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EFFECT OF A SUBMARINE SHAFT RESONANCE CHANGER IN THE PRESENCE OF FLUID FORCES

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ABSTRACT

The excitation of a submarine hull by the propeller through fluctuating fluid and shaft forces is investigated. The forces are due primarily to the operation of the propeller in a non-uniform wake and occur in the low frequency range. The resulting propeller pressure field can be represented by dipoles which are normal to and along the propeller axis, with the origin at the propeller hub. The hub forces act in the opposite direction to the fluid forces and are modified in transmission to the thrust block. Both fluid and shaft forces excite vibration of the hull. The axisymmetric vibration associated with accordion modes is a powerful source of sound radiation, so this work focuses on the effect of the axial propeller forces.

A simplified axisymmetric model of a submarine hull has been developed using the finite element method to represent the behaviour of the structure and the boundary element method to represent the properties of the fluid domain. The model includes a rigid conical section at the aft end of the pressure hull to represent the free-flood structure that supports the aft propeller shaft bearing. This is connected to a dynamic model of the pressure hull itself. It is shown that the conical tail section plays an important role in hull excitation through the fluid.

A resonance changer can be used to attenuate the vibration transmission through the propeller shaft, but not the excitation via the fluid. In this paper it is shown how the overall performance of the resonance changer is influenced by the fluid forces.

1. INTRODUCTION

The basic idea of a submarine is stealth, which is realised by submerging a marine vessel in order to prevent its detection due to visibility or by radar. However, water is an acoustic medium that can transmit sound for hundreds of kilometres, where the required sound power level is much smaller than in the case of air-borne sound. This property is utilised in sonar systems [1]. A distinction can be made between active and passive sonar. In the case of active sonar, sound is emitted by a speaker, where a target can be detected by its echo. To reduce the risk of detection by active sonar, submarines are usually covered with anechoic cladding [2]. Passive sonar aims to detect the target by its own sound radiation. Several sound sources can be identified for a submarine such as flow noise, propeller noise, machinery and crew noise, noise due to hull and panel vibrations and noise from exhaust systems.

This paper focuses on hull vibrations. They are of major importance as they can cause high levels of radiated noise at hull resonance frequencies [3]. The most important excitation mechanism for hull vibrations is the propulsion system. Excitation occurs due to the fact that the propeller operates in a non-uniform velocity field, leading to axial and radial fluctuating forces on the propeller shaft and dipole sound radiation at the propeller blades [4, 5].

The point dipole approximation can be used, because the wavelength is large relative to the propeller diameter. Also, the propeller itself is small relative to the overall dimensions of the submarine. The net fluctuating force, resulting from integration of pressure fluctutations over the propeller blades, defines the pressure field away from the immediate vicinity of the blades. The pressure field due to a dipole is described by Ross [1]. Figure 1 shows the nature of the pressure field, where the amplitude varies as $\cos\theta$ at a given distance r from the source, where θ is the direction of the observer relative to the dipole direction. The amplitude varies as $1/r^2$ in the near field, but as $1/r$ in the far field. The transition between the near and far field occurs at $\lambda/2\pi$ from the dipole.

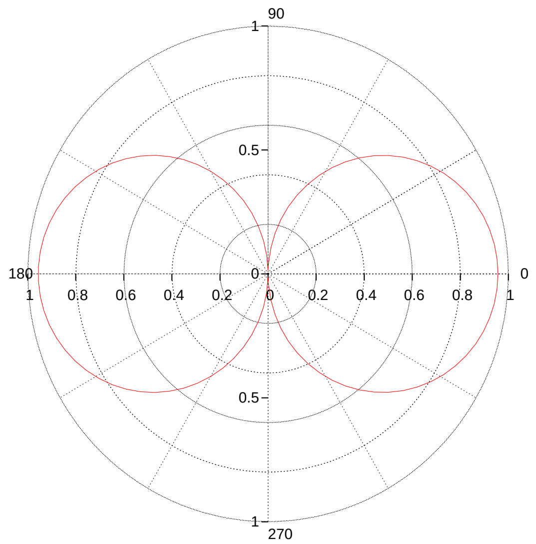

Figure 1: Polar distribution of pressure amplitude at a given distance from the source, relative to the force direction

The principal excitation occurs at the blade-passing frequency and its multiples, where the shaft forces are transmitted through the propulsion system, the thrust bearing and the foundation to the hull. The dipole field results in an acoustic excitation of the hull surface. Acoustic excitation was often ignored in the past. Chertock [6] concluded that it is only about 6–8% of the corresponding stuctural excitation in magnitude, where the Laplace equation was used to model the fluid. However, recent work has shown that the acoustic excitation is much stronger [7, 8] and can be similar to the structural excitation in magnitude. In this case the Helmholtz equation was used instead of the Laplace equation to take into account the finite speed of sound. The propeller blades are assumed to be light and rigid in this analysis, so the hub forces are equal and opposite to the fluid forces.

The structural and acoustic excitations lead to vibrations correlated to the global accordion and bending modes of the submarine hull. The axial shaft forces have been addressed previously by implementing a hydraulic vibration absorber known as a resonance changer [9, 10]. Its parameters can be optimised using different cost functions such as minimisation of the force transmission through the propeller-shafting system, the drive-point hull velocity, and the structure-borne radiated noise. However, acoustic excitation of the hull due to dipole forces has been ignored. Therefore, the excitation of a submarine hull by propeller forces, transmitted to the hull via the combination of the propeller shaft and the external pressure field, is investigated in this paper. The resonance changer has been included in the dynamic model of the propeller-shafting system. Results show that the dipole excitation significantly decreases the efficiency of the resonance changer if its parameters are optimised for structural excitation only.

