\Gamma} [u(\mathbf{y}) - u(\mathbf{x})]\, \frac{\partial}{\partial \nu_{\mathbf{y}}} K(\mathbf{x}-\mathbf{y})\, d\sigma_{\mathbf{y}}, \qquad x \in \Gamma, \end{aligned} \tag{2.1.12}$$

and by (2.1.10),

$$\mathcal{W}u(\mathbf{z}) = \begin{cases} u(\mathbf{x}) + \int_{\Gamma} [u(\mathbf{y}) - u(\mathbf{x})]\, \frac{\partial}{\partial \nu_{\mathbf{y}}} K(\mathbf{z}-\mathbf{y})\, d\sigma_{\mathbf{y}}, & \mathbf{z} \in \Omega^+, \\ \int_{\Gamma} [u(\mathbf{y}) - u(\mathbf{x})]\, \frac{\partial}{\partial \nu_{\mathbf{y}}} K(\mathbf{z}-\mathbf{y})\, d\sigma_{\mathbf{y}}, & \mathbf{z} \in \Omega^-. \end{cases} \tag{2.1.13}$$

Therefore, we have

$$(\mathcal{W}u)^+(\mathbf{x}) = u(\mathbf{x}) + \int_{\Gamma} [u(\mathbf{y}) - u(\mathbf{x})]\, \frac{\partial}{\partial \nu_{\mathbf{y}}} K(\mathbf{x}-\mathbf{y})\, d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Gamma, \tag{2.1.14}$$

$$(\mathcal{W}u)^-(\mathbf{x}) = \int_{\Gamma} [u(\mathbf{y}) - u(\mathbf{x})]\, \frac{\partial}{\partial \nu_{\mathbf{y}}} K(\mathbf{x}-\mathbf{y})\, d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Gamma. \tag{2.1.15}$$

By the definition of the linear operator T in (2.1.8), the double layer boundary integral operator T can be written as

$$Tu(\mathbf{x}) = u(\mathbf{x}) + 2\int_\Gamma [u(\mathbf{y}) - u(\mathbf{x})] \frac{\partial}{\partial \nu_\mathbf{y}} K(\mathbf{x}-\mathbf{y})\, d\sigma_\mathbf{y}. \tag{2.1.16}$$

Moreover, we have

$$\int_\Gamma \frac{\partial}{\partial \nu_\mathbf{y}} K(\mathbf{x}-\mathbf{y})\, d\sigma_\mathbf{y} = \begin{cases} 1 & \text{if } \mathbf{x} \in \Omega^+, \\ \frac{1}{2} & \text{if } \mathbf{x} \in \Gamma \text{ (and } \Gamma \text{ is smooth)}, \\ 0 & \text{if } \mathbf{x} \in \Omega^-. \end{cases} \tag{2.1.17}$$

Thus, if Γ is smooth in a neighbourhood of $\mathbf{x}$ then $(Tu)(\mathbf{x})$ can be also written as

$$(Tu)(\mathbf{x}) = 2\int_\Gamma \frac{\partial}{\partial \nu_\mathbf{y}} K(\mathbf{x}-\mathbf{y})\, u(\mathbf{y})\, d\sigma_\mathbf{y}, \qquad \mathbf{x} \in \Gamma.$$

With the above definitions of the single and double layer potentials, Costabel [14] shows that the following mapping properties hold, even for a general Lipschitz domain Ω^+.

Theorem 2.1.2 *The linear operators*

$$\begin{aligned} \mathcal{V}:\ & H^{-1/2}(\Gamma) \longrightarrow H^1(\Omega^+), \\ \mathcal{W}:\ & H^{1/2}(\Gamma) \longrightarrow H^1(\Omega^+), \\ A:\ & H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \\ T:\ & H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma) \end{aligned}$$

are continuous and bounded.

We denote the normal derivatives of a function ϕ by

$$\phi_\nu^\pm \equiv \nu \cdot (\nabla \phi)^\pm \equiv \left(\frac{\partial \phi}{\partial \nu}\right)^\pm.$$

The single and the double layer potentials have following main properties.

Theorem 2.1.3 *Let $\mathcal{V}u$ and $\mathcal{W}u$ be the single and double layer potentials, respectively, with density u.*

i) *If $u \in H^{-\frac{1}{2}}(\Gamma)$, then $\mathcal{V}u$ satisfies*

$$(\mathcal{V}u)^+ - (\mathcal{V}u)^- = 0, \tag{2.1.18}$$

$$(\mathcal{V}u)^+_\nu - (\mathcal{V}u)^-_\nu = -u. \tag{2.1.19}$$

ii) *If $u \in H^{\frac{1}{2}}(\Gamma)$, then $\mathcal{W}u$ satisfies*

$$(\mathcal{W}u)^+_\nu - (\mathcal{W}u)^-_\nu = 0, \tag{2.1.20}$$

$$(\mathcal{W}u)^+ - (\mathcal{W}u)^- = u. \tag{2.1.21}$$

The boundary integral equation (2.1.3) follows at once from the results in Theorem 2.1.3.

Theorem 2.1.4 *Let $g \in H^{\frac{1}{2}}(\Gamma)$ and $f = -\frac{1}{2}g + \frac{1}{2}Tg$.*

i) *If $\phi \in H^1(\Omega^+)$ is the weak solution of* (2.1.1) *and* (2.1.2)*, then the normal derivative $\phi^+_\nu \in H^{-\frac{1}{2}}(\Gamma)$ is a solution of the boundary integral equation*

$$A\phi^+_\nu = f \qquad \text{on } \Gamma.$$

ii) *Conversely, if $u \in H^{-\frac{1}{2}}(\Gamma)$ is a solution of $Au = f$, then $\phi = \mathcal{W}g - \mathcal{V}u$ is a weak solution of* (2.1.1) *and* (2.1.2)*, and $u = \phi^+_\nu$.*

The solution technique based on Theorem 2.1.4 is called the direct boundary integral method. Written out in full, the boundary integral equation $Au = f$ is

$$\int_\Gamma K(\mathbf{x}-\mathbf{y})u(\mathbf{y})\,d\sigma_{\mathbf{y}} = \int_\Gamma [g(\mathbf{y}) - g(\mathbf{x})]\,\frac{\partial}{\partial \nu_{\mathbf{y}}}K(\mathbf{x}-\mathbf{y})\,d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Gamma, \tag{2.1.22}$$

or

$$\int_\Gamma K(\mathbf{x}-\mathbf{y})u(\mathbf{y})\,d\sigma_\mathbf{y} = -\frac{1}{2}\,g(\mathbf{x}) + \int_\Gamma g(\mathbf{y})\,\frac{\partial}{\partial \nu_\mathbf{y}} K(\mathbf{x}-\mathbf{y})\,d\sigma_\mathbf{y}, \qquad \mathbf{x}\in\Gamma. \quad (2.1.23)$$

We now define the bilinear form associated with the boundary integral operator A,

$$a(u,v) = \langle Au, v\rangle \;=\; \int_\Gamma\int_\Gamma K(\mathbf{x}-\mathbf{y})\,u(\mathbf{y})\,v(\mathbf{x})\,d\sigma_\mathbf{x}\,d\sigma_\mathbf{y} \qquad (2.1.24)$$

so that the weak form of the integral equation (2.1.3) is, if $f \in H^{1/2}(\Gamma)$,

$$a(u,v) = \langle f, v\rangle \qquad \text{for } v \in H^{-1/2}(\Gamma), \qquad (2.1.25)$$

where the inner product is defined as

$$\langle v, w\rangle \;=\; \langle v, w\rangle_0 \;=\; \int_\Gamma v\,w\,d\sigma_\mathbf{x}.$$

Recall from (2.1.4) that cap(Γ) denote the logarithmic capacity of Γ.

Theorem 2.1.5 *The formula* (2.1.24) *defines a bounded bilinear form*

$$a : H^{-1/2}(\Gamma)\times H^{-1/2}(\Gamma) \to \mathbb{R},$$

i.e., there is a constant C such that

$$|a(u,v)| \le C\|u\|_{H^{-1/2}(\Gamma)}\|v\|_{H^{-1/2}(\Gamma)} \qquad \forall u,\, v \in H^{-1/2}(\Gamma).$$

Furthermore, if $b >$cap(Γ), then a is $H^{-1/2}(\Gamma)$ – elliptic, i.e., there is a constant $\gamma > 0$ such that

$$a(u,u) \ge \gamma\|u\|^2_{H^{-1/2}(\Gamma)} \qquad \forall\, u \in H^{-1/2}(\Gamma). \qquad (2.1.26)$$

The existence and uniqueness of a weak solution of $Au = f$ is guaranteed by the Riesz representation theorem [30, Theorem 4.8] applied to the space $H^{-1/2}(\Gamma)$ and the energy inner product $a(u,v)$.

Theorem (Riesz representation theorem) 2.1.6 *Let H be a Hilbert space. Then for each bounded linear functional $F : H \to \mathbb{R}$ there exists a unique element $u \in H$ such that for all $v \in H$ there holds*

$$F(v) = \langle u, v \rangle.$$

It was proved in [21] that

Theorem 2.1.7 *If Γ is C^∞, then*

$$A: \ H^{s-1/2}(\Gamma) \longrightarrow H^{s+1/2}(\Gamma)$$

is invertible for all $s \in \mathbb{R}$.

2.2 Boundary Element Spaces

For the remainder of this Chapter, we assume that the dimension $n = 2$, and that the boundary Γ is a smooth, closed curve in $\mathbb{R}^2$.

Suppose that

$$\mathbf{F} : \mathbb{R} \to \Gamma \tag{2.2.1}$$

is a smooth, 1-periodic map that satisfies

$$|\mathbf{F}'(\tau)| \geq C > 0 \qquad \text{for } \ 0 \leq \tau \leq 1, \tag{2.2.2}$$

and gives Γ a counterclockwise orientation. We choose points in the interval $[0, 1]$,

$$0 = \tau_0 < \tau_1 < \cdots < \tau_{N-1} < \tau_N = 1.$$

Put

$$h_k = \tau_k - \tau_{k-1},$$

and let

$$h = \max_{1 \leq k \leq N} h_k.$$

For a uniform mesh, $h = h_k$ for all k.

Under the parametric representation $\mathbf{F}$ of Γ, for any $\mathbf{x} \in \Gamma$, we have

$$\mathbf{x} = \mathbf{F}(\tau). \tag{2.2.3}$$

The curve interval $\triangle_k$ is defined by

$$\triangle_k = \{\mathbf{F}(\tau) : \ \tau_{k-1} \leq \tau \leq \tau_k\}, \qquad k = 1, \ldots, N,$$

so that

$$\Gamma = \cup_{k=1}^{N} \triangle_k.$$

In order to define the boundary interpolations we use in the next Chapter, we introduce a specific mapping on the arc of boundary. We have, for each $\triangle_k$,

$$\mathbf{m}_k : [0,1] \longrightarrow \triangle_k, \qquad k = 1, \ldots, N,$$

defined by

$$\begin{aligned}
\mathbf{m}_k(s) &= \mathbf{F}[(1-s)\tau_{k-1} + s\tau_k] \\
&= \mathbf{F}[\tau_{k-1} + s(\tau_k - \tau_{k-1})] \\
&= \mathbf{F}(\tau_{k-1} + sh_k), \qquad s \in [0,1].
\end{aligned} \tag{2.2.4}$$

Let us introduce the definition of the Sobolev spaces $H^k(\Omega^+)$ and $H^s(\Gamma)$. Detailed studies of Sobolev spaces can be found in [1].

Suppose now that $k \geq 1$ is an integer. The Sobolev space $H^k(\Omega^+)$ will be defined in the norm $\|\cdot\|_{H^k(\Omega^+)}$, which is

$$\|u\|^2_{H^k(\Omega^+)} = \sum_{0\leq|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(\Omega^+)},$$

where D^α is viewed as a distribution on Ω^+ and $L^2(\Omega^+)$ is the usual Lebesgue space of square-integrable functions, with norm

$$\|u\|^2_{L^2(\Omega^+)} = \int_{\Omega^+} u^2 dx.$$

For an integer $r \geq 0$, the norm in the Sobolev space $H^r(\Gamma)$ is defined by

$$\|u\|^2_{H^r(\Gamma)} = \sum_{i=0}^{r} \int_0^1 \left| \left(\frac{d}{d\tau}\right)^i [(u \circ \mathbf{F})(\tau)] \right|^2 d\tau. \tag{2.2.5}$$

For $s = r + \mu$, $0 < \mu < 1$, the norm in the Sobolev spaces $H^s(\Gamma)$ is defined by

$$\|u\|^2_{H^s(\Gamma)} = \|u\|^2_{H^r(\Gamma)} + [(u \circ \mathbf{F})(\tau)]^2_\mu, \tag{2.2.6}$$

where

$$\begin{aligned} [v]^2_\mu &= \int_0^1 \int_0^1 \frac{|v(\tau) - v(\tau')|^2}{|e^{i2\pi\tau} - e^{i2\pi\tau'}|^{1+2\mu}} d\tau\, d\tau' \\ &= \int_0^1 \int_0^1 \frac{|v(\tau) - v(\tau')|^2}{|2\sin\pi(\tau - \tau')|^{1+2\mu}} d\tau\, d\tau'. \end{aligned}$$

The norm in the Sobolev space $H^{-s}(\Gamma)$ is defined by duality, i.e.,

$$\|u\|_{H^{-s}(\Gamma)} = \sup_{\phi\in H^s(\Gamma)} \frac{|\langle u, \phi\rangle_0|}{\|\phi\|_{H^s(\Gamma)}}. \tag{2.2.7}$$

Note that if $s = 0$, $H^0(\Gamma)$ coincides with $L^2(\Gamma)$. Also,

$$H^s(\Gamma) = \{u|_\Gamma : u \in H^{s+1/2}(\mathbb{R}^2)\} \qquad \text{for } s > 0.$$

In the approximation of a variational problem, we choose generally piecewise polynomial functions which can form finite dimensional subspaces of a Sobolev space

$H^s(\Gamma)$. These subspaces are referred to as boundary element spaces, and within them one seeks the approximate solutions of the integral equation. We define boundary element spaces $S_h^{r,e} \subseteq H^s(\Gamma)$ as follows:

i) For $r > e = 0$,

$$S_h^{r,0} = \{u \;:\; \text{for each } k, \text{ the function } s \mapsto |\mathbf{m}_k'(s)|\, u[\mathbf{m}_k(s)] \text{ is a polynomial of degree } \leq r-1 \text{ for } 0 \leq s \leq 1\}.$$

ii) For $r > e = 1$,

$$S_h^{r,1} = S_h^{r,0} \cap C(\Gamma).$$

Here, $C(\Gamma)$ is the space of continuous functions on Γ.

One sees that,

1. A function $u \in S_h^{r,0}$ need not be continuous at the mesh breakpoints τ_k, $k = 1, \ldots, N$;
2. The functions in $S_h^{r,1}$ are continuous on Γ;
3. If $s < e + \frac{1}{2}$, then $S_h^{r,e} \subset H^s(\Gamma)$;
4. The dimension of $S_h^{r,e}$ is $(r-e)N$ for $e = 0$ or 1.

The following properties (see, for example [2]) will be frequently used in the error estimates for the Galerkin method.

Definition 2.2.1 *The mesh is said to be quasi-uniform if there exists $C > 0$ such that*

$$\max_k h_k \leq C \min_k h_k.$$

Theorem 2.2.1 *Suppose that the boundary* Γ *is* C^r.

i) (Approximation property) *Let* $\beta < e + 1/2$. *There exists a family of approximation operators* $P_h : H^\beta(\Gamma) \to S_h^{r,e}(\Gamma)$ *such that if* $\alpha \le \beta \le t \le r$, *then*

$$\|u - P_h u\|_{H^\alpha(\Gamma)} \le Ch^{t-\alpha}\|u\|_{H^t(\Gamma)} \qquad \text{for all } u \in H^t(\Gamma). \tag{2.2.8}$$

Thus,

$$\inf_{v_h \in S_h^{r,e}(\Gamma)} \|v_h - u\|_{H^\alpha(\Gamma)} \le Ch^{t-\alpha}\|u\|_{H^t(\Gamma)} \qquad \forall u \in H^t(\Gamma). \tag{2.2.9}$$

ii) (Inverse property) *If the mesh is quasi-uniform, then for* $\alpha \le t < e + 1/2$, *we have*

$$\|v\|_{H^t(\Gamma)} \le Ch^{\alpha-t}\|v\|_{H^\alpha(\Gamma)} \qquad \forall v \in S_h^{r,e}. \tag{2.2.10}$$

Remark: Throughout all the following, C denotes a generic constant which can take different values at different occurrences.

2.3 The Galerkin Method

Using $S_h^{r,e} \subset H^{-1/2}(\Gamma)$ as the boundary element space, the Galerkin method for the integral equation $Au = f$ is : find an approximate solution $u_h \in S_h^{r,e}$ such that

$$a(u_h, v_h) = \langle f, v_h\rangle \qquad \forall v_h \in S_h^{r,e}. \tag{2.3.1}$$

The equation (2.3.1) is called the Galerkin equation.

For any $k, l = 1, \ldots, N$, put $\mathbf{x} = \mathbf{m}_l(t) \in \triangle_l$, and $\mathbf{y} = \mathbf{m}_k(s) \in \triangle_k$, where $0 \le s, t \le 1$. This parametric representation leads to the formula (recall $n = 2$)

$a(u_h, v_h)$

$$= \frac{1}{2\pi} \int_\Gamma \int_\Gamma \log\left(\frac{b}{|\mathbf{x}-\mathbf{y}|}\right) u_h(\mathbf{y})\, v_h(\mathbf{x})\, d\sigma_{\mathbf{x}}\, d\sigma_{\mathbf{y}}$$

$$= \frac{1}{2\pi} \sum_{k=1}^{N}\sum_{l=1}^{N} \int_0^1 \int_0^1 \log\left(\frac{b}{|\mathbf{m}_k(s)-\mathbf{m}_l(t)|}\right) u_h[\mathbf{m}_k(s)]\, v_h[\mathbf{m}_l(t)]\, |\mathbf{m}_k'(s)|\, |\mathbf{m}_l'(t)|\, ds\, dt.$$

Let us choose a basis $\{P_1 \dots, P_r\}$ for the space of polynomials of degree $\le r-1$, and write then u_h and v_h as

$$u_h[\mathbf{m}_k(s)] = \sum_{i=1}^{r} U_{k,i} \frac{P_i(s)}{|\mathbf{m}_k'(s)|}, \qquad k = 1, \dots, N, \tag{2.3.2}$$

$$v_h[\mathbf{m}_l(t)] = \sum_{j=1}^{r} V_{l,j} \frac{P_j(t)}{|\mathbf{m}_l'(t)|}, \qquad l = 1, \dots, N \tag{2.3.3}$$

for $0 \le s \le 1$, $0 \le t \le 1$. Expressing the bilinear form in terms of coefficients in the expansions (2.3.2) and (2.3.3), we have

$$a(u_h, v_h)$$

$$= \frac{1}{2\pi} \sum_{k=1}^{N}\sum_{l=1}^{N} \int_0^1 \int_0^1 \log\left(\frac{b}{|\mathbf{m}_k(s)-\mathbf{m}_l(t)|}\right) \left[\sum_{i=1}^{r} U_{k,i}\, P_i(s)\right] \left[\sum_{j=1}^{r} V_{l,j}\, P_j(t)\right] ds\, dt$$

$$= \sum_{k=1}^{N}\sum_{l=1}^{N} [V_{l,1}, \dots, V_{l,r}]\, A^{(k,l)} \begin{bmatrix} U_{k,1} \\ \vdots \\ U_{k,r} \end{bmatrix}$$

$$= \sum_{k=1}^{N}\sum_{l=1}^{N} \mathbf{V}^{(l)}\, A^{(k,l)}\, \mathbf{U}^{(k)},$$

where

$$\mathbf{U}^{(k)} = [U_{k,1}, \dots, U_{k,r}]^T,$$

$$\mathbf{V}^{(l)} = [V_{l,1}, \dots, V_{l,r}],$$

and

$$A^{(k,l)} = \left[a_{ij}^{(k,l)}\right]_{1 \le i,j \le r}, \qquad k, l = 1, \dots, N \tag{2.3.4}$$

is the $r \times r$ element stiffness matrix for $\triangle_k \times \triangle_l$ whose entries are

$$a_{ij}^{(k,l)} = \frac{1}{2\pi} \int_0^1 \int_0^1 \log\left(\frac{b}{|\mathbf{m}_k(s) - \mathbf{m}_l(t)|}\right) P_i(s) P_j(t)\, ds\, dt. \qquad (2.3.5)$$

For the case $e = 0$, the trial function $u_h \in S_h^{r,0}$ may be discontinuous at the breakpoints τ_k, $k = 1, \ldots, N$. There are Nr unknowns $U_{k,i}$ and N^2 element stiffness matrices, each $r \times r$, that combine to yield an $(Nr) \times (Nr)$ system of linear equations $A\mathbf{U} = \mathbf{F}$, where $A, \mathbf{F}$ and $\mathbf{U}$ will be constructed below.

Let $\dot{A}^{(k,l)}$ be the unique $Nr \times Nr$ matrix such that

$$\mathbf{V}^{(l)T} A^{(k,l)} \mathbf{U}^{(k)} = \mathbf{V}^T \dot{A}^{(k,l)} \mathbf{U}, \qquad k, l = 1, \ldots, N$$

for any

$$\begin{aligned} \mathbf{U} &= [U_{1,1}, \ldots, U_{1,r}, U_{2,1}, \ldots, U_{2,r}, \ldots, U_{N,1}, \ldots, U_{N,r}]^T, \\ \mathbf{V} &= [V_{1,1}, \ldots, V_{1,r}, V_{2,1}, \ldots, V_{2,r}, \ldots, V_{N,1}, \ldots, V_{N,r}]^T. \end{aligned}$$

For example: If $e = 0, r = 2, N = 3$, then $\dot{A}^{(k,l)} \in \mathbb{R}^{6\times 6}$ is a 6×6 matrix, containing 2×2 element stiffness matrix $A^{(k,l)}$. Writing $a_{ij} = a_{ij}^{(k,l)}$, we illustrate $\dot{A}^{(k,l)}$ and $A^{(k,l)}$ in Table 2.1.

The right-hand side of equation (2.3.1) can be written

$$\begin{aligned} \langle f, v_h \rangle &= \int_\Gamma f(\mathbf{x})\, v_h(\mathbf{x})\, d\sigma_{\mathbf{x}} \\ &= \sum_{l=1}^{N} \int_0^1 f[\mathbf{m}_l(t)]\, v_h[\mathbf{m}_l(t)]\, |\mathbf{m}_l'(t)|\, dt \\ &= \sum_{l=1}^{N} [V_{l,1}, \ldots, V_{l,r}]\, \mathbf{F}^{(l)}. \end{aligned}$$

$$\dot{A}^{(1,1)} = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(1,2)} = \begin{bmatrix} 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(1,3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & a_{21} & a_{22} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dot{A}^{(2,1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(2,2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(2,3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dot{A}^{(3,1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ a_{11} & a_{12} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(3,2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & a_{11} & a_{12} & 0 & 0 \end{bmatrix} \quad \dot{A}^{(3,3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & a_{11} & a_{12} \end{bmatrix}.$$

Table 2.1: Case of $e = 0, r = 2$ and $N = 3$.