Numerical methods have been used to solve the problem as it is not straightforward to model the strong structure/fluid interaction between the hull and the water using analytical methods [11]. Furthermore, these methods allow more flexibility regarding the geometry of the submarine. The finite element method (FEM) [12] has been used to model the structure as well as the fluid loading effects for some models, whereas the direct boundary element method (DBEM) [13, 14] has been used to model the external fluid domain and the external acoustical sources.

Application of each of the methods results in a system of equations. These systems can be solved simultanously after coupling the structure/fluid interfaces for the combined problem. For the presented models ANSYS 11.0 and Sysnoise 5.6 were used for the FE and BE modelling, respectively.

2. MODELLING OF THE SUBMARINE HULL

A submarine hull consists of a pressure hull stiffened by additional elements, usually ring stiffeners and bulkheads to withstand the hydrostatic water pressure. Furthermore, the submarine possesses end caps attached to the stern and bow, where the bow side end cap is a hemisphere and the stern side end cap is a truncated cone. Both are free flooded as they are not part of the pressure hull. In this work, the submarine pressure hull was simplified as a thin-walled cylinder with ring-stiffeners, bulkheads and rigid end plates. The stern side end cap was modelled as a rigid cone under the assumption that a cone is stiffer than a cylinder of similar dimensions due to its geometry. Preliminary investigation using a segmented model of the cone and internal water suggested that this is a good approximation in the frequency range of interest. Modelling the cone as flexible would require modelling of the internal water. This was not possible when using the FE/BE approach due to software limitations. The internal water and the ballast tank were considered as a lumped mass at the stern side pressure hull end plate. The necessity for modelling the stern side end cap arises from the acoustic dipole excitation due to the propeller pressure field being located close to it. The front end cap was not modelled geometrically, but its mass was considered as a lumped mass at the bow side end plate. The simplified physical model of the submarine is shown in Figure 2.

The on-board machinery and structure were represented as an added mass attached to the cylindrical shell. The added distributed mass was adjusted to maintain neutral buoyancy of the submarine. The structural hull excitation was applied at the stern side end plate of the pressure hull.

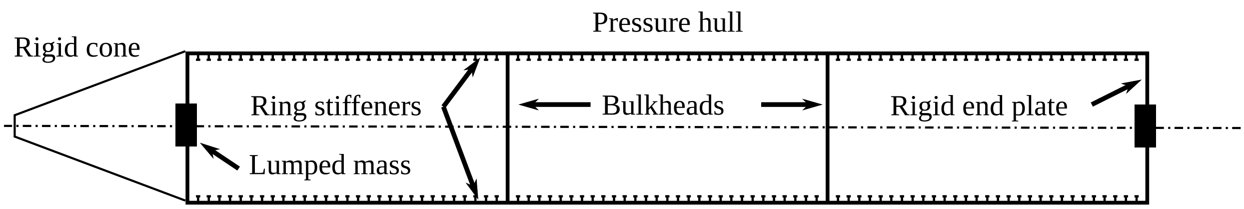

Figure 2: Simplified physical model of the submarine hull

3. MODELLING OF THE PROPELLER/SHAFTING SYSTEM

The propeller/shafting system consists of the propeller, shaft, thrust bearing, resonance changer and foundation as depicted in Figure 3. A low frequency dynamic model of the propeller/shafting system as depicted in Figure 5 is presented in [15], where the propeller was simplified as a lumped mass. The shaft was represented by a rod, where the thrust bearing is attached before the shaft end. This means that a part of the shaft merely acts as another lumped mass. The thrust bearing is represented by a spring-mass-damper system. It is attached to the resonance changer, a hydraulic device that incorporates a cylinder, a pipe and a reservoir as depicted in Figure 4. Under the assumptions made by Goodwin [9], the virtual mass, stiffness and damping parameters for the RC can be defined as:

$$m = \frac{\rho A_0^2 L}{A_1}; \quad c = 8\pi\mu L \frac{A_0^2}{A_1^2}; \quad k = \frac{A_0^2 B}{V}, \tag{1}$$

where ρ is the density, μ is the dynamic viscosity and B is the bulk modulus of the oil in the RC. The last element in the propeller/shafting system is the foundation that connects the system to the pressure hull. It should be noted that the foundation is not an axisymmetric structure, however it is shell-like and stiff in comparison to the other elements of the propeller/shafting system. As an approximation, it was modelled as a tapered cylindrical shell.

4. APPLICATION OF NUMERICAL METHODS

To investigate the sensitivity of the system to excitation parameters, a harmonic response analysis was conducted. A fully coupled system was modelled in order to include the fluid loading effects. The software packages Sysnoise and ANSYS were used, where Sysnoise focuses on acoustics and utilises the finite element method as well as the boundary element method. ANSYS is a general purpose finite element software. Both software packages provide elements for axisymmetric analysis when modelling thin-walled structures.

Each package could be applied to solve this problem solely, but with some restrictions. In the case of Sysnoise, the finite element (FE) method is used for the structural part of the model and the direct boundary element method (DBEM) is used to model the fluid domain. For ANSYS, both the structure and the fluid are modelled using finite elements. Desired capabilities/features of Sysnoise are higher acoustical accuracy, external acoustic sources such as dipoles and computation of far-field pressure. Features that are available in ANSYS are constraint equations and

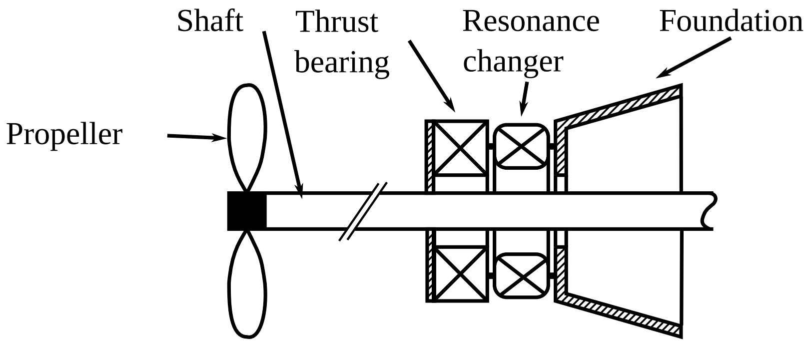

Figure 3: Propeller/shafting system

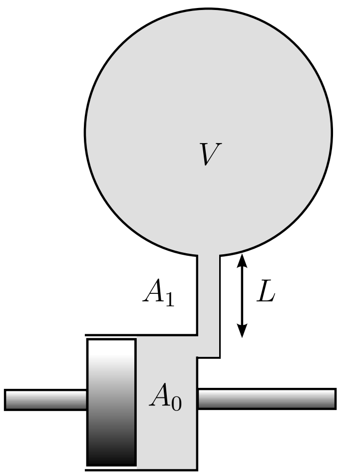

Figure 4: Resonance changer

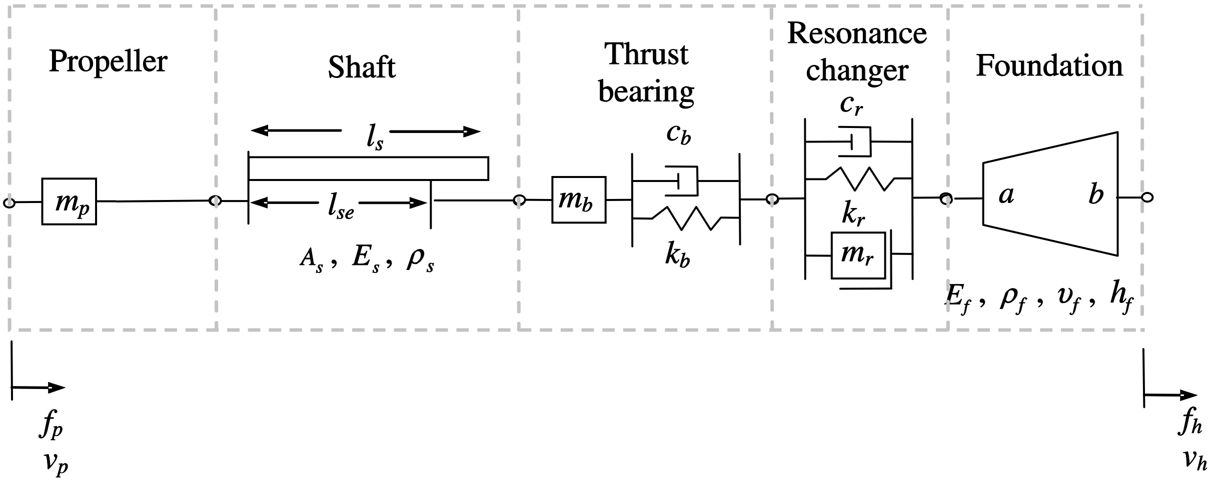

Figure 5: Simplified model of the propeller/shafting system [10]

parametrical modelling

For a fully-coupled approach, all the capabilities are required simultaneously. As this is not possible, a semi-coupled model was used. The analysis was undertaken in three steps: (i) A fully coupled analysis for the entire model in ANSYS but without the dipole excitation was initially conducted, to obtain the force transmissibility of the propeller/shafting system. (ii) A modal basis of the submarine hull was obtained for later use in Sysnoise. (iii) A coupled analysis including dipole excitation in Sysnoise was conducted, using the modal basis for the submarine hull from (ii) and the force transmissibility data for the propeller/shafting system obtained from ANSYS (step (i)). It must be noted that an error will be introduced as the force transmissibility does not consider the acoustic dipole excitation and therefore the drive point impedance of the submarine hull will differ slightly.