The vector $\mathbf{F}^{(l)} = \left[\mathbf{F}_j^{(l)}\right]_{1\le j\le r} \in \mathbb{R}^r$ is called the element load vector for $\triangle_l$, where

$$\mathbf{F}_j^{(l)} = \int_0^1 f[\mathbf{m}_l(t)]\, P_j(t)dt, \qquad j = 1, \ldots, r.$$

Define

$$n_k(s) = \left|\mathbf{m}_k'(s)\right| \,\nu[\mathbf{m}_k(s)] = \left[-(\mathbf{m}_k')_2(s), (\mathbf{m}_k')_1(s)\right],$$

where $\nu[\mathbf{m}_k(s)]$ is the inward unit normal via the counter-clockwise orientation of the curve. If f is the right hand side from (2.1.22) then we have

$$\begin{aligned} f[\mathbf{m}_l(t)] &= \frac{1}{2\pi}\sum_{k=1}^{N}\int_0^1 \frac{\nu[\mathbf{m}_k(s)]\cdot[\mathbf{m}_l(t)-\mathbf{m}_k(s)]}{|\mathbf{m}_k(s)-\mathbf{m}_l(t)|^2}\left(g[\mathbf{m}_k(s)] - g[\mathbf{m}_l(t)]\right)\left|\mathbf{m}_k'(s)\right| ds \\ &= \frac{1}{2\pi}\sum_{k=1}^{N}\int_0^1 \frac{n_k(s)\cdot[\mathbf{m}_l(t)-\mathbf{m}_k(s)]}{|\mathbf{m}_k(s)-\mathbf{m}_l(t)|^2}\left(g[\mathbf{m}_k(s)] - g[\mathbf{m}_l(t)]\right) ds, \end{aligned}$$

or if f is from (2.1.23) then

$$\begin{aligned} f[\mathbf{m}_l(t)] &= -\frac{1}{2}g[\mathbf{m}_l(t)] \\ &\quad +\frac{1}{2\pi}\sum_{k=1}^{N}\int_0^1 \frac{n_k(s)\cdot[\mathbf{m}_l(t)-\mathbf{m}_k(s)]}{|\mathbf{m}_k(s)-\mathbf{m}_l(t)|^2} g[\mathbf{m}_k(s)]\, ds. \end{aligned}$$

Similarly, let $\dot{\mathbf{F}}^{(l)}$ be the unique vector in $\mathbb{R}^{Nr}$ such that

$$\mathbf{V}^{(l)^T}\mathbf{F}^{(l)} = \mathbf{V}^T\dot{\mathbf{F}}^{(l)}, \qquad l = 1, \ldots, N$$

for

$$\mathbf{V}^T = [V_{1,1}, \ldots, V_{1,r}, V_{2,1}, \ldots, V_{2,r}, \ldots, V_{N,1}, \ldots, V_{N,r}].$$

Synthesising the expressions above,

$$\begin{aligned} & a(u_h, v_h) = \langle f, v_h\rangle \\ \iff & \sum_{k=1}^{N}\sum_{l=1}^{N} \mathbf{V}^{(l)T} A^{(k,l)}\,\mathbf{U}^{(k)} = \sum_{l=1}^{N} \mathbf{V}^{(l)T}\,\mathbf{F}^{(l)} \end{aligned}$$

$$\begin{aligned}
&\iff \mathbf{V}^T \sum_{k=1}^{N}\sum_{l=1}^{N} \dot{A}^{(k,l)}\, U = \mathbf{V}^T \sum_{l=1}^{N} \dot{\mathbf{F}}^{(l)} \\
&\iff \mathbf{V}^T A \mathbf{U} = \mathbf{V}^T \mathbf{F},
\end{aligned}$$

where

$$A = \sum_{k=1}^{N}\sum_{l=1}^{N} \dot{A}^{(k,l)} \tag{2.3.6}$$

is an $Nr \times Nr$ matrix, and

$$\mathbf{F} = \sum_{l=1}^{N} \dot{\mathbf{F}}^{(l)} \tag{2.3.7}$$

is a vector in $\mathbb{R}^{Nr}$.

We call A the global stiffness matrix for Γ, and $\mathbf{F}$ the global load vector for Γ.

Thus, we conclude that the Galerkin equation (2.3.1) holds for all $v_h \in S_h^{r,0}$ if and only if

$$\mathbf{V}^T A\, \mathbf{U} = \mathbf{V}^T \mathbf{F} \qquad \text{for all } V \in \mathbb{R}^{Nr},$$

or equivalently

$$A\mathbf{U} = \mathbf{F}. \tag{2.3.8}$$

That is, finding u_h reduces to seeking $U_{k,i}$ for $k = 1, \ldots, N$ and $i = 1, \ldots, r$ by solving the system of linear equations (2.3.8).

We now consider the case $e = 1$, in which a function $u_h \in S_h^{r,1}$ is continuous at the breakpoints τ_k ($k = 1, \ldots, N$). Let $\{P_i\}$, $i = 1, \ldots, r$, be a basis of the space of polynomials of degree $\leq r - 1$ satisfying

$$\begin{aligned}
P_i(1) &= \begin{cases} 0 & 1 \leq i \leq r-1, \\ 1 & i = r, \end{cases} \\
P_i(0) &= \begin{cases} 0 & 2 \leq i \leq r, \\ 1 & i = 1. \end{cases}
\end{aligned}$$

From the parametric representation (2.2.4), we have

$$\mathbf{m}_k'(s) = h_k \mathbf{F}'(\tau_{k-1} + s h_k), \qquad s \in [0,1]. \tag{2.3.9}$$

Moreover,

$$\begin{aligned} \mathbf{m}_k'(1) &= h_k \mathbf{F}'(\tau_k), \\ \mathbf{m}_{k+1}'(0) &= h_{k+1} \mathbf{F}'(\tau_k). \end{aligned}$$

From expression (2.3.2), we obtain

$$u_h[\mathbf{m}_k(1)] = \frac{U_{k,r}}{|\mathbf{m}_k'(1)|}, \tag{2.3.10}$$

$$u_h[\mathbf{m}_{k+1}(0)] = \frac{U_{k+1,1}}{|\mathbf{m}_{k+1}'(0)|}. \tag{2.3.11}$$

Similarly for $v_h[\mathbf{m}_l(1)]$ and $v_h[\mathbf{m}_{l+1}(0)]$.

Assuming the function u_h is continuous at τ_k,

$$\begin{aligned} u_h[\mathbf{m}_k(1)] &= u_h[\mathbf{m}_{k+1}(0)], \qquad 1 \le k \le N-1, \\ u_h[\mathbf{m}_N(1)] &= u_h[\mathbf{m}_1(0)], \end{aligned}$$

with

$$U_{k,r} = \alpha_k U_{k+1,1}, \qquad 1 \le k \le N-1, \tag{2.3.12}$$

$$U_{N,r} = \alpha_N U_{1,1}, \tag{2.3.13}$$

where

$$\alpha_k = \frac{|\mathbf{m}_k'(1)|}{|\mathbf{m}_{k+1}'(0)|} \qquad \text{and} \qquad \alpha_N = \frac{|\mathbf{m}_N'(1)|}{|\mathbf{m}_1'(0)|}.$$

In this case, we take for the solution vector $U \in \mathbb{R}^{N(r-1)}$,

$$\mathbf{U} = [U_{1,1}, U_{1,2}, \ldots, U_{1,r-1}, U_{2,1}, \ldots, U_{2,r-1}, \ldots, U_{N,1}, \ldots, U_{N,r-1}]^T$$

which satisfies an $N(r-1) \times N(r-1)$ system of linear equations $A\mathbf{U} = \mathbf{F}$.

We now construct the matrix A and right-hand side $\mathbf{F}$.

If $1 \leq j \leq N-1$, let

$$U_{j,1} = U_{r(j-1)-(j-2)}, \quad U_{j,2} = U_{r(j-1)-(j-3)}, \quad \ldots, \quad \frac{1}{\alpha_j} U_{j,r} = U_{j+1,1} = U_{j(r-1)+1}, \tag{2.3.14}$$

and if $j = N$, let

$$U_{N,1} = U_{r(N-1)-(N-2)}, \quad U_{N,2} = U_{r(N-1)-(N-3)}, \quad \ldots, \quad \frac{1}{\alpha_N} U_{N,r} = U_{1,1} = U_1. \tag{2.3.15}$$

By applying (2.3.14) and (2.3.15), the left-hand side of the Galerkin equation has the following expression,

$$\begin{aligned}
a(u_h, v_h) &= \sum_{k=1}^{N} \sum_{l=1}^{N} [V_{l,1}, V_{l,2}, \ldots, V_{l,r}] \, A^{(k,l)} \begin{bmatrix} U_{k,1} \\ U_{k,2} \\ \vdots \\ U_{k,r} \end{bmatrix} \\
&= \sum_{k=1}^{N} \sum_{l=1}^{N} [V_{r(l-1)-(l-2)}, V_{r(l-1)-(l-3)}, \ldots, \alpha_l V_{l(r-1)+1}] \, A^{(k,l)} \begin{bmatrix} U_{r(k-1)-(k-2)} \\ U_{r(k-1)-(k-3)} \\ \vdots \\ \alpha_k U_{k(r-1)+1} \end{bmatrix} \\
&= \sum_{k=1}^{N} \sum_{l=1}^{N} [V_{r(l-1)-(l-2)}, V_{r(l-1)-(l-3)}, \ldots, V_{l(r-1)+1}] \, \hat{A}^{(k,l)} \begin{bmatrix} U_{r(k-1)-(k-2)} \\ U_{r(k-1)-(k-3)} \\ \vdots \\ U_{k(r-1)+1} \end{bmatrix}
\end{aligned}$$

$$= \sum_{k=1}^{N}\sum_{l=1}^{N} \mathbf{V}^{(l)^T} \hat{A}^{(k,l)} \mathbf{U}^{(k)},$$

provided

$$\mathbf{U}^{(k)} = \left[U_{r(k-1)-(k-2)}, U_{r(k-1)-(k-3)}, \ldots, U_{k(r-1)+1}\right]^T,$$

$$\mathbf{V}^{(l)^T} = \left[V_{r(l-1)-(l-2)}, V_{r(l-1)-(l-3)}, \ldots, V_{l(r-1)+1}\right],$$

and

$$\hat{A}^{(k,l)} = M_l\, A^{(k,l)}\, M_k = \left[\hat{a}_{ij}^{k,l}\right]_{1\le i,j\le r}, \qquad 1 \le k, l \le N,$$

where $A^{(k,l)}$ is defined by (2.3.4), and M_k is the $r \times r$ matrix

$$M_k = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_k \end{bmatrix}.$$

Let $\dot{A}^{(k,l)} \in \mathbb{R}^{N(r-1)\times N(r-1)}$ be the unique matrix such that

$$\mathbf{V}^{(l)T}\, \hat{A}^{(k,l)}\, \mathbf{U}^k = \mathbf{V}^T\, \dot{A}^{(k,l)}\, \mathbf{U}, \qquad k, l = 1, \ldots, N$$

for any

$$\mathbf{U} = [U_1, U_2, \ldots, U_{N(r-1)-1}, U_{N(r-1)}]^T,$$

$$\mathbf{V}^T = [V_1, V_2, \ldots, V_{N(r-1)-1}, V_{N(r-1)}].$$

We give an example of such matrix to assist with understanding of the above.

For example: we assume $e = 1, r = 3, N = 3$, then $\dot{A}^{(k,l)}$ is a 6×6 matrix. Writing $\hat{a}_{ij} = \hat{a}_{ij}^{(k,l)}$, the form of the matrix $\dot{A}^{(k,l)}$ is shown in Table 2.2.

$$\dot{A}^{(1,1)} = \begin{bmatrix} \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} & 0 & 0 & 0 \\ \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23} & 0 & 0 & 0 \\ \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(1,2)} = \begin{bmatrix} 0 & 0 & \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} & 0 \\ 0 & 0 & \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23} & 0 \\ 0 & 0 & \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(1,3)} = \begin{bmatrix} \hat{a}_{13} & 0 & 0 & 0 & \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{23} & 0 & 0 & 0 & \hat{a}_{21} & \hat{a}_{22} \\ \hat{a}_{33} & 0 & 0 & 0 & \hat{a}_{31} & \hat{a}_{32} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dot{A}^{(2,1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} & 0 & 0 & 0 \\ \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23} & 0 & 0 & 0 \\ \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(2,2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} & 0 \\ 0 & 0 & \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23} & 0 \\ 0 & 0 & \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(2,3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{a}_{13} & 0 & 0 & 0 & \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{23} & 0 & 0 & 0 & \hat{a}_{21} & \hat{a}_{22} \\ \hat{a}_{33} & 0 & 0 & 0 & \hat{a}_{31} & \hat{a}_{32} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dot{A}^{(3,1)} = \begin{bmatrix} \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} & 0 & 0 & 0 \\ \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23} & 0 & 0 & 0 \end{bmatrix} \quad \dot{A}^{(3,2)} = \begin{bmatrix} 0 & 0 & \hat{a}_{31} & \hat{a}_{32} & \hat{a}_{33} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{a}_{11} & \hat{a}_{12} & \hat{a}_{13} & 0 \\ 0 & 0 & \hat{a}_{21} & \hat{a}_{22} & \hat{a}_{23} & 0 \end{bmatrix} \quad \dot{A}^{(3,3)} = \begin{bmatrix} \hat{a}_{33} & 0 & 0 & 0 & \hat{a}_{31} & \hat{a}_{32} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hat{a}_{13} & 0 & 0 & 0 & \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{23} & 0 & 0 & 0 & \hat{a}_{21} & \hat{a}_{22} \end{bmatrix}.$$

Table 2.2: Case of $e = 1, r = 3$ and $N = 3$.

Reconsidering the right-hand side of the Galerkin equation (2.3.1), we have

$$\begin{aligned}\langle f, v_h \rangle &= \sum_{l=1}^{N} [V_{l,1}, V_{l,2}, \ldots, V_{l,r}]\, \mathbf{F}^{(l)} \\ &= \sum_{l=1}^{N} [V_{r(l-1)-(l-2)}, V_{r(l-1)-(l-3)}, \ldots, V_{l(r-1)+1}]\, \hat{\mathbf{F}}^{(l)},\end{aligned}$$

where

$$\hat{\mathbf{F}}^{(l)} = M_l\, \mathbf{F}^{(l)}.$$

Let $\dot{\mathbf{F}}^{(l)}$ be the unique vector in $\mathbb{R}^{N(r-1)}$ such that

$$\langle f, v_h \rangle = \sum_{l=1}^{N} \mathbf{V}^{(l)^T} \hat{\mathbf{F}}^{(l)} = \mathbf{V}^T \sum_{l=1}^{N} \dot{\mathbf{F}}^{(l)}$$

for

$$\mathbf{V}^T = [V_1, V_2, \ldots, V_{N(r-1)-1}, V_{N(r-1)}].$$

Therefore, the Galerkin equation is

$$\mathbf{V}^T \sum_{k=1}^{N} \sum_{l=1}^{N} \dot{A}^{(k,l)}\, \mathbf{U} = \mathbf{V}^T \sum_{l=1}^{N} \dot{\mathbf{F}}^{(l)} \qquad \text{for all } \mathbf{V} \in \mathbb{R}^{N(r-1)},$$

or equivalently

$$A\mathbf{U} = \mathbf{F},$$

where

$$A = \sum_{k=1}^{N} \sum_{l=1}^{N} \dot{A}^{(k,l)}$$

is a matrix in $\mathbb{R}^{N(r-1) \times N(r-1)}$, and

$$\mathbf{F} = \sum_{l=1}^{N} \dot{\mathbf{F}}^{(l)}$$

is a vector in $\mathbb{R}^{N(r-1)}$.

2.4 Error Estimates of the Galerkin Method

It follows from Theorem 2.1.7 that the single layer operator $A : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is invertible. A well known argument shows that the Galerkin method achieves optimal rates of convergence in the energy space $H^{-1/2}(\Gamma)$.

Theorem (Céa's Lemma) 2.4.1 *Let a be the bilinear form* (2.1.24), *and let $f \in H^{1/2}(\Gamma)$. Then the Galerkin equation* (2.3.1) *has a unique solution $u_h \in S_h^{r,e}$, and*

$$\|u_h - u\|_{H^{-1/2}(\Gamma)} \leq C \inf_{v_h \in S_h^{r,e}} \|v_h - u\|_{H^{-1/2}(\Gamma)}. \tag{2.4.1}$$

If $u \in H^r(\Gamma)$, then

$$\|u_h - u\|_{H^{-1/2}(\Gamma)} \leq C h^{r+1/2} \|u\|_{H^r(\Gamma)}. \tag{2.4.2}$$

Proof. We obtain $a(u_h - u, u_h - v_h) = 0$ by the Galerkin equation, then from a being $H^{-1/2}(\Gamma)$-*elliptic*,

$$\begin{aligned}
C\|u_h - u\|^2_{H^{-1/2}(\Gamma)} &\leq a(u_h - u, u_h - u) \\
&= a(u_h - u, v_h - u) + a(u_h - u, u_h - v_h) \qquad \forall v_h \in S_h^{r,e} \\
&= a(u_h - u, v_h - u) \\
&\leq C\|u_h - u\|_{H^{-1/2}(\Gamma)}\|v_h - u\|_{H^{-1/2}(\Gamma)}, \qquad \text{by Theorem 2.1.5}
\end{aligned}$$

and cancelling $\|u_h - u\|_{H^{-1/2}(\Gamma)}$ on both sides, we obtain

$$\|u_h - u\|_{H^{-1/2}(\Gamma)} \leq C\|v_h - u\|_{H^{-1/2}(\Gamma)} \qquad \forall\, v_h \in S_h^{r,e},$$

hence

$$\|u_h - u\|_{H^{-1/2}(\Gamma)} \leq C \inf_{v_h \in S_h^{r,e}(\Gamma)} \|v_h - u\|_{H^{-1/2}(\Gamma)},$$

which is (2.4.1). Moreover, by the approximation property in Theorem 2.2.1,

$$\|u_h - u\|_{H^{-1/2}(\Gamma)} \le Ch^{r+1/2}\|u\|_{H^r(\Gamma)} \qquad \text{for } u \in H^r(\Gamma). \tag{2.4.3}$$

□

Furthermore, if the error is measured in a more negative norm then the order of convergence of the Galerkin method can be even better than we obtain in (2.4.2). The best result in the $H^{-r-1}(\Gamma)$ norm is in following theorem. This super-convergence property of the boundary integral equation was first obtained in [23].

Theorem 2.4.2 *Let A be the single layer operator given by (2.1.9), and assume $u \in H^r(\Gamma)$. Then the Galerkin method with boundary element spaces $S_h^{r,e}$ has the following super-convergence property:*

$$\|u_h - u\|_{H^{-r-1}(\Gamma)} \le Ch^{2r+1}\|u\|_{H^r(\Gamma)}. \tag{2.4.4}$$

Proof. The result will be obtained by a duality argument known as "Nitsche's trick".

Let $\phi \in H^{r+1}(\Gamma)$ and let $v \in H^r(\Gamma)$ be the unique solution of

$$Av = \phi.$$

We have, from Theorem 2.1.7,

$$\|v\|_{H^r(\Gamma)} \le C\|\phi\|_{H^{r+1}(\Gamma)}. \tag{2.4.5}$$

By the dual definition of $\|u_h - u\|_{H^{-r-1}(\Gamma)}$, (see (2.2.7)) and because

$$\begin{aligned} a(u_h - u, v_h) &= a(u_h, v_h) - a(u, v_h) \\ &= \langle f, v_h\rangle - \langle f, v_h\rangle = 0 \qquad \text{for all } v_h \in S_h^{r,e} \end{aligned}$$

we obtain

$$
\begin{aligned}
\|u_h - u\|_{H^{-r-1}(\Gamma)} &= \sup_{\phi\in H^{r+1}(\Gamma)} \frac{|\langle u_h - u, \phi\rangle_0|}{\|\phi\|_{H^{r+1}(\Gamma)}} \\
&= \sup_{\phi\in H^{r+1}(\Gamma)} \frac{a(u_h - u, v)}{\|\phi\|_{H^{r+1}(\Gamma)}} \\
&= \sup_{\phi\in H^{r+1}(\Gamma)} \frac{a(u_h - u, v - v_h)}{\|\phi\|_{H^{r+1}(\Gamma)}} \\
&\leq \|u_h - u\|_{H^{-1/2}(\Gamma)} \sup_{\phi\in H^{r+1}(\Gamma)} \frac{\|v - v_h\|_{H^{-1/2}(\Gamma)}}{\|\phi\|_{H^{r+1}(\Gamma)}}, \qquad \text{by Theorem 2.1.5.}
\end{aligned}
$$

From the approximation property (2.2.9) and the error bound (2.4.2), we obtain

$$
\|u_h - u\|_{H^{-r-1}(\Gamma)} \leq Ch^{r+1/2}\|u\|_{H^r(\Gamma)} \sup_{\phi\in H^{r+1}(\Gamma)} \frac{Ch^{r+1/2}\|v\|_{H^r(\Gamma)}}{\|\phi\|_{H^{r+1}(\Gamma)}}, \tag{2.4.6}
$$

and so by (2.4.5),

$$
\|u_h - u\|_{H^{-r-1}(\Gamma)} \leq Ch^{2r+1}\|u\|_{H^r(\Gamma)}, \qquad u \in H^r(\Gamma).
$$

□

Corollary 2.4.1 *If the mesh is quasi-uniform, then*

$$
\|u_h - u\|_{L^2(\Gamma)} \leq Ch^r\|u\|_{H^r(\Gamma)}. \tag{2.4.7}
$$

Proof. If $v_h \in S_h^{r,e}(\Gamma)$, then by the inverse property (Theorem 2.2.1 (ii)) we have

$$
\begin{aligned}
\|u_h - u\|_{L^2(\Gamma)} &\leq \|u_h - v_h\|_{L^2(\Gamma)} + \|v_h - u\|_{L^2(\Gamma)} \\
&\leq Ch^{-1/2}\|u_h - v_h\|_{H^{-1/2}(\Gamma)} + \|v_h - u\|_{L^2(\Gamma)} \\
&\leq Ch^{-1/2}(\|u_h - u\|_{H^{-1/2}(\Gamma)} + \|u - v_h\|_{H^{-1/2}(\Gamma)}) + \|v_h - u\|_{L^2(\Gamma)} \\
&\leq Ch^r\|u\|_{H^r(\Gamma)} + C(h^{-1/2}\|v_h - u\|_{H^{-1/2}(\Gamma)} + \|v_h - u\|_{L^2(\Gamma)}).
\end{aligned}
$$

By choosing $v_h = P_h u$ where P_h is the approximation operator of Theorem 2.2.1 (i), we have

$$h^{-1/2}\|v_h - u\|_{H^{-1/2}(\Gamma)} + \|v_h - u\|_{L^2(\Gamma)} \leq Ch^r\|u\|_{H^r(\Gamma)} \tag{2.4.8}$$

giving the desired bound. $\square$

Chapter 3

PERTURBATION OF THE BILINEAR FORM

3.1 The Perturbed Bilinear Form a_h

We consider the Galerkin boundary element method (2.3.1) for a smooth curved boundary Γ, but now use an approximate parametric representation of Γ.

As we know, the boundary Γ is a union of the arcs $\triangle_k, k = 1, \dots, N$. A mapping

$$\mathbf{m}_k : [0,1] \to \triangle_k$$

is defined by (2.2.4), for each boundary element $\triangle_k$.

In the implementation of the Galerkin method, it is necessary to compute the stiffness matrix $A^{(k,l)}$ for $\triangle_k \times \triangle_l$, whose entries are

$$a_{ij}^{(k,l)} = \frac{1}{2\pi} \int_0^1 \int_0^1 \log\left(\frac{b}{|\mathbf{m}_k(s) - \mathbf{m}_l(t)|}\right) P_i(s)\, P_j(t)\, ds\, dt, \qquad i, j = 1, \dots, r, \tag{3.1.1}$$

see (2.3.4) and (2.3.5).

In a practical implementation of such a method, it will usually be convenient to approximate any curved boundaries using, for instance, some kind of piecewise polynomial or piecewise rational function. Thus, we try to replace the exact stiffness matrix $A^{(k,l)}$ by a perturbed matrix $\tilde{A}^{(k,l)} = \left[\tilde{a}_{ij}^{(k,l)}\right]_{1 \le i,j \le r}$ with entries

$$\tilde{a}_{ij}^{(k,l)} = \frac{1}{2\pi} \int_0^1 \int_0^1 \log\left(\frac{b}{|\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_l(t)|}\right) P_i(s)\, P_j(t)\, ds\, dt, \qquad i, j = 1, \dots, r, \tag{3.1.2}$$

where $\tilde{\mathbf{m}}_k(s) \approx \mathbf{m}_k(s)$ for $0 \le s \le 1$ and $1 \le k \le N$.