For the coupled FE model, the following equation has to be solved

$$\left(\begin{bmatrix}\mathbf{K}_\mathrm{s} & -\mathbf{R}\\ \mathbf{0} & \mathbf{K}_\mathrm{f}\end{bmatrix} + j\begin{bmatrix}\mathbf{C}_\mathrm{s} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_\mathrm{f}\end{bmatrix} - \omega^2\begin{bmatrix}\mathbf{M}_\mathrm{s} & \mathbf{0}\\ \rho\mathbf{R}^\mathrm{T} & \mathbf{M}_\mathrm{f}\end{bmatrix}\right)\begin{Bmatrix}\mathbf{u}\\ \mathbf{p}\end{Bmatrix} = \begin{Bmatrix}\mathbf{f}_\mathrm{s}\\ \mathbf{0}\end{Bmatrix}, \tag{2}$$

where $\mathbf{K}$, $\mathbf{C}$ and $\mathbf{M}$ denote the stiffness, damping and mass matrices, respectively. The subscript 's' indicates the structural part of the model and similarly, 'f' denotes the fluid part of the model, ρ is the density of the fluid and $\mathbf{R}$ is the fluid/structure coupling matrix. Solution of the system of equations for the angular frequency ω will yield the amplitude $\mathbf{u}$ for the nodal displacement degrees of freedom and the amplitude $\mathbf{p}$ for the nodal pressure degrees of freedom. $\mathbf{f}_\mathrm{s}$ is the structural load vector, containing merely the value f_1 at the node for the propeller mass. The derived data allows computation of the force f_2 at the submarine end plate. The force transmission for the propeller propulsion system is then given by

$$\Pi = \frac{f_2}{f_1} \tag{3}$$

The modal basis was obtained by a modal extraction of the undamped structure of the submarine hull. The following equation has to be fulfilled:

$$\left(\mathbf{K}_\mathbf{s} - \omega_i^2\mathbf{M}_\mathbf{s}\right)\boldsymbol{\phi}_{\boldsymbol{i}} = \mathbf{0}, \tag{4}$$

where $\boldsymbol{\phi}_{\boldsymbol{i}}$ is a vector representing the nodal displacements for the ith mode. The matrices $\mathbf{K}_\mathbf{s}$ and $\mathbf{M}_\mathbf{s}$ now do not include the propeller/shafting system. The eigenvalues ω_i^2 can be found by setting the characteristic equation of the system to zero:

$$\left|\mathbf{K_s} - \omega_i^2 \mathbf{M_s}\right| = 0. \tag{5}$$

Using equation (4), the mode shapes correlated to the eigenvalues can be found. It should be noted that for the harmonic response analysis, all mode shapes were considered up to a frequency 100% higher than the highest frequency of interest. The nodal displacements are then expressed as a superposition of the modes [16]:

$$\mathbf{u} = \sum_{i=1}^{q} \boldsymbol{\phi}_i d_i \,, \tag{6}$$

where q is the highest mode considered. The modal displacements d_i were obtained by solving the following coupled system of equations

$$\begin{bmatrix} \mathbf{K}_\mathrm{s}^* - \omega^2 \mathbf{I} & \boldsymbol{\Phi}^T \mathbf{D} \\ \mathbf{G}\mathbf{E}\boldsymbol{\Phi} & \mathbf{H} \end{bmatrix} \begin{Bmatrix} \mathbf{d} \\ \mathbf{p} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\Phi}^T \mathbf{f_s} \\ \mathbf{p}_\text{inc} \end{Bmatrix}, \tag{7}$$

where $\mathbf{K}_\mathrm{s}^*$ is the modal stiffness matrix, $\mathbf{I}$ is the unity matrix, $\mathbf{D}$ and $\mathbf{E}$ are fluid/structure coupling matrices and the matrices $\mathbf{G}$ and $\mathbf{H}$ are the DBEM influence matrices. The modal stiffness matrix is a diagonal matrix, where the elements i are given by $\omega_i^2(1 + j\eta_i)$ and η_i is the modal damping for the mode i. $\boldsymbol{\Phi}$ is a matrix containing the considered modeshapes $\boldsymbol{\phi_i}$. The structural load vector $\mathbf{f_s}$ contains only one value f_2 for the excitation of the stern side end plate, where the value is given by Πf_1. The acoustical load vector $\mathbf{p}_\text{inc}$ contains the nodal pressure values of the incident field, which is a dipole. The dipole can be described analytically as [7]

$$p_\text{inc}(r, \theta) = \frac{j\omega f_1}{4\pi r c}\, e^{j(\omega t - kr)} \left(1 - \frac{j}{kr}\right) \cos\theta, \tag{8}$$

where r is the distance to the node, θ is the angle of the node with respect to the axis of the fluctuating force f_1 correlated to the dipole, c is the speed of sound and k is the wave number. Equation (7) will also yield the vector $\mathbf{p}$ of total nodal pressures at the submarine surface.

5. RESULTS

Results were obtained for three different models. The two main models are shown in Figure 2 with a rigid cone at the stern, where Model 1 has a longer cone and Model 2 has a shorter cone. In this way the influence of the cone length on the excitation of the hull due to the dipole was observed. The model with the longer cone is partially depicted in Figure 6. In Model 3, the cone is absent. This represents the case where the cone is acoustically transparent. The lumped mass at the stern side end cap was adjusted to include the entire mass of the cone end and lumped mass. For the shell-structure, axisymmetric subparametric shell elements were used [17], where the shape functions for the displacements are cubic and the shape functions for the geometry are linear. The elements have radial, axial and rotational degrees of freedom. For the FE/FE models, the fluid was represented by isoparametric linear Helmholtz elements. The fluid domain was bounded by infinite elements to satisfy the Sommerfeldt radiation condition. In case of the FE/BE models, the fluid was represented by linear boundary elements using Galerkin collocation where the nodes coincide with the structural nodes. For the structure/fluid coupling, the displacement due to the rotational degree of freedom was neglected.

Model data is given in Table 1. For all models, the mobility of the stern side end plate of the pressure hull in the axial direction was the measure of hull response. This was chosen because

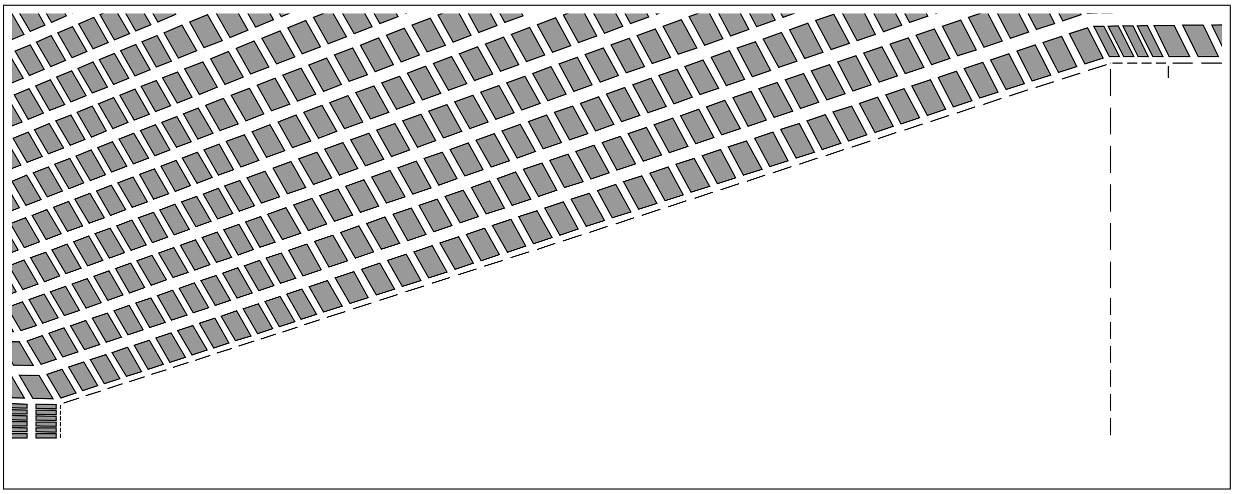

Figure 6: Discretisation of the structure and the fluid at the stern side end cap for the FE/FE model

Parameter	Value	Unit	Parameter	Value	Unit
Cylinder length	45.0	m	Stern lumped mass 1	188×10^3	kg
Cylinder radius	3.25	m	Stern lumped mass 2	191×10^3	kg
Shell thickness	0.04	m	Bow lumped mass	200×10^3	kg
Stiffener cross-sectional area	0.012	m^2	Cone half angle 1	18	deg
Stiffener spacing	0.5	m	Cone half angle 2	24	deg
Young's modulus of structure	210	GPa	Cone length 1	9.079	m
Poisson ratio of structure	0.3		Cone length 2	6.626	m
Density of structure	$7{,}800$	kg/m^3	Cone smaller radius	0.3	m
Structural loss factor	0.02		Density of fluid	$1{,}000$	kg/m^3
Added mass	678	kg/m^2	Speed of sound	$1{,}500$	m/s

Table 1: Model data for hull

axial motion of the ends can result in significant sound radiation, as in motion of a baffled piston.

The dipole was placed half way behind the small cone end plate and the apex of the cone for Models 1 and 2. For both models, the propeller/shafting system was implemented. A comparison was made between a system with and without a resonance changer and without the dipole excitation. A third result was obtained for the system with the RC, but additionally with the dipole excitation that corresponds to the fluctuating propeller force. The RC parameters were taken from [15], where they were optimised for a structure that has a similar drive point impedance to the submarine hull. The cost function was the minimisation of the maximum value of the weighted force transmissibility. Model data for the propeller/shafting system is given in Table 2.

To confirm the results for the propeller/shafting system, the force transmissibility was also obtained using an analytical model [10], but with a rigid termination. The results are shown in Figure 7. It can be seen that the numerical model yields almost the same result as the analytical model, except for slight differences in the phase at the hull resonance frequencies.

The mobility for Model 1 is shown in Figure 8. The first curve shows the results if there is no resonance changer implemented in the propeller/shafting system. Four major peaks can be identified. The first peak at about 20 Hz is the fundamental hull resonance frequency cor-

Parameter	Value	Unit	Parameter	Value	Unit
Propeller mass	$10{,}000$	kg	Resonance changer mass	$1{,}000$	kg
Shaft Young's modulus	200	GPa	Resonance changer stiffness	169	$^{\mathrm{MN}}/_{\mathrm{m}}$
Shaft Poisson's ratio	0.3		Resonance changer damping	287×10^3	$^{\mathrm{kg}}/_{\mathrm{s}}$
Shaft density	$7{,}800$	$^{\mathrm{kg}}/_{\mathrm{m}^3}$	Foundation major radius	1.25	m
Shaft cross-sect. area	0.071	m^2	Foundation minor radius	1.25	m
Shaft length	10.5	m	Foundation half angle	15	deg
Effective shaft length	9	m	Foundation thickness	10	mm
Bearing mass	200	kg	Foundation Young's modulus	200	GPa
Bearing stiffness	$20{,}000$	$^{\mathrm{MN}}/_{\mathrm{m}}$	Foundation density	$7{,}800$	$^{\mathrm{kg}}/_{\mathrm{m}^3}$
Bearing damping	$300{,}000$	$^{\mathrm{kg}}/_{\mathrm{s}}$			

Table 2: Model data for propeller/shafting system

responding to the accordion mode of a thin-walled cylinder. The second peak occurs at about 45 Hz and represents the second hull frequency. The third peak is the fundamental frequency of the propeller/propulsion system. This also results in a $180°$ phase shift. The peak indicating the third hull resonance frequency at about 75 Hz is strongly damped due to radiation damping of the surrounding fluid as the radiation efficiency increases with frequency. Additionally two minor peaks occur at about 8 Hz and 37 Hz, representing the natural frequencies of the bulkheads.