The arc $\triangle_k$ is approximated by an arc $\tilde{\triangle}_k$, which is formed by a smooth map

$$\tilde{\mathbf{m}}_k : [0,1] \to \tilde{\triangle}_k, \qquad k = 1, \dots, N. \tag{3.1.3}$$

Our purpose is to make the approximation $\tilde{\mathbf{m}}_k \approx \mathbf{m}_k$ sufficiently accurate so that the convergence rates presented in Theorem 2.4.1 and Theorem 2.4.2 are maintained. This approximation of $\mathbf{m}_k$ affects the Galerkin equation via the entries of the stiffness matrix on $\triangle_k \times \triangle_l$, and the components of the load vector on $\triangle_l$.

On the left-hand side of the integral equation (2.1.22) or (2.1.23), the weakly singular kernel K is approximated by the kernel K_h, defined as follows:

$$K_h(\mathbf{x}, \mathbf{y}) = K(\tilde{\mathbf{x}} - \tilde{\mathbf{y}}),$$

where

$$\mathbf{x} = \mathbf{m}_l(t) \in \triangle_l, \qquad \tilde{\mathbf{x}} = \tilde{\mathbf{m}}_l(t) \in \tilde{\triangle}_l, \tag{3.1.4}$$

$$\mathbf{y} = \mathbf{m}_k(s) \in \triangle_k, \qquad \tilde{\mathbf{y}} = \tilde{\mathbf{m}}_k(s) \in \tilde{\triangle}_k. \tag{3.1.5}$$

For $0 \leq s, t \leq 1$ and $u, v \in H^{-1/2}(\Gamma)$, a perturbed bilinear form a_h is defined as (cf. (2.1.24))

$$\begin{aligned} &a_h(u, v) \\ &= \int_\Gamma \int_\Gamma K_h(\mathbf{x}, \mathbf{y})\, u(\mathbf{y})\, v(\mathbf{x})\, d\sigma_{\mathbf{x}}\, d\sigma_{\mathbf{y}} \\ &= \sum_{k=1}^{N} \sum_{l=1}^{N} \int_0^1 \int_0^1 K[\tilde{\mathbf{m}}_l(t) - \tilde{\mathbf{m}}_k(s)]\, u[\mathbf{m}_k(s)]\, v[\mathbf{m}_l(t)] \left|\mathbf{m}_l'(t)\right| \left|\mathbf{m}_k'(s)\right| \, ds\, dt. \end{aligned} \tag{3.1.6}$$

Hence the perturbed weak formulation of the integral equation (2.1.3) is, (cf. (2.1.25))

$$a_h(u, v) = \langle f_h, v \rangle \qquad \forall v \in H^{-1/2}(\Gamma). \tag{3.1.7}$$

We assume that $f_h \approx f$, that is, f_h is a suitable approximation to f. The precise definition of f_h will be given later.

The trial space $S_h^{r,e}$ in the perturbed Galerkin method is the same as before, in the classical Galerkin method. Thus, the perturbed form of the Galerkin equation (2.3.1), for $u_h \in S_h^{r,e}$, is given by the expression:

$$a_h(u_h, v_h) = \langle f_h, v_h \rangle \qquad \forall v_h \in S_h^{r,e}. \tag{3.1.8}$$

The left-hand side of the perturbed Galerkin equation is given explicitly by

$$\begin{aligned} & a_h(u_h, v_h) \\ & = \int_\Gamma \int_\Gamma K_h(\mathbf{x}, \mathbf{y})\, u_h(\mathbf{y})\, v_h(\mathbf{x})\, d\sigma_{\mathbf{x}}\, d\sigma_{\mathbf{y}} \\ & = \sum_{k=1}^{N} \sum_{l=1}^{N} \int_0^1 \int_0^1 K\left[\tilde{\mathbf{m}}_l(t) - \tilde{\mathbf{m}}_k(s)\right] u_h[\mathbf{m}_k(s)]\, v_h[\mathbf{m}_l(t)] \left|\mathbf{m}_l'(t)\right| \left|\mathbf{m}_k'(s)\right| ds\, dt, \end{aligned} \tag{3.1.9}$$

and from (2.3.2) and (2.3.3),

$$u_h[\mathbf{m}_k(s)] = \sum_{i=1}^{r} U_{k,i} \frac{P_i(s)}{\left|\mathbf{m}_k'(s)\right|}, \qquad k = 1, \ldots, N, \tag{3.1.10}$$

$$v_h[\mathbf{m}_l(t)] = \sum_{j=1}^{r} V_{l,j} \frac{P_j(t)}{\left|\mathbf{m}_l'(t)\right|}, \qquad l = 1, \ldots, N \tag{3.1.11}$$

for $0 \le s \le 1$, $0 \le t \le 1$ and $i, j = 1, \ldots, r$. Substituting the expansions of u_h and v_h in (3.1.10) and (3.1.11), respectively, into (3.1.9), we have, in the two dimensional case,

$$a_h(u_h, v_h) = \frac{1}{2\pi} \sum_{k=1}^{N} \sum_{l=1}^{N} \sum_{i=1}^{r} \sum_{j=1}^{r} U_{k,i} V_{l,j} \int_0^1 \int_0^1 \log\left(\frac{b}{|\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_l(t)|}\right) P_i(s)\, P_j(t)\, ds\, dt,$$

which gives the formula (3.1.2) for the entries of the perturbed element stiffness matrix $\tilde{A}^{(k,l)}$.

3.2 An Estimate for $a_h - a$

We shall consider approximate local parametric representations of Γ satisfying the following conditions.

Definition 3.2.1 *The approximation $\tilde{\mathbf{m}}_k(s) \approx \mathbf{m}_k(s)$ is accurate to order $p \geq 2$, if*

1. *For $0 \leq s \leq 1$ and $0 \leq j \leq 2$,*

$$\left|\mathbf{m}_k^{(j)}(s) - \tilde{\mathbf{m}}_k^{(j)}(s)\right| \leq Ch_k^p \leq Ch^p, \tag{3.2.1}$$

   where C is independent of s, k, and h.

2. *The endpoints of $\tilde{\triangle}_k$ and $\triangle_k$ coincide:*

$$\begin{aligned}\tilde{\mathbf{m}}_k(0) = \mathbf{m}_k(0) = \mathbf{F}(\tau_{k-1}),\\ \tilde{\mathbf{m}}_k(1) = \mathbf{m}_k(1) = \mathbf{F}(\tau_k).\end{aligned}$$

Lemma 3.2.1 *If $k = 1, \ldots, N$ and $0 \leq s \leq 1$, then*

$$\left|\mathbf{m}_k^{(j)}(s)\right| = O\left(h^j\right) \qquad \textit{for } 0 \leq j. \tag{3.2.2}$$

Proof. Recalling that $\mathbf{m}_k(s)$ is defined in (2.2.4) by

$$\mathbf{m}_k(s) = \mathbf{F}(\tau_{k-1} + sh_k),$$

we have, by the chain rule,

$$\begin{aligned}\mathbf{m}_k^{(j)}(s) &= h_k^j \mathbf{F}^{(j)}(\tau_{k-1} + sh_k)\\ &= O\left(h^j\right).\end{aligned} \tag{3.2.3}$$

□

Lemma 3.2.2 *If $\tilde{\mathbf{m}}_k$ is an $O(h^p)$ approximation to $\mathbf{m}_k$, then*

$$\left|\tilde{\mathbf{m}}_k^{(j)}(s)\right| = O\left(h^j\right) \tag{3.2.4}$$

for $0 \leq s \leq 1$ and $0 \leq j \leq 2$.

Proof. By the triangle inequality and Definition 3.2.1,

$$\left||\tilde{\mathbf{m}}_k^{(j)}(s)| - |\mathbf{m}_k^{(j)}(s)|\right| \leq \left|\tilde{\mathbf{m}}_k^{(j)}(s) - \mathbf{m}_k^{(j)}(s)\right| \leq Ch^p,$$

hence, by Lemma 3.2.1,

$$\begin{aligned}\left|\tilde{\mathbf{m}}_k^{(j)}(s)\right| &\leq Ch^p + \left|\mathbf{m}_k^{(j)}(s)\right| \\ &= O\left(h^j\right).\end{aligned}$$

□

For polynomial interpolation, we have the following result from [32]:

Theorem 3.2.1 *If Γ is C^{p+q} and if $\tilde{\mathbf{m}}_k(s)$ is the unique polynomial of order p that interpolates $\mathbf{m}_k(s)$ at $s_1, \ldots, s_p$, for $0 = s_1 < s_2 < \cdots < s_p = 1$, then*

$$\left|\mathbf{m}_k^{(j)}(s) - \tilde{\mathbf{m}}_k^{(j)}(s)\right| \leq Ch^p \tag{3.2.5}$$

for $0 \leq j \leq q \leq p$ and $0 \leq s \leq 1$.

Proof. The error term in the polynomial interpolant can be written as

$$\mathbf{m}_k(s) - \tilde{\mathbf{m}}_k(s) = \mathbf{m}_k\left[s_1, s_2, \ldots, s_p, s\right] \prod_{n=1}^{p} (s - s_n),$$

where $\mathbf{m}_k[s_1, s_2, \ldots, s_p, s]$ denotes the vector divided difference which we can define componentwise. Thus, by Lemma 3.2.1, and for $j = 0, 1, \ldots, q$,

$$
\begin{aligned}
&\left|\mathbf{m}_k^{(j)}(s) - \tilde{\mathbf{m}}_k^{(j)}(s)\right| \\
= &\left|\left(\frac{\partial}{\partial s}\right)^{(j)}\left[\mathbf{m}_k[s_1, s_2, \ldots, s_p, s]\prod_{n=1}^{p}(s - s_n)\right]\right| \\
= &\left|\sum_{i=0}^{j}\binom{j}{i}\left(\frac{\partial}{\partial s}\right)^{(j-i)}\mathbf{m}_k[s_1, s_2, \ldots, s_p, s]\left(\frac{\partial}{\partial s}\right)^{(i)}\prod_{n=1}^{p}(s - s_n)\right| \\
\le\ & C\sum_{i=0}^{j}\binom{j}{i}\left|\left(\frac{\partial}{\partial s}\right)^{(j-i)}\mathbf{m}_k[s_1, s_2, \ldots, s_p, s]\right| \\
\le\ & C\sum_{i=0}^{j}\binom{j}{i}\frac{(j-i)!}{(p+j-i)!}\max_{0\le t\le 1}\left|\mathbf{m}_k^{(p+j-i)}(t)\right| \\
\le\ & C\sum_{i=0}^{j}h^{(p+j-i)} \\
\le\ & Ch^p.
\end{aligned}
$$

□

We give two simple examples that satisfy the Definition 3.2.1.

Example 1: The simplest scheme is linear interpolation, for which $p = 2$. In this case

$$
\begin{aligned}
\tilde{\mathbf{m}}_k(s) &= (1-s)\,\mathbf{m}_k(0) + s\,\mathbf{m}_k(1) \\
&= (1-s)\,\mathbf{F}(\tau_{k-1}) + s\,\mathbf{F}(\tau_k) \qquad \text{for } s \in [0,1].
\end{aligned}
$$

Example 2: To achieve order $p = 3$, we can use piecewise quadratic interpolation

$$\tilde{\mathbf{m}}_k(s) = (1-s)\,\mathbf{m}_k(0) + s\,\mathbf{m}_k(1) + \alpha\, s\,(1-s), \qquad 0 \le s \le 1,$$

choosing α so that

$$\tilde{\mathbf{m}}_k\left(\frac{1}{2}\right) = \mathbf{m}_k\left(\frac{1}{2}\right).$$

(Thus, in the notation of Theorem 3.2.1, we have $s_1 = 0, s_2 = \frac{1}{2}, s_3 = 1.$)

Since

$$\tilde{\mathbf{m}}_k\left(\frac{1}{2}\right) = \frac{1}{2}\,\mathbf{m}_k(0) + \frac{1}{2}\,\mathbf{m}_k(1) + \frac{1}{4}\,\alpha,$$

we see that

$$\begin{aligned}
\alpha &= -2\mathbf{m}_k(0) + 4\mathbf{m}_k\left(\frac{1}{2}\right) - 2\mathbf{m}_k(1) \\
&= -2\left[\mathbf{F}\left(\tau_{k-1}\right) - 2\mathbf{F}\left(\tau_{k-1} + \frac{1}{2}\,h_k\right) + \mathbf{F}(\tau_{k-1} + h_k)\right] \\
&= -2\mathbf{F}''\left(\tau_{k-1} + \frac{1}{2}\,h_k\right)\left(\frac{h_k}{2}\right)^2 + O\left(h_k^4\right).
\end{aligned}$$

Thus, we can interpret the term $\alpha s(1-s)$ as an $O\left(h_k^2\right)$ correction to the linear interpolant.

In what follows we will often use the Taylor expansion with the Integral Remainder,

$$f(s) = \sum_{j=0}^{p-1} \frac{f^{(j)}(t)}{j!}\,(s-t)^j + \frac{1}{p!}\int_t^s f^{(p)}(\xi)\,(s-\xi)^{p-1}\,d\xi, \qquad t < \xi < s.$$

Let $\xi = t + \eta\,(s-t)$, $0 \le \eta \le 1$, then

$$f(s) = \sum_{j=0}^{p-1} \frac{f^{(j)}(t)}{j!}\,(s-t)^j + \frac{1}{p!}\,(s-t)^p \int_0^1 f^{(p)}[t + \eta(s-t)](1-\eta)^{p-1}\,d\eta. \quad (3.2.6)$$

Next, we estimate the error between the bilinear form $a(u_h, v_h)$ in the Galerkin equation (2.3.1) and the bilinear form $a_h(u_h, v_h)$ in the perturbed Galerkin equation (3.1.8). That is, we estimate the difference

$$a(u_h, v_h) - a_h(u_h, v_h)$$

$$= \int_\Gamma \int_\Gamma [K(\mathbf{x}-\mathbf{y}) - K_h(\mathbf{x},\mathbf{y})]\, u_h(\mathbf{y})\, v_h(\mathbf{x})\, d\sigma_{\mathbf{x}}\, d\sigma_{\mathbf{y}}$$

$$= \sum_{k=1}^{N}\sum_{l=1}^{N} \int_0^1 \int_0^1 \left(K\left[\mathbf{m}_l(t) - \mathbf{m}_k(s)\right] - K\left[\tilde{\mathbf{m}}_l(t) - \tilde{\mathbf{m}}_k(s)\right]\right)$$

$$u_h[\mathbf{m}_k(s)]\, v_h[\mathbf{m}_l(t)]\, \left|\mathbf{m}_l'(t)\right| \left|\mathbf{m}_k'(s)\right|\, ds\, dt. \qquad (3.2.7)$$

In bounding this error, the crucial step is to investigate the error between the logarithmic kernel and perturbed logarithmic kernel.

Lemma 3.2.3 *For* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^2$ *with* $\mathbf{b} \neq 0$, *if*

$$\frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{b}|} < \frac{1}{2},$$

then

$$\left|\log \frac{|\mathbf{a}|}{|\mathbf{b}|}\right| \le 2\,\frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{b}|}. \qquad (3.2.8)$$

Proof. By the triangle inequality

$$|\mathbf{b}| - |\mathbf{a}-\mathbf{b}| \le |\mathbf{a}| \le |\mathbf{b}| + |\mathbf{a}-\mathbf{b}|,$$

so

$$1 - \frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{b}|} \le \frac{|\mathbf{a}|}{|\mathbf{b}|} \le 1 + \frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{b}|},$$

and hence

$$\begin{aligned}
\left|\log \frac{|\mathbf{a}|}{|\mathbf{b}|}\right| &= \left|\int_1^{|\mathbf{a}|/|\mathbf{b}|} \frac{dt}{t}\right| \\
&\le \frac{1}{1 - \frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{b}|}} \left|\int_1^{|\mathbf{a}|/|\mathbf{b}|} dt\right| = \frac{\left|\frac{|\mathbf{a}|}{|\mathbf{b}|} - 1\right|}{1 - \frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{b}|}} \\
&\le \frac{|\mathbf{a}-\mathbf{b}|/|\mathbf{b}|}{1 - |\mathbf{a}-\mathbf{b}|/|\mathbf{b}|} \le 2\,\frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{b}|}.
\end{aligned}$$

□

Lemma 3.2.4 *If $\tilde{\mathbf{m}}_k$ is an $O(h^p)$ order of approximation to $\mathbf{m}_k$, then*

$$|K(\mathbf{x}-\mathbf{y}) - K_h(\mathbf{x},\mathbf{y})| \le Ch^{p-1} \qquad \textit{for } \mathbf{x},\mathbf{y} \in \Gamma. \tag{3.2.9}$$

Proof. With the notation of (3.1.4) and (3.1.5), let

$$A = \left|\log\frac{|\tilde{\mathbf{x}}-\tilde{\mathbf{y}}|}{|\mathbf{x}-\mathbf{y}|}\right|,$$

and consider separately three cases.

Case 1: $k = l$, so that $\mathbf{x}$ and $\mathbf{y}$ are in the same boundary element $\triangle_k$. By Lemma 3.2.3 with $\mathbf{a} = \tilde{\mathbf{x}} - \tilde{\mathbf{y}}$ and $\mathbf{b} = \mathbf{x} - \mathbf{y}$,

$$A = \left|\log\frac{|\mathbf{a}|}{|\mathbf{b}|}\right| \le C\frac{|\mathbf{a}-\mathbf{b}|}{|\mathbf{b}|}. \tag{3.2.10}$$

By applying the Taylor's Theorem (3.2.6),

$$\begin{aligned}
|\mathbf{a}-\mathbf{b}| &= |[\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_k(t)] - [\mathbf{m}_k(s) - \mathbf{m}_k(t)]| \\
&= \left|(s-t)\int_0^1 \tilde{\mathbf{m}}_k'[t+\xi(s-t)]\,d\xi - (s-t)\int_0^1 \mathbf{m}_k'[t+\xi(s-t)]\,d\xi\right| \\
&= \left|(s-t)\int_0^1 [\tilde{\mathbf{m}}_k'[t+\xi(s-t)] - \mathbf{m}_k'[t+\xi(s-t)]]\,d\xi\right|,
\end{aligned}$$

so according to Definition 3.2.1,

$$|\mathbf{a}-\mathbf{b}| \le Ch_k^p\,|s-t|,$$

and since $\mathbf{m}_k(s) = \mathbf{F}(\tau_{k-1} + sh_k)$, the assumption (2.2.2) implies that

$$\begin{aligned}
|\mathbf{b}| &= |\mathbf{F}(\tau_{k-1}+sh_k) - \mathbf{F}(\tau_{k-1}+th_k)| \\
&= \left|\left[\mathbf{F}(\tau_{k-1}) + \mathbf{F}'(\tau_{k-1})sh_k + O\left((s-t)^2h_k^2\right)\right]\right. \\
&\qquad \left. - \left[\mathbf{F}(\tau_{k-1}) + \mathbf{F}'(\tau_{k-1})th_k + O\left((s-t)^2h_k^2\right)\right]\right| \\
&= |\mathbf{F}'(\tau_{k-1})(s-t)h_k|\,[1 + O(|s-t|h_k)] \\
&\ge C|s-t|h_k.
\end{aligned}$$

Hence,

$$A \leq C \frac{|s-t|\, h_k^p}{|s-t|\, h_k} \leq C h^{p-1}.$$

Case 2: $l = k+1$, so that $\mathbf{x}$ and $\mathbf{y}$ are in neighbouring boundary elements.

Thus, by condition 2. in Definition 3.2.1,

$$\begin{aligned}
|\mathbf{a}-\mathbf{b}| &= |\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_{k+1}(t) - \mathbf{m}_k(s) + \mathbf{m}_{k+1}(t)| \\
&= |[\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_k(1)] - [\tilde{\mathbf{m}}_{k+1}(t) - \tilde{\mathbf{m}}_{k+1}(0)] \\
&\quad -[\mathbf{m}_k(s) - \mathbf{m}_k(1)] + [\mathbf{m}_{k+1}(t) - \mathbf{m}_{k+1}(0)]| \\
&\leq \left|(s-1)\int_0^1 [\tilde{\mathbf{m}}_k'[1+\xi(s-1)] - \mathbf{m}_k'[1+\xi(s-1)]]\, d\xi\right| \\
&\quad + \left|t\int_0^1 \left[\tilde{\mathbf{m}}_{k+1}'(t\eta) - \mathbf{m}_{k+1}'(t\eta)\right]\, d\eta\right| \\
&\leq C\, h^p(1-s) + C\, h^p t \\
&\leq C\, h_k^p(1-s+t),
\end{aligned}$$

and

$$\begin{aligned}
|\mathbf{b}| &= |\mathbf{m}_k(s) - \mathbf{m}_{k+1}(t)| \\
&= \left|\int_{\tau_k-(1-s)h_k}^{\tau_k+th_{k+1}} \mathbf{F}'(\tau)\, d\tau\right| \\
&= \left|\int_{\tau_k-(1-s)h_k}^{\tau_k+th_{k+1}} [\mathbf{F}'(\tau_k) + O(h)]\, d\tau\right| \\
&\geq C\,[th_{k+1} + (1-s)h_k] \\
&\geq C\, h_k(1-s+t).
\end{aligned} \tag{3.2.11}$$

We therefore obtain,

$$A \leq C\, h^{p-1}.$$

The next case is a general one.

Case 3: When $\mathbf{x}$ and $\mathbf{y}$ are in the different boundary elements with $|k-l| \geq 2$, that means $|\mathbf{m}_k(s) - \mathbf{m}_l(t)| > Ch$, so

$$\begin{aligned} A &= \left|\log \frac{|\mathbf{a}|}{|\mathbf{b}|}\right| \\ &\leq C\, \frac{|\tilde{\mathbf{m}}_k(s) - \mathbf{m}_k(s)| + |\tilde{\mathbf{m}}_l(t) - \mathbf{m}_l(t)|}{h} \leq C\, h^{p-1}. \end{aligned}$$

The Lemma is now proved. $\square$

The error estimate $|a - a_h|$ follows from Lemma 3.2.4.

Theorem 3.2.2 *If the approximation $\tilde{\mathbf{m}}_k(s) \approx \mathbf{m}_k(s)$ is accurate to order p, then*

$$|a(u_h, v_h) - a_h(u_h, v_h)| \leq Ch^{p-1}\, \|u_h\|_0\, \|v_h\|_0 \qquad \forall u_h, v_h \in S_h^{r,e}. \tag{3.2.12}$$

Proof. From the expression (3.2.7) and Lemma 3.2.4, we have

$$\begin{aligned} &|a(u_h, v_h) - a_h(u_h, v_h)| \\ \leq\;& Ch^{p-1} \sum_{k=1}^{N} \sum_{l=1}^{N} \int_{\triangle_k} \int_{\triangle_l} |u_h(\mathbf{y})\, v_h(\mathbf{x})|\, d\sigma_{\mathbf{x}}\, d\sigma_{\mathbf{y}} \\ \leq\;& Ch^{p-1} \sum_{k=1}^{N} \int_{\triangle_k} |u_h(\mathbf{y})|\, d\sigma_{\mathbf{y}} \sum_{l=1}^{N} \int_{\triangle_l} |v_h(\mathbf{x})|\, d\sigma_{\mathbf{x}} \\ =\;& Ch^{p-1} \int_{\Gamma} |u_h(\mathbf{y})|\, d\sigma_{\mathbf{y}} \int_{\Gamma} |v_h(\mathbf{x})|\, d\sigma_{\mathbf{x}} \\ \leq\;& Ch^{p-1}\, \|u_h\|_0\, \|v_h\|_0. \end{aligned}$$

$\square$

3.3 A Sharper Estimate of the Bilinear Form

It turns out that in a certain sense, the estimate (3.2.12) can be improved upon using a trick from Le Roux [32] and Nedelec [36].

We remark that our technical formulation of the boundary approximation differs from the one in [32] in the following sense. In [32], the approximate solution of the integral equation is defined on the approximate boundary and is then mapped onto the exact boundary. In the approach we use in this thesis, the approximate solution is thought of as being defined directly on the real boundary.

We shall estimate $a\left(u_h^*, v_h^*\right) - a_h(u_h, v_h)$ for a special choice of $u_h^* \approx u_h$ and $v_h^* \approx v_h$. In this way, we shall obtain a better bound than the one in Theorem 3.2.2.