Using a resonance changer, the fundamental frequency of the propeller/shafting system is decreased to about 15 Hz. Due to the phase shift above this frequency, the excitation of the submarine hull is increased sigificantly by the dipole excitation above 40 Hz. If a shorter cone is used, the excitation of the hull increases slightly, as shown in Figure 9.

Figure 10 shows the mobility of the hull without the propeller/propulsion system. A force of unity strength is applied at the stern side end plate or a dipole is applied behind the stern side end plate. As the dipole is at the same locations as for Models 1 and 2, it is clear that the cone end plays an important role for the acoustic excitation of the submarine hull. The excitation by the dipole is now less significant than in case of the models that include the cone.

6. CONCLUSIONS

A model of an axially excited submarine has been developed, where the excitation occurs due to fluctuating propeller forces. A resonance changer was implemented to attenuate the forces that are transmitted to the submarine hull through the propeller/shafting system, where the parameters for the RC were obtained by neglecting forces that are transmitted to the hull via the fluid. It was shown that these forces negatively influence the efficiency of the RC at frequencies higher than 40 Hz. This means that the acoustic excitation of the submarine hull has to be taken into account in order to obtain better parameters for the RC. Furthermore it was shown that the cone end has a significant influence on the magnitude of the excitation of the submarine hull through the fluid forces.

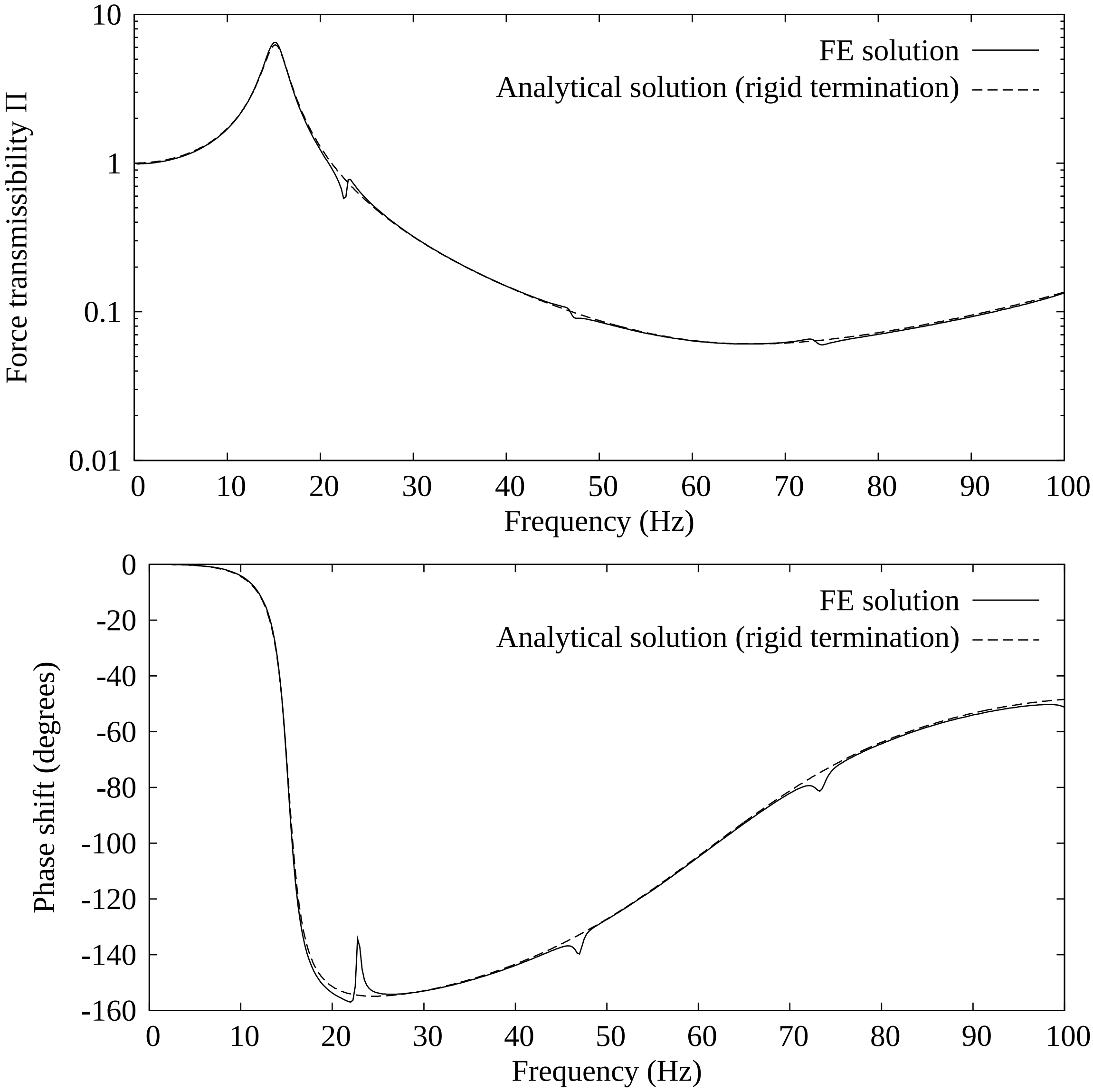

Figure 7: Force transmissibility of propeller/shafting system with a resonance changer

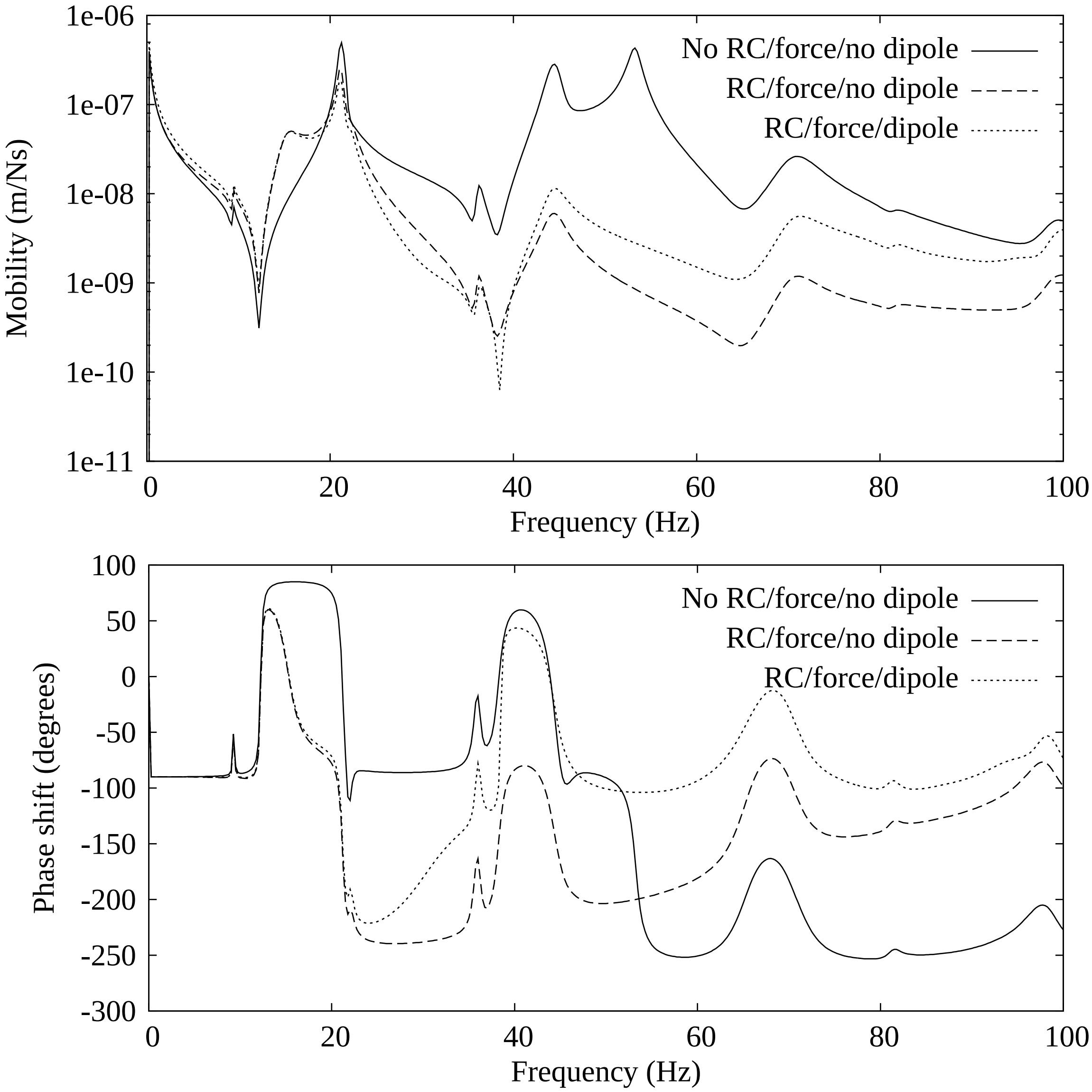

Figure 8: Mobility for Model 1 (long cone).