Define a neighbourhood of Γ,

$$N_\delta = \left\{ \mathbf{z} \in \mathbb{R}^2 : \operatorname{dist}(\mathbf{z}, \Gamma) < \delta \right\}.$$

Assuming Γ is C^∞ and δ is sufficiently small, there exists for each $\mathbf{z} \in N_\delta$ a unique point $\Psi(\mathbf{z}) \in \Gamma$ satisfying

$$|\mathbf{z} - \Psi(\mathbf{z})| = \min_{\mathbf{y} \in \Gamma} |\mathbf{z} - \mathbf{y}|.$$

For each k, if the curve $\tilde{\triangle}_k$ is contained in N_δ, then Ψ determines a smooth diffeomorphism of $\tilde{\triangle}_k$ onto $\triangle_k$, and given a point

$$\mathbf{y} = \mathbf{m}_k(s) \in \triangle_k,$$

we write

$$\tilde{\mathbf{y}} = \tilde{\mathbf{m}}_k(s) \in \tilde{\triangle}_k \qquad \text{and} \qquad \mathbf{y}^* = \Psi(\tilde{\mathbf{y}}) = \mathbf{m}_k(s^*) \in \triangle_k.$$

(Remember that, by assumption, $\triangle_k$ and $\tilde{\triangle}_k$ have the same end-points.)

In this way, we obtain a smooth bijection

$$g_k : [0,1] \to [0,1]$$

given by

$$g_k(s) = s^* \qquad \text{for } 0 \leq s \leq 1.$$

Moreover, the line $\tilde{\mathbf{y}} - \Psi(\tilde{\mathbf{y}})$ is orthogonal to the tangent to Γ at $\Psi(\tilde{\mathbf{y}})$, so

$$\mathbf{m}_k'(s^*) \cdot [\mathbf{m}_k(s^*) - \tilde{\mathbf{m}}_k(s)] = 0. \tag{3.3.1}$$

Lemma 3.3.1 *The norm of the first derivative of $\mathbf{m}_k$ satisfies the lower bound*

$$|\mathbf{m}_k'(s)| \geq Ch_k \qquad \textit{for } 0 \leq s \leq 1,$$

and if $\tilde{\mathbf{m}}_k \approx \mathbf{m}_k$ to order $p \geq 2$, then for h_k sufficiently small,

$$|\tilde{\mathbf{m}}_k'(s)| \geq Ch_k.$$

Proof. The lower bound for $|\mathbf{m}_k'|$ follows at once from (2.2.2) and (3.2.3). Next, by (3.2.1),

$$\begin{aligned} |\tilde{\mathbf{m}}_k'(s)| &= |\mathbf{m}_k'(s) + \tilde{\mathbf{m}}_k'(s) - \mathbf{m}_k'(s)| \\ &\geq |\mathbf{m}_k'(s)| - |\tilde{\mathbf{m}}_k'(s) - \mathbf{m}_k'(s)| \\ &\geq Ch_k - Ch_k^2 \\ &\geq Ch_k. \end{aligned}$$

Lemma 3.3.2 *If $\tilde{\mathbf{m}}_k$ is an $O(h^p)$ order of approximation to $\mathbf{m}_k$, for $s \in [0,1]$, we have*

$$s^* - s = O\left(h^{p-1}\right). \tag{3.3.2}$$

Moreover, if $t \in [0,1]$ *such that* $t \neq s$, *then*

$$\frac{|s^* - t^*|}{|s - t|} = 1 + O(h^{p-1}). \tag{3.3.3}$$

Proof. Using Taylor approximation and Lemma 3.2.2,

$$\begin{aligned}
&\mathbf{m}_k'(s^*) \cdot [\mathbf{m}_k(s^*) - \tilde{\mathbf{m}}_k(s)] \\
&= \mathbf{m}_k'(s^*) \cdot [\mathbf{m}_k(s^*) - \tilde{\mathbf{m}}_k(s^*)] + \mathbf{m}_k'(s^*) \cdot [\tilde{\mathbf{m}}_k(s^*) - \tilde{\mathbf{m}}_k(s)] \\
&= O(h) \cdot O(h^p) + \mathbf{m}_k'(s^*) \cdot \left[(s^* - s) \int_0^1 \tilde{\mathbf{m}}_k' [s + \eta(s^* - s)] \, d\eta\right] \\
&= O\left(h^{p+1}\right) + I_h(s, s^*) \, (s^* - s),
\end{aligned}$$

where $|I_h(s, s^*)| \geq Ch^2$, by Lemma 3.3.1. Since

$$\mathbf{m}_k'(s^*) \cdot [\mathbf{m}_k(s^*) - \tilde{\mathbf{m}}_k(s)] = 0,$$

we deduce that

$$s^* - s = \frac{O\left(h^{p+1}\right)}{I_h(s, s^*)} = O\left(h^{p-1}\right). \tag{3.3.4}$$

Moreover, by the Mean Value Theorem, from $|s^* - t^*| = |g_k(s) - g_k(t)|$ we have

$$\frac{|s^* - t^*|}{|s - t|} = g_k'(\xi)$$

for some ξ between s and t. Let

$$\Phi(s, s^*) = \mathbf{m}_k'(s^*) \cdot [\mathbf{m}_k(s^*) - \tilde{\mathbf{m}}_k(s)] \tag{3.3.5}$$

so that (3.3.1) is

$$\Phi[s, g_k(s)] = 0,$$

giving

$$\frac{\partial \Phi}{\partial s} [s, g_k(s)] + \frac{\partial \Phi}{\partial s^*} [s, g_k(s)] \, g_k'(s) = 0,$$

and hence

$$g_k'(s) = -\frac{\partial \Phi / \partial s \, [s, g_k(s)]}{\partial \Phi / \partial s^* \, [s, g_k(s)]} \qquad \text{for } \frac{\partial \Phi}{\partial s^*} [s, g_k(s)] \neq 0.$$

From (3.3.5), we have

$$
\begin{aligned}
-\frac{\partial \Phi}{\partial s}[s, g_k(s)] &= \mathbf{m}_k'(s^*) \cdot \tilde{\mathbf{m}}_k'(s) \\
&= -\mathbf{m}_k'(s^*) \cdot [\mathbf{m}_k'(s^*) - \tilde{\mathbf{m}}_k'(s)] + |\mathbf{m}_k'(s^*)|^2 \\
&= -\mathbf{m}_k'(s^*) \cdot [\mathbf{m}_k'(s^*) - \tilde{\mathbf{m}}_k'(s^*)] \\
&\quad -\mathbf{m}_k'(s^*) \cdot [\tilde{\mathbf{m}}_k'(s^*) - \tilde{\mathbf{m}}_k'(s)] + |\mathbf{m}_k'(s^*)|^2 \\
&= O(h \cdot h^p) + O(h) \cdot \left[(s^* - s) \int_0^1 \tilde{\mathbf{m}}_k''[s + \eta(s^* - s)]\, d\eta \right] + |\mathbf{m}_k'(s^*)|^2 \\
&= O\left(h^{p+1}\right) + O(h^3)\,(s^* - s) + |\mathbf{m}_k'(s^*)|^2 \\
&= |\mathbf{m}_k'(s^*)|^2 \left[1 + O\left(h^{p-1}\right) + O(h)\,(s^* - s)\right] \\
&= |\mathbf{m}_k'(s^*)|^2 \left[1 + O\left(h^{p-1}\right) + O(h \cdot h^{p-1})\right] \qquad \text{by (3.3.4)} \\
&= |\mathbf{m}_k'(s^*)|^2 \left[1 + O\left(h^{p-1}\right)\right],
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \Phi}{\partial s^*}[s, g_k(s)] &= \mathbf{m}_k''(s^*) \cdot [\mathbf{m}_k(s^*) - \tilde{\mathbf{m}}_k(s)] + |\mathbf{m}_k'(s^*)|^2 \\
&= \mathbf{m}_k''(s^*) \cdot [\mathbf{m}_k(s^*) - \tilde{\mathbf{m}}_k(s^*)] \\
&\quad +\mathbf{m}_k''(s^*) \cdot [\tilde{\mathbf{m}}_k(s^*) - \tilde{\mathbf{m}}_k(s)] + |\mathbf{m}_k'(s^*)|^2 \\
&= O\left(h^2 \cdot h^p\right) + O(h^3)\,(s^* - s) + |\mathbf{m}_k'(s^*)|^2 \\
&= |\mathbf{m}_k'(s^*)|^2 \left[1 + O(h^p) + O(h)\,(s^* - s)\right] \\
&= |\mathbf{m}_k'(s^*)|^2 \left[1 + O(h^p) + O(h^p)\right] \\
&= |\mathbf{m}_k'(s^*)|^2 \left[1 + O(h^p)\right],
\end{aligned}
$$

hence

$$
g_k'(s) = \frac{1 + O\left(h^{p-1}\right)}{1 + O\left(h^p\right)} = 1 + O\left(h^{p-1}\right). \tag{3.3.6}
$$

□

Using Lemma 3.3.2, we can show

Theorem 3.3.1 *If the approximation* $\tilde{\mathbf{m}}_k(s) \approx \mathbf{m}_k(s)$ *is accurate to order* $p \geq 2$, *then, for* $0 \leq s \leq 1$,

$$|\mathbf{m}_k(s^*) - \tilde{\mathbf{m}}_k(s)| \leq Ch^p, \tag{3.3.7}$$

$$\left|\mathbf{m}_k^{'}(s^*) - \tilde{\mathbf{m}}_k^{'}(s)\right| \leq Ch^p, \tag{3.3.8}$$

where C *is independent of* s, s^*, k *and* h.

Proof. We use the result of Theorem 3.2.1 and Lemma 3.3.2,

$$\begin{aligned} |\mathbf{m}_k(s^*) - \tilde{\mathbf{m}}_k(s)| &\leq |\mathbf{m}_k(s^*) - \mathbf{m}_k(s)| + |\mathbf{m}_k(s) - \tilde{\mathbf{m}}_k(s)| \\ &= O(h(s^* - s)) + O\left(h^p\right) = O\left(h^p\right), \end{aligned}$$

and

$$\begin{aligned} \left|\mathbf{m}_k^{'}(s^*) - \tilde{\mathbf{m}}_k^{'}(s)\right| &\leq \left|\mathbf{m}_k^{'}(s^*) - \mathbf{m}_k^{'}(s)\right| + \left|\mathbf{m}_k^{'}(s) - \tilde{\mathbf{m}}_k^{'}(s)\right| \\ &= \left(h^2(s^* - s)\right) + O\left(h^p\right) = O\left(h^p\right). \end{aligned}$$

□

We now define u_h^* by

$$u_h^*\left[\mathbf{m}_k(s^*)\right] = u_h\left[\mathbf{m}_k(s)\right] \frac{\left|\mathbf{m}_k^{'}(s)\right|}{\left|\mathbf{m}_k^{'}(s^*)\right|} \frac{ds}{ds^*}, \tag{3.3.9}$$

where $u_h \in S_h^{r,e}$ is the perturbed Galerkin approximation given by (3.1.8), and think about the error in the approximation $a_h(u_h, v_h) \approx a\left(u_h^*, v_h^*\right)$.

Let $\mathbf{y}^* = \mathbf{m}_k(s^*) \in \Delta_k$ and define the bilinear form by

$$\begin{aligned} &a\left(u_h^*, v_h^*\right) \\ &= \int_\Gamma \int_\Gamma K(\mathbf{x}^* - \mathbf{y}^*)\, u_h^*(\mathbf{y}^*)\, v_h^*(\mathbf{x}^*)\, d\sigma_{\mathbf{x}^*}\, d\sigma_{\mathbf{y}^*} \end{aligned}$$

$$= \sum_{k=1}^{N}\sum_{l=1}^{N}\int_0^1\int_0^1 K\left[\mathbf{m}_l(t^*) - \mathbf{m}_k(s^*)\right]\, u_h^*[\mathbf{m}_k(s^*)]\, v_h^*[\mathbf{m}_l(t^*)]\, \left|\mathbf{m}_l'(t^*)\right|\, \left|\mathbf{m}_k'(s^*)\right|\, ds^*\, dt^*$$

$$= \sum_{k=1}^{N}\sum_{l=1}^{N}\int_0^1\int_0^1 K\left[\mathbf{m}_l(t^*) - \mathbf{m}_k(s^*)\right]\, u_h[\mathbf{m}_k(s)]\, v_h[\mathbf{m}_l(t)]$$

$$\frac{\left|\mathbf{m}_k'(s)\right|}{\left|\mathbf{m}_k'(s^*)\right|}\, \frac{\left|\mathbf{m}_l'(t)\right|}{\left|\mathbf{m}_l'(t^*)\right|}\, \frac{ds}{ds^*}\, \frac{dt}{dt^*}\, \left|\mathbf{m}_k'(s^*)\right|\, \left|\mathbf{m}_l'(t^*)\right|\, ds^*\, dt^*$$

$$= \sum_{k=1}^{N}\sum_{l=1}^{N}\int_0^1\int_0^1 K\left[\mathbf{m}_l(t^*) - \mathbf{m}_k(s^*)\right]\, u_h[\mathbf{m}_k(s)]\, v_h[\mathbf{m}_l(t)]\, \left|\mathbf{m}_l'(t)\right|\, \left|\mathbf{m}_k'(s)\right|\, ds\, dt.$$

One therefore deduces

$$\begin{aligned} & a\left(u_h^*, v_h^*\right) - a_h(u_h, v_h) \\ & = \sum_{k=1}^{N}\sum_{l=1}^{N}\int_0^1\int_0^1 \left(K\left[\mathbf{m}_l(t^*) - \mathbf{m}_k(s^*)\right] - K\left[\tilde{\mathbf{m}}_l(t) - \tilde{\mathbf{m}}_k(s)\right]\right) \\ & \quad u_h[\mathbf{m}_k(s)]\, v_h[\mathbf{m}_l(t)]\, \left|\mathbf{m}_l'(t)\right|\, \left|\mathbf{m}_k'(s)\right|\, ds\, dt. \end{aligned} \tag{3.3.10}$$

To be able to compare the bilinear form a and a_h, it is thus necessary to study the error between $K(\mathbf{x}^* - \mathbf{y}^*)$ and $K_h(\mathbf{x}, \mathbf{y})$.

Lemma 3.3.3 *For h sufficiently small,*

$$|\mathbf{m}_k(s^*) - \mathbf{m}_k(t^*)| \geq C h_k |s - t|, \tag{3.3.11}$$

and

$$|\mathbf{m}_k(s^*) - \mathbf{m}_{k+1}(t^*)| \geq C h_k (1 - s + t). \tag{3.3.12}$$

Proof. By Lemma 3.3.2, we have

$$\begin{aligned} |\mathbf{m}_k(s^*) - \mathbf{m}_k(t^*)| &= \left|(s^* - t^*)\int_0^1 \mathbf{m}_k'[t^* + \zeta(s^* - t^*)]\, d\zeta\right| \\ &\geq C h_k\, |s^* - t^*| \end{aligned}$$

$$\begin{aligned} &= Ch_k\,|s-t|\,\frac{|s^*-t^*|}{|s-t|} \\ &= Ch_k|s-t| + Ch^{p-1}|s-t| \\ &\geq Ch_k|s-t|. \end{aligned}$$

We rewrite $|\mathbf{m}_k(s^*) - \mathbf{m}_{k+1}(t^*)|$ as $|\mathbf{m}_k(s^*) - \mathbf{m}_k(1) + \mathbf{m}_{k+1}(0) - \mathbf{m}_{k+1}(t^*)|$, and use the same techniques as above to obtain

$$|\mathbf{m}_k(s^*) - \mathbf{m}_{k+1}(t^*)| \geq Ch_k(1-s+t).$$

$\square$

In addition to (3.1.4) and (3.1.5), we now introduce the notation

$$\mathbf{x}^* = \mathbf{m}_l(t^*) \in \triangle_l \qquad \text{and} \qquad \mathbf{y}^* = \mathbf{m}_k(s^*) \in \triangle_k.$$

Lemma 3.3.4 *If $\tilde{\mathbf{m}}_k$ is an $O(h^p)$ order of approximation to $\mathbf{m}_k$ with $p \geq 2$ and if (3.3.1) holds, then*

$$|K(\mathbf{x}^* - \mathbf{y}^*) - K_h(\mathbf{x},\mathbf{y})| \leq Ch^p \qquad \textit{for } \mathbf{x},\mathbf{y} \in \Gamma. \tag{3.3.13}$$

Proof. Using Lemma 3.2.3 with

$$\begin{aligned} \mathbf{a} &= \tilde{\mathbf{x}} - \tilde{\mathbf{y}} = \tilde{\mathbf{m}}_l(t) - \tilde{\mathbf{m}}_k(s) \\ \mathbf{b} &= \mathbf{x}^* - \mathbf{y}^* = \mathbf{m}_l(t^*) - \mathbf{m}_k(s^*), \end{aligned}$$

we reduce (3.3.13) to

$$\left||\mathbf{a}|^2 - |\mathbf{b}|^2\right| \leq Ch^p\,|\mathbf{b}|^2 \tag{3.3.14}$$

with

$$\left|\log\frac{|\mathbf{a}|^2}{|\mathbf{b}|^2}\right| \leq 2\frac{\left||\mathbf{a}|^2 - |\mathbf{b}|^2\right|}{|\mathbf{b}|^2}.$$

One sees that

$$\begin{aligned}
|\mathbf{a}|^2 - |\mathbf{b}|^2 &= \mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b} \\
&= (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) + \mathbf{b}\cdot(\mathbf{a}-\mathbf{b}) + (\mathbf{a}-\mathbf{b})\cdot\mathbf{b} \\
&= 2\,(\mathbf{a}-\mathbf{b})\cdot\mathbf{b} + |\mathbf{a}-\mathbf{b}|^2 \\
&= 2I + II,
\end{aligned}$$

where

$$I = ([\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_l(t))] - [\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)]) \cdot (\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*))\,,$$
$$II = |[\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_l(t)] - [\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)]|^2\,.$$

We consider I and II separately. Firstly we decompose I into two terms, i.e.,

$$\begin{aligned}
I &= (\tilde{\mathbf{m}}_k(s) - \mathbf{m}_k(s^*)) \cdot (\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)) \\
&\quad - (\tilde{\mathbf{m}}_l(t) - \mathbf{m}_l(t^*)) \cdot (\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*))\,.
\end{aligned}$$

Let W_p be the projection of the vector $[\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)]$ on the tangent line to Γ at the point $\mathbf{m}_k(s^*)$, then $W_p \cdot [\tilde{\mathbf{m}}_k(s) - \mathbf{m}_k(s^*)] = 0$, so we can rewrite the first term of I as

$$\begin{aligned}
&(\tilde{\mathbf{m}}_k(s) - \mathbf{m}_k(s^*)) \cdot (\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)) \\
&= (\tilde{\mathbf{m}}_k(s) - \mathbf{m}_k(s^*)) \cdot (\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*) - W_p)\,.
\end{aligned}$$

Since the curve Γ is smooth,

$$|\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*) - W_p| \le C\,|\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)|^2,$$

so by using the Cauchy-Schwarz inequality and Theorem 3.3.1, we then obtain for the first term of I,

$$|(\tilde{\mathbf{m}}_k(s) - \mathbf{m}_k(s^*)) \cdot (\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*))|$$

$$\leq |\tilde{\mathbf{m}}_k(s) - \mathbf{m}_k(s^*)|\ |\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*) - W_p|$$
$$\leq Ch^p\,|\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)|^2.\mathbf{m}_k(s) - \tilde{\mathbf{m}}_l(t)|^2.$$

Similarly,

$$|(\tilde{\mathbf{m}}_l(t) - \mathbf{m}_l(t^*)) \cdot (\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*))| \leq Ch^p\,|\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)|^2.$$

Now, we consider the different cases for II.

Case i. If $k = l$. From Taylor's formula (3.2.6), we have

$$\begin{aligned} II &= |[\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_k(t)] - [\mathbf{m}_k(s^*) - \mathbf{m}_k(t^*)]|^2 \\ &= \left|(s-t)\int_0^1 \tilde{\mathbf{m}}_k'[t+\xi(s-t)]\,d\xi + (s^*-t^*)\int_0^1 \mathbf{m}_k'[t^*+\xi^*(s^*-t^*)]\,d\xi^*\right|^2, \end{aligned}$$

and by Lemma 3.3.2, Theorem 3.3.1 and (3.3.11) in Lemma 3.3.3,

$$\begin{aligned} II &\leq |s-t|^2\left|\int_0^1 (\tilde{\mathbf{m}}_k'[t+\xi(s-t)] - \mathbf{m}_k'[t+\xi(s-t)])\,d\xi\right|^2 \\ &\leq Ch_k^{2p}\,|s-t|^2 \\ &\leq Ch_k^{2p-2}\,|\mathbf{m}_k(s^*) - \mathbf{m}_k(t^*)|^2 \\ &\leq Ch^p\,|\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)|^2 \end{aligned}$$

Case ii. If $l = k+1$. Using similar techniques as in Case i, we obtain

$$\begin{aligned} II &= |[\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_l(t)] - [\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)]|^2 \\ &= |[\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_{k+1}(t)] - [\mathbf{m}_k(s^*) - \mathbf{m}_{k+1}(t^*)]|^2 \\ &= |[\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_k(1)] - [\tilde{\mathbf{m}}_{k+1}(t) - \tilde{\mathbf{m}}_{k+1}(0)] \\ &\quad -[\mathbf{m}_k(s^*) - \mathbf{m}_k(1)] + [\mathbf{m}_{k+1}(t^*) - \mathbf{m}_{k+1}(0)]|^2 \\ &\leq C\,h_k^{2p}(1-s+t)^2, \end{aligned}$$

by (3.3.12) in Lemma 3.3.3,

$$\begin{aligned} II &\le Ch_k^{2p-2} \left|\mathbf{m}_k(s^*) - \mathbf{m}_{k+1}(t^*)\right|^2 \\ &\le Ch^p \left|\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)\right|^2. \end{aligned}$$

Case iii. If $|k-l| \ge 2$, then $|\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)| \ge Ch_k$,

$$\begin{aligned} II &= \left|\tilde{\mathbf{m}}_k(s) - \tilde{\mathbf{m}}_l(t) - [\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)]\right|^2 \\ &\le \left||\tilde{\mathbf{m}}_k(s) - \mathbf{m}_k(s^*)| + |\mathbf{m}_l(t^*) - \tilde{\mathbf{m}}_l(t)|\right|^2 \\ &\le Ch_k^{2p} \\ &\le Ch_k^{2p-2} \left|\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)\right|^2 \\ &\le Ch^p \left|\mathbf{m}_k(s^*) - \mathbf{m}_l(t^*)\right|^2. \end{aligned}$$

Hence, we finally deduce

$$\left|\log \frac{|\tilde{\mathbf{x}} - \tilde{\mathbf{y}}|}{|\mathbf{x}^* - \mathbf{y}^*|}\right| = O\left(h^p\right).$$

□

Compare the following estimate with the one in Theorem 3.2.2.

Theorem 3.3.2 *If Γ is approximated to order $p \ge 2$, then we have the improved upper bound*

$$|a(u_h^*, v_h^*) - a_h(u_h, v_h)| \le Ch^p \, \|u_h\|_{H^0(\Gamma)} \, \|v_h\|_{H^0(\Gamma)} \qquad \forall u_h, v_h \in S_h^{r,e}. \tag{3.3.15}$$

Proof. By (3.3.10) and Lemma 3.3.4,

$$\begin{aligned} &|a(u_h^*, v_h^*) - a_h(u_h, v_h)| \\ &\le Ch^p \sum_{k=1}^{N} \sum_{l=1}^{N} \int_{\triangle_k} \int_{\triangle_l} |u_h(\mathbf{y})\, v_h(\mathbf{x})| \, d\sigma_{\mathbf{x}} \, d\sigma_{\mathbf{y}} \end{aligned}$$

$$
\begin{aligned}
&\leq Ch^p \sum_{k=1}^{N} \int_{\triangle_k} |u_h(\mathbf{y})| \, d\sigma_{\mathbf{y}} \sum_{l=1}^{N} \int_{\triangle_l} |v_h(\mathbf{x})| \, d\sigma_{\mathbf{x}} \\
&= Ch^p \int_{\Gamma} |u_h(\mathbf{y})| \, d\sigma_{\mathbf{y}} \int_{\Gamma} |v_h(\mathbf{x})| \, d\sigma_{\mathbf{x}} \\
&\leq Ch^p \, \|u_h\|_{H^0(\Gamma)} \, \|v_h\|_{H^0(\Gamma)}.
\end{aligned}
$$

□

There is a consequence which follows from above immediately,

Corollary 3.3.1 *For $u_h \in S_h^{r,e}$, then*

$$|a(u_h^*, u_h^*) - a_h(u_h, u_h)| \leq Ch^{p-1} \, \|u_h\|^2_{H^{-1/2}(\Gamma)}. \tag{3.3.16}$$

The proof is straightforward by applying the inverse property (2.2.10).