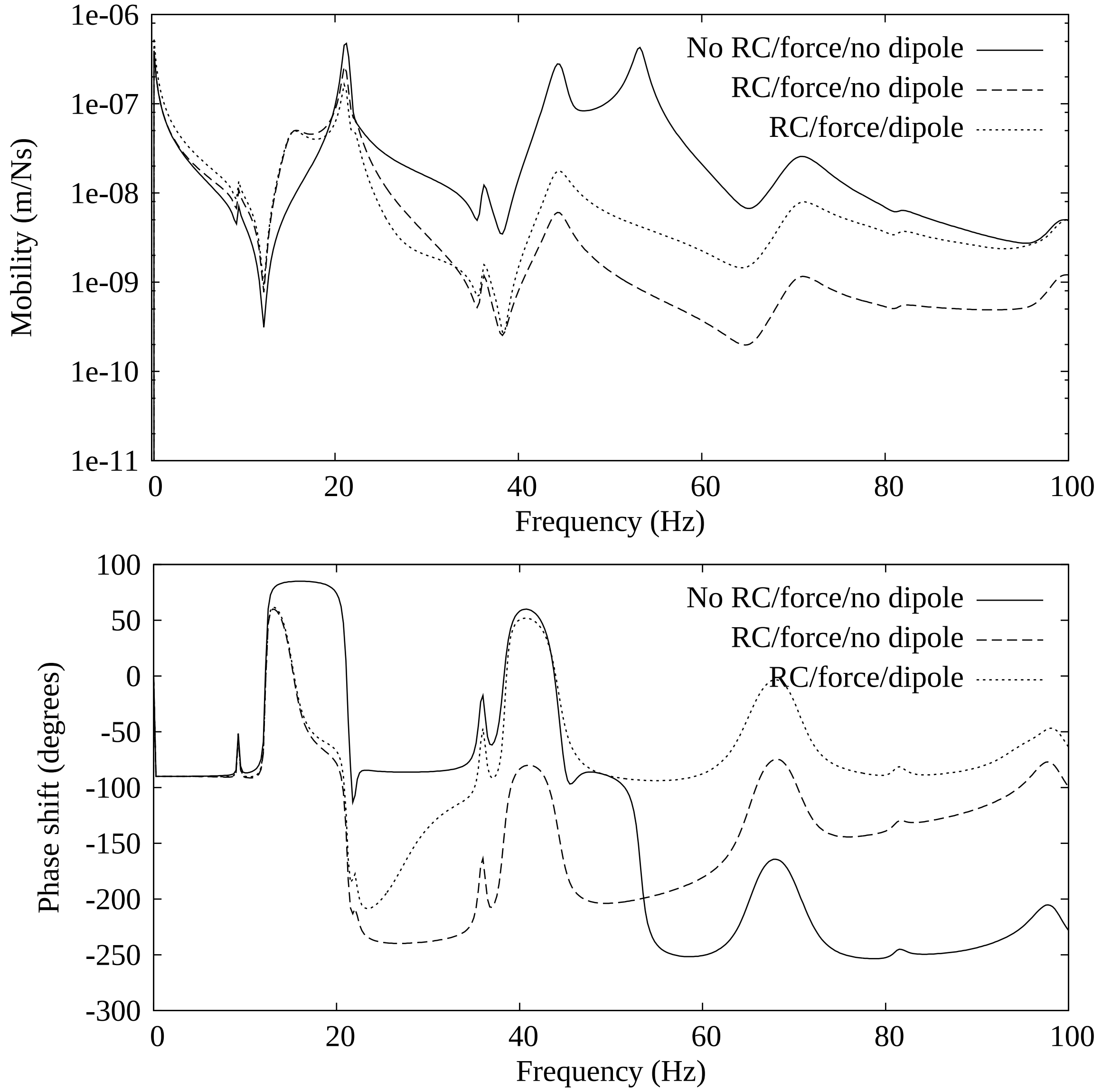

Figure 9: Mobility for Model 2 (short cone).

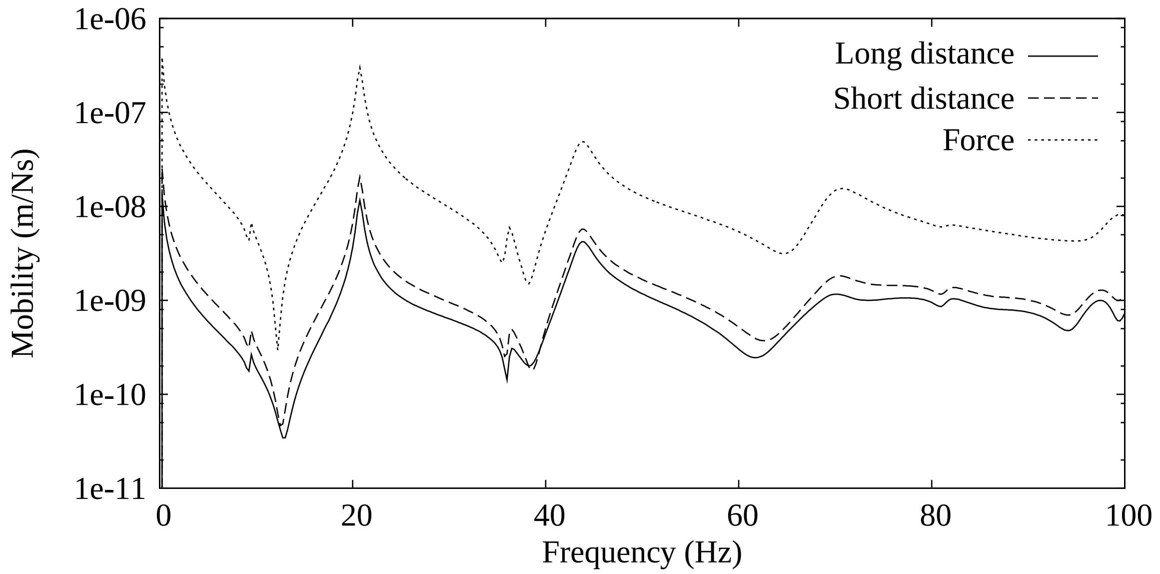

Figure 10: Mobility for a model without a resonance changer and a cone. Shown is the excitation for either a force or a dipole, where the distance of the dipole from the stern end plate is the same as for the models with the long and short cones.

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