3.4 Other Types of Boundary Approximation

In Section 3.2, two examples of polynomial boundary interpolation were given, satisfying the conditions in Definition 3.2.1. We now give some other types of boundary approximation that also satisfy the conditions required for our theory.

Homogeneous Coordinates

In computer graphics, it is common to represent curves using homogeneous coordinates; see Farin [51, Chapter 15]. A boundary approximation using a piecewise polynomial representation in terms of homogeneous coordinates is equivalent, in Cartesian coordinates, to a piecewise rational approximation.

Indeed, if $\mathbf{F} : [0,1] \to \mathbb{R}^2$ is a parametric representation of Γ as in (2.2.1), then for any function $\Lambda : [0,1] \to \mathbb{R}$ with $|\Lambda(\tau)| \geq C > 0$ for $\tau \in [0,1]$, we can think of $\boldsymbol{\Theta}(\tau) = [\Lambda(\tau)\mathbf{F}(\tau), \Lambda(\tau)]$ as homogeneous coordinates for $\mathbf{F}(\tau)$. If the components of $\boldsymbol{\Theta}(\tau)$ are approximated by piecewise polynomials, then in effect we are approximating

$$\mathbf{F}(\tau) = \left(\frac{\boldsymbol{\Theta}_1(\tau)}{\Lambda(\tau)}, \frac{\boldsymbol{\Theta}_2(\tau)}{\Lambda(\tau)}, \frac{\boldsymbol{\Theta}_3(\tau)}{\Lambda(\tau)}\right)$$

by piecewise rational functions.

Note that with homogeneous coordinates, we can parameterise any conic section exactly using quadratic polynomials [51, Chapter 14].

Shape-Preserving Approximation

Gregory and Delbourgo [52] discussed the use of piecewise-rational interpolants that preserve convexity and monotonicity. Their schemes are quadratic/quadratic, i.e., they use quadratics in the numerator and denominator, and depend on certain derivative parameters. An $O(h^4)$ convergence result can be obtained when accurate derivative values are available. Otherwise, $O(h^3)$ convergence can be obtained when derivative values are determined by local approximations.

Circular Arcs

We can achieve an $O(h^3)$ approximation of the boundary by making $\tilde{\Delta}_h$ in (3.1.3) the arc of the unique circle that passes through the points $\mathbf{F}(\tau_{k-1}), \mathbf{F}(\tau_{k-\frac{1}{2}})$ and $\mathbf{F}(\tau_k)$. (If these three points are collinear, then $\tilde{\Delta}_h$ will have an infinite radius, i.e., $\tilde{\Delta}_h$ will be a line segment.) With such an approximation we can easily make the Jacobian $|\tilde{\mathbf{m}}_k'(t)|, t \in [0,1]$, constant, and can conveniently evaluate the unit normal.

Polar Coordinates

If the parametric representation of the boundary is given in polar form,

$$\Gamma : r = \rho(\theta),$$

then

$$\mathbf{m}_k(s) = \rho(\theta)(\cos\theta, \sin\theta),$$

where $\theta = (1-s)\theta_{k-1} + s\theta_k$, $s \in [0,1]$, and the θ_k play the role of the τ_k, i.e., $0 = \theta_0 < \theta_1 < \cdots < \theta_N = 1$ and $h_k = \theta_k - \theta_{k-1}$. In this case, it is natural to seek $\tilde{\mathbf{m}}_k(s)$ in the form

$$\tilde{\mathbf{m}}_k(s) = \tilde{\rho}(\theta)(\cos\theta, \sin\theta).$$

To achieve $O(h^3)$ accuracy, we can take $\tilde{\rho}$ to be piecewise quadratic with

$\tilde{\rho}(\theta) = \rho(\theta)$ at $\theta_{k-1}, \frac{1}{2}(\theta_{k-1}+\theta_k), \theta_k$. Thus,

$$\begin{aligned} |\tilde{\mathbf{m}}_k(s) - \mathbf{m}_k(s)| &= |[\tilde{\rho}(\theta) - \rho(\theta)]\,(\cos\theta, \sin\theta)| \\ &= O\left(h^3\right), \end{aligned}$$

and also, since

$$\mathbf{m}_k'(s) = [\rho'(\theta)(\cos\theta, \sin\theta) + \rho(\theta)(-\sin\theta, \cos\theta)]\,\frac{d\theta}{ds},$$

where $d\theta/ds = h_k$, we see that

$$\begin{aligned} |\tilde{\mathbf{m}}_k'(s) - \mathbf{m}_k'(s)| &= |[\tilde{\rho}'(\theta) - \rho'(\theta)](\cos\theta, \sin\theta)h_k + [\tilde{\rho}(\theta) - \rho(\theta)](-\sin\theta, \cos\theta)h_k| \\ &= O\left(h^3\right) \end{aligned}$$

Furthermore,

$$\mathbf{m}_k''(s) = [\rho''(\theta)(\cos\theta, \sin\theta) + 2\rho'(\theta)(-\sin\theta, \cos\theta) + \rho(\theta)(-\cos\theta, -\sin\theta)]\,h_k^2$$

and so

$$\begin{aligned} |\tilde{\mathbf{m}}_k''(s) - \mathbf{m}_k''(s)| &= \big|[\tilde{\rho}''(\theta) - \rho''(\theta)]\,(\cos\theta, \sin\theta)h_k^2 + [\rho(\theta) - \tilde{\rho}(\theta)]\,(\cos\theta, \sin\theta)h_k^2 \\ &\quad +2\,[\rho'(\theta) - \tilde{\rho}'(\theta)]\,(\sin\theta, -\cos\theta)h_k^2\big| \\ &= O\left(h^3\right) \end{aligned}$$

Chapter 4

ERROR ESTIMATES

4.1 Stability Property

The convergence of $u_h \to u$ can be established if the perturbed Galerkin equation is stable in an appropriate sense. In fact, we will show (in Corollary 4.1.1) that $\|u_h\|_{H^{-1/2}(\Gamma)} \le C\,\|f_h\|_{H^{1/2}(\Gamma)}$.

Define linear operators R_h and R'_h by

$$R_h\phi[\mathbf{m}_k(s)] = \phi[\mathbf{m}_k(s^*)] \tag{4.1.1}$$

for any $\phi \in H^0(\Gamma)$ and

$$R'_h u[\mathbf{m}_k(s^*)] = u[\mathbf{m}_k(s)]\,\frac{|\mathbf{m}'_k(s)|}{|\mathbf{m}'_k(s^*)|}\,\frac{ds}{ds^*},$$

and let $u^* = R'_h u$. The notation is consistent with (3.3.9), i.e.

$$u^*_h = R'_h u_h, \tag{4.1.2}$$

and in fact R'_h is the transpose of R_h, because for any $\phi \in H^0(\Gamma)$,

$$\begin{aligned}
\langle R'_h u, \phi\rangle &= \sum_{k=1}^{N}\int_0^1 R'_h u[\mathbf{m}_k(s^*)]\,\phi[\mathbf{m}_k(s^*)]\;|\mathbf{m}'_k(s^*)|\;ds^* \\
&= \sum_{k=1}^{N}\int_0^1 u[\mathbf{m}_k(s)]\,\phi[\mathbf{m}_k(s^*)]\;|\mathbf{m}'_k(s)|\;ds \\
&= \langle u, R_h\phi\rangle.
\end{aligned} \tag{4.1.3}$$

In the usual way, we extend R_h and R'_h to act on distributions when necessary.

Lemma 4.1.1 *For $0 \le \alpha \le 1$, the operator R'_h satisfies the uniform bound,*

$$\|R'_h u\|_{H^{-\alpha}(\Gamma)} \le C\|u\|_{H^{-\alpha}(\Gamma)}, \qquad u \in H^{-\alpha}(\Gamma). \tag{4.1.4}$$

Proof. We use the formula (cf. (2.2.5))

$$\|v\|^2_{H^1(\Gamma)} \equiv \|v\|^2_{H^0(\Gamma)} + \|Dv\|^2_{H^0(\Gamma)}.$$

Recall that, see (2.2.4)

$$\mathbf{F}(\tau) = \mathbf{m}_k(s) \qquad \text{for } \tau = (1-s)\,\tau_{k-1} + s\,\tau_k,$$

and therefore,

$$\mathbf{F}(\tau^*) = \mathbf{m}_k(s^*) \qquad \text{for } \tau^* = (1-s^*)\,\tau_{k-1} + s^*\,\tau_k,$$

so,

$$\frac{d\tau}{ds} = \tau_k - \tau_{k-1} = h_k, \qquad \frac{d\tau^*}{ds^*} = \tau_k - \tau_{k-1} = h_k.$$

By the definition of the $H^0(\Gamma)$ norm (see (2.2.5))

$$\begin{aligned}
\|R_h\phi\|^2_{H^0(\Gamma)} &= \int_0^1 |R_h\phi \circ \mathbf{F}(\tau)|^2 \, d\tau \\
&= \sum_{k=1}^{N} \int_0^1 |R_h\phi[\mathbf{m}_k(s)]|^2 \, h_k \, ds \\
&= \sum_{k=1}^{N} \int_0^1 |\phi[\mathbf{m}_k(s^*)]|^2 \, h_k \, \frac{ds}{ds^*} \, ds^* && \text{by (4.1.1)} \\
&\leq C \sum_{k=1}^{N} \int_0^1 |\phi[\mathbf{m}_k(s^*)]|^2 \, h_k \, ds^*,
\end{aligned}$$

since $ds/ds^* = 1 + O(h^{p-1}) = O(1)$,

$$\|R_h\phi\|^2_{H^0(\Gamma)} \leq C \int_0^1 |\phi \circ \mathbf{F}(\tau^*)|^2 \, d\tau^* = C\,\|\phi\|^2_{H^0(\Gamma)},$$

hence

$$\|R_h\phi\|_{H^0(\Gamma)} \leq C\,\|\phi\|_{H^0(\Gamma)}. \tag{4.1.5}$$

Moreover,

$$\begin{aligned}
\|DR_h\phi\|^2_{H^0(\Gamma)} &= \int_0^1 \left|\frac{d}{d\tau}[R_h\phi\circ\mathbf{F}(\tau)]\right|^2 d\tau \\
&= \sum_{k=1}^{N}\int_0^1 \left|\frac{ds}{d\tau}\,\frac{d}{ds}R_h\phi[\mathbf{m}_k(s)]\right|^2 \frac{d\tau}{ds}\,ds \\
&= \sum_{k=1}^{N}\int_0^1 \left|\frac{1}{h_k}\,\frac{d}{ds}R_h\phi[\mathbf{m}_k(s)]\right|^2 h_k\,ds \\
&= \sum_{k=1}^{N}\int_0^1 \left|\frac{1}{h_k}\,\frac{ds^*}{ds}\,\frac{d}{ds^*}\phi[\mathbf{m}_k(s^*)]\right|^2 h_k\,\frac{ds}{ds^*}\,ds^* \\
&\leq C\sum_{k=1}^{N}\int_0^1 \left|\frac{1}{h_k}\,\frac{d}{ds^*}\phi[\mathbf{m}_k(s^*)]\right|^2 h_k\,ds^* \\
&= C\sum_{k=1}^{N}\int_0^1 \left|\frac{1}{h_k}\,\frac{d\tau^*}{ds^*}\,\frac{d}{d\tau^*}\phi[\mathbf{m}_k(s^*)]\right|^2 h_k\,ds^* \\
&= C\sum_{k=1}^{N}\int_0^1 \left|\frac{d}{d\tau^*}[\phi\circ\mathbf{F}(\tau^*)]\right|^2 d\tau^* \\
&= C\,\|D\phi\|^2_{H^0(\Gamma)},
\end{aligned}$$

so

$$\begin{aligned}
\|R_h\phi\|^2_{H^1(\Gamma)} &= \|R_h\phi\|^2_{H^0(\Gamma)} + \|DR_h\phi\|^2_{H^0(\Gamma)} \\
&\leq C\left(\|\phi\|^2_{H^0(\Gamma)} + \|D\phi\|^2_{H^0(\Gamma)}\right) \\
&= C\,\|\phi\|^2_{H^1(\Gamma)},
\end{aligned}$$

giving

$$\|R_h\phi\|_{H^1(\Gamma)} \leq C\,\|\phi\|_{H^1(\Gamma)}. \tag{4.1.6}$$

The estimates (4.1.5) and (4.1.6) show that the operator of R_h is bounded, uniformly in h, on $H^\alpha(\Gamma)$ for $\alpha = 0$ and $\alpha = 1$. Hence, by interpolation, see [6], we have

$$\|R_h\phi\|_{H^\alpha(\Gamma)} \leq C\,\|\phi\|_{H^\alpha(\Gamma)}, \qquad 0 \leq \alpha \leq 1. \tag{4.1.7}$$

Finally, from the definition of the negative norm and from (4.1.3),

$$\begin{aligned}
\|R_h' u\|_{H^{-\alpha}(\Gamma)} &= \sup_{\phi \in H^{\alpha}(\Gamma)} \frac{|\langle R_h' u, \phi\rangle|}{\|\phi\|_{H^{\alpha}(\Gamma)}} \\
&= \sup_{\phi \in H^{\alpha}(\Gamma)} \frac{|\langle u, R_h \phi\rangle|}{\|\phi\|_{H^{\alpha}(\Gamma)}} \\
&\leq C \sup_{R_h\phi \in H^{\alpha}(\Gamma)} \frac{|\langle u, R_h \phi\rangle|}{\|R_h\phi\|_{H^{\alpha}(\Gamma)}} && \text{by (4.1.7)} \\
&= C\,\|u\|_{H^{-\alpha}(\Gamma)}.
\end{aligned}$$

□

Recall from Theorem 2.1.5 that the original bilinear form a is $H^{-1/2}(\Gamma)$-elliptic, i.e., for an $\alpha_0 > 0$,

$$a(u,u) \geq \alpha_0 \,\|u\|^2_{H^{-1/2}(\Gamma)} \qquad \forall u \in H^{-1/2}(\Gamma).$$

Now we present a stability property of the perturbed boundary element Galerkin method.

Theorem 4.1.1 *If Γ is approximated to order $p \geq 2$, then there exists an $h_0 > 0$ and a positive constant α such that*

$$a_h(u_h,u_h) \geq \alpha\,\|u_h\|^2_{H^{-1/2}(\Gamma)} \quad \textit{and} \quad a_h(u_h,u_h) \geq \alpha\,\|u_h^*\|^2_{H^{-1/2}(\Gamma)}$$

for $u_h \in S_h^{r,e}$ and $h \leq h_0$.

Proof. From the inequality (3.3.16) and the $H^{-1/2}(\Gamma)$-ellipticity of a,

$$\begin{aligned}
a_h(u_h,u_h) &= a(u_h^*,u_h^*) - [a(u_h^*,u_h^*) - a_h(u_h,u_h)] \\
&\geq \alpha_0\,\|u_h^*\|^2_{H^{-1/2}(\Gamma)} - C_1 h_0^{p-1}\,\|u_h\|^2_{H^{-1/2}(\Gamma)} \\
&\geq C_2\,\alpha_0\,\|u_h\|^2_{H^{-1/2}(\Gamma)} - C_1\,h_0^{p-1}\,\|u_h\|^2_{H^{-1/2}(\Gamma)} \\
&= \alpha\|u_h\|^2_{H^{-1/2}(\Gamma)},
\end{aligned}$$

where $\alpha = C_2\,\alpha_0 - C_1\,h_0^{p-1}$. Since $p \geq 2$, we can chose h_0 such that $\alpha > 0$. □

Corollary 4.1.1 *The Galerkin method* (3.1.8) *is stable:*

$$\|u_h\|_{H^{-1/2}(\Gamma)} \leq C\,\|f_h\|_{H^{1/2}(\Gamma)}. \tag{4.1.8}$$

Proof.

$$\begin{aligned}
\alpha\,\|u_h\|^2_{H^{-1/2}(\Gamma)} &\leq a_h(u_h, u_h) \\
&= \langle f_h, u_h\rangle \\
&\leq C\,\|f_h\|_{H^{1/2}(\Gamma)}\,\|u_h\|_{H^{-1/2}(\Gamma)},
\end{aligned}$$

hence, the result follows on cancelling $\|u_h\|_{H^{-1/2}(\Gamma)}$. □

4.2 Error Estimates of the Perturbed Galerkin Method

We are now able to establish the error estimates of the perturbed Galerkin method.

Lemma 4.2.1 *The operator R_h' satisfies*

$$\|u - R_h' u\|_{H^0(\Gamma)} \leq Ch^{p-1}\|u\|_{H^1(\Gamma)}, \qquad u \in H^1(\Gamma). \tag{4.2.1}$$

Proof. Recall that $u^* = R_h' u$. We have

$$\|u - u^*\|^2_{H^0(\Gamma)} = \sum_{k=1}^{N} \int_0^1 |u[\mathbf{m}_k(s^*)] - u^*[\mathbf{m}_k(s^*)]|^2\,h_k\,ds^*$$

$$
\begin{aligned}
&= \sum_{k=1}^{N} \int_0^1 \left| u[\mathbf{m}_k(s^*)] - u[\mathbf{m}_k(s)] \frac{|\mathbf{m}_k'(s)|}{|\mathbf{m}_k'(s^*)|} \frac{ds}{ds^*} \right|^2 h_k \, ds^* \\
&\leq \sum_{k=1}^{N} \int_0^1 |u[\mathbf{m}_k(s^*)] - u[\mathbf{m}_k(s)]|^2 \, h_k \, ds \\
&\quad + \sum_{k=1}^{N} \int_0^1 |u[\mathbf{m}_k(s)]|^2 \left| 1 - \frac{|\mathbf{m}_k'(s)|}{|\mathbf{m}_k'(s^*)|} \frac{ds}{ds^*} \right|^2 h_k \, ds.
\end{aligned}
$$

The first term on the right satisfies

$$
\begin{aligned}
&\sum_{k=1}^{N} \int_0^1 |u[\mathbf{m}_k(s^*)] - u[\mathbf{m}_k(s)]|^2 \, h_k \, ds \\
&= \sum_{k=1}^{N} \int_0^1 \left| \int_s^{s^*} |(u \circ \mathbf{m}_k)'(t)| \, dt \right|^2 h_k \, ds \\
&\leq \sum_{k=1}^{N} \int_0^1 |s^* - s| \left| \int_s^{s^*} |(u \circ \mathbf{m}_k)'(t)|^2 \, h_k \, dt \right| ds \\
&\leq C \sum_{k=1}^{N} h^{2(p-1)} \int_0^1 |(u \circ \mathbf{m}_k)'(t)|^2 \, h_k \, dt \qquad \text{by (3.3.2)} \\
&\leq C h^{2(p-1)} \|u\|_{H^1(\Gamma)}^2,
\end{aligned}
$$

and the second term satisfies

$$
\begin{aligned}
&\sum_{k=1}^{N} \int_0^1 |u[\mathbf{m}_k(s)]|^2 \left| 1 - \frac{|\mathbf{m}_k'(s)|}{|\mathbf{m}_k'(s^*)|} \frac{ds}{ds^*} \right|^2 h_k \, ds \\
&\leq \sum_{k=1}^{N} \int_0^1 |u[\mathbf{m}_k(s)]|^2 \left| 1 - \frac{|\mathbf{m}_k'(s)|}{|\mathbf{m}_k'(s^*)|} + \frac{|\mathbf{m}_k'(s)|}{|\mathbf{m}_k'(s^*)|} O\left(h^{p-1}\right) \right|^2 h_k \, ds \\
&= \sum_{k=1}^{N} \int_0^1 |u[\mathbf{m}_k(s)]|^2 \left| \frac{|\mathbf{m}_k'(s^*)| - |\mathbf{m}_k'(s)| + |\mathbf{m}_k'(s)| \, O\left(h^{p-1}\right)}{|\mathbf{m}_k'(s^*)|} \right|^2 h_k \, ds \\
&\leq \sum_{k=1}^{N} \int_0^1 |u[\mathbf{m}_k(s)]|^2 \left| \frac{|\mathbf{m}_k'(s^*) - \mathbf{m}_k'(s)| + |\mathbf{m}_k'(s)| \, O\left(h^{p-1}\right)}{|\mathbf{m}_k'(s^*)|} \right|^2 h_k \, ds \\
&\leq C \sum_{k=1}^{N} \int_0^1 |u[\mathbf{m}_k(s)]|^2 \left| \frac{O\left(h^2(s^* - s)\right) + O\left(h^p\right)}{h_k} \right|^2 h_k \, ds
\end{aligned}
$$

$$
\begin{aligned}
&= C\sum_{k=1}^{N}\int_0^1 |u[\mathbf{m}_k(s)]|^2 \left|\frac{O\left(h^{p+1}\right)+O\left(h^p\right)}{h_k}\right|^2 h_k\,ds \\
&\leq C\sum_{k=1}^{N}\int_0^1 |u[\mathbf{m}_k(s)]|^2 \left|\frac{O(h^{p+1})+O(h^p)}{h_k}\right|^2 h_k\,ds \\
&\leq Ch^{2(p-1)}\|u\|^2_{H^0(\Gamma)},
\end{aligned}
$$

so

$$
\|u-u^*\|_{H^0(\Gamma)} \leq Ch^{p-1}\|u\|_{H^1(\Gamma)}. \tag{4.2.2}
$$

□

Theorem 4.2.1 *Suppose that u is the exact solution of the equation* (2.1.25), *and $u_h \in S_h^{r,e}$ is the perturbed Galerkin approximation satisfying* (3.1.8). *If Γ is approximated to order $p \geq 2$, then*

$$
\|u-u_h^*\|_{H^{-1/2}(\Gamma)} \leq C\left(\|f_h - R_h f\|_{H^{1/2}(\Gamma)} + h^{p-1/2}\|u\|_{H^0(\Gamma)} + h^{r+1/2}\|u\|_{H^r(\Gamma)}\right), \tag{4.2.3}
$$

where R_h is given by (4.1.1).

Proof. For all $v_h \in S_h^{r,e}$,

$$
\begin{aligned}
&a_h(u_h - v_h, u_h - v_h) \\
&= a_h(u_h, u_h - v_h) - a_h(v_h, u_h - v_h) \\
&\quad + a(v_h^*, (u_h - v_h)^*) - a(v_h^*, (u_h - v_h)^*) \\
&\quad + a(u, (u_h - v_h)^*) - a(u, (u_h - v_h)^*) \\
&= \langle f_h, u_h - v_h\rangle - \langle f, R_h'(u_h - v_h)\rangle \qquad \text{by (4.1.2)} \\
&\quad + a(v_h^*, (u_h - v_h)^*) - a_h(v_h, u_h - v_h) \\
&\quad + a(u, (u_h - v_h)^*) - a(v_h^*, (u_h - v_h)^*)
\end{aligned}
$$

$$
\begin{aligned}
&= \langle f_h, u_h - v_h\rangle - \langle R_h f, u_h - v_h\rangle \qquad\qquad \text{by (4.1.3)}\\
&\quad + a(v_h^*, (u_h - v_h)^*) - a_h(v_h, u_h - v_h)\\
&\quad + a(u, (u_h - v_h)^*) - a(v_h^*, (u_h - v_h)^*).
\end{aligned}
$$

If we apply Theorem 2.2.1, Theorem 3.3.2 and Lemma 4.1.1 to above, then

$$
\begin{aligned}
|a_h(u_h - v_h, u_h - v_h)| &\le |\langle f_h, u_h - v_h\rangle - \langle R_h f, u_h - v_h\rangle|\\
&\quad + |a(v_h^*, (u_h - v_h)^*) - a_h(v_h, u_h - v_h)|\\
&\quad + |a(u - v_h^*, (u_h - v_h)^*)|\\
&\le |\langle f_h - R_h f, u_h - v_h\rangle|\\
&\quad + Ch^p \|v_h\|_{H^0(\Gamma)} \|u_h - v_h\|_{H^0(\Gamma)}\\
&\quad + C \|u - v_h^*\|_{H^{-1/2}(\Gamma)} \|(u_h - v_h)^*\|_{H^{-1/2}(\Gamma)}\\
&\le C\,[\|f_h - R_h f\|_{H^{1/2}(\Gamma)} \|(u_h - v_h)^*\|_{H^{-1/2}(\Gamma)}\\
&\quad + h^{p-1/2} \|v_h\|_{H^0(\Gamma)} \|(u_h - v_h)^*\|_{H^{-1/2}(\Gamma)}\\
&\quad + \|u - v_h^*\|_{H^{-1/2}(\Gamma)} \|(u_h - v_h)^*\|_{H^{-1/2}(\Gamma)}].
\end{aligned}
$$

Recall that by Theorem 4.1.1,

$$
\alpha \,\|(u_h - v_h)^*\|^2_{H^{-1/2}(\Gamma)} \le a_h(u_h - v_h, u_h - v_h), \qquad \alpha > 0,
$$

so by cancellation

$$
\|u_h^* - v_h^*\|_{H^{-1/2}(\Gamma)} \le \frac{C}{\alpha}\left(\|f_h - R_h f\|_{H^{1/2}(\Gamma)} + h^{p-1/2}\|v_h\|_{H^0(\Gamma)} + \|u - v_h^*\|_{H^{-1/2}(\Gamma)}\right). \tag{4.2.4}
$$

Let P_h be the operator of orthogonal projection of $L^2(\Gamma)$ in the subspace $S_h^{r,e}$, and choose $v_h = P_h u$, then

$$
\|v_h\|_{H^0(\Gamma)} \le \|u\|_{H^0(\Gamma)},
$$

and

$$
\begin{aligned}
\|u - v_h^*\|_{H^{-1/2}(\Gamma)} &\leq C\|(R_h')^{-1}u - v_h\|_{H^{-1/2}(\Gamma)} \\
&\leq Ch^{r+1/2}\left(\sum_{k=1}^{N}\|(R_h')^{-1}u\|^2_{H^r(\triangle_k)}\right)^{1/2} \\
&\leq Ch^{r+1/2}\|u\|_{H^r(\Gamma)}, \qquad (4.2.5)
\end{aligned}
$$

so

$$
\|u_h^* - v_h^*\|_{H^{-1/2}(\Gamma)} \leq C\left(\|f_h - R_h f\|_{H^{1/2}(\Gamma)} + h^{p-1/2}\|u\|_{H^0(\Gamma)} + h^{r+1/2}\|u\|_{H^r(\Gamma)}\right). \qquad (4.2.6)
$$

Finally, the error estimate is obtained as follows:

$$
\begin{aligned}
\|u - u_h^*\|_{H^{-1/2}(\Gamma)} &\leq \|u - v_h^*\|_{H^{-1/2}(\Gamma)} + \|u_h^* - v_h^*\|_{H^{-1/2}(\Gamma)} \\
&\leq C\left(\|f_h - R_h f\|_{H^{1/2}(\Gamma)} + h^{p-1/2}\|u\|_{H^0(\Gamma)} + h^{r+1/2}\|u\|_{H^r(\Gamma)}\right).
\end{aligned}
$$

□

In our numerical experiments, it will be more convenient to measure errors in the L^2 norm, for which the following bound holds.

Corollary 4.2.1 *Suppose that u is the exact solution of the equation* (2.1.25), *and $u_h \in S_h^{r,e}$ is the perturbed Galerkin approximation satisfying* (3.1.8). *If Γ is approximated to order $p \geq 2$, and if the mesh is quasi-uniform then*

$$
\|u - u_h\|_{H^0(\Gamma)} \leq C\left(h^{-1/2}\|f_h - R_h f\|_{H^{1/2}(\Gamma)} + h^{p-1}\|u\|_{H^1(\Gamma)} + h^r\|u\|_{H^r(\Gamma)}\right), \qquad (4.2.7)
$$

where R_h is given by (4.1.1).

Proof. By triangle inequality and Lemma 4.2.1,

$$\begin{aligned}\|u-u_h\|_{H^0(\Gamma)} &\le \|u_h-u_h^*\|_{H^0(\Gamma)}+\|u-u_h^*\|_{H^0(\Gamma)}\\ &\le Ch^{p-1}\|u_h\|_{H^1(\Gamma)}+\|u-u_h^*\|_{H^0(\Gamma)},\end{aligned}$$

and from the inverse property in Theorem 2.2.1 and (4.2.6),

$$\begin{aligned}\|u-u_h^*\|_{H^0(\Gamma)} &\le \|u-v_h^*\|_{H^0(\Gamma)}+\|u_h^*-v_h^*\|_{H^0(\Gamma)}\\ &\le Ch^r\|u\|_{H^r(\Gamma)}+Ch^{-1/2}\|u_h^*-v_h^*\|_{H^{-1/2}(\Gamma)}\\ &\le C\left(h^{-1/2}\|f_h-R_hf\|_{H^{1/2}(\Gamma)}+h^{p-1}\|u\|_{H^1(\Gamma)}+h^r\|u\|_{H^r(\Gamma)}\right),\end{aligned}$$

hence, the proof is completed. □

The perturbed Galerkin solution also converges faster in a more negative norm, resulting in faster convergence of the potential. This is an example of a super-convergence property.

Theorem 4.2.2 *Let A be the single layer operator given by (2.1.9), and suppose that the $u_h \in S_h^{r,e}$ is the perturbed Galerkin approximation to u, given by (3.1.8). If Γ is approximated to order $p \ge 2$, then*

$$\begin{aligned}\|u-u_h^*\|_{H^{-r-1}(\Gamma)} &\le Ch^{r+1/2}\|f_h-R_hf\|_{H^{1/2}(\Gamma)}+Ch^{r+p}\|u\|_{H^0(\Gamma)}+Ch^{2r+1}\|u\|_{H^r(\Gamma)}\\ &\quad +Ch^p\|u\|_{H^1(\Gamma)}+\|f_h-R_hf\|_{H^0(\Gamma)}, \qquad (4.2.8)\end{aligned}$$

where R_h is given by (4.1.1).

Proof. The operator $A : H^r(\Gamma) \to H^{r+1}(\Gamma)$ has a bounded inverse, so for any $\phi \in H^{r+1}(\Gamma)$, there exists a unique $\psi \in H^r(\Gamma)$ satisfying

$$A\psi = \phi,$$

and we have

$$\|\psi\|_{H^r(\Gamma)} \le C\,\|\phi\|_{H^{r+1}(\Gamma)}.$$

By definition of the negative norm,

$$\|u-u_h^*\|_{H^{-r-1}(\Gamma)} = \sup_{\phi\in H^{r+1}(\Gamma)} \frac{|\langle u-u_h^*,\phi\rangle_0|}{\|\phi\|_{H^{r+1}(\Gamma)}}. \tag{4.2.9}$$

Let P_h be the orthogonal projection operator of $L^2(\Gamma)$ in $S_h^{r,e}$, and choose $\psi_h = P_h\psi$, then

$$\|\psi_h\|_{H^0(\Gamma)} \le \|\psi\|_{H^0(\Gamma)} \le \|\psi\|_{H^r(\Gamma)} \tag{4.2.10}$$

and by (4.2.5),

$$\begin{aligned}
|\langle u-u_h^*,\phi\rangle_0| &= |\langle u-u_h^*,A\psi\rangle_0| \\
&= |a(u-u_h^*,\psi)| \\
&= |a(u-u_h^*,\psi-\psi_h^*)+a(u-u_h^*,\psi_h^*)| \\
&\le \|u-u_h^*\|_{H^{-1/2}(\Gamma)}\,\|\psi-\psi_h^*\|_{H^{-1/2}(\Gamma)} + |a(u-u_h^*,\psi_h^*)| \\
&\le \|u-u_h^*\|_{H^{-1/2}(\Gamma)}\,Ch^{r+1/2}\|\psi\|_{H^r(\Gamma)} + |a(u-u_h^*,\psi_h^*)| \\
&\le \|u-u_h^*\|_{H^{-1/2}(\Gamma)}\,Ch^{r+1/2}\|\phi\|_{H^{r+1}(\Gamma)} + |a(u-u_h^*,\psi_h^*)|\,,
\end{aligned}$$

and by Theorem 3.3.2,

$$\begin{aligned}
|a(u-u_h^*,\psi_h^*)| &\le |a_h(u_h,\psi_h)-a(u_h^*,\psi_h^*)| + |a(u,\psi_h^*)-a_h(u_h,\psi_h)| \\
&\le Ch^p\|u_h\|_{H^0(\Gamma)}\|\psi_h\|_{H^0(\Gamma)} + \langle f,\psi_h^*\rangle - \langle f_h,\psi_h\rangle \\
&\le Ch^p\|u\|_{H^0(\Gamma)}\|\psi\|_{H^r(\Gamma)} + \langle R_h f,\psi_h\rangle - \langle f_h,\psi_h\rangle \\
&\le Ch^p\|u\|_{H^1(\Gamma)}\|\phi\|_{H^{r+1}(\Gamma)} + \|R_h f - f_h\|_{H^0(\Gamma)}\|\psi_h\|_{H^0(\Gamma)} \\
&\le Ch^p\|u\|_{H^1(\Gamma)}\|\phi\|_{H^{r+1}(\Gamma)} + \|R_h f - f_h\|_{H^0(\Gamma)}\|\phi\|_{H^{r+1}(\Gamma)},
\end{aligned}$$

hence, from Theorem 4.2.1, we obtain the error estimate in the $H^{-r-1}(\Gamma)$ norm,

$$\|u-u_h^*\|_{H^{-r-1}(\Gamma)} \le Ch^{r+1/2}\|u-u_h^*\|_{H^{-1/2}(\Gamma)} + Ch^p\|u\|_{H^1(\Gamma)} + \|R_h f - f_h\|_{H^0(\Gamma)}$$

$$
\begin{aligned}
&\le \; Ch^{r+1/2}\|f_h - R_h f\|_{H^{1/2}(\Gamma)} + Ch^{r+p}\|u\|_{H^0(\Gamma)} \\
&\quad + Ch^{2r+1}\|u\|_{H^r(\Gamma)} + Ch^p\|u\|_{H^1(\Gamma)} + \|f_h - R_h f\|_{H^0(\Gamma)}.
\end{aligned}
$$

□

In the next result, we think of ϕ_h as a numerical approximation of ϕ. Note that evaluation of the single or double layer potential of u at a point in Ω^+ requires the calculation on an integral of the form $\langle \phi, u \rangle$, where ϕ involves the fundamental solution K.

Corollary 4.2.2 *Suppose that u is the exact solution of the equation* (2.1.25), *and $u_h \in S_h^{r,e}$ is the perturbed Galerkin approximation satisfying* (3.1.8). *If Γ is approximated to order $p \ge 2$, then*

$$
|\langle u, \phi \rangle - \langle u_h, \phi_h \rangle| \le C\|u - u_h^*\|_{H^{-r-1}(\Gamma)}\|\phi\|_{H^{r+1}(\Gamma)} + C\|\phi_h - R_h\phi\|_{H^0(\Gamma)}\|u\|_{H^0(\Gamma)}, \tag{4.2.11}
$$

where R_h is given by (4.1.1), ϕ is any function in $H^{r+1}(\Gamma)$, and ϕ_h is any function in $H^0(\Gamma)$.

Proof.

$$
\begin{aligned}
|\langle u, \phi \rangle - \langle u_h, \phi_h \rangle| &= |\langle u, \phi \rangle - \langle u_h^*, \phi \rangle + \langle u_h, R_h\phi \rangle - \langle u_h, \phi_h \rangle| \\
&\le |\langle u - u_h^*, \phi \rangle| + |\langle u_h, R_h\phi - \phi_h \rangle| \\
&\le C\|u - u_h^*\|_{H^{-r-1}(\Gamma)}\|\phi\|_{H^{r+1}(\Gamma)} + C\|\phi_h - R_h\phi\|_{H^0(\Gamma)}\|u_h\|_{H^0(\Gamma)}.
\end{aligned}
$$

□

If, in the above Theorems, we have

$$
\|f_h - R_h f\|_{H^{1/2}(\Gamma)} = O\left(h^{p-1/2}\right) \tag{4.2.12}
$$

and

$$\|f_h - R_h f\|_{H^0(\Gamma)} = O\left(h^p\right), \tag{4.2.13}$$

then Corollary 4.2.1 gives

$$\|u - u_h\|_{H^0(\Gamma)} = O\left(h^{\min(r,p-1)}\right),$$

and the super-convergence property is given by the error estimate

$$\|u - u_h^*\|_{H^{-r-1}(\Gamma)} = O\left(h^{\min(2r+1,p)}\right).$$

If, in addition,

$$\|\phi_h - R_h \phi\|_{H^0(\Gamma)} = O\left(h^p\right),$$

then from Corollary 4.2.2, we have

$$\langle u, \phi\rangle - \langle u_h, \phi_h\rangle = O\left(h^{\min(2r+1,p)}\right).$$

4.3 Approximating the Potentials

Let $u_h \in S_h^{r,e}$ be the solution of the perturbed Galerkin equation (3.1.8). We can compute the single layer and double layer potentials $\mathcal{V}u_h$ and $\mathcal{W}g$, given by

$$\begin{aligned}
\mathcal{V}u_h(\mathbf{z}) &= \int_\Gamma K(\mathbf{z}-\mathbf{y})\, u_h(\mathbf{y})\, d\sigma_{\mathbf{y}} \\
&= \sum_{k=1}^{N} \int_0^1 K[\mathbf{z} - \mathbf{m}_k(s)]\, u_h[\mathbf{m}_k(s)]\; |\mathbf{m}_k'(s)|\, ds
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{W}g(\mathbf{z}) &= \int_\Gamma \frac{\partial}{\partial \nu_{\mathbf{y}}} K(\mathbf{z}-\mathbf{y})\, g(\mathbf{y})\, d\sigma_{\mathbf{y}} \\
&= \sum_{k=1}^{N} \int_0^1 \frac{\partial}{\partial \nu_{\mathbf{y}}} K[\mathbf{z} - \mathbf{m}_k(s)]\, g[\mathbf{m}_k(s)]\; |\mathbf{m}_k'(s)|\, ds \\
&= \sum_{k=1}^{N} \int_0^1 \mathbf{n}_k(s) \cdot K[\mathbf{z} - \mathbf{m}_k(s)]\, g[\mathbf{m}_k(s)]\, ds
\end{aligned}$$

for $\mathbf{z} \in \Omega^+$, where $\mathbf{n}_k(s) = [-(\mathbf{m}_k')_2(s), (\mathbf{m}_k')_1(s)]$.

We introduce the notation for the kernel of the perturbed double layer potential $\mathcal{W}_h$. Let

$$L(\mathbf{z}, \mathbf{y}) = \frac{\partial}{\partial \nu_{\mathbf{y}}} K(\mathbf{z} - \mathbf{y}) = \nu_{\mathbf{y}} \cdot \nabla K(\mathbf{z} - \mathbf{y}),$$

and define

$$L_h(\mathbf{z}, \mathbf{y}) = \nu_{\mathbf{y}} \cdot \nabla K(\mathbf{z} - \tilde{\mathbf{y}}).$$

After interpolating the curved boundary Γ, we obtain the perturbed single layer potential $\mathcal{V}_h u_h(\mathbf{z})$ and double layer potential $\mathcal{W}_h g(\mathbf{z})$, defined by

$$\begin{aligned}
\mathcal{V}_h u_h(\mathbf{z}) &= \int_\Gamma K(\mathbf{z} - \tilde{\mathbf{y}})\, u_h(\mathbf{y})\, d\sigma_{\mathbf{y}} \\
&= \sum_{k=1}^{N} \int_0^1 K[\mathbf{z} - \tilde{\mathbf{m}}_k(s)]\, u_h[\mathbf{m}_k(s)]\, |\mathbf{m}_k'(s)|\, ds \\
&= \sum_{k=1}^{N} \int_0^1 K[\mathbf{z} - \tilde{\mathbf{m}}_k(s)] \sum_{i=1}^{r} U_{k,i}\, P_i(s)\, ds \\
&= \sum_{k=1}^{N} \sum_{i=1}^{r} U_{k,i} \int_0^1 K[\mathbf{z} - \tilde{\mathbf{m}}_k(s)]\, P_i(s)\, ds
\end{aligned} \tag{4.3.1}$$

and

$$\begin{aligned}
\mathcal{W}_h g(\mathbf{z}) &= \int_\Gamma L_h(\mathbf{z}, \mathbf{y})\, g(\mathbf{y})\, d\sigma_{\mathbf{y}} \\
&= \sum_{k=1}^{N} \int_0^1 \mathbf{n}_k(s) \cdot \nabla K[\mathbf{z} - \tilde{\mathbf{m}}_k(s)]\, g[\mathbf{m}_k(s)]\, ds.
\end{aligned} \tag{4.3.2}$$

Chapter 5

IMPLEMENTATION AND NUMERICAL EXPERIMENTS

5.1 Strategies of Implementation

In a practical implementation of the Galerkin method, it is usually necessary to use numerical quadratures to evaluate the coefficients and right-hand sides of the linear algebraic equations $A\mathbf{U} = \mathbf{F}$.

We now focus on the coefficients of such a system of linear equations.

As in Chapter 2, we denote the $r \times r$ element stiffness matrix for $\Delta_k \times \Delta_l$ by

$$A^{(k,l)} = \left[a_{ij}^{(k,l)}\right]_{1\le i,j\le r}, \qquad 1 \le k, l \le N,$$

where

$$a_{ij}^{(k,l)} = \frac{1}{2\pi}\int_0^1\int_0^1 \log\left(\frac{b}{|\mathbf{m}_k(s)-\mathbf{m}_l(t)|}\right) P_i(s)\, P_j(t)\, ds\, dt, \qquad 0 \le s, t \le 1. \tag{5.1.1}$$

Note that

$$A^{(l,k)} = \left[A^{(k,l)}\right]^T, \qquad 1 \le k \le l \le N.$$

We define the $r \times r$ matrix

$$P(s,t) = [P_i(s)\, P_j(t)]_{1\le i,j\le r},$$

and rewrite the double integral in (5.1.1) as

$$A^{(k,l)} = \frac{1}{2\pi}\log(b)\int_0^1\int_0^1 P(s,t)\, ds\, dt - E \tag{5.1.2}$$

where

$$E = \frac{1}{2\pi}\int_0^1\int_0^1 \log\left(|\mathbf{m}_k(s)-\mathbf{m}_l(t)|\right) P(s,t)\, ds\, dt$$

depends on k and l.

In general it is not possible to evaluate the integral E analytically. However E can be obtained approximately with the help of the Q-point Gauss-Legendre quadrature rule

$$\int_0^1 f(s)\,ds \approx \sum_{q=1}^{Q} w_q\, f(\xi_q)$$

for an appropriate choice of Q. The details of computing w_q and ξ_q can be found in [5, Theorem 5.3].

In order to reduce the cost of the quadrature so that the integrand evaluations are cheaper, we consider evaluating E under several different conditions. That means, we do not apply the global singularity subtraction to the whole operator but only where it is necessary.

Firstly, when $k = l$, we use the splitting

$$\log|\mathbf{m}_k(s) - \mathbf{m}_l(t)| = \log|s-t| + \log\frac{|\mathbf{m}_k(s) - \mathbf{m}_l(t)|}{|s-t|}$$

and put

$$\begin{aligned} E_I^{[Q]} &= \frac{1}{2\pi}\int_0^1\int_0^1 \log|s-t|\, P(s,t)\, ds\, dt \\ &+\frac{1}{2\pi}\sum_{p=1}^{Q}\sum_{q=1}^{Q} w_p\, w_q \log\left(\frac{|\mathbf{m}_k(\xi_p) - \mathbf{m}_l(\xi_q)|}{|\xi_p - \xi_q|}\right) P(\xi_p, \xi_q) \end{aligned} \tag{5.1.3}$$

with the understanding that

$$\frac{|\mathbf{m}_k(\xi_p) - \mathbf{m}_l(\xi_q)|}{|\xi_p - \xi_q|} = |\mathbf{m}_k'(\xi_p)| \qquad \text{if } p = q.$$

Secondly, if $\triangle_k$ and $\triangle_l$ are neighbouring elements, for example, if $l = k+1$, then a special singularity subtraction technique is used:

$$\log|\mathbf{m}_k(s) - \mathbf{m}_l(t)| = \log|\mathbf{m}_k'(1)(s-1) - \mathbf{m}_l'(0)t| + \log\frac{|\mathbf{m}_k(s) - \mathbf{m}_l(t)|}{|\mathbf{m}_k'(1)(s-1) - \mathbf{m}_l'(0)t|}.$$

The quantity $\frac{|\mathbf{m}_k(s)-\mathbf{m}_l(t)|}{|\mathbf{m}_k'(1)(s-1)-\mathbf{m}_l'(0)t|}$ is smooth for $0 \leq s \leq 1$ and $0 \leq t \leq 1$ because

$$\mathbf{m}_k(1) = \mathbf{m}_l(0),$$

so we put

$$\begin{aligned} E_{II}^{[Q]} &= \frac{1}{2\pi}\int_0^1\int_0^1 \log\left(|\mathbf{m}_k'(1)(s-1) - \mathbf{m}_l'(0)t|\right) P(s,t)\,ds\,dt \\ &\quad + \frac{1}{2\pi}\sum_{p=1}^{Q}\sum_{q=1}^{Q} w_p\, w_q \log\left(\frac{|\mathbf{m}_k(\xi_p) - \mathbf{m}_l(\xi_q)|}{|\mathbf{m}_k'(1)(\xi_p - 1) - \mathbf{m}_l'(0)\xi_q|}\right) P(\xi_p, \xi_q). \qquad (5.1.4)\end{aligned}$$

For the special case of $k = N$ and $l = 1$, the second term of $E_{II}^{[Q]}$ takes the form

$$\frac{1}{2\pi}\sum_{p=1}^{Q}\sum_{q=1}^{Q} w_p w_q \log\left(\frac{|\mathbf{m}_N(\xi_p) - \mathbf{m}_1(\xi_q)|}{|\mathbf{m}_N'(1)(\xi_p - 1) - \mathbf{m}_1'(0)\xi_q|}\right) P(\xi_p, \xi_q). \qquad (5.1.5)$$

The first terms of $E_I^{[Q]}$ and $E_{II}^{[Q]}$ can be computed analytically, as explained in the Appendix.

Finally, in the case when $\triangle_k$ and $\triangle_l$ are not neighbours, that is, neither $k = l$ nor $k = l \pm 1$, we put

$$E_{III}^{[Q]} = \frac{1}{2\pi}\sum_{p=1}^{Q}\sum_{q=1}^{Q} w_p\, w_q \log\left(|\mathbf{m}_k(\xi_p) - \mathbf{m}_l(\xi_q)|\right) P(\xi_p, \xi_q) \qquad (5.1.6)$$

by applying the Gauss rule straight away.

Next we consider the right-hand side of the system of linear equations.

If f_h is the L^2-projection of f onto $S_h^{r+1,e}$ as we assumed in the last Chapter, then $\langle f - f_h, v_h\rangle = 0$ for all $v_h \in S_h^{r+1,e}$, i.e.,

$$\langle f_h, v_h\rangle = \langle f, v_h\rangle.$$

The element load vector $\mathbf{F}^{(l)}$ in $\triangle_l, l = 1, \ldots, N$, is denoted by $\mathbf{F}^{(l)} = \left[\mathbf{F}_j^{(l)}\right]_{1\leq j\leq r}$, where

$$\mathbf{F}_j^{(l)} = \frac{1}{2\pi}\sum_{k=1}^{N}\int_0^1\int_0^1 \frac{\mathbf{n}_k(s)\cdot[\mathbf{m}_l(t) - \mathbf{m}_k(s)]}{|\mathbf{m}_k(s) - \mathbf{m}_l(t)|^2}\left(g[\mathbf{m}_k(s)] - g[\mathbf{m}_l(t)]\right) P_j(t)\,ds\,dt$$

or

$$
\begin{aligned}
\mathbf{F}_j^{(l)} &= -\frac{1}{2}\int_0^1 g[\mathbf{m}_l(t)]\, P_j(t)\, dt \\
&\quad +\frac{1}{2\pi}\sum_{k=1}^{N}\int_0^1\int_0^1 \frac{\mathbf{n}_k(s)\cdot[\mathbf{m}_l(t)-\mathbf{m}_k(s)]}{|\mathbf{m}_l(t)-\mathbf{m}_k(s)|^2}\, g[\mathbf{m}_k(s)]\, P_j(t)\, ds\, dt
\end{aligned}
$$

with $\mathbf{n}_k(s) = [-(\mathbf{m}_k')_2(s), (\mathbf{m}_k')_1(s)]$.

We evaluate $\mathbf{F}_j^{(l)}$ using Gaussian quadrature. Write $\mathbf{F}^{[Q]}$ (dependent on j and l) as an approximation of $\mathbf{F}_j^{(l)}$, define

$$
P(s) = [P_i(s)]_{1\le i\le r}
$$

and put

$$
\mathbf{F}^{[Q]} = \frac{1}{2\pi}\sum_{k=1}^{N}\sum_{p=1}^{Q}\sum_{q=1}^{Q} w_p\, w_q\, \frac{\mathbf{n}_k(\xi_p)\cdot[\mathbf{m}_l(\xi_q)-\mathbf{m}_k(\xi_p)]}{|\mathbf{m}_l(\xi_q)-\mathbf{m}_k(\xi_p)|^2}\, (g[\mathbf{m}_k(\xi_p)] - g[\mathbf{m}_l(\xi_q)])\; P(\xi_q).
$$

or

$$
\begin{aligned}
\mathbf{F}^{[Q]} &= -\frac{1}{2}\sum_{q=1}^{Q} w_q\, g[\mathbf{m}_l(\xi_q)]\, P(\xi_q) \\
&\quad +\frac{1}{2\pi}\sum_{k=1}^{N}\sum_{p=1}^{Q}\sum_{q=1}^{Q} w_p\, w_q\, \frac{\mathbf{n}_k(\xi_p)\cdot[\mathbf{m}_l(\xi_q)-\mathbf{m}_k(\xi_p)]}{|\mathbf{m}_l(\xi_q)-\mathbf{m}_k(\xi_p)|^2}\, g[\mathbf{m}_k(\xi_p)]\, P(\xi_q).
\end{aligned}
$$

If $k = l$ and $\xi_p = \xi_q$, then in the second term we use the limiting value

$$
\begin{aligned}
&\lim_{\xi_q\to\xi_p} \frac{\mathbf{n}_k(\xi_p)\cdot[\mathbf{m}_l(\xi_q)-\mathbf{m}_k(\xi_p)]}{|\mathbf{m}_l(\xi_q)-\mathbf{m}_k(\xi_p)|^2} \\
&= \lim_{\xi_q\to\xi_p} \frac{\mathbf{n}_k(\xi_p)\cdot\mathbf{m}_l'(\xi_q)}{2|\mathbf{m}_l'(\xi_q)|\,|\mathbf{m}_l(\xi_q)-\mathbf{m}_k(\xi_p)|} \\
&= \frac{\mathbf{n}_k(\xi_p)\cdot\mathbf{m}_l''(\xi_p)}{2|\mathbf{m}_l'(\xi_p)|^2}.
\end{aligned}
$$

Remark: In the numerical experiments, we shall be interested in the effect of replacing $\mathbf{m}_k(s)$ by $\tilde{\mathbf{m}}_k(s)$ in the quantities above.

5.2 Numerical Potentials

For $\mathbf{z} \in \Omega^+$, the perturbed potential is given by $\phi_h(\mathbf{z}) = \mathcal{W}_h g(\mathbf{z}) - \mathcal{V}_h u_h(\mathbf{z})$, where $\mathcal{V}_h u_h(\mathbf{z})$ and $\mathcal{W}_h g(\mathbf{z})$ are defined in (4.3.1) and (4.3.2) respectively. The numerical perturbed single layer potential $\mathcal{V}_h u_h(\mathbf{z})$ and the perturbed double layer potential $\mathcal{W}_h g(\mathbf{z})$ will be computed by using the Gauss-Legendre quadrature rule. They are

$$\begin{aligned}\mathcal{V}_h u_h(\mathbf{z}) &\approx \sum_{k=1}^{N}\sum_{i=1}^{r} U_{k,i} \sum_{p=1}^{Q} w_p\, K[\mathbf{z} - \tilde{\mathbf{m}}_k(\xi_p)]\, P_i(\xi_p)\\ &= \frac{1}{2\pi}\sum_{k=1}^{N}\sum_{i=1}^{r} U_{k,i} \sum_{p=1}^{Q} w_p \log\left(\frac{b}{|\mathbf{z} - \tilde{\mathbf{m}}_k(\xi_p)|}\right) P_i(\xi_p),\end{aligned}$$

and

$$\mathcal{W}_h g(\mathbf{z}) \approx \frac{1}{2\pi}\sum_{k=1}^{N}\sum_{i=1}^{r} g[\mathbf{m}_k(\xi_p)] \sum_{p=1}^{Q} w_p \frac{\mathbf{n}_k(s)\cdot[\mathbf{z} - \tilde{\mathbf{m}}_k(\xi_p)]}{|\mathbf{z} - \tilde{\mathbf{m}}_k(\xi_p)|^2},$$

where Q is number of the Gauss points, ξ_p is the Gauss point, w_p is the Gauss weight, and $\mathbf{n}_k(s) = [-(\mathbf{m}'_k)_2(s), (\mathbf{m}'_k)_1(s)]$.

Note that, the above kernels of the integrals are not singular as $\mathbf{z} \notin \Gamma$.

5.3 Numerical Experiments

We now present errors and convergence rates for some numerical experiments. The code was written in FORTRAN 90 and run on a DEC alpha.

Recall from (2.1.22) that the boundary integral equation is

$$\int_\Gamma K(\mathbf{x}-\mathbf{y})u(\mathbf{y})\, d\sigma_{\mathbf{y}} = \int_\Gamma [g(\mathbf{y}) - g(\mathbf{x})] \frac{\partial}{\partial \nu_{\mathbf{y}}} K(\mathbf{x}-\mathbf{y})\, d\sigma_{\mathbf{y}}, \qquad \mathbf{x} \in \Gamma, \quad (5.3.1)$$

or

$$\int_\Gamma K(\mathbf{x}-\mathbf{y})u(\mathbf{y})\,d\sigma_{\mathbf{y}} = -\frac{1}{2}\,g(\mathbf{x}) + \int_\Gamma g(\mathbf{y})\,\frac{\partial}{\partial\nu_{\mathbf{y}}}K(\mathbf{x}-\mathbf{y})\,d\sigma_{\mathbf{y}}, \qquad \mathbf{x}\in\Gamma, \quad (5.3.2)$$

where $K(\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi}\log\frac{b}{|\mathbf{x}-\mathbf{y}|}$. Equation (5.3.1) or (5.3.2) arises from the Dirichlet problem for Laplace's equation

$$\Delta\phi = 0 \qquad \text{in } \Omega^+,$$
$$\phi^+ = g \qquad \text{on } \Gamma,$$

using the direct boundary integral method, where Ω^+ is a bounded domain with a curved boundary Γ, $u = \phi_\nu^+$ and ν is the unit inward normal to Ω^+.

The boundary Γ is parameterised by a smooth function $\mathbf{x} = \mathbf{F}(\tau), 0 \le \tau \le 1$. We approximate $\mathbf{F}$ using polynomial interpolation of two different orders.

In our experiments, the function $g(\mathbf{x}), \mathbf{x} = (x_1, x_2) \in \Gamma$, in the right-hand side of (5.3.1) or (5.3.2) is chosen to be the restriction to Γ of a known harmonic function, which is thus the solution ϕ of the Dirichlet problem. The exact solution u of equation (5.3.1) or (5.3.2) is given by

$$u = \nu(\mathbf{x}) \cdot \nabla\phi(\mathbf{x}).$$

For finding an approximate solution to the boundary integral equation (5.3.1) or (5.3.2), the boundary element space $S_h^{r,e}$ is always chosen as the piecewise constant space defined in Chapter 2 with $r = 1$ and $e = 0$.

For $k, l = 1, \ldots, N$, in the general case of the point $\mathbf{x}$ in $\triangle_l$ and point $\mathbf{y}$ in $\triangle_k$ with $|k-l| \ge 2$, the number of Gauss points chosen for computing $E_{III}^{[Q]}$ in (5.1.6) depends on the distance between $\mathbf{x}$ and $\mathbf{y}$. The idea is to make computation cheaper by using fewer Gauss points where possible rather than the same number of Gauss points everywhere, while maintaining the order of convergence. Our strategy employs four different bands of the distance $|\mathbf{x}-\mathbf{y}|$, with more Gauss points at smaller distances, and fewer Gauss points for larger distances.

However, in the special cases when the boundary elements $\triangle_k$ and $\triangle_l$ coincide ($k = l$) or are neighbours ($k = l \pm 1$), we use only a few Gauss points to compute $E_I^{[Q]}$ and $E_{II}^{[Q]}$ in (5.1.3) and (5.1.4) respectively, since we have already eliminated the singularities in the integrands.

In numerical experiments, we compute the relative errors and the orders of convergence in the L^2 norm, and also the errors and orders of convergence for point evaluations of the potentials, both for an exact parametric representation of the boundary, and for linear ($p = 2$) or quadratic ($p = 3$) interpolation.

Let ϕ be the solution of the Dirichlet problem and ϕ_h be the perturbed potential given by $\phi_h(\mathbf{z}) = \mathcal{W}_h g(\mathbf{z}) - \mathcal{V}_h u_h(\mathbf{z})$, where $\mathcal{V}_h u_h(\mathbf{z})$ and $\mathcal{W}_h g(\mathbf{z})$ are defined in (4.3.1) and (4.3.2) respectively. Since $r = 1$, the theoretical error bounds for the exact boundary are (see Corollary 2.4.1)

$$\|u - u_h\|_{H^0(\Gamma)} \leq Ch\|u\|_{H^1(\Gamma)}, \tag{5.3.3}$$

$$|\phi(\mathbf{z}) - \phi_h(\mathbf{z})| \leq C(\mathbf{z})\|u - u_h\|_{H^{-2}(\Gamma)} \leq C(\mathbf{z})h^3\|u\|_{H^1(\Gamma)}. \tag{5.3.4}$$

For the approximate boundary, we expect

$$\|u - u_h\|_{H^0(\Gamma)} \leq Ch^{\min(1,p-1)}, \tag{5.3.5}$$

$$|\phi(\mathbf{z}) - \phi_h(\mathbf{z})| \leq C(\mathbf{z})h^{\min(3,p)}, \tag{5.3.6}$$

see Corollary 4.2.1 and 4.2.2.

5.3.1 Smooth Boundary

Numerical results for an integral equation on a smooth, closed boundary Γ in $\mathbb{R}^2$ are presented here. Specifically, the curved boundary Γ is an ellipse with semi-axes a_1 and a_2,

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1. \tag{5.3.7}$$

It follows that the parametric representation is expressed as

$$\mathbf{x} = \mathbf{F}(\tau) = (a_1 \cos 2\pi\,\tau, a_2 \sin 2\pi\,\tau), \qquad 0 \le \tau \le 1. \tag{5.3.8}$$

For the harmonic function $\phi(\mathbf{x})$ we chose

$$\phi(\mathbf{x}) = e^{x_1} \cos x_2 + x_1 + x_2, \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2, \tag{5.3.9}$$

so the exact solution u of the integral equation is

$$\begin{aligned} u &= \nu(\mathbf{x}) \cdot \nabla\phi(\mathbf{x}) \\ &= \frac{-x_2'\, e^{x_1} \cos(x_2) - x_1'\, e^{x_1} \sin(x_2) + x_1' - x_2'}{\left((x_1')^2 + (x_2')^2\right)^{1/2}}, \end{aligned} \tag{5.3.10}$$

where

$$\nu(\mathbf{x}) = \frac{(-x_2', x_1')}{|\mathbf{x}'|}, \quad \mathbf{x} = (x_1, x_2) \quad \text{and} \quad \mathbf{x}' = (x_1', x_2').$$

Since the logarithmic capacity of the ellipse (5.3.7) is equal to the arithmetic mean of its major and minor semi-axes, we choose the constant $b > \frac{a_1+a_2}{2}$ to ensure that the stiffness matrix is positive definite. The test point $\mathbf{z} \in \Omega^+$ is taken to be $(0.50, 0.50)$ in our calculation of the errors in the potentials.

We compute the numerical solution of the integral equation not only with the exact elliptical boundary, but also with approximate boundaries using piecewise linear and piecewise quadratic interpolation.

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.85824E+00		0.85659E+00		0.85779E+00	
16	0.40728E+00	1.08	0.39857E+00	1.10	0.40703E+00	1.08
32	0.20455E+00	0.99	0.20249E+00	0.98	0.20454E+00	0.99
64	0.10155E+00	1.01	0.10123E+00	1.00	0.10155E+00	1.10
128	0.50632E-01	1.00	0.50589E-01	1.00	0.50632E-01	1.00
256	0.25295E-01	1.00	0.25290E-01	1.00	0.25295E-01	1.00
512	0.12645E-01	1.00	0.12644E-01	1.00	0.12645E-01	1.00
1024	0.63221E-02	1.00	0.63221E-02	1.00	0.63221E-02	1.00

Table 5.1: Relative errors and orders of convergence in L^2 norm for Example 1.

Example 1. Consider the integral equation in the form of (5.3.2) for a smooth boundary Γ. The curve boundary Γ is an ellipse with $a_1 = 4$ and $a_2 = 2$, and the scaling parameter is $b = 4$. The distance bands for the quadrature are set as $1.0, 2.0, 3.0, 4.0$ with the corresponding numbers of Gauss points $6, 5, 4, 3$. A 3-point Gauss rule is used in the other situations. The results are given in Tables 5.1 and 5.2.

The Tables 5.1 and 5.2 present the errors and rates of convergence in both the L^2 norm of u and the pointwise value $\phi(\mathbf{z})$.

By comparing results among the cases, we see that the errors in L^2 norm for the cases of the linear and quadratic interpolation are close to those for the case of the exact boundary, and their orders of convergence are stable as well. Table 5.1 shows that we have a good correspondence between the experimental results and the theoretical results (5.3.5). In Table 5.2, the orders of convergence of potentials for linear and quadratic function interpolation on a boundary are 2 and 3 respectively,

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.27510E+00		0.83200E-01		0.26140E+00	
16	0.48152E-02	5.84	0.12529E+00	-0.59	0.51084E-02	5.68
32	0.54225E-03	3.15	0.31427E-01	2.00	0.52415E-03	3.28
64	0.50860E-04	3.41	0.80545E-02	1.96	0.50024E-04	3.39
128	0.57104E-05	3.15	0.20349E-02	1.98	0.56955E-05	3.13
256	0.68563E-06	3.06	0.51123E-03	1.99	0.68941E-06	3.05
512	0.84317E-07	3.02	0.12811E-03	2.00	0.85145E-07	3.02
1024	0.10465E-07	3.01	0.32066E-04	2.00	0.10590E-07	3.01

Table 5.2: Errors and orders of convergence of potentials for Example 1.

these results approach the prediction (5.3.6).

5.3.2 Corner Problem

The error estimates and the rates of convergence proved in section 4.2 are valid for the first kind integral equation over a smooth boundary Γ.

In the current section, we explore the case when the boundary Γ is not smooth, that is, when the domain has corners. In the case of a non-smooth boundary, singularities in the solution u will generally be produced at the corners. These singularities will degrade the rates of convergence when the Galerkin method is applied with uniform meshes. In order to restore optimal orders of convergence, a mesh grading technique is considered.

In our numerical experiments, the boundary Γ is given by a "teardrop-shaped" region which contains only one corner (described in [15, 16]). We use the parameterisation

$$\mathbf{F} : [0,1] \to \Gamma,$$

defined by

$$\mathbf{F}(\tau) = [\sin(\tau\pi)\, \cos(1-\chi)\,(\tau\pi),\ \sin(\tau\pi)\, \sin(1-\chi)\,(\tau\pi)], \qquad (5.3.11)$$

where the corner is at $\tau = 0$ or $\tau = 1$, and the interior angle between the tangent at $\tau = 0$ is $(1-\chi)\,\pi$, $0 < |\chi| < 1$.

We define graded meshes by choosing $q \geq 1$ and putting

$$\tau_k = \begin{cases} (\frac{1}{2})^{1-q}\,(\frac{k}{N})^q, & k \in \left[0, \frac{N}{2}\right], \\ 1 - (\frac{1}{2})^{1-q}\,(1-\frac{k}{N})^q, & k \in \left(\frac{N}{2}, N\right]. \end{cases} \qquad (5.3.12)$$

Note that these meshes are uniform when $q = 1$.

Another problem occurs when the direct method is applied in the formulation of the boundary integral equation, because of the behaviour of the right-hand side of the boundary integral equation

$$f(\mathbf{x}) = -\frac{1}{2}\,g(\mathbf{x}) + \frac{1}{2}\,Tg(\mathbf{y}), \qquad \mathbf{x} \in \Gamma. \qquad (5.3.13)$$

Quadrature errors lead to a relatively poor convergence rate because a fixed singularity arises in the kernel of double layer operator T at the corner point. The tricky point is how to smooth out the singularity in the kernel of the double layer potential. Jaswon and Symm in the book [28] address some general techniques by which the kernel of the double layer potential can be integrated analytically along a smooth arc on the boundary. We have instead used a singularity subtraction method which we already mentioned in Chapter 2, that weakens the singularity in the double layer operator, making the integrals easier to tackle. That is, when a boundary Γ is not smooth, we use a singularity subtraction method to write the operator T as

$$Tu(\mathbf{x}) = u(\mathbf{x}) + 2\int_\Gamma [u(\mathbf{y}) - u(\mathbf{x})]\,\frac{\partial}{\partial \nu_\mathbf{y}} K(\mathbf{x}-\mathbf{y})\,d\sigma_\mathbf{y}$$

so that the denominator of $\frac{\partial}{\partial \nu_\mathbf{y}} K(\mathbf{x}-\mathbf{y})$ can be cancelled with $[u(\mathbf{y}) - u(\mathbf{x})]$ as $\mathbf{x} \to \mathbf{y}$. In other words, we work with the integral equation in the form (5.3.1) instead of (5.3.2), so that the right-hand side is in the form

$$f(\mathbf{x}) = \int_\Gamma [g(\mathbf{y}) - g(\mathbf{x})]\,\frac{\partial K(\mathbf{x}-\mathbf{y})}{\partial \nu_\mathbf{y}}\,d\sigma_\mathbf{y} \tag{5.3.14}$$

This singularity subtraction method reduces the quadrature errors from the double layer potential operator without appreciably increasing the amount of computation involved.

- **Results for a Smooth Potential**

In this experiment, we consider the problem over the "teardrop-shaped" boundary Γ with a smooth potential ϕ on Γ.

Example 2. Consider the integral equation (5.3.1), and suppose $g = \phi|_\Gamma$ when

$$\phi(\mathbf{x}) = e^{x_1}\cos(x_2) + x_1 + x_2, \qquad \mathbf{x} = (x_1, x_2) \in \Gamma,$$

is harmonic and smooth. The exact solution u of the integral equation (5.3.1) is computed by formula (5.3.10). The Gauss rules with $6, 5, 4, 3$ points are used when the distances $|\mathbf{x}-\mathbf{y}|$ are $0.1, 0.2, 0.3, 0.4$ respectively, and the 3-point Gauss rule is used in other places. This example contains cases with different value of χ, they are,

(a) $\chi = 0.75, q = 3$ and $\mathbf{z} = (0.7, 0.25)$;

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.45614E+00		0.45352E+00		0.45308E+00	
16	0.26174E+00	0.80	0.25542E+00	0.83	0.26097E+00	0.80
32	0.13381E+00	0.97	0.13208E+00	0.95	0.13376E+00	0.96
64	0.66833E-01	1.00	0.66537E-01	0.99	0.66831E-01	1.00
128	0.33378E-01	1.00	0.33336E-01	1.00	0.33378E-01	1.00
256	0.16682E-01	1.00	0.16676E-01	1.00	0.16682E-01	1.00
512	0.83399E-02	1.00	0.83392E-02	1.00	0.83399E-02	1.00
1024	0.41698E-02	1.00	0.41697E-02	1.00	0.41698E-02	1.00

Table 5.3: Relative errors and orders of convergence in L^2 norm for case **(a)** in Example 2.

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.23325E-01		0.61348E+00		0.27579E-01	
16	0.30158E-03	6.27	0.10478E+00	2.55	0.83268E-03	5.05
32	0.88215E-04	1.77	0.32864E-01	1.67	0.11803E-03	2.82
64	0.82860E-05	3.41	0.81128E-02	2.02	0.12265E-04	3.27
128	0.99929E-06	3.05	0.20282E-02	2.00	0.12626E-05	3.28
256	0.12390E-06	3.01	0.50708E-03	2.00	0.14142E-06	3.16
512	0.15455E-07	3.00	0.12677E-03	2.00	0.16679E-07	3.08
1024	0.19309E-08	3.00	0.31692E-04	2.00	0.20234E-08	3.04

Table 5.4: Errors and orders of convergence of potentials for case **(a)** in Example 2.

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.48474E+00		0.44720E+00		0.47915E+00	
16	0.26046E+00	0.90	0.25389E+00	0.82	0.25991E+00	0.88
32	0.13064E+00	1.00	0.12955E+00	0.97	0.13061E+00	0.99
64	0.65154E-01	1.00	0.64996E-01	1.00	0.65153E-01	1.00
128	0.32543E-01	1.00	0.32522E-01	1.00	0.32543E-01	1.00
256	0.16266E-01	1.00	0.16264E-01	1.00	0.16266E-01	1.00
512	0.81326E-02	1.00	0.81322E-02	1.00	0.81326E-02	1.00
1024	0.40662E-02	1.00	0.40661E-02	1.00	0.40662E-02	1.00

Table 5.5: Relative errors and orders of convergence in L^2 norm for case **(b)** in Example 2.

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.40230E-02		0.31083E+00		0.63365E-01	
16	0.47130E-03	3.09	0.14180E+00	1.13	0.35483E-02	4.16
32	0.52705E-04	3.16	0.46779E-02	4.92	0.37129E-03	3.26
64	0.71806E-05	2.88	0.16293E-02	1.52	0.53041E-04	2.81
128	0.91606E-06	2.97	0.40179E-03	2.02	0.63509E-05	3.06
256	0.11509E-06	2.99	0.10040E-03	2.00	0.77444E-06	3.04
512	0.14405E-07	3.00	0.25104E-04	2.00	0.95686E-07	3.02
1024	0.18011E-08	3.00	0.62770E-05	2.00	0.11893E-07	3.01

Table 5.6: Errors and orders of convergence of potentials for case **(b)** in Example 2.

(b) $\chi = -0.75, q = 3$ and $\mathbf{z} = (0.2, 0.4)$.

Qualitatively, the order of convergence in Tables 5.3-5.6 are similar to those obtained for a smooth boundary, suggesting that the mesh grading restores the rate of convergence. In relation to the smooth potential ϕ on boundary Γ in example 2, in Table 5.3 and Table 5.5, one sees that the relative errors of piecewise linear and piecewise quadratic interpolation are just slightly different whether value of χ is chosen to be positive or not. In particular, the errors for the potentials in Table 5.4 and Table 5.6 for cases of piecewise linear and piecewise quadratic boundary approximation are quite different, but the rates of convergence of potentials hold for the predicted value p.

• Results for a Non-smooth Potential

In all the cases above, the potential ϕ is smooth. We now consider what happens when ϕ is not smooth at the corner point.

If a harmonic function ϕ takes prescribed continuous values on the boundary Γ a corner of angle α, the function ϕ may in general have the form

$$\phi(\rho, \theta) = A\,\rho^{\frac{\pi}{\alpha}}\,\sin(\frac{\pi}{\alpha}\,\theta) + \phi_0, \tag{5.3.15}$$

where ρ, θ are polar coordinates relative to the corner, A is a constant, $\alpha = (1-\chi)\,\pi$ is the size of the angle of the corner, $0 < \theta < \alpha$ and ϕ_0 is a smoother term.

If χ is chosen to be negative, then the derivatives of ϕ becomes infinite in magnitude as ρ approaches to zero. Thus, it is difficult to compute ϕ accurately in this region when such singularities arise. The book [28] gives some treatments to tackle this problem, but they are often complicated in practice. Instead, we use singularity subtraction and mesh grading.

Example 3. We are concerned with the integral equation (5.3.1), and give a specific harmonic function

$$\phi(\mathbf{x}) = \mathrm{Im}\left[(x_1 + ix_2)^{\frac{\pi}{\alpha}}\right] + e^{x_1}\cos(x_2) + x_1 + x_2, \tag{5.3.16}$$

where $\alpha = (1-\chi)\,\pi$ and $\mathbf{x} = (x_1, x_2) \in \Gamma$. The gradient of ϕ is

$$\nabla\phi(\mathbf{x}) = [-\frac{1}{\alpha}\rho^{(\frac{1}{\alpha}-1)}\sin(\theta - \frac{1}{\alpha}\theta) + \phi_1,\ \frac{1}{\alpha}\rho^{(\frac{1}{\alpha}-1)}\cos(\theta - \frac{1}{\alpha}\theta) + \phi_2],$$

where $\rho = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan(x_2/x_1) \in (0, \alpha)$, $\phi_1 = e^{x_1}\cos(x_2) + 1$ and $\phi_2 = -\ e^{x_1}\sin(x_2) + 1$.

The other conditions given are as same as in Example 2. We also consider different cases when χ is positive and negative:

(a) $q = 3, \chi = 0.75$ and $\mathbf{z} = (0.7, 0.25)$;

(b) $q = 3, \chi = -0.3$ and $\mathbf{z} = (0.2, 0.6)$.

The results for those cases are given in Tables 5.7-5.10.

The results in Table 5.7-5.8 are again in satisfactory agreement with expectations, even though the function ϕ is not smooth. When we take $\chi < 0$, from Tables 5.9-5.10 the convergence rates are slightly worse than when $\chi > 0$, which is not surprising in view of the stronger singularity in the solution.

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.66942E+00		0.18456E+01		0.66831E+00	
16	0.33997E+00	0.98	0.90973E+00	1.02	0.33934E+00	0.98
32	0.16660E+00	1.03	0.44312E+00	1.04	0.16651E+00	1.03
64	0.82823E-01	1.01	0.22236E+00	0.99	0.82820E-01	1.01
128	0.41286E-01	1.00	0.11126E+00	1.00	0.41286E-01	1.00
256	0.20624E-01	1.00	0.55641E-01	1.00	0.20624E-01	1.00
512	0.10309E-01	1.00	0.27822E-01	1.00	0.10309E-01	1.00
1024	0.51544E-02	1.00	0.13911E-01	1.00	0.51544E-02	1.00

Table 5.7: Relative errors and orders of convergence in L^2 norm for case **(a)** in Example 3.

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.76876E-01		0.80326E+00		0.78638E-01	
16	0.25398E-02	4.92	0.13389E+00	2.58	0.30525E-02	4.69
32	0.47244E-03	2.43	0.41585E-01	1.69	0.51181E-03	2.58
64	0.52002E-04	3.18	0.10286E-01	2.02	0.56977E-04	3.17
128	0.63787E-05	3.03	0.25697E-02	2.00	0.67070E-05	3.09
256	0.79363E-06	3.01	0.64217E-03	2.00	0.81551E-06	3.04
512	0.99088E-07	3.00	0.16051E-03	2.00	0.10062E-06	3.02
1024	0.12382E-07	3.00	0.40122E-04	2.00	0.12499E-07	3.01

Table 5.8: Errors and orders of convergence of potentials for case **(a)** in Example 3.

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.46071E+00		0.11886E+01		0.46024E+00	
16	0.22233E+00	1.05	0.59798E+00	0.99	0.22206E+00	1.05
32	0.11179E+00	0.99	0.30468E+00	0.97	0.11177E+00	0.99
64	0.56514E-01	0.98	0.15464E+00	0.98	0.56513E-01	0.98
128	0.28711E-01	0.98	0.78653E-01	0.98	0.28711E-01	0.98
256	0.14657E-01	0.97	0.40164E-01	0.97	0.14657E-01	0.97
512	0.75225E-02	0.96	0.20615E-01	0.96	0.75225E-02	0.96
1024	0.38844E-02	0.95	0.10645E-01	0.95	0.38844E-02	0.95

Table 5.9: Relative errors and orders of convergence in L^2 norm for case **(b)** in Example 3.

N	Exact boundary		Linear interpolation		Quadratic interpolation	
8	0.28599E-01		0.27114E+01		0.18902E-01	
16	0.12051E-02	4.57	0.10191E+00	4.73	0.38863E-02	2.28
32	0.38369E-03	1.65	0.23674E-01	2.11	0.40437E-03	3.26
64	0.41168E-04	3.22	0.59306E-02	2.00	0.38245E-04	3.40
128	0.50976E-05	3.01	0.14799E-02	2.00	0.49119E-05	2.96
256	0.63891E-06	3.00	0.36971E-03	2.00	0.62709E-06	2.97
512	0.80525E-07	2.99	0.92401E-04	2.00	0.79776E-07	2.97
1024	0.10210E-07	2.98	0.23097E-04	2.00	0.10162E-07	2.97

Table 5.10: Errors and orders of convergence of potentials for case **(b)** in Example 3.

Appendix: Evaluation of Some Integrals

Evaluating E has been discussed in the section 5.1. We now deal with the analytical computation of the first term of (5.1.2),

$$\log(b) \int_0^1 \int_0^1 P(s,t)\, ds\, dt, \tag{5.3.17}$$

the first term of $E_I^{[Q]}$,

$$\int_0^1 \int_0^1 \log|s-t|\, P(s,t)\, ds\, dt, \tag{5.3.18}$$

and the first term of $E_{II}^{[Q]}$,

$$\int_0^1 \int_0^1 \log\left(|\mathbf{m}_k'(1)(s-1) - \mathbf{m}_l'(0)t|\right)\, P(s,t)\, ds\, dt, \tag{5.3.19}$$

where $P(s,t) = [P_i(s)P_j(t)]_{1\le i,j\le r}$ and b is a constant. Actually, it is straightforward to compute (5.3.17) and (5.3.18), but some strategies will be used to compute (5.3.19).

For brevity, we use the notations

$$\mathbf{a} = \mathbf{m}_k'(1), \quad \mathbf{b} = \mathbf{m}_l'(0), \qquad\qquad \mathbf{a}, \mathbf{b} \in \mathbb{R}^2,$$

to rewrite (5.3.19) as

$$\mathbf{F}(\mathbf{a}, \mathbf{b}) \quad = \quad \int_0^1 \int_0^1 \log|(1-s)\mathbf{a} + t\mathbf{b}|\; P(s,t)\, ds\, dt.$$

We shall discuss three cases where $\mathbf{a}$ and $\mathbf{b}$ are of different values.

1. If $\mathbf{a} = 0$, then

$$\begin{aligned} \mathbf{F}(0, \mathbf{b}) &= \int_0^1 \int_0^1 \log|t\mathbf{b}|\; P(s,t)\, ds\, dt \\ &= \log|\mathbf{b}| \int_0^1 \int_0^1 P(s,t)\, ds\, dt + \int_0^1 \int_0^1 \log t\, P(s,t)\, ds\, dt. \end{aligned}$$

2. If $\mathbf{b}=0$, then

$$\begin{aligned}\mathbf{F}(\mathbf{a},0) &= \int_0^1\int_0^1 \log|(1-s)\mathbf{a}|\; P(s,t)\,ds\,dt \\ &= \log|\mathbf{a}|\int_0^1\int_0^1 P(s,t)\,ds\,dt + \int_0^1\int_0^1 \log(1-s)\,P(s,t)\,ds\,dt.\end{aligned}$$

3. If $\mathbf{a}\neq 0$ and $\mathbf{b}\neq 0$, then

$$\begin{aligned}\mathbf{F}(\mathbf{a},\mathbf{b}) &= \int_0^1\int_0^1 \log|s\mathbf{a}+t\mathbf{b}|\; P(1-s,t)\,ds\,dt \\ &= \mathbf{F}_1(\mathbf{a},\mathbf{b})+\mathbf{F}_2(\mathbf{a},\mathbf{b}),\end{aligned}$$

where

$$\mathbf{F}_1(\mathbf{a},\mathbf{b}) = \int_0^1\int_s^1 \log|s\mathbf{a}+t\mathbf{b}|\; P(1-s,t)\,dt\,ds,$$

$$\mathbf{F}_2(\mathbf{a},\mathbf{b}) = \int_0^1\int_0^s \log|s\mathbf{a}+t\mathbf{b}|\; P(1-s,t)\,dt\,ds.$$

We take $s=tx$, then

$$\begin{aligned}&\mathbf{F}_1(\mathbf{a},\mathbf{b}) \\ &= \int_0^1\int_0^t \log|s\mathbf{a}+t\mathbf{b}|\; P(1-s,t)\,ds\,dt \\ &= \int_0^1\int_0^1 \log|t\,(x\mathbf{a}+\mathbf{b})|\; P(1-tx,t)\,t\,dx\,dt \\ &= \int_0^1\int_0^1 t\,\log t\,P(1-tx,t)\,dx\,dt + \int_0^1\int_0^1 t\,\log|x\mathbf{a}+\mathbf{b}|\; P(1-tx,t)\,dx\,dt,\end{aligned}$$

and

$$\begin{aligned}&\mathbf{F}_2(\mathbf{a},\mathbf{b}) \\ &= \int_0^1\int_0^s \log|s\mathbf{a}+t\mathbf{b}|\; P(1-s,t)\,dt\,ds \\ &= \int_0^1\int_t^1 \log|s\mathbf{a}+t\mathbf{b}|\; P(1-s,t)\,ds\,dt \\ &= \int_0^1\int_s^1 \log|s\mathbf{a}+t\mathbf{b}|\; P(1-t,s)\,dt\,ds \\ &= \int_0^1\int_0^1 t\,\log t\,P(1-t,tx)\,dx\,dt + \int_0^1\int_0^1 t\,\log|x\mathbf{b}+\mathbf{a}|\; P(1-t,tx)\,dx\,dt.\end{aligned}$$

We firstly select $\{P_1, \ldots, P_r\}$ as a basis of the test space of piecewise constant functions, that is, $r = 1$ and $P(s,t) = 1$. We obtain,

$$\log(b) \int_0^1 \int_0^1 P(s,t)\, ds\, dt = \log(b),$$

and

$$\int_0^1 \int_0^1 \log|s-t|\, P(s,t)\, ds\, dt = -\frac{2}{3}.$$

For (5.3.19),

$$\mathbf{F}(0, \mathbf{b}) = \log|\mathbf{b}| - 1,$$

$$\mathbf{F}(\mathbf{a}, 0) = \log|\mathbf{a}| + 1,$$

and

$$\begin{aligned} \mathbf{F}_1(\mathbf{a}, \mathbf{b}) &= \int_0^1 \int_0^1 t \log t\, dx\, dt + \int_0^1 \int_0^1 t \log|x\mathbf{a} + \mathbf{b}|\, dx\, dt \\ &= -\frac{1}{4} + \frac{1}{2} \int_0^1 \log|x\mathbf{a} + \mathbf{b}|\, dx. \end{aligned}$$

Furthermore,

$$\begin{aligned} (xa_1 + b_1)^2 + (xa_2 + b_2)^2 &= |\mathbf{a}|^2 x^2 + 2|\mathbf{a}|\, |\mathbf{b}| \cos\theta\, x + |\mathbf{b}|^2 \\ &= (|\mathbf{a}|\, x + |\mathbf{b}| \cos\theta)^2 + (|\mathbf{b}| \sin\theta)^2 \end{aligned}$$

with

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\, |\mathbf{b}|}.$$

Let $u = |\mathbf{a}|\, x + |\mathbf{b}| \cos\theta$ and $c = |\mathbf{b}| \sin\theta$, we have

$$\begin{aligned} &\int_0^1 \log|x\mathbf{a} + \mathbf{b}|\, dx \\ &= \frac{1}{2} \frac{1}{|\mathbf{a}|} \int_{|\mathbf{b}| \cos\theta}^{|\mathbf{a}|+|\mathbf{b}| \cos\theta} \log\left(u^2 + c^2\right) du \\ &= \frac{1}{2} \frac{1}{|\mathbf{a}|} \left[G(|\mathbf{a}| + |\mathbf{b}| \cos\theta,\ |\mathbf{b}| \sin\theta) - G(|\mathbf{b}| \cos\theta,\ |\mathbf{b}| \sin\theta)\right], \end{aligned}$$

where

$$\begin{aligned} G(u,c) &= \int \log\left(u^2+c^2\right)\, du \\ &= u\,\log\left(u^2+c^2\right) - 2u + 2c\,\arctan\left(\frac{u}{c}\right). \end{aligned}$$

On other hand, we have

$$\begin{aligned} \mathbf{F}_2(\mathbf{a},\mathbf{b}) &= \int_0^1\int_0^1 t\,\log t\, dx\, dt + \int_0^1\int_0^1 t\,\log|x\mathbf{b}+\mathbf{a}|\, dx\, dt \\ &= \mathbf{F}_1(\mathbf{b},\mathbf{a}). \end{aligned}$$

Finally,

$$\begin{aligned} \mathbf{F}(\mathbf{a},\mathbf{b}) &= \mathbf{F}_1(\mathbf{a},\mathbf{b}) + \mathbf{F}_1(\mathbf{b},\mathbf{a}) \\ &= -\frac{1}{2} + \frac{1}{2}\int_0^1 \log|x\mathbf{a}+\mathbf{b}|\, dx + \frac{1}{2}\int_0^1 \log|x\mathbf{b}+\mathbf{a}|\, dx \end{aligned}$$

for $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$.

If the basis $\{P_1,\ldots,P_r\}$ spans the space of piecewise linear functions, that is, $r=2$, and $P_1(s)=1-s$ and $P_2(s)=s$, then

$$P(s,t) = \begin{bmatrix} (1-s)(1-t) & (1-s)t \\ s(1-t) & st \end{bmatrix}.$$

We have, for (5.3.17),

$$\log(b)\int_0^1\int_0^1 P(s,t)\, ds\, dt = \frac{1}{4}\log(b)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

for (5.3.18),

$$\int_0^1\int_0^1 \log|s-t|\, P(s,t)\, ds\, dt = -\frac{1}{16}\begin{bmatrix} 7 & 5 \\ 5 & 7 \end{bmatrix},$$

and for (5.3.19), if $\mathbf{a} = 0$, then

$$\begin{aligned}
&\mathbf{F}(0,\mathbf{b}) \\
&= \frac{1}{4}\log|\mathbf{b}| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \int_0^1\int_0^1 \begin{bmatrix} \log(t)\,(1-s)\,(1-t) & \log(t)\,(1-s)\,t \\ \log(t)\,s\,(1-t) & \log(t)\,st \end{bmatrix} ds\,dt \\
&= \frac{1}{4}\log|\mathbf{b}| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix},
\end{aligned}$$

and if $\mathbf{b} = 0$, then

$$\begin{aligned}
&\mathbf{F}(\mathbf{a},0) \\
&= \frac{1}{4}\log|\mathbf{a}| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&+ \int_0^1\int_0^1 \begin{bmatrix} \log(1-s)\,(1-s)\,(1-t) & \log(1-s)\,(1-s)\,t \\ \log(1-s)\,s\,(1-t) & \log(1-s)\,st \end{bmatrix} ds\,dt \\
&= \frac{1}{4}\log|\mathbf{a}| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \frac{1}{8}\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}.
\end{aligned}$$

For the general case $|\mathbf{a}| \neq 0$ and $|\mathbf{b}| \neq 0$, the piecewise linear functions in the basis of boundary element space $P_i(s)$ times $P_j(t)$, $i,j = 1,2$, is

$$P(1-tx,t) = \begin{bmatrix} (t-t^2)\,x & t^2\,x \\ (1-t)+(t^3-t^2)\,x & t-t^2\,x \end{bmatrix},$$

hence,

$$\begin{aligned}
\mathbf{F}_1(\mathbf{a},\mathbf{b}) &= \int_0^1\int_0^1 t\,\log(t)\,P(1-tx,t)\,dx\,dt \\
&\quad + \int_0^1\int_0^1 t\,\log|x\mathbf{a}+\mathbf{b}|\;P(1-tx,t)\,dx\,dt \\
&= -\frac{1}{288}\begin{bmatrix} 7 & 9 \\ 33 & 27 \end{bmatrix} + \begin{bmatrix} d_{11}(\mathbf{a},\mathbf{b}) & d_{12}(\mathbf{a},\mathbf{b}) \\ d_{21}(\mathbf{a},\mathbf{b}) & d_{22}(\mathbf{a},\mathbf{b}) \end{bmatrix},
\end{aligned} \tag{5.3.20}$$

where

$$\begin{aligned}
d_{11}(\mathbf{a},\mathbf{b}) &= \int_0^1 (t^2-t)\,dt \int_0^1 x\,\log|x\mathbf{a}+\mathbf{b}|\;dx \\
&= \frac{1}{12}\int_0^1 x\,\log|x\mathbf{a}+\mathbf{b}|\;dx,
\end{aligned}$$

$$\begin{aligned}
d_{12}(\mathbf{a},\mathbf{b}) &= \int_0^1 t^3\,dt \int_0^1 x\,\log|x\mathbf{a}+\mathbf{b}|\;dx \\
&= \frac{1}{4}\int_0^1 x\,\log|x\mathbf{a}+\mathbf{b}|\;dx,
\end{aligned}$$

$$\begin{aligned}
d_{21}(\mathbf{a},\mathbf{b}) &= \int_0^1\int_0^1 (t-t^2)\,\log|x\mathbf{a}+\mathbf{b}|\;dx\,dt + \int_0^1\int_0^1 (t^3-t^2)\,x\,\log|x\mathbf{a}+\mathbf{b}|\;dx\,dt \\
&= \frac{1}{6}\int_0^1 \log|x\mathbf{a}+\mathbf{b}|\;dx - \frac{1}{12}\int_0^1 x\,\log|x\mathbf{a}+\mathbf{b}|\;dx,
\end{aligned}$$

and

$$\begin{aligned}
d_{22}(\mathbf{a},\mathbf{b}) &= \int_0^1\int_0^1 t^2\,\log|x\mathbf{a}+\mathbf{b}|\;dx\,dt - \int_0^1\int_0^1 t^3\,x\,\log|x\mathbf{a}+\mathbf{b}|\;dx\,dt \\
&= \frac{1}{3}\int_0^1 \log|x\mathbf{a}+\mathbf{b}|\;dx - \frac{1}{4}\int_0^1 x\log|x\mathbf{a}+\mathbf{b}|\;dx.
\end{aligned}$$

Recall that we let $u = |\mathbf{a}|\;x + |\mathbf{b}|\;\cos\theta$ and $c = |\mathbf{b}|\;\sin\theta$, therefore,

$$\begin{aligned}
&\int_0^1 x\,\log|x\mathbf{a}+\mathbf{b}|\;dx \\
&= \int_0^1 x\,\log\left[(|\mathbf{a}|\;x+|\mathbf{b}|\;\cos\theta)^2 + (|\mathbf{b}|\;\sin\theta)^2\right]\,dx \\
&= \frac{1}{2}\frac{1}{|\mathbf{a}|}\int_{|\mathbf{b}|\;\cos\theta}^{|\mathbf{a}|+|\mathbf{b}|\;\cos\theta} \left(\frac{u}{|\mathbf{a}|} - \frac{|\mathbf{b}|\;\cos\theta}{|\mathbf{a}|}\right)\log\left(u^2+c^2\right)\,du \\
&= \frac{1}{2}\frac{1}{|\mathbf{a}|^2}\left[H\left(|\mathbf{a}|+|\mathbf{b}|\;\cos\theta,\;|\mathbf{b}|\;\sin\theta\right) - H\left(|\mathbf{b}|\;\cos\theta,\;|\mathbf{b}|\;\sin\theta\right)\right]
\end{aligned}$$

$$-\frac{1}{2}\frac{|\mathbf{b}|\,\cos\theta}{|\mathbf{a}|^2}\left[G\left(|\mathbf{a}|+|\mathbf{b}|\,\cos\theta,\ |\mathbf{b}|\,\sin\theta\right)-G\left(|\mathbf{b}|\,\cos\theta,\ |\mathbf{b}|\,\sin\theta\right)\right],$$

where

$$\begin{aligned} H(u,c) &= \int u\,\log\left(u^2+c^2\right)\,du \\ &= \frac{1}{2}\left(u^2+c^2\right)\,\log\left(u^2+c^2\right)-\frac{1}{2}u^2-\frac{1}{2}c^2. \end{aligned}$$

In addition, let $l=|\mathbf{a}|+|\mathbf{b}|\,\cos\theta$ and $m=|\mathbf{b}|\,\cos\theta$, then the elements of the second matrix in (5.3.20) are expressed as following:

$$d_{11}\left(\mathbf{a},\mathbf{b}\right)=\frac{1}{24}\frac{1}{|\mathbf{a}|^2}\left[H(l,c)-H(m,c)\right]-\frac{1}{24}\frac{m}{|\mathbf{a}|^2}\left[G(l,c)-G(m,c)\right],$$

$$d_{12}\left(\mathbf{a},\mathbf{b}\right)=\frac{1}{8}\frac{1}{|\mathbf{a}|^2}\left[H(l,c)-H(m,c)\right]-\frac{1}{8}\frac{m}{|\mathbf{a}|^2}\left[G(l,c)-G(m,c)\right],$$

$$d_{21}\left(\mathbf{a},\mathbf{b}\right)=-\frac{1}{24}\frac{1}{|\mathbf{a}|^2}\left[H(l,c)-H(m,c)\right]+\left(\frac{1}{24}\frac{m}{|\mathbf{a}|^2}+\frac{1}{12}\frac{1}{|\mathbf{a}|}\right)\left[G(l,c)-G(m,c)\right],$$

and

$$d_{22}\left(\mathbf{a},\mathbf{b}\right)=-\frac{1}{8}\frac{1}{|\mathbf{a}|^2}\left[H(l,c)-H(m,c)\right]+\left(\frac{1}{8}\frac{m}{|\mathbf{a}|^2}+\frac{1}{6}\frac{1}{|\mathbf{a}|}\right)\left[G(l,c)-G(m,c)\right].$$

For $\mathbf{F}_2\left(\mathbf{a},\mathbf{b}\right)$, the multiplication of the piecewise linear functions in basis of test space likes

$$P(1-t,tx)=\begin{bmatrix} t-t^2x & t^2x \\ (1-t)+(t^3-t^2)\,x & (t-t^2)\,x \end{bmatrix},$$

we apply the same manner of discussing as $\mathbf{F}_1\left(\mathbf{a},\mathbf{b}\right)$, $\mathbf{F}_2\left(\mathbf{a},\mathbf{b}\right)$ is obtained as

$$\mathbf{F}_2\left(\mathbf{a},\mathbf{b}\right)=-\frac{1}{288}\begin{bmatrix} 27 & 9 \\ 33 & 7 \end{bmatrix}+\begin{bmatrix} d_{22}\left(\mathbf{b},\mathbf{a}\right) & d_{12}\left(\mathbf{b},\mathbf{a}\right) \\ d_{21}\left(\mathbf{b},\mathbf{a}\right) & d_{11}\left(\mathbf{b},\mathbf{a}\right) \end{bmatrix}.$$

In fact, the exact result in (5.3.19) has been obtained analytically for $r = 2$ as follows:

$$\mathbf{F}(\mathbf{a},\mathbf{b}) = -\frac{1}{144}\begin{bmatrix} 17 & 9 \\ 33 & 17 \end{bmatrix} + \begin{bmatrix} d_{11}(\mathbf{a},\mathbf{b}) + d_{22}(\mathbf{b},\mathbf{a}) & d_{12}(\mathbf{a},\mathbf{b}) + d_{12}(\mathbf{b},\mathbf{a}) \\ d_{21}(\mathbf{a},\mathbf{b}) + d_{21}(\mathbf{b},\mathbf{a}) & d_{22}(\mathbf{a},\mathbf{b}) + d_{11}(\mathbf{b},\mathbf{a}) \end{bmatrix}.$$

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